Stochastic Control in Optimal Insurance and Investment with Regime Switching

by

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Abstract

Motivated by the financial crisis of 2007-2009 and the increasing demand for portfolio and risk management, we study optimal insurance and investment problems with regime switching in this thesis.

We incorporate an insurable risk into the classical consumption and investment framework and consider an investor who wants to select optimal consumption, investment and insurance policies in a regime switching economy. We allow not only the financial market but also the insurable risk to depend on the regime of the economy. The objective of the investor is to maximize his/her expected total discounted utility of consumption over an infinite time horizon. For the case of hyperbolic absolute risk aversion (HARA) utility functions, we obtain the first explicit solutions for simultaneous optimal consumption, investment and insurance problems when there is regime switching.

Next we consider an insurer who wants to maximize his/her expected utility of terminal wealth by selecting optimal investment and risk control policies. The insurer's risk is modeled by a jump-diffusion process and is negatively correlated with the capital gains in the financial market. In the case of no regime switching in the economy, we apply the martingale approach to obtain optimal policies for HARA utility functions, constant absolute risk aversion (CARA) utility functions, and quadratic utility functions. When there is regime switching in the economy, we apply dynamic programming to derive the associated Hamilton-Jacobi-Bellman (HJB) equation. Optimal investment and risk control policies are then obtained in explicit forms by solving the HJB equation.

We provide economic analyses for all optimal control problems considered in this thesis. We study how optimal policies are affected by the economic conditions, the financial and insurance markets, and investor's risk preference.

Preface

The research conducted for this thesis forms of a research collaboration with Professor Abel Cadenillas at the University of Alberta. I was the key investigator of all the research projects in Chapters 2-4.

The main results of Chapter 2 of this thesis have been published as B. Zou and A. Cadenillas, "Explicit Solutions of Optimal Consumption, Investment and Insurance Problems with Regime Switching", *Insurance: Mathematics and Economics*, vol 58, 159-167.

The main results of Chapter 3 of this thesis have been published as B. Zou and A. Cadenillas, "Optimal Investment and Risk Control Policies for an Insurer: Expected Utility Maximization", *Insurance: Mathematics and Economics*, vol 58, 57-67.

I was responsible for both theoretical proofs and economic analyses as well as the manuscript composition of the above two published papers. Professor Abel Cadenillas was the supervisory author. He was involved in the economic analyses. He also provided feedback and suggestions to improve the manuscripts, and contributed to the final editing and corrections of the papers.

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Chapter 1

Introduction

As a field of applied mathematics, mathematical finance has been experiencing a rapid and fascinating development in the last a few decades. The origin of mathematical finance dates back to Louis Bachelier's Ph.D thesis in 1900 (see Bachelier [5]). In his fundamental work, Bachelier applied Winer processes (Brownian motion) to model the dynamics of stock prices and studied option pricing problems, which was five years earlier than Albert Einstein's celebrated work on Brownian motion (see Einstein [26]). But Bachelier's work was not recognized by either academic field or industries at that time, it rather took about another half a century until the world finally realized the importance of the applications of mathematics to finance.

Modern finance is built upon two revolutions, mainly by mathematicians. The first revolution was led by Harry Markowitz, with the publication of the paper *Portfolio Selection* in 1952. Markowitz argued that all rational investors should select mean-variance efficient portfolios which minimize variance under a given expected return or maximize expected return under a given variance (see Markowitz [60, 61]). Building on Markowitz's original work, Sharpe [82] developed the *Capital Asset Pricing Model*, which was also proposed by Lintner [57] and Mossin [67] independently. Markowitz and Sharpe, jointly with Miller, received the 1990 Nobel Prize in Economics for their contributions to modern portfolio theory.

The second revolution began with the Black-Scholes model on option pricing in

1973. Fischer Black and Myron Scholes derived a partial differential equation that yields the explicit solution of European call/put option price (see Black and Scholes [7]). Black-Scholes model also provides an important risk management procedure, called delta hedging. Robert Merton worked closely with Black and Scholes on the development of the option pricing formula and extended the original work of Black and Scholes [7] in Merton [65]. Merton and Scholes were awarded the Nobel Price in Economics in 1997 for their work on options pricing. Sadly, Black passed away in 1995 and then was only memorized as a contributor.

In this thesis, we concern not only investment problems but also risk management problems. In particular, we deal with risk management problems with insurance/reinsurance. Arrow's paper in 1963 (see Arrow [2]) draw the attention of many researchers to risk management with insurance. In general, insurance is considered as a risk transfer tool. But insurance can also be used to prevent risk. Ehrlich and Becker [25] were the first to propose how insurance can be used as a risk prevention tool. Early contributions to insurance/reinsurance problems can be found in Louberge [59] and Dionne [22].

1.1 Objectives of the Thesis

The financial crisis of 2007-2009 caused a severe recession in global economy, considered by many economists to be the worst financial crisis since the Great Depression of the 1930s. It resulted in the threat of bankruptcy of large financial institutions, the bailout of banks, and downturns in stock markets around the world (see IMF World Economic Outlook [45], Stein [88], and Zou and Cadenillas [103] among others). To understand the causes of this financial crisis and manage portfolio and risk with securities, derivatives and other financial products, we focus on the case of American International Group, Inc. (AIG), whose severe liquidity crisis put the entire financial system on the brink of collapse.

AIG, once the largest insurance company in the United States with a triple-A credit rating, collapsed within a few months in 2008. The stock price of AIG was traded at over $50 per share in February, but plunged down to less than $2 per share in the last quarter of 2008. The severity of AIG's liquidity crisis led to an initial rescue of $85 billion and a total of $182 billion bailout by the U.S. government, the largest government bailout in history (See Stein [88, Chapter 6] and Sjostrom [83] for detailed discussions on AIG bailout case). According to Stein [88, Chapter 6], AIG made several major mistakes which together contributed to its sudden collapse. First, AIG did not take business cycles into consideration and expected the housing price index and capital gains to continue to grow when making risk management decisions. Second, AIG misunderstood the influence of derivatives trading on the company's capital structure and failed to regulate risk using a proper model. Third, AIG underpriced the risk of writing Credit Default Swap (CDS) contracts since it ignored the negative correlation between its liabilities and the capital gains in the financial market. Therefore, our research aims to address two important questions that arose from the AIG case:

- Problem (i)

  How can we find optimal consumption, investment and insurance strategies for an investor who is trading securities, derivatives and insurance in an economy with business cycles?

- Problem (ii)

  How can we regulate risk for an insurer who makes investment decisions and controls its liabilities simultaneously in a market with regime switching?

1.2 Literature Review

In this thesis, we study two major topics: optimal investment problems, and risk management problems with insurance/reinsurance. We review literature on these two topics in this section.

The theory of portfolio selection started with the mean-variance framework of Markowitz [60]. Markowitz used variance as a measure of risk and expected return of a portfolio as a selection criterion, and then treated portfolio selection and risk management problems as mathematical optimization problems. A major drawback of mean-variance model is its inconsistency with second order stochastic dominance, see Rothschild and Stiglitz [77]. Built as a static model, mean-variance framework has limitations when applied to dynamic settings.

Next milestone in portfolio management attributed to the application of stochastic calculus and control theory in financial economics, see Samuelson [78] and Merton [63, 64]. Their work can be described as dynamic portfolio selection, since the price processes of risky assets are modeled by geometric Brownian motions and portfolio decisions are made dynamically. Merton [63] was the first to obtain explicit solutions to consumption and investment problems in continuous time using dynamic programming. Many generalizations to Merton's work can be found in Karatzas [52], Karatzas and Shreve [53], Sethi [81], et cetera. Another major approach to solve dynamic portfolio selection problems is the martingale approach, see, for instance, Cox and Huang [19], and Karatzas et al. [50].

Portfolio selection in incomplete markets has attracted more attention recently. In incomplete markets, explicit solutions of portfolio strategies may not be obtained, and Monte Carlo simulations are then needed to obtain numerical solutions. Thanks to the Girsanov Theorem (see Girsanov [34]), researchers can apply stochastic interest rate models and deal with changes of numeraire in economics and finance, see Geman et al. [32] for their original work on this topic. Applications with changes of numeraire on portfolio selection can be found in Munk and Sorensen [69, 70]. Implementing constraints on portfolio strategies, such as short-selling forbiddance, will make markets incomplete. Examples of research on portfolio selection with constraints can be found in Karatzas et al. [51], and Cvitanic and Karatzas [20]. Another direction is to include jump components in the modelling, which leads to incomplete markets as well. Optimal portfolio selection with jumps have been studied by Aase [1], Jeanblant-Picque and Pontier [46], Jin and Zhang [48] and many others. A recent review on portfolio selection can be found in Detemple [21].

The initial optimal insurance problem studies an individual who is subject to an insurable risk and seeks the optimal amount of insurance under the utility maximization criterion. Using the expected value principle for premium, Arrow [2] found the optimal insurance is deductible insurance in discrete time, see also Arrow [3]. The work of Arrow [2] has been generalized with state dependent utility functions in Arrow [4]. Mossin [68] showed that full coverage is never optimal if the insurance premium is not equal to its actuarial value (in other words, there exists a strictly positive loading in the premium). Smith [84] extended Arrow's research in medical insurance to casualty insurance and liability insurance. Raviv [76] formally explained the cause of optimal insurance being deductible insurance, and considered the case of multiply losses. Doherty and Schlesinger [23] considered the presence of an uninsurable risk and found the sufficient conditions for optimal insurance to be full coverage or deductible insurance. Promislow and Young [75] reviewed optimal insurance problems (without investment and consumption). They proposed a general market model and obtained explicit solutions to optimal insurance problems under different premium principles, such as variance principle, equivalent utility principle, Wang's principle, et cetera. Most research on optimal insurance use the criterion of utility maximization, such as Arrow [2, 4], Mossin [68]. Doherty and Eeckhoudt [24] applied the dual theory (also called rank-dependence, see Yaari [95]) as the criterion to obtain optimal insurance contracts.

Optimal reinsurance problem considers an insurer who wants to select the optimal reinsurance contract to insure against its risk (liabilities) under certain criteria. Borch [9] was the first to study Pareto optimal problems with reinsurance in a equilibrium model. Early contributions on optimal reinsurance are summarized in the books of Buhlmann [12] and Gerber [33]. Common optimization criteria include mean-variance principle, see Kaluszka [49], maximizing expected value of discounted reserve, see Hojgaard and Taksar [44], and minimizing the ruin probability, see Schmidli [79].

In traditional financial modeling, the market parameters, like the risk-free interest rate, stock returns and volatility, are assumed to be independent of general macroeconomic activities. Examples can be found extensively in the literature, such as option pricing models (see Black and Scholes [7], and Merton [63]) and interest rate models (see Cox et. al [18], and Vasicek [91]). However, historical data and empirical research both show that the market behavior is affected by long-term economic factors, which may change dramatically as time evolves. Interested readers may refer to Chen [16], Fama [27, 29], Lee [56], Schwert [80], and the references therein, for detailed discussions on the relationship between the financial market and macroeconomic activities. All those research show that there exist business cycles (regime switching features) in the financial market.

In the insurance market, insurance policies depend on the regime of the economy as well. In the case of traditional insurance, the underwriting cycle has been well documented in the literature. Indeed, empirical research provides evidence for the dependence of insurance policies' underwriting performance on external economic conditions (see for instance Grace and Hotchkiss [35], Haley [37] on property-liability insurance, and Chung and Weiss [17] on reinsurance). In the case of non-traditional insurance, by investigating the comovements of credit default swap (CDS) and the bond/stock markets, Norden and Weber [71] found that CDS spreads are negatively correlated with the price movements of the underlying stocks and such cointegration is affected by the corporate bond volume.

Regime switching models have been developed to capture the uncertainty of those long-term economic factors by a continuous-time Markov chain with finite states. Hence, regime switching models can be used to capture business cycles in the economy. Hamilton [38] was the first to introduce a regime switching model for postwar real GNP in the U.S. and showed that the regime switching model captures the movements of GNP in the long run better than the models with deterministic coefficients.

Thereafter, regime switching has been applied to model many important problems in economics, finance, actuarial science, operation research and other fields. Bollen [8], Buffington and Elliott [11], and Zhang and Guo [99] considered option pricing problems in regime switching models. Portfolio optimization problems have been studied in regime switching models by Bauerle and Rieder [6], Sotomayor and Cadenillas [86], Yin and Zhou [96], Zariphopoulou [98], Zhang and Yin [100], Zhou and Yin [101], Zou and Cadenillas [103], and many others. Regime switching models in optimal dividend policy problems can be found in Jiang and Pistorius [47], Sotomayor and Cadenillas [87], and Wei et al. [93]. Cadenillas et al. [14] studied optimal production management in a regime switching framework. Optimal investment-reinsurance problems have been studied by Liu et al. [58] in a regime switching model.

1.3 Organization of the Thesis

This thesis consists of three research projects on optimal investment and insurance problems conducted by the candidate under the supervision of Professor Abel Cadenillas during his PhD studies at the University of Alberta.

In Chapter 2, we study Problem (i) introduced in Section 1.1. We assume that an investor faces an insurable risk and can purchase insurance to insure against such risk. Motivated by new insurance products, we allow not only the financial market but also the insurable loss to depend on the regime of the economy. The investor wants to select optimal consumption, investment and insurance policies to maximize his/her expected total discounted utility of consumption. In the case of hyperbolic absolute risk aversion (HARA) utility functions, via applying classical stochastic control theory and solving the Hamilton-Jacobi-Bellman (HJB) equations, we obtain the first explicit solutions for simultaneous optimal consumption, investment, and insurance problems when there is regime switching. We determine that the optimal insurance contract is either no-insurance or deductible insurance, and calculate when it is optimal to buy insurance. The optimal policy depends strongly on the regime of the economy. Through an economic analysis, we calculate the advantage of buying insurance.

In Chapter 3, we study Problem (ii) under the assumption of no regime switching in the economy. As discussed in Section 1.1, we consider an insurer whose risk process is modeled by a jump-diffusion process and is negatively correlated with the capital gains in the financial market. The objective of the insurer is to select optimal investment and risk control strategies to maximize his/her expected utility of terminal wealth. We apply the martingale approach to obtain explicit optimal strategies in the cases of logarithmic utility, power utility, exponential utility and quadratic utility. Through an economic analysis, we investigate the impact of two factors on optimal strategies: the negative correlation between the risk process and the capital gains, and the jump intensity of the risk.

In Chapter 4, we continue our studies on Problem (ii), and generalize the results obtained in Chapter 3 in a regime switching model. We assume both the financial market and the insurance market depend on the same regime. In this chapter, we apply dynamic programming to derive the HJB equation, and then solve the HJB equation to obtain explicit optimal investment and risk control strategies for logarithmic utility, power utility and exponential utility. An economic analysis is also provided to study the impact of markets on optimal strategies.

We summarize the results of this thesis in Chapter 5.

Chapter 2

Optimal Consumption, Investment and Insurance Policies with Regime Switching

In classical consumption and investment problems (see, e.g., Merton [63, 65]), a risk-averse investor wants to select optimal consumption and investment policies in order to maximize his/her expected discounted utility of consumption. In the traditional models for consumption and investment problems, there is only one source of risk that comes from the uncertainty of the stock prices in the financial market. But in real life, apart from the risk exposure in the financial market, investors often face other random risks, such as property-liability risk and credit default risk. Thus, it is more realistic and practical to extend the traditional models by incorporating an insurable risk. When an investor is subject to an additional insurable risk, buying insurance is a trade-off decision. On one hand, insurance can provide the investor with compensation and then offset capital losses if the specified risk events occur. On the other hand, the cost of insurance diminishes the investor's ability to consume and therefore reduces the investor's expected utility of consumption.

Moore and Young [66] incorporated an insurable loss and a random horizon into Merton's framework in continuous time. They found explicit or numerical solutions for different utility functions, but they did not verify rigourously that the obtained policies are indeed optimal. Perera [73] revisited Moore and Young's work by considering the same problem in a more general Levy market, and applied the martingale approach to obtain explicit optimal policies for exponential utility functions. Pirvu and Zhang [74] considered the insurable risk to be mortality risk and studied optimal investment, consumption and life insurance problems in a financial market in which the stock price is modeled by a mean-reverting process.

As discussed in Section 1.1 and Section 1.2, both the financial market and the insurance market are affected by long-term economic factors that present regime switching features, see also Sotomayor and Cadenillas [86], Sotomayor and Cadenillas [87], and Zou and Cadenillas [103]. Hence in this chapter, we use an observable continuous-time finite-state Markov chain to model the regime of the economy, and allow both the financial market and the insurance market to depend on the regime. Our objective is to obtain simultaneously optimal consumption, investment and insurance policies for a risk-averse investor who wants to maximize his/her expected total discounted utility of consumption over an infinite time horizon. We extend Sotomayor and Cadenillas [86] by including a random loss in the model and an insurance policy in the control. The most important difference between the model of Moore and Young [66] and ours is that they do not allow regime switching, while we allow regime switching in both the financial market and the insurance market.

2.1 The Model

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which a standard Brownian motion W and an observable continuous-time, stationary, finite-state Markov chain ϵ are defined. Denote by $\mathcal{S} = \{1, 2, \cdots, S\}$ the state space of this Markov chain, where S is the number of regimes in the economy. The matrix $Q = (q_{ij})_{S\times S}$ denotes the strongly irreducible generator of ϵ, where $\forall\, i \in \mathcal{S}$, $\sum\limits_{j\in\mathcal{S}} q_{ij} = 0$, $q_{ij} > 0$ when $j \neq i$ and $q_{ii} = -\sum\limits_{j\neq i} q_{ij}$.

We consider a financial market consisting of two assets, a bond with price P_0 (riskless asset) and a stock with price P_1 (risky asset), respectively. Their price processes are driven by the following Markov-modulated stochastic differential equations:

$$
\begin{aligned}
dP_0(t) &= r_{\epsilon(t)} P_0(t) dt, \\
dP_1(t) &= P_1(t)(\mu_{\epsilon(t)} dt + \sigma_{\epsilon(t)} dW(t)),
\end{aligned}
$$

with initial conditions $P_0(0) = 1$ and $P_1(0) > 0$. The coefficients r_i, μ_i and σ_i, $i \in \mathcal{S}$, are all positive constants.

An investor chooses $\pi = \{\pi(t), t \geq 0\}$, the proportion of wealth invested in the stock, and a consumption rate process $c = \{c(t), t \geq 0\}$. We require the consumption rate c to be non-negative, but allow the investment proportion π in the stock to take all real values. In other words, short-selling of the stock is allowed in the financial market and we assume the stock is infinitely divisible. We assume the investor is subject to an insurable loss $L(t, \epsilon(t), X(t))$, where $X(t)$ denotes the investor's wealth at time t. We shall use the short notation L_t (or $L(t)$) to replace $L(t, \epsilon(t), X(t))$ if there is no confusion. We use a Poisson process N with intensity $\lambda_{\epsilon(t)}$, where $\lambda_i > 0$ for every $i \in \mathcal{S}$, to model the occurrence of this insurable loss. In the insurance market, there are insurance policies available to insure against the loss L_t. We further assume the investor can control the payout amount $I(t)$, where $I(t) : [0, \infty) \times \Omega \mapsto [0, \infty)$ and $I(t, \omega) := I_t(L(t, \epsilon(t, \omega), X(t, \omega)))$, or in short, $I(t) = I_t(L_t)$. For example, if $\Delta N(t_0) = 1$, then at time t_0 the investor suffers a loss of amount L_{t_0} but receives a compensation of amount $I_{t_0}(L_{t_0})$ from the insurance policy, so the investor's net loss is $L_{t_0} - I_{t_0}(L_{t_0})$. Following the premium setting used in Moore and Young [66] (the famous expected value principle), we assume investors pay premium continuously at the rate P given by

$$P(t) = \lambda_{\epsilon(t)}(1 + \theta_{\epsilon(t)})E[I_t(L_t)],$$

where the positive constant θ_i, $i \in \mathcal{S}$, is known as the loading factor in the insurance industry. Such extra positive loading comes from insurance companies' administrative cost, tax, profit, et cetera.

Following Sotomayor and Cadenillas [86], we assume the Brownian motion W, the Poisson process N and the Markov chain ϵ are mutually independent. We also assume that the loss process L is independent of N. We take the $\mathbb{P}-$augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by W, N, L and ϵ as our filtration and define $\mathcal{F} := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

For an investor with triplet strategies $u(t) := (\pi(t), c(t), I(t))$, the associated wealth process X is obtained as

$$\begin{aligned} dX(t) = \big(&r_{\epsilon(t)}X(t) + (\mu_{\epsilon(t)} - r_{\epsilon(t)})\pi(t)X(t) - c(t) - \lambda_{\epsilon(t)}(1 + \theta_{\epsilon(t)}) \\ &\cdot E[I_t(L_t)]\big)dt + \sigma_{\epsilon(t)}\pi(t)X(t)dW(t) - (L_t - I_t(L_t))\, dN(t), \end{aligned} \tag{2.1}$$

with initial conditions $X(0) = x > 0$ and $\epsilon(0) = i \in \mathcal{S}$.

We define the criterion function J as

$$J(x, i; u) := E_{x,i}\left[\int_0^{+\infty} e^{-\delta t}U(c(t), \epsilon(t))dt\right], \tag{2.2}$$

where $\delta > 0$ is the discount rate and $E_{x,i}$ means conditional expectation given $X(0) = x$ and $\epsilon(0) = i$. We assume that for every $i \in \mathcal{S}$, the utility function $U(\cdot, i)$ is $C^2(0, +\infty)$, strictly increasing and concave, and satisfies the linear growth condition

$$\exists K > 0 \text{ such that } U(y,i) \leq K(1+y),\, \forall\, y > 0, i \in \mathcal{S}.$$

Besides, we use the notation $U(0,i) := \lim_{y \to 0^+} U(y,i)$, $\forall\, i \in \mathcal{S}$.

Remark 2.1. *The strict concavity implies that the investors considered here are risk averse. Furthermore, we assume the utility function is regime dependent. Such assumption is supported by extensive literature, see, for instance, Koszegi and Rabin [54, 55] and Sugden [89]. Sotomayor [85], and Sotomayor and Cadenillas [86] gave a detailed explanation and reference regarding the validness of this assumption.*

We define the bankruptcy time as

$$\Theta := \inf\{t \geq 0 : \; X(t) \leq 0\}.$$

Since an investor can consume only when his/her wealth is strictly positive, we define

$$R(\Theta) := \int_\Theta^\infty e^{-\delta t} U(c(t), \epsilon(t)) dt = \int_\Theta^\infty e^{-\delta t} U(0, \epsilon(t)) dt.$$

A control $u := (\pi, c, I)$ is called admissible if $\{u_t\}_{t \geq 0}$ is predictable with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and satisfies, $\forall\, t \geq 0$

$$E_{x,i}\left[\int_0^t c(s) ds\right] < +\infty, \tag{2.3}$$

$$E_{x,i}\left[\int_0^t \sigma^2_{\epsilon(s)} \pi^2(s) ds\right] < +\infty, \tag{2.4}$$

$$E_{x,i}\left[\int_0^\Theta e^{-\delta s} U^+(c(s), \epsilon(s)) ds\right] < +\infty, \tag{2.5}$$

and $I(t) \in \mathcal{I}_t := \{I : \; 0 \leq I(Y) \leq Y, \;$ where Y is $\mathcal{F}_t$-measurable$\}$.

The set of all admissible controls with initial conditions $X(0) = x$ and $\epsilon(0) = i$

is denoted by $\mathcal{A}_{x,i}$. We study the following problem.

Problem 2.1. *Select an admissible control $u^* = (\pi^*, c^*, I^*) \in \mathcal{A}_{x,i}$ that maximizes the criterion function J. In addition, find the value function*

$$V(x,i) := \sup_{u \in \mathcal{A}_{x,i}} J(x,i;u).$$

The control u^* is called an optimal control or an optimal policy.

Moore and Young [66] also incorporated an insurable risk into the consumption and investment framework. However, they did not consider a regime switching model, or equivalently they assumed that there is only one regime in the economy. Nevertheless, the insurable risk and the coefficients of the financial market most likely depend on the regime of the economy. Hence, in the above regime switching model, we assume that the insurance market (insurable loss and insurance performance) and the financial market are regime dependent. Furthermore, we assume these two markets depend on the same regime. We mention three examples below to support such assumption. First, the assumption that the financial market and the insurance market depend on the same regime is supported by the bailout case of AIG (see Sjostrom [83] for details) and new financial derivatives traded in the insurance market. Before the crash of the U.S. housing market in 2007, many investors, banks and financial institutions bought obligations constructed from mortgage payments or made loans to the housing agencies. To insure against the credit risk that the obligations or loans may default, they purchased credit default swap (CDS) contracts from insurance companies like AIG. In a CDS contract, the buyer makes periodic payments to the seller, and in return, receives the par value of the underlying obligation or loan in the event of a default. Apparently, the credit default risk insured by CDS contracts is negatively correlated with the reference entity's stock performance (see Norden and Weber [71] for empirical evidence). Second, generated by the financial engineering on derivatives, insurance companies have created numerous equity-linked products, such as equity-linked life insurance (see Hardy [41] for more details on such insurance policy). If the insured of an equity-linked life insurance policy survives to the expiration, then the beneficiary receives investment benefit that depends upon the market value of the reference equity. Hence, equity-linked life insurance and its reference equity are affected by the same long-term economic factors. Third, even in traditional insurance products like property-liability insurance, there is empirical evidence (see, for instance, Grace and Hotchkiss [35]) that the loading factor θ depends on the regime of the economy. Indeed, in those traditional insurance products, $\lambda_{\epsilon(t)}$ and $L(t, \epsilon(t, \omega), X(t, \omega))$ might be independent of $\epsilon(t)$ but θ depends on $\epsilon(t)$.

2.2 Verification Theorems

Let $\psi : (0, \infty) \times \mathcal{S} \to \mathbb{R}$ be a function with $\psi(\cdot, i) \in C^2(0, \infty), \forall\, i \in \mathcal{S}$. We define the operator $\mathcal{L}_i^u$ by

$$\mathcal{L}_i^u(\psi) := (r_i x + (\mu_i - r_i)\pi x - c - \lambda_i(1 + \theta_i)E[I(L)])\psi' + \frac{1}{2}\sigma_i^2 \pi^2 x^2 \psi'' - \delta\psi,$$

where $\psi' = \frac{\partial \psi}{\partial x}$ and $\psi'' = \frac{\partial^2 \psi}{\partial x^2}$.

Theorem 2.1. *Suppose $U(0, i)$ is finite, $\forall\, i \in \mathcal{S}$. Let $v(\cdot, i) \in C^2(0, \infty)$ be an increasing and concave function such that $v(0, i) = \frac{U(0,i)}{\delta}$ for every $i \in \mathcal{S}$. If $v = v(\cdot, \cdot)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_u \left\{ \mathcal{L}_i^u v(x,i) + U(c,i) + \lambda_i E[v(x - L + I(L), i) - v(x,i)] \right\}$$
$$= -\sum_{j \in \mathcal{S}} q_{ij} \left(v(x,j) - \frac{U(0,j)}{\delta} \right) \tag{2.6}$$

for every $x > 0, i \in \mathcal{S}$, and the control $u^* = (\pi^*, c^*, I^*)$ defined by

$$u_t^* := \arg\sup_u \Big(\mathcal{L}_{\epsilon(t)}^u v(X_t^*, \epsilon_t) + U(c, \epsilon_t) + \lambda_{\epsilon(t)} E[v(X_t^* - L_t + I(L_t), \epsilon_t) - v(X_t^*, \epsilon_t)] \Big) \mathbf{1}_{0 \le t < \Theta}$$

is admissible, then u^* is an optimal control to Problem 2.1. In addition, the value function is given by

$$V(x,i) = v(x,i) + \frac{1}{\delta} E_{x,i} \left[\int_0^\infty e^{-\delta s} dU(0, \epsilon_s) \right],$$

where $dU(0,\epsilon_s) := \sum_{j \in \mathcal{S}} q_{\epsilon_s,j} U(0,j) ds$.

Furthermore, if the utility function does not depend on the regime, namely $U(y,i) = U(y)$, for every $i \in \mathcal{S}$, then the value function $V(x,i) = v(x,i)$.

Proof. $\forall\, u \in \mathcal{A}_{x,i}$, consider $f(t, X_t, \epsilon_t) := e^{-\delta t}(v(X_t, \epsilon_t) - \frac{U(0,\epsilon_t)}{\delta})$. By applying Ito's formula for Markov-modulated processes (see, for instance, Buffington and Elliott [11], and Sotomayor and Cadenillas [86]), we get

$$\begin{aligned} f(t, X_t, \epsilon_t) = \int_0^t e^{-\delta s} \Big(& \mathcal{L}_{\epsilon(s)}^{u(s)} v(X_s, \epsilon_s) + \lambda_{\epsilon(s)} \left[v(X_s - L_s + I_s, \epsilon_s) - v(X_s, \epsilon_s) \right] \\ & + \sum_{j \in \mathcal{S}} q_{\epsilon_s, j} \big(v(X_s, j) - \frac{U(0,j)}{\delta} \big) + U(0, \epsilon_s) \Big) ds \\ & + v(X_0, \epsilon_0) - \frac{U(0,\epsilon_0)}{\delta} + m_t^f, \end{aligned} \tag{2.7}$$

where $\{m_t^f\}_{t \ge 0}$ is a $\mathbb{P}-$martingale with $m_0^f = 0$.

Let $0 < a < X_0 = x < b < \infty$ and define a stopping time $\tau := \inf\{t \ge$

$0 : X_t \leq a$ or $X_t \geq b\}$. Then by replacing t by $t \wedge \tau$ in (2.7), taking conditional expectation and applying the HJB equation (2.6), we obtain

$$E_{x,i}[f(t\wedge\tau, X_{t\wedge\tau}, \epsilon_{t\wedge\tau})] \leq -E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s}(U(c_s,\epsilon_s) - U(0,\epsilon_s))ds\right] + v(x,i) - \frac{U(0,i)}{\delta}.$$

Let $a \downarrow 0$, $b \uparrow +\infty$ and $t \to \infty$. Then $t \wedge \tau \to \Theta$. Since f is continuous, we obtain

$$f(t\wedge\tau, X_{t\wedge\tau}, \epsilon_{t\wedge\tau}) \to f(\Theta, 0, \epsilon_\Theta) = 0,$$

when $a \downarrow 0, b \uparrow +\infty, t \to \infty$.

Then, we get

$$v(x,i) - \frac{U(0,i)}{\delta} - E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}(U(c_s,\epsilon_s) - U(0,\epsilon_s))ds\right] \geq 0. \tag{2.8}$$

Define $g(t,\epsilon_t) := -e^{-\delta t}\frac{U(0,\epsilon_t)}{\delta}$. Applying Ito's formula to $g(t,\epsilon_t)$ yields

$$g(t,\epsilon_t) - g(0,\epsilon_0) = \int_0^t e^{-\delta s}\left(U(0,\epsilon_s) - \frac{1}{\delta}\sum_{j\in\mathcal{S}} q_{\epsilon_s,j}U(0,j)\right)ds + m_t^g,$$

where $\{m_t^g\}_{t\geq 0}$ is a square-integrable martingale with $m_0^g = 0$.

Taking conditional expectation and applying the monotone convergence theorem to the above equality, we get

$$\frac{U(0,i)}{\delta} = E_{x,i}\left[\int_0^\infty e^{-\delta s}U(0,\epsilon_s)ds\right] - \frac{1}{\delta}E_{x,i}\left[\int_0^\infty e^{-\delta s}dU(0,\epsilon_s)\right],$$

and then

$$
\begin{aligned}
& v(x,i) - \frac{U(0,i)}{\delta} - E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}(U(c_s,\epsilon_s) - U(0,\epsilon_s))ds\right] \\
=& v(x,i) - E_{x,i}\left[\int_0^{\infty} e^{-\delta s}U(0,\epsilon_s)ds - \frac{1}{\delta}\int_0^{\infty} e^{-\delta s}dU(0,\epsilon_s)\right] \\
& - E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}(U(c_s,\epsilon_s) - U(0,\epsilon_s))ds\right] \\
=& v(x,i) + \frac{1}{\delta}E_{x,i}\left[\int_0^{\infty} e^{-\delta s}dU(0,\epsilon_s)\right] - E_{x,i}\left[\int_0^{\Theta} e^{-\delta s}U(c_s,\epsilon_s)ds\right] \\
& - E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta s}U(0,\epsilon_s))ds\right].
\end{aligned}
$$

Therefore, the inequality (2.8) can be rearranged as

$$
(x,i) + \frac{1}{\delta}E_{x,i}\left[\int_0^{\infty} e^{-\delta s}dU(0,\epsilon_s)\right] \geq E_{x,i}\left[\int_0^{\infty} e^{-\delta s}U(c_s,\epsilon_s)ds\right] = J(x,i;u),
$$

and the equality will be achieved when $u = u^*$.

