EXTENDED AFFINE LIE ALGEBRAS AND DESCENT THEORY

by

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Abstract

This thesis gives two realizations of fgc extended affine Lie algebras, as fixed point subalgebras and as descended objects. Fgc stands for "finitely generated over the centroid " . All extended affine Lie algebras are fgc except for a well understood family of type A. In the process, the Lie algebra of derivations of certain Lie algebras (including multiloop algebras) are also studied from the twisted forms point of view.

Given an extended affine Lie algebra E, it is known that E can be realized in the form $E = L \oplus D^{gr*} \oplus D$, where L is a centreless Lie torus, D is a graded subalgebra of the derivation algebra $\mathrm{Der}_k(L)$, and D^{gr*} is the graded dual of D. For fgc extended affine Lie algebras, L can be choosen to be a multiloop algebra. In chapter 3, we show that every fgc extended affine Lie algebra E can be realized as a fixed point subalgebra of some untwisted extended affine Lie algebra. This generalizes known results of affine Kac-Moody algebras.

Descent theory has been used with great success to study several aspects of infinite dimensional Lie theory. In particular, multiloop algebras have been studied extensively from the descent point of view. The motivation of studying the whole E as a descended object leads naturally to our study of $\mathrm{Der}_k(L)$ as a twisted form. In chapter 5, we first study Lie algebra of derivations of certain Lie algebras under both étale cover and Galois cases. More precisely, if L is determined by some 1-cocycle, we construct a 1-cocycle that leads to $\mathrm{Der}_k(L)$ explicitly. Furthermore, we get maps between the corresponding (non-abelian) cohomology sets. Once this

is done, we give a realization of fgc E as a descended algebra of some untwisted extended affine Lie algebra.

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Contents

Chapter 1

Introduction

Extended affine Lie algebras (EALAs for short) are natural axiomatic generalizations of affine Kac-Moody Lie algebras and finite-dimensional simple Lie algebras. They were first considered under the name of irreducible quasisimple Lie algebras in the paper [H-KT]. It was in [AABGP] that the authors gave a mathematical foundation of these Lie algebras. Since then, a lot of work has been done for the study of EALAs (c.f. [ABFP2], [ABP1], [ABP4], [AF], [N2], [N3], [N4], etc.).

Since EALAs are defined via abstract axioms, to better understand them, it is important to realize them explicitly. Associated to every EALA there is an invariant named nullity; it is a non-negative integer. EALAs of nullity 0 are just finite-dimensional simple Lie algebras (c.f. [ABP4]). We are interested in EALAs with positive nullity. Neher gives an explicit procedure of how to construct EALAs from centreless Lie tori. The realization of EALAs thus reduces to the realization of centreless Lie tori. Explicitly, let k be an algebraically closed field of characteristic 0. Every extended affine Lie algebra E over k can be constructed in the form ([N1], [N2])

$$E = L \oplus D^{gr*} \oplus D,$$

for some centreless Lie torus L, where D is a graded subalgebra of the skew-centroidal derivation algebra $\mathrm{SCDer}_k(L)$ of L, D^{gr*} is the graded dual of D and plays the role of central extension (see Section 2.2 for detail).

The motivation for realizing centreless Lie tori as multiloop Lie algebras comes from affine Kac-Moody Lie theory. Multiloop Lie algebra is defined with respect to $\mathfrak{g}$, $\boldsymbol{\sigma}$ and $\boldsymbol{m}$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra over k, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ is an n-tuple of commuting finite order automorphisms of $\mathfrak{g}$ and $\boldsymbol{m} = (m_1, \dots, m_n)$ is an n-tuple of positive integers with $\sigma_i^{m_i} = id$, $i = 1, \dots, n$. It is denoted by $L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ (see Section 2.3 for its definition). If $n = 1$, then $L(\mathfrak{g}, \sigma, m)$ is called a loop algebra. Every affine Kac-Moody Lie algebra $\widehat{\mathcal{L}}$ over k can be realized as a direct sum ([K], [ABP4])

$$\widehat{\mathcal{L}} = L \oplus kc \oplus kd,$$

for some loop algebra L, $L \oplus kc$ is an universal central extension of L, and d is the degree derivation of L (see Section 3.3). Affine Kac-Moody Lie algebras are EALAs of nullity 1 ([ABGP], [ABP4]). In the paper [ABFP2], the authors successfully realized every fgc centreless Lie torus (see its definition in Chapter 2) as a multiloop Lie algebra. From the classification of centreless Lie tori, we know that there is only one family of centreless Lie tori that is not fgc, i.e. the family of type $A_\ell(\ell \geq 1)$ associated with a quantum torus $k_{\mathbf{q}}$, where $\mathbf{q}$ is a quantum matrix containing an entry that is not a root of unity (c.f. Remark 1.4.3 of [ABFP2]). Depending on whether the centreless Lie torus in the construction of EALAs is fgc or not, EALAs are divided into two families, fgc EALAs and non-fgc EALAs. In this thesis, we will focus on the study of fgc EALAs.

For an fgc EALA E, L is of the form of a multiloop Lie algebra (see Section 2.2). It is known that multiloop Lie algebras can be viewed as fixed point subalgebras of certain untwisted multiloop Lie algebras. For example, let $L = L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ be a multiloop algebra, let $S_n = k[t_1^{\pm\frac{1}{m_1}}, \dots, t_n^{\pm\frac{1}{m_n}}]$ be a Laurent polynomial ring, then L is the fixed point subalgebra of the untwisted multiloop Lie algebra $\mathfrak{g} \otimes S_n$ under a group action by the group $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}$ (see Section 3.1), i.e.

$$L = (\mathfrak{g} \otimes_k S_n)^\Gamma.$$

It is also known that affine Kac-Moody Lie algebras are fixed point subalgebras of untwisted affine Kac-Moody Lie algebras (c.f. Chapter 8 of [K]). Note that starting from $\mathfrak{g} \otimes_k S_n$, one can construct untwisted EALAs (see its definition in Section 2.2) which are of the form

$$E_{S_n} = (\mathfrak{g} \otimes_k S_n) \oplus \tilde{D}^{gr*} \oplus \tilde{D},$$

for any suitable $\tilde{D} \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes_k S_n)$. Then a natural question to ask is if the group Γ also acts naturally on the algebra $\tilde{D}$ and its graded dual $\tilde{D}^{gr*}$, is the fixed point subalgebra $(E_S)^\Gamma$ also an EALA? Conversely, can every fgc EALA be realized as a fixed point subalgebra of some untwisted EALA?

To solve such questions, one first needs to know that the derivation algebra D that is used in the construction of an EALA E is actually a graded subalgebra of the skew-centroidal derivation algebra $\mathrm{SCDer}_k(L)$ of L. Denote by $R_n = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ another Laurent polynomial ring. An important observation is that S_n is an étale ring extension of R_n (in fact, S_n/R_n is Galois with finite Galois group Γ), which makes it possible to identify the skew-centroidal derivation algebra $\mathrm{SCDer}_k(L)$ as a subalgebra of $\mathrm{SCDer}_k(\mathfrak{g} \otimes S_n)$. These two derivation algebras can be further identified with the skew-symmetric derivation algebras $\mathrm{SDer}_k(R_n)$, $\mathrm{SDer}_k(S_n)$ respectively, as one sees from Section 4.2 of [CNPY], i.e.

$$\mathrm{SDer}_k(R_n) = \mathrm{SCDer}_k(L) \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes S_n) = \mathrm{SDer}_k(S_n).$$

We will use these identifications to get the results that for every graded subalgebra $\tilde{D} \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes S_n)$, Γ acts on $\tilde{D}$ and its graded dual $\tilde{D}^{gr*}$, so the Γ-action on $\mathfrak{g} \otimes_k S_n$ induces a Γ-action on every untwisted EALA E_{S_n} and the fixed point subalgebra of E_{S_n} under Γ is also an EALA. We also show that every fgc EALA can be constructed as a fixed point subalgebra of some untwisted EALA in Chapter 3. But, except for the nullity 1 case (i.e. $n = 1$), the untwisted EALA may not be unique. We give an explanation and construct an explicit example in Section 3.2. For EALAs of nullity 1, it is known that they are just affine Kac-Moody algebras. We will use the identifications that

$$\mathrm{SCDer}_k(\mathfrak{g} \otimes k[z^{\pm 1}]) = kd = \mathrm{SCDer}_k(L(\mathfrak{g}, \sigma, m)),$$

where $d = z\frac{d}{dz}$ is the degree derivation, $L(\mathfrak{g}, \sigma, m)$ is any loop algebra, and the requirements of ingredients that are used in the construction of EALAs (see Section 2.2 for the requirements), to explain that up to isomorphism every affine Kac-Moody algebra can be realized as a fixed point subalgebra of a unique untwisted affine Kac-Moody algebra.

The second topic of the thesis is about the application of descent theory to the study of Lie algebras of derivations, and in particular, to the Lie algebra of derivations that arise in the construction of EALAs. Descent theory has been used to study certain Lie algebras and their different structures. The basic idea of descent is that a (complicated) object can be viewed as a "twisted form" of a much simpler object. Knowledge of this simple object can in certain cases "descend" to the object that we had originally set out to understand. It is known that multiloop Lie algebras can also be viewed as Lie algebras over certain rings. For example, the multiloop Lie algebra $L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ is a Lie algebra over the ring R_n above. And when we view it as a Lie algebra over R_n, there is the following Lie algebra isomorphism

$$L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) \otimes_{R_n} S_n \cong (\mathfrak{g} \otimes_k R_n) \otimes_{R_n} S_n \cong \mathfrak{g} \otimes_k S_n,$$

which tells us that $L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ can be viewed as a twisted form of $\mathfrak{g} \otimes_k R_n$.

In general, let R be a commutative unital ring, and let N be any R-module. A *twisted form* of N is by definition an R-module N' such that

$$N' \otimes_R S \cong N \otimes_R S$$

as S-modules for some ring extension S/R, where "S/R is a ring extension " means that there exists a ring homomorphism $R \longrightarrow S$ that takes the identity element of R to the identity element of S. We are most interested in the case when the ring extension is faithfully flat and finitely presented. If N is also equipped with other structures, for example, if N is an R-Lie algebra, then we say an R-Lie algebra N' is a twisted form of the Lie algebra N if $N' \otimes_R S \cong N \otimes_R S$ as S-Lie algebras. The R-isomorphism classes of such modules can be computed by means of cocycles, just as one does in Galois cohomology. There exists a one-to-one correspondence between the following two sets,

$$\text{Isomorphism classes of } \; S/R - \text{forms of } N \longleftrightarrow H^1(S/R, \mathbf{Aut}(N)),$$

where $H^1(S/R, \mathbf{Aut}(N))$ is a certain (non-abelian) first cohomology set that can be defined explicitly (see Section 4.1). The elements in $H^1(S/R, \mathbf{Aut}(N))$ are equivalence classes of 1-cocycles, where equivalent 1-cocycles will give isomorphic twisted forms. From the sheaf viewpoint it is Čech cohomology for the covering Spec $S \longrightarrow$ Spec R. The automorphism group functor $\mathbf{Aut}(N)$ is a functor from the category of ring extensions of R to the category of groups with

$$\mathbf{Aut}(N) : R'/R \mapsto \mathbf{Aut}(N)(R') := \text{Aut}_{R'-mod}(N \otimes_R R'),$$

$$\mathbf{Aut}(N)(f) : \text{Aut}_{R'-mod}(N \otimes_R R') \mapsto \text{Aut}_{R''-mod}(N \otimes_R R'')$$

for $f : R^{'} \longrightarrow R^{''}$ an arbitrary ring homomorphism over R. If N is an R-Lie algebra, then $\mathbf{Aut}(N)(R^{'}) = \mathrm{Aut}_{R^{'}-Lie}(N \otimes_R R^{'})$. For any 1-cocycle φ which is a representative of an element in $H^1(S/R, \mathbf{Aut}(N))$, it determines a descended object

$$(N \otimes_R S)_\varphi = \left\{ \sum n_i \otimes s_i \in N \otimes_R S \mid \varphi(\sum n_i \otimes s_i \otimes 1_S) = \sum n_i \otimes 1_S \otimes s_i \right\},$$

which is the corresponding twisted form of N. For any $N^{'} \cong (N \otimes_R S)_\varphi$, we refer to φ as the descent data that leads to the form $N^{'}$. When S/R is Galois with finite Galois group Γ (see Section 5.2), which is the case we are interested in, $H^1(S/R, \mathbf{Aut}(N))$ can be identified with the usual (non-abelian) Galois cohomology set $H^1(\Gamma, \mathbf{Aut}(N)(S))$ (c.f. [W], [S]). And analogously there is a one-to-one correspondence between the following two sets,

$$\text{Isomorphism classes of } \; S/R - \text{forms of } N \longleftrightarrow H^1(\Gamma, \mathbf{Aut}(N)(S)).$$

Similarly, for any Galois 1-cocycle $\mu = (\mu_\gamma)_{\gamma \in \Gamma}$ which is a representative of an element in $H^1(\Gamma, \mathbf{Aut}(N)(S))$, its corresponding descended object is defined as

$$(N \otimes_R S)_\mu = \{ m \in N \otimes_R S \mid \mu_\gamma({}^\gamma m) = m, \; \forall \gamma \in \Gamma \},$$

where Γ acts on $N \otimes_R S$ via $id \otimes_R \gamma$ for any $\gamma \in \Gamma$.

It is well known how descent theory can be used to study the nature of the Lie tori that appeared in extended affine Lie algebras (more precisely, multiloop Lie algebras). Let $\mathbf{Aut}(\mathfrak{g})$ be the k-algebraic group of automorphisms of $\mathfrak{g}$. The R-group functor $\mathbf{Aut}(\mathfrak{g})_R$ obtained by base change is clearly isomorphic to $\mathbf{Aut}(\mathfrak{g} \otimes_k R)$. It is an affine, smooth, and finitely presented group scheme over R whose functor of points is given by

$$\mathbf{Aut}(\mathfrak{g} \otimes_k R)(S) = \mathrm{Aut}_S(\mathfrak{g} \otimes_k R \otimes_R S) \cong \mathrm{Aut}_S(\mathfrak{g} \otimes_k S) = \mathbf{Aut}(\mathfrak{g})(S).$$

For the multiloop Lie algebra $\mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ and rings R_n, S_n defined as above, S_n/R_n is Galois with Galois group $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}$, where the Γ-action on S_n

is given in Section 3.1. The corresponding Galois 1-cocycle is

$$u : \Gamma \longrightarrow \mathbf{Aut}(\mathfrak{g})(S_n),$$

$$(\overline{i_1}, \ldots, \overline{i_n}) \mapsto (\sigma_1^{-i_1} \otimes_k id_{S_n}) \cdots (\sigma_n^{-i_n} \otimes_k id_{S_n}),$$

i.e. $\mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) = (\mathfrak{g} \otimes_k S_n)_u$. On the other hand, one can also look at twisted forms from the torsors viewpoint (c.f. [GP1], [GP2], [GP3], [GP4], [P1], [P3], [P4], etc.). Aside from these, the idea of descent has been used in the study of the structure of Lie algebras. For example, let $\mathcal{L}$ be a multiloop Lie algebra. The derivation algebra $\mathrm{Der}_k(\mathcal{L})$ is determined using descent theory in [P5]. Galois descent of central extensions of $\mathcal{L}$ has been studied in [PPS]. Invariant bilinear forms on $\mathcal{L}$ have been studied in [NPPS].

In [KP], the authors developed a theory of relative Kähler differentials for Lie algebras. Therein they define a module of R/k-differentials $\Omega_{R,L/k}$ for any R-Lie algebra L, where R is a k-algebra and k is a commutative unital ring. $\Omega_{R,L/k}$ is an object that represent the functor $\mathrm{Der}_{R/k}(L, \cdot)$. When k is a field of characteristic 0, L is an R-form of some finite dimensional perfect Lie algebra $\mathfrak{g}$ over k (by this we mean it is a twisted form of $\mathfrak{g} \otimes_k R$ under faithfully flat and finitely presented ring extensions), $\Omega_{R,L/k}$ behaves very well under étale base change. The results in [KP] tell us that there exists a canonical isomorphism

$$\mathrm{Der}_k(L) \otimes_R S \cong \mathrm{Der}_k(L \otimes_R S)$$

for any S/R étale. Hence for any S/R faithfully flat and étale (S/R is then an étale cover, see Section 4.2 for its definition), if $L \otimes_R S \cong (\mathfrak{g} \otimes_k R) \otimes_R S$, then we can get an isomorphism

$$\mathrm{Der}_k(L) \otimes_R S \cong \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S.$$

So the derivation algebra $\mathrm{Der}_k(L)$ of such an L can be viewed as an R-form of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$ under étale covers, and then descent consideration for derivation algebras of such Lie algebras follows. Since $\mathrm{Der}_k(L)$ is an R-form of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$, there exists a 1-cocycle that leads to it. We will construct explicitly the descent data in Chapter 5.

Because of the one-to-one correspondence between the set of isomorphism classes of twisted forms and the corresponding non-abelian first cohomology set, and there exists a natural map between the isomorphism classes of Lie algebras and their derivation algebras, it follows that there exists a natural map between the corresponding (non-abelian) first cohomology sets under étale covers:

$$H^1(S/R, \mathbf{Aut}(L)) \longrightarrow H^1(S/R, \mathbf{Aut}(\mathrm{Der}_k(L))).$$

We will construct the explicit map between these two cohomology sets. Actually, we will construct a natural transformation between the two automorphism group functors

$$\mathbf{Aut}(L) \longrightarrow \mathbf{Aut}(\mathrm{Der}_k(L)),$$

where now the automorphism group functors are restricted to the category of étale ring extensions. Then for any equivalence class of 1-cocycle $[\varphi]$ in $H^1(S/R, \mathbf{Aut}(\mathfrak{g}_R))$, where we denote $\mathfrak{g} \otimes_k R = \mathfrak{g}_R$, we construct an equivalence class of 1-cocycle $[\widetilde{\varphi}]$ in $H^1(S/R, \mathbf{Aut}(\mathrm{Der}_k(\mathfrak{g}_R)))$. These classes determine two descended objects $(\mathfrak{g} \otimes_k S)_\varphi$ and $(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}}$. We have shown that there exists an isomorphism

$$(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \cong \mathrm{Der}_k\left((\mathfrak{g} \otimes_k S)_\varphi\right).$$

Since Galois ring extensions are faithfully flat and étale, the same considerations apply to the corresponding (non-abelian) Galois cohomology sets. For any S/R Galois with finite Galois group Γ, we construct a well-defined map between Galois cohomology sets

$$H^1(\Gamma, \mathbf{Aut}(\mathfrak{g}_R)(S)) \longrightarrow H^1(\Gamma, \mathbf{Aut}(\mathrm{Der}_k(\mathfrak{g}_R))(S)),\ [\mu] \mapsto [\widetilde{\mu}],$$

and show that there exists an isomorphism

$$(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\mu}} \cong \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_\mu).$$

At the end, we apply the idea of descent to study extended affine Lie algebras. Explicitly, if E is an fgc EALA, then E is of the form $E = \mathcal{L} \oplus D^{gr*} \oplus D$, where $\mathcal{L}$ is a twisted form of $\mathfrak{g} \otimes_k R$ under Galois ring extensions for some finite dimensional simple Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0, R is a Laurent polynomial ring over k. We know the descent data that leads to $\mathcal{L}$. D is a graded subalgebra of the skew-centroidal derivation algebra $\mathrm{SCDer}_k(\mathcal{L})$, and when D is taken to be the whole of $\mathrm{SCDer}_k(\mathcal{L})$, then $\mathcal{L} \oplus \mathrm{SCDer}_k(\mathcal{L})^{gr*}$ is a universal central extension of $\mathcal{L}$. Descent construction of such Lie algebras has been studied in [PPS]. We will show generally that for every central extension $\mathcal{L} \oplus D^{gr*}$ of $\mathcal{L}$, its descent data can be constructed. More generally, we will use the results obtained in Chapter 3 to construct the descent data that leads to the whole extended affine Lie algebra E, which makes it possible to use descent theory to study extended affine Lie algebras. For the Galois 1-cocycle u defined above that determines the multiloop Lie algebra $\mathcal{L}$, we construct a 1-cocycle $\widetilde{u}$ in Section 5.3 such that the descended object of some untwisted EALA under $\widetilde{u}$ is the EALA E.

This thesis is organized as follows. In Chapter 2, we will review some basic notions and results of extended affine Lie algebras, Lie tori and multiloop Lie algebras. We also recall the process of how to construct extended affine Lie algebras from centreless Lie tori. In Chapter 3, we show that there exists an action of some finite abelian group on any untwisted EALA and the fixed point subalgebra is also an EALA. Conversely, we give the result that every fgc EALA can be constructed as a fixed point subalgebra of some untwisted EALA, and we apply the results to the study of affine Kac-Moody algebras. In Chapter 4, we first introduce the notion of twisted forms that will be used, the definition of automorphism group functors, the first non-abelian cohomology sets and the results about one-to-one correspondences between twisted forms and non-abelian cohomology sets. Then we review derivations and differentials for both Lie algebras and associative algebras. We also present step by step all the maps that will be used in the construction of an important isomorphism, which leads to the result that derivation algebras of certain Lie algebras can also be viewed as twisted forms. In Chapter 5, we will study the derivation algebras as twisted forms from both étale cover and Galois descent points of view. At the end, we apply the above obtained results to construct the descent data that leads to every fgc extended affine Lie algebra.

In this thesis, R is a k-algebra means R is a commutative unital ring with a fixed ring homomorphism $k \longrightarrow R$. All rings are assumed to be commutative with unit, and ring homomorphisms take unit to unit.

Chapter 2

Extended affine Lie algebras and multiloop Lie algebras

In this chapter, we will recall the definitions and related results of extended affine Lie algebras, Lie tori and multiloop Lie algebras, and also review the construction of EALAs from centreless Lie tori introduced by Neher.

2.1 Extended affine Lie algebras (EALAs)

The definition of extended affine Lie algebras over the field $\mathbb{C}$ of complex numbers was first introduced in [AABGP]. EALAs over an arbitrary field of characteristic 0 are introduced in [N2] which are defined similarly to the definition of EALAs over $\mathbb{C}$ but with slight modifications. Let k be a field of characteristic 0.

Definition 2.1. An *extended affine Lie algebra*, or EALA for short, is a Lie algebra E over k satisfying the conditions (EA1)-(EA6) below:
(**EA1**) E has a nondegenerate invariant symmetric bilinear form $(\mid)$.
(**EA2**) E contains a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra H.

Because of (**EA2**), E has a root space decomposition

$$E = \bigoplus_{\alpha \in H^*} E_\alpha, \quad E_0 = H,$$

where $E_\alpha = \{x \in E \mid [h, x] = \alpha(h)x \text{ for all } h \in H\}$.

Due to the property (**EA1**) and $H = E_0$, the restriction of $(\mid)$ to H is also nondegenerate. We can therefore represent any $\alpha \in H^*$ by a unique vector $t_\alpha \in H$ via $\alpha(h) = (t_\alpha \mid h)$ for all $h \in H$. Transfer $(\mid)$ to H^* by $(\alpha \mid \beta) = (t_\alpha \mid t_\beta)$ for all

$\alpha, \beta \in H^*$. Define

$$R = \{\alpha \in H^* : E_\alpha \neq 0\} \quad \text{(root system of } E\text{)},$$

$$R^0 = \{\alpha \in R : (\alpha \mid \alpha) = 0\} \quad \text{(isotropic roots)},$$

$$R^{an} = \{\alpha \in R : (\alpha \mid \alpha) \neq 0\} \quad \text{(anisotropic roots)},$$

so that $R = R^0 \cup R^{an}$. Let E_c be the subalgebra of E generated by all subspaces $E_\alpha, \alpha \in R^{an}$. E_c is called the *core* of E. Now we state the remaining axioms:
(**EA3**) For any $\alpha \in R^{an}$, $x_\alpha \in E_\alpha$, $ad\ x_\alpha$ is locally nilpotent.
(**EA4**) R^{an} is *irreducible*, i.e. there does not exist R_1, R_2 such that $R^{an} = R_1 \cup R_2$ and $(R_1 \mid R_2) = 0$.
(**EA5**) E is *tame* in the sense that $\{x \in E \mid [x, E_c] = 0\} \subset E_c$.
(**EA6**) The $\mathbb{Z}$-span of R^0 is a free abelian group of finite rank.

