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Fractional Order Transmission Line Modeling and Parameter Identification

by

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To my wonderful parents and my beloved wife Moutushi

Abstract

Fractional order calculus (FOC) has wide applications in modeling natural behavior of systems related to different areas of engineering including bioengineering, viscoelasticity, electronics, robotics, control theory and signal processing. This thesis aims at modeling a lossy transmission line using fractional order calculus and identifying its parameters.

A lossy transmission line is considered where its behavior is modeled by a fractional order transfer function. A semi-infinite lossy transmission line is presented with its distributed parameters R, L, C and ordinary AC circuit theory is applied to find the partial differential equations. Furthermore, applying boundary conditions and the Laplace transformation a generalized fractional order transfer function of the lossy transmission line is obtained. A finite length lossy transmission line terminated with arbitrary load is also considered and its fractional order transfer function has been derived.

Next, the frequency responses of lossy transmission lines from their fractional order transfer functions are also derived. Simulation results are presented to validate the frequency responses. Based on the simulation results it can be concluded that the derived fractional order transmission line model is capable of capturing the phenomenon of a distributed parameter transmission line.

The achievement of modeling a highly accurate transmission line requires that a realistic account needs to be taken of its parameters. Therefore, a parameter identification technique to identify the parameters of the fractional order lossy transmission line is introduced.

Finally, a few open problems are listed as the future research directions.

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Sincere apology for omissions of any contribution, which may have occurred unintentionally.

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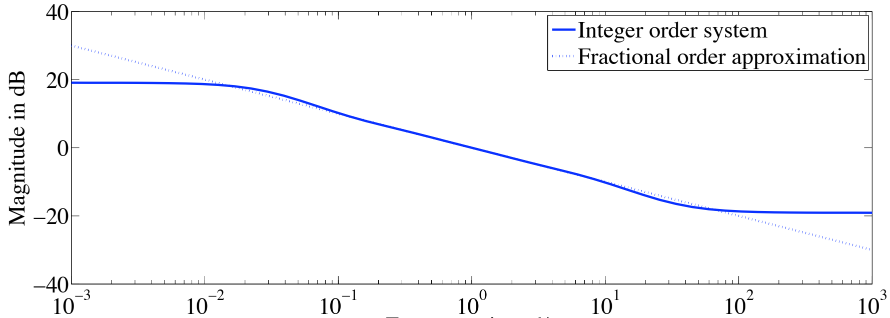
Chapter 1

Introduction

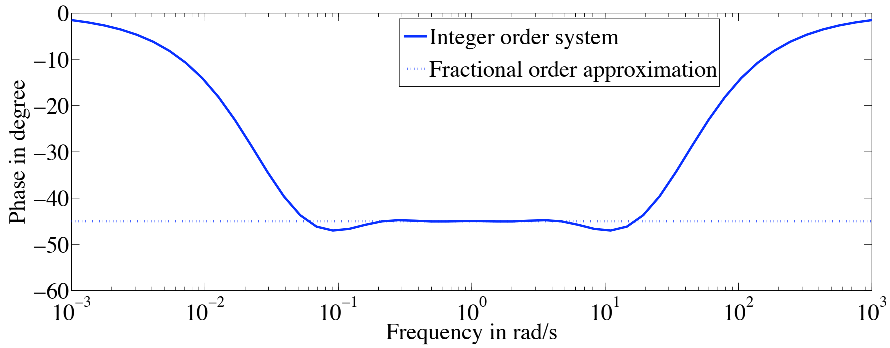
1.1 Motivation

Fractional order calculus (FOC) was not much explored in engineering due to its inherent complexity and the fact that it does not have a fully acceptable physical or geometrical interpretation [1], [2]. Despite of these complexities, it can be used to describe some natural behavior of systems related to different engineering areas including bioengineering [3], [4], viscoelasticity [5], [6], robotics [7], [8], [9], electronics [10], [11], control theory [12], [13] and signal processing [14], [15].

Fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using fewer coefficients [6], [16], [17], since the arbitrary order of the derivatives gives an additional degree of freedom to fit a specific behavior. A noteworthy merit of fractional derivative is that it may still apply to the functions which are not differentiable in the classical sense. The fractional derivatives of these functions depend on the domain and boundary conditions. The domain and boundary conditions of the functions must be chosen and restricted in order to find the fractional derivatives. Therefore, unlike the integral order derivative, the fractional order derivative at a point x is not determined by an arbitrary small neighborhood of x. In other words, fractional derivative is not a local property of the function. This characteristic is useful when the system has long-term memory and any evaluation point depends on the past values of the function. As an instance, a model of the 4^{th}



(a) Magnitude in dB



(b) Phase in degree

Figure 1.1: Comparison between a high order integer system and its approximation by fractional one.

order [18],

$$G(s) = \frac{s^4 + 36s^3 + 126s^2 + 84s + 9}{9s^4 + 84s^3 + 126s^2 + 36s + 1}$$

can be approximated by $G(s) \approx 1/s^{0.5}$, which is a compact fractional order system with just one parameter, valid in the frequency range from 0.01265 rad/s to 80 rad/s, as shown in Figure 1.1.

Another example of the fractional order formulation is presented in [19], where the authors modeled a complex system, a flexible structure with five vibration modes which is an infinite dimensional system, by a fractional order system with fewer parameters. They proposed the transfer function:

$$G(s) = \frac{\sum_{i=0}^{m} a_i (s^\alpha)^i}{(s^\alpha)^n + \sum_{j=0}^{n-1} b_j (s^\alpha)^j}$$

with $\alpha = 0.5$, $i = (0, 1, \ldots, m)$, $j = (0, 1, \ldots, n-1)$ and $n, m \in \mathbb{Z}$. A real-valued α

models the damping behavior without increasing the order of the system and maintaining a compact expression which is valid for the frequency range from 0.1 to 200 Hz. The authors showed that this transfer function is an optimal candidate for obtaining useful finite dimensional model of the infinite dimensional system. This fact leads us to consider fractional calculus as an appropriate tool to model more accurately the dynamics of distributed parameter systems which are infinite dimensional in nature. Many real systems properties such as viscoelastic material properties [20], [21] can be better identified by fractional order equations than integer order ones.

Transmission line theory is well known in physics and electrical engineering. The availability of an accurate transmission line model is desirable for many applications including power engineering, microwave engineering and telecommunication. The achievement of modeling a highly accurate model requires that a realistic account needs to be taken for the transmission line parameters [22]. The difficulty for this attainment is the variance of the parameters. Some of the parameters vary in a complex way with frequency. In particular, inductive and capacitive elements can be strongly frequency dependant due to the result of the skin effect of the conductor.

Transmission lines are distributed parameter systems and thus the dynamics are described by parabolic partial differential equations [23]. Due to the distributive nature, it is not possible to obtain rational transfer functions with limited number of parameters. Fractional order calculus has the property of infinite dimensionality due to its fractional order. This property of fractional order calculus might be useful to describe the dynamics of distributive parameter systems such as transmission lines. It might be possible to obtain a transfer function of the transmission line with a small number of parameters using fractional order calculus theory.

Abnormal diffusion process appears in the transmission lines. The voltage (current) wave of a transmission line is composed of incident and reflected waves. The reflected wave is an abnormal diffusion wave of the incident wave. Classical calculus theory cannot accurately describe this diffusion phenomenon. Fractional calculus is capable of better capturing this phenomenon due to its infinite dimensional structure [24]. Hence, fractional calculus can be used to model a distributed parameter transmission line.

In classical power engineering textbooks, transmission lines are modeled using classical partial differential equations where they simply overlooked the fractional order behavior of the system. In [25], the author developed a semi-infinite fractional order lossy transmission line model. In that model, the author only considered parameters R (resistance) and C (capacitance) of the system. In this thesis, we extended the model into a general case considering the parameters R (resistance), L (inductance) and C (capacitance). We also derived a model for finite length transmission line terminated by arbitrary load at the receiving terminal.

1.2 Objectives

The main objectives of this thesis lie in the following aspects:

1. To derive a generalized fractional order transfer function of a lossy transmission line considering the parameters R, L, and C.

2. Finding out the frequency response expressions of the fractional order transfer functions in order to identify the parameters of the transmission line from the frequency responses.

3. Identify the parameters of the line using parameter estimation techniques.

1.3 Scope of the Thesis

In chapter 2, we review the concept of fractional order calculus. Fractional order calculus theory will be briefly discussed. Definitions of fractional derivatives/integrals and the Laplace transformation of fractional derivatives are given. Analysis of control properties such as stability, controllability and observability criteria for fractional order systems are also presented. Properties of fractional calculus have been discussed with functions useful in the analysis of fractional calculus. Traditional and a new modified approximation techniques to find a rational transfer function are also reviewed.

Chapter 3 aims at development of a fractional order model of a lossy transmission line. An extension of a fractional order model in [25] is proposed as a more generalized model of a lossy transmission line with R, L and C parameters.

In chapter 4, frequency responses of fractional order transfer functions are obtained. Frequency responses for both finite and semi-infinite length lossy transmission lines have been derived. Simulation results have been presented to validate the obtained frequency response models.

Chapter 5 is dedicated to the introduction of parameter estimation techniques for fractional order systems. Linear and nonlinear parameter estimation techniques have been discussed. A nonlinear estimation technique has been applied to identify the parameters of the transmission line from its frequency response data. Estimation results are presented to show that it is possible to identify the parameters of the line from the fractional order model.

The last chapter summarizes the work in the thesis, and outlines some possible future research directions.

1.4 Notation

The notation used through out the thesis is fairly standard. The superscript 'T' stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{C}$ are the set of all real, integer and complex numbers, respectively.

Chapter 2

Introduction to Fractional Calculus

2.1 Introduction

In 1695, L' Hospital asked Leibniz that he had used a notation for the n^{th} derivative of a function,

$$\frac{d^n f(x)}{dx^n}$$

and asked what would be the result if $n = 1/2$? In response Leibniz replied, "An apparent paradox from which one day useful consequences will be drawn". In these words, fractional calculus was born around 300 years ago [25]. Fractional calculus is a natural generalization of the calculus theory. Before the 19th century, fractional calculus was developed as a pure mathematical theory without any physical significance. But unfortunately it was not much popular in the engineering community due to its inherent complexity, the apparent self-sufficiency of the integer order calculus, and the fact that it does not have a fully acceptable geometrical or physical interpretation [1]. Due to the growing advancement in computing science, fractional calculus becomes an increasingly interesting topic of research in the field of science and technology. The essence of this subject is that fractional derivatives and integrals have a distributive property. Therefore, it provides an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with the classical integer-order ones, in which such effects are neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of materials as well as in the theory of control of dynamic systems. It has been found that behavior of many physical systems can be properly described by using the fractional order system theory. For example, heat conduction [26], dielectric polarization, electromagnetic waves [27], electrode-electrolyte polarization [28], visco-elastic systems [6], quantum evolution of complex systems [29], quantitative finance [30] and diffusion waves [31] are among the known dynamic systems that were modeled using fractional order equations. The special issue of signal processing [32] discusses many of its applications in detail.

This chapter discusses the basic concepts of the fractional order calculus. The chapter provides the background on the mathematical knowledge of fractional order calculus theory.

2.2 Definitions of Fractional Derivatives/Integrals in Fractional Calculus

Fractional calculus is a generalization of integration and differentiation to non-integer orders [33], [34]. The continuous integro-differential operator is defined as

$$
{}_a\mathrm{D}_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases} \tag{2.1}
$$

The three equivalent definitions most frequently used for the general fractional differintegral (a term that was coined to avoid the cumbersome alternate "derivatives or integrals to arbitrary order") are the Riemann-Liouville (RL) [33], the Grünwald-Letnikov (GL) and the Caputo (1967) definitions.

2.2.1 Riemann-Liouville Definition

The RL definition for a function $f(t)$ is given as

$$
{}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.2}
$$

where $t > 0$, $\alpha \in \mathbb{R}_+$ and n is an integer such that $(n-1) \leq \alpha < n$. $\Gamma(.)$ is the well known Euler's *Gamma* function.

2.2.2 Caputo Definition

The Caputo definition for a function $f(t)$ can be given as

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{2.3} $$

where $(n-1) \leq \alpha < n$, n is an integer and α is a real number. This definition is very famous and is used in many literature to define the fractional order derivatives and integrals.

2.2.3 Grünwald-Letnikov Definition

The GL definition is the most popular definition for the general fractional derivatives and integrals because of its discrete nature. It is very useful for simulation of fractional derivatives and integrals. According to GL, the fractional derivative of a function $f(t)$ is given as

$$ {}_a\mathrm{D}^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{i=0}^{[\frac{t-a}{h}]} (-1)^i \begin{pmatrix} \alpha \\ i \end{pmatrix} f(t - ih) \tag{2.4} $$

where

$$ \begin{pmatrix} \alpha \\ i \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha - i + 1)} $$

is the generalization of Newton's binomial $\begin{pmatrix} \alpha \\ i \end{pmatrix}$ by the use of Euler's Gamma (Γ) function where h is the sampling period and [.] means the integer part.

2.3 Laplace Transform of Fractional Order Derivatives

The analysis of dynamical behavior in systems theory often uses transfer functions. In this respect, introduction to the Laplace transform for non-integer order derivatives is necessary. Fortunately, the Laplace transforms of fractional order derivatives are straightforward like the classical case and very useful in the study of fractional order systems. The inverse Laplace transformation is also useful for time domain representations of systems for which only the frequency response is known. The formula for the Laplace transform of the fractional derivative has the following form [35]:

$$\int_0^\infty e^{-st} \mathrm{D}_t^\alpha f(t)dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \mathrm{D}_t^{\alpha-k-1} f(t)|_{t=0} \tag{2.5}$$

where n is an integer such that $n-1 < \alpha \leq n$ and s denotes the Laplace transform variable. The above expression becomes very simple when all the initial conditions are zero:

$$L\{\mathrm{D}_t^\alpha f(t)\} = s^\alpha L\{f(t)\} \tag{2.6}$$

2.4 General Properties of Fractional Calculus

The main properties of fractional derivatives and integrals described in [36] are as follows:

- For $\alpha = n$, where n is an integer, the operation $\mathrm{D}_t^\alpha f(t)$ gives the same result as the classical differentiation of integer order n.

- For $\alpha = 0$, the operation $\mathrm{D}_t^\alpha f(t)$ is the identity operator:

$$\mathrm{D}_t^0 f(t) = f(t)$$

- Fractional differentiations and integrations are linear operations:

$$a\mathrm{D}_t^\alpha f(t) + b\mathrm{D}_t^\alpha g(t) = a_0\mathrm{D}_t^\alpha f(t) + b_0\mathrm{D}_t^\alpha g(t)$$

- The additive index law (semigroup property)

$$\mathrm{D}_t^\alpha \mathrm{D}_t^\beta f(t) = \mathrm{D}_t^\beta \mathrm{D}_t^\alpha f(t) = \mathrm{D}_t^{\alpha+\beta} f(t)$$

  holds under some reasonable constraints on the function $f(t)$. The fractional-order derivative commutes with integer order derivative,

$$\frac{d^n}{dt^n}({}_a\mathrm{D}_t^\alpha f(t)) = {}_a\mathrm{D}_t^\alpha (\frac{d^n f(t)}{dt^n}) = {}_a\mathrm{D}_t^{\alpha+n} f(t)$$

  under the condition that if $t = a$, we have $f^{(k)}(a) = 0$, $(k = 0, 1, 2, \ldots, n-1)$. The above relationship says the operators $\frac{d^n}{dt^n}$ and ${}_a\mathrm{D}_t^\alpha$ commute.