If the utility function does not depend on the regime, then $dU(0,\epsilon_s) = U(0) \cdot \sum_{j\in\mathcal{S}} q_{ij}ds = 0$, and so $V(x,i) = v(x,i)$. □

$U(\cdot,i)$ is an increasing function for every $i \in \mathcal{S}$, so if $U(0,i)$ is not finite, then $U(0,i) = -\infty$. The following theorem deals with the case when $U(0,i) = -\infty$, $\forall\, i \in \mathcal{S}$.

Theorem 2.2. *Suppose $U(0,i) = -\infty$ for every $i \in \mathcal{S}$. Let $v(\cdot,i) \in C^2(0,\infty)$ be an increasing and concave function such that $v(0,i) = -\infty$ for every $i \in \mathcal{S}$. If $v = v(\cdot,\cdot)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$
\begin{aligned}
& \sup_u \left\{\mathcal{L}_i^u v(x,i) + U(c,i) + \lambda_i E[v(x - L + I(L), i) - v(x,i)]\right\} \\
& \qquad = -\sum_{j\in\mathcal{S}} q_{ij} v(x,j) \qquad (2.9)
\end{aligned}
$$

for every $x > 0, i \in \mathcal{S}$, and the control $u^* = (\pi^*, c^*, I^*)$ defined by

$$u_t^* := \arg\sup_u \Big(\mathcal{L}^u_{\epsilon(t)} v(X_t^*, \epsilon_t) + U(c, \epsilon_t) + \lambda_{\epsilon(t)} E[v(X_t^* - L_t + I(L_t), \epsilon_t) - v(X_t^*, \epsilon_t)] \Big) \mathbf{1}_{0 \le t < \Theta}$$

is admissible, then u^* is an optimal control to Problem 2.1 and the value function is $V(x, i) = v(x, i)$.

Proof. Define $h(t, X_t, \epsilon_t) := e^{-\delta t} v(X_t, \epsilon_t)$. For any admissible control u, by following a similar argument as in Theorem 2.1, we obtain

$$\begin{aligned} E_{x,i}\left[h(t \wedge \tau, X_{t\wedge\tau}, \epsilon_{t\wedge\tau})\right] &= E_{x,i}\Big[\int_0^{t\wedge\tau} e^{-\delta s} \Big(\mathcal{L}^{u(s)}_{\epsilon(s)} v(X_s, \epsilon_s) + \sum_{j \in \mathcal{S}} q_{\epsilon_s, j} v(X_s, j) \\ &\quad + \lambda_i E[v(X_s - L_s + I_s, \epsilon_s) - v(X_s, \epsilon_s)] \Big) ds \Big] + v(x, i) \\ &\le v(x, i) - E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s} U(c_s, \epsilon_s) ds \right]. \end{aligned}$$

$E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s} U(c(s), \epsilon(s)) ds\right]$ is well defined and finite, because u is an admissible control and U satisfies the linear growth condition. Then the above inequality becomes

$$v(x, i) \ge E_{x,i}\left[h(t \wedge \tau, X_{t\wedge\tau}, \epsilon_{t\wedge\tau})\right] + E_{x,i}\left[\int_0^{t\wedge\tau} e^{-\delta s} U(c_s, \epsilon_s) ds \right].$$

By assumption, $v(\cdot, i)$ is increasing in $(0, \infty)$ and $v(0, i) = \frac{U(0,i)}{\delta} = -\infty$ for every $i \in \mathcal{S}$, so

$$E_{x,i}[h(t \wedge \tau, X_{t\wedge\tau}, \epsilon_{t\wedge\tau})] \ge E_{x,i}\left[\int_{t\wedge\tau}^{\infty} e^{-\delta s} U(0, \epsilon_s) ds \right].$$

By letting $a \downarrow 0$, $b \uparrow +\infty$ and $t \to \infty$, and applying the monotone convergence theorem, we obtain

$$v(x, i) \ge E_{x,i}\left[\int_0^{\Theta} e^{-\delta s} U(c_s, \epsilon_s) ds \right] + E_{x,i}\left[\int_{\Theta}^{\infty} e^{-\delta s} U(0, \epsilon_s) ds \right] = J(x, i; u),$$

and the equality holds when $u = u^*$. □

2.3 Explicit Solutions of Value Function and Optimal Policies

In this section, we obtain explicit solutions to optimal consumption, investment and insurance problems when there is regime switching in the economy. We assume the utility function is of HARA (Hyperbolic Absolute Risk Aversion) type and the insurable loss L is proportional to the investor's wealth, $L(t, \epsilon(t), X(t)) = \eta_{\epsilon(t)}\, l_t\, X_t$. Here for every $i \in \mathcal{S}$, $\eta_i > 0$ measures the intensity of the insurable loss in regime i, and for every $t \geq 0$, l_t denotes the loss proportion at time t. We assume that l_t is $\mathcal{F}_t-$measurable and $l_t \in (0, 1)$ for all $t \geq 0$.

To obtain optimal policy, we first construct a candidate policy at time t, which is a function of (x, i, l), namely, $\pi^* = \pi^*(x, i, l)$, $c^* = c^*(x, i, l)$ and $I^* = I^*(x, i, l)$ (In fact, we find π^* and c^* are independent of l). The candidate policy is indeed optimal once we can prove it is admissible.

We rewrite the HJB equation (2.6) as

$$\begin{aligned}
&\max_{\pi}\left[(\mu_i - r_i)\pi x v'(x,i) + \frac{1}{2}\sigma_i^2\pi^2x^2v''(x,i)\right] + \max_{c}\left[U(c,i) - cv'(x,i)\right] \\
&+ \lambda_i \max_{I}\left[Ev(x - \eta_i l x + I(\eta_i l x), i) - (1+\theta_i)E(I(\eta_i l x))v'(x,i)\right] \\
&= (\delta + \lambda_i)v(x,i) - r_i x v'(x,i) - \sum_{j\in\mathcal{S}} q_{ij}\left(v(x,j) - \frac{U(0,j)}{\delta}\right), \qquad (2.10)
\end{aligned}$$

and the HJB equation (2.9) as

$$\begin{aligned}
&\max_{\pi}\left[(\mu_i - r_i)\pi x v'(x,i) + \frac{1}{2}\sigma_i^2\pi^2x^2v''(x,i)\right] + \max_{c}\left[U(c,i) - cv'(x,i)\right] \\
&+ \lambda_i \max_{I}\left[Ev(x - \eta_i l x + I(\eta_i l x), i) - (1+\theta_i)E(I(\eta_i l x))v'(x,i)\right] \\
&= (\delta + \lambda_i)v(x,i) - r_i x v'(x,i) - \sum_{j\in\mathcal{S}} q_{ij}v(x,j). \qquad (2.11)
\end{aligned}$$

We conjecture that $v(\cdot, i)$ is strictly increasing and concave for every $i \in \mathcal{S}$. Then a candidate for π^* is given by

$$\pi^*(x,i) = -\frac{(\mu_i - r_i)v'(x,i)}{\sigma_i^2 x v''(x,i)}. \qquad (2.12)$$

Since U' is strictly decreasing, the inverse of U' exists. Then a candidate for c^* is given by

$$c^*(x,i) = (U')^{-1}(v'(x,i), i). \qquad (2.13)$$

For the optimal insurance, we have the following Lemma and Theorem.

Lemma 2.1. *$\forall\, x > 0$ and $i \in \mathcal{S}$, denote $z_0 := \eta_i l_0 x$, where constant $l_0 \in (0,1)$. We denote the optimal insurance policy by I^*. Then we have*

(a) $I^*(x,i,l_0) = 0$ if and only if

$$(1+\theta_i)v'(x,i) \geq v'(x - z_0, i).$$

(b) $0 < I^*(x,i,l_0) < z_0$ if and only if

$$(1+\theta_i)v'(x,i) = v'(x - z_0 + I^*(x,i;l_0), i).$$

Proof. $\forall\, i \in \mathcal{S}$, we use the notation $z := \eta_i l x$. We then break the proof into four steps.

Step 1: We show that $I^*(x,i,l) \neq z, \forall\, l \in (0,1)$.

Assume to the contrary that $\exists\, l_0 \in (0,1)$ such that $I^*(x,i,l_0) = I^*(z_0) = z_0$. Consider $\bar{I}(x,i,l) := I^*(x,i,l) - \zeta G(l)$, where $\zeta > 0$ and $G(l) = 1$ when $l_0 - \rho < l \leq l_0 + \rho$ and 0 otherwise, $\rho > 0$. Here we choose small ζ and ρ to ensure that $0 \leq \bar{I}(z) \leq z$. Let

$$f^I(x,i,l;I) := E\left[v(x - z + I(z), i)\right] - (1+\theta_i)E\left[I(z)\right]v'(x,i).$$

Since I^* is the maximizer of $f^I(x,i,l;I)$, we have

$$E\big[v(x - z + \bar{I}(z), i) - v(x - z + I^*(z), i)\big] \leq (1+\theta_i)E\big[\bar{I}(z) - I^*(z)\big]v'(x,i).$$

Using Taylor expansion and letting $\zeta \to 0^+$, we get

$$(1+\theta_i)E\big[G(l)\big]v'(x,i) \leq E\big[v'(x - z + I^*(z), i)G(l)\big].$$

Letting $\rho \to 0^+$ ($z \to z_0$) and applying the mean value theorem of integrals, we obtain

$$(1+\theta_i)v'(x,i) \leq v'(x - z_0 + I^*(z_0), i) = v'(x,i),$$

which is a contradiction since $v'(x,i) > 0$ and $\theta_i > 0$, $\forall i \in \mathcal{S}$.

Step 2: We show that $I^*(x,i,l_0) = 0 \Rightarrow (1+\theta_i)v'(x,i) \geq v'(x - z_0, i)$.

To this purpose, we consider $\bar{I}'(x,i,l) := I^*(x,i,l) + \zeta G(l)$. For small enough ζ and ρ, we have $0 \leq \bar{I}'(z) \leq z$. Then a similar argument as above gives the desired result

$$(1+\theta_i)v'(x,i) \geq v'(x - z_0 + I^*(x,i,l_0), i) = v'(x - z_0, i).$$

Step 3: We show that $0 < I^*(x,i,l_0) < z_0 \Rightarrow (1+\theta_i)v'(x,i) = v'(x - z_0 + I^*(x,i,l_0), i)$.

In this step, we consider $\bar{I}(x,i,l)$ and $\bar{I}'(x,i,l)$ constructed above. From the results in *Step 1* and *Step 2*, we obtain $(1+\theta_i)v'(x,i) \leq v'(x - z_0 + I^*(x,i,l_0), i)$

and $(1+\theta_i)v'(x,i) \geq v'(x-z_0+I^*(x,i,l_0),i)$ at the same time, and thus the equality is achieved.

Step 4: We show that $(1+\theta_i)v'(x,i) \geq v'(x-z_0,i) \Rightarrow I^*(x,i,l_0)=0$.

We assume to the contrary that $I^*(x,i,l_0) > 0$. Then the results above give $(1+\theta_i)v'(x,i) = v'(x-z_0+I^*(x,i,l_0),i) < v'(x-z_0,i)$, which is a contradiction to the given condition. A similar method also applies to the proof of $(1+\theta_i)v'(x,i) = v'(x-z_0+I^*(x,i,l_0),i) \Rightarrow 0 < I^*(x,i,l_0) < z_0$. □

Theorem 2.3. *The optimal insurance is either no insurance or deductible insurance (almost surely).*

(a) The optimal insurance is no insurance $I^*(x,i,l)=0, \forall i \in \mathcal{S}$, when

$$(1+\theta_i)v'(x,i) \geq v'\big((1-\eta_i \operatorname{ess\,sup}(l))\,x,i\big). \tag{2.14}$$

(b) The optimal insurance is deductible insurance $I^*(x,i,l)=(\eta_i l x - d_i)^+$, $\forall\, i \in \mathcal{S}$, when there exists $d_i := d_i(x) \in (0,x)$ satisfying

$$(1+\theta_i)v'(x,i) = v'(x-d_i,i). \tag{2.15}$$

Proof. We complete the proof in three steps.

Step 1: We show Case (a).

Assume there exists $l_0 \in (0,1)$ such that $0 < I^*(x,i,l_0) < z_0$. Then according to (b) in Lemma 2.1, we have $(1+\theta_i)v'(x,i) = v'(x-z_0+I^*(x,i,l_0),i)$. Define the set $N_l := \{\omega \in \Omega : l(\omega) > \operatorname{ess\,sup}(l)\}$ (we have $\mathbb{P}\{N_l\}=0$). If $l_0 \leq \operatorname{ess\,sup}(l)$ (on the set N_l^c), then $v'((1-\eta_i \operatorname{ess\,sup}(l))\,x,i) > v'(x-z_0+I^*(x,i,l_0),i) = (1+\theta_i)v'(x,i)$, which is a contradiction to the given condition. Therefore $I^*(x,i,l)=0$ on the set N_l^c.

Besides, if two policies I_1 and I_2 only differ on a negligible set, we have $f^I(i,l;I_1) = f^I(i,l;I_2)$, because the integration of a bounded function on a negligible set is zero.

Step 2: We show Case (b).

We notice that $v'(\cdot,i)$ is a strictly decreasing function, so if such d_i exists, it must be unique. We then break our discussion into two disjoint scenarios.

(i) $0 < l_0 \le \dfrac{d_i}{\eta_i x}$

In this scenario, $\eta_i l_0 x \le d_i$, so we have

$$v'((1-\eta_i l_0)x, i) \le v'(x - d_i, i) = (1+\theta_i)v'(x,i).$$

Then by part (a) of Lemma 2.1, we obtain

$$I^*(x,i,l_0) = 0 = (\eta_i l_0 x - d_i)^+.$$

(ii) $\dfrac{d_i}{\eta_i x} < l_0 < 1$

In this scenario, we have $0 < I^*(x,i,l_0) < z_0$ since $v'((1-\eta_i l_0)x,i) > v'(x - d_i, i) = (1+\theta_i)v'(x,i)$. Then the results in Lemma 2.1 shall give

$$(1+\theta_i)v'(x,i) = v'(x - z_0 + I^*(x,i,l_0), i) = v'(x - d_i, i).$$

Due to the monotonicity of $v'(\cdot,i)$, we must have

$$I^*(x,i,l_0) = \eta_i l_0 x - d_i = (\eta_i l_0 x - d_i)^+.$$

Step 3: We show that either (2.14) or (2.15) holds.

If condition (2.14) fails, then

$$v'(x,i) < (1+\theta_i)v'(x,i) < v'((1-\eta_i \operatorname{ess\,sup}(l))\, x, i) \le v'(0,i),$$

where $v'(0,i) := \lim_{x\to 0} v'(x,i)$. Since $v'(\cdot,i)$ is continuous and strictly decreasing in $[0, x]$, there must exist a unique $d_i \in (0, x)$ such that

$$(1+\theta_i)v'(x,i) = v'(x-d_i,i).$$

If (2.15) has no solution in $(0, x)$, then

$$(1+\theta_i)v'(x,i) \geq v'(0,i) \geq v'((1-\eta_i \operatorname{ess\,sup}(l))\,x,i).$$

Therefore, we conclude that the optimal insurance is either no insurance or deductible insurance. $\square$

Remark 2.2. *The optimal insurance I^* also satisfies the usual properties: $I_t^*(\cdot)$ is an increasing function of the loss and $I_t^*(0) = 0$.*

To find explicit solutions to the optimal consumption, investment and insurance problems, we consider four utility functions of HARA class. The first three utility functions do not depend on the market regimes:

1. $U(y,i) = \ln(y)$, $y > 0$,

2. $U(y,i) = -y^\alpha$, $y > 0$, $\alpha < 0$,

3. $U(y,i) = y^\alpha$, $y > 0$, $0 < \alpha < 1$.

The fourth utility function depends on the regime of the economy and we assume there are two regimes in the economy ($S = 2$).

4. $U(y,i) = \beta_i y^{1/2}$, $y > 0$, $\beta_i > 0, i = 1, 2$.

All these four utility functions are $C^2(0,\infty)$, strictly increasing and concave, and satisfy the linear growth condition. To be specific, we can take $K = 1$ for the first three utility functions and $K = \max\{\beta_1, \beta_2\}$ for the last one.

2.3.1 $U(y,i) = \ln(y), y > 0, \forall\, i \in \mathcal{S}$

In this case, a solution to the HJB equation (2.11) is given by

$$\hat{v}(x,i) = \frac{1}{\delta}\ln(\delta x) + \hat{A}_i,\ i \in \mathcal{S}, \tag{2.16}$$

where the constants $\hat{A}_i$, $i \in \mathcal{S}$, will be determined below.

Since $\hat{v}'(x,i) = \frac{1}{\delta x}$, $\hat{v}''(x,i) = -\frac{1}{\delta x^2}$ and $(U')^{-1}(y,i) = \frac{1}{y}$, we obtain from (2.12) and (2.13) that

$$\pi^*(x,i) = \frac{\mu_i - r_i}{\sigma_i^2} \quad \text{and} \quad c^*(x,i) = \delta x.$$

Solving $(1+\theta_i)\hat{v}'(x,i) = \hat{v}'(x - d_i, i)$ gives $d_i = \frac{\theta_i}{1+\theta_i}x \in (0,x)$. Then by Theorem 2.3,

$$I^*(x,i,l) = \left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+ x.$$

Therefore, the HJB equation (2.11) reads as

$$\frac{r_i}{\delta} + \frac{\gamma_i}{\delta} + \frac{\lambda_i}{\delta}\hat{\Lambda}_i - 1 = \delta\hat{A}_i - \sum_{j\in\mathcal{S}} q_{ij}\hat{A}_j,$$

where $\gamma_i := \frac{1}{2}\frac{(\mu_i - r_i)^2}{\sigma_i^2}$ and

$$\hat{\Lambda}_i := E\left[\ln\left(1 - \eta_i l + \left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right)\right] - (1+\theta_i)E\left[\left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right].$$

Let $\vec{\hat{A}} = (\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_S)'$, $\vec{r} = (r_1, r_2, \cdots, r_S)'$, $\vec{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_S)'$, $\lambda\vec{\hat{\Lambda}} = (\lambda_1\hat{\Lambda}_1, \lambda_2\hat{\Lambda}_2, \cdots, \lambda_S\hat{\Lambda}_S)'$, $\mathbf{1} = (1, 1, \cdots, 1)'_{S\times 1}$ and $\mathbb{I}$ be the $S \times S$ identity matrix. Then the constant vector $\vec{\hat{A}}$ satisfies the linear system

$$(\delta\mathbb{I} - Q)\vec{\hat{A}} = \frac{1}{\delta}\left(\vec{r} + \vec{\gamma} + \lambda\vec{\hat{\Lambda}} - \delta\mathbf{1}\right). \tag{2.17}$$

Proposition 2.1. *The function $\hat{v} = \hat{v}(\cdot,\cdot)$, given by*

$$\hat{v}(x,i) = \begin{cases} \frac{1}{\delta}\ln(\delta x) + \hat{A}_i, & x > 0 \\ -\infty, & x = 0 \end{cases}$$

where $\vec{\hat{A}} = (\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_S)'$ solves the linear system (2.17), is the value function of Problem 2.1. Furthermore, the policy given by

$$u^*(t) = (\pi^*(t), c^*(t), I^*(t)) = \left(\frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{\sigma^2_{\epsilon(t)}}, \delta X^*_t, \left(\eta_{\epsilon(t)}\, l_t - \frac{\theta_{\epsilon(t)}}{1+\theta_{\epsilon(t)}} \right)^+ X^*_t \right)$$

is an optimal policy of Problem 2.1.

Proof. The function $\hat{v}(\cdot, i)$ defined above is a smooth function which is strictly increasing and concave such that $\hat{v}(0,i) = -\infty$, for every $i \in \mathcal{S}$. By the construction of the vector $\vec{\hat{A}}$, $\hat{v}$ satisfies the HJB equation (2.11).

To show that the candidate policy is admissible, we consider an upper bound process Z of X^*

$$\frac{dZ_t}{Z_t} = \left(r_{\epsilon(t)} - \delta + 2\gamma_{\epsilon(t)}\right) dt + \frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{\sigma_{\epsilon(t)}} dW(t),$$

with initial value $Z(0) = x$.

Solving the above SDE gives

$$Z_t = x \exp\left\{ \int_0^t \left(r_{\epsilon(s)} - \delta + \gamma_{\epsilon(s)}\right) ds + \int_0^t \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{\sigma_{\epsilon(s)}} dW(s) \right\}.$$

By the definition of Z, we have $X^*_t \le Z_t$, $\forall t \ge 0$.

Notice that if $\epsilon(t) = i$ for $t \in (t_1, t_2]$, then

$$\int_{t_1}^{t_2} \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{\sigma_{\epsilon(s)}} dW(s) = \frac{\mu_i - r_i}{\sigma_i}(W(t_2) - W(t_1)).$$

So $\int_0^t \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{\sigma_{\epsilon(s)}} dW(s)$ is a linear combination of independent Brownian motions.

By the exponential martingale property of a Brownian motion, we have

$$E\left[\exp\left(\int_0^t \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{\sigma_{\epsilon(s)}} dW(s)\right)\right] = \exp\left(\int_0^t \gamma_{\epsilon(s)} ds\right).$$

For the candidate of optimal investment proportion π^*,

$$E_{x,i}\left[\int_0^t \sigma^2_{\epsilon(t)}(\pi^*(s))^2 ds\right] \leq 2\,\gamma_M\, t < \infty,\ \forall t \geq 0,$$

where $\gamma_M = \max_{i\in\mathcal{S}}\{\gamma_i\}$.

Since $c^*(t) = 0, \forall\, t > \Theta$, for the candidate of optimal consumption c^*,

$$\begin{aligned} E_{x,i}\left[\int_0^t c^*(s)ds\right] &= E_{x,i}\left[\int_0^t c^*(s)1_{s\leq\Theta}ds\right] \\ &\leq \delta E_{x,i}\left[\int_0^t X^*_s ds\right] \leq \delta E_{x,i}\left[\int_0^t Z_s ds\right] \\ &\leq \delta x \int_0^t e^{K_1 s}ds = \frac{\delta x}{K_1}\left(e^{K_1\, t} - 1\right) < \infty,\ \forall t \geq 0, \end{aligned}$$

where $K_1 = \max_{i\in\mathcal{S}}\{r_i - \delta + 2\gamma_i\}$.

For the candidate of optimal insurance I^*, $\forall\, \mathcal{F}_t$-measurable random variable Y, $0 \leq I^*_t(Y) = \left(Y - \frac{\theta_{\epsilon(t)}}{1+\theta_{\epsilon(t)}} X^*_t\right)^+ \leq Y$, so $I^*_t \in \mathcal{I}_t$.

Furthermore, we have

$$\begin{aligned} E_{x,i}\left[\int_0^\Theta e^{-\delta s}\ln^+(c^*_s)ds\right] &\leq E_{x,i}\left[\int_0^\infty e^{-\delta s}|\ln(\delta Z_s)|ds\right] \\ &\leq \frac{1}{\delta}|\ln(\delta x)| + K_1' \int_0^\infty e^{-\delta s} s ds \\ &\quad + 2\sqrt{\frac{\gamma_M}{\pi}}\int_0^\infty e^{-\delta s}\sqrt{s}\, ds \\ &= \frac{1}{\delta}|\ln(\delta x)| + \frac{K_1'}{\delta^2} + \frac{\sqrt{\gamma_M}}{\delta\sqrt{\delta}} < \infty, \end{aligned}$$

where $K_1' = \max_{i\in\mathcal{S}}|r_i - \delta + \gamma_i|$.

Therefore, $u^* = (\pi^*, c^*, I^*)$ is optimal policy of Problem 2.1, and by Theorem 2.2, $\hat{v}$ is the corresponding value function. □

Example 2.1. $S = 2$

In this example, we assume there are two regimes in the economy, where regime 1 represents a bull market and regime 2 represents a bear market. According to French et al. [31], the stock returns are higher in a bull market, so $\mu_1 > \mu_2$. Hamilton and Lin [39] found stock volatility is higher in a bear market, thus $\sigma_1 < \sigma_2$. The data of overnight financing rate and treasury bill rate (see, for instance, the statistical data from Bank of Canada) suggests the risk-free interest rate is higher in good economy, hence $r_1 > r_2$. Haley [37] found the underwriting margin is negatively correlated with the interest rate, which implies the loading factor is smaller in a bull market, $\theta_1 < \theta_2$. Norden and Weber [71] observed that CDS spreads (default risk) are negatively correlated with the stock prices. Equivalently, the default risk is higher in a bear market, that is, $\eta_1 < \eta_2$.

The generator matrix entries become

$$q_{11} = -\Pi_1,\ q_{12} = \Pi_1,\ q_{21} = \Pi_2,\ q_{22} = -\Pi_2,$$

with $\Pi_1, \Pi_2 > 0$, so the linear system (2.17) becomes

$$\begin{aligned}(\delta + \Pi_1)\hat{A}_1 - \Pi_1\hat{A}_2 &= \frac{1}{\delta}(r_1 + \gamma_1 - \delta + \lambda_1\hat{\Lambda}_1)\\ -\Pi_2\hat{A}_1 + (\delta + \Pi_2)\hat{A}_2 &= \frac{1}{\delta}(r_2 + \gamma_2 - \delta + \lambda_2\hat{\Lambda}_2)\end{aligned}$$

which gives a unique solution

$$\hat{A}_i = \frac{\Pi_i(r_j + \gamma_j - \delta + \lambda_j\hat{\Lambda}_j) + (\delta + \Pi_j)(r_i + \gamma_i - \delta + \lambda_i\hat{\Lambda}_i)}{\delta^2(\delta + \Pi_1 + \Pi_2)},$$

where $i, j = 1, 2$ and $i \neq j$.

From the above expression of $\hat{A}_i$, we notice that only $\hat{\Lambda}_i$ is not directly given by the market. To calculate $\hat{\Lambda}_i$, we assume the loss proportion l_t does not depend on *time t and we discuss the cases that l is constant or uniformly distributed on $(0, 1)$. We further assume $\frac{\theta_1}{\eta_1(1+\theta_1)} \leq \frac{\theta_2}{\eta_2(1+\theta_2)}$. If the opposite is true, then we switch the expressions when calculating $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$.*

1. *l is constant.*

   If $\left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+ \equiv 0$, $i = 1$ or 2, then

$$\hat{\Lambda}_i = \ln(1 - \eta_i l).$$

   Otherwise, we obtain

$$\hat{\Lambda}_i = -\ln(1+\theta_i) - \eta_i l(1+\theta_i) + \theta_i.$$

2. *l is uniformly distributed on $(0, 1)$.*

   If $\left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+ \equiv 0$, $i = 1$ or 2, then

$$\hat{\Lambda}_i = E\left[\ln(1 - \eta_i l)\right] = \left(1 - \frac{1}{\eta_i}\right)\ln(1 - \eta_i) - 1.$$

   Otherwise, through straightforward calculus, we obtain

$$E\left[\ln\left(1 - \eta_i l + \left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right)\right] = \left(\frac{1}{\eta_i} - 1\right)\ln(1+\theta_i) - \frac{\theta_i}{\eta_i(1+\theta_i)},$$
$$\text{and } E\left[\left(\eta_i l - \frac{\theta_i}{1+\theta_i}\right)^+\right] = \frac{\eta_i}{2} + \frac{\theta_i^2}{2\eta_i(1+\theta_i)^2} - \frac{\theta_i}{1+\theta_i}.$$

   Hence,

$$\hat{\Lambda}_i = \left(\frac{1}{\eta_i} - 1\right)\ln(1+\theta_i) - \frac{(\eta_i(1+\theta_i) - \theta_i)^2 + 2\theta_i}{2\eta_i(1+\theta_i)}.$$

2.3.2 $U(y,i) = -y^{\alpha}, y > 0, \alpha < 0, \forall i \in \mathcal{S}$

In this scenario, a solution to the HJB equation (2.11) is given by

$$\tilde{v}(x,i) = -\tilde{A}_i^{1-\alpha} x^{\alpha}, \tag{2.18}$$

where the constants $\tilde{A}_i > 0$, $i \in \mathcal{S}$, will be determined below.

From

$$\tilde{v}'(x,i) = -\alpha \tilde{A}_i^{1-\alpha} x^{-(1-\alpha)},$$

$$\tilde{v}''(x,i) = \alpha(1-\alpha)\tilde{A}_i^{1-\alpha} x^{-(2-\alpha)},$$

$$\text{and } (U')^{-1}(y,i) = (-\frac{\alpha}{y})^{\frac{1}{1-\alpha}},$$

we obtain

$$\pi^*(x,i) = \frac{\mu_i - r_i}{(1-\alpha)\sigma_i^2} \quad \text{and} \quad c^*(x,i) = \frac{x}{\tilde{A}_i} > 0.$$

Solving the equation

$$(1+\theta_i)\tilde{v}'(x,i) = \tilde{v}'(x - d_i, i)$$

gives $d_i = \nu_i x$, where

$$\nu_i := 1 - (1+\theta_i)^{-\frac{1}{1-\alpha}}.$$

Then, by Theorem 2.3,

$$I^*(x,i,l) = (\eta_i l - \nu_i)^+ x.$$

By plugging the candidate policy into the HJB equation (2.11), we find the constants $\tilde{A}_i$ should satisfy the following non-linear system

$$\left(\delta - \alpha r_i - \frac{\alpha}{1-\alpha}\gamma_i + \lambda_i(1-\tilde{\Lambda}_i)\right)\tilde{A}_i^{1-\alpha} - (1-\alpha)\tilde{A}_i^{-\alpha} = \sum_{j\in\mathcal{S}} q_{ij}\tilde{A}_j^{1-\alpha}, \tag{2.19}$$

where $\tilde{\Lambda}_i := E\big[(1-\eta_i l + (\eta_i l - \nu_i)^+)^{\alpha}\big] - \alpha(1+\theta_i)E\big[(\eta_i l - \nu_i)^+\big]$.

In order to guarantee the above non-linear system has a unique positive solution, we need the following technical condition

$$\delta > \max_{i \in \mathcal{S}} \left\{ \alpha r_i + \frac{\alpha}{1-\alpha}\gamma_i - \lambda_i(1-\tilde{\Lambda}_i) \right\}. \tag{2.20}$$

Lemma 2.2. *The non-linear system (2.19) has a unique positive solution $\tilde{A}_i$, $i \in \mathcal{S}$, if the condition (2.20) holds.*

Proof. Please refer to Lemma 4.1 in Sotomayor and Cadenillas [86] for proof. □

Proposition 2.2. *The function $\tilde{v} = \tilde{v}(\cdot,\cdot)$, given by*

$$\tilde{v}(x,i) = \begin{cases} -\tilde{A}_i^{1-\alpha} x^\alpha, & x > 0 \\ -\infty, & x = 0 \end{cases},$$

where $\tilde{A}_i$ is the unique solution to the non-linear system (2.19), is the value function of Problem 2.1. Furthermore, the policy given by

$$u^*(t) = \left(\frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{(1-\alpha)\sigma^2_{\epsilon(t)}}, \frac{X_t^*}{\tilde{A}_{\epsilon(t)}}, \left(\eta_{\epsilon(t)}\, l_t - \nu_{\epsilon(t)}\right)^+ X_t^* \right)$$

is an optimal policy of Problem 2.1.

Proof. To verify that the candidate policy is admissible, we consider an upper bound process $\tilde{Z}$ of X^* with the dynamics

$$\frac{d\tilde{Z}_t}{\tilde{Z}_t} = \left(r_{\epsilon(t)} + \frac{(\mu_{\epsilon(t)} - r_{\epsilon(t)})^2}{(1-\alpha)\sigma^2_{\epsilon(t)}} \right) dt + \frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{(1-\alpha)\sigma_{\epsilon(t)}} dW_t.$$

Given $\tilde{Z}_0 = X_0^* = x$, we can solve the above SDE to obtain

$$\tilde{Z}_t = x \exp\left\{ \int_0^t \left(r_{\epsilon(s)} + \frac{(1-2\alpha)(\mu_{\epsilon(s)} - r_{\epsilon(s)})^2}{2(1-\alpha)^2\sigma^2_{\epsilon(s)}} \right) ds + \int_0^t \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{(1-\alpha)\sigma_{\epsilon(s)}} dW_s \right\}.$$

We use this upper bound process $\tilde{Z}$ to verify that the conditions for an admissible control are satisfied. We have for every $t \geq 0$ that

$$
\begin{aligned}
E_{x,i}\left[\int_0^t \sigma^2_{\epsilon(t)}(\pi^*_s)^2 ds\right] &\leq \frac{2\gamma_M t}{(1-\alpha)^2} < \infty, \\
E_{x,i}\left[\int_0^t c^*_s ds\right] &\leq \frac{1}{\tilde{A}_m} E_{x,i}\left[\int_0^t \tilde{Z}_s ds\right] \\
&\leq \frac{x}{\tilde{A}_m}\int_0^t e^{K_2 s} ds < \infty, \\
E_{x,i}\left[\int_0^\Theta e^{-\delta t} U^+(c^*_t) dt\right] &\leq E_{x,i}\left[\int_0^\Theta e^{-\delta t}\left(-(\frac{\tilde{Z}_t}{\tilde{A}_{\epsilon(t)}})^\alpha\right)^+ dt\right] = 0,
\end{aligned}
$$

where $\tilde{A}_m := \min_{i\in\mathcal{S}}\{\tilde{A}_i\}$ and $K_2 = \max_{i\in\mathcal{S}}\{r_i + \frac{2\gamma_i}{1-\alpha}\}$.

