# Event Triggered Control of Nonlinear Systems

by

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# Abstract

Event-triggered control systems have emerged as an important alternative to classical digital control systems, in which the flow of information between sensors, controller and actuators takes place aperiodically in an event-based manner. Event-triggered control (ETC) has seen much attention from the research community in recent years resulting in a comprehensive theory which includes stability analysis, disturbance rejection, control design, etc. This thesis is concerned with important theoretical and practical aspects of event-triggered systems that can be divided into two main categories.

The first part includes the robust analysis of ETC systems involving different types of robustness measures. We start with designing a triggering condition (TC) for general nonlinear event-triggered systems in a way that an $\mathcal{L}_2$-type performance is guaranteed. The results are obtained in a local framework due to reliance on the assumption that the admissible disturbance is norm bounded by some function of the states. The results are then extended in two aspects. First, we study the $\mathcal{L}_p$-stability of nonlinear event-triggered systems and second, we relax the restriction on the class of disturbances. In addition, the TC is proposed using a unifying framework which includes several dynamic and static parameters to cover several existing TCs proposed earlier in the literature as special cases. More importantly, the approach solves the non-trivial problem of isolating the triggering instants in presence of arbitrary disturbances. As another extension, the more interesting scenario of jointly designing the TC and control law is studied for nonlinear Lipschitz systems. Our solution to this problem includes both state and output-based feedback laws and consists of assigning the dominant eigenvalues of the stability matrices according to desired control demands. We also consider the robust analysis of nonlinear input-affine systems and study the input-to-state stability of ETC systems with respect to actuator noise/error and exogenous disturbances. We consider both the design of the TC for a pre-design controller as well as the more challenging simultaneous design of controller and TCs. Finally, we consider the concept of dissipativity as general framework in the study of various forms of robust performance and system properties (including passivity, ISS, and

$\mathcal{L}_2$ gain performance), and study different forms of dissipativity for event-based network-communicated physical processes. The second category of results in this thesis focuses on the important problem of reducing the average sampling frequency for ETC systems. We study this problem from two points of view. First, we modify a pre-designed TC to effectively enlarge the intersampling intervals without violating the desired robust performance of the event-triggered system. Also, we obtain a lower bound on the amount of inter-event times extension. Moreover, for an ETC design to be successfully implemented in practice, the uniform isolation of triggering instants has to be guaranteed. This is even more challenging when disturbances are applied to the system. Our proposed triggering structure not only provides a general platform for the event design but also serves to the isolation of sampling instants in presence of arbitrary disturbances.

# Preface

Chapter 3 has been published in the article: M. Ghodrat and H. J. Marquez, "On the Local Input-Output Stability of Event-Triggered Control Systems", *IEEE transactions on Automatic Control*, vol. 64, no. 1, pp. 174-189, 2019. I was responsible for the main idea, analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez contributed in main the idea and also had the supervision role throughout the work. He was also involved with the paper composition and drafting.

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An early result of chapter 8 for event-triggered linear time-invariant (LTI) sytems has been published in the article: S. H. Mousavi and M. Ghodrat and H. J. Marquez, "A Novel Integral-Based Event Triggering Control for Linear Time-Invariant Systems", *Conference on Decision and Control (CDC)*, pp. 1239-1243, 2014. I was responsible for the main the idea and also assisted in the mathematical derivation. Seyed Hossein Mousavi contributed in the analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez had the supervision role throughout the work and was also involved with the paper composition and drafting.

The result of chapter 8 for general nonlinear event-triggered systems has been published in the article: S. H. Mousavi and M. Ghodrat and H. J. Marquez, "Integral-Based Event Triggered Control Scheme for a General Class of Non-Linear Systems", *IET Control Theory & Applications*, vol. 9, no. 13, pp. 1982-1988, 2015. I was responsible for the main the idea and also assisted in the mathematical derivation. Seyed Hossein Mousavi contributed in the analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez had the supervision role throughout the work and was also involved with the paper composition and drafting.

The result of chapter 8 for event-triggered analysis of distributed network systems has been published in the article: M. Ghodrat and H. J. Marquez, "An Integral Based Event Triggered Control Scheme of Distributed Network Systems", *European Control Conference (ECC)*, pp. 1724-1729, 2015. I was responsible for the main idea, analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez contributed in main the idea and also had the supervision role throughout the work. He was also involved with the paper composition and drafting.

*To my beloved parents, brothers and my wife*
*for their endless love and support*

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# Notation

| | |
|---|---|
| $\mathbb{R}$, $\mathbb{C}$ | The sets of real, integer and complex numbers |
| $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{R}_0^+$, $\mathbb{N}_0$ | The sets of positive and nonnegative real and integer numbers |
| $\mathbb{R}^n$ | The set of real $n$-dimensional vector |
| $\mathbb{R}^{n\times m}$ | The set of real $n \times m$ matrices |
| $\mathcal{Z}_0$ | Space of functions that are zero almost everywhere on $\mathbb{R}^+$ |
| $\mathcal{L}_p$ or $\mathcal{L}_p^n$ | Space of n-dimensional functions with well-defined $p$-norm |
| $\mathbf{C}^0$, $\mathbf{C}^1$ | Class of continuous, continuously differentiable functions |
| $\forall$ | Universal quantifier |
| $\lfloor\cdot\rfloor$ | Floor of a real number |
| $x \in X$ | $x$ is an element of set $X$ |
| $X \subset Y$ | $X$ is a subset of $Y$ |
| $A^\mathsf{T}$ | Transpose of matrix or vector $A$ |
| $A^\mathsf{H}$ | Conjugate transpose of matrix $A$ |
| $A^{-1}$ | Inverse of matrix $A$ |
| $A^\dagger$ | Moore–Penrose pseudo inverse of $A$ |
| $\lambda_i(A)$ | Eigenvalues of matrix $A$ with $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ when all are real |
| $\mathcal{S}_j(A)$ | Singular values of matrix $A$ with $\mathcal{S}_1(A) \geq \ldots \geq \mathcal{S}_n(A)$ |
| $\mathrm{Re}A$ | Hermitian part of matrix $A$ defined as $\mathrm{Re}A = \frac{1}{2}(A + A^\mathsf{H})$ |
| $A \preceq B$, $A \prec B$ | $B - A$ is positive semidefinite, positive definite |
| $I$ or $\mathbb{I}_n$ | Identity matrix of dimension n |
| $\lVert\cdot\rVert$ or $\lvert\cdot\rvert$ | Euclidean norm of a vector or matrix |
| $\lVert\cdot\rVert_\infty$ or $\lvert\cdot\rvert_\infty$ | $\infty$-norm of a vector |
| $\lVert\cdot\rVert_M$ | Weighted norm of a vector $x \in \mathbb{R}^n$ defined as $\lVert x\rVert_M = (x^\mathsf{T} M x)^{\frac{1}{2}}$ |
| $\langle\cdot,\cdot\rangle$ | Inner product of two vectors $x, y \in \mathbb{R}^n$ defined as $\langle x, y\rangle = x^\mathsf{T} y$ |
| $\lVert z\rVert_2$ or $\lVert z\rVert_{\mathcal{L}_2}$ | $\mathcal{L}_2$ norm of signal $z : \mathbb{R}_0^+ \to \mathbb{R}^n$, defined as $\lVert z\rVert_2 = (\int_0^\infty \lvert z(t)\rvert^2 dt)^{\frac{1}{2}}$ |
| $\lVert z\rvert_{[0,T]}\rVert$ | Truncated norm of signal $z : \mathbb{R}^+ \to \mathbb{R}^n$, defined as $\lVert z\rvert_{[0,T]}\rVert = (\int_0^T \lvert z(t)\rvert^2 dt)^{\frac{1}{2}}$ |
| $\lvert z\rvert_\infty$ or $\lVert z\rVert_\infty$ | $\infty$-norm of signal $z : \mathbb{R}_0^+ \to \mathbb{R}^n$, defined as $\lVert z\rVert_\infty = \sup\{\lVert z(t)\rVert : t \geq 0\}$ |
| $\chi_\mathcal{A}(\cdot)$ | Characteristic function of subset $\mathcal{A} \in \mathbb{R}$ defined as $\chi_\mathcal{A}(s) = 1$ for $s \in \mathcal{A}$ and $\chi_\mathcal{A}(s) = 0$ otherwise |

# Abbreviations

| | |
|---|---|
| LHS | Left Hand Side |
| RHS | Right Hand Side |
| LTI | Linear Time-Invariant |
| TC | Triggering Condition |
| ISS | Input-to-State Stability |
| ZOH | Zero-Order Hold |
| LMI | Linear Matrix Inequality |
| NLMI | Nonlinear Matrix Inequality |
| ETC | Event-Triggered Control |
| ETM | Event-Triggered Mechanism |
| MIET | Minimum Inter-Event Time |
| GES | Globally Exponentially Stable |

# Chapter 1

# Introduction

This thesis explores robust design and analysis of event-triggered control (ETC) systems subject to the exogenous disturbances. The purpose of this research is to offer solutions to several open problems using novel techniques for event-based controller design. In this chapter, we provide an overview of the subject along with some preliminary background, overview of the literature, define the research objectives, and summarize the main contributions.

## 1.1 Event-Triggered Control Systems

Modern feedback control systems are typically implemented digitally using a computer to realize the controller. Sample and hold devices provide the interface with the (analog) plant. In the classical approach data transmission between system components (such as actuator, sensor and plant) takes place periodically, regardless of whether or not changes in the measured output and/or commands require computation of a new control output. This approach, often referred to as *time-triggered control* is well understood and has led to several theories for control design of linear and nonlinear systems. See for example, [1–5]. The periodic exchange of information, however, imposes unnecessary communication demands that might become important and even critical in some systems such as distributed and networked systems, where optimal usage of communication network capacity is of great importance.

An alternative to time-driven systems is the so-called *event-triggered* approach, in which a new control action takes place only when changes in the measured outputs overpass a pre-established threshold. Event-based control systems has been an active area of research over the last decade. The primary characteristic of event-based controllers is that they can provide performance very similar to classical control approaches while reducing the transmission of information between plant and controller. The importance of this property is evidenced through several applications such as battery-operated systems with wireless transmission between plant and controller, which often have limited energy and/or memory supplies, or network control systems with shared wired or wireless communication channels, [6].

In an event-based scenario, the system decides when to update the control output, based on a so called *real time triggering condition* on the measured signals. This approach leads to aperiodic communication between plant and controller that only takes place when needed. In other words, the system components do not exchange information unless some TC is satisfied. This condition can be defined in different forms and varies depending on the nature of the system.

Mostly, the event-triggered mechanism (ETM) is designed to update the actuators whenever measurement error *i.e.,* the difference between current and most recent value of output, is above a pre-established threshold. The threshold can be a constant or a function of system's output or even a combination of them. Therefore, an event detector hardware is required to continuously monitor the system's output, compare it with the measurement error, and finally release the information if it is needed. As a consequence, the actuator receives an updated control signal at the triggering instants at which the TC is satisfied. A zero-order hold (ZOH) device serves to maintain the controller signal constant between events.

Two important aspects of an ETC are (i) the design should satisfy some form of closed-loop performance, and (ii) should guarantee that the execution times have enough separation to avoid excessive sampling. This second point is critical to any event design. Note that reducing communication between plant and controller is, in fact, the primary motivation behind event-based methods. However, since the execution time depends on the occurrence of a new event, the TC has to be designed in a way to avoid excessive triggering, particularly the existence of an accumulation point in which an infinite number of events are generated in finite-time (also known as Zeno phenomenon). In other words, the event-based controller has to be carefully planned to meet the hardware limitations associated with employed sampling devices since the sampling devices cannot sample the measurements unlimitedly fast. Therefore, a necessary practical requirement for designing and admissible TC is to prevent the triggering instants to be arbitrarily close to each other.

## 1.2 Literature review

Event-based systems have been used without theoretical supports for many years. The resurgence of interest in the subject began with the work reported in reference [7], that considers a first order stochastic system and shows that event-based sampling offers better performance than classical time-triggered control, in terms of closed-loop variance and sampling rate. Following publication of this work, event-triggered systems became a very active area of research and many important contributions have been reported addressing *stability* ( [8–13]), and *performance* ( [14–20]), to mention a few (see also the references therein).

Reference [8], one of the first references on stabilization of ETC systems, proposes an ETM for PID control. Reference [9], presents a clever and rather general solution to the stability problem of event-triggered systems. In this reference the author assumes the existence of a pre-designed continuous-time control law that results in input-to-state stability of a nonlinear plant, and shows that restricting the measurement error (*i.e.*, the difference between the system state and the last sampled value) to stay within a threshold which is a function of state, guarantees closed-loop global asymptotic stability.

Reference [9] has inspired much work and several event-based strategies have been proposed that extend this work (see [21] and the references therein). Reference [9] is restricted to state-feedback and therefore relies on full state measurement. This restriction is relaxed in [10, 11]. Reference [10], considers *periodic* ETC of linear systems, in which the TC is monitored at regular intervals instead of continuously, and can be viewed as a sampled data version of event-triggered systems. Reference [11] considers output feedback stabilization using the framework of passivity theory. References [12, 13] offer a unifying framework for the stability problem of nonlinear event-based in the context of hybrid systems.

All of the above mentioned works focus on stabilization. The effects of an ETM on control performance was first addressed in [15], which shows a trade-off between system performance and the complexity of the control law. A decentralized ETM is proposed in [22] for distributed linear systems. This reference considers an impulsive system approach to system stability and proposes an ETM that satisfies an $\mathcal{L}_\infty$ bound. References [16–18, 23–27] focus on the $\mathcal{L}_2$-gain. The $\mathcal{L}_2$-gain stability analysis of ETC systems was first investigated in [23] where a full-information $\mathrm{H}_\infty$ controller is proposed for LTI systems. [17, 24] continued the work of [23] in more details. [24] proposes an $\mathcal{L}_2$-gain performance-preserving TC for a class of nonlinear affine systems. [17] considers LTI systems and derives an explicit lower bound on the sampling periods. In this reference, the disturbance is assumed to be norm bounded by a linear function of the state norm. This condition is then relaxed in [18]. Reference [25] considers the $\mathcal{L}_2$-gain of distributed multi agent systems under event-triggered agreement protocols. Reference [16] proposes an ETM for distributed network linear systems and guarantees finite gain $\mathcal{L}_2$-stability in the presence of packet data dropouts. Reference [26] considers passive systems and proposes a TC that guarantees finite gain $\mathcal{L}_2$-stability when the external disturbance is bounded and shows that their approach preserves stability under constant network induced delays or delays with bounded jitters. Reference [27] extends the results of [26] to systems with constant network induced delays or time-varying delays with bounded jitters. [19] proposes a dynamic TC for the centralized state feedback ETC of nonlinear network control systems with guaranteed $\mathcal{L}_p$-stability. Reference [20] extends the work of [19] to the output feedback and decentralized case.

The TC should be designed properly to guarantee Zeno-free bahaviour for the ETC system. In this regard, most event-triggered laws define a threshold using the norm of a measured signal, typically, the state. Examples include [9, 13, 16, 17, 28]. Although this type of scheme has seen countless of successful applications and has provided an important place in the literature, it is, however, not free of limitations. Indeed, in [9], the author designed a TC departing from a continuous-time closed-loop ISS system (with respect to measurement error) to achieve an closed-loop stable ETC system, with Zeno-free behaviour. Similar rules can also guarantee other desirable performance measures such as $\mathcal{L}_2$ input-output bounds (*e.g.,* see [16, 17, 23]). However, it was recently shown in reference [19] that in the presence of disturbance or sensor noise, static TCs defined in terms of solely the state vector norm cannot guarantee positive minimum inter-event time (MIET), thus becoming vulnerable to Zeno behaviour. The same issue may be encountered when dealing with dynamic TCs (*e.g.,* [13, 28]), or output-based TCs (*e.g.,* [26, 27]).

As mentioned above, dealing with uncertainties in the event-triggered context is nontrivial. Indeed the ETM is designed to update the actuators whenever measurement errors are above a pre-established threshold. In the absence of disturbances, the error originates during the intersample as the difference between the present value of the state and its last sampled value. In the presence of exogenous disturbance, however, the error is also driven by the disturbance term making it difficult to design an effective TC, thus incorporating disturbance in event design is nontrivial.

The main problem is then to design the triggering mehanism in a way that the desired stability and/or performance for the event-triggered system is acheived. In the absence of disturbances, the sampling (measurement) error originates during the intersample as the difference between the present value of the state and its last sampled value. In the presence of exogenous disturbance, however, the error is also driven by the disturbance term making it difficult to design an effective TC. Thus, when designing an ETC law, two aspects need to be considered: first, the resulting feedback control must satisfy some form of performance criterion. Second, this performance must be satisfied when the control law is implemented in event-triggered form. This second aspect is nontrivial because ETC systems are inherently non-periodic, thus preventing the designer from using discrete-time models and forcing the use of emulation techniques to recover continuous-time performance when the controller is implemented in an event-triggered fashion. Indeed, the vast majority of the published work to-date follows an *emulation approach*, consisting of first designing a control law in continuous-time, ignoring implementation details, and often neglecting possible network constraints. A TC is then designed to meet as closely as possible the performance of the continuous-time design, possibly taking into account the effect of the communication network. This approach has been predominant in the research community and includes the majority of the works published up to date. Often control design is expressed in terms of linear matrix inequalities (LMIs), for which *feasibility* is a non-trivial issue. We refer the interested readers to [29–33] for full state feedback design and [22, 29, 30, 34–41] for output feedback design. Note that since the control law is originally designed for the network-free problem, the desired performance will not necessarily be optimal in presence of a network, [30]. A more recent approach consists of *jointly* designing the controller and TC and has recently seen attention, *e.g.,* see [29–31,33,39,42]. As demonstrated in these works, joint design can overcome possible deficiencies of the emulation approach by enhancing optimal performance.

Regardless of the particular method used in the design (*i.e.* emulation design or joint design), designing the TC also require careful attention. Indeed, a critical aspect of an ETC system is that since execution times depend on the occurrence of an event, the TC must be constructed in such a way to avoid events becoming excessively close. In this regard, disturbance rejection becomes a challenge, since the effect of disturbances may lead to execution times becoming arbitrarily close resulting in an accumulation point (maybe the only exception is the periodic event-triggered scheme, [10], in which the separation of triggering instants holds trivially). Therefore, constructing such TC in the presence of exogenous disturbances is non-trivial and has been the subject of much research. This problem is relatively well understood for linear ETC systems, for which several solutions have been proposed. See [17,18,22,23,29,30,43–46]. References [11,19,20,26,27,47–53] study the more general nonlinear case. A major trend in these works is to ensure the separation of execution times by enforcing a dwell-time between them, known as the *time-regularized* approach. In this method, the TC is only checked after a positive dwell-time since the last execution time. In this sense, the dwell-time can be seen as being inspired by classical periodic sampling (see [54]). See also [55], [56], [34] for a different approach. As pointed out in [19, 20], while offering guaranteed positive inter-event times, the time-regularized controller may reduce to a time-triggered (periodic) control in certain situations. This issue, however, has been avoided in the recent papers [20, 48] where a dynamic triggering scheme is incorporated with the time-regularization technique.

## 1.3 Research Motivation and Objectives

In this section we briefly discuss the motivation leading to the work presented and summarize the main contribution in this thesis. In the previous section, we discussed several aspects of ETC systems that have been well-studied in recent years. Despite major advances in the field, however, there are some fundamental open issues that requires careful attention.

### 1.3.1 Robust Analysis

An important problem in the realm of ETC systems is the solution to finite $\mathcal{L}_2$-gain stability for a wider class of systems and/or a less conservative set of assumptions. Indeed, in Chapter 3 we consider a rather general class of nonlinear system model with the sole assumption of satisfying a mild local Lipschitz continuity condition. Taking exogenous disturbances together with measurement errors as inputs, our proposed TC is obtained based on the assumption that the system is ISS. The ISS assumption implies working with bounded inputs and therefore suggests the need to consider *small signals* in some sense. To formalize this concept, we present our results using an extension of the classical input-output theory of systems with modified input spaces, referred to as *local* (or *small signal*) input-output stability introduced in [57]. It is then assumed that the disturbance term is originated from structural uncertainties in the system model and is norm bounded by some locally Lipschitz-continuous function of state. This assumption is rather mild and more general than previous references. For example, in the framework of self-triggered control, [17] considers a similar $\mathcal{L}_2$ problem to the one studied here, but assumes that the norm of disturbance is bounded by a linear function of the state norm. The results are then extended in Chapter 6, where the interest is to answer the question of whether or not the linearization of a nonlinear plant model can be rendered locally stable when the controller is implemented using an event-triggered approach. Note that this is different from the classical notion of local stability achieved via linearization in continuous-time, which simply ignores the event-triggered implementation. One can conjecture that the same principle holds, *i.e.* if the linearized model is stabilized via feedback, then the true nonlinear system is locally stable, even when the controller is implemented in event-triggered form. The results in Chapter 6, bring clarity to this conjecture using the fact that a wide range of nonlinearities satisfy a Lipschitz condition, at least, locally. Moreover, the results of Chapter 3 is improved in Chapter 6 from two main aspects. First, the restriction on the admissible input space is removed, *i.e.,* the disturbance here is not restricted to be norm bounded by some function of state's norm. Second, while in Chapter 3 the control and triggering laws are designed based on nonlinear model specifications, in Chapter 6, we explore the local $\mathcal{L}_2$ problem based on a linearized design. This problem is generally of more interest due to the existence of more developed tools/theoretical supports for linear ETC systems.

As an another generalization to the results of Chapter 3, where the local $\mathcal{L}_2$ stability of nonlinear ETC systems under state-dependent disturbances is studied, in Chapter 4 we generalize the problem to an $\mathcal{L}_p$ type performance and relax the restrictions on the set of admissible disturbances, stating the results in a global (non-local) framework. We consider a general class of nonlinear control-affine systems and a pre-designed state feedback controller whose continuous implementation satisfy some $\mathcal{L}_p$-gain performance level $\mu$. We provide a constructive TC design algorithm to achieve a new $\mathcal{L}_p$-performance level for some $\mu_d$. References [20, 48] follows a different approach. In comparison to these references, the results of Chapter 4 rely on a less conservative set of assumptions and a different approach that lead to a different structure for the TC design. In fact, assumptions made in [20, 48] require some sort of dissipativity property for ETC system, which we believe, is too strong when applied to the problem considered in Chapter 4. Moreover, the results in this chapter can be treated as a general framework for the construction of a dynamic TC, where several design parameters can be selected for specific purpose. The resulting design covers several well-known forms (namely, [9, 13, 19, 20, 28, 49, 58, 59]) as special cases.

While the majority of the literature on ETC design has focussed on the emulation method, consistent of finding a TC that closely resembles performance of an analog design, more recently some research has address the perhaps more important problem of jointly designing both controller and TC. Current solutions proposed controller design based mostly on the feasibility of LMIs. To improve these results, in Chapter 6 we propose a systematic mechanism to jointly design static and/or dynamic controller gains and the TC parameters to meet an $H_\infty$ performance. The approach is different from other joint design techniques in that our proposed method involves assigning dominant eigenvalues of the linear stability matrices, based on the desired performance and triggering specifications. This approach not only solves the intended $H_\infty$ performance objective, but also provides valuable insight into the design limitations. We consider the output feedback case via state feedback plus an observer, and discuss both the full and partial state feedback case.

Following the seminal work [9], most of the mentioned works on nonlinear ETC, design the TC assuming the system to be ISS with respect to measurement error and external disturbances. It is well-known for linear systems that a globally asymptotically stabilizing controller for continuous-time unperturbed (zero-disturbance) system renders the resulting perturbed ETC system ISS. More recently, [60] shows that instead of a primitive assumption, the ISS property with respect to disturbance can be taken as the consequence of applying the integral-based type of TC, [58], to linear output-based event-triggered systems. The generalization of the above results to nonlinear event-triggered systems, *i.e.,* building the stability analysis on the different assumption rather than the ISS condition or finding sufficient conditions for nonlinear event-triggered systems to have ISS property, has not seen much attention. This motivates the results of Chapter 5 where we provide sufficient conditions for input-to-state stability of the input-affine nonlinear sampled data systems with respect to actuator error and exogenous disturbance, based on the convex feasibility of nonlinear matrix inequalities (NLMIs). In addition, we propose a solution, independent of ISS assumption, to the $\mathcal{L}_2$-stabilizing ETC design. As a consequence, by utilizing the affine structure of the state space approach, the results obtained are built on different assumptions when compared to the related works [20, 48, 49]. We follow both joint design and emulation design approaches and express the robustness results in terms of convex feasibility of NLMIs. We show in Chapter 5 that while in the absence of a communication network, the theory of differential games, [61], can be effectively utilized to solve the desired $\mathcal{L}_2$ stability problem by finding the so-called *best strategies* for control and disturbance signals, this tool is not helpful when an event-based communication network is introduced.

