Variational methods for the dynamics of the porous medium

by

Tagir Farkhutdinov

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Abstract

In this thesis, I developed the ideas of applying the variational method in geometric mechanics to the porous media described as solid elastic materials with embedded ideal (incompressible) fluid, also known as Eulerian fluid. The work includes four chapters and a conclusion. In the Introduction, I familiarize the reader with basic variational principles and their applications. Chapter 2 has the statement and motivation for studying the porous medium, the derivation of the Lagrangian and the constraints, the geometric variational derivation of the equations of motion, and the discussion of the result and its applications. Chapter 3 contains the investigation of the properties of the poromechanics dynamical system, derived in the previous chapter, as well as the results of numerical simulations of the acoustic/seismic wave propagation. The comparison with the famous Biot poroelasticity equations shows the equivalence of the linearizations. Chapter 4 introduces the application of the previously developed theory to the living organisms and provides the results of computational experiments. Also, I present the investigation of the totally incompressible case of the porous medium as it may have broad physical applications.

The work concludes that the geometric variational method is an elegant and efficient instrument in the derivation of dynamics of complex multi-phase systems with an arbitrary number of incompressible components, including the poromechanics considered in my work. The derived system could explain physical phenomena that were previously attributed to unknown parameters of the porous medium, such as in Biot's theory. The nonlinear equations of poromechanics could be used in numerical modeling to explain the behavior of a wide range of physical phenomena, including biological systems.

Preface

The major part of the research conducted for this thesis forms part of an international research collaboration, with my supervisor Dr. Vakhtang Putkaradze (University of Albert and the ATCO Transformation Team), and Dr. François Gay-Balmaz (LMD, Ecole Normale Supérieure de Paris, and CNRS). While the project was strongly collaborative, I proposed, developed and implemented several ideas, such as the physical setting of the poromechanical variables, including the pore volume and the associated constraint, described in the second chapter, double incompressibility for biological systems, linear stability analysis and reduction and implementation of numerical simulations, in close collaboration with Drs. Gay-Balmaz and Putkaradze.

Chapter 2 and Chapter 3 of this thesis are based on a paper published as Farkhutdinov, T., Gay-Balmaz, F. & Putkaradze, V. "Geometric variational approach to the dynamics of the porous medium, filled with incompressible fluid", Acta Mechanica 231, 3897–3924 (2020) and reported by myself as "Variational Methods in the Dynamics of Porous Media" on several conferences, including SIAM Conference on Applications of Dynamical Systems 2019, Snowbird, UT, the U.S. in 2019, 2018 CMS Winter Meeting, Vancouver, BC, Canada in 2018, and PIMS Workshop on Stochastic and Deterministic Modelling in Biology, Jasper, AB, Canada, 2018.

Chapter 4 is based on a paper submitted for publication as an article "Actively deforming porous media in an incompressible fluid: a variational approach" and submitted in Physica D: Nonlinear Phenomena at the moment of the completion of this manuscript.

Initial stages of this work (Chapters 2-3) were done at the University of Alberta, and the work outlined in Chapter 4 was completed while I was PIMS-MITACS Intern at the ATCO's Transformation Team.

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Chapter 1

Introduction into the geometric variational approach and its applications

1.1 Introduction to geometric variational methods

This chapter will present the background of the geometric variational approach for the mechanics of physical systems, starting with the basic Hamilton's principle. The Euler-Poincaré theory for the reduction of left- and right- invariant physical systems will follow afterward. The handling of constraints and the presence of external forces will be covered further. Several examples of the derivation of equations of motion for various Lie groups using the Euler-Poincaré theory are provided. The applications of the method to incompressible fluid and elasticity theory are discussed as they are used in this thesis to obtain the equations of dynamics of elastic porous medium filled with an incompressible fluid.

1.1.1 Hamilton's principle

One of the most fundamental statements in classical mechanics is the principle of critical action or Hamilton's principle, according to which the motion of a mechanical system between two given positions is given by a curve that makes the action (integral of the Lagrangian) of the system critical (see, for instance, [51]).

Consider a mechanical system with configuration manifold Q and Lagrangian $L : TQ \to \mathbb{R}$ defined on the tangent bundle of Q. The Lagrangian L is usually given by the kinetic minus the potential energy of the system as $L(q, v) = K(q, v) - U(q)$. The

Hamilton principle reads

$$\delta \int_0^T L(q, \dot{q}) \mathrm{d}t = 0\,, \tag{1.1.1}$$

for arbitrary variations δq with $\delta q(0) = \delta q(T) = 0$, and yields the Euler-Lagrange equations, given in coordinates as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, ..., n. \tag{1.1.2}$$

The Hamilton principle has a natural extension to continuum systems, for which the configuration manifold becomes infinite dimensional, typically a manifold of maps, and which will be of crucial use in this chapter. For instance, let us assume that the motion of the continuum system is described by a curve of embeddings $\varphi_t : \mathcal{B} \to \mathbb{R}^3$, where $\mathcal{B}$ is the reference configuration of the continuum. The current position at time t of the particle with label $X \in \mathcal{B}$ is $x = \varphi_t(X) \in \mathbb{R}^3$. In this case, the configuration manifold Q of the system is infinite dimensional and given by all smooth embeddings of $\mathcal{B}$ into $\mathbb{R}^3$, i.e., $Q = \mathrm{Emb}(\mathcal{B}, \mathbb{R}^3)$. Given a Lagrangian $L : TQ \to \mathbb{R}$, Hamilton's principle formally takes the same form as equation (1.1.1), namely

$$\delta \int_0^T L(\varphi, \dot{\varphi}) \mathrm{d}t = 0\,, \tag{1.1.3}$$

for variations $\delta\varphi$ such that $\delta\varphi(0) = \delta\varphi(T) = 0$. A detailed account of Hamilton's principle and its symmetry reduced versions in continuum mechanics can be found in [35]. Hamilton's principle could be extended to include irreversible processes. For a detailed presentation, the reader could be referred to [36].

While the method described by Euler-Lagrange equations (1.1.2) is elegant and widely used, it often needs appropriate extensions and developments to become practical. In order to illustrate this point, let us start with the derivation of perhaps the simplest possible mechanical model, namely, the rigid body moving about its fixed center of mass in space. While such a model may seem quite detached from the scope of the paper, the reader will note that our approach uses essentially the same method in spirit, so the understanding of this problem is useful for further study. A rigid body position is described by a 3×3 orientation matrix Λ satisfying $\Lambda^T\Lambda = \Lambda\Lambda^T = \mathrm{Id}_{3\times 3}$, or, in other words, the configuration space Q of a rigid body is the group $SO(3)$ of rotation matrices. A Lagrangian depending on the configurations and velocities, can be constructed and has the form $L(\Lambda, \dot{\Lambda})$. A naive application of the Hamilton's principle that

handles the holonomic constraints coming from $\Lambda\Lambda^T = \mathrm{Id}_{3\times 3}$ using Lagrange multipliers will lead to the Euler-Lagrange equations for 9 matrix coordinates of Λ, coupled with 6 constraints. While the total number of equations is 3, as expected, the equations of motion obtained by this method are excessively complex. One can parameterize the group $SO(3)$ using, for example, three Euler angles, in which case the critical action principle will give highly non-intuitive equations for these angles. It is, however, known, since the time of Euler, that such an approach is not fruitful. Instead, Euler has derived elegant equations of motion by going to the variables of angular velocity, which we today call the symmetry-reduced variables. In 1901, Poincaré [66] carried out a modern derivation of these equations, which we will briefly outline here.

The key to Poincaré's method is to notice that since the whole system is invariant with respect to arbitrary rotations of space, the Lagrangian should also be invariant with respect to such rotations. More precisely, for any fixed rotation matrix $A \in SO(3)$, we have $L(A\Lambda, A\dot{\Lambda}) = L(\Lambda, \dot{\Lambda})$. The fact that Λ is multiplied from the left by A comes from physics; as a rule, the elastic and rigid bodies have a left-invariant dynamical description. Then, the Lagrangian can be brought to a form that depends on the single variable $\omega = \Lambda^{-1}\dot{\Lambda}$, called the angular velocity *in the body frame*. Poincaré's method works, in fact, for any Lie group, not necessarily $SO(3)$, and has been useful for deriving the equations for complex systems consisting of interacting parts, fluids, liquid crystals and other components. The principle remains the same; only the group and configuration manifolds change. In these notes, we shall only consider the case when the configuration manifold is itself the symmetry group.

1.1.2 Introduction of external forces through Lagrange-d'Alembert's principle

Suppose there is external non-potential force F^{ext}, such as friction, acting on the system. This force has to be defined *a priori* and not derived from any equations of motion. Such force should depend on the position and velocity, $F^{\text{ext}} = F^{\text{ext}}(q, \dot{q})$, and is mathematically given by a fiber preserving map $F^{\text{ext}} : TQ \to T^*Q$, where fiber preservation simply means that for each fixed $q \in Q$, it restricts to a map from the fiber T_qQ to the fiber T_q^*Q, at the same point q. In other words, F^{ext} takes the value in the cotangent space and depends only on the value of q and $\dot{q}$, as expected from physics. For example, we do

not consider physical forces that depend on $\ddot{q}$ and higher derivatives, as such forces are difficult to justify from postulates of classical mechanics such as Newton's second law of motion. The aforementioned external forces F^{ext} should be be included in the augmented Hamilton principle, also called the *Lagrange-d'Alembert's principle*, by considering the modified action

$$\delta \int_0^T L(q,\dot{q})\mathrm{d}t + \int_0^T \langle F^{\text{ext}}, \delta q\rangle \mathrm{d}t = 0\,. \tag{1.1.4}$$

1.1.3 Holonomic constraints

Hamilton's principle can be naturally extended to include constraints, should they be holonomic or not. In the holonomic case, which is the case that we will need, the constraint defines a submanifold $N \subset Q$ of the configuration manifold. Assuming that $N = \Phi^{-1}(0)$, for a submersion $\Phi : Q \to \mathbb{R}^r$, the equations of motion follow from the Hamilton principle with Lagrange multipliers

$$\delta \int_0^T \big[L(q,\dot{q}) + \lambda_\alpha \Phi^\alpha(q)\big]\mathrm{d}t = 0\,, \tag{1.1.5}$$

in which one considers arbitrary variations $\delta\lambda_\alpha$. In this holonomic case, the equations of motion can also be directly obtained by applying the Hamilton principle to the Lagrangian L restricted to TN, but in most examples in practice, as it will be the case for us, the constraint submanifold N takes such a complicated expression that it is impossible to avoid the use of (1.1.5).

1.1.4 Lagrangian reduction by symmetry

When symmetry is available in a mechanical system, it is often possible to exploit it in order to reduce the dimension of the system and thereby to facilitate its study. This process, called *reduction by symmetry*, is well developed both on the Lagrangian and Hamiltonian sides; see [58] for an introduction and references.

While on the Hamiltonian side, this process is based on the reduction of symplectic or Poisson structures; on the Lagrangian side, it is usually based on the reduction of variational principles, see [22], [59], [60]. Consider a mechanical system with configuration manifold Q and Lagrangian $L : TQ \to \mathbb{R}$ and also consider the action of a Lie group G on Q, denoted here simply as $q \mapsto g \cdot q$, for $g \in G$, $q \in Q$. This action naturally

induces an action on the tangent bundle TQ, denoted here simply as $(q, v) \mapsto (g \cdot q, g \cdot v)$, called the *tangent lifted action*. We say that the action is a symmetry for the mechanical system if the Lagrangian L is invariant under this tangent lifted action. In this case, L induces a *symmetry reduced Lagrangian* $\ell : (TQ)/G \to \mathbb{R}$ defined on the quotient space $(TQ)/G$ of the tangent bundle with respect to the action. The goal of the Lagrangian reduction process is to derive the equations of motion directly on the reduced space $(TQ)/G$. Under standard hypotheses on the action, this quotient space is a manifold and one obtains the *reduced Euler-Lagrange equations* by computing the *reduced variational principle* for the action integral $\int_{t_1}^{t_2} \ell \, \mathrm{d}t$ induced by Hamilton's principle (1.1.1) for the action integral $\int_{t_1}^{t_2} L \, \mathrm{dt}$. The main difference between the reduced variational principle and Hamilton's principle is the occurrence of constraints on the variations to be considered when computing the critical curves for $\int_{t_1}^{t_2} \ell \, \mathrm{d}t$. These constraints are uniquely associated with the reduced character of the variational principle but not with physical constraints.

1.2 Euler-Poincaré equation for an arbitrary Lie group

We shall only consider the case when then configuration manifold is a Lie group G, and the invariance of the manifold is given with respect to the same Lie group. Suppose $Q = G$, and $L = L(g, \dot{g}),\ g \in G$. In the *left-invariant systems*, the Lagrangian satisfies the following property

$$L(g, \dot{g}) = L(hg, h\dot{g}),\ g, h \in G, \tag{1.2.1}$$

and can be reduced to $l = l(\Omega)$, so that

$$L(g, \dot{g}) = l(\Omega),\ \text{with } \Omega = g^{-1}\dot{g} \in \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G. The similar reduction is applicable to the *right-invariant case*, where

$$L(g, \dot{g}) = L(gh, \dot{g}h) = l(\Omega), \text{ with } \Omega = \dot{g}g^{-1} \in \mathfrak{g}. \tag{1.2.2}$$

Let us consider the left-invariant case; the right-invariant case is computed equivalently. Suppose also there is a pairing $\langle\ ,\ \rangle$ between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

Suppose $\delta g(t) \in T_g G$ for every t. Let us define $\Sigma = g^{-1}\delta g \in \mathfrak{g}$. Then, since δ and

time derivatives are derivatives with respect to different independent variables, we write

$$\begin{aligned}\dot{\Sigma} &= -g^{-1}\dot{g}g^{-1}\delta g + g^{-1}\delta\dot{g} = -\Omega\Sigma + g^{-1}\delta\dot{g}\\ \delta\Omega &= -g^{-1}\delta g g^{-1}\dot{g} + g^{-1}\delta\dot{g} = -\Sigma\Omega + g^{-1}\delta\dot{g}\end{aligned} \tag{1.2.3}$$

Then, we have

$$\delta\Omega = \dot{\Sigma} + \Omega\Sigma - \Sigma\Omega = \dot{\Sigma} + \mathrm{ad}_\Omega\Sigma, \tag{1.2.4}$$

where the adjoint representation $\mathrm{ad}_\Omega\Sigma$ is derived as the matrix commutator

$$\mathrm{ad}_\Omega\Sigma := \Omega\Sigma - \Sigma\Omega. \tag{1.2.5}$$

Then, the critical action principle is

$$\begin{aligned}0 = \delta\int \ell(\Omega)\mathrm{d}t = \int\left\langle\frac{\partial\ell}{\partial\Omega}\,,\,\delta\Omega\right\rangle\mathrm{d}t &= \int\left\langle\frac{\partial\ell}{\partial\Omega}\,,\,\dot{\Sigma}+\mathrm{ad}_\Omega\Sigma\right\rangle\mathrm{d}t\\ &= \int\left\langle-\frac{d}{dt}\frac{\partial\ell}{\partial\Omega} + \mathrm{ad}^*_\Omega\frac{\partial\ell}{\partial\Omega}\,,\,\Sigma\right\rangle\mathrm{d}t\end{aligned} \tag{1.2.6}$$

Since $\Sigma(t)$ is arbitrary, we get the evolution equation for the momentum

$$\dot{\Pi} \mp \mathrm{ad}^*_\Omega\Pi = 0\,, \quad \Pi := \frac{\partial\ell}{\partial\Omega}\,. \tag{1.2.7}$$

The choice of the sign in the equations above is as follows: minus sign for left-invariant dynamical systems as defined in (1.2.1) (such as rigid bodies, elastic materials) and plus sign for right-invariant systems as defined in (1.2.2), such as fluids.

Passing from the Hamilton principle and Euler-Lagrange equations to their symmetry reduced versions corresponds in practical examples to pass from the *material* (or *Lagrangian*) description to either the *spatial* (or *Eulerian*) description (in case of symmetries associated to actions on the right), or to the *convective* (or *body*) description (in case of symmetries associated to actions on the left), see [35]. Mixing of the two descriptions could also arise in physical settings where both left- and right- invariant components are presented.

1.3 Equations of motion with advected parameters

Let us consider the evolution equations for the case when in addition to the dynamics quantities like Ω in (1.2.7), there is another advected quantity a. Passively advected

quantity means that the evolution of a is 'slaved' to the evolution of Ω. Using the notations for previous section, evolution equation for a is

$$\dot{a} + \mathcal{L}_\Omega a = 0\,, \tag{1.3.1}$$

where $\mathcal{L}_\Omega$ is the Lie derivative with respect to vector field Ω. In case of heavy top motion, a will be the vector of the vertical direction seen from the body frame. In the case of porous media, a will be density. Suppose we have $\ell = \ell(\Omega, a)$. Then, equations of motion are derived from the variational principle

$$\delta \int \ell(\Omega, a)\mathrm{d}t = 0\,, \quad \delta\Omega = \dot{\Sigma} + \mathrm{ad}_\Omega \Sigma\,, \quad \delta a + \mathcal{L}_\Sigma a = 0 \tag{1.3.2}$$

The last condition on variation is obtained from the Lagrange-d'Alembert's principle of computing variations from equations of motion, namely, 'dots to deltas'. Since $\Omega = g^{-1}\dot{g}$, and $\Sigma = g^{-1}\delta g$, then changing ∂_t to δ in (1.3.1), we get exactly the variational principle (1.3.2). Then, performing variations, we have

$$\begin{aligned} 0 &= \int \left\langle \frac{\partial \ell}{\partial \Omega}, \delta\Omega \right\rangle + \left\langle \frac{\partial \ell}{\partial \Omega}, \delta a \right\rangle \mathrm{d}t \\ &= \int \left\langle \partial_t \frac{\partial \ell}{\partial \Omega} - \mathrm{ad}^*_\Omega \frac{\partial \ell}{\partial \Omega}, \Sigma \right\rangle + \left\langle \frac{\partial \ell}{\partial \Omega} \diamond a, \Sigma \right\rangle \mathrm{d}t \end{aligned} \tag{1.3.3}$$

where we have defined the *diamond* operator for arbitrary $a, u \in \mathfrak{g}$ and $\Pi \in \mathfrak{g}^*$

$$\langle \Pi \diamond a\,,\, u \rangle := -\langle \Pi, \mathcal{L}_u a \rangle \tag{1.3.4}$$

The equations of motion are then

$$\dot{\Pi} - \mathrm{ad}^*_\Omega \Pi + \Pi \diamond a = 0, \quad \dot{a} + \mathcal{L}_\Omega a = 0 \tag{1.3.5}$$

Next, we discuss the applications of the geometric variational principle to mechanical systems, described by various Lie groups and other important applications of the theory. We start with the famous example of the rigid body motion.

1.3.1 Variational derivation of Euler's equations for rigid body

Normally, the equations of motion of a rigid body are obtained using the balance of angular momentum in the body frame, see, for example, [53]. We have also included this derivation in Appendix C for completeness. In this section, we show how to derive this equation using the variational principle using the Euler-Poincaré method.

Let us now return to the question of a rigid body dynamics and consider a left-invariant Lagrangian $L(\Lambda, \dot{\Lambda})$ with respect to arbitrary rotations of space $\Lambda \in SO(3)$. As we mentioned, we can rewrite this Lagrangian as a function of the angular velocity only, *i.e.*, we have $L(\Lambda, \dot{\Lambda}) = \ell\big((\Lambda^{-1}\dot{\Lambda})^\vee\big) = \ell(\boldsymbol{\Omega})$ for a function ℓ defined on 3-vectors and given by the kinetic energy: $\ell(\boldsymbol{\Omega}) = \frac{1}{2}\mathbb{I}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$. How do we write the analogue of the Euler-Lagrange equations for the Lagrangian $\ell(\boldsymbol{\omega})$? If we write the variations of the action as

$$\delta\int_{t_0}^{t_1} L(\Lambda,\dot{\Lambda})\mathrm{d}t = \delta\int_{t_0}^{t_1}\ell(\boldsymbol{\Omega})\mathrm{d}t = \int_{t_0}^{t_1}\frac{\partial\ell}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega}\mathrm{d}t\,,$$

we need to compute the variations $\delta\boldsymbol{\Omega}$ that are induced by the variations $\delta\Lambda$. Defining $\Sigma = \Lambda^T\delta\Lambda$ which is also an antisymmetric matrix or, equivalently, its associated vector $\boldsymbol{\Sigma} = \Sigma^\vee$, we compute

$$\begin{aligned}
\delta\Omega &= \delta\Lambda^{-1}\dot{\Lambda} = \delta\left(\Lambda^{-1}\right)\dot{\Lambda} + \Lambda^{-1}\delta\dot{\Lambda} = -\Lambda^{-1}\delta\Lambda\Lambda^{-1}\dot{\Lambda} + \Lambda^{-1}\delta\dot{\Lambda} = -\Sigma\Omega + \Lambda^{-1}\delta\dot{\Lambda}\\
\dot{\Sigma} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Lambda^{-1}\delta\Lambda\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Lambda^{-1}\right)\delta\Lambda + \Lambda^{-1}\delta\dot{\Lambda}\\
&\qquad = -\Lambda^{-1}\dot{\Lambda}\Lambda^{-1}\delta\Lambda + \Lambda^{-1}\delta\dot{\Lambda} = -\Omega\Sigma + \Lambda^{-1}\delta\dot{\Lambda}.
\end{aligned} \tag{1.3.6}$$

In (1.3.6), we have used the fact that the δ derivative and the time derivative commute and

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{-1} = -A^{-1}\dot{A}A^{-1}\,,\quad \text{consequently,}\quad \delta A^{-1} = -A^{-1}\left(\delta A\right)A^{-1}\,,$$

since the variation δ is, formally, the derivative with respect to some parameter before setting the value of that parameter to 0. Subtracting the equations (1.3.6) to eliminate the cross-derivatives $\delta\dot{\Lambda}$, we obtain the expression for the variation of ω in terms of Σ as

$$\delta\Omega = \dot{\Sigma} + \big[\Omega,\Sigma\big] \quad\Leftrightarrow\quad \delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}\,. \tag{1.3.7}$$

Substitution of (1.3.7) into the variational principle, integrating by parts once and using that $\boldsymbol{\Sigma}(t_0) = \boldsymbol{\Sigma}(t_1) = 0$ as a consequence of $\delta\Lambda(t_0) = \delta\Lambda(t_1) = 0$, gives

$$\begin{aligned}
\delta\int_{t_0}^{t_1}\ell(\boldsymbol{\Omega})\mathrm{d}t &= \int_{t_0}^{t_1}\frac{\partial\ell}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\omega}\mathrm{d}t = \int_{t_0}^{t_1}\frac{\partial\ell}{\partial\boldsymbol{\Omega}}\cdot\left(\dot{\boldsymbol{\Sigma}} + \boldsymbol{\omega}\times\boldsymbol{\Sigma}\right)\mathrm{d}t\\
&= -\int_{t_0}^{t_1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ell}{\partial\boldsymbol{\Omega}} + \boldsymbol{\Omega}\times\frac{\partial\ell}{\partial\boldsymbol{\Omega}}\right)\cdot\boldsymbol{\Sigma}\mathrm{d}t\,.
\end{aligned} \tag{1.3.8}$$

Since $\boldsymbol{\Sigma}(t)$ is an arbitrary function of time, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ell}{\partial\boldsymbol{\omega}} + \boldsymbol{\Omega}\times\frac{\partial\ell}{\partial\boldsymbol{\Omega}} = \mathbf{0} \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{I}\boldsymbol{\Omega} = \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega}\,, \tag{1.3.9}$$

which are the well-known Euler equations for the motion of a rigid body. We would like to draw the attention of the reader to the fact that the function multiplying $\mathbf{\Sigma}$ in (1.3.8) is *exactly the angular momentum balance*. We refer the reader to Appendix C for the connection between the vectors and antisymmetric matrices we have employed here. This connection is known as the *hat map*.

Thus, the advantage of the variational derivation is that the angular and, as we shall see, the linear momentum balance are computed automatically through a well-defined procedure, no matter how complex the Lagrangian may be. In contrast, trying to compute the angular and linear momentum balance equations by equating terms from Newton's laws is, perhaps, extremely difficult, if not impossible, when the system is highly complex.

1.4 Application of Euler-Poincare theory for other Lie groups

In this section, we include the applications of Euler-Poincare principle to the dynamics on several Lie groups, that have applications in physics to demonstrate the convenience of the variational approach for finite-dimensional systems. The examples are taken from Darryl D Holm's lecture notes on Geometric Mechanics, part II, [1], see also [47]. The groups covered below include the Heisenberg group, Elliptic group, $SO(n)$, and $SE(3)$. For each of the groups above, the overview will contain a brief description of

1. The multiplication of elements for a Lie group G,
2. The group automorphism $\mathrm{AD}_B A := BAB^{-1}, \quad A, B \in G$,
3. Derivation of the adjoint representations for a Lie group (Ad) and corresponding Lie algebra (ad), defined respectively as the following operators:

$$\mathrm{Ad}_A b := \frac{d}{dt}\,\mathrm{AD}_A B(t)|_{t=0} \qquad \dot{B}(0) = b, \quad B(0) = \mathrm{Id}_{\mathrm{g}}, \text{ and} \tag{1.4.1}$$

$$\mathrm{ad}_a b := \frac{d}{dt}\,\mathrm{Ad}_A(t) b|_{t=0} \qquad \dot{A}(0) = a, \quad A(0) = \mathrm{Id}_{\mathrm{g}}, \tag{1.4.2}$$

[1] http://wwwf.imperial.ac.uk/~dholm/classnotes/

4. Illustration of group coadjoint (Ad^*) and coadjoint (ad^*) representations, using the pairing between elements of the Lie algebra $\mathfrak{g}$, and its dual $\mathfrak{g}^*$

$$\langle \mathrm{Ad}^*_A b, \alpha \rangle = \langle b, \mathrm{Ad}_A \alpha \rangle \ , \quad \forall b \in \mathfrak{g}, \quad \alpha \in \mathfrak{g}^*, \quad A \in G, \text{ and}$$

$$\langle \mathrm{ad}^*_a b, \alpha \rangle = \langle b, \mathrm{ad}_a \alpha \rangle \ , \quad \forall (a, b) \in \mathfrak{g}, \quad \alpha \in \mathfrak{g}^*,$$

5. Computation of Euler-Poincare equations, that will be written in terms of ad^*.

1.4.1 The Heisenberg Lie group

Definitions for the Heisenberg group

1. The elements of the Heisenberg Lie group are 3×3 real upper triangular matrices of the form

$$H = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \ a, b, c \in \mathbb{R} \right\}. \tag{1.4.3}$$

The set $H \subset SL(3, \mathbb{R})$ defines a Lie group under matrix multiplication. Let $A, B \in H$, namely

$$A = \begin{bmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & b_1 & b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.4.4}$$

The **group commutator** of elements A and B is

$$[A, B] := ABA^{-1}B^{-1} = \begin{bmatrix} 1 & 0 & a_1 b_2 - b_1 a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.4.5}$$

Adjoint actions: AD, Ad, and ad

1. The **group automorphism** is given by

$$\mathrm{AD}_B A = BAB^{-1} = \begin{bmatrix} 1 & a_1 & a_3 - a_1 b_2 + b_1 a_2 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.4.6}$$

2. Linearizing the group automorphism $\mathrm{AD}_B A$ in A at the identity, as defined in (1.4.1), yields the Ad operation,

$$\mathrm{Ad}_B \xi = B\xi|_{\mathrm{Id}} B^{-1} = B \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} B^{-1} = \begin{bmatrix} 0 & \xi_1 & \xi_3 + b_1 \xi_2 - b_2 \xi_1 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.4.7}$$

This is the Ad operation of the Heisenberg group H on its Lie algebra $\mathfrak{h}(\mathbb{R}) \simeq \mathbb{R}^3$:

$$\mathrm{Ad}\colon H(\mathbb{R}) \times \mathfrak{h}(\mathbb{R}) \to \mathfrak{h}(\mathbb{R}). \tag{1.4.8}$$

3. One defines the right-invariant tangent vector,

$$\xi = \dot{A}A^{-1} = \begin{bmatrix} 0 & \dot{a_1} & \dot{a_3} - a_2\dot{a_1} \\ 0 & 0 & \dot{a_2} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}, \tag{1.4.9}$$

and the left invariant tangent vector,

$$\Xi = A^{-1}\dot{A} = \begin{bmatrix} 0 & \dot{a_1} & \dot{a_3} - a_1\dot{a_2} \\ 0 & 0 & \dot{a_2} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Xi_1 & \Xi_3 \\ 0 & 0 & \Xi_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}. \tag{1.4.10}$$

Next, we linearize $\mathrm{Ad}_B\xi$ in B around the identity as defined in (1.4.2) to find the adjoint representation, i.e. ad operator of the Heisenberg Lie algebra on itself,

$$\mathrm{ad} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}. \tag{1.4.11}$$

Omitting the details of the computation, the result is given explicitly by

$$\mathrm{ad}_\eta\xi = [\eta, \xi] := \eta\xi - \xi\eta = \begin{bmatrix} 0 & 0 & \eta_1\xi_2 - \xi_2\eta_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.4.12}$$

Coadjoint actions: $\mathbf{Ad}^*$ and $\mathbf{ad}^*$

The inner product on the Heisenberg Lie algebra $\mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ is defined by the matrix trace pairing

$$\langle \eta, \xi \rangle = \mathrm{Tr}\left(\eta^T\xi\right) = \boldsymbol{\eta} \cdot \boldsymbol{\xi}. \tag{1.4.13}$$

Thus, elements of the dual Lie algebra $\mathfrak{h}^*(\mathbb{R})$ may be represented as lower triangular matrices,

$$\mu = \begin{bmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_3 & \mu_2 & 0 \end{bmatrix} \in \mathfrak{h}^*(\mathbb{R}). \tag{1.4.14}$$

The Ad^* operation of the Heisenberg group $H(\mathbb{R})$ on its dual Lie algebra $\mathfrak{h}^* \simeq \mathbb{R}^3$ is defined in terms of the matrix pairing by

$$\langle \mathrm{Ad}^*_B\mu, \xi \rangle := \langle \mu, \mathrm{Ad}_B\xi \rangle. \tag{1.4.15}$$

Explicitly, one may compute

$$\begin{aligned}
\langle \mu, \mathrm{Ad}_B \xi \rangle &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_3 & \mu_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \xi_1 & \xi_3 + b_1\xi_2 - b_2\xi_1 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&= \boldsymbol{\mu} \cdot \boldsymbol{\xi} + \mu_3(b_1\xi_2 - b_2\xi_1) \\
&= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ \mu_1 - b_2\mu_3 & 0 & 0 \\ \mu_3 & \mu_2 + b_1\mu_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&= \langle \mathrm{Ad}_B^* \mu, \xi \rangle.
\end{aligned} \tag{1.4.16}$$

Thus, the formula for $\mathrm{Ad}_B^* \mu$ reads

$$\mathrm{Ad}_B^* \mu = \begin{bmatrix} 0 & 0 & 0 \\ \mu_1 - b_2\mu_3 & 0 & 0 \\ \mu_3 & \mu_2 + b_1\mu_3 & 0 \end{bmatrix}. \tag{1.4.17}$$

The ad^* operation of the Heisenberg Lie algebra $\mathfrak{h}$ on its dual $\mathrm{l}\mathfrak{h}$ is defined similarly in terms of the matrix pairing by

$$\langle \mathrm{ad}_\eta^* \mu, \xi \rangle := \langle \mu, \mathrm{ad}_\eta^* \xi \rangle. \tag{1.4.18}$$

Computing the expression on the right hand side of the definition above yields

$$\begin{aligned}
\langle \mu, \mathrm{ad}_\eta^* \xi \rangle &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_3 & \mu_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \eta_1\xi_2 - \xi_1\eta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&= \mu_3(\eta_1\xi_2 - \eta_2\xi_1) \\
&= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ -\eta_2\mu_3 & 0 & 0 \\ 0 & \eta_1\mu_3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&= \langle \mathrm{ad}_\eta^* \mu, \xi \rangle,
\end{aligned} \tag{1.4.19}$$

so the formula for $\mathrm{ad}_\eta^* \mu$ reads

$$\mathrm{ad}_\eta^* \mu = \begin{bmatrix} 0 & 0 & 0 \\ -\eta_2\mu_3 & 0 & 0 \\ 0 & \eta_1\mu_3 & 0 \end{bmatrix}. \tag{1.4.20}$$

Coadjoint motion equation for the path in the Heisenberg Lie group H

Let $A(t)$ be a path in the Heisenberg Lie group H and $\mu(t)$ be a path in $\mathfrak{h}^*$. Then we compute

$$\frac{d}{dt}\left(\mathrm{Ad}_{A(t)^{-1}}^* \mu(t)\right) = \mathrm{Ad}_{A(t)^{-1}}^* \left[\frac{d\mu}{dt} - \mathrm{ad}_{\eta(t)}^* \mu(t) \right] \tag{1.4.21}$$

with $\eta(t) = A(t)^{-1}\dot{A}(t)$. The differential equation for the coadjoint orbit reads

$$\mu(t) = \mathrm{Ad}^*_{A(t)}\mu(0), \tag{1.4.22}$$

and the desired coadjoint motion equation

$$\dot{\mu} = \mathrm{ad}^*_\eta \mu. \tag{1.4.23}$$

If we define the linear map $\mathfrak{h} \to \mathfrak{h}^* : (\mu_1, \mu_2) = (I_1\eta_1, I_2\eta_2)$, the equation takes the form

$$\frac{d}{dt}(\mu_1, \mu_2, \mu_3) = \left(-\frac{\mu_3\mu_2}{I_2}, \frac{\mu_1\mu_3}{I_1}, 0\right). \tag{1.4.24}$$

Therefore, planar isotropic harmonic oscillators describe coadjoint orbits on the Heisenberg Lie group.

1.4.2 Special orthogonal group SO(n)

To describe the dynamics of a system (further referred in this subsections as the *rigid body*) with configurations $O \in SO(n)$, we derive Euler's equations in the matrix commutator form, similar to regular rigid dynamics on $SO(3)$. The reduced Lagrangian of the rigid body, depending only on angular velocity Ω, has the form

$$l(\Omega) = -\frac{1}{2}\mathrm{Tr}\left(\Omega\mathbb{A}\Omega\right),$$

where $\Omega = O^{-1}\dot{O} \in \mathfrak{so}(n)$, a skew-symmetric matrix, and $\mathbb{A}$ is symmetric.

Using Hamilton's principle, we compute the variation of action,

$$\delta S = \delta\int_a^b l(\Omega)dt = -\frac{1}{2}\int_a^b \mathrm{Tr}\left(\delta\Sigma(\mathbb{A}\Omega + \Omega\mathbb{A})\right)dt = -\frac{1}{2}\int_a^b \mathrm{Tr}\left(\delta\Omega M\right)dt,$$

where we cyclically permuted the order of matrix multiplication under the trace and substituted $M \leftarrow \mathbb{A}\Omega + \Omega\mathbb{A}$. Using the variational formula (1.2.4) that applies for $\delta\Omega$ with $\Sigma = O^{-1}\delta O$ the variation takes the form

$$\delta S = -\frac{1}{2}\int_a^b \mathrm{Tr}\left((\dot{\Sigma} + \Omega\Sigma - \Sigma\Omega)M\right)dt.$$

Integrating by parts and permuting under the trace then yields the equation

$$\delta S = \frac{1}{2}\int_a^b \mathrm{Tr}\left(\Sigma(\dot{M} + \Omega M - M\Omega)\right)dt.$$

Finally, invoking stationary action principle for arbitrary Σ, implies the

Theorem 1.4.1 *(Manakov's proposition) Euler's equations for a rigid body on $SO(n)$ take the matrix commutator form,*

$$\frac{dM}{dt} = [M, \Omega] \text{ with } M = \mathbb{A}\Omega + \Omega\mathbb{A}. \tag{1.4.25}$$

The interesting fact is that the equations of motion in $SO(4)$ are integrable, according to Manakov. The reader can be referred to [57] to familiarize themselves with the proof. Manakov's method is extendable, and in fact, one can prove the integrability of all the rigid bodies on $SO(n)$. The moments of inertia of these bodies possess only $2n-3$ parameters, and they are algebraically solvable.

1.4.3 Special Euclidean group SE(3)

Definitions for the Special Euclidean group

The special Euclidean group corresponds to configurations of rigid bodies in three dimensions and is defined as the semi-direct product $\mathrm{SE}(3) := SO(3)\circledS\mathbb{R}^3$, i.e. it acts on $\mathbb{R}^3$ by rotations and translations $x \to Rx + v$, where $R \in SO(3)$, a matrix in the special orthogonal group, corresponding to rotational configuration of the rigid body and $v \in \mathbb{R}^3$ corresponds to translations in space. The general case of the special orthogonal group $SO(n)$ was discussed in the paragraph above, and the special case for 3 dimensions was considered in (1.3.1). The group action may be represented by the matrix multiplication as follows

$$(R, v)\begin{bmatrix} x \\ 1 \end{bmatrix} := \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + v \\ 1 \end{bmatrix}. \tag{1.4.26}$$

The group operation in $SE(3)$ is therefore equivalent to the matrix multiplication and in the so-called top-row notation reads

$$(R_2, v_2)(R_1, v_1) = (R_2R_1, R_2v_1 + v_2). \tag{1.4.27}$$

Adjoint actions: AD, Ad, and ad

The group adjoint operation

$$\mathrm{AD}\colon SE(3) \times SE(3) \to SE(3)$$

is conveniently expressed in the top-row notation as

$$\begin{aligned}\mathrm{AD}_{(R,v)}(\tilde{R},\tilde{v}) &= (R,v)(\tilde{R},\tilde{v})(R,v)^{-1}\\ &= (R,v)(\tilde{R},\tilde{v})(R^{-1},-R^{-1}v)\\ &= (R,v)(\tilde{R}R^{-1},\tilde{v}-\tilde{R}R^{-1}v)\\ &= (R\tilde{R}R^{-1},v+R\tilde{v}-R\tilde{R}R^{-1}v).\end{aligned} \tag{1.4.28}$$

The Ad operation is obtained by taking time derivatives of $\tilde{R}$ and $\tilde{v}$ at the identity $(\tilde{R(t)},\tilde{v(t)})|_{t=0} = (\,\mathrm{Id}\,,0)$. This yields

$$\mathrm{Ad}_{(R,v)}(\dot{\tilde{R}}(0),\dot{\tilde{v}}(0)) = (\mathrm{Ad}_R\dot{\tilde{R}}(0), -\mathrm{Ad}_R\dot{\tilde{R}}(0)v + R\dot{\tilde{v}}(0)). \tag{1.4.29}$$

Setting $\dot{\tilde{R}}(0) = \xi$ and $\dot{\tilde{v}}(0) = \alpha$ defines the Ad action of $SE(3)$ on its Lie algebra with elements $(\xi,\alpha) \in \mathfrak{se}(3)$ as $\mathrm{Ad} : SE(3)\times\mathfrak{se}(3)\to\mathfrak{se}(3)$,

$$\begin{aligned}\mathrm{Ad}_{(R,v)}(\xi,\alpha) &= (\mathrm{Ad}_R\xi, -\mathrm{Ad}_R\xi v + R\alpha)\\ &= (R\xi R^{-1}, -R\xi R^{-1}v + R\alpha),\end{aligned} \tag{1.4.30}$$

The adjoint action ad of $se(3)$ onto itself is expressed as the time derivatives of Ad at the identity,

$$\begin{aligned}&\mathrm{ad}_{(\dot{R}(0),\dot{v}(0))}(\xi,\alpha)\\ &= (\dot{R}\xi R^{-1} - \dot{R}\tilde{\xi}R^{-1}\dot{R}R^{-1} - \dot{R}\tilde{\xi}R^{-1}v + R\tilde{\xi}R^{-1}\dot{R}R^{-1}v - R\tilde{\xi}R^{-1}\dot{v} + \dot{R}\tilde{\alpha})|_{\,\mathrm{Id}}\,.\end{aligned} \tag{1.4.31}$$

As before, we set $\dot{R}(0) = \xi$, $\dot{v}(0) = \alpha$, with $(R(0), v(0) = (\,\mathrm{Id}\,,0)$. In this notation, the ad operation for the right-invariant Lie algebra action of $se(3)$ may be rewritten as

$$\begin{aligned}\mathrm{ad}_{(\xi,\alpha)}(\tilde{\xi},\tilde{\alpha}) &= (\xi\tilde{\xi} - \tilde{\xi}\xi, -(\xi\tilde{\xi} - \tilde{\xi}\xi)v - \tilde{\xi}(\xi v + \tilde{\alpha}) + \xi\tilde{\alpha})|_{\,\mathrm{Id}}\\ &= ([\xi,\tilde{\xi}], -\xi\tilde{\xi}v + \xi\tilde{\alpha} - \tilde{\xi}\tilde{\alpha})|_{\,\mathrm{Id}}\\ &= (\mathrm{ad}_\xi\tilde{\xi}, \xi\tilde{\alpha} - \tilde{\xi}\alpha).\end{aligned} \tag{1.4.32}$$

Coadjoint actions: Ad* and ad*

The pairing $\langle\cdot,\cdot\rangle : se(3)^*\times se(3) \Rightarrow \mathbb{R}$ is obtained by the identification $SE(3)\simeq SO(3)\times\mathbb{R}^3$ and taking the sum

$$\langle(\mu,\beta),(\xi,\alpha)\rangle := \frac{1}{2}\mathrm{Tr}\left(\mu^T\xi\right) + \beta\cdot\alpha, \tag{1.4.33}$$

with $\mu \in so(3)^*$, $\xi \in so(3)$, skew-symmetric matrices and $\beta, \alpha \in \mathbb{R}^3$, in vector notation. One computes Ad^* operation as

$$\begin{aligned}\left\langle \mathrm{Ad}^*_{(R,v)^{-1}}(\mu,\beta),(\xi,\alpha)\right\rangle &= \left\langle (\mu,\beta), \mathrm{Ad}_{(R,v)^{-1}}(\xi,\alpha)\right\rangle \\ &= \left\langle (\mu,\beta),(R^{-1}\xi R, R^{-1}\alpha + R^{-1}\xi v)\right\rangle \\ &= \left\langle \mu, R^{-1}\xi R\right\rangle + \left\langle \beta, R^{-1}\alpha + R^{-1}\xi v\right\rangle \\ &= \left\langle R\mu R^{-1}, \xi\right\rangle + \langle R\beta, \alpha + \xi v\rangle\end{aligned} \tag{1.4.34}$$

The second term of the last pairing in the equation above could be rewritten as

$$\langle R\beta, \xi v\rangle = \langle \mathrm{skew}(v \otimes R\beta), \xi\rangle\,, \tag{1.4.35}$$

where skew is the skew-symmetric part of the matrix

$$\mathrm{skew}A := (A - A^T)/2. \tag{1.4.36}$$

Coadjoint motion equation for the path in the $SE(3)$

One computes the ad^* action of $se(3)$ on its dual $se(3)^*$ by using the pairing,

$$\begin{aligned}\left\langle \mathrm{ad}^*_{(\xi,\alpha)}(\mu,\beta),(\tilde{\xi},\tilde{\alpha})\right\rangle &= \left\langle (\mu,\beta), \mathrm{ad}_{(\xi,\alpha)}(\tilde{\xi},\tilde{\alpha})\right\rangle \\ &= \left\langle (\mu,\beta),(\mathrm{ad}_\xi\tilde{\xi}, \xi\tilde{\alpha} - \tilde{\xi}\alpha)\right\rangle \\ &= \left\langle \mu, \mathrm{ad}_\xi\tilde{\xi}\right\rangle + \langle \beta, \xi\tilde{\alpha}\rangle - \left\langle \beta, \tilde{\xi}\alpha\right\rangle \\ &= \left\langle \mathrm{ad}^*_\xi\mu, \tilde{\xi}\right\rangle + \langle -\xi\beta, \tilde{\alpha}\rangle + \langle \beta\diamond\alpha, \tilde{\xi}\rangle \\ &= \left\langle (\mathrm{ad}^*_\xi\mu + \beta\diamond\alpha, \xi\beta),(\tilde{\xi},\tilde{\alpha})\right\rangle,\end{aligned} \tag{1.4.37}$$

where the **diamond operation** arises by the dual Lie algebra actions,

$$\langle \beta\diamond\alpha, \tilde{\xi}\rangle = -\langle \beta, \tilde{\xi}\alpha\rangle.$$

1.5 Continuum mechanics using geometric variational principle

Most of discussion in this thesis will pertain to the variational derivation of combined behavior of fluid and elastic media. For physical examples I consider in this thesis, it is reasonable to assume that the fluid is incompressible, such as water in normal conditions. We shall also later assume the incompressibility of elastic material for the biological media. For that reason, it is useful to develop the introductory exposition of an incompressible fluid, and elastic media, which we shall do in this Section.

