On intersection theory, Severi-Brauer varieties, and the intersection theory of Severi-Brauer varieties

by

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Abstract

This thesis investigates the Chow ring, and neighboring functors, of a Severi-Brauer variety. The approach taken here heavily depends on the computation of lower K-groups of a Severi-Brauer variety.

We construct a functor (for an arbitrary scheme essentially of finite type over a field) that is a universal target for additive Chern classes and we compare this functor to the associated graded for the gamma filtration on the Grothendieck group of locally free sheaves via a Grothendieck-Riemann-Roch type theorem. When the Chow ring is generated by Chern classes our theorem reduces to the standard Grothendieck-Riemann-Roch.

Following this we show that, for some Severi-Brauer varieties including the generic ones, the Chow ring is isomorphic with the associated graded of the gamma filtration on the Grothendieck ring. The theorem more generally involves Severi-Brauer varieties whose Chow rings are generated by Chern classes and whose associated algebra has index and exponent that differ very minimally (in the language of this section, for algebras of level 1). This prompts us to investigate the gamma filtration in its own right. We prove some results about the gamma filtration for a Severi-Brauer variety including results showing the gamma filtration depends only on primary division algebra factors of the central simple algebra of the Severi-Brauer variety.

Lastly, we continue work on the picture for the diagonal K-cohomology groups which can be considered in degree one higher than the Chow ring. By assuming the vanishing of reduced Whitehead groups for certain algebras with equal index and exponent, we provide a complete description of the coniveau filtration on the first K-group in some cases.

Preface

A word about the contents of the following thesis: most of what follows has been accepted for publication, and the rest has been submitted.

Chapter 3 is joint work with Nikita Karpenko and will appear in the Annals of K-theory published by Mathematical Sciences Publishers ©[2018]. In this chapter, the appendices (included in the print version) have been changed to sections, some equations have been "trimmed" to fit the format here, and some exposition has been changed for this thesis.

Chapter 5 has also been accepted for publication. This chapter was first published in the *Canadian Mathematical Bulletin* at https://doi.org/10.4153/S0008439518000073. ©[2018] Canadian Mathematical Society in partnership with Cambridge University Press. This article will appear in an edited form in press at a later date and the version here should not be redistributed by the end-user.

Chapters 2 and 4 have only been submitted.

The mathematical content of these works is mostly unchanged between the accepted for publication (or submitted) versions and the versions appearing here. However, small typos have been corrected and occasionally the formatting has been changed.

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Table of Contents

Chapter 1

Introduction

As challenging as it is to study the geometry of solutions to algebraic equations themselves, it is nearly as equally challenging to study the relations between such solutions. Algebraic geometry, however, is often solely dependent on such relations: to classify embeddings of a variety X into projective space it is equivalent to work with codimension one subvarieties up to an equivalence where such subvarieties can be moved inside of X; whether or not a surface is a minimal example in a family of blow-ups is equivalent to whether or not the surface contains a line with -1 self intersection. Both of these concepts are the first traces of a more interesting invariant one can assign to a variety in algebraic geometry called the Chow ring.

Definition 1.0.1. For any integer $k \geq 0$, the Chow group of k-dimensional cycles of a variety X is defined to be the quotient

$$\mathrm{CH}_k(X) = \mathrm{Z}_k(X)/\mathrm{R}_k(X)$$

where $\mathrm{Z}_k(X) = \bigoplus_{V \subset X} \mathbb{Z} \cdot V$ is the free abelian group generated by integral subvarieties V of X with $\dim(V) = k$, and $\mathrm{R}_k(X)$ is the subgroup of $\mathrm{Z}_k(X)$ generated by the (nonzero) divisors of rational functions f of function fields

$k(W)^\times$ of integral subvarieties W of X of dimension $k+1$.

The Chow ring is defined to be the sum of these groups,

$$\mathrm{CH}(X) = \bigoplus_{k \geq 0} \mathrm{CH}_k(X)$$

and it comes equipped with an intersection product when X is, for example, smooth. The name is motivated by the fact that the multiplication can, in nice situations, be defined as taking the product of equivalence classes of two subvarieties to the equivalence class of their intersection, i.e. $[V]\cdot[W] = [V \cap W]$.

The study of Chow rings is deeply interconnected with the study of algebraic geometry itself, as the examples above illuminate. The wealth of information one can gain by understanding in detail the Chow ring of a given variety is often too numerous to state, and this richness of information is typically directly related with the difficulty level of studying these objects. Some of the earliest examples of Chow rings that could be worked out in complete detail were, then, some of the most structured examples as well, e.g. for algebraic groups and their homogeneous spaces.

For split semisimple algebraic groups, the Chow groups of their projective homogeneous varieties are free groups generated by the Schubert varieties of their Bruhat decomposition, [Che94, Dem74]; see also [K̈91]. The Chow rings for these varieties are often more difficult to compute and, even when a description of this ring is known it can be difficult, at least in practice, to relate the structure of the Chow groups to the description of their Chow rings.

Although the picture in the case of a split semisimple algebraic group is incomplete, one can still ask if there's anything that can be said about the Chow groups/rings for a homogeneous variety under an arbitrary semisimple algebraic group. This problem is considerably harder and almost nothing is known in general about the structure of either the Chow groups or the Chow rings for these varieties. For example, since for projective homogeneous varieties under a nonsplit group there is no Bruhat decomposition, the Chow groups don't need to be free. Sometimes they do contain torsion, [Kar98], and sometimes this torsion isn't even finitely generated, [KM90].

This thesis takes a (small) step towards answering the question "to what extent can one describe the Chow ring or Chow groups of a projective homogeneous variety under a nonsplit semisimple algebraic group?" with a particular emphasis on the simplest class of these varieties: the Severi-Brauer varieties.

Definition 1.0.2. A Severi-Brauer variety X over a field k is a scheme that admits an isomorphism $X_{\overline{k}} \cong \mathbb{P}^n_{\overline{k}}$ with projective space of dimension $n \geq 0$ over an algebraic closure $\overline{k}$.

It's not a new question and the results themselves are not particularly unique. The entire thesis can be considered a generalization of techniques, theorems, and theories from places where the ideas were already known.

Before describing what is new here, I want to give some context on one approach that can be used to answer the question posed in the previous paragraph: the use of the Brown-Gersten-Quillen spectral sequence (henceforth called the BGQss, and whose pth row, qth column, on the rth page will be denoted $E_r^{p,q}$). This spectral sequence has played a critical role in a large number of computations involving the Chow group of a projective homogeneous variety for a nonsplit group and one could attribute a large part of the success that current programs have had in extending these results to the computability of the K-theory of these varieties, [Qui73, Pan94, LSW89].

The key observations are the following. The second page of the BGQss is a nice approximation to motivic cohomology, especially when the coordinates add to $0, -1, -2$ (e.g. $E_2^{p,-p}$ is isomorphic with the Chow group of codimension-

p cycles, and similarly $E_2^{p,-p-1}$ and $E_2^{p,-p-2}$ are isomorphic with other motivic cohomology groups). The converging terms $E_\infty^{*,*}$ are explicitly describable as graded pieces of a coniveau filtration and, for the $E_\infty^{p,-p}$ terms, this filtration is approximated by the even more computable γ-filtration (for definitions of both filtrations in the relevant degrees, see Section 4.3). Finally, the edge map $E_2^{p,-p} \to E_\infty^{p,-p}$ can be identified with the canonical surjection of the Grothendieck-Riemann-Roch without denominators (GRRwod):

Theorem 1.0.3 ([Ful98, Example 15.3.6])**.** *Let X be a smooth and connected variety. We write $\mathrm{gr}_\tau G(X)$ for the associated graded ring to the coniveau filtration on the Grothendieck ring of coherent sheaves $G_0(X)$. In the notation above, the degree p summand $\mathrm{gr}_\tau^p G(X)$ of $\mathrm{gr}_\tau G(X)$ is also the limiting term $E_\infty^{p,-p}$ of the BGQss for X. In this notation, there are canonical morphisms*

$$\varphi^p : \mathrm{CH}^p(X) \to \mathrm{gr}_\tau^p G(X) \quad \textit{and} \quad c_p : \mathrm{gr}_\tau^p G(X) \to \mathrm{CH}^p(X)$$

where φ^p takes the class of a integral subvariety $V \subset X$ to the class $[\mathcal{O}_V]$ and c_p is induced by the pth Chern class.

Moreover, the morphism φ^p is surjective for all $p \geq 0$ and the compositions

$$c_p \circ \varphi^p = (-1)^{p-1}(p-1)! \quad \textit{and} \quad \varphi^p \circ c_p = (-1)^{p-1}(p-1)!$$

are both multiplication by $(-1)^{p-1}(p-1)!$.

This means that, up to knowing the K-theory of a given variety and solving an extension problem, the BGQss can compute the Chow groups by analyzing the converging terms and a torsion subgroup of the Chow ring.

Since the K-theory of a Severi-Brauer variety has been computed, [Qui73], the BGQss effectively reduces the problem of computing the Chow groups to computing the associated graded ring for the coniveau filtration and computing the kernel of the canonical surjection of the GRRwod. And, still for Severi-Brauer varieties, both of these latter problems have been studied to a large degree. In the following paragraphs, we outline a program one can take towards solving these problems in general. At appropriate times we'll single out some papers that have impacted the work contained in this thesis and specify to what extent we've managed to solve these problems.

Severi-Brauer varieties are closely related to central simple algebras.

Definition 1.0.4. A central simple k-algebra A is a unital and associative, but not necessarily commutative, k-algebra that is finite dimensional as a k-vector space, is central over the k-subfield generated by its unit, and which has no nontrivial two-sided ideals.

These are objects that behave very unpredictably but closely depend on only a few invariants called the index, the degree, and the exponent. Each of these is a positive integer: the degree is the square-root of the dimension of the algebra, the index is the dimension of the largest subfield contained in a central division algebra contained in the central simple algebra, and the exponent is the smallest nonnegative integer such that taking the tensor power of the given algebra to the power of the exponent yields an algebra isomorphic to a matrix ring (nontrivially the exponent is always finite). Any central simple algebra A can be factored into a tensor product of a matrix ring and smaller division algebras each having degree a power of a prime dividing the index of A. The first step when studying the (Chow groups, K-theory, BGQss of the) Severi-Brauer variety of A, is a reduction (using [Kar00] and [Kar17a]) to the case A is a division algebra of prime power degree.

The problem of determining the Chow groups of a Severi-Brauer variety corresponding to a division algebra of prime power degree is much more subtle.

For these algebras one typically doesn't know whether or not the algebra will decompose as a tensor product of smaller algebras. The two extreme cases where one might expect to be able to say something would then be when this division algebra is indecomposable and has largest possible exponent, and when this division algebra is totally decomposable and has smallest possible exponent. In the indecomposable highest exponent case, the coniveau filtration on the Grothendieck group G_0 was computed by Karpenko, [Kar95b]. In the totally decomposable lowest exponent case, the coniveau filtration on G_0 was computed, again by Karpenko, under the assumptions that the degree of this division algebra is p^2 for a prime p and only two division algebras appear in the product, [Kar96]. In other cases very little is known.

Unfortunately, even when one knows the coniveau filtration on G_0 (and hence the converging terms $E_\infty^{p,-p}$ of the BGQss) as in the cases above, one can say very little about the kernel of the canonical surjection from the GRRwod, and hence one can't say anything complete regarding the Chow groups. In some highly decomposable cases, the kernel turns out to be nontrivial, [Mer95]. In other highly indecomposable examples, the kernel turns out to be trivial, [Kar17b]. Decomposability might be a red herring in these examples, however, since the former examples depend on arithmetic information contained in the Galois cohomology of the base field, while the latter examples depend on the lack of this information, in some sense.

More precisely, [Kar17b] shows the kernel of the GRRwod surjection vanishes for so called generic Severi-Brauer varieties. In this case, the coniveau filtration agrees with the more computable γ-filtration and it's through this equality that one can show the kernel is trivial. The equality between the γ-filtration and the coniveau filtration depends only on the associated graded ring for the coniveau filtration being generated by Chern classes and, when this happens, this ring depends only on the degree of the involved central simple algebra and the indices of its tensor powers.

The main results of this thesis could be considered the results of Chapter 3. This chapter is joint work with Nikita Karpenko. In it, we relate two conjectures around the triviality of the kernel of the canonical epimorphism of the GRRwod for some classes of projective homogeneous varieties. Specifically, we prove equivalence of the following two statements, both of which are stated as individual conjectures in Chapter 3:

Theorem 1.0.5 (See Theorem 3.3.3)**.** *The following two statements are equivalent:*

1. *the canonical surjection* $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$ *is an isomorphism for all varieties* X *that are isomorphic to a product of Severi-Brauer varieties and* $\mathrm{CH}(X)$ *is generated by Chern classes,*

2. *the canonical surjection* $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau G(E/P)$ *is an isomorphism for all varieties* E/P *where* E *is a versal* G*-torsor for a split semisimple algebraic group* G *satisfying the property that the Dynkin diagram of* G *is a union of diagrams of type* A *and/or type* C*, and* P *is a special parabolic subgroup of* G*.*

In the latter half of this chapter, Sections 3.4 and 3.5, I extend the results of [Kar17b] to a slightly larger class of Severi-Brauer varieties. This (reproves and) generalizes the results from [Kar17b] and proves a subcase of the two conjectures above; the techniques are exactly the same as before but the computations required are considerably more involved. The class of Severi-Brauer varieties I work with in this chapter are those associated to central simple algebras of level one. Formally, I prove:

Theorem 1.0.6 (See Theorem 3.4.15). *Suppose A is a central simple algebra which can be realized as a matrix ring over a division algebra D. Suppose $D = \bigotimes_{p\ prime} D_p$ a factorization into p-primary division algebras and, for each D_p there is at most one integer $r \geq 1$ such that there is an inequality of p-adic valuations*

$$v_p \mathrm{ind}(D_p^{\otimes p^r}) < v_p \mathrm{ind}(D_p^{\otimes p^{r-1}}) - 1.$$

Then, if X is the Severi-Brauer variety of A and $\mathrm{CH}(X)$ is generated by Chern classes, the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$ is an isomorphism.

The assumptions seem, at first, to be overly complicated. However, examples of such Severi-Brauer varieties naturally arise by base change of generic Severi-Brauer varieties to some function fields.

Chapters 2 and 4 stem from Chapter 3 in different ways. Chapter 2 is largely disjoint from the overall program related to Severi-Brauer varieties described here. The contents of this chapter investigate a universal theory for Chern classes and relate this theory to the associated graded ring for the γ-filtration on K_0, the Grothendieck ring of locally free sheaves. It can be considered an abstraction of the techniques used to study Chern classes that appear in the proofs of Chapter 3. Chapter 2 also provides a new GRRwod theorem that reduces to the usual one under the assumption the Chow ring is generated by Chern classes.

Theorem 1.0.7 (See Theorem 2.5.1). *Let B be the universal theory for additive Chern classes in the sense of Proposition 2.2.2. Let X be a variety in the sense of Chapter 2. We write $\mathrm{gr}_\gamma K(X)$ for the associated graded ring of the γ-filtration on the Grothendieck ring $K(X)$. We write $B^i(X)$ for the degree i summand of $B(X)$ and we write $\mathrm{gr}_\gamma^i K(X)$ for the degree i summand of*

$\mathrm{gr}^i_\gamma K(X)$. *In this notation, there are canonical morphisms*

$$b^i_\gamma : B^i(X) \to \mathrm{gr}^i_\gamma K(X) \quad \textit{and} \quad c^B_i : \mathrm{gr}^i_\gamma K(X) \to B^i(X)$$

where b^i_γ *takes universal Chern classes to Chern classes in* $\mathrm{gr}^i_\gamma K(X)$ *and* c^B_i *is induced by the* i*th Chern class.*

Moreover, the morphism b^i_γ *is surjective for all* $i \geq 0$ *and the compositions*

$$c^B_i \circ b^i_\gamma = (-1)^{i-1}(i-1)! \quad \textit{and} \quad b^i_\gamma \circ c^B_i = (-1)^{i-1}(i-1)!$$

are both multiplication by $(-1)^{i-1}(i-1)!$.

Chapter 4 works directly with the γ-filtration for an arbitrary Severi-Brauer variety. Here we show how to extend the results on the coniveau filtration, that allowed us to reduce to the case of a division algebra of prime power degree which depend on motivic information and hence are not accessible to the γ-filtration, to the γ-filtration. As a result, we get explicit computations of the Chow groups of generic Severi-Brauer varieties in high codimension and find they are torsion-free.

Theorem 1.0.8 (See Theorem 4.6.1)**.** *Suppose* X *is a Severi-Brauer variety such that* $\mathrm{CH}(X)$ *is generated by Chern classes and the canonical surjection* $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$ *is an isomorphism. Then* $\mathrm{CH}_i(X)$ *is torsion free for* $0 \leq i \leq p-2$ *where* p *is the smallest prime integer dividing* $\dim(X)+1$.

More generally, the above theorem follows from a complete computation of the γ-filtration in these low homological degrees for any Severi-Brauer variety X. We give the statement above due to its similarity to a conjecture that, if proved, would imply a conjecture of Suslin on the generic nontriviality of the reduced Whitehead group of a central simple algebra.

Finally, in Chapter 5, I generalize the description of the coniveau filtration on G_0 obtained in [Kar95b] to a description of the coniveau filtration on G_0 and G_1, the first G-theory group of the category of coherent sheaves, for Severi-Brauer varieties of central simple algebras satisfying a few conditions.

Theorem 1.0.9 (See Proposition 5.5.1)**.** *Let $i = 0$, or let $i = 1$ and assume the reduced Whitehead groups are trivial, $\mathrm{SK}_i(A^{\otimes r}) = 1$, for all $r \geq 0$. Assume A satisfies the condition that its index and exponent are equal over all finite extensions of the base field. Then there are isomorphisms*

$$K_i(X)^{j/j+1} \cong \mathrm{Nrd}_i(A^{\otimes j}),$$

where $K_i(X)^{j/j+1} = E_\infty^{j,-j-i}$ are the limiting terms of the BGQss and $\mathrm{Nrd}_i(A^{\otimes j})$ is the reduced norm group, for all $0 \leq j \leq \deg(A)-1$. For other j these groups vanish.

The techniques are quite different from [Kar95b] where one works by bounding the indices of certain subgroups because it's not necessarily clear whether they make sense for G_1. Here our proofs go by equating reduced norms to actual norms (or finite transfers) and giving some relationships between reduced norm subgroups of different tensor powers of a given algebra.

Each chapter is written as an independent article. Occasionally notation differs between the chapters (e.g. some chapters work primarily with G_0 or K_0 so we write G or K for simplicity; another chapter works primarily with K_0 so we just write K when we might mean, equivalently, G or G_0).

Chapter 2

Universal additive Chern classes and an integral GRR-type theorem

Conventions. In the following we say X is a *variety* to mean X is a scheme essentially of finite type over a field, i.e. a localization of a scheme of finite type over a field. In this way we can work not only with varieties proper but with their generic points as well.

For convenience, our fields all have continuum or countable cardinality.

We remark that the category of schemes essentially of finite type over a field is essentially small and, when necessary, we work only in a small equivalent category.

2.1 Introduction

A natural starting point for the investigation of the structure of the Chow ring for a smooth projective variety X is the structure of its Chern subring. Often this can be accomplished by understanding properties of the Grothendieck ring of X which is sometimes easier to compute.

This paper produces a functor which maps to any other functor having a suitable notion of Chern classes with an additive first Chern class. There are obvious extensions to a number of other situations (e.g. using a different formal group law in the definition, including an equivariant structure, or changing coefficients) that are not pursued in this text.

The second section outlines the construction of the functor for a given variety X. In this section we also prove a number of basic properties which are cohomological in nature. It should be noted this functor does not form a cohomology theory in any natural sense because it typically lacks pushforwards along a given finite morphism.

Section three reviews the relationship between the λ-ring structure on the Grothendieck ring and operations on polynomials of Chern classes. This material is well-known but we use it frequently in examples, appearing primarily in section four, and in the main theorem so it seemed fitting to include it.

In section five we prove our main theorem. More precisely, we prove that there are natural maps between our functor and the associated graded of the γ-filtration on the Grothendieck ring which are multiplication by a certain integer after composition.

2.2 Construction and fundamental properties

Fix a variety X over a field k. Let R_X be the set of symbols $\{c_i^B(\mathcal{F})\}$ varying over integers $i \geq 0$ and over an appropriate representative of all isomorphism classes of finite rank vector bundles $\mathcal{F}$ on X. The algebra $\mathbb{Z}[R_X]$ generated by these symbols is naturally graded with each $c_i^B(\mathcal{F})$ in degree i.

We define an ideal $I_X \subset \mathbb{Z}[R_X]$ having generators:

· $c_0^B(\mathcal{F}) - 1$ for all bundles $\mathcal{F}$

· $c_i^B(\mathcal{F})$ whenever i is greater than the rank of $\mathcal{F}$

· $c_i^B(\mathcal{G}) - \sum_{j=0}^{i} c_{i-j}^B(\mathcal{F})c_j^B(\mathcal{E})$ for any short exact sequence $0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$ (Whitney Sum)

· $c_1^B(\mathcal{L} \otimes \mathcal{L}') - c_1^B(\mathcal{L}) - c_1^B(\mathcal{L}')$ for any pair of line bundles $\mathcal{L}, \mathcal{L}'$.

We denote by $[c_i^B(\mathcal{F})]$ the class of $c_i^B(\mathcal{F})$ in $\mathbb{Z}[R_X]/I_X$. Note that I_X is a homogeneous ideal so that the quotient is graded with a well-defined notion of degree.

For any other variety Y and morphism $f : X \to Y$ there is a natural morphism

$$f^* : \mathbb{Z}[R_Y]/I_Y \to \mathbb{Z}[R_X]/I_X$$

defined by $f^*[c_i^B(\mathcal{F})] := [c_i^B(f^*\mathcal{F})]$.

We write $\mathbb{P}^X$ for the directed set of maps to X made up of isomorphism classes of chains of projective bundles. By this we mean $\mathbb{P}^X$ is the set of sequences of maps

$$X \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

where $P_1 \to X$ is a composition of projections from successive projective bundles, $P_2 \to P_1$ is likewise a chain of projective bundles over P_1, and so on; isomorphism classes of such chains are given by commutative ladders which are termwise isomorphic. One chain dominates another chain if there is a commutative ladder with each vertical arrow a chain of projective bundles.

$$\begin{array}{ccccccc} X & \longleftarrow & P_1 & \longleftarrow & P_2 & \longleftarrow & \cdots \\ \| & & \uparrow & & \uparrow & & \\ X & \longleftarrow & Q_1 & \longleftarrow & Q_2 & \longleftarrow & \cdots \end{array}$$

In the above diagram the bottom sequence, call it S_Q, dominates the top, S_P, and we would write $S_Q \geq S_P$. Any two chains have a chain that dominates them. To see this, let

$$X \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

$$X \leftarrow Q_1 \leftarrow Q_2 \leftarrow \cdots$$

be two such chains. Then by taking fiber products a third such chain that dominates the two given is

$$X \leftarrow P_1 \times_X Q_1 \leftarrow P_2 \times_{P_1 \times_X Q_1} Q_2 \leftarrow \cdots .$$

A chain P of chains of projective bundles

$$X \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

determines a directed system using the natural maps defined above

$$\mathbb{Z}[R_X]/I_X \to \mathbb{Z}[R_{P_1}]/I_{P_1} \to \mathbb{Z}[R_{P_2}]/I_{P_2} \to \cdots .$$

Denoting the limit of this directed system by $\mathbb{Z}[R_P] = \varinjlim \mathbb{Z}[R_{P_i}]/I_{P_i}$, we get a directed system of the $\mathbb{Z}[R_P]$ over all chains in the set $\mathbb{P}^X$.

Definition 2.2.1. We define a ring $B(X)$ as the quotient

$$\mathbb{Z}[R_X]/\ker(f_X)$$

where $f_X : \mathbb{Z}[R_X] \to \varinjlim_{P \in \mathbb{P}^X} \mathbb{Z}[R_P]$ is the canonical map.

The following proposition can be considered the defining quality of the rings $B(X)$ and it largely motivated its definition.

Proposition 2.2.2. *The rings given by B define a contravariant functor from the category of varieties over k to the category of graded rings with pullbacks along a morphism $f : X \to Y$ defined by $f^*[c_i^B(\mathcal{F})] = [c_i^B(f^*\mathcal{F})]$. There are natural transformations $c_i^B : K(-) \to B^i(-)$ defined by taking the class of a vector bundle $\mathcal{F}$ to the class $[c_i^B(\mathcal{F})]$. Moreover, the functor B and these c_i^B satisfy the universal property stated below.*

Let A be any other contravariant functor from the category of varieties to the category of graded rings which has a collection of natural transformations $c_i^A : K(-) \to A^i(-)$ for all $i \geq 0$. Assume $A(X)$ and these c_i^A satisfy the following properties for every variety X:

- $c_0^A(\mathcal{F}) = 1$ *for all vector bundles $\mathcal{F}$ on X*
- $c_i^A(\mathcal{F}) = 0$ *for all integers $i > \mathrm{rk}(\mathcal{F})$*
- $c_i^A(\mathcal{G}) = \sum_{j=0}^{i} c_{i-j}^A(\mathcal{F})c_j^A(\mathcal{E})$ *for any short exact sequence $0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$ (Whitney Sum)*
- $c_1^A(\mathcal{L} \otimes \mathcal{L}') = c_1^A(\mathcal{L}) + c_1^A(\mathcal{L}')$ *for any pair of line bundles $\mathcal{L}, \mathcal{L}'$*
- *For any projective bundle $P \to X$, the pullback map $A(X) \to A(P)$ is injective.*

Then there is a natural transformation $b_A : B(-) \to A(-)$ which is completely determined by the rule $b_A([c_i^B(\mathcal{F})]) = c_i^A(\mathcal{F})$.

Proof. That B is a contravariant functor with the defined pullback map is clear from the facts: to any chain of projective bundles

$$X \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

and any map $X \to Y$ one gets, via pullback, a chain

$$Y \leftarrow P_1 \times_X Y \leftarrow P_2 \times_{P_1} P_1 \times_X Y \leftarrow \cdots$$

and that pullbacks of the defining relations are defining relations.

That B comes equipped with natural transformations (of sets) c_i^B follows by defining the group homomorphisms, for any variety X,

$$K(X) \to 1 + B(X)[[t]]$$

which sends the class of a locally free sheaf $[\mathcal{F}]$ to the power series $1+[c_1^B(\mathcal{F})]t + \cdots + [c_i^B(\mathcal{F})]t^i + \cdots$, considered inside the set of all power series with leading term 1 and coefficients in $B(X)$, and extending linearly. By the Whitney sum relation such maps are well-defined. That this homomorphism commutes with the pullbacks on K and B follows from the fact both are defined by the pullback of sheaves (or vector bundles) f^*.