A very powerful tool in the study of system performance, is the so-called *dissipativity*, first introduced by Willems [62]. Generally speaking, a system is called dissipative with respect to a specific supply rate, if the energy stored by the system at any time $t$ does not exceed the energy externally supplied with the given supply rate. In other words, the system is dissipative if there is no internal production of energy. Dissipativity provides a general framework in the study of various forms of performance and system properties, including passivity, input-to-state stability, and $\mathcal{L}_2$ gain. Extensions of the notion of dissipativity include the concepts of *quasi dissipativity* and *weak quasi dissipativity*, introduced in [63,64]. These notions extend the original definition to include the possible existence of internal generation of energy with finite power. Chapter 7 deals with the preservation of some forms of dissipativity when a continuous-time plant is connected to a discrete-time controller via a communication network. It is well established that, under continuous communication, the feedback interconnection of a dissipative plant and controller results in a dissipative closed-loop system [5]. When the communication between plant and controller is done via periodic sampling, however, closed-loop dissipativity does not hold. It was shown in [65], however that in this case the interconnection results in a quasi-dissipative closed-loop. The analysis of dissipative systems under event-based communication is in fact, well understood (see for example [11, 26, 27, 51, 52]). One issue, however, critical to any event-triggered system, is ensuring that there is a time-separation between triggering events. The above mentioned references prove the non-existence of Zeno behaviour by restricting disturbances to Sobolev spaces, *i.e.* continuously differentiable signals with bounded derivative, and showing that an accumulation point can only occur as time tends to infinity. Although this approach indeed prevents the existence of Zeno behaviour in finite-time, it does not however constraint the number of triggering events over finite intervals and therefore may fail to serve the main purpose of a triggering system; namely, limit the transmission of information between subsystems. This is mainly because the obtained lower bound on intersampling times is not constant and may shrink to zero over time. In Chapter 7, we study dissipativity properties of event-triggered feedback control systems by generalizing the results given in [65] and show that under event-triggered sampling, different variations of dissipativity property may hold. In particular, it is proved that the dissipativity property in its classical sense or in its more general forms of quasi or weak dissipativity, in the spirit of [63], may hold depending on how the triggering parameters are selected. Our solution excludes the existence of Zeno behaviour using an approach different from that in [51, 52] and addresses the above mentioned shortcomings associated with these references. Our approach employs a *fully event-based* scenario in which the TC is monitored continuously without the use of dwell-time, as opposed to time-regularization, thus avoiding the use of dwell-time restrictions, [66]. Additionally, we propose a general framework to design the parameters of the TC and show how the proper selection of these parameters leads to different forms of dissipativity. We emphasize that our analysis and results do not require input signals to be differentiable, thus relaxing to conditions used in references [51, 52].

### 1.3.2 Sampling Frequency Improvement

The main objective of ETC is to reduce the communication traffic among the components of a control system. However, the majority of the obtained results to date rely on the Lyapunov theory and might provide conservative results in terms of transmission rates. Therefore, reducing the associated conservatisms by modifying the existing TCs has been at the centre of much research in recent years.

To reduce the conservatism associated with the application of the Lyapunov theory in the analysis of ETC systems, in [67], a new TC based on a logic function of the Lyapunov function is introduced. The result weakens the assumptions in [9] by considering an ISS approach and assuming a lower bound on the derivative of the Lyapunov function. In Chapter 8, we propose a different approach and re-examine the paper [9]. Assuming that an analog controller has been designed and satisfies an ISS condition, we propose an alternative, less conservative, approach for the construction of the TC. The main idea consists of using an integral-based ETM that allows the Lyapunov function to be non-decreasing between triggering instants (thus allowing the time derivative of the Lyapunov function to have instantaneous positive values between triggering instants). Unlike the work [67], our proposed integral-based TC is a function of the state values and measurement error and relies only on the ISS assumption for the original analog system to prove the results. The existence of a lower bound for the inter-event times is also proved and an explicit value for this bound is provided for a specific class of nonlinear systems. The proposed method is shown to be more efficient than the existing results in terms of communication exchanged between plant and controller.

Limiting the number of triggerings over a time interval (when necessary) may be of higher importance than decreasing their number over an infinite time span. For example, high data load on a communication channel over a finite time interval may result in undesirable effects such as data packet drop out and/or transmission delay. Therefore, instead of improving all inter-event times, we focus our study on controlling the number of samples over a time interval in which the triggering frequency may become critical. This is addressed in Chapter 3 where our proposed strategy improves existing results, see, *e.g.*, [13, 19, 20, 28, 58, 67], in that the increase in inter sampling-times can be designed a-priori, at least for a desired period of time (or a desired number of triggering iterations). By contrast, in [13, 19, 20, 28, 58, 67] the intersampling increase is not estimated quantitatively. We also show that there is a trade-off between intersampling improvement and stability of the zero-input system, in the sense that enlarging intersampling periods results in practical sense stability rather than the classical notion of stability.

One issue, however, critical to any event-triggered system, is ensuring that there is a time-separation between triggering events. Although this issue has been extensively studied in multiple references, recent research has shown that event-triggered systems can be critically sensitive in the presence of external disturbances [59], possibly resulting in Zeno behaviour, unless the TC is properly constructed. A recent line of research, known as *time-regularization*, is recently proposed to address the separation of event times in presence of exogenous disturbances, where the separation of events is guaranteed by imposing a positive dwell-time after each triggering instant, during which the TC monitoring is paused. It is well-known that as states converge towards the origin, the time-regularization approach converges to a time-triggered sampling and hence produce unnecessary samplings. Our approach to design TCs in this research offers an independent solution of the event separation problem mentioned above. Indeed, in the time-regularized approach proposed in [59], time-regularization holds by construction, by forcing a minimum dwell-time between events. Our approach, however, is purely event-based, suggesting that the pre-designed dwell-time assumption can be relaxed and can potentially offer better performance compared to the time-regularized method.

## 1.4 Thesis Outline

The rest of this thesis is organized as follows.

**Chapter 3**: This chapter studies the performance preserving event design in nonlinear ETC systems based on a local $\mathcal{L}_2$-type performance criterion. Considering a finite gain local $\mathcal{L}_2$-stable disturbance driven continuous-time system, we propose a ETM so that the resulting sampled-data system preserves similar disturbance attenuation local $\mathcal{L}_2$-gain property. The results are applicable to nonlinear systems with exogenous disturbances bounded by some Lipschitz-continuous function of state. Also, it is shown that an exponentially decaying function of time, combined with the proposed TC, extends the inter-event periods, which compared to the existing works, analytically estimates the increase in intersampling periods at least for an arbitrary period of time. We also propose a so-called *discrete triggering condition* to quantitatively find the improvement in inter-event times at least for an arbitrary number of triggering iterations.

**Chapter 4**: In this chapter, we focus on generalizing the results in Chapter 3 from several aspects. First, we propose a framework to design the TC while keeping global $\mathcal{L}_p$

performance within desired limits. Also, our general framework captures several existing TCs as a special case, and can achieve the performance objectives while reducing transmission rate. Indeed, this general structure is shown to enlarge the intersampling periods by a specified amount, for a desired period of time. Moreover, compared to Chapter 3, we consider a dynamic approach to design triggering condition.

**Chapter 5**: The results in Chapters 3, 4 proved that The ISS condition is a powerful tool in designing TCs especially when dealing with nonlinear systems. In this chapter, we seek to propose sufficient conditions for the ISS condition to be hold when nonlinearites has the particular input-affine structure. Moreover, we propose an $\mathcal{L}_2$ stabilizing event-triggered controller which guarantees the isolation of sampling instants in presence of arbitrary disturbances. While our proposed design does not rely on the ISS assumption, it covers both emulation and joint design approaches.

**Chapter 6**: While the majority of ETC literature, including our results in previous chapter, concentrates on designing TC assuming control input to emulate an analog design, in this chapter, both state and output feedback laws are jointly synthesized with the triggering law for nonlinear Lipschitz systems. In the proposed method, the dominant eigenvalues of the linear stability matrices are assigned according to desired performance and triggering specifications. Moreover, the results serve as a local framework for stability of general nonlinear ETC systems. In addition, it is shown for the output-based feedback case that under the fast sampling at the controller-to-actuator channel, the separation principle for designing the controller and observer gains hold.

**Chapter 7**: This chapter studies different forms of dissipativity property for the network-communicated physical processes. The results are then a generalization of the ines in previous chapters as several stability and robustness properties of control systems can be unified under the notion of dissipativity. While this concept has been recently studied for network control systems with communication constraints, the obtained results suffers from a concrete proof of Zeno-freeness property for the sampling times. In this chapter, we address this issue and prove the triggering instants to guarantee the well-known separation property.

**Chapter 8**: In this chapter, an integral-based event-driven mechanism is proposed for a general class of nonlinear systems. The proposed scheme is less conservative than earlier work on the subject and achieves asymptotic stability without forcing the derivative of the Lyapunov function to be negative between samples. additionally, the results are applied as an event-triggered solution to the consensus problem of multi-agent systems.

# Chapter 2

# Mathematical Background

This chapter provides some technical definitions and preliminaries that will be used throughout the rest of the thesis.

**Definition 2.1** *A function $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ is said to be locally Lipschitz-continuous in an open set $B$, if for each $z \in B$ there exist $L_f \in \mathbb{R}^+$ and $r \in \mathbb{R}^+$ such that $|f(x) - f(\tilde{x})| \leq L_f |x - \tilde{x}|$ for all $x, \tilde{x} \in \{y \in B : |y - z| < r\}$. We also say that $f$ is Lipschitz-continuous in a set $D$ if there exists $L_f \in \mathbb{R}^+$ (called the Lipschitz constant of $f$ on $D$) such that $|f(x) - f(\tilde{x})| \leq L_f |x - \tilde{x}|$ for all $x, \tilde{x} \in D$.*

**Definition 2.2** *A function $\alpha : [0, a) \mapsto \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A class $\mathcal{K}$ function $\alpha$ belongs to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\eta : \mathbb{R}^p \to \mathbb{R}^p$ is of class $\mathcal{L}$ ($\eta \in \mathcal{L}$) if it is decreasing and $\eta(s) \to 0$ as $s \to \infty$. A function $\beta: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$ and for each $r \geq 0$, $\beta(r, \cdot) \in \mathcal{L}$. A function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{N}$ ($\gamma \in \mathcal{N}$), if it is continuous and nondecreasing.*

**Definition 2.3** *A sequence $\{x_i : i \in \mathbb{N}_0\}$ is said to be uniformly isolated iff there exists some $r \in \mathbb{R}^+$ so that $|x_i - x_j| > r$ for any $i, j \in \mathbb{N}_0$ with $i \neq j$.*

**Definition 2.4** *Every sequence $\mathbb{T} = \{t_k : k \in \mathbb{N}_0\}$ of positive real numbers is called partition, if $t_0 = 0$, $t_k < t_{k+1}$ and $t_k \to \infty$ as $k \to \infty$.*

**Lemma 2.1 (Barbalat's Lemma)** *Let $f : \mathbb{R} \to \mathbb{R}$ be an upper bounded function, i.e., $|f(t)| \leq c$ for some $c \in \mathbb{R}$. Suppose $\dot{f}$ is positive semi-definite and is uniformly continuous (satisfied if $\ddot{f}$ is finite). Then $\dot{f} \to 0$ as $t \to \infty$.*

The following lemma is a consequence of Schur complement.

**Lemma 2.2** *For any vectors $x, y$ and matrices $A, B, C$ of appropriate dimensions with $C \prec 0$, we have*

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} < x^{\mathsf{T}}(A - BC^{-1}B^{\mathsf{T}})x. \tag{2.1}$$

The Cauchy-Schwarz (C-S) inequality is referred to as the special case of the following Holder's integral inequality

$$\int_{\mathcal{T}} \|x(t)y(t)\| dt \leq (\int_{\mathcal{T}} \|x(t)\|^p dt)^{\frac{1}{p}} (\int_{\mathcal{T}} \|y(t)\|^q dt)^{\frac{1}{q}}$$

for scalar signlas $x$, $y$ when $p = q = 2$.

## 2.1 Stability Criteria

Input-output stability is a key tool in this research to study the robustness performance of the following nonlinear system $\mathscr{G}$:

$$\mathscr{G} : \begin{cases} \dot{x} = f(x, u, w) \\ z = h(x, w) \end{cases} \tag{2.2}$$

where $x \in \mathbb{R}^n$ represents the state, $u \in \mathscr{U} \subseteq \mathbb{R}^m$ the control input, $w \in \mathscr{W} \subseteq \mathbb{R}^q$ the exogenous disturbance, and $z \in \mathbb{R}^p$ the measured output.

The classical definitions of the input-output stability can be found in many references, see, *e.g.,* [68]. However, the results are not applicable to the systems with norm bounded input space. Instead, we build our theory using the local version of input-output stability introduced in [57].

In the next definitions we exploit the concept of *relations* as a traditional tool to state the local stability criteria. Equivalently, one can define the input-output stability as a property of the operators. We recall that given two nonempty sets $A_1$ and $A_2$, a relation $\mathscr{R}$ on $A_1 \times A_2$ is any subset of the Cartesian product $A_1 \times A_2$.

**Definition 2.5** *Let $A_1 \times A_2$ be the Cartesian product of two sets $A_1$ and $A_2$. We denote by $P_i : A_1 \times A_2 \to A_i$, $i = 1, 2$ the* evaluation map *at $i$ defined as $P_i(x_1, x_2) = x_i$, $i = 1, 2$.*

**Definition 2.6** *We define the set $\mathscr{W}_Q \subset \mathcal{L}_2$ as follows:*

$$\mathscr{W}_Q = \{w \in \mathcal{L}_2 : \|w\|_\infty < Q\}, \tag{2.3}$$

*where $Q \in \mathbb{R}^+$. We note that $\mathscr{W}_Q$, which is a subset of $\mathcal{L}_2 \cap \mathcal{L}_\infty$, is not a linear space in general since there exists elements $x, y \in \mathscr{W}_Q$ such that $x + y \notin \mathscr{W}_Q$.*

We remark that the triplet $(\mathcal{L}_2, \|\cdot\|_2, \|\cdot\|_\infty)$ consisting of linear space $\mathcal{L}_2$ and the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ is a *binormed linear space*, where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are the *primary* and *secondary* norms of the space $\mathcal{L}_2$. $\mathscr{W}_Q$ is then the subset of $\mathcal{L}_2$ consisting of functions with secondary norm less than $Q \in \mathbb{R}^+$.

**Definition 2.7** *A relation $\mathscr{R}$ on $\mathcal{L}_2 \times \mathcal{L}_2$ is said to be $\mathscr{W}_Q$-stable if the evaluation map at 2 is a bounded subset of $\mathcal{L}_2$ whenever the evaluation map at 1 belongs to the set $\mathscr{W}_Q$.*

**Definition 2.8** *The system $\mathscr{G}$ defined in (2.2) is said to be locally $\mathcal{L}_2$-stable if for any $w \in \mathscr{W}_Q$, the relation $\mathscr{R} \doteq \{(w, z) \in \mathcal{L}_2 \times \mathcal{L}_2\}$ is $\mathscr{W}_Q$-stable.*

In the next definition, we provide a local version of finite gain $\mathcal{L}_2$-stability [1], a deviation from the classical definition by restricting the spaces of admissible inputs and initial conditions to the sets $\mathscr{W}_Q$ (defined in Definition 2.6) and

$$\mathscr{X}_0 \doteq \{r \in \mathbb{R}^n : |r| \leq \varepsilon \in \mathbb{R}^+\}, \tag{2.4}$$

respectively.

**Definition 2.9** *The system $\mathscr{G}$ described in (2.2) is said to be finite gain locally $\mathcal{L}_2$-stable and has the local $\mathcal{L}_2$-gain less than or equal to $\Gamma$, if it is locally $\mathcal{L}_2$-stable and there exist finite constants $\eta \in \mathbb{R}_0^+$, $\Gamma \in \mathbb{R}^+$ and positive semi-definite $\mathbf{C}^0$ function $\mu$ such that for any $T, t_0 \in \mathbb{R}_0^+$, any $w \in \mathscr{W}_Q$ and any $x_0 \in \mathscr{X}_0 \subset \mathbb{R}^n$*

$$\int_{t_0}^{T} |z(s)|^2 ds \leq \Gamma^2 \int_{t_0}^{T} |w(s)|^2 ds + \mu(x_0) + \eta. \tag{2.5}$$

*We shall denote the local $\mathcal{L}_2$-gain of system $\mathscr{G}$ by $\|\mathscr{G}\|_{\mathcal{L}_2}$. We also say that $\mathscr{G}$ is finite gain locally $\mathcal{L}_2$-stable with zero bias if $\eta = 0$ in (2.5).*

The following theorem provides a sufficient condition to estimate an upper bound on the local disturbance attenuation $\mathcal{L}_2$-gain of system $\mathscr{G}$ in the context of dissipative systems theory introduced by [62].

**Theorem 2.1** *The nonlinear system $\mathscr{G}$ is finite gain locally $\mathcal{L}_2$-stable with zero bias and has $\|\mathscr{G}\|_{\mathcal{L}_2} \leq \Gamma$, provided there exist a positive definite $\mathbf{C}^1$ function $V$ and a control input $u \in \mathscr{U}$ such that for all $w \in \mathscr{W}_Q$*

$$H_\Gamma(V, u) \doteq \nabla V(x) \cdot f(x, u, w) - \Gamma^2 |w|^2 + |h(x, w)|^2 \leq 0. \tag{2.6}$$

[1] See [68] for the classical finite gain $\mathcal{L}_2$-stability definition.

**Proof.** The result is readily obtained by integration of (2.6), positive definiteness of $V(x)$ and Definition 2.9. ■

**Remark 2.1** *If system $\mathscr{G}$ is reachable from $x_0$, condition (2.6) is necessary and sufficient for finite gain local $\mathcal{L}_2$-stability of $\mathscr{G}$ with zero bias and $\|\mathscr{G}\|_{\mathcal{L}_2} \leq \Gamma$.*

**Proof.** The result follows directly from ( [69], Theorem 2.1). ■

Next we investigate the input-to-state stability of the the system $\mathscr{G}$.

**Definition 2.10** *The $\mathbf{C}^1$ function $V : \mathbb{R}^n \mapsto \mathbb{R}_0^+$ is an ISS Lyapunov function for system $\mathscr{G}$ defined in (2.2) if there exist class $\mathcal{K}_\infty$ functions $\sigma$, $\sigma_i$, $\gamma_i$ $(i = 1, 2)$ such that*

$$\sigma_1(|\xi|) \leq V(\xi) \leq \sigma_2(|\xi|) \tag{2.7}$$

*holds for all $\xi \in \mathbb{R}^n$, and*

$$\nabla V(\xi) \cdot f(\xi, \mu, w) \leq -\sigma(|\xi|) \tag{2.8}$$

*for any $\xi \in \mathbb{R}^n$, any $\mu \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $|\xi| \geq \gamma_1(|\mu|) + \gamma_2(|w|)$.*

The next theorem suggests an equivalent condition to the above given inequality (2.8). We will use this theorem later to develop our main theorem in Section 3.2.

**Theorem 2.2** *The $\mathbf{C}^1$ function $V$ is an ISS Lyapunov function for system $\mathscr{G}$ if and only if (2.7) holds and there exist class $\mathcal{K}_\infty$ functions $\bar{\sigma}$ and $\beta_i$ $(i = 1, 2)$ so that*

$$\nabla V(\xi) \cdot f(\xi, \mu, w) \leq -\bar{\sigma}(|\xi|) + \beta_1(|\mu|) + \beta_2(|w|) \tag{2.9}$$

*for any $\xi \in \mathbb{R}^n$, any $\mu \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$.*

**Proof.** We need to show that if there exist class $\mathcal{K}_\infty$ functions $\sigma$, $\gamma_i$ $(i = 1, 2)$ so that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\sigma(|\xi|)$ holds for any $\xi \in \mathbb{R}^n$, any $\mu \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $|\xi| \geq \gamma_1(|\mu|) + \gamma_2(|w|)$ then one can find class $\mathcal{K}_\infty$ functions $\bar{\sigma}$, $\beta_i$ $(i = 1, 2)$ such that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\bar{\sigma}(|\xi|) + \beta_1(|\mu|) + \beta_2(|w|)$ and vice versa. Let us start by assuming $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\sigma(|\xi|)$ for $|\xi| \geq \gamma_1(|\mu|) + \gamma_2(|w|)$. Then we can say that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) + \sigma(|\xi|) \leq \bar{\beta}(|\mu|, |w|)$ where

$$\begin{aligned}\bar{\beta}(|\mu|, |w|) = \max\{\nabla V(\xi) \cdot f(\xi, k(\xi + r), s) + \sigma(|\xi|) \mid\, & |r| \leq |\mu|, \\ & |s| \leq |w|, |\xi| \leq \gamma_1(|r|) + \gamma_2(|s|)\}.\end{aligned}$$

Defining class $\mathcal{K}_\infty$ functions $\beta_1(|\mu|) \doteq \bar{\beta}(|\mu|, |\mu|)$ and $\beta_2(|w|) \doteq \bar{\beta}(|w|, |w|)$ it is not difficult to verify that $\bar{\beta}(|\mu|, |w|) \leq \beta_1(|\mu|)$ for $|\mu| \geq |w|$ and $\bar{\beta}(|\mu|, |w|) \leq \beta_2(|w|)$ otherwise. Therefore we conclude that $\bar{\beta}(|\mu|, |w|) \leq \beta_1(|\mu|) + \beta_2(|w|)$ that proves one part of the claim. To prove the other side, assume that we have $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\bar{\sigma}(|\xi|) + \beta_1(|\mu|) + \beta_2(|w|)$. Then we can write $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\bar{\sigma}(|\xi|)/2$ for $\bar{\sigma}(|\xi|)/4 \geq \beta_1(|\mu|)$ and $\bar{\sigma}(|\xi|)/4 \geq \beta_2(|w|)$. Finally defining $\gamma_i \doteq \bar{\sigma}^{-1}(4\beta_i)$ $(i = 1, 2)$, we may conclude that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\bar{\sigma}(|\xi|)/2$ for $|\xi| \geq \gamma_1(|\mu|) + \gamma_2(|w|)$ which completes the proof. ■

Definition 2.10 provides a characterization of the notion of ISS property, rather than its definition, using Lyapunov-like conditions. Next theorem shows that these conditions are necessary and sufficient for input-to-state stability.

**Theorem 2.3** *The closed-loop system $\mathscr{G}$ defined in (2.2) is ISS with respect to inputs $u$ and $w$ iff there exists an ISS Lyapunov function $V$ satisfying (2.7), (2.8).*

**Proof.** The proof follows from Theorem 2.2 and ( [70], Theorem 1). ■

**Remark 2.2** *Later in Section 3.2 our study will focus on the systems with disturbances norm bounded by some function of state , i.e., $|w(t)| \leq \gamma_3(|x(t)|)$. This assumption seems to be implied in Definition 2.10 as condition (2.8) is valid for $\gamma_2(|w(t)|) \leq |x(t)| - \gamma_1(|e(t)|)$. Thus to prevent any possible redundancy of these conditions, we will unify them later in section 3.2.*

**Remark 2.3** *When $Q = \mathbb{R}^+$, the classical input-output stability can be extracted from the above definitions and results.*

## 2.2 Graph Theory

The following definitions and notation will mostly be used in Chapter 8. Consider a team of $n$ vehicles. A directed graph is a pair $(V_n, E_n)$ where $V_n = \{1, \ldots, n\}$ is a finite nonempty node set and $E_n \subseteq V_n \times V_n$ is a set of ordered pairs of nodes, called edges. Existence of edge $(i, j)$ in the edge set of a directed graphs shows that vehicle $j$ can obtain information from vehicle $i$, but not necessarily vise versa. In contrast to a directed graph, the pairs of nodes in an undirected graph are unordered, where the edge $(i, j)$ denotes that vehicles $i$ and $j$ can obtain information from each other. We call node $i$ to be a neighbor of node $j$ if an edge $(i, j) \in E_n$ exists. We show the set of neighbors of node $i$ by $N_i \subseteq \{1, \ldots, n\}$. A directed graph is strongly connected if there is a directed path from every node to every other node. An undirected graph is connected if there is an undirected path between every pair of nodes. The adjacency matrix $A_n = [a_{ij}] \in \mathbb{R}^{n \times n}$ of a directed graph $(V_n, E_n)$ is defined such that $a_{ij}$ is a positive constant if $(j,i) \in E_n$, and $a_{ij} = 0$ if $(j,i) \notin E_n$. The adjacency matrix of an undirected graph is defined analogously except that $a_{ij} = a_{ji}$ for all $i \neq j$, because $(j,i) \in E_n$ implies $(i,j) \in E_n$. $a_{ij}$ denotes the weight for the edge $(j,i) \in E_n$. If the weight is not relevant, then $a_{ij}$ is set equal to 1 if $(j,i) \in E_n$. Matrix $L_n = [l_{ij}] \in \mathbb{R}^{n\times n}$ defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$, and $l_{ij} = -a_{ij}$, $i \neq j$. If $(j,i) \notin E_n$, then $l_{ij} = -a_{ij} = 0$. Matrix $L_n$ satisfies $l_{ij} \leq 0$, $i \neq j$, and $\sum_{j=1}^{n} l_{ij} = 0$, $i = 1,\ldots,n$. For undirected graph, $L_n$ is symmetrical and is called the Laplacian matrix. However, for a directed graph, $L_n$ is not necessarily symmetrical and sometimes called the nonsymmetrical Laplacian matrix or directed Laplacian matrix. Note the $L_n$ can be equivalently defined as $L_n \triangleq D - A_n$ where $D = [d_{ij}] \in \mathbb{R}^{n\times n}$ is the in-degree matrix given as $d_{ij} = 0$, $i \neq j$, and $d_{ii} = \sum_{j=1}^{n} a_{ij}$, $i = 1,\ldots,n$. In both the undirected and directed cases, because $L_n$ has zero row sums, 0 is an eigenvalue of $L_n$ with the associated eigenvector $\mathbf{1}_n$, the $n \times 1$ column vector of ones. $L_n$ is diagonally dominant and has nonnegative diagonal entries. For an undirected graph, let $\lambda_i(L_n)$ be the $i$th eigenvalue of $L_n$ with $\lambda_1(L_n) \leq \lambda_2(L_n) \leq \ldots \leq \lambda_i(L_n)$, so that $\lambda_1(L_n) = 0$. For an undirected graph, $\lambda_2(L_n)$ is the algebraic connectivity, which is positive if and only if the directed graph is connected. The algebraic connectivity quantifies the convergence rate of consensus algorithms. To simplify our notation, we denote $L_n$ simply by $L$.