1.5.1 Arnold's derivation of inviscid, incompressible Euler equations

We follow the framework developed by V. I. Arnold [1], see also [48] for a more detailed and expanded interpretation and other examples of fluid models. The fluid flow is defined as a smooth invertible time-dependent transformation of initial conditions $\boldsymbol{X} \in M$ at $t = 0$, regarded as fluid particle labels taking values in a configuration manifold $M \subseteq \mathbb{R}^3$ acted on by smooth invertible maps $\mathbf{Diff}(M)$. Thus, we lift the motion of fluid parcels $\mathbf{x} \in M$ with initial condition $\boldsymbol{X} \in M$ to the manifold of diffeomorphisms by identifying it with a time-dependent curve $\boldsymbol{\varphi}_t \in \mathbf{Diff}(M)$ with $\boldsymbol{\varphi}_t|_{t=0} = \mathrm{Id}$, whose action from the left generates the motion $\mathbf{x} = \boldsymbol{\varphi}_t \boldsymbol{X}$. We can introduce an alternative notation using the transformation $\boldsymbol{\varphi}_t$, as the "embeddings" $\varphi(\boldsymbol{X}, t) := \boldsymbol{\varphi}_t \boldsymbol{X}$. So, the equation

$$\mathbf{x} = \varphi(\boldsymbol{X}, t),$$

is the Lagrangian mapping starting at $\boldsymbol{X}$. The incompressibility constraint preserves volumes, so the Jacobian of the transformation satisfies the condition

$$J(\mathbf{x}, t) := \det \left| \frac{\partial \varphi}{\partial \boldsymbol{X}} \circ \varphi^{-1}(\mathbf{x}, t) \right| = 1 \,, \tag{1.5.1}$$

where φ^{-1} is the inverse transformation, i.e. $\varphi\left(\varphi^{-1}(\mathbf{x}, t), t\right) = \mathrm{Id}$. Suppose $p = p(\mathbf{x}, t)$ is the Lagrange multiplier for the incompressibility constraint (1.5.1).

The Eulerian velocity, also known as the spatial velocity of the fluid is given by

$$\boldsymbol{u}(\mathbf{x}, t) = \dot{\varphi} \circ \varphi^{-1}(\mathbf{x}, t) \tag{1.5.2}$$

and its variation reads $\boldsymbol{\eta}(\mathbf{x}, t) = \delta\varphi \circ \varphi^{-1}$. Notice that if we replace $\boldsymbol{\varphi}_t$ by $\boldsymbol{\varphi}_t \circ \boldsymbol{\varphi}_0$ for a fixed (time-independent) $\boldsymbol{\varphi}_0 \in \mathbf{Diff}(M)$, then Eulerian velocity will not change; this reflects the right invariance of the Eulerian description ($\boldsymbol{u}$ is invariant under composition of $\boldsymbol{\varphi}_t$ by an element of $\mathbf{Diff}(M)$ on the right). This is also called the particle relabeling symmetry of fluid dynamics.

The variations of $\boldsymbol{\eta}$ and $\boldsymbol{u}$ are given by

$$\delta J + \mathrm{div}(\eta J) = 0 \,, \quad \delta \boldsymbol{u} = \boldsymbol{\eta}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{u} \,. \tag{1.5.3}$$

Equations of motion are obtained from the variational principle. The action integral S is written in the same way as in [46] and uses the augmented Lagrangian, consisting

of the kinetic energy and the incompressibility constraint, multiplied by p,

$$S[\boldsymbol{u}, p, J] := \int_{t_0}^{t_1} \mathrm{d}t \int_M \frac{1}{2}\rho J(\mathbf{x},t)|\boldsymbol{u}(\mathbf{x},t)|^2 - p(\mathbf{x},t)(J(\mathbf{x},t)-1)\mathrm{d}\mathbf{x}, \quad \rho = \text{const.} \tag{1.5.4}$$

Here, we have assumed $\rho = \text{const}$ which makes $\boldsymbol{u}$ satisfy the incompressibility condition

$$\mathrm{div}\boldsymbol{u} = 0\,. \tag{1.5.5}$$

The variation of the action according to the Lagrange-d'Alembert principle is

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} \mathrm{d}t \int_M \frac{1}{2}\rho J\boldsymbol{u}\cdot\delta\boldsymbol{u} - (J-1)\delta p + \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 - p\right)\,\delta J\,\mathrm{d}\mathbf{x} \\
&= \int_{t_0}^{t_1} \mathrm{d}t \int_M \frac{1}{2}\rho J\boldsymbol{u}\cdot(\boldsymbol{\eta}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{u}) \\
&\qquad\qquad - (J-1)\delta p - \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 - p\right)\,\mathrm{div}(J\boldsymbol{\eta})\,\mathrm{d}\mathbf{x} \\
&= \int_{t_0}^{t_1} \mathrm{d}t \int_M -\rho\cdot\left((J\boldsymbol{u})_t + \nabla_j\left(u^j u^i J\right) + \frac{1}{2}\rho J\nabla|\boldsymbol{u}|^2\right) \\
&\qquad\qquad - (J-1)\delta p + \left(\frac{1}{2}\rho J\nabla|\boldsymbol{u}|^2 - \nabla p\right)\cdot\boldsymbol{\eta}\,\mathrm{d}\mathbf{x} \\
&= \int_{t_0}^{t_1} \mathrm{d}t \int_M -\boldsymbol{\eta}\rho\cdot(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\mathrm{div}\boldsymbol{u} + \nabla p)\cdot\boldsymbol{\eta} - (J-1)\delta p\,\mathrm{d}\mathbf{x} = 0.
\end{aligned} \tag{1.5.6}$$

Setting the term proportional to δp to vanish simply enforces $J = 1$ constraint. Setting $J = 1$ and enforcing incompressibility conditions (1.5.5), and further collecting the coefficients proportional to $\boldsymbol{\eta}$ gives the Euler equations for ideal incompressible fluid

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p\,, \quad \mathrm{div}\boldsymbol{u} = 0\,. \tag{1.5.7}$$

Notice, that external forces can be trivially included in the equation (1.5.7) above, as they do not affect the dynamic terms from the variational principle. In addition to this, one can include additional dissipation due to viscosity.

1.5.2 Variational derivation of Elasticity equations in spatial coordinates.

Elastic bodies are usually computed in a left-invariant framework, as opposed to right-invariant systems used for fluid dynamics. Physically, the equivalent statement is that the equations of the elastic body are usually written in the Lagrangian and not the Eulerian frame. The equations of motion for a fluid (1.5.7), on the other hand, are

written in the Eulerian frame. We will use the Eulerian description to compute equations of a porous media containing fluid since the equations of motion of the fluid are most conveniently written in the spatial frame. Thus, the derivation of equations of motion for an elastic body is of relevance to the main topic of this thesis.

The configuration of elastic body is described by embeddings $\Psi = \Psi(\mathbf{x}, t)$. The spatial velocity is expressed as

$$\boldsymbol{u}_s(t, \mathbf{x}) = \partial_t \boldsymbol{\Psi}\big(t, \boldsymbol{\Psi}^{-1}(t, \mathbf{x})\big)\,, \tag{1.5.8}$$

and the density ρ_s is advected by the motion of solid.

The Lagrangian is given by

$$\ell(\boldsymbol{u}_s, \rho_s, b) = \int_{\mathcal{B}} \left[\frac{1}{2}\rho_s |\boldsymbol{u}_s|^2 - V(b)\right] \mathrm{d}^3\mathbf{x}\,. \tag{1.5.9}$$

In the right-invariant framework, we need to use Finger tensor b. Then, using the definition of the diamond,

$$\int_{\mathcal{B}_t} \left\langle \frac{\delta \ell}{\delta b}, \delta b \right\rangle \mathrm{d}^3\mathbf{x}\,\mathrm{d}t = \int_{\mathcal{B}_t} \left\langle \frac{\delta \ell}{\delta b}, -\pounds_{\boldsymbol{\eta}} b \right\rangle \mathrm{d}^3\mathbf{x}\,\mathrm{d}t = \int_{\mathcal{B}_t} \left\langle \frac{\delta \ell}{\delta b} \diamond b, \boldsymbol{\eta} \right\rangle \mathrm{d}^3\mathbf{x}\,\mathrm{d}t.$$

We compute the diamond operator explicitly using the expression for Lie derivatives of $(2,0)$ tensor b in Cartesian coordinates

$$(\Pi \diamond b)_k = -\Pi_{ij} \frac{\partial b^{ij}}{\partial x^k} - 2\frac{\partial}{\partial x^i}\left(\Pi_{kj} b^{ij}\right) \tag{1.5.10}$$

whose coordinate-free form reads

$$\Pi \diamond b = -\Pi : \nabla b - 2\mathrm{div}\left(\Pi \cdot b\right)\,. \tag{1.5.11}$$

The equations of motion also naturally involve the expression of the Lie derivative of a momentum density, whose global and local expressions are

$$\begin{aligned} \pounds_{\boldsymbol{u}} \boldsymbol{m} &= \boldsymbol{u} \cdot \nabla \boldsymbol{m} + \nabla \boldsymbol{u}^{\mathsf{T}} \cdot \boldsymbol{m} + \boldsymbol{m}\,\mathrm{div}\,\boldsymbol{u} \\ (\pounds_{\boldsymbol{u}} \boldsymbol{m})_i &= \partial_j m_i u^j + m_j \partial_i u^j + m_i \partial_j u^j\,. \end{aligned} \tag{1.5.12}$$

Proceeding with the computation of variations according to the Lagrange-d'Alembert principle, we derive

$$\delta S = \int_{\mathcal{B}_t} \left[\frac{\delta \ell}{\delta \boldsymbol{u}_s} \cdot \delta \boldsymbol{u}_s + \frac{\delta \ell}{\delta \rho_s}\delta\rho_s + \frac{\delta \ell}{\delta b} : \delta b\right] \mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0\,. \tag{1.5.13}$$

After substitution of functional derivatives and corresponding variations, the equation of dynamics takes the form

$$\partial_t \frac{\delta \ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s} \frac{\delta \ell}{\delta \boldsymbol{u}_s} = \rho_s \nabla \frac{\delta \ell}{\delta \rho_s} + \frac{\delta \ell}{\delta b} \diamond b. \tag{1.5.14}$$

The equations of motion can be further simplified to

$$\begin{cases} \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = \nabla V + 2 \operatorname{div} \dfrac{\partial V}{\partial b} \cdot b \\ \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0. \end{cases} \tag{1.5.15}$$

Equations (1.5.15) are equations of motion for elastic body in the spatial coordinates (Eulerian frame).

One can write (1.5.15) in a more familiar form by defining a tensor

$$\sigma = V \operatorname{Id} + 2 \frac{\partial V}{\partial b} \cdot b \,. \tag{1.5.16}$$

The physical meaning of σ is the stress tensor written in the Eulerian frame. Then, equations of motion for an elastic body (1.5.15) become

$$\begin{cases} \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = \operatorname{div} \sigma \\ \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0. \end{cases} \tag{1.5.17}$$

External forces can be included in the equation above, as they do not affect the dynamic terms from the variational principle. In addition to this, one can include dissipation in the resulting system (1.5.16) using viscosity assumptions.

Chapter 2

Geometric variational approach to the dynamics of porous medium, filled with incompressible fluid

This chapter represents the derivation of the equations of motion for the dynamics of a porous media filled with an incompressible fluid. The presentation follows the approach outlined in [32], where we use the geometric variational method. The Lagrangian of the porous media is the sum of terms representing the kinetic and potential energy of the elastic matrix, and the kinetic energy of the fluid, coupled through the constraint of incompressibility. As an illustration of the method, the equations of motion for both the elastic matrix and the fluid are derived in the spatial (Eulerian) frame. Such an approach is of relevance *e.g.* for biological problems, such as sponges in water, which is discussed further in this thesis. In biological applications, the elastic porous media is highly flexible, and the motion of the fluid has a 'primary' role in the dynamics of the whole system.

2.1 Introduction

The coupled dynamics of porous media filled with fluid, also known as *poromechanics*, has been the subject of an active research for many decades. The foundational works in the area were driven by applications to soil dynamics and geophysics, whereas in the latter years the applications also included biomedical fields. The earlier developments were associated with the works of K. von Terzaghi [77] and M. Biot [9], [10], [15] in the consolidation of porous media, and subsequent works by M. Biot which derived the

time-dependent equations of motion for poromechanics, based on certain assumptions on the media, and considered the wave propagation in both low and high wavenumber regime [11]–[14]. There has been substantial amount of new work in the field of porous media, see [19], [21], [30], [31], [44], [49] and subsequent mathematical analysis of the models [6], [16], [76]. We refer the reader interested in the history of the field to the review [67] for a more detailed exposition of the literature.

While Biot's equations, especially with respect to acoustic propagation in porous media, still remain highly influential today, subsequent investigations have revealed difficulties in the interpretation of various terms through the general principles of mechanics, such as material objectivity, frequency-dependent permeability and changes of porosity in the model, as well as the need to describe large deformations of the model [82]. The above-cited paper then proceeds in outlining a detailed derivation for the modern approach to saturated porous media equations which does not have the limitations of the Biot's model. We shall also mention here two recent papers [23], [24] where the equations for saturated porous media were further developed based on the general thermodynamics principles of mechanics.

By their very nature, variational methods involve fully nonlinear treatment of the inertial terms. The mainstream approach to the porous media has been to treat the dynamics as being friction-dominated by dropping the inertial terms from the equations. The equations we will derive here, without the viscous terms, will be of infinite-dimensional Hamiltonian type. On the other hand, the friction-dominated approach gives equations of motion that are of gradient flow type. The seminal book of Coussy [26] contains a lot of background information and analysis. For more recent work, we will refer the reader to, for example, the studies of multi-component porous media flow [70], as well as the gradient approach to the thermo-poro-visco-elastic processes [18].

Fluid-filled elastic porous media, by its very nature, is a highly complex object involving both the individual dynamics of fluid and media, and a highly nontrivial interactions between them. The pores in the elastic matrix, and the fluid motion inside them, are micro-structured elements that contribute to the macro-structured dynamics. Thus, the porous media must include the interaction between the large scale dynamics and an accurate, and yet treatable, description of micro-structures. It has long been known in mechanics that variational principles are ideally suited to treat complex, multi-

component systems. Variational methods proceed formally by describing the Lagrangian of the system on an appropriate configuration manifold, and proceeding with variations to obtain the equations of motion in a systematic way. The advantage of the variational methods is their consistency, as opposed to the theories based on balancing the conservation laws for a given point, or volume, of fluid. In a highly complex system like poromechanics, especially when written in the non-inertial Lagrangian frame associated with the matrix, writing out all the forces and torques to obtain correct equations is very difficult. In contrast, the equations of motion, as well as the conservation laws, come out of variational methods automatically without the need to find all the forces and torques involved. Thus, porous media looks like an ideal application for applications of variational principles. Before we proceed further, however, we would like to give a verbatim quote of an inspiring sentence from the conclusion of [82]:

It seems to be also clear that it is a waste of effort to try to construct a true variational principle as the Biot model contains a nonequilibrium variable, the increment of fluid contents which rules out the existence of such a principle.

In spite of this difficulty, variational methods were actively applied to the field poromechanics. One of the earliest papers papers in the field was [7] where the kinetic energy of expansion was incorporated into the Lagrangian to obtain the equations of motion. In that work, several Lagrange multipliers were introduced to enforce the continuity equation for both solid and fluid. The works [3], [4] use variational principles for explanation of the Darcy-Forchheimer law. Furthermore, [54], [55] derive the equations of porous media using additional terms in the Lagrangian coming from the kinetic energy of the microscopic fluctuations. Of particular interest to us are the works on the Variational Macroscopic Theory of Porous Media (VMTPM) which was formulated in its present form in [2], [28], [29], [68], [69], [71]–[73], [75], [78], also summarized in a recent book [74]. In these works, the microscopic dynamics of capillary pores is modelled by a second grade material, i.e. such material, where the internal energy of the fluid depends on both the deformation gradient of the elastic media, and the gradients of local fluid content. In contrast to this approach, we tried to avoid the second-grade assumptions in our research and used only explicit dependencies. The study of a pre-stressed system using variational principles and subsequent study of propagation of sound waves was undertaken in [65].

One of the main assumptions of the VMTPM is the dependence of the internal energy of the fluid on the quantity measuring the micro-strain of the fluid, or, alternatively, the fluid content or local density of fluid, including, in some works, the gradients of that quantity. This assumption is physically relevant for compressible fluid, but, in our view, for an incompressible fluid (which, undoubtedly, is a mathematical abstraction), such dependence is difficult to interpret. For example, for geophysical applications, fluids are usually considered compressible because of the large pressures involved. In contrast, for biological applications like the dynamics of highly porous sponges in water, the compressibility effects can be neglected. For a truly incompressible fluid, it is difficult to assign a physical meaning to the dependence of internal energy of the fluid on the parameters of the porous media. We refer the reader to the the classical Arnold's description of incompressible fluid [1] as geodesic motion on the group of volume-preserving diffeomorphism in the three-dimensional space, in the absence of external forces. In that theory the Lagrangian is simply the kinetic energy, as the potential energy of the fluid is absent, and the fluid pressure enters the equations from the incompressibility condition. The main result of the present paper is to extend this geometric description to the motion of the fluid-filled porous media, for the case when the fluid inside the pores is incompressible, and, neglecting all thermal effects, without considering the internal energy of fluid.

Before we delve into detailed derivations, it is useful to have a discussion on the physics of what is commonly considered the saturated porous media. In most, if not all, previous works, the saturated porous media is a combined object consisting of an (elastic) dense matrix, and a network of small connected pores filled with fluid. The fluid encounters substantial resistance when moving through the pores due to viscosity and the no-slip condition on the boundary. In such a formulation, it is easier to consider the motion of the porous matrix to be 'primary', and the motion of the fluid to be computed with respect to the porous matrix itself. Because the motion of the elastic matrix is 'primary', the equations are written in the system of coordinates consistent with the description of the elastic media, which is the material frame associated with the media. In this paper, we take an alternative view where we choose the same coordinate system of the stationary observer (Eulerian frame) for the description of both the fluid and the elastic media. Such system is more frequently used in the classical fluid description,

but is less common in the description of elastic media. Physically, our description is more relevant for the case of a porous media consisting of a dense network of elastic 'threads' positioned inside the fluid, which is a case that has not been considered before. In our formulation, we choose the Eulerian description for both the fluid and the elastic matrix. It is worth noting that the combined Eulerian description is also applicable to the regular porous media with a 'dense' matrix, and is also well suited for the description of wave propagation in such media. Finally, we shall also point out that our theory can be reformulated and is applicable for the familiar choice of the Lagrangian material description with respect to the elastic porous matrix. These descriptions are completely equivalent from the mathematical point of view, and this is rigorously justified by using the process of Lagrangian reduction by symmetry in continuum mechanics [38].

2.2 Equations of motion for porous media in spatial coordinates

In this Section we derive the equations of motion for a porous medium filled with an incompressible fluid by using a variational formulation deduced from Hamilton's principle.We will follow the description of both fluid and elastic matrix, individually, as outlined in the book by Marsden and Hughes [62], where the reader can find the background and fill in technical details of the description of each media. This derivation closely follows the approach developed in [32] to which the reader is referred for technical details. This derivation is presented here, first, to make this paper self-consistent, and second, in order to introduce the definitions of the variables. We start with some necessary background information on the description of the combined dynamics of the elastic media and fluid that is contained in it.

2.2.1 Definition of variables

We shall remark that the preferred description for the motion of an elastic body is achieved through the Lagrangian coordinates of the media as being the independent variables, and balancing the forces in the spatial frame or the frame attached to the media. On the other hand, the description of the fluid equation is traditionally done in the Eulerian (spatial) frame. The combined mixed fluid-material motion for porous

media can thus be described in either frame. In order to connect with the earlier works by Biot and subsequent analysis of wave propagation in the porous media, we compute the equations of motion in spatial coordinates throughout the paper.

Configuration of the elastic body and the fluid. Suppose that at $t = 0$ the fluid and the elastic body occupy completely a given volume $\mathcal{B} \subset \mathbb{R}^3$. The motion of the elastic body (indexed by s) and the fluid (indexed by f) is defined by two time dependent maps $\boldsymbol{\Psi}(t, \_) : \mathcal{B}_s \to \mathbb{R}^3$ and $\boldsymbol{\varphi}(t, \_) : \mathcal{B}_f \to \mathbb{R}^3$ with variables denoted as

$$\mathbf{x} = \boldsymbol{\Psi}(t, \boldsymbol{X}) \quad \text{and} \quad \mathbf{x} = \boldsymbol{\varphi}(t, \boldsymbol{Y}) \,.$$

Here $\mathcal{B}_s$ and $\mathcal{B}_f$ denote the reference configurations containing the elastic and fluid labels $\boldsymbol{X}$ and $\boldsymbol{Y}$. We assume that there is no fusion of either fluid or elastic body particles, so the map $\boldsymbol{\Psi}$ and $\boldsymbol{\varphi}$ are embeddings for all times t, defining uniquely the *back-to-labels maps* $\boldsymbol{X} = \boldsymbol{\Psi}^{-1}(t, \mathbf{x})$ and $\boldsymbol{Y} = \boldsymbol{\varphi}^{-1}(t, \mathbf{x})$.

By default, we are working with a three-dimensional system, although the equations of motion reduce trivially to the two- and one-dimensional cases. The motion of the elastic body (indexed by s) and the fluid (indexed by f) is defined by two time dependent maps $\boldsymbol{\Psi}$ and $\boldsymbol{\varphi}$ defined on $\mathcal{B}$ with values in $\mathbb{R}^3$, with variables denoted as $\mathbf{x} = \boldsymbol{\Psi}(t, \boldsymbol{X}_s)$ and $\mathbf{x} = \boldsymbol{\varphi}(t, \boldsymbol{X}_f)$. We assume that there is no fusion of either fluid or elastic body particles, so the map $\boldsymbol{\Psi}$ and $\boldsymbol{\varphi}$ are embeddings for all times t, defining uniquely the mappings $\boldsymbol{X}_s = \boldsymbol{\Psi}^{-1}(t, \mathbf{x})$ and $\boldsymbol{X}_f = \boldsymbol{\varphi}^{-1}(t, \mathbf{x})$ [1].

We also assume, for now, that the fluid cannot escape the porous medium or create voids, so at all times t, the domains occupied in space by the fluid $\mathcal{B}_{t,f} = \boldsymbol{\varphi}(t, \mathcal{B})$ and the elastic body $\mathcal{B}_{t,s} = \boldsymbol{\Psi}(t, \mathcal{B})$ coincide: $\mathcal{B}_{t,f} = \mathcal{B}_{t,s} = \mathcal{B}_t$. Finally, we shall assume for simplicity that the domain $\mathcal{B}_t$ does not change with time, and will simply call it $\mathcal{B}$, hence both $\boldsymbol{\varphi} : \mathcal{B}_f \to \mathcal{B}$ and $\boldsymbol{\Psi} : \mathcal{B}_s \to \mathcal{B}$ are diffeomorphisms for all time t. An extension to the case of the fluid escaping the boundary is possible, although it will require appropriate modifications in the variational principle and we shall not consider it in general for now, but only in a specific case later.

[1] We prefer to use the term *embedding* as in [62] which is correct for arbitrary setting, such as three-dimensional media, and also the motion of two-dimensional shells and rods in three-dimensional space. In contrast, the notion 'homeomorphism' is only applicable to the motion of three-dimensional bodies in three-dimensional space.

Velocities of the elastic body and the fluid. The fluid velocity $\boldsymbol{u}_f$ and elastic solid velocity $\boldsymbol{u}_s$, measured relative to the fixed coordinate system, *i.e.*, in the Eulerian representation, are given as usual by

$$\boldsymbol{u}_f(t,\mathbf{x}) = \partial_t\boldsymbol{\varphi}\big(t,\boldsymbol{\varphi}^{-1}(t,\mathbf{x})\big)\,,\quad \boldsymbol{u}_s(t,\mathbf{x}) = \partial_t\boldsymbol{\Psi}\big(t,\boldsymbol{\Psi}^{-1}(t,\mathbf{x})\big)\,, \tag{2.2.1}$$

for all $\mathbf{x}\in\mathcal{B}$. Note that since $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ keep the boundary $\partial\mathcal{B}$ invariant, the vector fields $\boldsymbol{u}_f$ and $\boldsymbol{u}_s$ are tangent to the boundary, *i.e.*,

$$\boldsymbol{u}_f\cdot\boldsymbol{n} = 0\,,\quad \boldsymbol{u}_s\cdot\boldsymbol{n} = 0\,, \tag{2.2.2}$$

where $\boldsymbol{n}$ is the unit normal vector field to the boundary. One can alternatively impose that $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ (or only $\boldsymbol{\Psi}$) keeps the boundary $\partial\mathcal{B}$ pointwise fixed. In this case, one gets no-slip boundary conditions

$$\boldsymbol{u}_f|_{\partial\mathcal{B}} = 0\,,\quad \boldsymbol{u}_s|_{\partial\mathcal{B}} = 0\,,\quad \text{(or only } \boldsymbol{u}_s|_{\partial\mathcal{B}} = 0\text{)}. \tag{2.2.3}$$

Elastic deformations of the dry media. In order to incorporate the description of the elastic deformations of the media in the potential energy, we consider the deformation gradient of $\boldsymbol{\Psi}$ denoted

$$\mathbb{F}(t,\boldsymbol{X}_s) = \nabla\boldsymbol{\Psi}(t,\boldsymbol{X}_s)\,. \tag{2.2.4}$$

In the spatial frame, we consider the Finger deformation tensor $b(t,\mathbf{x})$ defined by

$$b(t,\mathbf{x}) = \mathbb{F}\,\mathbb{F}^{\mathsf{T}}(t,\boldsymbol{X}_s)\,, \tag{2.2.5}$$

where $\mathbf{x} = \boldsymbol{\Psi}(t,\boldsymbol{X}_s)$, see the paragraph below for the intrinsic geometric definition of b. In coordinates, we have

$$\mathbb{F}^i_{\ A} = \frac{\partial\boldsymbol{\Psi}^i}{\partial X_s^A},\qquad b^{ij} = \frac{\partial\boldsymbol{\Psi}^i}{\partial X_s^A}\frac{\partial\boldsymbol{\Psi}^j}{\partial X_s^A}$$

with the summation over A is assumed.

In general the deformation of an elastic media *without fluid* leads to $b\neq$ Id (the unit tensor). The potential energy V of deformation of the dry media thus depends on b. However, in our case there is another part that leads to the elastic potential energy, namely, the microscopic deformations of the pores that we shall describe below.

Internal deformation of the pores, microscopic volume and concentration. Out of many microscopic variables presented in the solid, the geometric shape of pores and their connectivity are most important for computing the volume occupied by the fluid. In my thesis, as well as several previous works in literature, c.f. [65], [71], [74] and others, the internal 'microscopic' volume of the pores is chosen as an important variable affecting potential energy of the solid. The further text closely follows the corresponding description in our paper [32]. This choice is true for the case when the pores' geometry will be roughly similar throughout the material. The model will need to be corrected when there is a drastic change of pores' geometry (e.g. from roughly spherical to elliptical pores, for merging of pores *etc*). For now, we consider that the locally averaged internal volume of the pores is represented by the local variable $v(t, \mathbf{x})$ in the Eulerian description. We will take into account that the pores themselves can expand and contract, which one can understand as modeling the pores through infinitesimally small elastic volumes filled with fluid. When the pores expand, they generate stress in the material; however, the stress averaged over any volume that is much larger than the size of the pores, is going to vanish. We thus introduce an additional dependence of the elastic part of the media on the infinitesimal volume denoted

$$\mathcal{V}(t, \boldsymbol{X}) = v(t, \boldsymbol{\Psi}(t, \boldsymbol{X})) \tag{2.2.6}$$

in the Lagrangian description. Since we are concentrating on the Eulerian description, we will focus on $v(t, \mathbf{x})$. Thus, in our model, the elastic energy of the solid will depend on the Finger deformation tensor b and the infinitesimal pore volume v. Physically, this assumption is equivalent to stating that the internal volume variable v will encompass all the effects of microscopic deformations on the elastic energy.

Let us now consider the volume occupied by the fluid in a given spatial domain. We assume that the fluid fills the pores completely, so the volume occupied by the fluid in any given spatial domain is equal to the net volume of pores in that volume. Let us take the infinitesimal Eulerian volume $\mathrm{d}^3\mathbf{x}$ and define the pore volume fraction $g(t, \mathbf{x})$, so that the volume of fluid is given by $g(t, \mathbf{x})\mathrm{d}^3\mathbf{x}$. Then, one must take into account the available volume to the fluid, namely, the local concentration of pores $c(t, \mathbf{x})$ and the infinitesimal pore volume $v(t, \mathbf{x})$. This consideration leads to the following constraint on

the total volume of pores, which is more easily written in the spatial description:

$$g(t,\mathbf{x}) = c(b(t,\mathbf{x}))v(t,\mathbf{x})\,. \tag{2.2.7}$$

If, for example, the pores are "frozen" in the material, they simply move as material moves. Then, the change of the local concentrations of pores $c(t,\mathbf{x})$ due to deformations is given by

$$c\big(t,\boldsymbol{\Psi}(t,\boldsymbol{X})\big)J_{\boldsymbol{\Psi}}(t,\boldsymbol{X}) = c_0(\boldsymbol{X})\,,\quad J_{\boldsymbol{\Psi}} = |\det(\nabla\boldsymbol{\Psi})| = |\det(\mathbb{F})|\,, \tag{2.2.8}$$

where $c_0(\boldsymbol{X})$ is the initial concentration of pores in the Lagrangian point $\boldsymbol{X}$. Using the definition (2.2.5) of the Finger tensor b gives $\det b(t,\mathbf{x}) = |\det \mathbb{F}(t,\boldsymbol{X}_s)|^2$, hence we can rewrite the previous relation as

$$c(t,\mathbf{x})\sqrt{\det b(t,\mathbf{x})} = c_0(\boldsymbol{X}).$$

In the case of an initially uniform porous media, *i.e.*, $c_0 = \text{const}$, this formula shows that the concentration $c(t,\mathbf{x})$ is a function of the value $b(t,\mathbf{x})$ of the Finger deformation tensor

$$c(b) = \frac{c_0}{\sqrt{\det b}}\,. \tag{2.2.9}$$

Note that from (2.2.8), the concentration of pores satisfies

$$\partial_t c + \operatorname{div}(c\boldsymbol{u}_s) = 0\,.$$

Conservation law for the fluid. In what follows, we will consider an incompressible fluid, as that case has not been studied in the literature in sufficient details. The density of the fluid itself is denoted as $\rho_f^0 = \text{const}$, therefore the discussion involves the conservation of the volume of fluid rather than its mass. Let us now look at the volume of fluid $g(t,\mathbf{x})\mathrm{d}^3\mathbf{x}$ from a different point of view. The fluid must fill all the available volume completely, and it must have come from the initial point $\boldsymbol{X}_f = \boldsymbol{\varphi}^{-1}(t,\mathbf{x})$. If the initial volume fraction at that point was $g_0(\boldsymbol{X}_f)\mathrm{d}^3\boldsymbol{X}_f$, then at a point t in time we have

$$g(t,\mathbf{x}) = g_0\big(\boldsymbol{\varphi}^{-1}(t,\mathbf{x})\big)J_{\boldsymbol{\varphi}^{-1}}(t,\mathbf{x})\,,\quad J_{\boldsymbol{\varphi}^{-1}} := \det\big(\nabla\boldsymbol{\varphi}^{-1}\big)\,. \tag{2.2.10}$$

Differentiating (2.2.10), we obtain the conservation law for $g(t,\mathbf{x})$ written as

$$\partial_t g + \operatorname{div}(g\,\mathbf{u}_f) = 0\,. \tag{2.2.11}$$

The mass of the fluid in the given volume is $\rho_f^0 g \mathrm{d}^3\mathbf{x}$. Note that the incompressibility condition of the fluid *does not* mean that $\operatorname{div} \boldsymbol{u}_f = 0$. That statement is only true for the case where no elastic matrix is present, *i.e.*, for pure fluid. In the porous media case, a given spatial volume contains both fluid and elastic parts. The conservation of volume available to the fluid is thus given by (2.2.11).

Conservation law for the elastic body. The mass density of the elastic body, denoted ρ_s, satisfies an equation analogous to (2.2.10), namely,

$$\rho_s(t,\mathbf{x}) = \rho_{s,0}\big(\boldsymbol{\Psi}^{-1}(t,\mathbf{x})\big) J_{\boldsymbol{\Psi}^{-1}}(t,\mathbf{x})\,, \tag{2.2.12}$$

where $\rho_{s,0}(\boldsymbol{X}_s)$ is the mass density in the reference configuration. The corresponding differentiated form is

$$\partial_t \rho_s + \operatorname{div}(\rho_s \mathbf{u}_s) = 0\,. \tag{2.2.13}$$

Intrinsic geometric formulation. To understand the transport equation of the Finger deformation tensor, it is advantageous to reformulate geometrically its definition. We assume that the reference configuration $\mathcal{B}$ is endowed with a reference Riemannian metric G, locally denoted $G = G_{AB} dX_s^A dX_s^B$ and we consider its inverse G^{-1}. It is a symmetric two-contravariant tensor locally denoted $G^{-1} = G^{AB} \frac{\partial}{\partial X_s^A} \frac{\partial}{\partial X_s^B}$ with $G^{AB} G_{BC} = \delta^A_C$. Then, the Finger deformation tensor is the symmetric two-contravariant tensor obtained by pushing forward G^{-1} by the elastic configuration $\boldsymbol{\Psi}$, namely

$$b = \boldsymbol{\Psi}_* G^{-1}\,. \tag{2.2.14}$$

For a domain in three-dimensional Euclidean space, the Riemannian metric is simply an identity, and is often not included in the considerations. However, the differential-geometric considerations here are important, *e.g.* for evolution of porous shells, which we do not consider here. The geometric description presented here is explained in details in [62]. Using local coordinates, one notes that when G is the Euclidean metric, (2.2.14) reduces to (2.2.5). Using (2.2.14) and (2.2.1), we get the transport equation for b as

$$\partial_t b + \pounds_{\boldsymbol{u}_s} b = 0\,,$$

where $\pounds_{\boldsymbol{u}_s}$ denotes the Lie derivative of a two-contravariant tensor, given in coordinates by

$$(\pounds_{\boldsymbol{u}_s} b)^{ij} = \frac{\partial b^{ij}}{\partial x^k} u_s^k - b^{kj} \frac{\partial u_s^i}{\partial x^k} - b^{ik} \frac{\partial u_s^j}{\partial x^k}\,. \tag{2.2.15}$$

Let us now formulate (2.2.9) intrinsically *i.e.*, without the use of the local coordinates. Given a Riemanian metric γ on the spatial domain, the Jacobian $J_{\boldsymbol{\Psi}}$ of $\boldsymbol{\Psi}$ is defined by $\boldsymbol{\Psi}^*\mu_\gamma = J_{\boldsymbol{\Psi}}\mu_G$, where $\mu_\gamma = \sqrt{\det\gamma}\,\mathrm{d}^3\mathbf{x}$ and $\mu_G = \sqrt{\det G}\,\mathrm{d}^3\boldsymbol{X}_s$ are the Riemannian volume forms. From this, one expresses intrinsically the Jacobian of $\boldsymbol{\Psi}$ in terms of the Finger deformation tensor as

$$J_{\boldsymbol{\Psi}} \circ \boldsymbol{\Psi}^{-1} = \frac{\mu_\gamma}{\mu_{b^{-1}}},$$

where the Riemannian metric b^{-1} is the inverse of b, and $f \circ g$ denotes the composition of mappings or functions.

Since equation (2.2.8) can be written intrinsically as $(c \circ \boldsymbol{\Psi})J_{\boldsymbol{\Psi}} = c_0$, we get

$$c = \frac{c_0 \circ \boldsymbol{\Psi}^{-1}}{J_{\boldsymbol{\Psi}} \circ \boldsymbol{\Psi}^{-1}} = (c_0 \circ \boldsymbol{\Psi}^{-1})\frac{\mu_{b^{-1}}}{\mu_\gamma}.$$

If $c_0 = \text{const}$, we get the expression

$$c(b) = c_0\frac{\mu_{b^{-1}}}{\mu_\gamma}$$

which is the intrinsic version of (2.2.9).

Summary of the variables in the Lagrangian and Eulerian descriptions. From the discussion above, the independent variables in the Lagrangian descriptions are the two embeddings and the infinitesimal volume, *i.e.*,

$$\boldsymbol{\Psi}(t, \boldsymbol{X}_s), \qquad \boldsymbol{\varphi}(t, \boldsymbol{X}_f), \qquad \mathcal{V}(t, \boldsymbol{X}_s). \tag{2.2.16}$$

In the Eulerian description the variables are

$$\boldsymbol{u}_f(t, \mathbf{x}), \quad \boldsymbol{u}_s(t, \mathbf{x}), \quad v(t, \mathbf{x}), \quad g(t, \mathbf{x}), \quad \rho_s(t, \mathbf{x}), \quad b(t, \mathbf{x})\,, \tag{2.2.17}$$

defined from the Lagrangian variables in (2.2.1), (2.2.6), (2.2.10), (2.2.12), (2.2.14), respectively.