Besides, we can verify that $I^*_t \in \mathcal{I}_t$ since $0 \leq I^*_t(Y) = (Y - d_{\epsilon(t)})^+ \leq Y$, for every $\mathcal{F}_t$-measurable random variable Y.

Therefore, u^* defined above is admissible and then is optimal policy of Problem 2.1. By definition, smooth function $\tilde{v}(\cdot, i)$ is strictly increasing and concave, and satisfies $\tilde{v}(0,i) = -\infty$, $\forall i \in \mathcal{S}$. From the construction of $\tilde{A}_i$, the HJB equation (2.11) holds for all $i \in \mathcal{S}$. Therefore, according to Theorem 2.2, $\tilde{v}$ is the value function of Problem 2.1. □

Example 2.2. $S = 2$

To solve the non-linear system (2.19), we need to find $\tilde{\Lambda}_i$ first. In this example, we show how to find $\tilde{\Lambda}_i$ when l is constant or is uniformly distributed on $(0,1)$. Without loss of generality, we assume $\frac{\nu_1}{\eta_1} \leq \frac{\nu_2}{\eta_2}$. If the opposite holds, we switch the formulas for $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$. The results will be used for economic analysis in the next section.

1. *l is constant.*

   If $(\eta_i l - \nu_i)^+ \equiv 0$, $i = 1$ or 2, then $\tilde{\Lambda}_i = (1-\eta_i l)^\alpha$.

   Otherwise, we obtain

$$\tilde{\Lambda}_i = (1-\nu_i)^\alpha - \alpha(1+\theta_i)(\eta_i l - \nu_i).$$

2. *l is uniformly distributed on $(0,1)$.*

If $(\eta_i l - \nu_i)^+ \equiv 0$, $i = 1$ or 2, then

$$\tilde{\Lambda}_i = E\left[(1-\eta_i l)^\alpha\right] = \begin{cases} -\frac{1}{\eta_i}\ln(1-\eta_i), & \alpha = -1 \\ \frac{1}{\eta_i(1+\alpha)}(1-(1-\eta_i)^{1+\alpha}), & \alpha \neq -1 \end{cases}.$$

Otherwise, we obtain

$$E[(\eta_i l - \nu_i)^+] = \int_{\frac{\nu_i}{\eta_i}}^{1} (\eta_i l - \nu_i) dl = \frac{(\eta_i - \nu_i)^2}{2\eta_i},$$

and when $\alpha = -1$,

$$E[(1-\eta_i l + (\eta_i l - \nu_i)^+)^\alpha] = (1-\nu_i)^{-1}\left(1 - \frac{\nu_i}{\eta_i}\right) - \frac{1}{\eta_i}\ln(1-\nu_i),$$

and when $\alpha \neq -1$,

$$E[(1-\eta_i l + (\eta_i l - \nu_i)^+)^\alpha] = (1-\nu_i)^\alpha\left(1 - \frac{\nu_i}{\eta_i} - \frac{1-\nu_i}{\eta_i(1+\alpha)}\right) + \frac{1}{\eta_i(1+\alpha)}.$$

Therefore, if $(\eta_i l - \nu_i)^+ \not\equiv 0$, $i = 1$ or 2, and $\alpha = -1$, then

$$\tilde{\Lambda}_i = (1-\nu_i)^{-1}\left(1 - \frac{\nu_i}{\eta_i}\right) - \frac{1}{\eta_i}\ln(1-\nu_i) + (1+\theta_i)\frac{(\eta_i - \nu_i)^2}{2\eta_i};$$

and if $(\eta_i l - \nu_i)^+ \not\equiv 0$, $i = 1$ or 2, and $\alpha \neq -1$, then

$$\tilde{\Lambda}_i = (1-\nu_i)^\alpha\left(1 - \frac{\nu_i}{\eta_i} - \frac{1-\nu_i}{\eta_i(1+\alpha)}\right) + \frac{1}{\eta_i(1+\alpha)} - \alpha(1+\theta_i)\frac{(\eta_i - \nu_i)^2}{2\eta_i}.$$

2.3.3 $U(y,i) = y^\alpha, y > 0, 0 < \alpha < 1, \forall\, i \in \mathcal{S}$

In this case, a solution to the HJB equation (2.10) has the form

$$\bar{v}(x,i) = \bar{A}_i^{1-\alpha} x^\alpha, \tag{2.21}$$

where the constants $\bar{A}_i > 0$, $i \in \mathcal{S}$, will be determined below.

Then we can find the candidate for π^* and c^* as

$$\pi^*(x,i) = \frac{\mu_i - r_i}{(1-\alpha)\sigma_i^2} \quad \text{and} \quad c^*(x,i) = \frac{x}{\bar{A}_i}.$$

From $(1+\theta_i)\bar{v}'(x,i) = \bar{v}'(x-d_i,i)$, we can solve to obtain $d_i = \nu_i x$ with $\nu_i := 1 - (1+\theta_i)^{-\frac{1}{1-\alpha}}$. By Theorem 2.3, we have

$$I^*(x,i) = (\eta_i l - \nu_i)^+ x.$$

Plugging the candidate policy into the HJB equation (2.10) yields

$$\left(\delta - \alpha r_i - \frac{\alpha}{1-\alpha}\gamma_i + \lambda_i(1-\bar{\Lambda}_i)\right)\bar{A}_i^{1-\alpha} - (1-\alpha)\bar{A}_i^{-\alpha} = \sum_{j\in\mathcal{S}} q_{ij}\bar{A}_j^{1-\alpha}, \tag{2.22}$$

where $\bar{\Lambda}_i := E\big[(1-\eta_i l + (\eta_i l - \nu_i)^+)^\alpha\big] - \alpha(1+\theta_i)E\big[(\eta_i l - \nu_i)^+\big]$.

We need to impose an extra requirement for δ

$$\delta > \max_{i\in\mathcal{S}}\left\{\alpha r_i + \frac{\alpha}{1-\alpha}\gamma_i\right\}. \tag{2.23}$$

Lemma 2.3. *The non-linear system (2.22) has a unique positive solution $\bar{A}_i$, $i \in \mathcal{S}$, if the condition (2.23) is satisfied.*

Proof. See Lemma 4.2 in Sotomayor and Cadenillas [86]. □

Proposition 2.3. *The function $\bar{v}(x,i) = \bar{A}_i^{1-\alpha}x^\alpha$, $x \geq 0$, where $\bar{A}_i$ is the unique solution to the non-linear system (2.22), is the value function of Problem 2.1. Fur-*

thermore, the policy given by

$$u^*(t) := \left(\frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{(1-\alpha)\sigma^2_{\epsilon(t)}}, \frac{X^*_t}{\bar{A}_{\epsilon(t)}}, (\eta_{\epsilon(t)}\, l_t - \nu_{\epsilon(t)})^+ X^*_t \right)$$

is an optimal policy of Problem 2.1.

Proof. We use the same upper bound process $\tilde{Z}$ defined in Subsection 2.3.2. By following a similar argument as in the previous proposition, we can easily verify $E_{x,i}\left[\int_0^t \sigma^2_{\epsilon(t)}(\pi^*_s)^2 ds\right] < \infty$, $E_{x,i}\left[\int_0^t c^*_s ds\right] < \infty$ and $I^*_t \in \mathcal{I}_t$, $\forall t \geq 0$.

Besides, we have

$$\begin{aligned} E_{x,i}\left[\int_0^{\Theta} e^{-\delta t} U^+(c^*_t) dt\right] &\leq E_{x,i}\left[\int_0^{\infty} e^{-\delta t}\left(\frac{\tilde{Z}_t}{\bar{A}_{\epsilon(t)}}\right)^{\alpha} dt\right] \\ &= \frac{x^{\alpha}}{\bar{A}^{\alpha}_m}\int_0^{\infty} e^{-\delta t}\exp\left(\int_0^t \left(\alpha r_{\epsilon(s)} + \frac{\alpha}{1-\alpha}\gamma_{\epsilon(s)}\right) ds\right) dt \\ &\leq \frac{x^{\alpha}}{K_3 \bar{A}^{\alpha}_m} < \infty, \end{aligned}$$

where $\bar{A}_m = \min_{i\in\mathcal{S}} \bar{A}_i$ and $K_3 = \min_{i\in\mathcal{S}}(\delta - \alpha r_i - \frac{\alpha}{1-\alpha}\gamma_i) > 0$ ($K_3 > 0$ is because of the condition (2.23)).

By definition, $\bar{v}(\cdot, i) \in C^2(0,\infty)$ is strictly increasing and concave, and satisfies $\bar{v}(0,i) = \frac{U(0,i)}{\delta} = 0$ for all $i \in \mathcal{S}$. By the construction of constants $\bar{A}_i$, the HJB equation (2.11) holds for all $i \in \mathcal{S}$.

Therefore, u^* is admissible and then is optimal policy of Problem 2.1. Furthermore, by Theorem 2.1, $\bar{v}$ defined above is the value function of Problem 2.1.

□

Example 2.3. $S = 2$

We notice that the non-linear systems (2.19) and (2.22) are identical expect that α is negative in (2.19) while in (2.22), $\alpha \in (0,1)$. Hence in a two-regime economy, we shall obtain $\bar{\Lambda}_i$ in the same form of $\tilde{\Lambda}_i$ as in Example 2.2.

2.3.4 $U(y,i) = \beta_i y^{1/2}, y > 0, \beta_i > 0, i = 1, 2$

In this case, a solution to the HJB equation (2.10) is given by

$$\breve{v}(x,i) = (\breve{A}_i x)^{1/2}, \tag{2.24}$$

where the constants $\breve{A}_i > 0, i = 1, 2$, will be determined below.

From $\breve{v}'(x,i) = \frac{1}{2}\breve{A}_i^{\frac{1}{2}} x^{-\frac{1}{2}}$, $\breve{v}''(x,i) = -\frac{1}{4}\breve{A}_i^{\frac{1}{2}} x^{-\frac{3}{2}}$ and $(U')^{-1}(y,i) = (\frac{\beta_i}{2y})^2$, we obtain the candidate for π^* and c^*

$$\pi^*(x,i) = \frac{2(\mu_i - r_i)}{\sigma_i^2} \text{ and } c^*(x,i) = \frac{\beta_i^2 x}{\breve{A}_i}.$$

Solving $(1+\theta_i)\breve{v}'(x,i) = \breve{v}'(x - d_i, i)$ gives $d_i = \breve{\nu}_i x$ where $\breve{\nu}_i := 1 - \frac{1}{(1+\theta_i)^2}$. Thus a candidate for optimal insurance is

$$I^*(x,i) = (\eta_i l - \breve{\nu}_i)^+ x.$$

From the HJB equation (2.10), we obtain the following nonlinear system

$$\left(\delta - \frac{1}{2}r_i - \gamma_i + \lambda_i(1 - \breve{\Lambda}_i)\right)\breve{A}_i^{1/2} - \frac{1}{2}\frac{\beta_i^2}{\breve{A}_i^{1/2}} = \sum_{j\in\mathcal{S}} q_{ij}\breve{A}_j^{1/2},$$

where $\breve{\Lambda}_i := E\left[(1 - \eta_i l + (\eta_i l - \breve{\nu}_i)^+)^{1/2}\right] - \frac{1}{2}(1+\theta_i)E\left[(\eta_i l - \breve{\nu}_i)^+\right]$.

Since $S = 2$, so we have $q_{11} = -\Pi_1$, $q_{12} = \Pi_1$, $q_{21} = \Pi_2$, $q_{22} = -\Pi_2$ with $\Pi_1, \Pi_2 > 0$. Thus we can rewrite the above system as

$$\breve{\xi}_i \breve{A}_i - \frac{\beta_i^2}{2\Pi_i} = (\breve{A}_1 \breve{A}_2)^{1/2}, \tag{2.25}$$

where $\breve{\xi}_i := \frac{1}{\Pi_i}\left[\delta + \Pi_i - \frac{1}{2}r_i - \gamma_i + \lambda_i(1 - \breve{\Lambda}_i)\right]$.

Lemma 2.4. *The non-linear system (2.25) has a real solution $\breve{A}_i \geq \frac{\beta_i^2}{2\Pi_i\breve{\xi}_i} > 0$, $i = 1, 2$, if $\delta > \max_{i=1,2}\left\{\frac{1}{2}r_i + \gamma_i, \frac{1}{2}r_i + \gamma_i - \lambda_i(1 - \breve{\Lambda}_i)\right\}$.*

Proof. The non-linear system (2.25) is equivalent to

$$\check{\xi}_1\check{A}_1 - \frac{\beta_1^2}{2\Pi_1} = \sqrt{\check{A}_1\check{A}_2} = \check{\xi}_2\check{A}_2 - \frac{\beta_2^2}{2\Pi_2}.$$

Solving this system for $\check{A}_1$ gives

$$\left(\frac{\check{\xi}_1}{\check{\xi}_2} - \check{\xi}_1^2\right)\check{A}_1^2 - \left(\frac{\beta_1^2}{2\Pi_1\check{\xi}_2} - \frac{\beta_2^2}{2\Pi_2\check{\xi}_2} - \frac{\check{\xi}_1\beta_1^2}{\Pi_1}\right)\check{A}_1 - \frac{\beta_1^4}{4\Pi_1^2} = 0.$$

The discriminant of the above quadratic equation is

$$\Delta = \left(\frac{\beta_1^2}{2\Pi_1\check{\xi}_2} - \frac{\beta_2^2}{2\Pi_2\check{\xi}_2}\right)^2 + \frac{\check{\xi}_1\beta_1^2\beta_2^2}{\Pi_1\Pi_2\check{\xi}_2}.$$

Since $\delta > \frac{1}{2}r_i + \gamma_i - \lambda_i(1-\check{\Lambda}_i)$, we have $\check{\xi}_i > 1$, $i = 1, 2$ and then $\Delta > 0$, which implies $\check{A}_1$ has a real solution. Besides, $\sqrt{\check{A}_1\check{A}_2} \geq 0$, so $\check{A}_1 \geq \dfrac{\beta_1^2}{2\Pi_1\check{\xi}_1} > 0$. Similar analysis also applies to $\check{A}_2$. □

Proposition 2.4. *The function $\check{v}$ defined by $\check{v}(x,i) = (\check{A}_i x)^{1/2}$, $x > 0$, where $\check{A}_i$ is the positive solution to the non-linear system (2.25), is the value function of Problem 2.1. Furthermore, the policy given by*

$$u^*(t) := \left(\frac{2(\mu_{\epsilon(t)} - r_{\epsilon(t)})}{\sigma_{\epsilon(t)}^2}, \frac{\beta_{\epsilon(t)}^2 X_t^*}{\check{A}_{\epsilon(t)}}, (\eta_{\epsilon(t)}\, l_t - \check{\nu}_{\epsilon(t)})^+ X_t^*\right)$$

is an optimal policy of Problem 2.1.

Proof. We consider an upper bound process $\check{Z}$ of X^* to verify that the candidate policy is admissible. The dynamics of $\check{Z}$ is given by

$$\frac{d\check{Z}_t}{\check{Z}_t} = \left(r_{\epsilon(t)} + \frac{2(\mu_{\epsilon(t)} - r_{\epsilon(t)})^2}{\sigma_{\epsilon(t)}^2}\right)dt + \frac{2(\mu_{\epsilon(t)} - r_{\epsilon(t)})}{\sigma_{\epsilon(t)}}dW(t),$$

with initial condition $\check{Z}_0 = X_0^* = x$.

The solution to the above SDE is

$$\check{Z}_t = x \cdot \exp\left\{\int_0^t r_{\epsilon(s)}ds + 2\int_0^t \frac{\mu_{\epsilon(s)} - r_{\epsilon(s)}}{\sigma_{\epsilon(s)}}dW(s)\right\}.$$

Since $X_t^* \leq \check{Z}_t, \forall\, t \geq 0$, we have

$$\begin{aligned} E_{x,i}\left[\int_0^t c^*(s)ds\right] &\leq \frac{\beta_M^2}{\check{A}_m}E_{x,i}\left[\int_0^t \check{Z}_s ds\right] \\ &= \frac{\beta_M^2 x}{\check{A}_m}\int_0^t \exp\left(\int_0^s (r_{\epsilon(v)} + 4\gamma_{\epsilon(v)})dv\right)ds \\ &\leq \frac{\beta_M^2 x}{\check{A}_m}\int_0^t e^{K_4 s} = \frac{\beta_M^2 x}{K_4\check{A}_m}(e^{K_4 t} - 1) < \infty, \end{aligned}$$

where $\beta_M = \max\{\beta_1, \beta_2\}$, $\check{A}_m = \min\{\check{A}_1, \check{A}_2\}$ and $K_4 = \max_{i=1,2}\{r_i + 4\gamma_i\}$.

Furthermore, we calculate

$$\begin{aligned} E_{x,i}\left[\int_0^\Theta e^{-\delta s}U^+(c_s^*, \epsilon_s)ds\right] &\leq E_{x,i}\left[\int_0^\infty e^{-\delta s}\beta_{\epsilon(s)}\frac{\beta_{\epsilon(s)}(X_s^*)^{1/2}}{\check{A}_{\epsilon(s)}^{1/2}}ds\right] \\ &\leq E_{x,i}\left[\int_0^\infty e^{-\delta s}\beta_{\epsilon(s)}\frac{\beta_{\epsilon(s)}\check{Z}_s^{1/2}}{\check{A}_{\epsilon(s)}^{1/2}}ds\right] \\ &= \frac{\beta_M^2 x^{1/2}}{\check{A}_m^{1/2}}\int_0^t e^{-\delta s}\exp\left(\int_0^s (\frac{1}{2}r_{\epsilon(v)} + \gamma_{\epsilon(v)})dv\right)ds \\ &\leq \frac{\beta_M^2 x^{1/2}}{\check{A}_m^{1/2}}\int_0^\infty e^{-K_5 s}ds = \frac{\beta_M^2 x^{1/2}}{K_5\check{A}_m^{1/2}} < \infty, \end{aligned}$$

where $K_5 := \min_{i\in\mathcal{S}}(\delta - \frac{1}{2}r_i - \gamma_i) > 0$ (Notice $K_5 > 0$ due to the assumption that $\delta > \frac{1}{2}r_i + \gamma_i, \forall\, i \in \mathcal{S}$).

Besides, $\forall\, t \geq 0$,

$$E_{x,i}\left[\int_0^t \sigma_{\epsilon(s)}^2(\pi_s^*)^2 ds\right] \leq 8\gamma_M t < +\infty,$$

and $0 \leq I_t^*(Y) = (Y - \check{\nu}_{\epsilon(t)}X_t^*)^+ \leq Y$, for every $\mathcal{F}_t$-measurable random variable Y.

We have proved u^* is admissible and thus u^* is optimal policy of Problem 2.1.

By definition, $\check{v}(\cdot, i) \in C^2(0, \infty)$ is strictly increasing and concave, and satisfies $\check{v}(0, i) = \frac{U(0,i)}{\delta}$, $i = 1, 2$. By the construction of $\check{A}_i$, the HJB equation (2.10) is satisfied for $i = 1, 2$. Therefore, by Theorem 2.1, the value function is given by $\check{v}(x, i) + \frac{1}{\delta} E_{x,i}[\int_0^\infty e^{-\delta s} dU(0, \epsilon_s)] = \check{v}(x, i)$ because $dU(0, \epsilon_s) = 0$. □

2.4 Impact of Markets and Risk Aversion on Optimal Policies

In this section, we analyze the impact of market parameters and the investor's risk aversion on optimal policies. To conduct the economic analysis, we assume there are two regimes in the economy, like in Example 2.1, Example 2.2, and Example 2.3: regime 1 represents a bull market while regime 2 represents a bear market. We only consider the first three utility functions that do not depend on the regime of the economy in the economic analysis.

According to the results obtained in Section 2.3, we write the optimal proportion invested in the stock in an uniform expression

$$\pi_t^* = \frac{1}{1-\alpha} \frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{\sigma_{\epsilon(t)}^2}, \tag{2.26}$$

where $\alpha = 0$ when $U(y, i) = \ln(y)$.

During any given regime, the optimal investment proportion in the stock π^* is constant, and is independent of the investor's wealth. This result is consistent with the findings in the classical work of Merton [63], which also obtained constant optimal investment proportion under HARA utility functions.

The dependency of π^* on market parameters (expected excess return over variance) is evident. Through empirical research, French et al. [31] found that the expected excess return over variance is higher in good economy. $\alpha < 1$, and then $\frac{1}{1-\alpha} > 0$, so we have

$$\pi(\epsilon(t) = 2) < \pi(\epsilon(t) = 1), \text{ for all } \alpha < 1.$$

Hence all investors should invest a greater proportion of their wealth on the stock in a bull market.

Expression (2.26) shows that π^* is inversely proportional to the relative risk aversion $1 - \alpha$. Thus, investors with higher risk tolerance (greater α) will invest a larger proportion of their wealth on the stock in both regimes.

For all three cases, the optimal consumption rate process is proportional to the wealth process and such ratio $\kappa(t) := \frac{c^*(t)}{X^*(t)}$ is given by

$$\kappa(t) = \begin{cases} \delta, & \text{if } U(y,i) = \ln(y), \alpha = 0; \\ \dfrac{1}{\tilde{A}_{\epsilon(t)}}, & \text{if } U(y,i) = -y^\alpha, \alpha < 0; \\ \dfrac{1}{\bar{A}_{\epsilon(t)}}, & \text{if } U(y,i) = y^\alpha, 0 < \alpha < 1. \end{cases}$$

Since $\kappa(t)$ is positive in all three cases, investors will consume more when they become wealthier. To examine the dependency of the optimal consumption to wealth ratio $\kappa(t)$ on α, we separate our discussion into the following three cases.

For moderate risk-averse investors ($\alpha = 0$), $\kappa(t)$ is constant regardless of the market regimes, so moderate risk-averse investors consume the same proportion of their wealth in both bull and bear markets.

For high risk-averse investors ($\alpha < 0$), their optimal consumption to wealth ratio is given by $1/\tilde{A}_i, i = 1, 2$, where $\tilde{A}_i$ can be obtained from the system (2.19). To find a numerical solution to the system (2.19), we set market parameters as $\mu_1 = 0.2, \mu_2 = 0.15, r_1 = 0.08, r_2 = 0.03, \sigma_1 = 0.25, \sigma_2 = 0.6, \theta_1 = 0.15, \theta_2 =$

$0.25, \eta_1 = 0.8, \eta_2 = 1, \lambda_1 = 0.1, \lambda_2 = 0.2, \Pi_1 = 6.04, \Pi_2 = 6.4$, and $\delta = 0.15$ (for the convenience of citation thereafter, we denote the choice of market parameters here as Parameter Set I). Notice that these parameters satisfy the technical condition (2.20). We draw graphs in Figure 2.1 for the optimal consumption to wealth ratio when $-1 < \alpha < 0$ and $l = 0.3$, $l = 0.5$, and $l = 0.7$. We see that the optimal consumption to wealth ratio is an increasing function of α. Thus, the higher the risk tolerance, the higher the proportion of consumption over wealth. For the above parameter values, we find $1/\tilde{A}_1 > 1/\tilde{A}_2$, which can be seen from Figure 2.1. Hence

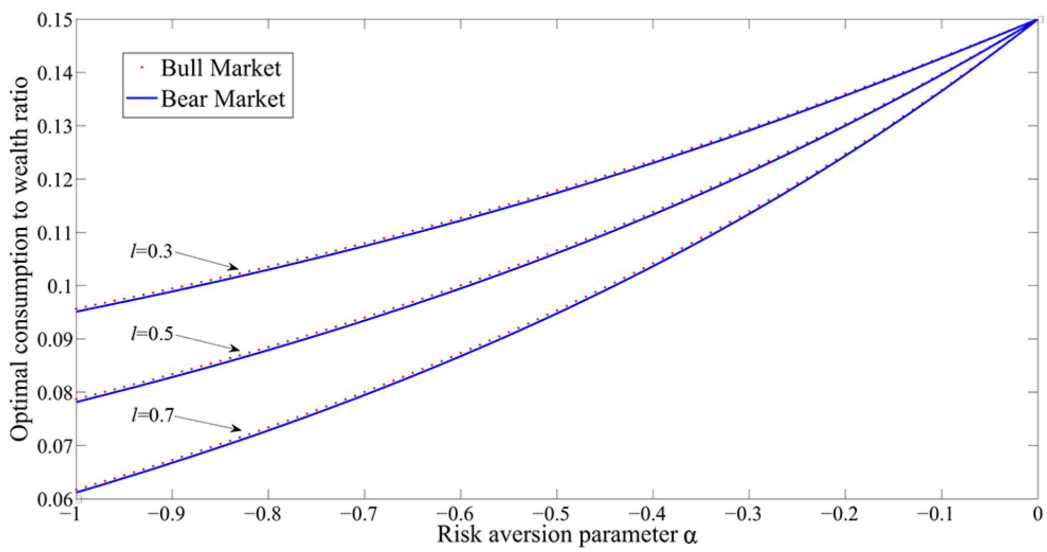

Figure 2.1: Optimal Consumption to Wealth Ratio When $\alpha < 0$

investors should allocate a higher proportion of their wealth to consumption in a bull market. For any chosen investor (fixed α), she/he will behave more conservatively by reducing the proportion spent in consumption when facing larger losses (greater l). This behavior was not noticed in Sotomayor and Cadenillas [86], because they did not incorporate an insurable loss in their model. Besides, from a mathematical point of view, the ratios all converge to 0.15 when α approaches 0, which is exactly the same optimal consumption to wealth ratio when $\alpha = 0$ ($\delta = 0.15$).

For low risk-averse investors ($0 < \alpha < 1$), the optimal consumption to wealth ratio is given by $1/\bar{A}_i, i = 1, 2$, where $1/\bar{A}_i$ can be calculated from the system (2.22). We set market parameters to be $\mu_1 = 0.2, \mu_2 = 0.15, r_1 = 0.15, r_2 = 0.1, \sigma_1 = 0.4, \sigma_2 = 0.6, \theta_1 = 0.15, \theta_2 = 0.25, \eta_1 = 0.8, \eta_2 = 1, \lambda_1 = 0.1, \lambda_2 = 0.2, \Pi_1 = 6.04, \Pi_2 = 6.4$, and $\delta = 0.2$ (denoted as Parameter Set II). For these parameters values, the corresponding technical condition (2.23) is satisfied. Figure

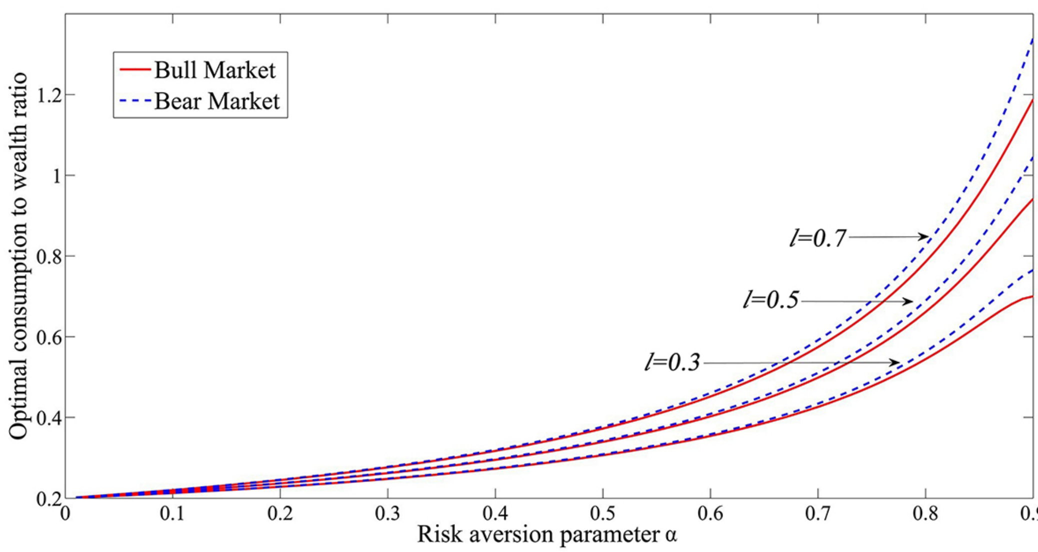

Figure 2.2: Optimal Consumption to Wealth Ratio When $0 < \alpha < 1$

2.2 shows that the optimal consumption to wealth ratio when $l = 0.3$, 0.5, and 0.7. Similar to the previous case, we also observe that the optimal consumption to wealth ratio is an increasing function of α. However, contrary to the previous case, we have $1/\bar{A}_1 < 1/\bar{A}_2$ when $0 < \alpha < 1$. This means low risk-averse investors ($0 < \alpha < 1$) spend a smaller proportion of their wealth on consumption in a bull market. We notice that for very low risk-averse investors (α close to 1), the optimal consumption to wealth ratio is even greater than 1, meaning they finance consumption by borrowing.

By comparing all three cases, we conclude that investors with high risk tolerance (large α) consume a large proportion of their wealth in every market regime. However, investors' consumption decision depends on the market regimes, and investors with different risk aversion attitudes behave differently in bull and bear markets. High risk-averse investors consume proportionally more in a bull market, but low risk-averse investors spend more proportion of wealth on consumption in a bear market.

The optimal insurance for all three utility functions is deductible insurance, and is given in the form of

$$I_t^* = \left[\eta_{\epsilon(t)}\, l - 1 + (1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}}\right]^+ X_t^*.$$

We observe that, for each fixed regime, the optimal insurance is proportional to the investor's optimal wealth X^*. We note that it is optimal to buy insurance if and only if

$$\eta_{\epsilon(t)}\, l - 1 + (1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}} > 0,$$

or equivalently if and only if

$$\eta_{\epsilon(t)} l > 1 - (1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}}. \tag{2.27}$$

Thus, it is optimal to buy insurance if and only if, relative to the other variables, $\eta_{\epsilon(t)}$ is large, l is large, $\theta_{\epsilon(t)}$ is small, and α is small (we recall that $\alpha \in (-\infty, 1)$). That is, it is optimal to buy insurance if the insurable loss is large, the cost of insurance is low, and the investor is very risk averse. It is surprising that the variable $\lambda_{\epsilon(t)}$ does not appear explicitly in this expression. Our explanation is that $\lambda_{\epsilon(t)}$ is implicitly incorporated in X_t^*, so $\lambda_{\epsilon(t)}$ is important as well to determine the optimal insurance.

If it is optimal to buy insurance, or equivalently, the condition (2.27) is satisfied, then

$$I_t^* = \left[\eta_{\epsilon(t)}\, l - 1 + (1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}}\right] X_t^*.$$

Thus, as expected, the optimal insurance is proportional to $\eta_{\epsilon(t)}$ and l. Furthermore,

$$\frac{\partial I_t^*}{\partial \theta_{\epsilon(t)}} = -\left[\frac{1}{1-\alpha}(1+\theta_{\epsilon(t)})^{-\frac{2-\alpha}{1-\alpha}}\right] X_t^* < 0,$$
$$\frac{\partial^2 I_t^*}{\partial \theta_{\epsilon(t)}^2} = \left[\frac{2-\alpha}{(1-\alpha)^2}(1+\theta_{\epsilon(t)})^{\frac{2\alpha-3}{1-\alpha}}\right] X_t^* > 0.$$

The above results of partial derivatives indicate that the optimal insurance is a decreasing and convex function of θ. The decreasing property means that, as the premium loading θ increases, it is optimal to reduce the purchase of insurance. The convexity indicates the amount of reduction in insurance decreases as the premium loading increases.