# Chapter 3

# Local Input-Output Stability of Event-Triggered Control Systems

## 3.1 Problem Definition

In this chapter[1] we study the input-to-state stability of the system $\mathscr{G}$ defined in (2.2). We assume that $f$ and $h$ are class $\mathbf{C}^0$ and $f(0,0,0) = 0$, $h(0,0) = 0$ so that $x = 0$ is an equilibrium point of zero-input system. Moreover, we will assume the state $x$ evolves on an open subset of $\mathbb{R}^n$ containing the origin. We also assume that $\mathscr{G}$ is driven from initial conditions $x_0 = x(t_0)$ and the inputs $u$ and $w$ are applied at time $t = t_0^+$. We shall assume that the measurement of state is affected by an error $e$. As a result, designing the state feedback controller $u = k(x)$, where $k$ is of class $\mathbf{C}^0$ and satisfy $k(0) = 0$, the implemented control law will be $k(x + e)$. The corresponding closed-loop system with perturbed measurement is therefore

$$\mathscr{G}_e : \begin{cases} \dot{x} = f(x, k(x + e), w), \\ z = h(x, w). \end{cases} \tag{3.1}$$

To state our problem we shall need to define a continuous-time version of system $\mathscr{G}_e$ defined by assuming measurement error to be zero all the time. This system will be referred as $\mathscr{G}_c$ throughout the rest of this chapter. Now assume the existence of a positive definite $\mathbf{C}^1$ function $V$ and a $\mathbf{C}^0$ function $k : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that $H_\Gamma(V, k(x)) \leq 0$, *i.e.*, the state feedback control law $u = k(x)$ renders the continuous-time system $\mathscr{G}_c$ finite gain locally $\mathcal{L}_2$-stable with zero bias and $\|\mathscr{G}_c\|_{\mathcal{L}_2} \leq \Gamma$. We also assume the implementation of the control law to be performed in an event-based scheme in which an event detector decides when to update the control signal. As a consequence, the actuator receives an updated control

[1]The results of this chapter have been published in the article: M. Ghodrat and H. J. Marquez, “On the Local Input-Output Stability of Event-Triggered Control Systems”, *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 174-189, 2019.

signal at triggering instants $\{t_i : i \in \mathbb{N}_0\}$, at which a TC is satisfied. The first sampling instant can always be assumed to coincide with initial time $t_0$. A ZOH device serves to maintain the controller signal constant between two successive sampling instants. Thus, between time instants $t_i$ and $t_{i+1}$, the controller signal is $k(x(t_i))$ and remains unchanged. This enables us to define the measurement error $e(t)$ as the difference between the current value of state at the event detector, $x(t)$, and the last triggered value of state, $x(t_i)$, *i.e.*,

$$e(t) = x(t_i) - x(t), \quad t \in [t_i, t_{i+1}). \tag{3.2}$$

It follows that the measurement error is zero at each sampling instants and its value is continuously monitored to check a TC which, as we will see later, sets an upper bound on the norm of admissible measurement error. Once the condition holds, the system sends an updated signal to the actuator and resets the measurement error to zero.

In [9] it is shown that in presence of an execution rule that restricts the measurement error to satisfy

$$\beta_1(|e|) \leq c\bar{\sigma}(|x|), \tag{3.3}$$

where $c \in (0, 1)$, and if there exists an ISS Lyapunov function $V$ so that

$$\nabla V(x) \cdot f(x, k(x+e), 0) \leq -\bar{\sigma}(|x|) + \beta_1(|e|), \tag{3.4}$$

the system $\mathscr{G}_e$ with zero-input is globally asymptotically stable.

In general, the aforementioned ETM (3.3) guarantees closed-loop stability. However, it is by no means clear how it affects the *input/output* performance of the system. More specifically, in this chapter, we are concerned with finite gain $\mathcal{L}_2$-stability performance. The purpose of this chapter is then to present an input-output stability analysis of ETC systems. Departing from the TC offered in [9], we propose a condition which guarantees the finite gain local $\mathcal{L}_2$-stability of the system.

## 3.2 $\mathcal{L}_2$-Gain Performance of Event Triggered Nonlinear Systems

In this section we present a novel TC that ensures finite gain local $\mathcal{L}_2$-stability of the ETC system $\mathscr{G}_e$. The design of such a sampling rule is based on the following assumptions.

**Assumption 3.1** *There exist a positive definite* $\mathbf{C}^1$ *function* $W$ *and some* $Q \in \mathbb{R}^+$ *such that*

$$H_\Gamma(W, k(x)) \leq 0, \tag{3.5}$$

*for all* $w \in \mathscr{W}_Q$*, where* $\mathscr{W}_Q$ *is defined in (2.3).*

**Assumption 3.2** *There exist a radially unbounded positive definite* $\mathbf{C}^1$ *function* $V$ *and class* $\mathcal{K}_\infty$ *functions* $\bar{\sigma}$, $\beta_1$ *satisfying*

$$\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\bar{\sigma}(|\xi|) + \beta_1(|\mu|) \tag{3.6}$$

*for any* $\xi \in \mathbb{R}^n$, *any* $\mu \in \mathbb{R}^n$ *and any* $w \in \mathscr{W}_Q$.

Recalling Theorem 2.1, condition (3.5) ensures that the continuous-time system $\mathscr{G}_c$ is finite gain locally $\mathcal{L}_2$-stable with zero bias and has $\|\mathscr{G}_c\|_{\mathcal{L}_2} \leq \Gamma$. The following lemma describes the connection between Assumption 3.2 and the previously defined ISS concept. Indeed, we show that this assumption can be used to deal with unmodeled parameter uncertainties.

**Lemma 3.1** *(a) Assumption 3.2 holds if and only if there exists a radially unbounded positive definite* $\mathbf{C}^1$ *function* $V$ *satisfying* $\nabla V(\xi)\cdot f(\xi, k(\xi+\mu), w) \leq -\sigma(|\xi|)$ *for any* $\xi \in \mathbb{R}^n$, *any* $\mu \in \mathbb{R}^n$ *and any* $w \in \mathscr{W}_Q$ *such that* $|\xi| \geq \gamma(|\mu|)$ *for some* $\sigma, \gamma \in \mathcal{K}_\infty$. *(b) The later condition is satisfied when for any* $w \in \mathscr{W}_Q$ *the followings hold:*

*(I)* $V$ *is an ISS Lyapunov function for the system* $\mathscr{G}_e$,

*(II) there exist solutions* $\gamma_3$, $\gamma_4 \in \mathcal{K}_\infty$ *to the inequality*

$$\gamma_4 \circ (\gamma_{id} - \gamma_2 \circ \gamma_3)(r) \geq r, \tag{3.7}$$

*for all* $r \in \mathbb{R}_0^+$, *where* $\gamma_{id}$ *is the identity function and* $\gamma_2 \in \mathcal{K}_\infty$ *is defined in Definition 2.10,*

*(III) disturbance is bounded through*

$$|w(t)| \leq \gamma_3(|x(t)|) \tag{3.8}$$

*for all* $t \in \mathbb{R}_0^+$ *where* $x$ *denotes the state of system* $\mathscr{G}_e$ *defined in (3.1).*

**Proof.** (a) This is an immediate consequence of Theorem 2.2. (b) We need to show that under conditions I-III, there exists a class $\mathcal{K}_\infty$ function $\gamma$ so that $\nabla V(\xi) \cdot f(\xi, k(\xi+\mu), w) \leq -\sigma(|\xi|)$ for $|\xi| \geq \gamma(|\mu|)$. To this end, let us start with conditions II and III that suggest $\gamma_4 \circ (|\xi| - \gamma_2(|w|)) \geq \gamma_4 \circ (\gamma_{id} - \gamma_2 \circ \gamma_3)(|\xi|) \geq |\xi|$. Now taking $\gamma = \gamma_4 \circ \gamma_1$ we can say that if $|\xi| \geq \gamma(|\mu|)$, we have $|\xi| - \gamma_2(|w|) \geq \gamma_1(|\mu|)$ which, in view of condition I, implies that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \leq -\sigma(|\xi|)$. ■

Note that condition (3.7) is similar to $\delta$-admissible perturbation provided in ( [47], Definition 2).

We will need the following technical lemma to prove our main result. This lemma sets the stage for the design of the TC required to achieve disturbance attenuation bound $\Gamma$ for the ETC system.

**Lemma 3.2** *Assumption 3.2 holds if and only if there exist a radially unbounded positive definite* $\mathbf{C}^1$ *function* $V$ *and class* $\mathcal{K}_\infty$ *functions* $\hat{\sigma}$, $\sigma_0$, $\beta_0$, $\psi$, $\bar{\beta}_1$ *and some* $c \in (0,1)$ *satisfying*

$$\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \le -\hat{\sigma}(|\xi|) - \sigma_0(|\xi|)\beta_0(|\mu|) \tag{3.9}$$

*for any* $\xi \in \mathbb{R}^n$*, any* $\mu \in \mathbb{R}^n$ *and any* $w \in \mathscr{W}_Q$ *such that* $c\psi(|\xi|) \ge \bar{\beta}_1(|\mu|)$.

**Proof.** (if) From (3.9) we may conclude that $\nabla V(\xi) \cdot f(\xi, k(\xi + \mu), w) \le -\hat{\sigma}(|\xi|)$ for $c\psi(|\xi|) \ge \bar{\beta}_1(|\mu|)$. Then taking $\sigma = \hat{\sigma}$, $\gamma = \psi^{-1}(\bar{\beta}_1/c)$ and applying Lemma 3.1 part (a), the desired result is obtained. (only if) Starting from Assumption 3.2, by adding and subtracting $\sigma_0(|\xi|)\beta_0(\mu)$ term to the right hand side of inequality (3.6), we may write $\nabla V(\xi) \cdot f(\xi, k(\xi+\mu), w) \le -(1-c)\bar{\sigma}(|\xi|) - \sigma_0(|\xi|)\beta_0(|\mu|)$ for $\beta_1(\mu) + \sigma_0(|\xi|)\beta_0(|\mu|) \le c\bar{\sigma}(|\xi|)$. Now defining functions $\psi(r) \doteq \bar{\sigma}(r)/(1 + \sigma_0(r))$, $\bar{\beta}_1(r) \doteq \max\{\beta_1(r), \beta_0(r)\}$, we claim that if $c\psi(|\xi|) \ge \bar{\beta}_1(|\mu|)$ we have $\beta_1(\mu) + \sigma_0(|\xi|)\beta_0(|\mu|) \le c\bar{\sigma}(|\xi|)$. This is true since $c\bar{\sigma}(|\xi|) \ge (1 + \sigma_0(|\xi|)) \cdot \max\{\beta_1(\mu), \beta_0(\mu)\} \ge \beta_1(\mu) + \sigma_0(|\xi|)\beta_0(|\mu|)$. Therefore, if $c\psi(|\xi|) \ge \bar{\beta}_1(|\mu|)$, (3.9) holds for $\hat{\sigma} = (1 - c)\bar{\sigma}$ and hence the proof is complete. ■

*Triggering Condition:* Let $t_i$, $i \in \mathbb{N}_0$, be the most recent sampling instant, the control signal is updated again at $t_{i+1}$ defined by the following rule:

$$t_{i+1}^- = \inf\left\{t \in \mathbb{R}_0^+ : t > t_i \bigwedge \bar{\beta}_1(|e(t)|) \ge c\psi(|x(t)|)\right\}, \tag{3.10}$$

where $c \in (0,1)$ and $\psi$, $\bar{\beta}_1$ are defined as

$$\psi(r) \doteq \frac{\bar{\sigma}(r)}{1 + \sigma_0(r)}, \ \bar{\beta}_1(r) \doteq \max\{\beta_1(r), \beta_0(r)\}. \tag{3.11}$$

for $\sigma_0(r) = L_f L_k \sigma_3(r)$ and $\beta_0(r) = r$ with $L_f$, $L_k$ defined in Remark 3.4. Note that we assume that the update of the control task is done at $t_{i+1}$, shortly after the given inequality in (3.10) is satisfied at $t_{i+1}^-$ The following theorem states that if the continuous-time system has some local $\mathcal{L}_2$-gain property, it is always possible to guarantee the same disturbance attenuation level for the ETC system by applying the above ETM.

**Theorem 3.1** *Let us consider Assumptions 3.1, 3.2 and the following conditions:*

*(i)* $|\nabla W(x)| \le \sigma_3(|x|)$ *for some class* $\mathcal{K}_\infty$ *function* $\sigma_3$*, locally Lipschitz-continuous in* $\mathbb{R}_0^+$*,*

*(ii)* $\bar{\sigma}^{-1}$, $\beta_1$, $\gamma_3$ *are locally Lipschitz-continuous in* $\mathbb{R}_0^{+}$ [2],

*(iii)* $k$ *and* $f$ *are locally Lipschitz-continuous in* $\mathbb{R}^n$ *and* $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$, *respectively* [2].

*Then the system* $\mathscr{G}_e$ *driven from initial conditions* $x_0 \in \mathscr{X}_0$, *defined in (2.4), is finite gain locally* $\mathcal{L}_2$*-stable with zero bias and has* $\|\mathscr{G}_e\|_{\mathcal{L}_2} \leq \Gamma$ *if the control signal is executed under rule (3.10).*

**Proof.** Let us start with Assumption 3.2 which, in view of proof of Lemma 3.2, implies the existence of $\mathbf{C}^1$ function $V$ such that

$$\nabla V(x) \cdot f(x, k(x+e), w) \leq -(1-c)\bar{\sigma}(|x|) - \sigma_0(|x|)\beta_0(|e|) \tag{3.12}$$

for any $x \in \mathbb{R}^n$, any $e \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $c\psi(|x|) \geq \bar{\beta}_1(|e|)$. Now consider positive definite $\mathbf{C}^1$ function $U = V + W$, where $W$ is a positive definite $\mathbf{C}^1$ function that, in view of Assumption 3.1, guarantees the finite gain local $\mathcal{L}_2$-stability of continuous-time system $\mathscr{G}_c$. We can easily write

$$\begin{aligned}\dot{U}(x) &= \nabla V(x) \cdot f(x, k(x+e), w) + \nabla W(x) \cdot f(x, k(x), w) \\ &\quad + \nabla W(x) \cdot (f(x, k(x+e), w) - f(x, k(x), w)).\end{aligned} \tag{3.13}$$

Also applying condition (i) and inequality (3.17) gives $\nabla W(x) \cdot (f(x, k(x+e), w) - f(x, k(x), w)) \leq \sigma_0(|x|)\beta_0(|e|)$. As a consequence, in view of (3.5), (3.12) and (3.13) we can write

$$\dot{U}(x) \leq -(1-c)\bar{\sigma}(|x|) + \Gamma^2|w|^2 - |h(x,w)|^2 \tag{3.14}$$

for any $x \in \mathbb{R}^n$, any $e \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $c\psi(|x|) \geq \bar{\beta}_1(|e|)$. Thus under TC (3.10) we obtain $H_\Gamma(U, k(x+e)) \leq 0$, *i.e.*, the ETC system $\mathscr{G}_e$ has the disturbance attenuation local $\mathcal{L}_2$-gain $\|\mathscr{G}_e\|_{\mathcal{L}_2} \leq \Gamma$. ■

It is worth remarking that Theorem 3.1 is stated in local form. Note that condition (3.8) which restricts $w$ to be norm bounded by some Lipschitz-continuous function of state, plays an essential role in satisfying Assumption 3.2. This assumption is not consistent with classical input-output stability notion that requires $w$ to be *any* perturbation in $\mathcal{L}_2$. Thus it remains to define $Q$ such that for any given initial conditions in $\mathscr{X}_0$, $w$ is guaranteed to be in the set $\mathscr{W}_Q$. Condition (3.8) is a key tool to define such an admissible inputs set. Indeed, later in view of Lemma 3.3, condition (3.8) and Lipschitz-continuity of $\gamma_3$ with Lipschitz constant $L_{\gamma_3}$ defined in Remark 3.4, one can choose $Q = L_{\gamma_3}\bar{\varepsilon}$.

[2] This condition can be relaxed in the proof of Theorem 3.1, however, is needed in the proof of Theorem 3.2.

**Remark 3.1** *The assumed dependence of $\gamma_3$ on the state of the system in (3.8) is a generalization of the assumption of state-dependent disturbance made in ( [17], Assumption 6.1). Indeed, the Assumption 6.1 in [17] can be extracted from (3.8) by choosing $\gamma_3$ to be a linear function of state, i.e., $\gamma_3(|x|) = c_0|x|$, for all $x \in \mathbb{R}^n$ and some $c_0 \in \mathbb{R}^+$. This generalization has to be considered more carefully as it gives more flexibility in choosing function $\gamma_2$ in (3.7), e.g., for $\gamma_2(r) = \sqrt{r}$, (3.7) does not provide any solution for possible linear functions $\gamma_3$. However, it is not difficult to verify that the solution to this inequality exists assuming $\gamma_3$ to be locally Lipschitz-continuous in $\mathbb{R}_0^+$.*

**Remark 3.2** *Using the same discussion as in ( [17], Remark 6.2), it is more precise to state condition (3.8) as $|w(t, x(t))| \leq \gamma_3(|x(t)|)$ for all $t \in \mathbb{R}_0^+$ to emphasize the state dependence of exogenous disturbance. To simplify our notation, we write $w(t)$ instead of $w(t, x(t))$ throughout the rest of this chapter.*

**Remark 3.3** *The TC (3.3) proposed in [9] can be extracted from the one we proposed in (3.10). Indeed, between consecutive sampling instants, (3.10) suggests*

$$\begin{aligned} c\bar{\sigma}(|x|) &\geq \max\{\beta_1(|e|), \beta_0(|e|)\}(1 + \sigma_0(|x|)) \\ &\geq \beta_1(|e|) + \beta_0(|e|)\sigma_0(|x|) \end{aligned} \tag{3.15}$$

*and hence we conclude that $\beta_1(|e|) \leq c\bar{\sigma}(|x|)$. This consequence simply suggests that under the conditions assumed in this chapter, in order to preserve system performance (in $\mathcal{L}_2$ sense) along with asymptotic stability provided in [9], a more conservative execution rule than the one proposed in [9] is needed.*

Our next Lemma shows that the state of the ETC system $\mathscr{G}_e$ is constrained to some compact set. The result is fundamental in the rest of this section.

**Lemma 3.3** *Under the assumptions of Theorem 3.1, $\mathscr{X} \doteq \{r \in \mathbb{R}^n : |r| \leq \bar{\varepsilon}\}$ for $\bar{\varepsilon} = \sigma_1^{-1}(\sigma_2(\varepsilon))$ is a positive invariant set for the trajectories of system $\mathscr{G}_e$ driven from any $x_0 \in \mathscr{X}_0$.*

**Proof.** We deduce from inequality (3.15) that $\bar{\sigma}(|x|) - \beta_1(|e|) - \beta_0(|e|)\sigma_0(|x|) \geq (1-c)\bar{\sigma}(|x|)$ and hence $\bar{\sigma}(|x|) - \beta_1(|e|) \geq 0$. Thus we conclude from (3.6) that $\dot{V}(x) \leq 0$ and consequently $V(x(t)) \leq V(x(0))$ for all $t \in \mathbb{R}_0^+$. Since $V$ is a radially unbounded positive definite function, we conclude that there exists $\sigma_1$, $\sigma_2 \in \mathcal{K}_\infty$ so that (2.7) holds and hence $\sigma_1(|x(t)|) \leq V(x(t)) \leq V(x(0)) \leq \sigma_2(|x(0)|)$. Then we can write $|x(t)| \leq \sigma_1^{-1}(\sigma_2(x_0))$ and since $x_0 \in \mathscr{X}_0$ and $\sigma_1^{-1}$, $\sigma_2$ are class $k_\infty$ functions, the desired result is obtained. ■

**Remark 3.4** *We now show how this analysis can be applied to find upperbounds on the norm of* $\dot{x}$*, something needed later to exclude Zeno-behaviour for the system* $\mathscr{G}_e$*. Lemma 3.3 suggests that* $x(t)$ *remains in the compact set* $\mathscr{X}$ *for all* $t \in \mathbb{R}_0^+$*. Moreover, in view of definition of e given in (3.2) we have* $|e(t)| \leq 2\bar{\varepsilon}$ *for all* $t \in \mathbb{R}_0^+$*. Thus we may conclude that* $e(t) \in \mathscr{X}_e \doteq \{r \in \mathbb{R}^n : r/2 \in \mathscr{X}\}$ *for all* $t \in \mathbb{R}_0^+$*. Also the control signal* $u = k(x+e)$ *does not leave the compact set* $\mathscr{X}_u \doteq \{r \in \mathbb{R}^n : r/k \in \mathscr{X}\}$ *since* $|u(t)| \leq k|x(t_i)| = k\bar{\varepsilon}$ *for all* $t \in \mathbb{R}_0^+$*. Now consider compact sets* $\mathscr{B}_x \subset \mathscr{X}$ *and* $\mathscr{B}_e \subset \mathscr{X}_e$*. In the view of Lipschitz-continuity of function k, one can define the compact set* $\mathscr{B}_u \subset \mathscr{X}_u$ *of all points* $u \in \mathbb{R}^m$ *satisfying* $|u| \leq |k(x+e)|$ *for all* $x \in \mathscr{B}_x$ *and* $e \in \mathscr{B}_e$*. Similarly, we can define the compact set* $\mathscr{B}_w \subset \mathscr{W}_Q$ *containing all points* $w \in \mathbb{R}^q$ *satisfying* $|w| \leq \gamma_3(|x|)$ *for all* $x \in \mathscr{B}_x$*. Now using the Lipschitz-continuity of function f with respect to* $(x^\mathsf{T}\ u^\mathsf{T}\ w^\mathsf{T})^\mathsf{T}$ *in compact set* $\mathscr{B}_x \times \mathscr{B}_u \times \mathscr{B}_w$ *with* $L_f$ *is the Lipschitz constant of the function f on* $\mathscr{X} \times \mathscr{X}_u \times \mathscr{W}_Q$ *and applying triangle inequality* $|f(x,u,w) - f(\tilde{x},\tilde{u},\tilde{w})| \leq |f(x,u,w) - f(x,\tilde{u},w)| + |f(x,\tilde{u},w) - f(\tilde{x},\tilde{u},\tilde{w})|$*, it is not difficult to confirm the Lipschitz-continuity of function* $\bar{f}(x,e,w) \doteq f(x,k(x+e),w)$ *in any compact set* $\mathscr{B}_x \times \mathscr{B}_e \times \mathscr{B}_w$ *with Lipschitz constant* $L_f(L_k+1)$*. It is also straight forward to check*

$$|\dot{x}| \leq L_f(L_k+1)|x| + L_f L_k |e| + L_f |w|, \tag{3.16}$$

$$|f(x,k(x+e),w) - f(x,k(x),w)| \leq L_f L_k |e|, \tag{3.17}$$

*that will be used further. Also inequality (3.16) in view of condition (3.8) in Theorem 3.1 and Lipschitz-continuity of* $\gamma_3$ *in the compact set* $\{r \in \mathbb{R}_0^+ : r \leq \max_{x \in \mathscr{B}_x} |x|\}$ *with Lipschitz constant* $L_{\gamma_3}$ *(defined on* $[0,\bar{\varepsilon}]$*), reads as*

$$|\dot{x}| \leq L_f(L_k + L_{\gamma_3} + 1)|x| + L_f L_k |e|. \tag{3.18}$$

In the next theorem we show that the sequence of triggering instants is a uniformly isolated set and hence there always exists a non-zero lower bound $\tau$ on the intersampling times. This feature guarantees the non-existence of accumulation points and is thus critical to the successful implementation of the proposed ETM.

**Theorem 3.2** *If the hypotheses of Theorem 3.1 hold, the inter sampling periods are lower bounded by some* $\tau \in \mathbb{R}^+$*, i.e.,* $t_i \geq t_{i-1} + \tau$ *for all* $i \in \mathbb{N}$*.*

Proof of Theorem 3.2 relies on Properties 3.1-3.2 outlined below.