An EALA is usually denoted by $(E, H, (\ \mid\ ))$. The finite rank in (**EA6**) is called the *nullity* of the extended affine Lie algebra E. Denote by $Z(E_c)$ the center of E_c. Since E_c is a perfect Lie algebra, the quotient $E_{cc} = E_c/Z(E_c)$ is centreless. It is called the *centreless core* of the extended affine Lie algebra E.

Example 2.2. Let $\mathfrak{g}$ be any finite dimensional split simple Lie algebra over k with H a splitting Cartan subalgebra (c.f. [C], [H], [MP], etc.). Let $(\ \mid\ )$ be the Killing form, i.e.

$$(x|y) = tr(ad\ x \circ ad\ y), \quad \text{for any } x, y \in \mathfrak{g}.$$

Let R be the root system of $\mathfrak{g}$. Then $R^{an} = R\backslash\{0\}$ and $R^0 = \{0\}$. Hence the $\mathbb{Z}$-span of R^0 is of rank 0. $(\mathfrak{g}, H, (\ \mid\ ))$ is an EALA over k of nullity 0.

We are concerned with Lie algebras up to isomorphism, so recall the definition of two EALAs being isomorphic (c.f. [AF], [ABFP2], etc.).

Definition 2.3. Let $(E, H, (\ \mid\ ))$ and $(E', H', (\ \mid\ )')$ be extended affine Lie algebras. We say $(E, H, (\ \mid\ ))$ and $(E', H', (\ \mid\ )')$ are *isomorphic*, denoted by $(E, H, (\ \mid\ )) \cong (E', H', (\ \mid\ )')$, if there exists a Lie algebra isomorphism $\varphi : E \longrightarrow E'$ such that

$$\varphi(H) = H', \quad \text{and} \quad (\varphi(x) \mid \varphi(y))' = a(x \mid y) \ \text{ for some } 0 \neq a \in k,\ \forall x, y \in E.$$

Before giving the notion of an EALA being fgc, we first recall the general definition of the centroid of a Lie algebra. The *centroid* of a Lie algebra L over any commutative ring R is by definition the set of all R-linear endomorphisms of L that commute with left and right multiplication by elements of L. It is denoted by $\mathrm{Ctd}_R(L)$, i.e.

$$\mathrm{Ctd}_R(L) = \{\chi \in \mathrm{End}_R(L) \mid \chi([x, y]) = [\chi(x), y] = [x, \chi(y)],\ \forall x, y \in L\}.$$

By the skew-symmetry of Lie bracket, it is the same as

$$\mathrm{Ctd}_R(L) = \{\chi \in \mathrm{End}_R(L) \mid \chi([x, y]) = [\chi(x), y],\ \forall x, y \in L\},$$

or

$$\mathrm{Ctd}_R(L) = \{\chi \in \mathrm{End}_R(L) \mid \chi([x, y]) = [x, \chi(y)],\ \forall x, y \in L\}.$$

If L is a perfect Lie algebra, then $\mathrm{Ctd}_R(L)$ is a commutative ring (c.f. Lemma 4.1 of [GP2], Lemma 2.3 of [P5]). L can be naturally viewed as a module over $\mathrm{Ctd}_R(L)$ via $\chi.x = \chi(x)$ for all $x \in L$, $\chi \in \mathrm{Ctd}_R(L)$.

Definition 2.4. An EALA is *fgc* if its centreless core is fgc, i.e. the centreless core is finitely generated as a module over its centroid.

Remark 2.5. From the definition of EALAs, we see that the structure of an EALA is determined by the triple $(E, H, (\ |\ ))$. But for fgc EALAs, it is proved in Theorem 7.6 of [CNPY] that different subalgebras H in E are conjugate. Therefore, up to isomorphism of EALAs, the choice of H has no effect. Similarly, fgc EALAs can be constructed from multiloop Lie algebras as we will explain below, and it is proved in Corollary 7.4 of [NPPS] that the invariant bilinear forms $(\ |\ )$ on multiloop Lie algebras are unique up to nonzero scalars, so the choice of the invariant bilinear form is also not important (see also Remark 2.9 of [CNPY]).

Remark 2.6. For non-fgc EALAs, there is also some study for conjugacy in [CNP].

2.2 The general construction

In [N2], Neher gives a construction of extended affine Lie algebras from centreless Lie tori. In this section, we will review his construction. Let us first recall some terminology and results on Lie tori ([N1], [Y]). By a *finite irreducible root system* we mean a finite subset Δ of a finite dimensional vector space (equipped with a symmetric bilinear form $(\ ,\ )$) over the field k of characteristic 0 such that $0 \in \Delta$ and $\Delta^\times := \Delta \backslash \{0\}$ is a finite irreducible root system in the usual sense (see [B], chap. VI, § 1, Définition 1). Denote by $\Delta_{ind} = \{0\} \cup \{\alpha \in \Delta \mid \frac{1}{2}\alpha \notin \Delta\}$ the subsystem of indivisible roots. Then it is known that Δ has one of the following types ([B], chap. VI, § 4):

$$A_\ell(\ell \geq 1), B_\ell(\ell \geq 2), C_\ell(\ell \geq 3), D_\ell(\ell \geq 4), E_6, E_7, E_8, F_4, G_2, BC_\ell(\ell \geq 1).$$

Δ is said to be *reduced* if $2\alpha \notin \Delta^\times$ for $\alpha \in \Delta^\times$. Then, type $BC_\ell(\ell \geq 1)$ is not reduced; all others are reduced.

Definition 2.7. Let Λ be an abelian group. A Λ-*graded algebra* is an algebra A over k together with a family $(A^\lambda :\ \lambda \in \Lambda)$ of subspaces A^λ of A such that

$$A = \bigoplus_{\lambda \in \Lambda} A^\lambda \ \text{ and } \ A^\lambda A^\mu \subseteq A^{\lambda+\mu} \ \text{ for all } \lambda, \mu \in \Lambda.$$

If $A = \oplus_{\lambda\in\Lambda} A^\lambda$ is a Λ-graded algebra we let $\mathrm{supp}_\Lambda(A) = \{\lambda \in \Lambda \mid A^\lambda \neq 0\}$ denote the Λ-*support* of A.

If S is a subset of a group, we denote by $\langle S \rangle$ the subgroup generated by S.

We now recall the definition of a Lie torus. Lie tori were introduced by Y. Yoshii in [Y] to give a characterization of EALAs and were further studied by E. Neher in [N1].

Let Λ be a free abelian group of finite rank. Let Δ be a finite irreducible root system. The span $Q := \mathrm{span}_{\mathbb{Z}}(\Delta)$ is the *root lattice* of Δ. For $\alpha, \beta \in \Delta$, $\check{\alpha}$ is the coroot of α, and $\langle \beta, \check{\alpha} \rangle$ is the Cartan integer of (β, α) (c.f. [B], [N1]).

Definition 2.8. A *Lie* Λ*-torus of type* Δ (or a *Lie torus of type* (Δ, Λ)) is a Lie algebra L over k satisfying the following conditions $(\mathbf{LT1}) - (\mathbf{LT4})$.
$(\mathbf{LT1})$ L is a $Q \times \Lambda$-graded Lie algebra

$$L = \bigoplus_{(\alpha,\lambda)\in Q\times\Lambda} L_\alpha^\lambda, \ \ [L_\alpha^\lambda, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu} \ \text{ such that } L_\alpha^\lambda = 0 \text{ if } \alpha \notin \Delta.$$

$(\mathbf{LT2})$ For $\alpha \in \Delta^\times$ and $\lambda \in \Lambda$ we have

(i) dim $L_\alpha^\lambda \leq 1$, with dim $L_\alpha^0 = 1$ if $\alpha \in \Delta_{ind}$,

(ii) if dim $L_\alpha^\lambda = 1$ then there exist elements $e_\alpha^\lambda \in L_\alpha^\lambda$ and $f_\alpha^\lambda \in L_{-\alpha}^{-\lambda}$ such that

$$L_\alpha^\lambda = ke_\alpha^\lambda, \quad L_{-\alpha}^{-\lambda} = kf_\alpha^\lambda,$$

and

$$[[e_\alpha^\lambda, f_\alpha^\lambda], x_\beta] = \langle \beta, \check{\alpha} \rangle x_\beta$$

for all $x_\beta \in L_\beta, \beta \in \Delta$.

(**LT3**) For $\lambda \in \Lambda$ we have $L_0^\lambda = \sum_{(\alpha,\mu)\in\Delta^\times\times\Lambda} [L_\alpha^\mu, L_{-\alpha}^{\lambda-\mu}]$.

(**LT4**) $\Lambda = \langle \text{supp}_\Lambda(L) \rangle$, where $\text{supp}_\Lambda(L) = \{\lambda \in \Lambda \mid L_\alpha^\lambda \neq 0 \text{ for some } \alpha \in \Delta\}$.

The rank of Λ is called the *nullity* of L. We say L is a Lie torus means that it is a Lie torus of type (Δ, Λ) for some Δ, Λ. Denote by

$$L^\lambda = \bigoplus_{\alpha\in Q} L_\alpha^\lambda \ \text{ for } \lambda \in \Lambda \ \text{ and } \ L_\alpha = \bigoplus_{\lambda\in\Lambda} L_\alpha^\lambda \ \text{ for } \alpha \in Q,$$

then $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ is Λ-graded and $L = \bigoplus_{\alpha\in Q} L_\alpha$ is Q-graded.

It is known that if L is an arbitrary Lie torus, then $L/Z(L)$ is in a natural way a centreless Lie torus (i.e. Lie torus with center 0) of the same type and nullity as L (see Lemma 1.4 of [Y]).

Example 2.9. Let $\mathbf{q} = (q_{ij}) \in \text{Mat}_n(k)$ be a *quantum matrix*, i.e. $q_{ij}q_{ji} = 1 = q_{ii}$ for all $1 \leq i, j \leq n$. The *quantum torus associated to* $\mathbf{q}$ is the unital associative k-algebra $k_\mathbf{q}$ generated by $t_i^{\pm 1}$, $i = 1, \ldots, n$, modulo relations

$$t_i t_i^{-1} = 1 = t_i^{-1} t_i \ \text{ and } \ t_i t_j = q_{ij} t_j t_i \ \text{ for } 1 \leq i, j \leq n.$$

On $k_\mathbf{q}$, we have a natural $\mathbb{Z}^n$-grading

$$k_\mathbf{q} = \bigoplus_{\lambda\in\mathbb{Z}^n} kt^\lambda \ \text{ where } \ t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n} \ \text{ for } \lambda = (\lambda_1, \ldots, \lambda_n).$$

Denote by $[k_\mathbf{q}, k_\mathbf{q}]$ the span of all commutators $[a, b] = ab - ba$ with $a, b \in k_\mathbf{q}$. Then $sl_{l+1}(k_\mathbf{q}) = \{x \in \text{Mat}_{l+1}(k_\mathbf{q}) \mid tr(x) \in [k_\mathbf{q}, k_\mathbf{q}]\}$ is a Lie torus of type A_l, $l \geq 1$, and is of nullity n. If all $q_{ij} = 1$, then $k_\mathbf{q} = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the *Laurent polynomial ring* in n variables and $sl_{l+1}(k_\mathbf{q}) = sl_{l+1}(k) \otimes k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

The following result is obtained from Theorem 2.2 and Theorem 7.1 of [Y].

Lemma 2.10. *Let L be a Lie torus with $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$. Then L has a nonzero invariant symmetric bilinear form $(\ |\ )$, Λ-graded in the sense that $(L^\lambda|L^\mu) = 0$ if $\lambda + \mu \neq 0$. And any such form is unique up to a nonzero scalar. If furthermore, $\mathcal{L}$ is centreless, then $(\ |\ )$ is nondegenerate.*

Extended affine Lie algebras are closely related to Lie tori (c.f. Proposition 12 of [N2]).

Proposition 2.11. *Let E be an extended affine Lie algebra of nullity n. Then its core E_c is a Lie torus of nullity n, and its centreless core E_{cc} is a centreless Lie torus of nullity n.*

The following definition is given in Section 4 of [CNPY].

Definition 2.12. An EALA is called *untwisted* if its centreless core, as a Lie torus, is of the form $\mathfrak{g} \otimes_k R$ for some finite dimensional simple Lie algebra $\mathfrak{g}$ over k and some Laurent polynomial ring R in finitely many variables.

Let L be a Lie torus over k of nullity n with grading Λ. Then its centroid $\mathrm{Ctd}_k(L)$ is also Λ-graded

$$\mathrm{Ctd}_k(L) = \bigoplus_{\lambda\in\Lambda} \mathrm{Ctd}_k(L)^\lambda, \quad \text{with } \dim_k \mathrm{Ctd}_k(L)^\lambda \leq 1,$$

where $\mathrm{Ctd}_k(L)^\lambda = \{\chi \in \mathrm{Ctd}_k(L) \mid \chi(L^\mu) \subseteq L^{\lambda+\mu} \text{ for all } \mu \in \Lambda\}$. Let $\Xi = \{\lambda \in \Lambda \mid \mathrm{Ctd}_k(L)^\lambda \neq 0\}$. The following results from Theorem 7 of [N1] give justification to call Ξ the *centroid grading group* of L.

Theorem 2.13. *Let L be a centreless Lie Λ-torus of type Δ. Then*

(a) Ξ is a subgroup of Λ, and $Ctd_k(L)$ is isomorphic to the group ring $k[\Xi]$, hence to a Laurent polynomial ring in finitely many variables.

(b) L is a free $Ctd_k(L)$-module, and if $\Delta \neq A_l$, then L has finite rank as a $Ctd_k(L)$-module.

In the following, L will be a centreless Lie Λ-torus of type Δ with centroid grading group Ξ. We can thus write

$$\mathrm{Ctd}_k(L) = \bigoplus_{\mu\in\Xi} k\chi^\mu,$$

where χ^μ acts on L as endomorphisms of degree μ and $\chi^\mu\chi^\nu = \chi^{\mu+\nu}$.

For any $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$, define ∂_θ of L by

$$\partial_\theta(x^\lambda) = \theta(\lambda)x^\lambda \text{ for } \lambda \in \Lambda, x^\lambda \in L^\lambda.$$

∂_θ is called a *degree derivation* of L. Put

$$\mathcal{D} = \{\partial_\theta \mid \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)\}$$

the set of all degree derivations. Denote by

$$\mathrm{CDer}_k(L) := \mathrm{Ctd}_k(L)\mathcal{D} = \bigoplus_{\mu\in\Xi} \chi^\mu \mathcal{D}$$

the *centroidal derivations* of L. It is a Ξ-graded subalgebra of the derivation algebra $\mathrm{Der}_k(L)$ with

$$[\chi^\mu\partial_\theta, \chi^\nu\partial_\psi] = \chi^{\mu+\nu}(\theta(\nu)\partial_\psi - \psi(\mu)\partial_\theta).$$

Fix a nondegenerate invariant Λ-graded bilinear form $(\ |\ )_L$ on L (recall Lemma 2.10). Let $\mathrm{SCDer}_k(L)$ be the set of all *skew-centroidal derivations* of L. It is a Ξ-graded subalgebra of $\mathrm{CDer}_k(L)$ consisting of all skew derivations with respect to $(\ |\ )_L$, i.e.

$$\begin{aligned}\mathrm{SCDer}_k(L) &= \{d \in \mathrm{CDer}_k(L) \mid (d(x) \mid x)_L = 0 \text{ for all } x \in L\} \\ &= \bigoplus_{\mu\in\Xi} \mathrm{SCDer}_k(L)^\mu = \bigoplus_{\mu\in\Xi} \chi^\mu\{\partial_\theta \mid \theta(\mu) = 0\}. \qquad (2.2.1)\end{aligned}$$

Note that $\mathrm{SCDer}_k(L)^0 = \mathcal{D}$.

Now we are in a position to list two ingredients that appear in the construction of EALAs.

(i) Let $D = \bigoplus_{\mu\in\Xi} D^\mu$ be a Ξ-graded subalgebra of $\mathrm{SCDer}_k(L)$ such that the canonical evaluation map

$$\mathrm{ev} : \Lambda \longrightarrow (D^0)^*$$

defined by

$$ev(\lambda)(\partial_\theta) = \theta(\lambda), \quad \lambda \in \Lambda$$

is injective, where $(D^0)^*$ is the dual space of D^0. Note that $D^0 \subseteq \mathrm{SCDer}_k(L)^0 = \mathcal{D}$.

(ii) Let $\tau : D \times D \longrightarrow D^{gr*}$ be a 2-cocycle which is graded and invariant, i.e.

$$\tau(D^{\mu_1}, D^{\mu_2}) \subseteq (D^{-\mu_1-\mu_2})^* \text{ and } \tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1),$$

and further require $\tau(D^0, D) = 0$. Here $D^{gr*} = \bigoplus_{\mu\in\Xi}(D^\mu)^*$ is the graded dual of D with grading $(D^{gr*})^\mu = (D^{-\mu})^*$, and it is viewed as a D-module by the contragredient action, i.e.

$$(d.\varphi)(d^{'}) = \varphi([d^{'}, d]) \text{ for } d, d^{'} \in D, \varphi \in D^{gr*},$$

where $\varphi \in (D^\mu)^*$ is viewed as a linear form on D by $\varphi|_{D^\nu} = 0$ for $\nu \neq \mu$.

Remark 2.14. Clearly, $\tau = 0$ always satisfies the condition. It is important to know that there do exist nontrivial τ, see Remark 3.71 of [BGK]. For (i), the whole skew-centroidal derivations $\mathrm{SCDer}_k(L)$ always satisfies the condition and in that case $\mathrm{SCDer}_k(L)^{gr*}$ leads to a universal central extension of L (the Remarks after Theorem 14 of [N2]).

Denote by

$$P(L) = \{(D, \tau) \mid D, \tau \text{ satisfying properties (i) and (ii) above}\}.$$

From the above Remark we know that $P(L)$ is a nonempty set.

Then starting from L, for any $(D, \tau) \in P(L)$, let

$$E = L \oplus D^{gr*} \oplus D$$

be a Lie algebra with Lie bracket

$$\begin{aligned}[x_1 + c_1 + d_1, x_2 + c_2 + d_2] &= ([x_1, x_2]_L + d_1(x_2) - d_2(x_1)) \\ &+ (\sigma_D(x_1, x_2) + d_1. c_2 - d_2. c_1 + \tau(d_1, d_2)) \\ &+ [d_1, d_2] \end{aligned} \qquad (2.2.2)$$

for $x_1, x_2 \in L$, $c_1, c_2 \in D^{gr*}$, $d_1, d_2 \in D$, where $[\ ,\ ]_L$ denotes the Lie bracket of L,

$[d_1, d_2] = d_1 d_2 - d_2 d_1$, and $\sigma_D : L \times L \longrightarrow D^{gr*}$ is defined by

$$\sigma_D(x, y)(d) = (d(x) \mid y)_L \text{ for all } x, y \in L, d \in D. \tag{2.2.3}$$

Note that σ_D is a 2-cocycle with D^{gr*} viewed as a trivial L-module. E has a non-degenerate invariant symmetric bilinear form defined by

$$(x_1 + c_1 + d_1 \mid x_2 + c_2 + d_2) = (x_1 \mid x_2)_L + c_1(d_2) + c_2(d_1). \tag{2.2.4}$$

Let

$$H = \mathfrak{h} \oplus (D^0)^* \oplus D^0$$

where

$$\mathfrak{h} = L_0^0 = \text{span}_k\{[e_\alpha^\lambda, f_\alpha^\lambda],\ \alpha \in \Delta^\times, \lambda \in \Lambda\}$$

for $e_\alpha^\lambda, f_\alpha^\lambda$ as in (**LT2**).

The following Theorem is given in Theorem 14 of [N2].

Theorem 2.15. *(a) For any (D, τ) in $P(L)$, the triple $(E, H, (\ \mid\ ))$ constructed above is an extended affine Lie algebra, denoted by $E(L, D, \tau)$. Its core is $L \oplus D^{gr*}$, and centreless core is L.*

(b) Conversely, for any extended affine Lie algebra $(E, H, (\ \mid\ ))$, let $L = E_{cc}$ be its centreless core. Then there exists $(D, \tau) \in P(L)$ such that $(E, H, (\ \mid\ )) \cong E(L, D, \tau)$ as extended affine Lie algebras.

2.3 Multiloop Lie algebras

By now, we know that every extended affine Lie algebra can be constructed from a centreless Lie torus, so the realization of extended affine Lie algebras reduces to the realization of centreless Lie tori. In this section, we will explain the realization of fgc centreless Lie tori as multiloop Lie algebras.

Firstly, we will recall the construction of multiloop Lie algebras and related results (c.f. [ABFP1], [ABFP2]). Let now k be an algebraically closed field of characteristic 0. For any positive integer m, fix once and for all a choice of m-th primitive root of unity ξ_m compatible in the sense that $\xi_{m\ell}^\ell = \xi_m$ for all $\ell \in \mathbb{Z}_{\geq 1}$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over k. Aut($\mathfrak{g}$) is the automorphism group of $\mathfrak{g}$. For n a positive integer, denote by

$$\text{cfo}_n(\mathfrak{g}) = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \text{Aut}(\mathfrak{g})^n \mid |\sigma_i| < \infty,\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for all } i, j\}$$

the set of n-tuples of commuting finite order automorphisms of $\mathfrak{g}$.

Let $\boldsymbol{m} = (m_1, \ldots, m_n)$ be a sequence of positive integers such that $\sigma_i^{m_i} = id$ for all $i = 1, \ldots, n$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \text{cfo}_n(\mathfrak{g})$. For simplicity, we will write $\boldsymbol{\sigma}^{\boldsymbol{m}} = \boldsymbol{Id}$. Let

$$R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}],\ \ S = k[t_1^{\pm \frac{1}{m_1}}, \ldots, t_n^{\pm \frac{1}{m_n}}]$$

be two Laurent polynomial rings, and $R \subset S$ as a subring. Set $z_i = t_i^{\frac{1}{m_i}}$, thus $t_i = z_i^{m_i}$, $i = 1, \ldots, n$, and

$$R = k[z_1^{\pm m_1}, \ldots, z_n^{\pm m_n}],\ \ S = k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}].$$

Then $\mathfrak{g} \otimes_k S$ is a Lie algebra under

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg, \ \ \text{for } x, y \in \mathfrak{g},\ f, g \in S. \tag{2.3.1}$$

Let $\Lambda = \mathbb{Z}^n$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, denote by

$$\boldsymbol{z}^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n} = t_1^{\frac{\lambda_1}{m_1}} \cdots t_n^{\frac{\lambda_n}{m_n}}. \tag{2.3.2}$$

Let $^- : \Lambda \longrightarrow \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z} = \bar{\Lambda}$ be the canonical map. With respect to $\boldsymbol{\sigma}$, $\mathfrak{g}$ has an eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{\bar{\lambda} \in \bar{\Lambda}} \mathfrak{g}_{\bar{\lambda}}$$

with $\mathfrak{g}_{\bar{\lambda}} = \{x \in \mathfrak{g} \mid \sigma_i(x) = \xi_{m_i}^{\lambda_i} x,\ 1 \le i \le n\}$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$.