In addition, if it is optimal to buy insurance (when condition (2.27) is satisfied), then

$$\frac{\partial I_t^*}{\partial \alpha} = -\left[\frac{1}{(1-\alpha)^2}\ln(1+\theta_{\epsilon(t)})(1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}}\right] X_t^* < 0,$$
$$\frac{\partial^2 I_t^*}{\partial \alpha^2} = \left[\frac{\ln(1+\theta_{\epsilon(t)})}{(1-\alpha)^3}\left(\frac{\ln(1+\theta_{\epsilon(t)})}{1-\alpha} - 2\right)(1+\theta_{\epsilon(t)})^{-\frac{1}{1-\alpha}}\right] X_t^*.$$

Hence, the optimal insurance is a decreasing function of α, which implies the higher the risk tolerance, the smaller amount spent on insurance.

The sign of $\frac{\partial^2 I_t^*}{\partial \alpha^2}$ depends on α, investor's risk aversion parameter. We observe that $\frac{\partial^2 I_t^*}{\partial \alpha^2}$ and $\frac{\ln(1+\theta_{\epsilon(t)})}{1-\alpha} - 2$ have the same sign. Recall that θ is the premium loading, which usually does not exceed 100%. We make the discussions based on the following two cases.

(i) $\alpha \leq 0$

In this case, we have $\frac{\partial^2 I_t^*}{\partial \alpha^2} < 0$. This indicates that for high and moderate risk-averse investors ($\alpha \le 0$), the reduction in insurance is more significant when α is greater (or when α is closer to 0).

(ii) $0 < \alpha < 1$

In this case, we define $\tilde{\alpha}$ by

$$\tilde{\alpha} := 1 - \frac{1}{2}\ln(1 + \theta_{\epsilon(t)}).$$

Notice the result

$$\frac{\partial^2 I_t^*}{\partial \alpha^2} = 0 \text{ at the point } \alpha = \tilde{\alpha}.$$

We find that $\frac{\partial^2 I_t^*}{\partial \alpha^2} < 0$ when $\alpha < \tilde{\alpha}$ and $\frac{\partial^2 I_t^*}{\partial \alpha^2} > 0$ when $\alpha > \tilde{\alpha}$. So for low risk-averse investors ($0 < \alpha < 1$), the magnitude of reduction in insurance depends on the risk aversion attitude.

2.5 Comparison with No Insurance Case

In this section, we want to find out the advantage of buying insurance for investors who face a random insurable risk. To this purpose, we assume that insurance policies are not available in the market to insure against the random loss L. We calculate the value function with the constraint of no insurance, denoted by $V_1(x, i)$, and compare $V_1(x, i)$ with $V(x, i)$ (the value function of Problem 2.1 with insurance).

Under the constraint of no insurance, the dynamics of the wealth process X_1 changes accordingly from (2.1) to

$$\begin{aligned} dX_1(t) = &\big(r_{\epsilon(t)}X_1(t) + (\mu_{\epsilon(t)} - r_{\epsilon(t)})\pi(t)X_1(t) - c(t)\big)\,dt \\ &+ \sigma_{\epsilon(t)}\pi(t)X_1(t)dW(t) - L(t)dN(t), \end{aligned}$$

where the insurable loss is $L(t) = \eta_{\epsilon(t)}\, l(t)X_1(t)$ and the initial condition is given by $X_1(0) = x$.

We denote the control by $u_1 = (\pi_1, c_1)$ in this section and the set of all admissible sets with initial conditions $X_1(0) = x, \epsilon(0) = i$ by $\mathcal{A}_1$. For every $u_1 = (\pi_1, c_1) \in \mathcal{A}_1$, π_1 and c_1 need to satisfy all the conditions (conditions (2.3), (2.4) and (2.5)) that π and c satisfy, where $(\pi, c, I) \in \mathcal{A}_{x,i}$.

We formulate the constrained version of Problem 2.1 as follows.

Problem 2.2. *Select an admissible policy $u_1^* := (\pi_1^*, c_1^*)$ that maximizes the criterion function J, defined by (2.2). In addition, find the value function*

$$V_1(x, i) := \sup_{u_1 \in \mathcal{A}_1} J(x, i; u_1).$$

Since for any $u_1 = (\pi_1, c_1) \in \mathcal{A}_1$, we have $(\pi_1, c_1, I \equiv 0) \in \mathcal{A}_{x,i}$. Therefore, $V(x, i) \geq V_1(x, i)$ for all $x > 0$ and $i \in \mathcal{S}$.

We provide a verification theorem to Problem 2.2 (see Theorems 2.1 and 2.2 for proofs) when the utility function does not depend on the regime , that is, $U(y, i) = U(y)$ for every $i \in \mathcal{S}$.

Theorem 2.4. *Suppose $U(0)$ is finite or $U(0) = -\infty$. Let $v(\cdot, i) \in C^2(0, \infty)$ be an increasing and concave function such that $v(0, i) = \frac{U(0)}{\delta}$ for every $i \in \mathcal{S}$. If $v = v(\cdot, \cdot)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_{(\pi_1, c_1)} \left\{ \mathcal{G}_i^{\pi_1, c_1} v(x, i) + U(c_1) + \lambda_i E[v(x - L, i)] \right\} = -\sum_{j \in \mathcal{S}} q_{ij} v(x, j), \qquad (2.28)$$

where the operator $\mathcal{G}$ is defined as

$$\mathcal{G}_i^{\pi_1, c_1}(\psi) := (r_i x + (\mu_i - r_i)\pi x - c_1)\psi' + \frac{1}{2}\sigma_i^2 \pi^2 x^2 \psi'' - (\delta + \lambda_i)\psi,$$

and the control $u_1^* := (\pi_1^*, c_1^*)$ defined by

$$u_1^*(t) := \left(\frac{1}{1 - \alpha} \frac{(\mu_{\epsilon(t)} - r_{\epsilon(t)})}{\sigma_{\epsilon(t)}^2}, \, (U')^{-1}(v'(X_1^*(t), \epsilon(t))) \right)$$

is admissible, then u_1^* is an optimal control to Problem 2.2.

2.5.1 $U(y) = \ln(y),\ y > 0$

Under the logarithmic utility, we find the value function to Problem 2.2 is given by

$$\hat{v}_1(x,i) = \frac{1}{\delta}\ln(\delta x) + \hat{a}_i,$$

where the constants $\hat{a}_i$ satisfy the following linear system

$$\frac{r_i}{\delta} + \frac{\gamma_i}{\delta} + \frac{\lambda_i}{\delta}\hat{\Upsilon}_i - 1 = \delta\hat{a}_i - \sum_{j\in\mathcal{S}} q_{ij}\hat{a}_j, \tag{2.29}$$

with $\hat{\Upsilon}_i$ defined by $\hat{\Upsilon}_i := E[\ln(1-\eta_i l)]$.

To compare the value functions $\hat{v}$ and $\hat{v}_1$, we assume there are two regimes ($S = 2$) in the economy. Under this assumption, we find $\hat{a}_i$ given by

$$\hat{a}_i = \frac{\Pi_i(r_j + \gamma_j - \delta + \lambda_j\hat{\Upsilon}_j) + (\delta + \Pi_j)(r_i + \gamma_i - \delta + \lambda_i\hat{\Upsilon}_i)}{\delta^2(\delta + \Pi_1 + \Pi_2)},$$

where $i, j = 1, 2$ and $i \neq j$.

We then calculate

$$\hat{v}(x,i) - \hat{v}_1(x,i) = \frac{\Pi_i\lambda_j(\hat{\Lambda}_j - \hat{\Upsilon}_j) + \lambda_i(\delta + \Pi_j)(\hat{\Lambda}_i - \hat{\Upsilon}_i)}{\delta^2(\delta + \Pi_1 + \Pi_2)}, \tag{2.30}$$

where $i, j = 1, 2$ and $i \neq j$.

To facilitate our scenario analysis, we assume $\frac{\theta_1}{\eta_1(1+\theta_1)} \leq \frac{\theta_2}{\eta_2(1+\theta_2)}$ and l is either constant or uniformly distributed on $(0, 1)$.

- Case 1: l is constant.

  In this case, $\hat{\Upsilon}_i = \ln(1 - \eta_i l)$, $i = 1, 2$.

  (i) Optimal insurance is no insurance for both regimes.

From Example 2.1, we notice when the optimal insurance I^* is no insurance, we have $\hat{\Lambda}_i = \ln(1-\eta_i l)$, $i = 1, 2$. Then, we obtain $\hat{\Lambda}_i - \hat{\Upsilon}_i = 0$ for both regimes. Hence $\hat{v}(x, i) = \hat{v}_1(x, i)$ for all $x > 0$ and $i = 1, 2$.

(ii) Optimal insurance is strictly positive in at least one regime.

When the optimal insurance I^* is strictly positive in at least one regime, we must have at least one $\hat{\Lambda}_i$ in the form of $\hat{\Lambda}_i = -\ln(1+\theta_i) - \eta_i l(1+\theta_i) + \theta_i$. Without loss of generality, we assume $I^* > 0$ in regime 1, or equivalently, $\eta_1 l - \frac{\theta_1}{1+\theta_1} > 0$. Then we obtain

$$\begin{aligned}\hat{\Lambda}_1 - \hat{\Upsilon}_1 &= -\ln(1+\theta_1) - \eta_1 l(1+\theta_1) + \theta_1 - \ln(1-\eta_1 l) \\ &> -\eta_1 l(1+\theta_1) - \ln(1-\eta_1 l),\end{aligned}$$

where the second inequality comes from $-\ln(1+\theta_1) + \theta_1 > 0$.

Consider $w(l) := -\eta_1 l(1+\theta_1) - \ln(1-\eta_1 l)$. We have $w(0) = 0$ and

$$w'(l) = \frac{\eta_1}{(1-\eta_1 l)(1+\theta_1)} \left(\eta_1 l - \frac{\theta_1}{1+\theta_1}\right) > 0.$$

This implies $w(l) > 0$ for all $l \in (0, 1)$, and then $\hat{\Lambda}_1 - \hat{\Upsilon}_1 > 0$. Together with the result above, we can claim that $\hat{\Lambda}_2 - \hat{\Upsilon}_2 \geq 0$.

Hence, regardless of the optimal insurance I^* in regime 2, we have $\hat{v}(x, i) > \hat{v}_1(x, i)$ for both regimes according to (2.30). Even when $I^*(x, i = 2, l) = 0$, buying insurance in regime 1 increases the value function in regime 2.

To further study the advantage of buying insurance, we define the increase ratio of the value function by

$$m(x, i) := \left| \frac{V(x, i) - V_1(x, i)}{V_1(x, i)} \right|, \; i = 1, 2,$$

where $V(x, i)$ and $V_1(x, i)$ are the value functions to Problem 2.1 and Problem

2.2, respectively.

Without loss of generality, we assume $x = \frac{1}{\delta}$ (such assumption makes the constant $\frac{1}{\delta}\ln(\delta x)$ be 0). Hence, we have $V(x,i) = \hat{v}(x,i) = \hat{A}_i$ and $V_1(x,i) = \hat{v}_1(x,i) = \hat{a}_i$, $i = 1,2$. Then we obtain for $i = 1,2$ that

$$m(x,i) = \frac{\Pi_i\lambda_j(\hat{\Lambda}_j - \hat{\Upsilon}_j) + \lambda_i(\delta + \Pi_j)(\hat{\Lambda}_i - \hat{\Upsilon}_i)}{|\Pi_i(r_j + \gamma_j - \delta + \lambda_j\hat{\Upsilon}_j) + (\delta + \Pi_j)(r_i + \gamma_i - \delta + \lambda_i\hat{\Upsilon}_i)|}.$$

To analyze the impact of the insurable loss on the ratio m, we keep l as a variable and choose Parameter Set I but $\delta = 0.2$. Notice that for the chosen parameters, our assumption is satisfied

$$\frac{\theta_1}{\eta_1(1+\theta_1)} = 0.16 < \frac{\theta_2}{\eta_2(1+\theta_2)} = 0.2.$$

Since we assume $I^* > 0$ in regime 1, $l \in (0.16, 1)$. We draw the graph of the increase ratio of the value function in Figure 2.3. As expected, the advantage of buying insurance increases when the insurable loss becomes larger in both regimes. But surprisingly, we find that buying insurance benefits investors more in a bull market, especially when the insurable loss is large.

- Case 2. l is uniformly distributed on $(0,1)$.

  In this case, $\hat{\Upsilon}_i = \int_0^1 \ln(1-\eta_i l)dl = (1 - \frac{1}{\eta_i})\ln(1-\eta_i) - 1$, $i = 1,2$.

  (i) Optimal insurance is no insurance for both regimes.

  In this scenario, it is obvious that $\hat{\Lambda}_i = \hat{\Upsilon}_i$ and then $\hat{v}(x,i) = \hat{v}_1(x,i)$, for all $x > 0$ and $i = 1,2$.

  (ii) Optimal insurance is strictly positive in at least one regime.

  Again we assume $I^* > 0$ in regime 1. Then we have

$$\hat{\Lambda}_1 - \hat{\Upsilon}_1 = (\frac{1}{\eta_1} - 1)\ln((1+\theta_1)(1-\eta_1)) + 1 - \frac{(\eta_1(1+\theta_1) - \theta_1)^2 + 2\theta_1}{2\eta_1(1+\theta_1)}.$$

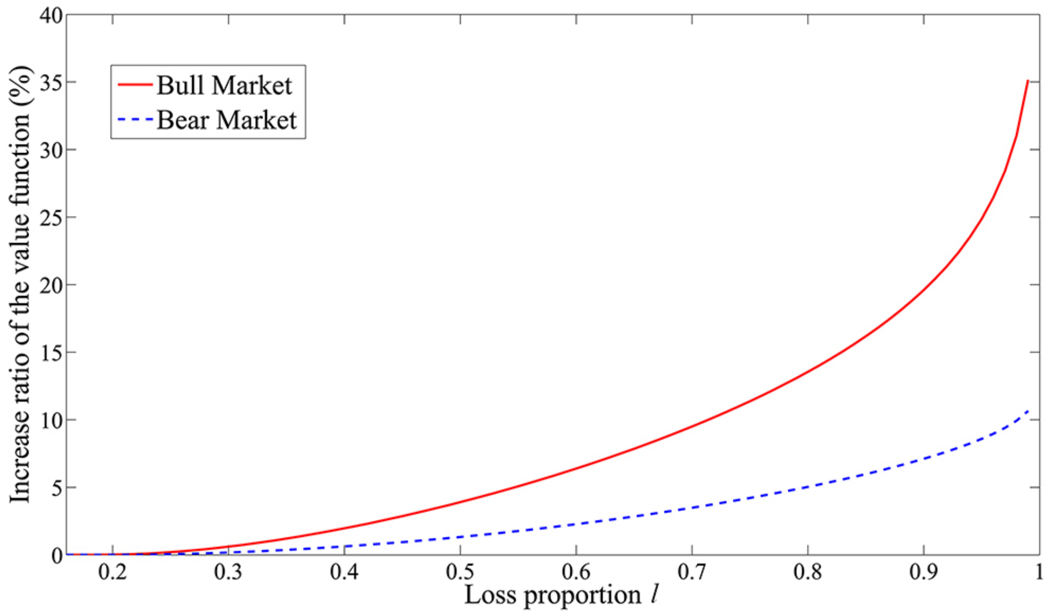

Figure 2.3: Increase Ratio of the Value Function When l Is Constant

Here $\hat{\Lambda}_1 - \hat{\Upsilon}_1$ depends on the premium loading θ and loss intensity η in regime 1. To investigate such dependency, we conduct a numerical calculation. Notice that η_1 must satisfy the condition $\eta_1 \geq \frac{\theta_1}{1+\theta_1}$. We draw the difference $\hat{\Lambda}_1 - \hat{\Upsilon}_1$ in Figure 2.4 when $\theta_1 = 0.01, 0.1, 0.2, 0.5, 0.8, 0.99$.

We observe that $\hat{\Lambda}_1 - \hat{\Upsilon}_1$ is strictly positive and therefore $\hat{v}(x,i) > \hat{v}_1(x,i)$ for both regimes, which is consistent with our findings in the previous case. Furthermore, as θ increases (which means the cost of insurance policy increases), the difference of $\hat{\Lambda}_i - \hat{\Upsilon}_i$ becomes smaller, so the benefit of purchasing insurance policy decreases accordingly. We also observe that investors gain more advantage from insurance when the insurable loss becomes larger (that is, the loss intensity η increases).

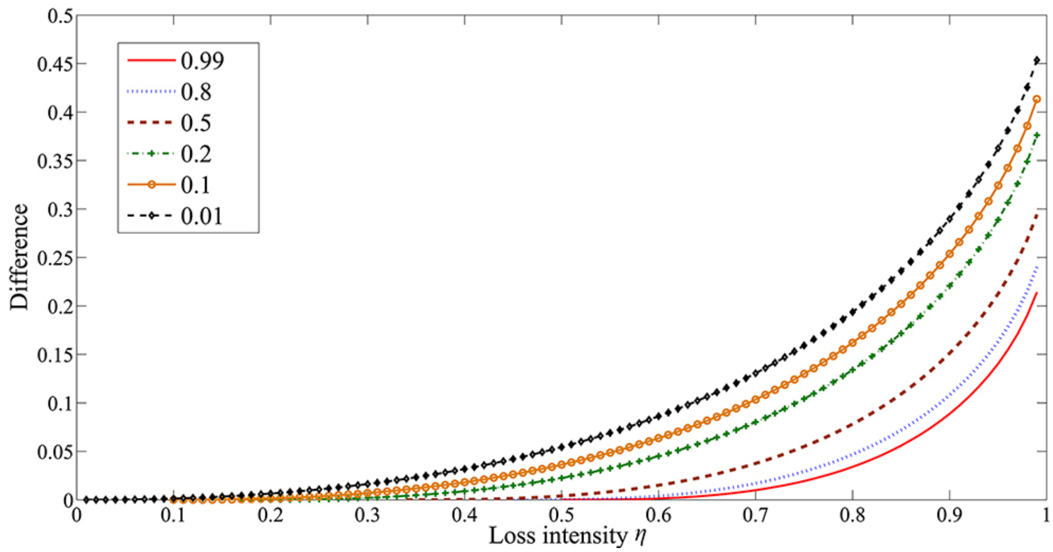

Figure 2.4: Difference of $\hat{\Lambda}_i - \hat{\Upsilon}_i$

2.5.2 $U(y) = -y^\alpha$, $\alpha < 0$

In the case of negative power utility, the value function to Problem 2.2 is given by

$$\tilde{v}_1(x,i) = -\tilde{a}_i^{1-\alpha} x^\alpha,$$

where positive constants $\tilde{a}_i$ satisfy

$$\left(\delta - \alpha r_i - \frac{\alpha}{1-\alpha}\gamma_i + \lambda_i(1-\tilde{\Upsilon}_i)\right)\tilde{a}_i^{1-\alpha} - (1-\alpha)\tilde{a}_i^{-\alpha} = \sum_{j\in\mathcal{S}} q_{ij}\tilde{a}_j^{1-\alpha}, \quad (2.31)$$

with $\tilde{\Upsilon}_i := E\big[(1-\eta_i l)^\alpha\big]$.

Comparing with the value function we found in Section 2.3.2, we have

$$\tilde{v}(x,i) - \tilde{v}_1(x,i) = -(\tilde{A}_i^{1-\alpha} - \tilde{a}_i^{1-\alpha})x.$$

We assume there are two regimes in the economy and the loss proportion l is constant. We skip the trivial case of $I^* \equiv 0$, in which $\tilde{v}(x,i) = \tilde{v}_1(x,i)$ in both regimes. We then carry out a numerical calculation to study the non-trivial case, that is $I^*(x, i, l) > 0$ in at least one regime.

To solve the systems (2.19) and (2.31) numerically, we choose Parameter Set I but $\delta = 0.25$. For the chosen parameters, it is more reasonable to consider the case when $l \in (\frac{\nu_2}{\eta_2}, 1)$ (Since both $\frac{\nu_1}{\eta_1}$ and $\frac{\nu_2}{\eta_2}$ are small). In Table 2.1 we calculate $\tilde{v}(x, i) - \tilde{v}_1(x, i)$ for various values of α (when calculating $\tilde{v}(x, i) - \tilde{v}_1(x, i)$, we take $x = 1$).

risk aversion α	loss proportion l	$\tilde{v}(x, 1) - \tilde{v}_1(x, 1)$	$\tilde{v}(x, 2) - \tilde{v}_1(x, 2)$
-0.01	$l = 0.30$	7.7176×10^{-5}	7.4304×10^{-5}
	$l = 0.60$	9.8981×10^{-4}	9.5315×10^{-4}
	$l = 0.90$	0.0041	0.0039
-0.5	$l = 0.15$	0.0015	0.0015
	$l = 0.35$	0.0797	0.0781
	$l = 0.50$	0.3010	0.2961
-1	$l = 0.20$	0.1675	0.1653
	$l = 0.30$	0.8116	0.8036
	$l = 0.40$	2.6969	2.6841
-2	$l = 0.08$	0.2454	0.2441
	$l = 0.10$	0.9421	0.9381
	$l = 0.12$	2.2418	2.2344

Table 2.1: $\tilde{A}_i - \tilde{a}_i$ When Loss Proportion $l \in (\frac{\nu_2}{\eta_2}, 1)$

The result clearly confirms that $\tilde{v}(x, i) > \tilde{v}_1(x, i)$ in both regimes. We also observe that the advantage of buying insurance is greater for investors with higher risk aversion. The size of the insurable loss l affects the advantage of buying insurance as well. When the insurable loss increases (loss proportion l increases), buying insurance will give investors more advantage. We obtain $\tilde{v}(1,1)-\tilde{v}_1(1,1) > \tilde{v}(1,2)-\tilde{v}_1(1,2)$, meaning buying insurance is more advantageous in a bull market.

2.5.3 $U(y) = y^\alpha$, $0 < \alpha < 1$

We find the corresponding value function to Problem 2.2 given by

$$\bar{v}_1(x,i) = \bar{a}_i^{1-\alpha} x^\alpha,$$

where the constants $\bar{a}_i$ satisfy the system (2.31) with $0 < \alpha < 1$.

From the discussion in Section 2.3.3, we obtain $\bar{v}(x,i) - \bar{v}_1(x,i) = (\bar{A}_i^{1-\alpha} - \bar{a}_i^{1-\alpha})x^\alpha$. We then follow all the assumptions made in Section 2.5.2 including $x = 1$ and conduct a numerical analysis by choosing Parameter Set II. In this numerical example, we have $\frac{\nu_1}{\eta_1} \leq \frac{\nu_2}{\eta_2} < 1$ when $\alpha \in (0, 0.8672]$, $\frac{\nu_2}{\eta_2} \leq \frac{\nu_1}{\eta_1} < 1$, when $\alpha \in (0.8672, 0.9132]$, and $\frac{\nu_2}{\eta_2} \leq 1 < \frac{\nu_2}{\eta_2}$ when $\alpha \in (0.9132, 1)$. We consider the first scenario: $\frac{\nu_1}{\eta_1} \leq \frac{\nu_2}{\eta_2} < 1$ since it includes most low risk-averse investors. We are interested in the case of $I^* > 0$ in at least one regime. For the chosen parameters, we find $\frac{\nu_1}{\eta_1}$ is so small that the case of $l \in (0, \frac{\nu_1}{\eta_1})$ is rare. So we further assume constant loss proportion $l \in (\frac{\nu_1}{\eta_1}, \frac{\nu_2}{\eta_2}]$. Notice that when $l \in (\frac{\nu_1}{\eta_1}, \frac{\nu_2}{\eta_2}]$, we have $I^*(x,1,l) > 0$ but $I^*(x,2,l) = 0$.

From solving the non-linear systems (2.22) and (2.31), we draw $\bar{v}(x,i)-\bar{v}_1(x,i)$ for $l = l_M := \frac{1}{2}(\frac{\nu_1}{\eta_1} + \frac{\nu_2}{\eta_2})$ and $l = l_m := \frac{\nu_2}{\eta_2} - 0.01$ in Figure 2.5. It is obvious that $\bar{v}(1,i)-\bar{v}_1(1,i) > 0$ in both regimes. As have seen in the previous cases, the benefit of buying insurance in a bull market strictly outperforms that in a bear market. We also observe a surprising result that the difference of the value functions (advantage

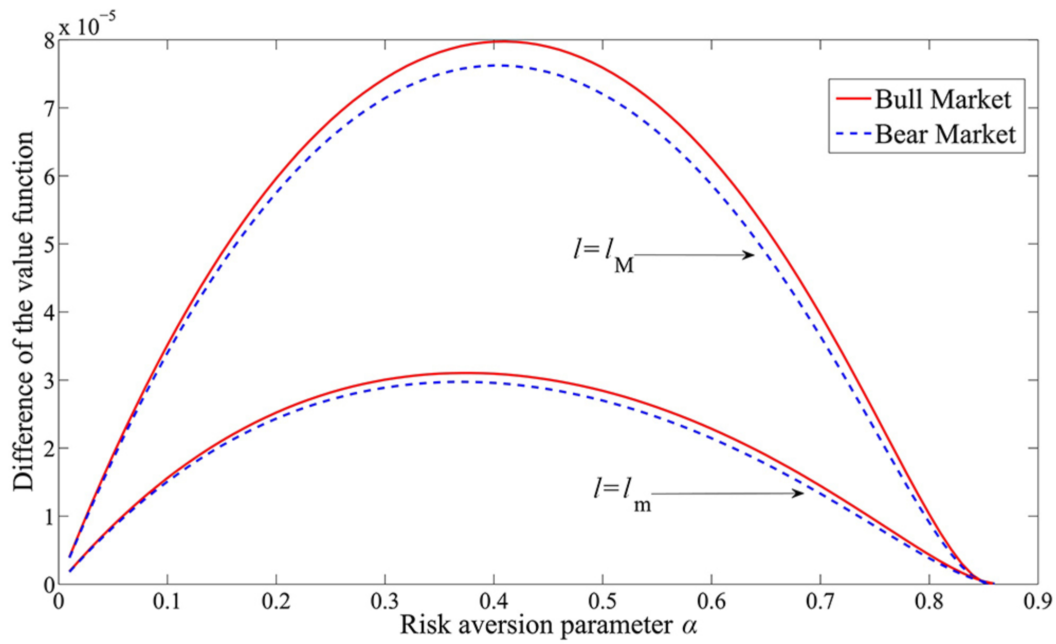

Figure 2.5: $\bar{v}(1,i) - \bar{v}_1(1,i)$ When $l \in (\frac{\nu_1}{\eta_1}, \frac{\nu_2}{\eta_2}]$

of buying insurance) is not an increasing function of α, which is different from the result in Section 2.5.2. But the difference is a concave function of α.

2.6 The Case of No Regime Switching

In this section, we consider a special case of Problem 2.1 when there is no regime switching in the economy, or equivalently, we assume there is only one regime in the economy ($S = 1$). Since $S = 1$, we remove the Markovian component ϵ from the definitions of the criterion function J, the utility function U, and the operator $\mathcal{L}$.

In the financial market, there is a bond with a fixed yield rate $r > 0$, and a stock, whose price process P_1 is modeled by a geometric Brownian motion with expected return μ and volatility σ. Both μ and σ are strictly positive. Equivalently, their price processes are given by

$$dP_0(t) = rP_0(t)dt,$$
$$dP_1(t) = P_1(t)(\mu dt + \sigma dW(t)),$$

with initial conditions $P_0(0) = 1$ and $P_1(t) > 0$.

We denote the set of all admissible controls by $\mathcal{A}_x$. For any $u \in \mathcal{A}_x$, all the conditions (2.3-2.5) and $I \in \mathcal{I}$ are satisfied.

For an investor with triplet policies u, his/her wealth process X_2 is governed by the following dynamics:

$$dX_2(t) = \big(rX_2(t) + (\mu - r)\pi(t)X_2(t) - c(t) - \lambda(1+\theta)E[I(t)]\big)dt + \sigma\pi(t)X_2(t)dW(t) - (L(t) - I(t))dN(t),$$

where the insurable loss is assumed as $L(t) = lX_2(t)$, where random variable $l \in (0, 1)$. Notice that since there is no regime switching in the economy, we drop the intensity constant of the regime η from the definition of $L(t) = \eta_{\epsilon(t)}l(t)X(t)$ used in Section 2.3.

In the case of no regime switching, Problem 2.1 reads as follows.

Problem 2.3. *Find an admissible control $u^* \in \mathcal{A}_x$ that maximizes the criterion function*

$$u^* = \arg\sup_{u \in \mathcal{A}_x} J(x; u)$$

and the value function

$$V(x) := J(x; u^*).$$

The control u^* is called an optimal control or optimal policy.

To study Problem 2.3, we rewrite the verification theorem without regime switching.

Theorem 2.5. *Suppose $U(0)$ is finite or $U(0) = -\infty$. Let $v(\cdot) \in C^2(0,\infty)$ be an increasing and concave function such that $v(0) = \frac{U(0)}{\delta}$. If $v(\cdot)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_u \left\{ \mathcal{L}^u v(x) + U(c) + \lambda E[v(x - lx + I(lx)) - v(x)] \right\} = 0, \tag{2.32}$$

for $\forall\, x > 0$ and $u^* = (\pi^*, c^*, I^*)$ defined by

$$u_t^* := \arg\sup_u \Big(\mathcal{L}^{u_t} v(X_2^*(t)) + U(c_t) + \lambda E[v(X_2^*(t) - lX_2^*(t) + I_t(lX_2^*(t))) - v(X_2^*(t))] \Big)$$

is admissible, then u^* is an optimal control to Problem 2.3.

2.6.1 Optimal Consumption, Investment and Insurance Policies

We consider the first three utility functions of HARA class that do not depend on the regime of the economy for Problem 2.3. We directly present the results here (please refer to Propositions 2.1, 2.2 and 2.3 for proofs).

(i) $U(y) = \ln(y)$, $y > 0$

Define two constants

$$\hat{\Lambda} := E\left[\ln\left(1 - l + \left(l - \frac{\theta}{1+\theta}\right)^+\right)\right] - (1+\theta)E\left[\left(l - \frac{\theta}{1+\theta}\right)^+\right] \tag{2.33}$$

and

$$\hat{A} := \frac{1}{\delta^2}\left[r - \delta + \frac{(\mu - r)^2}{2\sigma^2} + \lambda\hat{\Lambda}\right]. \tag{2.34}$$

Proposition 2.5. *The function defined by*

$$\hat{v}_2(x) := \begin{cases} \dfrac{1}{\delta}\ln(\delta x) + \hat{A}, & x > 0 \\ -\infty, & x = 0 \end{cases}$$

is the value function of Problem 2.3. Furthermore, the control given by

$$u^*(t) := \left(\frac{\mu - r}{\sigma^2}, \delta X_2^*(t), \left(l - \frac{\theta}{1+\theta}\right)^+ X_2^*(t)\right)$$

is an optimal control to Problem 2.3.

In the above Proposition , we obtain the value function $\hat{v}_2$ in explicit expression with $\hat{A}$ determined by (2.34), which depends on the distribution of l.

Example 2.4. *In this example, we calculate $\hat{A}$ under two special scenarios.*

(1) Loss proportion l is constant.

If $l \in (0, \frac{\theta}{1+\theta}]$, then

$$\hat{A} = \frac{1}{\delta^2}\left[r - \delta + \frac{(\mu - r)^2}{2\sigma^2} + \lambda \ln(1 - l)\right].$$

Otherwise, we obtain

$$\hat{A} = \frac{1}{\delta^2}\left[r - \delta + \frac{(\mu - r)^2}{2\sigma^2} - \lambda \ln(1+\theta) - \lambda l(1+\theta) + \lambda\theta\right].$$

(2) Loss proportion l is uniformly distributed on $(0, 1)$.

In this scenario, we obtain

$$E\left[\ln\left(1 - l + \left(l - \frac{\theta}{1+\theta}\right)^+\right)\right] = -\frac{\theta}{1+\theta},$$
$$E\left(l - \frac{\theta}{1+\theta}\right)^+ = \int_{\frac{\theta}{1+\theta}}^1 \left(w - \frac{\theta}{1+\theta}\right) dw = \frac{1}{2(1+\theta)^2}.$$

Therefore, constant $\hat{A}$ is calculated as

$$\hat{A} = \frac{1}{\delta^2}\left[r - \delta + \frac{(\mu - r)^2}{2\sigma^2} - \frac{\lambda(1+2\theta)}{2(1+\theta)}\right].$$

(ii) $U(y) = -y^\alpha$, $\alpha < 0$

Define three constants

$$\nu := 1 - (1+\theta)^{-\frac{1}{1-\alpha}} \in (0,1),$$
$$\tilde{\Lambda} := E\Big[\left(1 - l + (l-\nu)^+\right)^\alpha\Big] - \alpha(1+\theta)E\Big[(l-\nu)^+\Big],$$

and

$$\tilde{A} := \frac{1-\alpha}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda(1-\tilde{\Lambda})}. \tag{2.35}$$

Assume the cumulative distribution function of l is F_l, then

$$\begin{aligned} 1 - \tilde{\Lambda} &= 1 - \int_0^\nu (1-w)^\alpha dF_l - \int_\nu^1 (1-\nu)^\alpha dF_l + \alpha(1+\theta)\int_\nu^1 (l-\nu)dF_l \\ &\geq 1 - (1-\nu)^\alpha F_l(\nu) - (1-\nu)^\alpha(1-F_l(\nu)) + \alpha(1-\nu)(1+\theta)(1-F_l(\nu)) \\ &\geq 1 - (1-\nu)^\alpha + \alpha(1-\nu)(1+\theta). \end{aligned}$$

We impose the following technical condition for the discount rate δ

$$\delta > \alpha r + \frac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} - \lambda(1 - (1-\nu)^\alpha + \alpha(1-\nu)(1+\theta)). \tag{2.36}$$

Due to the technical condition (2.36) above, the constant $\tilde{A}$ defined by (2.35) is strictly positive.