Finally, to see that B satisfies the universal property stated we define a map $\mathbb{Z}[R_X] \to A(X)$, again for any given variety X, taking the symbol $c_i^B(\mathcal{F})$ to the Chern class $c_i^A(\mathcal{F})$. To finish the claim, it suffices to show this map descends to a ring map $B(X) \to A(X)$. Let r be a relation in the kernel $\ker(f_X)$ as defined above. As this element is zero when mapped to a direct limit, of a direct limit, of rings there is a chain of projective bundles $P \to X$ so that $\mathbb{Z}[R_X] \to \mathbb{Z}[R_P]/I_P$ contains r in its kernel. There is a canonical map from $\mathbb{Z}[R_P]/I_P$ to $A(P)$ sending symbols to Chern classes and it follows there is a commuting diagram as below.

$$\begin{array}{ccc} \mathbb{Z}[R_X] & \longrightarrow & \mathbb{Z}[R_P]/I_P \\ \big\downarrow & & \big\downarrow \\ A(X) & \longrightarrow & A(P) \end{array}$$

Since the pullback $A(X) \to A(P)$ commutes with Chern classes and is injective, a diagram chase shows r is 0 in $A(X)$. In this way we get a map $B(X) \to A(X)$ having all of the specified properties. $\square$

Although B turns out not to be a cohomology theory, it does share a number of properties that are typical of a cohomology theory. As an example of this, we'll show that B could reasonably be said to satisfy homotopy invariance, weak-localization, and continuity. Our main observation is the following lemma.

Lemma 2.2.3. *Let $f : X \to Y$ be a morphism of varieties such that $f^* : K(Y) \to K(X)$ is surjective. Then $f^* : B(Y) \to B(X)$ is surjective.*

Proof. It suffices to show each class $[c_i^B(\mathcal{F})]$ is in the image of f^* as $\mathcal{F}$ ranges over vector bundles on X. Since the following diagram commutes for any $i \geq 0$,

$$\begin{array}{ccc} K(Y) & \xrightarrow{f^*} & K(X) \\ \big\downarrow{\scriptstyle c_i^B} & & \big\downarrow{\scriptstyle c_i^B} \\ B(Y) & \xrightarrow{f^*} & B(X) \end{array}$$

the lemma follows from observing that there is a class x in $K(Y)$ mapping to $\mathcal{F}$ under f^*. $\square$

Lemma 2.2.4. *Assume either X is reduced and quasi-projective or smooth and separated. Then the pullback $\pi^* : B(X) \to B(X \times \mathbb{A}^n)$ along the projection $\pi : X \times \mathbb{A}^n \to X$ is an isomorphism.*

Proof. It suffices to treat the case $n = 1$. Letting $\sigma : X \to X \times \mathbb{A}^1$ be the zero section, the composite $\pi \circ \sigma$ is the identity on X. By functorality $\sigma^* \circ \pi^*$ is the identity on $B(X)$ and the map π^* is therefore injective.

To show surjectivity of π^*, we can apply Lemma 2.2.3 to π noting that, with the given assumptions on X, the induced map $K(X) \to K(X \times \mathbb{A}^1)$ is surjective. □

Lemma 2.2.5. *If $i : U \to X$ is the inclusion of an open subvariety $U \subset X$, then the restriction $i^* : B(X) \to B(U)$ is surjective.*

Proof. Immediate from Lemma 2.2.3. □

Lemma 2.2.6. *Let x be a point of X. There are isomorphisms*

$$\varinjlim_{x \in U} B(U) = B(\varprojlim_{x \in U} U) = B(\mathrm{Spec}(\mathcal{O}_{X,x}))$$

where the limits are along Zariski opens containing x.

Proof. Since all projective modules over a local ring are free, we have

$$K(\mathrm{Spec}(\mathcal{O}_{X,x})) = \mathbb{Z}$$

with generator the class of $\mathcal{O}_{X,x}$. Thus for every open U the canonical map $K(U) \to K(\mathrm{Spec}(\mathcal{O}_{X,x}))$ is surjective and the surjectivity of the canonical map

$$\varinjlim_{x \in U} B(U) \to B(\varprojlim_{x \in U} U)$$

follows by Lemma 2.2.3.

To show injectivity of this map, it suffices to show every Chern class of positive degree is trivial over some open set around x. But this is true for every vector bundle on X so it is also true for every Chern class. □

2.3 Chern classes and λ-rings

The functor B defined in the previous section is closely determined by the λ-ring structure of the Grothendieck ring. Since most of our examples depend on this relation, we provide reference in this section. For further properties of the objects in this section one can consult the relevant literature (cf. [MR071, Expose 0], [Man69], or [FL85]).

We continue to work over a fixed field k. For any variety X over this field, the Grothendieck ring $K(X)$ is equipped with a canonical structure of a λ-ring. That is to say, there are natural transformations $\lambda^i : K(-) \to K(-)$ defined so that $\lambda^i([\mathcal{F}]) = [\Lambda^i(\mathcal{F})]$ for a vector bundle $\mathcal{F}$. These natural transformations satisfy the following properties:

- $\lambda^0(x) = 1$ for all x in $K(X)$
- $\lambda^1(x) = x$ for all x in $K(X)$
- $\lambda^i(x+y) = \sum_{j=0}^{i} \lambda^{i-j}(x)\lambda^j(y)$
- $\lambda^i(xy) = P_i(\lambda^1(x), ..., \lambda^i(x), \lambda^1(y), ..., \lambda^i(y))$ for certain universal polynomials P_i
- $\lambda^i(\lambda^j(x)) = P_{i,j}(\lambda^1(x), ..., \lambda^{ij}(x))$ for certain universal polynomials $P_{i,j}$.

Remark 2.3.1. For any λ-ring R, there are well-defined Schur operations $S^\mu : R \to R$ for any partition $\mu = (\mu_1, ..., \mu_n)$ defined by

$$S^\mu(x) = \det(\lambda^{\mu_i+j-i}(x))_{1\leq i,j\leq n}.$$

If $\epsilon \subset \mu$ is another partition, one can define an operation $S^{\mu/\epsilon} : R \to R$ for the skew diagram μ/ϵ as

$$S^{\mu/\epsilon}(x) = \sum_{\nu} c^{\mu}_{\epsilon,\nu} S^\nu(x)$$

where $c^{\mu}_{\epsilon,\nu}$ is a Littlewood-Richardson coefficient. These operations satisfy the formula

$$S^{\mu/\epsilon}(x+y) = \sum_{\epsilon\subset\nu\subset\mu} S^{\nu/\epsilon}(x)S^{\mu/\nu}(y)$$

generalizing that for the λ-operations.

Example 2.3.2. Let $\mathrm{Gr}(m,n)$ be the Grassmannian of m-planes in an n-dimensional vector space. Then $K(\mathrm{Gr}(m,n))$ is additively generated by the classes $S^{\mu}(Q)$ where Q is the universal quotient bundle on $\mathrm{Gr}(m,n)$ of rank $n-m$ and μ ranges over partitions which fit inside a box of size $(n-m)\times m$.

One also has the γ-operations $\gamma^i : K(-) \to K(-)$ defined by the formula

$$\gamma^i(x) = \lambda^i(x+i-1).$$

To define the γ-filtration on $K(X)$ for a smooth variety X, one lets $\gamma^0 = K(X)$, $\gamma^1 = \ker(\mathrm{rk})$ where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the rank homomorphism, and γ^i is defined to be the ideal generated by monomials $\gamma^{i_1}(x_1)\cdots\gamma^{i_j}(x_j)$ where $x_1, ..., x_j$ are elements of γ^1 and $i_1 + \cdots + i_j \geq i$.

Denote by $1 + tB(X)[[t]]$ the set of invertible power series with coefficients in $B(X)$.

Definition 2.3.3. The total Chern class is the homomorphism

$$c_t^B : K(X) \to 1 + tB(X)[[t]]$$

defined by $c_t^B(x) = 1 + c_1^B(x)t + c_2^B(x)t^2 + \cdots$.

The total Chern class commutes with the pullbacks on K and B hence it defines a natural transformation of some type. By composing with the universal homomorphism b_A of Proposition 2.2.2 one also gets total Chern classes with values in A. When no confusion will arise, we will omit the superscript B in the notation. We'll show later (see Proposition 2.4.7) that whenever X has an ample line bundle, the total Chern class is a polynomial.

Lemma 2.3.4. *For any vector bundles $\mathcal{F}, \mathcal{G}$ of ranks n, m respectively there are polynomials $Q_{n,m,i}$ so that*

$$c_t(\mathcal{F} \otimes \mathcal{G}) = 1 + \sum_{i \geq 1} Q_{n,m,i}(c_1(\mathcal{F}), ..., c_i(\mathcal{F}), c_1(\mathcal{G}), ..., c_i(\mathcal{G}))t^i.$$

Proof. It suffices to work over a chain of projective bundles $\pi : P \to X$ where the classes of $\mathcal{F}, \mathcal{G}$ split into a sum of line bundles in $K(P)$. If $\pi^*\mathcal{F} = \mathcal{L}_1 + \cdots + \mathcal{L}_n$ and $\pi^*\mathcal{G} = \mathcal{L}'_1 + \cdots + \mathcal{L}'_m$ then

$$c_t(\pi^*\mathcal{F} \otimes \pi^*\mathcal{G}) = \prod_{1 \leq i \leq n, 1 \leq j \leq m} \left(1 + (c_1(\mathcal{L}_i) + c_1(\mathcal{L}'_j))t\right).$$

Since the latter is symmetric in the $c_1(\mathcal{L})$'s and in the $c_1(\mathcal{L}')$'s, the claim follows by choosing $Q_{n,m,i}$ to be the homogeneous polynomial expressing the weight i part of this product as a polynomial of in elementary symmetric polynomials e_i in these variables.

In more details, one can write

$$c_t(\pi^*\mathcal{F}) = c_t(\mathcal{L}_1 + \cdots + \mathcal{L}_n) = \prod_{i=1}^{n} (1 + c_1(\mathcal{L}_i)t) \tag{E}$$

as c_t is a group homomorphism. Then comparing coefficients of degree i, one finds an equality

$$c_i(\pi^*\mathcal{F}) = \prod_{1 \leq j_1 < \cdots < j_i \leq n} c_1(\mathcal{L}_{j_i})$$

by expanding the expression on the right side of (E). This last product being, equivalently, the ith elementary symmetric polynomial e_i in the variables

$c_1(\mathcal{L}_1), ..., c_1(\mathcal{L}_n)$. A similar formula holds for the Chern classes $c_i(\pi^*\mathcal{G})$ as elementary symmetric polynomials f_i in the $c_1(\mathcal{L}')$. Now as $c_i(\pi^*\mathcal{F} \otimes \pi^*\mathcal{G})$ is expressible as a symmetric function in both of these sets of variables, and since the ring of functions symmetric in both sets of variables is generated integrally by both sets of elementary symmetric polynomials, there is a polynomial $Q_{n,m,i}$ with equality

$$c_i(\pi^*\mathcal{F} \otimes \pi^*\mathcal{G}) = Q_{n,m,i}(e_1, ..., e_i, f_1, ..., f_i).$$

Hence, this claim is then because the difference

$$c_i(\mathcal{F} \otimes \mathcal{G}) - Q_{n,m,i}(c_1(\mathcal{F}), ..., c_i(\mathcal{F}), c_1(\mathcal{G}), ..., c_i(\mathcal{G}))$$

is an element of the kernel of $B(X) \to B(P)$, which is trivial. $\square$

Example 2.3.5 (cf. [Ful98, Example 3.2.2])**.** If $\mathcal{F}$ is a vector bundle of rank n and $\mathcal{L}$ is a line bundle then

$$c_j(\mathcal{F} \otimes \mathcal{L}) = \sum_{i=0}^{j} \binom{n-i}{j-i} c_i(\mathcal{F}) c_1(\mathcal{L})^{j-i}.$$

Equivalently,

$$c_t(\mathcal{F} \otimes \mathcal{L}) = c_t(\mathcal{L})^n c_\tau(\mathcal{F})$$

where $\tau = t/c_t(\mathcal{L})$.

Example 2.3.6 (cf. [Ful98, Remark 3.2.3 (a)])**.** If $\mathcal{F}$ is a vector bundle of rank n, then

$$c_j(\mathcal{F}^\vee) = (-1)^j c_j(\mathcal{F}).$$

Lemma 2.3.7. *For any vector bundle $\mathcal{F}$, the Chern class $c_j(\lambda^i(\mathcal{F}))$ is a polynomial in the Chern classes of $\mathcal{F}$.*

Proof. Let $x = [\mathcal{F}]$. Again we work over a chain of projective bundles P where $x = x_1 + \cdots + x_n$ for the class of some invertible sheaves $x_1, ..., x_n$. Then

$$c_t(\Lambda^j \mathcal{F}) = \prod_{i_1 < \cdots < i_j} (1 + c_1(x_{i_1}) + \cdots + c_1(x_{i_j}))$$

is symmetric in the $c_1(x_k)'s$ which proves the claim. $\square$

2.4 Examples

The main purpose of this section is to compute some examples to illustrate how one might go about studying the functor B.

Example 2.4.1. $B(\mathrm{Spec}(k)) = \mathbb{Z}$.

Example 2.4.2. $B(\mathbb{P}^n) = \mathbb{Z}[x]/(x^{n+1})$ where $x = c_1(\mathcal{O}(1))$. To see this, one observes $K(X)$ is generated as a ring by $\mathcal{O}(1)$ so that $B(\mathbb{P}^n)$ is generated by x because of Lemma 2.3.4. To get the relation $x^{n+1} = 0$, one can apply the total Chern class to the Euler exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

and note the tangent bundle has vanishing $(n+1)$th Chern class. It follows $B(\mathbb{P}^n)$ is a quotient of $\mathbb{Z}[x]/(x^{n+1})$. To complete the proof, it's sufficient to find a cohomology theory A such that $A(\mathbb{P}^n) = \mathbb{Z}[x]/(x^{n+1})$ (e.g. $A = \mathrm{CH}$ or $A = \mathrm{gr}_\gamma K$).

Example 2.4.3. By Example 2.3.2, the Grothendieck ring $K(\mathrm{Gr}(n, m))$ is generated by polynomials in the λ-operations of the universal quotient bundle Q. By Lemma 2.3.7, this means the ring $B(\mathrm{Gr}(m, n))$ is generated by the Chern classes of Q, call them $c_1, ..., c_{n-m}$.

We get relations in $B(\mathrm{Gr}(m,n))$ from the exact sequence of the universal sub and quotient bundles

$$0 \to S \to \mathcal{O}_{\mathrm{Gr}(m,n)}^{\oplus n} \to Q \to 0.$$

If $m \leq n-m$, then from this exact sequence we find $c_t(S) = 1/c_t(Q)$. Let $t_{m+1}, ..., t_n$ be the polynomials in the Chern classes of Q which are the coefficients of $t^{m+1}, ..., t^n$ in the expansion of $1/c_t(Q)$ as a power series in t. If $m > n-m$, then let us rename $c_1, ..., c_m$ to be the Chern classes of S, which evidently also generate $B(\mathrm{Gr}(m,n))$ due to the exact sequence above, and name $t_{n-m+1}, ..., t_n$ to be the coefficients of $t^{n-m+1}, ..., t^n$ in the expansion of $1/c_t(S)$ as a power series in t.

We claim that $B(\mathrm{Gr}(m,n)) = \mathbb{Z}[c_1, ..., c_{n-m}]/(t_{m+1}, ..., t_n)$ if $m \leq n-m$ or $B(\mathrm{Gr}(m,n)) = \mathbb{Z}[c_1, ..., c_m]/(t_{n-m+1}, ..., t_n)$ if $m > n-m$. Indeed, to complete the proof it's sufficient to find a cohomology theory A such that $A(\mathrm{Gr}(n,m))$ is the desired ring. Taking $A = \mathrm{CH}$ suffices (see [EH16, Theorem 5.26]).

Example 2.4.4. Let X be a smooth projective curve. Then there is an isomorphism $K(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$ and any class x in $K(X)$ can be written $x = \mathrm{rk}(x) + \det(x)$. It follows that $B(X)$ is generated by $\mathbb{Z}$ and the first Chern classes.

In fact, there is an isomorphism $B(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$. We've shown there is a natural surjection from the right side of this equality to the left. And to show that this map is an injection, we compose it with the map $b_{CH} : B(X) \to \mathrm{CH}(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$.

A similar argument shows that $B(X) \cong \mathrm{CH}(X)$ for a smooth projective surface X.

Knowing λ-ring generators for the Grothendieck ring of a variety X allows one to determine generators for the ring $B(X)$ using Lemma 2.3.7.

Definition 2.4.5. The *level* of a variety X, shorthand $\text{lev}(X)$, is defined to be the minimal number of elements that generate $K(X)$ as a λ-ring. If no such number exists, the level is said to be infinite.

Example 2.4.6. If $X = \mathbb{P}^n$ or $X = \text{Gr}(m, n)$ then $\text{lev}(X) = 1$.

For any sequence $0 = n_0 < n_1 < \cdots < n_k = n$, let $X(n_0, ..., n_k)$ be the variety of $(n_0, ..., n_k)$-flags in a vector space of dimension n. Then

$$\text{lev}(X(n_0, ..., n_k)) = k - 1.$$

To see this, note that there are k tautological vector bundles which generate the ring $K(X)$ of ranks $n_1, ..., n_k$ respectively with one linear relation between them. Thus $\text{lev}(X) \leq k - 1$. Conversely, $\text{lev}(X) \geq \text{rk}_{\mathbb{Z}}\text{Pic}(X)$ and the latter of these equals $k - 1$ as well.

If X is a Severi-Brauer variety, then the level of X is determined by a sequence of indices of tensor powers of the associated algebra of X, cf. [KM18a, Lemma A.6].

We conclude this section by showing the total Chern class of any element of $K(X)$ is a polynomial if X has an ample line bundle.

Proposition 2.4.7. *Suppose X is a variety with ample line bundle $\mathcal{L}$. Then for any vector bundle $\mathcal{F}$ on X, the Chern class $c_i(\mathcal{F})$ is nilpotent for all $i \geq 1$. Moreover, for any element x in $K(X)$, the total Chern class $c_t(x)$ is a polynomial in t.*

Proof. If $\mathcal{F}$ is globally generated, then there is a morphism $f : X \to \text{Gr}(m, n)$ for some m, n such that $f^*Q = \mathcal{F}$. Since the Chern classes of Q are nilpotent due to Example 2.4.3, the same statement follows for this $\mathcal{F}$.

In the general case, since X has an ample line bundle, there is some product $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ that is globally generated. By Example 2.3.5 and induction, the jth Chern class of $\mathcal{F}$ can be written as a polynomial in the Chern classes of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ and $\mathcal{L}^{\otimes n}$. Since both of these latter bundles are globally generated, their Chern classes are nilpotent and thus so are the Chern classes of $\mathcal{F}$.

For the final statement, we write $x = [\mathcal{F}] - [\mathcal{G}]$ and observe for sufficiently large Chern classes of x there are sufficiently large powers of the Chern classes of $\mathcal{F}$ or $\mathcal{G}$ involved. Eventually then these terms must vanish. □

2.5 An integral GRR theorem

In this final section, we show how one can relate B with another functor, $\mathrm{gr}_\gamma K$, which associates to any variety X the associated graded ring of the γ-filtration on the Grothendieck ring $K(X)$. Recall $\mathrm{gr}_\gamma K(-)$ is equipped with a collection of Chern classes c_i^γ, in the spirit of Proposition 2.2, determined by the rule $c_i^\gamma(\mathcal{F}) = \gamma^i(\mathrm{rk}(\mathcal{F}) - [\mathcal{F}^\vee])$ for any vector bundle $\mathcal{F}$. Our main theorem is the following:

Theorem 2.5.1. *Let X be a variety and write*

$$b_\gamma^i : B^i(X) \to \mathrm{gr}_\gamma^i K(X)$$

for the ith summand of the canonical morphism of Proposition 2.2.2 applied to $(\mathrm{gr}_\gamma K, c_i^\gamma)$. Then the Chern classes c_i^B induce well-defined maps

$$c_i^B : \mathrm{gr}_\gamma^i K(X) \to B^i(X)$$

such that the compositions are both multiplication by $(-1)^{i-1}(i-1)!$,

$$c_i^B \circ b_\gamma^i = (-1)^{i-1}(i-1)! \quad \textit{and} \quad b_\gamma^i \circ c_i^B = (-1)^{i-1}(i-1)!.$$

The proof can be reduced to an essentially combinatorial argument. We present the main computations as two separate lemmas below.

Remark 2.5.2. Theorem 2.5.1 recovers the integral Grothendieck-Riemann-Roch for smooth varieties X with $\mathrm{CH}(X)$ generated by Chern classes. To see this, denote by $G(X)$ the Grothendieck ring of coherent sheaves on X and by $\mathrm{gr}_\tau G(X)$ the associated graded of the coniveau filtration on $G(X)$.

There is a canonical map $\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\tau G(X)$ given by comparing the γ and coniveau filtrations. When $\mathrm{CH}(X)$ is generated by Chern classes, this comparison morphism is an isomorphism, see [KM18b, Proposition 3.3]. To get the statement for the integral Grothendieck-Riemann-Roch in this case one observes the following square

$$\begin{tikzcd}
B^i(X) \arrow[r, "b^i_{CH}"] \arrow[d, shift left, "b^i_\gamma"] & \mathrm{CH}^i(X) \arrow[d, shift left] \\
\mathrm{gr}^i_\gamma K(X) \arrow[u, shift left, "c_i^B"] \arrow[r, equal] & \mathrm{gr}^i_\tau G(X) \arrow[u, shift left]
\end{tikzcd}$$

is commutative in more than one way where the arrow $\mathrm{CH}^i(X) \to \mathrm{gr}^i_\tau G(X)$ is the canonical epimorphism from the Chow ring to the associated graded of the coniveau filtration and the arrow $\mathrm{gr}^i_\tau G(X) \to \mathrm{CH}^i(X)$ can be defined by going around the outside of the square. This latter map coincides with the usual Chern class map from the Grothendieck-Riemann-Roch and composition in either direction is multiplication by $(-1)^{i-1}(i-1)!$ by Theorem 2.5.1.

Lemma 2.5.3. *Let* $\mathcal{L}_1, ..., \mathcal{L}_{i+1}$ *be* $i+1$ *line bundles on some variety* P *which* *can be realized as a chain of projective bundles over X. Then*

$$c_k^B\left(\prod_{j=1}^{i+1}(\mathcal{L}_j - 1)\right) = 0$$

inside of $B(P)$ for all $k \leq i$.

Proof. We proceed by induction on the length of the product $i+1$. For our base case, we observe that

$$\begin{aligned} c_1^B((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)) &= c_1^B(\mathcal{L}_1 \otimes \mathcal{L}_2 - \mathcal{L}_1 - \mathcal{L}_2 + 1) \\ &= c_1^B(\mathcal{L}_1) + c_1^B(\mathcal{L}_2) - c_1^B(\mathcal{L}_2) - c_1^B(\mathcal{L}_1) + c_1^B(1) = 0. \end{aligned}$$

For our induction hypothesis, we assume the Chern class c_k^B of any product of such elements of length i vanishes for all $k \leq i-1$. Let $\prod_{j=1}^{i}(\mathcal{L}_j - 1) = [\mathcal{F}] - [\mathcal{G}]$. Then

$$\begin{aligned} c_t^B\left(\prod_{j=1}^{i+1}(\mathcal{L}_j - 1)\right) &= c_t^B((\mathcal{F} - \mathcal{G})(\mathcal{L}_{i+1} - 1)) \\ &= \frac{c_t^B(\mathcal{F} \otimes \mathcal{L}_{i+1})}{c_t^B(\mathcal{G} \otimes \mathcal{L}_{i+1}) c_t^B(\mathcal{F} - \mathcal{G})} \\ &= \frac{c_\tau^B(\mathcal{F})}{c_\tau^B(\mathcal{G}) c_t^B(\mathcal{F} - \mathcal{G})} \\ &\qquad \left(\text{using } \tau = \tfrac{t}{1 + c_1^B(\mathcal{L}_{i+1})t}, \text{ cf. Example 2.3.5}\right) \\ &= \frac{c_\tau^B(\mathcal{F} - \mathcal{G})}{c_t^B(\mathcal{F} - \mathcal{G})} \\ &= \frac{1 + c_i(\mathcal{F} - \mathcal{G})\tau^i + c_{i+1}(\mathcal{F} - \mathcal{G})\tau^{i+1} + \cdots}{1 + c_i(\mathcal{F} - \mathcal{G})t^i + c_{i+1}(\mathcal{F} - \mathcal{G})t^{i+1} + \cdots} \\ &\qquad (\text{by induction hypothesis}) \\ &= 1 - i c_i(\mathcal{F} - \mathcal{G}) c_1(\mathcal{L}_{i+1}) t^{i+1} + \cdots . \end{aligned}$$

$\square$

Lemma 2.5.4. *Let $\mathcal{L}_1, ..., \mathcal{L}_i$ be i line bundles on some variety P which can be realized as a chain of projective bundles over X. Then*

$$c_i^B\left(\prod_{j=1}^{i}(\mathcal{L}_j - 1)\right) = (-1)^{i-1}(i-1)!\prod_{j=1}^{i} c_1^B(\mathcal{L}_j)$$

inside of $B(P)$.

Proof. Expanding the Chern class $c_i(\mathcal{F} - \mathcal{G})$ in the last expression in the proof of Lemma 2.5.3 gives the result. $\square$

Proof of Theorem 2.5.1. Let x be an element of γ^i, the ith piece of the γ-filtration on $K(X)$. The proof will be complete if we can show there is a variety P which can be realized as a chain of projective bundles over X such that the pullback of x to $K(P)$ can be written as a sum or difference of monomials of the form $(\mathcal{L}_1 - 1)\cdots(\mathcal{L}_j - 1)$ with $j \geq i$. Indeed, assuming this is the case, there is a commuting square

$$\begin{array}{ccc} \gamma^i & \longrightarrow & K(P) \\ \big\downarrow{\scriptstyle c_i^B} & & \big\downarrow{\scriptstyle c_i^B} \\ B^i(X) & \longrightarrow & B^i(P) \end{array}$$

where the horizontal pullback morphisms are injections. We find

$$c_t^B(x) = c_t^B\left(\sum_{m=0}^{k}\left(\pm\prod_{j=1}^{n_m}(\mathcal{L}_{m_j} - 1)\right)\right) = \prod_{m=0}^{k} c_t^B(\prod_{j=1}^{n_m}(\mathcal{L}_{m_j} - 1))^{\pm 1}.$$

The latter factors vanish whenever $n_m > i$ by Lemma 2.5.3 while the latter factors are equal

$$1 + (-1)^{i-1}(i-1)!\left(\pm\prod_{j=1}^{n_m} c_1^B(\mathcal{L}_{m_j})\right)t^i + \cdots$$

whenever $n_m = i$ by Lemma 2.5.4. Since

$$b^i_\gamma(c^B_1(\mathcal{L}_j)) = b^i_\gamma(-c^B_1(\mathcal{L}_j^\vee)) = -b^i_\gamma(c^B_1(\mathcal{L}_|{}^\vee)) = \mathcal{L}_j - 1,$$

where we use Example 2.3.6 for the first equality, the proof is completed once we can show our starting assumption.

To do this, we start by writing

$$x = \sum_{m=0}^{k} \left(\pm \prod_{j=1}^{n_m} \gamma^{m_j}(x_{m_j}) \right)$$

for some elements x_{m_j} in γ^1. Note that we can focus on a single monomial since if we prove a monomial can be written in the desired way then the same follows for the sum. So assume $x = \gamma^{n_1}(x_1) \cdots \gamma^{n_j}(x_j)$ for some $n_1 + \cdots + n_j \geq i$. Each x_k, belonging to γ^1, can be written as

$$x_k = [\mathcal{F}] - [\mathcal{G}] = [\mathcal{F}] - \mathrm{rk}(\mathcal{F}) - ([\mathcal{G}] - \mathrm{rk}(\mathcal{G}))$$

for some $\mathcal{F}, \mathcal{G}$ that depend on k.