**Property 3.1** *Function* $\psi^{-1}$ *defined in (3.11) is Lipschitz-continuous in any compact set* $\mathscr{D}_x \subset \mathbb{R}_0^+$*.*

**Proof.** Let us define $\varepsilon_m \doteq \max_{r\in\mathscr{D}_x}\{r\}$. Also let $L_{\sigma_0}$ and $L_{\bar{\sigma}^{-1}}$ be the Lipschitz constants of functions $\sigma_0$ and $\bar{\sigma}^{-1}$ on compact sets $\{\psi^{-1}(r) : r \in \mathscr{D}_x\} = [0, \psi^{-1}(\varepsilon_m)]$ and $\{\bar{\sigma}(\psi^{-1}(r)) : r \in \mathscr{D}_x\} = [0, \bar{\sigma}(\psi^{-1}(\varepsilon_m))]$, respectively. Using the fact that $\bar{\sigma}$ and $\sigma_0$ are class $\mathcal{K}_\infty$ functions, one can write

$$\begin{aligned}|\psi(r) - \psi(\tilde{r})| &= \Big|\frac{\bar{\sigma}(r)}{1+\sigma_0(r)} - \frac{\bar{\sigma}(\tilde{r})}{1+\sigma_0(\tilde{r})}\Big| \\ &\geq \Big|\frac{(1+\sigma_0(r))\Delta_{r,\tilde{r}}(\bar{\sigma}) - \bar{\sigma}(r)\Delta_{r,\tilde{r}}(\sigma_0)}{(1+\sigma_0(\varepsilon_m))^2}\Big| \\ &\geq \frac{(1+\sigma_0(r))|\Delta_{r,\tilde{r}}(\bar{\sigma})| - \bar{\sigma}(r)|\Delta_{r,\tilde{r}}(\sigma_0)|}{(1+\sigma_0(\varepsilon_m))^2}\end{aligned}$$

for any $r, \tilde{r} \in \{s : \psi(s) \in \mathscr{D}_x\}$, where functional $\Delta_{r,\tilde{r}}$ is defined as $\Delta_{r,\tilde{r}}(\varphi) \doteq \varphi(r) - \varphi(\tilde{r})$ for some function $\varphi$. The Lipschitz-continuity of functions $\sigma_0$ and $\bar{\sigma}^{-1}$ imply that $|\Delta_{r,\tilde{r}}(\sigma_0)|\leq L_{\sigma_0}|r-\tilde{r}|$ and $|r-\tilde{r}|\leq L_{\bar{\sigma}^{-1}}|\Delta_{r,\tilde{r}}(\bar{\sigma})|$ which together with Lemma 3.3 reduces the above inequality to

$$|\psi(r) - \psi(\tilde{r})| \geq \frac{L_{\bar{\sigma}^{-1}}^{-1} - L_{\sigma_0}\bar{\sigma}(\varepsilon_m)}{(1+\sigma_0(\varepsilon_m))^2}|r-\tilde{r}|. \tag{3.19}$$

■

**Property 3.2** *The function $\bar{\beta}_1(r)$ defined in (3.11) is of class $\mathcal{K}_\infty$ and locally Lipschitz-continuous in $\mathbb{R}_0^+$. Also if the Lipschitz constant of function $\beta_1$ is $L_{\beta_1}$ on some compact set $\mathscr{D}_e \subset \mathbb{R}_0^+$, then $L_{\bar{\beta}_1} = \max\{L_{\beta_1}, 1\}$ is the Lipschitz constant of $\bar{\beta}_1$ on this set.*

**Proof of Theorem 3.2.** From Lemma 3.3, we have $x(t) \in \mathscr{X}$ for all $t \in \mathbb{R}_0^+$. Now in view of Properties 3.1, 3.2 it can be inferred that function $\psi^{-1}(\bar{\beta}_1/c)$ is Lipschitz-continuous in any compact set in $\mathbb{R}_0^+$. Let us denote by $\bar{L}$ the Lipschitz constant of this function on set $\mathscr{D}_e$ defined as $\mathscr{D}_e = \{\bar{\beta}_1^{-1}(c\psi(s)) : s \in [0, \bar{\varepsilon}]\} = [0, \bar{\beta}_1^{-1}(c\psi(\bar{\varepsilon}))]$. Thus we have $\psi^{-1}(\bar{\beta}_1(|e|)/c) \leq \bar{L}|e|$ which suggests that a more conservative lower bound on inter-event times can be achieved when instead of (3.10), the next triggering of control task occurs when $\bar{L}|e|\geq |x|$. Following the same procedure as in ( [9], Theorem III.1), we can upper bound the dynamics of $y \doteq |e|/|x|$ as $\dot{y} \leq (1+y)|\dot{x}|/|x|$, which using (3.18) reads as

$$\dot{y} \leq L_f\Big(1+y\Big)\Big(L_k + L_{\gamma_3} + 1 + L_k y\Big). \tag{3.20}$$

Thus the inter-execution times are lower bounded by the solution $\tau$ of $y(\tau) = 1/\bar{L}$, where $y$ is the solution to

$$\dot{y} = L_f(1+y)(L + L_k y),\ y(0) = 0 \tag{3.21}$$

with $L = L_k + L_{\gamma_3} + 1$. It then follows that the lower bound on inter-event times is

$$0 < \tau = \frac{1}{L_f(L - L_k)} \ln\left(1 + \frac{L - L_k}{L\bar{L} + L_k}\right). \tag{3.22}$$

■

Proof of Theorem 3.2 implies that $\tau$ is a function of $L_f$, $L_k$, $L_{\gamma_3}$. Applying Lemma 3.3 and Remark 3.4, we conclude that these constants are defined on invariant sets and hence are valid for all initial conditions.

The proof of Property 3.1 suggests that function $\psi^{-1}$ is Lipschitz-continuous in $\mathscr{D}_x$ with Lipschitz constant $L_{\psi^{-1}} = (1+\sigma_0(\varepsilon_m))^2\{L_{\bar{\sigma}^{-1}}^{-1} - L_{\sigma_0}\bar{\sigma}(\varepsilon_m)\}^{-1}$, where $\varepsilon_m = \max_{r \in \mathscr{D}_x}\{r\}$ and $L_{\sigma_0} = L_f L_k L_{\sigma_3}$. To make sure $L_{\psi^{-1}}$ is positive, $\sigma_3$ which is the upper bound on the norm of Lyapunov function $W$, has to be chosen so that $L_f L_k L_{\sigma_3} L_{\bar{\sigma}^{-1}} \bar{\sigma}(\varepsilon_m) < 1$. This condition depends on the set $\mathscr{D}_x$. In design procedure, however, one can choose $\sigma_3$ such that

$$L_f L_k L_{\sigma_3} L_{\bar{\sigma}^{-1}} \bar{\sigma}(\bar{\varepsilon}) < 1, \tag{3.23}$$

where $\bar{\varepsilon}$ is defined in Lemma 3.3. To see this, let us assume that system starts from initial condition $x(0) = x_0$. Then in view of Lemma 3.3, we have $|x(t)| \leq \bar{\varepsilon}$ and hence for any compact set $\mathscr{D}_x \subseteq [0, \bar{\varepsilon}]$ we have $\varepsilon_m \leq \bar{\varepsilon}$. Thus since $\bar{\sigma}$ is a class $\mathcal{K}_\infty$ function, we will have $\bar{\sigma}(\varepsilon_m) \leq \bar{\sigma}(\bar{\varepsilon})$ and hence (3.23) ensures that $L_f L_k L_{\sigma_3} L_{\bar{\sigma}^{-1}} \bar{\sigma}(\varepsilon_m) < 1$.

We finish our discussions in this section by showing the global asymptotic stability property for the ETC system $\mathscr{G}_e$ in the absence of disturbances.

**Corollary 3.1** *Under the assumptions of Theorem 3.1, the zero-input ETC system $\mathscr{G}_e$ has a global asymptotically stable point at $0 \in \mathbb{R}^n$.*

**Proof.** From Remark 3.3 we conclude that $\beta_1(|e|) \leq c\bar{\sigma}(|x|)$ between triggering instants. Then assuming $w = 0$ and taking (3.6) into account, we can write $\nabla V(x) \cdot f(x, k(x+e), 0) \leq -(1-c)\bar{\sigma}(|x|) < 0$, *i.e.,* $x = 0$ is an asymptotically stable point for disturbance-free system $\mathscr{G}_e$. The above argument is global since from Assumption 3.2, $V$ is radially unbounded. ■

Although the $\mathcal{L}_2$-stability results are provided locally, the above result is global. Recall that the local character of the results arises from the restrictions placed on the input space. Thus, in the absence of disturbances, the result becomes global.

## 3.3 Improving Average Sampling Frequency

In this section, we are concerned with the problem of decreasing the average sampling rate for the proposed ETM in Section 3.2. Our solution consists of modifying the TC (3.10) of section 3.2 by adding an exponentially time decaying term to the right hand side. We show that following this idea, the ETC system enjoys the same $\mathcal{L}_p$-gain performance as in section 3.2, however, the zero-input system is stable in practical sense as opposed to asymptotically stable.

The results of this section can then be applied to limit high triggering during transient response. In a regulation problem, after an initial transition, the state remains near the equilibrium, possibly continuously excited by a disturbance or noise. Focusing on practical stability of such problems, where the state is required to enter a stability bound, it is reasonable to assume that the triggering frequency reduces when the transient response vanishes. Note that when the state is near the equilibrium, the control action is only required to keep the state within the desired bound. As a result, the system can be controlled with much less attention and hence the number of triggering instants drops significantly. Similar behaviour is expected when tackling regulation problems with non-persistent disturbances. In such case, while in transient both disturbance and the change in the state's norm affect sampling frequency, during steady state a lower triggering rate is expected due to the non-existence of disturbance.

We now state the main problem to be solved in this section. Note that we implicitly assume that the system experiences finite transition interval over which the sampling frequency exceeds a critical level. Without loss of generality, we assume that only one such interval exists. Generalization to several transition intervalsdiscrete is discussed later.

*Problem 1:* Modify the proposed TC (3.10) so that while the resulting ETC system is finite gain locally $\mathcal{L}_2$-stable with the same disturbance rejection bound $\Gamma$, the average sampling frequency does not exceed $f_{cr}$ at least for

(A) a desired period of time, *i.e.,* $t \in [0, \bar{T}]$.

(B) a desired number of triggerings, *i.e.,* $1 \leq i \leq N$.

### 3.3.1 Continuous Triggering Condition Scenario

We begin our study of Problem 1-(A) by modifying rule (3.10) as

$$t_{i+1}^{-} = \inf \Big\{ t \in \mathbb{R}_0^+ : t > t_i \bigwedge \bar{\beta}_1(|e(t)|) \geq c\tilde{\psi}(|x(t)|) \Big\}, \tag{3.24}$$

where $\tilde{\psi} \doteq \tilde{\sigma}(t,r)/(1+\sigma_0(r))$ and $\tilde{\sigma}$ is an exponentially time-decaying perturbation of function $\bar{\sigma}$ defined in Assumption 3.2, i.e.,

$$\tilde{\sigma}(t,r) \doteq \bar{\sigma}(r) + \frac{\kappa}{c} e^{-\zeta t}. \tag{3.25}$$

Also $\kappa$ and $\zeta$ are positive parameters to be designed.

**Stability Analysis**

The following theorem shows that the time-decaying perturbation of function $\bar{\sigma}$ introduces a non-zero bias term (see Definition 2.9) but does not affect the $\mathcal{L}_2$ bound with respect to the input.

**Theorem 3.3** *Under the hypotheses of Theorem 3.1, the system $\mathscr{G}_e$ is finite gain locally $\mathcal{L}_2$-stable and has $\|\mathscr{G}_e\|_{\mathcal{L}_2} \leq \Gamma$ if the control signal is updated under the execution rule (3.24).*

**Proof.** Following similar lines as in the proof of Theorem 3.1, we can upper bound $\dot{U}$ as

$$\dot{U}(x) \leq -(1-c)\bar{\sigma}(|x|) + \kappa e^{-\zeta t} + \Gamma^2|w|^2 - |h(x,w)|^2 \tag{3.26}$$

for any $x \in \mathbb{R}^n$, any $e \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $c\tilde{\psi}(|x|) \geq \bar{\beta}_1(|e|)$. Integrating (3.26) from 0 to $T \in \mathbb{R}^+$ and using the positive definiteness of $U$ we obtain $\int_0^T |h(x(t),w(t))|^2 dt \leq \Gamma^2 \int_0^T |w(t)|^2 ds + \kappa(1-e^{-\zeta T})/\zeta + U(x_0)$, which by applying Definition 2.9 with $\eta = \kappa/\zeta$ and $\mu = U$, completes the proof. ■

**Remark 3.5** *It can be readily inferred from the proof of Theorem 3.3 that the exponential time decaying term in (3.24) does not affect the finite gain local $\mathcal{L}_2$-stability of the ETC system $\mathscr{G}_e$ as its integral from 0 to any $T \in \mathbb{R}^+$ is finite, independent of $T$ and hence can be considered as the bias term $\eta$ in Definition 2.9.*

**Corollary 3.2** *Under the assumptions of Theorem 3.3, trajectories of the system $\mathscr{G}_e$ converge to $0 \in \mathbb{R}^n$.*

**Proof.** It can be inferred from execution rule (3.24) that between successive triggering instants we have $\beta_1(|e|) \leq \bar{\beta}_1(|e|) \leq c\bar{\sigma}(|x|) + \kappa e^{-\zeta t}$. Then from (3.6), we can upper bound $\dot{V}$ as

$$\dot{V}(x) \leq -(1-c)\bar{\sigma}(|x|) + \kappa e^{-\zeta t}. \tag{3.27}$$

Defining $\bar{c} = 1-c$, we conclude that $\dot{V}(x) < 0$ for $|x| > \bar{\sigma}^{-1}(\kappa e^{-\zeta t}/\bar{c})$. Now define compact set $\Lambda_i = \{x \in \mathbb{R}^n : |x| \leq \bar{\sigma}^{-1}(\kappa e^{-\zeta i}/\bar{c})\}$ for $i \in \mathbb{N}_0$. Also the set of boundary points of $\Lambda_i$ is defined as $\partial\Lambda_i = \{x \in \mathbb{R}^n : |x| = \bar{\sigma}^{-1}(\kappa e^{-\zeta i}/\bar{c})\}$ for $i \in \mathbb{N}_0$. We denote by $m_i$ the argument of maximum value of $V(x)$ over the set $\partial\Lambda_i$, *i.e.*, $m_i = \arg\max_{x\in\partial\Lambda_i} V(x)$. Next define compact set $\Omega_i \doteq \{x \in \mathbb{R}^n : V(x) \leq V(m_i)\}$ for $i \in \mathbb{N}_0$. Clearly $\Omega_i$ is positive invariant under the dynamics of ETC system $\mathscr{G}_e$ for $t \geq i$. We claim that $\Omega_i$ is the global attracting set of system $\mathscr{G}_e$ for $t \geq i$. To see this, let us define the complement of $\Omega_i$ in $\mathbb{R}^n$ as $\Omega_i^c = \{x \in \mathbb{R}^n : V(x) > V(m_i)\}$. If $x \in \Omega_i^c$, we conclude that $|x| > m_i$ and since $m_i \in \partial\Lambda_i$ we deduce that $|x| > \bar{\sigma}^{-1}(\kappa e^{-\zeta i}/\bar{c})$. Then since $t \geq i$ it follows that $|x| > \bar{\sigma}^{-1}(\kappa e^{-\zeta t}/\bar{c})$ and consequently $\dot{V}(x) < 0$ for all $x \in \Omega_i^c$ which confirms our claim. For $t \geq i+1$, however, $\Omega_{i+1} \subset \Omega_i$ is the new global attracting set of the ETC system $\mathscr{G}_e$. Thus the sequence of positive invariant attracting sets $\{\Omega_i\}_{i\in\mathbb{N}_0}$ with $\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_i \supset \cdots$ shrinks to the origin as $i \to \infty$ (since $m_i$ converges to 0) which confirms the convergence of trajectories of system $\mathscr{G}_e$ to the origin. ■

**Remark 3.6** *Note that Corollary 3.2 proves that trajectories converge to the origin, but* <u>*does not*</u> *imply that the origin is asymptotically stable for the disturbance-free system. Asymptotic stability does not follow from this corollary since the result falls short of proving* stability *of the origin of the zero-input ETC system $\mathscr{G}_e$. This situation may occur, for example, when the trajectories of the zero-input system that start from certain neighbourhood of the origin, diverge from origin temporarily, but finally converge to it. In such situations, the system may still be finite gain locally $\mathcal{L}_2$-stable, however, the zero-input system is not necessarily stable since there exist neighbourhoods of the origin such that any trajectory starting there, can not stay there forever. This happens for system $\mathscr{G}_e$ under TC (3.24) as the proof of Corollary 3.2 suggests that in the absence of disturbances we have $\dot{V}(x) \geq 0$ for $|x(t)| \leq \bar{\sigma}^{-1}(\kappa e^{-\zeta t}/(1-c))$, i.e., trajectories starting within this bound diverge from origin at first but finally converge as the area of positive $\dot{V}$ shrinks to zero.*

The analysis, however, can be extended a bit further than the classical notion of stability. Indeed, we now show that in the absence of disturbances, the ETC system $\mathscr{G}_e$ is practically stable in the sense of following definition cited from [71]:

**Definition 3.1** *Given $\varsigma > \rho \in \mathbb{R}_0^+$, the origin of the system $\dot{x} = f(x,t)$ is $(\varsigma \to \rho)$-stable if*

*(a) for any $\epsilon > \rho$ there exists $\delta(\epsilon) \in \mathbb{R}^+$ such that if $|x_0| \leq \delta(\epsilon)$, then $|x(t)| < \epsilon$ for all $t \in \mathbb{R}_0^+$,*

*(b) for a given $r \in (0,\varsigma)$ there exists a finite $\upsilon(r) \in \mathbb{R}^+$ such that if $|x_0| \leq r$, then $|x(t)| < \upsilon(r)$ for all $t \in \mathbb{R}_0^+$,*

*(c) for a given $r \in (0,\varsigma)$ and $\epsilon > \rho$ there exists a finite $T(r,\epsilon) \in \mathbb{R}^+$ such that if $|x_0| \leq r$, then $|x(t)| < \epsilon$ for all $t \geq T(r,\epsilon)$.*

If we set $\varsigma = \infty$ and $\rho = 0$ in the above definition, we obtain the familiar uniform global asymptotic stability, ( [71], Remark 2.1). it is worth mentioning that the stability in the above-mentioned $\rho$-practical sense guarantees convergence of trajectories of the system $\dot{x} = f(x,t)$ to the set $\{x \in \mathbb{R}^n : |x| \leq \rho\}$ through condition 3.1 in Definition 3.1. The converse, however, is not generally true.

**Theorem 3.4** *Under the assumptions of Theorem 3.3, the zero-input ETC system $\mathscr{G}_e$ is $(\infty \to \bar{\sigma}^{-1}(\kappa/(1-c)))$-stable.*

**Proof.** First we apply (3.27) to conclude that $\dot{V}(x) \leq -\bar{c}\bar{\sigma}(|x|) + \kappa$ and hence $\dot{V}(x) < 0$ for $|x| > \bar{\sigma}^{-1}(\kappa/\bar{c})$, where $\bar{c} = 1 - c$. Then we just need to show conditions (a)-(c) in Definition 3.1 hold. To satisfy condition (a) we can choose $\delta(\epsilon)$ such that $0 < \delta(\epsilon) < \epsilon$. For condition (b) one can choose $\upsilon(r) = \bar{\sigma}^{-1}(\kappa/\bar{c})$ for $r \leq \bar{\sigma}^{-1}(\kappa/\bar{c})$ and some $\upsilon(r) > r$ for $r > \bar{\sigma}^{-1}(\kappa/\bar{c})$. Finally, to satisfy condition (c) we consider two cases. For $r \leq \bar{\sigma}^{-1}(\kappa/\bar{c})$ we can choose $T(r,\epsilon)$ to be any positive number since trajectories of the system $\mathscr{G}_e$ do not leave the ball $\{x \in \mathbb{R}^n : |x| \leq \bar{\sigma}^{-1}(\kappa/\bar{c})\}$ and hence $|x(t)| < \epsilon$ for all $t \geq 0$ and all $\epsilon > \bar{\sigma}^{-1}(\kappa/\bar{c})$. However, for $r > \bar{\sigma}^{-1}(\kappa/\bar{c})$ we need a more detailed argument. Let us choose $T'$ such that $|x(T')| = \epsilon$ and integrate (3.27) from 0 to $T'$ to obtain $V(\epsilon) - V(x_0) \leq -\bar{c}\int_0^{T'} \bar{\sigma}(|x(t)|)dt + \kappa\int_0^{T'} e^{-\zeta t}dt \leq -\bar{c}T'\bar{\sigma}(|\epsilon|) + \frac{\kappa}{\zeta}(1 - e^{-\zeta T'})$. Since $V(\epsilon) - V(r) \leq V(\epsilon) - V(x_0)$ we can upperbound $T'$ as the solution to inequality $\bar{c}T'\bar{\sigma}(|\epsilon|) + \kappa e^{-\zeta T'} \leq V(r) - V(\epsilon) + \kappa$. One can find a more conservative upper bound on $T'$ by neglecting the exponential term in left hand side, *i.e.*, $\bar{c}T'\bar{\sigma}(|\epsilon|) \leq V(r) - V(\epsilon) + \kappa - \kappa e^{-\zeta T'} \leq V(r) - V(\epsilon) + \kappa$ and obtain $T' \leq (V(r) - V(\epsilon) + \kappa)/(\bar{c}\bar{\sigma}(|\epsilon|))$. This is exactly what if we integrate the more conservative inequality $\dot{V}(x) \leq -\bar{c}\bar{\sigma}(|x|) + \kappa$ instead. Choosing $T(r,\epsilon) > (V(r) - V(\epsilon) + \kappa)/(\bar{c}\bar{\sigma}(|\epsilon|))$ completes the proof. ■

### Inter-Event Lower Bound Comparison

We recall from Theorem 3.2 that the lower bound on intersampling periods of ETC system $\mathscr{G}_e$ under execution rule (3.10) is $\tau$ and given in (3.22). Also by $\tau_1$ we denote the lower bound on intersampling periods of this system under execution rule (3.24). We show that one can design parameters $\kappa$ and $\zeta$ in (3.24) such that for a given $\bar{T} > 0$ and $\tau^* > f_{cr}^{-1}$, we have $\tau_1 \geq \tau + \tau^*$ at least for $t \in [0, \bar{T}]$. This guarantees that the average sampling frequency is less than $f_{cr}$ for $t \in [0, \bar{T}]$. To this end, defining $\bar{\kappa} = \kappa/(1 + \sigma_0(\bar{\varepsilon}))$, we assume the updation of the control task is decided based on the following TC

$$\bar{\beta}_1(|e|) \geq c\psi(|x|) + \bar{\kappa}e^{-\zeta t} \tag{3.28}$$

which is more conservative than the one proposed in (3.24) and hence gives a lower bound on $\tau_1$. Let $L_{\psi^{-1}}$ and $L_{\bar{\beta}_1}$ be the Lipschitz constants of functions $\psi^{-1}$ and $\bar{\beta}_1$, respectively. A more conservative TC than (3.28) can be obtained from $L_{\bar{\beta}_1}|e|\geq cL^{-1}_{\psi^{-1}}|x|+\bar{\kappa}e^{-\zeta t}$. In fact, if this condition is not satisfied, we have

$$\bar{\beta}_1(|e|) \leq L_{\bar{\beta}_1}|e|< cL^{-1}_{\psi^{-1}}|x|+\bar{\kappa}e^{-\zeta t} \leq c\psi(|x|) + \bar{\kappa}e^{-\zeta t} \tag{3.29}$$

and hence (3.28) will not be satisfied too. This TC restricts measurement error $e$ to satisfy

$$cL^{-1}_{\psi^{-1}}|x|(\hat{L}\frac{|e|}{|x|} - 1) \leq \bar{\kappa}e^{-\zeta t}, \tag{3.30}$$

where $\hat{L} = c^{-1}L_{\psi^{-1}}L_{\bar{\beta}_1}$. We remark that $\hat{L} \geq \bar{L}$, where $\bar{L}$ is the Lipschitz constant of function $\psi^{-1}(\bar{\beta}_1/c)$. From the proof of Theorem 3.2 it follows that $|e|/|x| \geq 1/\bar{L}$ shortly before the execution instant $t_i$ and hence we have $\hat{L}|e(t_i^-)|/|x(t_i^-)| > 1$. We can even express the TC more conservatively, by virtue of Lemma 3.3, so that the control signal is updated at sampling instant $t_i$ when the following condition is satisfied

$$c\bar{\varepsilon}L^{-1}_{\psi^{-1}}(\hat{L}\frac{|e(t_i^-)|}{|x(t_i^-)|} - 1) \geq \bar{\kappa}e^{-\zeta t_i^-}. \tag{3.31}$$

We now define $L^*$ so that $y(\tau + \tau^*) = 1/L^*$ where $y$ is the solution to (3.21). Thus our aim is to design $\kappa$ and $\zeta$ such that the solution $|e(t_i^-)|/|x(t_i^-)|$ to inequality (3.31) satisfy $L^*|e(t_i^-)|\geq |x(t_i)|$ for all execution instants $t_i \leq \bar{T}$, $i \in \mathbb{R}_0^+$, *i.e.*, until $t = \bar{T}$ the intersampling intervals are lower bounded by the solution $\tau_1$ of $y(\tau_1) = 1/L^*$. This means that the lower bound on inter-event times increases to $\tau_1 \geq \tau + \tau^*$ at least until instant $t = \bar{T}$. Finally it suffices to choose $\kappa$ and $\zeta$ so that

$$\kappa = c\bar{\varepsilon}L^{-1}_{\psi^{-1}}(\frac{\hat{L}}{L^*} - 1)(1 + \sigma_0(\bar{\varepsilon}))e^{\zeta\bar{T}}. \tag{3.32}$$

Then the lower bounds on intersampling periods are the solutions $\tau_1$ and $\tau$ to

$$\begin{cases} y(\tau_1) = \frac{1}{L^*}, & \text{for } 0 \leq t \leq \bar{T} \\ y(\tau) = \frac{1}{\bar{L}}, & \text{for } t > \bar{T} \end{cases} \tag{3.33}$$

**Remark 3.7** *Our result in section 3.3.1 is far more general than that of ( [9], Theorem III.1, when the delay between state measurement and actuator updating is nonzero). In [9], it is shown that the lower bound on intersampling times, $\tau$, can be extended (due to the time required to read state measurement, compute the control signal and update actuators) to the solution $\tau'$ of $y(\tau') = 1/\bar{L}'$, where $\bar{L}'$ is the Lipschitz constant of function $\psi^{-1}(\bar{\beta}_1/c')$*

*on compact set $\mathscr{D}_e$ defined in the proof of Theorem 3.2, where $c' \in (c, 1)$. Following this approach, the lower bound on intersampling intervals is restricted (through the upper bound limit on $c'$) to $\tau' < \tau_{\max}$, where $\tau_{\max}$ is the solution to $y(\tau_{\max}) = 1/\bar{L}_{\min}$ with $\bar{L}_{\min}$ as the Lipschitz constant of function $\psi^{-1}(\bar{\beta}_1)$. This limitation, however, is relaxed in our proposed method by introducing the exponentially decaying term $\kappa e^{-\zeta t}$ which allows taking $L^*$ smaller than $\bar{L}_{\min}$.*

### 3.3.2 Discrete Triggering Condition Scenario

In this section we address Problem 1-(B). In section 3.3.1 we showed that an exponentially time decaying term added to execution rule (3.10) enables us to affect average sampling frequency arbitrarily at least for the period $[0, \bar{T}]$. Here, we address the problem of improving the average sampling frequency for the first $N$ iterative triggerings of the control task using a discrete version of the TC (3.24). Let $i \in \mathbb{N}$ denote the number of triggerings completed up until time $t$ assuming the first triggering occurs at $t_0 = 0$. As a consequence, $t_i$ and $t_{i-1}$ denote the upcoming and the most recent execution instants, respectively. We denote by $t'_i > t_{i-1}$, $i \in \mathbb{N}$, just a moment after the following so called discrete TC holds

$$t_i'^{-} = \inf\left\{t \in \mathbb{R}_0^+ : t > t_{i-1} \bigwedge \bar{\beta}_1(|e(t)|) \geq c\check{\psi}(|x(t)|)\right\}, \tag{3.34}$$

where $\check{\psi} \doteq \check{\sigma}(t, r)/(1 + \sigma_0(r))$ and $\check{\sigma}$ is a discrete decaying perturbation of $\bar{\sigma}$ defined in Assumption 3.2 defined as

$$\check{\sigma}(t, r) \doteq \bar{\sigma}(r) + \frac{\hat{\kappa} e^{\theta^i}}{ci!}, \tag{3.35}$$

where $\hat{\kappa}$ and $\theta$ are positive parameters to be designed. We refer to (3.34) as the discrete TC as it depends on index $i$ which changes non-continuously between successive triggerings.