Proposition 2.6. *The function defined by*

$$\tilde{v}_2(x) := \begin{cases} -\tilde{A}^{1-\alpha}x^\alpha, & x > 0 \\ -\infty, & x = 0 \end{cases}$$

where $\tilde{A}$ is given by (2.35), is the value function of Problem 2.3. Furthermore, the control u^* given by

$$u_t^* := \left(\frac{\mu - r}{(1-\alpha)\sigma^2}, \frac{1}{\tilde{A}}X_2^*(t), (l-\nu)^+ X_2^*(t)\right)$$

is an optimal control to Problem 2.3.

Example 2.5. *In this example, we calculate $\tilde{A}$ under two special scenarios.*

(1) Loss proportion l is constant.

If $l \in (0, \nu]$, then

$$\tilde{A} = \frac{1-\alpha}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda(1-(1-l)^\alpha)}.$$

Otherwise,

$$\tilde{A} = \frac{1-\alpha}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda(1-(1-\nu)^\alpha + \alpha(1+\theta)(l-\nu))}.$$

(2) Loss proportion l is uniformly distributed on $(0, 1)$.

In this scenario, we calculate

$$E(l-\nu)^+ = \int_\nu^1 (w-\nu)dw = \frac{1}{2}(1-\nu)^2,$$

and

$$E(1-l+(l-\nu)^+)^\alpha = \begin{cases} \dfrac{1+\alpha(1-\nu)^{1+\alpha}}{1+\alpha} & \text{if } \alpha \neq -1, \\ 1-\ln(1-\nu) & \text{if } \alpha = -1. \end{cases}$$

Therefore, we obtain when $\alpha \neq -1$,

$$\tilde{A} = \frac{1-\alpha}{\delta - \alpha r - \dfrac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda\left[\dfrac{\alpha}{1+\alpha}(1-(1-\nu)^{1+\alpha}) + \dfrac{1}{2}\alpha(1+\theta)(1-\nu)^2\right]},$$

and when $\alpha = -1$,

$$\tilde{A} = \frac{1-\alpha}{\delta - \alpha r - \dfrac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda\left[\ln(1-\nu) + \dfrac{1}{2}\alpha(1+\theta)(1-\nu)^2\right]}.$$

(iii) $U(y) = y^\alpha$, $0 < \alpha < 1$

We denote $\bar{A} = \tilde{A}$, as defined by (2.35). The technical condition needed in this case is

$$\delta > \alpha r + \frac{\alpha(\mu - r)^2}{2(1-\alpha)\sigma^2}. \tag{2.37}$$

The above condition (2.37) guarantees $\bar{A} > 0$ when $0 < \alpha < 1$.

Proposition 2.7. *The function defined by*

$$\bar{v}_2(x) = \bar{A}^{1-\alpha} x^\alpha,\ x \geq 0,$$

is the value function of Problem 2.3. Furthermore, the control u^* given by

$$u^*(t) := \left(\frac{\mu - r}{(1-\alpha)\sigma^2}, \frac{1}{\bar{A}} X_2^*(t), (l-\nu)^+ X_2^*(t) \right)$$

is an optimal control to Problem 2.3.

Example 2.6. *Similar to Example 2.4 and Example 2.5, we calculate $\bar{A}$ in the following two scenarios.*

(1) Loss proportion l is constant.

In this case, $\bar{A}$ has the exactly same formula as $\tilde{A}$ in Example 2.5.

(2) Loss proportion l is uniformly distributed on $(0, 1)$

In this scenario, we find $\bar{A}$ given by

$$\bar{A} = \frac{1-\alpha}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)\sigma^2} + \lambda \left[\frac{\alpha}{1+\alpha}(1-(1-\nu)^{1+\alpha}) + \frac{1}{2}\alpha(1+\theta)(1-\nu)^2 \right]}.$$

2.6.2 Economic Analysis

We have investigated the impact of market parameters and investor's risk aversion on optimal policies when there is regime switching in Section 2.4. Here we conduct a similar economic analysis to analyze how various factors affect the value function and the optimal policies.

The optimal investment proportion in the stock π^* is given by

$$\pi^*(t) = \frac{\mu - r}{(1-\alpha)\sigma^2},$$

which is similar to (2.26), the optimal investment proportion π^* with regime switching. Hence, the impact of the market and risk aversion on π^* follows from the discussions Section 2.4.

The optimal consumption rate is proportional to the investor's optimal wealth in all three cases, see Propositions 2.5-2.7. In the case of logarithmic utility function (moderate risk-averse investors), the proportion is constant,

$$\frac{c^*(t)}{X_2^*(t)} = \delta.$$

Thus, the investor's consumption decision is not affected by the market or individual risk aversion.

In the case of negative power utility function (high risk-averse investors), the optimal consumption to wealth ratio is given by

$$\frac{c^*(t)}{X_2^*(t)} = \frac{1}{\tilde{A}}, \text{ where } \tilde{A} \text{ is defined by (2.35).}$$

In the following analysis, we assume the market parameters are given by the ones of the bull market in *Parameter Set I*, namely, $\mu = 0.2$, $r = 0.08$, $\sigma = 0.25$, $\lambda = 0.1$, $\theta = 0.15$ and $\delta = 0.15$. Under the given parameters, we call this market a *good market*. In the two-regime case (Section 2.4), the insurable loss is $L(t) = \eta_{\epsilon(t)} l(t) X(t)$ and $\eta_1 = 0.8$ in a bull market. But in this case (one regime only), we have $L(t) = lX_2(t)$. Hence, if we choose $l = 0.5$ in the two-regime case, we need to assign $l = 0.4$ in the one-regime case. Next we repeat the same analysis in a different market, called *bad market*, with all the parameters given by a bear market in *Parameter Set I*. In other words, we set: $\mu = 0.15$, $r = 0.03$, $\sigma = 0.6$, $\lambda = 0.2$, $\theta = 0.25$ and $\delta = 0.15$. Since $\eta_1 = 1$ in a bear market, so loss proportion l is the same for both one-regime case and two-regime case. In the analysis below, the value of l is assigned for the two-regime case. In Table 2.2, we calculate the optimal consumption to wealth ratio for one-regime case and two-regime case.

The results obtained in Table 2.2 confirm our conclusions made in Section 2.4 that $\frac{1}{\tilde{A}_1} > \frac{1}{\tilde{A}_2}$, that is, investors consume proportionally more in a bull market that in a bear market. Similarly, we observe that more risk-averse investors (smaller α) spend a lower proportion of their wealth on consumption. When comparing the optimal consumption to wealth ratio in the last four columns in Table 2.2, we obtain the relationship of the optimal consumption to wealth ratio in different markets as follows:

$$\text{Good Market} > \text{Bull Market} > \text{Bear Market} > \text{Bad Market}.$$

Hence, we come to the conclusion that investors spend proportionally more when economy is stronger.

In the case of positive power utility function (low risk-averse investors), the optimal consumption to wealth ratio is given by

$$\frac{c^*(t)}{X_2^*(t)} = \frac{1}{\bar{A}}, \text{ where } \bar{A} = \tilde{A} \text{ defined by (2.35).}$$

We follow the analysis above, but take the market parameters from *Parameter Set II*. In a *good market*, we set parameters as: $\mu = 0.2$, $r = 0.15$, $\sigma = 0.4$, $\lambda = 0.1$, $\theta = 0.15$ and $\delta = 0.2$. In a *bad market*, we choose: $\mu = 0.15$, $r = 0.1$, $\sigma = 0.6$, $\lambda = 0.2$, $\theta = 0.25$ and $\delta = 0.2$. The results are obtained in Table 2.3 and indicate the relationship of the optimal consumption to wealth ratio in different markets as follows:

$$\text{Bad Market} > \text{Bear Market} > \text{Bull Market} > \text{Good Market},$$

which is exactly the opposite of that in the case of negative power utility.

In all three cases, the optimal insurance is deductible insurance with the form

$$I^*(t) = \left(l - 1 + (1+\theta)^{-\frac{1}{1-\alpha}}\right)^+ X_2^*(t),$$

where $\alpha = 0$ when $U(y) = \ln(y)$. The above optimal insurance is a special example of the result obtained in Section 2.4, so all our analysis applies here.

		One Regime		Two Regime	
Risk Aversion Parameter α	Loss Proportion l	Bad Market $\frac{1}{\tilde{A}}$	Good Market $\frac{1}{\tilde{A}}$	Bull Market $\frac{1}{\tilde{A}_1}$	Bear Market $\frac{1}{\tilde{A}_2}$
$\alpha = -1$	$l = 0.3$	0.058893	0.130262	0.095672	0.095129
	$l = 0.5$	0.033893	0.121062	0.078703	0.078156
	$l = 0.7$	0.008893	0.111862	0.061691	0.061185
$\alpha = -0.5$	$l = 0.3$	0.090668	0.143298	0.117902	0.117409
	$l = 0.5$	0.074001	0.137164	0.106638	0.106103
	$l = 0.7$	0.057334	0.131031	0.095359	0.094800
$\alpha = -0.1$	$l = 0.3$	0.134372	0.150733	0.142874	0.142688
	$l = 0.5$	0.129827	0.149060	0.139818	0.139604
	$l = 0.7$	0.125282	0.147387	0.136760	0.136520

Table 2.2: Optimal consumption to wealth ratio without regime switching when $\alpha < 0$

		One Regime		Two Regime	
Risk Aversion Parameter α	Loss Proportion l	Bad Market $\frac{1}{\tilde{A}}$	Good Market $\frac{1}{\tilde{A}}$	Bull Market $\frac{1}{\tilde{A}_1}$	Bear Market $\frac{1}{\tilde{A}_2}$
$\alpha = 0.1$	$l = 0.3$	0.218358	0.207532	0.212698	0.212880
	$l = 0.5$	0.223914	0.209576	0.216415	0.216660
	$l = 0.7$	0.229469	0.211621	0.220130	0.220441
$\alpha = 0.5$	$l = 0.3$	0.358392	0.260019	0.306538	0.308913
	$l = 0.5$	0.408056	0.278418	0.339521	0.342982
	$l = 0.7$	0.458056	0.296818	0.372568	0.377285
$\alpha = 0.7$	$l = 0.3$	0.553623	0.314163	0.426220	0.434223
	$l = 0.5$	0.662612	0.356064	0.498579	0.510526
	$l = 0.7$	0.779169	0.398997	0.574526	0.591543

Table 2.3: Optimal consumption to wealth ratio without regime switching when $0 < \alpha < 1$

2.7 Concluding Remarks

We have considered simultaneous optimal consumption, investment and insurance problems in a regime switching model which enables the regime of the economy to affect not only the financial but also the insurance market. A risk-averse investor facing an insurable risk wants to obtain the optimal consumption, investment and insurance policy that maximizes his/her expected total discounted utility of consumption over an infinite time horizon.

We have presented the first version of verification theorems for simultaneous optimal consumption, investment and insurance problems when there is regime switching. We have also obtained explicitly the optimal policy and the value function when the utility function belongs to the HARA class.

The optimal proportion of wealth invested in the stock is constant in every regime, and is greater in a bull market regardless of the investor's risk aversion attitude. We observe that investors with high risk tolerance invest a large proportion of wealth in the stock.

The optimal consumption to wealth ratio is a strictly increasing function of the investor's risk aversion parameter (α). Moderate risk-averse investors ($\alpha = 0$) consume at a constant proportion in both regimes. High risk-averse investors ($\alpha < 0$) consume a higher proportion of their wealth in a bull market. In contrast, low risk-averse investors ($0 < \alpha < 1$) consume proportionally more in a bear market.

The optimal insurance is proportional to the investor's wealth and such proportion depends on the premium loading θ and the investor's risk aversion parameter α. As the loading θ increases, the demand for insurance decreases. This decrease of the demand for insurance is more significant when θ is small. We observe that investors who are very risk tolerant (that is, investors with large α) spend a small amount of wealth in insurance. For high and moderate risk-averse investors ($\alpha \leq 0$), the amount of reduction in insurance is greater when α is far away from 0. However, low risk-averse investors ($0 < \alpha < 1$) reduce the amount of insurance in different magnitudes depending on the value of α.

We have obtained the conditions under which it is optimal to buy insurance, and analyzed their dependence on the different parameters.

We have calculated the advantage of buying insurance. Based on a comparative analysis, we find the value function $V(x, i)$ to Problem 2.1 is strictly greater than the value function $V_1(x, i)$ to Problem 2.2 when the optimal insurance is not equal to 0 in all regimes. We also observe that the advantage of buying insurance is greater in a bull market. Investors who face a large random loss, gain more benefits from purchasing insurance.

We have also studied optimal consumption, investment and insurance problems under the special case of no regime switching in the economy. Optimal consumption, investment and insurance policies are obtained in explicit forms.

Chapter 3

Optimal Investment and Risk Control Policies without Regime Switching

As discussed in Chapter 1 (see also Stein [88, Chapter 6]), AIG ignored the negative correlation between its liabilities and the capital gains in the financial market, and applied a questionable model for risk management. To address these issues, we assume that the insurer's risk is modeled by a jump-diffusion process, and is negatively correlated with the capital gains in the financial market. We then seek optimal investment and risk control policies for an insurer who wants to maximize its expected utility of terminal wealth. Our research has two roots in the literature: optimal investment problems and optimal reinsurance (risk control) problems.

Merton [63] was the first to apply stochastic control theory to solve consumption and investment problems in continuous time. Motivated by Merton's work on consumption/investment problems, many researchers added an uncontrollable risk process to Merton's model and then sought to obtain optimal investment policies (mostly without consumption) under certain criteria. For instance, Browne [10] modeled the risk by a continuous diffusion process and studied optimal investment problems under two different criteria: maximizing expected exponential utility of terminal wealth and minimizing the probability of ruin. Wang et al. [92] applied a jump-diffusion model for the risk process and considered optimal investment problems under the utility maximization criterion.

The second root of our research is optimal reinsurance problems, which consider an insurer who wants to control the reinsurance payout to achieve certain objectives. Reinsurance is an important tool for insurance companies to manage their risk exposure. A classical risk model in the insurance literature is Cramér-Lundberg model, which uses a compound Poisson process to model risk (claims). The Cramér-Lundberg model was introduced by Lundberg in 1903 and then republished by Cramér in 1930s. Since the limiting process of a compound Poisson process is a diffusion process, see, e.g., Taksar [90], recent research began to model risk using a diffusion process or a jump-diffusion process, see, e.g., Wang et al. [92]. Hojgaard and Taksar [44] assumed the reserve of an insurance company is governed by a diffusion process and considered optimal proportional reinsurance problems under the criterion of maximizing expected utility of running reserve up to bankruptcy. Kaluszka [49] studied optimal reinsurance problems in discrete time under the mean-variance criterion for both proportional reinsurance and step-loss reinsurance. Schmidli [79] considered both the Cramér-Lundberg model and the diffusion model, and obtained optimal proportional reinsurance policies when the insurer's objective is to minimize the probability of ruin. Cai and Tan [15] considered optimal reinsurance under VaR (value-at-risk) and CTE (conditional tail expectation) measures. Recent generalizations of modeling for optimal reinsurance problems include incorporating regime switching, see Zhuo et al. [102], and interest rate risk and inflation risk, see Guan and Liang [36].

In mathematics, there are two main tools for solving stochastic control problems. The first tool is dynamic programming and maximum principle, see, for instance, Fleming and Soner [30], Cadenillas [13], and Yong and Zhou [97]. The second tool is martingale approach, based on equivalent martingale measures and martingale representation theorems. The martingale approach and its application in continuous time finance was developed by Harrison and Kreps [42]. Thereafter, the martingale method has been applied to solve numerous important problems in economics and finance, such as option pricing problem in Harrison and Pliska [43], optimal consumption and investment problem in Karatzas et al. [50], Karatzas et al. [51], optimal consumption, investment and insurance problem in Perera [73] and optimal investment problem in Wang et al. [92]. In this chapter, we also apply the martingale approach to solve our stochastic control problem.

Our model and optimization problem are different from the existing ones in the literature in several directions. Comparing with Merton's framework and its generalizations, we add a controllable jump-diffusion process into the model, which will be used to model the insurer's risk. We then regulate the insurer's risk by controlling the insurer's total liabilities or the ratio of the liabilities over wealth. Our model is also different from the ones considered in optimal reinsurance problem and its variants, in which the risk is managed through purchasing reinsurance policies from another insurer. As suggested in Stein [88, Chapter 6], we assume there exists negative correlation between the insurer's risk and the capital returns in the financial market. Stein [88, Chapter 6] considered a similar risk regulation problem as ours, but in his model, the investment strategy is fixed, and the insurer's risk is modeled by a diffusion process. To generalize Stein's work, we use a jump-diffusion process to model the insurer's risk and allow the insurer to select investment strategies continuously. Stein [88, Chapter 6] considered the problem with logarithmic utility function only, which can be easily solved using classical optimization method. We obtain explicit solutions of optimal investment and risk control policies for various utility functions, including hyperbolic absolute risk aversion (HARA) utility function (logarithmic function and power function), constant absolute risk aversion (CARA) utility function (exponential function), and quadratic utility function.

3.1 The Model

In the financial market, there are two assets available for investment, a riskless asset with price process P_0 and a risky asset (stock) with price process P_1. On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, the dynamics of P_0 and P_1 are given by

$$\begin{aligned} dP_0(t) &= r(t)P_0(t)dt, \\ dP_1(t) &= P_1(t)(\mu(t)dt + \sigma(t)dW^{(1)}(t)), \end{aligned}$$

where r, μ and σ are positive bounded functions and $W^{(1)}$ is a standard Brownian motion. The initial conditions are $P_0(0) = 1$ and $P_1(0) > 0$.

For an insurer like AIG, its main liabilities (risk) come from writing insurance policies, and we denote the insurer's total liabilities at time t by $L(t)$. In the actuarial industry, the premium is usually precalculated, which means insurance companies charge premium based on historical data and estimation models. For example, regarding auto insurance policies, insurance companies consider several main factors, such as the insured's demographic information, previous driving record, coverage needs, type of vehicle, et cetera, and then use an actuarial model to calculate the premium for the insured. Therefore, it is reasonable to assume the premium per dollar amount of liabilities is a fixed constant for a certain type of insurance contracts and a given group of the insured. To simplify our analysis, we further assume the average premium per dollar amount of liabilities for the insurer is p, so the revenue from selling insurance policies over the time period $(t, t + dt)$ is given by $pL(t)dt$.

A commonly used risk model for claims in the actuarial industry is compound Poisson model (Cramér-Lundberg model), in which the risk is modelled by $\sum_{i=1}^{N(t)} Y_i$, where $\{Y_i\}$ is a series of independent and identically distributed random variables, and $N(t)$ is a Poisson process independent of Y_i (see, e.g., Melnikov [62, Chapter 3]). If the mean of Y_i and the intensity of $N(t)$ are finite, then such compound Poisson process is a Levy process with finite Levy measure. According to Oksendal and Sulem [72, Theorem 1.7], a Levy process can be decomposed into three components, a linear drift part, a Brownian motion part and a pure jump part. Based on this result, we assume the insurer's risk (per dollar amount of liabilities) is given by

$$dR(t) = a dt + b d\bar{W}(t) + \gamma dN(t), \; R(0) = 0,$$

where $\bar{W}$ is a standard Brownian motion and N is a Poisson process defined on the given filtered space, respectively. We assume a, b, γ are all positive constants.

Stein [88] argues that a major mistake AIG made during the financial crisis is ignore the negative correlation between its liabilities and the capital returns in the financial market. So we assume

$$\bar{W}(t) = \rho W^{(1)}(t) + \sqrt{1-\rho^2} W^{(2)}(t),$$

where $-1 \leq \rho < 0$ and $W^{(2)}$ is another standard Brownian motion, independent of $W^{(1)}$. We also assume the Poisson process N has a constant intensity λ, and is independent of both $W^{(1)}$ and $W^{(2)}$.

At time t, an insurer (AIG) chooses $\tilde{\pi}(t)$, the dollar amount invested in the risky asset, and total liabilities $L(t)$. For a control $\tilde{u} := (\tilde{\pi}, L)$, the corresponding wealth process (surplus process) $X^{\tilde{u}}$ is driven by the following SDE:

$$\begin{aligned} dX^{\tilde{u}}(t) = &\left(r(t)X^{\tilde{u}}(t) + (\mu(t) - r(t))\tilde{\pi}(t) + (p-a)L(t)\right) dt - \gamma L(t) dN(t) \\ &+ (\sigma(t)\tilde{\pi}(t) - \rho b L(t)) dW^{(1)}(t) - b\sqrt{1-\rho^2} L(t) dW^{(2)}(t), \end{aligned} \tag{3.1}$$

with initial wealth $X^{\tilde{u}}(0) = x > 0$.

Following Stein [88, Chapter 6], we define the ratio of liabilities over wealth as $\kappa(t) := \frac{L(t)}{X(t)}$ (which is called liability ratio). We denote $\pi(t)$ as the proportion of wealth invested in the risky asset at time t. Then for a control $u(t) := (\pi(t), \kappa(t))$, we have $\tilde{u}(t) = X(t)u(t)$. We then rewrite SDE (3.1) as

$$\begin{aligned}\frac{dX^u(t)}{X^u(t-)} &= (r(t) + (\mu(t) - r(t))\pi(t) + (p-a)\kappa(t))dt - \gamma\kappa(t)dN(t) \\ &\quad + (\sigma(t)\pi(t) - \rho b\kappa(t))dW^{(1)}(t) - b\sqrt{1-\rho^2}\kappa(t)dW^{(2)}(t),\end{aligned} \tag{3.2}$$

with $X^u(0) = x > 0$.

Remark 3.1. *In a financial market, it is universally acknowledged that extra uncertainty (risk) must be compensated by extra return. So in our model, we impose further conditions on the coefficients:* $\mu(t) > r(t) \geq 0$ *and* $p > a > 0$.

We define the criterion function as

$$J(x; u) = E_x\left[U(X^u(T))\right],$$

where $T > 0$ is the terminal time, and E_x means conditional expectation under probability measure $\mathbb{P}$ given $X^u(0) = x$. The utility function U is assumed to be a strictly increasing and concave function. The common choices for the utility function in economics and finance are $U(x) = \ln(x)$, $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$, and $U(x) = \frac{1}{\alpha}x^\alpha$, where $\alpha < 1$ and $\alpha \neq 0$.

We denote $\mathcal{A}(x)$ as the set of all admissible controls with initial wealth $X(0) = x$. Depending on the choice of the utility function, we choose in Sections 3.2, 3.3, 3.4, and 3.5 either u or $\tilde{u}$ to be our control and then define $\mathcal{A}(x)$ formally.

The value function is defined by

$$V(x) := \sup_{u \in \mathcal{A}(x)} J(x; u),$$

where u will be changed accordingly if the control we choose is $\tilde{u}$.

We then formulate optimal investment and risk control problems as follows.

Problem 3.1. *Select an admissible control $u^* = (\pi^*, \kappa^*) \in \mathcal{A}(x)$ (or $\tilde{u}^* = (\tilde{\pi}^*, L^*) \in \mathcal{A}(x)$) that attains the value function $V(x)$. The control u^* (or $\tilde{u}^*$) is called an optimal control or an optimal policy.*

3.2 The Analysis for $U(x) = \ln(x)$, $x > 0$

We first consider Problem 3.1 when the utility function is given by $U(x) = \ln(x)$, $x > 0$, which belongs to the class of hyperbolic absolute risk aversion (HARA) utility functions.

We choose u as control and denote $\mathcal{A}_1$ as the set of all admissible controls when $U(x) = \ln(x)$. For every $u \in \mathcal{A}_1$, $\{u(t)\}_{0 \le t \le T}$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ and satisfies the following conditions, $\forall\, t \in [0, T]$,

$$E\left[\int_0^t \pi(s)^2 ds\right] < \infty, \text{ and } E\left[\int_0^t \kappa(s)^2 ds\right] < \infty,\ \kappa(t) \ge 0.$$

Furthermore, to avoid the possibility of bankruptcy at jumps, we assume $\kappa(t) < \frac{1}{\gamma}$ if $u \in \mathcal{A}_1$.

Notice that $\forall\, u \in \mathcal{A}_1$, the SDE (3.2) satisfies the linear growth condition and Lipschitz continuity condition. Thus by Theorem 1.19 in Oksendal and Sulem [72], there exists a unique solution X^u such that

$$E[|X^u(t)|^2] < \infty \quad \text{for all } t \in [0, T].$$

Proposition 3.1. *Under optimal control u^* of Problem 3.1, the associated optimal terminal wealth $X^{u^*}(T)$ is strictly positive with probability 1.*

Proof. Notice that $u_0 := (\pi \equiv 0, \kappa \equiv 0) \in \mathcal{A}_1$ is an admissible control, and the associated wealth X^{u_0} is given by

$$X^{u_0}(t) = x\, e^{\int_0^t r(s)ds} > 0,\ \forall\, t \in [0, T].$$

Thus, $E[U(X^{u_0}(T))]$ is finite. Since u^* is optimal control of Problem 3.1, we have $E[U(X^{u^*}(T))] \geq E[U(X^{u_0}(T))]$, and in particular $E[U(X^{u^*}(T))]$ is bounded from below. Since U is the logarithmic utility, this implies that $X^{u^*}(T) > 0$ with probability 1. □

3.2.1 Method 1: Optimization Method in Calculus

Under the logarithmic utility assumption, we can apply the classical optimization method in calculus to solve Problem 3.1. For more details on using this method to solve stochastic control problems, please see Stein [88, Chapters 4,5,6].

Applying Ito's formula to $\ln(X_t)$, we obtain

$$\begin{aligned}\ln \frac{X_t^u}{X_0} = \int_0^t \Big(& r_s + (\mu_s - r_s)\pi_s + (p-a)\kappa_s - \frac{1}{2}\sigma_s^2\pi_s^2 + \rho b \sigma_s \pi_s \kappa_s \\ & - \frac{1}{2} b^2 \kappa_s^2 + \lambda \ln(1 - \gamma\kappa_s)\Big) ds + \int_0^t (\sigma_s \pi_s - b\rho\kappa_s)\, dW_s^{(1)} \\ & - \int_0^t b\sqrt{1-\rho^2}\kappa_s dW_s^{(2)} + \int_0^t \ln(1-\gamma\kappa_s) dM_s,\end{aligned}$$

where $M_t := N_t - \lambda t$ is the compensated Poisson process of N and is a martingale under $\mathbb{P}$.

For any given $u \in \mathcal{A}_1$, we have

$$\begin{aligned}\int_0^t (\sigma_s\pi_s - b\rho\kappa_s)^2\, ds \leq K_1 \int_0^t \pi_s^2 ds + K_2 \int_0^t \kappa_s^2 ds < \infty, \\ \int_0^t b^2(1-\rho^2)\kappa_s^2 ds \leq K_3 \int_0^t \kappa_s^2 ds < \infty,\end{aligned}$$

for some positive constants K_i, $i = 1, 2, 3$.

Therefore, we obtain

$$E_x\left[\int_0^t (\sigma_s \pi_s - b\rho\kappa_s)\, dW_s^{(1)}\right] = E_x\left[\int_0^t b\sqrt{1-\rho^2}\kappa_s dW_s^{(2)}\right] = 0.$$

Since κ is a bounded predictable process, so is $\ln(1 - \gamma\kappa)$ and this implies the stochastic integral $\int_0^t \ln(1 - \gamma\kappa_s)dM_s$ is again a $\mathbb{P}$-martingale with the initial value being 0. So we obtain

$$E_x\left[\int_0^t \ln(1-\gamma\kappa_s)dM_s\right] = 0.$$

The above analysis yields

$$E_x\left[\ln\frac{X_T^u}{X_0^u}\right] = E_x\left[\int_0^T f(\pi(t), \kappa(t))dt\right],$$

where $f(\pi(t), \kappa(t)) := r(t) + (\mu(t) - r(t))\pi(t) + (p - a)\kappa(t) - \frac{1}{2}\sigma(t)^2\pi(t)^2 + \rho b\sigma(t)\pi(t)\kappa(t) - \frac{1}{2}b^2\kappa(t)^2 + \lambda\ln(1 - \gamma\kappa(t))$.

Hence we obtain the optimization condition as follows

$$u^*(t) = \arg\sup_{u\in\mathcal{A}_1} J(x; u) = \arg\sup_{u\in\mathcal{A}_1} f(\pi(t), \kappa(t)).$$

The first-order condition is then given by

$$\begin{aligned} \mu(t) - r(t) - \sigma^2(t)\pi^*(t) + \rho b\sigma(t)\kappa^*(t) &= 0, \\ p - a + \rho b\sigma(t)\pi^*(t) - b^2\kappa^*(t) - \frac{\lambda\gamma}{1-\gamma\kappa^*(t)} &= 0. \end{aligned} \tag{3.3}$$

We next obtain a candidate for π^* as

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{\rho b}{\sigma(t)}\kappa^*(t), \tag{3.4}$$

where $\kappa^*(t)$ is the solution to the following quadratic equation

$$A(\kappa^*(t))^2 - B(t)\,\kappa^*(t) + C(t) = 0, \tag{3.5}$$

with coefficients defined as

$$\begin{aligned} A &:= b^2(1-\rho^2)\gamma, \\ B(t) &:= b^2(1-\rho^2) + \gamma\left(p - a + \rho\, b\,\frac{\mu(t)-r(t)}{\sigma(t)}\right), \\ C(t) &:= p - a + \rho\, b\,\frac{\mu(t)-r(t)}{\sigma(t)} - \lambda\gamma. \end{aligned}$$

It is easy to check that

$$\Delta(t) := B^2(t) - 4A\,C(t) = (b^2(1-\rho^2) - \gamma(C(t)+\lambda\gamma))^2 + 4\lambda b^2(1-\rho^2)\gamma^2 > 0.$$

So the quadratic system (3.5) has two solutions. One of them is given by

$$\kappa_+(t) = \frac{B(t) + \sqrt{\Delta(t)}}{2A} > \frac{1}{\gamma},$$

which is not included in the admissible set $\mathcal{A}_1$.

To ensure the existence of a non-negative $\kappa^* \in [0, \frac{1}{\gamma})$, we impose the technical condition $\min_{t\in[0,T]} C(t) > 0$, which is equivalent to

$$p - a + \rho\, b\,\frac{\mu(t)-r(t)}{\sigma(t)} > \lambda\gamma,\ \forall\, t \in [0,T]. \tag{3.6}$$

If the technical condition (3.6) holds, we have

$$\kappa^*(t) = \kappa_-(t) := \frac{B(t) - \sqrt{\Delta(t)}}{2A}. \tag{3.7}$$

Notice that a sufficient condition for a regular interior maximizer and the first-order condition to hold is

$$f_{\pi\pi} < 0,\ f_{\kappa\kappa} < 0,\ \text{and}\ f_{\pi\pi} f_{\kappa\kappa} - f_{\pi\kappa}^2 > 0. \tag{3.8}$$

We then calculate those partial derivatives and verify that the above condition (3.8) is satisfied.

$$\begin{aligned} f_{\pi\pi} &= -\sigma^2(t) < 0, \\ f_{\kappa\kappa} &= -b^2 - \frac{\lambda\gamma^2\kappa(t)}{(1-\gamma\kappa(t))^2} < 0, \\ f_{\pi\pi}f_{\kappa\kappa} - f_{\pi\kappa}^2 &= (1-\rho^2)b^2\sigma^2(t) + \frac{\lambda\gamma^2\sigma^2(t)\kappa(t)}{(1-\gamma\kappa(t))^2} > 0. \end{aligned}$$

Theorem 3.1. *When $U(x) = \ln(x)$, and the technical condition (3.6) holds, $u^*(t) = (\pi^*(t), \kappa^*(t))$, where $\pi^*(t)$ and $\kappa^*(t)$ are given by (3.4) and (3.7), respectively, is optimal control to Problem 3.1 with the admissible set $\mathcal{A}_1$.*

Proof. $\forall\, u = (\pi, \kappa) \in \mathcal{A}_1$, since u^* defined above is the maximizer of f, we have

$$f(\pi^*(t), \kappa^*(t)) \geq f(\pi(t), \kappa(t)),\ \forall\, t \in [0, T],$$

and then

$$\int_0^T f(\pi^*(t), \kappa^*(t))dt \geq \int_0^T f(\pi(t), \kappa(t))dt,$$

which implies $J(x, u^*) \geq J(x, u)$. Due to the arbitrariness of u, we obtain $J(x, u^*) \geq V(x)$.