Now there is a variety P which can be realized as a chain of projective bundles over X such that each of the $\mathcal{F}, \mathcal{G}$'s can be written

$$x_k = [\mathcal{F}] - \mathrm{rk}(\mathcal{F}) - ([\mathcal{G}] - \mathrm{rk}(\mathcal{G})) = (\mathcal{L}_1 + \cdots + \mathcal{L}_n - n) - (\mathcal{L}'_1 + \cdots + \mathcal{L}'_n - n)$$

with the $(\mathcal{L})$'s and $(\mathcal{L}')$'s depending on k still. Another way to say this is that we can find such a P so that for every k we have an expression like

$$x_k = (\mathcal{L}_1 - 1) + \cdots + (\mathcal{L}_n - 1) - (\mathcal{L}'_1 - 1) - \cdots - (\mathcal{L}'_n - 1).$$

Finally, applying the operation $\gamma_t = \sum_{j\geq 0} \gamma^j t^j$ we find

$$\begin{aligned}\gamma_t(x_k) &= \gamma_t\left(\sum_{j=1}^{n}(\mathcal{L}_j - 1) - \sum_{j=1}^{n}(\mathcal{L}' - 1)\right)\\ &= \frac{\gamma_t\left(\sum_{j=1}^{n}(\mathcal{L}_j - 1)\right)}{\gamma_t\left(\sum_{j=1}^{n}(\mathcal{L}' - 1)\right)}\\ &= \frac{\sum_{j\geq 0}\sigma_j t^j}{\sum_{j\geq 0}\sigma'_j t^j}\end{aligned}$$

where σ_j is the jth elementary symmetric polynomial in the variables $(\mathcal{L}_1 - 1), ..., (\mathcal{L}_n - 1)$ and similarly for σ'_j with $(\mathcal{L}'_1 - 1), ..., (\mathcal{L}'_n - 1)$. Expanding this series in t we find $\gamma^{m_k}(x_k)$ is a polynomial, homogeneous and symmetric in variables like $(\mathcal{L}-1)$, of degree m_k. This completes the proof since we've shown there is a variety P which can be realized as a chain of projective bundles over X such that x can be written in the desired form. $\square$

Chapter 3

On the $\mathbb{K}$-theory coniveau epimorphism for products of Severi-Brauer varieties

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise. Sometimes we use k as an index when no confusion will occur.

For any field F, we fix an algebraic closure $\overline{F}$.

A variety X is a separated scheme of finite type over a field.

Let $X = X_1 \times \cdots \times X_r$ be a product of varieties with projections $\pi_i : X \to X_i$. Let $\mathcal{F}_1, ..., \mathcal{F}_r$ be sheaves of modules on $X_1, ..., X_r$. We use $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_r$ for the external product $\pi_1^*\mathcal{F}_1 \otimes \cdots \otimes \pi_r^*\mathcal{F}_r$.

For a ring R with a $\mathbb{Z}$-indexed descending filtration $F_\nu^\bullet$, (e.g. $\nu = \gamma$ or τ as in Section 3.2), we write $\mathrm{gr}_\nu^i R$ for the corresponding quotient F_ν^i / F_ν^{i+1}. We write $\mathrm{gr}_\nu R = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}_\nu^i R$ for the associated graded ring.

A semisimple algebraic group G is of type AC if its Dynkin diagram is a union of diagrams of type A and type C. Similarly a semisimple group G is of type AA if its Dynkin diagram is a union of diagrams of type A.

For an index set $\mathcal{I}$, two elements $i, j \in \mathcal{I}$, we write δ_{ij} for the function which is 0 when $i \neq j$ and 1 if $i = j$.

Given two r-tuples of integers, say I, J, we write $I < J$ if the ith component of I is less than the ith component of J for any $1 \leq i \leq r$.

3.1 Introduction

For any smooth variety X, the coniveau spectral sequence for algebraic K-theory induces a canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ from the Chow ring of X to the associated graded ring of the coniveau filtration on the Grothendieck ring of X (for notation related to Grothendieck rings see Section 3.2). The kernel of this epimorphism is torsion, as can be seen using the Grothendieck-Riemann-Roch without denominators. In general this can't be refined: there are examples of smooth varieties where the kernel of the K-theory coniveau epimorphism is nontrivial. With this in mind, a particularly difficult problem has been finding families of varieties where this epimorphism is, or fails to be, an isomorphism. In this direction we propose the following:

Conjecture 3.1.1. *Let X be a product of Severi-Brauer varieties. If the Chow ring* $\mathrm{CH}(X)$ *is generated by Chern classes, then the canonical epimorphism* $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Since the ring $\mathrm{gr}_\tau \mathrm{G}(X)$ is computable for such X (see Section 3.2 for recollections on the Grothendieck rings of Severi-Brauer varieties and their products), a positive answer to Conjecture 3.1.1 could then be interpreted as a method for computing the Chow ring of such varieties. This is carried out, for instance, in [Kar17c, Theorem 3.1] where Karpenko shows a special case of Conjecture 3.1.1 and, using this, is able to compute the Chow ring of certain generic Severi-Brauer varieties.

In Section 3.3, we give some evidence that a positive answer to Conjecture 3.1.1 is a likely one. The main result of this section, Theorem 3.3.3, shows that Conjecture 3.1.1 is equivalent to a particular case of an older conjecture of Karpenko's: [1]

Conjecture 3.1.2. *Let G be a split semisimple algebraic group, E a standard generic G-torsor, and P a special parabolic subgroup of G. Then the canonical epimorphism* $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ *is an isomorphism.*

The proof uses an analysis of the products of Severi-Brauer varieties one obtains from a standard generic G-torsor for algebraic groups of type AA along with various specialization maps.

In Section 3.4, we introduce the notion of the level of a central simple algebra. We show how the level gives a useful description of the Grothendieck ring of a Severi-Brauer variety and use this description in the main result of this section, Theorem 3.4.15, where we prove Conjecture 3.1.1 for a single Severi-Brauer variety associated to a central simple algebra of level 1. This generalizes the previously known results obtained in [Kar17c, Theorem 3.1].

3.2 Grothendieck rings of Severi-Brauer varieties

By $\mathrm{K}(X)$, we mean the Grothendieck ring of locally free sheaves (equivalently vector bundles) on a variety X; by $\mathrm{G}(X)$ we mean the Grothendieck group of

[1] In its original formulation [Kar17b, Conjecture 1.1], Conjecture 3.1.2 only asserts there is an isomorphism in the case P is a Borel subgroup. However, to prove Conjecture 3.1.2 for all special parabolic subgroups of G it suffices to check the result holds for a particular choice of special parabolic subgroup P. These two forms of Conjecture 3.1.2 are then equivalent since a Borel subgroup is special.

coherent sheaves on X. The ith term of the γ-filtration on $\mathrm{K}(X)$ is denoted $F^i_\gamma(X)$; the ith term of the coniveau filtration on $\mathrm{G}(X)$ is denoted $F^i_\tau(X)$.

There's a canonical map $\varphi_X : \mathrm{K}(X) \to \mathrm{G}(X)$ taking the class $[\mathcal{L}] \in \mathrm{K}(X)$ of a locally free sheaf $\mathcal{L}$ to the class $[\mathcal{L}] \in \mathrm{G}(X)$. When X is smooth, φ_X is an isomorphism giving $\mathrm{G}(X)$ the structure of a ring. The coniveau filtration is compatible with the ring structure on $\mathrm{G}(X)$, and $\varphi_X(F^i_\gamma(X)) \subset F^i_\tau(X)$. Moreover, if the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes, then $\varphi_X(F^i_\gamma(X)) = F^i_\tau(X)$, cf. [Kar98, Proof of Theorem 3.7].

We will often be working with the rings $\mathrm{K}(X)$ for X a Severi-Brauer variety and for X a product of Severi-Brauer varieties.

In the case X is a Severi-Brauer variety, $\mathrm{K}(X)$ has been determined by Quillen. To state this result, recall that X is the variety of right ideals of dimension $\deg(A)$ in the central simple algebra A associated with X. The tautological vector bundle ζ_X on X is a right A-module.

For any central simple algebra B, let us define $\mathrm{K}(B)$ as the Grothendieck group of the category of finitely generated left B-modules. The group $\mathrm{K}(B)$ is infinite cyclic with a canonical generator given by the class of a (unique up to isomorphism) simple B-module.

Theorem 3.2.1 ([Qui73, §8, Theorem 4.1])**.** *Let X be the Severi-Brauer variety of a central simple algebra A. The group homomorphism*

$$\bigoplus_{i=0}^{\deg(A)-1} \mathrm{K}(A^{\otimes i}) \to \mathrm{K}(X),$$

mapping the class of a left $A^{\otimes i}$-module M to the class of $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$, is an isomorphism.

Note that if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects the decomposition of Theorem 3.2.1, is injective, and the image

$$\mathrm{K}(A^{\otimes i}) \subset \mathrm{K}(A_F^{\otimes i}) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A^{\otimes i})/\mathrm{ind}(A_F^{\otimes i})$. For $i \geq 0$, let us write $\zeta_X(i)$ for the tensor product (over $A^{\otimes i}$) of $\zeta_X^{\otimes i}$ by a simple $A^{\otimes i}$-module. This is a vector bundle of rank $\mathrm{ind}(A^{\otimes i})$ and $\zeta_X^{\otimes i}$ decomposes into a direct sum of $\deg(A^{\otimes i})/\mathrm{ind}(A^{\otimes i})$ copies of $\zeta_X(i)$.

A similar description is afforded to the rings $\mathrm{K}(X)$ for products $X = X_1 \times \cdots \times X_r$ of Severi-Brauer varieties:

Theorem 3.2.2 (cf. [Pey95, Corollary 3.2])**.** *Let $X = X_1 \times \cdots \times X_r$ be a product of Severi-Brauer varieties $X_1, ..., X_r$ corresponding to central simple algebras $A_1, ..., A_r$ respectively. Then the group homomorphism*

$$\bigoplus_{I < (\deg(A_1), ..., \deg(A_r))} \mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \to \mathrm{K}(X),$$

as $I = (i_1, ..., i_r)$ ranges over r-tuples of nonnegative integers, is an isomorphism. Here the class of a left $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$-module M is sent to the class $\zeta_{X_1}^{\otimes i_1} \boxtimes \cdots \boxtimes \zeta_{X_r}^{\otimes i_r} \otimes_{A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}} M$.

Similarly, if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects this decomposition, is injective, and the image

$$\mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \subset \mathrm{K}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})/\mathrm{ind}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F)$.

Given two products of Severi-Brauer varieties $X = X_1 \times \cdots \times X_r$ and $Y = Y_1 \times \cdots \times Y_r$, over possibly different fields F_1 and F_2 with $\dim(X_i) = \dim(Y_i)$ for every $1 \leq i \leq r$, let us identify $\mathrm{K}(X_{\overline{F_1}})$ with $\mathrm{K}(Y_{\overline{F_2}})$ via the isomorphism of Theorem 3.2.2. Let us also identify $\mathrm{K}(X)$ and $\mathrm{K}(Y)$ with their images in $\mathrm{K}(X_{\overline{F_1}}) = \mathrm{K}(Y_{\overline{F_2}})$. Note that we have $\mathrm{K}(X) = \mathrm{K}(Y)$ if and only if

$$\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \mathrm{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r})$$

for all integers $i_1, ..., i_r$, where $A_1, ..., A_r$ are the algebras associated to $X_1, ..., X_r$ and $B_1, ..., B_r$ are the algebras associated to $Y_1, ..., Y_r$.

The following statement shows that (unlike the coniveau filtration) the γ-filtration on $\mathrm{K}(X)$ is completely determined by $\mathrm{K}(X)$:

Theorem 3.2.3 ([IK99, Theorem 1.1 and Corollary 1.2])**.** *If* $\mathrm{K}(X) = \mathrm{K}(Y)$, *then* $F_\gamma^i(X) = F_\gamma^i(Y)$ *for all* $i \geq 0$.

3.3 Equivalence of the two conjectures

Let G be an affine algebraic group, let U be a non-empty open G-invariant subset of a G-representation V. If the fppf quotient U/G is representable by a scheme, and if U is a G-torsor over U/G, then U has the property that for any G-torsor H over an infinite field $F \supset k$, there is an F-point x of U/G so that H is isomorphic to the fiber of the morphism $U \to U/G$ over x, cf. [Ser03, §5]. The generic fiber E of the quotient map $U \to U/G$ is called a *standard generic* G-torsor.

Example 3.3.1. If $G = \mathrm{SL}_n$, then G acts on $V = \mathrm{End}(k^n)$ with $\mathrm{GL}_n \subset V$ an open, G-invariant subset. The generic fiber $E = \mathrm{SL}_{n,k(\mathbb{G}_m)}$ of the quotient $\mathrm{GL}_n \to \mathrm{GL}_n/G = \mathbb{G}_m$ is a standard generic G-torsor.

A standard generic G-torsor E exists for any affine algebraic group G: one can take E to be the generic fiber of the quotient morphism $\mathrm{GL}_n \to \mathrm{GL}_n/G$ for any embedding $G \hookrightarrow \mathrm{GL}_n$.

Now assume G is a split semisimple algebraic group, with P a special parabolic subgroup of G, and E a standard generic G-torsor. Recall an algebraic group H over a field k is *special* if every H-torsor over any field extension of k is trivial. The quotient E/P is a *generic flag variety*, which is moreover *generically split*, meaning that E becomes trivial after scalar extension to the function field $k(E/P)$, cf. [Kar18, Lemma 7.1].

Example 3.3.2. Let $G = \mathrm{SL}_n/\mu_m$, where m is a divisor of n. Then G acts on $X = \mathbb{P}^{n-1}$ and, if P is the stabilizer of a rational point in X, the quotient G/P is isomorphic to X. The parabolic P is special, it's conjugacy class is given by the subset of the Dynkin diagram of G corresponding to removing the first vertex, see [Kar18, §8].

If E is a standard generic G-torsor given as the generic fiber of a quotient map $U \to U/G$, then our identification of $G/P \cong X$ above shows that the generic flag variety E/P is a Severi-Brauer variety over the function field $k(U/G)$. The central simple $k(U/G)$-algebra associated to E/P is called a *generic central simple algebra of degree n and exponent m*. The index of such an algebra is equal to r where $n = rs$ is a factorization of n with r having the same prime factors as m and with s prime to m.

In [Kar17c], Karpenko proves Conjecture 3.1.1 for the Severi-Brauer variety of a generic central simple algebra of degree n and exponent m and, as a Corollary obtained by analysis similar to Example 3.3.2 above, proves Conjecture 3.1.2 for split semisimple almost-simple algebraic groups of type A and C. In this section we prove an equivalence between Conjecture 3.1.1 and Conjecture 3.1.2 for algebraic groups of type AC similar to that obtained in [Kar17c] for a single Severi-Brauer variety and for a split semisimple almost-simple group of type A or of type C:

Theorem 3.3.3. *The following statements are equivalent:*

(1) *Conjecture 3.1.1 holds for all X,*

(2) *Conjecture 3.1.2 holds for all G of type* AC *and P given by removing the first vertex from each of the connected components of the Dynkin diagram of G,*

(3) *Conjecture 3.1.2 holds for all G of type* AC *and arbitrary P,*

(4) *Conjecture 3.1.2 holds for all G of type* AA *and arbitrary P.*

The proof is given below Lemma 3.3.6, after some preparation. It proceeds by showing (1) implies (2) implies (3) implies (4) implies (1). The most difficult part of the proof is in showing the last step, (4) implies (1). To do this, one realizes a product of Severi-Brauer varieties $X = X_1 \times \cdots \times X_r$ as a specialization of a generic flag variety E/P for a certain choice of split semisimple algebraic group G of type AA, standard generic G-torsor E, and special parabolic P. With mild hypotheses, one can show that this will prove the claim:

Lemma 3.3.4. *Let G be a split semisimple algebraic group of type* AA*, E a standard generic G-torsor, and P a special parabolic subgroup of G. Let X be a product of Severi-Brauer varieties such that X is a specialization of E/P. Assume the following conditions hold:*

(1) $\mathrm{CH}(X)$ *is generated by Chern classes,*

(2) *the canonical surjection* $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ *is an isomorphism,*

(3) *the specialization* $\mathrm{K}(E/P) \to \mathrm{K}(X)$ *is an isomorphism.*

Then the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Proof. Since X is a specialization of E/P, there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}(E/P) \arrow[r, two heads] \arrow[d] & \mathrm{gr}_\tau \mathrm{G}(E/P) \arrow[d] \\ \mathrm{CH}(X) \arrow[r, two heads] & \mathrm{gr}_\tau \mathrm{G}(X) \end{tikzcd} \tag{D}$$

where the downward-pointing vertical arrows are specializations and the horizontal arrows are the canonical surjections.

In the diagram (D) above, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption and $\mathrm{CH}(X)$ is generated by Chern classes by assumption. Note that $\mathrm{CH}(E/P)$ is also generated by Chern classes, by [Kar18, Corollary 7.2 and Theorem 7.3]. Since the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism it follows the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is surjective.

The specialization $\mathrm{gr}_\tau \mathrm{G}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism: it fits into the commutative square below with the vertical arrows being specializations and the horizontal arrows being the canonical maps; the horizontal arrows are isomorphisms since the Chow rings $\mathrm{CH}(E/P)$ and $\mathrm{CH}(X)$ are generated by Chern classes, [Kar98, proof of Theorem 3.7]; the left-vertical arrow is an isomorphism since by Theorem 3.2.3 the isomorphism $\mathrm{K}(E/P) \to \mathrm{K}(X)$ induces a bijection $F^i_\gamma(E/P) \cong F^i_\gamma(X)$ for all i.

$$\begin{tikzcd} \mathrm{gr}_\gamma \mathrm{K}(E/P) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{gr}_\tau \mathrm{G}(E/P) \arrow[d] \\ \mathrm{gr}_\gamma \mathrm{K}(X) \arrow[r, "\sim"] & \mathrm{gr}_\tau \mathrm{G}(X) \end{tikzcd}$$

Hence the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is also an injection and therefore an isomorphism. It follows the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism as well, completing the proof. $\square$

The problem is to find the correct G, P, and E that satisfy the conditions of Lemma 3.3.4. The naïve method, taking $E/P = E_1/P_1 \times \cdots \times E_r/P_r$ to be a product of generic flag varieties with each E_i/P_i having X_i as a specialization fails in at least one regard: the algebras associated to such an E/P are usually too unrelated. That is to say, the specialization in (3) of Lemma 3.3.4 will typically not be a surjection.

The following result of Nguyen, giving a description to the central simple algebras obtained from a G-torsor for split semisimple algebraic groups G of type AA, provides at least one resolution to this problem.

Theorem 3.3.5 ([CR15, Theorem A.1]). *Let $\Gamma = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a product of r general linear groups for some integers $n_1, ..., n_r$. Let C be a central subgroup of Γ, and write $G = \Gamma/C$. Let $\pi : G \to \Gamma/Z(\Gamma)$ be the natural projection. Then, for every field extension F of k, π_* identifies $H^1(F, G)$ with the set of isomorphism classes of r-tuples $(A_1, ..., A_r)$ of central simple F-algebras such that the degree of each A_i is $\deg(A_i) = n_i$, and $A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r}$ is split over F for every r-tuple of*

$$\mathscr{X}^*(Z(\Gamma)/C) = \{(m_1, ..., m_r) \in \mathbb{Z}^r \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \ \forall (\tau_1, ..., \tau_r) \in C\}.$$

To apply the theorem above to get the same description for the algebras associated to a G-torsor for a split semisimple algebraic group G of type AA, one notes that such a G is isomorphic to a quotient of a product $G_{sc} = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$ by a central subgroup C of G_{sc}. One can then use the quotient $G' = G^{red}/C$ of the reductive group $G^{red} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and the canonical inclusion $\iota : G \to G'$, taking into account that the induced map on cohomology $\iota_* : H^1(F, G) \to H^1(F, G')$ is a surjection (with trivial kernel).

It turns out, with the description given in Theorem 3.3.5, one has sufficient control to ensure the conditions of Lemma 3.3.4 hold (up to introducing some additional factors, which won't matter in the end).

Lemma 3.3.6. *Let $X_1, ..., X_r$ be a finite number of Severi-Brauer varieties corresponding to central simple k-algebras $A_1, ..., A_r$ and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ for all $1 \leq i \leq r$. For every r-tuple of nonnegative integers $I = (i_1, ..., i_r)$, write D_I for the underlying division algebra of the product $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ and write $Y_I = \mathrm{SB}(D_I)$ for the associated Severi-Brauer variety. Let $Z = X \times \prod_{I<(n_1,...,n_r)} Y_I$.*

In this setting, there exists a split semisimple algebraic group G of type AA *and a special parabolic P of G so that for any standard generic G-torsor E, the variety Z is a specialization of E/P and the specialization map $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ is an isomorphism.*

Proof. For every such r-tuple $I = (i_1, ..., i_r)$ we set $m_I := \mathrm{ind}(D_I)$ to be the index of D_I. The group

$$G_{sc} = \prod_{j=1}^{r} \mathrm{SL}_{n_j} \times \prod_{I<(n_1,...,n_r)} \mathrm{SL}_{m_I}$$

is split, semisimple, and simply connected of type AA. We consider the quotient $G := G_{sc}/S$, where S is the subgroup of the center of G_{sc} consisting of those elements

$$(x_1, ..., x_r, x_{(0,...,0)}, ..., x_{(n_1-1,...,n_r-1)})$$

satisfying the relation $x_{(i_1,...,i_r)} = x_1^{i_1} \cdots x_r^{i_r}$ (when identified with elements of $\mathbb{G}_m$). Let E be a standard generic G-torsor. We let

$$\sigma : G \to G_{ad}, \quad \pi_i : G_{ad} \to \mathrm{PGL}_{n_i}, \quad \pi_I : G_{ad} \to \mathrm{PGL}_{m_I}$$

be the canonical isogeny, projection to the ith factor for $i \leq r$, and projection to the factor corresponding to the r-tuple I respectively.

Let G^{red} be the reductive group

$$G^{red} = \prod_{j=1}^{r} \mathrm{GL}_{n_j} \times \prod_{I<(n_1,...,n_r)} \mathrm{GL}_{m_I}$$

and set $G' = G^{red}/S$. Let T be the kernel of the quotient $G^{red} \to G_{ad}$. We fix the isomorphism of the character group $\mathscr{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ that identifies the character with weights $(i_1, ..., i_n)$ with the element $(i_1, ..., i_n)$. The subgroup S above is defined so that the inclusion $\mathscr{X}^*(T/S) \to \mathscr{X}^*(T)$ identifies $\mathscr{X}^*(T/S)$ with the sublattice generated by those elements

$$(i_1, ..., i_r, -\delta_{I(0,...,0)}, ..., -\delta_{I(n_1-1,...,n_r-1)}),$$

where $I = (i_1, ..., i_r) < (n_1, ..., n_r)$ is an r-tuple. For any field extension F of k, the map $\sigma_* : H^1(F, G) \to H^1(F, G_{ad})$ factors through the map $H^1(F, G) \to H^1(F, G')$, induced by the inclusion of G into G'; this puts us in position to apply the description in Theorem 3.3.5 of the algebras $B_i := (\pi_i \circ \sigma)_*(E)$, $C_I := (\pi_I \circ \sigma)_*(E)$. In particular, our choice of S implies $B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r}$ is Brauer equivalent with $C_{(i_1,...,i_r)}$.

Again by Theorem 3.3.5, each of the algebras A_i are specializations of the algebras B_i and, additionally, for every r-tuple $I = (i_1, ..., i_r)$ we have an equality

$$m_I = \mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \mathrm{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r})$$

since the underlying division algebra D_I of $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ is a specialization of C_I. The first claim then results from the fact the variety

$$\prod_{i=1}^{r} \mathrm{SB}(B_i) \times \prod_{I<(n_1,...,n_r)} \mathrm{SB}(C_I)$$

is isomorphic with E/P which has Z as a specialization. The second claim results from the description of the rings $\mathrm{K}(E/P)$ and $\mathrm{K}(Z)$ given in Theorem 3.2.2. $\square$

And now for the proof:

Proof of Theorem 3.3.3. We show (1) implies (2). To start, let G be a group of type AC and E be a standard generic G-torsor over a field extension F of our base k. Let G_{ad} be the adjoint group of G; it is isomorphic to a product

$$G_{ad} = \prod_{i=1}^{n} G_i$$

with each G_i a simple adjoint group of type A or type C. We write $\sigma : G \to G_{ad}$ for the canonical isogeny from G to its adjoint and $\pi_i : G_{ad} \to G_i$ for the projection to the ith factor of G_{ad}.

From the n maps $\pi_i \circ \sigma$ with varying i, we obtain n central simple F-algebras given by the images of E under the pushforwards on Galois cohomology

$$(\pi_i \circ \sigma)_*(E) \in \mathrm{im}(H^1(F, G) \to H^1(F, G_i)).$$

Let X be the product of the Severi-Brauer varieties associated to the n algebras $(\pi_i \circ \sigma)_*(E)$. Then X is isomorphic to E/P, where P is a parabolic subgroup of G whose conjugacy class is given by the subset of the set of vertices of the Dynkin diagram of G obtained by excluding the first vertex of each of its connected components. That the parabolic P obtained in this way is special is a consequence of Lemma 3.3.8 below since, by [Kar18, §8], the group $\sigma(P)$ is special. The claim now follows from [Kar18, Corollary 7.2 and Theorem 7.3], which shows $\mathrm{CH}(X)$ is generated by Chern classes, allowing us to apply (1) to $X \cong E/P$.

(2) implies (3) is a consequence of [Kar17c, Lemma 4.2].

(3) implies (4) is obvious.

We finish by showing (4) implies (1). Let $X_1, ..., X_r$ be Severi-Brauer varieties over a field k, corresponding to central simple algebras $A_1, ..., A_r$ respectively, and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ be the degree of the algebra A_i. For every r-tuple of nonnegative integers $I = (i_1, ..., i_r)$ we write D_I for the underlying division algebra of the tensor product $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$. We write $Y_I := \mathrm{SB}(D_I)$ for the associated Severi-Brauer variety and $Z = X \times \prod_{I<(n_1,\ldots,n_r)} Y_I$ for the product of these varieties.

Let G and P be respectively an algebraic group of type AA and its special parabolic subgroup, obtained from Z as in Lemma 3.3.6. Let E be a standard generic G-torsor. By Lemma 3.3.7 below, to show the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism, it's sufficient to show $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism since the projection $Z \to X$ factors

$$Z \to X \times \prod_{I<(n_1,\ldots,n_{r-1},n_r-1)} Y_I \to \cdots \to X \times Y_{(0,\ldots,0)} \to X$$

with each arrow a projective bundle. Finally, the arrow $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism by Lemma 3.3.4: $\mathrm{CH}(Z)$ is generated by Chern classes by repeated applications of the projective bundle formula and the assumption $\mathrm{CH}(X)$ is generated by Chern classes, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption, and the specialization $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ is an isomorphism. □

Lemma 3.3.7. *Assume Z is a projective bundle over a variety X. Then the canonical epimorphism $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism if, and only if, the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism.*

Proof. The pullback along the projection $Z \to X$ gives a commuting diagram

$$\begin{tikzcd} \mathrm{CH}(Z) \arrow[r, two heads] & \mathrm{gr}_\tau \mathrm{G}(Z) \\ \mathrm{CH}(X) \arrow[r, two heads] \arrow[u] & \mathrm{gr}_\tau \mathrm{G}(X) \arrow[u] \end{tikzcd}$$

with both vertical arrows injections. It follows if the top-horizontal arrow is an isomorphism, then the bottom-horizontal arrow is an isomorphism.