2.2.2 Lagrangian and variational principle in spatial variables

Lagrangian. For classical elastic bodies, the potential energy in the spatial description depends on the Finger deformation tensor b, *i.e.*, $V = V(b)$. As we discussed above, in the porous media case, we consider the potential energy to depend on b and v, and we write $V = V(b, v)$. Then, the Lagrangian of the porous medium is the sum of the

kinetic energies of the fluid and elastic body minus the potential energy of the elastic deformations:

$$\ell(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_s, b, g, v) = \int_{\mathcal{B}} \left[\frac{1}{2} \bar{\rho}_f^0 g |\boldsymbol{u}_f|^2 + \frac{1}{2} \rho_s |\boldsymbol{u}_s|^2 - V(b, v) \right] \mathrm{d}^3\mathbf{x} \,. \tag{2.2.18}$$

Note that the expression (2.2.18) explicitly separates the contribution from the fluid and the elastic body in simple physically understandable terms. The interaction between the fluid and the media comes from the critical action principle involving the incompressibility of the fluid. We shall derive the equations of motion for an arbitrary (sufficiently smooth) expression for $\ell(\boldsymbol{u}_s, \boldsymbol{u}_f, \rho_s, b, g, v)$, and will use the physical Lagrangian (2.2.18) for all computations in the paper.

Variational principle and incompressibility constraint. Condition (2.2.7) represents a scalar constraint for every point of an infinite-dimensional system. Formally, such constraint can be treated in terms of Lagrange multipliers. The application of the method of Lagrange multipliers for an infinite-dimensional system is quite challenging, see recent review papers [8], [27]. In terms of classical fluid flow, in the framework of Euler equations, the variational theory introducing incompressibility constraint has been developed by V. I. Arnold [1], with the Lagrange multiplier for incompressibility related to the physical pressure in the fluid. We will follow in the footsteps of Arnold's method and introduce a Lagrange multiplier for the incompressibility condition (2.2.7). By analogy with Arnold, we will also treat this Lagrange multiplier as related to pressure, as it has the same dimensions, and denote it p. Since (2.2.7) refers to the fluid content, the Lagrange multiplier p relates to the fluid pressure. This will be further justified by the equations of motion (2.2.32) below, connecting pressure with the derivatives of the potential energy with respect of the pores' volume. Note that p may be different from the actual physical pressure in the fluid depending on the implementation of the model. From the Lagrangian (2.2.18) and the constraint (2.2.7), we define the action functional in the Eulerian description as

$$S = \int_0^T \left[\ell(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_s, b, g, v) - \int_{\mathcal{B}_t} p\big(g - c(b)v\big) \mathrm{d}^3\mathbf{x} \right] \mathrm{d}t \,. \tag{2.2.19}$$

The equations of motion are obtained by computing the critical points of S with respect to constrained variations of the Eulerian variables induced by free variations of

the Lagrangian variables. Indeed, it is in the Lagrangian description that the variational principle is justified, as being given by the Hamilton principle with constraint. One also notes that the constraint (2.2.7) is holonomic when expressed in terms of the Lagrangian variables (2.2.16) via the relations (2.2.6), (2.2.10), (2.2.14). This justifies that this constraint can be incorporated via the introduction of a Lagrange multiplier. The constrained variations of the Eulerian variables induced by the free variations $\delta\boldsymbol{\Psi}$, $\delta\boldsymbol{\varphi}$ vanishing at $t = 0, T$ are computed by using the relations (2.2.1), (2.2.10), (2.2.12), (2.2.14). This yields

$$\begin{aligned}
\delta\boldsymbol{u}_f &= \partial_t\boldsymbol{\eta}_f + \boldsymbol{u}_f\cdot\nabla\boldsymbol{\eta}_f - \boldsymbol{\eta}_f\cdot\nabla\boldsymbol{u}_f\\
\delta\boldsymbol{u}_s &= \partial_t\boldsymbol{\eta}_s + \boldsymbol{u}_s\cdot\nabla\boldsymbol{\eta}_s - \boldsymbol{\eta}_s\cdot\nabla\boldsymbol{u}_s\\
\delta g &= -\operatorname{div}(g\boldsymbol{\eta}_f)\\
\delta\rho_s &= -\operatorname{div}(\rho_s\boldsymbol{\eta}_s)\\
\delta b &= -\pounds_{\boldsymbol{\eta}_s} b\,,
\end{aligned} \tag{2.2.20}$$

where $\boldsymbol{\eta}_f$ and $\boldsymbol{\eta}_s$ are defined

$$\boldsymbol{\eta}_f = \delta\boldsymbol{\varphi}\circ\boldsymbol{\varphi}^{-1}\,,\quad \boldsymbol{\eta}_s = \delta\boldsymbol{\Psi}\circ\boldsymbol{\Psi}^{-1} \tag{2.2.21}$$

and the variations δv and δp are arbitrary. In the case of the boundary conditions (2.2.2) it follows from (2.2.21) that $\boldsymbol{\eta}_f$ and $\boldsymbol{\eta}_s$ are arbitrary time dependent vector fields vanishing at $t = 0, T$ and tangent to the boundary $\partial\mathcal{B}$:

$$\boldsymbol{\eta}_s\cdot\boldsymbol{n} = 0\,,\quad \boldsymbol{\eta}_f\cdot\boldsymbol{n} = 0\,. \tag{2.2.22}$$

In the case of no-slip boundary conditions (2.2.3), we have

$$\boldsymbol{\eta}_f|_{\partial\mathcal{B}} = 0\,,\quad \boldsymbol{\eta}_s|_{\partial\mathcal{B}} = 0\,,\quad (\text{or only } \boldsymbol{\eta}_s|_{\partial\mathcal{B}} = 0). \tag{2.2.23}$$

Incorporation of external and friction forces. Frictions forces, or any other forces, acting on the fluid $\boldsymbol{F}_f$ and the media $\boldsymbol{F}_s$ can be incorporated into the variational formulation by using the Lagrange-d'Alembert principle for external forces. This principle reads

$$\delta S + \int_{\mathcal{B}_t}\left(\boldsymbol{F}_f\cdot\boldsymbol{\eta}_f + \boldsymbol{F}_s\cdot\boldsymbol{\eta}_s\right)\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0\,, \tag{2.2.24}$$

where S is defined in (2.2.19) and the variations are given by (2.2.20). Such friction forces are usually postulated from general physical considerations. If these forces are

due exclusively to friction, the forces acting on the fluid and media at any given point must be equal and opposite, *i.e.* $\boldsymbol{F}_f = -\boldsymbol{F}_s$, in the Eulerian treatment we consider here. For example, for porous media it is common to posit the friction law

$$\boldsymbol{F}_f = -\boldsymbol{F}_s = \mathbb{K}(\boldsymbol{u}_s - \boldsymbol{u}_f)\,, \tag{2.2.25}$$

with $\mathbb{K}$ being a positive definite matrix potentially dependent on material parameters and variables representing the media. In particular, the matrix $\mathbb{K}$ depends on the local porosity, composition of the porous media, deformation and other variables. The general functional form of dependence of $\mathbb{K}$ on the variables should be of the form $\mathbb{K} = \mathbb{K}(b, g)$. For example, when deformations of porous media are neglected, *i.e.*, assuming isotropic and a non-moving porous matrix with $b = \mathrm{Id}$, Kozeny-Carman equation is often used, which in our notation is written in the form $\mathbb{K} = \kappa g^3/(1-g)^2$, with κ being a constant, see [25] for discussion. In general, the derivation of the dependence of tensor $\mathbb{K}$ on variables g and b from the first principles is difficult, and should presumably be obtained from experimental observations. In general, the anisotropy of $\mathbb{K}$ is related to the geometry of the pores. The shape of the pores and their distribution in space will dictate the numerical values of $\mathbb{K}$ for each given point in space, and the deformation of the pores' geometry will determine the functional dependence $\mathbb{K} = \mathbb{K}(b, g)$. For the purpose of this paper, we will implicitly assume the dependence on flow variables without specifying them explicitly in the formulas. For computations in Section 3 dedicated to the description of propagation of linear disturbances about the steady state, such dependence of $\mathbb{K}$ on variables is not important. If there are other external forces acting on the system, then, in general, $\boldsymbol{F}_f + \boldsymbol{F}_s \neq 0$. This situation can happen, for example, if either the fluid or the media is either electrostatically charged or laden with magnetic particles, and is subjected to the electric or magnetic field. The equations that we derive in the general setting are valid for arbitrary external forces $\boldsymbol{F}_f$ and $\boldsymbol{F}_s$. For explicit computations, we assume the expression (2.2.25).

2.2.3 General form of the equations of motion

In order to derive the equations of motion, we take the variations in the Lagrange-d'Alembert principle (2.2.24) as

$$\begin{aligned}
&\delta S + \int_{\mathcal{B}_t} \left(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s \cdot \boldsymbol{\eta}_s\right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \\
&= \int_{\mathcal{B}_t} \Big[\frac{\delta \ell}{\delta \boldsymbol{u}_f} \cdot \delta \boldsymbol{u}_f + \frac{\delta \ell}{\delta \boldsymbol{u}_s} \cdot \delta \boldsymbol{u}_s + \frac{\delta \ell}{\delta \rho_s} \delta \rho_s + \left(\frac{\delta \ell}{\delta b} + pv \frac{\partial c}{\partial b} \right) : \delta b \\
&\qquad + \left(\frac{\delta \ell}{\delta g} - p \right) \delta g + \left(\frac{\delta \ell}{\delta v} + pc(b) \right) \delta v + \big(g - c(b)v\big) \delta p \\
&\qquad + \boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s \cdot \boldsymbol{\eta}_s \Big] \, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0 \,.
\end{aligned} \tag{2.2.26}$$

The symbol " : " denotes the contraction of tensors on both indices. Substituting the expressions for variations (2.2.20), integrating by parts to isolate the quantities $\boldsymbol{\eta}_f$ and $\boldsymbol{\eta}_s$, and dropping the boundary terms leads to the expressions for the balance of the linear momentum for the fluid and porous medium, respectively, written in the Eulerian frame. This calculation is tedious yet straightforward for most terms and we omit it here. The main difficulty is the calculation of the terms related to the evolution of the tensor b, which we now show in some details.

Denoting by Π the 2-covariant symmetric tensor field $\frac{\delta \ell}{\delta b} + pv\frac{\partial c}{\partial b}$, we compute the fourth term on the right hand side of (2.2.26) by using (2.2.15) as follows:

$$\begin{aligned}
\int_{\mathcal{B}} \Pi : \delta b &= -\int_{\mathcal{B}} (\Pi : \pounds_{\boldsymbol{\eta}} b) \mathrm{d}^3\mathbf{x} \\
&= -\int_{\mathcal{B}} \Pi_{ij} \left(\frac{\partial b^{ij}}{\partial x^k} \eta^k - b^{kj} \frac{\partial \eta^i}{\partial x^k} - b^{ik} \frac{\partial \eta^j}{\partial x^k} \right) \mathrm{d}^3\mathbf{x} \\
&= -\int_{\mathcal{B}} \left(\Pi_{ij} \frac{\partial b^{ij}}{\partial x^k} \eta^k + \eta^i \frac{\partial}{\partial x^k} \left(\Pi_{ij} b^{kj} \right) + \eta^j \frac{\partial}{\partial x^k} \left(\Pi_{ij} b^{ik} \right) \right) \mathrm{d}^3\mathbf{x} \\
&\quad + \int_{\mathcal{B}} \frac{\partial}{\partial x^k} \left(\Pi_{ij} b^{kj} \eta^i + \Pi_{ij} b^{ik} \eta^j \right) \mathrm{d}^3\mathbf{x} \\
&= -\int_{\mathcal{B}} \left(\Pi_{ij} \frac{\partial b^{ij}}{\partial x^k} + \frac{\partial}{\partial x^i} \left(\Pi_{kj} b^{ij} \right) + \frac{\partial}{\partial x^j} \left(\Pi_{ik} b^{ij} \right) \right) \eta^k \mathrm{d}^3\mathbf{x} \\
&\quad + 2 \int_{\partial \mathcal{B}} \Pi_{ij} b^{kj} \eta^i n_k \mathrm{d}s \\
&= -\int_{\mathcal{B}} \left(\Pi_{ij} \frac{\partial b^{ij}}{\partial x^k} + 2 \frac{\partial}{\partial x^i} \left(\Pi_{kj} b^{ij} \right) \right) \eta^k \mathrm{d}^3\mathbf{x} + 2 \int_{\partial \mathcal{B}} (\Pi_{ij} b^{kj} n_k) \eta^i \mathrm{d}s \,,
\end{aligned} \tag{2.2.27}$$

where in three-dimensional case, n_i are the components of the normal vector $\boldsymbol{n}$ [2]. For

[2] For a general metric G, the rigorous statement is that n_i is the one form associated to the normal vector field $\boldsymbol{n}$ via the Riemannian metric.

compactness of notation, we denote the one-form appearing in the first term above with the *diamond* operator

$$(\Pi \diamond b)_k = -\Pi_{ij}\frac{\partial b^{ij}}{\partial x^k} - 2\frac{\partial}{\partial x^i}\left(\Pi_{kj}b^{ij}\right) \tag{2.2.28}$$

whose coordinate-free form reads

$$\Pi \diamond b = -\Pi : \nabla b - 2\mathrm{div}\left(\Pi \cdot b\right) . \tag{2.2.29}$$

The result of (2.2.27) thus reads

$$\int_{\mathcal{B}} \Pi : \delta b = \int_{\mathcal{B}} (\Pi \diamond b) \cdot \boldsymbol{\eta}\, \mathrm{d}^3\mathbf{x} + 2\int_{\partial\mathcal{B}} [(\Pi \cdot b) \cdot \mathbf{n}] \cdot \boldsymbol{\eta}\, \mathrm{d}s. \tag{2.2.30}$$

The equations of motion also naturally involve the expression of the Lie derivative of a momentum density, whose global and local expressions are

$$\begin{aligned} \pounds_{\boldsymbol{u}}\boldsymbol{m} &= \boldsymbol{u}\cdot\nabla\boldsymbol{m} + \nabla\boldsymbol{u}^{\mathsf{T}}\cdot\boldsymbol{m} + \boldsymbol{m}\,\mathrm{div}\,\boldsymbol{u} \\ (\pounds_{\boldsymbol{u}}\boldsymbol{m})_i &= \partial_j m_i u^j + m_j\partial_i u^j + m_i\partial_j u^j\,. \end{aligned} \tag{2.2.31}$$

With these notations, the Lagrange-d'Alembert principle (2.2.26) yields the system of equations

$$\begin{cases} \partial_t \dfrac{\delta\ell}{\delta \boldsymbol{u}_f} + \pounds_{\boldsymbol{u}_f}\dfrac{\delta\ell}{\delta \boldsymbol{u}_f} = g\nabla\left(\dfrac{\delta\ell}{\delta g} - p\right) + \boldsymbol{F}_f \\ \partial_t \dfrac{\delta\ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s}\dfrac{\delta\ell}{\delta \boldsymbol{u}_s} = \rho_s\nabla\dfrac{\delta\ell}{\delta\rho_s} + \left(\dfrac{\delta\ell}{\delta b} + pv\dfrac{\partial c}{\partial b}\right)\diamond b + \boldsymbol{F}_s \\ \dfrac{\delta\ell}{\delta v} = -pc(b)\,, \qquad g = c(b)v \\ \partial_t g + \mathrm{div}(g\boldsymbol{u}_f) = 0\,, \qquad \partial_t\rho_s + \mathrm{div}(\rho_s\boldsymbol{u}_s) = 0\,, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0\,. \end{cases} \tag{2.2.32}$$

When the boundary conditions (2.2.3) are used, no additional boundary condition arise from the variational principle. In the case of the free slip boundary condition (2.2.2), the variational principle yields the condition

$$[\sigma_p \cdot \mathbf{n}] \cdot \boldsymbol{\eta} = 0, \quad \text{for all } \boldsymbol{\eta} \text{ parallel to } \partial\mathcal{B}, \tag{2.2.33}$$

where

$$\sigma_p := -2\left(\frac{\delta\ell}{\delta b} + pv\frac{\partial c}{\partial b}\right)\cdot b. \tag{2.2.34}$$

This is shown by using (2.2.30). Physically, the condition (2.2.33) states that the force $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$ exerted at the boundary must be normal to the boundary (free slip).

The first equation arises from the term proportional to $\boldsymbol{\eta}_f$ in the application of the Lagrange-d'Alembert principle. The second condition and the boundary condition (2.2.33) arise from the term proportional to $\boldsymbol{\eta}_s$ and via the use of (2.2.30). The third and fourth equations arise from the variations δv and δp. The last three equations follow from the definitions (2.2.10), (2.2.12), (2.2.14), respectively. In the derivation of (2.2.32), we have used the fact that on the boundary $\partial\mathcal{B}$, $\boldsymbol{\eta}_s$ and $\boldsymbol{\eta}_f$ satisfy the boundary condition (2.2.22).

Remark 2.2.1 (Discussion of the Lagrangian) Equations (2.2.32) allow for an arbitrary form of the dependence of the Lagrangian on the variables. The derivatives of the Lagrangian with respect to the variables entering (2.2.32) should be considered to be variational derivatives. For example, if the integrand of the Lagrangian depends on both ρ_s and its spatial derivatives $\nabla\rho_s$, *e.g.*

$$\ell = \int_{\mathcal{B}} \ell_0(\rho_s, \nabla\rho_s, \boldsymbol{u}_s, \ldots)\mathrm{d}\mathbf{x}$$

then

$$\frac{\delta\ell}{\delta\rho_s} = \frac{\partial\ell_0}{\partial\rho_s} - \operatorname{div}\frac{\partial\ell_0}{\partial\nabla\rho_s},$$

and similarly with other variables such as $\boldsymbol{u}_s$, $\boldsymbol{\rho}_f$, v *etc.* Thus, equations (2.2.32) are capable of incorporating very general physical models of the porous media. However, it is important to note that in our model, we do not assume that the energy of the fluid depends on any kind of strain measure of the solid or the fluid. These energy considerations only refer to the fluid; the energy of the solid, of course, depends on b, the strain measure of the solid. The pressure p in (2.2.32) is obtained purely from the action principle with the action (2.2.19). In that sense, our paper follows the framework of fluid description due to Arnold [1].

Specific form of the equations. We now use the Lagrangian function ℓ defined in (2.2.18) and compute the derivatives

$$\begin{cases} \dfrac{\delta\ell}{\delta\boldsymbol{u}_f} = \rho_f g\boldsymbol{u}_f\,, & \dfrac{\delta\ell}{\delta\boldsymbol{u}_s} = \rho_s\boldsymbol{u}_s\,, & \dfrac{\delta\ell}{\delta\rho_s} = \dfrac{1}{2}|\boldsymbol{u}_s|^2\,, \\ \dfrac{\delta\ell}{\delta b} = -\dfrac{\partial V}{\partial b}\,, & \dfrac{\delta\ell}{\delta g} = \dfrac{1}{2}\rho_f|\boldsymbol{u}_f|^2\,, & \dfrac{\delta\ell}{\delta v} = -\dfrac{\partial V}{\partial v}\,. \end{cases} \tag{2.2.35}$$

For the Lagrangian in (2.2.18), using (2.2.29) and the third and fourth equations in (2.2.32), the diamond term in (2.2.32) simplifies as

$$\begin{aligned}\left(-\frac{\partial V}{\partial b}+pv\frac{\partial c}{\partial b}\right)\diamond b &= -\left(pv\frac{\partial c}{\partial b}-\frac{\partial V}{\partial b}\right):\nabla b-2\,\mathrm{div}\left[\left(pv\frac{\partial c}{\partial b}-\frac{\partial V}{\partial b}\right)\cdot b\right]\\ &= g\nabla p+\nabla\left(V-\frac{\partial V}{\partial v}v\right)-2\,\mathrm{div}\left[\left(pv\frac{\partial c}{\partial b}-\frac{\partial V}{\partial b}\right)\cdot b\right].\end{aligned}$$

Then, the equations of motions (2.2.32) become

$$\begin{cases}\rho_f(\partial_t\boldsymbol{u}_f+\boldsymbol{u}_f\cdot\nabla\boldsymbol{u}_f)=-\nabla p+\dfrac{1}{g}\boldsymbol{F}_f\\ \rho_s(\partial_t\boldsymbol{u}_s+\boldsymbol{u}_s\cdot\nabla\boldsymbol{u}_s)=g\nabla p+\nabla\left(V-\dfrac{\partial V}{\partial v}v\right)\\ \qquad\qquad -2\,\mathrm{div}\left[\left(pv\dfrac{\partial c}{\partial b}-\dfrac{\partial V}{\partial b}\right)\cdot b\right]+\boldsymbol{F}_s\\ \dfrac{\partial V}{\partial v}=pc(b),\qquad g=c(b)v\\ \partial_t g+\mathrm{div}(g\boldsymbol{u}_f)=0,\qquad \partial_t\rho_s+\mathrm{div}(\rho_s\boldsymbol{u}_s)=0,\qquad \partial_t b+\pounds_{\boldsymbol{u}_s}b=0.\end{cases}\tag{2.2.36}$$

together with the boundary condition (2.2.33) in which the stress tensor σ_p in (2.2.34) reads

$$\sigma_p=-2\left(pv\frac{\partial c}{\partial b}-\frac{\partial V}{\partial b}\right)\cdot b\,,\qquad (\sigma_p)^i_k=-2\left(pv\frac{\partial c}{\partial b^{kj}}-\frac{\partial V}{\partial b^{kj}}\right)b^{ij}\,.\tag{2.2.37}$$

The divergence term in the media momentum equation (second equation above) is the analogue of the divergence of the stress tensor for an ordinary elastic media: This term, however, contains the contribution from both the potential energy and the fluid pressure.

These equations define the coupled motion of an incompressible fluid and porous media. We are not aware of these equations having been derived before.

Remark 2.2.2 (Equations of motion with external equilibrium pressure) If the media is subjected to a uniform external pressure p_0, then the equations of motion are derived by changing the Lagrangian to $\ell_p\to\ell+(p-p_0)(g-c(b)v)$. In that case, equations (2.2.32), and, similarly, (2.2.36) are altered by simply substituting $p-p_0$ instead of p. In what follows, we shall put $p_0=0$.

2.2.4 Energy dissipation

We are now going to proceed to prove that our model yields strict dissipation of mechanical energy in the presence of friction forces. This is important in order to demonstrate that our derivation is physically consistent. Fortunately, variational methods are guaranteed to provide energy conservation for the absence of friction, and when the friction forces are introduced correctly, also guaranteed to provide energy dissipation. Let us consider the energy density associated to the Lagrangian ℓ given by

$$e = \boldsymbol{u}_f \cdot \frac{\delta \ell}{\delta \boldsymbol{u}_f} + \boldsymbol{u}_s \cdot \frac{\delta \ell}{\delta \boldsymbol{u}_s} + \dot{v}\frac{\delta \ell}{\delta \dot{v}} - \mathcal{L}\,, \tag{2.2.38}$$

where $\mathcal{L}$ denotes the integrand of ℓ. Note that in our case ℓ does not depend on $\dot{v}$ hence the third term vanishes. For the general system (2.2.32), and its explicit form (2.2.36), to be physically consistent, we need to prove that in the absence of forces $\boldsymbol{F}_s$ and $\boldsymbol{F}_f$, the total energy $E = \int_{\mathcal{B}} e \,\mathrm{d}^3\mathbf{x}$ is conserved. When these forces are caused by friction, we must necessarily have $\dot{E} \leq 0$.

We begin by noticing the formula

$$\boldsymbol{u} \cdot \pounds_{\boldsymbol{u}} \boldsymbol{m} = \boldsymbol{u} \cdot \big(\boldsymbol{u} \cdot \nabla \boldsymbol{m} + \nabla \boldsymbol{u}^{\mathsf{T}} \cdot \boldsymbol{m} + \boldsymbol{m} \operatorname{div} \boldsymbol{u}\big) = \operatorname{div}\big(\boldsymbol{u}\,(\boldsymbol{m} \cdot \boldsymbol{u})\big)\,, \tag{2.2.39}$$

which easily follows from its coordinates expression in (2.2.31). Then, using equation (2.2.39) and system (2.2.32), we compute

$$\begin{aligned}
\partial_t e &= \boldsymbol{u}_f \cdot \frac{\partial}{\partial t}\frac{\delta \ell}{\delta \boldsymbol{u}_f} + \boldsymbol{u}_s \cdot \frac{\partial}{\partial t}\frac{\delta \ell}{\delta \boldsymbol{u}_s} - \frac{\delta \ell}{\delta \rho_s}\partial_t \rho_s - \frac{\delta \ell}{\delta b} : \partial_t b - \frac{\delta \ell}{\delta g}\partial_t g - \frac{\delta \ell}{\delta v}\partial_t v \\
&= -\operatorname{div}\left[\boldsymbol{u}_f \left(\boldsymbol{u}_f \cdot \frac{\delta \ell}{\delta \boldsymbol{u}_f}\right) + \boldsymbol{u}_s \left(\boldsymbol{u}_s \cdot \frac{\delta \ell}{\delta \boldsymbol{u}_s}\right) - \left(\frac{\delta \ell}{\delta g} - p\right) g \boldsymbol{u}_f \right. \\
&\qquad\qquad \left. - \frac{\delta \ell}{\delta \rho_s}\rho_s \boldsymbol{u}_s + 2\boldsymbol{u}_s \cdot \left(\frac{\delta \ell}{\delta b} + pv\frac{\partial c}{\partial b}\right) \cdot b\right] \\
&\quad + \left(\frac{\delta \ell}{\delta g} - p\right)\partial_t g + \frac{\delta \ell}{\delta \rho_s}\partial_t \rho_s + \left(\frac{\delta \ell}{\delta b} + pv\frac{\partial c}{\partial b}\right)\partial_t b \\
&\quad - \frac{\delta \ell}{\delta \rho_s}\partial_t \rho_s - \frac{\delta \ell}{\delta b} : \partial_t b - \frac{\delta \ell}{\delta g}\partial_t g - \frac{\delta \ell}{\delta v}\partial_t v + \boldsymbol{u}_s \cdot \boldsymbol{F}_s + \boldsymbol{u}_f \cdot \boldsymbol{F}_f \\
&= -\operatorname{div}\boldsymbol{J} - p\partial_t g + pv\frac{\partial c}{\partial b} : \partial_t b - \frac{\delta \ell}{\delta v}\partial_t v + \boldsymbol{u}_s \cdot \boldsymbol{F}_s + \boldsymbol{u}_f \cdot \boldsymbol{F}_f\,,
\end{aligned} \tag{2.2.40}$$

where we denoted by $\boldsymbol{J}$ the vector field in the brackets inside the div operator. The last term in these brackets has the local expression

$$\left(2\boldsymbol{u}_s \cdot \left(\frac{\delta \ell}{\delta b} + pv\frac{\partial c}{\partial b}\right) \cdot b\right)^k = 2\boldsymbol{u}_s^i \left(\frac{\delta \ell}{\delta b_{ij}} + pv\frac{\partial c}{\partial b_{ij}}\right) b^{jk} = -\sigma_p \cdot \mathbf{u}_s\,.$$

The sum of the second, third, and fourth terms in last line of (2.2.40) cancel thanks to the third and fourth equations in (2.2.32). We thus get the energy balance

$$\partial_t e + \operatorname{div} \boldsymbol{J} = \boldsymbol{u}_s \cdot \boldsymbol{F}_s + \boldsymbol{u}_f \cdot \boldsymbol{F}_f \,.$$

Thus, the balance of total energy is

$$\dot{E} = \int_{\mathcal{B}} (\boldsymbol{u}_s \cdot \boldsymbol{F}_s + \boldsymbol{u}_f \cdot \boldsymbol{F}_f) \, \mathrm{d}^3\mathbf{x} - \int_{\partial\mathcal{B}} \boldsymbol{J} \cdot \boldsymbol{n} \, \mathrm{d}s \,. \tag{2.2.41}$$

From the boundary conditions (2.2.2) and (2.2.33) we have $\mathbf{u}_s \cdot \mathbf{n} = 0$, $\mathbf{u}_f \cdot \mathbf{n} = 0$, and $[\sigma_p \cdot \mathbf{n}] \cdot \mathbf{u}_s = 0$ on the boundary $\partial\mathcal{B}$, so that $\mathbf{J} \cdot \mathbf{n} = \mathbf{0}$ at the boundary. In the case of the boundary conditions (2.2.3), we have $\mathbf{J}|_{\partial\mathcal{B}} = 0$. In the absence of external forces, when $\boldsymbol{F}_f$ and $\boldsymbol{F}_s$ are caused exclusively by the friction between the porous media and the fluid, we have $\boldsymbol{F}_f = -\boldsymbol{F}_s$. Since in that case $\dot{E} \leq 0$, we must necessarily have

$$\dot{E} = \int_{\mathcal{B}} \boldsymbol{F}_s \cdot (\boldsymbol{u}_s - \boldsymbol{u}_f) \, \mathrm{d}^3\mathbf{x} \leq 0 \,. \tag{2.2.42}$$

If one assumes (2.2.25) for the friction, *i.e.*, $\boldsymbol{F}_s = \mathbb{K}(\boldsymbol{u}_s - \boldsymbol{u}_f)$, then $\mathbb{K}$ must be a positive operator, *i.e.*, $\mathbb{K}\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$, for all $\boldsymbol{v} \in \mathbb{R}^3$ and for any point $\mathbf{x} \in \mathcal{B}$.

2.3 Connection with the previously derived models of porous media

2.3.1 The case of a compressible porous media filled with compressible fluid

Let us start with connecting to the case considered frequently in the literature, namely, the case of a compressible fluid moving inside a matrix made out of elastic compressible material. In this case, the fluid pressure is no longer a Lagrange multiplier, but has to be found from the identities regarding the internal energy of the fluid as a function of its density. We refer the reader to [52] for background in classical thermodynamics. If the volume fraction occupied by the fluid is ϕ, the volume fraction of the elastic matrix is then $1 - \phi$. In the general thermodynamic description, the specific internal energy of the material e is a function of its density ρ and specific entropy S, with the pressure being given as $p = \rho^2 \frac{\partial e}{\partial \rho}$. This formula is correct whether the thermodynamics effects are considered, *i.e.* S is varying, or ignored, *i.e.* S =const. If the effective density of

the fluid is ρ_f, and its volume fraction is ϕ, then the microscopic density of the fluid is $\bar{\rho}_f = \rho_f/\phi$, so the internal energy of the fluid is a function of $\bar{\rho}_f$, *i.e.*, $e_f = e_f(\bar{\rho}_f)$. Similarly, the microscopic density of the solid is $\bar{\rho}_s = \rho_s/(1-\phi)$. It is natural to assume that the internal energy of the elastic solid depends on both $\bar{\rho}_s$ and the Finger deformation tensor b, $e_s = e_s(\bar{\rho}_s, b)$. Thus, the physically relevant Lagrangian takes the form

$$\begin{aligned}\ell(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_f, \rho_s, b, \phi) = \int_{\mathcal{B}} \Big[&\frac{1}{2}\rho_f|\boldsymbol{u}_f|^2 + \frac{1}{2}\rho_s|\boldsymbol{u}_s|^2 \\ &-\rho_f e_f\left(\frac{\rho_f}{\phi}\right) - \rho_s e_s\left(\frac{\rho_s}{1-\phi}, b\right)\Big]\,\mathrm{d}x.\end{aligned} \tag{2.3.1}$$

Proceeding as in the derivation of (2.2.32), we obtain the following system, written in terms of a general Lagrangian:

$$\begin{cases} \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} + \pounds_{\boldsymbol{u}_f} \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} = \rho_f \nabla \dfrac{\delta \ell}{\delta \rho_f} \\ \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s} \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} = \rho_s \nabla \dfrac{\delta \ell}{\delta \rho_s} - \dfrac{\delta \ell}{\delta b} : \nabla b - 2\,\mathrm{div}\,\dfrac{\delta \ell}{\delta b} \cdot b \\ \partial_t \rho_f + \mathrm{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \mathrm{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \dfrac{\delta \ell}{\delta \phi} = 0 \end{cases} \tag{2.3.2}$$

When the particular form of the Lagrangian (2.3.1) is assumed, the equations take the form:

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\rho_f \nabla\left(e_f + \bar{\rho}_f \dfrac{\partial e_f}{\partial \bar{\rho}_f}\right) = -\phi \nabla\left(\bar{\rho}_f^2 \dfrac{\partial e_f}{\partial \bar{\rho}_f}\right) \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -\rho_s \nabla\left(e_s + \bar{\rho}_s \dfrac{\partial e_s}{\partial \bar{\rho}_s}\right) - \rho_s \dfrac{\partial e_s}{\partial b} : \nabla b \\ \qquad\qquad +2\,\mathrm{div}\,\dfrac{\partial e_s}{\partial b} \cdot b = -(1-\phi)\nabla\left(\bar{\rho}_s^2 \dfrac{\partial e_s}{\partial \bar{\rho}_s}\right) + 2\,\mathrm{div}\,\dfrac{\partial e_s}{\partial b} \cdot b \\ \partial_t \rho_f + \mathrm{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \mathrm{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \bar{\rho}_f^2 \dfrac{\partial e_f}{\partial \bar{\rho}_f} = \bar{\rho}_s^2 \dfrac{\partial e_s}{\partial \bar{\rho}_s} =: p \quad \text{where} \quad \bar{\rho}_f := \dfrac{\rho_f}{1-\phi}, \quad \bar{\rho}_s = \dfrac{\rho_s}{\phi}\,. \end{cases} \tag{2.3.3}$$

The last equation, coming from the variation in $\delta\phi$, states the equality of pressure in both elastic and fluid part of the system. We can transform the system to the following form:

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\phi \nabla p \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -(1-\phi)\nabla p + \mathrm{div}\,\sigma_{\mathrm{el}} \\ \partial_t \rho_f + \mathrm{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \mathrm{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \end{cases} \tag{2.3.4}$$

where

$$p := \bar{\rho}_f^2 \frac{\partial e_f}{\partial \bar{\rho}_f} = \bar{\rho}_s^2 \frac{\partial e_s}{\partial \bar{\rho}_s}, \qquad \sigma_{\text{el}} := 2 \frac{\partial e_s}{\partial b} \cdot b$$

Equations similar to (2.3.4) appear, for example in [23], with additional thermodynamical effects. These thermodynamics effects can be incorporated in our model as well if we allow the energies of the fluid and solid part in the Lagrangian (2.3.1) to depend on the entropies of fluid S_f and solid S_s, such as $e_f = e_f(\bar{\rho}_f, S_f)$ and $e_s = e_f(\bar{\rho}_s, S_s)$, with additional equations for advection of the entropy and heat exchange between the two phases. We shall postpone this discussion of thermal effect for our follow-up work in order not to distract from the main message of the paper. However, within the framework of this paper, it is worth noting that the internal energies of the fluid and solid are completely separated: the internal energy of the fluid depends only on the internal variables of the fluid, and, correspondingly, the internal energy of the elastic matrix depends only on the internal variables of the elastic material. The interaction between the terms comes from equality of pressure and follows from the equations of motion; it does not have to be assumed *a priori*. Thus, we believe, our approach is consistent with the classical Lagrangian approach of dealing with the systems with several interacting parts.

2.3.2 Compressible media with incompressible fluid

Let us now connect this description of compressible fluid and solid to the case of incompressible fluid and compressible solid. We shall keep the same variables as in the derivation of (2.3.3) to keep the notation consistent, and then show how to connect the resulting equations with (2.2.32). The difference between the cases of compressible and incompressible fluids comes to two fundamental restrictions:

1. Since the microscopic density of fluid $\bar{\rho}_f$, also denoted ρ_f^0 earlier, is constant, the internal energy of the fluid do not depend on $\bar{\rho}_f$.

2. There is an incompressibility condition $\phi = (\phi^0 \circ \boldsymbol{\varphi}^{-1}) J_{\boldsymbol{\varphi}^{-1}}$, equivalent to (2.2.10). We remind the reader that $\boldsymbol{\varphi}^{-1}(\mathbf{x}, t)$ is the inverse of the Lagrangian mapping for fluid particles, also known as the back-to-labels map. Physically, this law states that all the fluid in a given microscopic volume of porous media has appeared from its initial source at $t = 0$.

Note that the incompressibility condition presented above is similar to the conservation of mass in [7] (Eq. (10) taken for the case of fluid only). In spite of this similarity, there is an important difference to keep in mind: in [7], the conservation law is written for both *compressible* fluid and solid parts. In our case, no additional conservation laws are necessary in the case of compressible fluid and solid, so there is only one incompressibility condition for fluid for the incompressible fluid case, and none for the compressible fluid case. The conservation law for the compressible part in our theory is satisfied automatically, and no extra Lagrange multipliers are necessary. The action functional (2.2.19), incorporating the constraint with the Lagrange multiplier p, rewritten in the new variables, becomes

$$S_p = \int_0^T \left[\ell(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_f, \rho_s, b, \phi) + \int_{\mathcal{B}} p \left(\phi - (\phi^0 \circ \boldsymbol{\varphi}^{-1}) J_{\boldsymbol{\varphi}^{-1}} \right) \mathrm{d}x \right] \mathrm{d}t \,. \tag{2.3.5}$$

While the method works for an arbitrary Lagrangian, the physically relevant form of the Lagrangian to consider is given by

$$\ell(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_f, \rho_s, b, \phi) = \int_{\mathcal{B}} \left[\frac{1}{2} \rho_f |\boldsymbol{u}_f|^2 + \frac{1}{2} \rho_s |\boldsymbol{u}_s|^2 - \rho_s e_s \left(\frac{\rho_s}{1-\phi}, b \right) \right] \mathrm{d}x. \tag{2.3.6}$$

Note that compared to the previous form for compressible fluid case (2.3.1), the term $\rho_f e_f \left(\bar{\rho}_f\right)$ is now absent from (2.3.6). Using the identity

$$\delta \left[(\phi^0 \circ \boldsymbol{\varphi}^{-1}) J_{\boldsymbol{\varphi}^{-1}} \right] = - \operatorname{div} \left((\phi^0 \circ \boldsymbol{\varphi}^{-1}) J_{\boldsymbol{\varphi}^{-1}} \boldsymbol{\eta}_f \right) \,, \tag{2.3.7}$$

we get the following set of equations written for a general Lagrangian ℓ:

$$\begin{cases} \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} + \pounds_{\boldsymbol{u}_f} \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} = \rho_f \nabla \dfrac{\delta \ell}{\delta \rho_f} - \phi \nabla p \\ \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s} \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} = \rho_s \nabla \dfrac{\delta \ell}{\delta \rho_s} - \dfrac{\delta \ell}{\delta b} : \nabla b - 2 \operatorname{div} \dfrac{\delta \ell}{\delta b} \cdot b \\ \partial_t \rho_f + \operatorname{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \phi = (\phi^0 \circ \varphi_f^{-1}) J_{\varphi_f^{-1}}, \qquad \dfrac{\delta \ell}{\delta \phi} + p = 0. \end{cases} \tag{2.3.8}$$

In the case of the physically relevant Lagrangian (2.3.6), we obtain

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\phi \nabla p \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -\rho_s \nabla \left(e_s + \bar{\rho}_s \dfrac{\partial e_s}{\partial \bar{\rho}_s} \right) - \rho_s \dfrac{\partial e_s}{\partial b} : \nabla b + 2 \operatorname{div} \dfrac{\partial e_s}{\partial b} \cdot b \\ \qquad\qquad\qquad\qquad = -(1-\phi) \nabla \left(\bar{\rho}_s^2 \dfrac{\partial e_s}{\partial \bar{\rho}_s} \right) + 2 \operatorname{div} \dfrac{\partial e_s}{\partial b} \cdot b \\ \partial_t \rho_f + \operatorname{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \partial_t \phi + \operatorname{div}(\phi \boldsymbol{u}_f) = 0, \qquad \bar{\rho}_s^2 \dfrac{\partial e_s}{\partial \bar{\rho}_s} = p. \end{cases} \tag{2.3.9}$$

Equations $\partial_t \rho_f + \operatorname{div}(\rho_f \boldsymbol{u}_f) = 0$ and $\partial_t \phi + \operatorname{div}(\phi \boldsymbol{u}_f) = 0$ imply that $\rho_f = \bar{\rho}_f \phi$ with $\bar{\rho}_f$ a constant. Note that the last equation of (2.3.9), states that the thermodynamic pressure in the solid, defined through the derivatives of the internal energy function e_s, is equal to the Lagrange multiplier p. Thus, physically, the Lagrange multiplier p is equal to the pressure inside the solid, so it also acquires the physical meaning of the pressure in the fluid. However, that physical meaning is elucidated only *after* the equations of motion (2.3.9) are derived and cannot be inferred *a priori*.

A quick calculation shows that the system (2.3.9) is equivalent to the equations (2.2.32) derived earlier, under the change of variables

$$g = \phi, \quad c(b) = \rho_s, \quad v = \frac{1}{\rho_s} - \frac{1}{\bar{\rho}_s}. \tag{2.3.10}$$

That equivalence is proved by assuming the internal energy of the solid in the form

$$V(b, v) = \rho_s(b) e_s\left(\bar{\rho}_s, b\right), \quad \bar{\rho}_s := \frac{\rho_s(b)}{1 - \rho_s(b) v}. \tag{2.3.11}$$

Substitution of that expression for the internal energy of the solid into (2.2.32) gives (2.3.9). We believe that such calculation is useful since it connects our earlier derivation (2.2.32) with the information on the compressible case, and also elucidates the nature of the variable ϕ. It is useful to recall the quote from [82] mentioned in the Introduction, where the nature of this variable was suggested to preclude the existence of a variational principle. Our theory presented here shows that the variable describing the fluid content has to be considered carefully in the variational principle (2.3.5), or, equivalently, in (2.2.19) earlier, as a constraint through the geometric variational formulation presented here. The understanding of the role of this variable, we believe, is the key to the

derivation of the variational principle for porous media, and was perhaps the source of difficulty in explaining the incompressible fluid case in previous works.