2.5 Functions Used in Fractional Calculus

There are a number of functions that have been found to be useful in fractional calculus. The Gamma function, which generalizes factorial ($n!$) expressions and allow n to be non-integer values. The Mittag-Leffler function is a basis function in fractional calculus as the exponential functions are in integer order calculus.

2.5.1 Gamma Function

The Gamma function is one of the basic functions in fractional order calculus. The function generalizes the the factorial $n!$ and allows n to be a real number. The Gamma function can be defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2.7}$$

which converges in the right half of the complex plane $\Re(z) > 0$. This statement implies that the Gamma function is defined continuously for positive real numbers. The basic property of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z). \tag{2.8}$$

Using the above property, we can find the values for $z = 1, 2, 3, \ldots$, as follows

$$\begin{aligned}
&\Gamma(1) = 1 \\
&\Gamma(2) = 1\Gamma(1) = 1! \\
&\Gamma(3) = 2\Gamma(2) = 2! \\
&\Gamma(4) = 3\Gamma(3) = 3! \\
&\vdots \qquad\quad \vdots \\
&\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!
\end{aligned}$$

2.5.2 Mittag-Leffler Function

Exponential functions play an important role in the integer order calculus, while Mittag-Leffler functions do the same in fractional order calculus. They are the generalization of exponential functions. Mittag-Leffler considered a parameter a to be a complex number, such as $a = |a|e^{j\phi}$ and defined a function as $E_q[az], q > 0$. When he studied the function, it became apparent that this function could be stable or unstable as z increases, depending upon how the parameters a and q are chosen. He found the function remain bounded for increasing z if $|\phi| \geq \frac{\pi}{2}$.

One Parameter Mittag-Leffler Function

The one parameter Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{2.9}$$

where $\alpha > 0$.

Two Parameter Mittag-leffler Function

The two-parameter Mittag-Leffler function was introduced by Agarwal and Erdelyi in 1953-1954 [37]. It plays an important role in the fractional order calculus. The two-parameter function is defined as follows

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.10}$$

where $\alpha > 0, \beta > 0$. If $\beta = 1$ it becomes the one parameter Mittag-Leffler function.

Laplace Transform of Mittag-Leffler Function

The Laplace transformation is a very useful tool for finding the solution of fractional order differential equations. The following expression gives the identities for Laplace transform pairs of Mittag-Leffler functions:

$$t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(at^\alpha) \Longleftrightarrow \frac{s^{\alpha-\beta} k!}{(s^\alpha - a)^{k+1}} \tag{2.11}$$

where

$$E^{(k)}_{\alpha,\beta} = \frac{d^{(k)}}{dt^{(k)}} E_{\alpha,\beta}$$

and α and β are Mittag-Leffler function parameters. The operation is differentiation of the Mittaf-Leffler function for $k > 0$ and integration for $k < 0$.

2.6 Fractional Order Systems Representation

A fractional mathematical model can be described by the following fractional order differential equation [35]:

$$y(t)+b_1D^{\beta_1}y(t)+\ldots+b_{m_B}D^{\beta_{m_B}}y(t) = a_0D^{\alpha_0}u(t)+a_1D^{\alpha_1}u(t)+\ldots+a_{m_A}D^{\alpha_{m_A}}u(t) \tag{2.12}$$

where differentiation orders $\beta_1 < \beta_2 < \ldots < \beta_{m_B}$ and $\alpha_0 < \alpha_1 < \ldots < \alpha_{m_A}$ are allowed to be non-integer positive numbers. This fractional differential equation can be represented more concisely using algebraic tools, e.g., the Laplace transformation [38]. The Laplace transform notation is

$$\mathcal{L}\{D^\alpha x(t)\} = s^\alpha X(s), \quad \text{if} \quad x(t) = 0\,\forall\, t < 0.$$

So a generic single-input single-output (SISO) fractional order system representation in the Laplace domain can be given as

$$F(s) = \frac{\sum_{i=0}^{m_A} a_i s^{\alpha_i}}{1+\sum_{j=1}^{m_B} b_j s^{\beta_j}} \tag{2.13}$$

where, $(a_i, b_j) \in \mathbb{R}^2$, $(\alpha_i, \beta_j) \in \mathbb{R}_+^2$, $\forall\, i = 0, 1, \ldots, m_A$ and $\forall\, j = 1, 2, \ldots, m_B$.

The transfer function given by equation (2.13) can be classified as either a commensurate or a non-commensurate transfer function. A transfer function, $F(s)$ is commensurate order of α if and only if it can be written as $F(s) = S(s^\alpha)$, where $S = T/R$ is a rational function, with T and R as two co-prime polynomials. The commensurate order α is defined as the biggest real number such that all differentiation orders are integer multiples of α. The transfer function $F(s)$ is non-commensurate if α_i, β_j can take arbitrary values.

To obtain a discrete model of the fractional-order system in (2.13), we have to use the discrete approximation of the fractional-order integro-differential operators to get a general expression for the discrete transfer function of the system [39]:

$$F(z) = \frac{a_{m_A}\big(w(z^{-1})\big)^{\alpha_{m_A}}+\ldots+a_0\big(w(z^{-1})\big)^{\alpha_0}}{b_{m_B}\big(w(z^{-1})\big)^{\beta_{m_B}}+\ldots+b_1\big(w(z^{-1})\big)^{\beta_1}+1} \tag{2.14}$$

where $\big(w(z^{-1})\big)$ denotes the discrete equivalent of the Laplace operator s, expressed as a functions of the complex variable z or the shift operator z^{-1}.

2.7 Stability, Controllability and Observability of Fractional Order Systems

A commensurable fractional order linear time invariant (LTI) system can also be represented by state-space model:

$$\begin{aligned} {}_0\mathrm{D}_t^{\alpha} x(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) \end{aligned} \tag{2.15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $y \in \mathbb{R}^p$ are the state, input and output vectors of the system and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ are the system matrices. α is the fractional commensurable order.

An integer order LTI system is stable if the roots of the characteristic polynomial are negative or have negative real parts if they are complex. This means, they are located on the left half of the complex plane. It is different for fractional order LTI systems. A fractional order stable transfer function may have roots in the right half of the complex plane. Matignon [40] has established the stability condition of any commensurable explicit fractional order system. *The fractional transfer function* $G(s) = Z(s)/P(s)$ *is stable if and only if the following condition is satisfied in the* σ *plane*:

$$|\arg(\sigma)| > \alpha\frac{\pi}{2} \qquad \forall\sigma \in \mathbb{C}, P(\sigma) = 0 \tag{2.16}$$

where $\sigma := s^{\alpha}$*. When* $\sigma = 0$ *is a single root of* $P(s)$*, the system cannot be stable. For* $\alpha = 1$*, this is the classical theorem of pole location in the complex plane.*

Figure 2.1 shows the stable and unstable regions of this case.

The controllability and observability of system (2.15) are defined by Matignon [41]. System (2.15) is *controllable* if the controllability matrix,

$$C_a = [BABA^2B\ldots A^{n-1}B]$$

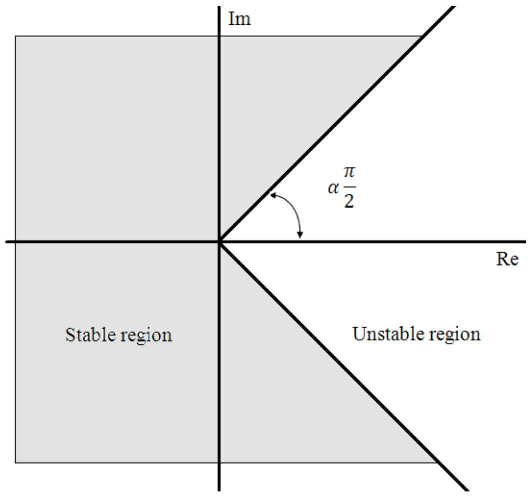


Figure 2.1: Stability region of fractional order LTI system (2.15) with order $0 < \alpha < 1$.

has rank n. Similarly, system (2.15) is *observable* if the observability matrix,

$$O_a = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n.

2.8 Approximation Methods of Fractional Order Systems

A fractional order system is an infinite dimensional system due to its fractional order while integer order systems are finite dimensional with limited memory. The characteristic equation of a fractional LTI system is not a polynomial but a pseudo-polynomial function of the fractional power of the complex variable s [42]. A frequency band limited implementation technique is required for fractional order systems. The finite dimensional approximation should be done in the range of frequency band and it is an active research topic in the community. Direct discretization methods were proposed by Chen et al. [43], while in [44], a new infinite impulse response (IIR) type digital fractional differentiator is proposed. In [45], continued fraction expansion is used for approximation due to the fact that the continued fraction expansion often converges much more rapidly than power series expansion. In [46], it was proposed to design digital fractional differentiator and integrator by recursive filtering via power series expansion and Prony's approximation.

2.8.1 Oustaloup's Approximation Algorithm

Oustaloup's approximation algorithm [47] is widely used, where a frequency domain response is fitted by a bank of integer order filters to the fractional order derivatives within a band of frequency range. The output $s(t)$ of a non-integer differentiator is the non-integer order derivative of its input $e(t)$ multiplied by a factor, i.e.,

$$s(t) = \tau^{\alpha}\left(\frac{d}{dt}\right)^{\alpha} e(t) \tag{2.17}$$

where τ is real and positive differentiation time constant and α is the non-integer order of differentiation. Taking the Laplace transform of equation (2.17) in which $\omega_\mu = 1/\tau$, also known as *unit gain frequency or transition frequency*, we obtain

$$S(s) = \left(\frac{s}{\omega_\mu}\right)^{\alpha} E(s). \tag{2.18}$$

The fractional order differentiation is carried out by limiting differentiation transfer function $H(s) = \left(\frac{s}{\omega_\mu}\right)^{\alpha}$ within the frequency range $[\omega_A, \omega_B]$, replacing by the frequency bounded differentiation transfer function

$$C_0 \frac{1 + s/\omega_b}{1 + s/\omega_h} \tag{2.19}$$

where

$$\sqrt{\omega_b \omega_h} = \omega_u \tag{2.20}$$

and

$$C_0 = \frac{\omega_b}{\omega_u} = \frac{\omega_u}{\omega_h} \tag{2.21}$$

distributed geometrically around ω_μ (2.20). The high and low transitional frequencies ω_h and ω_b are such that $\omega_b \ll \omega_A$ and $\omega_h \gg \omega_B$.

The synthesis of such a differentiator results from an intuitive approach based on the concept of fractal through recursivity [48]. A recursive distribution of real zeros and poles is used

$$H(s) = \lim_{N\to\infty} \hat{H}(s)$$

with

$$\hat{H}(s) = \left(\frac{\omega_u}{\omega_h}\right)^{\alpha} \prod_{k=-N}^{N} \frac{1 + s/\omega_k^{'}}{1 + s/\omega_k} \tag{2.22}$$

where

$$\omega_k^{'} = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+1/2-\alpha/2}{2N+1}} \tag{2.23}$$

and

$$\omega_k = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+1/2+\alpha/2}{2N+1}} \tag{2.24}$$

are respectively the zeros and poles of rank k, and $2N+1$ are the total number of zeros and poles.

The quality of the Oustaloup's approximation suffers from the inaccuracy problem near the low and/or high frequency bands. The fitting around this band may not provide satisfactory result.

2.8.2 Modified Approximation Method

A new modified approximation algorithm is presented by Xue et. al. [49] to overcome the boundary fitting problem in using the Oustaloup's algorithm. The approximation is almost perfect within the whole pre-specified frequency range of interest.

Assuming the pre-specified frequency range to be fit is defined as (ω_b, ω_h), the fractional operator s^{α} can be approximated by the fractional order transfer function as

$$K(s) = \left(\frac{1 + \frac{s}{\frac{d}{b}\omega_b}}{1 + \frac{s}{\frac{b}{d}\omega_h}}\right)^{\alpha} \tag{2.25}$$

where $0 < \alpha < 1$, $s = j\omega$, $b > 0$, $d > 0$, Thus

$$K(s) = \left(\frac{bs}{d\omega_b}\right)^{\alpha} \left(1 + \frac{-ds^2 + d}{ds^2 + b\omega_h s}\right)^{\alpha}. \tag{2.26}$$

Now, in the frequency range $\omega_b < \omega < \omega_h$, the Taylor series expansion will result

$$K(s) = \left(\frac{bs}{d\omega_b}\right)^{\alpha} \left[1 + \alpha p(s) + \frac{\alpha(\alpha-1)}{2} p^2(s) + \dots\right] \tag{2.27}$$

with

$$p(s) = \frac{-ds^2 + d}{ds^2 + b\omega_h s} \tag{2.28}$$

We can get

$$s^\alpha \approx \frac{(d\omega_b)^\alpha}{\left[1 + \alpha p(s) + \frac{\alpha(\alpha-1)}{2} p^2(s) + \ldots\right]} \left(\frac{1 + \frac{s}{\frac{d}{b}\omega_b}}{1 + \frac{s}{\frac{b}{d}\omega_h}} \right)^\alpha, \tag{2.29}$$

Truncating the Taylor series to first order leads to

$$s^\alpha \approx \frac{(d\omega_b)^\alpha}{b^\alpha(1 + \alpha p(s))} \left(\frac{1 + \frac{s}{\frac{d}{b}\omega_b}}{1 + \frac{s}{\frac{b}{d}\omega_h}} \right)^\alpha \tag{2.30}$$

Thus

$$s^\alpha \approx \left(\frac{d\omega_b}{b}\right)^\alpha \left(\frac{ds^2 + b\omega_h s}{d(1-\alpha)s^2 + b\omega_h s + d\alpha}\right) \left(\frac{1 + \frac{s}{\frac{d}{b}\omega_b}}{1 + \frac{s}{\frac{b}{d}\omega_h}} \right)^\alpha. \tag{2.31}$$

Expression (2.31) is stable if and only if all the poles are on the left hand side of the complex s-plane. It can be seen that expression (2.31) has 3 poles. That is,

- One of the poles is located at $-b\omega_h/d$, which is a negative real pole since $\omega_h > 0$, $b > 0$ and $d > 0$;

- Two other poles are the roots of the equation

$$d(1-\alpha)s^2 + \alpha\omega_h s + d\alpha = 0 \tag{2.32}$$

  whose real parts are negative since $0 < \alpha < 1$.