Then we can define a Λ-graded Lie algebra

$$\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) = \bigoplus_{\lambda \in \Lambda} (\mathfrak{g}_{\bar{\lambda}} \otimes \boldsymbol{z}^\lambda) \subseteq \mathfrak{g} \otimes_k S. \tag{2.3.3}$$

$\mathfrak{g} \otimes_k S$ is equipped with a Λ-grading by

$$(\mathfrak{g} \otimes_k S)^\lambda = \mathfrak{g} \otimes_k S^\lambda = \mathfrak{g} \otimes k\boldsymbol{z}^\lambda,\ \forall \lambda \in \Lambda,$$

so that $\mathcal{L}$ is a Λ-graded subalgebra of $\mathfrak{g} \otimes_k S$ with $\mathcal{L}^\lambda = \mathfrak{g}_{\bar{\lambda}} \otimes \boldsymbol{z}^\lambda \subset (\mathfrak{g} \otimes_k S)^\lambda$.

Definition 2.16. Suppose $\mathfrak{g}$ is a finite dimensional simple Lie algebra, $\boldsymbol{\sigma} \in \mathrm{cfo}_n(\mathfrak{g})$, and $\boldsymbol{m} \in \mathbb{Z}^n_{\geq 1}$ such that $\boldsymbol{\sigma}^{\boldsymbol{m}} = \boldsymbol{Id}$. Then the Lie algebra $\mathcal{L}$ defined above is called a *multiloop Lie algebra* determined by $\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}$. The Lie algebra $\mathfrak{g} \otimes S$ is called an *untwisted multiloop Lie algebra*. The positive integer n is called the *nullity* of both $\mathcal{L}$ and $\mathfrak{g} \otimes S$.

When $n = 1$, $\mathcal{L}$ and $\mathfrak{g} \otimes S$ are called *twisted loop algebra* and *untwisted loop algebra*, respectively (c.f. Chapter 8 of [K]).

Remark 2.17. The positive integer m_i is a peroid of σ_i. It may not be the order of σ_i, $i = 1, \ldots, n$. There are studies related to the effects of changing the three elements $\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}$ in the structures of multiloop Lie algebras. Since these are not relevant to our work, we will not go into details. The interested reader can refer to [ABFP2], [ABP1], [ABP2], [K], [Na].

Now we will review the realization of centreless Lie tori as multiloop Lie algebras ([ABFP2]). To be more precise, we need the following definition (c.f. Definition 2.1.1 of [ABFP2]).

Definition 2.18. If $\mathcal{L}$ is a Lie torus with grading $Q \times \Lambda$ and $\mathcal{L}'$ is a Lie torus with grading $Q' \times \Lambda'$, we say that $\mathcal{L}$ and $\mathcal{L}'$ are *bi-isomorphic* if there is an algebra isomorphism from $\mathcal{L}$ to $\mathcal{L}'$ that is isograded relative to the two gradings; this means that there is an algebra isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{L}'$, a group isomorphism $\varphi_r : Q \longrightarrow Q$, and a group isomorphism $\varphi_e : \Lambda \longrightarrow \Lambda'$ such that

$$\varphi(\mathcal{L}^\lambda_\alpha) = \mathcal{L}^{\varphi_e(\lambda)}_{\varphi_r(\alpha)}$$

for any $\alpha \in Q$ and $\lambda \in \Lambda$.

A multiloop Lie $\mathbb{Z}^n$-torus is a multiloop Lie algebra with a grading $\mathbb{Z}^n$ that is also a Lie torus. Note that not every multiloop Lie algebra with the defined grading (see (2.3.3)) is a Lie torus. For example, it is shown in Proposition 3.4.1 of [ABFP2] that if $n = 1$, then $\mathcal{L}(\mathfrak{g}, \sigma, m)$ with grading defined as in (2.3.3) is a centreless Lie torus if and only if σ is a diagram automorphism. The following important result is given in Theorem 3.3.1 of [ABFP2].

Theorem 2.19. *Let k be algebraically closed of characteristic 0. A centreless Lie Λ-torus $\mathcal{L}$ of nullity $n \geq 1$ is bi-isomorphic to a multiloop Lie $\mathbb{Z}^n$-torus if and only if $\mathcal{L}$ is fgc, i.e. it is finitely generated as a module over its centroid.*

Therefore, up to isomorphism, every fgc centreless Lie torus has the form of a multiloop Lie algebra.

Remark 2.20. The construction in Section 2.2 is given for centreless Lie tori. It is known that every fgc centreless Lie torus is a multiloop Lie algebra, but a multiloop Lie algebra may not be a Lie torus. However, starting from any multiloop Lie algebra $L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$, if the fixed point subalgebra $\mathfrak{g}^{\boldsymbol{\sigma}} = \mathfrak{g}_{\bar{0}} \neq 0$, then by the same construction process, one can get EALAs from multiloop Lie algebras (see Section 5 of [Na]).

Chapter 3

Fixed point subalgebra realization

It is known that every fgc centreless core can be realized as a fixed point subalgebra of some untwisted centreless core under a finite abelian group. In this chapter, we first show that the fixed point subalgebras of untwisted EALAs under a finite abelian group (which is determined by some commuting finite order automorphisms) are also (fgc) EALAs. Then we show that every fgc extended affine Lie algebra can be realized as a fixed point subalgebra of some untwisted EALAs, which generalizes the result of affine Kac-Moody Lie algebras as they are fgc extended affine Lie algebras (of nullity 1). At the end, we will give an explanation of our results to affine Kac-Moody Lie algebras. We also give a new proof of the equivalence between EALAs of nullity 1 and affine Kac-Moody Lie algebras. Throughout this chapter, k will be an algebraically closed field of characteristic 0.

3.1 Fixed point subalgebras of untwisted EALAs

Let E be an untwisted EALA of nullity n. Then, by definition, E_{cc} is of the form $\mathfrak{g} \otimes S$ with $\mathfrak{g}$ a finite dimensional simple Lie algebra, and S a Laurent polynomial ring in n variables. From Theorem 2.15, for any $(\tilde{D}, \tilde{\tau}) \in P(\mathfrak{g} \otimes S)$,

$$E_S = (\mathfrak{g} \otimes S) \oplus \tilde{D}^{gr*} \oplus \tilde{D} \tag{3.1.1}$$

is an untwisted EALA. For any sequence of positive integers $\boldsymbol{m} = (m_1, \ldots, m_n)$, let

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}.$$

Let S, R be the Laurent polynomial rings as in Section 2.3. For any $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n_{\geq 1}$, denote by $\boldsymbol{\xi}^{\boldsymbol{\ell}}_{\boldsymbol{m}} = \xi^{\ell_1}_{m_1} \cdots \xi^{\ell_n}_{m_n}$. For any $\boldsymbol{\sigma} \in \mathrm{cfo}_n(\mathfrak{g})$ such that $\mathfrak{g}^{\boldsymbol{\sigma}} \neq 0$. Suppose with $\boldsymbol{m} = (m_1, \ldots, m_n)$ is such that $\boldsymbol{\sigma}^{\boldsymbol{m}} = \boldsymbol{Id}$, consider the following Γ-action on

$\mathfrak{g} \otimes S$.

$$\gamma.(x \otimes s) = (\boldsymbol{\sigma}^{-\gamma} \otimes id_S) \circ (id_{\mathfrak{g}} \otimes \gamma)(x \otimes s) \ \text{ for any } \ \gamma \in \Gamma, x \in \mathfrak{g}, s \in S,$$

where $\gamma = \bar{\lambda} = (\bar{\lambda_1}, \ldots, \bar{\lambda_n}) \in \Gamma$ acts on the elements in S by

$$\gamma.\boldsymbol{z}^{\mu} = \boldsymbol{\xi}_{\boldsymbol{m}}^{\lambda\mu} \boldsymbol{z}^{\mu} \ \text{ for all } \mu \in \Lambda = \mathbb{Z}^n, \tag{3.1.2}$$

and $\boldsymbol{\sigma}^{-\gamma}$ means the composition $\sigma_1^{-\lambda_1} \circ \cdots \circ \sigma_n^{-\lambda_n}$. Denote by

$$h_\gamma = (\boldsymbol{\sigma}^{-\gamma} \otimes id_S) \circ (id_{\mathfrak{g}} \otimes \gamma) \tag{3.1.3}$$

for any $\gamma \in \Gamma$. Then $\gamma.(x \otimes s) = h_\gamma(x \otimes s)$.

Under this Γ-action, it is easy to see that the Γ-fixed point subalgebra of $\mathfrak{g} \otimes S$ is actually the multiloop Lie algebra $\mathcal{L}$ defined before, i.e.

$$\mathcal{L} = \mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) = (\mathfrak{g} \otimes S)^{\Gamma} = \{y \in \mathfrak{g} \otimes S \mid h_\gamma(y) = y, \ \ \forall \gamma \in \Gamma\}. \tag{3.1.4}$$

Now we are going to show that Γ actually acts on the untwisted EALA E_S and its fixed point subalgebra $(E_S)^{\Gamma}$ is also an EALA.

The Γ-action on $\mathfrak{g} \otimes S$ induces a Γ-action on $\mathrm{Der}_k(\mathfrak{g} \otimes S)$ by

$$\Gamma \times \mathrm{Der}_k(\mathfrak{g} \otimes S) \longrightarrow \mathrm{Der}_k(\mathfrak{g} \otimes S) \tag{3.1.5}$$

$$(\gamma, d) \mapsto \gamma.d = C_{h_\gamma}(d) = h_\gamma \circ d \circ h_\gamma^{-1}.$$

For any $(\tilde{D}, \tilde{\tau}) \in P(\mathfrak{g} \otimes S)$, to show that Γ acts on $\tilde{D}$ and $\tilde{D}^{gr*}$, we first need to explain several identifications (c.f. [CNPY], [GP2], [P5]). It follows from [GP2] that the centroids of $\mathcal{L}$, $\mathfrak{g} \otimes S$ can be identified with the rings R, S respectively, via

$$R \longrightarrow \mathrm{Ctd}_k(\mathcal{L}); \ \ r \mapsto \chi_r, \ \text{ where } \chi_r(x) = rx, \ \ \forall\, x \in \mathcal{L},$$

$$S \longrightarrow \mathrm{Ctd}_k(\mathfrak{g} \otimes S), \ \ s \mapsto \chi_s, \ \text{ where } \chi_s(y) = sy, \ \ \forall\, y \in \mathfrak{g} \otimes S.$$

Therefore, by Definition 2.4, every untwisted EALA is fgc. And the centroid grading groups of $\mathcal{L}$ and $\mathfrak{g} \otimes S$ (defined analogously as in the Lie torus case) are $\Xi = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and $\Lambda = \mathbb{Z}^n$ respectively.

The argument below follows from Section 4.2 of [CNPY]. Since S is étale over R, every k-linear derivation $\delta \in \mathrm{Der}_k(R)$ extends uniquely to a derivation $\hat{\delta} \in \mathrm{Der}_k(S)$. We identify $\mathrm{Der}_k(R) \subset \mathrm{Der}_k(S)$ by viewing $\delta = \hat{\delta}$. There is a natural symmetric bilinear form on S defined by

$$(\boldsymbol{z}^\mu \mid \boldsymbol{z}^\nu) = \delta_{\mu+\nu,0}, \tag{3.1.6}$$

and similarly on R. Then the skew-symmetric derivations on R can be identified as a graded-subalgebra of the skew-symmetric derivations on S in the following way:

$$\mathrm{SDer}_k(R) = \bigoplus_{\mu \in \Xi} z^\mu \{\partial_\theta \mid \theta(\mu) = 0\} \subset \bigoplus_{\mu \in \Lambda} z^\mu \{\partial_\theta \mid \theta(\mu) = 0\} = \mathrm{SDer}_k(S),$$

where ∂_θ is the degree derivation with respect to the natural gradings on R and S. Let κ be the Killing form on $\mathfrak{g}$. Then there is a natural bilinear form on $\mathfrak{g} \otimes S$ with

$$(x_1 \otimes \boldsymbol{z}^\mu \mid x_2 \otimes \boldsymbol{z}^\nu) = \kappa(x_1, x_2)\delta_{\mu+\nu,0}. \tag{3.1.7}$$

$(\ \mid\ )$ is invariant, nondegenerate and symmetric. Its restriction $(\ \mid\ )_{\mathcal{L}}$ to the multiloop Lie algebra $\mathcal{L}$ has the same properties and is up to a scalar the only such bilinear form (Corollary 7.4 of [NPPS]).

Clearly, the map $\delta \mapsto id_{\mathfrak{g}} \otimes \delta$ identifes $\mathrm{Der}_k(S)$ with the subalgebra $\mathrm{CDer}_k(\mathfrak{g} \otimes S)$ of centroidal derivations, and it maps $\mathrm{SDer}_k(S)$ onto $\mathrm{SCDer}_k(\mathfrak{g} \otimes S)$. Analogously, $\delta \mapsto (id_{\mathfrak{g}} \otimes \delta)|_{\mathcal{L}}$ identifies $\mathrm{Der}_k(R)$ with $\mathrm{CDer}_k(\mathcal{L})$ (c.f. [P5]) where δ is viewed as an element in $\mathrm{Der}_k(S)$ via the above identification $\mathrm{Der}_k(R) \subset \mathrm{Der}_k(S)$, and it maps $\mathrm{SDer}_k(R)$ onto $\mathrm{SCDer}_k(\mathcal{L})$. Via all these identifications, we get

$$\mathrm{SCDer}_k(\mathcal{L}) = \mathrm{SDer}_k(R) \subset \mathrm{SDer}_k(S) = \mathrm{SCDer}_k(\mathfrak{g} \otimes S).$$

Explicitly, we can write

$$\mathrm{SCDer}_k(\mathcal{L}) = \bigoplus_{\mu \in \Xi} (\mathrm{SCDer}_k(\mathcal{L}))^\mu = \bigoplus_{\mu \in \Xi} z^\mu \{\partial_\theta \in \mathcal{D} : \theta(\mu) = 0\},$$

$$\mathrm{SCDer}_k(\mathfrak{g} \otimes S) = \bigoplus_{\nu \in \Lambda} (\mathrm{SCDer}_k(\mathfrak{g} \otimes S))^\nu = \bigoplus_{\nu \in \Lambda} z^\nu \{\partial_\theta \in \mathcal{D} : \theta(\nu) = 0\},$$

so that for every $\mu \in \Xi \subset \Lambda$,

$$(\mathrm{SCDer}_k(\mathcal{L}))^\mu = (\mathrm{SCDer}_k(\mathfrak{g} \otimes S))^\mu.$$

We first show that the Γ-action preserves every graded subalgebra of the skew-centroidal derivation algebra and also preserves its grading so that it induces an action on its graded dual.

Lemma 3.1. Γ *acts on every graded subalgebra* $\tilde{D} \subseteq$ *SCDer*$_k(\mathfrak{g}\otimes S)$ *and its graded dual* $\tilde{D}^{gr*}$*. In particular,* $($*SCDer*$_k(\mathfrak{g}\otimes S))^\Gamma =$ *SCDer*$_k(\mathcal{L})$*, and* $(\tilde{D}^{gr*})^\Gamma = (\tilde{D}^\Gamma)^{gr*}$.

Proof. For any element $\boldsymbol{z}^\nu \partial_\theta \in (\mathrm{SCDer}_k(\mathfrak{g} \otimes S))^\nu \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes S)$, $\nu \in \Lambda$, it is easy to see, by the Γ-action defined on $\mathrm{Der}_k(\mathfrak{g} \otimes S)$, that

$$C_{h_\gamma}(\boldsymbol{z}^\nu \partial_\theta) = \boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma\nu} \boldsymbol{z}^\nu \partial_\theta \in (\mathrm{SCDer}_k(\mathfrak{g} \otimes S))^\nu, \quad \text{for any } \gamma \in \Gamma. \tag{3.1.8}$$

Therefore Γ acts on $\mathrm{SCDer}_k(\mathfrak{g} \otimes S)$ as an automorphism group and preserves its grading.

From (3.1.8), we see that actually Γ acts on the skew-centroidal derivations by scalars. Therefore for any graded subalgebra $\tilde{D} \subseteq \mathrm{SCDer}_k(\mathfrak{g}\otimes S)$, Γ acts on $\tilde{D}$ and preserves its grading. Write $\tilde{D} = \bigoplus_{\nu\in\Lambda} \tilde{D}^\nu$. Then for any $\nu \in \Lambda$, by definition,

$$\begin{aligned}(\tilde{D}^\nu)^\Gamma &= \{\boldsymbol{z}^\nu \partial_\theta \in \tilde{D}^\nu \mid \boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma\nu} \boldsymbol{z}^\nu \partial_\theta = \boldsymbol{z}^\nu \partial_\theta, \forall \gamma \in \Gamma\} \\ &= \{\boldsymbol{z}^\nu \partial_\theta \in \tilde{D}^\nu \mid \boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma\nu} = 1, \forall \gamma \in \Gamma\}.\end{aligned} \tag{3.1.9}$$

The ξ_{m_i}, $i = 1, \ldots, n$ are all primitive, so

$$(\tilde{D}^\nu)^\Gamma = \tilde{D}^\nu \ \text{ if } \nu \in \Xi,$$

$$(\tilde{D}^\nu)^\Gamma = 0 \ \text{ if } \nu \in \Lambda \backslash \Xi.$$

Hence

$$\tilde{D}^\Gamma = \bigoplus_{\nu\in\Lambda} (\tilde{D}^\nu)^\Gamma = \bigoplus_{\mu\in\Xi} \tilde{D}^\mu.$$

In particular,

$$
\begin{aligned}
(\text{SCDer}_k(\mathfrak{g}\otimes S))^\Gamma &= \bigoplus_{\nu\in\Lambda}((\text{SCDer}_k(\mathfrak{g}\otimes S))^\nu)^\Gamma \\
&= \bigoplus_{\mu\in\Xi}(\text{SCDer}_k(\mathfrak{g}\otimes S))^\mu \\
&= \bigoplus_{\mu\in\Xi}(\text{SCDer}_k(\mathcal{L}))^\mu = \text{SCDer}_k(\mathcal{L}). \qquad (3.1.10)
\end{aligned}
$$

Since the Γ-action preserves the grading, it induces a Γ-action on $(\tilde{D}^\nu)^*$ for any $\nu\in\Lambda$ by

$$(\gamma.\varphi)(d)=\varphi(\gamma^{-1}.d)=\varphi(C_{h_\gamma^{-1}}(d)),\ \forall\gamma\in\Gamma,\ \varphi\in(\tilde{D}^\nu)^*, d\in\tilde{D}^\nu, \quad (3.1.11)$$

which then induces a Γ-action on $\tilde{D}^{gr*}=\bigoplus_{\nu\in\Lambda}(\tilde{D}^\nu)^*$ that preserves the grading by definition. Simiarly, it is easy to show that

$$((\tilde{D}^\nu)^*)^\Gamma=(\tilde{D}^\nu)^* \ \text{ if } \nu\in\Xi,$$

$$((\tilde{D}^\nu)^*)^\Gamma=0 \ \text{ if } \nu\in\Lambda\backslash\Xi.$$

Hence

$$(\tilde{D}^{gr*})^\Gamma=\bigoplus_{\nu\in\Lambda}((\tilde{D}^\nu)^*)^\Gamma=\bigoplus_{\mu\in\Xi}(\tilde{D}^\mu)^*=(\tilde{D}^\Gamma)^{gr*}.$$

In particular,

$$((\text{SCDer}_k(\mathfrak{g}\otimes S))^{gr*})^\Gamma=(\text{SCDer}_k(\mathcal{L}))^{gr*}. \qquad (3.1.12)$$

□

Lemma 3.2. For any $(\tilde{D},\tilde{\tau})\in P(\mathfrak{g}\otimes_k S)$, the 2-cocycle $\tilde{\tau}$ is Γ-equivariant under the above Γ-actions.

Proof. Recall that $\tilde{\tau}:\tilde{D}\times\tilde{D}\longrightarrow\tilde{D}^{gr*}$ is graded, $\tilde{\tau}(\tilde{D}^{\nu_1},\tilde{D}^{\nu_2})\subset(\tilde{D}^{-\nu_1-\nu_2})^*$. For any $\gamma\in\Gamma$, $d_i\in\tilde{D}^{\nu_i}$, $i=1,2$, $d\in\tilde{D}^{-\nu_1-\nu_2}$, we have

$$\tilde{\tau}(\gamma.d_1,\gamma.d_2)(d)=\tilde{\tau}(\boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma\nu_1}d_1,\boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma\nu_2}d_2)(d)=\boldsymbol{\xi}_{\boldsymbol{m}}^{\gamma(\nu_1+\nu_2)}\tilde{\tau}(d_1,d_2)(d),$$

$$(\gamma.\tilde{\tau}(d_1,d_2))(d) = \tilde{\tau}(d_1,d_2)(\gamma^{-1}.d) = \tilde{\tau}(d_1,d_2)(\boldsymbol{\xi}_{\boldsymbol{m}}^{-\gamma(-\nu_1-\nu_2)}(d)).$$

They are equal, so $\tilde{\tau}$ is Γ-equivariant. □

Now
$$\tilde{D}^\Gamma \subseteq (\mathrm{SCDer}_k(\mathfrak{g}\otimes S))^\Gamma = \mathrm{SCDer}_k(\mathcal{L}),$$
and $(\tilde{D}^\Gamma)^0 = (\tilde{D}^0)^\Gamma = \tilde{D}^0$ (actually, Γ acts on degree derivations as the identity). If $ev : \Lambda \longrightarrow (\tilde{D}^0)^*$ is injective, then $ev : \Lambda \longrightarrow ((\tilde{D}^\Gamma)^0)^*$ is also injective. Hence $\tilde{D}^\Gamma$ satisfies property (i) in Section 2.2 for $L = \mathcal{L}$.

Since $\tilde{\tau} : \tilde{D} \times \tilde{D} \longrightarrow \tilde{D}^{gr*}$ is Γ-equivariant, $(\tilde{D}^{gr*})^\Gamma = (\tilde{D}^\Gamma)^{gr*}$, and $(\tilde{D}^\Gamma)^0 = \tilde{D}^0$. It is immediate to then see that the restriction map

$$\tilde{\tau}| : \tilde{D}^\Gamma \times \tilde{D}^\Gamma \longrightarrow (\tilde{D}^{gr*})^\Gamma = (\tilde{D}^\Gamma)^{gr*},$$

satisfies property (ii) in Section 2.2. Thus $(\tilde{D}^\Gamma, \tilde{\tau}|) \in P(\mathcal{L})$.

It then follows from Theorem 2.15 and Remark 2.20 that

Theorem 3.3. Let $E(\mathfrak{g}\otimes_k S, \tilde{D}, \tilde{\tau})$ be any untwisted EALA. For any $\boldsymbol{\sigma} \in \mathrm{cfo}_n(\mathfrak{g})$ with $\mathfrak{g}^{\boldsymbol{\sigma}} \neq 0$, $\boldsymbol{\sigma}^{\boldsymbol{m}} = \boldsymbol{Id}$, let $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$ act on $\tilde{D}$, $\tilde{D}^{gr*}$ as above. Then $(\tilde{D}^\Gamma, \tilde{\tau}|) \in P(\mathcal{L})$. In particular, the fixed point subalgebra $E(\mathfrak{g}\otimes_k S, \tilde{D}, \tilde{\tau})^\Gamma = E(\mathcal{L}, \tilde{D}^\Gamma, \tilde{\tau}|)$ is also an EALA.