The converse follows from the projective bundle formula: the groups $\mathrm{CH}(Z)$ and $\mathrm{gr}_\tau \mathrm{G}(Z)$ are direct sums of several copies of of the groups $\mathrm{CH}(X)$ and $\mathrm{gr}_\tau \mathrm{G}(X)$ respectively, and the coniveau epimorphism respects this direct sum decomposition. $\square$

Lemma 3.3.8. *Let G be a split semisimple algebraic group over a field F, and $\sigma : G \to G_{ad}$ the canonical isogeny with kernel C, the center of G. If P is a parabolic subgroup of G such that the image $\sigma(P)$ is special, then P is special.*

Proof. Let L be a Levi subgroup of P. By [Kar18, §3], P is special if and only if L is special. Since G is a split reductive group, P is also a split reductive group so that, by [Kar18, Theorem 2.1], L is special if and only if the semisimple commutator $L' \subset L$ is special. Similarly, $\sigma(P)$ is special if and only if $\sigma(L)'$ is special. Thus the proof of the lemma can be reduced to the following statement: if L' is a split semisimple algebraic group and $L' \to \sigma(L)'$ is an isogeny with $\sigma(L)'$ split, semisimple, and special, then L' is special. The result then follows from the fact a split semisimple algebraic group is special if and only if it is a product of special linear or symplectic groups and all such groups are simply connected. $\square$

We conclude this section with some remarks on, and special cases of, Conjectures 3.1.1 and 3.1.2.

Remark 3.3.9. One can construct a large class of products X of Severi-Brauer varieties which satisfy the condition $\mathrm{CH}(X)$ is generated by Chern classes. To do so, let $G = \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r}$ for some $n_1, .., n_r \geq 2$; let $A_1, ..., A_r$ be the central simple algebras associated to a standard generic G-torsor; let X be the product of the associated Severi-Brauer varieties. By [Kar18, Theorem 7.3], $\mathrm{CH}(X)$ has the desired property.

One can extend this class by base change: it's possible to lower the index of any tensor product $A = A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ by extending the base to the function field of any generalized Severi-Brauer variety of A. The new variety X obtained from these algebras also has the property $\mathrm{CH}(X)$ is generated by Chern classes, [Kar98, Theorem 3.7]. This procedure can be repeated indefinitely.

In fact, to prove Conjecture 3.1.1 for all products of Severi-Brauer varieties, it suffices to prove Conjecture 3.1.1 for the varieties obtained by the above procedure (one can even restrict to the class whose construction involves the function field of *usual* Severi-Brauer varieties only); to go from the above case to the general case, one can use the specialization argument as in Theorem 3.3.3.

Example 3.3.10 ($\mathrm{A}_1 \times \mathrm{A}_1$ and $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$)**.** In small rank cases, one can check Conjecture 3.1.2 for G of type AA by hand.

For G as in Conjecture 3.1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1$ one can observe: for any projective homogeneous variety X of dimension less or equal 2, the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism, cf. [CM06, Proposition 4.4].

For G as in Conjecture 3.1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, one can proceed by cases. If G is a product of groups of smaller rank, then [Kar17b, Proposition 4.1] proves the claim. Otherwise, G is a quotient of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ by the diagonal of the center $\mu_2 \times \mu_2 \times \mu_2$ or by the subgroup generated by the partial 2-diagonals. In the first case, the corresponding generic flag variety is a product $C \times C \times C$ of a fixed conic C and the claim follows. In the second case, the corresponding generic flag variety is a product $X = C_1 \times C_2 \times C_3$ where each C_i is the conic of a quaternion algebra Q_i; here the sum of the classes $[Q_1] + [Q_2] + [Q_3]$ is trivial in the Brauer group. Since X is a projective bundle over any two of the factors this proves the result by Lemma 3.3.7.

Example 3.3.11. Conjecture 3.1.2 holds for $G = \mathrm{SL}_n/\mu_m$ by [Kar17c, Theorem 1.1] and for products of such groups by [Kar17b, Proposition 4.1]. From this, one can show Conjecture 3.1.1 holds for products $X = X_1 \times \cdots \times X_r$ satisfying the following conditions:

(1) for each $1 \leq i \leq r$ there is a prime p_i so that the algebra A_i associated to the variety X_i has index $p_i^{n_i}$ and exponent $p_i^{m_i}$ for some integers $n_i \geq m_i \geq 1$,

(2) the algebras A_i satisfy $\mathrm{ind}(A_i^{\otimes p_i^{m_i-1}}) = \mathrm{ind}(A_i)/p_i^{m_i-1}$,

(3) the algebras A_i are disjoint in the sense there are equalities

$$\mathrm{ind}(A_1^{\otimes i_r} \otimes \cdots \otimes A_r^{i_r}) = \mathrm{ind}(A_1^{\otimes i_1}) \cdots \mathrm{ind}(A_r^{\otimes i_r})$$

for all integers $i_1, ..., i_r$.

To see this, one may assume that all A_i are division algebras and use Lemma 3.3.4. Property (2) allows one to realize such an X as a specialization of E/P where E is a standard generic $G = \prod_{1 \leq i \leq r} \mathrm{SL}_{p_i^{n_i}}/\mu_{p_i^{m_i}}$-torsor and $P \subset G$ is a special parabolic subgroup whose conjugacy class can be obtained by removing the first vertex from each of the connected components of the Dynkin diagram of G. The canonical map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ for this E/P is an isomorphism, as explained above. Now property (3), [Kar17b, Lemma 4.3], and Theorem 3.2.3 show the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism.

3.4 Algebras with level 1

In this section we introduce the level of a central simple k-algebra. The level is a nonnegative integer that measures, roughly speaking, how far away the algebra is from having its index equal to its exponent. It's related to, and depends on, the reduced behavior of the primary components of the algebra as defined in [Kar98]. The same concept was considered in [Bae15], there as the length of a reduced sequence obtained from the reduced behavior of a p-primary algebra for a prime p; the length of this reduced sequence as defined by Baek is equal to the level of the p-primary algebra as defined here.

It turns out the level of a central simple algebra A can be used to obtain detailed information on λ-ring generators for the Grothendieck ring of the Severi-Brauer variety X of A, see Lemma 3.4.6. A particular consequence of this is that the subring of $\mathrm{CH}(X)$ which is generated by Chern classes has an explicit and small set of generators that can be helpful for computational purposes. Using this more refined information based on the level, we're able to generalize the results of [Kar17c] to prove the main result, Theorem 3.4.15, that the K-theory coniveau epimorphism is an isomorphism for Severi-Brauer varieties whose Chow ring is generated by Chern classes and whose associated central simple algebra has level 1.

Throughout this section we work with a fixed prime p and we continue to work over the fixed but arbitrary field k. We write $v_p(-)$ for the p-adic valuation. We've relegated some computations needed in this section to Section 3.5.

Recall, the *reduced behavior* of an algebra A with index $\mathrm{ind}(A) = p^n$ and exponent $\exp(A) = p^m$, $0 < m \le n$, is defined to be the following sequence of p-adic orders of increasing p-primary tensor powers of A:

$$\begin{aligned} r\mathcal{B}eh(A) &= \left(v_p(\mathrm{ind}(A^{\otimes p^i}))\right)_{i=0}^{m} \\ &= \left(v_p(\mathrm{ind}(A)), v_p(\mathrm{ind}(A^{\otimes p})), ..., v_p(\mathrm{ind}(A^{\otimes p^m}))\right). \end{aligned}$$

The reduced behavior of A is strictly decreasing; it starts with $v_p(\mathrm{ind}(A)) = n$ and ends with $v_p(\mathrm{ind}(A^{\otimes p^m})) = 0$.

Definition 3.4.1. A is said to have *level* l, abbreviated $\mathrm{lev}(A) = l$, if there exist exactly l distinct integers $i_1, ..., i_l \ge 1$ with

$$v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1$$

for every $1 \le k \le l$. If no such integers exist, A is said to have level 0. An arbitrary central simple algebra B, not necessarily p-primary, is said to have *level* l if l is the maximum

$$l = \max_{q \text{ prime}} \{\mathrm{lev}(B_q)\}$$

of the levels of the q-primary components B_q of B.

Example 3.4.2. A central simple algebra A has level 0, i.e. $\mathrm{lev}(A) = 0$, if and only if the index and exponent of A coincide, $\mathrm{ind}(A) = \exp(A)$.

Example 3.4.3. If A is a generic algebra of degree p^n and exponent p^m with $m < n$, in the sense of Example 3.3.2, then the level of A is 1, i.e. $\mathrm{lev}(A) = 1$.

The reduced behavior for this algebra is

$$\begin{aligned} r\mathcal{B}eh(A) &= \big(v_p(\mathrm{ind}(A)), v_p(\mathrm{ind}(A^{\otimes p})), ..., v_p(\mathrm{ind}(A^{\otimes p^m}))\big) \\ &= (n, n-1, ..., n-m+1, 0). \end{aligned}$$

To see this, note that with a large enough field extension F of k one may find a central division F-algebra B with index p^n, exponent p^m, and reduced behavior $r\mathcal{B}eh(B) = (n, n-1, ..., n-m+1, 0)$, [Kar98, Lemma 3.10]. Since B is a specialization of A it follows

$$p^{n-i} \geq \mathrm{ind}(A^{\otimes p^i}) \geq \mathrm{ind}(B^{\otimes p^i}) = p^{n-i}$$

for $i = 0, \ldots, m-1$, so that equalities hold throughout.

We make the following definition for notational convenience.

Definition 3.4.4. The *Chern subring* of a smooth variety X, denoted $\mathrm{CS}(X)$, is the subring of $\mathrm{CH}(X)$ which is generated by all Chern classes of elements of $\mathrm{K}(X)$.

Proposition 3.4.5. *Let X be the Severi-Brauer variety of a central simple algebra A with* $\mathrm{ind}(A) = p^n$ *and* $\mathrm{lev}(A) = r$*. Then* $\mathrm{CS}(X)$ *is generated, as a ring, by the Chern classes of $r+1$ sheaves on X. Namely, the sheaves whose Chern classes generate* $\mathrm{CS}(X)$ *are:*

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \ldots,\ \zeta_X(p^{i_r}),$$

where $1 \leq i_1 < \cdots < i_r$ are the r distinct integers with $v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1$.

Proof. It suffices to show that $\mathrm{K}(X)$ is generated by the classes of

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \ldots,\ \zeta_X(p^{i_r})$$

as a λ-ring; this is because Chern classes of λ-operations of an element of $\mathrm{K}(X)$ are certain universal polynomials in the Chern classes of this element. This is done in the next lemma. $\square$

Lemma 3.4.6. *Let X be the Severi-Brauer variety of a central simple algebra A with $\mathrm{ind}(A) = p^n$ and $\mathrm{lev}(A) = r$. Then $\mathrm{K}(X)$ is generated, as a λ-ring, by $r+1$ elements. Namely, the sheaves whose classes generate $\mathrm{K}(X)$ are:*

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \ldots,\ \zeta_X(p^{i_r}),$$

where $1 \leq i_1 < \cdots < i_r$ are the r distinct integers with $v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1$.

Proof. Since the pullback $\pi^* : \mathrm{K}(X) \to \mathrm{K}(X_L)$ to a splitting field L of A is injective, we can work, instead of $\mathrm{K}(X)$ itself, with its image in $\mathrm{K}(X_L)$. We'll write ξ to denote the class of $\mathcal{O}(-1)$ in $\mathrm{K}(X_L)$. By the comments under Theorem 3.2.1 we have $\pi^*(\zeta_X(i)) = \mathrm{ind}(A^{\otimes i})\xi^i$. It follows that the elements $\mathrm{ind}(A^{\otimes i})\xi^i$ with $i \geq 0$ generate $\mathrm{K}(X)$ as an abelian group.

The λ-operations of any multiple of ξ^i are easy to compute:

$$\lambda^j(d\xi^i) = \binom{d}{j}\xi^{ij} \quad \text{for any } i, j, d \geq 0.$$

Let us first show that the elements $\mathrm{ind}(A^{\otimes p^j})\xi^{p^j}$ $(j \geq 0)$ generate $\mathrm{K}(X)$ as a λ-ring. Since the λ-subring generated by these elements contains powers of $\mathrm{ind}(A)\xi = p^n\xi$, we only need to check that, for every $i \geq 1$, this subring contains an integer multiple of ξ^i whose coefficient has p-adic valuation equal

$v_p(\mathrm{ind}(A^{\otimes i}))$. For this, given any $i \geq 1$, we write $i = p^j s$ with $j \geq 0$ and s prime-to-p. We set $p^v := \mathrm{ind}(A^{\otimes i}) = \mathrm{ind}(A^{\otimes p^j})$. Write further $s = s_0 p^v + s_1$ with $0 \leq s_1 < p^v$ and $s_0 \geq 0$. Then we have $\lambda^{p^v}(p^v \xi^{p^j}) = \xi^{p^j p^v}$ and $\lambda^{s_1}(p^v \xi^{p^j})$ is a multiple of $\xi^{p^j s_1}$ with p-adic valuation of the (binomial) coefficient of this multiple equal p^v, see [Kar98, Lemma 3.5]. The claim we are checking follows.

It remains to show if $v_p(\mathrm{ind}(A^{\otimes p^j})) \geq v_p(\mathrm{ind}(A^{\otimes p^{j-1}})) - 1$ for some $j \geq 1$, then the generator $\mathrm{ind}(A^{\otimes p^j})\xi^{p^j}$ can be omitted. Let us set $p^v := \mathrm{ind}(A^{\otimes p^{j-1}})$. If $v = 0$, then we get ξ^{p^j} as a pth power of $\xi^{p^{j-1}} = \mathrm{ind}(A^{\otimes p^{j-1}})\xi^{p^{j-1}}$. For $v > 0$, we consider the λ-operation $\lambda^p(p^v \xi^{p^{j-1}})$ which is a multiple of ξ^{p^j} with p-adic valuation of its coefficient equal $v - 1 \leq v_p(\mathrm{ind}(A^{\otimes p^j}))$. □

To systematically study the relations between the Chern classes of the sheaves appearing in Proposition 3.4.5, we introduce:

Definition 3.4.7. Let A be a central simple algebra and X the Severi-Brauer variety of A. We write $\mathrm{CT}(i_1, ..., i_r; X)$ for the graded subring of $\mathrm{CS}(X) \subset \mathrm{CH}(X)$ generated by the Chern classes of the sheaves $\zeta_X(i_1), ..., \zeta_X(i_r)$.

Proposition 3.4.8. *Let X be the Severi-Brauer variety of a central simple algebra A. Then, for any $i > 0$, $\mathrm{CT}(i; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Moreover, for $0 \leq j < \deg(A)$ the group $\mathrm{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ is additively generated by*

$$\tau_i(j) := c_{p^v}(\zeta_X(i))^{s_0} c_{s_1}(\zeta_X(i))$$

where p^v is the largest power of p dividing $\mathrm{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \leq s_1 < p^v$.

Proof. By first extending to a prime-to-p extension (which is an injection when $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ has $\mathbb{Z}_{(p)}$-coefficients) that splits the prime-to-p components of A, we can assume A is p-primary. We continue by reducing to the case $i = 1$.

Lemma 3.4.9. *Let X be the Severi-Brauer variety of a central simple algebra A, and let Y be the Severi-Brauer variety of $A^{\otimes i}$. Then there is a functorial surjection*

$$\mathrm{CT}(1;Y) \twoheadrightarrow \mathrm{CT}(i;X).$$

Proof. Let

$$X \to X^{\times i} \to Y$$

be the composition of the diagonal embedding and the twisted Segre embedding. The corresponding maps on Grothendieck groups can be determined by moving to a splitting field L of X. There is a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}(Y_L) & \longrightarrow & \mathrm{K}(X_L^{\times i}) & \longrightarrow & \mathrm{K}(X_L) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{K}(Y) & \longrightarrow & \mathrm{K}(X^{\times i}) & \longrightarrow & \mathrm{K}(X) \end{array}$$

defined so that under the top-horizontal maps we have

$$\mathcal{O}_{Y_L}(-1) \mapsto \mathcal{O}_{X_L}(-1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_L}(-1) \mapsto \mathcal{O}_{X_L}(-i).$$

Thus, the class of $\zeta_Y(1)$ on Y is mapped to the class of $\zeta_X(i)$ on X.

So under the composition of the diagonal $X \to X^{\times i}$ and the twisted Segre embedding $X^{\times i} \to Y$, there is a surjection $\mathrm{CT}(1;Y) \twoheadrightarrow \mathrm{CT}(i;X)$ induced by the pullback $\mathrm{CH}(Y) \to \mathrm{CH}(X)$. □

Next we reduce to the case our algebra is division. Let D be the underlying division algebra of A, and Y the Severi-Brauer variety of D. Fix an embedding $Y \to X$ so that, over a splitting field of both, the inclusion is as a linear subvariety. The pullback

$$\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(Y) \otimes \mathbb{Z}_{(p)}$$

is an isomorphism in degrees where both groups are nonzero. If the claim is true for $\mathrm{CH}(Y)\otimes\mathbb{Z}_{(p)}$ then, since the pullback is functorial for Chern classes, we find $\mathrm{CT}^j(1;X)\otimes\mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module of rank 1 in degrees $0 \leq j < \deg(D)$. That this holds is due to [Kar17c, Proposition 3.3], where it's shown $\mathrm{CT}(1;X)$ is free if A is division. This will serve as the base case for an induction proof.

In an arbitrary degree j between $\deg(D) \leq j < \deg(A)$, we assume the claim is true for all degrees $0 \leq k < j$. It suffices to show the multiplication by $\tau_1(p^v) = c_{p^v}(\zeta_X(1))$ map

$$\mathrm{CT}^{j-p^v}(1;X)\otimes\mathbb{Z}_{(p)} \to \mathrm{CT}^j(1;X)\otimes\mathbb{Z}_{(p)}$$

is surjective and, by Nakayama's Lemma, we can do this modulo p. Any element of $\mathrm{CT}^j(1;X)$ is a sum of monomials of the form $\tau_1(j-p^v)c_{i_1}^{n_1}\cdots c_{i_r}^{n_r}$ with $c_i = c_i(\zeta_X(1))$. We claim any such monomial which is not $\tau_1(j) = \tau_1(j-p^v)\tau_1(p^v)$ is congruent to 0 modulo p.

Indeed, if such a monomial was divisible by c_{i_1}, c_{i_2} then without loss of generality we can assume $v_p(i_2) \leq v_p(i_1) < v$. By [Kar17c, Proposition 3.5] there is a field F finite over the base so that $v_p\mathrm{ind}(A_F) = v_p(i_1)$, and $c_{i_1} = \pi_*(x)$ for an element x of $\mathrm{CH}(X_F)\otimes\mathbb{Z}_{(p)}$ and where $\pi : X_F \to X$ is the projection. Using the projection formula we find

$$c_{i_1}c_{i_2} = \pi_*(x)c_{i_2} = \pi_*(x\pi^*(c_{i_2})).$$

By Lemma 3.4.10 below, it follows $\pi^*(c_{i_2})$ is divisible by p which proves the claim.

To see the generators are as claimed for $i = 1$, one can compute the degrees of the images of the Chern classes of $\zeta_X(1)$ over an algebraic closure; for the other i, one can use Lemma 3.4.9. $\square$

Lemma 3.4.10. *Let X be the Severi-Brauer variety of a central simple algebra A with $\operatorname{ind}(A) = p^v$. Let F be a field with $p^{v-s} = \operatorname{ind}(A_F) < \operatorname{ind}(A) = p^v$ and let $\pi : X_F \to X$ be the projection. Then*

$$\pi^*(c_j(\zeta_X(1)) = 0 \pmod p$$

for all j not divisible by p^v.

Proof. We have $\pi^*(\zeta_X(1)) = \zeta_{X_F}(1)^{\oplus p^s}$ with $p^s = \operatorname{ind}(A)/\operatorname{ind}(A_F)$. By functorality we have

$$\pi^*(c_j(\zeta_X(1))) = c_j(\zeta_{X_F}(1)^{\oplus p^s}).$$

We're going to compute the total Chern polynomial of $\zeta_{X_F}(1)^{\oplus p^s}$ modulo p. If F splits A then $c_t(\zeta_{X_F}(1)^{\oplus p^s}) = (1-h)^{p^s} = 1 \pm h^{p^s} \pmod p$ where h is the class of a hyperplane in $\operatorname{CH}(X_F)$. Otherwise $v \neq s$ and we have

$$c_t(\zeta_{X_F}(1)^{\oplus p^s}) = c_t(\zeta_{X_F}(1))^{p^s} = (1 + c_1 t + \cdots + c_{p^{v-s}} t^{p^{v-s}})^{p^s}$$

with $c_i = c_i(\zeta_{X_F}(1))$. Using the multinomial formula, the latter expression can be rewritten

$$1 + \sum_{j=1}^{p^v} \left(\sum_{\substack{|I|=p^s \\ i_1+2i_2+\cdots+p^{v-s}i_{p^{v-s}}=j}} \binom{p^s}{i_0, i_1, \ldots, i_{p^{v-s}}} c_1^{i_1} \cdots c_{p^{v-s}}^{i_{p^{v-s}}} \right) t^j.$$

Here the notation means $\binom{n}{a_0,\ldots,a_i} = \frac{n!}{a_0!\cdots a_i!}$ and $I = (i_0, \ldots, i_{p^{v-s}})$ is a tuple of nonnegative integers with $|I| = i_0 + \cdots + i_{p^{v-s}}$.

By Lemma 3.5.3, p divides all of the coefficients $\binom{p^s}{i_0,\ldots,i_{p^{v-s}}}$ except when p^s divides one of $i_0, \ldots, i_{p^{v-s}}$. We are left to show $c_{i_k}^{p^s} = 0$ modulo p for any $k = 0, \ldots, p^{v-s} - 1$. Using [Kar17c, Proposition 3.5], we can find a finite field extension E/F lowering the index of A_F and such that $c_{i_k} = \rho_*(x)$ for some x in $\mathrm{CH}(X_E) \otimes \mathbb{Z}_{(p)}$ and for $\rho : X_E \to X_F$ the projection. The projection formula then gives

$$c_{i_k}^{p^s} = \rho_*(x(\rho^*\rho_*(x))^{p^s-1}) = 0 \pmod p$$

since $\rho^*\rho_* = [E : F]$. □

Corollary 3.4.11. *Let A be a central simple algebra and X its associated Severi-Brauer variety. The classes $\tau_i(j)$ of $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ satisfy the relations:*

(1) *for all $i \geq 1$, we have $\tau_i(0) = 1$,*

(2) *for any $j \geq 0$, we have $\tau_i(p^v)\tau_i(j) = \tau_i(p^v j)$, where $v = v_p(\mathrm{ind}(A^{\otimes i}))$,*

(3) *for any integers $a_1, ..., a_{p^v} \geq 0$, there is a relation*

$$\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}} = \alpha \tau_i(a_1 + 2a_2 + \cdots + p^v a_{p^v})$$

for some α in $\mathbb{Z}_{(p)}$ with

$$v_p(\alpha) = \begin{cases} 0 & \textit{if} \quad v = 0 \\ \sum_{k=1}^{p^v}(v - v_p(k))a_k & \textit{if } v > 0,\ j = 0 \pmod{p^v} \\ v_p(r) - v + \sum_{k=1}^{p^v}(v - v_p(k))a_k & \textit{if } v > 0,\ j \neq 0 \pmod{p^v} \end{cases}$$

where we write $j = a_1 + 2a_2 + \cdots + p^v a_{p^v}$ and $0 \leq r < p^v$ is the remainder in the division of j by p^v.

Proof. We remark that the definition of the classes $\tau_i(j)$ makes sense for any integer $j \geq 0$ but when $j > \deg(A)$ these classes are 0. For simplifications below, we don't put any upper bound on the value j may have.

The relation (1) is obvious from the definition. The relation (2) is also clear from the definition. So we're left proving the complicated relation (3). To do this, we pullback, to a splitting field L, the left and right side of the equation in (3) and compare p-adic valuations of their coefficients on the element h^j where h is the class of a hyperplane over L. Some immediate observations for the following: we can assume j isn't larger than the dimension of X and we can assume $v > 0$; otherwise the claim is trivial.

The pullback of $\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}}$ can be written βh^j where

$$v_p(\beta) = \sum_{k=1}^{p^v}(v - v_p(k) + v_p(i)k)a_k.$$

Similarly, the pullback of $\tau_i(a_1 + \cdots + p^v a_{p^v})$ can be written γh^j with

$$v_p(\gamma) = \begin{cases} v_p(i)p^v s_0 & \text{if} \quad j = 0 \pmod{p^v} \\ v_p(i)p^v s_0 + v - v_p(s_1) + v_p(i)s_1 & \text{if} \quad j \neq 0 \pmod{p^v} \end{cases}$$

where $j = s_0 p^v + s_1$ and $0 \leq s_1 < p^v$. Since $v_p(\gamma) \geq v_p(\beta)$ by Proposition 3.4.8, the result follows by subtracting. $\square$

Lemma 3.4.12. *Let A be a central simple algebra with* $\mathrm{ind}(A) = p^n$ *and* $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. *Let X be the Severi-Brauer variety of A. Then, for any pair of integers i, j with $0 \leq i \leq j \leq m$, the total Chern polynomial*

$$c_t(\zeta_X(p^j))^{p^{n_i - n_j - (j-i)}} = 1 + \sum_{k=1}^{p^{n_i-(j-i)}} \beta_k \tau_{p^j}(k) t^k$$

is a polynomial with coefficients in $\mathrm{CT}(p^i; X) \otimes \mathbb{Z}_{(p)}$.

Moreover, the p-adic valuation of the coefficient β_k equals

$$v_p(\beta_k) = \begin{cases} n_i - n_j - (j-i) - v_p(k/p^{n_j}) & \textit{if} \quad k = 0 \pmod{p^{n_j}} \\ n_i - n_j - (j-i) & \textit{if} \quad k \neq 0 \pmod{p^{n_j}}. \end{cases}$$

Proof. We identify $\mathrm{K}(X)$ with its image in $\mathrm{K}(X_L)$ for a splitting field L of X. We write ξ for the class of $\mathcal{O}(-1)$ in $\mathrm{K}(X_L)$. Then the class of $\zeta_X(p^i)$ is identified with $p^{n_i}\xi^{p^i}$ and the class of $\zeta_X(p^j)$ is identified with $p^{n_j}\xi^{p^j}$. We have

$$\lambda^{p^{j-i}}(p^{n_i}\xi^{p^i}) = \binom{p^{n_i}}{p^{j-i}}\xi^{p^j}.$$

It follows that

$$\begin{aligned} c_t(p^{n_i-(j-i)}\xi^{p^j}) &= c_t(p^{n_i-(j-i)-n_j}(p^{n_j}\xi^{p^j})) \\ &= c_t(\zeta_X(p^j))^{p^{n_i-n_j-(j-i)}} \\ &= (1 + \tau_{p^j}(1)t + \cdots + \tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} \end{aligned}$$

is a polynomial with coefficients contained in $\mathrm{CT}(p^i;X) \otimes \mathbb{Z}_{(p)}$. This proves the first claim.