Therefore, the poles of the expression in (2.31) are stable within the frequency range (ω_b, ω_h). This proposed approximation is only applicable to the case where $0 < \alpha < 1$. For a higher differentiation, e.g., $S^{2.3}$, the approximation should be made to $s^{0.3}$ such that the original differential can be written by $s^2 s^{0.3}$. The irrational fractional part of the expression (2.31) can be approximated by a continuous-time rational model

$$\left(\frac{1 + \frac{s}{\frac{d}{b}\omega_b}}{1 + \frac{s}{\frac{b}{d}\omega_h}} \right)^\alpha = \lim_{N\to\infty} \prod_{k=-N}^{N} \frac{1 + s/\omega_k'}{1 + s/\omega_k}. \tag{2.33}$$

According to recursive distribution of real zeros and poles, the zero and pole of rank k can be written as

$$\omega_k^{'} = \left(\frac{d\omega_b}{b}\right)^{\frac{\alpha-2k}{2N+1}} \tag{2.34}$$

and

$$\omega_k = \left(\frac{b\omega_h}{d}\right)^{\frac{\alpha+2k}{2N+1}} \tag{2.35}$$

Thus, the final continuous rational transfer function model can be obtained as

$$s^{\alpha} \approx K\Big[\frac{ds^2 + b\omega_h s}{d(1-\alpha)s^2 + b\omega_h s + d\alpha}\Big] \prod_{k=-N}^{N} \frac{1 + s/\omega_k^{'}}{1 + s/\omega_k} \tag{2.36}$$

where

$$K = \left(\frac{d\omega_b}{b}\right)^{\alpha} \prod_{k=-N}^{N} \frac{\omega_k}{\omega_k^{'}}. \tag{2.37}$$

Based on their numerical experiments, they suggest to use $b = 10$ and $d = 9$. The proposed scheme is only for a single term s^{α}. Obviously repeated application of the scheme will finally give a finite dimensional rational LTI transfer function. Unfortunately, the order of the transfer function might be high when the system has no commensurable order. In that case, they suggest an effective method to perform the reduction [50].

2.9 Conclusion

A brief introduction to fractional order calculus theory has been discussed in this chapter. Definitions of the fractional derivative or integral, the Laplace transformation technique and properties of fractional derivatives have been described. Furthermore, the system representation of a fractional order system with stability, controllability and observability criteria is presented. Approximation techniques for physical implementation are also illustrated in this chapter.

Chapter 3

Transmission Line Modeling

3.1 Introduction

The past decades have witnessed an increased efforts related to fractional calculus [51], [52] and its applications to modeling physical systems and to the control theory. The transmission line is a distributive parameter system. Modeling the dynamics of distributed parameter systems using fractional calculus [53] is a useful tool due to its infinite dimensionality.

In [25], a lossy semi-infinite fractional order transmission line was modeled considering the R and C parameters. In that model, the author showed that a lossy semi-infinite transmission line demonstrates fractional order behavior; he showed the current into the line is equal to the half order derivative of the applied voltage. In [53], the authors discussed fractional order dynamics of some distributed parameter systems; they introduced the half-order fractional capacitances and inductances which were overlooked in the classical textbooks. They showed the fractional order models capture phenomena and properties that classical integer order simply neglect. A lossy fractional order transmission line is also discussed in [54]; the authors presented a model of a fractional order lossy transmission line considering only R and C parameters. They showed the voltage and current of a lossy fractional order transmission line can be represented as

$$V(x,s) = e^{-x\sqrt{RCs}}V_I(s) \tag{3.1}$$

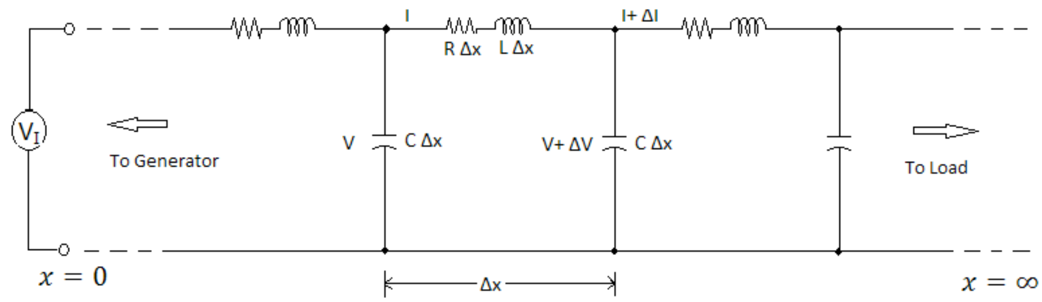


Figure 3.1: Circuit representation of uniform transmission line.

and

$$I(x,s) = \sqrt{\frac{Cs}{R}} e^{-x\sqrt{RCs}} V_I(s) \tag{3.2}$$

where $V(x,s)$, $I(x,s)$ are the voltage and current anywhere in the line respectively and $V_I(s)$ is the applied voltage in the Laplace domain. In [55], the authors showed the analysis of fractional order transmission lines based on Mittag-Leffler functions which yields the same result as the analysis based on standard methods.

In this chapter, we derived a more generalized lossy fractional order transmission line based on the steps of [25] considering the parameters R, L and C. It will be shown that the model in [25] and [54] are a special case of our derived model.

3.2 Problem Formulation

Consider an uniform single-phase two wire lossy semi-infinite transmission line. The transmission line has a loop inductance $L\ H/m$, line to line capacitance $C\ F/m$ and a series resistance $R\ \Omega/m$. Since the transmission line is of distributed nature, it can be modeled by the circuit representation as shown in Figure 3.1. The line is pictured as a cascade of identical sections, each Δx long. Each section consists of series inductance and resistance denoted by $L\Delta x$ and $R\Delta x$ respectively and shunt capacitance denoted by $C\Delta x$. Since the shunt conductance (G) is negligible compared to other parameters, it has been ignored in this derivation. As Δx is chosen small compared to the operating wavelength, an individual section of the line may be analyzed using ordinary AC circuit theory.

The main objective of this thesis is to derive a more generalized fractional order transfer function of the lossy RLC transmission line for matched and arbitrary load connected in the receiving terminal.

3.3 Fractional Order Transmission Line Modeling

As a lossy RLC transmission line is considered, the boundary value problem can be defined in terms of the current and voltage variables. The boundary conditions are $v(0,t) = v_I(t)$ and $v(\infty,t) = 0$, where v is the voltage, i is the current and $v_I(t)$ is a time-dependent input variable. At $x = \infty$, it is assumed that the line is short circuited. In terms of the voltage and current variables, the partial differential equation of the transmission line can be written as:

$$\frac{\partial v(x,t)}{\partial x} = Ri(x,t) + L\frac{\partial i(x,t)}{\partial t}, \tag{3.3}$$

$$\frac{\partial i(x,t)}{\partial x} = C\frac{\partial v(x,t)}{\partial t} \tag{3.4}$$

where R, L and C are the per unit resistance, inductance and capacitance of the line, respectively. Differentiating equation (3.3) with respect to x and then substituting in equation (3.4) we get

$$\frac{\partial^2 v(x,t)}{\partial x^2} = RC\frac{\partial v(x,t)}{\partial t} + LC\frac{\partial^2 v(x,t)}{\partial t^2} \tag{3.5}$$

Choosing $\alpha = RC$ and $\beta = LC$, equation (3.5) becomes

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \alpha\frac{\partial v(x,t)}{\partial t} + \beta\frac{\partial^2 v(x,t)}{\partial t^2}. \tag{3.6}$$

Now, taking the Laplace transformation of equation (3.6) with respect to time t and using s as the temporal Laplace variable gives:

$$\begin{aligned}\frac{d^2 V(x,s)}{dx^2} =& \alpha\{sV(x,s) - v(x,0)\} + \beta\{s^2V(x,s) - sv(x,0) - \frac{dv(x,0)}{dt}\} \\ =& \alpha\{sV(x,s) - v(x,0)\} + \beta\{s^2V(x,s) - sv(x,0) - V^*(x,0)\}\end{aligned} \tag{3.7}$$

where $\frac{dv(x,0)}{dt} = V^*(x,0)$ is the initial voltage distribution of the transmission line when $t = 0$. Again taking the Laplace transformation of equation (3.7) with respect to spatial position x and using p as the spatial Laplace variable gives

$$p^2V(p,s) - pV(0,s) - \frac{dV(0,s)}{dx} = \alpha\{sV(p,s) - V(p,0)\} + \beta\{s^2V(p,s) - sV(p,0) - V^*(p,0)\} \tag{3.8}$$

Substituting $\frac{dV(0,s)}{dx} = V^*(0,s)$ we get

$$p^2V(p,s) - pV(0,s) - V^*(0,s) = \alpha\{sV(p,s) - V(p,0)\} + \beta\{s^2V(p,s) - sV(p,0) - V^*(p,0)\}$$

$$\Rightarrow p^2V(p,s) - \alpha sV(p,s) - \beta s^2V(p,s) = pV(0,s) + V^*(0,s) - \alpha V(p,0) - \beta sV(p,0) - \beta V^*(p,0)$$

$$\Rightarrow V(p,s)(p^2 - \alpha s - \beta s^2) = pV(0,s) + V^*(0,s) - \alpha V(p,0) - \beta sV(p,0) - \beta V^*(p,0)$$

$$\Rightarrow V(p,s) = \Big[\frac{1}{p^2 - \alpha s - \beta s^2}\Big]\Big(pV(0,s) + V^*(0,s) - \alpha V(p,0) - \beta sV(p,0) - \beta V^*(p,0)\Big)$$

Thus

$$V(p,s) = \Big[\frac{1}{p^2 - \alpha s - \beta s^2}\Big]\Big(pV(0,s) + V^*(0,s)\Big) - \Big[\frac{1}{p^2 - \alpha s - \beta s^2}\Big]\Big(\alpha V(p,0) + \beta sV(p,0) + \beta V^*(p,0)\Big) \tag{3.9}$$

Here the first term represents the voltage present at $x = 0$ at the generator end and the second term represents the initial spatial voltage distribution. After doing partial fraction expansion of the equation (3.9), it can be expressed as

$$V(p,s) = \Big[\frac{1}{p^2 - \alpha s - \beta s^2}\Big]\Big(pV(0,s) + V^*(0,s)\Big) - \Big[\frac{1}{2\sqrt{\alpha s + \beta s^2}(p - \sqrt{\alpha s + \beta s^2})} - \frac{1}{2\sqrt{\alpha s + \beta s^2}(p + \sqrt{\alpha s + \beta s^2})}\Big] \Big(\alpha V(p,0) + \beta sV(p,0) + \beta V^*(p,0)\Big)$$

After rearrangement, we can write

$$\begin{aligned} V(p,s) =& \frac{pV(0,s)}{p^2 - (\sqrt{\alpha s + \beta s^2})^2} + \frac{\sqrt{\alpha s + \beta s^2}V^*(0,s)}{\sqrt{\alpha s + \beta s^2}\big(p^2 - (\sqrt{\alpha s + \beta s^2})^2\big)} \\ &- \frac{\alpha V(p,0)}{2\sqrt{\alpha s + \beta s^2}(p - \sqrt{\alpha s + \beta s^2})} + \frac{\alpha V(p,0)}{2\sqrt{\alpha s + \beta s^2}(p + \sqrt{\alpha s + \beta s^2})} \\ &- \frac{\beta sV(p,0)}{2\sqrt{\alpha s + \beta s^2}(p - \sqrt{\alpha s + \beta s^2})} + \frac{\beta sV(p,0)}{2\sqrt{\alpha s + \beta s^2}(p + \sqrt{\alpha s + \beta s^2})} \\ &- \frac{\beta V^*(p,0)}{2\sqrt{\alpha s + \beta s^2}(p - \sqrt{\alpha s + \beta s^2})} + \frac{\beta V^*(p,0)}{2\sqrt{\alpha s + \beta s^2}(p + \sqrt{\alpha s + \beta s^2})} \end{aligned} \tag{3.10}$$

We can take the inverse Laplace transform of equation (3.10) with respect to the variable p. The inverse Laplace transform of the first two terms can be done by standard transform pairs and the rest of the terms can be transformed using convolution. Taking the inverse Laplace transform of the equation (3.10), we have

$$
\begin{aligned}
V(x,s) = V(0,s)\cosh(x\sqrt{\alpha s+\beta s^2}) + \frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}}\sinh(x\sqrt{\alpha s+\beta s^2}) \\
-\int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{+(x-\lambda)\sqrt{\alpha s+\beta s^2}}\big[\alpha V(\lambda,0)\big]d\lambda + \int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{-(x-\lambda)\sqrt{\alpha s+\beta s^2}} \\
\big[\alpha V(\lambda,0)\big]d\lambda \\
-\int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{+(x-\lambda)\sqrt{\alpha s+\beta s^2}}\big[\beta s V(\lambda,0)\big]d\lambda + \int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{-(x-\lambda)\sqrt{\alpha s+\beta s^2}} \\
\big[\beta s V(\lambda,0)\big]d\lambda \\
-\int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{+(x-\lambda)\sqrt{\alpha s+\beta s^2}}\big[\beta V^*(\lambda,0)\big]d\lambda + \int_0^x \frac{1}{2\sqrt{\alpha s+\beta s^2}} e^{-(x-\lambda)\sqrt{\alpha s+\beta s^2}} \\
\big[\beta V^*(\lambda,0)\big]d\lambda
\end{aligned}
\tag{3.11}
$$

Equivalently,

$$
\begin{aligned}
V(x,s) = &\frac{V(0,s)}{2}\big[e^{+x\sqrt{\alpha s+\beta s^2}} + e^{-x\sqrt{\alpha s+\beta s^2}}\big] + \frac{V^*(0,s)}{2\sqrt{\alpha s+\beta s^2}}\big[e^{+x\sqrt{\alpha s+\beta s^2}} - e^{-x\sqrt{\alpha s+\beta s^2}}\big] \\
&-\int_0^x \frac{\alpha}{2\sqrt{\alpha s+\beta s^2}} e^{x\sqrt{\alpha s+\beta s^2}} e^{-\lambda\sqrt{\alpha s+\beta s^2}}\big[V(\lambda,0)d\lambda\big] \\
&\qquad+\int_0^x \frac{\alpha}{2\sqrt{\alpha s+\beta s^2}} e^{-x\sqrt{\alpha s+\beta s^2}} e^{\lambda\sqrt{\alpha s+\beta s^2}}\big[V(\lambda,0)d\lambda\big] \\
&-\int_0^x \frac{\beta s}{2\sqrt{\alpha s+\beta s^2}} e^{x\sqrt{\alpha s+\beta s^2}} e^{-\lambda\sqrt{\alpha s+\beta s^2}}\big[V(\lambda,0)d\lambda\big] \\
&\qquad+\int_0^x \frac{\beta s}{2\sqrt{\alpha s+\beta s^2}} e^{-x\sqrt{\alpha s+\beta s^2}} e^{\lambda\sqrt{\alpha s+\beta s^2}}\big[V(\lambda,0)d\lambda\big] \\
&-\int_0^x \frac{\beta}{2\sqrt{\alpha s+\beta s^2}} e^{x\sqrt{\alpha s+\beta s^2}} e^{-\lambda\sqrt{\alpha s+\beta s^2}}\big[V^*(\lambda,0)d\lambda\big] \\
&\qquad+\int_0^x \frac{\beta}{2\sqrt{\alpha s+\beta s^2}} e^{-x\sqrt{\alpha s+\beta s^2}} e^{\lambda\sqrt{\alpha s+\beta s^2}}\big[V^*(\lambda,0)d\lambda\big]
\end{aligned}
\tag{3.12}
$$