3.2 Fgc EALAs as fixed point subalgebras

Let E be an fgc EALA of nullity n. Then E_{cc} is an fgc centreless Lie torus of nullity n, so, by Theorem 2.19, E_{cc} is a multiloop Lie algebra. Suppose E_{cc} is of the form $\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ for some $\mathfrak{g}$, $\boldsymbol{\sigma}, \boldsymbol{m}$. Recall that (see 3.1.4)

$$\mathcal{L} = (\mathfrak{g}\otimes_k S)^\Gamma.$$

Now for any $(D, \tau) \in P(\mathcal{L})$, $E = \mathcal{L} \oplus D^{gr*} \oplus D$ is an fgc EALA.

Our question is, does there exist $(\tilde{D}, \tilde{\tau}) \in P(\mathfrak{g}\otimes_k S)$ such that the fixed point subalgebra under Γ of the obtained untwisted EALA $E_S = (\mathfrak{g}\otimes_k S) \oplus \tilde{D}^{gr*} \oplus \tilde{D}$ is the EALA E?

Note that for any $(\tilde{D}, \tilde{\tau}) \in P(\mathfrak{g} \otimes_k S)$, by Lemma 3.1 and Lemma 3.2, Γ acts on $\tilde{D}, \tilde{D}^{gr*}$ and $\tilde{\tau}$ is Γ-equivariant. In what follows, we will use the identifications explained in Section 3.1.

The answer to the question above is positive, as one example of such $(\tilde{D}, \tilde{\tau})$ is given in Lemma 4.5 of [CNPY]. One can just take $(\tilde{D}, \tilde{\tau}) = (D, \tau)$. Since $\mathrm{SCDer}_k(\mathcal{L}) \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes S)$, we have $(D, \tau) \in P(\mathfrak{g} \otimes_k S)$. So, by Theorem 2.15, $E_S = (\mathfrak{g} \otimes_k S) \oplus D^{gr*} \oplus D$ is an untwisted EALA.

By the Γ-actions defined above, we know that for any $D \subseteq \mathrm{SCDer}_k(\mathcal{L})$,

$$D^\Gamma = D, \ \text{ and } \ (D^{gr*})^\Gamma = D^{gr*}.$$

The following result is stated in Section 6 of [CNPY].

Theorem 3.4. *For any fgc EALA $E(\mathcal{L}, D, \tau)$ where $\mathcal{L} = (\mathfrak{g} \otimes_k S)^\Gamma$ for some Γ a finite abelian group, there exists an untwisted EALA $E_S(\mathfrak{g} \otimes_k S, D, \tau)$ such that the fixed point subalgebra of E_S under Γ is $E = E(\mathcal{L}, D, \tau)$, i.e.*

$$(E_S)^\Gamma = (\mathfrak{g} \otimes_k S)^\Gamma \oplus (D^{gr*})^\Gamma \oplus D^\Gamma = \mathcal{L} \oplus D^{gr*} \oplus D = E.$$

It is natural to ask then if such a $(\tilde{D}, \tilde{\tau})$ is unique? Generally, the answer is negative, the untwisted EALA need not be unique. For example, one can take any graded subalgebra $\tilde{D}$ of $\mathrm{SCDer}_k(\mathfrak{g} \otimes S)$ with the property that it contains D and $\tilde{D}^\mu = D^\mu$ for all $\mu \in \Xi$, so that it satisfies the injectivity condition and $\tilde{D}^\Gamma = D$. Then one defines

$$\tilde{\tau} : \tilde{D} \times \tilde{D} \longrightarrow (\tilde{D})^{gr*}$$

by $\tilde{\tau}|_{D \times D} = \tau$, and 0 on other elements. It is straightforward to show that such defined $\tilde{\tau}$ is indeed a 2-cocycle satisfying the property (ii) before. Hence

$$(\tilde{D}, \tilde{\tau}) \in P(\mathfrak{g} \otimes_k S)$$

with

$$(\tilde{D})^\Gamma = D \ \text{ and } \ \tilde{\tau}| = \tau.$$

Therefore, for any such EALA $E(\mathfrak{g} \otimes_k S, \tilde{D}, \tilde{\tau})$, its Γ-fixed point subalgebra is the EALA $E(\mathcal{L}, D, \tau)$. We give an explicit example of such $\tilde{D}$ that is not D.

Example 3.5. Suppose $n = 2$, $\boldsymbol{\sigma} = (id, \sigma_2)$ with $\sigma_2^2 = 1$, let $\boldsymbol{m} = (1, 2)$. Then

$\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\Xi = \mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = \Lambda$,

$$R = k[t_1^{\pm 1}, t_2^{\pm 1}] = k[z_1^{\pm 1}, z_2^{\pm 2}], \quad S = k[t_1^{\pm 1}, t_2^{\pm \frac{1}{2}}] = k[z_1^{\pm 1}, z_2^{\pm 1}],$$

$$\mathcal{L} = L(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) = (\mathfrak{g} \otimes_k S)^{\Gamma} \subset \mathfrak{g} \otimes_k S.$$

For any $D \subseteq \mathrm{SCDer}_k(\mathcal{L})$ satisfying property (i), note that D is $\mathbb{Z} \times 2\mathbb{Z}$-graded, consider

$$\tilde{D} = D \oplus D' \ \text{ with } \ D' = \bigoplus_{\nu \in \mathbb{Z} \times (2\mathbb{Z}+1)} k\boldsymbol{z}^{\nu} \partial_{\theta}^{\nu},$$

where

$$\boldsymbol{z}^{\nu} = z_1^m z_2^{2n+1}, \quad \partial_{\theta}^{\nu} = (2n+1) z_1 \frac{\partial}{\partial z_1} - m z_2 \frac{\partial}{\partial z_2}$$

for any $\nu = (m, 2n+1) \in \mathbb{Z} \times (2\mathbb{Z}+1)$. It is straightforward to show that $\tilde{D} \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes_k S)$ is a graded subalgebra satisfying the property (i) and $\tilde{D}^{\Gamma} = D$ with Γ-action defined as before (note Γ acts on the second variable of S).

We will explain below that for EALAs of nullity 1, the untwisted EALA is unique up to isomorphism.

3.3 Application to affine Kac-Moody Lie algebras

We now give a characterization of the equivalence of EALAs of nullity 1 and affine Kac-Moody Lie algebras which is different from the ones given in [ABGP] and [ABP4]. We also give an explanation of the above results in the affine case.

Theorem 3.6. *Up to natural isomorphism, extended affine Lie algebras of nullity 1 are the same as affine Kac-Moody Lie algebras.*

Proof. For any extended affine Lie algebra E of nullity 1, E_{cc} is a centreless Lie torus of nullity 1. By the classification of centreless Lie tori, we know that E_{cc} is always fgc. Therefore E_{cc} is a loop algebra, say $E_{cc} = L(\mathfrak{g}, \sigma, m)$ for some $(\mathfrak{g}, \sigma, m)$ where σ can be choosen to be a diagram automorphism (c.f. Theorem 2 of [P1]). Then up to isomorphism

$$E = L(\mathfrak{g}, \sigma, m) \oplus D^{gr*} \oplus D$$

for some $(D, \tau) \in P(L(\mathfrak{g}, \sigma, m))$. Given that

$$\mathrm{SCDer}_k(L(\mathfrak{g}, \sigma, m)) = kz\frac{d}{dz} = kt^{\frac{1}{m}}\frac{d}{dt^{\frac{1}{m}}}$$

only consists of degree derivations, where $z = t^{\frac{1}{m}}$, there is only one possibility for D, i.e. D can only be kd for d the degree derivation. And due to the requirement for τ that $\tau(D^0, D) = 0$, in this case, τ can only be identically 0. Since D^{gr*} is also one-dimensional, we denote it by kc with $c(d) = 1$. Hence

$$E = L(\mathfrak{g}, \sigma, m) \oplus kc \oplus kd,$$

with Lie bracket defined by the formula (2.2.2) (remember that $\tau = 0$, $[d, d] = 0$ and note that in this case the contragredient action of D on D^{gr*} is actually trivial). It follows from the realization of affine Kac-Moody Lie algebras (Chapter 7 and 8 of [K]) that E is an affine Kac-Moody Lie algebra.

Conversely, let E be an affine Kac-Moody Lie algebra. Then, again by its realization (Chapter 7 and 8 [K]), E is of the form

$$E = L(\mathfrak{g}, \sigma, m) \oplus kc \oplus kd,$$

for d the degree derivation, kc a one-dimensional center, and σ can be choosen to be a diagram automorphism of the finite dimensional simple Lie algebra $\mathfrak{g}$ of finite order m. Here $L(\mathfrak{g}, \sigma, m)$ is a loop algebra of nullity 1. By Proposition 3.4.1 of [ABFP2], $L(\mathfrak{g}, \sigma, m)$ is a centreless Lie torus and it is clear that

$$(kd, 0) \in P(L(\mathfrak{g}, \sigma, m)),$$

so E is an extended affine Lie algebra of nullity 1. □

By Theorem 3.6, affine Kac-Moody Lie algebras are extended affine Lie algebras of nullity 1 (see also Proposition 5.2.3 of [ABP4] or Theorem 2.32 of [ABGP]). Therefore, by Theorem 3.3, the fixed point subalgebra of an untwisted affine Kac-Moody Lie algebra is also an affine Kac-Moody Lie algebra. Conversely, by Section 3.2, we know that every affine Kac-Moody Lie algebra can be realized as a fixed point subalgebra of some untwisted affine Kac-Moody Lie algebra. We now explain why the untwised affine Kac-Moody Lie algebra is actually unique.

This is because

$$\mathrm{SCDer}_k(\mathfrak{g} \otimes k[z^{\pm 1}]) = kz\frac{d}{dz},$$

i.e. all skew-centroidal derivations are degree derivations, so

$$\mathrm{SCDer}_k(\mathfrak{g} \otimes k[z^{\pm 1}]) = kd = \mathrm{SCDer}_k(L(\mathfrak{g}, \sigma, m)),$$

where d is the degree derivation and $L(\mathfrak{g}, \sigma, m)$ is any loop algebra. Hence the graded subalgebra D has no choice but to be the whole set of skew-centroidal derivations, which justifies that in the affine case, the central extension is universal, and then τ has no choice but to be 0. Therefore

$$P(\mathfrak{g} \otimes k[z^{\pm 1}]) = \{(kd, 0)\} = P(L(\mathfrak{g}, \sigma, m))$$

only consists of one point, uniqueness follows (up to isomorphism).

Chapter 4

Descent theory and derivation algebras

Throughout this chapter, k will be a field of characteristic 0. We will first introduce some basic notions and terminology that will be used. The notion of (twisted) forms given here is suitable for our work. We also review the definition of a certain (non-abelian) first cohomology set and its relation with twisted forms (c.f. [W]). Then we recall related notions about differentials and derivations for Lie algebras and associative algebras from [KP], [M] respectively. At the end, we explain in detail how to get the result that derivation algebras of twisted forms of certain Lie algebras can also be viewed as twisted forms.

4.1 Faithfully flat descent

Definition 4.1. Let R be a ring, N and $N^{'}$ are R-modules. We say $N^{'}$ is an *R-form* (or *R-twisted form*) of N if

$$N^{'} \otimes_R S \cong N \otimes_R S$$

as S-modules for some faithfully flat ring extension S/R.

If N and $N^{'}$ are Lie algebras over R, then the isomorphism in the definition is required to be an S-Lie algebra isomorphism with the Lie bracket on the tensor product defined naturally.

By abuse of terminology and for simplicitly, if $\mathfrak{g}$ is a Lie algebra over k, we say an R-Lie algebra L is an *R-form* of $\mathfrak{g}$ if it is an R-form of the Lie algebra $\mathfrak{g} \otimes_k R$.

We now recall the group functor $\mathbf{Aut}(N)$ for an arbitrary R-module N. Let $R-$**alg** be the category of ring extensions of R, and **Gp** be the category of groups. Then $\mathbf{Aut}(N)$ is the functor, from the category $R-$**alg** to the category **Gp**, defined as follows:

$$\mathbf{Aut}(N) : R^{'}/R \mapsto \mathbf{Aut}(N)(R^{'}) := \text{Aut}_{R^{'}-mod}(N \otimes_R R^{'}),$$

and for every homomorphism $f : R^{'} \mapsto R^{''}$ in $R-$**alg**, the group homomorphism in $\mathbf{Gp}$ is given by

$$\mathbf{Aut}(N)(f) : \text{Aut}_{R^{'}-mod}(N \otimes_R R^{'}) \longrightarrow \text{Aut}_{R^{''}-mod}(N \otimes_R R^{''}),$$

$$\phi \mapsto \mathbf{Aut}(N)(f)(\phi) := \phi_{R^{''}},$$

where $\phi_{R^{''}}$ is the unique $R^{''}$-module automorphism of $N \otimes_R R^{''}$ such that for any $n \in N$,

$$\phi_{R^{''}}(n \otimes 1_{R^{''}}) = \sum n_i \otimes f(r_i^{'}) \;\text{ whenever }\; \phi(n \otimes 1_{R^{'}}) = \sum n_i \otimes r_i^{'}$$

where $n_i \in N$, $r_i^{'} \in R^{'}$. Similarly, if we are working with Lie algebras, then all module homomorphisms are required to be Lie algebra homomorphisms.

Let S/R be any ring extension, consider the following morphisms in $R-$**alg**:

$$S \otimes_R S \longrightarrow S \otimes_R S \otimes_R S$$
$$P_{12} : s \otimes t \mapsto s \otimes t \otimes 1_S,$$
$$P_{13} : s \otimes t \mapsto s \otimes 1_S \otimes t,$$
$$P_{23} : s \otimes t \mapsto 1_S \otimes s \otimes t,$$

$$P_1 : S \longrightarrow S \otimes_R S, \;\; s \mapsto s \otimes 1_S,$$

and

$$P_2 : S \longrightarrow S \otimes_R S, \;\; s \mapsto 1_S \otimes s.$$

Denote by $S^{''} = S \otimes_R S$ and $S^{'''} = S \otimes_R S \otimes_R S$. For any element φ in $\mathbf{Aut}(N)(S^{''}) = \text{Aut}_{S^{''}-mod}(N \otimes_R S^{''})$, denote by

$$\varphi_{ij} = \mathbf{Aut}(N)(P_{ij})(\varphi) \in \text{Aut}_{S^{'''}-mod}(N \otimes_R S^{'''})$$

for $1 \leq i < j \leq 3$. Consider the elements $\varphi \in \mathbf{Aut}(N)(S^{''})$ with $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$, they are called *1-cocycles*. Two 1-cocycles φ and $\varphi^{'}$ are called *cohomologous*, denoted by $\varphi \sim \varphi^{'}$, if

$$\varphi^{'} = (d^2\lambda)\varphi(d^1\lambda)^{-1}$$

for some $\lambda \in \mathbf{Aut}(N)(S)$, where $d^i\lambda = \mathbf{Aut}(N)(P_i)(\lambda)$ for $i = 1, 2$. Such defined $\sim$ is an equivalence relation. The set of equivalence classes is denoted by $H^1(S/R, \mathbf{Aut}(N))$, for every 1-cocycle φ, we denote by $[\varphi]$ its equivalence class. $H^1(S/R, \mathbf{Aut}(N))$ is a pointed set with a distinguished element, i.e. the equivalence class of the identity map. We call it the (non-abelian) first cohomology set.

When S/R is faithfully flat, every 1-cocycle φ determines a *descent datum*. Namely, φ determines a "descended object"M over R such that $M \otimes_R S \cong N \otimes_R S$. Up to isomorphism, M can be characterized as

$$\begin{aligned} M &= (N \otimes_R S)_\varphi \\ &= \{\sum n_i \otimes s_i \in N \otimes_R S \mid \varphi(\sum n_i \otimes s_i \otimes 1_S) = \sum n_i \otimes 1_S \otimes s_i\}, \end{aligned}$$

one verifies that the map

$$f : (N \otimes_R S)_\varphi \otimes_R S \longrightarrow N \otimes_R S;\ (\sum n_i \otimes s_i) \otimes s \mapsto \sum n_i \otimes ss_i$$

is an S-module isomorphism.

The above is a one-to-one correspondence (c.f. § 17.6 of [W]).

Theorem 4.2. *Let $R \longrightarrow S$ be faithfully flat. Then the set of isomorphism classes of S/R-forms of N is in one-to-one correspondence with $H^1(S/R, \mathbf{Aut}(N))$. For any 1-cocycle φ, the $(N \otimes_R S)_\varphi$ is the corresponding S/R-form of N.*

4.2 Derivations and differentials

We first recall some basic notions from [KP]. Let R be a k-algebra. L is an R-*Lie algebra* means that L is an R-module together with an R-bilinear Lie bracket $[\ ,\ ]$; note then L is also a Lie algebra over k. An L-*module* M is a k-module with compatible L-action, i.e. the map $L \otimes_k M \longrightarrow M$, $(x, m) \mapsto xm$ is k-linear such that $[x, y]m = xym - yxm$ for all $x, y \in L, m \in M$. If M is also an R-module, then M is called an $R - L$-*module* if in addition, $x(rm) = (rx)m = r(xm)$ for all $r \in R, x \in L, m \in M$. If M, N are $R - L$-modules, denote by $\mathrm{Hom}_{R-L}(M, N)$ the set of L-equivariant R-module maps. It has a natural R-module structure given by $(rf)(m) = rf(m)$ for all $r \in R$, $f \in \text{Hom}_{R-L}(M, N)$, $m \in M$.

Let L be a Lie algebra over k, and M be an L-module. A *k-derivation* of L with values in M is a k-linear map $\delta : L \longrightarrow M$ such that

$$\delta([x, y]) = x\delta(y) - y\delta(x), \quad \text{for all } x, y \in L.$$

Now let L be an R-Lie algebra. Consider the functor $\text{Der}_k(L, \cdot)$ (or $\text{Der}_{R/k}(L, \cdot)$ to emphasize the presence of R) which is defined from the category of $R - L$-modules to the category of R-modules, where for M an $R-L$-module, $\text{Der}_k(L, M)$ consists of all k-derivations of L with values in M. $\text{Der}_k(L, M)$ has a natural R-module structure via $(r\delta)(x) = r\delta(x)$ for all $r \in R$, $x \in L$, $\delta \in \text{Der}_k(L, M)$. When M is taken to be L, as an L-module via adjoint representation, then a k-derivation of L with values in L is the same as the notion of a derivation on L, and $\text{Der}_k(L, L)$ is the same as the algebra of derivations $\text{Der}_k(L)$ on the Lie algebra L.

In the paper [KP], the authors showed that the functor $\text{Der}_k(L, \cdot)$ is representable, i.e. there exists a unique (up to isomorphism) module of R/k-differentials $\Omega_{R,L/k}$ for L with a universal derivation $d_{R,L,k} : L \longrightarrow \Omega_{R,L/k}$ such that for any $R - L$-module M, the following map is an R-module isomorphism,

$$\psi_{R,L,M} : \text{Hom}_{R-L}(\Omega_{R,L/k}, M) \longrightarrow \text{Der}_k(L, M); \quad \alpha \mapsto \alpha \circ d_{R,L,k}.$$

In other words, any derivation $\delta : L \longrightarrow M$ corresponds to a unique $R-L$-module homomoprhism $\sigma_\delta : \Omega_{R,L/k} \longrightarrow M$ such that $\sigma_\delta \circ d_{R,L,k} = \delta$.

In particular, when M is taken to be L with the adjoint action (note that L is naturally an $R - L$-module), we denote by

$$\psi_{R,L} : \text{Hom}_{R-L}(\Omega_{R,L/k}, L) \longrightarrow \text{Der}_k(L); \quad \alpha \mapsto \alpha \circ d_{R,L,k} \tag{4.2.1}$$

the R-module isomorphism.

We also recall some results on Kähler differentials (c.f. Chapter 10 of [M]). Let R be a k-algebra, and M be an R-module. A *k-derivation* of R into M is an additive map $d : R \longrightarrow M$ such that

$$da = 0 \;\text{ for all } a \in k, \quad \text{and} \quad d(rs) = rds + sdr \;\text{ for all } r, s \in R.$$

There exists a module of differentials (or of Kähler differentials) $\Omega_{R/k}$ of R over k, together with a k-derivation $d_{R/k} : R \longrightarrow \Omega_{R/k}$, which satisfies the following universal property: for any R-module M and for any k-derivation $d : R \longrightarrow M$, there exists a unique R-module homomorphism $f : \Omega_{R/k} \longrightarrow M$ such that $d = f \circ d_{R/k}$.

$\Omega_{R/k}$ can be constructed in the following way: it is generated as a free R-module by formal generators $d_{R/k}(r)$ for all $r \in R$, modulo the relations

$$d_{R/k}(a) = 0, \;\; d_{R/k}(r+s) = d_{R/k}(r) + d_{R/k}(s),$$

$$d_{R/k}(rs) = rd_{R/k}(s) + sd_{R/k}(r), \forall a \in k, r, s \in R.$$

In the following, we will explain in detail how to get that derivation algebras can also be viewed as twisted forms. Let R be a k-algebra, $f : R \longrightarrow T$ be an étale ring extension. Then the following map

$$d\Phi_T : \Omega_{R/k} \otimes_R T \longrightarrow \Omega_{T/k}; \; d_{R/k}(r) \otimes t \mapsto td_{T/k}(f(r)) \tag{4.2.2}$$

is a T-module isomorphism (see for example [EGAIV] Corollary 20.5.8), and its inverse $d\Phi_T^{-1} : \Omega_{T/k} \longrightarrow \Omega_{R/k} \otimes_R T$ induces a T-module isomorphism

$$\epsilon_T^{-1} : \Omega_{T/k} \otimes_T L \otimes_R T \longrightarrow \Omega_{R/k} \otimes_R L \otimes_R T. \tag{4.2.3}$$

Specifically, if $d\Phi_T^{-1}(d_{T/k}(t)) = \sum_j d_{R/k}(r_j) \otimes t_j$, then $\epsilon_T^{-1}(d_{T/k}(t) \otimes (x \otimes u)) = \sum_j d_{R/k}(r_j) \otimes x \otimes t_j u$ for $x \in L$, $t, u \in T$.

When L is a perfect Lie algebra, the map $\sigma : \Omega_{R/k} \otimes_R L \longrightarrow \Omega_{R,L/k}$ defined by $\sigma(d_{R/k}(r) \otimes x) = d_{R,L,k}(rx) - rd_{R,L,k}(x)$ for all $r \in R$, $x \in L$, is an R-module homomorphism. And then for any T/R a ring extension, σ induces a T-module homomorphism which we denoted by σ_T,

$$\sigma_T = \sigma \otimes_R id_T : \Omega_{R/k} \otimes_R L \otimes_R T \longrightarrow \Omega_{R,L/k} \otimes_R T. \tag{4.2.4}$$

Combining them together we get a T-module homomorphism

$$\sigma_T \circ \epsilon_T^{-1} : \Omega_{T/k} \otimes_T L \otimes_R T \longrightarrow \Omega_{R,L/k} \otimes_R T.$$

For any ring extension T/R, denote by $L_T = L \otimes_R T$. The Lie bracket on L induces a Lie bracket on L_T by

$$[x \otimes r, y \otimes t] = [x, y] \otimes st, \quad x, y \in L, \ s, t \in T,$$

which makes L_T a T-Lie algebra as well as an R-Lie algebra, and then

$$\varphi_T : L \longrightarrow L_T; \quad x \mapsto x \otimes 1_T$$

is a homomorphism of R-Lie algebras. The composition

$$d_{T,L_T,k} \circ \varphi_T : L \longrightarrow \Omega_{T,L_T/k}; \quad x \mapsto d_{T,L_T,k}(x \otimes 1_T)$$

is a k-derivation of L with values in $\Omega_{T,L_T/k}$, where $\Omega_{T,L_T/k}$ as an L-module via φ_T. Then by the universal property of $\Omega_{R,L/k}$ we get an induced $R - L$-linear map

$$\alpha : \Omega_{R,L/k} \longrightarrow \Omega_{T,L_T/k}$$

such that $\alpha \circ d_{R,L,k} = d_{T,L_T,k} \circ \varphi_T$. Finally, the following composition

$$\Omega_{R,L/k} \otimes_R T \xrightarrow{\alpha \otimes_R id_T} \Omega_{T,L_T/k} \otimes_R T \longrightarrow \Omega_{T,L_T/k}$$

is a $T - L_T$-linear map, which we denote by $d\varphi_T$, i.e.

$$d\varphi_T : \Omega_{R,L/k} \otimes_R T \longrightarrow \Omega_{T,L_T/k}; \quad d_{R,L,k}(x) \otimes t \mapsto t d_{T,L_T,k}(x \otimes 1_T) \tag{4.2.5}$$

for $x \in L, t \in T$.