To complete the proof, we verify that u^* defined above is admissible.

Since $\frac{C(t)}{A} > 0$ and $\kappa_+(t) > 0$, we have $\kappa^*(t) = \kappa_-(t) > 0$.

To show $\kappa^*(t) < \frac{1}{\gamma}$, it is equivalent to show

$$\Delta(t) > (\gamma(C(t) + \lambda\gamma) - b^2(1-\rho)^2)^2,$$

which is always satisfied if we recall the definition of $\Delta(t)$.

So we have $0 \leq \kappa^*(t) < \frac{1}{\gamma}$, which in turn implies

$$\int_0^t (\kappa^*(s))^2 ds < \infty, \forall\, t \in [0, T].$$

From our assumption, $\mu(t)$, $r(t)$ and $\sigma(t)$ are all positive and bounded functions, for all $t \in [0, T]$, we obtain

$$\int_0^t (\pi^*(s))^2 ds \leq K_4 t + K_5 \int_0^t (\kappa^*(s))^2 ds < \infty,$$

for some positive constants K_4 and K_5.

Therefore u^* defined above is an admissible control and then is optimal control to Problem 3.1. □

3.2.2 Method 2: Martingale Method

In this subsection, we apply the martingale method to solve Problem 3.1. To begin with, we give two important Lemmas, which are Proposition 2.1 and Lemma 2.1 in Wang et al. [92], respectively. Lemma 3.1 gives the condition that optimal control must satisfy. Lemma 3.2 is a generalized version of martingale representation theorem.

Lemma 3.1. *If there exists a control $u^* \in \mathcal{A}(x)$ such that*

$$E\left[U'(X^{u^*}(T))\, X^u(T)\right] \text{ is constant over all admissible controls,} \tag{3.9}$$

then u^* is optimal control to Problem 3.1.

Lemma 3.2. *For any $\mathbb{P}$-martingale Z, there exists predictable processes $\theta = (\theta_1, \theta_2, \theta_3)$ such that*

$$Z_t = Z_0 + \int_0^t \theta_1(s) dW_s^{(1)} + \int_0^t \theta_2(s) dW_s^{(2)} + \int_0^t \theta_3(s) dM_s,$$

for all $t \in [0, T]$.

We obtain optimal control to Problem 3.1 through the following three steps.

Step 1. We conjecture candidates for optimal strategies π^* and κ^*.

Define

$$Z_T := \frac{(X_T^{u^*})^{-1}}{E[(X_T^{u^*})^{-1}]}, \text{ and } Z_\eta := E[Z_T|\mathcal{F}_\eta] \tag{3.10}$$

for any stopping time $\eta \leq T$ almost surely. Recall Proposition 3.1, the process Z is a strictly positive (square-integrable) martingale under $\mathbb{P}$ with $E[Z_t] = 1$, for all $t \in [0, T]$. We define a new measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} := Z_T$.

From the SDE (3.1), we have

$$\begin{aligned} X_t^u = X_t^{\tilde{u}} = xe^{rt} + \int_0^t e^{r(t-s)}((\mu_s - r_s)\tilde{\pi}_s + (p-a)L_s)ds - \int_0^t e^{r(t-s)}\gamma L_s dN_s \\ + \int_0^t e^{r(t-s)}(\sigma_s\tilde{\pi}_s - \rho b L_s)dW_s^{(1)} - \int_0^t e^{r(t-s)} b\sqrt{1-\rho^2} L_s dW_s^{(2)}. \end{aligned}$$

Using the above expression of X and Lemma 3.1, for all admissible controls, we have

$$\begin{aligned} E_{\mathbb{Q}}\Big[\int_0^t e^{-rs}\Big(((\mu_s - r_s)\tilde{\pi}_s + (p-a)L_s)ds + (\sigma_s\tilde{\pi}_s - \rho b L_s)dW_s^{(1)} \\ - b\sqrt{1-\rho^2}L_s dW_s^{(2)} - \gamma L_s dN_s\Big)\Big] \text{ is constant.} \end{aligned} \tag{3.11}$$

We define

$$K_t := \int_0^t \frac{1}{Z_{s-}} dZ_s, \; t \in [0, T].$$

Since Z is a $\mathbb{P}$-martingale, so is K.

By Lemma 3.2, there exist predictable processes $(\theta_1, \theta_2, \theta_3)$ such that (see Wang et al. [92] for measurability and integrability conditions that θ should satisfy)

$$dK_t = \theta_1(t)dW_t^{(1)} + \theta_2(t)dW_t^{(2)} + \theta_3(t)dM(t).$$

Then by the Doleans-Dade exponential formula, we have

$$Z_t = Z_0 \exp\Big\{ \int_0^t (\theta_1(s)dW_s^{(1)} + \theta_2(s)dW_s^{(2)} + \ln(1+\theta_3(s))dN_s) \\ - \frac{1}{2}\int_0^t (\theta_1^2(s) + \theta_2^2(s) + 2\lambda\theta_3(s))ds\Big\}. \tag{3.12}$$

By Girsanov's Theorem, $W^{(i)}(t) - \int_0^t \theta_i(s)ds$, $i = 1, 2$, is a Brownian Motion under $\mathbb{Q}$ and $N(t) - \int_0^t \lambda(1+\theta_3(s))ds$ is a martingale under $\mathbb{Q}$.

For any stopping time $\eta \le T$, we choose $\tilde{\pi}(t) = 1_{t\le\eta}$ and $L(t) = 0$, which is apparently an admissible control. By substituting this control into (3.11), we obtain

$$E_{\mathbb{Q}}\left[\int_0^\eta e^{-rs}(\mu_s - r_s)ds + \int_0^\eta e^{-rs}\sigma_s dW_s^{(1)}\right] \text{ is constant over all } \eta \le T,$$

which implies

$$\int_0^t e^{-rs}(\mu_s - r_s)ds + \int_0^t e^{-rs}\sigma_s dW_s^{(1)} \text{ is a } \mathbb{Q}\text{-martingale.} \tag{3.13}$$

Therefore, θ_1 must satisfy the equation

$$\mu(t) - r(t) + \sigma(t)\theta_1(t) = 0,$$

or equivalently,

$$\theta_1(t) = -\frac{\mu(t) - r(t)}{\sigma(t)},\ t \in [0, T]. \tag{3.14}$$

Next we choose $\tilde{\pi}(t) = 0$ and $L(t) = 1_{t\le\eta}$. By following a similar argument as above, we have

$$\int_0^t e^{-rs}((p-a)ds - \rho b dW_s^{(1)} - b\sqrt{1-\rho^2}dW_s^{(2)} - \gamma dN_s) \text{ is a } \mathbb{Q}\text{-martingale,}$$

which in turn yields

$$p - a - \rho b\theta_1(t) - b\sqrt{1-\rho^2}\theta_2(t) - \lambda\gamma(1+\theta_3(t)) = 0,\ t \in [0, T].$$

By (3.14), we can rewrite the above equation as

$$p - a + \rho b\,\frac{\mu(t) - r(t)}{\sigma(t)} - b\sqrt{1-\rho^2}\,\theta_2(t) - \lambda\gamma(1 + \theta_3(t)) = 0,\ t \in [0, T]. \quad (3.15)$$

Remark 3.2. *Notice that the above analysis holds for all utility functions except that the definition of Z in (3.10) changes accordingly. More importantly, we emphasize that the conditions (3.14) and (3.15) are satisfied for all utility functions, although θ_2 and θ_3 will be different for different utility functions. We shall use the conclusion in this Remark when applying the martingale approach to solve Problem 3.1 for different utility functions thereafter.*

From the SDE (3.2), we can solve to get $(X_T^{u^*})^{-1}$

$$\begin{aligned}(X_T^{u^*})^{-1} = x^{-1}\exp\Big\{ &- \int_0^T f(\pi_t^*, \kappa_t^*)dt - \int_0^T (\sigma_t\pi_t^* - \rho b\kappa_t^*)dW_t^{(1)} \\ &+ \int_0^T b\sqrt{1-\rho^2}\kappa_t^* dW_t^{(2)} - \int_0^T \ln(1 - \gamma\kappa_t^*)dM_t\Big\}.\end{aligned} \quad (3.16)$$

By comparing the $dW^{(1)}$, $dW^{(2)}$ and dN terms in (3.12) and (3.16), we obtain

$$\begin{aligned}\theta_1(t) &= -(\sigma(t)\pi^*(t) - \rho b\kappa^*(t)),\\ \theta_2(t) &= b\sqrt{1-\rho^2}\kappa^*(t), \\ \ln(1 + \theta_3(t)) &= -\ln(1 - \gamma\kappa^*(t)).\end{aligned} \quad (3.17)$$

By plugging (3.17) into (3.14) and (3.15), we obtain the same system (3.3) as in Method 1. Hence, the candidates of the optimal policies π^* and κ^* are given by (3.4) and (3.7).

Step 2. For θ_i given in (3.17) and $u^* = (\pi^*, \kappa^*)$ defined by (3.4) and (3.7), we verify that Z_T defined by (3.12) is consistent with its definition.

We first rewrite (3.16) as

$$\frac{1}{X_T^{u^*}} = I_T H_T,$$

where

$$\begin{aligned}
I_T &:= \frac{1}{x}\exp\left\{\int_0^T (-f(\pi_s^*,\kappa_s^*) + \lambda\ln(1-\gamma\kappa_s^*))ds\right\},\\
H_T &:= \exp\Big\{ -\int_0^T (\sigma_s\pi_s^* - \rho b\kappa_s^*)dW_s^{(1)} + \int_0^T b\sqrt{1-\rho^2}\kappa_s^* dW_s^{(2)}\\
&\qquad - \int_0^T \ln(1-\gamma\kappa_s^*)dN_s\Big\}.
\end{aligned}$$

By substituting (3.17) back into (3.12), we obtain

$$Z_T = J_T H_T,$$

where

$$J_T := \exp\left\{\int_0^T (-\frac{1}{2}\sigma_s^2(\pi_s^*)^2 + \rho b\sigma_s\pi_s^*\kappa_s^* - \frac{1}{2}b^2(\kappa_s^*)^2 + \lambda(\frac{1}{1-\gamma\kappa_s^*} - 1))ds\right\}$$

is constant.

By definition, we know Z is a $\mathbb{P}$-martingale and $E[Z_T] = 1$, and then

$$E[H_T] = \frac{1}{J_T}.$$

Therefore, we obtain

$$Z_T = \frac{(X_T^{u^*})^{-1}}{E\left[(X_T^{u^*})^{-1}\right]} = \frac{I_T H_T}{I_T E[H_T]} = \frac{H_T}{J_T^{-1}} = J_T H_T,$$

which shows Z given by (3.12) with θ_i provided by (3.17) is the same as the definition: $Z_T = \frac{(X_T^{u^*})^{-1}}{E\left[(X_T^{u^*})^{-1}\right]}$.

Step 3. We verify that π^* and κ^*, given by (3.4) and (3.7), respectively, are indeed optimal strategies. Equivalently, we verify the condition (3.9) is satisfied for $u^* = (\pi^*, \kappa^*)$.

For any $u \in \mathcal{A}_1$, we define a new process Y^u as follows

$$\begin{aligned}
Y_t^u &:= \int_0^t e^{-rs} X_s^u \left((\mu_s - r_s)\pi_s + (p-a)\kappa_s\right) ds - \int_0^t e^{-rs} X_s^u \gamma \kappa_s dN_s \\
&\quad + \int_0^t e^{-rs} X_s^u \left((\sigma_s \pi_s - \rho b \kappa_s) dW_s^{(1)} - b\sqrt{1-\rho^2}\kappa_s dW_s^{(2)}\right) \\
&= \int_0^t e^{-rs} X_s^u \kappa_s \left(p - a + \rho b \frac{\mu_s - r_s}{\sigma_s} - b^2(1-\rho^2)\kappa_s^* - \frac{\lambda\gamma}{1-\gamma\kappa_s^*}\right) ds \\
&\qquad + \text{ local } \mathbb{Q}\text{-martingale.}
\end{aligned}$$

Due to the first-order condition (3.3), the above ds term will be 0, and then Y^u is a local $\mathbb{Q}$-martingale.

Since u^* is deterministic and bounded, Z is a square-integrable martingale under $\mathbb{P}$, which implies $E[(Z_T)^2] < \infty$ or equivalently, $Z \in L^2(\mathcal{F})$. Furthermore, for any $u \in \mathcal{A}_1$, we have $X^u \in L^2(\mathcal{F})$, so is Y^u. Therefore, we have

$$E_{\mathbb{Q}}\left[\sup_{0\le t\le T} |Y_t^u|\right] \le \sqrt{E[(Z_T)^2]}\sqrt{E\left[\sup_{0\le t\le T} |Y_t^u|^2\right]} < \infty,$$

which enables us to conclude that the family

$$\{Y_\eta^u : \text{ stopping time } \eta \le T\} \text{ is uniformly integrable under } \mathbb{Q}.$$

Hence Y^u is indeed a martingale under $\mathbb{Q}$ with $E_{\mathbb{Q}}[Y_t^u] = 0$ for any $u \in \mathcal{A}_1$. This result verifies the condition (3.9) is satisfied.

Therefore, Lemma 3.1 together with the above three steps lead to Theorem 3.1.

□

3.3 The Analysis for $U(x) = \frac{1}{\alpha}x^\alpha$, $\alpha < 1$, $\alpha \neq 0$

The second utility function we consider is power function, which also belongs to HARA class. Here, we choose $\mathcal{A}_1$ as the admissible set for Problem 3.1.

Since $U'(X_T^{u^*}) = (X_T^{u^*})^{\alpha-1}$, we define Z as

$$Z_T := \frac{\left(X_T^{u^*}\right)^{\alpha-1}}{E\left[\left(X_T^{u^*}\right)^{\alpha-1}\right]}, \text{ and } Z_\eta := E[Z_T|\mathcal{F}_\eta], \tag{3.18}$$

where η is a stopping time and $\eta \leq T$ almost surely. With the help of Z, we define a new probability measure $\mathbb{Q}$ by $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.

From the SDE (3.2), we obtain

$$\begin{aligned}\left(X_T^{u^*}\right)^{\alpha-1} = \text{constant} \cdot \exp\Bigg\{ \int_0^T (\alpha-1)\Big((\sigma_t\pi_t^* - \rho b\kappa_t^*)dW_t^{(1)} \\ - b\sqrt{1-\rho^2}\kappa_t^* dW_t^{(2)} + \ln(1-\gamma\kappa_t^*)dN_t\Big)\Bigg\}.\end{aligned} \tag{3.19}$$

Thanks to Remark 3.2, Z_T also bears the expression (3.12). So by comparing the terms of $dW^{(1)}$, $dW^{(2)}$ and dN in (3.12) and (3.19), we obtain

$$\begin{aligned}\theta_1(t) &= (\alpha-1)(\sigma(t)\pi^*(t) - \rho b\kappa^*(t)), \\ \theta_2(t) &= -b(\alpha-1)\sqrt{1-\rho^2}\kappa^*(t), \\ \ln(1+\theta_3(t)) &= (\alpha-1)\ln(1-\gamma\kappa^*(t)).\end{aligned} \tag{3.20}$$

Substituting θ_1 in (3.20) into (3.14), we obtain optimal proportion π^* of investment in the risky asset

$$\pi^*(t) = -\frac{\mu(t) - r(t)}{(\alpha-1)\sigma^2(t)} + \frac{\rho\, b}{\sigma(t)}\kappa^*(t), \tag{3.21}$$

where κ^* will be determined below by (3.22).

Due to Remark 3.2, θ_2 and θ_3 defined above should satisfy the equation (3.15).

We then plug (3.20) into (3.15), and obtain

$$p-a+\rho b\,\frac{\mu(t)-r(t)}{\sigma(t)}+(\alpha-1)b^2(1-\rho^2)\kappa^*(t)-\lambda\gamma(1-\gamma\kappa^*(t))^{\alpha-1}=0. \quad (3.22)$$

Define

$$\begin{aligned}
\phi(t) &:= 1-\gamma\kappa^*(t),\\
B_1 &:= \frac{(\alpha-1)b^2(1-\rho^2)}{\lambda\gamma^2},\\
C_1(t) &:= -\frac{1}{\lambda\gamma}\left[p-a+\rho b\frac{\mu(t)-r(t)}{\sigma(t)}+\frac{(\alpha-1)b^2(1-\rho^2)}{\gamma}\right].
\end{aligned}$$

Then the equation (3.22) for optimal liability ratio κ^* can be rewritten as

$$(\phi(t))^{\alpha-1}+B_1\,\phi(t)+C_1(t)=0. \quad (3.23)$$

Lemma 3.3. *If the condition* (3.6) *holds, then there exists a unique solution* $\phi(t)\in(0,1)$ *to the equation* (3.23)*, equivalently, a unique solution* $\kappa^*(t)\in(0,\frac{1}{\gamma})$ *to the equation* (3.22)*.*

Proof. Define $h(x):=x^{\alpha-1}+B_1\,x+C_1(t)$. It is easy to check $h'(x)=(\alpha-1)x^{\alpha-2}+B_1<0$ since $\alpha-1<0$. Besides, $\lim_{x\to 0^+}h(x)=+\infty$. Due to the technical condition (3.6), we have

$$h(1)=1-\frac{1}{\lambda\gamma}\left(p-a+\rho b\,\frac{\mu(t)-r(t)}{\sigma(t)}\right)<0.$$

Hence, there exists a unique solution in $(0,1)$ to the equation (3.23) for all $t\in[0,T]$. Recall the definition of ϕ, if $\phi\in(0,1)$, then $\kappa^*\in(0,\frac{1}{\gamma})$, and so the equation (3.22) bears a unique solution in $(0,\frac{1}{\gamma})$. □

Theorem 3.2. *When* $U(x)=\frac{1}{\alpha}x^\alpha$, $\alpha<1$, $\alpha\neq 0$, *and the technical condition* (3.6) *holds,* $u^*(t)=(\pi^*(t),\kappa^*(t))$, *with* π^* *and* κ^* *given by* (3.21) *and* (3.22)*, respectively, is optimal control to Problem 3.1 with the admissible set* $\mathcal{A}_1$.

Proof. Because of Lemma 3.3, π^* and κ^* given by (3.21) and (3.22) are well-defined if the condition (3.6) is satisfied. By following Steps 2 and 3 as in Section 3.2, we can verify that the condition (3.9) holds for the above defined $u^* = (\pi^*, \kappa^*)$. Then it remains to show that u^* is admissible.

By Lemma 3.3, we have $\kappa^*(t) \in (0, \frac{1}{\gamma})$, and then the square integrability condition for κ^* follows. Recall (3.21) and the assumption that μ, r, σ are all bounded, π^* is also square-integrable. Therefore, $u^* \in \mathcal{A}_1$ and then u^* defined above is optimal control to Problem 3.1. $\square$

3.4 The Analysis for $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, $\alpha > 0$

In this section, we consider Problem 3.1 for exponential utility function, which is of constant absolute risk aversion (CARA) class. In this case, we use control $\tilde{u}$ and define the admissible set $\mathcal{A}_2$ as follows: for any admissible control $\tilde{u} = (\tilde{\pi}, L) \in \mathcal{A}_2$, $\{\tilde{u}_t\}_{0\le t\le T}$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, and satisfies the integrability conditions

$$E\left[\int_0^t (\tilde{\pi}(s))^2 ds\right] < \infty,\; E\left[\int_0^t (L(s))^2 ds\right] < \infty,$$

and $L(t) \ge 0, \forall\, t \in [0, T]$.

By Lemma 3.1, optimal control $\tilde{u}^*$ should satisfy the following condition

$$E\left[\exp\{-\alpha X_T^{\tilde{u}^*}\}\, X_T^{\tilde{u}}\right] \text{ is constant for all } u \in \mathcal{A}_2. \tag{3.24}$$

So we define the Radon-Nikodym process by

$$Z_T := \frac{e^{-\alpha X_T^{u^*}}}{E[e^{-\alpha X_T^{u^*}}]},\text{ and } Z_\eta := E[Z_T|\mathcal{F}_\eta], \tag{3.25}$$

for any stopping time $\eta \le T$, and a new probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.

Since Z is a martingale under $\mathbb{P}$, there exist progressively measurable processes θ_i, $i = 1, 2, 3$, such that Z is in the form of (3.12).

From the SDE (3.1), we can calculate

$$\exp\left\{-\alpha X_T^{\tilde{u}^*}\right\} = \text{constant} \cdot \exp\Bigg\{ - \int_0^T \alpha e^{r(T-t)} \Big((\sigma_t \tilde{\pi}_t^* - \rho b L_t^*) dW_t^{(1)} - b\sqrt{1-\rho^2} L_t^* dW_t^{(2)} - \gamma L_t^* dN_t \Big) \Bigg\}. \tag{3.26}$$

Comparing (3.12) and (3.26) gives

$$\begin{aligned} \theta_1(t) &= -\alpha e^{r(T-t)} (\sigma(t) \tilde{\pi}^*(t) - \rho b L^*(t)), \\ \theta_2(t) &= \alpha e^{r(T-t)} b \sqrt{1-\rho^2} L^*(t), \\ \ln(1 + \theta_3(t)) &= \alpha \gamma e^{r(T-t)} L^*(t). \end{aligned} \tag{3.27}$$

By (3.14), we have

$$\tilde{\pi}^*(t) = e^{-r(T-t)} \frac{\mu(t) - r(t)}{\alpha \sigma(t)^2} + \frac{\rho b}{\sigma(t)} L^*(t). \tag{3.28}$$

Substituting (3.27) into (3.15), we obtain

$$\lambda \gamma \, e^{A_3(t) L^*(t)} + B_3(t) \, L^*(t) - C_3(t) = 0, \tag{3.29}$$

with A_3, B_3 and C_3 defined by

$$\begin{aligned} A_3(t) &:= \alpha \, \gamma \, e^{r(T-t)}, \\ B_3(t) &:= \alpha e^{r(T-t)} b^2 (1 - \rho^2), \\ C_3(t) &:= p - a + \rho b \, \frac{\mu(t) - r(t)}{\sigma(t)}. \end{aligned}$$

Lemma 3.4. *If the condition (3.6) holds, then there exists a (unique) positive solution to the equation (3.29).*

Proof. We define $\tilde{h}(x) := \lambda \, \gamma \, e^{A_3(t) x} + B_3(t) \, x - C_3(t)$. Since $A_3(t) > 0$, $B_3(t) > 0$

for all $t \in [0, T]$, and

$$\tilde{h}'(x) = \lambda\,\gamma\,A_3(t)e^{A_3(t)x} + B_3(t),$$

we have $\tilde{h}'(x) > 0$. Because the condition (3.6) holds, we obtain $\tilde{h}(0) = \lambda\gamma - C_3(t) < 0$ for all $t \in [0, T]$. Besides, $C_3(t)$ is a bounded function on $[0, t]$ and then has a finite maximum, which implies $\tilde{h}(x) > 0$ when x is large enough. Therefore, as a continuous and strictly increasing function, $\tilde{h}(x)$ has a (unique) positive zero point. $\square$.

Theorem 3.3. *When $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, $\alpha > 0$, and the condition (3.6) holds, $\tilde{u}^*(t) = (\tilde{\pi}^*(t), L^*(t))$, where $\tilde{\pi}^*$ and L^* are defined by (3.28) and (3.29), respectively, is optimal control to Problem 3.1 with the admissible set $\mathcal{A}_2$.*

Proof. Please refer to Theorem 3.1 for proof. $\square$

3.5 The Analysis for $U(x) = x - \dfrac{\alpha}{2}x^2$, $\alpha > 0$

As pointed in Wang et al. [92], to find a mean-variance portfolio strategy is equivalent to maximize expected utility for a quadratic function. So in this section, we consider a quadratic utility function, and solve Problem 3.1 with admissible set $\mathcal{A}(x) = \mathcal{A}_2$. Notice that the quadratic utility function considered in this section is not strictly increasing for all x, but rather has a maximum point at $x = \frac{1}{\alpha}$. This means if investor's wealth is greater than the maximum point, he/she will experience a decreasing utility as wealth increases. Such result is consistent with the famous efficient frontier theory (discovered by Markowitz [60]).

Since $U'(x) = 1 - \alpha x$, our objective is to find $\tilde{u}^* \in \mathcal{A}_2$ such that

$$E[(1 - \alpha X_T^{\tilde{u}^*})X_T^{\tilde{u}}] \text{ is constant for all } u \in \mathcal{A}_2. \tag{3.30}$$

Define $Z_T := 1 - \alpha X_T^{\tilde{u}^*}$ and $Z_t := E[Z_T|\mathcal{F}_t]$. Since $\tilde{u}^* \in \mathcal{A}_2$, Z is a square-integrable martingale under $\mathbb{P}$. By Lemma 3.2, there exist progressively measurable processes θ_i, $i = 1, 2, 3$, such that

$$dZ(t) = \theta_1(t)dW^{(1)}(t) + \theta_2(t)dW^{(2)}(t) + \theta_3(t)dM(t).$$

Define the process $\tilde{Y}^{\tilde{u}}$ by

$$\tilde{Y}^{\tilde{u}}(t) := \int_0^t e^{-rs}\big[((\mu_s - r_s)\tilde{\pi}_s + (p-a)L_s)ds + (\sigma_s\tilde{\pi}_s - \rho b L_s)dW_s^{(1)} \\ - b\sqrt{1-\rho^2}L_s dW_s^{(2)} - \gamma L_s dN_s\big].$$

Then we can write $X^{\tilde{u}}$ as $X^{\tilde{u}}(t) = e^{rt}(x + \tilde{Y}^{\tilde{u}}(t))$ and obtain a sufficient condition for (3.30)

$$\{\tilde{Y}^{\tilde{u}}(t)Z(t)\}_{t\in[0,T]} \text{ is a martingale under measure } \mathbb{P}.$$

By Ito's formula, we have

$$\begin{aligned} d\tilde{Y}_t^{\tilde{u}} Z_t &= \tilde{Y}_{t-}^{\tilde{u}} dZ_t + Z_{t-} d\tilde{Y}_t^{\tilde{u}} + d[\tilde{Y}^{\tilde{u}}, Z](t) \\ &= \tilde{Y}_{t-}^{\tilde{u}} dZ_t + Z_{t-}e^{-rt}\big((\mu_t - r_t)\tilde{\pi}_t + (p-a)L_t\big)dt \\ &\quad + Z_{t-}e^{-rt}(\sigma_t\tilde{\pi}_t - \rho b L_t)dW_t^{(1)} - Z_{t-}e^{-rt}b\sqrt{1-\rho^2}L_t dW_t^{(2)} \\ &\quad - Z_{t-}e^{-rt}\gamma L_t dN_t + \theta_1(t)e^{-rt}(\sigma_t\tilde{\pi}_t - \rho b L_t)dt \\ &\quad - \theta_2(t)e^{-rt}b\sqrt{1-\rho^2}L_t dt - \theta_3(t)e^{-rt}\gamma L_t dN_t. \end{aligned}$$

Then a necessary condition for $\tilde{Y}^{\tilde{u}}Z$ to be a $\mathbb{P}$-martingale is

$$Z_{t-}((\mu_t - r_t)\tilde{\pi}_t + (p-a)L_t - \lambda\gamma L_t) + \theta_1(t)(\sigma_t\tilde{\pi}_t - \rho b L_t) \\ - \theta_2(t)b\sqrt{1-\rho^2}L_t - \theta_3(t)\lambda\gamma L_t = 0.$$

By considering two admissible controls $(\tilde{\pi} = 1, L = 0)$ and $(\tilde{\pi} = 0, L = 1)$, we obtain

$$Z_{t-}(\mu(t) - r(t)) + \sigma(t)\theta_1(t) = 0 \ \Rightarrow \theta_1(t) = -\frac{\mu(t) - r(t)}{\sigma(t)} Z_{t-}. \tag{3.31}$$

$$Z_{t-}(p - a - \lambda\gamma) - \rho b\theta_1(t) - b\sqrt{1-\rho^2}\theta_2(t) - \lambda\gamma\theta_3(t) = 0. \tag{3.32}$$

Define $P(t) := \exp\{\int_0^t \xi(s)ds\}$, $t \in [0, T]$, where ξ is a deterministic function that will be determined later. Applying Ito's formula to $P_t Z_t$ gives

$$\begin{aligned} P_T Z_T &= Z_0 + \int_0^T P_t dZ_t + \int_0^T Z_{t-} dP_t \\ &= Z_0 + \int_0^T Z_{t-}\xi_t P_t dt - \int_0^T \frac{\mu_t - r_t}{\sigma_t} Z_{t-} P_t dW_t^{(1)} \\ &\quad + \int_0^T P_t\theta_2(t) dW_t^{(2)} + \int_0^T P_t\theta_3(t) dN_t - \int_0^T \lambda P_t \theta_3(t) dt. \end{aligned}$$

Recall the definition of Z_T, we obtain $X_T^{\tilde{u}^*} = \frac{1-Z_T}{\alpha} = \frac{1}{\alpha} - \frac{P_T Z_T}{\alpha P_T}$ and

$$\begin{aligned} X_T^{\tilde{u}^*} &= \frac{1}{\alpha} - \frac{Z_0}{\alpha P_T} - \frac{1}{\alpha P_T}\int_0^T Z_{t-}\xi_t P_t dt \\ &\quad + \frac{1}{\alpha P_T}\int_0^T \frac{\mu_t - r_t}{\sigma_t} Z_{t-} P_t dW_t^{(1)} - \frac{1}{\alpha P_T}\int_0^T P_t\theta_2(t) dW_t^{(2)} \\ &\quad - \frac{1}{\alpha P_T}\int_0^T P_t\theta_3(t) dN_t + \frac{1}{\alpha P_T}\int_0^T \lambda P_t\theta_3(t) dt. \end{aligned} \tag{3.33}$$

By substituting optimal control $\tilde{u}^*$ into the SDE (3.1), we solve to get

$$\begin{aligned} X_T^{\tilde{u}^*} &= xe^{rT} + \int_0^T e^{r(T-t)}\big(((\mu_s - r_s)\tilde{\pi}_s^* + (p-a)L_s^*\big)ds - \int_0^T e^{r(T-t)}\gamma L_t^* dN_t \\ &\quad + \int_0^T e^{r(T-t)}\left((\sigma_t\tilde{\pi}_t^* - \rho b L_t^*)dW_t^{(1)} - b\sqrt{1-\rho^2}L_t^* dW_t^{(2)}\right). \end{aligned} \tag{3.34}$$

Apparently, the above two expressions of $X_T^{\tilde{u}^*}$ should match, and hence

$$\begin{aligned} \frac{1}{\alpha}\frac{\mu(t)-r(t)}{\sigma(t)}\frac{P_t}{P_T}Z_{t-} &= e^{r(T-t)}(\sigma(t)\tilde{\pi}^*(t)-\rho bL^*(t)), \\ \frac{1}{\alpha}\frac{P_t}{P_T}\theta_2(t) &= e^{r(T-t)}b\sqrt{1-\rho^2}L^*(t), \\ \frac{1}{\alpha}\frac{P_t}{P_T}\theta_3(t) &= e^{r(T-t)}\gamma L^*(t). \end{aligned} \tag{3.35}$$

By (3.35), we can rearrange (3.33) as

$$\begin{aligned} X_T^{\tilde{u}^*} &= \frac{1}{\alpha} - \frac{Z_0}{\alpha P_T} - \frac{1}{\alpha P_T}\int_0^T Z_{t-}\xi_t P_t dt + \int_0^T e^{r(T-t)}\lambda\gamma L_t^* dt \\ &+ X_T^{\tilde{u}^*} - xe^{rT} - \int_0^T e^{r(T-t)}\big((\mu_t - r_t)\tilde{\pi}_t^* + (p-a)L_t^*\big)dt. \end{aligned} \tag{3.36}$$

From the systems of (3.31) and (3.32) along with the above conditions (3.35), we find optimal control as

$$\tilde{\pi}^*(t) = e^{-r(T-t)}\frac{1}{\alpha}\frac{\mu(t)-r(t)}{\sigma(t)^2}\frac{P_t}{P_T}Z_{t-} + \frac{\rho b}{\sigma(t)}L^*(t), \tag{3.37}$$

$$L^*(t) = e^{-r(T-t)}\frac{1}{\alpha}\frac{p-a-\lambda\gamma+\rho b\frac{\mu(t)-r(t)}{\sigma(t)}}{b^2(1-\rho^2)+\lambda\gamma^2}\frac{P_t}{P_T}Z_{t-}. \tag{3.38}$$

To ensure the equation (3.36) holds, we choose ξ to be

$$\xi(t) = -\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 - \varphi(t), \tag{3.39}$$

with φ defined by

$$\varphi(t) := \frac{\left(p-a-\lambda\gamma+\rho b\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2}{b^2(1-\rho^2)+\lambda\gamma^2},$$

and Z_0 as

$$Z_0 = (1-\alpha xe^{rT})P_T = (1-\alpha xe^{rT})\exp\left\{\int_0^T \xi(t)dt\right\}. \tag{3.40}$$

Now we substitute optimal L^* into (3.35) and obtain the expressions of θ_2 and

θ_3 in Z as

$$\theta_2(t) = b\sqrt{1-\rho^2}\Phi(t)Z_{t-}, \tag{3.41}$$

$$\theta_3(t) = \gamma\Phi(t)Z_{t-}, \tag{3.42}$$

where Φ is defined as

$$\Phi(t) := \frac{p - a - \lambda\gamma + \rho b\frac{\mu(t)-r(t)}{\sigma(t)}}{b^2(1-\rho^2)+\lambda\gamma^2}.$$

Therefore, we obtain the dynamics of Z as

$$dZ_t = Z_{t-}\left(-\frac{\mu_t - r_t}{\sigma_t}dW_t^{(1)} + b\sqrt{1-\rho^2}\Phi_t dW_t^{(2)} + \gamma\Phi_t dM_t\right),$$

which yields a unique solution

$$\begin{aligned} Z_t =& Z_0 \exp\left\{ -\int_0^t \left(\frac{1}{2}\left(\frac{\mu_s - r_s}{\sigma_s}\right)^2 + \frac{1}{2}b^2(1-\rho^2)\Phi_s^2 + \lambda\gamma\Phi_s\right) ds \right. \\ & \left. \int_0^t \left(-\frac{\mu_s - r_s}{\sigma_s}dW_s^{(1)} + b\sqrt{1-\rho^2}\Phi_s dW_s^{(2)} + \ln(1+\gamma\Phi_s)dN_s\right)\right\}, \end{aligned} \tag{3.43}$$

where Z_0 is given by (3.40).