The physical meaning of v becomes clear from the last formula of (2.3.10). Indeed, choose m_s to be a given mass of elastic solid, then $\frac{m_s}{\rho_s}$ is the volume occupied by the porous elastic solid, and $\frac{m_s}{\bar{\rho}_s}$ is the volume occupied by the (imaginary) elastic solid without any porosity. Thus, the quantity $m_s\left(\frac{1}{\rho_s}-\frac{1}{\bar{\rho}_s}\right)$ is the volume occupied by the fluid per unit mass of the solid, and therefore the quantity $v=\frac{1}{\rho_s}-\frac{1}{\bar{\rho}_s}$ is the physical meaning of specific volume of the fluid's content, measured per unit mass of the elastic solid.

2.3.3 Static media with ideal gas and connection to the porous medium equation (PME)

Let us consider a physical system that describes a polytropic flow of an ideal gas through a homogeneous static porous medium. For such a system, the kinetic and potential energies of the solid matrix are no longer relevant and we only have the terms, describing the gas component. The state equation for pressure has the form

$$p=p_0\rho^\gamma, \tag{2.3.12}$$

where $\gamma\geq 1$ is the polytropic exponent. The choice of the exponent could describe among others isentropic (adiabatic and reversible, $\gamma=\gamma_0>0$) and isothermal (with $\gamma=1$) flows. To apply the variational geometric formulation to such system, we need to compute the potential energy of the gas. In case $p_0(\mathbf{x})=\text{const}$ we simply have

$$V(\rho,T)=V_0\rho^\gamma. \tag{2.3.13}$$

The corresponding Lagrangian takes the form

$$\ell(\boldsymbol{u},\rho)=\int_{\mathcal{B}}\left[\frac{1}{2}\rho|\boldsymbol{u}|^2-V_0\rho^\gamma\right]\mathrm{d}\mathbf{x}. \tag{2.3.14}$$

With these notations, the variations in the Lagrange-d'Alembert principle (2.2.24) yields the system of equations

$$\begin{cases}\partial_t\dfrac{\delta\ell}{\delta\boldsymbol{u}}+\pounds_{\boldsymbol{u}}\dfrac{\delta\ell}{\delta\boldsymbol{u}}=\rho\nabla\dfrac{\delta\ell}{\delta\rho}+\boldsymbol{F}\\ \partial_t\rho+\operatorname{div}(\rho\boldsymbol{u})=0\,,\end{cases} \tag{2.3.15}$$

where for the friction force we posit $\boldsymbol{F} := -\mu\rho\boldsymbol{u}$, according to the linear Darcy's law.

After computing the explicit form of functional derivatives and collecting the terms, the equations of motion (2.3.15) become

$$\begin{cases} \rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\rho\nabla\dfrac{\partial V}{\partial\rho} - \mu\rho\boldsymbol{u} \\ \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0\,. \end{cases} \tag{2.3.16}$$

Let us consider the special case of a degenerate Lagrangian without kinetic energy term

$$\ell(\boldsymbol{u},\rho) = -\int_{\mathcal{B}} V_0\rho^\gamma \mathrm{d}\mathbf{x}. \tag{2.3.17}$$

The application of Lagrange-d'Alembert's principle to (2.3.17) will formally yield a system with no dynamic terms in the left hand side of the first equation of (2.3.15), namely

$$\begin{cases} \mu\boldsymbol{u} = -\nabla\dfrac{\partial V}{\partial\rho} = -\gamma V_0\rho^{\gamma-1}\nabla\rho = -\dfrac{V_0}{p_0}\nabla p \\ \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0\,. \end{cases} \tag{2.3.18}$$

We substitute the velocity from the first equation into the continuity equation in the system (2.3.18) to get a closed form equation for the density

$$\partial_t\rho = \frac{V_0}{\mu}\operatorname{div}(\rho\nabla\rho^\gamma). \tag{2.3.19}$$

The equation (2.3.19) is known as the porous medium equation (PME).

We should notice, that while this result was obtained simply by dropping the inertial term $\frac{1}{2}\rho|\boldsymbol{u}|^2$ from the Lagrangian, the correctness of this approach goes beyond the scope of this paper, as it employs a degenerate Lagrangian. The use of a small parameter $\epsilon \to 0$ as a coefficient for the inertial term would lead to the singular perturbations and will not yield the same result immediately. The solutions of systems with singular perturbations may employ asymptotic methods, see for example [80].

The porous medium equation (2.3.19) could be nondimensionalized by scaling out the constant and rewritten in the form

$$\partial_t u = \Delta(u^m), \quad m := 1+\gamma > 1, \tag{2.3.20}$$

which is a nonlinear heat equation, formally of parabolic type [79]. The interesting property of this porous medium equation, that does not hold for the linear heat equation

$\partial_t u = \Delta u$ is that (2.3.20) has finite speed for the propagation of disturbances from the level $u = 0$.

The PME (2.3.20) has a special solution representing mass or heat release from a point source [79], that was obtained by Zeldovich, Kompaneets and Barenblatt, and the terms *source solution*, *Barenblatt-Pattle solution*, *Barenblatt solution* or *ZKB solution* are often used for this solution. The ZKB solution has the self-similar form: $u = t^{-\alpha} f(\xi)$ with $\xi = \|\mathbf{x}\| t^{-\beta}$, see [5]. For equation (2.3.20), this self-similar solution can be written explicitly as

$$u(\mathbf{x}, t) := t^{-\alpha} \left(C - k\|\mathbf{x}\|^2 t^{-2\beta} \right)_+^{\frac{1}{m-1}}, \tag{2.3.21}$$

where $(f)_+ := \max\{f, 0\}$,

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{\alpha}{d}, \quad k = \frac{\alpha(m-1)}{2md}, \tag{2.3.22}$$

$C > 0$ is an arbitrary constant and d is the dimension of the space. The constant C is chosen in such a way that the mass of the solution M is equal to a given value. The initial data in (2.3.21) as $t \to 0$ is a Dirac mass $u(\mathbf{x}, t) \to M\delta(\mathbf{x})$, where $M = M(C, m, d)$. The source solution has compact support in space for every fixed time, in other words, the solution is non-zero only for $\|\mathbf{x}\| \le R(t)$, with the radius of the support $R(t) = t^\beta \sqrt{C/k}$, and the mass of the solution conserved for all times, $\int_{\mathbb{R}^d} u(\mathbf{x}, t) \mathrm{d}\mathbf{x} = M = \text{const}$.

2.3.4 Connections with (quasi-static) equations of porous media (Poroelasticity Equations)

Another important equation related to porous media is the so-called poroelasticity equations [16], [17]. which are sometimes also called the equations for the porous media, or the (nonlinear) Biot model for quasi-static porous media. See also [76] for literature review and mathematical analysis of solutions for related models. These equations describe slow, inertia-less deformation of the porous media filled with an incompressible fluid. If we neglect the kinetic energy terms in the Lagrangian and assume

$$\ell = \int_{\mathcal{B}} e_s(b, \phi) + p \left(\phi - \phi_0 \circ \boldsymbol{\varphi}^{-1} J_{\boldsymbol{\varphi}^{-1}} \right) \mathrm{d}\mathbf{x} \tag{2.3.23}$$

the equations of motion in our case become

$$\begin{cases} 0 = -\phi\nabla p + \mathbb{K}(\boldsymbol{u}_s - \boldsymbol{u}_f) \\ 0 = -(1-\phi)\nabla p + \mathrm{div}\sigma_e + \mathbb{K}(\boldsymbol{u}_f - \boldsymbol{u}_s) \\ 0 = \dfrac{\partial \ell}{\partial \phi}(\phi, b) - p. \end{cases} \tag{2.3.24}$$

If we linearize the elastic stress tensor σ_e in above equations as

$$\sigma_e \simeq \Lambda\epsilon + G\mathrm{div}\epsilon - (1-\phi)p\mathrm{Id}, \quad \epsilon \simeq \frac{1}{2}(b - \mathrm{Id}) \tag{2.3.25}$$

and neglect the gradients of $\nabla\phi$ in the second equation of (2.3.24), the first two equations demonstrating the balance of momenta for the fluid and the solid, reduce to the quasi-static porous media equations from [16] (without the viscoelastic terms) and [17] as

$$\mu\nabla p = \mathbb{K}(\boldsymbol{u}_f - \boldsymbol{u}_s)\,, \quad \mathrm{div}\sigma_{\mathrm{tot}} + \mathbb{K}(\boldsymbol{u}_f - \boldsymbol{u}_s) = 0\,, \tag{2.3.26}$$

The last equation of our reduced equations (2.3.24), obtained with respect to variations with respect to ϕ, connects the pressure, which is the Lagrange multiplier for incompressibility, with the fluid content and deformations. In contrast, the nonlinear Biot model uses an alternative *Bio-Willis* relationship for the quantity $\zeta = \phi(\mathbf{x}, t) - \phi(\mathbf{x}, 0)$, which in our notation reads:

$$\zeta = c_0 p + \alpha\,\mathrm{div}\boldsymbol{\Psi} \circ \boldsymbol{\Psi}^{-1}(\mathbf{x}, t) \tag{2.3.27}$$

For incompressible fluid, one takes $c_0 = 0$ and $\alpha = 1$, attributing all the change in available volume to the dilation of the media. In our opinion, that is a highly simplified assumption which is difficult to justify mathematically, especially for the elastic media that is incompressible as well, so, technically speaking, $\mathrm{div}\boldsymbol{\Psi} = 0$, leading to $\zeta = 0$. Thus, in case of both the fluid and solid being incompressible, the equation (2.3.27) states that $\phi(\mathbf{x}, t) = \phi(\mathbf{x}, 0)$ which is in general incorrect.

In contrast, we do not need that additional constraint (2.3.27) as the relationship connecting ζ, p and automatically follows from the last equation of (2.3.24). Moreover, we have automatic dissipation of energy quantity due to the existence of variational principle. In contrast, finding the energy-like quantity for the nonlinear porous media (Biot) model is highly nontrivial, as the works [16], [17] illustrate. We thus believe that in spite of higher apparent complexity compared to the simplified models used in

the literature in the field, our equations are actually mathematically simpler to analyze from the point of view of functional analysis, and hope that experts in analysis of PDEs will have an opportunity to perform rigorous analysis of existence and uniqueness of solutions to our equations.

Chapter 3

Linear stability analysis

This chapter is dedicated to the analysis of the linearized equations of porous dynamics (2.2.36), derived in the previous chapter. The linearization describes the propagation of waves through the media, which is essential for acoustic and seismic applications. In particular, the presentation includes the derivation of the propagation of S-waves and P-waves in isotropic media. The analysis includes the stability criteria for the wave equations and demonstration of their equivalence to the physicality conditions of the elastic matrix. The final part of the chapter shows that the celebrated Biot's equations for waves in porous media are obtained for certain values of parameters in our models.

3.1 Derivation of the linearized equations of motion

We linearize equations (2.2.36) about the equilibrium state

$$(\boldsymbol{u}_f, \boldsymbol{u}_s, \rho_s, b, g, v, p) = (\mathbf{0}, \mathbf{0}, \rho_s^0, b_0, g_0, v_0, p_0)\,, \tag{3.1.1}$$

where each component on the right-hand side of (3.1.1) with a subscript 0 is a constant. The equilibrium condition reads

$$\left.\frac{\partial V}{\partial v}\right|_0 = p_0 c_0\,. \tag{3.1.2}$$

where $F|_0$ denotes the value of a function F taken at the equilibrium (3.1.1). We consider the potential $V(b, v)$ to be general and assume, for simplicity, an unstressed state $b_0 = \mathrm{Id}$ and $p_0 = 0$. Throughout this section, we shall assume friction forces of the form (2.2.25) with a given constant general permeability tensor $\mathbb{K}$. For simplicity of computations, we will eventually further assume isotropic and uniform media, so the permeability tensor $\mathbb{K}$ will be taken proportional to a unity matrix.

Notation. In this chapter on linearization, we denote the value of a variable f evaluated at the equilibrium with the index 0, *i.e.*, f_0. The spatiotemporal deviation from the equilibrium is then denoted as $\delta f(\mathbf{x},t) \simeq f(\mathbf{x},t) - f_0(\mathbf{x},t)$, with δf assumed small. Note that this is the same notation δ as for the variations used in the previous chapter. We hope that no confusion arises due to that clash of notation.

Expression of the stress tensor. The full stress tensor computed from (2.2.37) is

$$\sigma_p = \sigma_{\rm el} + \frac{c_0 v p}{J}\,\mathrm{Id} = \sigma_{\rm el} + cvp\,\mathrm{Id} = \sigma_{\rm el} + gp\,\mathrm{Id}\,, \qquad J = \sqrt{\det b}\,,$$

where for $c(b) = c_0/J$, we used

$$\frac{\partial c}{\partial b} = -\frac{c_0}{2J} b^{-1}$$

and where

$$\sigma_{\rm el} = 2\frac{\partial V}{\partial b}\cdot b$$

is the elastic stress tensor associated to the potential V. The linearization of the full stress tensor is

$$\delta\sigma_p = \delta\sigma_{\rm el} + g_0\delta p\,\mathrm{Id}\,, \tag{3.1.3}$$

where we recall that we chose $p_0 = 0$ and that $b_0 = \mathrm{Id}$, so $J|_0 = 1$. The linearization of the elastic stress tensor is written as

$$\delta\sigma_{\rm el} = \left.\frac{\partial\sigma_{\rm el}}{\partial b}\right|_0 : \delta b + \left.\frac{\partial\sigma_{\rm el}}{\partial v}\right|_0 \delta v = 2\left.\frac{\partial^2 V}{\partial b^2}\right|_0 : \delta b + 2\left.\frac{\partial V}{\partial b}\right|_0 \cdot \delta b + 2\left.\frac{\partial^2 V}{\partial b \partial v}\right|_0 \delta v. \tag{3.1.4}$$

Linearization. The system (2.2.36) is linearized as follows:

$$\begin{cases} g_0\rho_f \partial_t \delta\boldsymbol{u}_f = -g_0\nabla\delta p + \mathbb{K}(\delta\boldsymbol{u}_s - \delta\boldsymbol{u}_f) \\ \rho_s^0 \partial_t \delta\boldsymbol{u}_s = \nabla\left(\left.\dfrac{\partial V}{\partial b}\right|_0 : \delta b\right) + \operatorname{div}\delta\sigma_p + \mathbb{K}(\delta\boldsymbol{u}_f - \delta\boldsymbol{u}_s) \\ \left.\dfrac{\partial^2 V}{\partial v^2}\right|_0 \delta v + \left.\dfrac{\partial^2 V}{\partial v \partial b}\right|_0 : \delta b = c_0\delta p\,, \qquad \delta g = -\dfrac{c_0}{2}\mathrm{Tr}\,(\delta b)\, v_0 + c_0\delta v \\ \partial_t\delta g + \operatorname{div}(g_0\delta\boldsymbol{u}_f) = 0\,, \qquad \partial_t\delta\rho_s + \operatorname{div}(\rho_s^0\delta\boldsymbol{u}_s) = 0\,, \\ \partial_t\delta b - 2\,\mathrm{Def}\,\delta\boldsymbol{u}_s = 0\,, \text{ where } \mathrm{Def}\,\delta\boldsymbol{u}_s := \frac{1}{2}\left(\nabla\delta\boldsymbol{u}_s + [\nabla\delta\boldsymbol{u}_s]^T\right). \end{cases} \tag{3.1.5}$$

To get the linearized balance of elastic momentum, we used the fact that the linearization of the term $\nabla(V - v\frac{\partial V}{\partial v}) = \nabla(V - pg)$ for $p_0 = 0$ in the second equation of (2.2.36) is

computed as

$$\delta\nabla\,(V - pg) = \nabla\Big(\underbrace{\frac{\partial V}{\partial v}\Big|_0}_{=p_0 c_0 = 0}\,\delta v + \frac{\partial V}{\partial b}\Big|_0 : \delta b\Big) - g_0 \nabla \delta p\,. \tag{3.1.6}$$

The last term in (3.1.6) then cancels with the linearization of the first term on the right hand side of (2.2.36), thus yielding the second equation in (3.1.5).

To get the last equation in (3.1.5) we used that the linearization of the Lie derivative $\pounds_{\boldsymbol{u}_s} b$ at $\boldsymbol{u}_{s,0} = 0$ and $b_0 = \mathrm{Id}$ is $-2\,\mathrm{Def}\,\delta\boldsymbol{u}_s$ as a direct computation using (2.2.15) shows.

For the linearized equations, we shall only need the coefficients of the linear and the quadratic expansions of the potential $V(b, v)$ about the equilibrium. We thus define the coefficients:

$$\sigma_0 = \frac{\partial V}{\partial b}\Big|_0\,, \quad \zeta = \frac{v_0}{c_0}\,\frac{\partial^2 V}{\partial v^2}\Big|_0\,, \quad \mathbb{C} = \frac{\partial^2 V}{\partial b^2}\Big|_0\,, \quad \mathbb{D} = \frac{\partial^2 V}{\partial v \partial b}\Big|_0\,. \tag{3.1.7}$$

The coefficient ζ, from its definition, has the order of magnitude of the bulk modulus of the microscopic material itself, although it can depend on the concentration of pores and their arrangement in the matrix. Using this, the potential energy of the elastic deformation $V(b, v)$ about the equilibrium, up to the second order in deviations from equilibrium, and assuming $V(b_0, v_0) = 0$, is represented as

$$\begin{aligned} V(b, v) \simeq \sigma_0 : (b - b_0) + \frac{1}{2}(b - b_0) : \mathbb{C} : (b - b_0) \\ + \frac{c_0 \zeta}{2 v_0}(v - v_0)^2 + \mathbb{D} : (b - b_0)(v - v_0)\,. \end{aligned} \tag{3.1.8}$$

From (3.1.7) and (3.1.4), we have

$$\delta\sigma_p = \delta\sigma_{\mathrm{el}} + g_0 \delta p\,\mathrm{Id} \;= 2\mathbb{C} : \delta b + 2\sigma_0 \cdot \delta b + 2\mathbb{D}\delta v + g_0 \delta p\,\mathrm{Id}\,. \tag{3.1.9}$$

The first term identifies the Hooke law connecting the linearized stress and linearized strain ϵ as follows

$$\sigma_1 := 2\,\frac{\partial^2 V}{\partial b^2}\Big|_0 : \delta b = 2\mathbb{C} : \delta b = 4\mathbb{C} : \epsilon\,, \qquad \epsilon := \frac{1}{2}\delta b \simeq \frac{1}{2}(b - b_0)\,, \tag{3.1.10}$$

where the definition of ϵ above is understood as a linearization of b about the equilibrium. We have intentionally denoted this linearized part of Finger tensor as ϵ since it happens to be exactly the standard linear strain used in elasticity, see (3.1.11) below.

We shall now assume an isotropic and uniform material, which will be the case of study for the remainder of the paper. Then, the tensor $\mathbb{C}$ in (3.1.10) has only two independent coefficients, and (3.1.10) becomes the familiar Hooke law for isotropic uniform materials, *i.e.*,

$$\sigma_1(\epsilon) = 4\mathbb{C} : \epsilon = 2G\epsilon + \Lambda \mathrm{Tr}\,(\epsilon)\ \mathrm{Id}\,, \tag{3.1.11}$$

where Λ and G are well known as Lamé parameters for isotropic materials in continuum mechanics [1]. Furthermore, the tensors σ_0 and $\mathbb{D}$ in (3.1.7) are proportional to the unit tensor, and it is convenient to express them as follows:

$$\sigma_0 = \frac{1}{2} g_0 \mu\ \mathrm{Id}\ , \quad \mathbb{D} = \frac{1}{2} c_0 \xi\ \mathrm{Id}\ ,\ \text{with } \mu, \xi = \text{const.} \tag{3.1.12}$$

The constants μ and ξ, defined above, as well as coefficient ζ defined by (3.1.7), have the dimension of Young's modulus, *i.e.*, pressure. With all these assumptions (3.1.9) becomes

$$\delta\sigma_p = \frac{\Lambda}{2}\mathrm{Tr}\,(\delta b)\ \mathrm{Id}\ + (G + g_0\mu)\delta b + c_0\xi\delta v\ \mathrm{Id}\ + g_0\delta p\,\mathrm{Id}\ . \tag{3.1.13}$$

Linear stability. We now set

$$\begin{aligned} &\delta \boldsymbol{u}_s = \lambda \boldsymbol{v} e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \quad \delta \boldsymbol{u}_f = \lambda \boldsymbol{u} e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \quad \delta\rho_s = \rho_{s,1} e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \quad \delta b = b_1 e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \\ &\delta g = g_1 e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \qquad \delta v = v_1 e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,, \qquad \delta p = p_1 e^{\lambda t + i\mathbf{k}\cdot\mathbf{x}}\,. \end{aligned} \tag{3.1.14}$$

and equations (3.1.5) become

$$\begin{cases} g_0\rho_f\lambda^2\boldsymbol{u} = -g_0 p_1 i\mathbf{k} + \lambda\mathbb{K}(\boldsymbol{v} - \boldsymbol{u}) \\ \rho_s^0\lambda^2\boldsymbol{v} = \dfrac{1}{2}(\Lambda + g_0\mu)\mathrm{Tr}\,(b_1)\, i\mathbf{k} + i(G + g_0\mu) b_1\cdot\mathbf{k} \\ \qquad\qquad\qquad\qquad + i\mathbf{k}(g_0 p_1 + c_0\xi v_1) + \lambda\mathbb{K}(\boldsymbol{u} - \boldsymbol{v}) \\ \dfrac{\zeta}{v_0} v_1 + \dfrac{\xi}{2}\mathrm{Tr}\,(b_1) = p_1\,, \qquad g_1 = -\dfrac{c_0}{2}\mathrm{Tr}\,(b_1)\, v_0 + c_0 v_1\,, \\ g_1 + g_0 i(\boldsymbol{u}\cdot\mathbf{k}) = 0\,, \qquad \rho_{s,1} + \rho_s^0 i(\boldsymbol{v}\cdot\mathbf{k}) = 0\,, \\ b_1 - i(\boldsymbol{v}\otimes\mathbf{k} + \mathbf{k}\otimes\boldsymbol{v}) = 0\,. \end{cases} \tag{3.1.15}$$

By using the expression $c(b) = c_0/J$ and the last equation in (3.1.15), we get $c_1 = -c_0\mathrm{Tr}\,(b_1)/2 = -c_0 i(\boldsymbol{v}\cdot\mathbf{k})$ so, from the constraint $g = cv$ we have

$$g_1 = c_0 v_1 + c_1 v_0 = c_0(v_1 - i v_0(\boldsymbol{v}\cdot\mathbf{k}))\,. \tag{3.1.16}$$

[1] Sometimes Lamé coefficients are denoted λ and μ. We will avoid that notation since it clashes with the notation used here, where λ denotes the growth rate of the disturbances as defined in (3.1.14), and μ characterizing the residual stress according to (3.1.12). We hope no confusion arises from our use of notation for Lamé coefficients.

From the linearized continuity equation for g in (3.1.15) we get $g_1 = -ig_0(\boldsymbol{u}\cdot\mathbf{k})$, so combining this with (3.1.16) we obtain the expression of v_1 as

$$v_1 = iv_0\left(\boldsymbol{v}\cdot\mathbf{k} - \boldsymbol{u}\cdot\mathbf{k}\right), \tag{3.1.17}$$

and then, substituting this result into the third equation of (3.1.15), we deduce

$$p_1 = i\left((\zeta+\xi)\left(\boldsymbol{v}\cdot\mathbf{k}\right) - \zeta(\boldsymbol{u}\cdot\mathbf{k})\right). \tag{3.1.18}$$

From (3.1.17) and (3.1.18), we get

$$g_0 p_1 + c_0\xi v_1 = ig_0\left((\zeta+2\xi)(\boldsymbol{v}\cdot\mathbf{k}) - (\zeta+\xi)(\boldsymbol{u}\cdot\mathbf{k})\right), \tag{3.1.19}$$

and the first two equations in (3.1.15) become

$$\begin{cases} g_0\rho_f\lambda^2\boldsymbol{u} = g_0\mathbf{k}\left((\zeta+\xi)\left(\boldsymbol{v}\cdot\mathbf{k}\right) - \zeta(\boldsymbol{u}\cdot\mathbf{k})\right) + \lambda\mathbb{K}(\boldsymbol{v}-\boldsymbol{u}) \\ \rho_s^0\lambda^2\boldsymbol{v} = -(\Lambda+g_0\mu)(\boldsymbol{v}\cdot\mathbf{k})\mathbf{k} - (G+g_0\mu)(\boldsymbol{v}|\mathbf{k}|^2 + \mathbf{k}(\boldsymbol{v}\cdot\mathbf{k})) \\ \qquad\qquad +g_0\mathbf{k}[(\zeta+\xi)(\boldsymbol{u}\cdot\mathbf{k}) - (\zeta+2\xi)(\boldsymbol{v}\cdot\mathbf{k})] + \lambda\mathbb{K}(\boldsymbol{u}-\boldsymbol{v}). \end{cases} \tag{3.1.20}$$

Let us now compute the dispersion relation explicitly for the case when the dissipation in the media is isotropic, so $\mathbb{K} = \beta\,\mathrm{Id}$ for some $\beta > 0$. In that case, we obtain the dispersion relation $\det\mathbb{S} = 0$ with the matrix $\mathbb{S}$ of the form

$$\begin{aligned} \mathbb{S} = \begin{bmatrix} \lambda^2\rho_f g_0\,\mathrm{Id} & 0 \\ 0 & \lambda^2\rho_s^0 \end{bmatrix} + \lambda\beta\begin{bmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{bmatrix} \\ + \begin{bmatrix} g_0\zeta\mathbb{A} & -g_0(\zeta+\xi)\mathbb{A} \\ -g_0(\zeta+\xi)\mathbb{A} & g_0(\zeta+\xi)\mathbb{A}+\mathbb{B} \end{bmatrix}, \end{aligned} \tag{3.1.21}$$

where

$$\mathbb{A} := \mathbf{k}\otimes\mathbf{k}, \quad \mathbb{B} := (\Lambda+G+g_0(2\mu+\xi))\mathbb{A} + (G+g_0\mu)|\mathbf{k}|^2\,\mathrm{Id}\,. \tag{3.1.22}$$

Remark 3.1.1 (On formal equivalence of Lamé coefficients) One can notice that in (3.1.21), G, Λ and μ only enter in combinations $G+g_0\mu$ and $\Lambda+g_0\mu$. Therefore, the acoustic properties of the media with Lamé coefficients G, Λ and with $\frac{\partial V}{\partial b}\big|_0 = \frac{1}{2}\mu g_0 \neq 0$ are the same as the acoustic properties of the media with the Lamé coefficients replaced by the shifted values

$$G \to G+g_0\mu\,, \quad \Lambda\to\Lambda+g_0\mu, \quad \text{and with} \quad \left.\frac{\partial V}{\partial b}\right|_0 = 0\,. \tag{3.1.23}$$

Since $G > 0$ for consistency of the media, we must have

$$G + g_0\mu > 0\,. \tag{3.1.24}$$

Or, more generally, two medias with G_i, Λ_i and $\frac{\partial V}{\partial b}\big|_0 = \frac{1}{2}\mu_i g_0$, $i = 1, 2$ are equivalent if $G_1 + g_0\mu_1 = G_2 + g_0\mu_2$ and $\Lambda_1 + g_0\mu_1 = \Lambda_2 + g_0\mu_2$.

Nondimensionalization. It is convenient to define the dimensionless growth rates and wavenumbers by choosing the length scales L and time scales T:

$$\lambda_* = T\lambda\,, \quad \mathbf{k}_* = L\mathbf{k}\,. \tag{3.1.25}$$

and $\delta = \rho_f g_0/\rho_s^0$ is the ratio between the effective equilibrium density of the fluid and the equilibrium density of the elastic material.

Let us define the following dimensionless matrices

$$\mathbb{A}_* = \mathbf{k}_* \otimes \mathbf{k}_*\,, \quad \mathbb{B}_* = \left(1 + \frac{\Lambda + g_0(2\mu+\xi)}{G}\right)\mathbb{A}_* + \left(1 + \frac{g_0\mu}{G}\right)|\mathbf{k}_*|^2\ \mathrm{Id}\ . \tag{3.1.26}$$

Then, dividing $\mathbb{S}$ defined by (3.1.21) by ρ_s^0, we obtain the following dimensionless dispersion matrix defining the equation for nonlinear eigenvalues (growth rates) λ_*

$$\begin{aligned}\mathbb{S}_* = \lambda_*^2 \begin{bmatrix} \delta\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} + \lambda_* \frac{\beta T}{\rho_s^0}\begin{bmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{bmatrix} \\ + \frac{g_0 T^2}{\rho_s^0 L^2}\begin{bmatrix} \zeta\mathbb{A}_* & -(\zeta+\xi)\mathbb{A}_* \\ -(\zeta+\xi)\mathbb{A}_* & (\zeta+\xi)\mathbb{A}_* \end{bmatrix} + \frac{GT^2}{\rho_s^0 L^2}\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{B}_* \end{bmatrix}.\end{aligned} \tag{3.1.27}$$

We are free to choose the time and length scales T and L, and we choose them in such a way that the coefficients of λ_* (friction term) and the last term in (3.1.27) are equal to unity. This corresponds to choosing

$$T = \frac{\rho_s^0}{\beta}, \quad L = T\sqrt{\frac{G}{\rho_s^0}}\,. \tag{3.1.28}$$

Physically, T is the typical relaxation time in the porous media; L is the distance the elastic sound waves in the matrix filled with fluid propagate during that relaxation time. We then define the dimensionless quantities

$$\zeta_* = \frac{g_0 T^2}{\rho_s^0 L^2}\zeta = g_0\frac{\zeta}{G}\,, \quad \xi_* = \frac{g_0 T^2}{\rho_s^0 L^2}\xi = g_0\frac{\xi}{G}\,, \quad \mu_* = \frac{g_0\mu}{G}\,. \tag{3.1.29}$$

With these definitions, the nondimensionalized dispersion matrix takes the form:

$$\mathbb{S}_* = \begin{bmatrix} \mathrm{Id}\,(\delta\lambda_*^2 + \lambda_*) & -\mathrm{Id}\,\lambda_* \\ -\mathrm{Id}\,\lambda_* & \mathrm{Id}\,(\lambda_*^2 + \lambda_*) \end{bmatrix} + \begin{bmatrix} \zeta_*\mathbb{A}_* & -(\zeta_*+\xi_*)\mathbb{A}_* \\ -(\zeta_*+\xi_*)\mathbb{A}_* & (\zeta_*+\xi_*)\mathbb{A}_* + \mathbb{B}_* \end{bmatrix}. \tag{3.1.30}$$

Equation $\det\mathbb{S}_* = 0$ defines a 12-th order polynomial in λ_*, and thus there are exactly 12 roots $\lambda_* = \lambda_*(\mathbf{k}_*)$ in the complex plane. We now show that given $\mathbf{k}_*$, all these roots can be computed as S- and P-waves by considering subspaces parallel and orthogonal to a given $\mathbf{k}_*$.

3.2 S-waves

Let us consider the case in which $(\boldsymbol{u}, \boldsymbol{v}) \perp \mathbf{k}_*$. Since $\mathbb{A}_* = \mathbf{k}_* \otimes \mathbf{k}_*$, we have $\mathbb{A}_* \boldsymbol{u} = \mathbb{A}_* \boldsymbol{v} = \mathbf{0}$. In other words, we only consider the displacements orthogonal to the wave vector $\mathbf{k}_*$, which is exactly the definition of an S-wave. We can set $\boldsymbol{u}_\perp$ and $\boldsymbol{v}_\perp$ to be parallel to a given vector $\boldsymbol{\xi}$ in the plane $\mathbf{k}_*^\perp$, *i.e.*, $\boldsymbol{u}_\perp = u\boldsymbol{\xi}$ and $\boldsymbol{v}_\perp = v\boldsymbol{\xi}$. The eigenvalues have multiplicity 1 and are computed from the 2×2 matrix:

$$\mathbb{S}_{*,s} = \begin{bmatrix} (\delta\lambda_*^2 + \lambda_*) & -\lambda_* \\ -\lambda_* & (\lambda_*^2 + \lambda_*) \end{bmatrix} + |\mathbf{k}_*|^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 + \mu_* \end{bmatrix} . \tag{3.2.1}$$

Since the space $\mathbf{k}_*^\perp$ is two-dimensional, all the eigenvalues of $\det \mathbb{S}_* = 0$ given by (3.1.30) with $\boldsymbol{u} \parallel \boldsymbol{\xi}$ and $\boldsymbol{v} \parallel \boldsymbol{\xi}$ have multiplicity 2. The equation $\det \mathbb{S}_{*,s} = 0$ given by (3.2.1) defines a fourth-order polynomial having 4 roots. Because of the multiplicity 2 of the S-waves, the total number of roots for S-waves is 8.

The condition $\det \mathbb{S}_{*,s} = 0$ gives either $\lambda_* = 0$, or λ_* satisfying the following cubic equation:

$$\delta\lambda_*^3 + \lambda_*^2(1 + \delta) + \lambda_* k_*^2(1 + \mu_*)\delta + k_*^2(1 + \mu_*) = 0 , \quad k_* := \|\mathbf{k}_*\| . \tag{3.2.2}$$

By Routh-Hurwitz' stability criterion (for the detailed description, see [34]), the polynomial $s^3 + a_2 s^2 + a_1 s + a_0$ is stable if $a_2 a_1 > a_0$. Thus, (3.2.2) is stable, *i.e.*, for any real k_*, $\operatorname{Re} \lambda_* < 0$, as long as $\delta > 0$ (which is natural since δ is the ratio of densities), and $\mu_* > -1$. Note that this is exactly the requirement (3.1.24) for consistency of the media.

Alternatively, instead of the dispersion relation $\lambda_* = \lambda_*(\mathbf{k}_*)$, it is common in the literature to compute the attenuation of harmonic signals in porous media, in other words, $k_*(\omega_*)$ when $\lambda_* = i\omega_*$, with $\omega_* \in \mathbb{R}$ being the frequency of forcing. In that case, from (3.2.2) we obtain

$$k_*(\omega_*) = \pm\omega_* \sqrt{\frac{1 + \delta + i\delta\omega_*}{(1 + \mu_*)(1 + i\delta\omega_*)}} . \tag{3.2.3}$$

As one can see, for $\delta > 0$ and $\mu_* > -1$, $\operatorname{Im} k_* \to 0$ when $\omega_* \to 0$, so the attenuation of low-frequency waves decreases with decreasing frequency, which is physically reasonable. If one considers propagation of waves for $x > 0$, one needs to choose the sign in the equation for $k_*(\omega_*)$ in such a way that $\operatorname{Im} k_*(\omega_*) > 0$, so the waves will be decaying as $x \to \infty$.

3.3 P-waves

Consider the case $(\boldsymbol{u}, \boldsymbol{v}) \parallel \mathbf{k}$. In other words, we consider the disturbances parallel to the wave vector $\mathbf{k}$, which is the definition of a P-wave. Then $\mathbb{A}_*\boldsymbol{u} = (\mathbf{k}_* \cdot \boldsymbol{u})\mathbf{k}_* = |\mathbf{k}_*|^2\boldsymbol{u}$, and $\mathbb{A}_*\boldsymbol{v} = (\mathbf{k}_* \cdot \boldsymbol{v})\mathbf{k}_* = |\mathbf{k}_*|^2\boldsymbol{v}$, and the dispersion relation $\det \mathbb{S}_* = 0$ takes the form $\det \mathbb{S}_{*,p} = 0$ for the 2×2 matrix

$$\mathbb{S}_{*,p} = \begin{bmatrix} \delta\lambda_*^2 + \lambda_* & -\lambda_* \\ -\lambda_* & \lambda_*^2 + \lambda_* \end{bmatrix} + k_*^2 \begin{bmatrix} \zeta_* & -(\zeta_* + \xi_*) \\ -(\zeta_* + \xi_*) & \zeta_* + 2\xi_* + Z \end{bmatrix}, \tag{3.3.1}$$

where we defined for shortness

$$k_* := \|\mathbf{k}\|, \quad Z := 2 + \frac{\Lambda}{G} + 3\mu_* \tag{3.3.2}$$

and we used $\frac{g_0\xi}{G} = \xi_*$ by (3.1.28) and (3.1.29). We rewrite this dispersion relation as

$$\det \begin{bmatrix} \delta\lambda_*^2 + \lambda_* + k_*^2\zeta_* & -\lambda_* - (\zeta_* + \xi_*)k_*^2 \\ -\lambda_* - k_*^2(\zeta_* + \xi_*) & \lambda_*^2 + \lambda_* + (\zeta_* + 2\xi_* + Z)k_*^2 \end{bmatrix} = 0. \tag{3.3.3}$$

Equation (3.3.3) defines a fourth-order polynomial for λ_*, thus, for a given $\mathbf{k}_*$ there are 4 roots corresponding to the P-waves. Combining with 8 roots for S-waves, we get the total number of roots found being equal to 12, which is exactly the number of solutions for $\lambda_*(\mathbf{k}_*)$ expected from (3.1.30). Thus, we have found all the roots of the equation (3.1.30). After computing the determinant in (3.3.3) we get the following polynomial

$$\delta\lambda_*^4 + \lambda_*^3(\delta + 1) + \lambda_*^2 k_*^2 \left(\zeta_* + \delta(Z + 2\xi_* + \zeta_*)\right) + \lambda_* k_*^2 Z + k_*^4(\zeta_* Z - \xi_*^2) = 0\,. \tag{3.3.4}$$

For the stability of polynomial (3.3.4) we investigate the principal minors Δ_i, $i = 1, \dots 4$ of the Hurwitz matrix associated with the polynomial (Liénard-Chipart form of the criterion), for the detailed description, see Gantmacher's book [34]. The Hurwitz matrix corresponding to the polynomial (3.3.4) has the form

$$\begin{bmatrix} \delta + 1 & Zk_*^2 & 0 & 0 \\ \delta & K_1 & K_2 & 0 \\ 0 & \delta + 1 & Zk_*^2 & 0 \\ 0 & \delta & K_1 & K_2 \end{bmatrix} \tag{3.3.5}$$

where we have defined

$$K_1 = \left(\zeta_*(\delta + 1) + \delta(Z + 2\xi_*)\right) k_*^2, \qquad K_2 = \left(\zeta_* Z - \xi_*^2\right) k_*^4\,. \tag{3.3.6}$$

All Δ_i (with their exact forms given below) must be positive for stability. First, we notice that the conditions $\Delta_1 > 0$ and $\Delta_3 > 0$ read

$$\Delta_1 = \delta + 1 > 0 \text{ and } \Delta_3 = (\delta Z + (\delta+1)\xi_*)^2 k_*^4 > 0, \tag{3.3.7}$$

and are trivially satisfied.

Next, we study the condition $\Delta_4 > 0$. Since $\Delta_3 > 0$, we can write, equivalently,

$$\frac{\Delta_4}{\Delta_3 k_*^4} = \zeta_* Z - \xi_*^2 > 0 \quad \Leftrightarrow \quad \zeta_* \left(\frac{\Lambda}{G} + 2 + 3\mu_* \right) > \xi_*^2 \,. \tag{3.3.8}$$

Finally, we compute the condition $\Delta_2 > 0$:

$$\frac{\Delta_2}{k_*^2} = \delta^2 Z + 2\delta(\delta+1)\xi_* + (\delta+1)^2 \zeta_*. \tag{3.3.9}$$

For stability of the steady state, we must have $\zeta_* > 0$, otherwise $v = v_0$ is not a stable equilibrium. Multiplying condition (3.3.9) by ζ_*, and adding/subtracting the term $\delta^2 \xi_*^2$, we obtain an equivalent formulation

$$\frac{\Delta_2}{k_*^2}\zeta_* = \delta^2 \left(Z\zeta_* - \xi_*^2 \right) + \left(\delta\xi_* + (\delta+1)\zeta_* \right)^2 > 0 \,. \tag{3.3.10}$$

which is satisfied as long as (3.3.8) is true. Since for physical reasons we necessarily have $G > 0$, the stability condition for the P-waves can be rewritten as

$$\zeta_* > 0 \quad \text{and} \quad 2\left(G + G\mu_*\right) + \left(\Lambda + G\mu_* - G\frac{\xi_*^2}{\zeta_*} \right) > 0 \,. \tag{3.3.11}$$

Using the conditions (3.1.28) and (3.1.29), we can transform (3.3.11) to the following form which will be useful for using the Sylvester criterion (3.3.20) below:

$$2\left(G + g_0\mu\right) + \left(\Lambda + g_0\mu - g_0\frac{\xi^2}{\zeta} \right) > 0 \,. \tag{3.3.12}$$

We shall now show that the condition for the stability of the P-waves (3.3.8) is exactly equivalent to the requirement for consistency of modified P-wave modulus in an isotropic medium.