To prove the second claim, we write

$$= (1 + \tau_{p^j}(1)t + \cdots + \tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} = 1 + \sum_{k=1}^{p^{n_i-(j-i)}} \beta_k \tau_{p^j}(k)t^k$$

using Proposition 3.4.8. Explicitly there are equalities

$$\beta_k \tau_{p^j}(k) = \sum_I \binom{p^{n_i-(j-i)-n_j}}{I} \tau_{p^j}^I$$

where the sum runs over tuples $I = (a_0, ..., a_{p^{n_j}})$ such that $a_0 + \cdots + a_{p^{n_j}} =$

$p^{n_i-(j-i)-n_j}$ and $a_1 + 2a_2 + \cdots + p^{n_j} a_{p^{n_j}} = k$; here we're using the notation

$$\binom{p^{n_i-(j-i)-n_j}}{I} = \binom{p^{n_i-(j-i)-n_j}}{a_0, ..., a_{p^{n_j}}} = \frac{p^{n_i-(j-i)-n_j}!}{a_0! \cdots a_{p^{n_j}}!}$$

and

$$\tau_{p^j}^I = \tau_{p^j}(0)^{a_0} \tau_{p^j}(1)^{a_1} \cdots \tau_{p^j}(p^{n_j})^{a_{p^{n_j}}}$$

for a tuple $I = (a_0, ..., a_{p^{n_j}})$. Thus

$$v_p(\beta_k) = v_p\left(\sum_I \binom{p^{n_i-(j-i)-n_j}}{I} \alpha_I\right) \geq \min\left\{v_p\left(\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I\right)\right\}$$

where α_I is the coefficient in $\tau_{p^j}^I = \alpha_I \tau_{p^j}(k)$ from Corollary 3.4.11. In fact, the above inequality is an equality if there is a unique minimum over the given tuples I. The p-adic valuation of any coefficient $\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I$ can be found using Corollary 3.4.11 and Lemma 3.5.2; the p-adic valuation of any coefficient $\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I$ can also be bounded below using Corollary 3.4.11 and Lemma 3.5.3. With this bound, one can show there is a unique minimum among the $v_p(\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I)$: set $s = n_i - (j-i)$ and $r = n_j$ in Lemma 3.5.4. Finally, using Lemma 3.5.2 to compute the valuation explicitly and using Lemma 3.5.5, setting $s = n_i - (j-i)$ and $r = n_j$, shows the p-adic valuation of β_k is as claimed. $\square$

The lemma above provides a collection of numbers β_k with $\beta_k \mathrm{CT}^k(p^j; X) \subset \mathrm{CT}^k(1; X)$. Using a technique developed in [Kar17c], we can reduce the size of the β_j further. We assume A is a division algebra in the following as this is the only case we will need.

Corollary 3.4.13. *Let A be a division algebra with $\mathrm{ind}(A) = p^n$ and $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Let X be the Severi-Brauer variety of A. Pick an integer $0 \leq j \leq m$, and let $0 \leq i \leq p^n - 1$ be a second integer.*

There exists a number α_i in $\mathbb{Z}_{(p)}$ so that $\alpha_i \tau_{p^j}(i)$ is contained in $\mathrm{CT}(1;X)\otimes \mathbb{Z}_{(p)}$*. Moreover, the p-adic valuation of the α_i we find equals*

$$v_p(\alpha_i) = \begin{cases} n-j-n_j & \textit{if} \quad 1 \le i \le p^{n_j} \\ n-j-n_j - \lfloor \log_p(i/p^{n_j}) \rfloor & \textit{if} \quad p^{n_j} < i \le p^{n-j} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let L be a maximal subfield of A, of degree p^n over the base, and let N be the image of the pushforward $\pi_* : \mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ along the projection $\pi : X_L \to X$. By [Kar17c, Proposition 3.5], the image N is contained in $\mathrm{CT}(1;X) \otimes \mathbb{Z}_{(p)}$. Recall also the pullback π^* followed by the pushforward π^* is multiplication by p^n, the degree of L over the base. The proof of the corollary mimics that of [Kar17c, Proposition 3.12]; the idea of the proof is to use the explicit bounds of Lemma 3.4.12 and the projection formula to get the result for any i. Note that the claim is trivial for $j = 0$ (or we can just set $\alpha_i = 1$ in this case) so, throughout the proof, it's safe to assume $j > 0$.

We first show, for $i \le p^{n-j}$ and using β_i for the coefficient such that $\beta_i \mathrm{CT}^i(p^j;X) \subset \mathrm{CT}^i(1;X)$ found in Lemma 3.4.12, that $p^{v_p(\beta_i)}\tau_{p^j}(i)$ is in the image of the map π_*. Write $i = s_0 p^{n_j} + s_1$ with $0 \le s_1 < p^{n_j}$. The image of $\tau_{p^j}(i)$ in $\mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)}$ is equal, up to prime-to-p parts, to

$$\pi^*(\tau_{p^j}(i)) = \begin{cases} p^{ij} h^i & \text{if} \quad s_1 = 0 \\ p^{ij+n_j-v_p(s_1)} h^i & \text{if} \quad s_1 > 0. \end{cases}$$

By Lemma 3.4.12, the multiple $\beta_i \tau_{p^j}(i)$ has image, up to prime-to-p parts,

$$\pi^*(\beta_i \tau_{p^j}(i)) = p^{n+(i-1)j-v_p(i)} h^i$$

regardless of s_1. Thus,

$$\begin{aligned} p^{v_p(\beta_i)}\tau_{p^j}(i) &= \frac{1}{p^n}\pi_*\pi^*(p^{v_p(\beta_i)}\tau_{p^j}(i)) \\ &= \pi_*(\frac{1}{p^n}(\pi^*(p^{v_p(\beta_i)}\tau_{p^j}(i)))) \\ &= \pi_*(p^{(i-1)j-v_p(i)}h^i). \end{aligned}$$

Since $(i-1)j - v_p(i) \geq 0$, we find $p^{v_p(\beta_i)}\tau_{p^j}(i)$ is in N as claimed.

Now let i be an integer with $1 \leq i \leq p^n - 1$ and set $\ell = \lfloor \log_p(i/p^{n_j}) \rfloor$. To get the bounds on the p-adic valuation in the corollary statement, we work in cases. We first assume $\ell \geq n - j - n_j$ or equivalently $i \geq p^{n-j}$. By the above and Lemma 3.4.12, we can find an element x of $\mathrm{CH}(X_L)$ with

$$\pi_*(x) = \tau_{p^j}(p^{n-j}).$$

Set $k = i - p^{n-j}$. Then, using (2) and (3) of Corollary 3.4.11,

$$\begin{aligned} \tau_{p^j}(i) &= \tau_{p^j}(p^{n_j})^{n-j-n_j}\tau_{p^j}(k) \\ &= \tau_{p^j}(p^{n-j})\tau_{p^j}(k) \\ &= \pi_*(x)\tau_{p^j}(k) \\ &= \pi_*(x\pi^*(\tau_{p^j}(k))). \end{aligned}$$

By [Kar17c, Proposition 3.5], it follows $\tau_{p^j}(i)$ is contained in $N \subset \mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$ for all $i \geq p^{n-j}$.

For the other i, we act similarly. If $p^{n_j} < i \leq p^{n-j}$ then set $k = i - p^{n_j+\ell}$. Then there is a (different) element x with $\pi_*(x) = p^r\tau_{p^j}(p^{\ell+n_j})$ where $r =$

$v_p(\beta_{p^{\ell+n_j}})$. Then

$$\begin{aligned} p^r\tau_{p^j}(i) &= p^r\tau_{p^j}(p^{n_j})^{\ell}\tau_{p^j}(k) \\ &= p^r\tau_{p^j}(p^{\ell+n_j})\tau_{p^j}(k) \\ &= \pi_*(x)\tau_{p^j}(k) \\ &= \pi_*(x\pi^*(\tau_{p^j}(k))) \end{aligned}$$

and the claim follows as before.

For the remaining i, when $i \le p^{n_j}$, the claim is actually immediate from Lemma 3.4.12. □

We can do better still if we multiply the classes $\tau_1(i)$ and $\tau_{p^j}(k)$ for some integers $i, k \ge 0$.

Corollary 3.4.14. *Let A be a division algebra with $\operatorname{ind}(A) = p^n$ and $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Let X be the Severi-Brauer variety of A. Pick an integer $0 \le j \le m$, and let $1 \le i, k \le p^n - 1$ be two integers with $i + k \le p^n - 1$.*

There exists a number $\beta_{i,k}$ in $\mathbb{Z}_{(p)}$ so that $\beta_{i,k}\tau_1(i)\tau_{p^j}(k)$ is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. Moreover, the p-adic valuation of the $\beta_{i,k}$ we find equals

$$v_p(\beta_{i,k}) = \begin{cases} \max\{v_p(i) - j - n_j, 0\} & \textit{if} \quad 1 \le k \le p^{n_j} \\ \max\{v_p(i) - j - n_j - \lfloor \log_p(k/p^{n_j}) \rfloor, 0\} & \textit{if} \quad p^{n_j} < k \le p^{n-j} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is the same as Corollary 3.4.13 except that we use the equality, up to prime-to-p parts,

$$\pi^*(\beta_k\tau_1(i)\tau_{p^j}(k)) = p^{n+(k-1)j-v_p(k)+n-v_p(i)}h^{i+k}$$

to find $p^{v_p(\beta_{i,k})}\tau_1(i)\tau_{p^j}(k)$ is contained in N. $\square$

As an application, the above can be used to settle the particular case of Conjecture 3.1.1 when X is the Severi-Brauer variety of an algebra A with level 1:

Theorem 3.4.15. *Let A be a central simple k-algebra of level 1 and let X be the Severi-Brauer variety of A. Assume $\mathrm{CH}(X)$ is generated by Chern classes. Then the* K*-theory coniveau epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism.*

Proof. It's sufficient to show the claim when A is a division algebra of index p^n. In this case the kernel of the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is p-primary-torsion so we can work with $\mathbb{Z}_{(p)}$ coefficients throughout the proof. Let L be a splitting field for A. Since $\mathrm{CT}(1;X) \otimes \mathbb{Z}_{(p)}$ is p-torsion free, the composition

$$\mathrm{CT}(1;X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)}$$

is injective; we denote by C the image of this composition. We have an inequality

$$[\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} : \mathrm{CT}(1;X) \otimes \mathbb{Z}_{(p)}] \geq [\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C]. \qquad \text{(in)}$$

We're going to use the bounds from Corollary 3.4.14 to get an upper bound on the left of (in). We'll also bound the right of (in), by computing

$$[\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C] = \frac{[\mathrm{gr}_\tau \mathrm{G}(X_L) : C]}{[\mathrm{K}(X_L) : \mathrm{K}(X)]}$$

precisely; the equality of the ratio of these indices can be found in [Kar17c, proof of Theorem 3.1]. The proof will be completed once we show these two bounds are equal.

To get an upper bound on the left of (in), we sum the maximums of the p-adic valuations occurring in Corollaries 3.4.13 and 3.4.14. Plainly said, we compute an upper bound on p-adic valuations of the orders of the elements $\tau_1(i)\tau_{p^r}(k)$, where r is the (unique since A has level 1) smallest positive integer with $v_p(\mathrm{ind}(A^{\otimes p^r})) < v_p(\mathrm{ind}(A^{\otimes p^{r-1}})) - 1$, in the group $\mathrm{CH}(X)/\mathrm{CT}(1;X)$. Note that, by Proposition 3.4.5 and Proposition 3.4.8 the elements $\tau_1(i)\tau_{p^r}(k)$ are exactly the generators of this quotient group so that by computing an upper bound on their orders and raising p to this upper bound, we also compute an upper bound on the index in the left of (in). Once we have this upper bound, we'll move on to give a lower bound for the right hand side of (in). These two bounds turn out to be equal, showing our upper bound on the orders were in fact their precise order.

Set $n_r = v_p(\mathrm{ind}(A^{\otimes p^r}))$ and $\ell = n - r - n_r$. When $i = 0$, we sum the contributions from Corollary 3.4.13,

$$\sum_{a=1}^{p^{n_r}-1} n - r - n_r + \sum_{a=p^{n_r}}^{p^{n-r}-1} n - r - n_r - \lfloor \log_p(a/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)\ell + \sum_{b=0}^{\ell-1} \varphi(p^{n_r+b+1})(\ell - b)$$

where φ is the Euler totient function (we use this function to combine those terms a that have the same value of $\lfloor \log_p(a/p^{n_r}) \rfloor$; there are exactly

$$\varphi(p^{n_r+b+1}) = p^{n_r+b+1} - p^{n_r+b}$$

such terms with value b, i.e. p^{n_r+b},..., $p^{n_r+b+1} - 1$). When $i > 0$, we only need to account for the terms with $v_p(i) > n - \ell$, (note if $\ell = 1$ then $r + n_r = n - 1$

and there are no terms of this kind),

$$\sum_{b=1}^{p^{n_r}-1} v_p(i) - r - n_r + \sum_{b=p^{n_r}}^{p^{v_p(i)-r}-1} v_p(i) - r - n_r - \lfloor \log_p(b/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)(v_p(i) - r - n_r) + \sum_{b=0}^{v_p(i)-r-n_r-1} \varphi(p^{n_r+b+1})(v_p(i) - r - n_r - b).$$

Of the integers i satisfying $1 \leq i < p^n$ there are $\varphi(p^{\ell-1})$ integers i with $v_p(i) = n - \ell + 1$, there are $\varphi(p^{\ell-2})$ integers i with $v_p(i) = n - \ell + 2$, and so on to $\varphi(p)$ integers i with $v_p(i) = n - \ell + (\ell - 1)$. Summing over all such i with $v_p(i) > n - \ell$ we get

$$\sum_{a=1}^{\ell-1} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

Combining both the $i = 0$ and $i > 0$ contributions gives a definitive upper bound of

$$S = \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

To get a lower bound on the right of (in), we calculate $[\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C]$ precisely. Since this index equals

$$\frac{[\mathrm{gr}_\tau \mathrm{G}(X_L) : C]}{[\mathrm{K}(X_L) : \mathrm{K}(X)]},$$

it's sufficient to calculate the numerator and denominator of this fraction. The numerator depends only on the dimension of X and equals

$$\prod_{i=1}^{p^n} (p^{n-v_p(i)}) = \prod_{j=1}^{n-1} (p^{n-j})^{\varphi(p^{n-j})}.$$

The denominator depends on the reduced behavior of A and equals

$$\prod_{i=0}^{p^n-1} \mathrm{ind}(A^{\otimes i}) = \left(\prod_{j=0}^{r-1}(p^{n-j})^{\varphi(p^{n-j})}\right)\left(\prod_{j=r}^{n_r+r}(p^{n_r+r-j})^{\varphi(p^{n-j})}\right)$$

Dividing the two gives

$$P = \left(\prod_{i=r}^{n_r+r}(p^{\ell})^{\varphi(p^{n-i})}\right)\left(\prod_{i=n_r+r+1}^{n}(p^{n-i})^{\varphi(p^{n-i})}\right).$$

What remains to be shown is the equality $\log_p(P) = S$. A computation of the logarithm gives

$$\begin{aligned}
\log_p(P) &= \log_p\left(\prod_{i=r}^{n_r+r}(p^{\ell})^{\varphi(p^{n-i})}\prod_{i=n_r+r+1}^{n}(p^{n-i})^{\varphi(p^{n-i})}\right)\\
&= \sum_{i=r}^{n_r+r}\ell\varphi(p^{n-i}) + \sum_{i=n_r+r+1}^{n}(n-i)\varphi(p^{n-i})\\
&= \ell(p^{n-r}-p^{\ell-1}) + \sum_{i=1}^{n-r-n_r-1} i\varphi(p^i)\\
&= \ell(p^{n-r}-p^{\ell-1}) + \frac{(\ell-1)p^{\ell}-\ell p^{\ell-1}+1}{p-1}\\
&= \ell p^{n-r} - \frac{p^{\ell}-1}{p-1}.
\end{aligned}$$

And by simplifying the sum S we find

$$\begin{aligned}
S &= \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a-b) \right) \\
&= \sum_{a=1}^{\ell} \varphi(p^{\ell-a})(p^{n_r} - 1)a + \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a-b) \\
&= \frac{p^{n-r} - p^{n_r}}{p-1} - \frac{p^{\ell} - 1}{p-1} + \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \left(\frac{p^{n_r}(p^{a+1} - (a+1)p + a)}{(p-1)} \right) \\
&= \frac{p^{n-r} - p^{n_r}}{p-1} - \frac{p^{\ell} - 1}{p-1} + \frac{\ell p^{n-r+1} - (\ell+1)p^{n-r} + p^{n_r}}{p-1} \\
&= \ell p^{n-r} - \frac{p^{\ell} - 1}{p-1}
\end{aligned}$$

as desired. □

3.5 On p-adic valuations

Fix a prime p to be used throughout this section. For any integer $n \geq 0$ we use $S_p(n)$ to denote the sum of the base-p digits of n. In other words, if $n = a_0 + a_1 p + \cdots + a_r p^r$ with $0 \leq a_0, ..., a_r \leq p-1$ then $S_p(n) = a_0 + a_1 + \cdots + a_r$. This section proves some simple results on the function S_p and on p-adic valuations involving this function.

Lemma 3.5.1. *Let $n \geq 0$ be an integer.*

(1) $S_p(p^n) = 1$

(2) $S_p(p^n a) = S_p(a)$ *for any integer* $a \geq 0$

(3) $S_p(p^n - 1) = n(p-1)$

(4) *If* $0 \leq k \leq n$ *then* $S_p(p^n - p^k) = (n-k)(p-1)$

(5) *If* $0 \leq a \leq p^n$ *then* $S_p(p^n - a) + S_p(a) = (n - v_p(a))(p-1) + 1$

(6) *If* $0 \le a \le p^n - 1$ *then* $S_p(p^n - 1 - a) + S_p(a) = n(p-1)$

Proof. The proofs for (1)-(6) are elementary and omitted. □

We use the notation

$$\binom{n}{a_0, ..., a_r} = \frac{n!}{a_0! \cdots a_r!}.$$

If $a_0 + \cdots + a_r = n$ then we have the following:

Lemma 3.5.2. *Let* $n = a_0 + \cdots + a_r$ *with* $n, a_0, ..., a_r \ge 0$. *Then*

$$v_p\left(\binom{n}{a_0, ..., a_r}\right) = \frac{1}{p-1}\left(\left(\sum_{i=0}^{r} S_p(a_i)\right) - S_p(n)\right).$$

Proof. See for example [Mer03, Lemma 11.2]. □

Lemma 3.5.3. *Let* $n > 0$ *be an integer. Let* $a_0, ..., a_r \ge 0$ *be integers with* $a_0 + \cdots + a_r = n$. *Then*

$$v_p\left(\binom{n}{a_0, ..., a_r}\right) \ge v_p(n) - \min_{0 \le i \le r}\{v_p(a_i)\}.$$

Proof. See for example [Mer03, Lemma 11.3]. □

Lemma 3.5.4. *Let* $0 \le r \le s$ *be integers. Fix an integer* $0 < j \le p^s$. *Let* $a_0, ..., a_{p^r} \ge 0$ *be integers with* $a_0 + \cdots + a_{p^r} = p^{s-r}$ *and* $a_1 + 2a_2 + \cdots + p^r a_{p^r} = j$. *Write* $j = s_0 p^r + s_1$ *with* $0 \le s_1 < p^r$. *Then if* $s_1 = 0$ *there is an inequality*

$$s - r - \min_{0 \le k \le p^r}\{v_p(a_k)\} + \sum_{i=1}^{p^r}(r - v_p(i))a_i \ge s - r - v_p(s_0)$$

and if $s_1 > 0$ *there is an inequality*

$$s - r - \min_{0 \le k \le p^r}\{v_p(a_k)\} - (r - v_p(s_1)) + \sum_{i=1}^{p^r}(r - v_p(i))a_i \ge s - r.$$

If $s_1 = 0$*, then equality holds if and only if* $a_0 = p^{s-r} - s_0$ *and* $a_{p^r} = s_0$*. If* $s_1 > 0$*, then equality holds if and only if* $a_0 = p^{s-r} - s_0 - 1$*,* $a_{s_1} = 1$*, and* $a_{p^r} = s_0$*.*

Proof. We first assume $s_1 = 0$. If $\ell = \min\{v_p(a_k)\}$ is 0, then the inequality clearly holds since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$. If $\ell > 0$ and $r = 0$, then $j = a_1$ and $j = s_0$. So ℓ is either $v_p(a_0) = v_p(p^s - j)$ or $v_p(a_1) = v_p(j) = v_p(s_0)$. Since $j \leq p^s$, it follows $\ell = v_p(s_0)$ and the claim follows with equality in this case. If $\ell = \min\{v_p(a_k)\} > 0$, then since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$, the inequality also holds if $r \neq 0$ and if there is a nonzero a_i with $i \neq 0, p^r$ as $(r - v_p(i))a_i - \ell \geq 0$.

Thus, to prove that the inequality holds in general (for $s_1 = 0$), it suffices to assume $\ell > 0$, $r > 0$, and $a_i = 0$ unless $i = 0$ or $i = p^r$. Assuming this is the case, it follows from the assumption $p^r a_{p^r} = j$ that $a_{p^r} = s_0$ and from the assumption $a_0 + a_{p^r} = p^{s-r}$ that $a_0 = p^{s-r} - s_0$. Since $s_0 \leq p^{s-r}$, we also have $v_p(a_{p^r}) \leq s - r$ so that $v_p(a_0) = v_p(a_{p^r})$ unless $a_{p^r} = p^{s-r}$ (in which case $v_p(a_0) = \infty$ and the claim is clear). Thus $\ell = v_p(s_0)$, the inequality holds, and it is even an equality in this case.

To see $a_0 = p^{s-r} - s_0$ and $a_{p^r} = s_0$ is the only case the inequality is an equality, one can work through the same cases. If $\ell = 0$ and there is equality, then $v_p(s_0) = 0$ and the large summation must equal 0. Hence $p^r a_{p^r} = j$ and the claim follows. If $\ell > 0$, then either $r = 0$ or $r > 0$. If $r = 0$, the claim follows from the first paragraph. If $r > 0$, then either all a_i with $i \neq 0, p^r$ vanish or there is at least one $0 < i < p^r$ with $a_i \neq 0$. We can assume the latter case where the inequality is a strict inequality since $(r - v_p(i))a_i - \ell \geq a_i - \ell > 0$.

To show the claim when $s_1 > 0$, we work through cases similar to before. Note now $r > 0$ holds always, as otherwise we'd have $s_1 = 0$. If $\ell = \min\{v_p(a_k)\} = 0$ then since $r - v_p(i) \geq 0$, we're left to show the summation

$$\sum_{i=1}^{p^r}(r - v_p(i))a_i$$

is greater or equal $r - v_p(s_1) \leq r$. Since $s_1 > 0$, there is a smallest integer k with $0 \leq k \leq r-1$, $a_{bp^k} \neq 0$, and b relatively prime to p. It follows that p^k divides s_1 and $-(r - v_p(s_1)) \geq -r + k$. Since $(r - v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k)$ we find that the inequality holds by summing $(r - v_p(bp^k))a_{bp^k} - (r - v_p(s_1)) \geq (r-k) - (r-k) = 0$.

Thus to prove the inequality holds in general, it suffices to assume $\ell > 0$. Under our assumptions $\ell > 0$, $r > 0$, and $j \neq p^r a_{p^r}$ we have that there exists at least one i with $i \neq 0, p^r$ such that $a_i \neq 0$. Let k be the smallest integer between $0 \leq k < r$ such that $a_{bp^k} \neq 0$ for some b relatively prime to p. It follows p^k divides s_1 hence $-(r - v_p(s_1)) \geq -r + k$. Now

$$\begin{aligned}
(r - v_p(bp^k))a_{bp^k} - r + v_p(s_1) - \ell &\geq (r-k)p^\ell - r + v_p(s_1) - \ell \\
&= (r-k)(p^\ell - 1) - \ell + v_p(s_1) \\
&\geq (p^\ell - 1 - \ell) + v_p(s_1) \\
&\geq 0.
\end{aligned}$$

We end by showing that equality holds, assuming $s_1 > 0$, only in the specified case (it's clear equality holds in this case). We first assume $\ell = 0$. For equality to hold, we must have

$$\sum_{i=1}^{p^r}(r - v_p(i))a_i = r - v_p(s_1).$$

Again there is a minimal $0 \leq k < r$ with $a_{bp^k} \neq 0$ for some b relatively prime to p. We also get that p^k divides s_1. It follows

$$(r - v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k) \geq r - v_p(s_1)$$

must be an equality. Hence $a_{bp^k} = 1$ and we are in the specified case.

We next assume $\ell > 0$ and show our inequality is strict. Let k with $0 \leq k < r$ be minimal with $a_{bp^k} \neq 0$ for some b relatively prime to p. Then

$$\sum_{i=1}^{p^r} (r - v_p(i))a_i \geq (r-k)p^\ell.$$

Since $\ell + r - v_p(s_1) \leq \ell + r - k$ it suffices to check $(r-k)p^\ell > \ell + r - k$ holds for all $(r-k), \ell > 0$ in order to show this is a strict inequality in this case. But this is true since dividing by $r-k$ yields $p^\ell > \ell/(r-k) + 1$; making another estimate we can show $p^\ell > \ell + 1$ for all ℓ and this is always true for $\ell > 0$ and $p \geq 2$. $\square$

Lemma 3.5.5. *Let $0 \leq r \leq s$ be integers. Fix an integer $1 \leq j \leq p^s$ and write $j = s_0 p^r + s_1$ with $0 \leq s_1 < p^r$.*

If $s_1 = 0$, let $I = (a_0, ..., a_{p^r})$ be the tuple with $a_0 = p^{s-r} - s_0$, $a_{p^r} = s_0$ and $a_i = 0$ for all other i. Then,

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0) + S_p(a_{p^r}) - S_p(p^{s-r})) = s - r - v_p(s_0).$$

If $s_1 > 0$, let $I = (a_0, ..., a_{p^r})$ be the tuple with $a_0 = p^{s-r} - s_0 - 1$, $a_{s_1} = 1$, $a_{p^r} = s_0$ and $a_i = 0$ for all other i. Then,

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0) + S_p(a_{s_1}) + S_p(a_{p^r}) - S_p(p^{s-r})) = s - r.$$

Proof. The first equality follows from Lemma 3.5.2 and Lemma 3.5.1 (1) and

(5). The second equality follows from Lemma 3.5.2 and Lemma 3.5.1 (1) and (6). $\square$

Chapter 4

On the gamma filtration for a Severi-Brauer variety

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise.

If p is a prime, then v_p is the p-adic valuation.

$\#A$ denotes the cardinality of the set A.

4.1 Introduction

Chow rings of Severi-Brauer varieties have been the subject of a number of articles over the years. One attempt at studying these rings that has been particularly fruitful is Karpenko's use of the γ-filtration and the coniveau filtration on the Grothendieck ring. Much of the material in this article lends itself to the ideas contained in this work, particularly [Kar17a, Kar95b, Kar98].

The organization is as follows: sections 2 and 3 recall some background information on the Grothendieck groups we study. Section 4 is more involved and I've decided to give it a certain amount more of attention than it might deserve. There we introduce the notion of a τ-functorial replacement for a

Severi-Brauer variety X. This object is another Severi-Brauer variety, over possibly a different field, that computes the γ-filtration of X but has the enjoyable property the γ and coniveau filtrations agree. The existence of this object was known before but a proof in the general case is not in the literature.

Still in section 4, we use our τ-functorial replacements to give some functorial statements about the γ-filtration which were only known to hold for the τ-filtration. In particular, we show how one can reduce certain results about the γ-filtration of a Severi-Brauer variety of a central simple algebra to the Severi-Brauer varieties of the primary components of the underlying division algebra. I expect this idea could be used for a number of more general varieties, specifically when there is a decomposition of the Grothendieck ring of the variety into a sum of Grothendieck rings of central simple algebras (or, when there is a decomposition of the motive of this variety into a sum of motives of separable algebras in the sense of [Mer05]). This line of thought isn't pursued here, however.