We have (c.f. Lemma 5.3 of [KP])

Lemma 4.3. Let L be a perfect Lie algebra over R and $f : R \longrightarrow T$ an étale ring extension. Then $d\varphi_T$ is an isomorphism with inverse

$$(d\varphi_T)^{-1} : \Omega_{T,L_T/k} \longrightarrow \Omega_{R,L/k} \otimes_R T$$

a $T - L_T$-linear map such that

$$(d\varphi_T)^{-1}(d_{T,L_T,k}(x \otimes t)) = d_{R,L,k}(x) \otimes t + \sigma_T \circ \epsilon_T^{-1}(d_{T/k}(t) \otimes (x \otimes 1_T))$$

for all $x \in L$, $t \in T$. (Note $d_{T/k}(1_T) = 0$ by the definition of k-derivation.)

The isomorphism $(d\varphi_T)^{-1}$ induces a T-module isomorphism

$$\varphi_T^* : \mathrm{Hom}_{T-L_T}(\Omega_{R,L/k} \otimes_R T, L_T) \longrightarrow \mathrm{Hom}_{T-L_T}(\Omega_{T,L_T/k}, L_T) \qquad (4.2.6)$$

$$\alpha \mapsto \alpha \circ (d\varphi_T)^{-1}.$$

Recall that we have the canonical T-module isomorphism

$$\psi_{T,L_T} : \mathrm{Hom}_{T-L_T}(\Omega_{T,L_T/k}, L_T) \longrightarrow \mathrm{Der}_k(L_T);\ \alpha \mapsto \alpha \circ d_{T,L_T,k}. \qquad (4.2.7)$$

Note that if L is an R-form of a perfect, finite dimensional Lie algebra $\mathfrak{g}$ over k, then L is a perfect Lie algebra and is finitely generated as an R-module (c.f. Lemma 3.4 of [P5], Lemma 4.6 of [GP2]). The following result is Lemma 6.5 of [KP].

Lemma 4.4. *Let R be a k-algebra of finite type. Let L be an R-form of a perfect, finite dimensional Lie algebra $\mathfrak{g}$ over k, let T/R be a flat ring extension, then the canonical map*

$$\rho_{T,L_T} : \mathit{Hom}_{R-L}(\Omega_{R,L/k}, L) \otimes_R T \longrightarrow \mathit{Hom}_{T-L_T}(\Omega_{R,L/k} \otimes_R T, L_T) \qquad (4.2.8)$$

$$\alpha \otimes t \mapsto \alpha \otimes t(id_T)$$

is a T-module isomorphism.

Therefore, the following result is obtained.

Proposition 4.5. *Let k be a field of characteristic 0, let R be a k-algebra of finite type and L an R-form of a perfect, finite dimensional Lie algebra $\mathfrak{g}$ over k. Let T/R be an étale ring extension (so is flat by definition). Then we have a canonical T-module isomorphism*

$$\beta_T : \mathit{Der}_k(L) \otimes_R T \longrightarrow \mathit{Der}_k(L_T) \qquad (4.2.9)$$

which is obtained via $\beta_T = \psi_{T,L_T} \circ \varphi_T^* \circ \rho_{T,L_T} \circ ((\psi_{R,L})^{-1} \otimes id_T)$.

The following definition is given in Section 2.3 of [NP].

Definition 4.6. Let S/R be a ring extension. S/R is called an *étale cover* if S/R is étale and the induced map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is surjective.

Recall that S/R is faithfully flat is equivalent to S/R is flat and Spec$(S) \longrightarrow$ Spec(R) is surjective (c.f. § 13.2 of [W]). And S/R étale implies that S/R is flat (c.f. Chapter I, § 3 of [Mil]). Therefore, an étale cover is the same as a faithfully flat and étale ring extension.

Now we give our main result of this chapter.

Proposition 4.7. *Let k be a field of characteristic 0, let R be a k-algebra of finite type, let $\mathfrak{g}$ be a perfect, finite dimensional Lie algebra over k. If R-Lie algebra L is an R-form of $\mathfrak{g}$ under étale cover, then Der$_k(L)$ is an R-form of Der$_k(\mathfrak{g} \otimes_k R)$ as modules.*

Proof. Let S/R be an étale cover such that $L \otimes_R S \cong \mathfrak{g} \otimes_k R \otimes_R S$ as S-Lie algebras. Then

$$\mathrm{Der}_k(L \otimes_R S) \cong \mathrm{Der}_k(\mathfrak{g} \otimes_k R \otimes_R S)$$

as S-modules (obtained by conjugation). On the other hand, by Proposition 4.5, there exist canonical S-module isomorphisms

$$\mathrm{Der}_k(L) \otimes_R S \cong \mathrm{Der}_k(L \otimes_R S)$$

and

$$\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S \cong \mathrm{Der}_k(\mathfrak{g} \otimes_k R \otimes_R S).$$

Hence we get an S-module isomorphism

$$\mathrm{Der}_k(L) \otimes_R S \cong \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S, \tag{4.2.10}$$

which by definition means that Der$_k(L)$ is an R-form of Der$_k(\mathfrak{g} \otimes_k R)$. □

It is easy to see that if two Lie algebras L_1, L_2 are isomorphic as R-Lie algebras (so also as k-Lie algebras), then Der$_k(L_1)$ and Der$_k(L_2)$ are isomorphic as R-modules. So we also have the following result.

Theorem 4.8. *Under the hypothesis of Proposition 4.7, there exists a well-defined map from the set of isomorphism classes of R-forms of $\mathfrak{g}$ to the set of isomorphism classes of R-forms of Der$_k(\mathfrak{g} \otimes_k R)$ under étale covers, where the isomorphism class $[L]$ goes to the isomorphism class $[$Der$_k(L)]$.*

Chapter 5

Descent construction of certain derivation algebras and EALAs

In this chapter, we will study derivation algebras of certain Lie algebras and extended affine Lie algebras from the descent point of view. We first consider k is a field of characteristic 0, R is a k-algebra of finite type, $\mathfrak{g}$ is a perfect, finite dimensional Lie algebra over k, and we study the descent theory of the derivation algebras of any R-forms of $\mathfrak{g}$ in both étale cover and Galois cases. Then considering k an algebraically closed field of characteristic 0, $\mathfrak{g}$ is a finite dimensional simple Lie algebra over k, E is an arbitrary fgc EALA, we will construct explicitly the descent data that leads to E from some untwisted EALAs. This gives a descent construction of fgc extended affine Lie algebras.

5.1 Étale cover descent

If L is an R-form of $\mathfrak{g}$ under some étale cover, then by Proposition 4.7, $\mathrm{Der}_k(L)$ is an R-form of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$. By Theorem 4.2, L is determined up to isomorphism by some descent data, say a 1-cocycle φ, i.e. $L \cong (\mathfrak{g} \otimes_k S)_\varphi$ as R-Lie algebras. Also by Theorem 4.2, there exists some descent data that leads to $\mathrm{Der}_k(L)$. We will construct the descent data explicitly below. On the other hand, the isomorphism in Proposition 4.5 exists for any étale ring extensions. We observed that if we restrict our ring extensions to be étale, then there exists a morphism between the automorphism group functor of L and the automorphism group functor of the derivation algebra of L, which then induces a map between the corresponding cohomology sets. We will show that the map obtained is exactly the one implied by Theorem 4.8 and Theorem 4.2 under an étale cover.

We first study the existence of a morphism between two automorphism group functors. For this part, L is an R-form of a perfect, finite dimensional Lie algebra

$\mathfrak{g}$. Let R_{et}-**alg** be the category of all étale ring extensions over R. Consider the following two group functors defined in the same way as in Section 4.1,

$$\mathbf{Aut}(L) : R_{et} - \mathbf{alg} \longrightarrow \mathbf{Gp},$$

$$\mathbf{Aut}(\mathrm{Der}_k(L)) : R_{et} - \mathbf{alg} \longrightarrow \mathbf{Gp}.$$

Let $f_1 : R \longrightarrow T_1$ and $f_2 : R \longrightarrow T_2$ be two étale ring extensions. Let $f : T_1 \longrightarrow T_2$ be a ring homomorphism such that $f_2 = f \circ f_1$. Then f is also étale (c.f. Corollary 3.6 of [Mil]). Hence we have three isomorphisms as explained in Section 4.2:

$$d\Phi_{T_1} : \Omega_{R/k} \otimes_R T_1 \longrightarrow \Omega_{T_1/k};\ d_{R/k}(r) \otimes t_1 \mapsto t_1 d_{T_1/k}(f_1(r)),$$

$$d\Phi_{T_2} : \Omega_{R/k} \otimes_R T_2 \longrightarrow \Omega_{T_2/k};\ d_{R/k}(r) \otimes t_2 \mapsto t_2 d_{T_2/k}(f_2(r)),$$

$$\tilde{f} : \Omega_{T_1/k} \otimes_{T_1} T_2 \longrightarrow \Omega_{T_2/k};\ d_{T_1/k}(t_1) \otimes t_2 \mapsto t_2 d_{T_2/k}(f(t_1)).$$

Then we have the following Lemma.

Lemma 5.1.

$$(d\Phi_{T_2})^{-1} \circ d_{T_2/k} \circ f = (id_{\Omega_{R/k}} \otimes f) \circ (d\Phi_{T_1})^{-1} \circ d_{T_1/k}.$$

Proof. It suffices to show that

$$d_{T_2/k} \circ f = (d\Phi_{T_2}) \circ (id_{\Omega_{R/k}} \otimes f) \circ (d\Phi_{T_1})^{-1} \circ d_{T_1/k}.$$

For any $t \in T_1$, say $(d\Phi_{T_1})^{-1}(d_{T_1/k}(t)) = \sum_i d_{R/k}(r_i) \otimes t_i$, then

$$d_{T_1/k}(t) = d\Phi_{T_1}(\sum_i d_{R/k}(r_i) \otimes t_i) = \sum_i t_i d_{T_1/k}(f_1(r_i)),$$

and

$$(d\Phi_{T_2}) \circ (id_{\Omega_{R/k}} \otimes f) \circ (d\Phi_{T_1})^{-1} \circ d_{T_1/k}(t) = \sum_i f(t_i) d_{T_2/k}(f_2(r_i)).$$

On the other hand,

$$\begin{aligned}
&d_{T_2/k} \circ f(t) = d_{T_2/k}(f(t)) = \tilde{f}(d_{T_1/k}(t) \otimes 1_{T_2}) \\
&= \tilde{f}(\sum_i t_i d_{T_1/k}(f_1(r_i)) \otimes 1_{T_2}) \\
&= \tilde{f}(\sum_i d_{T_1/k}(f_1(r_i)) \otimes f(t_i)) \\
&= \sum_i f(t_i) d_{T_2/k}(f(f_1(r_i))) = \sum_i f(t_i) d_{T_2/k}(f_2(r_i)),
\end{aligned}$$

so they are equal. □

Recall that for F and G two group functors between two categories C and D, a *natural transformation* or *morphism* η from F to G is a family of morphisms that satisfies the following two requirements.

(a) The natural transformation must associate to every object X in C a group homomorphism

$$\eta_X : F(X) \longrightarrow G(X)$$

between objects of D. The morphism η_X is called the *component of* η *at* X.

(b) Components must be such that for every morphism $f : X \longrightarrow Y$ in C we have:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X,$$

where $F(f) : F(X) \longrightarrow F(Y)$, $G(f) : G(X) \longrightarrow G(Y)$ are group homomorphisms in D.

We will show that there is a morphism (natural transformation) between the above two group functors $\mathbf{Aut}(L)$, $\mathbf{Aut}(\mathrm{Der}_k(L))$. For any T an object in the category R_{et}-**alg** and for any $\phi \in \mathbf{Aut}(L)(T)$, there is a natural T-module isomorphism

$$C_\phi : \mathrm{Der}_k(L \otimes_R T) \longrightarrow \mathrm{Der}_k(L \otimes_R T);\ d \mapsto \phi \circ d \circ \phi^{-1},$$

named the conjugation map. Define

$$\tilde{\phi} = \beta_T^{-1} \circ C_\phi \circ \beta_T,$$

where β_T is the one obtained by Proposition 4.5. It is easy to see that such defined

$\tilde{\phi}$ is indeed an element in $\mathbf{Aut}(\mathrm{Der}_k(L))(T) = \mathrm{Aut}_{T-mod}(\mathrm{Der}_k(L) \otimes_R T)$. So we get a map between the two group functors defined on every object in the category R_{et}-**alg**. To show that it is well-defined on every morphism in R_{et}-**alg**, we will need the following equation. The way we get it is by using their definitions to check their actions on elements, it is a rather complicated calculation, and we will need to use Lemma 5.1.

Lemma 5.2. *For any $f : T_1 \longrightarrow T_2$ a morphism in the category R_{et}-**alg** and for any $\phi \in \mathbf{Aut}(L)(T_1)$, the following equation holds:*

$$\mathbf{Aut}(Der_k(L))(f)(\tilde{\phi}) = \beta_{T_2}^{-1} \circ C_{\mathbf{Aut}(L)(f)(\phi)} \circ \beta_{T_2},$$

where $\tilde{\phi} = \beta_{T_1}^{-1} \circ C_\phi \circ \beta_{T_1}$, and

$$C_{\mathbf{Aut}(L)(f)(\phi)} : Der_k(L_{T_2}) \longrightarrow Der_k(L_{T_2}),$$

is a T_2-module isomorphism defined as $d \mapsto \mathbf{Aut}(L)(f)(\phi) \circ d \circ \mathbf{Aut}(L)(f)(\phi^{-1})$, for any $d \in Der_k(L_{T_2})$.

Proof. Since both sides are T_2-module homomorphisms, we only need to consider their action on the element of the form $d \otimes_R 1_{T_2}$, for $d \in \mathrm{Der}_k(L)$, $1_{T_2} \in T_2$ the identity element:

$$\begin{aligned}
&\mathbf{Aut}(\mathrm{Der}_k(L))(f)(\tilde{\phi})(d \otimes 1_{T_2}) = (id \otimes f)(\tilde{\phi}(d \otimes 1_{T_1})) \\
&= (id \otimes f)(\beta_{T_1}^{-1} \circ C_\phi \circ \beta_{T_1}(d \otimes 1_{T_1})) \\
&= (id \otimes f)(\beta_{T_1}^{-1}(\phi \circ \beta_{T_1}(d \otimes 1_{T_1}) \circ \phi^{-1})) \in Der_k(L) \otimes_R T_2.
\end{aligned}$$

Denote by $(id \otimes f)(\beta_{T_1}^{-1}(\phi \circ \beta_{T_1}(d \otimes 1_{T_1}) \circ \phi^{-1})) = X$. On the other hand,

$$\begin{aligned}
&\beta_{T_2}^{-1} \circ C_{\mathbf{Aut}(L)(f)(\phi)} \circ \beta_{T_2}(d \otimes 1_{T_2}) \\
&= \beta_{T_2}^{-1}\left(\mathbf{Aut}(L)(f)(\phi) \circ \beta_{T_2}(d \otimes 1_{T_2}) \circ \mathbf{Aut}(L)(f)(\phi)^{-1}\right).
\end{aligned}$$

So it suffices to show that

$$\beta_{T_2}(X) \circ \mathbf{Aut}(L)(f)(\phi) = \mathbf{Aut}(L)(f)(\phi) \circ \beta_{T_2}(d \otimes 1_{T_2}). \tag{5.1.1}$$

Say $(\psi_{R,L})^{-1}(d) = \alpha$, then $\alpha \circ d_{R,L,k} = d$, so

$$\beta_{T_2}(d \otimes 1_{T_2}) = \varphi^*_{T_2}(\alpha \otimes id_{T_2}) \circ d_{T_2,L_{T_2},k},$$

then

$$\text{RHS of (5.1.1)} = \mathbf{Aut}(L)(f)(\phi) \circ \varphi^*_{T_2}(\alpha \otimes id_{T_2}) \circ d_{T_2,L_{T_2},k},$$

and

$$\text{LHS of (5.1.1)} = \varphi^*_{T_2}\left[\rho_{T_2,L_{T_2}}\left((\psi_{R,L})^{-1} \otimes id_{T_2})(X)\right)\right] \circ d_{T_2,L_{T_2},k} \circ \mathbf{Aut}(L)(f)(\phi).$$

Since $d_{T_2,L_{T_2},k}$ is k-linear, not T_2-linear, let's consider the action of two sides on the element of the form $x \otimes t \in L \otimes T_2$, where $x \in L$, $t \in T_2$. Note

$$\mathbf{Aut}(L)(f)(\phi)(x \otimes t) = (id_L \otimes tf)(\phi(x \otimes 1_{T_1})).$$

Say $\phi(x \otimes 1_{T_1}) = \sum\limits_{l} x_l \otimes t_l \in L \otimes_R T_1$, then

$$\mathbf{Aut}(L)(f)(\phi)(x \otimes t) = \sum_{l} x_l \otimes f(t_l)t.$$

Say

$$\beta^{-1}_{T_1}(\phi \circ \beta_{T_1}(d \otimes 1_{T_1}) \circ \phi^{-1}) = \sum_{m} d_m \otimes t_m \in Der_k(L) \otimes_R T_1, \tag{5.1.2}$$

then $X = \sum\limits_{m} d_m \otimes f(t_m)$ and so

$$((\psi_{R,L})^{-1} \otimes id_{T_2})(X) = \sum_{m} \psi^{-1}_{R,L}(d_m) \otimes f(t_m).$$

Say $\psi^{-1}_{R,L}(d_m) = \alpha_m$, then $\alpha_m \circ d_{R,L,k} = d_m, \forall\, m$ and so

$$((\psi_{R,L})^{-1} \otimes id_{T_2})(X) = \sum_{m} \alpha_m \otimes f(t_m).$$

By applying both sides of (5.1.1) to $x \otimes_R t$, we get

$$\begin{aligned}
&\text{LHS of } (5.1.1)(x \otimes t) \\
&= \varphi_{T_2}^*(\sum_m \alpha_m \otimes f(t_m) id_{T_2})(\sum_l d_{T_2,L_{T_2},k}(x_l \otimes f(t_l)t)) \\
&= (\sum_m \alpha_m \otimes f(t_m) id_{T_2}) \circ (d\varphi_{T_2})^{-1}(\sum_l d_{T_2,L_{T_2},k}(x_l \otimes f(t_l)t)),
\end{aligned}$$

and

$$\begin{aligned}
&\text{RHS of } (5.1.1)(x \otimes t) \\
&= \mathbf{Aut}(L)(f)(\phi) \circ \varphi_{T_2}^*(\alpha \otimes id_{T_2}) \circ d_{T_2,L_{T_2},k}(x \otimes t) \\
&= \mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2}) \circ (d\varphi_{T_2})^{-1} \circ d_{T_2,L_{T_2},k}(x \otimes t).
\end{aligned}$$

By definition of $(d\varphi_{T_2})^{-1}$ (Lemma 4.3) we have

$$\begin{aligned}
&(d\varphi_{T_2})^{-1} \circ d_{T_2,L_{T_2},k}(x \otimes t) \\
&= d_{R,L,k}(x) \otimes t + \sigma_{T_2} \circ \epsilon_{T_2}^{-1}\left(d_{T_2/k}(t) \otimes_{T_2} (x \otimes 1_{T_2})\right),
\end{aligned}$$

we denote the two sums above by A and B respectively, i.e.

$$(d\varphi_{T_2})^{-1} \circ d_{T_2,L_{T_2},k}(x \otimes t) = A + B.$$

Similarly,

$$\begin{aligned}
&(d\varphi_{T_2})^{-1}(\sum_l d_{T_2,L_{T_2},k}(x_l \otimes f(t_l)t)) \\
&= \sum_l d_{R,L,k}(x_l) \otimes f(t_l)t \\
&\quad + \sum_l \sigma_{T_2} \circ \epsilon_{T_2}^{-1}(d_{T_2/k}(f(t_l)t) \otimes (x_l \otimes 1_{T_2})),
\end{aligned}$$

and we denote the two sums above by C and D respectively, i.e.

$$(d\varphi_{T_2})^{-1}(\sum_l d_{T_2,L_{T_2},k}(x_l \otimes f(t_l)t)) = C + D.$$

Now

$$\text{LHS of (5.1.1)}(x \otimes t) = \sum_m (\alpha_m \otimes f(t_m) id_{T_2})(C + D),$$

$$\text{RHS of (5.1.1)}(x \otimes t) = \mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2})(A + B).$$

Consider A and C terms first,

$$\begin{aligned}
&\mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2})(A) \\
&= \mathbf{Aut}(L)(f)(\phi)\left((\alpha \circ d_{R,L,k})(x) \otimes t\right) \\
&= \mathbf{Aut}(L)(f)(\phi)(d(x) \otimes t) \\
&= (id_L \otimes tf) \circ \phi(d(x) \otimes 1_{T_1}),
\end{aligned}$$

$$\begin{aligned}
&(\sum_m \alpha_m \otimes f(t_m) id_{T_2})(C) \\
&= \sum_{m,l} (\alpha_m \circ d_{R,L,k})(x_l) \otimes f(t_m) f(t_l) t \\
&= \sum_{m,l} d_m(x_l) \otimes f(t_m t_l) t.
\end{aligned}$$

By (5.1.2) we have

$$\phi \circ \beta_{T_1}(d \otimes 1_{T_1}) = \beta_{T_1}(\sum_m d_m \otimes t_m) \circ \phi,$$

and by the definition of β_{T_1} we have

$$\begin{aligned}
&\beta_{T_1}(d \otimes 1_{T_1}) = \psi_{T_1, L_{T_1}}\left[(\alpha \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1}\right] \\
&= (\alpha \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1} \circ d_{T_1, L_{T_1}, k},
\end{aligned}$$

so

$$\phi \circ (\alpha \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1} \circ d_{T_1, L_{T_1}, k} = \beta_{T_1}(\sum_m d_m \otimes t_m) \circ \phi. \tag{5.1.3}$$

Again by the definition of β_{T_1} we have

$$\begin{aligned}\beta_{T_1}(d_m \otimes 1_{T_1}) &= \psi_{T_1,L_{T_1}}\left[(\alpha_m \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1}\right] \\ &= (\alpha_m \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1} \circ d_{T_1,L_{T_1},k}.\end{aligned}$$

Applying both sides of (5.1.3) to the element $x \otimes 1_{T_1}$, recall $\phi(x \otimes 1_{T_1}) = \sum_l x_l \otimes t_l$, β_{T_1} is T_1-module homomorphism, and by the definition of $(d\varphi_{T_1})^{-1}$ we get

$$\begin{aligned}&\text{RHS of (5.1.3)}(x \otimes 1_{T_1}) = \sum_m (\alpha_m \otimes t_m id_{T_1}) \circ (d\varphi_{T_1})^{-1} \circ d_{T_1,L_{T_1},k}(\sum_l x_l \otimes t_l) \\ &= \sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[d_{R,L,k}(x_l) \otimes t_l + \sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1}))\right] \\ &= \sum_{m,l} d_m(x_l) \otimes t_m t_l \\ &\quad + \sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1}))\right].\end{aligned}$$