Now, collecting the similar exponentials give the following expression

$$
\begin{aligned}
V(x,s) =& \frac{e^{+x\sqrt{\alpha s+\beta s^2}}}{2}\Bigg[V(0,s) + \frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}} - \frac{\alpha}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\
& - \frac{\beta s}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda - \frac{\beta}{\sqrt{\alpha s+\beta s^2}}e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda\Bigg] \\
& + \frac{e^{-x\sqrt{\alpha s+\beta s^2}}}{2}\Bigg[V(0,s) - \frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}} + \frac{\alpha}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\
& + \frac{\beta s}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda + \frac{\beta}{\sqrt{\alpha s+\beta s^2}}e^{\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda\Bigg]
\end{aligned}
\tag{3.13}
$$

It can be seen from equation (3.13) that using the limit $x = \infty$, the second term goes to zero due to its exponential behavior. From the boundary condition, $v(\infty, t) = 0$, equation (3.13) is simplified to

$$
\begin{aligned}
V(\infty,s) =& \frac{e^{\infty\sqrt{\alpha s+\beta s^2}}}{2}\Bigg[V(0,s) + \frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}} - \lim_{x\to\infty}\frac{\alpha}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\
& - \lim_{x\to\infty}\frac{\beta s}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\
& - \lim_{x\to\infty}\frac{\beta}{\sqrt{\alpha s+\beta s^2}}e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda\Bigg] = 0
\end{aligned}
\tag{3.14}
$$

Equivalently,

$$
\begin{aligned}
V(0,s) + \frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}} - \left[\frac{\alpha}{\sqrt{\alpha s+\beta s^2}} + \frac{\beta s}{\sqrt{\alpha s+\beta s^2}}\right]\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\
- \frac{\beta}{\sqrt{\alpha s+\beta s^2}}\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda = 0
\end{aligned}
\tag{3.15}
$$

Taking the Laplace transformation of equation (3.3), we can find out the current $I(x,s)$ anywhere in the line which is given by

$$
I(x,s) = \frac{1}{R+sL}\frac{dV(x,s)}{dx} \tag{3.16}
$$

Evaluating equation (3.16) at $x = 0$, the source current can be found as

$$
I(0,s) = \frac{1}{R+sL}\frac{dV(0,s)}{dx} = \frac{V^*(0,s)}{R+sL} \tag{3.17}
$$

Solving for voltage in terms of source current gives

$$\begin{aligned} V(0,s) = -\frac{(R+sL)I(0,s)}{\sqrt{\alpha s+\beta s^2}} + \left[\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\right]\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\ + \frac{\beta}{\sqrt{\alpha s+\beta s^2}}\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda \end{aligned} \tag{3.18}$$

Setting the initial condition as no voltage distribution in the line gives the driving point impedance of the transmission line which is of fractional order. The driving point impedance function $Z(s)$ is defined as,

$$Z(s) = \frac{V(0,s)}{I(0,s)}$$

So, the driving point impedance of the lossy RLC transmission line is given by

$$Z(s) = \frac{R+sL}{\sqrt{\alpha s+\beta s^2}}$$

or as $\alpha = RC$ and $\beta = LC$,

$$Z(s) = \frac{R+sL}{\sqrt{RCs+LCs^2}} \tag{3.19}$$

It can be seen that the integrals of equation (3.15) are equivalent to a Laplace transform integral with,

$$s \to q = \sqrt{\alpha s+\beta s^2}$$

Thus, the Laplace transform table can be used to simplify the evaluation of these integral terms as follows:

$$\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda = \Big[V(q,0)\Big]_{q=\sqrt{\alpha s+\beta s^2}}$$
$$\int_0^\infty e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda = \Big[V^*(q,0)\Big]_{q=\sqrt{\alpha s+\beta s^2}}$$

The notation here on the right hand side of these equations is used to indicate the evaluation procedure. The initial spatial voltage distribution $v(x,0)$ is Laplace transformed with respect to the spatial Laplace variable p to give $V(p,0)$. The integral on the left side of the above equations are then easily calculated by replacing the spatial variable p with

$$p = q = \sqrt{\alpha s+\beta s^2}$$

So, equation (3.15) can be written as

$$V(0,s)+\frac{V^*(0,s)}{\sqrt{\alpha s+\beta s^2}}=\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\Big[V(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}}+\frac{\beta}{\sqrt{\alpha s+\beta s^2}}\Big[V^*(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}} \tag{3.20}$$

Now, using equation (3.20) and since $V(0,s)=V_I(s)$, we can write equation (3.13) as

$$\begin{aligned} V(x,s)=&\frac{e^{+x\sqrt{\alpha s+\beta s^2}}}{2}\Bigg[\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\Big[V(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}}+\frac{\beta}{\sqrt{\alpha s+\beta s^2}}\Big[V^*(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}} \\ &-\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda-\frac{\beta}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{-\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda\Bigg] \\ &+\frac{e^{-x\sqrt{\alpha s+\beta s^2}}}{2}\Bigg[2V_I(s)-\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\Big[V(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}}-\frac{\beta}{\sqrt{\alpha s+\beta s^2}} \\ &\Big[V^*(p,0)\Big]_{p=\sqrt{\alpha s+\beta s^2}}+\frac{\alpha+\beta s}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{\lambda\sqrt{\alpha s+\beta s^2}}V(\lambda,0)d\lambda \\ &+\frac{\beta}{\sqrt{\alpha s+\beta s^2}}\int_0^x e^{\lambda\sqrt{\alpha s+\beta s^2}}V^*(\lambda,0)d\lambda\Bigg] \end{aligned} \tag{3.21}$$

This is the final expression of voltage anywhere in the line as a function of applied voltage and the initial voltage distribution of the line. This expression represents a general case of the expression given in [25].

For the case of transmission lines with zero initial spatial voltage distribution $V(x,0)=0$, the above expression can be reduced to

$$V(x,s)=e^{-x\sqrt{\alpha s+\beta s^2}}V_I(s) \tag{3.22}$$

where, $V(x,s)$ is the voltage anywhere in the line and $V_I(s)$ is the applied voltage at the entrance in the Laplace domain. Considering $V(x,s)$ as the output at a distance x from the entrance of the line and $V_I(s)$ as input, we can find out the transfer function of the semi-infinite lossy transmission line with zero initial spatial voltage distribution as follows:

$$G_v(s)=\frac{V(x,s)}{V_I(s)}=e^{-x\sqrt{\alpha s+\beta s^2}}=e^{-x\sqrt{RCs+LCs^2}} \tag{3.23}$$

Similarly we can find out the current of the transmission line at a distance x from the entrance from equation (3.16):

$$\begin{aligned} I(x,s) =& \frac{1}{R+sL}\frac{dV(x,s)}{dx} \\ =& \frac{\sqrt{Cs}}{\sqrt{R+sL}}e^{-x\sqrt{RCs+LCs^2}}V_I(s) \end{aligned} \tag{3.24}$$

where $I(x,s)$ is the current anywhere in the line and $V_I(s)$ is the applied voltage at the entrance in the Laplace domain. Considering $I(x,s)$ as the output at a distance x from the entrance of the line and $V_I(s)$ as input, we can also find out the transfer function of the semi-infinite uniform lossy transmission line with zero initial spatial voltage distribution:

$$G_i(s) = \frac{I(x,s)}{V_I(s)} = \frac{\sqrt{Cs}}{\sqrt{R+sL}}e^{-x\sqrt{RCs+LCs^2}} \tag{3.25}$$

The above transmission line transfer functions are fractional order in nature with a few number of parameters. The transfer functions are generalized case of the model described in [25], [54]. Considering the line inductance $L = 0$, equation (3.23) and (3.25) becomes

$$V(x,s) = e^{-x\sqrt{RCs}}V_I(s) \tag{3.26}$$

and

$$I(x,s) = \sqrt{\frac{Cs}{R}}e^{-x\sqrt{RCs}}V_I(s) \tag{3.27}$$

which match with the model described in [25], [54].

The characteristic impedance of a transmission line is defined as the ratio of the voltage to current of the line. So, the characteristic impedance of the semi-infinite lossy transmission line is

$$Z_0(s) = \frac{V(x,s)}{I(x,s)} = \sqrt{\frac{R+sL}{Cs}} \tag{3.28}$$

It can be seen that the characteristic impedance leads us to a fractional order expression. For a lossless line where $R = 0$ the characteristic impedance in the frequency domain leading to two port network with integer order elements:

$$Z_0(s) = \sqrt{\frac{L}{C}}$$

But for the case of a lossy line with R, L, C $\in \mathbb{R}^+$, we can have half-order fractional inductances and half-order fractional capacitances, that is, $-\pi/4 \le \arg(Z_0(s)) \le \pi/4$. For an example, if $L = 0$

$$Z_0(j\omega) = [(j\omega)^{-1} R C^{-1}]^{1/2}$$

These results were overlooked in the classical textbooks [23].

3.4 Terminated Transmission Lines

3.4.1 Lines Terminated with Characteristic Impedance $Z_0(s)$

Figure 3.1 showed a semi-infinite transmission line driven by a voltage source V_I. For the infinite line only forward traveling waves exist in the line because there is no reflection waves. Thus the driving point impedance is equal to the characteristic impedance of the line.

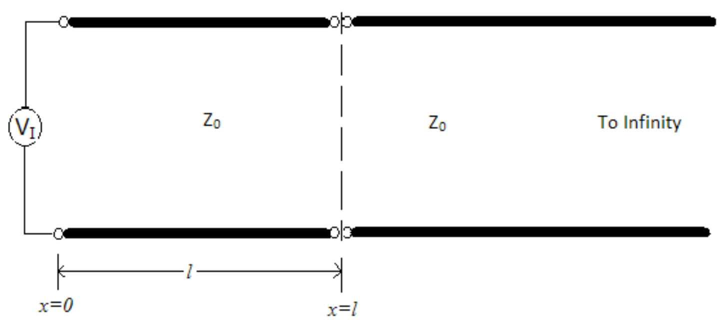


Figure 3.2: Infinite length transmission line.

Suppose the semi-infinite line is broken at $x = l$ as shown in Figure 3.2. Since the length of the line to the right of $x = l$ is still infinite therefore replacing it by a load impedance with its input impedance does not change any of the condition to the left of $x = l$. This means that a finite length transmission line terminated by its characteristic impedance is equivalent to an infinitely long line. Like the infinite case, a finite length line terminated with $Z_0(s)$ shown in Figure 3.3, has no reflection waves and the characteristic impedance is independent of the length of the line.

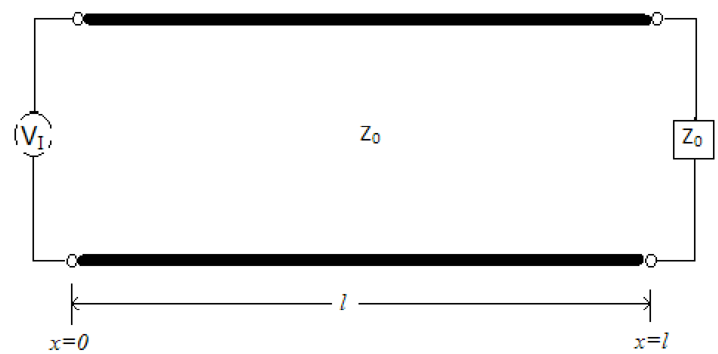


Figure 3.3: Equivalent line.

Since the line characteristic impedance is known, we can find out the voltage and current of the terminated (with characteristic impedance) line using equations (3.22) and (3.24) respectively. For an example at $x = l$, the load voltage and current will be

$$V(l, s) = V_L = e^{-l\sqrt{\alpha s + \beta s^2}} V_I(s) \tag{3.29}$$

and

$$I(l, s) = I_L = \frac{V_I(s)}{Z_0(s)} e^{-l\sqrt{\alpha s + \beta s^2}} \tag{3.30}$$

The above equations show that a finite length lossy transmission line terminated with its characteristic impedance is equivalent to a semi-infinite lossy transmission line. Therefore, a finite length lossy transmission line with matched load (terminated by its characteristic impedance) can also be modeled by fractional order calculus.

3.4.2 Lines Terminated with General Load, Z_L

A finite length transmission line with matched load is an ideal case. It is not always possible to terminate a transmission line with a matched load in practice. The load of the transmission line always varies. So, a fractional order model for finite length transmission line with arbitrary load is necessary. A finite length transmission line terminated with an arbitrary load Z_L will have the reflection waves coming from the load end. This reflection occurs in the transmission line when $Z_L \neq Z_0$. The general expressions of the voltage and current of this transmission line have two traveling waves as forward and reflected waves. The forward traveling waves are defined in the previous section and the reflected waves can be found by putting the sign of x as negative since the direction of these waves are opposite. So, the voltage and current of the line can be written as

$$V(x,s) = V_+ + V_- = V_0^+(s)e^{-x\sqrt{\alpha s+\beta s^2}} + V_0^-(s)e^{+x\sqrt{\alpha s+\beta s^2}} \tag{3.31}$$

$$\begin{aligned} I(x,s) =& I_+ - I_- = I_0^+(s)e^{-x\sqrt{\alpha s+\beta s^2}} - I_0^-(s)e^{+x\sqrt{\alpha s+\beta s^2}} \\ &= \frac{1}{Z_0(s)}\left[V_0^+(s)e^{-x\sqrt{\alpha s+\beta s^2}} - V_0^-(s)e^{+x\sqrt{\alpha s+\beta s^2}}\right] \end{aligned} \tag{3.32}$$

Let a lossy RLC transmission line with length l be terminated with a load impedance of Z_L shown in Figure 3.4. The characteristic impedance of the line is defined as equation (3.28).

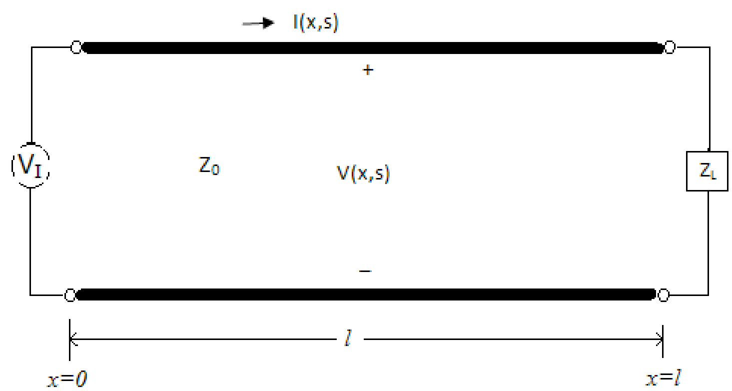


Figure 3.4: A lossy transmission line terminated by a load impedance Z_L.