Sections 5 and 6 are computational. The main result of these sections is that the associated graded ring of the γ filtration is torsion free for Severi-Brauer varieties associated to p-primary central simple algebras, for a prime p, in (homological) degrees less than or equal $p-2$.

4.2 Grothendieck groups of Severi-Brauer varieties

Throughout this section we fix a central simple algebra A of degree n and let $X = \mathrm{SB}(A)$ be the Severi-Brauer variety of A of dimension $n-1$. We write ζ_X for the tautological sheaf on X. For any k-algebra R and any point x of $X(R)$ corresponding to a right ideal $I \subset A \otimes_k R$, the sheaf $x^*\zeta_X$ is canonically identified with I; in particular, ζ_X is a right module over the constant sheaf A.

By $K(X)$ we mean the Grothendieck ring of locally free sheaves on X. By $G(X)$ we mean the Grothendieck ring of coherent sheaves on X. The two groups are canonically isomorphic via the morphism sending the class of a locally free sheaf in $K(X)$ to the class of itself in $G(X)$. These groups have been computed, in the following sense:

Theorem 4.2.1 ([Qui73, §8, Theorem 4.1])**.** *The homomorphism of K-groups*

$$\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X)$$

sending the class of a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

In particular, $K(X)$ is free of rank $\deg(A)$ generated additively by the classes of

$$\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$$

as $0 \leq i < \deg(A)$ and for a simple $A^{\otimes i}$-module M_i. For any splitting field F of A, the extension of scalars map $K(X) \to K(X_F)$ is injective, and identifies $K(X)$ as a subring of $K(X_F)$. More precisely, we have:

Theorem 4.2.2. *In the setting above, let ξ denote the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$. There is a ring isomorphism*

$$\mathbb{Z}[x]/(1-x)^n \xrightarrow{\sim} K(X_F)$$

sending x to ξ.

Under this isomorphisms $K(X)$ identifies with the subring of $\mathbb{Z}[x]/(1-x)^n$ generated by $\operatorname{ind}(A^{\otimes i})x^i$.

Proof. The isomorphism is well-known, see [Man69]. Finally, we use that $\zeta_X \otimes_k F$ has class $\deg(A)\xi$ in $K(X_F)$ to get the remaining claim by computing the ranks of the $\zeta_X(i)$. $\square$

When working with $K(X)$, it's often more helpful to work with a covering of this ring (e.g. this is done in [Kar98, Section 4]).

Lemma 4.2.3. *Consider the subring $S \subset \mathbb{Z}[x]$ generated by the elements*

$$\mathrm{ind}(A^{\otimes i})x^i \quad \textit{for all } 1 \leq i \leq \exp(A).$$

Then the image of S in $\mathbb{Z}[x]/(1-x)^n$ is isomorphic with $K(X)$.

In particular we'll need the following lemma from [Kar98, Lemma 4.5]. The proof is short and goes by induction on the coefficients.

Lemma 4.2.4. *Let f, g, h be polynomials in $\mathbb{Z}[x]$ and assume $g(0) = \pm 1$. Assume both $f = gh$ and f is contained in S. Then h is also contained in S.*

We include here as well the following formulas. The first is just the binomial theorem, and the second follows from the first by a change of coordinates.

Lemma 4.2.5. *In any commutative ring there are equalities*

$$(1-x)^i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} x^j \quad \textit{and} \quad x^i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} (1-x)^j.$$

4.3 The gamma and coniveau filtrations

In this section we recall some results on the γ-filtration of $K(X)$ and of the coniveau (or topological or Chow) filtration on $G(X)$ for a smooth variety X.

For the first, recall there are γ-operations defined on $K(X)$ as follows. The ith-exterior power operation induces a well-defined map $\lambda^i : K(X) \to K(X)$

which is uniquely determined by sending the class of a locally free sheaf $\mathcal{F}$ to the class of $\wedge^i \mathcal{F}$. The ith γ operation $\gamma^i : K(X) \to K(X)$ is defined by sending an element x to the coefficient of t^i in the formal series

$$\gamma_t(x) = \sum_{j \geq 0} \lambda^j(x) \left(\frac{t}{1-t} \right)^j .$$

The γ-filtration on $K(X)$ is defined as $\gamma^0 = K(X)$, $\gamma^1 = \ker(\mathrm{rk})$ where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the map sending the class of a locally free sheaf $\mathcal{F}$ to its rank, and γ^i for $i \geq 0$ is generated by monomials $\gamma^{i_1}(x_1) \cdots \gamma^{i_r}(x_r)$ for any $r \geq 0$, $i_1 + \cdots + i_r \geq i$ and $x_1, ..., x_r$ elements of γ^1. We use the notation

$$\mathrm{gr}^i_\gamma K(X) := \gamma^{i/i+1} := \gamma^i / \gamma^{i+1} \quad \text{and} \quad \mathrm{gr}_\gamma K(X) := \bigoplus_{i \geq 0} \mathrm{gr}^i_\gamma K(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. When we need to be precise about which variety the γ-filtration is being considered for, we will specify by writing $\gamma^i(X)$ to mean the ith piece of the γ-filtration for the variety X. For further properties of these operations we refer to the references [Man69, MR071].

For the second, recall the coniveau filtration on $G(X)$ is defined by setting τ^i, for any $i \geq 0$, to be the ideal generated by

$$\tau^i := \sum_{x \in X^{(j)}} \ker \left(G(X) \to G(X \setminus \overline{x}) \right)$$

where $j \geq i$, $X^{(j)}$ denotes the set of codimension j points of X, and the arrows are flat pullbacks with respect to the inclusion. We use the notation

$$\mathrm{gr}^i_\tau G(X) := \tau^{i/i+1} := \tau^i / \tau^{i+1} \quad \text{and} \quad \mathrm{gr}_\tau G(X) := \bigoplus_{i \geq 0} \mathrm{gr}^i_\tau G(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. Sometimes when more precision is needed, we include the variety in our notation for the coniveau filtration, i.e. $\tau^i(X)$ for the ith piece of the coniveau filtration of X.

The two filtrations are related:

Theorem 4.3.1. *We identify $K(X)$ with its image in $G(X)$ under the canonical isomorphism. For any $i \geq 0$ we have $\gamma^i \subset \tau^i$. Moreover, if the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes then the two filtrations are equal, i.e. $\gamma^i = \tau^i$ for all $i \geq 0$.*

Proof. For the first claim, see [Man69]. The second claim originally appears in [Kar98] and is updated in [KM18b, Proposition 3.3]. □

Remark 4.3.2. Slightly more generally, if the canonical morphism $B(X) \to \mathrm{gr}_\tau G(X)$ is a surjection, then there is also an equality $\gamma^i = \tau^i$ for all $i \geq 0$. Here $B(X)$ is the universal source of Chern classes on X constructed in [Mac18].

4.4 Reductions

We specialize to the case A is a central simple algebra and $X = \mathrm{SB}(A)$. The main purpose of this section is to provide a way to reduce to computations of the associated graded for the γ-filtration to the case A is a p-primary division algebra. In this regard we utilize heavily the motivic techniques of Karpenko (e.g. [Kar95a, Corollary 1.3.2],[Kar17a, Lemma 3.5]). The reason we can use these results is due to an observation (also Karpenko's) that for any Severi-Brauer variety X associated to an algebra A, there is a Severi-Brauer variety Y so that the γ-filtrations of X and Y are equal and the γ-filtration and coniveau filtration for this Y are also equal. This allows us to prove results about X by first replacing it with the functorially-nicer Y and then reducing to previously known results. This observation seems nice enough to name it.

Definition 4.4.1. Let X be an arbitrary Severi-Brauer variety associated to A. We say that a Severi-Brauer variety Y associated to a central simple algebra B is a *τ-functorial replacement for X* if the following conditions hold:

1. $\deg(A) = \deg(B)$
2. for every prime p, the p-behavior of A, B are the same $\mathcal{B}eh(p, A) = \mathcal{B}eh(p, B)$
3. the filtration comparison map $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an isomorphism.

Here we're using the definition:

Definition 4.4.2. For an arbitrary central simple algebra A with primary decomposition $A = M_n(k) \otimes \left(\bigotimes_{p \text{ prime}} A_p\right)$, the *behavior* of A is the sequence

$$\mathcal{B}eh(A) = \left(\mathrm{ind}(A), \mathrm{ind}(A^{\otimes 2}), \ldots, \mathrm{ind}(A^{\otimes \exp(A)})\right).$$

The *p-behavior* is the sequence

$$\mathcal{B}eh(p, A) = \left(\mathrm{ind}(A_p), \mathrm{ind}(A_p^{\otimes p}), \ldots, \mathrm{ind}(A_p^{\otimes \exp(A_p)})\right).$$

The *reduced p-behavior* of A is the sequence

$$r\mathcal{B}eh(p, A) = \left(v_p\mathrm{ind}(A_p), v_p\mathrm{ind}(A_p^{\otimes p}), \ldots, v_p\mathrm{ind}(A_p^{\otimes \exp(A_p)})\right).$$

If A is a p-primary algebra then we will call the reduced p-behavior simply the reduced behavior of A, and write $r\mathcal{B}eh(A)$ for the reduced behavior.

Remark 4.4.3. Note that a τ-functorial replacement doesn't necessarily need to exist over the same base field. In fact, it often doesn't.

Remark 4.4.4. The reduced behavior is a strictly descending sequence ending in 0. Conversely, for every prime p and for every strictly descending sequence ending in 0 there is a p-primary algebra with reduced behavior the given sequence, see [Kar98, Lemma 3.10]. Note that it's possible to reconstruct the behavior of A from the p-behavior (or the reduced p-behavior) as p ranges over all primes.

The first two conditions our τ-functorial replacements are required to have insure that we haven't changed the γ-filtration on replacement.

Lemma 4.4.5. *The ring* $\mathrm{gr}_\gamma K(X)$ *depends only on the integers* $\mathrm{ind}(A^{\otimes i})$ *for* $0 \leq i < \deg(A)$. *In particular, if B is another central simple algebra of the same degree as A with Severi-Brauer variety* $Y = \mathrm{SB}(B)$ *and if there are equalities*

$$\mathrm{ind}(A^{\otimes i}) = \mathrm{ind}(B^{\otimes i})$$

for all $i \geq 0$ then the rings $\mathrm{gr}_\gamma K(X)$ *and* $\mathrm{gr}_\gamma K(Y)$ *are isomorphic (but maybe not naturally).*

Proof. This is the content of [IK99, Theorem 1.1 and Corollary 1.2]. $\square$

Modifying a proof from [Kar98], it's possible to show a τ-functorial replacement exists for any Severi-Brauer variety of a division algebra.

Proposition 4.4.6. *If A is a division algebra then there exists a division algebra B, possibly over a different field than k, with*

$$\mathrm{ind}(A^{\otimes i}) = \mathrm{ind}(B^{\otimes i})$$

for all $i \geq 0$ satisfying the property that the canonical morphism comparing the γ and coniveau filtrations for $Y = \mathrm{SB}(B)$ is an isomorphism, i.e. $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\tau G(Y)$.

Proof. The construction of B is given in [Kar98, Lemma 3.10] for A of p-primary index. The proof that $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\tau G(Y)$ in this case follows from [Kar98, Theorem 3.7]. The following proof is a simple generalization of these two references.

We first construct B. One can find a field F and a division algebra B_0 with $\mathrm{ind}(A) = \mathrm{ind}(B_0) = \exp(B_0)$. Let $B_0 = \bigotimes_{p \text{ prime}} B_{0,p}$ be a p-primary decomposition of B_0. Let q be the smallest prime appearing among the indices of these factors. We consider the reduced q-behavior

$$r\mathcal{B}eh(q, A) = (n_0, n_1, ..., 0).$$

Set $\tilde{B_0} = B_0^{q^{n_1}}$ with Severi-Brauer variety $Y_1 = \mathrm{SB}(\tilde{B_0})$ and $B_1 = B_{0,F(Y_1)}$. Then using index reduction formulas, see [SVdB92, Theorem 1.3], we find

$$r\mathcal{B}eh(q, B_1) = (n_0, n_1, n_1 - 1, ..., 0).$$

By repeating this process finitely many times, we can construct an algebra B with the same reduced q-Behavior as A. Doing this procedure for the other primes allows us to find B satisfying the restriction on its indices as in the proposition statement.

It remains to show the γ-filtration and the coniveau filtration agree for $Y = \mathrm{SB}(B)$. For the Severi-Brauer variety of any algebra with equal index and exponent, for example B_0, these two filtrations coincide, [Kar98, Corollary 3.6]. In the general case, we note that $X \times Y_1$ is a projective bundle over X

since $\tilde{B}_0$ is in the subgroup generated by B_0. The commuting diagram

$$\begin{array}{ccc} B(X \times Y_1) & \longrightarrow & B(X_{F(Y_1)}) \\ \downarrow & & \downarrow \\ \mathrm{gr}_\tau G(X \times Y_1) & \longrightarrow & \mathrm{gr}_\tau G(X_{F(Y_1)}) \end{array}$$

has surjective left vertical arrow by Lemma 4.4.7 below. The bottom horizontal arrow is also surjective by localization. Hence the right vertical arrow is surjective, which implies the γ and coniveau filtrations coincide for $X_{F(Y_1)}$ by Remark 4.3.2. Continuing this process for each modification of B_i shows B has the specified properties. $\square$

Lemma 4.4.7. *Suppose X is a variety (a scheme essentially smooth and essentially of finite type over k) with $\gamma^i = \tau^i$ for all $i \geq 0$. Then, for any chain of morphisms*

$$Y_r \to \cdots \to Y_1 \to Y_0 = X$$

with each Y_j the projective bundle of some vector bundle over Y_{j-1}, the γ and coniveau filtrations for Y_r coincide.

Proof. It suffices to assume $r = 1$ and work by induction. As there is equality of the γ and coniveau filtrations for X, we get a commuting diagram like the one below, using the universal maps from B and the filtration comparison map.

$$\begin{tikzcd} & B(X) \arrow[dl] \arrow[dr] & \\ \mathrm{gr}_\gamma K(X) \arrow[rr, equal] & & \mathrm{gr}_\tau G(X) \end{tikzcd}$$

As the left diagonal map is always a surjection since $\mathrm{gr}_\gamma K(X)$ is generated by Chern classes, it follows that the right diagonal map is also a surjection.

Now for any projective bundle $Y_1 \to X$, the Grothendieck ring $K(Y_1)$ is generated as a $K(X)$-algebra by $K(X)$ and a single element t which is the class of a rank one locally free sheaf on Y_1. In particular, $B(Y_1)$ is generated as an algebra by the image of $B(X)$ under the pullback of the projection and powers of the first chern class of t. Since $\mathrm{gr}_\tau G(Y_1)$ is generated as an algebra by $\mathrm{gr}_\tau G(X)$ and the chern classes of t, it follows the canonical map $B(Y_1) \to \mathrm{gr}_\tau G(Y_1)$ is a surjection. We conclude by Remark 4.3.2. $\square$

The remainder of this section is devoted to showing the use in τ-functorial replacements. First, we show that the associated graded for the γ-filtration depends only on the underlying division algebra of A.

Proposition 4.4.8. *Let A be an arbitrary central simple algebra. If D is the underlying division algebra of A with $X_D = \mathrm{SB}(D)$, then the morphism*

$$\bigoplus_{i=1}^{\deg(A)/\deg(D)} \mathrm{gr}_\gamma K(X_D) \to \mathrm{gr}_\gamma K(X)$$

taking the element $(x_1, ..., x_r)$ to $x_1 + x_2 c + \cdots + x_r c^{r-1}$, where c is the top Chern class of $\zeta_X(1)$, is an isomorphism.

We'll need some lemmas.

Lemma 4.4.9. *If D is the underlying division algebra of A with $X_D = \mathrm{SB}(D)$, then the map induced by the pullback of the inclusion*

$$\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\gamma K(X_D)$$

is an isomorphism in degrees where both groups are nonzero.

Proof. As the morphism of the lemma statement depends only on the behavior of A and the degree of A, we can first make a τ-functorial replacement, Proposition 4.4.6, to assume the γ and coniveau filtrations agree for X_D. Now the diagram

$$\begin{tikzcd}
\mathrm{gr}_\gamma K(X) \arrow[r] \arrow[d] & \mathrm{gr}_\gamma K(X_D) \arrow[d] \\
\mathrm{gr}_\tau G(X) \arrow[r] & \mathrm{gr}_\tau G(X_D)
\end{tikzcd}$$

is commutative, with vertical arrows the comparison maps and horizontal arrows the pullbacks. The top horizontal arrow is surjective because the pullback $K(X) \to K(X_D)$ is surjective and the associated graded for the γ-filtration is generated by Chern classes, cf. [Mac18, Lemma 2.3]. The bottom horizontal arrow is an isomorphism in degrees where both groups are nonzero by [Kar95a, Corollary 1.3.2]. And the right vertical arrow is an isomorphism because of our replacement.

Now from the commutative ladder with exact rows below

$$\begin{tikzcd}
0 \arrow[r] & \gamma^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \gamma^i \arrow[r] \arrow[d, "f_i"] & \gamma^{i/i+1} \arrow[r] \arrow[d, "f_{i/i+1}"] & 0 \\
0 \arrow[r] & \tau^{i+1} \arrow[r] & \tau^i \arrow[r] & \tau^{i/i+1} \arrow[r] & 0
\end{tikzcd}$$

we get short exact sequences (using the snake lemma)

$$0 \to \ker(f_{i/i+1}) \to \mathrm{coker}(f_{i+1}) \to \mathrm{coker}(f_i) \to 0$$

for all $0 \le i < n = \mathrm{ind}(A)$. To complete the proof, it suffices to show $f_{n/n+1}$ is a surjection and $\mathrm{coker}(f_n) = 0$. These are both shown in the next lemma. $\square$

Lemma 4.4.10. *If A is a central simple algebra and* $\mathrm{ind}(A) = n$*, then the group* $\mathrm{gr}^n_\gamma K(X)$ *is torsion free and there are equalities*

$$\gamma^n = \tau^n = (\xi - 1)^n K(X) \quad \textit{and} \quad \mathrm{gr}^n_\gamma K(X) = \mathrm{gr}^n_\tau G(X) = (\xi - 1)^n \mathbb{Z}.$$

Proof. Let F be a splitting field for A and identify $K(X)$ with its image in $K(X_F)$ under the extension of scalars map. We set ξ to be the class of $\mathcal{O}_{X_F}(-1)$. Our first goal is to show the inclusions

$$(\xi - 1)^n K(X) \subset \gamma^n \subset \tau^n \subset (\xi - 1)^n G(X)$$

which will imply equalities hold throughout. Note that the left of these is immediate, as we have $(\xi - 1)^n = \gamma^n(n(\xi - 1))$.

As in Lemma 4.2.2, let $S \subset \mathbb{Z}[x]$ be the subring generated as an algebra by the elements $\mathrm{ind}(A^{\otimes i})x^i$. The preimage of τ^i in S under the surjection

$$S \to \mathbb{Z}[x] \to K(X) = G(X)$$

is always composed of polynomials in $(1 - x)$ of degree greater or equal i; this is because, if F were a splitting field for X then $\tau^i(X) \subset \tau^i(X_F) \subset G(X_F)$ and the preimage of $\tau^i(X_F)$ is the ideal $(1 - x)^i \subset \mathbb{Z}[x]$. We know, from the inclusion $\gamma^n \subset \tau^n$ that the preimage of τ^n contains x^n. We want to show that this preimage is actually also contained in the ideal $S \cap (x^n) \subset S$. It would then follow $\tau^n \subset (1 - \xi)^n G(X) = \gamma^n$ and this would complete this part of the proof.

To proceed, suppose f is a polynomial in the preimage of τ^n. Note this implies f is in S. Assuming $f \neq 0$, we can write $f = (1 - x)^n g$ for some polynomial g of $\mathbb{Z}[x]$. It suffices to check that g is in S as well and this is true by Lemma 4.2.4.

Next we compute the quotients. The rank map $\mathrm{rk} : K(X) \to \mathbb{Z}$ is surjective and provides a splitting $K(X) = \gamma^1 \oplus \mathbb{Z}$ given by $x \mapsto (x - \mathrm{rk}(x), \mathrm{rk}(x))$. We have a commuting diagram of free abelian groups

$$\begin{tikzcd}
& \gamma^{n+1} \oplus (\xi-1)^n\mathbb{Z} \arrow[dr, "\pi"] & \\
\gamma^1 \oplus \mathbb{Z} \arrow[ur, "\cdot(\xi-1)^n"] \arrow[rr, "\cdot(\xi-1)^n"] & & \gamma^n
\end{tikzcd}$$

with the canonical inclusion $\pi : \gamma^{n+1} \oplus (\xi - 1)^n\mathbb{Z} \subset \gamma^n$. The bottom row of this diagram is surjective by the description of γ^n. Hence π must also be a surjection which, since both domain and target are free abelian groups of the same rank (see Lemma 4.4.11), must be an isomorphism. This shows $\mathrm{gr}^n_\gamma K(X) = (\xi - 1)^n\mathbb{Z}$ which happens to be the same as $\mathrm{gr}^n_\tau G(X)$ by the same proof replacing everywhere γ appears with τ. $\square$

Lemma 4.4.11. *The groups $\gamma^i \subset K(X)$ and $\tau^i \subset G(X)$ are free abelian of rank $n - i$ for any $0 \leq i \leq n = \deg(A)$.*

Proof. For any such i, γ^i (resp. τ^i) is a free abelian group as its a subgroup of $K(X)$ (resp. $G(X)$). For the claim on the rank we go by induction. For any i there is an exact sequence

$$0 \to \gamma^{i+1} \to \gamma^i \to \gamma^{i/i+1} \to 0$$

and $\gamma^{i/i+1}$ has rank 1 by one variant of the Riemann-Roch theorem (resp. with τ^i's). For large enough i we have $\gamma^{i+1} = 0$, $\gamma^i \neq 0$ and $\gamma^{i/i+1}$ of rank 1 (resp. with τ^i's). $\square$

Proof of Proposition 4.4.8. Using Proposition 4.4.6 on D, one can find a τ-functorial replacement B. In particular,

$$\mathcal{B}eh(p, A) = \mathcal{B}eh(p, A)$$

for all primes p satisfying the property the coniveau filtration and γ-filtration agree for the Severi-Brauer variety of B. Set $C = M_r(B)$, the ring of square matrices of size $r = \deg(A)/\deg(D)$, set $Z = \mathrm{SB}(C)$, and set $Z_B = \mathrm{SB}(B)$.

There's a canonical morphism

$$\bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\gamma K(Z_B) \to \mathrm{gr}_\gamma K(Z).$$

To see it, label a basis of the left sum by e_i, $1 \leq i \leq \deg(C)/\deg(B)$. The canonical morphism is the map that sends xe_i to xc^{i-1} where c is the top Chern class of $\zeta_X(1)$; here x is considered in $\mathrm{gr}_\gamma K(Z)$ via the isomorphism of Lemma 4.4.9. We compose this morphism with the maps

$$\mathrm{gr}_\gamma K(Z) \to \mathrm{gr}_\tau G(Z) \to \bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\tau G(Z_B)$$

where the right arrow is an the inverse of the isomorphism

$$\bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\tau G(Z_B) \to \mathrm{gr}_\tau G(Z)$$

appearing from [Kar95a, Corollary 1.3.2].

The composition

$$\bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\gamma K(Z_B) \to \bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\tau G(Z_B)$$

is an isomorphism due to our choice of Z_B. Hence there is a surjection

$$\mathrm{gr}_\gamma K(Z) \to \mathrm{gr}_\tau G(Z).$$

The filtration comparison map has the nice property that surjectivity implies injectivity, [KM18b, Proposition 3.3 (2)], so it's an isomorphism here. Thus the map

$$\bigoplus_{i=1}^{\deg(C)/\deg(B)} \mathrm{gr}_\gamma K(Z_B) \to \mathrm{gr}_\gamma K(Z)$$

is both injective and surjective. Since these rings are isomorphic when replacing Z_B by X_D and Z by X the claim follows. $\square$

As a corollary to the above proof we get:

Theorem 4.4.12. *For any arbitrary central simple algebra A, there exists a τ-functorial replacement of $X = \mathrm{SB}(A)$.*

Proof. Let D be the underlying division algebra of A. There is a τ-functorial replacement B of D. The proof of Proposition 4.4.8 shows that taking a matrix ring over B with the same degree of A satisfies all the required properties of a τ-functorial replacement of A. $\square$

A τ-functorial replacement also allows us to characterize the torsion in the associated graded for the γ-filtration of the Severi-Brauer variety of a central simple algebra in terms of the Severi-Brauer variety associated to its underlying division algebra.

Lemma 4.4.13. *If A is a central simple algebra of p-primary index for some prime p, then $\mathrm{gr}_\gamma K(X)$ and $\mathrm{gr}_\tau G(X)$ contain only p-primary torsion.*

Additionally, for every finite field extension F/k of degree prime-to-p, the extension of scalars map $\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\gamma K(X_F)$ is an isomorphism.

Proof. The first claim is known for the associated graded of the coniveau filtration where it follows from a restriction-corestriction argument. The first claim for associated graded of the γ-filtration then follows from the existence of a τ-functorial replacement of X.

For the second claim, it suffices to note the extension of scalars map along a prime-to-p induces a natural isomorphism between $K(X)$ and $K(X_F)$ and then apply Lemma 4.4.5. $\square$

Lemma 4.4.14. *For an arbitrary central simple algebra A, we write $A = \bigotimes_{p \text{ prime}} A_p \otimes M_r(k)$ for a decomposition of A into p-primary division algebras A_p and a matrix ring $M_r(k)$. Then, for any prime p, for any integer $0 \leq j < \deg(A)$, and for j' the remainder after dividing j by $\mathrm{ind}(A_p)$, there are isomorphisms*

$$\mathrm{gr}^j_\gamma K(X) \otimes \mathbb{Z}_{(p)} \cong \mathrm{gr}^{j'}_\gamma K(X) \otimes \mathbb{Z}_{(p)} \quad \textit{and} \quad \mathrm{gr}^j_\tau G(X) \otimes \mathbb{Z}_{(p)} \cong \mathrm{gr}^{j'}_\tau G(X) \otimes \mathbb{Z}_{(p)}.$$

Proof. We use [Kar17a, Lemma 3.5] to get the claim involving the coniveau filtration and Theorem 4.4.12 to get the claim for the γ-filtration. $\square$

4.5 Generating the γ-filtration

Again A is a central simple algebra, and $X = \mathrm{SB}(A)$ its Severi-Brauer variety. In this section we describe the γ-filtration for X when A is a p-primary division algebra. The most distinguishing property for this purpose is the p-level of A.

Definition 4.5.1. The *p-level* of A is defined to be the level of A_p where A_p is the p-primary division algebra occurring as a factor of A. We write $\mathrm{lev}(p, A)$ for the p-level of A. Recall that the *level* of a p-primary algebra A, written $\mathrm{lev}(A)$, as defined in [KM18a], is the largest number of distinct integers $1 \leq i_1, ..., i_l \leq \exp(A)$ with

$$v_p\mathrm{ind}(A^{\otimes p^{i_k}}) < v_p\mathrm{ind}(A^{\otimes p^{i_k - 1}}) - 1$$

for every $1 \leq k \leq l$. In other words, the level of A is the number of places the reduced behavior decreases by more than one from one position to the next.