A digression: Linear stability of purely elastic media. Let us now elucidate the physical meaning of (3.3.8), which, as we show, is simply the condition on the stability of propagation of P-waves in an elastic media. Suppose a wave is propagating in an

elastic media with Lamé coefficients (Λ, G) in accordance with (3.1.11). The linearized equation for wave propagation is

$$\rho_s^0 \partial_t \delta \boldsymbol{u}_s = \operatorname{div} \sigma_1(\epsilon) \quad \Leftrightarrow \quad \lambda^2 \boldsymbol{v} = -G\boldsymbol{v}|\mathbf{k}|^2 - (\Lambda + G)(\mathbf{k} \cdot \boldsymbol{v})\mathbf{k}\,, \tag{3.3.13}$$

where we assumed unstressed or relaxed elastic media *i.e.* $\sigma_0 = \frac{\partial V}{\partial b}\big|_0 = 0$ so that $\mu = 0$.

For S-waves, $\boldsymbol{v} \perp \mathbf{k}$, and λ is purely imaginary if and only if $G > 0$. For P-waves, λ is purely imaginary if $2G + \Lambda > 0$. The coefficient $2G + \Lambda$ is also known as the P-wave modulus of the elastic media. As we shall see, the condition of positive P-wave modulus will play the crucial part in the stability considerations.

For further discussion, it is interesting to compute the general condition on the convexity of the potential energy in the purely elastic case. In this case (3.1.8) reduces to

$$\begin{aligned} V_0(b) &\simeq \frac{1}{2}(b - b_0) : \mathbb{C} : (b - b_0) \simeq G\epsilon : \epsilon + \frac{1}{2}\Lambda(\operatorname{Tr}(\epsilon))^2 \\ &= 2G \sum_{i>j} \epsilon_{ij}^2 + \frac{1}{2}\mathbf{X}_0^T \mathbb{Q}_0 \mathbf{X}_0\,, \quad \mathbf{X}_0 := (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}), \end{aligned} \tag{3.3.14}$$

and we have defined the quadratic form $\mathbb{Q}_0$ to be

$$\mathbb{Q}_0 := \begin{bmatrix} 2G + \Lambda & \Lambda & \Lambda \\ \Lambda & 2G + \Lambda & \Lambda \\ \Lambda & \Lambda & 2G + \Lambda \end{bmatrix}. \tag{3.3.15}$$

Assuming that the coefficients ϵ_{ij} are independent numbers for a given deformations, the condition on V_0 to be positive definite is equivalent to the condition that the quadratic form $\mathbb{Q}_0$ is positive definite. By the Sylvester criterion, the quadratic form is positive definite if and only if all the leading principal minors are positive, leading to

$$\text{a)}\ \ 2G + \Lambda > 0\,, \quad \text{b)}\ \ 2G + 2\Lambda > 0\,, \quad \text{c)}\ \ 2G + 3\Lambda > 0\,. \tag{3.3.16}$$

The first minor, *i.e.*, condition a) is exactly the stability of P-waves. The third condition c) is equivalent to the positivity of the bulk modulus of the material. The second condition b) follows from the first and the third conditions.

As we shall see immediately below, the conditions for the well-posedness of the P-waves and positive definite nature of the potential energy for the porous media follows closely the purely elastic framework, with the appropriate corrections due to the dynamics of the pores v.

Justification of (3.3.8) from the potential energy considerations. Let us now consider the case of a general potential energy $V(b,v)$ locally expressed about the equilibrium according to the quadratic expansion (3.1.8). Without loss of generality we consider the case with no linear terms, *i.e.* $\mu = 0$, since the terms proportional to μ can be absorbed into G and Λ according to (3.1.23), with $G \to G + \mu$ and $\Lambda \to \Lambda + \mu$. We have

$$\begin{aligned} V(b,v) &\simeq \frac{1}{2}(b-b_0) : \mathbb{C} : (b-b_0) + \frac{c_0\zeta}{2v_0}(v-v_0)^2 + c_0\xi \mathrm{Tr}\,(\epsilon)\,(v-v_0) \\ &\simeq G\epsilon : \epsilon + \frac{1}{2}\Lambda(\mathrm{Tr}\,(\epsilon))^2 + \frac{c_0\zeta}{2v_0}(v-v_0)^2 + c_0\xi \mathrm{Tr}\,(\epsilon)\,(v-v_0), \end{aligned} \tag{3.3.17}$$

hence V is a quadratic form of 7 variables: (ϵ_{ij}) (6 elements from symmetry) and $(v-v_0)$. However, the off-diagonal elements of tensor ϵ, namely $(\epsilon_{12}, \epsilon_{23}, \epsilon_{13})$ enter only in terms of squares multiplied by $G > 0$. Thus, we rewrite (3.3.17) in the following form:

$$V(b,v) \simeq 2G\sum_{i>j}\epsilon_{ij}^2 + \frac{1}{2}\mathbf{X}^T \cdot \mathbb{Q} \cdot \mathbf{X}, \quad \mathbf{X} := (v-v_0, \epsilon_{11}, \epsilon_{22}, \epsilon_{33})^T\,, \tag{3.3.18}$$

where we have defined a 4×4 quadratic form $\mathbb{Q}$ as

$$\mathbb{Q} := \begin{bmatrix} \frac{c_0\zeta}{v_0} & c_0\xi & c_0\xi & c_0\xi \\ c_0\xi & 2G+\Lambda & \Lambda & \Lambda \\ c_0\xi & \Lambda & 2G+\Lambda & \Lambda \\ c_0\xi & \Lambda & \Lambda & 2G+\Lambda \end{bmatrix}. \tag{3.3.19}$$

Assuming the independence of all components of the strain tensor ϵ_{ij}, we see that V is a convex, positive definite function if and only if the quadratic form $\mathbb{Q}$ is positive definite.

The Sylvester criterion gives four stability conditions:

$$\begin{cases} \Delta_1 = \dfrac{c_0\zeta}{v_0} > 0 \\ \Delta_2 = \dfrac{c_0\zeta}{v_0}\left(2G+\Lambda - \dfrac{g_0\xi^2}{\zeta}\right) > 0 \\ \Delta_3 = \dfrac{4G\zeta c_0}{v_0}\left(G+\Lambda - \dfrac{g_0\xi^2}{\zeta}\right) > 0 \\ \Delta_4 = \det\mathbb{Q} = \dfrac{4c_0G^2\zeta}{v_0}\left[2G + 3\left(\Lambda - \dfrac{g_0\xi^2}{\zeta}\right)\right] > 0\,, \end{cases} \tag{3.3.20}$$

where we recall that $g_0 = c_0v_0$. The first condition of this system simply enforces the convexity of V with respect to the small changes in v about the equilibrium, and is thus very natural. To investigate the remaining three conditions, let us denote

$$\tilde{\Lambda} = \Lambda - \frac{g_0\xi^2}{\zeta}\,. \tag{3.3.21}$$

We notice that the conditions for $\Delta_2 > 0$, $\Delta_3 > 0$ and $\Delta_4 > 0$ in (3.3.20) are equivalent to the conditions (3.3.16) with the substitution $\Lambda \to \tilde{\Lambda}$. Thus, the new variable defined by (3.3.21) acquires the physical meaning of the effective value of the second Lamé coefficient for the porous media. We remind the reader that the coefficients ζ and ξ encode the values of the second derivatives of V with respect to v and (v, b) respectively, and are thus appearing only in the description of the porous media. No corresponding values exist for the purely elastic media. It is thus even more surprising that the stability criteria for the porous media can be written in the form very similar to the elastic media through the combination of variable (3.3.21).

Note also that the condition for the P-wave stability (3.3.11) for a general μ_* can now be written using the shift (3.1.23) as $\Delta_2 > 0$ in (3.3.20). The last condition of (3.3.20), *i.e.*, $\Delta_4 > 0$, is equivalent to the requirement that the effective bulk modulus of a dry porous matrix is positive, which is trivially satisfied for all materials unless they demonstrate strongly nonlinear or non quasi-stationary behaviour (for example local phase transitions).

3.4 Comparison with Biot's theory

The dispersion relation $\mathbb{S}(\boldsymbol{u}, \boldsymbol{v})^T$ described by (3.1.21) can be mapped to a system of linear PDEs. Let us assume, for simplicity, an isotropic media and take $\mathbb{K} = \beta \,\mathrm{Id}$. We use the mapping of powers of $\mathbf{k}$ to differential operators in Fourier space as $\mathbf{k} \otimes \mathbf{k} \to -\nabla\mathrm{div}$ and $|\mathbf{k}|^2 \to -\Delta$ to get

$$\begin{cases} \rho_f g_0 \dfrac{\partial^2}{\partial t^2}\boldsymbol{u} + \beta \dfrac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{v}) - g_0\zeta\nabla\mathrm{div}\boldsymbol{u} + g_0(\zeta + \xi)\nabla\mathrm{div}\boldsymbol{v} = \mathbf{0} \\ \rho_s^0 \dfrac{\partial^2}{\partial t^2}\boldsymbol{v} - \beta \dfrac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{v}) + g_0(\zeta + \xi)\nabla\mathrm{div}\boldsymbol{u} \\ \qquad - \left(g_0(\zeta + 2\xi + 2\mu) + \Lambda + G\right)\nabla\mathrm{div}\boldsymbol{v} - \left(G + g_0\mu\right)\Delta\boldsymbol{v} = \mathbf{0}\,. \end{cases} \tag{3.4.1}$$

Note that the contribution from pressure in our system exactly cancel, which is reasonable, as the pressure fluctuations generated by the motion of porous media in an internal force and thus must vanish. The corresponding Biot's system is given by

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}(\rho_{22}^{(f)}\boldsymbol{u} + \rho_{12}\boldsymbol{v}) + \beta \dfrac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{v}) - \nabla\mathrm{div}\left(R\boldsymbol{u} + Q\boldsymbol{v}\right) = \mathbf{0}, \\ \dfrac{\partial^2}{\partial t^2}(\rho_{11}^{(s)}\boldsymbol{v} + \rho_{12}\boldsymbol{u}) - \beta \dfrac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{v}) - \nabla\mathrm{div}\left(Q\boldsymbol{u} + P\boldsymbol{v}\right) + N\nabla\times\nabla\times\boldsymbol{v} = \mathbf{0} \end{cases} \tag{3.4.2}$$

with N being shear modulus of the skeleton and the fluid/elastic body, assumed to be the same. We shall note that Biot's equations is not directly applicable to an incompressible fluid, since the expressions for the variables P, Q and R in (3.4.2) involve explicitly the bulk modulus of the fluid. However, if we proceed formally and use the equations from the literature and put $K_f = \infty$ for an incompressible fluid, the expressions for P, Q and R in terms of the bulk moduli of the porous skeleton K_b and the elastic body itself K_s, see *e.g.*, [33] are given by

$$P = (1 - g_0)K_s + \frac{4}{3}N\,, \quad Q = g_0 K_s\,, \quad R = \frac{g_0^2 K_s}{1 - g_0 - K_b/K_s}\,. \tag{3.4.3}$$

Let us turn our attention to our theory described in (3.4.1), where we have set $\rho_{12} = \rho_{21} = 0$. The case of $\rho_{12} \neq 0$ and $\rho_{21} \neq 0$ can be easily incorporated by considering a more general inertia matrix in the Lagrangian. There is also an exact correspondence between the friction terms. Thus, we need to compare the coefficients of the spatial derivative terms. A direct comparison between Biot's linearized system (3.4.2) and (3.4.1) gives $R = g_0\zeta$ by observing the coefficients of the terms proportional to $\nabla \mathrm{div} \boldsymbol{u}$ from the equations (3.4.2). From the term proportional to $\nabla \mathrm{div} \boldsymbol{v}$ in the first equation of (3.4.2), we obtain $Q = -g_0(\xi + \zeta)$. Finally by using $\nabla \times \nabla \times \boldsymbol{v} = \nabla \mathrm{div}\, \boldsymbol{v} - \Delta \boldsymbol{v}$ we obtain the expressions of N and P. To summarize, the Biot's coefficients (P, Q, R, N) are given by

$$\begin{aligned} R &= g_0\zeta, \\ Q &= -g_0(\xi + \zeta), \\ N &= G + g_0\mu, \\ P &= (\Lambda + g_0\mu) + 2(G + g_0\mu) + g_0(\zeta + 2\xi)\,. \end{aligned} \tag{3.4.4}$$

Note that the expression $\Lambda + 2G$ is also known as the $P-$wave modulus. In our case, this P-wave modulus is modified by a shift of Lamé coefficients by $g_0\mu$ and additional terms ξ and ζ coming from the elasticity properties of the porous matrix.

3.5 Numerical investigation of phase and group velocities and attenuation

We investigate the non-dimensionalized dispersion relations (3.2.2) and (3.3.4) derived above in order to explore the phase velocity, group velocity, and attenuation coefficients

of wave propagation in porous media. Instead of computing the roots $\lambda = \lambda(k)$, we consider the response of the system to a fixed frequency, as is common in the literature. We shall notice, that due the linear stability of the wave propagation, the growth exponent, i.e. the real part $\mathfrak{Re}\lambda(k) \leq 0$ for all wave vectors k, so the system is well behaved. Thus, we take $\lambda = i\omega$ as a fixed parameter, and compute $k = k(\omega)$ from the dispersion relations. Then, the phase velocity is given by $v_p = \operatorname{Re}\omega/k(\omega)$. Once $k(\omega)$ is known, we compute the group velocity $v_g = \operatorname{Re} d\omega/dk = \operatorname{Re}(dk/d\omega)^{-1}$ by directly differentiating the dispersion relations as an implicit function and substituting $(\omega, k = k(\omega))$. We also present the attenuation coefficient for the wave $\operatorname{Im} k(\omega)$ and attenuation per cycle $\operatorname{Im} k(\omega)/\operatorname{Re} k(\omega)$.

According to (3.1.29), and the fact that ζ has the order of magnitude of the microscopic bulk modulus, most materials will have $\zeta_* \sim 1$, $\xi_* \sim 1$, $Z \sim 1$ at the order of magnitude. For biological materials, δ tends to be large whereas for porous media made out of dense materials conveying gas, δ is small. We thus explore both large and small values of δ in the simulations. In Figures 3.5.1-3.5.3 we present the results of computation of dispersion relation for the P-waves for a set of different parameters δ, Z, ζ_* and ξ_*. Only two roots of equation (3.3.4) are shown since the equation is a quadratic equation in k^2. The other roots correspond to the waves propagating with the same velocity and attenuation coefficient in the opposite direction. In Figures 3.5.4-3.5.6 we present the results of computation of dispersion relation for the S-waves for a set of different parameter δ. The axes variables in the figures are dimensionless, rescaled according to the time and length scales defined in (3.1.28).

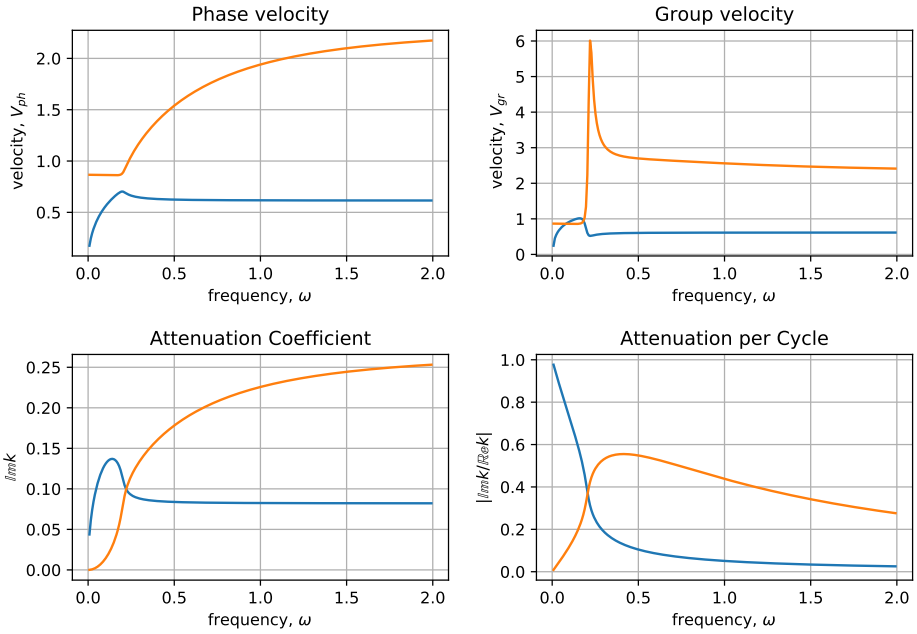

Figure 3.5.1: Velocities and attenuation coefficients for P-waves with $\delta = 3$, $Z = 3$, $\zeta_* = 2$ and $\xi_* = 0$.

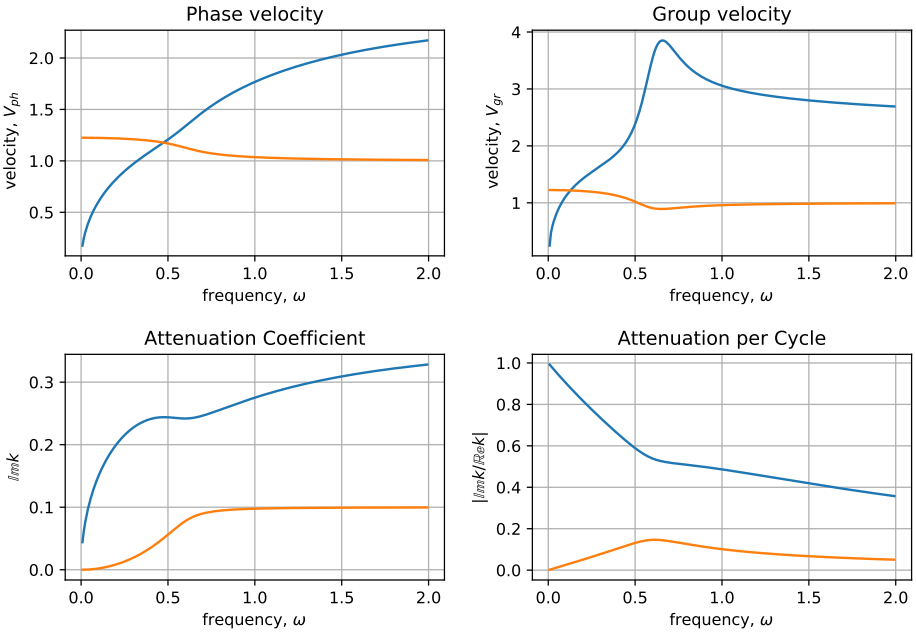

Figure 3.5.2: Velocities and attenuation coefficients for P-waves with $\delta = 1$, $Z = 3$, $\zeta_* = 2$ and $\xi_* = 0$.

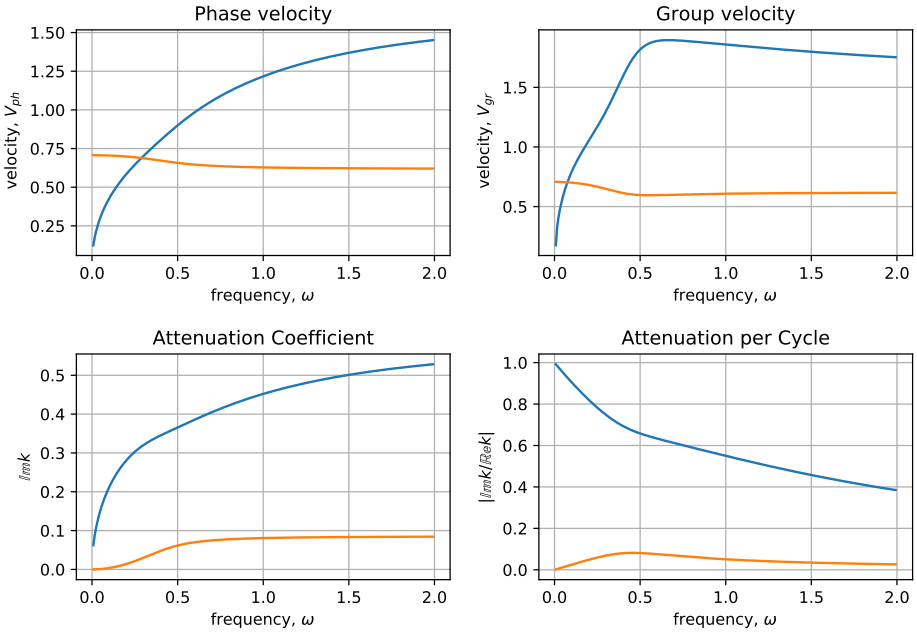

Figure 3.5.3: Velocities and attenuation coefficients for P-waves with $\delta = 1$, $Z = 1$, $\zeta_* = 1$ and $\xi_* = 1$.

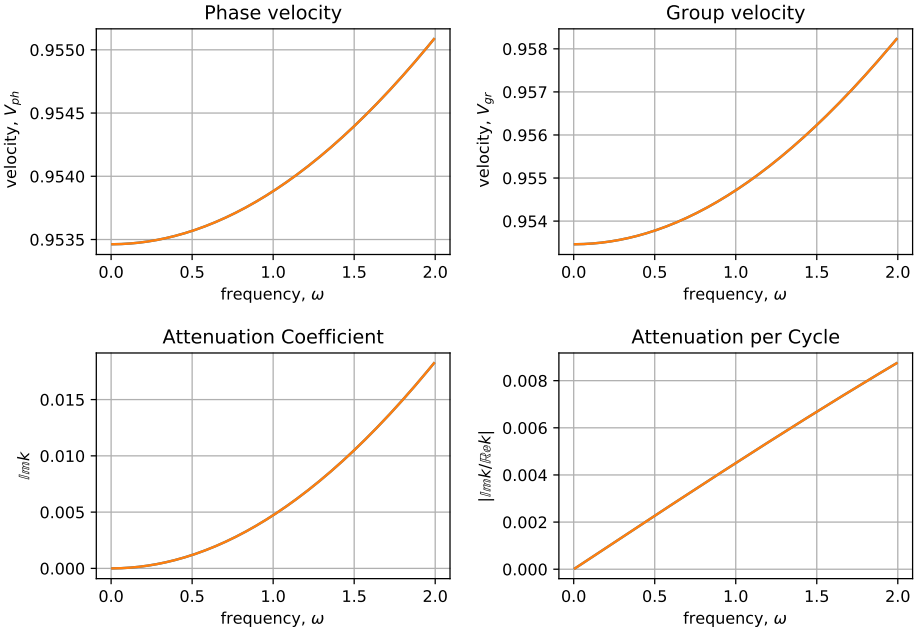

Figure 3.5.4: Velocities and attenuation coefficients for S-waves with $\delta = 0.1$ and $\mu_* = 0$.

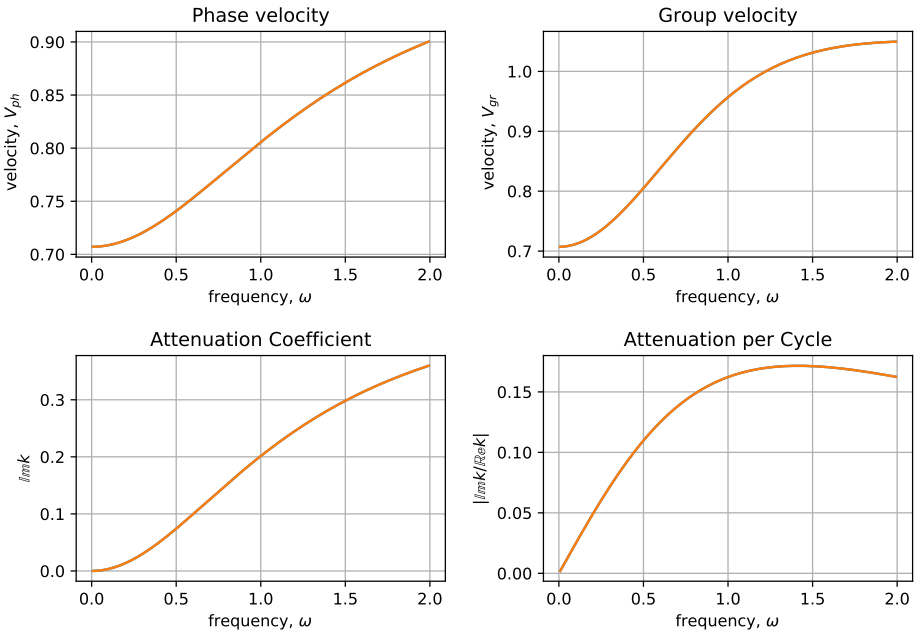

Figure 3.5.5: Velocities and attenuation coefficients for S-waves with $\delta = 1$ and $\mu_* = 0$.

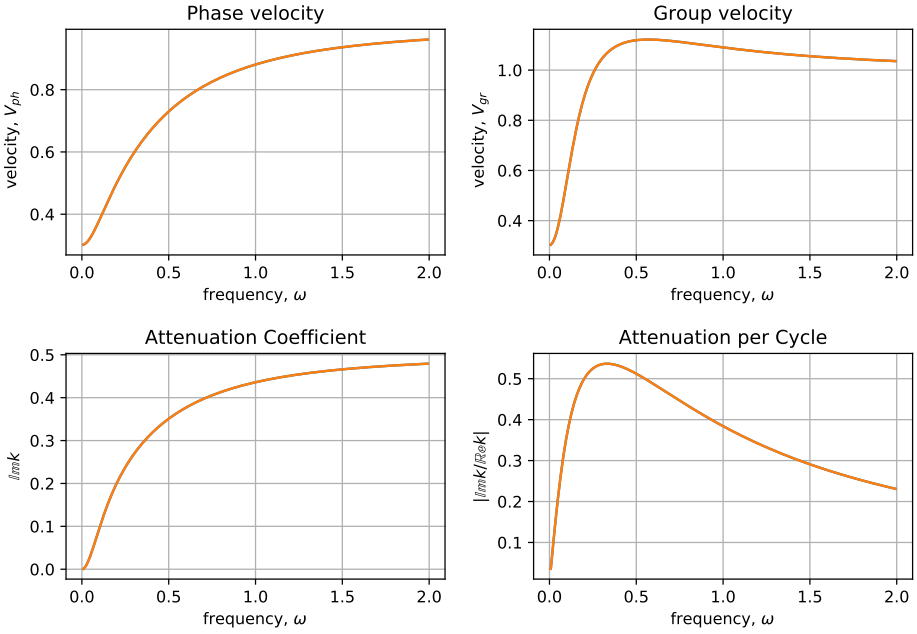

Figure 3.5.6: Velocities and attenuation coefficients for S-waves with $\delta = 10$ and $\mu_* = 0$.

Chapter 4

Actively deforming porous media

Many parts of biological organisms are comprised of deformable porous media. The biological media is both pliable enough to deform in response to an outside force and can deform by itself using the work of an embedded muscle. For example, the recent work [56] has demonstrated interesting 'sneezing' dynamics of a freshwater sponge, when the sponge contracts and expands to clear itself from surrounding polluted water.

In this Chapter, I present the derivation of the equations of motion for the dynamics of such an active porous media (*i.e.*, a deformable porous media that is capable of applying a force to itself with internal muscles), filled with an incompressible fluid. These equations of motion extend the earlier derived equation for a passive porous media filled with an incompressible fluid. We use a variational approach with a Lagrangian written as the sum of terms representing the kinetic and potential energy of the elastic matrix, and the kinetic energy of the fluid, coupled through the constraint of incompressibility. We then proceed to extend this theory by computing the case when both the active porous media and the fluid are incompressible, with the porous media still being deformable, which is often the case for biological applications. For the particular case of a uniform initial state, we rewrite the equations of motion in terms of two coupled telegraph-like equations for the material (Lagrangian) particles expressed in the Eulerian frame of reference, particularly suitable for numerical simulations, formulated for both the compressible media/incompressible fluid case and the doubly incompressible case. We derive interesting conservation laws for the motion, perform numerical simulations in both cases and show the possibility of self-propulsion of a biological organism due to particular running wave-like application of the muscle stress.

4.1 Introduction

The mechanics of deformable porous media filled with fluid, also known as *poromechanics*, plays an important role in understanding the dynamics of biological organisms. From arterial walls to fibrous spider silk and to sea sponges, multi-layer materials with the complex internal structure are ubiquitous in nature [63], [64]. When such porous materials are immersed in fluids, for example, as is the case for underwater organisms or various internal organs of the human body, the porous media interacts with the fluid permeating it [50]. Thus the understanding of the dynamics of such a system is important for our comprehension of the science of biological tissue. In many biological applications, there is an additional complication due to an internal muscle acting on the material and deforming it. In this case, the effect of the muscle stress should be combined with the dynamics of the elastic matrix (we call it 'solid' for briefness in this article) and of the fluid permeating the matrix. Finally, an essential feature of many biological materials is the large percentage of water in the elastic matrix itself, leading to virtual incompressibility of the elastic material. This paper addresses these challenges using a variational approach to the motion of active porous media.

It is useful to start with a short review of earlier works in poromechanics. Due to the large amount of work in the area, our description must necessarily be brief and only focus on the works essential to this discussion. The earlier developments in the field of poromechanics are due to K. von Terzaghi [77] and M. Biot [9], [10], [15] in the consolidation of porous media, and subsequent works by M. Biot which derived the time-dependent equations of motion for poromechanics, based on certain assumptions on the media. M. Biot also considered the wave propagation in both low and high wavenumber regime [11]–[14]. The amount of recent work in the field of porous media is vast, both in the field of model development [19], [21], [30], [31], [44], [49] and their subsequent mathematical analysis [6], [16], [76]. We refer the reader interested in the history of the field to the review [67] for a more detailed exposition of the literature.

Biot's work still remains highly influential today, especially in the field of acoustic propagation of waves through porous media. However, subsequent careful investigations have revealed difficulties in the interpretation of various terms through the general principles of mechanics, such as material objectivity, frequency-dependent permeability and

changes of porosity in the model, as well as the need to describe large deformations of the model [82]. Based on this criticism, [82] develops an alternative approach to saturated porous media equations which does not have the limitations of the Biot's model. Alternatively, [23], [24] develop equations for saturated porous media based on the general thermodynamics principles of mechanics.

The mainstream approach to the porous media has been to treat the dynamics as being friction-dominated by dropping the inertial terms from the equations. The seminal book of Coussy [26] contains a lot of background information and analysis on that approach. For more recent work, we will refer the reader to, for example, the studies of multi-component porous media flow [70], as well as the gradient approach to the thermo-poro-visco-elastic processes [18]. Our work, in contrast, is dedicated to the development of modes using variational principles of mechanics, which is a sub-field of all the approaches to porous media. The equations we will derive here, without the viscous terms, will be of an infinite-dimensional Hamiltonian type and approximate the inertia terms and large deformations consistently. On the other hand, the friction-dominated approach gives equations of motion that are of gradient flow type.

Fluid-filled elastic porous media, by its very nature, is a highly complex object involving both the individual dynamics of the fluid and the media and highly nontrivial interactions between them. It is not realistic to assume complete knowledge of the micro-structured geometry of pores in the elastic matrix and the details of fluid motion inside the pores. Hence, models of porous media must include interactions between the macroscopic dynamics and an accurate, and yet treatable, description of *relevant aspects* of the micro-structures. Because of the large variations of the geometry and dynamics of micro-structures between different porous media (e.g. biological materials vs geophysical applications), the task of deriving a detailed, unified theory of porous media suitable for all applications is most likely not possible. However, once a detailed set of assumptions is provided and their limitation is understood, deriving a consistent theory is possible. In such a framework, we believe, the variational theory is advantageous since it can develop a consistent mathematical model satisfying the physical assumptions on the system. Variational methods are developed by first describing the Lagrangian of the system on an appropriate configuration manifold, and then computing the critical curves of the associated action functional to obtain the equations of motion in a systematic way.

The advantage of the variational methods is their consistency, as opposed to approaches based on balancing the conservation laws for a given point, or volume, of fluid. In a highly complex system like poromechanics, primarily when written in the non-inertial Lagrangian frame associated with the matrix, writing out all the forces and torques to obtain correct equations is very difficult. In contrast, the equations of motion follow from variational methods automatically, and the conservation laws are obtained in a general setting, *i.e.* for arbitrary Lagrangians, as long as necessary symmetry arguments are satisfied.

One of the earliest papers in the field of variational methods applied to the porous media was [7] where the kinetic energy of microscopic expansion was incorporated into the Lagrangian to obtain the equations of motion. In that work, several Lagrange multipliers were introduced to enforce the continuity equation for both solid and fluid. The works [3], [4] use variational principles for explanation of the Darcy-Forchheimer law. Furthermore, [54], [55] derive the equations of porous media using additional terms in the Lagrangian coming from the kinetic energy of the microscopic fluctuations. Of particular interest to us are the works on the Variational Macroscopic Theory of Porous Media (VMTPM) which was formulated in its present form in [2], [28], [29], [68], [69], [71]–[73], [75], [78], also summarized in a recent book [74]. In these works, the microscopic dynamics of capillary pores is modelled by a second-grade material, where the internal energy of the fluid depends on both the deformation gradient of the elastic media and the gradients of local fluid content. The study of a pre-stressed system using variational principles and subsequent study of the propagation of sound waves was undertaken in [65].

One of the main assumptions of the VMTPM is the dependence of the internal energy of the fluid on the quantity measuring the micro-strain of the fluid, or, alternatively, the fluid content or local density of the fluid, including, in some works, the gradients of that quantity. This assumption is physically relevant for compressible fluid, but, in our view, for an incompressible fluid (which, undoubtedly, is a mathematical abstraction, as bulk moduli or inverse compressibilities of physical materials, including fluids can not be infinite), such dependence is difficult to interpret. For example, for geophysical applications, fluids are usually considered compressible because of the high pressures involved. In the biological applications like the dynamics of highly porous sponges in

the water we are interested in, the compressibility effects of the water, and, as we shall discuss here, of the sponge itself, can be neglected. For a truly incompressible fluid, it is difficult to assign physical meaning to the dependence of internal energy of the fluid on the parameters of the porous media. The paper [82] points out the difficulties of developing the true variational principle for Biot's model because of the difficulty of interpreting the fluid content through a variational principle. This difficulty in the interpretation of the fluid content was explained in [32], in terms of the fluid content being a constraint in the fluid incompressibility, and the fluid pressure being a Lagrange multiplier related to the incompressibility. Interestingly, Biot model for acoustic wave propagation in porous media was developed in that paper as the linearized case of wave propagation for certain Lagrangians.

In this paper, we go one step beyond the initial variational approach developed in [32] and show that the additional complexity of matrix incompressibility for biologically relevant materials can be treated similarly, as the solid's incompressibility constraint introduces another Lagrange multiplier related to the pressure inside the solid. Mathematically, we base our methods on the classical Arnold description of incompressible fluid [1] as geodesic motion on the group of volume-preserving diffeomorphisms of the fluid domain, in the absence of external forces. In Arnold's theory, the Lagrangian is simply the kinetic energy, as the potential energy of the fluid is absent, and the fluid pressure enters the equations from the incompressibility condition. This paper continues the initial derivation of [32], and achieves new results in the following directions:

1. We introduce incompressibility of both the fluid and the elastic matrix in the variational approach.

2. We also include the actions of the muscle by using a variational approach and show that the application of the muscle and its effect on the boundaries follows from a modified Lagrange-d'Alembert principle.

3. We show an exact reduction of the model for one-dimensional motion and derive integrals of motion, such as the net momenta and, in the case of double incompressibility, an affine relationship between the Lagrangian variables of the fluid and the solid.

4. We perform numerical simulations of the resulting reduced one-dimensional equations for incompressible fluid, for both compressible and incompressible matrix. We illustrate the difference between these cases and show the possibility of self-propulsion of the porous matrix (solid) while preserving the net-zero momentum of the fluid and solid.

It is also useful to have a short discussion on the choice of coordinates and physics of what is commonly considered the saturated porous media. In most, if not all, previous works, the saturated porous media is a combined object consisting of an (elastic) dense matrix, and a network of small connected pores filled with fluid. The fluid encounters substantial resistance when moving through the pores due to viscosity and the no-slip condition on the boundary. In such a formulation, it is easier to consider the motion of the porous matrix to be 'primary', and the fluid motion to be computed relative to the porous matrix coordinates. Because the motion of the elastic matrix is 'primary', the equations are written in the system of coordinates consistent with the description of the elastic media, which is the material frame associated with the media. In this paper, we take an alternative view where we choose the same coordinate system of the stationary observer (Eulerian frame) for the description of both the fluid and the elastic media. Such a system is more frequently used in the classical fluid description but is less common in the description of elastic media. Ironically, the Eulerian frame is also frequently used in the description of wave propagation in the media, in particular, classical Biot's theory [11]–[14]. Physically, our description is more relevant for the case of a porous media consisting of a dense network of elastic 'threads' positioned inside the fluid, which is a case that has not been considered before apart from [32]. In our formulation, we choose the Eulerian description for both the fluid and the elastic matrix. It is worth noting that the combined Eulerian description is also applicable to the traditional porous media with a 'dense' matrix, and is also well suited for the description of wave propagation in such media as shown in [32] comparing the results of variational models with that of Biot. We shall also point out that our theory can be reformulated and is applicable for the familiar choice of the Lagrangian material description of the elastic porous matrix. These descriptions are completely equivalent from the mathematical point of view, and this is rigorously justified by using the process of Lagrangian reduction by symmetry in continuum mechanics [38].

4.2 Elastic muscle stress in the case of living organisms

Let us now consider a model for a porous living organism that can generate internal stress in the solid matrix in addition to the elastic stress experienced by the matrix from the deformation. The physical context of this work are the sponges generating their own internal stress to contract and expunge water from themselves. The sponges generate their own internal muscle stress which we call $\bar{s}$, when we compute that stress in the Eulerian frame of reference. The forces $\operatorname{div}\bar{s}$ generated by this stress in the spatial frame have to be considered as external in Hamilton's principle (2.2.24), acting only on the solid part of the system.

In this work, we shall only consider the mechanical effect of the muscle stress, without getting into the details of the actual mechanism of generation of stress itself. The microscopic dynamics of the muscle itself is highly complex [43], and is not essential at this point. It may, however, become important later when thermodynamics effects are considered.

There are two ways to introduce the muscle force by modifying (2.2.24). One way is to consider the muscle stress through the typical way of adding external forces via the Lagrange-d'Alembert principle. In that modification, the muscle forces $\operatorname{div}\bar{s}$ are added to the force acting on the solid $\boldsymbol{F}_s$, as:

$$\delta S + \int_0^T \int_{\mathcal{B}} \left(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + (\boldsymbol{F}_s + \operatorname{div}\bar{s}) \cdot \boldsymbol{\eta}_s \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0\,. \tag{4.2.1}$$

Another way is to consider the muscle action as an *internal* force through an alternative formulation of the critical action principle as

$$\delta S + \int_0^T \int_{\mathcal{B}} \left(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s \cdot \boldsymbol{\eta}_s - \bar{s} : \nabla \boldsymbol{\eta}_s \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0\,. \tag{4.2.2}$$

At the first sight, both formulations are equivalent since application of integration over the volume to the last term brings (4.2.2) to (4.2.1), *except for the boundary terms*. The action (4.2.2) generates an additional force on the boundary given by $-\bar{s}\cdot\boldsymbol{n}$, with $\boldsymbol{n}(\mathbf{x})$ the normal to the boundary at $\mathbf{x} \in \partial\mathcal{B}$.

Let us now consider the following thought experiment: a volume of solid is acted upon with the uniform muscle stress $\bar{s} = \text{const}$. Then, $\operatorname{div}\bar{s} = 0$ inside the volume $\mathcal{B}$.

However, uniform stress will affect the boundary in the formulation (4.2.2) and not in the formulation (4.2.1). Thus, in our opinion, it is more natural to consider the approach (4.2.2) and not (4.2.1).

This variational principle leads to the modified version of (2.2.32) for the active porous media:

$$\begin{cases} \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} + \pounds_{\boldsymbol{u}_f} \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} = g \nabla \left(\dfrac{\delta \ell}{\delta g} - p \right) + \boldsymbol{F}_f \\ \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s} \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} = \rho_s \nabla \dfrac{\delta \ell}{\delta \rho_s} + \left(\dfrac{\delta \ell}{\delta b} + pv \dfrac{\partial c}{\partial b} \right) \diamond b + \operatorname{div} \bar{s} + \boldsymbol{F}_s \\ \dfrac{\delta \ell}{\delta v} = -pc(b)\,, \qquad g = c(b) v \\ \partial_t g + \operatorname{div}(g \boldsymbol{u}_f) = 0\,, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0\,, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0\,. \end{cases} \tag{4.2.3}$$

For the particular case of Lagrangian given by (2.2.18) the equations become

$$\begin{cases} \bar{\rho}_f^0 (\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\nabla p + \dfrac{1}{g} \boldsymbol{F}_f \\ \rho_s (\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = g \nabla p + \nabla \left(V - \dfrac{\partial V}{\partial v} v \right) + \operatorname{div} (\sigma_p + \bar{s}) + \boldsymbol{F}_s \\ \dfrac{\partial V}{\partial v} = pc(b), \qquad g = c(b) v \\ \partial_t g + \operatorname{div}(g \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0\,. \end{cases} \tag{4.2.4}$$

In the case of the free slip boundary condition (2.2.2), the variational principle yields boundary conditions similar to (2.2.33)

$$[(\sigma_p + \bar{s}) \cdot \boldsymbol{n}] \cdot \boldsymbol{\eta} = 0, \quad \text{for all } \boldsymbol{\eta} \text{ parallel to } \partial \mathcal{B}, \tag{4.2.5}$$

and the same definition of the stress tensor σ_p in (2.2.34). When the boundary conditions (2.2.3) are used, no additional boundary condition arise from the variational principle. Thus, physically, the muscle stress $\bar{s}$ is simply added to the effective stress σ_p in equations (4.2.4) and the boundary conditions (4.2.5).