Now suppose that the $i$-th execution of the control task happens at

$$t_i = \min\{t_{i-1} + \Delta, t'_i\}, \tag{3.36}$$

where $\Delta \in \mathbb{R}^+$ is an upper bound on intersampling intervals. The following theorem then states that discrete decaying perturbation of function $\bar{\sigma}$ given in (3.35) satisfies the same local $\mathcal{L}_2$-gain bound for the ETC system.

**Theorem 3.5** *Under the hypotheses of Theorem 3.1, the system $\mathscr{G}_e$ is finite gain $\mathcal{L}_2$-stable and has $\|\mathscr{G}_e\|_{\mathcal{L}_2} \leq \Gamma$ if the control signal is updated at triggering instants $\{t_i : i \in \mathbb{N}\}$ defined in (3.36).*

**Proof.** It can be inferred from (3.36) that $t_i \leq t'_i$ and hence we have $\bar{\beta}_1(|e(t)|) \leq c\check{\psi}(|x(t)|)$ for $t \in [t_{i-1}, t_i)$. Then following similar lines as the proof of Theorem 3.1, for $t \in [t_{i-1}, t_i)$ we obtain $\dot{U}(x) \leq -(1-c)\bar{\sigma}(|x|) + \hat{\kappa}\theta^i/i! + \Gamma^2|w|^2 - |h(x,w)|^2$ for any $x \in \mathbb{R}^n$, any $e \in \mathbb{R}^n$ and any $w \in \mathscr{W}_Q$ such that $c\check{\psi}(|x|) \geq \bar{\beta}_1(|e|)$. Integrating this inequality from 0 to some $T \geq 0$, we arrive at

$$U(x(T)) \leq U(x_0) + \int_0^T (\Gamma^2|w(t)|^2 - |h(x(t), w(t))|^2)dt$$
$$+\hat{\kappa}\Big\{\int_{t_0=0}^{t_1} \frac{\theta^1}{1!}dt + \cdots + \int_{t_{i-1}}^{t_i} \frac{\theta^i}{i!}dt + \cdots + \int_{t_{N-1}}^{T} \frac{\theta^N}{N!}dt\Big\},$$

where we assume $N$ triggering instants (including the first one at $t_0 = 0$) occur until $t = T$, *i.e.*, $t_{N-1} = \max_{t_i \leq T}\{t_i\}$. Now since $U(x(T)) \geq 0$ we conclude that $\int_0^T |h(x(t), w(t)|^2 dt \leq U(x_0) + \Gamma^2 \int_0^T |w(t)|^2 dt + \hat{\kappa} \max\limits_{1 \leq i \leq N}\{t_i - t_{i-1}\} \sum_{i=1}^N \theta^i/i!$ and hence

$$\int_0^T |h(x(t), w(t)|^2 dt \leq U(x_0) + \Gamma^2 \int_0^T |w(t)|^2 dt + \hat{\kappa}\Delta e^{\theta}. \tag{3.37}$$

We then choose $\eta = \hat{\kappa}\Delta e^{\theta}$ and $\mu = U$ in Definition 2.9 to obtain the desired result. ■

**Remark 3.8** *The $\Delta$ term in (3.36) imposes an upper bound on inter-event times. This restriction on intersampling intervals is necessary as it confirms the finiteness of the bias term in (3.37).*

**Corollary 3.3** *Under the assumptions of Theorem 3.5, trajectories of the ETC system $\mathscr{G}_e$ converge to $0 \in \mathbb{R}^n$.*

**Proof.** Following similar lines as the proof of Corollary 3.2 we deduce that $\beta_1(|e|) \leq c\bar{\sigma}(|x|) + \hat{\kappa}\theta^i/i!$ for $t \in [t_{i-1}, t_i)$ and hence from (3.6) it follows that

$$\dot{V} \leq -\bar{c}\bar{\sigma}(|x|) + \hat{\kappa}\frac{\theta^i}{i!} \tag{3.38}$$

for $t \in [t_{i-1}, t_i)$. As a consequence we conclude that $\dot{V}(x) < 0$ for $|x| \geq \bar{\sigma}^{-1}(\hat{\kappa}\theta^i/(\bar{c}i!))$ and $t \in [t_{i-1}, t_i)$. Now define compact set $\Lambda_i = \{x \in \mathbb{R}^n : |x| \leq \bar{\sigma}^{-1}(\hat{\kappa}\theta^i/(\bar{c}i!))\}$ for $i \in \mathbb{N}$. Then the set of boundary points of $\Lambda_i$ can be defined as $\partial\Lambda_i = \{x \in \mathbb{R}^n : |x| = \bar{\sigma}^{-1}(\hat{\kappa}\theta^i/(\bar{c}i!))\}$. We denote the argument of maximum value of $V(x)$ over set $\partial\Lambda_i$ by $m_i = \arg\max_{x \in \partial\Lambda_i} V(x)$. We remark that the discrete function $\theta^i/i!$ takes its maximum value at $i = \lfloor\theta\rfloor$ and is strictly decreasing over $i \geq \lfloor\theta\rfloor$. Now define compact set $\Omega_i = \{x \in \mathbb{R}^n : V(x) \leq V(m_i)\}$ for $i \in \mathbb{N}$. Following similar lines as the proof of Corollary 3.2, we can show that for $i \geq \lfloor\theta\rfloor$, $\Omega_i$ is positive invariant under dynamics of the ETC system $\mathscr{G}_e$ and moreover, is the global attracting set of this system for $t \geq t_i$. Since $t_i - t_{i-1} \leq \Delta$, we conclude that $i \to \infty$ as $t \to \infty$, *i.e.*, the triggering instants never terminate. Thus the sequence of positive invariant attracting sets $\{\Lambda_i\}_{\lfloor\theta\rfloor \leq i \in \mathbb{N}}$ with $\Lambda_{\lfloor\theta\rfloor} \supset \Lambda_{\lfloor\theta\rfloor+1} \supset \cdots \supset \Lambda_i \supset \cdots$ shrinks to the origin and hence completes the proof. ■

**Theorem 3.6** *Under the assumptions of Theorem 3.5 and in the absence of disturbances, the ETC system* $\mathscr{G}_e$ *is* $(\infty \to \bar{\sigma}^{-1}(\hat{\kappa}_\theta/(1-c)))$*-stable, where* $\hat{\kappa}_\theta \doteq \hat{\kappa}\theta^{\lfloor\theta\rfloor}/\lfloor\theta\rfloor!$.

**Proof.** In view of (3.38) which is valid in $[t_{i-1}, t_i)$, we conclude that $\dot{V}(x) \leq -\bar{c}\bar{\sigma}(|x|) + \hat{\kappa}_\theta$ for all $t \in \mathbb{R}_0^+$. Hence we have $\dot{V}(x) < 0$ for $|x| > \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$. Then we just need to show conditions (a)-(c) in Definition 3.1 hold. To satisfy condition (a) we can choose $\delta(\epsilon)$ such that $0 < \delta(\epsilon) < \epsilon$. To satisfy condition (b) we can choose $\upsilon(r) = \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$ for $r \leq \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$ and $\upsilon(r) > r$ for $r > \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$. For condition (c) we consider two cases. For $r \leq \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$ we can choose $T(r,\epsilon)$ to be any positive number since the trajectories do not leave the ball $\{x \in \mathbb{R}^n : |x| \leq \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})\}$ and hence $|x(t)| < \epsilon$ for all $t \geq 0$ and all $\epsilon > \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$. For $r > \bar{\sigma}^{-1}(\hat{\kappa}_\theta/\bar{c})$, choose $T'$ such that $|x(T')| = \epsilon$. Then integrating (3.38) from 0 to $T'$ gives

$$V(\epsilon) - V(x_0) \leq -\bar{c}\int_0^{T'} \bar{\sigma}(|x(t)|)dt + \hat{\kappa}\Big\{\int_{t_0=0}^{t_1} \frac{\theta^1}{1!}dt + \cdots + \int_{t_{i-1}}^{t_i} \frac{\theta^i}{i!}dt + \cdots + \int_{t_{N'-1}}^{T'} \frac{\theta^{N'}}{N'!}dt\Big\}.$$

Hence we have $V(\epsilon) - V(x_0) \leq -\bar{c}T'\bar{\sigma}(|\epsilon|) + \max_{1\leq i\leq N'}\{t_i - t_{i-1}\}\hat{\kappa}\sum_{i=1}^{N'} \frac{\theta^i}{i!} \leq -\bar{c}T'\bar{\sigma}(|\epsilon|) + \max_{1\leq i\leq N'}\{t_i - t_{i-1}\}\hat{\kappa}e^\theta$, where we assume $N'$ triggering instants (including the first one at $t_0 = 0$) occur until $t = T'$, *i.e.*, $t_{N'-1} = \max_{t_i \leq T'}\{t_i\}$. Then we can find an upper bound on $T'$ as the solution to inequality $\bar{c}T'\bar{\sigma}(|\epsilon|) \leq V(r) - V(\epsilon) + \max_{1\leq i\leq N'}\{t_i - t_{i-1}\}\hat{\kappa}e^\theta$ since $V(\epsilon) - V(r) \leq V(\epsilon) - V(x_0)$. We remark that $\max_{1\leq i\leq N'}\{t_i - t_{i-1}\}$ is a function of $\epsilon$ since $N'$ depends on $T'$ which is a function of $\epsilon$. Then one can choose $T(r,\epsilon)$ so that $T(r,\epsilon) > (V(r) - V(\epsilon) + \max_{1\leq i\leq N'}\{t_i - t_{i-1}\}\hat{\kappa}e^\theta)/(\bar{c}\bar{\sigma}(|\epsilon|))$. ■

In the rest of this section, we provide a discrete counterpart to the analysis given in section 3.3.1. Indeed, we design $\hat{\kappa}$ and $\theta$ in (3.34) so that given some $N \in \mathbb{N}$ and $\tau^* \in \mathbb{R}^+$, we have $\tau_2 \geq \tau + \tau^*$ at least for $t \in [0, \bar{T}]$, where $\tau_2$ denotes the lower bound on intersampling periods of system $\mathscr{G}_e$ under execution rule (3.34).

Choosing $\Delta > \tau + \tau^*$ in (3.36), it remains to consider the case where $t_i = t_i'$ and hence $t_i$ satisfy the TC (3.34). Even a more conservative TC can be obtained if the $i$-th execution of control task is fulfilled when the following holds

$$\bar{\beta}_1(|e|) \geq c\psi(|x|) + \tilde{\kappa}\frac{\theta^i}{i!}, \tag{3.39}$$

where $\tilde{\kappa} = \hat{\kappa}/(1+\sigma_0(\bar{\varepsilon}))$. Now using the same procedures as (3.29) and (3.30) were derived, we obtain a discrete version of TC (3.31):

$$c\bar{\varepsilon}L^{-1}_{\psi^{-1}}(\hat{L}\frac{|e(t_i^-)|}{|x(t_i^-)|} - 1) \geq \tilde{\kappa}\frac{\theta^i}{i!}. \tag{3.40}$$

Our goal is to design $\hat{\kappa}$ and $\theta$ such that the solution $|e(t_i^-)|/|x(t_i^-)|$ to the above inequality satisfies $L^*|e(t_i)| \geq |x(t_i)|$ for the first $N$ triggerings, where $L^*$ is defined such that $y(\tau+\tau^*) = 1/L^*$ and $y$ is the solution to (3.21). We now consider two cases.

Case 1: If $1 \leq N \leq N_\theta \doteq \max\{i : \theta^i/i! \geq \theta\}$ we have $\min_{1\leq i\leq N}\{\theta^i/i!\} = \theta$, *i.e.*, the discrete function $\theta^i/i!$ takes its minimum value at $i=1$, and hence we can choose $\hat{\kappa}$ and $\theta$ so that

$$\hat{\kappa} = \frac{c\bar{\varepsilon}}{\theta}L^{-1}_{\psi^{-1}}(\frac{\hat{L}}{L^*} - 1)(1+\sigma_0(\bar{\varepsilon})). \tag{3.41}$$

That is, for any $1 \leq N \leq N_\theta$, the first $N_\theta$ inter-event intervals are lower bounded by the solution $\tau_2$ of $y(\tau_2) = 1/L^*$.

Case 2: For $N > N_\theta$ we have $\min_{1\leq i\leq N}\{\theta^i/i!\} = \theta^N/N!$ and we can pick $\hat{\kappa}$ and $\theta$ such that

$$\hat{\kappa} = c\bar{\varepsilon}L^{-1}_{\psi^{-1}}(\frac{\hat{L}}{L^*} - 1)(1+\sigma_0(\bar{\varepsilon}))\frac{N!}{\theta^N}, \tag{3.42}$$

*i.e.*, for the first $N$ samplings, the inter-event times are lower bounded by the solution $\tau_2$ of $y(\tau_2) = 1/L^*$. Therefore, the lower bounds on intersampling periods are the solutions $\tau_2$ and $\tau$ to

$$\begin{cases} \left.\begin{matrix} y(\tau_2) = \frac{1}{L^*}, & \text{for } 0 \leq i \leq N_\theta \\ y(\tau) = \frac{1}{L}, & \text{for } i > N_\theta \end{matrix}\right\} \text{for } 1 \leq N \leq N_\theta \\ \\ \left.\begin{matrix} y(\tau_2) = \frac{1}{L^*}, & \text{for } 0 \leq i \leq N \\ y(\tau) = \frac{1}{L}, & \text{for } i > N \end{matrix}\right\} \text{for } N > N_\theta. \end{cases} \tag{3.43}$$

Note that while the continuous and discrete scenarios proposed in this section have similarities, they have different structures that lead to different properties. The primary difference between these methods is that while in continuous-time the decaying term is a function of time and will vanish as $t$ grows, this is not the case in discrete scenario. The decaying term in discrete approach is a function of the sampling instant and not time. Thus, if only a few triggering instants occur, the effect of perturbation term may be considerable, regardless of the time that has passed. This important feature of discrete scenario can be seen from the examples provided in next section and shows that in contrast to continuous counterpart, the decaying term may still be kept effective for a much longer time.

## 3.4 Illustrative Examples

In this section we illustrate the $\mathcal{L}_2$-stabilizing triggering design through several examples. Our examples are simple enough so that the $\mathcal{L}_2$-gain analysis can be done analytically, thus enabling us to provide further insight. We show in Example 3.2 that if some of the conditions of Theorem 3.1 are not satisfied, it may still be possible to relax these conditions by redefining TC. In Example 3.3, we replace the Euclidean vector norm with the infinity norm to obtain the $\mathcal{L}_2$-gain. This is important since this change facilitates the computation of the Lyapunov function.

We continue with the following remarks, containing important points regarding the simulations.

**Remark 3.9** *The examples are constructed according to our design principle, i.e. performance is defined in $\mathcal{L}_2$-sense and the design is such that preserves the $\mathcal{L}_2$ gain of the continuous-time design. In this approach, we have purposely ignored transient behaviour and pushed the design to the extreme to save communications during transient, something that should, of course, be corrected in a more realistic design. The simulations indeed show a deterioration of the transient response. This should be interpreted as indicative that, in general, $\mathcal{L}_2$ performance does not, in any way, imply good transient behaviour.*

**Remark 3.10** *Note that the plots for verification of $\mathcal{L}_2$-gain and system's trajectories are provided for one single initial condition. However, the discussion on number of samples and MIETs are provided based on averaging* 100 *initial conditions. Thus, one should be careful that since the $\mathcal{L}_2$-gain plots depend on initial condition, no general conclusion (such as comparing the $\mathcal{L}_2$-gain of continuous-time and ETC systems) other than verification of the proposed $\mathcal{L}_2$-gain for different scenarios can be made from them.*

**Example 3.1** *Consider the following first order system*

$$\dot{x} = -x^3 + xw + u, \quad z = x, \tag{3.44}$$

*where $x \in \mathbb{R}$, $u = -k(x+e)$ for some $k \in \mathbb{R}^+$ is the control input, $e$ is the measurement error and $w$ is the exogenous disturbance belongs to the set $\mathscr{W}_Q$ defined in (2.3). Choosing the Lyapunov function $V(x) = x^2/2$ it is straight forward to show that the system is ISS with respect to $e$ and $w$. Assuming $e$ to be zero all the time, the continuous-time system is finite gain locally $\mathcal{L}_2$-stable. To show this, we take $W(x) = \lambda V(x)$ for some $\lambda \in \mathbb{R}^+$. Now since $\dot{V}(x) = -kx^2 - x^4 + x^2 w \leq -kx^2 + (\gamma_1 - 1)x^4 + w^2/(4\gamma_1)$, where $\gamma_1 \in \mathbb{R}^+$,* *we will have* $\dot{W}(x) \leq -\lambda k x^2 + \lambda w^2/(4\gamma_1)$ *for* $\gamma_1 \leq 1$. *As a consequence, the minimum upper bound on the* $\mathcal{L}_2$*-gain of system (3.44) is* $1/(2\sqrt{\gamma_1 k})$ *(when* $\gamma_1 = 1$*). Finally by choosing* $U = V + W$ *we have* $\dot{U}(x) \leq \lambda w^2/4 - \lambda k z^2 - k x^2 + w^2/4 + (1+\lambda)k|x||e|$, *which by restricting* $e$ *and* $w$ *to satisfy* $|e| \leq c|x|/(1+\lambda)$ *and* $|w(t)| \leq \gamma_3(|x(t)|) = 2\sqrt{\bar{c}k}|x(t)|$, *reads as* $\dot{U}(x) \leq \lambda w^2/4 - \lambda k z^2 - (1 - c - \bar{c})k x^2$. *Thus we can design* $c$ *and* $\bar{c}$ *so that* $c + \bar{c} < 1$ *and hence ensure that the ETC system is finite gain* $\mathcal{L}_2$*-stable. Also it is not difficult to verify that* $\dot{U}(x) < 0$ *and hence* $|x|$ *monotonically converges to zero. This enables us to find* $Q$ *assuming* $x_0 \in \mathscr{X}_0$, *i.e.,* $|x_0| \leq \varepsilon$. *Indeed, we can write* $|w(t)| \leq 2\sqrt{\bar{c}k}|x(t)| \leq 2\sqrt{\bar{c}k}|x_0|$. *Hence taking* $Q = 2\sqrt{\bar{c}k}\varepsilon$ *guarantees* $w(t) \in \mathscr{W}_Q$.

*We continue the discussion carried out above, numerically. Taking* $k = 1$, $\varepsilon = 1$, $c = 0.5$, $\bar{c} = 0.45$, $Q = 1.34$, $\lambda = 0.5$, $\kappa = 15$, $\zeta = 1.6$, $\hat{\kappa} = 1.5$, $\theta = 1$ *and* $\Delta = 1.1$, *we arrive at the execution rule* $|e| = |x|/3$. *Consequently we have* $\dot{U}(x) \leq |w|^2/8 - |z|^2/2$, *where* $U(x) = 3x^2/4$. *It then follows that the ETC system is finite gain* $\mathcal{L}_2$*-stable with zero bias and has* $\mathcal{L}_2$*-gain less than or equal to* $1/2$. *To confirm the value of* $\mathcal{L}_2$*-gain numerically, we integrate* $\dot{U}(x) - |w|^2/8 + |z|^2/2 \leq 0$ *to get* $U(x) - U(x_0) - \frac{1}{8}\int_0^t |w(\tau)|^2 d\tau + \frac{1}{2}\int_0^t |z(\tau)|^2 d\tau \leq 0$ *which by defining* $\Gamma = \frac{1}{2}$, $\mu = 2U$ *and using positive definiteness of* $U$ *reduces to*

$$\frac{\int_0^t |z(\tau)|^2 d\tau}{\int_0^t |w(\tau)|^2 d\tau} \leq \Gamma^2 + \frac{\mu(x_0)}{\int_0^t |w(\tau)|^2 d\tau} \tag{3.45}$$

*and is verified in Fig. 3.1 for* $x_0 = 1$.

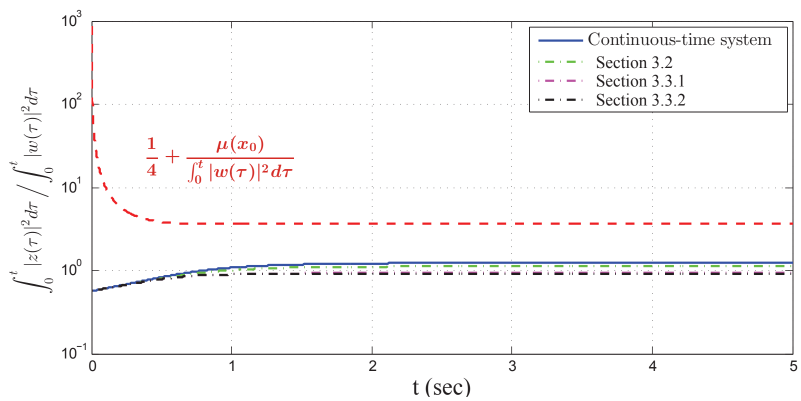



Figure 3.1: Verification of $\mathcal{L}_2$-gain.

*Also the corresponding state trajectory of the system is shown in Fig. 3.2.*

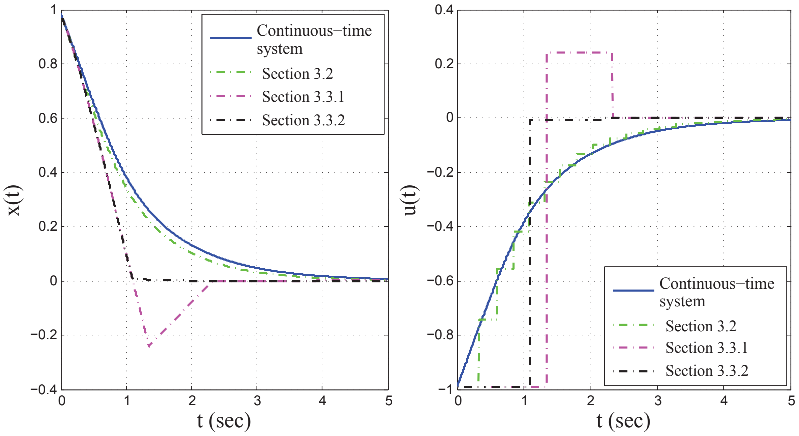


Figure 3.2: System's trajectory (Left). Actuator signal (Right).

*It is worth noticing that the $\mathcal{L}_2$-gain preserving nature of our proposed method can be inferred from Fig. 3.1 as the curves for the event-based scenarios lie under the one for continuous-time system. Also a comparison of the number of triggering instants and the MIET is given in the following table, where we average the results obtained from* 100 *initial conditions, uniformly distributed in* $[-1, 1]$. *The results of Table 3.1 clearly suggests that the effectiveness of the methods proposed in Section 3.3 on the sampling rate and intersampling interval diminishes with the passing of time.*

Table 3.1: Comparison of different scenarios.

| | Simulation time (sec) | Section 3.2 | Section 3.3 3.3.1 | Section 3.3 3.3.2 |
|---|---|---|---|---|
| Number of samples | 10 | 40 | 11 | 10 |
| | 30 | 120 | 48 | 38 |
| | 100 | 400 | 286 | 318 |
| Min inter-event time | 10 | 0.24 | 0.66 | 1.1 |
| | 30 | 0.24 | 0.46 | 0.27 |
| | 100 | 0.24 | 0.25 | 0.25 |

The proof of Theorem 3.2 suggests that nonzero inter-event times can be guaranteed if instead of condition (ii) in Theorem 3.1, the function $\psi^{-1}(\bar{\beta}_1/c)$ is Locally Lipschitz-continuous in $\mathbb{R}^n$. Neither of the these conditions hold in the next examples, however, we can still prove this important property for the ETC system through defining a new TC.