From Figure 3.4, the boundary condition at the receiving end is

$$V(l,s) = Z_L(s)I(l,s) \tag{3.33}$$

Now substituting equations (3.31) and (3.32) in equation (3.33), we have

$$V_0^+(s)e^{-l\sqrt{\alpha s+\beta s^2}} + V_0^-(s)e^{+l\sqrt{\alpha s+\beta s^2}} = \frac{Z_L(s)}{Z_0(s)}\left[V_0^+(s)e^{-l\sqrt{\alpha s+\beta s^2}} - V_0^-(s)e^{+l\sqrt{\alpha s+\beta s^2}}\right]$$

Solving for $V_0^-(s)$, gives

$$V_0^-(s) = \Gamma_L(s)V_0^+(s)e^{-2l\sqrt{\alpha s+\beta s^2}} \tag{3.34}$$

where $\Gamma_L(s)$ is the receiving end voltage (or current) reflection coefficient which is defined as the ratio of the reflected to forward voltages (or currents).

$$\Gamma_L(s) = \frac{\frac{Z_L(s)}{Z_0(s)} - 1}{\frac{Z_L(s)}{Z_0(s)+1}} \tag{3.35}$$

Using equation (3.34) in equations (3.31) and (3.32), gives

$$\begin{aligned} V(x,s) =& V_0^+(s)\Big[e^{-x\sqrt{\alpha s+\beta s^2}} + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}e^{+x\sqrt{\alpha s+\beta s^2}}\Big] \\ =& V_0^+(s)\Big[e^{-x\sqrt{\alpha s+\beta s^2}} + \Gamma_L(s)e^{(x-2l)\sqrt{\alpha s+\beta s^2}}\Big] \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} I(x,s) =& \frac{V_0^+(s)}{Z_0(s)}\Big[e^{-x\sqrt{\alpha s+\beta s^2}} - \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}e^{+x\sqrt{\alpha s+\beta s^2}}\Big] \\ =& \frac{V_0^+(s)}{Z_0(s)}\Big[e^{-x\sqrt{\alpha s+\beta s^2}} - \Gamma_L(s)e^{(x-2l)\sqrt{\alpha s+\beta s^2}}\Big] \end{aligned} \tag{3.37}$$

From Figure 3.4, the boundary condition at the sending end $V(0,s) = V_I(s)$ leads to

$$V_0^+(s)\Big[1 + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}\Big] = V_I(s)$$

Thus

$$V_0^+(s) = \frac{V_I(s)}{1 + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}} \tag{3.38}$$

Now substituting equation (3.38) in equations (3.36) and (3.37), we get

$$V(x,s) = V_I(s)\Big[\frac{e^{-x\sqrt{\alpha s+\beta s^2}} + \Gamma_L(s)e^{(x-2l)\sqrt{\alpha s+\beta s^2}}}{1 + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}}\Big] \tag{3.39}$$

and

$$I(x,s) = \frac{V_I(s)}{Z_0(s)}\Big[\frac{e^{-x\sqrt{\alpha s+\beta s^2}} - \Gamma_L(s)e^{(x-2l)\sqrt{\alpha s+\beta s^2}}}{1 + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}}\Big] \tag{3.40}$$

Considering $V(x,s)$ as the output voltage at a distance x from the entrance of the line and $V_I(s)$ as input voltage, we can find out the transfer function of the finite length lossy transmission line (terminated with general load impedance $Z_L(s)$) as

$$G_v(s) = \frac{V(x,s)}{V_I(s)} = \frac{e^{-x\sqrt{\alpha s+\beta s^2}} + \Gamma_L(s)e^{(x-2l)\sqrt{\alpha s+\beta s^2}}}{1 + \Gamma_L(s)e^{-2l\sqrt{\alpha s+\beta s^2}}} \tag{3.41}$$

Similarly, considering $I(x,s)$ as the output current at a distance x from the entrance of the line and V_G as input voltage, we can find out the transfer function of the finite length lossy transmission line (terminated with general load impedance $Z_L(s)$) as

$$G_i(s) = \frac{I(x,s)}{V_I(s)} = \frac{1}{Z_0(s)} \frac{e^{-x\sqrt{\alpha s+\beta s^2}} - \Gamma_L(s) e^{(x-2l)\sqrt{\alpha s+\beta s^2}}}{1+\Gamma_L(s) e^{-2l\sqrt{\alpha s+\beta s^2}}} \tag{3.42}$$

The above transfer functions is a generalized fractional order model of a finite length lossy transmission line for any arbitrary load connected at the receiving terminal. The models derived in the previous sections are special cases of this model. For example, if we consider a matched model ($Z_L(s) = Z_0(s)$), the receiving end voltage (or current) reflection coefficient $\Gamma_L(s)$ becomes zero. In that case, the models in equations (3.41) and (3.42) become

$$G_v(s) = \frac{V(x,s)}{V_I(s)} = e^{-x\sqrt{\alpha s+\beta s^2}} \tag{3.43}$$

and

$$G_i(s) = \frac{I(x,s)}{V_I(s)} = \frac{1}{Z_0(s)} e^{-x\sqrt{\alpha s+\beta s^2}} \tag{3.44}$$

which are the same models as in equations (3.23) and (3.25).

3.5 Conclusion

In this chapter we have derived a generalized fractional order transmission line model for a lossy transmission line. A finite length transmission line terminated with arbitrary load exhibits both forward and reflected traveling waves; on the other hand semi-infinite lines or equivalently terminated with matched loads have only forward traveling waves. The transfer functions for both of the cases are fractional order with infinite dimensions with a small number of parameter. It has been shown that distributed parameter systems like transmission lines can be modeled using the fractional calculus with a small number of parameters. Since the transfer functions that have been derived are infinite dimensional in nature, it is expected that they capture the transmission line phenomena better [24].

Chapter 4

Frequency Responses of Fractional Order Transmission Lines

4.1 Introduction

In the steady state, for an LTI system a sinusoidal input generates a sinusoidal output of the same frequency. Even though the frequency of the output is the same as the input, the phase and magnitude of the response differ from the input. These differences are functions of the frequency.

A sinusoidal signal can be represented by a complex number which is called a *phasor*. The magnitude of the complex number represents the amplitude of the sinusoid and the angle represents the phase of that sinusoid. This means that a sinusoid $A\cos(\omega t + \phi)$ can be presented by the phasor $A\angle\phi$, of the frequency ω.

Since an LTI system changes the amplitude and the phase of an input signal, we can represent the system itself as a complex number or phasor, so that the product of the input and system phasors will represent the output. In the system of Figure 4.1, the input signal is a sinusoid represented by phasor, $A_i(\omega)\angle\phi_i(\omega)$ having frequency ω and the steady state response of the system is also a sinusoid of same frequency represented by $A_o(\omega)\angle\phi_o(\omega)$. If the system phasor is represented by $G(\omega)\angle\phi(\omega)$, the output of the system will be the product of the system phasor and the input phasor. Thus, the steady state output of the system is

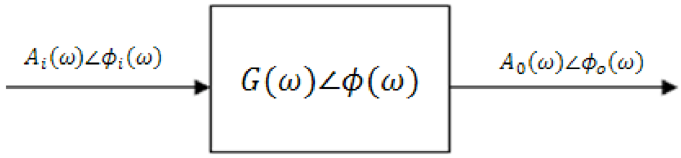


Figure 4.1: Sinusoidal response of an LTI system.

$$A_o(\omega)\angle\phi_o(\omega) = G(\omega)A_i(\omega)\angle\big[\phi(\omega) + \phi_i(\omega)\big] \tag{4.1}$$

From the above equation, we can find the magnitude and the phase of the system as

$$|G(\omega)| = \frac{A_o(\omega)}{A_i(\omega)} \tag{4.2}$$

and

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega) \tag{4.3}$$

Equation (4.2) and (4.3) are called the frequency response of the system. $|G(\omega)|$ is known as the magnitude/gain response and $\phi(\omega)$ is known as the phase response.

4.2 Frequency Responses of Fractional Order Transmission lines

4.2.1 Semi-infinite Line or Terminated Line with Characteristic Impedance

The transfer function of a semi-indefinite lossy transmission line has been derived in the previous chapter, which is equivalent to a finite length transmission line terminated by its characteristic impedance defined by equation (3.28). Taking $s = j\omega$ into the transfer function, we can obtain the frequency domain expression as

$$\begin{aligned} G_v(j\omega) &= e^{-x\sqrt{\alpha(j\omega)+\beta(j\omega)^2}} \\ &= e^{-x\sqrt{-\beta\omega^2+j\alpha\omega}} \end{aligned} \tag{4.4}$$

where $\sqrt{-\beta\omega^2 + j\alpha\omega}$ is a complex number. Let $p + jq = \sqrt{-\beta\omega^2 + j\alpha\omega}$, where p is the real part of this complex number and q is the imaginary part. After simple calculations, we can find the p and q as

$$p = \sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2 + \alpha^2} - \beta\omega^2)} \tag{4.5}$$

and

$$q = \sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2 + \alpha^2} + \beta\omega^2)}. \tag{4.6}$$

Using p and q in equation (4.4), the frequency response of the transfer function of the semi-infinite line or terminated line with characteristic impedance can be found as

$$\begin{aligned} G_v(j\omega) =& e^{-x\left[\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}-\beta\omega^2)}+j\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}+\beta\omega^2)}\right]} \\ =& \underbrace{e^{-x\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}-\beta\omega^2)}}}_{Gain} \underbrace{e^{-jx\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}+\beta\omega^2)}}}_{Phase}. \end{aligned} \tag{4.7}$$

The first exponential of the above expression represents the gain of the system, while the second one represents the phase response. So, the gain and phase responses of the fractional order transmission line are defined as

$$|G_v(\omega)| = e^{-x\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}-\beta\omega^2)}} \tag{4.8}$$

and

$$\phi_v(\omega) = -x\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2 + \alpha^2} + \beta\omega^2)} \tag{4.9}$$

The frequency response of the semi-infinite or matched line can be used to obtain the Bode plot of the infinite dimensional fractional order transmission line. The gain expression suggests that the gain of the transmission line exponentially decreases with the increase of distance and the frequency. The phase response is also a function of distance and the frequency of the transmission line.

4.2.2 Line Terminated by General Load Impedance

The transfer function of the finite length lossy transmission line terminated with general load Z_L has been defined by equation (3.41) in the previous chapter. This transfer function is cumbersome compared to the semi-infinite length transmission line because of the presence of reflecting traveling waves. For the sake of simplicity, we consider a load which is twice as the characteristic impedance of the line, that is $Z_L(s) = 2Z_0(s)$. In this case, the receiving end voltage reflection coefficient becomes

$$\Gamma_L(s) = \frac{\frac{Z_L(s)}{Z_0(s)} - 1}{\frac{Z_L(s)}{Z_0(s)+1}} = \frac{1}{3}$$

Evaluating the transfer function (in equation (3.41)) at the load end $x = l$ with the above load condition, we obtain

$$\begin{aligned} G_v(s) &= \frac{e^{-l\sqrt{\alpha s+\beta s^2}} + \frac{1}{3}e^{-l\sqrt{\alpha s+\beta s^2}}}{1 + \frac{1}{3}e^{-2l\sqrt{\alpha s+\beta s^2}}} \\ &= \frac{4e^{-l\sqrt{\alpha s+\beta s^2}}}{3 + e^{-2l\sqrt{\alpha s+\beta s^2}}} \end{aligned}$$

Now taking $s = j\omega$, we obtain the frequency domain expression as

$$G_v(j\omega) = \frac{4e^{-l\sqrt{-\beta\omega^2+j\alpha\omega}}}{3 + e^{-2l\sqrt{-\beta\omega^2+j\alpha\omega}}} \tag{4.10}$$

As $\sqrt{-\beta\omega^2 + j\alpha\omega} = p + jq$ defined earlier, equation (4.10) can be written as

$$\begin{aligned} G_v(j\omega) &= \frac{4e^{-l(p+jq)}}{3 + e^{-2l(p+jq)}} \\ &= \frac{4e^{-pl}e^{-jql}}{3 + e^{-2pl}e^{-j2ql}} \end{aligned} \tag{4.11}$$

Using the Euler's formula the above expression becomes

$$\begin{aligned} G_v(j\omega) &= \frac{4e^{-pl}\{\cos ql - j\sin ql\}}{3 + e^{-2pl}\{\cos 2ql - j\sin 2ql\}} \\ &= \frac{4e^{-pl}\cos ql - j4e^{-pl}\sin ql}{3 + e^{-2pl}\cos 2ql - je^{-2pl}\sin 2ql} \end{aligned} \tag{4.12}$$

Taking $a = 4e^{-pl}\cos ql$, $b = 4e^{-pl}\sin ql$, $c = 3 + e^{-2pl}\cos 2ql$ and $d = e^{-2pl}\sin 2ql$, the above equation simplifies to

$$\begin{aligned} G_v(j\omega) &= \frac{a - jb}{c - jd} \\ &= \left[\frac{ac + bd}{c^2 + d^2}\right] - j\left[\frac{bc - ad}{c^2 + d^2}\right] \end{aligned} \tag{4.13}$$

The first term of equation (4.13) is

$$\begin{aligned}\Big[\frac{ac+bd}{c^2+d^2}\Big] &= \frac{(4e^{-pl}\cos ql)(3+e^{-2pl}\cos 2ql)+(4e^{-pl}\sin ql)(e^{-2pl}\sin 2ql)}{(3+e^{-2pl}\cos 2ql)^2+(e^{-2pl}\sin 2ql)^2} \\ &= \frac{4e^{-pl}\cos ql(3+e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}\end{aligned} \tag{4.14}$$

Similarly the second term of the equation (4.13) becomes

$$\begin{aligned}\Big[\frac{bc-ad}{c^2+d^2}\Big] &= \frac{(4e^{-pl}\sin ql)(3+e^{-2pl}\cos 2ql)-(4e^{-pl}\cos ql)(e^{-2pl}\sin 2ql)}{(3+e^{-2pl}\cos 2ql)^2+(e^{-2pl}\sin 2ql)^2} \\ &= \frac{4e^{-pl}\sin ql(3-e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}\end{aligned} \tag{4.15}$$

Using equations (4.14) and (4.15), equation (4.13) becomes

$$G_v(j\omega) = \frac{4e^{-pl}\cos ql(3+e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql} - j\frac{4e^{-pl}\sin ql(3-e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql} \tag{4.16}$$

From the above equation the gain response can be found as

$$|G_v(\omega)| = \sqrt{\Big(\frac{4e^{-pl}\cos ql(3+e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}\Big)^2+\Big(\frac{4e^{-pl}\sin ql(3-e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}\Big)^2}.$$

After doing some simplifications (details are given in the Appendix A), $|G_v(\omega)|$ can be found as

$$|G_v(\omega)| = \frac{\sqrt{\{4e^{-pl}(3-e^{-2pl})\}^2+\{\sqrt{192}e^{-2pl}\cos ql\}^2}}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}. \tag{4.17}$$

Similarly the phase response can be found as (see the Appendix A for details)

$$\phi_v(\omega) = -\arctan\{\tan ql\Big(\frac{3-e^{-2pl}}{3+e^{-2pl}}\Big)\} \tag{4.18}$$

The above frequency response expressions of the finite length lossy fractional order transmission line can be used to obtain the Bode plot of the infinite dimensional transfer function.

4.3 Frequency Response Validation

To validate our derived frequency responses of the lossy fractional order transmission lines, we have simulated a transmission line model using the distributed parameter line block of the *SimPowerSystems* toolbox in SIMULINK. First, we simulated a transmission line terminated by its characteristic impedance which is equivalent to a semi-infinite length transmission line. Next, we simulated the second case where a finite length transmission line is considered with a load which is twice the characteristic impedance of the line. The line parameters are chosen as $l = 500$ km, $R = 6.86$ Ω/km, $C = 4.34$ pF/km, $L = 2.6$ mH/km. The simulations have been done for a range of frequencies ranging from 10 to 300 Hz. Using the simulation data, the gain and phase of the line for every frequency have been calculated and the Bode plots of the transmission lines are obtained.