Remark 4.2.1 (On the free flow of fluid through the boundary) Suppose the elastic solid is fixed in space in a domain $\mathcal{D}_s \subset \mathbb{R}^3$. If the fluid can leave or enter the domain, it is natural to assume that all fluid particles are included in a domain $\mathcal{D}_f$ that always contains the solid, $\mathcal{D}_s \subset \mathcal{D}_f$, for example, $\mathcal{D}_f = \mathbb{R}^3$, *i.e.* the fluid occupies the whole space. Then, the back-to-labels maps in that case are given by

$$\boldsymbol{X}(t, \_) : \mathcal{D}_s \to \mathcal{B}_s \quad \text{and} \quad \boldsymbol{Y}(t, \_) : \mathcal{D}_s \to \mathbb{R}^3\,, \tag{4.2.6}$$

where $\boldsymbol{X}(t,x) \in \mathcal{B}_s$ is a diffeomorphism with $\mathcal{B}_s$ the set of all solid labels as before, and $\boldsymbol{Y}(t,x)$ is an embedding with $\mathcal{B}(t) := \boldsymbol{Y}(t,\mathcal{D}_s)$ the set of all fluid labels corresponding to fluid particles that are in the elastic solid at time t. Consequently, fluid labels in $\mathbb{R}^3 - \mathcal{B}(t)$ correspond to fluid particles that are not in the elastic solid at time t. Such particle may, *e.g.*, have already left the solid at time t or will penetrate into the solid at a later time. We shall also note that the accurate representation of the moving material boundary needs to be treated carefully, see [29], as the boundary of the elastic matrix material presents a singularity. We shall postpone the quite complex and technical discussion of the free fluid outflow of the boundary to a follow up work.

Under the assumptions leading to the concentration dependence $c = c_0/\sqrt{\det b}$, equations (4.2.4) simplify further to

$$\begin{cases} \bar{\rho}_f^0(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\nabla p + \dfrac{1}{g}\boldsymbol{F}_f \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = g\nabla p + \operatorname{div}(\sigma_e + \bar{s}) + \boldsymbol{F}_s \\ \dfrac{\partial V}{\partial v} = pc(b), \qquad g = c(b)v \\ \partial_t g + \operatorname{div}(g\boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0\,, \end{cases} \tag{4.2.7}$$

where σ_e is the elastic stress

$$\sigma_e = 2\frac{\partial V}{\partial b} \cdot b + V\mathrm{Id}. \tag{4.2.8}$$

This formula follows from the equality $\nabla\left(V - \frac{\partial V}{\partial v}v\right) + \operatorname{div}(\sigma_p) = \operatorname{div}\sigma_e$ which holds when $c = c_0/\sqrt{\det b}$ since in this case $\frac{\partial c}{\partial b} \cdot b = -\frac{1}{2}c\mathrm{Id}$. We can further put $\boldsymbol{F}_f = -\boldsymbol{F}_s = \mathbb{K}(\boldsymbol{u}_s - \boldsymbol{u}_f)$ to introduce the friction according to the Darcy law.

For the developments below, we also need to introduce the equation for the back-to-labels maps. Recall that the Lagrangian label of the solid is denoted $\boldsymbol{X}$ and we use the same notation for the back-to-labels map defined by $\boldsymbol{X}(t,\mathbf{x}) = \boldsymbol{\Psi}^{-1}(t,\mathbf{x})$ where $\boldsymbol{\Psi}(t,\boldsymbol{X})$ denotes the configuration map for the solid. Recall also that the Eulerian velocity of the solid is given by

$$\boldsymbol{u}_s = \partial_t \boldsymbol{\Psi} \circ \boldsymbol{\Psi}^{-1} \quad \text{i.e.} \quad \boldsymbol{u}_s(t,\mathbf{x}) = \partial_t \boldsymbol{\Psi}\big(t, \boldsymbol{\Psi}^{-1}(t,\mathbf{x})\big)\,. \tag{4.2.9}$$

Differentiating the identity $\boldsymbol{\Psi} \circ \boldsymbol{X} = \mathrm{Id}$, or, in coordinates, $\boldsymbol{\Psi}(t, \boldsymbol{X}(t,\mathbf{x})) = \mathbf{x}$, with respect to time, we obtain:

$$\nabla\boldsymbol{\Psi} \circ \boldsymbol{X} \cdot \partial_t \boldsymbol{X} + \boldsymbol{u}_s = \mathbf{0} \quad \Leftrightarrow \quad \partial_t \boldsymbol{X} = -\left(\nabla\boldsymbol{\Psi} \circ \boldsymbol{X}\right)^{-1} \cdot \boldsymbol{u}_s \tag{4.2.10}$$

which can be written as

$$\partial_t \boldsymbol{X} = -\nabla \boldsymbol{X} \cdot \boldsymbol{u}_s \quad \Leftrightarrow \quad \boldsymbol{u}_s = -\left(\nabla \boldsymbol{X}\right)^{-1} \cdot \partial_t \boldsymbol{X}\,. \tag{4.2.11}$$

The strain tensor b and, correspondingly, stress tensor σ_e depend on the spatial gradients of that map, *i.e.*, $\nabla \boldsymbol{X}(t, \mathbf{x})$, as well as possibly $\mathbf{x}$ and v, since V depends on both b and v. Thus, the second equation of (4.2.7) becomes a set of coupled elliptic equations for $\nabla \boldsymbol{X}$ and v. On the other hand, the muscle action depends only on $\boldsymbol{X}$ and t.

4.2.1 Simplification: one-dimensional dynamics

Let us now compute the equations (4.2.7) for one dimensional reduction. The physical model is as follows: suppose there is an elastic porous media filled with an incompressible fluid, positioned inside a perfectly slippery one-dimensional pipe with a circular cross-section. The elastic media has a muscle inside which can contract and expand. Since the space occupied by the muscle is fixed in space, its back-to-labels map is a diffeomorphism of the form $X(t, \_) : [-L, L] \to [-L, L]$, $X = X(t, x)$. The fluid can leave or enter the muscle hence its back-to-labels map is an embedding $Y(t, \_) : [-L, L] \to \mathbb{R}$, $Y = Y(t, x)$.

Equation (4.2.11) for solid and fluid become simply

$$X_t(t, x) = -X_x(t, x) u_s(t, x)\,, \quad Y_t(t, x) = -Y_x(t, x) u_f(t, x). \tag{4.2.12}$$

Using the assumption of the Darcy law of friction in one dimension, we derive the one-dimensional version of (4.2.7) as

$$\begin{cases} \bar{\rho}_f^0(\partial_t u_f + u_f \partial_x u_f) = -\partial_x p + \dfrac{1}{g} K(u_s - u_f) \\ \rho_s(\partial_t u_s + u_s \partial_x u_s) = g \partial_x p + \partial_x \left(\sigma_e + \bar{s}\right) + K(u_f - u_s) \\ \dfrac{\partial V}{\partial v} = pc(b), \qquad g = c(b) v \\ \partial_t g + \partial_x (g u_f) = 0, \qquad \partial_t \rho_s + \partial_x (\rho_s u_s) = 0, \qquad \partial_t b + \pounds_{u_s} b = 0\,. \end{cases} \tag{4.2.13}$$

In one dimension, the Finger tensor becomes simply

$$b(t, x) = \left(X_x(t, x)\right)^{-2}. \tag{4.2.14}$$

The densities g and ρ_s with reference values $g^0(X)$ and $\rho_s^0(X)$ are expressed as

$$\rho_s(t, x) = \rho_s^0(X(t, x)) X_x(t, x) \quad \text{and} \quad g(t, x) = g^0(X(t, x)) X_x(t, x)\,. \tag{4.2.15}$$

The third equation in (4.2.13) expresses p as an explicit function of v and b hence from (4.2.14) the pressure becomes a function $p = p(X_x, v)$. Since $v = g/c(b)$ from the fourth equation in (4.2.13), we obtain explicitly the pressure as a function $p = p(X_x, g)$. In one dimension, the elastic stress tensor (4.2.8) is

$$\sigma_e = 2\frac{\partial V}{\partial b}(b, v)b + V(b, v). \tag{4.2.16}$$

From (4.2.14) and $v = g/c(b)$, we get an explicit expression

$$\sigma_e = \sigma_e(X_x, g).$$

4.2.2 A reduction of the equation of motion for Lagrangian variables

We now show how to reduce the system (4.2.13) to two coupled PDEs for the two variables describing the Lagrangian coordinates of the solid and fluid. Suppose that at $t = 0$, the state is non-deformed so $X(0, x) = x$ and $\rho_s(0, x) = \rho_s^0$ for all x, where ρ_s^0 is the reference density of the porous media assumed to be a constant in space. Thus from the first equation in (4.2.15), we have

$$\rho_s(t, x) = \rho_s^0 X_x(t, x)\,. \tag{4.2.17}$$

Similarly, we also assume that at $t = 0$, the fluid is undisturbed, $Y(0, x) = x$ and $g(0, x) = g_0$, for some constant g_0. From the second equation in (4.2.15), we have

$$g(t, x) = g_0 Y_x(t, x)\,. \tag{4.2.18}$$

On the other hand, the velocities can also be expressed from (4.2.12) as

$$u_s = -\frac{X_t}{X_x}\,, \qquad u_f = -\frac{Y_t}{Y_x}\,. \tag{4.2.19}$$

We also note that since the pressure depends on X_x and g, from (4.2.18) we have

$$p = p(X_x, g) = P(X_x, Y_x) \tag{4.2.20}$$

a given function of (X_x, Y_x).

Substitution of (4.2.19) together with (4.2.17) and (4.2.18) into the first two equations of (4.2.13) gives a closed system of two coupled PDEs for the back-to-labels maps X

and Y:

$$\begin{cases} \bar{\rho}_f^0 g_0 \left(-Y_{tt} + 2\dfrac{Y_t Y_{tx}}{Y_x} - \dfrac{Y_t^2 Y_{xx}}{Y_x^2} \right) \\ \qquad = -g_0 Y_x \partial_x \big(P(X_x, Y_x) \big) + K \left(\dfrac{Y_t}{Y_x} - \dfrac{X_t}{X_x} \right) \\ \rho_s^0 \left(-X_{tt} + 2\dfrac{X_t X_{tx}}{X_x} - \dfrac{X_t^2 X_{xx}}{X_x^2} \right) \\ \qquad = g_0 Y_x \partial_x \big(P(X_x, Y_x) \big) + \partial_x \big(\sigma_e(X_x, Y_x) + \overline{s} \big) + K \left(\dfrac{X_t}{X_x} - \dfrac{Y_t}{Y_x} \right) . \end{cases} \tag{4.2.21}$$

In addition to lowering the number of equations, in our opinion equation (4.2.21) is easier to implement than its Eulerian version since in nature, one would expect the muscle stress to be a function of the Lagrangian variable of the solid X, *i.e.*, a given muscle fiber, rather than the spatial coordinate x, so $\overline{s} = \overline{s}(X, t)$. It can be written as an explicit function of x only for given solution $X(t, x)$, which is awkward from the point of view of numerical solution, as the last equation of (4.2.13) describing evolution of X has to be explicitly computed at every time step. In the formulation (4.2.21), where the equation is formulated in terms of back-to-labels maps (X, Y), specifying $\overline{s} = \overline{s}(X, t)$ to be a given function of X does not present any fundamental problem for implementing the numerical solution.

Remark 4.2.2 (Conservation of total momentum) The total momentum of the fluid is $M_f = \int_{-L}^{L} \bar{\rho}_f^0 g u_f \mathrm{d}x = -\int_{-L}^{L} \bar{\rho}_f^0 g_0 Y_t \mathrm{d}x$, and the total momentum of the solid is $M_s = \int_{-L}^{L} \rho_s u_s \mathrm{d}x = -\int_{-L}^{L} \rho_s X_t \mathrm{d}x$. Then, the total fluid+solid momentum

$$M = \int_{-L}^{L} \left(\bar{\rho}_f^0 g u_f + \rho_s u_s \right) \mathrm{d}x = -\int_{-L}^{L} \left(\bar{\rho}_f^0 g_0 Y_t + \rho_s^0 X_t \right) \mathrm{d}x$$

is conserved for periodic boundary conditions. Indeed,

$$\begin{aligned} \dot{M} &= \int_{-L}^{L} \left(-2\rho_s^0 \frac{X_t X_{tx}}{X_x} + \rho_s^0 \frac{X_t^2 X_{xx}}{X_x^2} \right. \\ &\qquad \left. -2\bar{\rho}_f^0 g_0 \frac{Y_t Y_{tx}}{Y_x} + \bar{\rho}_f^0 g_0 \frac{Y_t^2 Y_{xx}}{Y_x^2} + \partial_x \left(\sigma_e + \overline{s} \right) \right) \mathrm{d}x \\ &= \int_{-L}^{L} \partial_x \left(-\rho_s^0 \frac{X_t^2}{X_x} - \bar{\rho}_f^0 g_0 \frac{Y_t^2}{Y_x} + \sigma_e + \overline{s} \right) \mathrm{d}x \\ &= \int_{-L}^{L} \partial_x \left(-\rho_s u_s^2 - \bar{\rho}_f^0 g u_f^2 + \sigma_e + \overline{s} \right) \mathrm{d}x \\ &= \left[-\rho_s u_s^2 - \bar{\rho}_f^0 g u_f^2 + \sigma_e + \overline{s} \right]_{-L}^{L} . \end{aligned} \tag{4.2.22}$$

The boundary terms vanish for periodic boundary conditions. For other cases, for example, fixed boundary conditions for the solid, $u_s(\pm L, t) = 0$, the conservation of momentum depends on the cancellation of the fluid momentum through the boundary, elastic stress and muscle stress contributions in the boundary term of (4.2.22).

4.2.3 Numerical solution of the equations (4.2.21)

Choice of potential. For the numerical solution, we postulate the following choice of potential energy:

$$V(X_x, g) = \frac{1}{2}\alpha\left(X_x - 1\right)^2 + \frac{1}{2}\beta\big(g + (1 - g_0)\, X_x - 1\big)^2. \tag{4.2.23}$$

The terms have the following physical sense: the expression proportional to α corresponds to the linear elasticity term (Hooke's law), the second term, proportional to β, is the difference of g and the "totally-incompressible porosity", a quantity, that would be equal to the porosity in the case if the solid was totally incompressible. The potential energy "penalizes" changes in microscopic volume of the solid.

Remark 4.2.3 Note that (4.2.23) is the physical description of the potential as a function of X_x and g, consistent with our description $V = V(b, v)$ in general case. For initially uniform system, using $g = g_0 Y_x$ we can express V as

$$V(X_x, Y_x) = \frac{1}{2}\alpha\left(X_x - 1\right)^2 + \frac{1}{2}\beta\big(g_0 Y_x + (1 - g_0)\, X_x - 1\big)^2 \tag{4.2.24}$$

in terms of X_x and Y_x, the latter related to the state of fluid. However, we do not assume that the potential energy of the solid is dependent on the state of the fluid here, it is an inference based on the properties of the solution. Thus, the potential energy defined by (4.2.23) depends only on the properties of the solid, consistent with our description. This is in contrast, for example, with works [68], [69] where the energy of the porous media is dependent on the states of fluid and solid. The expression (4.2.24) provides a formal connection between two approaches, as the solid energy *formally* depends on the state of the fluid after the substitution $g = g_0 Y_x$, even though the physics of two approaches is quite different.

In order to compute the associated stress σ_0 and pressure p, we need to express V as a function $V(b, v)$. Recalling that in one dimension, $b = X_x^{-2}$, and using

$$g = c(b)v = \frac{c_0}{\sqrt{b}}v$$

we can rewrite (4.2.23) as

$$V(b,v) = \frac{\alpha}{2}\left(\frac{1}{\sqrt{b}} - 1\right)^2 + \frac{\beta}{2}\left(\frac{c_0 v}{\sqrt{b}} + \frac{1-g_0}{\sqrt{b}} - 1\right)^2. \tag{4.2.25}$$

Now we are able to compute the stress terms from (4.2.16) as follows

$$\begin{aligned}\sigma_e &= 2b\frac{\partial}{\partial b}V(b,v) + V(b,v)\\ &= \left(1 - \frac{1}{\sqrt{b}}\right) - \frac{\beta}{\sqrt{b}}\left(c_0 v + 1 - g_0\right)\left(-1 + \frac{c_0 v}{\sqrt{b}} + \frac{1-g_0}{\sqrt{b}}\right) + V(b,v)\\ &= \frac{\alpha}{2}\left(1 - \frac{1}{b}\right) - \frac{\beta}{2b}\left(-b + (c_0 v + (1-g_0))^2\right)\end{aligned} \tag{4.2.26}$$

and the pressure is found as

$$pc(b) = \frac{\partial V}{\partial v} = \frac{\beta}{\sqrt{b}} c_0 \left(-1 + \frac{c_0 v}{\sqrt{b}} + \frac{1-g_0}{\sqrt{b}}\right). \tag{4.2.27}$$

Substituting $b = X_x^{-2}$ and $c_0 v = g\sqrt{b} = g_0 Y_x X_x^{-1}$ in (4.2.26) and (4.2.27) yields σ_e and p as functions of X_x and Y_x as

$$\sigma_e(X_x, Y_x) = -\frac{\alpha}{2}(X_x^2 - 1) - \frac{\beta}{2}\left((X_x(1-g_0) + Y_x g_0)^2 - 1\right) \tag{4.2.28}$$

$$p(X_x, Y_x) = \beta g_0 \left(g_0 Y_x + (1-g_0) X_x - 1\right). \tag{4.2.29}$$

We will use also the formula

$$\partial_x \sigma_e = -\alpha X_x X_{xx} - \beta \left(g_0 Y_x + (1-g_0)X_x\right)\left(g_0 Y_{xx} + (1-g_0)X_{xx}\right).$$

For simulations, it is useful to define $\xi(t,x)$ and $\phi(t,x)$ such that

$$X(t,x) = x + \xi(t,x), \qquad Y(t,x) = x + \phi(t,x). \tag{4.2.30}$$

Then V is a quadratic, positive definite function of (ξ_x, ϕ_x), as expected:

$$V(\xi_x, \phi_x) = \frac{1}{2}\alpha\xi_x^2 + \frac{1}{2}\beta\left(g_0\phi_x + (1-g_0)\,\xi_x\right)^2. \tag{4.2.31}$$

For the temporally and spatially bound muscle stress, we take

$$\overline{s}(t,X) = S_0 e^{-t/T - X^2/W^2}, \tag{4.2.32}$$

where S_0, T and W are the given parameters of amplitude, time scale and width of applied muscle stress.

Discretization. Let us consider the system (4.2.21) on a finite space interval $x \in [-L, L]$. We discretize the system by considering the $N+2$ discrete data points $\{x_0, \dots x_{N+1}\}$, with $x_0 = -L$ and $x_{N+1} = L$. We assume, for simplicity, a uniform discretization step $h = x_{i+1} - x_i$.

There are several types of boundary conditions: fixed, free or periodic. Let us for simplicity assume that the total extent of the x-domain occupied by the system is fixed, due to implemented boundary conditions holding the elastic material in a fixed place in space and not preventing the escape of fluid. Hence regarding the discretization of $X(t, x)$ we assume $X(t, -L) = X_1(t) = -L$ and $X(t, L) = X_{N+1}(t) = L$. Since X_0 and X_{N+1} do not have any dynamics, we only consider the dynamics of $X_1, X_2, \dots X_N$. Then, if the outflow of fluid from the boundaries is blocked, then we will have $Y(t, -L) = -L$ and $Y(t, L) = L$. If there is a fluid outflow from the boundaries, the nature of the boundary conditions will depend on many factors, *i.e.* the need to overcome external pressure of the outside fluid, and the exact nature of the outflow. Thus, even for the fixed boundary conditions for the solid, the setting of boundary conditions for the fluid is non-trivial. The simplest ones are periodic boundary conditions, where $\xi = X - x$ and $\phi = Y - x$ are periodic with period $2L$. We shall thus use periodic boundary conditions in our simulations.

For periodic boundary conditions, (X_x, X_t, Y_x, Y_t) and their spatial derivatives are periodic function in x with the period $2L$. The forward Δ_i^f and backward Δ_i^b derivatives of any periodic function F for the periodic boundary conditions is

$$\begin{cases} \Delta_i^f F = \dfrac{F_{i+1} - F_i}{h}, & 1 \le i < N, \quad \Delta_N^f F = \Delta_1^f F \\ \Delta_i^b F = \dfrac{F_i - F_{i-1}}{h}, & 1 < i \le N, \quad \Delta_1^b F = \Delta_N^b F \,. \end{cases} \tag{4.2.33}$$

Then, we can approximate the first derivative by $\Delta_i^0 = (\Delta_i^f + \Delta_i^b)/2$ and second derivative by $\Delta_i^2 = \Delta_i^f * \Delta_i^b$.

The results for numerical solution for (ξ, ϕ) with periodic boundary conditions are presented on Figure 4.2.1. An initial disturbance caused by the muscle action on the matrix in the center of the elastic body is propagating along the matrix, both for fluid and for elastic material, although the shape of wave propagation is different.

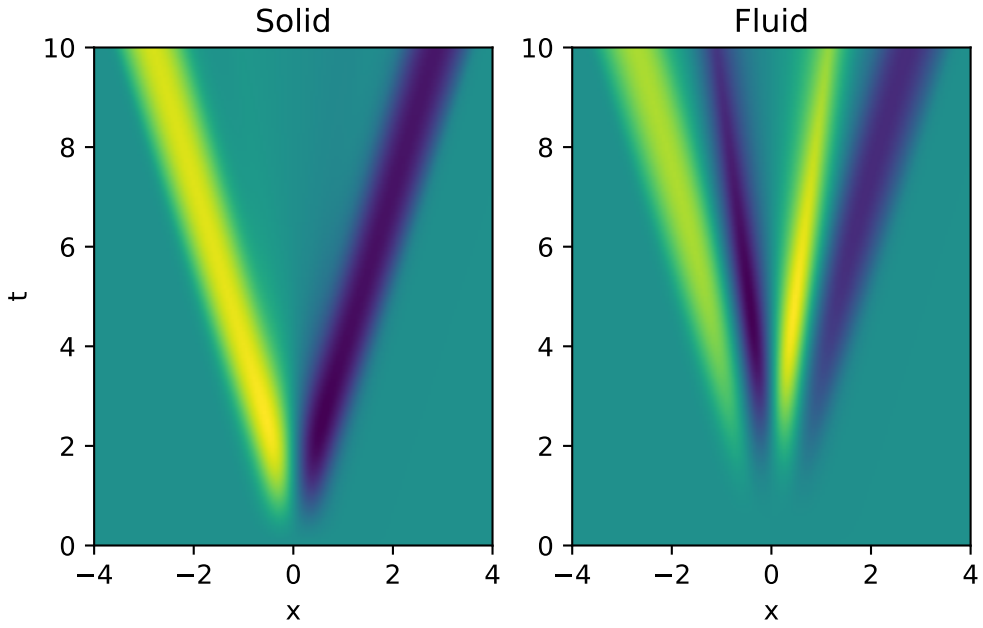

Figure 4.2.1: Example of numerical solution of (4.2.21). Left panel: solution for $\xi(t,x) = X(t,x) - x$ (solid), right panel: $\eta(t,x) = Y(t,x) - x$ (fluid). For simulations presented here, the parameters for the stress in (4.2.32) are $S_0 = 0.1$, $W = 1$, $T = 1$. The material parameters are $g_0 = 0.5$, $K = 1$ and $\alpha = 1$, $\beta = 1$ in (4.2.23).

4.2.4 Self-propulsion by periodic motion of the stress

Let us now consider periodic boundary conditions, $\xi(t, x+2L) = \xi(t, x)$ and $\phi(t, x+2L) = \phi(t, x)$, with $\xi(t, x) = X(t, x) - x$ and $\phi(t, x) = Y(t, x) - x$, and also consider the case when there is a periodic motion of the muscle's stress along the porous media. More precisely, let us consider the prescribed motion of the muscle stress in the form

$$\begin{aligned} &\overline{s}(t, X) = S_0 e^{-(X-Ut)^2_{\text{per}}/W^2}, \\ &(X - Ut)_{\text{per}} := (X - Ut \bmod 2L) - L. \end{aligned} \tag{4.2.34}$$

One can also express (4.2.34) by saying $(X - Ut)_{\text{per}}$ is a periodic function of X with the same period as X, with the values contained in the interval between $-L$ and L. Another interesting problem is to see if a periodic in time and space muscle stress along the porous body can create self-propulsion of the elastic matrix. Of course, due to the conservation of momentum given in Remark 4.2.2, equation (4.2.22), it is not possible to accelerate both the fluid and solid in the same direction. However, it is possible to have opposite, and non-zero net momenta of solid and fluid, as shown on Figure 4.2.2. One can see that the amplitude of the net momentum is quite small.

One could conjecture that the system is converging to a traveling wave solution, with small persistent oscillations about a steady state as illustrated on Figure 4.2.2. In the steady scenario, the average power of muscle stress balances the dissipation due to friction forces. Presumably, for biological applications, organisms would optimize the efficiency of motion and not smoothness. Thus, while considerations of traveling wave solutions are certainly possible, we will skip them here as they have, in our opinion, limited value for applications.

4.3 Equations for the case when both the fluid and the solid are incompressible

4.3.1 Physical justification and derivation of equations

From the physical point of view we may notice that for many biological materials the bulk modulus K has the same order of magnitude or sometimes higher than the bulk modulus of water (2.2 GPa). The physics of this effective incompressibility can be understood from the fact that the elastic matrix consists of cells which are en large

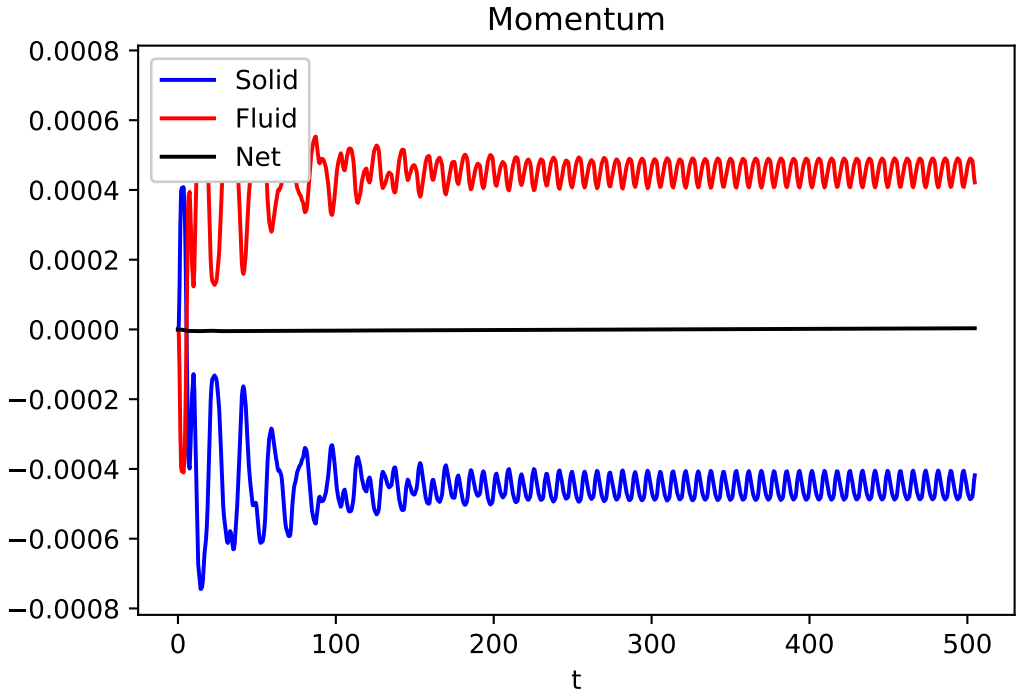

Figure 4.2.2: Momenta for solid $M_s = -\rho_s^0 \int_{-L}^{L} X_t \mathrm{d}x$ (blue line) and fluid $M_f = -\bar{\rho}_f^0 g_0 \int_{-L}^{L} Y_t \mathrm{d}x$ (red line) and total momentum $M = M_f + M_s$ (black line) for a given numerical solution with zero initial conditions and prescribed traveling muscle force given by (4.2.34) with parameters $S_0 = 1$, $W = 1$ and $U = 1$. The total momentum $M_s + M_f$ is close to 0 with expected accuracy throughout the simulation, starting at 10^{-6} and increasing to a fraction of 10^{-5} during the computation presented.

composed of incompressible water. Thus, the porous matrix can be effectively treated as incompressible elastic material. Physically, if we select an arbitrary region in porous media filled with fluid and 'lock' the fluid inside the porous matrix by an impermeable flexible membrane, the volume of such region shall not change under the assumption of total incompressibility. We can express the incompressibility of the solid as follows

$$1 - g(t, \mathbf{x}) = (1 - g_0)\big(\boldsymbol{\Psi}^{-1}(t, \mathbf{x})\big) J_{\boldsymbol{\Psi}^{-1}}(t, \mathbf{x}). \tag{4.3.1}$$

Notice the similarity with the incompressibility of fluid given by (2.2.10). One way to include the incompressibility of the solid given in (4.3.1) is by adding an extra term in the action enforcing this condition with a Lagrange multiplier. There is although a simpler way to reach the answer. We differentiate (4.3.1) with respect to time to get

$$\partial_t(1 - g) + \operatorname{div}\big((1 - g)\boldsymbol{u}_s\big) = 0. \tag{4.3.2}$$

(compare with (2.2.11) and (2.2.13)) which can be written as

$$\operatorname{div}(g\boldsymbol{u}_f + (1 - g)\boldsymbol{u}_s) = 0\,. \tag{4.3.3}$$

Since the constraint on the velocities is holonomic, we can also infer the following relationship between the variations $\boldsymbol{\eta}_s$ and $\boldsymbol{\eta}_f$

$$\operatorname{div}\big(g\boldsymbol{\eta}_f + (1 - g)\boldsymbol{\eta}_s\big) = 0\,. \tag{4.3.4}$$

We introduce a Lagrange multiplier μ for (4.3.4) and add it to the action S in (2.2.19) as

$$\begin{aligned} S_I &= S - \int_0^T \int_{\mathcal{B}} \mu \operatorname{div}(g\boldsymbol{\eta}_f + (1 - g)\boldsymbol{\eta}_s) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t \\ &= S + \int_0^T \int_{\mathcal{B}} \big(g\nabla\mu \cdot \boldsymbol{\eta}_f + (1 - g)\nabla\mu \cdot \boldsymbol{\eta}_s\big)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t\,. \end{aligned} \tag{4.3.5}$$

The Lagrange-d'Alembert principle with friction forces applied to S_I reads, similarly to (4.2.2),

$$\delta S_I + \int_0^T \int_{\mathcal{B}} \big(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s - \bar{s} : \nabla\boldsymbol{\eta}_s \cdot \boldsymbol{\eta}_s\big)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0 \tag{4.3.6}$$

and yields the system

$$\begin{cases} \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} + \pounds_{\boldsymbol{u}_f} \dfrac{\delta \ell}{\delta \boldsymbol{u}_f} = g\nabla\left(\dfrac{\delta \ell}{\delta g} - (p+\mu)\right) + \boldsymbol{F}_f \\ \partial_t \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} + \pounds_{\boldsymbol{u}_s} \dfrac{\delta \ell}{\delta \boldsymbol{u}_s} = -(1-g)\nabla\mu + \rho_s \nabla \dfrac{\delta \ell}{\delta \rho_s} \\ \qquad\qquad + \left(\dfrac{\delta \ell}{\delta b} + pv\dfrac{\partial c}{\partial b}\right) \diamond b + \operatorname{div} \bar{s} + \boldsymbol{F}_s \\ \dfrac{\delta \ell}{\delta v} = -pc(b)\,, \qquad g = c(b)v \\ \partial_t g + \operatorname{div}(g\boldsymbol{u}_f) = 0\,, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0\,, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \partial_t(1-g) + \operatorname{div}((1-g)\boldsymbol{u}_s) = 0 \;\Rightarrow\quad \operatorname{div}(g\boldsymbol{u}_f + (1-g)\boldsymbol{u}_s) = 0\,. \end{cases} \tag{4.3.7}$$

Using the physical Lagrangian (2.2.18), the expanded form of totally incompressible equations of motion becomes

$$\begin{cases} \bar{\rho}_f^0(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -\nabla(p+\mu) + \dfrac{1}{g}\boldsymbol{F}_f \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = g\nabla p - (1-g)\nabla\mu \\ \qquad\qquad + \nabla\left(V - \dfrac{\partial V}{\partial v} v\right) + \operatorname{div}(\sigma_p + \bar{s}) + \boldsymbol{F}_s \\ \dfrac{\partial V}{\partial v} = pc(b), \qquad g = c(b)v \\ \partial_t g + \operatorname{div}(g\boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0, \\ \operatorname{div}(g\boldsymbol{u}_f + (1-g)\boldsymbol{u}_s) = 0\,. \end{cases} \tag{4.3.8}$$

As before, under the assumptions leading to the concentration dependence $c = c_0/\sqrt{\det b}$ given by (2.2.9), the solid momentum equation in (4.3.8) simplifies further to

$$\rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = g\nabla p - (1-g)\nabla\mu + \operatorname{div}(\sigma_e + \bar{s}) + \boldsymbol{F}_s \tag{4.3.9}$$

with σ_e the elastic stress given in (4.2.8).

Note that in the above equations, the pressure, while being a Lagrange multiplier for the fluid incompressibility condition, is actually specified due to the δv condition, *i.e.*, the third equation of (4.3.8) for the pressure, exactly as in (2.2.36) and (4.2.4). In contrast, μ, the Lagrange multiplier for the incompressibility of elastic matrix, does not have an explicit expression and must be found so the solution satisfies the last equation of (4.3.8). The equation for μ can be derived explicitly by taking a time derivative of

the last equation of (4.3.8). For simplicity, let us rewrite the equations for velocities $\mathbf{u}_f$ and $\mathbf{u}_s$ as

$$\partial_t \mathbf{u}_f = \mathbf{R}_f - \frac{1}{\rho_f}\nabla\mu\,, \quad \partial_t \mathbf{u}_s = \mathbf{R}_s - \frac{1-g}{\rho_s}\nabla\mu\,, \tag{4.3.10}$$

where $\mathbf{R}_{f,s}$ are the right-hand sides of the fluid and solid equations excluding the μ-term, which depend on the variables $(\boldsymbol{u}_f, \boldsymbol{u}_s, b, g, \rho_s)$ but not on their time derivatives. Differentiating the last equation of (4.3.8) with respect to time, we obtain

$$\begin{aligned} \operatorname{div}\left[\left(\frac{g}{\bar{\rho}_f^0} + \frac{(1-g)^2}{\rho_s}\right)\nabla\mu\right] \\ = \operatorname{div}\big(-\operatorname{div}(g\mathbf{u}_f)(\mathbf{u}_f - \mathbf{u}_s) + g\mathbf{R}_f + (1-g)\mathbf{R}_s\big) \end{aligned} \tag{4.3.11}$$

which is an elliptic equation for μ, reminding of the regular pressure equation in a fluid. It is also useful to interpret the equation for the fluid pressure p in the doubly incompressible media. When interpreting the physical nature of the potential energy $V = V(b, v)$, one notices that if the solid is incompressible as well, then v is no longer a free variable, but has a dynamics slaved to that of b which, in the simplest case of uniform initial conditions, is written as $v = v(b)$, so $V(b, v) = V(b, v(b)) = W(b)$. Thus, the equation for the fluid pressure seemingly would give $p = 0$ since $W(b)$ does not depend on v. That conclusion, however, would be incorrect. One has to *first* write the expression for the potential energy in terms of the microscopic volume v, and *only then* connect v to the Finger tensor b after taking the derivative in the pressure equation. Thus, in general, the pressure in the fluid is not going to vanish. While this approach requires careful consideration, in our opinion, it does have merit since it is easier to compute $V(b, v)$ from general principles and then substitute $v = v(b)$. If one insists on using the expression for potential energy $W = W(b) = V(b, v(b))$ then one needs to accurately compute the derivatives of $W(b)$ as a complex function of b, leading to the same terms as in (4.3.8).

We shall further note that equations (4.3.8), while correct, are somewhat difficult to interpret physically because of the presence of two pressures, p and μ being the Lagrange multipliers for incompressibility of fluid and solid, respectively. With these two pressures, the interpretation of Terzaghi's principle of equating pressures within the matrix and the fluid becomes non-apparent.

4.3.2 Alternative derivation emphasizing the physical nature of pressure

We give an alternative derivation which will allow elucidating the physical meaning behind the pressure-like terms coming from the Lagrange multipliers. We show that these multipliers are related to the physical pressure in the fluid. While the resulting equations are the same as derived above, we believe that this derivation is useful since it appeals to the physical meaning behind the quantities, and not just their formal mathematical origin. Moreover, the derivation presented in this Section will be useful for further studies of thermodynamic effects in porous media which will be undertaken in follow-up work. Let us start by introducing the actual densities of the material

$$\overline{\rho}_f := \frac{\rho_f}{g}\,, \qquad \overline{\rho}_s := \frac{\rho_s}{1-g}\,. \tag{4.3.12}$$

These densities correspond to the physical density of the fluid or the matrix, taken by themselves, for example, water only, or the elastic matrix with all voids filled with the same material as the matrix itself. Next, the fluid and solid specific internal energy should depend only on the combinations $\overline{\rho}_f$ and $\overline{\rho}_s$, respectively, and not, for example, on the densities and g. In addition, the internal energy of the solid should also depend on the Finger deformation tensor b. Therefore, we put the specific internal energies of the fluid and solid to be $E_f = E_f(\overline{\rho}_f)$ and $E_s = E_s(\overline{\rho}_s, b)$ and take the Lagrangian

$$\begin{aligned}\ell(\mathbf{u}_f, \mathbf{u}_s, \rho_f, \rho_s, b, g) = \int_{\mathcal{B}} \Big[&\frac{1}{2}\rho_f|\mathbf{u}_f|^2 + \frac{1}{2}\rho_s|\mathbf{u}_s|^2 \\ &- \rho_f E_f(\bar{\rho}_f) - \rho_s E_s(\bar{\rho}_s, b)\Big]\mathrm{d}^3\mathbf{x}.\end{aligned} \tag{4.3.13}$$

Recall that the specific internal energy is the energy density per mass. Since the mass of the solid is proportional to the local density, we have $\mathrm{d}m_s = \rho_s \mathrm{d}\mathbf{x}$, so the energy density per volume is then $V_s = \rho_s E_s(\bar{\rho}_s, b)$, and not, for example, $V \neq \bar{\rho}_s E_s$. Similar consideration also applies to the fluid, with the energy density per volume is $V_f = \rho_f E_f(\bar{\rho}_f)$.

The case of fluid and matrix both being compressible. Neglecting the thermal effects, this is the simplest case to consider. The action is defined as

$$S = \int_0^T \ell(\mathbf{u}_f, \mathbf{u}_s, \rho_f, \rho_s, b, g)\mathrm{d}^3\mathbf{x}\,\mathrm{d}t \tag{4.3.14}$$

and we use the Lagrange-d'Alembert principle (4.2.2)

$$\delta S + \int_0^T \int_{\mathcal{B}} \left(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s \cdot \boldsymbol{\eta}_s - \bar{s} : \nabla \boldsymbol{\eta}_s \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0\,. \tag{4.3.15}$$

Here the variation δg is arbitrary and the variations of the other variables $(\delta \boldsymbol{u}_f, \delta \boldsymbol{u}_s, \delta\rho_s, \delta\rho_f)$ are given by

$$\begin{aligned}
\delta \boldsymbol{u}_f &= \partial_t \boldsymbol{\eta}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{\eta}_f - \boldsymbol{\eta}_f \cdot \nabla \boldsymbol{u}_f \\
\delta \boldsymbol{u}_s &= \partial_t \boldsymbol{\eta}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{\eta}_s - \boldsymbol{\eta}_s \cdot \nabla \boldsymbol{u}_s \\
\delta\rho_s &= -\operatorname{div}(\rho_s \boldsymbol{\eta}_s) \\
\delta\rho_f &= -\operatorname{div}(\rho_f \boldsymbol{\eta}_f) \\
\delta b &= -\pounds_{\boldsymbol{\eta}_s} b\,.
\end{aligned} \tag{4.3.16}$$

The variation with respect to g is particularly interesting and gives

$$\bar{\rho}_f^2 \frac{\partial E_f}{\partial \rho_f} - \bar{\rho}_s^2 \frac{\partial E_s}{\partial \rho_s} = 0 \quad \Rightarrow \quad \bar{\rho}_f^2 \frac{\partial E_f}{\partial \rho_f} = \bar{\rho}_s^2 \frac{\partial E_s}{\partial \rho_s} =: P\,. \tag{4.3.17}$$

Thus, from the variation with respect to g we obtain that the thermodynamic pressure in the fluid and solid are equal. The equations of motion are thus:

$$\begin{cases}
\rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -g\nabla \left(\bar{\rho}_f^2 \dfrac{\partial E_f}{\partial \bar{\rho}_f} \right) + \boldsymbol{F}_f \\
\rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -(1-g)\nabla \left(\bar{\rho}_s^2 \dfrac{\partial E_s}{\partial \bar{\rho}_s} \right) \\
\qquad\qquad +2 \operatorname{div} \left(\rho_s \dfrac{\partial E_s}{\partial b} \cdot b \right) + \operatorname{div} \bar{s} + \boldsymbol{F}_s \\
\partial_t \rho_f + \operatorname{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \operatorname{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\
\bar{\rho}_f^2 \dfrac{\partial E_f}{\partial \bar{\rho}_f} = \bar{\rho}_s^2 \dfrac{\partial E_s}{\partial \bar{\rho}_s} =: P\,.
\end{cases} \tag{4.3.18}$$

The variable g is determined in terms of the other variables from the last equality, which can be written as a condition $g = g(\rho_f, \rho_s, b)$.