**Example 3.2** *In the next example, we consider the following second order system*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -h(x_1) + u + w, \\ z = x_2. \end{cases} \tag{3.46}$$

*where* $u$ *is the control input,* $w$ *is the exogenous disturbance and is restricted to satisfy* $|w| \leq 1$ *and* $z$ *is the measured output. We design* $u = -k(x_2 + e)$ *where* $e$ *is the measurement error in* $x_2$*. The nonlinear function* $h : \mathbb{R} \mapsto \mathbb{R}$ *is assumed to be in the sector* $[c_1, c_2]$*, i.e.,* $c_1 r^2 \leq rh(r) \leq c_2 r^2$ *for any* $r \in \mathbb{R}$*. We first show that, in view of (2.9), the system is ISS with respect to* $e$ *and* $w$*. To this end, let us consider Lyapunov function* $V(x) = \frac{1}{2}x^\mathsf{T} Px + 2\int_0^{x_1} h(r)dr$*, where* $x = [x_1 \ x_2]^\mathsf{T}$ *and* $P = [1 \ 1; 1 \ 2]$*. Then by choosing* $k = 1$ *we have* $\dot{V}(x) = -x_1 h(x_1) - x_2^2 + 2x_2(-e + w) + x_1(-e + w)$*. Next we can rewrite the last two terms in* $\dot{V}$ *as* $2x_2(-e + w) = -\frac{1}{4}(x_2 + 4e)^2 - \frac{1}{4}(x_2 - 4w)^2 + \frac{1}{2}x_2^2 + 4e^2 + 4w^2$ *and* $x_1(-e + w) = -\frac{1}{4}(x_1 + 2e)^2 - \frac{1}{4}(x_1 - 2w)^2 + \frac{1}{2}x_1^2 + e^2 + w^2$*. Assuming* $h$ *to be in the sector* $[1, 2]$*, we conclude that* $r^2 \leq rh(r) \leq 2r^2$*. Taking this into account, we obtain* $\dot{V}(x) \leq -\bar{\sigma}(|x|) + \beta_1(|e|) + \beta_2(|w|)$*, where* $\bar{\sigma}(r) = r^2/2$ *and* $\beta_1(r) = \beta_2(r) = 5r^2$*.*

*We also claim that when* $e \equiv 0$*, the continuous-time system is finite gain locally* $\mathcal{L}_2$*-stable. To see this, consider the Lyapunov function* $W(x) = \lambda V(x)$ *for some* $\lambda \in \mathbb{R}^+$*. Then since* $\dot{V}(x) = -x_1 h(x_1) - x_2^2 + 2x_2 w + x_1 w = -x_1^2(1 - \epsilon_1) - x_2^2(1 - \epsilon_2) + (\frac{1}{4}\epsilon_1^{-1} + \epsilon_2^{-1})w^2 = -\epsilon_1(x_1 - \frac{1}{2}\epsilon_2^{-1}w)^2 - \epsilon_2(x_2 - \epsilon_2^{-1}w)^2$*, we conclude* $\dot{W}(x) \leq \lambda(1-\epsilon_2)z^2 + \lambda(\epsilon_1^{-1}/4 + \epsilon_2^{-1})w^2$*, i.e., the continuous-time system has local* $\mathcal{L}_2$*-gain less than or equal to* $\sqrt{(4\epsilon_1 + \epsilon_2)/(4\epsilon_1\epsilon_2(1 - \epsilon_2))}$*. The minimum value of this upper bound on the* $\mathcal{L}_2$*-gain of the system is* 4.4861 *and obtained by setting* $\epsilon_1 = 1$ *and* $\epsilon_2 = 0.4721$*.*

*To verify condition (3.8) we restrict* $w$ *to satisfy* $|w(t)| \leq \gamma_3(|x(t)|)$*, where* $\gamma_3(r) = \sqrt{\bar{c}/2}r$ *for some* $\bar{c} \in (0, 1)$ *is a solution to the inequality (3.7). So far we have showed that Assumptions 3.1, 3.2 hold. Therefore it suffices to verify conditions (i)-(iii) in Theorem 3.1 hold as well. Condition (iii) is readily hold for functions* $f$ *and* $k$*. Also, condition (i) holds for* $\sigma_3(r) = \lambda(\|P\| + 2c_2)r$ *since we have*

$$|\frac{\partial W}{\partial x}(x)| = \lambda \left| \begin{bmatrix} x_1 + x_2 + 2h(x_1) \\ x_1 + 2x_2 \end{bmatrix} \right| \leq \lambda(\|P\| + 2c_2)|x|.$$

*Condition (ii) in Theorem 3.1 is not satisfied for the given functions* $\bar{\sigma}$*,* $\beta_1$*. However, we will redefine functions* $\psi$ *and* $\bar{\beta}_1$ *in (3.10) and show the results of theorem are still valid. To this end, let us start with (2.9) which can be written as* $\nabla V(x) \cdot f(x, k(x + e), w) \leq -(1 - c_0)\bar{\sigma}(|x|) + \beta_1(|e|)$ *for some* $c_0 \in (0, 1)$ *when* $|w| \leq \gamma_3(|x|)$*. This is true since choosing* $c_0 \geq 5\bar{c}$ *ensures* $\beta_2(|w|) \leq c_0\bar{\sigma}(|x|)$*. Therefore, (3.13) reduces to* $\dot{U}(x) \leq -(1 - c_0)\bar{\sigma}(|x|) + \beta_1(|e|) + \sigma_0(|x|)\beta_0(|e|) + \Gamma^2|w|^2 - |z|^2$*. Using the definition of* $\beta_1$ *in this example, we can write* $\beta_1(|e|) + \sigma_0(|x|)\beta_0(|e|) = \beta_1(|e|) + L_f L_k|e|\sigma_3(|x|) = (\sqrt{5}|e| + L_f L_k\sigma_3(|x|)/(2\sqrt{5}))^2 -$

$L_f^2 L_k^2 \sigma_3^2(|x|)/20$ *and hence*

$$\dot{U}(x) \leq -(1-c_0)\bar{\sigma}(|x|) - \frac{L_f^2 L_k^2}{20}\sigma_3^2(|x|) + (\sqrt{5}|e| + \frac{L_f L_k}{2\sqrt{5}}\sigma_3(|x|))^2 + \Gamma^2|w|^2 - |z|^2. \quad (3.47)$$

*Therefore, we can define* $\psi(r) \doteq \sqrt{\bar{\sigma}(r) + L_f^2 L_k^2 \sigma_3{}^2(r)/20} - L_f L_k \sigma_3(r)/(2\sqrt{5})$ *and* $\bar{\beta}_1(r) \doteq \sqrt{5}r$. *Thus if for some* $c \in (0, 1-c_0)$ *the next triggering of control task occurs when*

$$\sqrt{5}|e| \geq \sqrt{c\bar{\sigma}(|x|) + \frac{L_f L_k^2}{20}\sigma_3^2(|x|)} - \frac{L_f L_k}{2\sqrt{5}}\sigma_3(|x|) \quad (3.48)$$

*we conclude that* $\dot{U}(x) \leq -(1-c_0-c)\bar{\sigma}(|x|) + \Gamma^2|w|^2 - |z|^2$. *As a consequence, the ETC system has the local* $\mathcal{L}_2$*-gain less than or equal to* 4.4861. *Also one can check the local Lipschitz-continuity of* $\psi^{-1}(\bar{\beta}_1/c)$ *in* $\mathbb{R}^n$ *which is necessary to prove Zeno-freeness property for the system.*

*To find* $Q$*, we have to find functions* $\sigma_1$ *and* $\sigma_2$ *so that (2.7) holds. Since* $x_1^2 \leq 2\int_0^{x_1} h(r)dr \leq 2x_1^2$ *and* $V(x) = (x^\top Px)/2 + 2\int_0^{x_1} h(r)dr$*, one can choose* $\sigma_1(r) = \Sigma_{min}(P_1)r^2/2$ *and* $\sigma_2(r) = \Sigma_{max}(P_2)r^2/2$*, where* $\Sigma_{max}(A)$ *(respectively* $\Sigma_{min}(A)$*) denotes maximum (respectively minimum) eigenvalue of matrix* $A$*, and* $P_1 = [3\ 1; 1\ 2]$*,* $P_2 = [5\ 1; 1\ 2]$*. Thus we can take* $Q = L_{\gamma_3}\bar{\varepsilon} = L_{\gamma_3}\sigma_1^{-1}(\sigma_2(\varepsilon)) = \varepsilon\sqrt{(\bar{c}\Sigma_{max}(P_2))/(2\Sigma_{min}(P_1))}$*, where* $\varepsilon$ *is the upper bound on the norm of admissible initial conditions. For numerical simulations we take* $\varepsilon = 1$*,* $\lambda = 10^{-3}$*,* $c = 0.7$*,* $\bar{c} = 0.05$*,* $\kappa = \hat{\kappa} = 50$*,* $\zeta = \theta = 1$*,* $\Delta = 4$ *and* $Q = 0.62$*. The verification of* $\mathcal{L}_2$*-gain of the system for* $x_0 = [0.87\ 0.5]^\top$ *is presented in Fig. 3.3 as it suggests (3.45) holds for* $\Gamma = 4.4861$*.*

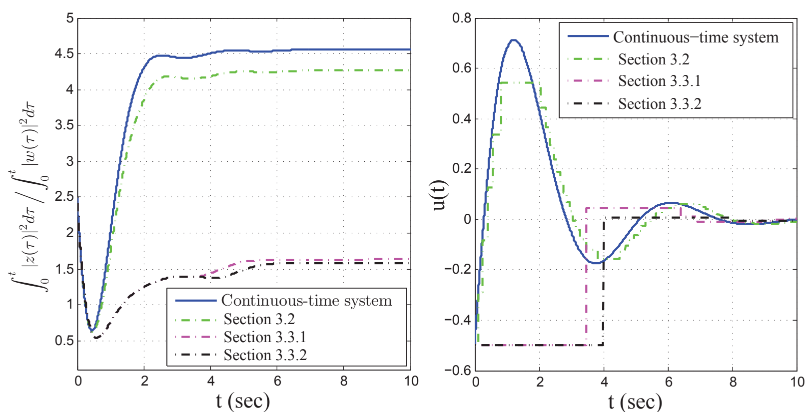



Figure 3.3: Verification of $\mathcal{L}_2$-gain (Left). Actuator signal (Right).

*The state trajectories of the system is also plotted in Fig. 3.4.*

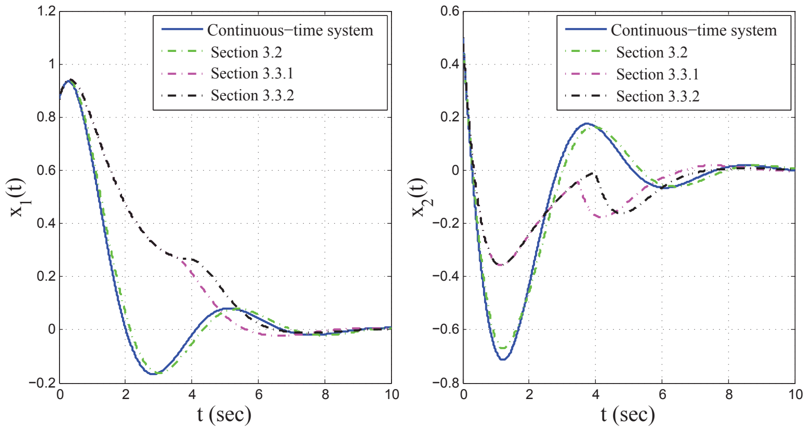



Figure 3.4: System's trajectories.

*Finally, a comparison of the number of triggering instants and the MIET is given in Table 3.2. To this end, we consider* 100 *initial conditions uniformly distributed in circle of radius* 1 *and average the obtained results.*

Table 3.2: Comparison of different scenarios.

| | Simulation time (sec) | Section 3.2 | Section 3.3 | |
|---|---|---|---|---|
| | | | 3.3.1 | 3.3.2 |
| Number of samples | 10 | 47 | 6 | 3 |
| | 30 | 139 | 89 | 8 |
| | 100 | 466 | 415 | 70 |
| Min inter-event time | 10 | 0.09 | 1.21 | 4 |
| | 30 | 0.09 | 0.1 | 4 |
| | 100 | 0.09 | 0.09 | 0.49 |

In the next example, we apply the results of Theorem 3.1 but replacing the Euclidean vector norm with the infinity norm.

**Example 3.3** *Using similar notations as in Example 3.2, we define the following second order system*

$$\begin{cases} \dot{x}_1 = x_2 - bx_1, \\ \dot{x}_2 = -ax_1^3 + u + w, \\ z = x_2, \end{cases} \tag{3.49}$$

*where* $|w| \leq 1$*. Defining Lyapunov function* $V(x) = ax_1^4/4 + |x|^2/2$*, where* $x = [x_1 \ x_2]^\mathsf{T}$*, we will have* $\dot{V}(x) = x_1x_2 - bx_1^2 - abx_1^4 + x_2u + x_2w$*, which by taking* $u = -(x_1+e_1) - (x_2+e_2)$ *can be written as*

$$\dot{V}(x) \leq -bx_1^2 - abx_1^4 - x_2^2 - \sqrt{2}|x|_\infty|e| + |x|_\infty|w| \tag{3.50}$$

*where $e_1$ and $e_2$ are the measurement errors in $x_1$ and $x_2$, respectively and $e = [e_1\ e_2]^\mathsf{T}$. Then in view of the following inequality*

$$bx_1^2 + abx_1^4 + \frac{1}{4}x_2^2 \geq \begin{cases} b|x|_\infty^2 + ab|x|_\infty^4, & \text{if } |x_1| > |x_2|, \\ \frac{1}{4}|x|_\infty^2, & \text{otherwise,} \end{cases}$$

*we conclude that $bx_1^2 + abx_1^4 + x_2^2/4 \geq \bar{\sigma}(|x|_\infty)$, where function $\bar{\sigma}(r) = \min\{br^2 + abr^4, r^2/4\}$ is of class $\mathcal{K}_\infty$. This enables us to write (3.50) as*

$$\dot{V}(x) \leq -\bar{\sigma}(|x|_\infty) + \sqrt{2}|x|_\infty|e| + |x|_\infty|w|. \tag{3.51}$$

*To show finite gain stability of continuous-time system, consider $W(x) = \lambda V(x)$ as the Lyapunov function. Thus for $e \equiv 0$, we have $\dot{V}(x) = -bx_1^2 - abx_1^4 - x_2^2 + x_2 w \leq -z^2 + zw$ which by using $zw = \hat{\epsilon}z^2/2 + w^2/(2\hat{\epsilon}) - \hat{\epsilon}(z - w/\hat{\epsilon})^2/2$ for some $\hat{\epsilon} \in \mathbb{R}^+$, gives $\dot{W}(x) \leq -\lambda(1 - \hat{\epsilon}/2)z^2 + \lambda w^2/(2\hat{\epsilon})$. As a consequence, it is not difficult to show that the minimum upper bound on the $\mathcal{L}_2$-gain of continuous-time system (3.49) can be achieved by choosing $\hat{\epsilon} = 1$ and is equal to 1. This value, however, may be improved by a different choice of Lyapunov function $W(x)$.*

*Defining $\sigma_3(r) \doteq \lambda r + a\lambda r^3$ for $r \in \mathbb{R}_0^+$, we have $|\frac{\partial W}{\partial x}(x)|_\infty \leq \sigma_3(|x|_\infty)$ since*

$$\begin{aligned} |\frac{\partial W}{\partial x}(x)|_\infty &= \lambda\left|\begin{bmatrix} ax_1^3 + x_1 \\ x_2 \end{bmatrix}\right|_\infty \\ &\leq \begin{cases} a\lambda|x|_\infty^3 + \lambda|x|_\infty, & \text{if } |x_1|(1 + ax_1^2) > |x_2|, \\ \lambda|x|_\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

*and hence $|\frac{\partial W}{\partial x}(x)|_\infty \leq \max\{\lambda|x|_\infty, \lambda|x|_\infty + a\lambda|x|_\infty^3\} = \lambda|x|_\infty + a\lambda|x|_\infty^3$. Therefore, by taking $U = V + W$ and following the similar lines as in deriving (3.14), we can write*

$$\dot{U}(x) \leq -\frac{\lambda}{2}z^2 + \frac{\lambda}{2}w^2 + |x|_\infty\Big(-|x|_\infty \min\{b + ab|x|_\infty^2, \frac{1}{4}\} + \sqrt{2}|e| + |w| + 2\lambda L_f L_k(1 + a|x|_\infty^2)|e|\Big),$$

*where we used the fact that*

$$\begin{aligned} &\frac{\partial W}{\partial x}(x)(f(x, k(x+e), w) - f(x, k(x), w)) \leq \\ &2|\frac{\partial W}{\partial x}(x)|_\infty |f(x, k(x+e), w) - f(x, k(x), w)|_\infty \leq \\ &2|\frac{\partial W}{\partial x}(x)|_\infty |f(x, k(x+e), w) - f(x, k(x), w)| \leq \\ &2L_f L_k|e||\frac{\partial W}{\partial x}(x)|_\infty. \end{aligned}$$

*We note that since $u = k(x) = -(x_1 + x_2)$, it can be easily inferred that $L_k = \sqrt{2}$. Now assuming $|w(t)| \leq \gamma_3(|x(t)|_\infty)$ where $\gamma_3(r) = \min\{br + abr^3, r/4\}\bar{c}$ and taking $c \in (0, 1 - \bar{c})$,* *we conclude that if the execution of control task occurs when*

$$|e| \geq c \frac{|x|_\infty \min\{b + ab|x|_\infty^2, \frac{1}{4}\}}{\sqrt{2}(1 + 2\lambda L_f(1 + a|x|_\infty^2))}, \tag{3.52}$$

*the system (3.49) is finite gain local $\mathcal{L}_2$-stable with zero bias and local $\mathcal{L}_2$-gain $\leq 1$. To find $Q$, let $\sigma_1(x) \leq V(x) \leq \sigma_2(x)$, where $\sigma_1(r) = r^2/2$ and $\sigma_2(r) = (2+a)r^2/4$. As a consequence, assuming initial conditions to be norm bounded by $\varepsilon$, we can take $Q = L_{\gamma_3}\bar{\varepsilon} = L_{\gamma_3}\sigma_1^{-1}(\sigma_2(\varepsilon)) = L_{\gamma_3}\varepsilon\sqrt{1+a/2}$, which by choosing $b > 1/4$, reduces to $Q = (\bar{c}\varepsilon\sqrt{1+a/2})/4$. In simulations, let $a = 1$, $b = 10$, $\epsilon = 1$, $c = 0.5$, $\bar{c} = 0.45$, $\kappa = 10$, $\bar{\kappa} = 10$, $\zeta = 1$, $\theta = 5$, $\Delta = 1$ and $Q = 0.138$. Therefore, the only parameter left to study the system's response is $\lambda$ which appears in TC (3.52). We start our simulation with $\lambda = 1$, however, the effect of this parameter on our results will be discussed later. Similar to the past examples, in the next table, we give a comparison of number of samplings and MIETs over different scenarios. The results are, indeed, the average over 100 initial conditions uniformly distributed in the circle of radius 1.*

Table 3.3: Comparison of different scenarios.

| | Simulation time (sec) | Section 3.2 | Section 3.3 | |
|---|---|---|---|---|
| | | | 3.3.1 | 3.3.2 |
| Number of samples | 10 | 420 | 8 | 10 |
| | 30 | 1190 | 23 | 30 |
| | 100 | 3882 | 75 | 100 |
| Min inter-event time | 10 | 0.007 | 0.61 | 1 |
| | 30 | 0.007 | 0.57 | 1 |
| | 100 | 0.007 | 0.53 | 1 |

*Recalling from Definition 2.9, the system (3.49) has local $\mathcal{L}_2$-gain $\leq \Gamma$ if for any $T$ we have (3.45). The local $\mathcal{L}_2$-gain of the system is then verified in Fig. 3.5 for $x_0 = [0.87\ 0.5]^\mathsf{T}$ and $\Gamma = 1$. Also the corresponding state trajectories is presented in Fig. 3.6.*

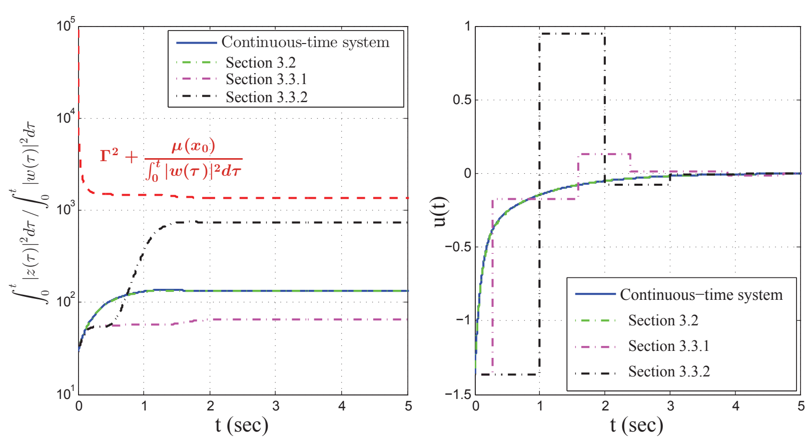



Figure 3.5: Verification of $\mathcal{L}_2$-gain (Left). Actuator signal (Right).

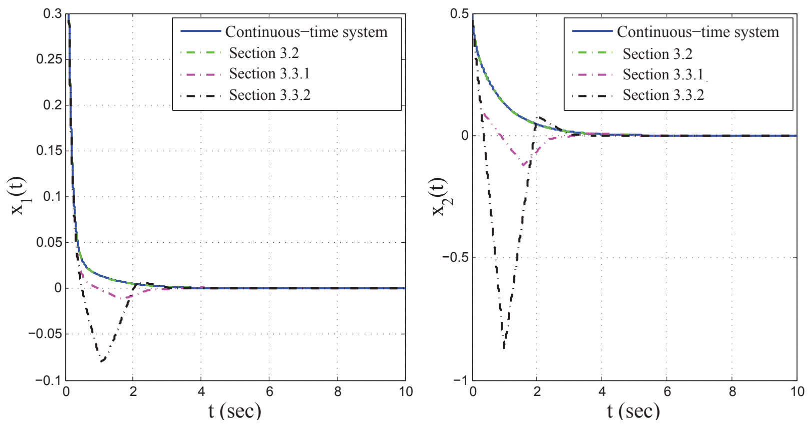



Figure 3.6: System's trajectories.

*Finally, the effect of parameter $\lambda$ on the above results in the first* 100 *seconds of response is investigated in Table 3.4. It suggests that $\lambda$ has negligible effect on the triggering numbers and MIETs using the methods of Sections 3.3.1, 3.3.2. However, choosing $\lambda > 10^{-2}$ degrades the efficiency of the results of Section 3.2, significantly.*

Table 3.4: Investigating the effect of parameter $\lambda$.

| | Method of Section | $\lambda$: $10^{-3}$ | $10^{-2}$ | $10^{-1}$ | 1 |
|---|---|---|---|---|---|
| Number of samples | 3.2 | 149 | 152 | 176 | 420 |
| | 3.3.1 | 8 | 8 | 8 | 8 |
| | 3.3.2 | 10 | 10 | 10 | 10 |
| Min inter-event time | 3.2 | 0.023 | 0.022 | 0.019 | 0.007 |
| | 3.3.1 | 0.670 | 0.669 | 0.680 | 0.610 |
| | 3.3.2 | 0.999 | 0.999 | 0.999 | 0.999 |

**Example 3.4** *The following example illustrates the necessity of using a local $\mathcal{L}_2$ theory. The example shows that while under arbitrary perturbations $w$ in $\mathcal{L}_2$ space, the event times are not necessarily guaranteed to be isolated, the local notion serves to exclude Zeno phenomenon. Consider the following linear example from [59]*

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad u(t) = Kx(t), \tag{3.53}$$

*where $A$, $B$, $K$ are matrices of appropriate dimensions and the controller is applied in an event-based fashion. The desired output is taken as $z(t) = x(t)$. Assume $t_0 = 0$, $x_0 \neq 0$, and the TC $|e(t)| \geq p|x(t)|$ for some $p \in \mathbb{R}^+$, it is shown in [59] that under the following choice of disturbance*

$$w(t) = ((t-1)A + (t_i - 1)BK)x_0 - x_0, \ t \in [t_i, t_{i+1}) \tag{3.54}$$

*for $t \in [0,1]$ and zero elsewhere (which is a signal in $\mathcal{L}_2$ space), the state and triggering instants are analytically given by $x(t) = (1-t)x_0$ and $t_i = 1-(1+p)^{-i}$, $i \in \mathbb{N}_0$, respectively. It is then obvious that event times has an accumulation point at $t = 1$. To address this issue, [59] suggests using the input-to-state practically stable (ISpS) property instead of ISS condition (2.8). The proposed method, however, is not applicable to the problem studied in this chapter since the $\mathcal{L}_2$-gain performance of the ETC system can not be guaranteed.*

*Note that the above discussion suggests that when $w$ is an arbitrary signal in $\mathcal{L}_2$, as in (3.54), the execution rules of the form (3.10) does not exclude the Zeno-behaviour. However, in this chapter our solution to this problem is to restrict $w$ to be in the admissible space $\mathscr{W}_Q$ and also satisfy condition (3.8) with $\gamma_3(r) \doteq \hat{c}r$, $\hat{c} \in \mathbb{R}^+$. The price we paid is then the local character of the results. We remark that $w$ defined in (3.54) does not satisfy (3.8), and hence is not a counter example of the local thoery. This is because (3.8) is violated near $t = 1$.*

*Indeed, applying the results of Theorem 3.2 one can show that limiting $w$ as above, the triggering instants are separated at least by*

$$\tau = \frac{1}{\|A\|+\hat{c}} \ln\left(1 + \frac{\|A\|+\hat{c}}{p(\|A\|+\|BK\|+\hat{c}) + \|BK\|}\right).$$

## 3.5 Summary

This chapter addresses the disturbance rejection problem of nonlinear ETC systems. Assuming the existence of a pre-designed control law with desirable local $\mathcal{L}_2$ performance characteristics, we propose a TC that preserves finite gain local $\mathcal{L}_2$-stability of the original continuous-time design. Our formulation is rather general; *i.e.* we consider a nonlinear plant and assume that disturbances are bounded by a Lipschitz-continuous function of the state. We also show that, in the absence of external disturbances, the control law render the origin asymptotically stable.

In addition to stability and disturbance rejection, we also study the intersampling behaviour of the proposed TC. First we show that the inter-event time period is lower bounded by a nonzero constant and focus on enlarging this constant. We show that, regardless of the construction of the ETM, the inter-event time period increase is actually lower bounded by a constant. Increasing the value of this constant can be done at the expense of relaxing the stability properties of the design.