The Bode plots of the transmission lines can also be obtained from the frequency response models that have been derived in this chapter. Comparison between these Bode plots can be used to validate the frequency response modeling of the fractional order transmission lines.

Comparison between the Bode plots of the transmission line terminated with its characteristic impedance is shown in Figure 4.2. Similarly, the comparison of the Bode plots of the transmission line terminated by a load which is twice the characteristic impedance is shown in Figure 4.3.

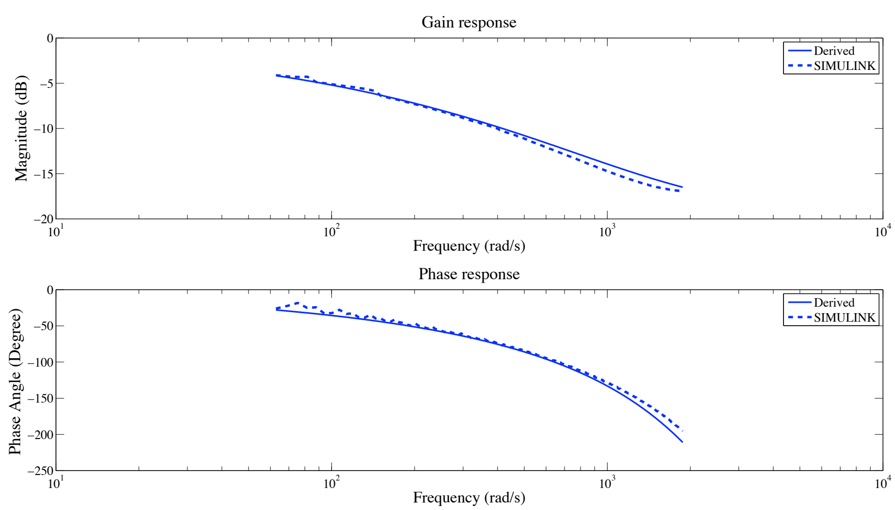


Figure 4.2: Comparison of the Bode plots of semi-infinite length or terminated line with matched load.

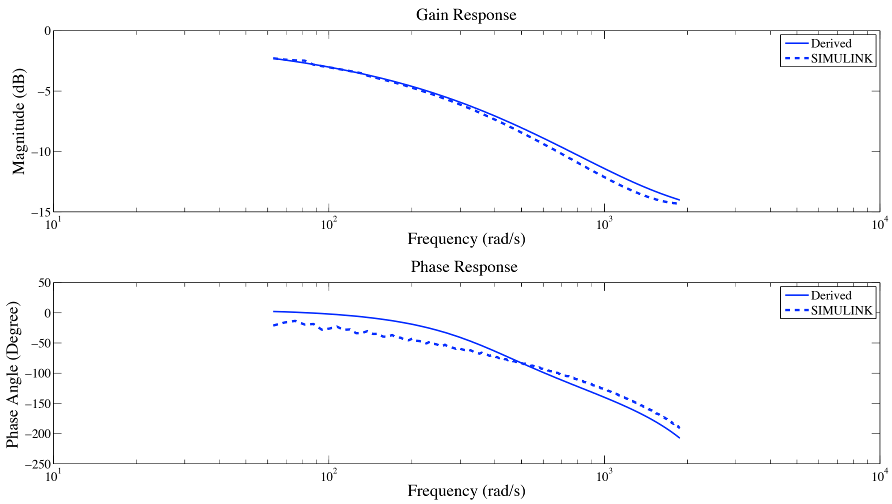


Figure 4.3: Comparison of the Bode plots of finite length line terminated with load $2Z_0(s)$.

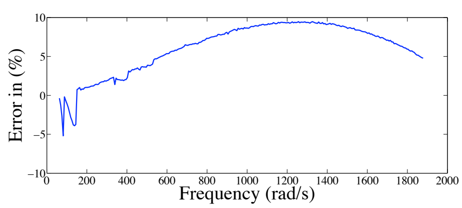
From Figures 4.2 and 4.3, we can see that the Bode plots of the transmission lines simulated by SIMULINK and the frequency response models closely match with each other. Gain responses for both of the cases are very similar while the phase responses have some discrepancies in the low frequency range. These discrepancies could be the result of simulation limitation in identifying the phase difference between the input and output. To identify the phase difference between the input and output sinusoidal signals, there are no functions or blocks presented in MATLAB or SIMULINK. The phase difference between two continuous time sinusoidal signals can be identified by finding out the lag between the signals in the time scale. The cross correlation between the signals can be used to identify the lag between the signals. But in our simulation the input and output signals are discrete in nature with nonuniform sampling periods. This is due to the fact that, the solver used by distributed parameter line block in SIMULINK works only with variable step sampling periods. For this reason, the exact lag between the input and output signals cannot be identified. So, the discrepancies in the phase responses are the result of this simulation limitation. Beside, the distributed parameter transmission line block is based on the Bergeron's traveling wave method used by the Electromagnetic Transient Program (EMTP)[56]. In this model the line losses are considered as lumped by lumping $R/4$ at both ends of the line and $R/2$ in the middle. Since our derived models have infinite dimension structure with distributed parameters, it is expected to have some discrepancies between the results.

To compare the gain and phase responses of both cases the Root-Mean-Square-Error (RMSE) and the Mean-Percentage-Error (MPE) of the responses are calculated and are given in the table 4.1.

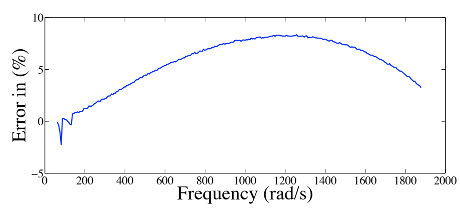
Table 4.1: RMSE and MPE of the Bode Plots

	RMSE	MPE
Gain response of semi-infinite length line	0.01381	6.1%
Gain response of finite length line	0.01664	5.6%
Phase response of semi-infinite length line	7.77801	4.7%
Phase response of finite length line	14.86291	1.9 %

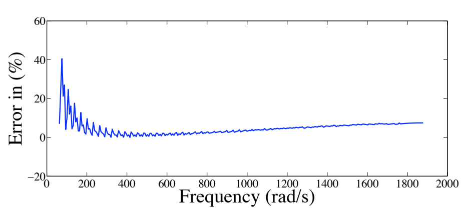
The above table shows that the RMSE of the phase responses are very high due to the large discrepancies in the low frequency ranges, but the MPE of the phase responses are very low which reflect a successful model validation. The percentage errors of gain and phase responses of both cases can be found in Figure 4.4.



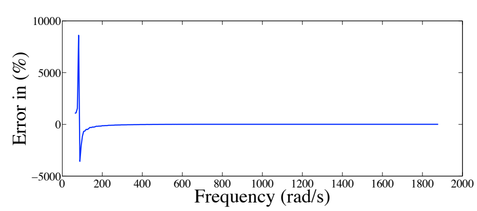
(a) Semi-infinite length line's gain response error in %



(b) Finite length line's gain response error in %



(c) Semi-infinite length line's phase response error in %



(d) Finite length line's phase response error in %

Figure 4.4: Percentage error of frequency response models.

From Figure 4.4, it can be seen that the percentage error of the phase responses are very high at the low frequency range while very low in the high frequency ranges. Due to this high percentage error in the low frequency range the RMSE of the phase responses are high which is stated above in the table. But the overall percentage error of the derived frequency response models are at the satisfactory level.

4.4 Conclusion

This chapter presented the frequency responses of the fractional order transmission line models that we have derived in the previous chapter. The gain and phase responses were derived for both semi-infinite and finite length transmission lines connected with loads. Simulations results have been presented to validate the derived model at the end. The simulation results show that our derived models are quite satisfactory.

Chapter 5

Parameter Identification of Fractional Order Transmission Lines

5.1 Introduction

System identification has become a standard tool for identifying the parameters of unknown systems in control engineering and scientific community. It is a necessary prerequisite for model based control to have a model of the system. Identifying a given system from the input output data becomes more difficult for fractional order systems. The use of fractional differential models was initiated in the late 1990s and the beginning of this century [57], [58], [59], [60]. Frequency domain system identification for fractional order systems was initiated by Lay in his PhD thesis [57]. Time domain system identification of fractional order systems was initiated by Lay [57], Cois [59], Lin [60]. Two classes of time domain identification were developed as Equation-Error-Based and Output-Error-Based methods.

Since a fractional order model of a lossy transmission line has an irrational transfer function with an exponential term in it, it is hard to find the time domain expressions of the system. Because of this difficulty a frequency domain identification method was developed in this thesis for a fractional order lossy transmission line model.

5.2 Parameter Identification Techniques

There are several parameter identification techniques that are discussed in the literature [61]. Regression analysis is one of the most used technique. Linear and nonlinear regression analysis for parameter identification will be discussed in the this chapter.

5.2.1 Linear Regression Analysis

A linear regression model with multiple regressor variables is known as multiple regression model and can be written as

$$y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k + e \tag{5.1}$$

where θ_0 is the intercept, $[\theta_1, \ldots, \theta_k]$ are regression coefficients and e is independent identically distributed (iid) noise. The dependent variable or response y is related to k independent variables or regressor variables.

To estimate the regression coefficients of the multiple regression model, a least square estimation technique can be used. Suppose N number of observations are available, and let x_{ij} denote the i^{th} observation of regressor x_j. So, the observations are

$$(x_{i1}, x_{i2}, \ldots, x_{ik}, y_i)$$

where $i = 1, 2, \ldots, N$ and $N > k$. Multiple linear regression data can be presented in a table form shown in table 5.1. Each of the observations $(x_{i1}, x_{i2}, \ldots, x_{ik}, y_i)$ satisfies

Table 5.1: Data for multiple linear regression

y	x_1	x_2	$\ldots$	x_k
y_1	x_{11}	x_{12}	$\ldots$	x_{1k}
y_2	x_{21}	x_{22}	$\ldots$	x_{2k}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
y_N	x_{N1}	x_{N2}	$\ldots$	x_{Nk}

the multiple regression model:

$$y_i = \theta_0 + \theta_1 x_{i1} + \theta_2 x_{i2} + \ldots + \theta_k x_{ik} + e_i.$$

The least square objective function can be written as

$$\begin{aligned} L &= \sum_{i=1}^{N} e_i^2 \\ &= \sum_{i=1}^{N} (y_i - \theta_0 - \sum_{j=1}^{k} \theta_j x_{ij})^2 \end{aligned} \tag{5.2}$$

The objective is to minimize the objective function with respect to $\theta_0, \theta_1, \ldots, \theta_k$. It is more convenient to use matrix notation for the multiple regression model. Let

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \qquad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{N1} & x_{N2} & \ldots & x_{Nk} \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix} \qquad \epsilon = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$$

So, the N equations of the multiple regression model are simplified to

$$Y = X\theta + \epsilon \tag{5.3}$$

where, Y is a $(N \times 1)$ vector of the observations or dependent variables, X is a $(N \times p)$ matrix of the levels of independent variables, θ is a $(p \times 1)$ vector of regression coefficients, and ϵ is a $(N \times 1)$ vector of random errors. The target is to find the least square vector estimator $\hat{\theta}$ that minimizes the the objective function

$$\begin{aligned} L &= \sum_{i=1}^{N} e_i^2 \\ &= \epsilon^T \epsilon \\ &= (Y - X\theta)^T (Y - X\theta) \end{aligned} \tag{5.4}$$

The least square estimator $\hat{\theta}$ can be found from the solution of the equation

$$\frac{\partial L}{\partial \theta} = 0 \tag{5.5}$$

The resulting equations that must be solved are

$$X^T X \hat{\theta} = X^T Y$$

The solution of these equations can be found as

$$\hat{\theta} = (X^T X)^{-1} X^T Y \tag{5.6}$$

where $\hat{\theta}$ is

$$\hat{\theta} = \begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \vdots \\ \hat{\theta}_k \end{bmatrix}$$

The fitted or estimated model can be written as

$$\hat{y}_i = \hat{\theta}_0 + \sum_{j=1}^{k} \hat{\theta}_j x_{i,j} \tag{5.7}$$

where $i = 1, 2, \ldots, N$. In matrix forms the fitted model is

$$\hat{Y} = X\hat{\theta} \tag{5.8}$$

5.2.2 Nonlinear Estimation

In the most general terms, nonlinear estimation will compute the relationship between a set of independent variables and a dependent variable. Nonlinear estimation is a general fitting procedure that will estimate a nonlinear relationship between a dependent and a list of independent variables. In general, a nonlinear model can be stated as

$$y = f(x_1, x_2, \ldots, x_k, \theta_0, \theta_1, \ldots, \theta_k) + e \tag{5.9}$$

where e is an independent identically distributed (iid) noise. The function $f(x_1, x_2, \ldots, x_k, \theta_0, \theta_1, \ldots, \theta_k)$ is known as the expectation function. For an example, an exponential model

$$y = \theta_0(1 - e^{-\theta_1 \xi}) + e$$

is a nonlinear regression model where θ_0, θ_1 are regression coefficients and ξ is an independent variable.

There are many estimation techniques available for nonlinear regression models such as the least square estimation, weighted least square, and maximum likelihood estimation. Among these, the least square estimation is the most used estimation technique. In the most general terms, least square estimation is aimed at minimizing the sum of squared deviations of the observed values for the dependent variable from those predicted by the model.

5.2.3 Nonlinear Least Square Estimation

To estimate a nonlinear regression model using least square estimation, Gauss suggested an approach to use a linear approximation of the expectation function to iteratively improve an initial guess θ^0 for θ and keep improving the estimates until there is no change. This is also known as the Gauss-Newton method. In this method, the the vector form of the parameters is taken as

$$\theta = [\theta_0, \theta_1, \ldots, \theta_k]^T$$

and the vector for the independent variables is taken as

$$\phi = [x_1, x_2, \ldots, x_k]^T$$

Then expanding the expectation function $f(\phi, \theta)$ in a first order Taylor series about the initial guess θ^0, we find

$$f(\phi, \theta) \approx f(\phi, \theta_0^0) + v_1(\theta_1 - \theta_1^0) + v_2(\theta_2 - \theta_2^0) + \ldots + v_k(\theta_k - \theta_k^0) \tag{5.10}$$

where $v_k = \frac{\partial f(\phi,\theta)}{\partial \theta_k}|_{\theta^0}$ is the Jacobian of the expectation function. Now assuming N number of experiments to run and collecting N sets of measurement of the independent and dependent variables such as $[\phi_1, y_1; \phi_2, y_2; \ldots; \phi_N, y_N]$, we can write for the i^{th} measurement

$$f(\phi_i, \theta) \approx f(\phi_i, \theta_0^0) + v_{i1}(\theta_1 - \theta_1^0) + v_{i2}(\theta_2 - \theta_2^0) + \ldots + v_{ik}(\theta_k - \theta_k^0)$$

where $v_{ik} = \frac{\partial f(\phi_i,\theta)}{\partial \theta_k}|_{\theta^0}$.