The case of a compressible solid and incompressible fluid. In this case, we need to augment the action principle by adding the fluid incompressibility condition with a Lagrange multiplier p_f. The Lagrange-d'Alembert principle now reads

$$\begin{aligned}
\delta \int_0^T \left[\ell(\mathbf{u}_f, \mathbf{u}_s, \rho_f, \rho_s, b, g) + \int_{\mathcal{B}} p_f \left(g - (g^0 \circ \varphi_f^{-1}) J_{\varphi_f^{-1}} \right) \mathrm{d}^3\mathbf{x} \right] \mathrm{d}t \\
+ \int_0^T \int_{\mathcal{B}} \left(\boldsymbol{F}_f \cdot \boldsymbol{\eta}_f + \boldsymbol{F}_s \cdot \boldsymbol{\eta}_s - \bar{s} : \nabla \boldsymbol{\eta}_s \right) \mathrm{d}^3\mathbf{x}\, \mathrm{d}t = 0\,,
\end{aligned} \tag{4.3.19}$$

with arbitrary variations δg and δp_f, the same Lagrangian (4.3.13) as before and variations of other variables given by (4.3.16). This yields the following system:

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -g\nabla\left(\bar\rho_f^2 \dfrac{\partial E_f}{\partial \bar\rho_f}\right) - g\nabla p_f + \boldsymbol{F}_f \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -(1-g)\nabla\left(\bar\rho_s^2 \dfrac{\partial E_s}{\partial \bar\rho_s}\right) \\ \qquad\qquad +2\,\mathrm{div}\left(\rho_s \dfrac{\partial E_s}{\partial b}\cdot b\right) + \mathrm{div}\,\bar s + \boldsymbol{F}_s \\ \partial_t \rho_f + \mathrm{div}(\rho_f \boldsymbol{u}_f) = 0, \qquad \partial_t \rho_s + \mathrm{div}(\rho_s \boldsymbol{u}_s) = 0, \qquad \partial_t b + \pounds_{\boldsymbol{u}_s} b = 0 \\ \partial_t g + \mathrm{div}(g\boldsymbol{u}_f) = 0 \\ \bar\rho_f^2 \dfrac{\partial E_f}{\partial \bar\rho_f} + p_f = \bar\rho_s^2 \dfrac{\partial E_s}{\partial \bar\rho_s} =: P\,. \end{cases} \tag{4.3.20}$$

Note the difference between the last equality in (4.3.20) and that of system (4.3.18). Note also that with the last equality, the first two equations of motion for fluid can be written as

$$\begin{cases} \rho_f(\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f) = -g\nabla P + \boldsymbol{F}_f \\ \rho_s(\partial_t \boldsymbol{u}_s + \boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s) = -(1-g)\nabla P + 2\,\mathrm{div}\left(\rho_s \dfrac{\partial E_s}{\partial b}\cdot b\right) + \mathrm{div}\,\bar s + \boldsymbol{F}_s\,, \end{cases} \tag{4.3.21}$$

with $P = \bar\rho_s^2 \frac{\partial E_s}{\partial \bar\rho_s}$. Hence the first six equations in (4.3.20), in which the first two equations are rewritten as (4.3.21) with $P = \bar\rho_s^2 \frac{\partial E_s}{\partial \bar\rho_s}$, can be solved for the six variables $\boldsymbol{u}_f$, $\boldsymbol{u}_s$, ρ_f, ρ_s, b, g. Then p_f is found from the last equality $p_f = \bar\rho_s^2 \frac{\partial E_s}{\partial \bar\rho_s} - \bar\rho_f^2 \frac{\partial E_f}{\partial \bar\rho_f}$.

This shows that the internal energy of the fluid $\rho_f E_f(\bar\rho_f)$ can be neglected in the Lagrangian without affecting the dynamics. Furthermore, notice that from the ρ_f and g equations, we get $\partial_t \bar\rho_f + \boldsymbol{u}_f \cdot \nabla \bar\rho_f = 0$, so $\bar\rho_f$ is a constant, if it is a constant $\bar\rho_f^0$ at the initial time $t = 0$. This corresponds to the case of a homogeneous incompressible fluid.

As we will show below, the link with our variable v describing internal volume, used in (2.2.18), is

$$\kappa v = v_s - \bar v_s = \frac{1}{\rho_s} - \frac{1}{\bar\rho_s}\,, \tag{4.3.22}$$

where the constant κ is given by $\kappa = c_0/\rho_s^0$. Thus, the volume v introduced in Section 2.2 is proportional to the effective specific volume of solid v_s minus the microscopic specific volume of solid $\bar v_s$. Note that v_s and $\bar v_s$ defined in (4.3.22) have the dimensions of inverse density, with the physical meaning of available free volume per mass. As we shall also explain below, in this description $\rho_s(b)$ takes place of the concentrations of pores $c(b)$ with the relation $c(b) = \kappa \rho_s(b)$.

The case when both fluid and solid are incompressible. In this case, we take the Lagrange-d'Alembert action principle to be enforcing both the incompressibility of the fluid and the solid using the Lagrange multipliers p_f and p_s as

$$\begin{aligned}\delta\int_0^T\Big[\ell(\mathbf{u}_f,\mathbf{u}_s,\rho_f,\rho_s,b,g)+\int_{\mathcal{B}}p_f\left(g-(g^0\circ\varphi_f^{-1})J_{\varphi_f^{-1}}\right)\mathrm{d}^3\mathbf{x}\\ +\int_{\mathcal{B}}p_s\left((1-g)-((1-g^0)\circ\varphi_s^{-1})J_{\varphi_s^{-1}}\right)\mathrm{d}^3\mathbf{x}\\ +\int_{\mathcal{B}}\left(\boldsymbol{F}_f\cdot\boldsymbol{\eta}_f+\boldsymbol{F}_s\cdot\boldsymbol{\eta}_s-\bar{s}:\nabla\boldsymbol{\eta}_s\right)\mathrm{d}^3\mathbf{x}\Big]\mathrm{d}t=0\,,\end{aligned}\tag{4.3.23}$$

with arbitrary variations δg, δp_f, and δp_s, and the same Lagrangian (4.3.13) as before. We get the following system of equations:

$$\begin{cases}\rho_f(\partial_t\boldsymbol{u}_f+\boldsymbol{u}_f\cdot\nabla\boldsymbol{u}_f)=-g\nabla\left(\bar{\rho}_f^2\dfrac{\partial E_f}{\partial\bar{\rho}_f}\right)-g\nabla p_f+\boldsymbol{F}_f\\ \rho_s(\partial_t\boldsymbol{u}_s+\boldsymbol{u}_s\cdot\nabla\boldsymbol{u}_s)=-(1-g)\nabla\left(\bar{\rho}_s^2\dfrac{\partial E_s}{\partial\bar{\rho}_s}\right)-(1-g)\nabla p_s\\ \qquad\qquad +2\,\mathrm{div}\left(\rho_s\dfrac{\partial E_s}{\partial b}\cdot b\right)+\mathrm{div}\,\bar{s}+\boldsymbol{F}_s\\ \partial_t\rho_f+\mathrm{div}(\rho_f\boldsymbol{u}_f)=0,\qquad\partial_t\rho_s+\mathrm{div}(\rho_s\boldsymbol{u}_s)=0,\qquad\partial_t b+\pounds_{\boldsymbol{u}_s}b=0\\ \partial_t g+\mathrm{div}(g\boldsymbol{u}_f)=0,\qquad\partial_t(1-g)+\mathrm{div}((1-g)\boldsymbol{u}_s)=0\\ \bar{\rho}_f^2\dfrac{\partial E_f}{\partial\bar{\rho}_f}+p_f=\bar{\rho}_s^2\dfrac{\partial E_s}{\partial\bar{\rho}_s}+p_s=:P\,.\end{cases}\tag{4.3.24}$$

The last equation, obtained from the variations δg, defines the effective pressure P expressed in terms of two Lagrange multipliers (p_f,p_s) enforcing the incompressibility of fluid and solid, respectively. We note that we can rewrite the system in an equivalent way as

$$\begin{cases}\rho_f(\partial_t\boldsymbol{u}_f+\boldsymbol{u}_f\cdot\nabla\boldsymbol{u}_f)=-g\nabla P+\boldsymbol{F}_f\\ \rho_s(\partial_t\boldsymbol{u}_s+\boldsymbol{u}_s\cdot\nabla\boldsymbol{u}_s)=-(1-g)\nabla P\\ \qquad +2\,\mathrm{div}\left(\rho_s\dfrac{\partial E_s}{\partial b}\cdot b\right)+\mathrm{div}\,\bar{s}+\boldsymbol{F}_s\\ \partial_t\rho_f+\mathrm{div}(\rho_f\boldsymbol{u}_f)=0,\quad\partial_t\rho_s+\mathrm{div}(\rho_s\boldsymbol{u}_s)=0,\quad\partial_t b+\pounds_{\boldsymbol{u}_s}b=0\\ \partial_t g+\mathrm{div}(g\boldsymbol{u}_f)=0\\ \mathrm{div}(g\boldsymbol{u}_f+(1-g)\boldsymbol{u}_s)=0\\ \bar{\rho}_f^2\dfrac{\partial E_f}{\partial\bar{\rho}_f}+p_f=\bar{\rho}_s^2\dfrac{\partial E_s}{\partial\bar{\rho}_s}+p_s=P\,.\end{cases}\tag{4.3.25}$$

The pressure P appearing in the first two equations is computed from the elliptic equation deduced from taking the time derivative of the next to last equation of (4.3.25)

which is the incompressibility constraint. The Lagrange multipliers p_f and p_s are then computed from the last equation. As earlier, the internal energy of the fluid can be neglected without changing the dynamics. From the ρ_f, ρ_s, g and $g-1$ equations, we get $\partial_t \bar{\rho}_f + \boldsymbol{u}_f \cdot \nabla \bar{\rho}_f = 0$ and $\partial_t \bar{\rho}_s + \boldsymbol{u}_s \cdot \nabla \bar{\rho}_s = 0$, so $\bar{\rho}_f$, $\bar{\rho}_s$ are constant, if they are constant at the initial time $t = 0$:

$$\bar{\rho}_f = \bar{\rho}_f^0 = \text{const}, \quad \bar{\rho}_s = \bar{\rho}_s^0 = \text{const}\,. \tag{4.3.26}$$

This corresponds to the case when both the incompressible fluid and solid are homogeneous.

Remark 4.3.1 (On the mixture of N incompressible materials) The variational approach presented above generalizes to a mixture of N incompressible or compressible fluids and solids, *e.g.* if a single incompressible matrix is filled with $N-1$ incompressible fluids like oil and water, or incompressible water and compressible air. In that case, we just consider $g_1, ..., g_N$ with $g_1 + ... + g_N = 1$ and as many constraints $p_k(...)$ as there are incompressible components.

Relation with the previous variational derivation. We shall now relate the variational treatment carried out in this paragraph with that of §4.2 and §4.3.1 which uses $c(b)$ and v. We focus on the concentration dependence given in (2.2.9). We start with the case of a compressible solid filled with an incompressible fluid and show that equations (4.2.7) reduce to (4.3.20) when V and E_s are related as

$$\begin{aligned} &V(b,v) = \rho_s(b) E_s(\bar{\rho}_s(b,v), b) \\ &\bar{\rho}_s(b,v) = \frac{\rho_s(b)}{1-\rho_s(b)\kappa v}\,, \quad \text{or} \quad \kappa v = \frac{1}{\rho_s} - \frac{1}{\bar{\rho}_s}\,, \end{aligned} \tag{4.3.27}$$

with $\kappa = c_0/\rho_s^0$. Then $c(b)$ and $\rho_s(b)$ take the form

$$c(b) = \frac{c^0}{\sqrt{\det b}}, \qquad \rho_s(b) = \frac{\rho_s^0}{\sqrt{\det b}} \tag{4.3.28}$$

hence they are related as

$$c(b) = \kappa \rho_s(b). \tag{4.3.29}$$

Additionally, ρ_f in (4.3.20) defined by $\rho_f := \bar{\rho}_f^0 g$. As we have seen, we can assume $E_f = 0$ without loss of generality. Using the formulas

$$\frac{\partial \rho_s}{\partial b} \cdot b = -\frac{1}{2}\rho_s \quad \text{and} \quad \frac{\partial \bar{\rho}_s}{\partial b} \cdot b = -\frac{1}{2}\frac{\bar{\rho}_s^2}{\rho_s}\,,$$

we have the relations

$$\sigma_e = 2\frac{\partial V}{\partial b}\cdot b + V\mathrm{Id} = -\bar{\rho}_s^2\frac{\partial E_s}{\partial \bar{\rho}_s}\mathrm{Id} + 2\rho_s\frac{\partial E_s}{\partial b}\cdot b \quad \text{and} \quad \frac{\partial V}{\partial v} = \kappa\rho_s\bar{\rho}_s^2\frac{\partial E_s}{\partial \bar{\rho}_s}. \tag{4.3.30}$$

From the second relation in (4.3.30), the third equation in (4.2.4) and (4.3.29) yield $p = \rho_s^2\frac{\partial E_s}{\partial \bar{\rho}_s}$ and then using the first relation in (4.3.30) and $\rho_f := \bar{\rho}_f^0 g$, we obtain that the two momentum equations in (4.2.7) reduce to those of (4.3.20). The equations for ρ_s, g, and b coincide while the equation for ρ_f in (4.3.20) follows from the definition $\rho_f := \bar{\rho}_f^0 g$ and the equation for g. In particular, we have the relations $p = \rho_s^2\frac{\partial E_s}{\partial \bar{\rho}_s} = p_f = P$ between the pressures appearing in (4.2.7) and (4.3.20) (recall that $E_f = 0$ here).

The case when both the solid and fluid are incompressible, *i.e.* the relation between equations (4.3.8) and (4.3.25) is shown in a similar way. In this case the pressures are related as $p_f = P = p + \mu$ and $p_s = \mu$ (recall again that $E_f = 0$ here).

Remark 4.3.2 (On neglecting internal energies in the incompressible case) We have seen in the above calculation that in the incompressible case, the internal energy of the fluid can be neglected, *i.e.* we can set $E_f = 0$. One could therefore be tempted to conclude that in the case of an incompressible solid and fluid, the corresponding spatial derivatives $\frac{\partial E_s}{\partial \bar{\rho}_s}$ can be neglected as well. Setting

$$\frac{\partial E_f(\bar{\rho}_f)}{\partial \bar{\rho}_f} = 0\,, \quad \frac{\partial E_s(\bar{\rho}_s, b)}{\partial \bar{\rho}_s} \stackrel{?}{=} 0 \tag{4.3.31}$$

in the last equation of (4.3.25) yields simply $p_f = p_s$ which gives, in the notation above, $\mu = P$ and $p = 0$. That assumption, however, is incorrect due to (4.3.30). This is true even when the solid is incompressible, and $\bar{\rho}_s$ is an advected quantity, and thus can be a constant in space. However, setting the partial derivative of $E_s(\bar{\rho}_s, b)$ with respect to $\bar{\rho}_s$ to zero in (4.3.31) is incorrect, even in that case, since E_s depends additionally on b. The derivatives $\frac{\partial E_f}{\partial \bar{\rho}_f}$ or $\frac{\partial E_s}{\partial \bar{\rho}_s}$ can be dropped only if there is no dependence of E_f or E_s on other quantities, like coordinates or, in our case, Finger's deformation tensor b. In our case, the derivative of E_s with respect to $\bar{\rho}_s$ has to be taken first and then evaluated at the physically relevant value $\bar{\rho}_s$, leading, in general, to $p_f \neq p_s$ in the last equation of (4.3.25). In general, the internal energies of either fluid E_f or solid E_s cannot be dropped if there is a dependence of the variables, such as b in our case, or the coordinates $\mathbf{x}$. An explicit dependence on coordinates for fluid E_f and solid E_s energies appears, for

example, from the introduction of the effects of gravity for fluid and solid, leading to terms proportional to the vertical coordinate x_3 for both fluid and solid. In that case, neither $\frac{\partial E_f}{\partial \bar{\rho}_f}$ nor $\frac{\partial E_s}{\partial \bar{\rho}_s}$ can be neglected in (4.3.25).

4.3.3 Reduction for 1D motion

In what follows, we shall proceed with further simplification of equation (4.3.8), see also (4.3.25), to one dimension and its subsequent numerical analysis. We follow the derivation of (4.2.21) applied now to the doubly incompressible system (4.3.8), with the solid momentum equation given in (4.3.9), which leads to

$$\begin{cases} \bar{\rho}_f^0 g_0 \left(-Y_{tt} + 2\dfrac{Y_t Y_{tx}}{Y_x} - \dfrac{Y_t^2 Y_{xx}}{Y_x^2} \right) \\ \qquad\qquad = -g_0 Y_x \partial_x (p+\mu) + K\left(\dfrac{Y_t}{Y_x} - \dfrac{X_t}{X_x} \right) \\ \rho_s^0 \left(-X_{tt} + 2\dfrac{X_t X_{tx}}{X_x} - \dfrac{X_t^2 X_{xx}}{X_x^2} \right) = g_0 Y_x \partial_x p - (1-g_0 Y_x)\partial_x \mu \\ \qquad\qquad + \partial_x \left(\sigma_e(X_x, Y_x) + \bar{s} \right) + K\left(\dfrac{X_t}{X_x} - \dfrac{Y_t}{Y_x} \right) \\ \partial_x \left(g_0 Y_t + (1-g_0 Y_x)\dfrac{X_t}{X_x} \right) = 0, \quad p = \dfrac{1}{c(b)} \dfrac{\partial V}{\partial v}, \quad b = X_x^{-2} \end{cases} \tag{4.3.32}$$

where, as before, $\sigma_e := 2\frac{\partial V}{\partial b} b + V$ is the elastic stress tensor reduced to the 1-dimensional case.

The last equation of (4.3.32) follows from the last equation of (4.3.8) since $g = g_0 Y_x$ by the incompressibility of fluid (2.2.10) in one dimension. We can further use the reduction of solid incompressibility (4.3.1) to one dimension to get

$$1 - g = 1 - g_0 Y_x = (1 - g_0) X_x \,, \tag{4.3.33}$$

so the last equation of (4.3.38) reduces to

$$\partial_x \left((1-g_0) X_t + g_0 Y_t \right) \quad \Rightarrow \quad (1-g_0) X_t + g_0 Y_t = C(t), \tag{4.3.34}$$

where the integration 'constant' in the right hand side can depend on time.

One can see that the net momentum defined as

$$M := \int_{-L}^{L} \left(\bar{\rho}_f^0 g u_f + \rho_s u_s \right) \mathrm{d}x = - \int_{-L}^{L} \left(\bar{\rho}_f^0 g_0 Y_t + \rho_s^0 X_t \right) \mathrm{d}x$$

is still conserved:

$$\begin{aligned}\dot{M} &= -\int_{-L}^{L} \left(\rho_f g_0 Y_{tt} + \rho_s^0 X_{tt}\right) \mathrm{d}x \\ &= \int_{-L}^{L} \partial_x \left(-\rho_s^0 \frac{X_t^2}{X_x} - \bar{\rho}_f^0 g_0 \frac{Y_t^2}{Y_x} - p + \sigma_e + \overline{s}\right) \mathrm{d}x \\ &= \left(-\rho_s^0 \frac{X_t^2}{X_x} - \bar{\rho}_f^0 g_0 \frac{Y_t^2}{Y_x} - p + \sigma_e + \overline{s}\right) \Big|_{-L}^{L},\end{aligned} \tag{4.3.35}$$

provided the boundary conditions are periodic, or chosen in such a way that the boundary terms in (4.3.35) vanish.

Using the initial conditions for the Lagrangian variables $X(t=0,x) = x$ and $Y(t=0,x) = x$, we obtain a connection between the Lagrangian variables *for all* x and time

$$(1-g_0)X(t,x) + g_0 Y(t,x) = D(t) + x\,, \quad D(t) = \int_0^t C(s)\mathrm{d}s\,. \tag{4.3.36}$$

This connection between the Lagrangian variables in porous media is, in our opinion, quite unexpected, as we started with the spatial description of the dynamics and the incompressibility.

To proceed, we also need to compute the pressure p which can be done from the double incompressibility condition, *i.e.*, the last equation of (4.3.32). We rewrite the first two equations of that system as

$$X_{tt} = R_X + \frac{1-g_0 Y_x}{\rho_s^0}\partial_x \mu\,, \quad Y_{tt} = R_Y + \frac{Y_x}{\bar{\rho}_f^0}\partial_x \mu\,. \tag{4.3.37}$$

where R_X and R_Y are the right-hand sides of the corresponding equations (4.3.32) without the p terms. Differentiating equation (4.3.34) with respect to time gives

$$\left[\frac{Y_x g_0}{\rho_f} + (1-g_0)\frac{(1-g_0 Y_x)}{\rho_s^0}\right]\mu_x = -\left(g_0 R_Y + (1-g_0)R_X\right) + C'(t)\,, \tag{4.3.38}$$

from which we express μ_x as

$$\begin{aligned}&\mu_x(X_t, X_x, X_{xt}, Y_x, Y_{xt}, R_X, R_Y, C'(t); g_0, \rho_s^0, \rho_f) = -\mathcal{A} + C'(t)\mathcal{B} \quad \text{with} \\ &\mathcal{A} := \frac{g_0 R_Y + (1-g_0)R_X}{\frac{Y_x g_0}{\rho_f} + (1-g_0)\frac{(1-g_0 Y_x)}{\rho_s^0}}, \quad \mathcal{B} := \frac{1}{\frac{Y_x g_0}{\rho_f} + (1-g_0)\frac{(1-g_0 Y_x)}{\rho_s^0}}\,.\end{aligned} \tag{4.3.39}$$

Note that (4.3.39) is the one-dimensional analogue of (4.3.11) which also uses (4.3.34) for the definition of $C(t)$.

The solution for μ_x computed from the condition (4.3.38) can be put back into the first two equations of (4.3.32) to form a closed system in terms of (X, Y, X_t, Y_t) and its

spatial derivatives. The value of $C'(t)$ is computed in such a way that the mean value of μ_x is zero for periodic boundary conditions, which gives:

$$C'(t) = \frac{\int_{-L}^{L} \mathcal{A} \mathrm{d}x}{\int_{-L}^{L} \mathcal{B} \mathrm{d}x} \,. \tag{4.3.40}$$

This adjustment is necessary since we have implicitly assumed that all functions, including the Lagrange multipliers, are periodic and thus all their derivatives have zero mean. The modification (4.3.40) is not necessary for simulations on the line.

To derive the potential, we consider the following physical realization of one dimensional, doubly incompressible porous media. Consider a tube filled with an incompressible fluid, and suppose there are elastic muscle threads of negligible volume that are running along the axis of the tube. On each thread, there are rigid (and hence incompressible) beads attached to a given point on a particular thread, as illustrated on Figure 4.3.1. When the threads are stretched, the beads move inside the fluid and change the local volume of the fluid g at the given Eulerian point. Then, the elastic energy is proportional to the deformation energy of the thread times the number of threads per given interval x.

Based on considerations above, we suggest using the following potential $V(b, v)$:

$$V = \frac{\alpha}{2}(X_x - 1)^2 = \frac{\alpha}{2}\left(\frac{1}{\sqrt{b}} - 1\right)^2 \,, \tag{4.3.41}$$

where in the physical realization presented in Figure 4.3.1, the constant α is dependent on the number of the strings for a cross-section of the tube and a typical elasticity of each string. This potential is just the first term of (4.2.23) with $\beta = 0$, since the second term in (4.2.23) proportional to β describes the elastic energy due to the deformation of the pores. With the potential (4.3.41), we obtain

$$\sigma_e = 2\frac{\partial V}{\partial b} b + V = -\frac{\alpha}{2}(X_x^2 - 1) \,, \quad p = \frac{1}{c}\frac{\partial V}{\partial v} = 0 \,. \tag{4.3.42}$$

We now present the results of numerical solutions obtained for the potential (4.3.41).

First, we present a doubly incompressible computation equivalent to the case of the compressible solid presented in Figure 4.2.1, for the same values of parameters except setting $\beta = 0$ in the the potential given by (4.2.23), *i.e.*, using the potential (4.3.41). Notice that the motion of the solid and the fluid acquire 'jerkiness' in the double incompressible case, since the motion is less smooth than that illustrated on the Figure 4.2.1.

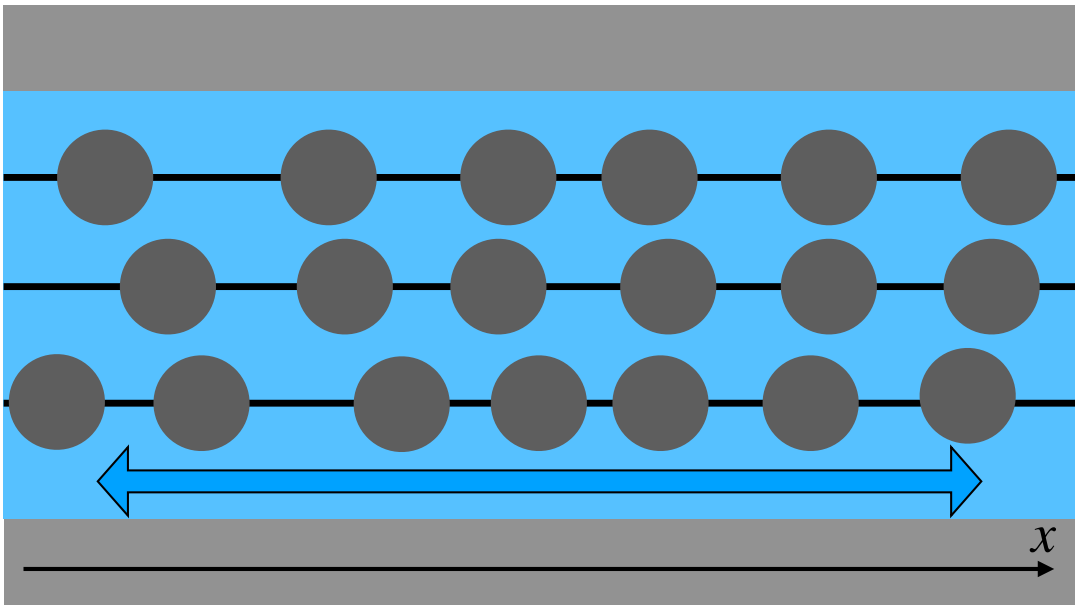

Figure 4.3.1: A sketch of the physical realization of a one dimensional, doubly incompressible case, justifying the potential (4.3.41). A channel is filled with incompressible fluid and holds solid particles on elastic strings running parallel to the axis of the channel (the x-axis). The space available to the fluid is dependent on the density of solid particles in a given interval $[x, x + \mathrm{d}x]$. The potential V depends on the local deformations of the strings caused by the motion of the particles, the elastic properties of the strings and the number of the strings running through the channel.

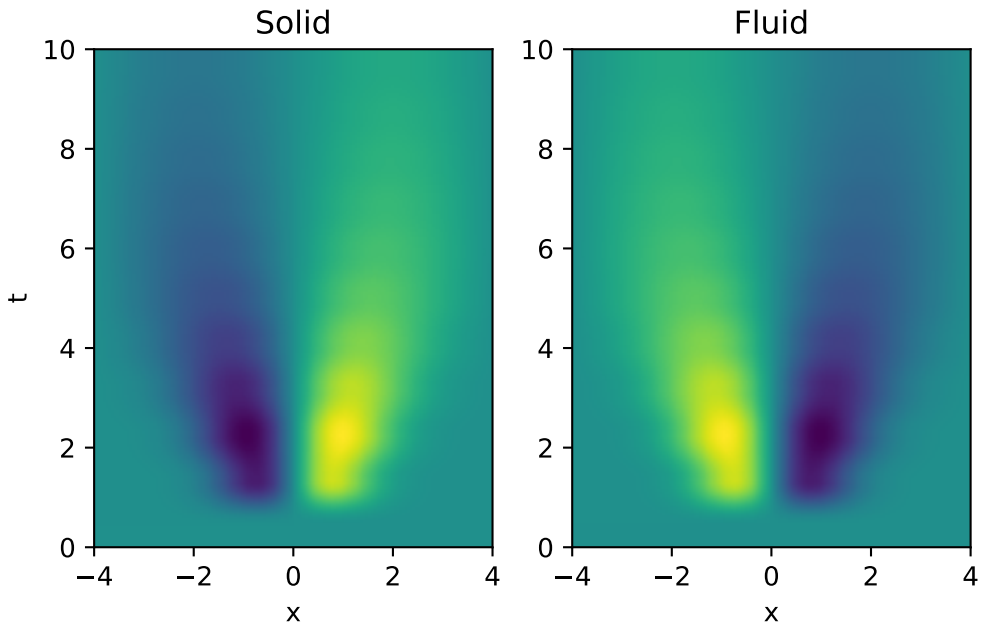

Figure 4.3.2: Example of numerical solution of (4.3.32). Left panel: solution for $X(t,x)-x$ (solid), right panel: $Y(t,x)-x$ (fluid). All parameters are as for solution presented in Figure (4.2.1) except for the extra incompressibility condition for the solid in (4.3.32) and change $\beta = 0$ in potential (4.2.23), *i.e.*, taking the the potential (4.3.41) with elastic stress σ_e given by (4.3.42).

Next, we show the self-propulsion due to the generation of momentum due to the traveling wave motion of the muscle stress as was discussed in Section 4.2.4 and further illustrated in (4.2.2). Figure 4.3.3 shows the possibility of generating self-propulsion of the solid from rest due to non-zero momenta of the solid and fluid, while the net momentum of both solid and fluid is conserved and equal to 0. Furthermore, on Figure 4.3.4,

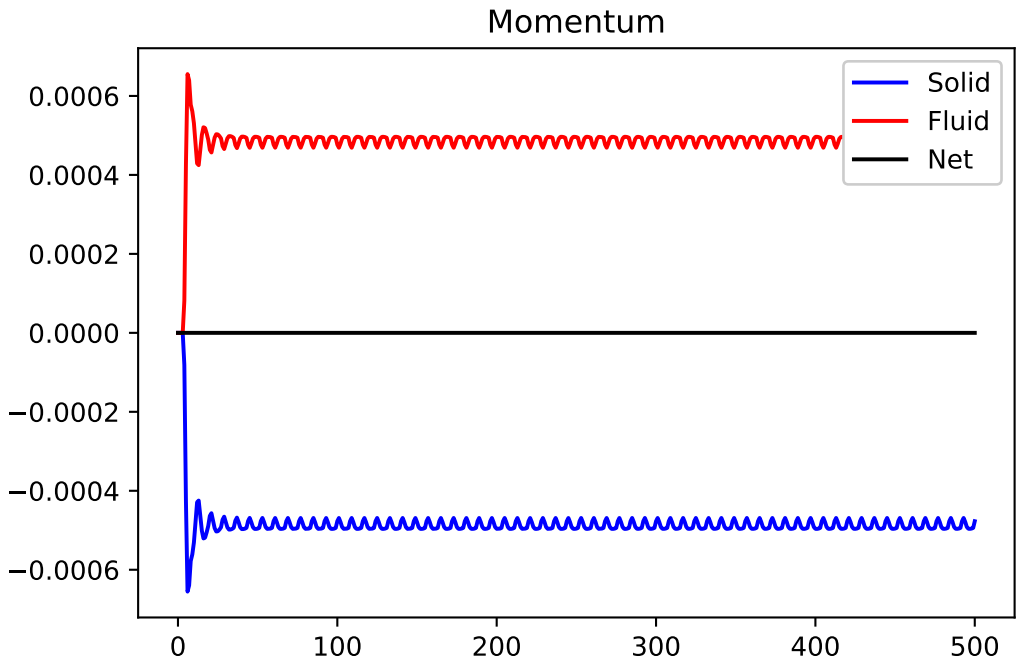

Figure 4.3.3: Momenta for solid $M_s = -\rho_s^0 \int_{-L}^{L} X_t \mathrm{d}x$ (blue line), $M_f = -\rho_f g_0 \int_{-L}^{L} Y_t \mathrm{d}x$ (red line) and the total $M_f + M_s$ (black line) for a given numerical solution with zero initial conditions and prescribed traveling muscle force given by (4.2.34) with parameters $S_0 = 0.1$, $W = 1$ and $U = 1$. As in Figure 4.3.3, the net momentum $M_s + M_f$ is close to 0 with expected accuracy throughout the simulation. All other parameters for simulations are taken exactly the same as in Figure 4.2.2, *i.e.* $g_0 = 0.5$, $K = 1$, with the same total time of simulation $t = 500$. We note that compared to (4.2.2), the stabilization of motion occurs on a much faster time scale compared with the compressible case.

we illustrate how well the incompressibility condition is satisfied. In other words, we plot the term $(1 - g_0)X + g_0 Y - x$ as a function of x for all available values of t, which, according to the last equation in (4.3.32), must be equal to $D(t)$, which is a constant as a function of x, but can vary in time. Finally, on Figure 4.3.5, we present the variable $D(t) = (1 - g_0)X + g_0 Y - x$, expected to be independent of x, as defined by the last equation of (4.3.32). Figure 4.3.4 confirms that this variable, taken as a function of x for

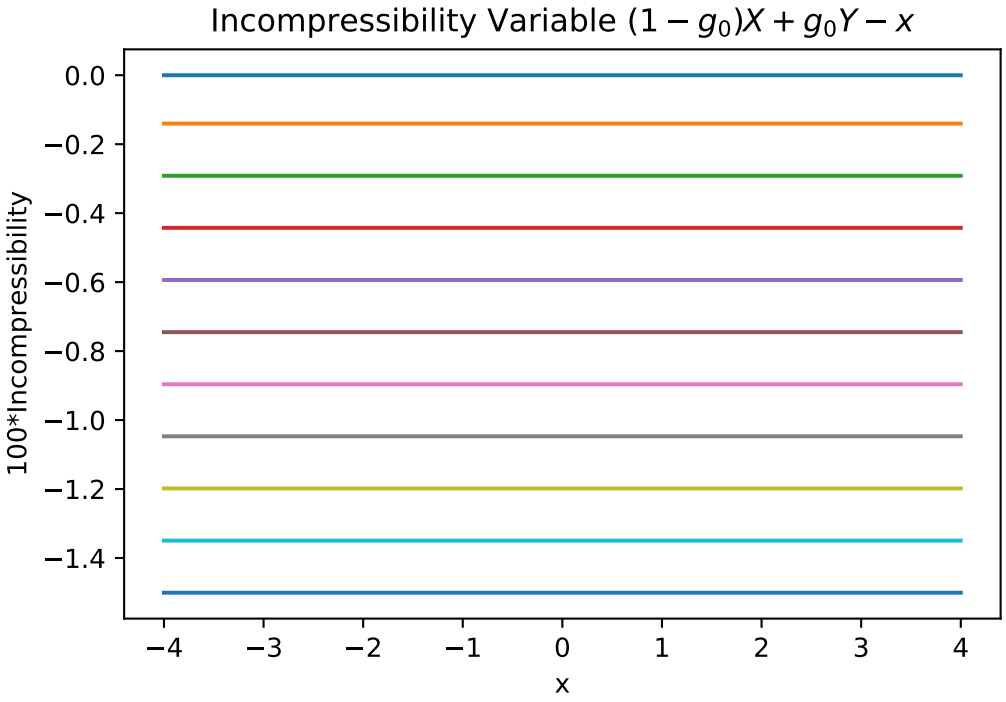

Figure 4.3.4: The value of $(1-g_0)X+g_0Y-x$ computed for a set of values of t in simulation output ($t=0,50,100,\ldots,500$), presented as a function of x. According to the equation (4.3.34), that expression should be independent of x, although can vary with t, which is consistent with results presented here. All solutions $(1-g_0)X+g_0Y-x$ presented as a function of x are close to horizontal lines. The distance of these horizontal lines from the x-axis is dependent on time as is permitted by (4.3.34).

a fixed t, is indeed almost a constant within numerical accuracy. Thus, we compute $D(t)$ as the mean value of $(1-g_0)X + g_0 Y - x$. It is worth noting that our numerical scheme

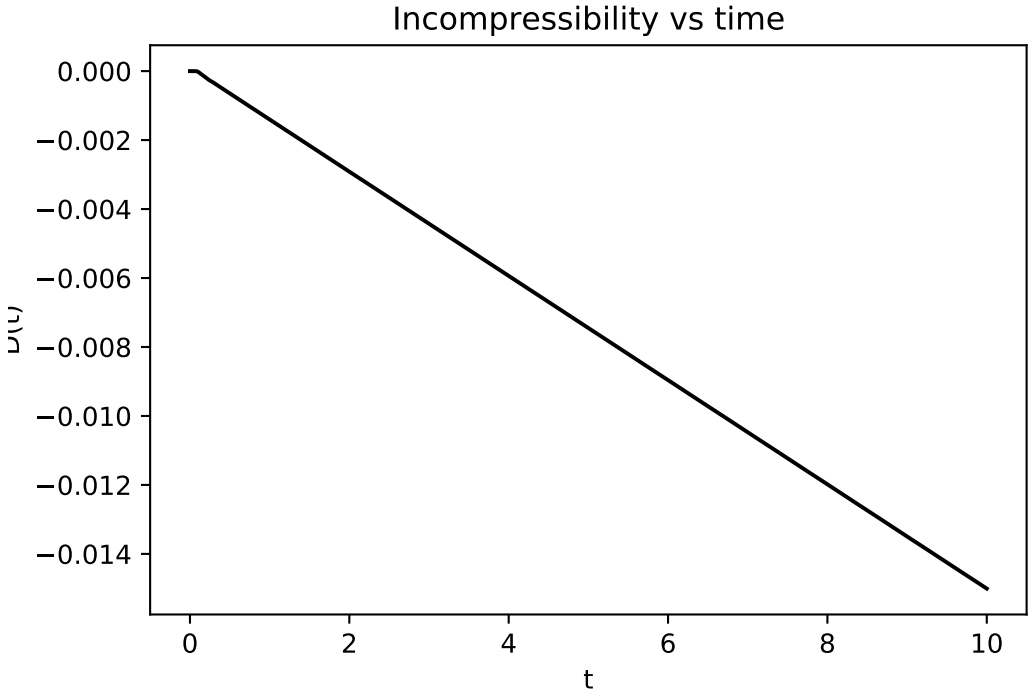

Figure 4.3.5: The value of $D(t)$, computed as the mean value of $(1-g_0)X + g_0 Y - x$ (mean computed with respect to the x-variable), as a function of time t, for the results presented in Figure 4.3.4.

does not take into account the variational structure of equations. For the time scales we have computed the solutions for, $t \sim 10^3$, no apparent loss of accuracy was observed in simulations. However, the computation of much longer times may need the use of variational integrators. Variational integrators conserve momenta-related quantities to machine precision and allow for accurate imposition of the friction. In our case, since the friction is strictly internal to the system, variational methods can still be beneficial for long-term preservation of momenta. We refer the reader to [61], [81] for the general theory of variational integrators and to [41], [42] for the derivations of variational integrators for fluid-structure interactions with incompressibility constraint, in particular, for the application to the motion of an elastic tube filled with an incompressible fluid.

Chapter 5

Conclusion

This thesis is a development of a fully variational geometric approach to handle the problem of elastic porous media filled with incompressible inviscid Eulerian fluid. The derivations of the dynamics equations cover two cases: the case when the solid material is compressible and where it is incompressible (total incompressibility). The analysis includes the linearization of the derived dynamical system to show the stability of acoustic wave propagation and proof of the conservation of energy to demonstrate that physical assumptions are satisfied in the resulting equations. The acoustic properties of S- and P- waves are documented for different non-dimensionalized system parameters.

For the derivation of the equations of motion for a porous media filled with an incompressible fluid, our research team has chosen to use the Eulerian frame for both the fluid and the porous media. Our equations are valid for arbitrary deformations and, as far as we are aware, are new. We have compared the linearized equations of motion to Biot's equations and found a correspondence between our equations and Biot's equations, with a clear and physical interpretation of the parameters.

The explicit reduction of the derived systems to the one-dimensional case allows to prove the conservation of total momenta, and perform numerical simulations, showing the potential of self-propulsion of a porous elastic body under moving muscle stress. An important application of this theory is the dynamics of active biological porous materials (such as sea sponges). The thesis illustrates the numerical simulations of the nonlinear dynamics in 1d and the nonlinear phenomena in the case of total incompressibility.

This research concludes that the geometric variational method is an elegant and efficient instrument in the derivation of equations of dynamics of complex multi-component systems with an arbitrary number of incompressible phases, including the poromechan-

ics system, studied in course of this work. The derived systems could explain various physical phenomena that were previously attributed to unknown parameters of the media. This is seen, for example, from the comparison with the linearization of Biot's poromechanics dynamics. The nonlinear dynamics equations of could be used in numerical experiments to explain and predict the behavior of a wide range of the multi-phase systems, including biological systems, that could activate endogenous muscle stress and respond to stimuli.