# Chapter 4

# Event-Triggered Design with Guaranteed Minimum Inter-Event Times and $\mathcal{L}_p$ Performance

## 4.1 Problem Statement

In this chapter[1] we consider the nonlinear systems of the following form:

$$\begin{cases} \dot{\xi} = f(\xi, d) + g(\xi)u, \\ z = h(\xi, d), \end{cases} \tag{4.1}$$

where $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathcal{L}_p^q$, $z \in \mathbb{R}^s$ represent the state, control input, exogenous disturbance and measured output. The functions $f$, $g$ and $h$ are locally Lipschitz-continuous and $f(0,0) = 0$, $h(0,0) = 0$ so that $\xi = 0$ is an equilibrium point of zero-input system. We will assume the state $\xi$ evolves from initial conditions $\xi_0 = \xi(t_0)$ on an open subset of $\mathbb{R}^n$ containing the origin. System (4.1) is said to be finite gain $\mathcal{L}_p$-stable and has an $\mathcal{L}_p$-gain $\leq \mu$ if there exist real numbers $\eta, t_0, T$, $\mu > 0$, $p \geq 1$ and positive semi-definite function $\beta$ such that for any $T > t_0$, any $d \in \mathcal{L}_p^n$ and any $\xi_0 \in \mathbb{R}^n$

$$\int_{t_0}^{T} \|z(s)\|^p ds \leq \mu^p \int_{t_0}^{T} \|d(s)\|^p ds + \beta(\xi_0) + \eta. \tag{4.2}$$

We assume plant and controller communicate apperiodically through a digital network and in an event-based manner. The ETC problem established in this chapter relies on the emulation of the analog design and consists of two steps:

First, we assume continuous data transmission between plant and a full information controller $u = \gamma(\xi)$, where $\gamma$ is locally Lipschitz-continuous. The resulting continuous-time

[1]The results of this chapter have been submitted for publication in the article: M. Ghodrat and H. J. Marquez, "Event-Triggered Design with Guaranteed Minimum Inter-Event Times and Lp-Performance", Submitted to *IEEE Trans. Autom. Control*, May 2018.

plant is then given by

$$\begin{cases} \dot{\xi} = f_c(\xi, d), \\ z = h(\xi, d), \end{cases} \tag{4.3}$$

where $f_c(\xi, d) := f(\xi, d) + g(\xi)\gamma(\xi)$. It is then assumed that the controller renders the closed-loop (4.3) finite gain $\mathcal{L}_p$-stable with disturbance attenuation level $\mu$.

Second, the communication between plant and controller occurs at the instants belong to the set $\{t_k : k \in \mathbb{K}\}$, where $\mathbb{K} = \{0, 1, 2, \ldots, K\}$. The sampling sequence is a monotone increasing set, starting at $t_0$ and implicitly defined through a TC. The actuator signal is held constant between events using a hold device $u(t) = u(t_k)$, $t \in [t_k, t_{k+1})$ where $t_{K+1} = \infty$ when $K$ is finite. The proposed TC is continuously monitored and once it is satisfied, the updated state is forwarded to the controller which computes the new control signal and send it to the actuator instantaneously. More specifically, let $t_k$ be the most recent sampling instant and TC be satisfied at some $\varpi_{k+1} > t_k$. Then the new control signal applied through the actuator at $t_{k+1} = \varpi_{k+1}^+$ and hence $u(t_{k+1}) = \gamma(\xi(t_{k+1}))$. Let $\varepsilon(t) := \xi(t_k) - \xi(t)$ represent the sampling error for $t \in [t_k, t_{k+1})$. $\varepsilon(t)$ is then a right-continuous signal with zero value at $t_k$. In our analysis we neglect practical issues such as transmission and computation delays, however, they can be readily addressed following the approach introduced in [9]. The resulting closed-loop ETC system is then described by

$$\begin{cases} \dot{\xi} = f_s(\xi, \varepsilon, d), \quad z = h(\xi, d), \\ t_{k+1} = \varpi_{k+1}^+, \ \varpi_{k+1} = \inf\{t \in \mathbb{R} : t > t_k \wedge \Phi(t) = 0\}, \end{cases} \tag{4.4}$$

where $f_s(\xi, \varepsilon, d) := f(\xi, d) + g(\xi)\gamma(\xi + \varepsilon)$ and $\Phi(t)$ is the TC to be designed.

Assuming the existence of an $\mathcal{L}_p$-stabilizing controller for (4.3), our main interest is to design an ETM that retains this input-output property of the network-free design for the resulting ETC system; perhaps with a worse disturbance attenuation level. The proposed ETM shall (1) exclude the Zeno behaviour and (2) serve as a general platform for TC design in ETC problems.

## 4.2 Event-Triggered Mechanism

In this section, we introduce a general structure to design $\Phi$ so that ETC system (4.4) has $\mathcal{L}_p$-gain $\leq \mu_d$. Consider the following TC structure:

$$\Phi(t) := \varphi(\xi(t), \varepsilon(t)) - \textstyle\sum_{i=1}^{2} k_i \phi_i(t) = 0 \tag{4.5}$$

where $k_1, k_2 > 0$, and the dynamic variables $\phi_1$, $\phi_2$ and function $\varphi$ are to be designed. We start with designing $\varphi$, for which the following assumption is required.

**Assumption 4.1** *There exist positive definite, radially unbounded functions $V_s$, $V_c$, positive constants $\mu$, $c_i$, $\bar{c}_i$ $i \in \{1,2,3\}$ and some $p \in [1,\infty)$ satisfying*

*(i)* $\nabla V_s(\xi) \cdot f_s(\xi, \varepsilon, d) \leq -c_1\|\xi\|^p + c_2\|\varepsilon\|^p + c_3\|d\|^p$,

*(ii)* $\nabla V_c(\xi) \cdot f_c(\xi, d) \leq \mu^p\|d\|^p - \|z\|^p$,

*(iii)* $V_s(\xi) \leq \bar{c}_1\|\xi\|^p$, $V_c(\xi) \leq \bar{c}_2\|\xi\|^p$, $\|\nabla V_c(\xi)\| \leq \bar{c}_3\|\xi\|^{p-1}$.

**Remark 4.1** *Assumption 4.1(i) implies that system (4.4) is ISS with respect to the inputs $\varepsilon$, $d$. Also Assumption 4.1(ii) implies that $u = \gamma(\xi)$ renders the continuous-time system (4.3) finite gain $\mathcal{L}_p$-stable with $\mathcal{L}_p$-gain $\leq \mu$.*

The function $\varphi$ is assumed to have the following form

$$\varphi(\xi, \varepsilon) = \varphi_1(\xi) + \varphi_2(\varepsilon) + \varphi_3(\xi, \varepsilon), \tag{4.6}$$

where $\varphi_1(r) = -c_1\sigma\|r\|^p$, $\varphi_2(r) = c_2\|r\|^p$,

$$\varphi_3(r, s) = \nabla V_{c,\lambda}(r) \cdot g(r)(\gamma(r+s) - \gamma(r))$$

and $\sigma < 1$, $p \in [1,\infty)$, $V_{c,\lambda}(r) = \lambda V_c(r)$ for some $\lambda \in \mathbb{R}^+$. We then continue with the design of $\phi_1$, $\phi_2$; dynamic parameters serve to enlarge the inter-event times and guarantee the event-separation property for ETC system (4.4). Consider the equations below for $t \in [t_k, t_{k+1})$

$$\frac{d}{dt}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} + \begin{pmatrix}\alpha_1(\phi_1) - k_2\phi_2 \\ \alpha_2(\phi_2)\end{pmatrix} = \begin{pmatrix}-\varphi \\ \bar{\varphi}\end{pmatrix}, \tag{4.7a}$$

$$\bar{\varphi}(t) = \begin{cases}\alpha_2(\bar{\delta}), & t \in [t_k, \hat{t}_k), \\ \dot{\delta}_k(t) + \alpha_2(\delta_k(t)), & t \in [\hat{t}_k, t_{k+1}),\end{cases} \tag{4.7b}$$

where $\bar{\delta}$ is a positive constant and $\delta_k$ is a positive, bounded and piecewise differentiable function defined over $[\hat{t}_k, t_{k+1})$ and satisfies $\sum_k \int_{\hat{t}_k}^{t_{k+1}} \delta_k(\tau)d\tau \leq \theta_1$ for some positive $\theta_1$. Also $\hat{t}_k = t_k + \hat{\tau}$, where $\hat{\tau}$ is a positive parameter and will be designed in the sequel. Note that function $\bar{\varphi}$ is defined such that $\bar{\delta}$ (resp. $\delta_k(t)$) is a solution of $\phi_2$ in (4.7a) over $[t_k, \hat{t}_k)$ (resp. $[\hat{t}_k, t_{k+1})$). Moreover $\alpha_2$ is an arbitrary class-$\mathcal{K}_\infty$ function and $\alpha_1 \in \mathcal{K}_\infty$ is designed based on the following assumption.

**Assumption 4.2** $\alpha_1(r) \geq \nu r$ *where* $\nu = c_1(1-\sigma)/(\bar{c}_1 + \bar{c}_2)$.

To solve (4.7a), (4.7b) the following initial values are assumed

$$\phi_1(t_k) = r_k,\ \phi_1(\hat{t}_k) = \hat{r}_k,\ \phi_2(t_k) = s_k,\ \phi_2(\hat{t}_k) = \hat{s}_k, \tag{4.8}$$

where $r_k$, $\hat{r}_k$, $s_k$, $\hat{s}_k$ are non-negative real numbers and are designed based on the following assumption.

**Assumption 4.3** *$r_k$ and $\hat{r}_k$ are chosen from sequences with convergent series, i.e., there exist finite numbers $\theta_2, \theta_3 \in \mathbb{R}^+$ so that $\sum_k r_k \leq \theta_2$, $\sum_k \hat{r}_k \leq \theta_3$. Moreover, $s_k$ and $\hat{s}_k$ satisfy $s_k \geq \bar{\delta}$ and $\hat{s}_k = \delta_k(\hat{t}_k)$.*

Dynamic rules have been previously studied in [13, 28]. The variable $\phi_1$ in (4.5) which satisfies the differential equation (4.7a), plays the role of dynamic parameter introduced in the above references. Here, we introduce an additional dynamic variable $\phi_2$; while both $\phi_1$, $\phi_2$ serve to extend the inter-event times, $\phi_2$ plays the fundamental role of guaranteeing event separation property for ETC system (4.4).

**Proposition 4.1** *Under TC (4.5) and Assumption 4.3, $\phi_1(t), \phi_2(t) \geq 0$ for all $t \geq t_0$. In detail, $\phi_2(t) \geq \bar{\delta}$ for $t \in [t_k, \hat{t}_k)$, $\phi_2(t) = \delta_k(t)$ for $t \in [\hat{t}_k, t_{k+1})$.*

**Proof.** From (4.5), (4.7a) $\phi_1$ satisfies $\dot{\phi}_1 + \alpha_1(\phi_1) + k_1\phi_1 \geq 0$ for $t \in [t_k, t_{k+1})$. Note that $\phi_1(t) \equiv 0$ is a solution to $\dot{\phi}_1 + \alpha_1(\phi_1) + k_1\phi_1 = 0$. Therefore, since $\phi_1(t_k), \phi_1(\hat{t}_k) \geq 0$ it follows that $\phi_1(t) \geq 0$ for all $t \geq t_0$. For the second part, since $\bar{\delta}$ (resp. $\delta_k(t)$) is a solution of $\phi_2$ in (4.7a) for $t \in [t_k, \hat{t}_k)$ (resp. $t \in [\hat{t}_k, t_{k+1})$) and $s_k \geq \bar{\delta}$ (resp. $\hat{s}_k = \delta_k(\hat{t}_k)$), it follows that $\phi_2(t) \geq \bar{\delta}$ (resp. $\phi_2(t) = \delta_k(t)$) over this interval. Finally, from the positiveness of $\bar{\delta}$ and $\delta_k(t)$, $\phi_2(t) \geq 0$ for all $t \geq t_0$. ■

Proposition 4.1 illustrates the previous claim that $\phi_1$, $\phi_2$ enlarge the inter-event times. In fact, in absence of $\phi_1$, $\phi_2$ triggering occurs when $\varphi(\xi, \varepsilon) = 0$. However, the positiveness of $\phi_1$, $\phi_2$ postpones the triggering to occur when $\varphi(\xi, \varepsilon) = k_1\phi_1 + k_2\phi_2$.

**Remark 4.2** *While TC (4.5) was originally proposed to address the state feedback problem with a guaranteed $\mathcal{L}_p$-gain level, the idea of introducing dynamic variables $\phi_1$, $\phi_2$ can readily be applied to the dynamic output feedback case. More specifically, let $\varphi = 0$ be a pre-designed output based TC. The ETC system then enjoys the benefits offered by $\phi_1$, $\phi_2$ under the modified condition $\varphi - k_1\phi_1 - k_2\phi_2 = 0$ with similar dynamics for $\phi_1$, $\phi_2$ as in (4.7a), (4.7b).*

To finish the design, it remains to define $\hat{\tau}$. Let us start with the following lemma.

**Lemma 4.1** *Under Assumptions 4.1-4.3 and if the control signal is updated under the TC (4.5), all the trajectories of the ETC system (4.4) starting from* $\mathscr{B}_\rho$ *will remain in* $\mathscr{B}_{\bar{\rho}}$*, where*

$$\bar{\rho} = \max\Big\{\|\xi\|\colon V_s(\xi) + V_{c,\lambda}(\xi) \le V_s(\xi_0) + V_{c,\lambda}(\xi_0) + \\ \frac{1}{\nu}(\lambda\mu_d^p\|d\|_\infty^p + k_2\|\phi_2\|_\infty) + \theta_2 + \theta_3, \xi, \xi_0 \in \mathbb{R}^n, \|\xi_0\| \le \rho\Big\}.$$

**Proof.** We shall need the following proposition whose proof can be obtained applying integration by parts and Assumption 4.2.

**Proposition 4.2** *For* $\phi_1$ *defined in (4.7a) and (4.8), we have*

$$\int_{t_k}^{\hat{t}_k} -e^{\nu\tau} d\phi_1(\tau) \le r_k e^{\nu t_k}, \quad \int_{\hat{t}_k}^{t_{k+1}} -e^{\nu\tau} d\phi_1(\tau) \le \hat{r}_k e^{\nu\hat{t}_k}.$$

Now we start from Assumption 4.1(ii) to write

$$\dot{V}_c(\xi) \le \mu^p\|d\|^p - \|z\|^p + \nabla V_c(\xi) g(\xi)(\gamma(\xi+\varepsilon) - \gamma(\xi))$$

Define $V(\xi) = V_s(\xi) + V_{c,\lambda}(\xi)$, one can apply Assumption 4.1(i), (4.5), (4.6) to get

$$\begin{aligned}\dot{V}(\xi) &\le -c_1(1-\sigma)\|\xi\|^p + (\lambda\mu^p + c_3)\|d\|^p + \varphi \le \\ &-c_1(1-\sigma)\|\xi\|^p + (\lambda\mu^p + c_3)\|d\|^p + k_2\phi_2 - \alpha_1(\phi_1) - \dot{\phi}_1\end{aligned}$$

where the second inequality is obtained using (4.7a). Let $A = \lambda\mu_d^p\|d\|_\infty^p + k_2\|\phi_2\|_\infty$, we can apply Proposition 4.1 to write

$$\dot{V}(\xi) + \nu V(\xi) \le A - \dot{\phi}_1,$$

where we used $V(\xi) \le (\bar{c}_1 + \bar{c}_2)\|\xi\|^p$ suggested by Assumption 4.1(iii). We then conclude from Proposition 4.2 that

$$\begin{aligned} V(\xi(\hat{t}_k))e^{\nu\hat{t}_k} &\le V(\xi(t_k))e^{\nu t_k} + r_k e^{\nu t_k} + A\int_{t_k}^{\hat{t}_k} e^{\nu\tau} d\tau, \\ V(\xi(t_{k+1}))e^{\nu t_{k+1}} &\le V(\xi(\hat{t}_k))e^{\nu\hat{t}_k} + \hat{r}_k e^{\nu\hat{t}_k} + A\int_{\hat{t}_k}^{t_{k+1}} e^{\nu\tau} d\tau. \end{aligned}$$

Adding the two inequalities and apply the result to the sampling intervals until $t \ge t_0$, Assumption 4.3 yields $V(\xi(t))e^{\nu t} \le V(\xi_0) + (\theta_2+\theta_3)e^{\nu t} + A\int_{t_0}^t e^{\nu\tau} d\tau$. Therefore,

$$\begin{aligned} V(\xi(t)) &\le V(\xi_0)e^{-\nu t} + \theta_2 + \theta_3 + A\int_{t_0}^t e^{-\nu(t-\tau)} d\tau \\ &\le V(\xi_0) + \theta_2 + \theta_3 + \nu^{-1}A \end{aligned}$$

which gives the desired result. ■

Since $\|\phi_2\|_\infty$ is limited by $\max\{s_k, \|\delta_k\|_\infty : k \in \mathbb{K}\}$ and hence is bounded, Lemma 4.1 suggests that the trajectories of the ETC system (4.4) are bounded by a non-decreasing function of $\|\xi_0\|$ and $\|d\|_\infty$. Next lemma employs the Lipschitz property of functions $f$, $g$, $\gamma$ to provide an upper bound on the norm of state dynamics.

**Lemma 4.2** *With the same conditions as in Lemma 4.1, there exist* $\lambda_i = \lambda_i(\|\xi_0\|, \|d\|_\infty)$, $i \in \{1, 2, 3\}$, *non-decreasing on their arguments, so that*

$$\|\dot{\xi}\| \leq \lambda_1\|\xi\| + \lambda_2\|\varepsilon\| + \lambda_3\|d\|.$$

**Sketch of the proof.** One can apply the Lipschitz property of functions $f$, $g$, $\gamma$ to get $\|\dot{\xi} - \dot{\tilde{\xi}}\| \leq \lambda_1\|\xi - \tilde{\xi}\| + \lambda_2\|\varepsilon - \tilde{\varepsilon}\| + \lambda_3\|d - \tilde{d}\|$ where $\lambda_i$'s are functions of $\|d\|_\infty$ and $\bar{\rho}$. The result then follows by applying Lemma 4.1. ■

**Remark 4.3** *As Lemma 4.2 suggests, since the Lipschitz properties of functions $f$, $g$ and $\gamma$ are only* local, *the Lipschitz coefficients $\lambda_i$ are bounded provided $\|\xi_0\| < \infty$, $\|d\|_\infty < \infty$.*

Let us define

$$\tau_i := \sup\left\{t \in \mathbb{R}_0^+ : \lambda_i^p \psi(t, \lambda_2) < \frac{B_i}{c}\right\}$$

for $i \in \{1, 3\}$ where $B_1 = c_1\sigma - (\bar{c}_3\lambda)^q/q$, $B_3 = \lambda(\mu_d^p - \mu^p) - c_3$, $c = c_2 + \lambda_2^p/p$ and

$$\psi(t, \lambda_2) = \frac{2^{2p}(p-1)^{p-1}}{\lambda_2^p p^p}(e^{\frac{\lambda_2 p}{2(p-1)}t} - 1)^{p-1}(e^{\frac{\lambda_2 p}{2}t} - 1).$$

$\hat{\tau}$ is then described by

$$\hat{\tau} = \min\{\tau_1, \tau_3\}. \tag{4.9}$$

Later in Lemma 4.3 we will see that $\hat{\tau} > 0$ is required to guarantee the isolation of triggering instants for ETC system (4.4). Moreover, $\tau_1$ (resp. $\tau_3$) is the elapsed time since the most recent triggering instant so that sampling error grows without violating stability (resp. desired $\mathcal{L}_2$ bound) of the ETC system (4.4) (see the proof of Theorem 4.1).

**Remark 4.4** *To design $\lambda$ one has to consider the restriction of having a positive $\hat{\tau}$. $\hat{\tau} > 0$ necessitates $\tau_1$, $\tau_3$ and hence $B_1$, $B_3$ to be positive. This gives the restriction on $\lambda$ as $\lambda < \bar{c}_3^{-1}(c_1\sigma q)^{\frac{1}{q}}$ and $\lambda > c_3(\mu_d^p - \mu^p)^{-1}$. The later condition implies $\mu_d > \mu$, i.e., the $\mathcal{L}_p$-stability of ETC system (4.4) is achieved at the expense of a larger rejection level. However, to minimize $\mu_d$, we may choose $c_3$ small enough by* scaling *Lyapunov function $V_s$ in Assumption 4.1 (refer to example section for more details). Obviously, one has to replace $c_i$, $i \in \{1, 2, 3\}$ by the corresponding scaled values in all of the discussions.*

## 4.3 Main Results

### 4.3.1 Uniform Isolation of Triggering Instants

One of the difficulties encountered in ETC systems is undesirable Zeno behaviour which happens when an infinite number of triggerings occur over a finite interval. This is even more challenging when the system of interest is exposed to exogenous disturbances or sensor noise, since in this case the sampling error is also driven by the disturbance/noise. As an example, while Zeno behavior is excluded in [9] for disturbance free systems, the same does not necessarily hold in presence of disturbance (see [59] for further discussion).

In the present chapter, we show that under TC (4.5), the ETC system (4.4) satisfies the following *robust event-separation* property defined in [59].

**Definition 4.1** *Let* $\tau_m = \inf\{t_{k+1} - t_k : k \in \mathbb{K}\}$ *be the MIET. ETC system (4.4) has the robust semi-global event-separation property if there exists* $\epsilon \in \mathbb{R}^+$ *so that for any compact set* $\Xi \subset \mathbb{R}^n$, $\inf\{\tau_m : \xi_0 \in \Xi, \|d\|_\infty \le \epsilon\} > 0$.

According to Definition 4.1, an ETC system has the robust semi-global event-separtion property if the sequence of sampling times $\{t_k : k \in \mathbb{K}\}$ is a uniformly isolated set provided that $\xi_0 \in \Xi$ and $\|d\|_\infty \le \epsilon$.

**Lemma 4.3** *Under Assumptions 4.1, 4.2, 4.3 the ETC system (4.4) with ETM (4.5)-(4.9) has the robust semi-global event-separation property. In detail,*

$$\tau_m = \min\{\tau^*(1), \hat{\tau}\},$$

*where* $m_1 = (\frac{B_1}{c})^{\frac{1}{p}}$, $m_2 = (\frac{k_2\bar{\delta}}{c})^{\frac{1}{p}}$, $\kappa = \max\left\{\frac{2\lambda_1}{m_1}, \frac{2\lambda_3\epsilon}{m_2}\right\}$ *and*

$$\tau^*(\chi) = \begin{cases} \frac{1}{\lambda_2 - m_1\frac{\kappa}{2}} \ln\left(\frac{\kappa+\lambda_2\chi}{\kappa(1+m_1\frac{\chi}{2})}\right), & \kappa \neq \frac{2\lambda_2}{m_1}, \\ \frac{m_1\chi}{\lambda_2(2+m_1\chi)}, & \kappa = \frac{2\lambda_2}{m_1}. \end{cases} \tag{4.10}$$

To prove Lemma 4.3 we report here two useful inequalities.

**Lemma 4.4** *For any* $p, q \ge 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and any* $r > 0$

$$(i)\ \|x+y\|^r \le 2^r\|x\|^r + 2^r\|y\|^r$$
$$(ii)\ \int_{\mathcal{T}} \|x(\tau)y(\tau)\| d\tau \le \Big(\int_{\mathcal{T}} \|x(\tau)\|^p d\tau\Big)^{\frac{1}{p}} \Big(\int_{\mathcal{T}} \|y(\tau)\|^q d\tau\Big)^{\frac{1}{q}}.$$

**Proof of Lemma 4.3.** Since any conservative TC than (4.5) gives rise to the lower bound on MIET, we aim to modify (4.5) to find such a condition. To begin, we first make the use of Proposition 4.1 which implies $\phi_1 \geq 0$, $\phi_2(t) \geq \bar{\delta}$ for $t \in [t_k, \hat{t}_k)$ and hence modify (4.5) as $\varphi = k_2\bar{\delta}$ in this interval. Note that we will assume $t_{k+1} \leq \hat{t}_k$ since otherwise $\tau_m = \hat{\tau}$ and the event-separation property holds trivially. From the inequality given in the sketch of proof of Lemma 4.2 with $\tilde{\xi} = \xi$, $\tilde{\varepsilon} = 0$, $\tilde{d} = d$, one can conclude $\|g(\xi)(\gamma(\xi + \varepsilon) - \gamma(\xi))\| \leq \lambda_2\|\varepsilon\|$ and hence modify condition $\varphi = k_2\bar{\delta}$ as

$$c_2\|\varepsilon\|^p + \lambda_2\|\nabla V_{c,\lambda}(\xi)\|\|\varepsilon\| = c_1\sigma\|\xi\|^p + k_2\bar{\delta}.$$

Next from Lemma 4.4(ii) and Assumption 4.1(iii), we find condition $c\|\varepsilon\|^p = B_1\|\xi\|^p + k_2\bar{\delta}$. Finally in view of Lemma 4.4(i) which suggests

$$(\frac{B_1^{\frac{1}{p}}}{2}\|\xi\| + \frac{(k_2\bar{\delta})^{\frac{1}{p}}}{2})^p \leq B_1\|\xi\|^p + k_2\bar{\delta}$$

the desired modification of (4.5) is given by

$$2\|\varepsilon\| = m_1\|\xi\| + m_2. \tag{4.11}$$

Define $\chi := 2\|\varepsilon\|/(m_1\|\xi\| + m_2)$, it can be concluded that

$$\dot{\chi} \leq \Big(1 + m_1\frac{\chi}{2}\Big)\Big(\frac{2\|\dot{\xi}\|}{m_1\|\xi\| + m_2}\Big) \leq \Big(1 + m_1\frac{\chi}{2}\Big)\Big(\kappa + \lambda_2\chi\Big)$$

where Lemma 4.2 is used to obtain the last inequality. Therefore, $\tau^* = t - t_k$ can be obtained as in (4.10) by solving

$$\dot{\chi} = \Big(1 + m_1\frac{\chi}{2}\Big)\Big(\kappa + \lambda_2\chi\Big)$$

from $t_k$ to $t$ with $\chi(t_k) = 0$. Event rule (4.11) suggests that triggering occurs when $\chi = 1$ and is given by $t_{k+1} = t_k + \tau^*(1)$. In addition, (4.10) implies that $\tau^*(1)$ is strictly nonzero since for $\xi_0 \in \Xi$ and $\|d\|_\infty \leq \epsilon$, Lemma 4.2 suggests that $\lambda_1$, $\lambda_2$, $\lambda_3$, and hence $\kappa$ are bounded. The robust semi-global event-separation property is then obtained from definition of $\tau_m$ and positiveness of $\hat{\tau}$. ■

### 4.3.2 Comparison with the Existing Strategies

In this subsection we study several popular existing ETMs that can be extracted as special cases of (4.5)-(4.7b). We emphasize that the design criteria in these references is not the same so our comparison is merely based on the structure of the TC with no reference to the relative merits or performance in each design, simply because there seems to be no fair way or value in such comparison. Moreover, since some of these works focus on output feedback, in our comparisons we assume the measurable output to be the full state vector.