Incorporating all N measurements, it will be

$$\eta(\theta) \approx \eta(\theta_0^0) + V^0(\theta - \theta^0) \tag{5.11}$$

where V^0 is the $(N \times k)$ derivative matrix with elements $\{v_{ik}\}$ and $\eta(\theta)$, $\eta(\theta_0^0)$ are

$$\eta(\theta) = \begin{bmatrix} f(\phi_1, \theta) \\ f(\phi_2, \theta) \\ \vdots \\ f(\phi_N, \theta) \end{bmatrix} \qquad \eta(\theta_0^0) = \begin{bmatrix} f(\phi_1, \theta_0^0) \\ f(\phi_2, \theta_0^0) \\ \vdots \\ f(\phi_N, \theta_0^0) \end{bmatrix}$$

Now denoting the dependent variables or observations as

$$Y = [y_1, y_2, \ldots, y_N]^T$$

We can write

$$\begin{aligned} Y &= \eta(\theta) + \epsilon \\ &\approx \eta(\theta_0^0) + V^0(\theta - \theta^0) + \epsilon \end{aligned} \tag{5.12}$$

where $\epsilon = [e_1, e_2, \ldots, e_N]^T$ is the noise vector of all measurements. Denoting $Z = Y - \eta(\theta_0^0)$ and $\tilde{\theta} = (\theta - \theta^0)$, we get the regression equation as

$$Z = V^0 \tilde{\theta} + \epsilon \tag{5.13}$$

This is a linear multiple regression model without intercept. Using the conventional linear least square method a correct $\tilde{\theta}$ can be calculated. Then correcting the initial guess θ^0, a new improved θ can be obtained. Iterating this procedure until we get a convergent solution will give us the estimation of the parameters. Nonlinear regression is an iterative procedure. The program must start with estimated initial values for each parameter. It then adjusts these values to improve the fit.

5.3 Parameter Identification of Fractional Models

Identification of the parameters of fractional order systems is more difficult than that of integer order systems. Two types of time domain identification models were discussed in [57], [59], [60] as the equation-error and output-error based methods. Equation-error based methods are suitable for systems which are linear in parameter while output-error based methods can estimate the orders of the system and model parameters simultaneously. Equation-error and output-error based models were briefly discussed by Malti et al. [62]. Only equation-error based method will be discussed in this thesis.

5.3.1 Equation-Error Based Method

The equation-error based method is able to estimate the parameters of fractional order systems which are linear in parameters. A fractional order model is based on fractional differential equation

$$y(t)+b_1D^{\beta_1}y(t)+\ldots+b_{m_B}D^{\beta_{m_B}}y(t)=a_0D^{\alpha_0}u(t)+a_1D^{\alpha_1}u(t)+\ldots+a_{m_A}D^{\alpha_{m_A}}u(t) \tag{5.14}$$

where differentiation orders $\beta_1<\beta_2<\ldots<\beta_{m_B}$ and $\alpha_0<\alpha_1<\ldots<\alpha_{m_A}$ are non-integer positive numbers. The input and output coefficients vector can be defined as

$$\theta=[a_0,a_1,\ldots,a_{m_A},b_1,b_2,\ldots,b_{m_B}]^T$$

Prior knowledge of the differential orders of the system is required in this method. Usually a commensurable order γ is chosen and then all its multiples are fixed up to a given order, say β_{m_B}. The order α_{m_A} generally set to $\beta_{m_B}-\gamma$ for a strictly proper system

$$F(s)=\frac{\sum_{k=0}^{\alpha_{m_A}/\gamma}a_ks^{k\gamma}}{1+\sum_{j=1}^{\beta_{m_B}/\gamma}b_js^{j\gamma}} \tag{5.15}$$

where the input and output data are $u(t)$ and $y^*(t)=y(t)+e(t)$ and $e(t)$ is the noise signal. If these data are sampled at regular interval of period T_s, it is possible to compute the fractional derivative of the input and output data by the GL (Grünwald-Letnikov) definition given in equation (2.4).

Then the output can be written in a regression form

$$y(t)=\phi^*(t)\theta \tag{5.16}$$

where the regression vector is given by

$$\phi^*(t)=[D^{\alpha_0}u(t)\ldots D^{\alpha_{m_A}}u(t)\ \ -D^{\beta_1}y^*(t)\ldots-D^{\beta_{m_B}}y^*(t)] \tag{5.17}$$

Estimated parameters $\hat{\theta}$ can be obtained by minimizing the quadratic norm of the error

$$J(\hat{\theta})=E^TE \tag{5.18}$$

where

$$E = [\epsilon(k_0 T_s), \epsilon((k_0+1)T_s), \ldots, \epsilon(k_0+K-1)T_s]^T$$

and

$$\epsilon(t) = y^*(t) - \phi^*(t)\hat{\theta}$$

The minimum of the error, $J(\hat{\theta})$, can be found by the least square estimation

$$\hat{\theta}_{opt} = (\Phi^{*T}\Phi^*)^{-1}\Phi^{*T}Y^* \tag{5.19}$$

where

$$\Phi^* = [\phi^{*T}(k_0 T_s), \phi^{*T}((k_0+1)T_s), \ldots, \phi^{*T}((k_0+K-1)T_s)]^T$$

Differentiation of a noisy signal amplifies the noise. Hence, a filter can be applied to minimize the noise. A linear transformation filter can be applied to equation (5.16) as to obtain a linear continuous regression of filtered input $u_f(t)$ and output $y_f^*(t)$ signals. After filtering we get

$$y_f(t) = \phi_f^*(t)\theta \tag{5.20}$$

where

$$\phi_f^*(t) = [D^{\alpha_0}u_f(t) \ldots D^{\alpha_{m_A}}u_f(t) \ \ -D^{\beta_1}y_f^*(t) \ldots - D^{\beta_{m_B}}y_f^*(t)]. \tag{5.21}$$

The filter is generally chosen to be causal, stationary and low pass. Among the possible filters, Cois et al. [63] extended the concept of state variable filters (SVFs) [64] to fractional differential systems. They proposed to use the following fractional filter:

$$H(s) = \frac{A}{\alpha_0 + \alpha_1 s^{\gamma} + \ldots + \alpha_{N_f-1}s^{\gamma(N_f-1)} + s^{\gamma N_f}}$$

where γN_f is the filter order. To design this filter the following specifications must be followed:

- $N_f > max(\beta_{m_B}, \alpha_{m_A})$.
- Coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{N_f-1}$ must be chosen such that $H(s)$ is stable.

A particular choice of SVF, proposed by Cois et al. [63] is the fractional Poisson filter

$$\begin{aligned} H(s) &= \frac{1}{\left(\left(\frac{s}{\omega_f}\right)^{\gamma}+1\right)^{N_f}} \\ &= \frac{\omega_f^{\gamma N_f}}{s^{\gamma N_f}+\binom{N_f}{1}\omega_f^{\gamma}s^{\gamma(N_f-1)}+\ldots+\binom{N_f}{N_f-1}\omega_f^{\gamma(N_f-1)}s^{\gamma}+\omega_f^{\gamma N_f}} \end{aligned} \tag{5.22}$$

which is an extension of the rational Poisson filter. The frequency ω_f is chosen according to the frequency characteristic of the system to be identified.

The state vector composed of fractional derivatives of filtered input and output signals is defined as

$$x_f = [D^{(N_f-1)\gamma}z_f(t), D^{(N_f-2)\gamma}z_f(t), \ldots, D^{\gamma}z_f(t), z_f(t)]^T \tag{5.23}$$

where $z_f(t)$ denotes either $u_f(t)$ or $y_f(t)$. The fractional state space representation of the filter is given by

$$D^{\gamma}x_f(t) = A_f x_f(t) + B_f z_f(t) \tag{5.24}$$

where

$$A_f = -\begin{bmatrix} \binom{N_f}{1}\omega_f^{\gamma} & \binom{N_f}{2}\omega_f^{2\gamma} & \ldots & \binom{N_f}{N_f-1}\omega_f{}^{\gamma}(N_f-1) & \omega_f^{\gamma N_f} \\ -1 & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 0 \end{bmatrix}$$

and

$$B_f = [\omega_f^{\gamma N_f}, 0, \ldots, 0]^T$$

Each state represents the derivative of a given order of input or output signals as shown in Figure 5.1. This fractional Poisson filter can be simulated using the definition of GL (Grünwald-Letnikov).

The estimated parameter vector, $\hat{\theta}$, is now obtained by minimizing the quadratic norm of the filtered equation error

$$J(\hat{\theta}) = E_f^T E_f \tag{5.25}$$

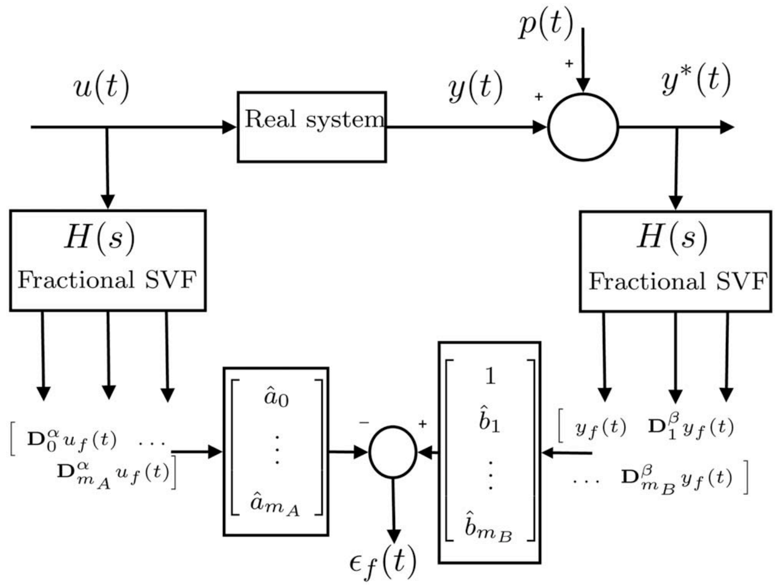


Figure 5.1: Fractional SVF

where

$$E_f = [\epsilon_f(k_0 T_s), \epsilon_f((k_0+1)T_s), \ldots, \epsilon_f((k_0+K-1)T_s)]^T$$

and

$$\epsilon_f(t) = y_f^*(t) - \phi_f^*(t)\hat{\theta}$$

where ϕ_f^* is defined by equation (5.21).

The solution is given by the classical least squares

$$\hat{\theta} = (\Phi_f^T \Phi_f)^{-1} \Phi_f^T Y_f^* \tag{5.26}$$

where

$$\Phi_f = [\phi_f^{*T}(k_0 T_s), \phi_f^{*T}((k_0+1)T_s), \ldots, \phi_f^{*T}((k_0+K-1)T_s)]$$

5.4 Parameter Identification of Fractional Order Transmission Lines

Different types of techniques for parameter identification have been discussed in the previous sections. Frequency responses of fractional order transmission lines have been derived in Chapter 4. The gain response of the finite length transmission line with the matched load has a nonlinear relationship between the response and the applied frequency. This implies that the parameters of the fractional order transmission line may be estimated via, e.g., the nonlinear regression analysis.

The frequency response of the fractional order finite length transmission line with the matched load is given by

$$G_v(j\omega) = e^{-x\left[\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}-\beta\omega^2)}+j\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}+\beta\omega^2)}\right]} \tag{5.27}$$

If we consider $|G_v(j\omega)|$ as the dependent variable y, $\theta = [\alpha, \beta]$ as regression coefficients and ω as the independent variable, the above equation can be presented as a nonlinear regression model

$$y = f(\theta, \omega) + e \tag{5.28}$$

where

$$f(\theta, \omega) = e^{-x\sqrt{\frac{1}{2}(\omega\sqrt{\beta^2\omega^2+\alpha^2}-\beta\omega^2)}} \tag{5.29}$$

is the expectation function and e is an additive normal disturbance. The finite length transmission line with the matched load has been simulated for a range of frequencies and the gain responses of the line have been calculated. Since simulation data has no disturbance or noise, different levels of noise have been added with the simulation data. Estimation of the parameters have been done for the deterministic case and also for the case when a zero mean white noise is added as a disturbance to the system with a signal to noise ratio (SNR) in dB defined by

$$SNR = 10\log\left(\frac{signal\ power}{noise\ power}\right) \tag{5.30}$$

We have corrupted the dependent variable y by additive Gaussian white noise with three different values of SNR as ∞, 30 dB, 10 dB. For each SNR level the model parameter α and β are estimated. Table 5.2 gives the results of the estimation. Since the nonlinear least square technique is highly dependent on the initial guess of the parameters, we have chosen the initial guess of the parameters in a neighborhood of the real parameters. The neighborhood of the parameter α was chosen in the range of (1×10^{-5} to 1×10^{-10}) while the real value of α was 2.9778×10^{-8} and the neighborhood of the parameter β was chosen in the range of (1×10^{-9} to 1×10^{-16}) while the real value of β was 1.1286×10^{-11}. In these neighborhood, the estimation of the parameters always converged into same result. This is the limitation of the nonlinear least square technique.

Table 5.2: Estimated parameters for different SNR level

Parameters	True Value	SNR=∞	SNR=30 dB	SNR=10 dB
		Estimated	Estimated	Estimated
α	2.9778×10^{-8}	2.9778×10^{-8}	2.9419×10^{-8}	2.6432×10^{-8}
β	1.1286×10^{-11}	1.1286×10^{-11}	1.0743×10^{-11}	0.6515×10^{-11}

Table 5.3: 95% Confidence intervals of estimated parameters

Parameter	SNR=∞		SNR= 30 dB		SNR=10 dB	
	95% CI		95% CI		95% CI	
	High	Low	High	Low	High	Low
α	2.9778×10^{-8}	2.9778×10^{-8}	2.9788×10^{-8}	2.9051×10^{-8}	2.9664×10^{-8}	2.322×10^{-8}
β	1.1286×10^{-11}	1.1286×10^{-11}	1.1269×10^{-11}	1.0218×10^{-11}	1.059×10^{-11}	0.2441×10^{-11}

To estimate the parameters of the line using the nonlinear regression analysis we have used 290 data points for each of the simulations. To show the closeness of the estimation, we have calculated the 95% confidence intervals of the estimated parameters which are given in Table 5.3. It can be seen that the estimated parameters are quite close to the true values for each of the SNR levels, thus indicating that the parameters of the fractional order transmission line can be estimated from its frequency response. The confidence intervals of the estimated parameters are quite low, which suggests the closeness of the estimation. To verify that the nonlinear regression converges, different initial guesses in the neighborhood of the real parameters are selected. In spite of different initial guesses, the estimates are found still the same. Further, more iteration steps are used to observe the change of parameters. With more iteration steps, the results remain the same. Based on these observations, we may conclude that the nonlinear regression does converge. Similarly, it is possible to identify the parameters of the fractional order transmission line from the phase response data using this estimation technique.

5.5 Conclusion

The objective of this chapter was to introduce the parameter identification techniques for fractional order systems and identify the parameters of the fractional order transmission line that has been derived in the previous chapters. We have successfully identified the parameters of the fractional order transmission line using the nonlinear regression method. A frequency domain identification method was used since it is hard to get the time domain expression for the transfer function we have derived. A time domain identification technique has been discussed in this chapter which can be investigated to identify the parameters of the line in the future.