Then we can rewrite optimal control in the following form

$$\tilde{\pi}^*(t) = \frac{1}{\alpha}e^{-r(T-t)}\left(\frac{\mu(t)-r(t)}{\sigma^2(t)} + \frac{\rho b}{\sigma(t)}\Phi(t)\right)\exp\left\{-\int_t^T \xi(s)ds\right\} Z_{t-}, \tag{3.44}$$

$$L^*(t) = \frac{1}{\alpha}e^{-r(T-t)}\Phi(t)\exp\left\{-\int_t^T \xi(s)ds\right\} Z_{t-}, \tag{3.45}$$

where ξ and Z are given by (3.39) and (3.43), respectively.

Theorem 3.4. *When* $U(x) = x - \frac{\alpha}{2}x^2$, $\alpha > 0$, *and the condition* (3.6) *holds,* $\tilde{u}^* = (\tilde{\pi}^*, L^*)$ *with* $\tilde{\pi}^*$ *and liabilities* L^* *given by* (3.44) *and* (3.45)*, respectively, is optimal investment to Problem 3.1 with the admissible control set* $\mathcal{A}_2$.

Proof. $\forall\, t \in [0,T]$, $\Phi(t)$ is a bounded deterministic function, so are $\frac{\theta_i}{Z_{t-}}$, with θ_i,

$i = 1, 2, 3$, defined by (3.31), (3.41) and (3.42), respectively. Hence Z, defined by (3.43), is indeed a square-integrable martingale. With our choices for ξ and Z_0, given by (3.39) and (3.40), we can verify that

$$-\frac{1}{\alpha P_T}\int_0^T Z_{t-}\xi_t P_t dt - \int_0^T e^{r(T-t)}\big((\mu_t - r_t)\tilde{\pi}_t^* + (p - a - \lambda\gamma)L_t^*\big)dt = 0,$$

and $\frac{1}{\alpha} - \frac{Z_0}{\alpha P_T} - xe^{rT} = 0$, which implies Z_T defined by (3.43) is equal to $1 - \alpha X_T^{\tilde{u}^*}$, with $X_T^{\tilde{u}^*}$ given by (3.34).

Provided $\tilde{u} \in \mathcal{A}_2$, $X^{\tilde{u}} \in L^2(\mathcal{F})$, then $\tilde{Y}Z \in L^2(\mathcal{F})$, which verifies $\tilde{Y}Z$ is indeed a martingale under $\mathbb{P}$. So the condition (3.30) holds.

In the last step, we show that $\tilde{u}^* = (\tilde{\pi}^*, L^*)$, with $\tilde{\pi}^*$ and L^* given by (3.44) and (3.45), is admissible. To that purpose, notice both $\Phi(t)$ and $\xi(t)$ are bounded for all $t \in [0, T]$. Hence, $\forall\, t \in [0, T]$, there exists a positive constant $\tilde{K}_t$ such that

$$\max\left\{(\tilde{\pi}^*(t))^2,\ (L^*(t))^2\right\} \le \tilde{K}_t\,(Z_{t-})^2.$$

Due to the fact that $Z \in L^2(\mathcal{F})$, we obtain $\tilde{\pi}^*$, $L^* \in L^2(\mathcal{F})$. When the condition (3.6) holds, $\Phi(t) \ge 0$ for all $t \in [0, T]$, which implies $L^*(t) \ge 0$, $\forall\, t \in [0, T]$. $\square$

3.6 Economic Analysis

In this section, we analyze the impact of the market parameters on the optimal policies in three cases: logarithmic utility function, power utility function and exponential utility function. To conduct the economic analysis, we assume the coefficients in the financial market are constants and select the market parameters as given in Table 3.1. Notice that in Table 3.1, three variables: ρ, γ and α, have not been assigned fixed values. We shall analyze the impact of those three variables on the optimal policies.

μ	r	σ	a	b	p	λ
0.05	0.01	0.25	0.08	0.1	0.15	0.1

Table 3.1: Market Parameters

First, we study the impact of ρ, the negative correlation coefficient between insurer's risk and stock returns, on the optimal policies. We fix $\gamma = 0.3$.

For the logarithmic utility function, the optimal policies of π^* and κ^* are given by (3.4) and (3.7), respectively. We draw the graph of optimal policies with respect to different values of ρ in Figure 3.1.

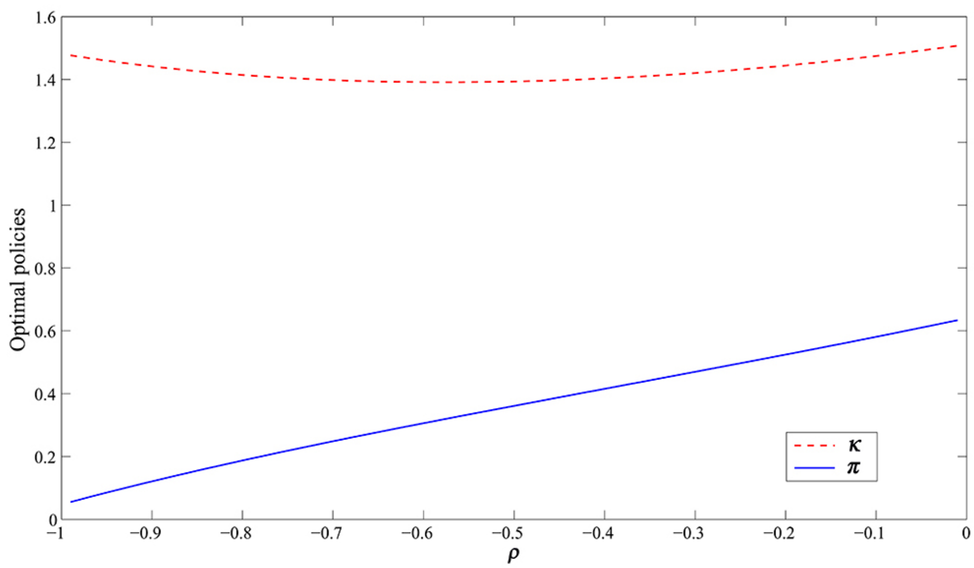

Figure 3.1: Impact of ρ on Optimal Policies When $U(x) = \ln(x)$

In the case of the power utility function, the optimal policies are obtained by (3.21) and (3.22). We notice that the optimal policies depend on α, the insurer's risk aversion parameter. The smaller the α, the higher the risk aversion. We divide all the insurers into two categories: high risk-averse insurers ($\alpha < 0$) and low risk-averse insurers ($0 < \alpha < 1$). For high risk-averse insurers, we pick $\alpha = -1$; for low risk-averse insurers, we consider $\alpha = 0.1$. The corresponding optimal policies are presented in Figure 3.2.

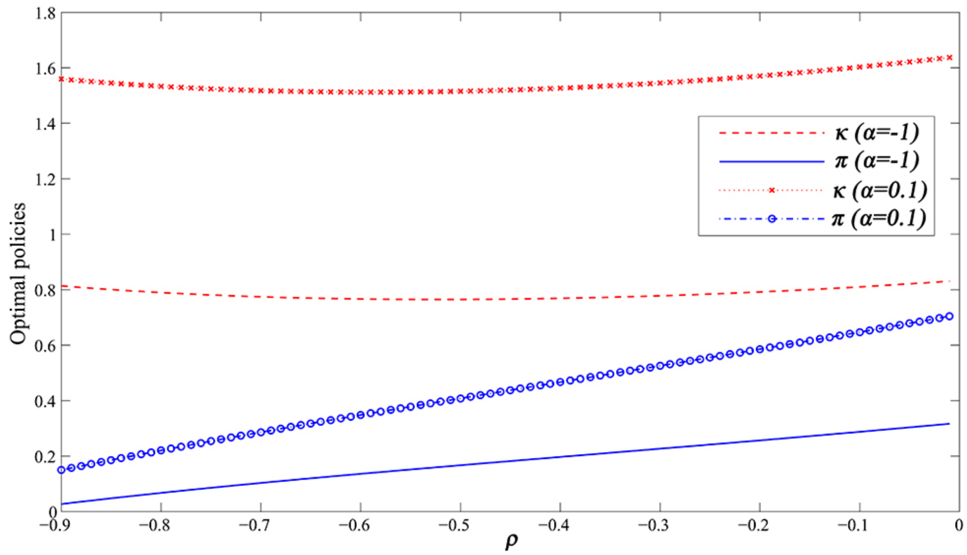

Figure 3.2: Impact of ρ on Optimal Policies When $U(x) = \frac{1}{\alpha}x^{\alpha}$

In the case of the exponential utility function, we obtain the optimal policies by (3.28) and (3.29). We notice that the optimal policies at time t depend on $T - t$, the duration to the terminal date. In the analysis below, we choose $T - t = 1$. We consider only risk averse insurers, so $\alpha > 0$. In this case, insurers with greater α are more risk averse. We draw the graph of optimal policies when $\alpha = 0.5$ and 1 in Figure 3.3.

Based on the results presented in Figure 3.1, Figure 3.2 and Figure 3.3, we observe that the optimal investment proportion π^* (or investment amount $\tilde{\pi}^*$ in the case of exponential utility) in the stock is an increasing function of ρ. However, the optimal liability ratio κ^* (or optimal liabilities L^* in the case of exponential utility) looks like a convex function of ρ. The explanation of this behavior comes from the equations of R and $\bar{W}$ in Section 2.1. We analyze first the case of power utility (logarithmic utility is a special case of power utility). For our parameter values, when ρ takes values around the middle of the interval $(-1, 0)$, there is a lot of uncertainty in the insurance market, and hence κ^* takes a minimum value in that

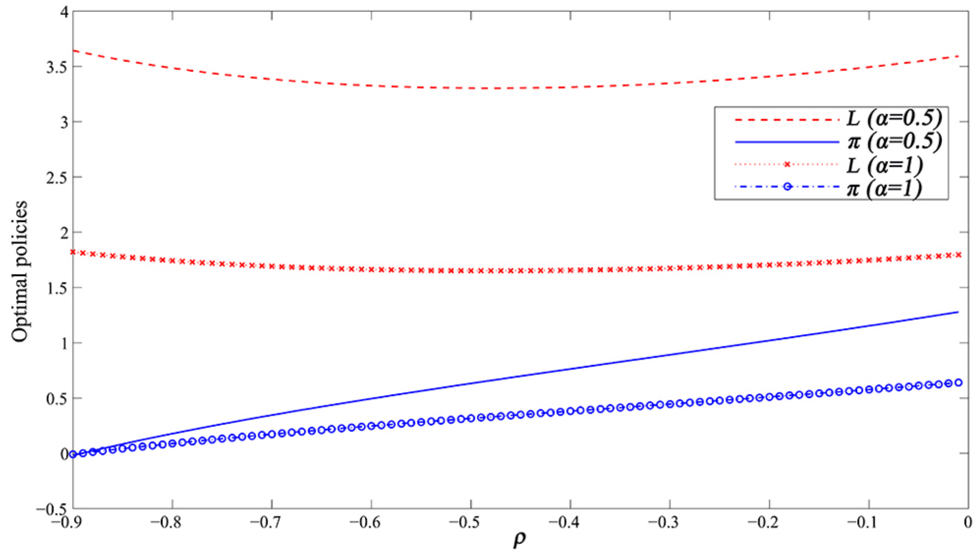

Figure 3.3: Impact of ρ on Optimal Policies When $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$

region. π^* is then calculated from equation (3.4) or (3.21). When ρ takes values near 0, there is little uncertainty in the insurance market, so κ^* takes a maximum value in that region. Furthermore, when ρ takes values near 0, the financial market and the insurance market are almost uncorrelated, so π^* takes a maximum value in that region, and indeed approaches the famous Merton proportion $\frac{\mu(t)-r(t)}{(1-\alpha)\sigma^2(t)}$. Similar comments can be made about the case of the exponential utility.

The second variable we analyze is γ, the jump intensity of the risk process. For this analysis, we fix $\rho = -0.5$ and consider $\gamma \in [0.2, 0.5]$, which includes the value 0.3 used in the previous analysis. We follow the same methodology used to analyze the impact of ρ on the optimal policies and obtain the results for logarithmic utility in Figure 3.4, for power utility in Figure 3.5, and for exponential utility in Figure 3.6.

From Figure 3.4, Figure 3.5 and Figure 3.6, we observe that the optimal investment proportion π^* (or investment amount $\tilde{\pi}^*$ in the case of exponential utility) in the stock is an increasing function of γ. These figures also show that the optimal

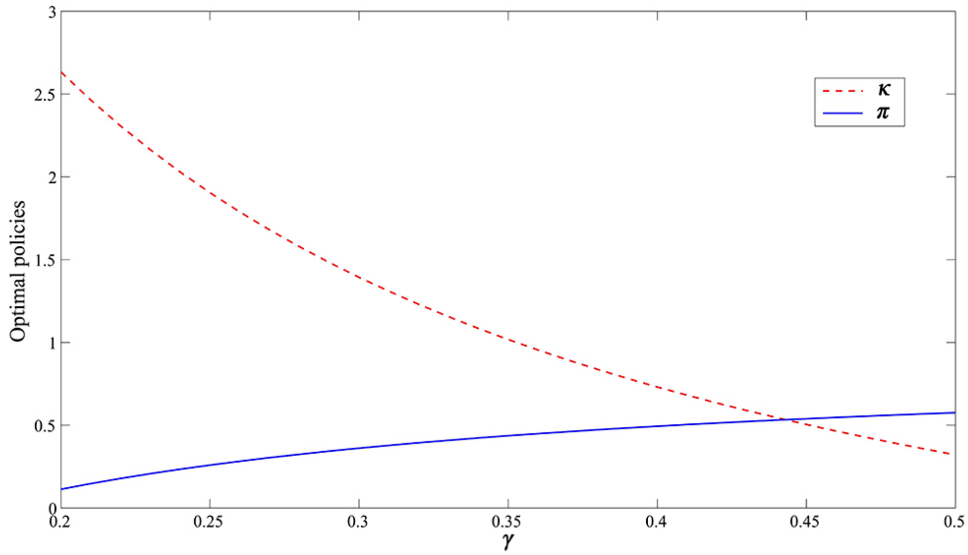

Figure 3.4: Impact of γ on Optimal Policies When $U(x) = \ln(x)$

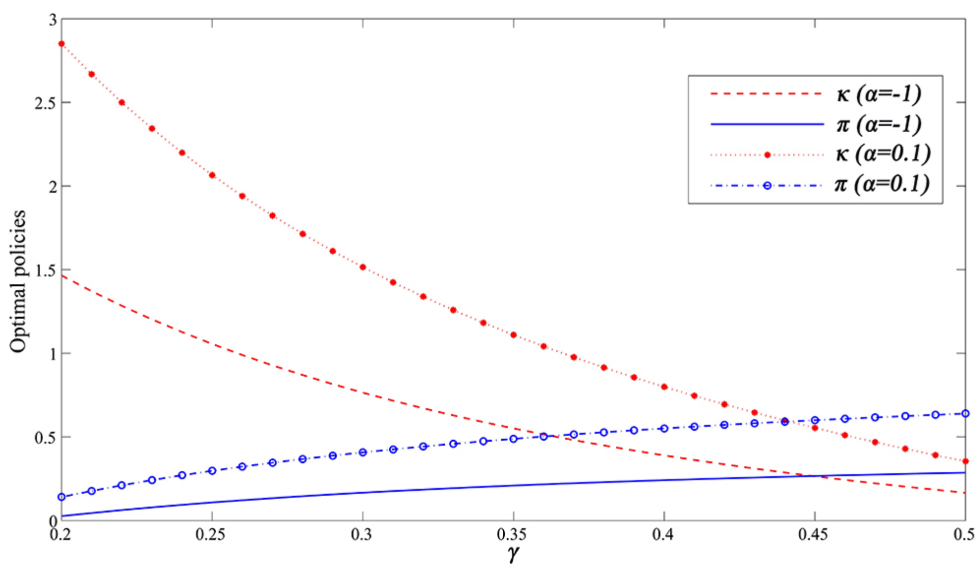

Figure 3.5: Impact of γ on Optimal Policies When $U(x) = \frac{1}{\alpha}x^{\alpha}$

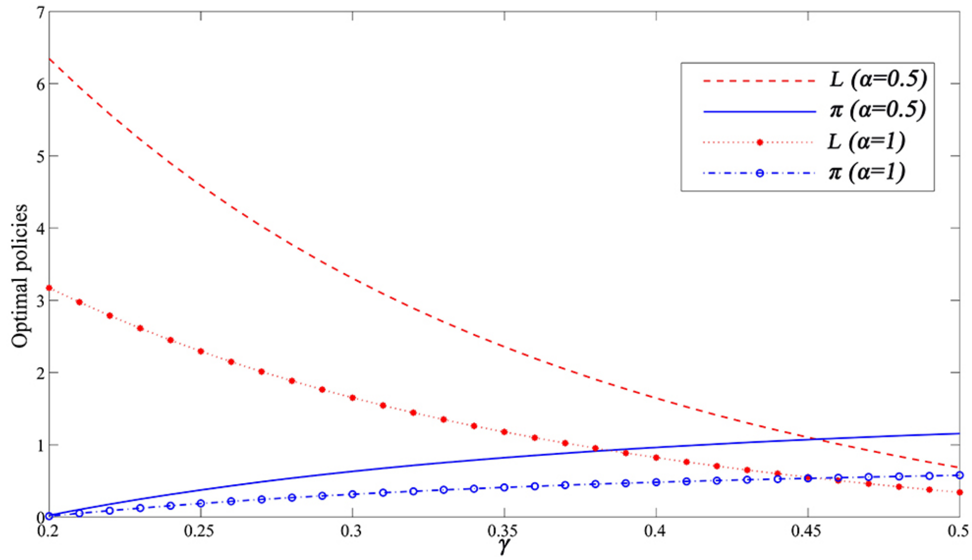

Figure 3.6: Impact of γ on Optimal Policies When $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$

liability ratio κ^* (or optimal liabilities L^* in the case of exponential utility) is a decreasing function of γ. This observation is supported by the economic interpretation of γ. In the modeling, greater γ means more risk for the insurers. Hence, when γ increases, a risk averse insurer reduces his/her optimal liability ratio (or optimal liabilities). From the optimal investment expressions (3.4), (3.21) and (3.28), we see that as γ increases, $\frac{\rho b}{\sigma(t)}\kappa^*$ (or $\frac{\rho b}{\sigma(t)}L^*$ in the exponential utility case) increases because $\rho < 0$. Hence when γ increases, π^* (or $\tilde{\pi}^*$) increases as well.

Last, we analyze the impact of α, the coefficient of risk aversion. We start with the power utility. From Figures 1 and 2 ($\alpha = 0$ in Figure 1), we see that for each fixed value of ρ, the larger the risk aversion (the lower the value of α), the lower the proportion invested in the risky asset (π^*) and the lower the liability ratio (κ^*). In Figures 4 and 5, we see that for each fixed value of γ, the larger the risk aversion (the lower the value of α), the lower the proportion invested in the risky asset (π^*) and the lower the liability ratio (κ^*). We consider now the exponential utility. In Figure 3, we see that for each fixed value of ρ, the larger the risk aversion (the higher the value of α), the lower the amount invested in the risky asset ($\tilde{\pi}^*$) and the lower the liabilities (L^*). In Figure 6, we see that for each fixed value of γ, the larger the risk aversion (the higher the value of α), the lower the amount invested in the risky asset ($\tilde{\pi}^*$) and the lower the liabilities (L^*). Summarizing, the impact of the risk aversion agrees with our intuition.

3.7 Concluding Remarks

Motivated by the bailout case of AIG in the financial crisis of 2007-2009 and the increasing demand on risk management in the insurance industry, we consider optimal investment and risk control problems for an insurer (like AIG). Since jump models can better capture the sudden spikes of risks (claims), we assume that the insurer's risk follows a jump-diffusion process in our model, which can be controlled proportionally by the insurer. As discussed in Stein [88, Chapter 6], one major mistake in AIG's modelling is missing the negative correlation between its liabilities (risk) and the capital gains in the financial market. Hence in our model, we assume the insurer's risk process is negatively correlated with the price process of the risky asset.

We consider an insurer who wants to maximize its expected utility of terminal wealth by selecting optimal investment and risk control (liabilities or liability ratio) strategies. We apply the martingale approach to solve our optimal control problem. We obtain explicit solutions of optimal investment and risk control strategies for logarithmic utility function, power utility function, exponential utility function and quadratic utility function, respectively.

Through an economic analysis, we investigate the impact of ρ (the negative correlation coefficient between insurer's risk and stock returns) and γ (jump intensity of insurer's risk) on optimal strategies. We observe that the optimal investment proportion (or optimal investment amount in the exponential utility case) in the stock is increasing with respect to both ρ and γ. The optimal liability ratio (or optimal liabilities in the exponential utility case) is a convex function of ρ and a decreasing function of γ. Furthermore, we find that insurers with high risk aversion invest a small proportion (small amount in the exponential utility case) in the stock and select a low liability ratio (or bear low liabilities in the exponential utility case).

Chapter 4

Optimal Investment and Risk Control Policies with Regime Switching

As agreed by most economists, the trigger of the 2007-2009 financial crisis is the crash of the housing market. But back at that time, most individual investors, companies, financial institutions and banks did not seriously take into account the business cycles in the U.S. housing market and made their decisions based on the false prediction of the housing price index. In the AIG case, AIG Financial Products Corp. (AIGFP), AIG's subsidiary, significantly underestimated the risk of writing CDS backed by mortgage payments. To manager the risk arising from business cycles, regime switching models should be applied (e.g., Sotomayor and Cadenillas [86], Zhou and Yin [101], Zou and Cadenillas [103] and the references therein for more details). Bauerle and Rieder [6] considered portfolio optimization problems in a regime switching market under the utility maximization criterion. In Sotomayor and Cadenillas [86], they further assumed the utility function is regime dependent and obtained explicit consumption and investment policies. Zhou and Yin [101] also studied Merton's problem in a regime switching model but under Markowitz's mean-variance criterion. Regime switching models can also be found in reinsurance problems, see, for instance, Liu et al. [58] and Zhuo et al. [102].

Motivated by the infamous AIG case discussed in Chapter 1, we propose a regime switching model which addresses several major mistakes AIG made in the financial crisis. We consider an insurer whose external risk (liabilities) can be modeled by a jump-diffusion process and assume that the insurer can control the risk process. The insurer makes investment decisions in a financial market which consists of a riskless asset and a risky asset. Following Stein [88, Chapter 6], we assume the insurer's risk process is negatively correlated with the price process of the risky asset. In our model, both the financial market and the risk process depend on the regime of the economy, see Section 2.1 for explanations. The objective of the insurer is to select the proportion of wealth invested in the risky asset and the liability ratio (which is defined as total liabilities over wealth) to maximize its expected utility of terminal wealth.

As far as we are concerned, this is by far the first work studying investment and liability ratio problems when there is regime switching in the economy. We also successfully obtain optimal investment and liability ratio policies in explicit forms for logarithmic utility, power utility and exponential utility. Stein [88, Chapter 6] considered a similar problem under the same criterion, but in a much simpler way compared with our work. First, he did not consider regime switching in the model. Second, the insurer does not control investment decisions and the risk is modeled by a diffusion process without jumps. Last, the only utility function considered in Stein [88, Chapter 6] is logarithmic function. Different from investment/consumption problems with regime switching, e.g., Bauerle and Rieder [6], Sotomayor and Cadenillas [86], Zhou and Yin [101], our model incorporates an external risk process. Our research also differs from recent work in reinsurance problems in several directions. For instance, in Liu et al. [58], the insurer's risk process is governed by a continuous diffusion process (without jumps) and is assumed to be independent of the price process of the securities. In Zhuo et al. [102], investment is not included and they only obtained numerical solutions in a regime switching model.

4.1 The Model

Comparing with the model in Section 3.1, the one used in this chapter presents regime switching features. The regime of the economy is represented by an observable, continuous and stationary Markov chain $\epsilon = \{\epsilon_t, 0 \leq t \leq T\}$ with finite state space $\mathcal{S} = \{1, 2, \ldots, S\}$, where $T \in (0, +\infty)$ is the terminal time and $S \in \mathbb{N}^+$ is the number of regimes in the economy. We assume the Markov Chain ϵ has a strongly irreducible generator $Q = (q_{ij})_{S\times S}$, where $\sum_{j\in\mathcal{S}} q_{ij} = 0$ for all $i \in \mathcal{S}$.

In the financial market, there are two trading assets, namely, a riskless asset and a risky asset. The price processes of the riskless asset and the risky asset are represented by P_0 and P_1, respectively, which satisfy the Markov-modulated stochastic differential equations:

$$dP_0(t) = r_{\epsilon(t)} P_0(t) dt,$$
$$dP_1(t) = P_1(t)(\mu_{\epsilon(t)} dt + \sigma_{\epsilon(t)} dW^{(1)}(t)),$$

where $t \in [0, T]$ and the initial conditions are $P_0(0) = 1$ and $P_1(0) > 0$. The coefficients r_i, μ_i, σ_i, $i \in \mathcal{S}$, are all positive constants, and $W^{(1)}$ is a standard one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume insurers are subject to a controllable external risk (liabilities). Following Wang et al. [92], we further assume the unit risk (risk per dollar amount of liabilities) is modelled by a jump-diffusion process

$$dR(t) = a_{\epsilon(t)} dt + b_{\epsilon(t)} d\bar{W}(t) + \gamma_{\epsilon(t)} dN(t),$$

where $\bar{W}$ is another standard one-dimensional Brownian motion and N is a Poisson process with constant intensity $\lambda > 0$. For all $i \in \mathcal{S}$, the coefficients a_i, b_i and γ_i

are positive constants. As discussed in Stein [88, Chapter 6], we assume the risk process R is negatively correlated with the capital gains in the financial market. Furthermore, we assume such negative correlation depends on the regime of the economy. Hence we can rewrite $\bar{W}$ as

$$\bar{W}(t) = \rho_{\epsilon(t)} W^{(1)}(t) + \sqrt{1 - \rho_{\epsilon(t)}^2} W^{(2)}(t),$$

where $\rho_i \in [-1, 0)$, $\forall i \in \mathcal{S}$, and $W^{(2)}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of $W^{(1)}$. We assume the unit premium (per dollar amount of liabilities) at time t is $p_{\epsilon(t)}$, where $p_i > 0$ for all $i \in \mathcal{S}$. Therefore, insurers' unit profit (loss if being negative) over the time period $(t, t + dt)$ is $p_{\epsilon(t)} dt - dR(t)$.

Denote the insurer's total liabilities at time t by $L(t)$. Then the dynamics of the insurer's total profit is given by

$$L(t) \left(p_{\epsilon(t)} dt - dR(t) \right).$$

Following Sotomayor and Cadenillas [86], we assume the Brownian motions $W^{(1)}$ and $W^{(2)}$, the Poisson process N and the Markov chain ϵ are mutually independent. We take the $\mathbb{P}-$augmented filtration generated by $W^{(1)}, W^{(2)}, N$ and ϵ as our filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$.

Remark 4.1. *The above model for risk process can be understood as a limiting process of the classic Cramér-Lundberg model in continuous time, see, e.g., Taksar [90], Wang et al. [92], Zhuo et al. [102].*

At time t, an insurer selects $\pi(t)$, fraction of wealth invested in the risky asset, and liability ratio $\kappa(t)$, defined as the ratio of total liabilities over wealth. Define control $u := \{(\pi(t), \kappa(t))\}_{t \in [0,T]}$. For any control u, we denote $X^u(t)$ as the insurer's wealth (surplus) at time t, and thus $L(t) = \kappa(t) X^u(t)$. Based on the above model setting, we have

$$dX^u(t) = r_{\epsilon(t)}(1-\pi(t))X^u(t)dt + \frac{\pi(t)X^u(t)}{P_1(t)}dP_1(t) + \kappa(t)X^u(t)(p_{\epsilon(t)}dt - dR(t)).$$

Hence the dynamics of $X^u(t)$ is given by

$$\begin{aligned}\frac{dX^u(t)}{X^u(t-)} &= (r_{\epsilon(t)} + (\mu_{\epsilon(t)} - r_{\epsilon(t)})\pi_t + (p_{\epsilon(t)} - a_{\epsilon(t)})\kappa_t)dt - \gamma_{\epsilon(t)}\kappa_t dN_t \\ &\quad + (\sigma_{\epsilon(t)}\pi_t - \rho_{\epsilon(t)}b_{\epsilon(t)}\kappa_t)dW_t^{(1)} - \sqrt{1-\rho_{\epsilon(t)}^2}\, b_{\epsilon(t)}\kappa_t dW_t^{(2)}, \qquad (4.1)\end{aligned}$$

with $X^u(0) = x > 0$.

We denote $\mathcal{A}_{x,i}$ as the set of all admissible controls under the initial conditions $X^u(0) = x$ and $\epsilon(0) = i$, where $x > 0$ and $i \in \mathcal{S}$. For all $u \in \mathcal{A}_{x,i}$, u is a predictable process with respect to the filtration $\{\mathcal{F}_t\}$, and satisfies

$$E\left[\int_0^T (\pi(t))^2 dt\right] < \infty, \; E\left[\int_0^T (\kappa(t))^2 dt\right] < \infty,$$

and

$$0 \le \kappa(t) < \frac{1}{\gamma_{\epsilon(t)}}, \text{ for all } t \in [0, T].$$

Since the coefficients of all dt, $dW^{(1)}$, $dW^{(2)}$ and dN terms are bounded almost surely for every $u \in \mathcal{A}_{x,i}$, by Oksendal and Sulem [72, Theorem 1.19], there exists a unique solution X_t^u to SDE (4.1) such that

$$E\left[|X_t^u|^2\right] < \infty \quad \text{for all } t \in [0, T].$$

We define the criterion function J by

$$J(x, i; u) := E_{x,i}\left[U(X^u(T))\right],$$

where the utility function U is strictly increasing and concave, and satisfies the linear growth condition. Notation $E_{x,i}$ means conditional expectation given $X^u(0) =$

x and $\epsilon(0) = i$ under the actual measure $\mathbb{P}$.

We then formulate optimal investment and liability ratio problem as follows.