Note $d_{T_1/k}(1_{T_1}) = 0$, we have

$$\begin{aligned}&\text{LHS of (5.1.3)}(x \otimes 1_{T_1}) \\ &= \phi \circ (\alpha \otimes id_{T_1}) \circ (d\varphi_{T_1})^{-1} \circ d_{T_1,L_{T_1},k}(x \otimes 1_{T_1}) \\ &= \phi \circ (\alpha \otimes id_{T_1})(d_{R,L,k}(x) \otimes 1_{T_1}) \\ &= \phi(d(x) \otimes 1_{T_1}).\end{aligned}$$

So from equation (5.1.3) we get that

$$\begin{aligned}&\phi(d(x) \otimes 1_{T_1}) \\ &= \sum_{m,l} d_m(x_l) \otimes t_m t_l \\ &\quad + \sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1}))\right], \end{aligned} \tag{5.1.4}$$

where $\phi(x \otimes 1_{T_1}) = \sum_l x_l \otimes t_l$. Hence

$$
\begin{aligned}
&\mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2})(A) \\
&= (id_L \otimes tf) \left[\sum_{m,l} d_m(x_l) \otimes t_m t_l \right] \\
&\quad + (id_L \otimes tf) \left[\sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1} (d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1})) \right] \right] \\
&= (\sum_m \alpha_m \otimes f(t_m) id_{T_2})(C) \\
&\quad + (id_L \otimes tf) \left[\sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1} (d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1})) \right] \right].
\end{aligned}
$$

Denote by

$$
Y = (id_L \otimes tf) \left[\sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1} (d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1})) \right] \right].
$$

Therefore, it now suffices to show

$$
(\sum_m \alpha_m \otimes f(t_m) id_{T_2})(D) = \mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2})(B) + Y. \tag{5.1.5}
$$

For D,

$$
\begin{aligned}
&\sum_l \sigma_{T_2} \circ \epsilon_{T_2}^{-1} \left[d_{T_2/k}(f(t_l)t) \otimes (x_l \otimes 1_{T_2}) \right] \\
&= \sum_l \sigma_{T_2} \circ \epsilon_{T_2}^{-1} \left[\left[f(t_l) d_{T_2/k}(t) + t d_{T_2/k}(f(t_l)) \right] \otimes (x_l \otimes 1_{T_2}) \right] \\
&= \sum_l \sigma_{T_2} \circ \epsilon_{T_2}^{-1} \left[d_{T_2/k}(t) \otimes (x_l \otimes f(t_l)) \right] \\
&\quad + \sum_l \sigma_{T_2} \circ \epsilon_{T_2}^{-1} \left[d_{T_2/k}(f(t_l)) \otimes (x_l \otimes t) \right],
\end{aligned}
$$

we denote by the two sums D_1 and D_2. Say $d\Phi_{T_2}^{-1}(d_{T_2/k}(t)) = \sum_q d_{R/k}(r_q) \otimes s_q$, then

$$\begin{aligned} D_1 &= \sum_l \sigma_{T_2}(\sum_q d_{R/k}(r_q) \otimes x_l \otimes s_q f(t_l)) \\ &= \sum_{l,q} (d_{R,L,k}(r_q x_l) - r_q d_{R,L,k}(x_l)) \otimes s_q f(t_l). \end{aligned}$$

Now for (5.1.5),

$$\begin{aligned} LHS &= \sum_{l,q,m} (d_m(r_q x_l) - r_q d_m(x_l)) \otimes s_q f(t_m t_l) \\ &+ (\sum_m \alpha_m \otimes f(t_m) id_{T_2})(D_2). \end{aligned} \tag{5.1.6}$$

For B,

$$\begin{aligned} &\sigma_{T_2} \circ \epsilon_{T_2}^{-1}(d_{T_2/k}(t) \otimes (x \otimes 1_{T_2})) \\ &= \sigma_{T_2}(\sum_q d_{R/k}(r_q) \otimes x \otimes s_q) \\ &= \sum_q (d_{R,L,k}(r_q x) - r_q d_{R,L,k}(x)) \otimes s_q, \end{aligned}$$

so

$$\mathbf{Aut}(L)(f)(\phi) \circ (\alpha \otimes id_{T_2})(B) = \sum_q (id_L \otimes s_q f) \left[\phi \left[(d(r_q x) - r_q d(x)) \otimes 1_{T_1} \right] \right].$$

Consider the term $\phi(d(r_q x) \otimes 1_{T_1}) - \phi(r_q d(x) \otimes 1_{T_1})$. Note that $\phi(x \otimes 1_{T_1}) = \sum_l x_l \otimes t_l$, and ϕ is a T_1-module homomorphism. We have

$$\phi(r_q x \otimes 1_{T_1}) = \sum_l r_q x_l \otimes t_l,$$

and then similar to how we obtained $\phi(d(x) \otimes 1_{T_1})$ in (5.1.4), we get that

$$\begin{aligned}
&\phi(d(r_q x) \otimes 1_{T_1}) \\
&= \sum_{m,l} d_m(r_q x_l) \otimes t_m t_l \\
&\quad + \sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(t_l) \otimes (r_q x_l \otimes 1_{T_1}))\right].
\end{aligned}$$

Note that since one can write $\phi(r_q x \otimes 1_{T_1}) = \sum_l x_l \otimes f_1(r_q) t_l$, then using the same considerations as above, we can write

$$\begin{aligned}
&\phi(d(r_q x) \otimes 1_{T_1}) \\
&= \sum_{m,l} d_m(x_l) \otimes f_1(r_q) t_m t_l \\
&\quad + \sum_{m,l} (\alpha_m \otimes t_m id_{T_1}) \left[\sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(f_1(r_q) t_l) \otimes (x_l \otimes 1_{T_1}))\right].
\end{aligned}$$

But note that by Lemma 5.1 we have

$$(d\Phi_{T_1})^{-1} \circ d_{T_1/k}(f_1(r_q)) = d_{R/k}(r_q) \otimes 1_{T_1},$$

so the two expressions are actually the same. Then from the above we get

$$\begin{aligned}
&\phi(d(r_q x) \otimes 1_{T_1}) - \phi(r_q d(x) \otimes 1_{T_1}) \\
&= \sum_{m,l} d_m(r_q x_l) \otimes t_m t_l - \sum_{m,l} r_q d_m(x_l) \otimes t_m t_l \\
&= \sum_{m,l} (d_m(r_q x_l) - r_q d_m(x_l)) \otimes t_m t_l. \qquad (5.1.7)
\end{aligned}$$

From (5.1.5), it now suffices to show that

$$(\sum_m \alpha_m \otimes f(t_m) id_{T_2})(D_2) = Y. \qquad (5.1.8)$$

Say $(d\Phi_{T_1})^{-1}(d_{T_1/k}(t_l)) = \sum_p d_{R/k}(r_{pl}) \otimes s_{pl}$, then

$$\begin{aligned}&\sigma_{T_1} \circ \epsilon_{T_1}^{-1}(d_{T_1/k}(t_l) \otimes (x_l \otimes 1_{T_1}))\\ &= \sum_p (d_{R,L,k}(r_{pl}x_l) - r_{pl}d_{R,L,k}(x_l)) \otimes s_{pl}.\end{aligned}$$

Hence the right hand side of (5.1.8) becomes

$$\text{RHS of (5.1.8)} = \sum_{m,p,l} (d_m(r_{pl}x_l) - r_{pl}d_m(x_l)) \otimes tf(t_m)f(s_{pl}). \qquad (5.1.9)$$

By Lemma 5.1 and

$$\begin{aligned}&(id_{\Omega_{R/k}} \otimes f) \circ (d\Phi_{T_1})^{-1} \circ d_{T_1/k}(t_l)\\ &= (id_{\Omega_{R/k}} \otimes f)(\sum_p d_{R/k}(r_{pl}) \otimes s_{pl}) = \sum_p d_{R/k}(r_{pl}) \otimes f(s_{pl}),\end{aligned}$$

we get that

$$(d\Phi_{T_2})^{-1}(d_{T_2/k}(f(t_l))) = \sum_p d_{R/k}(r_{pl}) \otimes f(s_{pl}).$$

Then by definition of σ_{T_2} and $\epsilon_{T_2}^{-1}$ we have

$$\sigma_{T_2} \circ \epsilon_{T_2}^{-1}(d_{T_2/k}(f(t_l)) \otimes (x_l \otimes t)) = \sum_p (d_{R,L,k}(r_{pl}x_l) - r_{pl}d_{R,L,k}(x_l)) \otimes tf(s_{pl}),$$

so the left hand side of (5.1.8) becomes

$$\begin{aligned}&\text{LHS of (5.1.8)}\\ &= \sum_m (\alpha_m \otimes f(t_m)id_{T_2})(\sum_{l,p}(d_{R,L,k}(r_{pl}x_l) - r_{pl}d_{R,L,k}(x_l)) \otimes tf(s_{pl})\\ &= \sum_{m,p,l}(d_m(r_{pl}x_l) - r_{pl}d_m(x_l)) \otimes tf(t_m)f(s_{pl}) = \text{RHS of (5.1.8)}, \qquad (5.1.10)\end{aligned}$$

which completes the proof. □

Using Lemma 5.2, it is straightforward to show the following.

Proposition 5.3. *Let* $\mathbf{Aut}(L)$ *and* $\mathbf{Aut}(Der_k(L))$ *be two group functors from* R_{et}-

alg *to* **Gp**. *Then there exists a natural transformation*

$$F : \mathbf{Aut}(L) \longrightarrow \mathbf{Aut}(Der_k(L)) \tag{5.1.11}$$

*such that for every object T in R_{et}-***alg***,*

$$F(T) : \mathbf{Aut}(L)(T) \longrightarrow \mathbf{Aut}(Der_k(L))(T); \quad \phi \mapsto \tilde{\phi} = \beta_T^{-1} \circ C_\phi \circ \beta_T.$$

Proof. Clearly, for any étale ring extension T/R, the above defined $F(T)$ is a group homomorphism in the category $\mathbf{Gp}$. Now for any homomorphism $f : T_1 \longrightarrow T_2$ in R_{et}-**alg** and for any $\phi \in \mathbf{Aut}(L)(T_1)$, we have

$$\left(F(T_2) \circ \mathbf{Aut}(L)(f)\right)(\phi) = \beta_{T_2}^{-1} \circ C_{\mathbf{Aut}(L)(f)(\phi)} \circ \beta_{T_2},$$

$$\left(\mathbf{Aut}(\mathrm{Der}_k(L))(f) \circ F(T_1)\right)(\phi) = \mathbf{Aut}(\mathrm{Der}_k(L))(f)(\beta_{T_1}^{-1} \circ C_\phi \circ \beta_{T_1}).$$

They are equal by Lemma 5.2. Hence F is a natural transformation. □

From Proposition 5.3 we obtained an induced map between the two corresponding cohomology sets. One can also show this directly using again Lemma 5.2.

Proposition 5.4. *For any R-form L of $\mathfrak{g}$, for any étale ring extension S/R. There exists a map of pointed sets*

$$\tilde{F} : H^1(S/R, \mathbf{Aut}(L)) \longrightarrow H^1(S/R, \mathbf{Aut}(Der_k(L))); \quad [\varphi] \mapsto [\tilde{\varphi}], \tag{5.1.12}$$

where $\tilde{\varphi} = \beta_{S''}^{-1} \circ C_\varphi \circ \beta_{S''}$, and

$$C_\varphi : Der_k(L_{S''}) \longrightarrow Der_k(L_{S''}),$$

is an S''-module isomorphism defined as $d \mapsto \varphi \circ d \circ \varphi^{-1}$, for any $d \in Der_k(L_{S''})$.

Proof. It suffices to show that $\tilde{\varphi} = \beta_{S''}^{-1} \circ C_\varphi \circ \beta_{S''}$ is a 1-cocycle. Recall

$$\tilde{\varphi}_{ij} = \mathbf{Aut}(\mathrm{Der}_k(L))(P_{ij})(\tilde{\varphi}), \quad 1 \le i < j \le 3.$$

Then by Lemma 5.2,

$$\tilde{\varphi}_{ij} = \beta_{S'''}^{-1} \circ C_{\mathbf{Aut}(L)(P_{ij})(\varphi)} \circ \beta_{S'''}, \quad 1 \leq i < j \leq 3,$$

where

$$C_{\mathbf{Aut}(L)(P_{ij})(\varphi)} : \mathrm{Der}_k(L_{S'''}) \longrightarrow \mathrm{Der}_k(L_{S'''}),$$

is an $S^{'''}$-module isomorphism defined as $d \mapsto \mathbf{Aut}(L)(P_{ij})(\varphi) \circ d \circ \mathbf{Aut}(L)(P_{ij})(\varphi^{-1})$, for any $d \in \mathrm{Der}_k(L_{S'''})$. So

$$\begin{aligned} & \tilde{\varphi}_{23} \circ \tilde{\varphi}_{12} \\ & = \beta_{S'''}^{-1} \circ C_{\mathbf{Aut}(L)(P_{23})(\varphi)} \circ C_{\mathbf{Aut}(L)(P_{12})(\varphi)} \circ \beta_{S'''} \\ & = \beta_{S'''}^{-1} \circ C_{\mathbf{Aut}(L)(P_{23})(\varphi) \circ \mathbf{Aut}(L)(P_{12})(\varphi)} \circ \beta_{S'''} \\ & = \beta_{S'''}^{-1} \circ C_{\mathbf{Aut}(L)(P_{13})(\varphi)} \circ \beta_{S'''} \\ & = \tilde{\varphi}_{13}, \end{aligned} \tag{5.1.13}$$

by definition, $\tilde{\varphi}$ is a 1-cocycle.

□

Now if L is an R-form of $\mathfrak{g}$ for some étale cover, say S/R, then by Theorem 4.2, L is given by some 1-cocycle $\varphi \in \mathbf{Aut}(\mathfrak{g}_R)(S^{''})$ (up to isomorphism), i.e. $L = (\mathfrak{g} \otimes_k S)_\varphi$ as R-Lie algebras, where $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$. By Proposition 5.4, we know that $\widetilde{\varphi} = \beta_{S''}^{-1} \circ C_\varphi \circ \beta_{S''} \in \mathbf{Aut}(\mathrm{Der}_k(\mathfrak{g}_R))(S^{''})$ is also a 1-cocycle, where $C_\varphi : \mathrm{Der}_k(\mathfrak{g}_{S''}) \longrightarrow \mathrm{Der}_k(\mathfrak{g}_{S''})$, $d \mapsto \varphi \circ d \circ \varphi^{-1}$, is an $S^{''}$-module isomorphism. S/R is faithfully flat, so $\widetilde{\varphi}$ determines a descended object defined by

$$\begin{aligned} & (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \\ & = \{\sum_n d_n \otimes s_n \in \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S \mid \\ & \widetilde{\varphi}(\sum_n d_n \otimes s_n \otimes 1_S) = \sum_n d_n \otimes 1_S \otimes s_n\}. \end{aligned}$$

The following result also contains some complicated calculation.

Proposition 5.5. *Let S/R be an étale cover and let L be an S/R-form of $\mathfrak{g}$ determined up to isomorphism by an 1-cocycle φ. Then there exists an R-module* *isomorphism*

$$(\boldsymbol{Der}_k(\mathfrak{g}\otimes_k R)\otimes_R S)_{\widetilde{\varphi}} \cong \boldsymbol{Der}_k((\mathfrak{g}\otimes_k S)_{\varphi}) = \boldsymbol{Der}_k(L)$$

which is induced from the isomorphism β_S, where $\tilde{\varphi} = \beta_{S''}^{-1}\circ C_{\varphi}\circ \beta_{S''}$.

Proof. For

$$\beta_S : \mathrm{Der}_k(\mathfrak{g}\otimes_k R)\otimes_R S \longrightarrow \mathrm{Der}_k((\mathfrak{g}\otimes_k R)\otimes_R S)$$

the canonical S-module isomorphism defined before (i.e. in Proposition 4.5 we take L to be the trivial form $\mathfrak{g}\otimes_k R$), we first show that β_S induces a map from $(\mathrm{Der}_k(\mathfrak{g}\otimes_k R)\otimes_R S)_{\widetilde{\varphi}}$ to $\mathrm{Der}_k((\mathfrak{g}\otimes_k S)_{\varphi})$.

For any $\sum_n d_n\otimes s_n \in (\mathrm{Der}_k(\mathfrak{g}\otimes_k R)\otimes_R S)_{\widetilde{\varphi}}$, since $\widetilde{\varphi} = \beta_{S''}^{-1}\circ C_{\varphi}\circ\beta_{S''}$ we have

$$\varphi\circ\beta_{S''}(\sum_n d_n\otimes s_n\otimes 1_S) = \beta_{S''}(\sum_n d_n\otimes 1_S\otimes s_n)\circ\varphi, \tag{5.1.14}$$

and $\beta_S(\sum_n d_n\otimes s_n)$ is a derivation from $(\mathfrak{g}\otimes_k R)\otimes_R S$ to $(\mathfrak{g}\otimes_k R)\otimes_R S$. Consider its restriction on the elements $(\mathfrak{g}\otimes_k S)_{\varphi}\subseteq \mathfrak{g}\otimes_k S$, where we identify $(\mathfrak{g}\otimes_k R)\otimes_R S$ with $\mathfrak{g}\otimes_k S := \mathfrak{g}_S$, so there is the map

$$\beta_S(\sum_n d_n\otimes s_n)|_{(\mathfrak{g}_S)_{\varphi}} : (\mathfrak{g}\otimes_k S)_{\varphi} \longrightarrow (\mathfrak{g}\otimes_k R)\otimes_R S.$$

Now we are going to show that this map actually maps to $(\mathfrak{g}\otimes_k S)_{\varphi}$. For any $\sum_u x_u\otimes s_u \in (\mathfrak{g}\otimes_k S)_{\varphi}$, we have

$$\varphi(\sum_u x_u\otimes s_u\otimes 1_S) = \sum_u x_u\otimes 1_S\otimes s_u. \tag{5.1.15}$$

We want to show that

$$\beta_S(\sum_n d_n\otimes s_n)|_{(\mathfrak{g}_S)_{\varphi}}(\sum_u x_u\otimes s_u) \in (\mathfrak{g}\otimes_k S)_{\varphi}. \tag{5.1.16}$$

Applying (5.1.14) to the element $\sum_u x_u \otimes s_u \otimes 1_S$ and using (5.1.15) we get

$$\varphi \circ \beta_{S''}(\sum_n d_n \otimes s_n \otimes 1_S)(\sum_u x_u \otimes s_u \otimes 1_S)$$
$$= \beta_{S''}(\sum_n d_n \otimes 1_S \otimes s_n)(\sum_u x_u \otimes 1_S \otimes s_u). \tag{5.1.17}$$

Say $\psi^{-1}_{R,\mathfrak{g}_R}(d_n) = \alpha_n$ for all n. Then $\alpha_n \circ d_{R,\mathfrak{g}_R,k} = d_n$ and

$$\beta_{S''}(d_n \otimes 1_{S''}) = (\alpha_n \otimes id_{S''}) \circ (d\varphi_{S''})^{-1} \circ d_{S'',\mathfrak{g}_{S''},k}.$$

$\beta_{S''}$ is an S''-module homomorphism, so

$$\beta_{S''}(\sum_n d_n \otimes s_n \otimes 1_S)(\sum_u x_u \otimes s_u \otimes 1_S)$$
$$= \sum_n (\alpha_n \otimes (s_n \otimes 1_S) id_{S''}) \circ (d\varphi_{S''})^{-1} \circ d_{S'',\mathfrak{g}_{S''},k}(\sum_u x_u \otimes s_u \otimes 1_S)$$
$$= \sum_{n,u} d_n(x_u) \otimes s_n s_u \otimes 1_S$$
$$+ \sum_{n,u} (\alpha_n \otimes (s_n \otimes 1_S) id_{S''}) \circ \sigma_{S''} \circ \epsilon^{-1}_{S''}(d_{S''/k}(s_u \otimes 1_S) \otimes (x_u \otimes 1_{S''})).$$

Similarly one can get the equation for $\beta_{S''}(\sum_n d_n \otimes 1_S \otimes s_n)(\sum_u x_u \otimes 1_S \otimes s_u)$. Denote by

$$X_1 = \sum_{n,u} (\alpha_n \otimes (s_n \otimes 1_S) id_{S''}) \circ \sigma_{S''} \circ \epsilon^{-1}_{S''}(d_{S''/k}(s_u \otimes 1_S) \otimes (x_u \otimes 1_{S''})),$$

$$X_2 = \sum_{n,u} (\alpha_n \otimes (1_S \otimes s_n) id_{S''}) \circ \sigma_{S''} \circ \epsilon^{-1}_{S''}(d_{S''/k}(1_S \otimes s_u) \otimes (x_u \otimes 1_{S''})).$$

Then

$$\beta_{S''}(\sum_n d_n \otimes s_n \otimes 1_S)(\sum_u x_u \otimes s_u \otimes 1_S)$$
$$= \sum_{n,u} d_n(x_u) \otimes s_n s_u \otimes 1_S + X_1,$$

$$\beta_{S''}(\sum_n d_n \otimes 1_S \otimes s_n)(\sum_u x_u \otimes 1_S \otimes s_u)$$
$$= \sum_{n,u} d_n(x_u) \otimes 1_S \otimes s_n s_u + X_2,$$

and by (5.1.17) we get that

$$\varphi(X_1) + \varphi(\sum_{n,u} d_n(x_u) \otimes s_n s_u \otimes 1_S)$$
$$= X_2 + \sum_{n,u} d_n(x_u) \otimes 1_S \otimes s_n s_u. \tag{5.1.18}$$

By definition, $\beta_S(d_n \otimes 1_S) = (\alpha_n \otimes id_S) \circ (d\varphi_S)^{-1} \circ d_{S,\mathfrak{g}_S,k}$ for all n. Denote by

$$X_3 = \sum_{n,u} (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s_u) \otimes (x_u \otimes 1_S)).$$

Then

$$\beta_S(\sum_n d_n \otimes s_n)|_{(\mathfrak{g}_R)_\varphi}(\sum_u x_u \otimes s_u) = X_3 + \sum_{n,u} d_n(x_u) \otimes s_n s_u. \tag{5.1.19}$$

We now investigate the expression of X_1, X_2 and X_3. Say $(d\Phi_S)^{-1}(d_{S/k}(s_u)) = \sum_v d_{R/k}(r_{uv}) \otimes s_{uv}$ for all u, then $d_{S/k}(s_u) = \sum_v s_{uv} d_{S/k}(r_{uv})$. Note that to be more precise we should write $d_{S/k}(f(r_{uv}))$ on the right hand side where $f : R \longrightarrow S$, but as one can see this makes no difference for our final answer, we will just write as $d_{S/k}(r_{uv})$. On the other hand, since S/R is faithfully flat, the map $f : R \longrightarrow S$ is actually injective, so we can view the elements of R as elements of S. Then

$$\epsilon_S^{-1}(d_{S/k}(s_u) \otimes (x_u \otimes 1_S)) = \sum_v d_{R/k}(r_{uv}) \otimes x_u \otimes s_{uv},$$

so

$$X_3 = \sum_{n,u,v} (d_n(r_{uv} x_u) - r_{uv} d_n(x_u)) \otimes s_n s_{uv}.$$

For X_2, take $T_1 = S, T_2 = S''$, $f = P_2$ in Lemma 5.1. We get

$$d\Phi_{S''}^{-1}(d_{S''/k}(1_S \otimes s_u)) = \sum_v d_{R/k}(r_{uv}) \otimes 1_S \otimes s_{uv},$$

so that

$$X_2 = \sum_{n,u,v} (d_n(r_{uv}x_u) - r_{uv}d_n(x_u)) \otimes 1_S \otimes s_n s_{uv}.$$