In the future, it will be interesting to combine this research to include additional biologically relevant problems. For example, one could introduce the variational approach in combination with the previous work by the research team on the geometric variational approach to elastic tubes conveying fluid [37], [39], [40]. Making the tube's wall porous will be relevant to other engineering [45] and biological applications like arterial flow [20]. For the successful implementation of variational methods for the fluid-structure interaction problems, the boundary conditions for moving boundaries need to be considered in more detail, as outlined in Remark 4.2.1. In general, further applications of variational methods for fluid-structure interactions look promising and warrant future studies.

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Appendix A

Geometric structure of classical mechanics

A.1 Manifolds

The contents of this appendix refer to definitions and examples of smooth manifolds that are collected here for ready reference. We closely follow D. D. Holm's Lecture notes [1], see also [47], [48].

Definition A.1.1 *A smooth (i.e., differentiable) manifold M is a set of points together with a finite (or perhaps countable) set of subsets $U_\alpha \subset M$ and one-to-one mappings $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that*

- $\bigcup_\alpha U_\alpha = M$;
- *for every nonempty intersection $U_\alpha \cap U_\beta$, the set $\phi_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$ and the one-to-one mapping $\phi \circ \phi_\alpha^{-1}$ is a smooth function $\phi_\alpha(U_\alpha \cap U_\beta)$.*

Remark A.1.2 *The sets U_α in the definition are called **coordinate charts**. The mappings ϕ_α are called **coordinate functions** or **local coordinates**. A collection of charts satisfying both conditions is called an **atlas**.*

Definition A.1.3 (Tangent vector) *The tangent space T_xM at a point x of a manifold M is a vector space. The elements of this space are called **tangent vectors** (Figure A.3).*

Definition A.1.4 (Tangent bundle) *The tangent bundle of a manifold M, denoted by TM, is the smooth manifold whose underlying set is the disjoint union of the tangent spaces to M at the points $x \in M$ (Figure A.4); that is,*

$$TM = \bigcup_{x \in M} T_xM.$$

Thus, a single point of TM is (x, v) where $x \in M$ and $v \in T_xM$.

Definition A.1.5 (Lie group and Lie subgroup manifolds) *A Lie group is a group that is also a manifold. A Lie subgroup is a submanifold that is invariant under group operations. That is, Lie subgroups are injective immersions.*

[1] http://wwwf.imperial.ac.uk/ dholm/classnotes/

A.2 Motion: Tangent vectors and flows

Envisioning our later considerations of dynamical systems, we shall consider motion along curves $c(t)$ parameterized by time t on a smooth manifold M. Suppose these curves are trajectories of a flow ϕ_t along the tangent vectors of the manifold. We anticipate this means $\phi_t(c(0)) = c(t)$ and $\phi_t \circ \phi_s = \phi_{t+s}$ (flow property). The flow will be tangent to M along the curve. To deal with such flows, we will need to know more about tangent vectors.

A.2.1 Vector fields, integral curves and flows

Definition A.2.1 *A **vector field** on a manifold M is a map $X : M \to TM$ that assigns a vector $X(x)$ at each point $x \in M$. This implies that $\tau_M \circ X = id$.*

Definition A.2.2 *An **integral curve** of X with initial conditions x_0 at $t = 0$ is a differentiable map $c :]a, b[\to M$, where $]a, b[$ is an open interval containing 0, such that $c(0) = x_0$ and $c'(t) = X(c(t))$ for all $t \in]a, b[$.*

A.2.2 Differentials of functions: The cotangent bundle

We are now ready to define differentials of smooth functions and the cotangent bundle.

Let $f : M \to R$ be a smooth function. We differentiate f at $x \in M$ to obtain $T_x f : T_x M \to Tf(x)R$. As is standard, we identify $Tf(x)R$ with R itself, thereby obtaining a linear map $df(x) : T_x M \to R$. The result $df(x)$ is an element of the cotangent space $T_x^* M$, the dual space of the tangent space $T_x M$. The natural pairing between elements of the tangent space and the cotangent space is denoted as $\langle \cdot, \cdot \rangle : T_x^* M \times T_x M \mapsto R$.

In coordinates, the linear map $df(x) : T_x M \to R$ may be written as the directional derivative,

$$\boxed{\langle df(x), x \rangle = df(x) \cdot v = \frac{\partial f}{\partial x^i} \cdot v^i, \text{ for all } v \in T_x M.}$$

(Reminder: The summation convention is applied over repeated indices.) Hence, elements $df(x) \in T_x^* M$ are dual to vectors $v \in T_x M$ with respect to the pairing $\langle \cdot, \cdot \rangle$.

Definition A.2.3 *df is the **differential** of the function f.*

Definition A.2.4 *The dual space of the tangent bundle TM is the **cotangent bundle** T^*M. That is,*

$$(T_x M)^* = T_x^* M \text{ and } T^* M = \bigcup_x T_x^* M.$$

Thus, replacing $v \in T_x M$ with $df \in Tx^* M$, for all $x \in M$ and for all smooth functions $f : M \to R$, yields the cotangent bundle T^*M.

A.3 Tangent and cotangent lifts

We next define derivatives of differentiable maps between manifolds (tangent lifts).

We expect that a smooth map $f : U \to V$ from a chart $U \subset M$ to a chart $V \subset N$ will lift to a map between the tangent bundles TM and TN so as to make sense from the viewpoint of ordinary calculus,

$$U \times \mathbb{R} \subset TM \longrightarrow V \times \mathbb{R}^n \subset TN$$

$$(q^1, \ldots, q^m; X^1, \ldots, X^m) \mapsto (Q^1, \ldots, Q^m; Y^1, \ldots, Y^m).$$

Namely, the relations between the vector field components should be obtained from the differential of the map $f : U \to V$. Perhaps not unexpectedly, these vector field components will be related by

$$Y^i \frac{\partial}{\partial Q^i} = X^j \frac{\partial}{\partial q^j}, \text{ so } Y_i = \frac{\partial Q^i}{\partial q_j} X^j,$$

in which the quantity called the **tangent lift**

$$Tf = \frac{\partial Q}{\partial q}$$

of the function f arises from the chain rule and is equal to the Jacobian for the transformation $Tf : TM \mapsto TN$.

The dual of the tangent lift is the cotangent lift. Roughly speaking, the **cotangent lift** of the function f,

$$T^*f = \frac{\partial q}{\partial Q},$$

arises from

$$\beta_i dQ^i = \alpha_j dq^j, \text{ so } \beta_i = \alpha_j \frac{\partial q^j}{\partial Q^i},$$

and $T^*f : T^*N \mapsto T^*M$. Note the directions of those maps.

A.3.1 Summary of derivatives on manifolds

Definition A.3.1 (Differentiable map) *A map $f : M \to N$ from manifold M to manifold N is said to be **differentiable** (resp. C^k) if it is represented in local coordinates on M and N by differentiable (resp. C^k) functions.*

Definition A.3.2 (Derivative of a differentiable map) *The **derivative** of a differentiable map*

$$f : M \to N$$

at a point $x \in M$ is defined to be the linear map

$$T_x f : T_x M \to T_x N,$$

constructed as follows. For $v \in T_x M$, choose a curve that maps an open interval $(-\epsilon, \epsilon)$ around the point $t = 0$ to the manifold M,

$$c : (-\epsilon, \epsilon) \Rightarrow M,$$
$$\text{with } c(0) = x,$$
$$\text{and velocity vector } \left.\frac{dc}{dt}\right|_{t=0} = v.$$

Then $T_x f \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$. That is,

$$\boxed{T_x f \cdot v = \left.\frac{d}{dt} f(c(t))\right|_{t=0}}.$$

Definition A.3.3 *The union $Tf = \bigcup_x T_x f$ of the derivatives $T_x f : T_x M \to T_x N$ over points $x \in M$ is called the **tangent lift** of the map $f : M \to N$.*

Appendix B

Lie groups, algebras and their applications

Similar to Appendix A, the contents of this appendix refer to definitions and examples of Lie groups that are collected here for reference purposes. Again, we closely follow Lecture notes by D. D. Holm [1], see also [47], [48].

B.1 Matrix Lie groups

Definition B.1.1 (Group) *A group G is a set of elements possessing*

- *A binary product (multiplication), $G \times G \to G$, such that the following properties hold:*
  - *The product of g and h is written gh.*
  - *The product is associative, $(gh)k = g(hk)$.*
- *Identity element $e : eg = g$ and $ge = g, \forall g \in G$.*
- *Inverse operation $G \to G$, so that $gg^{-1} = g^{-1}g = e$.*

Definition B.1.2 (Lie group) *A Lie group is a smooth manifold G which is also a group and for which the group operations of multiplication, $(g, h) \to gh$ for $g, h \in G$, and inversion, $g \to g^{-1}$ with $gg^{-1} = g^{-1}g = e$, are smooth functions.*

Definition B.1.3 *A matrix Lie group is a set of invertible $n \times n$ matrices which is closed under matrix multiplication and which is a submanifold of $\mathbb{R}^{n\times n}$.*

Example B.1.4 (The general linear group $GL(n, \mathbb{R})$) *The matrix Lie group $GL(n, \mathbb{R})$ is the group of linear isomorphisms of $\mathbb{R}^n$ to itself. The dimension of the $n \times n$ matrices in $GL(n, \mathbb{R})$ is n^2, the number of independent elements.*

B.2 Defining matrix Lie algebras

Definition B.2.1 (Matrix Lie algebra) *A matrix Lie algebra $\mathfrak{g}$ is a set of $n \times n$ matrices which is a vector space with respect to the usual operations of matrix addition and multiplication by real numbers and which is closed under the matrix commutator $[\cdot, \cdot]$.*

[1] http://wwwf.imperial.ac.uk/ dholm/classnotes/

B.3 Examples of matrix Lie groups

Example B.3.1 (The orthogonal group $O(n)$) *The set of $n \times n$ real matrices satisfying $A^T A = AA^T = 1$ defines the orthogonal group $O(n)$, the group of $n \times n$ orthogonal matrices.*

Example B.3.2 *(The special orthogonal group $SO(n)$) If, in addition to the condition $A^T A = AA^T = 1$ above we require $\det A = 1$, we get the Special Orthogonal group, called $SO(n)$. This group plays a particular important role in mechanics.*

Example B.3.3 (The general linear group $GL(n, \mathbb{R})$) *The set of all $n \times n$ real, invertible matrices is called $GL(n, \mathbb{R})$, General Linear Group.*

Example B.3.4 (The special linear group $SL(n, \mathbb{R})$) *The subgroup of $GL(n, \mathbb{R})$ with $\det U = 1$ is called $SL(n, \mathbb{R})$.*

Example B.3.5 *(Tangent space of SO(n) at the identity) Suppose $A(t)$ is a smooth curve in $SO(n)$, with $A(0) = \mathrm{Id}$, $-\epsilon < t < \epsilon$. Differentiating the condition $A^T(t)A(t) = \mathrm{Id}$ with respect to t and setting t to 0 gives $\dot{A}^T + \dot{A} = 0$. Thus, $\dot{A}(0)$ are antisymmetric matrices. Lying in the tangent space at the identity of a matrix Lie group, this linear vector space forms the matrix Lie algebra $\mathfrak{so}(n)$.*

Example B.3.6 (The special unitary group SU(n)) *The Lie group $SU(n)$ comprises complex $n \times n$ unitary matrices U with $U^\dagger U = \mathrm{Id}$ and unit determinant $\det U = 1$. An element u in its tangent space at the identity satisfies $u + u^\dagger = 0$ for $u \in T_I SU(n)$, so that $u \in \mathfrak{su}(n)$ is an $n \times n$ traceless skew Hermitian matrix.*

Example B.3.7 (The symplectic group $Sp(l)$) *Suppose $n = 2l$ (that is, let n be even) and consider the nonsingular skew-symmetric matrix*

$$J = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix} \tag{B.3.1}$$

where $\mathbb{I}$ is the $l \times l$ identity matrix and $\mathbb{O}$ is the $l \times l$ zero matrix.

One may verify that the matrices $U \in Sp(l)$, where

$$Sp(l) := \{U \in GL(2l, \mathbb{R}) \mid U^T J U = J\}$$

form a group. The group of these matrices is called the symplectic group. Reasoning as before, the matrix algebra $T_I Sp(l)$ is defined as the set of $n \times n$ matrices A satisfying $JA^T + AJ = 0$. This matrix Lie algebra is denoted as $\mathfrak{sp}(l)$.

Example B.3.8 (The special Euclidean group $SE(3)$) *Consider the Lie group of 4×4 matrices of the form*

$$A = \begin{pmatrix} R & \mathbf{v} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{B.3.2}$$

where $R \in SO(3)$ and $\mathbf{v} \in \mathbb{R}^3$. and $\mathbf{0}^T$ is a length 3 row vector. This is the special Euclidean group, denoted $SE(3)$. The special Euclidean group is of central interest in mechanics since it describes the set of rigid motions of objects in the three-dimensional space.

B.4 Lie group actions

The action of a Lie group G on a manifold M is a group of transformations of M associated with elements of the group G, whose composition acting on M corresponds to group multiplication in G.

Definition B.4.1 (Left action of a Lie group) *Let M be a manifold and let G be a Lie group. A left action of a Lie group G on M is a smooth mapping $\Phi\colon G \times M \to M$ such that*

- $\Phi(e, x) = x \quad \forall x \in M$;
- $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ *for all* $g, h \in G$ *and* $x \in M$; *and*
- $\Phi(g, \cdot)$ *is a diffeomorphism on M for each $g \in G$.*

We often use the convenient notation gx for $\Phi(g, x)$ and think of the group element g acting on the point $x \in M$. The associativity condition (ii) then simply reads $(gh)x = g(hx)$. Similarly, one can define a right action, which is a map $\Psi : M \times G \to M$ satisfying $\Psi(x, e) = x$ and $\Psi(\Psi(x, g), h) = \Psi(x, gh)$. The convenient notation for right action is xg for $\Psi(x, g)$, the right action of a group element g on the point $x \in M$. Associativity $\Psi(\Psi(x, g), h) = \Psi(x, gh)$ may then be expressed conveniently as $(xg)h = x(gh)$.

Example B.4.2 (Properties of Lie group actions) *The action $\Phi : G \times M \to M$ of a group G on a manifold M is said to be*

1. *transitive, if for every $x, y \in M$ there exists a $g \in G$, such that $gx = y$;*
2. *free, if it has no fixed points, that is, $\Phi g(x) = x$ implies $g = e$; and*
3. *proper, if whenever a convergent subsequence $\{x_n\}$ in M exists, and the mapping $g_n x_n$ converges in M, then $\{g_n\}$ has a convergent subsequence in G.*

Definition B.4.3 (Group orbits) *Given a Lie group action of G on M, for a given point $x \in M$, the subset $\operatorname{Orb} x = \{gx \mid g \in G\} \subset M$ is called the group orbit through x.*

B.4.1 Left and right translations on a Lie group

Left and right translations on the group are denoted L_g and R_g, respectively. For example, $L_g : G \to G$ is the map given by $h \to gh$, while $R_g : G \to G$ is the map given by $h \to hg$, for $g, h \in G$. Left translation $Lg : G \to G; h \to gh$ defines a transitive and free action of G on itself. Right multiplication $Rg : G \to G; h \to hg$ defines a right action, while $h \to hg^{-1}$ defines a left action of G on itself.

G acts on G by conjugation, $g \to Ig = Rg - 1 \circ Lg$. The map $I_g : G \to G$ given by $h \to ghg^{-1}$ is the inner automorphism associated with g. Orbits of this action are called conjugacy classes.

Differentiating conjugation at the identity e gives the adjoint action Ad_g of G on $\mathfrak{g}$. Explicitly, the adjoint action of G on g is given by Ad_g

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}g(\xi) = T_e(R_{g^{-1}} \circ L_g)\xi.$$

We have used the adjoint action for matrix Lie groups acting on matrix Lie algebras, when we defined Ad_g for matrix Lie groups in 1.4.

The coadjoint action of G on g_*, the dual of the Lie algebra $\mathfrak{g}$ of G, is defined using the above definition of Ad_g and pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $\mathrm{Ad}_g^* : \mathfrak{g}^* \to g^*$ be the dual of Ad_g, defined by

$$\langle Ad_g^* \alpha, \xi \rangle = \langle \alpha, \mathrm{Ad}_g \xi \rangle$$

or $\alpha \in \mathfrak{g}^*, \xi \in \mathfrak{g}$ and pairing $\langle \cdot, \cdot \rangle : g_* \times g \to \mathbb{R}$. Then the map $\Phi^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by $(g, \alpha) \to Ad_{g^{-1}}^* \alpha$ is the coadjoint action of G on $\mathfrak{g}^*$.

B.5 Lie derivative and Jacobi–Lie bracket

The physical meaning of Lie derivative $\mathcal{L}_{\boldsymbol{X}} T$ of a geometric quantity T with respect to vector field $\boldsymbol{X}$ is the rate of change of this quantity when the underlying coordinates are flowing by the observer with the speed defined by the vector field $\boldsymbol{X}$. There are several definitions of Lie derivative which are equivalent in the case when the manifold M, on which $\boldsymbol{X}$ is defined, is finite-dimensional. One definition is computed through the flow of $\boldsymbol{X}$. Another definition is axiomatic, simply stating that

1. For any scalar function f,
$$\mathcal{L}_{\boldsymbol{X}} f = (\boldsymbol{X} \cdot \nabla) f$$
An equivalent notation for the directional derivative is $(\boldsymbol{X} \cdot \nabla) f = \boldsymbol{X}(f)$.

2. For vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$,
$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}] = (\boldsymbol{X} \cdot \nabla) \boldsymbol{Y} - (\boldsymbol{Y} \cdot \nabla) \boldsymbol{X} \tag{B.5.1}$$

3. For one-form α and arbitrary vector field $\boldsymbol{Y}$ we have
$$\langle \mathcal{L}_{\boldsymbol{X}} \alpha, \boldsymbol{Y} \rangle = \langle \alpha, [\boldsymbol{X}, \boldsymbol{Y}] \rangle + (\boldsymbol{Y} \cdot \nabla) \langle \alpha, \boldsymbol{X} \rangle$$

4. For arbitrary tensor fields S and T we have Leibnitz rule:
$$\mathcal{L}_{\boldsymbol{X}} (S \otimes T) = \mathcal{L}_{\boldsymbol{X}} \otimes T + S \otimes \mathcal{L}_{\boldsymbol{X}} T \tag{B.5.2}$$

This Lie derivative (B.5.1), known as a commutator of vector fields, plays an important role in the derivation of continuum mechanics. Lie derivaties of arbitrary tensor fields can be then computed using (B.5.2).

Appendix C

Rigid body dynamics

C.1 Notation: vectors as antisymmetric matrices and vice versa

Here, we explain the notation related to rigid body rotations related to the motion on the group $SO(3)$. A careful reader has noticed that the object $\omega = \Lambda^{-1}\dot{\Lambda}$, that we have called the angular velocity, is an antisymmetric 3×3 matrix. This can be seen by differentiating the identity for orientation matrices:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^T\Lambda = \mathrm{Id}_{3\times 3} \quad \Rightarrow \quad \dot{\Lambda}^T\Lambda + \Lambda^T\dot{\Lambda} = 0 \quad \Rightarrow \quad \omega^T + \omega = 0\,. \tag{C.1.1}$$

As it turns out, these matrices are equivalent to vectors in three-dimensional space through the so-called hat map, which is defined as follows. To a given antisymmetric 3×3 matrix ω, we associated a vector $\boldsymbol{\omega}$ according to the following rule:

$$\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{C.1.2}$$

Then, for any column vector $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, we have

$$\omega\mathbf{v} = \begin{pmatrix} \omega_2 v_3 - \omega_3 v_2 \\ \omega_3 v_1 - \omega_1 v_3 \\ \omega_1 v_2 - \omega_2 v_1 \end{pmatrix} = \boldsymbol{\omega} \times \mathbf{v}\,. \tag{C.1.3}$$

Thus, to every antisymmetric 3×3 matrix ω we can associate a vector $\boldsymbol{\omega}$ through the rule (C.1.2). The mapping from vectors to antisymmetric matrices is called the *hat map*, and we use the notation $\widehat{\boldsymbol{\omega}} = \omega$. The inverse procedure, taking an antisymmetric matrix and producing a vector, is called the *inverse hat map* and is denoted as $\omega^\vee = \boldsymbol{\omega}$. In coordinates we have $\omega_{ij} = -\epsilon_{ijk}\omega_k$ where ϵ_{ijk} is the completely antisymmetric tensor with $\epsilon_{123} = 1$. Because of this property, the notation $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}\times$ is also used, although we will not employ it here. Another useful property of the hat map relates the commutator of matrices a and b to the cross product of vectors $\mathbf{a} = a^\vee$ and $\mathbf{b} = b^\vee$ as

$$(ab - ba)^\vee = [a, b]^\vee = \mathbf{a} \times \mathbf{b} \quad \Leftrightarrow \quad ab - ba = [a, b] = \widehat{\mathbf{a} \times \mathbf{b}}\,. \tag{C.1.4}$$

Thus, we can treat the angular velocity ω to be both an antisymmetric matrix when it is defined as $\omega = \Lambda^{-1}\dot{\Lambda}$, and, in the same time, a 3-vector using $\boldsymbol{\omega} = \omega^{\vee} = (\Lambda^{-1}\dot{\Lambda})^{\vee}$ through the hat map. These representations are completely equivalent and are fundamental for our further discussions.

In addition, it is also useful to review the concept of differentiation with respect to vectors and matrices, in order to make the meaning of equations more precise. Clearly, the derivative of a scalar function, such as the Lagrangian, with respect to a column vector is a row vector, and their product can be computed using either the dyadic algebra or scalar product. In other words, for column vectors $\mathbf{a}$ and $\mathbf{b}$, and a function $F(\mathbf{a})$, we have

$$\frac{\partial F}{\partial \mathbf{a}}\mathbf{b} = \left(\frac{\partial F}{\partial \mathbf{a}}\right)^{T} \cdot \mathbf{b} = \sum \frac{\partial F}{\partial a_i} b_i = (\text{row})(\text{vector}) = (\text{scalar}). \tag{C.1.5}$$

The equivalent representation of derivatives in terms of matrices is less straightforward. First, we need to introduce the pairing (scalar product) between two 3×3 matrices A and B

$$\big\langle A\,,\, B\big\rangle = \frac{1}{2}\mathrm{tr}\big(A^T B\big)\,. \tag{C.1.6}$$

We will typically take derivatives of functions of the type $F(a) = \frac{1}{2}\big\langle \mathbb{D}a, a\big\rangle$ for antisymmetric matrices a and a diagonal matrix $\mathbb{D} = \mathrm{diag}(d_1, d_2, d_3)$, having the physical meaning of the inertia matrix. One can readily check that the matrix $\frac{\partial F}{\partial a} = \mathbb{D}a$ is, in general, not antisymmetric so it cannot be directly interpreted as a vector. However, for any antisymmetric matrix b, the product $\big\langle \frac{\partial F}{\partial a}\,,\, b\big\rangle$ only depends on the antisymmetric part of $\frac{\partial F}{\partial a}$. Thus, the following quantity is readily interpreted as a vector

$$\frac{\partial F}{\partial \mathbf{a}} = \frac{1}{2}\left[\frac{\partial F}{\partial a} - \left(\frac{\partial F}{\partial a}\right)^{T}\right]^{\vee}. \tag{C.1.7}$$

Because of the apparent complexity of (C.1.7), we shall mostly use vector derivatives (C.1.5) in the formulas in this thesis.

C.1.1 Rotation matrices

Suppose a rigid body is fixed at point O which we choose as the origin. Let us mark a point $\boldsymbol{X}$ on the rigid body and observe where this point moves after time t. We mark this point as $\mathbf{x}(t, \boldsymbol{X})$. The key assumption of the rigid body is that all points of the rigid body move in unison, *i.e.*

$$\mathbf{x}(t, \boldsymbol{X}) = \mathbb{A}(t)\boldsymbol{X}\,, \quad \mathbb{A}(t) \text{ is a } 3 \times 3 \text{ matrix}. \tag{C.1.8}$$

The fundamental assumption about the rigid body is that the distance between any two points remains the same under the evolution. Thus, if $\mathbf{x} = \mathbb{A}(t)\boldsymbol{X}$ and $\mathbf{y} = \mathbb{A}(t)\boldsymbol{Y}$. Then, remembering that $\mathbf{a}\cdot\mathbf{a} = \mathbf{a}^T\mathbf{a}$ for any vector $\mathbf{a} \in \mathbb{R}^3$, we compute

$$|\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = (\boldsymbol{X}-\boldsymbol{Y})\mathbb{A}^T\mathbb{A}(\boldsymbol{X}-\boldsymbol{Y}) = |\boldsymbol{X}-\boldsymbol{Y}|^2 \tag{C.1.9}$$

for any $\boldsymbol{X}$ and $\boldsymbol{Y}$ in the body, so $\mathbb{A}^T\mathbb{A} = \mathrm{Id}$. Such matrices are called *orthogonal*.

We remember that $\det(\mathbb{A}\mathbb{B}) = \det A \det B$, and $\det\mathbb{A}^T = \det\mathbb{A}$. so if matrices are orthogonal,

$$\det(\mathbb{A}^T\mathbb{A}) = (\det\mathbb{A})^2 = \det\mathrm{Id} = 1\,,$$

so there are two options, $\det\mathbb{A} = 1$ and $\det\mathbb{A} = -1$. If we choose $\det\mathbb{A} = 1$, the matrices form a group under multiplication, which has the name of $SO(3)$ - special orthogonal group, or the group of rotations. Let us choose a fixed bases $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ frozen in the body, and the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ that is stationary in the fixed frame. We define the α-th column of $\mathbb{A}$ as the coordinates of the vectors $\mathbb{E}_\alpha$ in the fixed spatial frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

C.1.2 Description of rotations: Euler's angles

Let us now describe rotations using different methods. We start first with the description using Euler's angles. In description of rotation matrices, it is useful to remember the following

Theorem C.1.1 (Euler's theorem on rotations) *Any matrix in $SO(3)$ can be obtained as a rotation about a fixed axis $\mathbf{n}$ by some angle α.*

In what follows, we shall denote rotation matrices by $R(\alpha, \mathbf{n})$. Usually, one uses the right-hand rule to determine the direction of rotations about the given axis. We can see that $\mathbb{R}(\varphi, \mathbf{n}) = -\mathbb{R}(-\varphi, -\mathbf{n})$, so the same matrix can be represented in two different ways.

In the literature, it is also common to denote the body axes $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ as (X, Y, Z), and denote the rotation about these axes by a given angle φ as $\mathbb{R}_X(\varphi), \mathbb{R}_Y(\varphi)$ and $\mathbb{R}_Z(\varphi)$. The matrices are written in coordinates as follows:

$$\begin{aligned}
\mathbb{R}_X(\varphi) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \\
\mathbb{R}_Y(\varphi) &= \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \\
\mathbb{R}_Z(\varphi) &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{C.1.10}$$

The Euler's angles (or generalized Euler's angles, or Tait-Bryan angles, or other names associated with these rotations) consist of combinations of 3 rotations about the body (sometimes called intrinsic) axes. The sequence of rotations is denoted by a sequences of three letters, for example, $Z - X - Z$.

There is some ambiguity in the literature as to which matrices represent the rotations, and what is the order of notation. We shall use the notation matrices being operations on vectors, so if there is a fixed vector $\boldsymbol{v}$, the first rotation changes it to $\mathbb{R}_1\boldsymbol{v}$, the second rotation as $\mathbb{R}_2\mathbb{R}_1\boldsymbol{v}$ and the third rotation as $\mathbb{R}_3\mathbb{R}_2\mathbb{R}_1\boldsymbol{v}$. We shall denote the rotations in the order they multiply matrices, *i.e.*, from right to left. For example, ZYX means first rotation about X by some angle α, second rotation about Y by another angle β and third rotation about Z by angle γ. The rotation matrix for ZYX is, in our notation, given by $\mathbb{R}_Z(\gamma)\mathbb{R}_Y(\beta)\mathbb{R}_X(\alpha)$. To find the corresponding rotation matrix, you would multiply three matrices as given by (C.1.10).

C.1.3 Angular velocity expressed through the rate of change of a matrix

During rotations, the orientation matrix of the rigid body $\mathbb{R}(t)$ changes with time. Let us consider the matrix $\dot{\mathbb{R}}$, and notice that the following two matrices are antisymmetric

$\widehat{\omega} = \dot{\mathbb{R}}\mathbb{R}^T$ and $\widehat{\Omega} = \mathbb{R}^T\dot{\mathbb{R}}$ (hats above letters denote antisymmetric matrices). This can be seen by differentiating the condition on the $SO(3)$ matrices $\mathbb{R}\mathbb{R}^T = \mathrm{Id}$:

$$\frac{d}{dt}\mathbb{R}\mathbb{R}^T = 0\,, \quad \dot{\mathbb{R}}\mathbb{R}^T + \mathbb{R}\dot{\mathbb{R}}^T = \dot{\mathbb{R}}\mathbb{R}^T + \left(\dot{\mathbb{R}}\mathbb{R}^T\right)^T = 0 \quad \Rightarrow \widehat{\omega} + (\widehat{\omega})^T = 0 \tag{C.1.11}$$

and similarly with $\widehat{\Omega}$ by differentiating $\mathbb{R}^T\mathbb{R} = \mathrm{Id}$.

Antisymmetric matrices and cross-product Suppose $\widehat{b}$ is a 3×3 antisymmetric matrix. Because of antisymmetry, the diagonal elements are 0, and the matrix can be written as

$$\widehat{b} = \begin{pmatrix} 0 & -b^3 & b^2 \\ b^3 & 0 & -b^1 \\ -b^2 & b^1 & 0 \end{pmatrix}\,, \quad \text{or} \quad \widehat{b}_{ij} = -\epsilon_{ijk}b_k\,. \tag{C.1.12}$$

Then, one can see that for any vector $\boldsymbol{v} = (v^1, v^2, v^3)^T$, equation (C.1.12) gives $\widehat{b}\boldsymbol{v} = \mathbf{b}\times\boldsymbol{v}$, with the vector $\mathbf{b}$ defined as $\mathbf{b} = (b^1, b^2, b^3)^T$. Thus, 3×3 antisymmetric matrices and 3-vectors are equivalent through (C.1.12). We will sometime write $\widehat{b} = \mathbf{b}\times$. The mapping from vectors $\mathbf{b} = (b^1, b^2, b^3)^T$ to antisymmetric matrices $\widehat{b}$ defined by (C.1.12) is called the *hat map*.

There is an inverse mapping from antisymmetric matrices to vectors. It is obtained by reading the off-diagonal elements in the matrix $\widehat{b}$ and inserting them in the vector elements of $\mathbf{b}$. That map is denoted by the inverted hat, and we write $b^\vee = \mathbf{b}$ for any antisymmetric matrix $\widehat{b}$.

Coming back to the rotations, we can define two useful quantities.

1. $\widehat{\omega} = \dot{\mathbb{R}}\mathbb{R}^T$ is the angular velocity of rotation expressed in the spatial frame. The corresponding vector $\boldsymbol{\omega} = \left(\dot{\mathbb{R}}\mathbb{R}^T\right)^\vee$ is the vector of angular velocities observed from the spatial frame.
2. $\widehat{\Omega} = \mathbb{R}^T\dot{\mathbb{R}}$ is the angular velocity of rotations expressed in the body frame. The corresponding vector $\boldsymbol{\Omega} = \left(\mathbb{R}^T\dot{\mathbb{R}}\right)^\vee$ is the vector of angular velocities observed from the body frame.

bf Notation. Here and below, we have used capital letters to denote quantities as seen in body frame, and script letters to determine the quantities seen in body frame.

Let us determine the rule of transformation of vectors between body and spatial frames. Suppose A is any antisymmetric matrix, and $\boldsymbol{A} = A^\vee$ is its vector equivalent under the inverse hat map. Let us take any vector $\boldsymbol{v}$ and write $A\boldsymbol{v} = \boldsymbol{A}\times\boldsymbol{v}$. If we apply any rotation matrix $\mathbb{R}$ to this statement, and remember that $\mathbb{R}(\mathbf{a}\times\mathbf{b}) = \mathbb{R}\mathbf{a}\times\mathbb{R}\mathbf{b}$, we get

$$\mathbb{R}A\boldsymbol{v} = \mathbb{R}\left(\boldsymbol{A}\times\boldsymbol{v}\right) = \mathbb{R}\boldsymbol{A}\times\mathbb{R}\boldsymbol{v}\,. \tag{C.1.13}$$

On the other hand, we can notice that $A' = \mathbb{R}A\mathbb{R}^T$ is an antisymmetric matrix, and $\mathbb{R}A\boldsymbol{v} = \mathbb{R}A\mathbb{R}^T\mathbb{R}\boldsymbol{v} = A'\times\mathbb{R}\boldsymbol{v}$. Remembering (C.1.13), we obtain the law of transformations of matrices and vectors under multiplication by $\mathbb{R}$

$$A \to \mathbb{R}A\mathbb{R}^T\,, \quad \boldsymbol{A} = A^\vee \to \mathbb{R}A^\vee\,. \tag{C.1.14}$$

Remember the expressions of the spatial $\widehat{\omega} = \dot{\mathbb{R}}\mathbb{R}^T$ and body $\widehat{\Omega} = \mathbb{R}^T\dot{\mathbb{R}}$ angular velocities. Thus, we have the following correspondence between these matrices and vectors

$$\widehat{\omega} = \mathbb{R}\widehat{\Omega}\mathbb{R}^T \quad \text{by definition}, \quad \boldsymbol{\omega} = \mathbb{R}\boldsymbol{\Omega} \quad \text{by (C.1.13)}\ . \tag{C.1.15}$$

Similarly, any vector vector can be seen in both spatial and body frames. If the vector in spatial frame is $\boldsymbol{v}$, the corresponding vector seen in body frame will be $\boldsymbol{V} = \mathbb{R}^T\boldsymbol{v}$, and, vice versa, $\boldsymbol{v} = \mathbb{R}\boldsymbol{V}$.

Remark C.1.2 (On a common misconception) One often hears that the angular velocity as seen in body frame must be zero, since the body is not rotating with respect to itself. That is incorrect. The vector of angular velocity in spatial frame $\boldsymbol{\omega}$ is clearly is non-zero, and the same vector, seen from the body frame, cannot be zero since it is obtained by the application of a rotation matrix.

C.1.4 Euler's equations of motion for a rigid body

Suppose we have a rigid body consisting of N material particles with masses m_i and positions, in body frame, $\boldsymbol{X}_i$. The corresponding positions in the spatial frame are $\mathbf{x}_i = \mathbb{R}\boldsymbol{X}_i$, with $\boldsymbol{X}_i$ fixed. The velocity of these particles in spatial frame is then

$$\boldsymbol{v}_i = \dot{\mathbf{x}}_i = \dot{\mathbb{R}}\boldsymbol{X}_i = \mathbb{R}\left(\mathbb{R}^T\dot{\mathbb{R}}\boldsymbol{X}_i\right) = \mathbb{R}\left(\boldsymbol{\Omega}\times\boldsymbol{X}_i\right) \tag{C.1.16}$$

The total angular momentum of these particles in spatial frame is computed as

$$\mathbf{l} = \sum_i m_i\mathbf{x}_i\times\boldsymbol{v}_i = \sum_i m_i\mathbb{R}\boldsymbol{X}_i\times\mathbb{R}\left(\boldsymbol{\Omega}\times\boldsymbol{X}_i\right) = \sum_i m_i\boldsymbol{X}_i\times\left(\boldsymbol{\Omega}\times\boldsymbol{X}_i\right) \tag{C.1.17}$$

Note that the angular momentum in the spatial frame depends on the orientation. However, the angular momentum in the body frame is independent of orientation, and is a linear operator in the *body* angular velocity. That operator is represented by a 3×3 matrix called the moment of inertia $\mathbb{I}$, and is only constant when computed in the body frame. This matrix is symmetric, $\mathbb{I}^T = \mathbb{I}$, and positive definite.

Let us also notice that the matrix $\mathbb{I}$ is real, symmetric, and positive definite, so the eigenvalues of this matrix (I_1, I_2, I_3) are real and positive. We can choose the axes $\mathbf{E}_i$ to coincide with the eigenvector directions of $\mathbb{I}$. Then, the matrix $\mathbb{I}$ is diagonal in that basis:

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{C.1.18}$$

In almost all work on the subject, the matrix $\mathbb{I}$ is assumed in the diagonal form (C.1.18), since this form can be achieved without the loss of generality. We shall assume that diagonal form of $\mathbb{I}$ as well.

Therefore, we write the expressions for the angular momenta in the spatial $\mathbf{l}$ and in the body $\mathbf{L}$ frames

$$\mathbf{l} = \mathbb{R}\mathbb{I}\boldsymbol{\Omega}\,, \quad \mathbf{L} = \mathbb{I}\boldsymbol{\Omega}\,. \tag{C.1.19}$$

Given the torque in the spatial frame $\mathbf{t}$, the conservation of angular momentum in the spatial frame is written as

$$\frac{d\mathbf{l}}{dt} = \mathbf{t}\,. \tag{C.1.20}$$

Equation (C.1.20) is rather awkward to use since it involves, explicitly, the rotation matrices $\mathbb{R}$. We want to rewrite it in the body frame, so we multiply (C.1.20) by $\mathbb{R}^T$ and use (C.1.19):

$$\mathbb{R}^T\frac{d}{dt}\left(\mathbb{R}\mathbb{I}\boldsymbol{\Omega}\right) = \mathbb{I}\dot{\boldsymbol{\Omega}} + \mathbb{R}^T\dot{\mathbb{R}}\mathbb{I}\boldsymbol{\Omega} = \mathbb{I}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}\times\mathbb{I}\boldsymbol{\Omega} = \mathbb{R}^T\mathbf{t} = \mathbf{T}\,. \tag{C.1.21}$$

This equation, deserves to be written explicitly, as it forms the foundation of all work on the theory of rotation of rigid bodies and heavy tops:

$$\mathbb{I}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbb{I}\boldsymbol{\Omega} = \mathbf{T}\,. \tag{C.1.22}$$

Note that the equation for the rigid body is written in the *body frame*. Unless you have a very good reason, there is no point in writing these equations in the spatial frame: the moment of inertia of a rigid body in the spatial frame is not constant, but depends explicitly on the orientation matrix, more precisely, $\mathbb{I}_{sp} = \mathbb{R}^T\mathbb{I}\mathbb{R}$. It makes the computations very awkward. The only potential exception is the case when the matrix $\mathbb{I}$ is proportional to the identity matrix, and the body is dynamically spherically symmetric. In that case, $\mathbb{I}_{sp} = \mathbb{I}$ =const. However, this is a very particular case, rarely encountered in practice.

C.1.5 Euler's equations for the rigid body motion

One of the particular cases of the rigid body motion is given when the motion is free, so there is no external torque on the body, *i.e.*, $\mathbf{T} = \mathbf{0}$ in (C.1.22). In that particular case, the equations of motion reduce to the Euler's equations of motion:

$$\mathbb{I}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbb{I}\boldsymbol{\Omega} = \mathbf{0}\,. \tag{C.1.23}$$

These are called Euler's equations for a rigid body.

Constants of motion There are two constants of motion for equations (C.1.23). One is energy $\frac{1}{2}\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega}$:

$$\frac{d}{dt}\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} = \boldsymbol{\Omega}\cdot\mathbb{I}\dot{\boldsymbol{\Omega}} + \dot{\boldsymbol{\Omega}}\cdot\mathbb{I}\boldsymbol{\Omega} = \boldsymbol{\Omega}\cdot\mathbb{I}\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}\cdot\mathbb{I}^T\boldsymbol{\Omega} = -2\boldsymbol{\Omega}\cdot(\boldsymbol{\Omega}\times\mathbb{I}\boldsymbol{\Omega}) = 0\,. \tag{C.1.24}$$

The second constant of motion is the absolute value of the angular momentum $|\mathbb{I}\boldsymbol{\Omega}|^2$:

$$\frac{d}{dt}|\mathbb{I}\boldsymbol{\Omega}|^2 = 2\mathbb{I}\boldsymbol{\Omega}\cdot\mathbb{I}\dot{\boldsymbol{\Omega}} = -2\mathbb{I}\boldsymbol{\Omega}\cdot(\boldsymbol{\Omega}\times\mathbb{I}\boldsymbol{\Omega}) = 0\,. \tag{C.1.25}$$

Physically, this constant of motion can be understood as the consequence of the conservation of angular momentum in spatial frame, $\mathbf{l} = \mathbb{R}\mathbb{I}\boldsymbol{\Omega}$. Taking $|\mathbf{l}|^2$ yields exactly $|\mathbb{I}\boldsymbol{\Omega}|^2$ and hence the result (C.1.25).

These two constants are ellipsoids in the three-dimensional $\boldsymbol{\Omega}$ space. The intersection of these ellipsoids selects a closed curve in that space. Thus, all solutions of (C.1.23) are periodic, and the motion is integrable. It is also known that the solutions can be expressed in terms of elliptic integrals.

For future reference, it is also useful to write (C.1.23) explicitly in terms of the components of $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$:

$$\begin{cases} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 = 0 \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 = 0 \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = 0 \end{cases} \tag{C.1.26}$$