Problem 4.1. *Select an admissible control* $u^* = (\pi^*, \kappa^*)$ *that attains the value function* V*, defined by*

$$V(x, i) := \sup_{u \in \mathcal{A}_{x,i}} J(x, i; u).$$

The control u^* *is called an optimal control or optimal strategies of investment and liability ratio.*

To apply stochastic control method to solve Problem 4.1, we consider a modified version of Problem 4.1 (See, e.g., Bauerle and Rieder [6], Fleming and Soner [30, Chapter III] for details)

$$V(t, x, i) := \sup_{u \in \mathcal{A}_{t,x,i}} E_{t,x,i}[U(X_T^u)],$$

where $\mathcal{A}_{t,x,i}$ is defined similarly as $\mathcal{A}_{x,i}$ except for the starting point of time being t instead of 0. The notation $E_{t,x,i}$ means conditional expectation under $X_t^u = x$ and $\epsilon(t) = i$.

4.2 The Analysis

Let $\psi(\cdot, \cdot, i)$ be a $C^{1,2}$ function for all $i \in \mathcal{S}$. Define operator $\mathcal{Q}_i^u$ by

$$\begin{aligned} \mathcal{Q}_i^u \psi := \ & \psi_t(t, x, i) + [r_i + (\mu_i - r_i)\pi + (p_i - a_i)\kappa] x \psi_x(t, x, i) \\ & + \frac{1}{2}[(\sigma_i \pi - \rho_i b_i \kappa)^2 + (1 - \rho_i^2) b_i^2 \kappa^2] x^2 \psi_{xx}(t, x, i), \end{aligned}$$

where $\pi \in \mathbb{R}$ and $0 \leq \kappa < \frac{1}{\gamma_i}$.

Theorem 4.1. *Suppose* $v(\cdot, \cdot, i) \in C^{1,2}$ *and* $v(t, \cdot, i)$ *be an increasing and concave function for all* $t \in [0, T]$ *and* $i \in \mathcal{S}$*. If* $v(t, x, i)$ *satisfies the Hamilton-Jacobi-*

Bellman equation

$$\sup_{u\in\mathbb{R}\times[0,\frac{1}{\gamma_i})}\Big\{\mathcal{Q}_i^u v(t,x,i)+\lambda[v(t,(1-\gamma_i\kappa)x,i)-v(t,x,i)]\Big\}=-\sum_{j\in\mathcal{S}}q_{ij}v(t,x,j) \tag{4.2}$$

and the boundary condition

$$v(T,x,i)=U(x) \tag{4.3}$$

for every $x>0$, $i\in\mathcal{S}$, and the control $u^*=(\pi^*,\kappa^*)$ defined by

$$u^*=\arg\sup_{u\in\mathbb{R}\times[0,\frac{1}{\gamma_i})}\Big\{\mathcal{Q}_i^u v(t,x,i)+\lambda[v(t,(1-\gamma_i\kappa)x,i)-v(t,x,i)]\Big\}$$

is admissible, then u^* is an optimal control to Problem 4.1 and $v(t,x,i)$ is the associated value function.

Proof. $\forall\, u\in\mathcal{A}_{t,x,i}$, by applying Markov-modulated Ito's formula (see, e.g., Sotomayor and Cadenillas [86]), we obtain

$$\begin{aligned}
v(w,X_w^u,\epsilon_w)=v(t,X_t^u,\epsilon_t)&+\int_t^w\left(\mathcal{Q}_{\epsilon(s)}^{u(s)}v(s,X_s^u,\epsilon_s)+\sum_{j\in\mathcal{S}}q_{\epsilon(s),j}v(s,X_s^u,j)\right)ds\\
&+\int_t^w X_{s-}^u v_x(s,X_{s-}^u,\epsilon_s)(\sigma_{\epsilon(s)}\pi_s-\rho_{\epsilon(s)}b_{\epsilon(s)}\kappa_s)dW_s^{(1)}\\
&-\int_t^w X_{s-}^u v_x(s,X_{s-}^u,\epsilon_s)\sqrt{1-\rho_{\epsilon(s)}^2}\,b_{\epsilon(s)}\kappa_s dW_s^{(2)}\\
&+\int_t^w\Big(v(s,(1-\gamma_{\epsilon(s)}\kappa_s)X_{s-}^u,\epsilon_s)-v(s,X_{s-}^u,\epsilon_s)\Big)dN_s+m_w^v,
\end{aligned}$$

where m^v is a square-integrable martingale and $m_0^v=0$.

For $u\in\mathcal{A}_{t,x,i}$, X^u, π and κ are all square integrable. By assumption, v_x is bounded on $[t,T]$. Hence we have

$$E_{t,x,i}\left[\int_t^w X^u_{s-}v_x(X^u_{s-},\epsilon_s)(\sigma_{\epsilon(s)}\pi_s - \rho_{\epsilon(s)}b_{\epsilon(s)}\kappa_s)dW^{(1)}_s\right] = 0,$$
$$E_{t,x,i}\left[\int_t^w X^u_{s-}v_x(X^u_{s-},\epsilon_s)\sqrt{1-\rho^2_{\epsilon(s)}}b_{\epsilon(s)}\kappa_s dW^{(2)}_s\right] = 0.$$

The function $v(s,(1-\gamma_{\epsilon(s)}\kappa_s)X^u_{s-},\epsilon_s) - v(s,X^u_{s-},\epsilon_s)$ is left continuous and bounded, thus

$$E_{t,x,i}\left[\int_t^w \Big(v(s,(1-\gamma_{\epsilon(s)}\kappa_s)X^u_{s-},\epsilon_s) - v(s,X^u_{s-},\epsilon_s)\Big)dM_s\right] = 0,$$

where M, defined as $M_t = N_t - \lambda t$, is the compensated Poisson process of N, and then a true martingale under measure $\mathbb{P}$.

Hence, taking conditional expectation for $v(w,X^u_w,\epsilon_w)$ yields

$$E_{t,x,i}\big[v(w,X^u_w,\epsilon_w)\big] = v(t,x,i) + \int_t^w \Big(\mathcal{Q}^u_i v(s,X^u_s,i) + \lambda[v(s,(1-\gamma_i\kappa)X^u_s,i) - v(s,X^u_s,i)]\Big)ds + \int_t^w \sum_{j\in\mathcal{S}} q_{\epsilon(s),j}v(s,X^u_s,j)ds,$$

which directly implies the HJB equation (4.2). Then v defined in Theorem 4.1 is the value function to modified Problem 4.1. Given u^* is admissible, u^* is an optimal control to Problem 4.1 (See, e.g., Fleming and Soner [30, Chapter III], Oksendal and Sulem [72, Chapter 3] for analysis). □

4.3 Construction of Explicit Solutions

In this section, we obtain explicit solutions to Problem 4.1 in a regime switching market. Our strategy is to conjecture a strictly increasing and strictly concave candidate for the value function and then obtain candidate for optimal control. We next verify that such candidate control is admissible and then indeed optimal.

To obtain candidate for optimal control, we separate the optimization problem in the HJB equation (4.2) into two sub optimization problems

$$\max_{\pi\in\mathbb{R}}\left[xv_x(t,x,i)(\mu_i-r_i)\pi+\frac{1}{2}x^2v_{xx}(t,x,i)(\sigma_i^2\pi^2-2\rho_i\sigma_ib_i\pi\kappa)\right]$$

for investment portfolio π, and

$$\max_{\kappa\in[0,\frac{1}{\gamma_i})}\Big\{xv_x(t,x,i)(p_i-a_i)\kappa+\frac{1}{2}x^2v_{xx}(t,x,i)(-2\rho_i\sigma_ib_i\pi\kappa+b_i^2\kappa^2)\\ +\lambda v(t,(1-\gamma_i\kappa)x,i)\Big\}$$

for liability ratio κ.

Under the assumption that $v(t,\cdot,i)$ is strictly increasing and strictly concave, we obtain the candidate of π^* as

$$\pi^*=-\frac{v_x(t,x,i)(\mu_i-r_i)}{xv_{xx}(t,x,i)\sigma_i^2}+\rho_i\frac{b_i}{\sigma_i}\kappa^*, \tag{4.4}$$

where the candidate of κ^* satisfies

$$\begin{aligned} &xv_{xx}(t,x,i)(1-\rho_i^2)b_i^2\kappa^*-\lambda\gamma_iv_x(t,(1-\gamma_i\kappa^*)x,i)\\ &+v_x(t,x,i)(p_i-a_i+\rho_i\frac{b_i}{\sigma_i}(\mu_i-r_i))=0. \end{aligned} \tag{4.5}$$

To guarantee the equation (4.5) has a unique solution, we impose a technical condition

$$p_i-a_i+\rho_i\frac{b_i}{\sigma_i}(\mu_i-r_i)>\lambda\gamma_i \text{ for all } i\in\mathcal{S}. \tag{4.6}$$

To obtain the value function v and optimal control $u^*=(\pi^*,\kappa^*)$ in explicit forms, we consider three utility functions:

1. $U(x)=\ln(x)$, $x>0$,

2. $U(x)=\frac{1}{\alpha}x^\alpha$, $x>0$, where $\alpha<1$ and $\alpha\neq 0$,

3. $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$.

4.3.1 $U(x) = \ln(x),\ x > 0$

In this case, we find the solution to the HJB equation (4.2) is

$$v(t,x,i) = \ln(x) + g(t,i),$$

where $g(t,i)$ will be determined below.

We obtain $v_x(t,x,i) = \frac{1}{x}$ and $v_{xx}(t,x,i) = -\frac{1}{x^2}$, then

$$\pi^* = \frac{\mu_i - r_i}{\sigma_i^2} + \rho_i \frac{b_i}{\sigma_i}\kappa^*, \tag{4.7}$$

and

$$A_i(\kappa^*)^2 - B_i\,\kappa^* + C_i = 0, \tag{4.8}$$

with

$$\begin{aligned} A_i &:= (1-\rho_i^2)b_i^2\gamma_i, \\ B_i &:= (1-\rho_i^2)b_i^2 + \gamma_i\left(p_i - a_i + \rho_i\frac{b_i}{\sigma_i}(\mu_i - r_i)\right), \\ C_i &:= p_i - a_i + \rho_i\frac{b_i}{\sigma_i}(\mu_i - r_i) - \lambda\gamma_i. \end{aligned} \tag{4.9}$$

Lemma 4.1. *If the technical condition (4.6) holds, or equivalently, $C_i > 0$ for all $i \in \mathcal{S}$, then there exists a unique solution in $[0, \frac{1}{\gamma_i})$ to the equation (4.8).*

Proof. We calculate the discriminant of the quadratic equation system (4.8)

$$\Delta_i := B_i^2 - 4A_iC_i = (b_i^2(1-\rho_i^2) - \gamma_i(C_i + \lambda\gamma_i))^2 + 4\lambda b_i^2(1-\rho_i^2)\gamma_i^2 > 0.$$

Therefore there are two solutions to the equation (4.8), and one of them is given by

$$\kappa_i^+ = \frac{B_i + \sqrt{\Delta_i}}{2A_i} > \frac{1}{\gamma_i},$$

while the other is given by

$$\kappa_i^- = \frac{B_i - \sqrt{\Delta_i}}{2A_i}. \tag{4.10}$$

For all $i \in \mathcal{S}$, $A_i > 0$. Along with the assumption $C_i > 0$ and the result $\kappa_i^+ > \frac{1}{\gamma_i} > 0$, we obtain $\kappa_i^- > 0$.

To show $\kappa_i^- < \frac{1}{\gamma_i}$, it is equivalent to show that

$$\Delta_i > \left(B_i - \frac{2A_i}{\gamma_i}\right)^2 = (\gamma_i(C_i + \lambda\gamma_i) - (1-\rho_i^2)b_i^2)^2.$$

Recall the definition of Δ_i, the above inequality is always satisfied.

Denote $\kappa^*(x,i) := \kappa_i^-$ as the unique solution in $[0, \frac{1}{\gamma_i})$ to the equation (4.8) when $C_i > 0$ for all $i \in \mathcal{S}$. □

By substituting candidate strategies π^* and κ^*, given by (4.7) and (4.10), into the HJB equation (4.2), we obtain the following system of linear differential equations:

$$g_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij} g(t,j) + \Pi_i = 0 \tag{4.11}$$

with boundary condition

$$g(T,i) = 0 \text{ for all } i \in \mathcal{S}.$$

In (4.11), Π_i is defined as $\Pi_i := r_i + \frac{(\mu_i - r_i)^2}{2\sigma_i^2} + \left(p_i - a_i + \rho_i \frac{b_i}{\sigma_i}(\mu_i - r_i)\right)\kappa^* + \lambda \ln(1 - \gamma_i\kappa^*) - \frac{1}{2}(1-\rho_i^2)b_i^2(\kappa^*)^2$.

Theorem 4.2. *When $U(x) = \ln(x)$, $u^* = (\pi^*, \kappa^*)$, where*

$$\pi^*(t) = \frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{\sigma_{\epsilon(t)}^2} + \rho_{\epsilon(t)} \frac{b_{\epsilon(t)}}{\sigma_{\epsilon(t)}} \kappa^*(t),$$

and $\kappa^*(t)$ is the unique solution to the equation

$$A_{\epsilon(t)}(\kappa^*(t))^2 - B_{\epsilon(t)}\kappa^*(t) + C_{\epsilon(t)} = 0,$$

is an optimal control to Problem 4.1.

Proof. First notice that the ODE system (4.11) has a unique solution (See Bauerle and Rieder [6]) and then $v(t,x,i) = \ln(x) + g(t,i)$ is the value function to modified Problem 4.1. By Lemma 4.1, there exists a unique solution in $[0, \frac{1}{\gamma_{\epsilon(t)}})$ to the equation (4.8). The boundedness of π^* and κ^* implies that both are square integrable in $[0,T]$. Hence, $u^* = (\pi^*, \kappa^*)$ defined above is admissible, and then an optimal control to Problem 4.1. □

4.3.2 $U(x) = \frac{1}{\alpha}x^\alpha,\ x > 0,$ where $\alpha < 1$ and $\alpha \neq 0$

In this case, the utility function is given by $U(x) = \frac{1}{\alpha}x^\alpha$, where $\alpha < 1$, $\alpha \neq 0$. This utility function is of constant relative risk aversion (CRRA) type and the relative risk aversion coefficient is $1-\alpha$.

The solution to the HJB (4.2) is given by

$$v(t,x,i) = \frac{1}{\alpha}x^\alpha \cdot \hat{g}(t,i),$$

where $\hat{g}(t,i) > 0$ for all $i \in \mathcal{S}$ will be determined below.

Next, we obtain the candidate for optimal control

$$\pi^* = \frac{\mu_i - r_i}{(1-\alpha)\sigma_i^2} + \rho_i \frac{b_i}{\sigma_i}\kappa^*, \tag{4.12}$$

and

$$(\alpha-1)(1-\rho_i^2)b_i^2\kappa^* - \lambda\gamma_i(1-\gamma_i\kappa^*)^{\alpha-1} + p_i - a_i + \rho_i\frac{b_i}{\sigma_i}(\mu_i - r_i) = 0. \tag{4.13}$$

Lemma 4.2. *If the condition (4.6) holds, then there exists a unique solution in $[0, \frac{1}{\gamma_i})$ to the equation (4.13).*

Proof. Please see Lemma 3.3 for proof. □

By plugging candidate control into the HJB equation (4.2), we obtain

$$\hat{g}_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij}\hat{g}(t,j) + \alpha\,\hat{\Pi}_i\,\hat{g}(t,i) = 0 \tag{4.14}$$

with the boundary condition

$$\hat{g}(T,i) = 1 \text{ for all } i \in \mathcal{S}.$$

Here $\hat{\Pi}_i$ is defined by $\hat{\Pi}_i := r_i + \frac{(\mu_i - r_i)^2}{2(1-\alpha)\sigma_i^2} + \left(p_i - a_i + \rho_i \frac{b_i}{\sigma_i}(\mu_i - r_i)\right)\kappa^* + \lambda[(1 - \gamma_i\kappa^*)^\alpha - 1] - \frac{1}{2}(1-\alpha)(1-\rho_i^2)b_i^2(\kappa^*)^2$.

We remark that the above ODE system has a unique solution. Furthermore, to verify our conjecture that $v(t,\cdot,i)$ is strictly increasing and concave, we need to show $\hat{g}(t,i)$ is strictly positive for all $i \in \mathcal{S}$. The lemma below provides the proof for $\hat{g}(t,i) > 0$.

Lemma 4.3. *The function $\hat{g}(t,i)$, which is the unique solution to the system (4.14), is strictly positive.*

Proof. Using Ito's formula for Markov-modulated process, we obtain

$$\hat{g}(T,\epsilon_T) = \hat{g}(t,\epsilon_t) + \int_t^T \hat{g}_t(s,\epsilon_s)ds + \int_t^t \sum_{j\in\mathcal{S}} q_{\epsilon_s,j}\hat{g}(s,j)ds + m_T^{\hat{g}},$$

where $m^{\hat{g}}$ is a square integrable martingale with $E[m_t^{\hat{g}}] = 0$ for all $t \in [0,T]$.

Taking conditional expectation and using the equation (4.14), we get

$$E_{t,x,i}[\hat{g}(T,\epsilon_T)] = \hat{g}(t,\epsilon_t) - E_{t,x,i}\left[\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}\hat{g}(s,\epsilon_s)ds\right],$$

which is equivalent to (Recall the boundary condition $\hat{g}(T,i) = 1$)

$$\hat{g}(t,i) = 1 + E_{t,x,i}\left[\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}\hat{g}(s,\epsilon_s)ds\right].$$

Solving the above equation yields

$$\hat{g}(t,i) = E_{t,x,i}\left[\exp\left\{\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}ds\right\}\right].$$

Hence, the positiveness of $\hat{g}(t,i)$ follows. $\square$

From the construction of $\hat{g}(t,i)$ and Lemma 4.3, $v(t,x,i) = \frac{1}{\alpha}x^\alpha \cdot \hat{g}(t,i)$ is the associated value function to the modified Problem 4.1. Thanks to Lemma 4.2, $u^* = (\pi^*, \kappa^*)$, with π^* and κ^* given by (4.12) and (4.13), is admissible. Hence Theorem 4.3 follows accordingly.

Theorem 4.3. *When $U(x) = \frac{1}{\alpha}x^\alpha$, where $\alpha < 1$ and $\alpha \neq 0$, an optimal control to Problem 4.1 is given by $u^* = (\pi^*, \kappa^*)$, where*

$$\pi^*(t) = \frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{(1-\alpha)\sigma^2_{\epsilon(t)}} + \rho_{\epsilon(t)}\frac{b_{\epsilon(t)}}{\sigma_{\epsilon(t)}}\kappa^*(t),$$

and $\kappa^*(t)$ is the unique solution to the equation

$$(\alpha-1)(1-\rho^2_{\epsilon(t)})b^2_{\epsilon(t)}\kappa^*(t) - \lambda\gamma_{\epsilon(t)}(1-\gamma_{\epsilon(t)}\kappa^*(t))^{\alpha-1} + p_{\epsilon(t)} - a_{\epsilon(t)}$$
$$+ \rho_{\epsilon(t)}\frac{b_{\epsilon(t)}}{\sigma_{\epsilon(t)}}(\mu_{\epsilon(t)} - r_{\epsilon(t)}) = 0.$$

4.3.3 $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$

In this subsection, we consider exponential utility function, which belongs to the class of constant absolute risk aversion (CARA) utility functions. We find the solution to the HJB (4.2) is of the form

$$v(t,x,i) = -\frac{1}{\alpha}\exp\left\{-\alpha e^{r_i(T-t)}x + \tilde{g}(t,i)\right\},$$

where $\tilde{g}(t,i)$ will be determined below.

For the above solution, we calculate that

$$
\begin{aligned}
v_t(t,x,i) &= \left[\alpha\, r_i x e^{r_i(T-t)} + \tilde{g}_t(t,i)\right] v(t,x,i), \\
v_x(t,x,i) &= -\alpha\, e^{r_i(T-t)} v(t,x,i), \\
v_{xx}(t,x,i) &= \alpha^2\, e^{2r_i(T-t)} v(t,x,i).
\end{aligned}
$$

Hence, we obtain the candidate for π^*

$$
\pi^* = e^{-r_i(T-t)} \frac{\mu_i - r_i}{\alpha x \sigma_i^2} + \rho_i \frac{b_i}{\sigma_i} \kappa^*. \tag{4.15}
$$

Apparently, in this case, it is more convenient to use the actual amount instead of the proportion as the control, see, e.g., Browne [10], Wang et al. [92], Yang and Zhang [94]. We then define $\theta(t)$ as the amount of money invested in the risky asset and $L(t)$ as the total liabilities at time t. By definition, we have $\tilde{u} := (\theta, L) = X^u \cdot u$. If both θ and L are predictable and square integrable in $[t,T]$, then (θ, L) is an admissible control. Denote $\tilde{\mathcal{A}}_{x,i}$ as the admissible set given $X(0) = x, \epsilon(0) = i$.

By (4.15) and (4.5), we obtain the candidate for θ^*

$$
\theta^* = e^{-r_i(T-t)} \frac{\mu_i - r_i}{\alpha \sigma_i^2} + \rho_i \frac{b_i}{\sigma_i} L^*, \tag{4.16}
$$

and the candidate for L^*, which satisfies

$$
\lambda \gamma_i e^{\tilde{A}_i L^*} + \tilde{B}_i L^* - \tilde{C}_i = 0, \tag{4.17}
$$

where

$$
\begin{aligned}
\tilde{A}_i &:= \alpha \gamma_i e^{r_i(T-t)}, \\
\tilde{B}_i &:= \alpha e^{r_i(T-t)} b_i^2 (1 - \rho_i^2), \\
\tilde{C}_i &:= p_i - a_i + \rho_i b_i \frac{\mu_i - r_i}{\sigma_i}.
\end{aligned}
$$

Lemma 4.4. *If the condition (4.6) holds, then there exists a unique solution to the equation (4.17).*

Proof. Please refer to Lemma 3.4 for a similar proof. □

Next, we rewrite the HJB equation (4.2) as follows

$$\tilde{g}_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij} \exp\left\{-\alpha x e^{(r_j-r_i)(T-t)}\right\} e^{\tilde{g}(t,j)-\tilde{g}(t,i)} + \tilde{\Pi}_i = 0, \tag{4.18}$$

where $\tilde{\Pi}_i := -\alpha e^{r_i(T-t)}[(\mu_i-r_i)\theta^*+(p_i-a_i)L^*]+\frac{1}{2}\alpha^2 e^{2r_i(T-t)}[\sigma_i^2(\theta^*)^2-2\rho_i b_i\sigma_i\theta^* L^*+ b_i^2(L^*)^2] + \lambda(\exp\left(\alpha\gamma_i e^{r_i(T-t)}L^*\right) - 1)$.

Let $\tilde{q}_{ij} := q_{ij}\exp\left\{-\alpha x e^{(r_j-r_i)(T-t)}\right\}$ and $\Phi(t,i) := \exp\{\tilde{g}(t,i)\}$. Then equation (4.18) becomes

$$\Phi_t(t,i) + \sum_{j\in\mathcal{S}} \tilde{q}_{ij}\Phi(t,j) + \tilde{\Pi}_i\Phi(t,i) = 0,$$

which, similar to the system (4.14), bears a unique solution. Hence, there exists a unique solution $\tilde{g}(t,i)$ to the system (4.18).

By Lemma 4.4, the solution L^* to the equation (4.17) is finite for all $i\in\mathcal{S}$, which implies L^* is finite and square integrable on $[0,T]$. Hence, by (4.15), θ^* is also finite and square integrable on $[0,T]$. In conclusion, (θ^*, L^*) is admissible, and then indeed an optimal control to Problem 4.1 over the admissible set $\tilde{\mathcal{A}}_{x,i}$.

Theorem 4.4. *When utility function is $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$, an optimal control to Problem 4.1 over the admissible set $\tilde{\mathcal{A}}_{x,i}$ is $\tilde{u}^*(t) := (\theta^*(t), L^*(t))$, where*

$$\theta^*(t) = e^{-r_{\epsilon(t)}(T-t)}\frac{\mu_{\epsilon(t)} - r_{\epsilon(t)}}{\alpha\sigma^2_{\epsilon(t)}} + \rho_{\epsilon(t)}\frac{b_{\epsilon(t)}}{\sigma_{\epsilon(t)}}L^*(t),$$

and $L^*(t)$ is the unique solution to the equation

$$\lambda\gamma_{\epsilon(t)}\exp\{\tilde{A}_{\epsilon(t)}L^*(t)\} + \tilde{B}_{\epsilon(t)}L^*(t) - \tilde{C}_{\epsilon(t)} = 0.$$

4.4 Economic Analysis

In this section, we study the impact of the economy and the insurer's risk attitude on optimal policy. To this purpose, we assume there are two regimes in the economy. Regime 1 represents a bull market, in which the economy is booming. Regime 2 represents a bear market, meaning the economy is in recession. For comparative analysis, we consider HARA utility functions, namely, $U(x) = \frac{1}{\alpha}x^{\alpha}$, where $\alpha < 1$ ($\alpha = 0$ is associated with the case of logarithmic utility function $U(x) = \ln(x)$). When $\alpha < 0$, insurers are high risk-averse, when $\alpha = 0$, insurers are moderate risk-averse, when $0 < \alpha < 1$, insurers are low risk-averse.

Following Fama and French [28], we assume $\mu_i > r_i > 0$ and $p_i > a_i > 0$, $i = 1, 2$. French et al. [31] find that capital returns are higher in a bull market, hence we assume $\mu_1 > \mu_2$ and $r_1 > r_2$. Hamilton and Lin [39] show that the stock volatility is greater when the economy is in recession, which implies $\sigma_1 < \sigma_2$. Furthermore, we assume $\frac{\mu_1 - r_1}{\sigma_1^2} > \frac{\mu_2 - r_2}{\sigma_2^2}$, as supported by French et al. [31]. In the insurance market, the risk process (claims) is negatively correlated with the stock returns and interest rate, see, e.g., Haley [37], Norden and Weber [71]. This conclusion leads to the assumption that $a_2 > a_1$, $b_2 > b_1$, and $\gamma_2 > \gamma_1$. When the economy is in recession, the insurance companies charge a higher premium, hence $p_2 > p_1$. In the analysis, we assume that ρ is same in both regimes ($\rho_1 = \rho_2$). We also notice that the coefficient we choose should satisfy the technical condition (4.6). Based on the above argument, we choose the parameters and list in Table 4.1.

Regime	μ	r	σ	a	b	γ	p	ρ	λ
1 (bull market)	0.1	0.03	0.15	0.04	0.05	0.2	0.1	-0.5	0.01
2 (bear market)	0.05	0.01	0.25	0.08	0.1	0.5	0.2		

Table 4.1: Market Parameters

By Theorem 4.2, we calculate the optimal policy for moderate risk-averse insurers (that is, $\alpha = 0$). For both high risk-averse and low risk-averse insurers, we obtain the corresponding optimal policy through Theorem 4.3. The results are listed in Table 4.2.

α	Regime	π^* (Investment)	κ^* (Liability Ratio)
-5	1	0.2211	1.7843
	2	-0.0407	0.7370
-2	1	0.5364	3.0040
	2	-0.0308	1.2207
-1	1	0.9240	3.7895
	2	0.0148	1.5262
-0.01	1	2.2906	4.7383
	2	0.2548	1.8945
0	1	2.3200	4.7464
	2	0.2605	1.8977
0.01	1	2.3501	4.7544
	2	0.2663	1.9009
0.1	1	2.6532	4.8218
	2	0.3256	1.9275
0.2	1	3.0747	4.8854
	2	0.4094	1.9529
0.5	1	5.3906	4.9895
	2	0.8809	1.9954
0.7	1	9.5371	4.9999
	2	1.7333	1.9990

Table 4.2: Impact of α on Optimal Policies

According to the results obtained in Table 4.2, we observe that both the optimal investment proportion in the risky asset π^* and the optimal liability ratio κ^* are increasing functions of the risk aversion parameter α. Hence less risk-averse insurers (that is, insurers with large α) invest proportionally more in the risky asset and choose a higher liability ratio.

As pointed out in Stein [88, Chapter 6], a major mistake that contributed significantly to AIG's sudden collapse is the negligence of the negative correlation between the risk and the capital returns (or equivalently, $\rho < 0$). Hence in the next analysis, we calculate the optimal policy for different values of ρ. We still keep all the other parameters unchanged as in Table 4.1, but consider $\rho = -0.9$, -0.5, -0.2. We obtain optimal strategies in Table 4.3.

ρ	α	Regime	π^* (Investment)	κ^* (Liability Ratio)
-0.9	-1	1	0.4122	3.8112
		2	-0.2388	1.5521
	0	1	1.6927	4.7279
		2	-0.0447	1.9019
	0.5	1	4.7264	4.9860
		2	0.5617	1.9954
-0.5	-1	1	0.9240	3.7895
		2	0.0148	1.5262
	0	1	2.3200	4.7464
		2	0.2605	1.8977
	0.5	1	5.3906	4.9895
		2	0.8809	1.9954
-0.2	-1	1	1.2998	3.8363
		2	0.1982	1.5221
	0	1	2.7930	4.7721
		2	0.4881	1.8986
	0.5	1	5.8894	4.9918
		2	1.1203	1.9957

Table 4.3: Impact of ρ on Optimal Policies

Based on the results in Table 4.3, we find the optimal proportion invested in the risky asset π^* is an increasing function of ρ. However, the relation between κ^* and ρ is more complicated, some show convexity while other show monotonicity. Those observations can be seen in the case of no regime switching, see Figures 3.1, 3.2, and 3.3. Hence the explanations made there shall apply here as well.

Furthermore, the dependency of the optimal policy on the regime of the economy is evident. We notice from Table 4.2 and Table 4.3 that $\pi_1^* > \pi_2^*$ and $\kappa_1^* > \kappa_2^*$ for all insurers (all α). This result shows that all insurers take more risk in a bull market by spending a greater proportion on the risky asset and selecting a higher liability ratio.

4.5 Concluding Remarks

The 2007-2009 financial crisis brought new challenges on risk management to all market participants. There are two major contributors to AIG's sudden collapse. First, AIG did not pay full attention to the business cycles in the U.S. housing market, which directly caused a significant underestimation of the risk involved in the CDS trading. Second, AIG ignored the negative correlation between its liabilities and the capital gains in the financial market. Such ignorance indicates that AIG was not fully aware of the impact of derivatives trading on its capital structure.

To address these two problems in the AIG case, we set up a regime switching model from an insurer's perspective and assume not only the financial market but also the insurer's risk process depend on the regime of the economy. An insurer makes investment decisions in a financial market which consists of a riskless asset and a risky asset, and faces an external risk that is negatively correlated with the price of the risky asset. The insurer wants to maximize its expected utility of terminal wealth by selecting optimal investment proportion in the risky asset and liability ratio simultaneously. We obtain explicit solutions of optimal investment and liability ratio policies when the insurer's utility is given by logarithmic, power and exponential utility functions.

Through an economic analysis, we find the optimal policy depends on the regime of the economy. All insurers spend a greater proportion in the risky asset and choose a higher liability ratio when the market is in bull regime. We also observe that the optimal proportion invested in the risky asset is increasing with respect to both α and ρ. In the meantime, the optimal liability ratio also increases when α increases, but its relation with ρ is not monotone.

Chapter 5

Conclusions

The financial crisis of 2007-2009 caused severe recession in the global economy, and brought new challenges on portfolio selection and risk management. Motivated by the financial crisis of 2007-2009, we incorporate new features into market modeling. One major mistake in the financial crisis is the ignorance of business cycles in the economy. Hence, we include regime switching in our model, which is modeled by an observable continuous-time Markov chain. We apply stochastic control theory to study optimal insurance and investment problems.

We first incorporate the presence of an insurance risk and regime switching of economy into Merton's consumption/investment framework. We seek to find optimal consumption, investment and insurance policies for an investor who wants to maximize his/her expected total discounted utility of consumption over an infinite time horizon. We provide rigourous proof to the verification theorems and obtain explicit solutions to the associated Hamilton-Jacobi-Bellman equations in the case of HARA utility functions. We conduct an economic analysis to study the impact of market parameters and investor's risk preference on optimal policies. The advantage of buying insurance is also calculated.

Next, we consider an insurer who makes investment decisions and controls liabilities dynamically in a regime switching economy. The objective of the insurer is to select optimal investment and risk control policies that maximize his/her expected utility of terminal wealth. In the special case when there is only one regime in the economy (no regime switching), we apply the martingale approach to obtain explicit optimal policies for many important utility functions. When there is regime switching in the economy, classical dynamic programming is used to derive the Hamilton-Jacobi-Bellman (HJB) equation. We obtain explicit optimal policies by solving the HJB equation. Economic analysis is provided to study the impact of economic factors and investor's risk preference on optimal policies in both cases.

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