Chapter 6

Conclusion

6.1 Summary

In summary, the contributions and results presented in this thesis are as follows.

A new generalized lossy fractional order transmission line model has been obtained which is an extension of the model described in [25]. For modeling the fractional order transmission line, the Laplace transformation technique has been used which is the same as classical textbook derivations. In classical textbooks the fractional behavior of the line has been neglected, but it should be considered for describing the distributive nature of the transmission line. After deriving the new fractional order model, frequency responses of the lines have been derived and simulated to validate the derived model. A parameter identification technique has been used to identify the parameters of the fractional order transmission line from the frequency response data.

The main contribution of this thesis is to model the fractional order lossy transmission line transfer functions and to derive the frequency responses of the line. The transfer functions have an infinite dimensional nature due to its fractional order which is expected since the transmission line is a distributed parameter system. It has been shown in this thesis that it is possible to identify the unknown parameters of the transmission line using fractional order models from the frequency response by classical parameter identification techniques.

To validate the derived frequency response expressions, a lossy transmission line has been simulated in SIMULINK using distributed parameter line blocks of the *SimPowerSystems*. The comparisons between the simulation and derived expressions were quite satisfactory though there were some discrepancies found. The source of these variances has been explained in the thesis.

6.2 Future Work

It should be noted that the above identification of the model parameters was done from the frequency response data. It is quite hard to come up with a time domain expression of the infinite dimensional transfer function with an exponential term in it. One direction of the future work is to find out the time domain expression of the fractional order transmission line using some approximation techniques. A possible approximation would be the Padé approximation since it provides the best approximation of a function by a rational function of a given order. Finding out a rational transfer function of the fractional order model will give a time domain expression by taking the inverse Laplace transformation which will result in a fractional order differential equation.

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Appendix A

Frequency Response Derivation

Frequency expression for the finite length fractional order transmission line with a general load is given as

$$G_v(j\omega) = \frac{4e^{-pl}\cos ql(3 + e^{-2pl})}{9 + e^{-4pl} + 6e^{-2pl}\cos 2ql} - j\frac{4e^{-pl}\sin ql(3 - e^{-2pl})}{9 + e^{-4pl} + 6e^{-2pl}\cos 2ql} \tag{A.1}$$

From the above equation the gain response can be found by

$$|G_v(\omega)| = \sqrt{\left(\frac{4e^{-pl}\cos ql(3 + e^{-2pl})}{9 + e^{-4pl} + 6e^{-2pl}\cos 2ql}\right)^2 + \left(\frac{4e^{-pl}\sin ql(3 - e^{-2pl})}{9 + e^{-4pl} + 6e^{-2pl}\cos 2ql}\right)^2}$$

$$= \sqrt{\frac{\left(4e^{-pl}\cos ql(3 + e^{-2pl})\right)^2 + \left(4e^{-pl}\sin ql(3 - e^{-2pl})\right)^2}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}\{\cos^2 ql(3 + e^{-2pl})^2 + \sin^2 ql(3 - e^{-2pl})^2\}}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}\{\cos^2 ql(9 + e^{-4pl} + 6e^{-2pl}) + \sin^2 ql(9 + e^{-4pl} - 6e^{-2pl})\}}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}\{9(\cos^2 ql + \sin^2 ql) + e^{-4pl}(\cos^2 ql + \sin^2 ql) + 6e^{-2pl}(\cos^2 ql - \sin^2 ql)\}}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}(9 + e^{-4pl} + 6e^{-2pl}(2\cos^2 ql - 1))}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}(9 + e^{-4pl} + 12e^{-2pl}\cos^2 ql - 6e^{-2pl})}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{16e^{-2pl}(9 + e^{-4pl} - 6e^{-2pl}) + 192e^{-4pl}\cos^2 ql}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

$$= \sqrt{\frac{\{4e^{-pl}(3 - e^{-2pl})\}^2 + \{\sqrt{192}e^{-2pl}\cos ql\}^2}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)^2}}$$

Thus

$$|G_v(\omega)| = \frac{\sqrt{\{4e^{-pl}(3 - e^{-2pl})\}^2 + \{\sqrt{192}e^{-2pl}\cos ql\}^2}}{\left(9 + e^{-4pl} + 6e^{-2pl}\cos 2ql\right)} \tag{A.2}$$

Similarly the phase response can be found as

$$\phi_v(\omega) = \arctan - \frac{\frac{4e^{-pl}\sin ql(3-e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}}{\frac{4e^{-pl}\cos ql(3+e^{-2pl})}{9+e^{-4pl}+6e^{-2pl}\cos 2ql}}$$

$$= \arctan - \frac{4e^{-pl}\sin ql(3 - e^{-2pl})}{4e^{-pl}\cos ql(3 + e^{-2pl})}$$

$$= \arctan - \frac{\sin ql(3 - e^{-2pl})}{\cos ql(3 + e^{-2pl})}$$

Thus

$$\phi_v(\omega) = -\arctan\{\tan ql(\frac{3 - e^{-2pl}}{3 + e^{-2pl}})\} \tag{A.3}$$

Appendix B

Codes and Simulink Blocks

B.1 MATLAB Codes

B.1.1 Dis_Line_Sim.m

```
clc
clear all
close all

R=6.86;
L=0.0026;
C=4.3409e-9;
a=L/C;

i=0;

for p=10:1:299
    f=p
    i=i+1;
b=R/(2*pi*f*C);
Rl=(1/sqrt(2))*sqrt(sqrt(a^2+b^2)+a);
Xc=(1/sqrt(2))*sqrt(sqrt(a^2+b^2)-a);
Cl=1/(Xc*2*pi*f);
sim(dis_tx.mdl);
load input.mat;
load output.mat;
Ip_Op_data;
Xcor_Phase;
end

save GAIN_SAME.mat

f=10:1:299;
w=2*pi*f;
```

B.1.2 Ip_Op_Data.m

```
x=find(ip(1,:)>0.1);

time_ip = ip(1,x);
time_ip=time_ip-0.1;


data_ip = ip(2,x);


y=find(op(1,:)>0.1);

time_op = op(1,y);
time_op=time_op-0.1;

data_op = op(2,y);
```

B.1.3 Xcor_Phase.m

```
t = [0 time_ip];

s1=[0 data_ip];
s2=[0 data_op];

x = xcorr(s1,s2,coeff);
tx =- time_ip(end):1e-5:time_ip(end);

[mx,ix] = max(x);
lag = (tx(ix));

ph_same(1,i)=lag*f*360;
```

B.1.4 Derived_Gain_Phase.m

```
clc
close all
clear all

f=10:1:299;
w=2*pi.*f;
R=4+2.86;
L=0.0026;
C=4.3409e-9;
l=500;

alpha=R*C;
beta=L*C;

p=sqrt(1/2)*sqrt(w.*sqrt(beta^2*w.^2+alpha^2)-beta*w.^2);
q=sqrt(1/2)*sqrt(w.*sqrt(beta^2*w.^2+alpha^2)+beta*w.^2);

%% gain for double load
gain3=20*log10((sqrt((4*exp(-p*l).*(3-exp(-2*p*l))).^2+(sqrt(192)\\
.*exp(-2*p*l).*cos(q*l)).^2)./(9+exp(-4*p*l)+6*exp(-2*p*l).*cos(2*q*l))));
```

```
phase3=zeros(1,290);

for i=10:1:299
    w1=2*pi*i;
    p1=sqrt(1/2)*sqrt(w1.*sqrt(beta^2*w1.^2+alpha^2)-beta*w1.^2);
    q1=sqrt(1/2)*sqrt(w1.*sqrt(beta^2*w1.^2+alpha^2)+beta*w1.^2);
phase3(1,i-9)=-(atan(-tan(q1*l).*((3-exp(2*p1*l))./(3+exp(2*p1*l)))))\\
*(180/pi));
phase4(1,i-9)=-l*q1*(180/pi);
if (phase3(1,i-9)>2.5)&(phase3(1,i-9)<180)
    phase3(1,i-9)=phase3(1,i-9)-180;
end

if phase4(1,i-9)>0
   phase4(1,i-9)=phase4(1,i-9)-360;
end
end

%% Gain for same load

gain4=(20*log10(exp(-p*l)));

save SIMULATED.mat
```

B.1.5 Bode_Comparison.m

```
clc
clear all
close all

load GAIN_SAME.mat
load PHASE_SAME.mat
load SIMULATED.mat
load GAIN_DOUBLE.mat
load PHASE_DOUBLE.mat

f=10:1:299;
w=2*pi*f;
for i=1:1:290
    if ph_same(i)>0
        ph_same(i)=ph_same(i)-360;
    end
end

for i=1:1:290
    if ph(i)>0
        ph(i)=ph(i)-360;
    end
end

for i=1:1:290
    if PHASE(i)>0
        PHASE(i)=PHASE(i)-360;
    end
```

```
end

for i=1:1:290
    if PHASE1(i)>0
        PHASE1(i)=PHASE1(i)-360;
    end
end
figure
subplot(2,1,1)
semilogx(w,gain4)
hold on
semilogx(w,20*log10(gain_same),r:)
legend(Derived,SIMULINK)
xlabel(Frequency (rad/s));
ylabel(Magnitude (dB));
title(Gain Response)

subplot(2,1,2)
semilogx(w,phase4)
hold on
semilogx(w,ph_same,r:)
legend(Derived,SIMULINK)
xlabel(Frequency (rad/s))
ylabel(Phase Angle (Degree))
title(Phase Response)

figure
subplot(2,1,1)
semilogx(w,gain3);
hold on
semilogx(w,20*log10(gain_double),r)
legend(Derived,SIMULINK)
xlabel(Frequency (rad/s));
ylabel(Magnitude (dB));
title(Gain Response);

subplot(2,1,2)
semilogx(w,phase3);
hold on
semilogx(w,ph,r);
legend(Derived,SIMULINK)
xlabel(Frequency (rad/s))
ylabel(Phase Angle (Degree))
title(Phase Response)

gain_same_sim=10.^(gain4/20);

for i=1:1:290
error_g_s(i)=((gain_same_sim(i)-gain_same(i))/gain_same_sim(i))*100;
end

figure
subplot(2,2,1)
plot(w,error_g_s)
xlabel(Frequency (rad/s))
ylabel(Error (%))
title(Percentage error of semi-infinite line gain response)
```

```
gain_double_sim=10.^(gain3/20);
for i=1:1:290
error_g_d(i)=((gain_double_sim(i)-gain_double(i))/gain_double_sim(i))*100;
end

subplot(2,2,2)
plot(w,error_g_d)
xlabel(Frequency (rad/s))
ylabel(Error (%))
title(Percentage error of finite length terminated line gain response)

for i=1:1:290
error_p_s(i)=((phase4(i)-ph_same(i))/phase4(i))*100;
end
subplot(2,2,3)
plot(w,error_p_s)
xlabel(Frequency (rad/s))
ylabel(Error (%))
title(Percentage error of semi-inifite line phase response)

for i=1:1:290
error_p_d(i)=((phase3(i)-ph(i))/phase3(i))*100;
end
subplot(2,2,4)
plot(w,error_p_d)
xlabel(Frequency (rad/s))
ylabel(Error (%))
title(Percentage error of finite length terminated line phase response)

mse_g_same=sqrt(mean((gain_same-gain_same_sim).^2))
mpe_g_same=mean((gain_same_sim-gain_same)./gain_same_sim)*100

mse_g_double=sqrt(mean((gain_double-gain_double_sim).^2))
mpe_g_double=mean((gain_double_sim-gain_double)./gain_double_sim)*100

mse_p_s=sqrt(mean((ph_same-phase4).^2))
mpe_p_s=mean((phase4-ph_same)./phase4)*100

mse_p_d=sqrt(mean((phase3-ph).^2))
mpe_p_d=mean((phase3-ph)./phase3)*100
```

B.1.6 Gain_Id_Same.m

```
close all
clear all
clc

format long
f=10:1:299;
w=2*pi.*f;
R=4+2.86;
L=0.0026;
C=4.3409e-9;
l=500;
```

```
alpha=R*C;
beta=L*C;

load GAIN_SAME.mat

p=sqrt(1/2)*sqrt(w.*sqrt(beta^2*w.^2+alpha^2)-beta*w.^2);
q=sqrt(1/2)*sqrt(w.*sqrt(beta^2*w.^2+alpha^2)+beta*w.^2);

gain4=(exp(-p*l));


Y=[gain4];
X=[w];
n=length(Y);

signal_power=mean((Y\).^2);
snr=30;
noise_power=signal_power./(10.^(snr/10));
randn(state,4);
noise=sqrt(noise_power)*(randn(n,1));
noisy_y=noise+Y;
Y=noisy_y;

fun=inline(exp(-500*sqrt(1/2)*sqrt(X.*sqrt(k(1)^2*X.^2+k(2)^2)-k(1)*X.^2)),k,X);
beta0=[1e-11;1e-8];
p=2;

[hat_theta_nl,e,J]=nlinfit(X,Y,fun,beta0);

MSE=e*e/(n-p);
C=(J*J)^(-1);

d_e=e/sqrt(MSE);

hat_Y_nl=nlpredci(fun,X,hat_theta_nl,e,J);
t1=tinv(0.975,n-p);

CI_beta_nls_low=hat_theta_nl-t1*sqrt(MSE*diag(C))
CI_beta_nls_high=hat_theta_nl+t1*sqrt(MSE*diag(C))

CI_theta_nl=nlparci(hat_theta_nl,e,J)
```

B.2 Simulink Blocks

The simulation blocks used in the SIMULINK are given below:

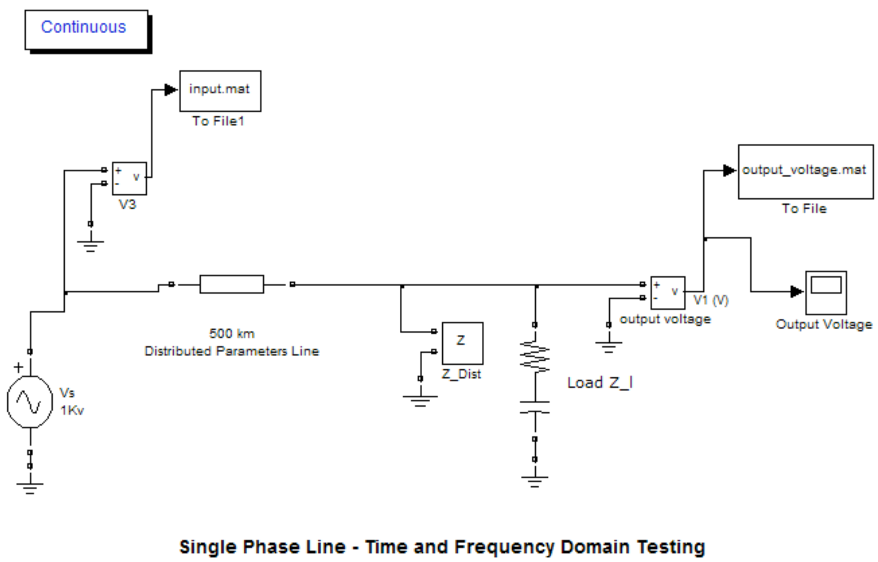

Figure B.1: Simulink block of a distributed parameter transmission line