Lemma 4.5.2. *Let A be a central simple algebra with p-primary index for a prime p. We assume also that $\operatorname{lev}(A) = r$, and that $i_1, ..., i_r$ are the distinct integers satisfying*

$$v_p\operatorname{ind}(A^{\otimes p^{i_k}}) < v_p\operatorname{ind}(A^{\otimes p^{i_k - 1}}) - 1$$

for all $1 \leq k \leq r$. Then $\gamma^i \subset K(X)$ is generated additively by products

$$\gamma^{j_1}(x_1 - \operatorname{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \operatorname{rk}(x_r))$$

where $j_1 + \cdots + j_r \geq i$ and $x_1, ..., x_r$ are elements of $\{\zeta_X(p^{i_k})\}_{k=1}^{l}$.

Proof. In this setting, the ring $\operatorname{gr}_\gamma K(X)$ is generated as an algebra by the Chern classes of the $\zeta_X(p^{i_k})^\vee$. This is because: $K(X)$ is generated as a λ-ring by $\zeta_X(p^{i_k})$, see [KM18a, Lemma A.6], Chern classes of λ-operations of a vector bundle are polynomials in the Chern class of this bundle, see [Mac18, Lemma 3.7], and Chern classes of the dual of a bundle are Chern classes of this bundle up to a sign [Mac18, Example 3.6].

Since these Chern classes in $\operatorname{gr}_\gamma K(X)$ are defined as

$$c_i^\gamma(x) = \gamma^i(\operatorname{rk}(x) - x^\vee) \mod \gamma^{i+1},$$

it follows that γ^i is generated by the lifts of monomials of degree i in these Chern classes and γ^{i+1}. By induction we can assume γ^{i+1} is generated by similarly defined elements but of degree $i+1$ and γ^{i+2}. Eventually, for large enough d, we have $\gamma^d = 0$ and it follows γ^i is generated by the lifts of these monomials of degree i or larger.

To complete the claim then, we only need to show that $\gamma^i(x)$ is a polynomial in the $\gamma^j(-x)$. This follows as $1 = \gamma_t(x - x) = \gamma_t(x)\gamma_t(-x)$ implies

$$\gamma_t(x) = \frac{1}{\gamma_t(-x)}$$

and the right hand side is a series in t with coefficients polynomials in the γ-operations of $-x$. $\square$

Lemma 4.5.3. *Let A be a central simple algebra with p-primary index for some prime p. Assume A has reduced behavior $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Fix a splitting field F of A and identify $K(X)$ with its image in $K(X_F)$ under the extension of scalars map. Let ξ be the class of $\mathcal{O}_{X_F}(-1)$.*

Then

$$\gamma^i(\zeta_X(p^j) - p^{n_j}) = \binom{p^{n_j}}{i}(\xi^{p^j} - 1)^i.$$

Proof. This is computed in [Kar98]. It's done by

$$\gamma_t(p^{n_j}\xi^{p^j} - p^{n_j}) = \gamma_t(p^{n_j}(\xi^{p^j} - 1)) = \gamma_t(\xi^{p^j} - 1)^{p^{n_j}} = (1 + (\xi^{p^j} - 1)t)^{p^{n_j}}.$$

$\square$

For future reference we provide the formula below.

Lemma 4.5.4.

$$x^n - 1 = \sum_{i=1}^{n}(-1)^i\binom{n}{i}(1 - x)^i$$

Proof. Note $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$. Now apply Lemma 4.2.5 to the latter sum and combine. $\square$

4.6 Comparison between the γ, τ, and η filtrations

For this section, fix a central division algebra A of index p^n for some prime p and $n \geq 0$. We write $X = \mathrm{SB}(A)$ as before. We also assume that A is chosen so that $\gamma^i = \tau^i$ for all $i \geq 0$, applying Theorem 4.4.12 if needed (and possibly renaming our base field).

We're going to compute the γ-filtration on X in degrees greater $p^n - p$. In some ways, this computation is facilitated by the fact that most of the terms in an element of the γ-filtration start to vanish in these large degrees. The restriction to degree greater $p^n - p$ in particular means we'll be doing computations with polynomials that can be written as sums of monomials of length at most $p - 1$. After making this observation, it only takes some rudimentary approximations on the divisibility properties of these sums to get our main result:

Theorem 4.6.1. *For an arbitrary central simple algebra A with $\mathrm{ind}(A) = p^n$ and $X = \mathrm{SB}(A)$ we have*

$$\mathrm{gr}_\gamma^{p^n - i} K(X) = p^n(\xi - 1)^{p^n - i}\mathbb{Z}$$

for all $1 \leq i \leq p - 1$.

In the above we're identifying $K(X)$ with its image in $K(X_F)$ for some splitting field F of A and we are setting ξ to be the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$.

Before giving the proof, we give some lower bounds on the size of the γ^i. Strictly speaking these bounds aren't needed and the interested reader can go straight to the proof of Theorem 4.6.1. We take the time to work through these bounds because it was consideration of these bounds that led to the description of the γ-filtration in these degrees.

So, we introduce a new filtration on $K(X)$ using the equality $\gamma^i = \tau^i$. Up to making a prime-to-p extension of the base field, we can assume there are finite field extensions $k \subset L_0 \subset L_1 \subset \cdots \subset L_n$ with $[L_i : L_{i+1}] = p$ and $\mathrm{ind}(A_{L_i}) = p^{n-i}$ for each i. For any j with $(j, p^n) = p^{n-i}$ Consider the composition

$$\gamma^j(X_{L_i}) \to \tau^j(X_{L_i}) \xrightarrow{N_{L_i/k}} \tau^j(X) = \gamma^j(X).$$

The leftmost of these groups is equal (over $\overline{k}$) to the ideal in $K(X)$ generated by $(\xi - 1)^j$ by Lemma 4.4.10. The image of this element under the composition is equal $[L_i : k](\xi - 1)^j = a_j p^i(\xi - 1)^j$ for some a_j coprime to p.

Definition 4.6.2. We define $\eta^i \subset \gamma^i \subset K(X)$ to be the group generated by the elements

$$p^{n - v_p(j)}(\xi - 1)^j \quad \text{for all } j \geq i.$$

That these elements exist inside of γ^i follows because

$$\gamma^1(\zeta_X(1) - p^n)^j = p^{nj}(\xi - 1)^j$$

is an element of γ^i and $(a_j p^{n-v_p(j)}, p^n) = p^{n-v_p(j)}$.

Alternatively, η^i can be described as the ideal generated by the degree j products of γ-operations of $p^n(\xi - 1)$ for all $j \geq i$.

We denote by $\eta^{i/i+1} := \eta^i/\eta^{i+1}$ and $\mathrm{gr}_\eta K(X)$ for the associated graded pieces and the associated graded ring respectively. The following corollary comes from the existence of the η-filtration; it can also be deduced, at least when $i = 0$, from [Kar95b, Proposition 2 and Lemma 3].

Corollary 4.6.3. *There's an inequality*

$$\# \operatorname{Tors} \bigoplus_{j=i}^{p^n-1} \operatorname{gr}_\gamma^j K(X) \leq \prod_{j=i}^{p^n-1} p^{n-v_p(j)}$$

for any $0 \leq i \leq p^n - 1$*. When* $i = 0$ *or* $i = 1$ *we have*

$$\prod_{j=1}^{p^n} p^{n-v_p(j)} = np^n - (p^{n-1} + p^{n-2} + \cdots + 1).$$

Proof. The ladder below is commuting and has exact rows for every $i \geq 0$.

$$\begin{tikzcd}
0 \arrow[r] & \eta^{i+1} \arrow[r] \arrow[d] & \eta^i \arrow[r] \arrow[d] & \eta^{i/i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \gamma^{i+1} \arrow[r] & \gamma^i \arrow[r] & \gamma^{i/i+1} \arrow[r] & 0
\end{tikzcd}$$

As $\eta^{i/i+1}$ is torsion free, all of the vertical arrows are injections. Since η^j and γ^j have the same rank for every j, it follows the cokernels of these vertical arrows are torsion. Using the snake lemma we get short exact sequences

$$0 \to \gamma^{i+1}/\eta^{i+1} \to \gamma^i/\eta^i \to \gamma^{i/i+1}/\eta^{i/i+1} \to 0.$$

Setting $A = \operatorname{Tors} \gamma^{i/i+1}$ we can write

$$\gamma^{i/i+1} = A \oplus \gamma^{i/i+1}/A \quad \text{and} \quad \gamma^{i/i+1}/\eta^{i/i+1} = A \oplus (\gamma^{i/i+1}/A)/\eta^{i/i+1}.$$

Now

$$\# \operatorname{Tors} \bigoplus_{j=i}^{p^n-1} \operatorname{gr}_\gamma^j K(X) \leq \# \prod_i \gamma^{i/i+1}/\eta^{i/i+1} = \frac{\prod_{i \geq i} \#\gamma^i/\eta^i}{\prod_{i \geq i+1} \#\gamma^i/\eta^i} = \#\gamma^i/\eta^i.$$

Considering the natural inclusions of free abelian groups $\eta^i \subset \gamma^i \subset K(X)$ we get inequalities

$$\#\,\mathrm{Tors}\,\gamma^i/\eta^i \leq \#\,\mathrm{Tors}\,K(X)/\eta^i = \prod_{j=i}^{p^n-1} p^{n-v_p(j)}$$

which proves the corollary. $\square$

Our main theorem says the bound above is far from sharp. The remainder of the section is devoted to this proof.

Proof of Theorem 4.6.1. It suffices by Lemma 4.4.8 to assume A is a division algebra. Our proof works by showing p^n divides the coefficient of every element of $\gamma^{p^n-p+1} \supset \gamma^{p^n-i}$ when each of these elements is written as polynomial in $1-\xi$. Note since there are inclusions

$$\gamma^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X_F) = (1-\xi)^{p^n-p+1}K(X_F),$$

we can write every element y of γ^{p^n-p+1} as a sum

$$y = \sum_{j=p^n-p+1}^{p^n-1} a_j(1-\xi)^j$$

for some integers a_j. After we show p^n divides each of these a_j, it follows that we have inclusions

$$\eta^{p^n-p+1} \subset \gamma^{p^n-p+1} \subset \eta^{p^n-p+1}$$

and this will end the proof.

Suppose then

$$y = \gamma^{j_1}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \mathrm{rk}(x_r))$$

is an arbitrary monomial generating γ^{p^n-p+1} like those described in Lemma

4.5.2. We can work in the two cases: each of $x_1, ..., x_k$ equal $\zeta_X(1)$ for some $1 \leq k \leq r$ (since $v_p(j) \geq \min\{v_p(j_1), ..., v_p(j_k)\}$ and $v_p\binom{p^n}{i} = n - v_p(i)$, we can even assume $k = 1$) or $\zeta_X(1)$ does not appear among the $x_1, ..., x_r$.

Assuming we're in the former case, we can expand y as

$$
\begin{aligned}
y &= \binom{p^n}{j_1}(\xi - 1)^{j_1}\binom{p^{n-t_2}}{j_2}(\xi^{p^{s_2}} - 1)^{j_2} \cdots \binom{p^{n-t_r}}{j_r}(\xi^{p^{s_r}} - 1)^{j_r} \\
&= \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r}(\xi - 1)^{j_1}(\xi^{p^{s_2}} - 1)^{j_2} \cdots (\xi^{p^{s_r}} - 1)^{j_r}.
\end{aligned}
$$

Note also that $s_2, ..., s_r \geq 1$.

Now by Lemma 4.5.4, there is an expansion, for each $2 \leq l \leq r$,

$$
\xi^{p^{s_l}} - 1 = \sum_{i=1}^{p^{s_l}} (-1)^i \binom{p^{s_l}}{i}(1 - x)^i.
$$

We set $x_{low}(l) = \sum_{i=1}^{p-1}(-1)^i\binom{p^{s_l}}{i}(1-x)^i$ and $x_{high}(l) = \sum_{i=p}^{p^{s_l}}(-1)^i\binom{p^{s_l}}{i}(1-x)^i$. Note that p divides $x_{low}(l)$ for every $2 \leq l \leq r$. Rewriting y in terms of x_{low}'s and x_{high}'s gives

$$
y = \binom{p^n}{j_1}(\xi - 1)^{j_1}(x_{low}(2) + x_{high}(2))^{j_2} \cdots (x_{low}(r) + x_{high}(r))^{j_r}.
$$

The lowest degree of any x_{high} is p, while the lowest degree of any x_{low} is 1. This means, applying the binomial theorem and expanding, the lowest degree of $(1 - \xi)$ in any monomial containing an x_{high} is $j_1 + j_2 + \cdots + j_r - 1 + p \geq p^n - p + 1 - 1 + p \geq p^n$. Hence all of these summands are 0.

Thus we find

$$
\begin{aligned}
y &= \binom{p^n}{j_1}(\xi - 1)^{j_1} x_{low}(2)^{j_2} \cdots x_{low}(r)^{j_r} \\
&= \binom{p^n}{j_1} p^{j_2 + \cdots + j_r}(\xi - 1)^{j_1}\left(\frac{x_{low}(2)}{p}\right)^{j_2} \cdots \left(\frac{x_{low}(r)}{p}\right)^{j_r}
\end{aligned}
$$

since each x_{low} is divisible by p.

The p-adic valuation of the coefficient leading this product is exactly $n - v_p(j_1) + j_2 + \cdots + j_r$. We finish by showing $n - v_p(j_1) + j_2 + \cdots + j_r \geq n$ for all possible $j_1, ..., j_r$ or, equivalently, assuming $j_1 + \cdots + j_r = p^n - i$ with $0 < i < p$ we finish by showing

$$p^n - i \geq j_1 + v_p(j_1).$$

Assuming i is largest possible we can also show $p^n - p + 1 \geq j_1 + v_p(j_1)$. We can assume $v_p(j_1) > 0$ as otherwise p^n divides $\binom{p^n}{j_1}$. Hence we can assume $j_1 = a_1 p^{n-1} + \cdots + a_{n-r} p^r$ with $0 \leq a_1, ..., a_{n-r} < p$ and some minimal $r \geq 1$. This inequality becomes

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r} p^r + r.$$

We make one last approximation, and assume all $a_1, ..., a_{n-r}$ are equal $(p-1)$, as this is the largest they can be. We're left checking

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r} p^r + r = p^n - p^r + r.$$

Rearranging, we check

$$p^r - p \geq r - 1$$

which is clear if $r = 1$ and is the same as

$$\frac{p^r - p}{r - 1} \geq 1$$

for $r > 1$. Using the mean value theorem, the left of this inequality equals $f'(c)$ for some c in the interval $[1, r]$ and $f(x) = p^x$. Since $f'(c) = \log(p) p^c \geq \log(p) p \geq \log(2) 2 > 1$ we've completed this case.

We still need to check the second case, when $\zeta_X(1)$ is not a part of the γ-operations of our monomial. Following the same process as before, we're left to check the inequality $p^n - i \geq n$ for $0 < i < p$. But this is also readily checked to be true: we can assume we want to show $p^n - p + 1 \geq n$; and $p^n - p \geq n - 1$ is the same (ignoring the $n = 1$ case which is trivial) as $\frac{p^n - p}{n-1} \geq 1$ which by the mean value theorem equals $f'(c)$ for some c in the interval $[1, n]$ and $f(x) = p^x$; for all such c we have $f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1$. □

We end with some more general statements that can be obtained from Theorem 4.6.1.

Corollary 4.6.4. *Let B be a central simple algebra, and Y the Severi-Brauer variety of B. Suppose $\operatorname{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization of B with $p_1 < \cdots < p_r$. Then for all $1 \leq i \leq p_1 - 1$*

$$\operatorname{gr}_\gamma^{d-i} K(Y) = d(1 - \xi)^{d-i}\mathbb{Z},$$

where ξ is the class of $\mathcal{O}_{X_F}(-1)$ when identifying $K(X) \subset K(X_F)$ for a splitting field F of X.

Proof. Apply Lemma 4.4.14. □

Corollary 4.6.5. *Suppose B is generic central simple algebra of index p^n and exponent p^m in the sense of [Kar17a, Example 2.2] and set $X = \operatorname{SB}(B)$. Then*

$$\operatorname{CH}_j(X) = p^n\mathbb{Z} \quad \textit{for all } 0 \leq j \leq p - 2.$$

More generally, suppose B is a central simple algebra with $\operatorname{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ a prime factorization of d ordered like $p_1 < \cdots < p_r$. Suppose the p_i-level of B is less or equal 1 for all $1 \leq i \leq r$ and suppose $\operatorname{CH}(X)$ is generated *by Chern classes where* $X = \mathrm{SB}(B)$*. Then*

$$\mathrm{CH}_j(X) = d\mathbb{Z} \quad \textit{for all } j \leq p_k - 2$$

where k *is the smallest number with* $\mathrm{lev}(p_k, B) = 1$*; if no* k *exists then* $\mathrm{CH}(X)$ *is torsion free.*

Proof. In the former case, the rings $\mathrm{CH}(X)$ and $\mathrm{gr}_\gamma K(X)$ are isomorphic, [Kar17c, Theorem 3.1]. In the latter case, we use the same fact as before but for these algebras [KM18a, Theorem A.15]. □

Chapter 5

The coniveau filtration on K_1 for some Severi-Brauer varieties

Notation and conventions.

We work over a fixed base field k.

A variety is a separated scheme of finite type over a field.

For a prime p, we write $v_p(-)$ for the p-adic valuation.

5.1 Introduction

Some K-cohomology groups were studied, and computed, for Severi-Brauer varieties associated to algebras with square-free degree in [MS82]. As an application of these computations one can compute the Chow groups of these Severi-Brauer varieties and find they are torsion free. Chow groups of arbitrary Severi-Brauer varieties X have been studied in depth and, in certain degrees, are known to be torsion free (e.g. $\mathrm{CH}^0(X)$ is free trivially, $\mathrm{CH}^1(X)$ is torsion free by [Art82], $\mathrm{CH}_0(X)$ is torsion free by [CM06], if X is associated to an algebra whose index equals its exponent then $\mathrm{CH}^2(X)$ is torsion free by [Kar98]).

The Chow groups of Severi-Brauer varieties are not always torsion free. Their torsion subgroups have also been studied in depth. In [Kar98], Karpenko shows, if X is a Severi-Brauer variety associated to an algebra with differing index and exponent, $\mathrm{CH}^2(X)$ sometimes contains a nontrivial torsion subgroup which surjects onto torsion in the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$. In a different direction, Merkurjev [Mer95] has shown that there is sometimes nontrivial torsion in the Chow groups of Severi-Brauer varieties which occurs in codimension 3 or higher; this torsion can't be detected by Karpenko's methods since it's contained in the kernel of the canonical epimorphism from $\mathrm{CH}(X)$ onto the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$.

Recently, Karpenko has computed the Chow ring of a Severi-Brauer variety associated to a central simple algebra with equal index and exponent under the assumption the Chow ring is generated by Chern classes, [Kar17a]. In this computation, the Chow ring turns out to be torsion free. Without the assumption the Chow ring is generated by Chern classes, any nontrivial torsion in the Chow ring of such a Severi-Brauer variety will come from nontrivial differentials in the K-theory coniveau, or Brown-Gersten-Quillen, spectral sequence.

This article stemmed from exploring the possibility of torsion in the Chow group of a Severi-Brauer variety associated to an algebra A with index equal to its exponent. Hopefully, it will be of use in further study of this problem.

Section 2 is mainly for reference and introducing notation. In Section 3 we prove a series of lemmas that will be used for the main results of Sections 4 and 5.

In Section 4, we compute the $E_\infty^{m,-m-1}$ terms of the K-theory coniveau spectral sequence for any Severi-Brauer variety X associated to an algebra A

satisfying the properties: the index of A is a power of a prime p, the exponent of A equals the index of A over all finite extensions of the center of A, and the reduced Whitehead groups $\mathrm{SK}_1(A^{\otimes r}) = 1$ vanish for all $r \geq 1$. This result is a direct generalization of the known computation for the terms $E_\infty^{m,-m}$ and the proof of the main theorem manages to describe both simultaneously. The main result is Theorem 5.4.2; it's proof is elementary but, it requires some involved arguments comparing the reduced norms of certain tensor powers of a given algebra.

In Section 5, we show how to prove the general case stated using the primary case of Section 4.

5.2 On the K-theory of a Severi-Brauer variety

The material in this section has been developed in detail by Quillen, [Qui73]. The K-theory coniveau spectral sequence, or the Brown-Gersten-Quillen spectral sequence, is a fourth quadrant cohomological spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} \mathrm{K}_{-p-q}(k(x)) \implies \mathrm{G}_{-p-q}(X)$$

where $X^{(p)}$ denotes the set of codimension p points of X. For a variety X, the spectral sequence converges, and for a regular variety X one can identify the E_2-terms with K-cohomology groups

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{K}_{-q}) \implies \mathrm{G}_{-p-q}(X).$$

Recall the K-cohomology groups $\mathrm{H}^p(X, \mathcal{K}_q)$ are defined to be the homology of a complex

$$\coprod_{x \in X^{(p-1)}} \mathrm{K}_{q-p+1}(k(x)) \to \coprod_{x \in X^{(p)}} \mathrm{K}_{q-p}(k(x)) \to \coprod_{x \in X^{(p+1)}} \mathrm{K}_{q-p-1}(k(x)).$$

In particular, the groups $\mathrm{H}^p(X, \mathcal{K}_q) = 0$ whenever $p > q$ or $p > \dim(X)$.

The coniveau filtration is the filtration appearing in the abutment of the K-theory coniveau spectral sequence. If X is a regular variety (which is all that is worked with in this note), then there are natural isomorphisms $\mathrm{K}_i(X) \cong \mathrm{G}_i(X)$ and by transporting the filtration on G-theory to K-theory we get a coniveau filtration on the groups $\mathrm{K}_i(X)$. The jth term of this filtration on $\mathrm{K}_i(X)$ is denoted $\mathrm{K}_i(X)^j$ below. We write $\mathrm{K}_i(X)^{j/j+1}$ for the quotient $\mathrm{K}_i(X)^j/\mathrm{K}_i(X)^{j+1}$.

The K-theory of a Severi-Brauer variety X associated to a central simple algbera A was computed by Quillen in terms of the tautological bundle ζ_X on X:

Theorem 5.2.1 ([Qui73, §8, Theorem 4.1])**.** *Let X be the Severi-Brauer variety of a central simple algebra A. Then, for every $i \geq 0$ the group homomorphism*

$$\bigoplus_{j=0}^{\deg(A)-1} \mathrm{K}_i(A^{\otimes j}) \to \mathrm{K}_i(X)$$

induced by the exact functor that takes a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

Crucial in our computation will be the reduced norm subgroups of a central simple k-algebra. For this, let L be a Galois splitting field for A. The *first reduced norm* of A is defined to be the unique map making the following diagram commutative.

$$\begin{array}{ccc} \mathrm{K}_1(A_L) & \xrightarrow{\det} & \mathrm{K}_1(L) \\ \uparrow & & \uparrow \\ \mathrm{K}_1(A) & \xrightarrow{\mathrm{Nrd}_1} & \mathrm{K}_1(k) \end{array}$$

The vertical arrows in this diagram are induced by extension of scalars. Similarly we define the *zeroth reduced norm* of A to be the map $\mathrm{Nrd}_0 : \mathrm{K}_0(A) \to \mathrm{K}_0(k)$ taking the class of an A-module M to the k-vector space of dimension $\mathrm{rdim}_A(M)$, the reduced dimension of M. For $i = 0, 1$ we will often use the abbreviation $\mathrm{Nrd}_i(\mathrm{K}_i(A)) := \mathrm{Nrd}_i(A)$.

The kernel of the map Nrd_i is called the *ith reduced Whitehead group* and denoted $\mathrm{SK}_i(-)$. Note the group $\mathrm{SK}_0(A)$ necessarily vanishes since Nrd_0 is injective with image the subgroup generated by the index of A, $\mathrm{ind}(A)\mathbb{Z} \subset \mathrm{K}_0(k) = \mathbb{Z}$. The group $\mathrm{SK}_1(A)$ doesn't vanish in general.

For any finite field extension E of k, the extension of scalars map $\rho^*_{E/k} : \mathrm{K}_i(X) \to \mathrm{K}_i(X_E)$ is the sum of the maps $\mathrm{K}_i(A^{\otimes j}) \to \mathrm{K}_i(A_E^{\otimes j})$ in the decomposition of Theorem 5.2.1. In the other direction, the pushforward $\rho_{E/k*} : \mathrm{K}_i(X_E) \to \mathrm{K}_i(X)$ is given by the sum of the norm maps $\mathrm{K}_i(A_E^{\otimes j}) \to \mathrm{K}_i(A^{\otimes j})$ in the same decomposition. If $i = 0$ then the norm map is characterized componentwise by having image the number

$$\rho_{E/k*}(\mathrm{K}_0(A_E)) = [E : k]\frac{\mathrm{rdim}_{A_E}(M)}{\mathrm{rdim}_A(N)} \subset \mathrm{K}_0(A) = \mathbb{Z}$$

where M, N are simple modules under A_E, A respectively. The image of the norm maps when $i = 1$ are more complicated to describe. In the simple situation we work in, these images can be described fairly explicitly. We do this in detail in the next section.

5.3 Relations between reduced norms

In this section we fix a central simple algebra A over k and we set X to be the Severi-Brauer variety associated with X.

Our first objective is to describe the image of the reduced norm using splitting fields of A:

Lemma 5.3.1. *Let A be a central simple algebra. Then, for every finite field extension L of k and for $i = 0, 1$, the following diagram commutes*

$$\begin{tikzcd} \mathrm{K}_i(A_L) \arrow[r, "\mathrm{Nrd}_i"] \arrow[d, "N_{A_L/A}"] & \mathrm{K}_i(L) \arrow[d, "N_{L/k}"] \\ \mathrm{K}_i(A) \arrow[r, "\mathrm{Nrd}_i"] & \mathrm{K}_i(k) \end{tikzcd}$$

where both $N_{A_L/A}$ and $N_{L/k}$ are the norm maps induced by restriction of scalars.

Moreover, the subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extensions of k that split A. This can be reduced further: the subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extensions of k that are maximal subfields of the underlying division algebra of A.

Proof. The commutativity of the digram is clear when $i = 0$, and is well-known (see [GS06, Proposition 2.8.11]) when $i = 1$.

The only claim that needs to be proved is the last one: the subgroup $\mathrm{Nrd}_i(A)$ is generated by norms of maximal subfields of the underlying division algebra of A. In the case $i = 0$, the claim follows from the fact such a field has degree $\mathrm{ind}(A)$ over k so we are left proving the case $i = 1$.

For the proof when $i = 1$, we'll use Morita invariance to reduce to the case A is a division algebra and we'll use [GS06, Proposition 2.6.3] which says

$\mathrm{Nrd}_1(x) = N_{K/k}(x)$ for any element x of a maximal subfield K contained in A. Any element x of A is contained in some maximal subfield (indeed, if F is a maximal element in the collection of subfields of A containing $k(x)$, then the centralizer of F in A is F itself – this is known to be equivalent to being a maximal subfield) so taking the composition

$$A^\times \twoheadrightarrow \mathrm{K}_1(A) \xrightarrow{\mathrm{Nrd}_1} \mathrm{K}_1(k)$$

of the natural surjection and the reduced norm gives the result by the commutativity of the given diagram. □

The K-theory of the Severi-Brauer variety X relies heavily on the tensor powers of the algebra A due to the decomposition of Theorem 5.2.1. Because of this, we'll need to investigate certain relations between the reduced norms $\mathrm{Nrd}_i(A)$ and $\mathrm{Nrd}_i(A^{\otimes r})$ for varying $r \geq 0$. It will be necessary in our formulation of these relations to introduce some condition on the index of A over finite extensions. From now on we'll say an algebra A satisfies condition (C) if:

$$\mathrm{ind}(A_E) = \exp(A_E) \text{ for any finite extension } E/k. \tag{C}$$

Example 5.3.2. Any central simple algebra of square-free index satisfies condition (C) trivially. Any central simple algebra over a finite extension of $\mathbb{Q}_p$ satisfies condition (C). Central simple algebras over function fields of surfaces, with base a separably closed field, having index coprime to the characteristic of the base also satisfy condition (C), see [dJ04].