Then

$$X_2 = (id \otimes P_2)(X_3). \tag{5.1.20}$$

For X_1, take $T_1 = S$, $T_2 = S''$, $f = P_1$ in Lemma 5.1. We get

$$d\Phi_{S''}^{-1}(d_{S''/k}(s_u \otimes 1_S)) = \sum_v d_{R/k}(r_{uv}) \otimes s_{uv} \otimes 1_S,$$

so that

$$X_1 = \sum_{n,u,v} (d_n(r_{uv}x_u) - r_{uv}d_n(x_u)) \otimes s_n s_{uv} \otimes 1_S = X_3 \otimes 1_S. \tag{5.1.21}$$

With (5.1.18), (5.1.19), (5.1.20), (5.1.21) and the definition of $(\mathfrak{g} \otimes_k S)_\varphi$, we get that (5.1.16) holds, i.e.

$$\beta_S(\sum_n d_n \otimes s_n)|_{(\mathfrak{g}_S)_\varphi}(\sum_u x_u \otimes s_u) \in (\mathfrak{g} \otimes_k S)_\varphi,$$

hence

$$\beta_S(\sum_n d_n \otimes s_n)|_{(\mathfrak{g}_S)_\varphi} : (\mathfrak{g} \otimes_k S)_\varphi \longrightarrow (\mathfrak{g} \otimes_k S)_\varphi \tag{5.1.22}$$

is a well-defined map. It is clearly a derivation, so

$$\beta_S(\sum_n d_n \otimes s_n)|_{(\mathfrak{g}_S)_\varphi} \in Der_k((\mathfrak{g} \otimes_k S)_\varphi).$$

Therefore, β_S induces a well-defined map

$$\beta_S| : (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \longrightarrow \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_\varphi),$$

To avoid confusion, for any $y \in (\mathrm{Der}_k(\mathfrak{g} \otimes_k R))_{\widetilde{\varphi}}$, we denote by $\beta_S|(y) = \beta_S(y)|$ to mean that it is restricted to $(\mathfrak{g} \otimes_k S)_\varphi$. By definition of $\beta_S|$, it is an R-module homomorphism. Then we can get an S-module homomorphism

$$\beta_S| \otimes_R id_S : (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \otimes_R S \longrightarrow \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_{\varphi}) \otimes_R S.$$

Next we will show that $\beta_S| \otimes_R id_S$ is actually an isomorphism. Let

$$f_{\varphi} : (\mathfrak{g} \otimes_k S)_{\varphi} \otimes_R S \longrightarrow (\mathfrak{g} \otimes_k R) \otimes_R S$$

be the S-module isomorphism defined by $(\sum_i y_i \otimes s_i) \otimes s \mapsto \sum_i y_i \otimes ss_i$, note $y_i \in \mathfrak{g} \otimes_k R$, so it induces an S-module isomorphism (conjugation by f_{φ}) on the derivations

$$C_{f_{\varphi}} : \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_{\varphi} \otimes_R S) \longrightarrow \mathrm{Der}_k((\mathfrak{g} \otimes_k R) \otimes_R S) \tag{5.1.23}$$

$$d \mapsto f_{\varphi} \circ d \circ f_{\varphi}^{-1}.$$

Let

$$\beta_{\varphi,S} : \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_{\varphi}) \otimes_R S \longrightarrow \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_{\varphi} \otimes_R S)$$

be the canonical S-module isomorphism obtained by Proposition 4.5. Then the composition $C_{f_{\varphi}} \circ \beta_{\varphi,S}$ gives an S-module isomorphism

$$C_{f_{\varphi}} \circ \beta_{\varphi,S} : \mathrm{Der}_k((\mathfrak{g} \otimes_k S)_{\varphi}) \otimes_R S \longrightarrow \mathrm{Der}_k((\mathfrak{g} \otimes_k R) \otimes_R S). \tag{5.1.24}$$

On the other hand, since $(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}}$ is a descended module,

$$f_{\widetilde{\varphi}} : (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \otimes_R S \longrightarrow \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S$$

$$(\sum_i d_i \otimes s_i) \otimes s \mapsto \sum_i d_i \otimes s_i s$$

is an S-module isomorphism and the composition $\beta_S \circ f_{\widetilde{\varphi}}$ gives an S-module isomorphism

$$\beta_S \circ f_{\widetilde{\varphi}} : (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\varphi}} \otimes_R S \longrightarrow \mathrm{Der}_k(\mathfrak{g} \otimes_k R \otimes_R S). \tag{5.1.25}$$

In the following we will show that

$$\beta_S \circ f_{\widetilde{\varphi}} = C_{f_{\varphi}} \circ \beta_{\varphi,S} \circ (\beta_S| \otimes_R id_S),$$

which gives that $\beta_S|\otimes_R id_S$ is an S-module isomorphism. With S/R being faithfully flat we will get that $\beta_S|$ is an R-module isomorphism and then we are done.

Since all maps are S-module homomorphisms, we only need to check their action on the element of form $\sum_n d_n \otimes s_n \otimes 1_S$, where $\sum_n d_n \otimes s_n \in (Der_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\tilde{\varphi}}$ as before. Now

$$\begin{aligned}\beta_S \circ f_{\tilde{\varphi}}(\sum_n d_n \otimes s_n \otimes 1_S) &= \beta_S(\sum_n d_n \otimes s_n)\\ &= \sum_n (\alpha_n \otimes s_n id_S) \circ (d\varphi_S)^{-1} \circ d_{S,\mathfrak{g}_S,k}\end{aligned}$$

which we denote by d. Then

$$\begin{aligned}&C_{f_\varphi} \circ \beta_{\varphi,S} \circ (\beta_S| \otimes id_S)(\sum_n d_n \otimes s_n \otimes 1_S)\\ &= C_{f_\varphi} \circ \beta_{\varphi,S}(d| \otimes 1_S) = f_\varphi \circ \beta_{\varphi,S}(d| \otimes 1_S) \circ f_\varphi^{-1}.\end{aligned}$$

Note $d|$ is the same as d, just that its action is restricted on $(\mathfrak{g}_S)_\varphi$, so it suffices to show that

$$d \circ f_\varphi = f_\varphi \circ \beta_{\varphi,S}(d| \otimes 1_S). \tag{5.1.26}$$

For simplicity, we denote by $\mathcal{L} = (\mathfrak{g}_S)_\varphi$, say $\psi_{R,\mathcal{L}}^{-1}(d|) = \alpha$, then $\alpha \circ d_{R,\mathcal{L},k} = d|$, so

$$\beta_{\varphi,S}(d| \otimes 1_S) = (\alpha \otimes id_S) \circ (d\varphi_{\mathcal{L}})^{-1} \circ d_{S,\mathcal{L}_S,k},$$

where $d\varphi_{\mathcal{L}} : \Omega_{R,\mathcal{L}/k} \otimes_R S \longrightarrow \Omega_{S,\mathcal{L}_S/k}$ is an isomorphism defined in the same way as $d\varphi_S$, i.e.

$$d\varphi_{\mathcal{L}}(d_{R,\mathcal{L},k}(y) \otimes s) = s d_{S,\mathcal{L}_S,k}(y \otimes 1_S), \forall\, y \in \mathcal{L}, s \in S$$

with inverse

$$(d\varphi_{\mathcal{L}})^{-1}(d_{S,\mathcal{L}_S,k}(y \otimes s)) = d_{R,\mathcal{L},k}(y) \otimes s + \sigma_{\mathcal{L}} \circ \epsilon_{\mathcal{L}}^{-1}(d_{S/k}(s) \otimes (y \otimes 1_S))$$

for all $y \in \mathcal{L}$, $s \in S$, and $\sigma_{\mathcal{L}}$, $\epsilon_{\mathcal{L}}$ defined in the same way as σ_S, ϵ_S.

Hence it suffices to show

$$\sum_n (\alpha_n \otimes s_n id_S) \circ (d\varphi_S)^{-1} \circ d_{S,\mathfrak{g}_S,k} \circ f_\varphi = f_\varphi \circ (\alpha \otimes id_S) \circ (d\varphi_{\mathcal{L}})^{-1} \circ d_{S,\mathcal{L}_S,k}. \tag{5.1.27}$$

Consider their action on an element of the form $(\sum_u x_u \otimes s_u) \otimes s \in \mathcal{L} \otimes S$, where $\sum_u x_u \otimes s_u \in \mathcal{L}$. Applying both sides of (5.1.27) to $(\sum_u x_u \otimes s_u) \otimes s$ we get

$$\begin{aligned} LHS &= \sum_n (\alpha_n \otimes s_n id_S) \circ (d\varphi_S)^{-1} \circ d_{S,\mathfrak{g}_S,k}(\sum_u x_u \otimes s s_u) \\ &= \sum_{n,u} d_n(x_u) \otimes s_n s s_u + \sum_{n,u} (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s s_u) \otimes (x_u \otimes 1_S)). \end{aligned}$$

We denote the two terms in the last equation by A and B respectively.

$$\begin{aligned} RHS &= f_\varphi \circ (\alpha \otimes id_S) \circ (d\varphi_{\mathcal{L}})^{-1} \circ d_{S,\mathcal{L}_S,k}((\sum_u x_u \otimes s_u) \otimes s) \\ &= f_\varphi \circ (\alpha \otimes id_S) \circ \sigma_{\mathcal{L}} \circ \epsilon_{\mathcal{L}}^{-1} \left[d_{S/k}(s) \otimes ((\sum_u x_u \otimes s_u) \otimes 1_S) \right] \\ &\quad + f_\varphi(d|(\sum_u x_u \otimes s_u) \otimes s), \end{aligned}$$

and we denote the two terms in the last equation by C and D respectively. Then

$$\begin{aligned} D &= sd|(\sum_u x_u \otimes s_u) \\ &= \sum_{n,u} d_n(x_u) \otimes s_n s s_u + \sum_{n,u} (\alpha_n \otimes s s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s_u) \otimes (x_u \otimes 1_S)). \end{aligned}$$

Denote by

$$\sum_{n,u} (\alpha_n \otimes s s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s_u) \otimes (x_u \otimes 1_S)) = C_1,$$

so

$$D = A + C_1.$$

Since $d_{S/k}(ss_u) = sd_{S/k}(s_u) + s_u d_{S/k}(s)$,

$$B = C_1 + \sum_n (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s) \otimes (\sum_u x_u \otimes s_u)).$$

Hence now it suffices to show

$$\begin{aligned} &\textstyle\sum_n (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s) \otimes (\sum_u x_u \otimes s_u)) = \\ &f_\varphi \circ (\alpha \otimes id_S) \circ \sigma_{\mathcal{L}} \circ \epsilon_{\mathcal{L}}^{-1}(d_{S/k}(s) \otimes ((\textstyle\sum_u x_u \otimes s_u) \otimes 1_S)). \end{aligned} \tag{5.1.28}$$

Say $d\Phi_S^{-1}(d_{S/k}(s)) = \sum_v d_{R/k}(r_v) \otimes s_v$, then $d_{S/k}(s) = \sum_v s_v d_{S/k}(r_v)$, and

$$\epsilon_{\mathcal{L}}^{-1}(d_{S/k}(s) \otimes ((\sum_u x_u \otimes s_u) \otimes 1_S)) = \sum_v d_{R/k}(r_v) \otimes (\sum_u x_u \otimes s_u) \otimes s_v,$$

so

$$\text{RHS of (5.1.28)} = \sum_v s_v \left[d|(r_v(\sum_u x_u \otimes s_u)) - r_v d|(\sum_u x_u \otimes s_u) \right].$$

Note $\sum_u x_u \otimes s_u \in \mathcal{L}$, and it may not that each term $x_u \otimes s_u \in \mathcal{L}$, so we can't take the sum $\sum_u$ to be in front. Now

$$\begin{aligned} &d|(r_v(\sum_u x_u \otimes s_u)) \\ &= \sum_{u,n} d_n(x_u) \otimes s_n r_v s_u + \sum_{n,u} (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(r_v s_u) \otimes (x_u \otimes 1_S)), \end{aligned}$$

and

$$\begin{aligned} &r_v d|((\sum_u x_u \otimes s_u)) \\ &= \sum_{u,n} d_n(x_u) \otimes r_v s_n s_u + \sum_{n,u} (\alpha_n \otimes r_v s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1}(d_{S/k}(s_u) \otimes (x_u \otimes 1_S)). \end{aligned}$$

Using

$$d_{S/k}(r_v s_u) = r_v d_{S/k}(s_u) + s_u d_{S/k}(r_v)$$

we get

$$\text{RHS of (5.1.28)} = \sum_n (\alpha_n \otimes s_n id_S) \circ \sigma_S \circ \epsilon_S^{-1} (\sum_v s_v d_{S/k}(r_v) \otimes (\sum_u x_u \otimes s_u)).$$

Therefore, (5.1.28) holds (note $d_{S/k}(s) = \sum_v s_v d_{S/k}(r_v)$). □

Proposition 5.5 implies that $\widetilde{\varphi} = \beta_{S''}^{-1} \circ C_\varphi \circ \beta_{S''}$ is the descent data that leads to the twisted form $\mathrm{Der}_k(L)$ up to isomorphism. It also shows that the map defined in Proposition 5.4 for $L = \mathfrak{g} \otimes_k R$ is indeed the one implied by Theorem 4.8 and Theorem 4.2 under étale covers.

5.2 Galois descent

In this Section, we will review some notions and results related to Galois descent theory, and we will study analogous results for derivation algebras of R-forms of $\mathfrak{g}$ under Galois ring extensions, where $\mathfrak{g}$ is again a finite dimensional perfect Lie algebra over a field k of characteristic 0, and R is a k-algebra of finite type.

We first recall some concepts of (non-abelian) cohomology set ([S], [W]). Let Γ be any group acting as automorphisms of a group G. The maps $\mu : \Gamma \longrightarrow G,\ \gamma \mapsto \mu_\gamma$ satisfying

$$\mu_{\sigma\tau} = \mu_\sigma \cdot^\sigma \mu_\tau$$

are *1-cocycles*, where ${}^\sigma\mu_\tau$ denotes the Γ-action on G. Two 1-cocycles μ, μ' are said to be *cohomologous* if there exists some $\lambda \in G$ such that

$$\mu'_\sigma = \lambda \cdot \mu_\sigma \cdot ({}^\sigma\lambda)^{-1}$$

for all $\sigma \in \Gamma$. It is an equivalence relation and the set of equivalence classes of 1-cocycles from Γ to G is denoted by $H^1(\Gamma, G)$.

Recall some general results of Galois descent theory. Let S be in R-**alg**, and denote by $\mathrm{Aut}_R(S)$ the group of automorphisms of S which fix R.

Definition 5.6. Let Γ be a finite subgroup of $\mathrm{Aut}_R(S)$. We say that S/R is *Galois with Galois group* Γ if the following conditions hold

(i) S/R is faithfully flat.

(ii) The map

$$\rho : S \otimes_R S \longrightarrow \prod_{\Gamma} S;\ a \otimes b \mapsto (\gamma(a)b)_{\gamma \in \Gamma}$$

is an isomorphism.

Suppose that S/R is Galois with finite Galois group Γ. Then S/R is an étale cover (c.f. Remark 2.24 of [Mil]). Let N be any R-module. Let $\mathbf{Aut}(N)$ be the group functor defined as before. In this Section, we fix the Γ-action on the group $\mathbf{Aut}(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S)$ by

$$^{\gamma}f = (id_N \otimes \gamma) \circ f \circ (id_N \otimes \gamma^{-1}) = C_{id_N \otimes \gamma}(f)$$

for all $\gamma \in \Gamma, f \in \mathbf{Aut}(N)(S)$. The maps $\mu : \Gamma \longrightarrow \mathbf{Aut}(N)(S), \quad \gamma \mapsto \mu_\gamma$ satisfying

$$\mu_{\gamma_1 \gamma_2} = \mu_{\gamma_1} \cdot^{\gamma_1} \mu_{\gamma_2}$$

are called *Galois 1-cocycles*, which we denote by $\mu = (\mu_\gamma)_{\gamma \in \Gamma}$. Similarly, two Galois 1-cocycles $\mu = (\mu_\gamma)_{\gamma \in \Gamma}$ and $\mu' = (\mu'_\gamma)_{\gamma \in \Gamma}$ are said to be *cohomologous* if there exists some $\lambda \in \mathbf{Aut}(N)(S)$ such that

$$\lambda \cdot \mu_\gamma \cdot (^{\gamma}\lambda)^{-1} = \mu'_\gamma,\ \forall \gamma \in \Gamma.$$

The set of equivalence classes of Galois 1-cocycles is denoted by $H^1(\Gamma, \mathbf{Aut}(N)(S))$ and we call it the first (non-abelian) Galois cohomology set.

Fix the Γ-action on $N \otimes_R S$ by acting on S, i.e. for any $m \in N \otimes_R S$, and for any $\gamma \in \Gamma$, $^{\gamma}m = (id_N \otimes \gamma)(m)$. For a Galois 1-cocycle $\mu = (\mu_\gamma)_{\gamma \in \Gamma}$, its *descended module* can be characterized as

$$(N \otimes_R S)_\mu = \{m \in N \otimes_R S \mid \mu_\gamma(^{\gamma}m) = m,\ \forall \gamma \in \Gamma\}$$

such that

$$f : (N \otimes_R S)_\mu \otimes_R S \longrightarrow N \otimes_R S;\ m \otimes s \mapsto sm$$

is an S-module isomorphism. Similarly as in the faithfully flat descent theory, one has (c.f. § 17.7 of [W]) the following one-to-one correspondence.

Theorem 5.7. *Let S/R be Galois with finite Galois group Γ. Then the isomorphism classes of S/R-forms of N correspond to $H^1(\Gamma, \mathbf{Aut}(N)(S))$, where for any Galois 1-cocycle $\mu = (\mu_\gamma)_{\gamma\in\Gamma}$, the above defined $(N \otimes_R S)_\mu$ is the corresponding S/R-form of N.*

Let S/R be Galois with finite Galois group Γ. Let $P_\gamma : \prod_\Gamma S \longrightarrow S$ be the standard projection to the γ-component, for any $\gamma \in \Gamma$. For $\mu \in \mathbf{Aut}(N)(S'')$, let

$$\mu_\gamma := \mathbf{Aut}(N)(P_\gamma \rho)(\mu) \in \mathbf{Aut}(N)(S), \ \ \forall\, \gamma \in \Gamma.$$

If μ is a 1-cocycle, i.e. a representative of an element in $H^1(S/R, \mathbf{Aut}(N))$, then it can be checked that the above defined $(\mu_\gamma)_{\gamma\in\Gamma}$ is a Galois 1-cocycle, and the two first cohomology sets $H^1(S/R, \mathbf{Aut}(N))$ and $H^1(\Gamma, \mathbf{Aut}(N)(S))$ can be identified (c.f. § 17.7 of [W]).

Now we are going to study the Galois descent theory of derivation algebras of R-forms of $\mathfrak{g}$. Let L be an S/R-form of $\mathfrak{g}$ for S/R Galois with finite Galois group Γ. Suppose $\mu = (\mu_\gamma)_{\gamma\in\Gamma}$ is the Galois 1-cocycle that determines L, i.e. $L = (\mathfrak{g} \otimes_k S)_\mu$. Again by Proposition 4.7, $\mathrm{Der}_k(L)$ is an S/R-form of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$. We will first construct the Galois 1-cocycle that leads to $\mathrm{Der}(L)$ and then get maps between the two corresponding Galois cohomology sets.

For any $\gamma \in \Gamma$, denote by $h_\gamma = \mu_\gamma \circ (id_{\mathfrak{g}_R} \otimes_R \gamma)$, $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$, so $h_\gamma \in \mathrm{Aut}_{k-Lie}(\mathfrak{g}_R \otimes_R S)$. There is a natural Γ-action on $\mathrm{Der}_k(\mathfrak{g}_R \otimes_R S)$ given by C_{h_γ}, for all $\gamma \in \Gamma$, i.e. $C_{h_\gamma}(d) = h_\gamma \circ d \circ h_\gamma^{-1}$, $d \in \mathrm{Der}_k(\mathfrak{g}_R \otimes_R S)$. There is then an induced Γ-action on $\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S$ via $\beta_S^{-1} \circ C_{h_\gamma} \circ \beta_S$ for all $\gamma \in \Gamma$. Define

$$\widetilde{\mu} : \Gamma \longrightarrow \mathrm{Aut}_{S-mod}(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S), \ \gamma \mapsto \widetilde{\mu}_\gamma, \tag{5.2.1}$$

where

$$\widetilde{\mu}_\gamma = \beta_S^{-1} \circ C_{h_\gamma} \circ \beta_S \circ (id \otimes \gamma^{-1}) \tag{5.2.2}$$

for all $\gamma \in \Gamma$. Consider the following defined R-module determined by $\widetilde{\mu}$

$$\begin{aligned}&(\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\mu}} \\ &= \{m \in \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S \mid \widetilde{u}_\gamma(id \otimes \gamma)(m) = m, \ \forall \gamma \in \Gamma\}.\end{aligned} \tag{5.2.3}$$

Similarly as in Proposition 5.5, one has

Proposition 5.8. *Let S/R be Galois with finite Galois group Γ.*

(1) For any Galois 1-cocycle $\mu : \Gamma \longrightarrow \mathbf{Aut}(\mathfrak{g}_R)(S))$, the above defined $\widetilde{\mu} : \Gamma \longrightarrow \mathbf{Aut}(Der_k(\mathfrak{g}_R))(S)$ is also a Galois 1-cocycle.

(2) There exists an R-module isomorphism

$$(Der_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\mu}} \cong Der_k((\mathfrak{g} \otimes_k S)_\mu) = Der_k(L)$$

which is again induced from the isomorphism β_S.

Proof. (1) For any $\gamma_1, \gamma_2 \in \Gamma$, note $h_{\gamma_1\gamma_2} = h_{\gamma_1} \circ h_{\gamma_2}$. And

$$\widetilde{\mu}_{\gamma_1\gamma_2} = \beta_S^{-1} \circ C_{h_{\gamma_1\gamma_2}} \circ \beta_S \circ (id \otimes (\gamma_1\gamma_2)^{-1})$$

$$\widetilde{\mu}_{\gamma_1} \cdot^{\gamma_1} \widetilde{\mu}_{\gamma_2} = \beta_S^{-1} \circ C_{h_{\gamma_1}} \circ C_{h_{\gamma_2}} \circ \beta_S \circ (id \otimes \gamma_2^{-1}) \circ (id \otimes \gamma_1^{-1}),$$

so

$$\widetilde{\mu}_{\gamma_1\gamma_2} = \widetilde{\mu}_{\gamma_1} \cdot^{\gamma_1} \widetilde{\mu}_{\gamma_2},$$

$\widetilde{\mu}$ is a Galois 1-cocycle.

(2) Now it is easier to check than the one in Proposition 5.5 that the map induced by β_S, which we denoted by $\beta_S|$, is well-defined. So

$$\beta_S| : (\mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S)_{\widetilde{\mu}} \longrightarrow \mathrm{Der}_k(L), \quad \delta \mapsto \beta_S(\delta)|_L$$

is an R-mdoule homomorphism. Then one can use the same method as the proof of Proposition 5.5 to show that $\beta_S| \otimes id_S$ is an isomorphism. Hence $\beta_S|$ is an isomorphism as S/R is faithfully flat.