Moreover, if a central simple algebra A satisfies condition (C) then so do the tensor powers of A. This is because, given a central simple algebra A with equal index and exponent, the indices of all tensor powers of A can be explicitly determined. If the index of A was a power of a prime p, say p^n, then

$A^{\otimes p}$ has index p^{n-1}, cf. [Kar98, Example 3.9]. The general case follows easily from this one.

Remark 5.3.3. There exists a cyclic algebra A of index and exponent 4, over a field F of characteristic 2, along with a finite purely inseparable field extension E/F with $[E:F]=2$ and such that $\mathrm{ind}(A_E)=4$ and $\exp(A_E)=2$ (cf. [Per41, Theorem 4]).

Lemma 5.3.4. *Let A be a central simple k-algebra with* $\mathrm{ind}(A)=p^n$ *for some $n\geq 0$ and let $i=0$ or $i=1$. Then*

$$\mathrm{Nrd}_i(A^{\otimes j})=\mathrm{Nrd}_i(A^{\otimes p^{v_p(j)}})$$

for any $j>0$.

Proof. By Lemma 5.3.1 the subgroup $\mathrm{Nrd}_i(A^{\otimes j})\subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extension of k splitting $A^{\otimes j}$. The set of such fields is the same for $A^{\otimes j}$ and $A^{\otimes p^{v_p(j)}}$, which proves the claim. □

Lemma 5.3.5. *Let A be a central simple k-algebra with* $\mathrm{ind}(A)=p^n=\exp(A)$ *for some prime p and some $n\geq 0$. Assume A satisfies condition* (C). *Then for $i=0,1$ the containments*

$$\mathrm{Nrd}_i(A^{\otimes p^a})\supset \mathrm{Nrd}_i(A^{\otimes p^b})\supset \mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}}$$

hold for all $a\geq b\geq 0$.

Proof. By Lemma 5.3.1 the subgroup $\mathrm{Nrd}_i(A^{\otimes j})\subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extension of k splitting $A^{\otimes j}$. If such an L would split $A^{\otimes p^b}$, then L would also split $A^{\otimes p^a}$. Hence we have the inclusion $\mathrm{Nrd}_i(A^{\otimes p^b})\subset \mathrm{Nrd}_i(A^{\otimes p^a})$.

To show the inclusion $\mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset \mathrm{Nrd}_i(A^{\otimes p^b})$, we work in two cases. If $a \geq n$, then $A^{\otimes p^a}$ is split; if L is a maximal subfield of the underlying division algebra of $A^{\otimes p^b}$, then $[L : k] = p^{n-b}$ (see Example 5.3.2) and

$$\mathrm{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset p^{n-b}\mathrm{K}_i(k) = N_{L/k}(\mathrm{K}_i(k)) \subset \mathrm{Nrd}_i(A^{\otimes p^b}).$$

Otherwise, when $a < n$, let L be a maximal subfield of the underlying division algebra of $A^{\otimes p^a}$. Then L has degree $[L : k] = p^{n-a}$, the algebra A_L has exponent dividing p^a and, since we're assuming condition (C), index dividing p^a. If E is a maximal subfield of the underlying division algebra of $A_L^{\otimes p^b}$ then $[E : L]$ divides p^{a-b}. Again by Lemma 5.3.1 we have the inclusion

$$N_{E/k}(\mathrm{K}_i(E)) \subset \mathrm{Nrd}_i(A^{\otimes p^b})$$

since E splits $A^{\otimes p^b}$. It follows that for any element x of $\mathrm{K}_i(L) \subset \mathrm{K}_i(E)$ we have

$$N_{E/k}(x) = N_{L/k}(N_{E/L}(x)) = N_{L/k}(x^{[E:L]}) = N_{L/k}(x)^{[E:L]}$$

is contained in $\mathrm{Nrd}_i(A^{\otimes p^b})$. The proof is then complete since we've shown the collection of elements $N_{L/k}(x)^{p^{a-b}}$, as L varies over all maximal subfields of the underlying division algebra of $A^{\otimes p^a}$ and x varies over $\mathrm{K}_i(L)$, are contained in $\mathrm{Nrd}_i(A^{\otimes p^b})$ and these form a generating set by Lemma 5.3.1. □

Lemma 5.3.6. *Let A be a central simple k-algebra with $\mathrm{ind}(A) = p^n = \exp(A)$ for some prime p and some $n \geq 0$. Assume A satisfies condition* (C). *Then for $i = 0, 1$ there is containment*

$$\mathrm{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \mathrm{Nrd}_i(A^{\otimes b})$$

for all $a \geq b > 0$.

Proof. The proof continues by working in cases: assuming either $v_p(a) \leq v_p(b)$ or $v_p(a) > v_p(b)$. In the first case, $v_p(a) \leq v_p(b)$, we appeal to Lemma 5.3.4 and Lemma 5.3.5 to find

$$\mathrm{Nrd}_i(A^{\otimes a}) = \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}}) \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b}).$$

In the second case, $v_p(a) > v_p(b)$, we appeal to the second containment of Lemma 5.3.5. That is to say, by Lemma 5.3.7 below we find $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$ so that

$$\mathrm{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}})^{p^{v_p(a)-v_p(b)}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b})$$

by applying Lemma 5.3.4 for the first inclusion, Lemma 5.3.5 for the second inclusion, and Lemma 5.3.4 for the last equality. □

The lemma needed for the above is:

Lemma 5.3.7. *Assume $a > b$ and $v_p(a) > v_p(b)$. Then $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$.*

Proof. More generally, for any pair of integers $a > b$, one can show $\frac{a}{(a,b)}$ divides the binomial coefficient $\binom{a}{b}$. The claim follows from noting

$$v_p\left(\frac{a}{(a,b)}\right) = v_p(a) - v_p((a,b)) = v_p(a) - v_p(b).$$

First, write $(a,b) = na + mb$ with n, m both integers. Then

$$\frac{(a,b)}{a}\binom{a}{b} = \frac{(na+mb)}{a}\binom{a}{b} = n\binom{a}{b} + \frac{mb}{a}\binom{a}{b} = n\binom{a}{b} + m\binom{a-1}{b-1}$$

with the latter sum an integer. □

To go from an algebra of p-primary index to an arbitrary central simple algebra A, see Proposition 5.5.1, we'll need a characterization of $\mathrm{Nrd}_i(A)$ in terms of the primary components of A when A is a division algebra. For this, we fix a primary decomposition

$$A \cong A_{p_1} \otimes \cdots \otimes A_{p_s}$$

with $p_1, ..., p_s$ the primes dividing $\operatorname{ind}(A)$ (such decompositions exist with the factors unique up to isomorphism, see [GS06, Proposition 4.5.16]). For each algebra A_{p_j} we fix a maximal subfield F_{p_j} of its underlying division algebra, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, ..., F_{p_{j-1}}, F_{p_{j+1}}, ..., F_{p_s}$, the jth field being omitted, contained in some fixed algebraic closure L.

Lemma 5.3.8. *In the notation above, and for $i = 0, 1$,*

$$\operatorname{Nrd}_i(A) = \bigcap_{j=1}^{s} \operatorname{Nrd}_i(A_{F^{p_j}})$$

inside of $\mathrm{K}_i(L)$.

Proof. If $s = 1$, the lemma is trivial so we can assume $s > 1$.

The inclusion $\subset$ is immediate from Lemma 5.3.1 since a field E splitting A also necessarily splits each of the $A_{F^{p_j}}$.

For the other inclusion, $\supset$, we let x be an element of the intersection. By Lemma 5.3.1 this means we have equalities

$$x = N_{E_{1,1}/F^{p_1}}(y_{1,1}) \cdots N_{E_{1,r_1}/F^{p_1}}(y_{1,r_1})$$

$$\vdots$$

$$x = N_{E_{s,1}/F^{p_s}}(y_{s,1}) \cdots N_{E_{s,r_s}/F^{p_s}}(y_{s,r_s})$$

for some elements $y_{j,k}$ of fields $E_{j,k}$ splitting $A_{F^{p_j}}$ respectively. It follows from these equalities that x is an element of $B = \mathrm{K}_i(F^{p_1}) \cap \cdots \cap \mathrm{K}_i(F^{p_s})$. If $i = 0$, then B is just $\mathrm{ind}(A)\mathbb{Z}$. If $i = 1$, then, since by construction the degrees $[F^{p_j} : k]$ are divisible by all primes dividing $\mathrm{ind}(A)$ except for p_j, we have $\gcd([F^{p_1} : k], ..., [F^{p_s} : k]) = 1$ and $B = k^\times$.

Applying the norm, from F^{p_j} to k, to the corresponding expression above for x, we find the elements

$$N_{F^{p_j}/k}(x) = N_{E_{j,1}/k}(y_{j,1}) \cdots N_{E_{j,r_j}/k}(y_{j,r_j})$$

are contained in $\mathrm{Nrd}_i(A)$, for every $1 \leq j \leq s$, since each $E_{j,k}$ splits $A_{F^{p_j}}$ and so necessarily also splits A. Since x is already contained in $\mathrm{K}_i(k)$, taking the norm also yields equalities

$$N_{F^{p_j}/k}(x) = x^{[F^{p_j}:k]}.$$

Finally, as x is in the subgroup spanned by these powers, x is contained in $\mathrm{Nrd}_i(A)$, completing the proof. $\square$

5.4 The coniveau filtration on K_i for a p-primary algebra

We fix a prime p throughout. We fix a central simple algebra A with index $\mathrm{ind}(A) = p^n$ and exponent $\exp(A) = p^n$ for some $n > 0$. We write X for the Severi-Brauer variety of A.

This section describes the groups $\mathrm{K}_i(X)^j$ and $\mathrm{K}_i(X)^{j/j+1}$ for $j \geq 0$ assuming A satisfies condition (C) and either $i = 0$ or, $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \geq 1$. In the case $i = 0$, this result was shown in [Kar98, Proposition 3.3] (condition (C) is not needed in this result). Although the only new result is when $i = 1$, the proof does not depend on this assumption.

We note that the assumption $\mathrm{SK}_1(A^{\otimes r})$ is trivial for all powers r is another way of stating that $\mathrm{K}_1(X) \to \mathrm{K}_1(X_L)$ is injective for a splitting field L of A. The reason the latter, more natural, assumption is not given is because it's often easier to check that the groups $\mathrm{SK}_1(A^{\otimes r})$ are trivial. Note the analogous statement is also true replacing $i = 1$ with $i = 0$ in the above so that the map $\mathrm{K}_0(X) \to \mathrm{K}_0(X_L)$ is always injective. Formally:

Lemma 5.4.1. *Suppose B is an arbitrary central simple algebra and let Y be the Severi-Brauer variety of B. Let L be a splitting field for B. Then, for $i = 0, 1$ the pullback $\mathrm{K}_i(Y) \to \mathrm{K}_i(Y_L)$ is injective if, and only if, the groups $\mathrm{SK}_i(B^{\otimes j})$ are trivial for all $j \geq 0$.*

Proof. The diagram

$$\begin{tikzcd} \mathrm{K}_i(B_L^{\otimes r}) \arrow[r, equal, "\mathrm{Nrd}_i"] & \mathrm{K}_i(L) \\ \mathrm{K}_i(B^{\otimes r}) \arrow[u, "\pi_r^*"] \arrow[r, "\mathrm{Nrd}_i"] & \mathrm{K}_i(k) \arrow[u, hook] \end{tikzcd}$$

commutes where the vertical arrows are the extension of scalars maps. Since the right-vertical arrow is always an injection we find $\mathrm{SK}_i(B^{\otimes r}) = \ker(\pi_r^*)$. The claim then follows from Theorem 5.2.1 by summing over all $r \geq 0$. $\square$

As in the above lemma, let B be an arbitrary central simple algebra and Y the associated Severi-Brauer variety. If L is a splitting field for B, then $\mathrm{K}_0(Y_L)$ is generated as a group by the powers γ^i, from $i = 0$ to $\deg(B) - 1$, of the element γ representing the class of $\mathcal{O}_{Y_L}(-1)$. By Lemma 5.4.1, the pullback $\mathrm{K}_0(Y) \to \mathrm{K}_0(Y_L)$ is injective and we identify $\mathrm{K}_0(Y)$ with its image in $\mathrm{K}_0(Y_L)$. Similarly, the group $\mathrm{K}_1(Y_L)$ is a sum of groups $L^\times\gamma^i$ as i ranges from $i = 0$ to $i = \deg(B) - 1$. If $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 1$, then the pullback $\mathrm{K}_1(Y) \to \mathrm{K}_1(Y_L)$ is injective and we identify $\mathrm{K}_1(Y)$ with its image in $\mathrm{K}_1(Y_L)$.

Theorem 5.4.2. *Assume A satisfies condition* (C)*. Let L be a splitting field for A. If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \geq 1$, then there is an equality (with notation as above)*

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}$$

for all $0 \leq j \leq \deg(A) - 1$. For $j < 0$, or for $j > \deg(A) - 1$, the groups $\mathrm{K}_i(X)^j = 0$ vanish.

Proof. The claim when $j < 0$ or $j > \deg(A) - 1$ is immediate: the first of these is by definition, the second follows from the fact $(\gamma - 1)^{\deg(A)} = 0$ in $\mathrm{K}_0(X)$. Recall (cf. [Pey95, Proposition 3.6]) the coniveau filtration on $\mathrm{K}_i(X_L)$ is given by

$$\mathrm{K}_i(X_L)^j = \mathrm{K}_i(A_L^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{K}_i(A_L^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}$$

where $\gamma = [\mathcal{O}(-1)]$ is the class of the tautological line bundle in $\mathrm{K}_0(X_L)$. Under the pullback $\mathrm{K}_i(X) \to \mathrm{K}_i(X_L)$ the groups $\mathrm{K}_i(A^{\otimes j})$ are identified with the subgroups $\mathrm{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(L)$. Hence, we identify

$$\mathrm{K}_i(X) = \mathrm{Nrd}_i(k) \cdot 1 + \mathrm{Nrd}_i(A)\gamma + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})\gamma^{\deg(A)-1}.$$

We claim

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}. \quad (*)$$

The proof utilizes the following lemmas:

Lemma 5.4.3. *Let A and L be as in Theorem 5.4.2. Fix an element b in $\mathrm{Nrd}_i(A^{\otimes k})$ with $k \geq 0$ and $i = 0$ or $i = 1$. Then, for any sequence of integers*

$(n_j)_{j\geq 0}$ *an equality*

$$bx^k = \sum_{j\geq 0} a_j(x+n_j)^j$$

inside of the free $\mathrm{K}_i(L)$*-module* $\mathrm{K}_i(L)[x]$ *implies* a_j *is contained in* $\mathrm{Nrd}_i(A^{\otimes j})$ *for all* $j \geq 0$.

Proof. By assumption $a_k = b$ is contained in $\mathrm{Nrd}_i(A^{\otimes k})$. By descending induction on j, we assume each a_j is contained in $\mathrm{Nrd}_i(A^{\otimes j})$ for all j larger than some fixed $l \geq 0$. Then by expanding the right side of the given equality and comparing coefficients yields

$$a_l = -\sum_{j=l+1}^{k} n_j^{j-l}\binom{j}{l}a_j$$

which is contained in $\mathrm{Nrd}_i(A^{\otimes l})$ due to Lemma 5.3.6 applied to each $\binom{j}{l}a_j$. □

Lemma 5.4.4. *Keeping notation as above, we have*

$$\sum_{j\geq 0}\mathrm{Nrd}_i(A^{\otimes j})\gamma^j = \sum_{j\geq 0}\mathrm{Nrd}_i(A^{\otimes j})(\gamma-1)^j$$

inside of $\mathrm{K}_i(X_L)$.

Proof. Setting $n_j = -1$ for all $j \geq 0$ in Lemma 5.4.3, and setting $x = \gamma$, shows the forward containment. Setting $n_j = 1$ for all $j \geq 0$, and setting $x = \gamma - 1$, shows the reverse containment. □

Continuing with the proof of Theorem 5.4.2, we have

$$\begin{aligned}
\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j &= \sum_{n \geq 0} \mathrm{Nrd}_i(A^{\otimes n})\gamma^n \cap \sum_{n \geq j} \mathrm{K}_i(L)(\gamma - 1)^n \\
&= \sum_{n \geq 0} \mathrm{Nrd}_i(A^{\otimes n})(\gamma - 1)^n \cap \sum_{n \geq j} \mathrm{K}_i(L)(\gamma - 1)^n \\
&= \sum_{n \geq j} \mathrm{Nrd}_i(A^{\otimes n})(\gamma - 1)^n
\end{aligned}$$

as claimed. Here we used Lemma 5.4.4 to go from the first line to the second.

□

Corollary 5.4.5. *Let L be an algebraic closure of k. Assume A satisfies condition* (C)*. Let $i = 0$ or $i = 1$ and assume $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 1$. Then we have an equality*

$$\mathrm{K}_i(X)^j = \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$$

for all $j \geq 0$.

Proof. It's clear we have the inclusion $\mathrm{K}_i(X)^j \subset \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$. By Theorem 5.4.2, there is an equality

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}.$$

To show the reverse containment $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X)^j$ we go by induction on the index. That is to say: if E is a finite extension of k splitting A then we have containment $\mathrm{K}_i(X_E) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X_E)^j$ and for our induction hypothesis we assume this containment holds for all fields E with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$.

If E is a finite extension of k with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$ then, using our induction hypothesis and the assumption A satisfies condition (C), we have

$$
\begin{aligned}
\mathrm{K}_i(X)^j &= \rho^*_{L/k}(\mathrm{K}_i(X)^j) \\
&\supset \rho^*_{L/k}(\rho_{E/k*}(\mathrm{K}_i(X_E)^j)) \\
&= \rho_{E/k_*}\left(\mathrm{Nrd}_i(A_E^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A_E^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}\right).
\end{aligned}
$$

Expanding a product $(\gamma-1)^r$ and taking $\rho_{E/k*}$ shows

$$\rho_{E/k*}(a(\gamma-1)^r) = N_{E/k}(a)(\gamma-1)^r.$$

Since all elements of $\mathrm{Nrd}_i(A^{\otimes r})$ are norms from finite extensions E of k splitting $A^{\otimes r}$ by Lemma 5.3.1, it follows $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$ is generated by the groups on the right of the containment above. □

Corollary 5.4.6. *Let $i = 0$, or $i = 1$ and $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 0$. Assume A satisfies condition* (C)*. Then there is an isomorphism*

$$\mathrm{K}_i(X)^{j/j+1} \cong \mathrm{Nrd}_i(A^{\otimes j})$$

for all $0 \leq j \leq \deg(A) - 1$. For other j these groups vanish.

Proof. This follows immediately from Theorem 5.4.2 and Corollary 5.4.5. □

5.5 The coniveau filtration on K_i for a central simple algebra

In this section we assume B is a central simple algebra with $\mathrm{ind}(B_E) = \exp(B_E)$ for all finite field extensions E/k. We let Y be the Severi-Brauer variety of B.

Proposition 5.5.1. *If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 0$, then there is an isomorphism*

$$\mathrm{K}_i(Y)^{j/j+1} \cong \mathrm{Nrd}_i(B^{\otimes j})$$

for all $0 \leq j \leq \deg(B) - 1$. For other j these groups vanish.

Proof. Using a result of Karpenko, [Kar00, Example 10.20], we can assume B is a division algebra throughout the proof.

Fix a primary decomposition

$$B \cong B_{p_1} \otimes \cdots \otimes B_{p_s}$$

with $p_1, ..., p_s$ the primes dividing $\mathrm{ind}(B)$. We can assume $s > 1$, as the result has been proved above otherwise. For each algebra B_{p_j} we fix a maximal subfield F_{p_j} of its underlying division algebra, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, ..., F_{p_{j-1}}, F_{p_{j+1}}, ..., F_{p_s}$, the jth field being omitted, contained in some fixed algebraic closure L of k

We first observe an equality

$$\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j = \mathrm{Nrd}_i(B^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}.$$

Indeed, by Lemma 5.3.8 and the explicit description of $\mathrm{K}_i(Y)$ given by Lemma 5.4.1, we have

$$\mathrm{K}_i(Y) = \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}})$$

inside of $\mathrm{K}_i(Y_L)$. Hence we get equalities

$$\begin{aligned}
&\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j \\
&= \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}}) \cap \mathrm{K}_i(Y_L)^j \\
&= \bigcap_{r=1}^{s} \left(\mathrm{K}_i(Y_{F^{p_r}}) \cap \mathrm{K}_i(Y_L)^j\right) \\
&= \bigcap_{r=1}^{s} \left(\mathrm{Nrd}_i(B_{F^{p_r}})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B_{F^{p_r}}^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}\right) \\
&= \mathrm{Nrd}_i(B^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}.
\end{aligned}$$

A careful reading of the proof of Corollary 5.4.5 shows that the assumption A has p-primary index was unnecessary. Hence the corollary can be applied to B as well to show $\mathrm{K}_i(Y) = \mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j$ and the result follows. □

Chapter 6

Conclusion

In this conclusion we discuss some of the possible avenues of further study that one can take to continue the work presented above. We do this by presenting questions, differing in specificity and detail, with motivation whenever possible.

The first question we ask is about extending results on the functor B introduced in Chapter 2. There are a number of results about B that one might expect to be true and that are not included here. Questions 1 and 2 pertain to results that I'd say are expected to be true but, throughout the course of trying to find an answer to said questions, single out the difficulties one may encounter when using this functor.

Question 1: Let X be a variety and $P = \mathbb{P}(E)$ the projective bundle of a vector bundle $\pi : E \to X$. Let ξ be the class of the tautological line bundle on P. Then $B(P)$ is generated by $B(X) = \pi^* B(X)$ and $c_1(\xi)$. Can one determine relations on these generators?

If A is a cohomology theory with the correct form of Chern classes, and the natural map $B(X) \to A(X)$ is an isomorphism, then it follows from the Projective bundle theorem for $A(P)$ that $B(P)$ is isomorphic to a sum of copies of $B(X)$. However, it's been extremely difficult, for the author, to find ways to compute relations for B even in the case of a projective bundle. As a more difficult question, still having to do with determining relations between the elements of B, would be the following.

Question 2: Is $B(X)$ torsion free for a Severi-Brauer variety associated to a central simple algebra A with equal index and exponent?

Setting $G_A = \mathrm{Gr}(\deg(A), A)$ to be the Grassmannian of $\deg(A)$-dimensional k-planes in A, there is a natural closed immersion $\iota : X \to G_A$ which realizes X as the subvariety of G_A of planes which are also left ideals of A (or right ideals, depending one's convention). The pullback $\iota^* : K(G_A) \to K(X)$ takes the universal sub-bundle S on G_A to the tautological bundle ζ_X on X. In particular, if A is division then, since $K(G_A)$ is generated by λ-operations of S and $K(X)$ is generated by λ-operations of $\zeta_X(1) \subset \zeta_X$, the map ι^* is surjective. It seems reasonable to believe that one can show precisely what the kernel of this map is. The answer might only involve computing precise relations between Schubert classes of G_A of the same codimension when intersected with X and these relations can even be computed: they should be the same relations one gets between the Schur operations of the universal quotient bundle Q on G_A from the kernel of ι^*.

The next question is more subtle and involves the level of a Severi-Brauer variety. In Section 3.4 we define the level of a central simple algebra (Definition 3.4.1). In Chapter 2 there is the definition of the level of an arbitrary variety (Definition 2.4.5). The following question is about the relationship between these two definitions.

Question 3: If A is a central simple algebra, what is the relation between $\mathrm{lev}(A)$ and $\mathrm{lev}(X)$ for X the Severi-Brauer variety of A?

It's clear from Proposition 3.4.5 that, if A has p-primary degree then $\mathrm{lev}(X) \leq \mathrm{lev}(A) + 1$. The reverse inequality, $\mathrm{lev}(X) \geq \mathrm{lev}(A) + 1$, is a much more difficult statement to prove (if it's at all possible). A direct proof would involve computing λ-operations for an arbitrary element of $K(X)$ which seems unreasonably difficult.

More fitting to the theme of the rest of the thesis, I ask:

Question 4: For a p-primary central division algebra A of degree p^n with Severi-Brauer variety X there is a decomposition

$$\mathrm{gr}_\gamma K(X) = \mathbb{Z}^{\oplus p^n} \oplus (\mathbb{Z}/p^n\mathbb{Z})^{\oplus \mu_n} \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus \mu_1}.$$

From the reduced behavior of A, can one determine formulas for $\mu_1, ..., \mu_n$?

This question is related to computations of Chow groups of Severi-Brauer varieties. There is always a canonical map $\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\tau G(X)$ by comparing the γ and τ filtrations. One knows that if this map is either injective, or surjective, then it is in fact bijective (and this claim only involves an induction argument). Since the Chow ring surjects onto $\mathrm{gr}_\tau G(X)$, one can hope to use the latter object to obtain information about the Chow ring in general. But even without considering the general case, in some particular cases where the filtration comparison map is an isomorphism it's also known the surjection from the Chow ring is an isomorphism. Hence results in this direction would provide a complete description of Chow groups for some class of Severi-Brauer varieties (cf. Example 3.3.9).

Our last question is about extending these results from Severi-Brauer varieties to other projective homogeneous varieties. Essentially this entire thesis can be reworked from the point of view of an arbitrary projective homogeneous variety; one can ask about computations for the associated graded rings for the γ and τ filtrations and for how these relate to the Chow rings of such varieties. Explicitly, the following question outlines a program for doing just this.

Questions 5+: For any projective homogeneous variety X under a semisimple algebraic group G, there's a separable algebra A and a natural isomorphism $K(X) \simeq K(A)$. Since A is separable, it's a sum of central simple algebras A_i over finite extensions F_i of the base field. If, over an algebraic closure $\overline{F}$, there is an isomorphism $X_{\overline{F}} = G_{\overline{F}}/P$ for a parabolic $P \subset G_{\overline{F}}$ then one can ask:

I. Can one give a formula for $\mathrm{lev}(X)$ in terms of the $\mathrm{lev}(A_i)$ and the subset of the vertices of the Dynkin Diagram corresponding to P?

II. Can one determine generators for $B(X)$?

III. For any such X, does there exist a τ-functorial replacement for X? In other words, does there exist a variety Y with a natural equality $\mathrm{gr}_\gamma K(X) = \mathrm{gr}_\gamma K(Y)$ and such that the filtration-comparison morphism is an isomorphism, $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\tau G(Y)$?

IV. If the answer to III. is yes, then, in the notation of that question, is the canonical surjection $\mathrm{CH}(Y) \to \mathrm{gr}_\tau G(Y)$ an isomorphism?

V. If the A_i are p-primary algebras for some prime p, then there is a decomposition

$$\mathrm{gr}_\gamma K(X) = \mathbb{Z}^{\oplus n} \oplus \bigoplus_{i=1}^{\infty} (\mathbb{Z}/p^i\mathbb{Z})^{\mu_i}$$

for some integer $n \geq 1$. Can one determine formulas for the μ_i?

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