□

It is straightforward to check that if

$$\mu \sim \mu' \;\text{ via }\; \lambda \in \mathbf{Aut}(\mathfrak{g}_R)(S),$$

then

$$\widetilde{\mu} \sim \widetilde{\mu'} \;\text{ via }\; \beta_S^{-1} \circ C_\lambda \circ \beta_S \in \mathbf{Aut}(\mathrm{Der}_k(\mathfrak{g}_R))(S),$$

where

$$C_\lambda : \mathrm{Der}_k(\mathfrak{g}_S) \longrightarrow \mathrm{Der}_k(\mathfrak{g}_S), \quad d \mapsto \lambda \circ d \circ \lambda^{-1},$$

is an S-module isomorphism. Therefore we get

Proposition 5.9. *Let S/R be Galois with finite Galois group Γ. Then there is a well-defined map between Galois cohomology sets*

$$H^1(\Gamma, \mathbf{Aut}(\mathfrak{g}_R)(S)) \longrightarrow H^1(\Gamma, \mathbf{Aut}(Der_k(\mathfrak{g}_R))(S)), \; [\mu] \mapsto [\widetilde{\mu}],$$

where $\widetilde{\mu}_\gamma = \beta_S^{-1} \circ C_{h_\gamma} \circ \beta_S \circ (id \otimes \gamma^{-1})$, and $h_\gamma = \mu_\gamma \circ (id_{\mathfrak{g}_R} \otimes \gamma)$, for all $\gamma \in \Gamma$.

Remark 5.10. Again Proposition 5.8 shows that $\widetilde{\mu}$ is the Galois 1-cocycle that determines $\mathrm{Der}_k(L)$ and the map defined in Proposition 5.9 is the one implied by Theorem 4.8 and Theorem 5.7 in Galois case.

5.3 Descent construction of extended affine Lie algebras

In this Section, we use the Galois descent data of multiloop Lie algebras to construct the descent data that leads to extended affine Lie algebras, more precisely, fgc EALAs. We will take G to be a group generated by elements acting on extended affine Lie algebras defined below and study the descent construction of extended affine Lie algebras.

In the following, k will be an algebraically closed field of characteristic 0, and $\mathfrak{g}$ will be a finite dimensional simple Lie algebra over k. Let E be an extended affine Lie algebra, then its centreless core E_{cc} is a centreless Lie torus. By Theorem 2.19, if E_{cc} is fgc, then E_{cc} is of the form of a multiloop Lie algebra. Multiloop Lie algebras can be viewed as twisted forms under Galois ring extensions, and we know the Galois descent data that leads to multiloop Lie algebras. For fgc EALA, we will use its realization of multiloop Lie algebra and the descent data of multiloop Lie algebra to construct the descent data (1-cocycle) that leads to itself.

We first look at multiloop Lie algebras from twisted forms point of view. Let

$$\mathcal{L} = \mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} \mathfrak{g}_{\overline{i_1},\ldots,\overline{i_n}} \otimes_k t_1^{\frac{i_1}{m_1}} \cdots t_n^{\frac{i_n}{m_n}} \subset \mathfrak{g} \otimes_k S \qquad (5.3.1)$$

be a multiloop Lie algebra determined by $\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}$. Let S, R be the Laurent polynomial rings as in Section 2.3. Then $\mathcal{L}$ is an R-Lie algebra and $\mathcal{L} \otimes_R S \cong \mathfrak{g} \otimes_k S$ as S-Lie algebras, so $\mathcal{L}$ is an S/R-form of $\mathfrak{g}$. Actually, $\mathcal{L}$ is determined by the Galois 1-cocycle

$$u : \Gamma \longrightarrow \mathbf{Aut}(\mathfrak{g})(S) \tag{5.3.2}$$

$$(\overline{i_1}, \dots, \overline{i_n}) \mapsto (\sigma_1^{-i_1} \otimes id_S) \cdots (\sigma_n^{-i_n} \otimes id_S).$$

Write $\mathcal{L} = (\mathfrak{g} \otimes_k S)_u$, where

$$u_\gamma = \boldsymbol{\sigma}^{-\gamma} \otimes id_S = (\sigma_1^{-i_1} \otimes id_S) \cdots (\sigma_n^{-i_n} \otimes id_S)$$

for $\gamma = (\overline{i_1}, \dots, \overline{i_n}) \in \Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}$.

Now let E be any fgc EALA. Then E_{cc} is of the form of a multiloop Lie algebra, say $E_{cc} = \mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m}) := \mathcal{L}$ for some $\mathfrak{g}$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, $\boldsymbol{m} = (m_1, \dots, m_n)$. And E can be constructed in the form

$$E = \mathcal{L} \oplus D^{gr*} \oplus D \tag{5.3.3}$$

for some (unique) graded subalgebra $D \subset \mathrm{SCDer}_k(\mathcal{L})$ of the skew centroidal derivations of $\mathcal{L}$. The descent data of $\mathcal{L}(\mathfrak{g}, \boldsymbol{\sigma}, \boldsymbol{m})$ is known to be the u defined as above. We are going to construct the descent data that leads to E.

Let $E_S = (\mathfrak{g} \otimes_k S) \oplus D^{gr*} \oplus D$ for the same D as above. By the identifications explained in Section 3.1 (c.f. Section 4.2 of [CNPY]), E_S is also an EALA and it is an untwisted EALA. We know that the descended object of $\mathfrak{g} \otimes_k S$ determined by the descent data u is the multiloop Lie algebra $\mathcal{L}$. Let now

$$h_\gamma = u_\gamma \circ (id_{\mathfrak{g}} \otimes \gamma),$$

for any $\gamma \in \Gamma$, where each u_γ is defined as above. For any $\gamma \in \Gamma$, we denote by

$$\hat{\gamma} : D^{gr*} \longrightarrow D^{gr*}, \quad f \mapsto \hat{\gamma}.f,$$

where

$$(\hat{\gamma}.f)(d) = f(C_{h_\gamma^{-1}}(d)),$$

for any $d \in D$. Then $\hat{\gamma}_1 \circ \hat{\gamma}_2 = (\widehat{\gamma_1\gamma_2})$ for $\gamma_1, \gamma_2 \in \Gamma$. Note our u_γ above actually satisifes $u_{\gamma_1} \circ u_{\gamma_2} = u_{\gamma_1\gamma_2}$ for $\gamma_1, \gamma_2 \in \Gamma$ (remember that Γ is an additive group).

Consider the group G generated, under composition, by triples $(u_\gamma, \hat{\gamma}, C_{h_\gamma})$ for all $\gamma \in \Gamma$. Let G act on the elements in E_S by acting on each component, i.e. for any $x \oplus f \oplus d \in E_S$, define

$$(u_\gamma, \hat{\gamma}, C_{h_\gamma}).(x \oplus f \oplus d) = u_\gamma(x) \oplus \hat{\gamma}.f \oplus C_{h_\gamma}(d)$$

for all $\gamma \in \Gamma$, $f \in D^{gr*}$, $d \in D$. It is a well-defined action as proved in Lemma 3.1. Actually, from Lemma 3.1 we see that for any element $d \in D \subset \text{SCDer}_k(\mathcal{L})$, any $f \in D^{gr*}$, $C_{h_\gamma}(d) = d$ and $\hat{\gamma}.f = f$, i.e. the actions on D^{gr*} and D defined above are actually trivial because $D \subset \text{SCDer}_k(\mathcal{L})$.

It is straightforward to check that G is a subgroup of the k-Lie algebra automorphism group $\text{Aut}_{k-Lie}(E_S)$ of the Lie algebra E_S.

Consider the following action of the finite abelian group $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \mathbb{Z}/m_n\mathbb{Z}$ on G:

$$^{\gamma'}(u_\gamma, \hat{\gamma}, C_{h_\gamma}) = (C_{id_\mathfrak{g} \otimes \gamma'}(u_\gamma), \hat{\gamma}, C_{h_\gamma}) \tag{5.3.4}$$

for any $\gamma, \gamma' \in \Gamma$, where

$$C_{id_\mathfrak{g} \otimes \gamma'}(u_\gamma) = (id_\mathfrak{g} \otimes \gamma') \circ u_\gamma \circ (id_\mathfrak{g} \otimes \gamma')^{-1}.$$

Again by our definition of u_γ we actually have $C_{id_\mathfrak{g} \otimes \gamma'}(u_\gamma) = u_\gamma$ for any $\gamma, \gamma' \in \Gamma$. Define a map

$$\widetilde{u} : \Gamma \longrightarrow G, \quad \gamma \mapsto \widetilde{u}_\gamma = (u_\gamma, \hat{\gamma}, C_{h_\gamma}). \tag{5.3.5}$$

Proposition 5.11. *Let S, R, Γ be defined as above. Let E, E_S be extended affine Lie algebras as above.*

(1) Let $u : \Gamma \longrightarrow \mathbf{Aut}(\mathfrak{g})(S) = Aut_{S-Lie}(\mathfrak{g} \otimes_k S)$ be the 1-cocycle that determines the multiloop Lie algebra $\mathcal{L}$. Then the above defined $\widetilde{u} : \Gamma \longrightarrow G$ is also a 1-cocycle with the above defined Γ-action on G.

(2) Consider the Γ-action on E_S by $(id_\mathfrak{g} \otimes \gamma, id, id)$. Then the descended object of E_S determined by the 1-cocycle $\tilde{u}$ in (1) is the extended affine Lie algebra

E, *i.e.*

$$(E_S)_{\widetilde{u}} := \{x \oplus f \oplus d \in E_S \mid \widetilde{u}_\gamma \cdot^\gamma (x \oplus f \oplus d) = x \oplus f \oplus d,\ \forall \gamma \in \Gamma\} = E.$$

Proof. (1). For any $\gamma_1, \gamma_2 \in \Gamma$,

$$\widetilde{u}_{\gamma_1\gamma_2} = (u_{\gamma_1\gamma_2}, \gamma_1\hat{\gamma}_2, C_{h_{\gamma_1\gamma_2}})$$

and

$$\begin{aligned}
\widetilde{u}_{\gamma_1} \circ^{\gamma_1} \widetilde{u}_{\gamma_2} &= (u_{\gamma_1}, \hat{\gamma}_1, C_{h_{\gamma_1}}) \circ (C_{id_{\mathfrak{g}} \otimes_k \gamma_1}(u_{\gamma_2}), \hat{\gamma}_2, C_{h_{\gamma_2}}) \\
&= (u_{\gamma_1} \circ C_{id_{\mathfrak{g}} \otimes_k \gamma_1}(u_{\gamma_2}), \hat{\gamma}_1 \circ \hat{\gamma}_2, C_{h_{\gamma_1}} \circ C_{h_{\gamma_2}}) \\
&= (u_{\gamma_1} \circ^{\gamma_1} u_{\gamma_2}, \gamma_1\hat{\gamma}_2, C_{h_{\gamma_1} \circ h_{\gamma_2}}) \\
&= (u_{\gamma_1\gamma_2}, \gamma_1\hat{\gamma}_2, C_{h_{\gamma_1\gamma_2}}) \\
&= \widetilde{u}_{\gamma_1\gamma_2},
\end{aligned}$$

so the defined $\widetilde{u}$ is a 1-cocycle.

(2). Now for any $x \oplus f \oplus d \in (E_S)_{\widetilde{u}} = (\mathfrak{g} \otimes_k S \oplus D^{gr*} \oplus D)_{\widetilde{u}}$ we have

$$\begin{aligned}
x \oplus f \oplus d &= \widetilde{u}_\gamma \cdot^\gamma (x \oplus f \oplus d) \\
&= \widetilde{u}_\gamma((id_{\mathfrak{g}} \otimes \gamma)(x) \oplus f \oplus d) \\
&= (u_\gamma \circ (id_{\mathfrak{g}} \otimes \gamma)(x) \oplus \hat{\gamma}.f \oplus C_{h_\gamma}(d)).
\end{aligned}$$

Since $\hat{\gamma}(f) = f$, $C_{h_\gamma}(d)) = d$ automatically hold, we actually get that x must satisfy $u_\gamma \circ (id_{\mathfrak{g}} \otimes \gamma)(x) = x$ for any $\gamma \in \Gamma$, which implies $x \in \mathcal{L}$. Hence $x \oplus f \oplus d \in E = \mathcal{L} \oplus D^{gr*} \oplus D$. Conversely, for any $x \oplus f \oplus d \in E$, $x \in \mathcal{L}$, we have

$$u_\gamma \circ (id_{\mathfrak{g}} \otimes \gamma)(x) = x$$

for any $\gamma \in \Gamma$. Again by that $D \subset \mathrm{SCDer}_k(\mathcal{L})$ and the definition of the Γ-actions on D^{gr*} and D, we have

$$\hat{\gamma}.f = f,\ \ C_{h_\gamma}(d) = d,\ \ \gamma \in \Gamma,\ f \in D^{gr*},\ d \in D.$$

So

$$\widetilde{u}_\gamma \cdot^\gamma (x \oplus f \oplus d) = x \oplus f \oplus d$$

for any $\gamma \in \Gamma$, i.e. $x \oplus f \oplus d \in (E_S)_{\widetilde{u}}$. Therefore $(E_S)_{\widetilde{u}} = E$. □

In summary, for any multiloop Lie algebra $\mathcal{L} \subseteq \mathfrak{g} \otimes_k S$, and for any $D \subseteq \mathrm{SCDer}_k(\mathcal{L})$ which satisfies the condition in the construction of EALAs, if $u = (u_\gamma)_{\gamma\in\Gamma} = (\boldsymbol{\sigma}^{-\gamma} \otimes id_S)_{\gamma\in\Gamma}$ is the descent data such that the descended object of $\mathfrak{g} \otimes_k S$ is $\mathcal{L}$, then the above defined $\widetilde{u}$ is the descent data such that the descended object of the untwisted EALA $E_S = \mathfrak{g} \otimes_k S \oplus D^{gr*} \oplus D$ is the extended affine Lie algebra $E = \mathcal{L} \oplus D^{gr*} \oplus D$.

Remark 5.12. (i) Let $\mathcal{L}$ be the multiloop algebra defined in (5.3.1). Let E be the fgc EALA given in (5.3.3). We can also consider

$$E_S = (\mathfrak{g} \otimes_k S) \oplus \widetilde{D}^{gr*} \oplus \widetilde{D}$$

for any $\widetilde{D}$ a graded subalgebra of the skew-centroidal derivation algebra $\mathrm{SCDer}_k(\mathfrak{g} \otimes_k S)$ with the property that it contains D as a graded subalgebra and $\widetilde{D}^\mu = D^\mu$ for all $\mu \in \Xi$, where Ξ is the centroid grading group of $\mathcal{L}$. Let $\widetilde{\tau} : \widetilde{D} \times \widetilde{D} \longleftarrow \widetilde{D}^{gr*}$ be a map defined by $\widetilde{\tau}|_{D\times D}$ is the 2-cocycle τ that appeared E and $\widetilde{\tau} = 0$ on all other elements. Then this E_S (with the defined $\widetilde{\tau}$) is also an EALA (Section 3.2). Consider the same action of G on such E_S. It is well-defined (Lemma 3.1), and now the actions on $\widetilde{D}^{gr*}$ and $\widetilde{D}$ may not be trivial. With the same Γ-action on G, one can get the same results as in Proposition 5.11. But note now G is not a subgroup of $\mathrm{Aut}_{k-Lie}(E_S)$ since the Lie bracket on this E_S may not be G-equivariant.

(ii) If we take $\widetilde{D}$ to be the whole skew-centroidal derivation algebra $\mathrm{SCDer}_k(\mathfrak{g}\otimes_k S)$, then $(\mathfrak{g}\otimes_k S)\oplus\widetilde{D}^{gr*}$ is the universal central extension of $\mathfrak{g}\otimes_k S$ with Lie bracket

$$[x_1 \oplus f_1, x_2 \oplus f_2] = [x_1, x_2] \oplus \sigma_D(x_1, x_2)$$

for any $x_1, x_2 \in \mathfrak{g} \otimes_k S$, $f_1, f_2 \in (\mathrm{SCDer}_k(\mathfrak{g} \otimes_k S))^{gr*}$. Then consider the group G generated by two-tuples $(u_\gamma, \hat{\gamma})$ for all $\gamma \in \Gamma$ (i.e. removing the action on $\mathrm{SCDer}_k(\mathfrak{g}\otimes_k S)$). Define all actions analogously. Then $\widehat{u} = (\widehat{u}_\gamma)_{\gamma\in\Gamma}$ with $\widehat{u}_\gamma = (u_\gamma, \hat{\gamma})$, is also a 1-cocycle from Γ to the defined G. And the descended object of $(\mathfrak{g} \otimes_k S) \oplus (\mathrm{SCDer}_k(\mathfrak{g} \otimes_k S))^{gr*}$ determined by $\widehat{u}$ is actually the universal central extension $\mathcal{L} \oplus (\mathrm{SCDer}_k(\mathcal{L}))^{gr*}$ of $\mathcal{L}$.

Chapter 6

Conclusion

This thesis studies extended affine Lie algebras (EALAs for short) from both the point of vew of fixed point subalgebras and of descended objects of some untwisted extended affine Lie algebras. It also studies the descent considerations necessary to study the Lie algebra of derivations of certain Lie algebras (which include multiloop algebras).

The two most important results used in the fixed point subalgebra part are the construction of EALAs introduced by Neher, and the identification of the skew-centroidal derivation algebras of multiloop Lie algebras as subalgebras of the corresponding one of untwisted multiloop Lie algebras.

On the one hand, let $\mathfrak{g}$ be any finite dimensional simple Lie algebra over an algegraically closed field k of characteristic 0. Let Γ be a finite abelian group which is determined by some finite order commuting automorphisms of $\mathfrak{g}$. Then there exists a Laurent polynomial ring S and an action of Γ on $\mathfrak{g} \otimes_k S$ such that the fixed point subalgebra of the defined action is actually a multiloop Lie algebra (see Section 3.1). By Neher's construction, starting from $\mathfrak{g} \otimes_k S$, there are untwisted EALAs

$$E_S = (\mathfrak{g} \otimes_k S) \oplus \tilde{D}^{gr*} \oplus \tilde{D}$$

for any suitable graded subalgebra $\tilde{D}$ of the skew-centroidal derivation algebra $\mathrm{SCDer}_k(\mathfrak{g} \otimes_k S)$. We showed that Γ also acts on $\tilde{D}$ and $\tilde{D}^{gr*}$, and the fixed point subalgebra

$$(E_S)^\Gamma = (\mathfrak{g} \otimes_k S)^\Gamma \oplus (\tilde{D}^{gr*})^\Gamma \oplus \tilde{D}^\Gamma$$

is also an EALA; these are results of Lemma 3.1 and Theorem 3.3.

On the other hand, any multiloop Lie algebra $\mathcal{L}$ can be viewed as a fixed point subalgebra of some untwisted multiloop Lie algebra $\mathfrak{g} \otimes_k S$ for some finite abelian group Γ:

$$\mathcal{L} = (\mathfrak{g} \otimes_k S)^\Gamma.$$

And we know that for any suitable $D \subset \mathrm{SCDer}_k(\mathcal{L})$, the direct sum

$$E = \mathcal{L} \oplus D^{gr*} \oplus D$$

is an EALA. The result in Theorem 3.4 tells us that there exists an untwisted EALA such that the fixed point subalgebra of it is the EALA E above. But this untwisted EALA may not be unique unless the nullity $= 1$ as explained in Section 3.2 and Section 3.3. The results in this part also tell us how to construct the descent data so that the descended object of the untwisted EALA is also an EALA, which is the result of Proposition 5.11.

The second part is about descent theory in the study of Lie algebra of derivations of certain Lie algebras and descent construction of extended affine Lie algebras. Descent theory and non-abelian cohomology are important and effective tools to study infinite dimensional Lie theory, especially in the study of multiloop Lie algebras. Multiloop Lie algebras can not only be viewed as fixed point subalgebras, but they can also be viewed as twisted forms. More precisely, for any multiloop Lie algebra $\mathcal{L}$ over k, it is also a Lie algebra over some Laurent polynomial ring R. And when we view $\mathcal{L}$ as a Lie algebra over R, there exists a ring extension S/R such that

$$\mathcal{L} \otimes_R S \cong \mathfrak{g} \otimes_k S \cong (\mathfrak{g} \otimes_k R) \otimes_R S$$

for some finite dimensional simple Lie algebra $\mathfrak{g}$ over k. This tells us that $\mathcal{L}$ is a twisted form of the untwisted multiloop Lie algebra $\mathfrak{g} \otimes_k R$. Hence descent theory comes into the picture. By classical results, isomorphism classes of twisted forms can be classified by non-abelian first cohomology sets. The classification of multiloop Lie algebras is a particular example of this, see [GP2], [GP4]. From EALA theory, we know that besides the centreless core itself (which is a multiloop Lie algebra in our considerations), its derivation algebra, central extensions of it and invariant bilinear forms also play important roles. The derivation algebras of multiloop Lie algebras have been determined using descent theory in [P5]. Descent construction of central extensions of certain Lie algebras has been studied in [PPS]. Invariant bilinear forms were studied in [NPPS] using descent. Hence, by looking at multiloop Lie algebras as twisted forms, and then using descent theory and non-abelian cohomology, one can get many results about the structure of the Lie algebras.

Now from the work in [KP], we know that derivation algebras of twisted forms of certain Lie algebras are also twisted forms under suitable ring extensions. This motivates our study of derivation algebras of such Lie algebras from the twisted forms point of view. Generally, for any perfect finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0, for any k-algebra R of finite type, if L is a twisted form of $\mathfrak{g} \otimes_k R$, it can be deduced from the results in [KP] that there exists an S-module isomorphism

$$\mathrm{Der}_k(L) \otimes_R S \cong \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \otimes_R S$$

for some étale cover S/R. So $\mathrm{Der}_k(L)$ is a twisted form of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$. We showed the results regarding $\mathrm{Der}_k(L)$ from the descent point of view in Proposition 5.5. As Galois ring extensions are always faithfully flat and étale, and the Laurent polynomial ring extensions we are interested in are Galois, we also considered the questions in the Galois case. The result is Proposition 5.8. Besides, Proposition 5.4 and Proposition 5.9 tell us that the relation between the isomorphism of Lie algebras and the isomorphism of their derivation algebras can be characterized by the property of the maps between corresponding non-abelian cohomology sets, i.e. the map

$$H^1(S/R, \mathbf{Aut}(L)) \longrightarrow H^1(S/R, \mathbf{Aut}(\mathrm{Der}_k(L))); \quad [\varphi] \mapsto [\tilde{\varphi}]$$

for S/R an étale cover, where

$$\tilde{\varphi} = \beta_{S''}^{-1} \circ C_{\varphi} \circ \beta_{S''},$$

and the map

$$H^1(\Gamma, \mathbf{Aut}(\mathfrak{g}_R)(S)) \longrightarrow H^1(\Gamma, \mathbf{Aut}(\mathrm{Der}_k(\mathfrak{g}_R))(S)), \; [\mu] \mapsto [\widetilde{\mu}]$$

for S/R Galois with Galois group Γ, where $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$,

$$\widetilde{\mu}_{\gamma} = \beta_S^{-1} \circ C_{h_{\gamma}} \circ \beta_S \circ (id \otimes_R \gamma^{-1}),$$

$h_{\gamma} = \mu_{\gamma} \circ (id_{\mathfrak{g}_R} \otimes_R \gamma)$, for all $\gamma \in \Gamma$. Of course, to better understand these non-abelian cohomology sets, one needs to know the automorphism group functors, which are unknown except for some special cases.

In summary, our results of fgc extended affine Lie algebras ("almost all" extended affine Lie algebras) reduces the study of extended affine Lie algebras to the study of untwisted extended affine Lie algebras and their fixed point subalgebras, or to the study of untwisted extended affine Lie algebras and their descended objects. The structure of untwisted extended affine Lie algebras is much simpler.

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