Our proposed ETM is *dynamic* due to the existence of the dynamic variable $\phi_1$. See [13,28] for discussions regarding the effect of this variable. To the best of our knowledge, the parameter $\phi_2$ has not been introduced before. Thus, we provide the following observations regarding $\phi_2$.

(i) The inter-event expansion that originates from $\phi_2$ can be quantified for a desired period of time, or a desired number of trigger instants (see Section 3.3).

(ii) As shown in the examples provided in Chapter 3, $\phi_2$ serves to avoid redundant samplings when the norm of state is close to 0. This is important since as a primary pitfall, TCs based on the norm of the state tend to increase triggering as the state approaches the origin.

(iii) The primary functionality of $\phi_2$ is to exclude Zeno behaviour as suggested by the proof of Lemma 4.3.

(iv) While the approach in the present article is considered purely event-based, an appropriate choice of parameters in the dynamics of $\phi_2$ enables the TC (4.5) to capture the time-regularization strategies.

We conclude this subsection by extracting several TCs proposed in the literature from (4.5).

- [49] (Chapter 3): For $k_1 = 0$ and $s_k = \bar{\delta}$, TC (4.5) reduces to the one proposed in [49]. In the rest of our comparisons we assume $\varphi_3 = 0$ in (4.6).

- [59]: For $k_1 = 0$ and $\delta_k(t) = s_k = \hat{s}_k = \bar{\delta}$ we obtain $\phi_2 = \bar{\delta}$. Hence, the TC becomes $\varphi(\xi, \varepsilon) = k_2\bar{\delta}$.

- [28]: Take $k_2 = 0$, (4.5), (4.7a) reduce to $\varphi(\xi, \varepsilon) = k_1\phi_1$, $\dot{\phi}_1 + \alpha_1(\phi_1) = -\varphi$.

- [58]: Taking $k_2 = 0$, $k_1 = \infty$ and $\alpha_1(r) = 0$ for any $r$ reduce (4.5) to $\phi_1 = 0$, where $\phi_1(t) = -\int_{t_k}^{t} \varphi(\xi(s), \varepsilon(s))ds$, *i.e.*, the integral-based TC.

- [9]: Substitute $k_1 = k_2 = 0$ in (4.5) one can extract the TC $\varphi = 0$.

- [19, 20]: Define $\hat{t}_k = t_k + \tau_m$ where $\tau_m = \min\{\tau^*, \hat{\tau}\}$. This guarantees no triggering of the control task occurs over $[t_k, \hat{t}_k)$. Then, if we set $k_1 = \infty$ for $t \in [\hat{t}_k, t_{k+1})$, $\varpi_{k+1}$ in (4.4) can be written in a time-regularization fashion as $\varpi_{k+1} = \inf\{t \in \mathbb{R} : t > t_k + \tau_m \bigwedge \ \phi_1 = 0\}$ where $\dot{\phi}_1 = -\varphi$ by setting $k_2 = 0$ and $\alpha_1(r) = 0$ for any $r$ in (4.7a).

- [48]: Set $k_1 = k_2 = 0$ and follow similar lines as in comparison with [19, 20], we get $\varpi_{k+1} = \inf\{t \in \mathbb{R} : t > t_k + \tau_m \bigwedge \ \varphi(\xi(t), \varepsilon(t)) = 0\}$.

- [13]: Let $k_1 = 0$, $\bar{\varphi}(t) = 0$ for all $t \in \mathbb{R}$ and $\hat{s}_k = \phi_2(\hat{t}_k^-)$. Choose $s_k \geq 0$ we have $\phi_2(t) \geq 0$ for all $t \geq t_0$. Then (4.5), (4.7a) reduce to $\varphi_2(\varepsilon) = -\varphi_1(\xi) + k_2\phi_2$, where $\dot{\phi}_2 + \alpha_2(\phi_2) = 0$. In this case, $\phi_2$ plays the role of threshold variable defined in [13]. However, unlike the present work where $\phi_2$ appears in the TC as a positive term that is added to some functions of state's norm, in [13], the admissible measurement error is bounded by the maximum of these two.

### 4.3.3 $\mathcal{L}_p$-Gain Performance

We start with a useful lemma which is an application of Lemma 4.2.

**Lemma 4.5** *Let $a = \lambda_1^p \psi(\hat{\tau}, \lambda_2)$, $b = \lambda_3^p \psi(\hat{\tau}, \lambda_2)$. Then*

$$\int_{t_k}^{\hat{t}_k} \|\varepsilon(\tau)\|^p d\tau \leq a \int_{t_k}^{\hat{t}_k} \|\xi(\tau)\|^p d\tau + b \int_{t_k}^{\hat{t}_k} \|d(\tau)\|^p d\tau. \tag{4.12}$$

**Proof.** We first define the notations below:

$$\mathcal{I}(x) = \int_{t_k}^{s} e^{\lambda_2(s-\tau)} \|x(\tau)\| d\tau, \;\; \mathcal{Q}(s) = (\int_{t_k}^{s} e^{\frac{\lambda_2 q}{2}(s-\tau)} d\tau)^{\frac{p}{q}},$$
$$\mathcal{J}(x) = \int_{t_k}^{s} e^{\frac{\lambda_2 p}{2}(s-\tau)} \|x(\tau)\|^p d\tau, \;\; \mathcal{P}(s) = \int_{t_k}^{s} e^{\frac{\lambda_2 p}{2}(s-t_k)} ds.$$

From definition of $\varepsilon$ and Lemma 4.2 we have

$$\frac{d\|\varepsilon\|}{dt} \leq \|\dot{\varepsilon}\| = \|\dot{\xi}\| \leq \lambda_1\|\xi\| + \lambda_2\|\varepsilon\| + \lambda_3\|d\|,$$

solving which for $\varepsilon(t_k) = 0$ and $s \geq t_k$ gives $\|\varepsilon(s)\| \leq \lambda_1 \mathcal{I}(\xi) + \lambda_3 \mathcal{I}(d)$. Then from Lemma 4.4(i) we conclude

$$\|\varepsilon(s)\|^p \leq 2^p(\lambda_1^p \mathcal{I}^p(\xi) + \lambda_3^p \mathcal{I}^p(d)) \leq 2^p \mathcal{Q}(s)(\lambda_1^p \mathcal{J}(\xi) + \lambda_3^p \mathcal{J}(d))$$

where the last inequality is obtained using Lemma 4.4(ii). It is then straight forward to check that for $t \geq t_k$, $\int_{t_k}^t \mathcal{J}(\xi) ds \leq \mathcal{P}(t) \int_{t_k}^t \|\xi(\tau)\|^p d\tau$ and hence conclude

$$\begin{aligned} \int_{t_k}^{t} \|\varepsilon(s)\|^p ds &\leq 2^p \int_{t_k}^{t} \mathcal{Q}(s)(\lambda_1^p \mathcal{J}(\xi) + \lambda_3^p \mathcal{J}(d)) ds \\ &\leq 2^p \mathcal{Q}(t)\mathcal{P}(t)(\lambda_1^p \int_{t_k}^{t} \|\xi(\tau)\|^p d\tau + \lambda_3^p \int_{t_k}^{t} \|d(\tau)\|^p d\tau). \end{aligned}$$

The proof is then complete taking $t = \hat{t}_k$ since $\psi(\hat{\tau}, \lambda_2) = 2^p \mathcal{Q}(\hat{t}_k)\mathcal{P}(\hat{t}_k)$. ■

**Remark 4.5** *In view of the definition of $a$, $b$ and $\hat{\tau}$, one can verify that $a \leq \lambda_1^p \psi(\tau_1, \lambda_2)$, $b \leq \lambda_3^p \psi(\tau_3, \lambda_2)$. Also from definition of $\tau_1$, $\tau_3$ we conclude $ac < B_1$, $bc \leq B_3$; inequalities that will be used later in the proof of main results.*

Next theorem states our primary result where the finite gain $\mathcal{L}_p$-stability of continuous-time system (4.3) is shown to be preserved under the event-based execution of control task. Compared to [20, 48], our result relies on a less conservative set of assumptions.

**Theorem 4.1** *Under Assumptions 4.1, 4.2, 4.3 and ETM (4.5)-(4.9) the ETC system (4.4) is finite gain $\mathcal{L}_p$-stable with $\mathcal{L}_p$-gain $\leq \mu_d$. In addition, the origin $\xi = 0$ is globally asymptotically stable.*

**Proof.** For $t \in [t_k, \hat{t}_k)$, Assumption 4.1(ii) suggests

$$\dot{V}_c(\xi) \leq \mu^p \|d\|^p - \|z\|^p + \nabla V_c(\xi) \cdot g(\xi)(\gamma(\xi + \varepsilon) - \gamma(\xi)) \tag{4.13}$$

which further reduces to

$$\dot{V}_c(\xi) \leq \mu^p \|d\|^p - \|z\|^p + \lambda_2 \|\nabla V_c(\xi)\| \; \|\varepsilon\|$$

by applying $\|g(\xi)(\gamma(\xi + \varepsilon) - \gamma(\xi))\| \leq \lambda_2 \|\varepsilon\|$ (that is already proven in the proof of Lemma 4.3). Thus, from Lemma 4.4(ii) and Assumption 4.1(iii), we get

$$\dot{V}_{c,\lambda(\xi)} \leq \frac{\lambda^q \bar{c}_3^q}{q} \|\xi\|^p + \frac{\lambda_2^p}{p} \|\varepsilon\|^p + \lambda \mu^p \|d\|^p - \lambda \|z\|^p.$$

As a consequence, for $V(\xi) = V_s(\xi) + V_{c,\lambda}(\xi)$ it follows from Assumption 4.1, (4.12) and (4.13) that

$$\begin{aligned} V(\xi(\hat{t}_k) - V(\xi(t_k)) &\leq -(c_1 - \frac{\lambda^q \bar{c}_3^q}{q} - ac) \int_{t_k}^{\hat{t}_k} \|\xi(\tau)\|^p d\tau \\ &+ (\lambda \mu^p + c_3 + bc) \int_{t_k}^{\hat{t}_k} \|d(\tau)\|^p d\tau - \lambda \int_{t_k}^{\hat{t}_k} \|z(\tau)\|^p d\tau \\ &\leq \lambda \mu_d^p \int_{t_k}^{\hat{t}_k} \|d(\tau)\|^p d\tau - \lambda \int_{t_k}^{\hat{t}_k} \|z(\tau)\|^p d\tau, \end{aligned}$$

where the last inequality follows from Remark 4.5. For $t \in [\hat{t}_k, t_{k+1})$ one can apply the TC (4.5) to calculate an upper bound on $\dot{V}$ as

$$\dot{V}(\xi) \leq -c_1(1 - \sigma)\|\xi\|^p + (\lambda \mu^p + c_3)\|d\|^p - \lambda \|z\|^p + k_2 \phi_2 - \dot{\phi}_1$$

where $-\alpha_1(\phi_1)$ term is eliminated from the right hand side since $\phi_1$ is non-negative. It then follows that

$$\dot{V}(\xi) \leq \lambda \mu_d^p \|d\|^p - \lambda \|z\|^p + k_2 \delta_k - \dot{\phi}_1$$

and hence

$$V(\xi(t_{k+1})) - V(\xi(\hat{t}_k)) \leq \lambda\mu_d^p \int_{\hat{t}_k}^{t_{k+1}} \|d(\tau)\|^p d\tau$$
$$-\lambda \int_{\hat{t}_k}^{t_{k+1}} \|z(\tau)\|^p d\tau + k_2 \int_{\hat{t}_k}^{t_{k+1}} \delta_k(\tau) d\tau + \hat{r}_k.$$

Therefore, we may conclude

$$V(\xi(t_{k+1})) - V(\xi(t_k)) \leq \lambda\mu_d^p \int_{t_k}^{t_{k+1}} \|d(\tau)\|^p d\tau$$
$$-\lambda \int_{t_k}^{t_{k+1}} \|z(\tau)\|^p d\tau + k_2 \int_{\hat{t}_k}^{t_{k+1}} \delta_k(\tau) d\tau + \hat{r}_k.$$

Apply this inequality to the sampling intervals until $t \geq t_0$, the positive definiteness of $V$ can be employed to write

$$\int_{t_0}^{t} \|z(\tau)\|^p d\tau \leq \mu_d^p \int_{t_0}^{t} \|d(\tau)\|^p d\tau + \frac{1}{\lambda}(k_2\theta_1 + \theta_3 + V(\xi_0)).$$

This proves $\mathcal{L}_p$-stability of ETC system (4.4) with $\mathcal{L}_p$-gain $\leq \mu_d$. To show asymptotic stability, let $d = 0$. Using a similar process as we prove of $\mathcal{L}_p$-stability, it can be shown then suggests that for any for $\lambda = 0$, any $t \geq t_0$,

$$V_s(\xi(t)) \leq -c_1(1-\sigma) \int_{t_0}^{t} \|\xi(\tau)\|^p d\tau + k_2\theta_1 + \theta_3 + V_s(\xi_0).$$

This proves the ultimate boundedness of trajectories of system (4.4). However, global asymptotic stability is postponed to show that for any $\epsilon \in \mathbb{R}^+$ there exists some $\delta \in \mathbb{R}^+$ such that if $\|\xi_0\| \leq \delta$, $\|\xi(t)\| \leq \epsilon$ for all $t \geq t_0$ and $\lim_{t\to\infty} \xi(t) = 0$. This is achieved by redefining $\delta_k(t)$ (resp. $\hat{r}_k$) as $\lambda_0 V_s(\xi_0)\delta_k(t)$ (resp. $\lambda_0 V_s(\xi_0)\hat{r}_k$) for some $\lambda_0 \in \mathbb{R}^+$. Thus by choosing

$$\delta = V_s^{-1}\Big(\frac{V_s(\epsilon)}{1 + \lambda_0(k_2\theta_1 + \theta_3)}\Big)$$

for a given $\epsilon$, we have

$$V_s(\xi(t)) \leq -c_1(1-\sigma) \int_{t_0}^{t} \|\xi(\tau)\|^p d\tau + V_s(\epsilon),$$

*i.e.*, $\|\xi(t)\| \leq \epsilon$ for all $t \geq t_0$. Convergence of $\xi$ to zero is easy to show, thus the details are left to the interested readers. ■

### 4.3.4 Inter-Event Time Enlargement

In the sequel, we present an important feature of TC (4.5) on extending inter-event times. For this purpose, we define

$$\tau^*_{max} \doteq \max\{\tau^* : \bar{\rho}, \chi \in \mathbb{R}_0^+\},$$

which in view of the following theorem, upperbounds the new extended inter-event times. Note that in this definition $\tau^*$ is assumed to be a function of $\bar{\rho}$ and $\chi$, a fact suggested by (4.10) and dependence of $\lambda_i$, $i \in \{1, 2, 3\}$ on $\bar{\rho}$ (that is defined in Lemma 4.1).

**Theorem 4.2** *For any $T^\circ \in \mathbb{R}^+$ and $\tau^\circ \in [0, \tau^*_{max}]$, $\bar{\varphi}$ in (4.7b) can be designed in a way that $\tau_m \geq \tau^\circ$ at least for $t \leq T^\circ$.*

**Proof.** To find a lower bound on inter-event times, let us restrict the TC (4.5) to $\varphi(\xi, \varepsilon) = k_2\phi_2$ by taking $k_1 = 0$. Recalling the proof of Lemma 4.3 where the triggering happens when $\chi = 1$, our goal here is to design $\phi_2$ so that the triggering occurs for some $\chi > 1$. Note that $\tau^*_{max} \geq \tau^*(1)$ by definition. Due to continuity of $\tau^*$ in (4.10), for any $\tau^\circ \in [0, \tau^*_{max}]$ one can find $\chi^\circ$ (obviously $\geq 1$) so that $\tau^\circ = \tau^*(\chi^\circ)$. It only remains to choose the TC such that $\chi \geq \chi^\circ$ at sampling instants. With the same notation as in Lemma 4.3, let $\delta^* := {\chi^*}^2\bar{\delta}$ where $\chi^* = \chi^\circ + \frac{m_1}{m_2}\bar{\rho}(\chi^\circ - 1)$. We redefine $\bar{\varphi}$ in (4.7b) as

$$\bar{\varphi}(t) = \begin{cases} \alpha_2(\delta^*), & t \in [0, T^\circ), \\ 0, & \text{elsewhere.} \end{cases}$$

This implies $\phi_2(t) = \delta^*$ for $t \in [0, T^\circ)$. Then following similar lines as we derived (4.11), the lower bound on the inter-event times can be calculated by assuming the TC

$$2\|\varepsilon\| = m_1\|\xi\| + \chi^* m_2.$$

From definition of $\chi$ given in the proof of Lemma 4.3 it is easy to verify that

$$\chi = \frac{m_1\|\xi\| + \chi^* m_2}{m_1\|\xi\| + m_2} \geq \chi^\circ$$

at triggering instants and hence inter-event times are lower bounded by $\tau^\circ$ for $t \leq T^\circ$. ■

**Remark 4.6** *Theorem 4.2 explores one of the advantages of our proposed strategy where the inter-event times are extended to $\tau^\circ$ for $t \in [0, T^\circ]$. The numerical example in section 3.4 suggests that the average sampling time is also improved in this interval. Note that while the results are not explicitly applicable to $t > T^\circ$, numerical examples in Chapter 3 verify the efficiency of this technique for all $t \geq t_0$.*

## 4.4 Example

### 4.4.1 System Model

Consider the system (4.3) with $\xi = [\xi_1 \ \xi_2]^\mathsf{T}$ and

$$f(\xi, d) = \begin{pmatrix} \xi_2 \\ -H(\xi_1) + d \end{pmatrix}, \quad g(\xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h(\xi, d) = \xi_1,$$

$u(t) = \gamma(\xi(t)) = -\xi_2(t))$. The piecewise linear function $H : \mathbb{R} \mapsto \mathbb{R}$ is given by: $H(r) = 2r$ for $|r| \leq h^*$, $H(r) = h^* + r$ for $r \geq h^*$ and $H(r) = -h^* + r$ for $r \leq -h^*$, some $h^* \in \mathbb{R}_0^+$ for some non-negative $h^*$. Note that $H$ satisfies $r^2 \leq rH(r) \leq 2r^2$ for any $r \in \mathbb{R}$. In the following, we study the finite gain $\mathcal{L}_2$-stability this system under event-based implementation of control law.

### 4.4.2 Verification of Assumption 4.1

Taking $V_s(\xi) = \frac{\upsilon_1}{2}\xi^\top P\xi + 2\upsilon_1 \int_0^{\xi_1} H(r)dr$, $P = [1\ 1; 1\ 2]$, it is straight forward to see that item (i) holds for $c_1 = \frac{\upsilon_1}{2}$, $c_2 = c_3 = 5\upsilon_1$ (see Example 3.2 for an in-depth analysis). We remark that $\upsilon_1$ is the scaling factor discussed in Remark 4.4. To show item (ii), we start with $\frac{1}{\upsilon_1}\dot{V}_s(\xi) = -\xi_1 H(\xi_1) - \xi_2^2 + (\xi_1 + 2\xi_2)d \leq -(1-n_1)\xi_1^2 - (1-n_2)\xi_2^2 + (\frac{1}{4n_1} + \frac{1}{n_2})d^2 - n_1(\xi_1 - \frac{d}{2\sigma})^2 - n_2(\xi_2 - \frac{d}{n_2})^2$ for some positive $n_1$, $n_2$. Choosing $V_c(x) = \frac{1}{\upsilon_1(1-n_2)}V_s(x)$ yields $\dot{V}_c(x) \leq |z|^2 - \mu^2|d|^2$ where $\mu^2 = \frac{1}{1-n_2}(\frac{1}{4n_1} + \frac{1}{n_2})$. The minimum value of $\mu$ is 4.49 and is obtained for $n_1 = 1$, $n_2 = 0.47$. Note that one may find a less conservative bound from a more suitable choice of $V_c$. Finally, it is easy to see that item (iii) holds for $\bar{c}_1 = \frac{\upsilon_1}{2}\lambda_{max}([5\ 1; 1\ 2])$, $\bar{c}_2 = \frac{1}{\upsilon_1(1-n_2)}\bar{c}_1$ and $\bar{c}_3 = \frac{1}{1-n_2}(\|P\| + 4)$, where $\lambda_{max}(\cdot)$ stands for the maximum eignevalue of a desired real matrix.

### 4.4.3 Triggering Condition

Our design criteria is to guarantee $\mu_d \leq 5$. For this purpose, we consider here two scenarios for $\delta_k$ in (4.7b):

$$\delta_k^1(t) = D_1 e^{-\varrho_1 t}, \quad \delta_k^2(t) = D_2 \frac{\varrho_2^n}{n!}, \ n = \lceil \frac{t}{\bar{n}} \rceil,$$

where $D_1 = 10$, $D_2 = 2$, $\varrho_1 = 0.05$, $\varrho_2 = 3$, $\bar{n} = 10$. Also, we consider $\alpha_1(r) = \alpha_2(r) = r$ in (4.7a). To cover all strategies discussed in Section 4.3.2, we categorize our analysis into six possible cases, depending on the values of the parameters $k_1$, $k_2$, $\delta_k^1$, $\delta_k^2$.

| case: | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---|---|---|---|---|---|---|
| $(k_1, k_2)$ | $(1, 1)$ | $(1, 1)$ | $(1, 0)$ | $(0, 1)$ | $(0, 1)$ | $(0, 0)$ |
| $\delta_k$ | $\delta_k^1$ | $\delta_k^2$ | $n/a$ | $\delta_k^1$ | $\delta_k^2$ | $n/a$ |

Cases (i), (ii) are the general dynamic triggering scenarios with both $\phi_1$, $\phi_2$ effective in condition (4.5). The role of $\phi_1$ (resp. $\phi_2$) is studied in case (iii) (resp. cases (iv), (v)). Also, case (vi) results in static TC since both $\phi_1$, $\phi_2$ are absent.

It is not difficult to verify $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 1$ in Lemma 4.2. Therefore, we may choose $\lambda = 4.7 \times 10^{-3}$, $\upsilon_1 = 3.6 \times 10^{-3}$ (which satisfy the required bounds on $\lambda$ given in Remark

4.4) and obtain $\hat{\tau} = 8.9 \times 10^{-3}$ from (4.9). Finally, we take $\bar{\delta} = 10$, $r_k = 0$, $\hat{r}_k = \phi_1(\hat{t}_k^-)$, $s_k = 12.5$.

### 4.4.4 Numerical Simulation

Signal $d(t)$ follows a zero mean Gaussian distribution with variance 1 over $t \in [0, 100)$ and zero everywhere else. We also take $h^* = 0.3$ and run the simulation for 100 initial conditions uniformly distributed in a circle of radius 1 over 100 seconds and finally average the results. The plots are provided for initial condition $\xi_0 = (\sin(\frac{\pi}{3}), \cos(\frac{\pi}{3}))$.

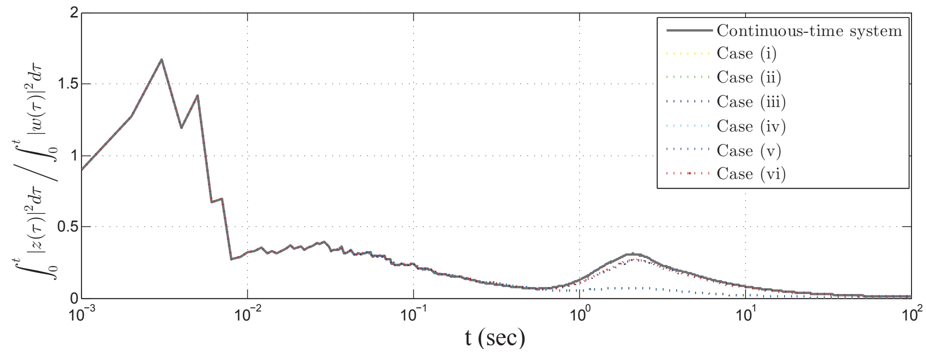



Figure 4.1: Verification of $\mathcal{L}_2$-gain.

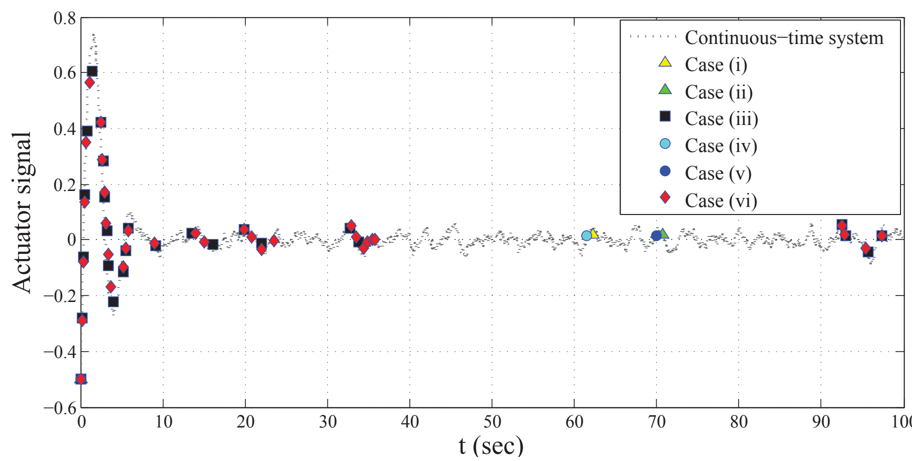



Figure 4.2: Actuator signal at the triggering instants

Table 4.1: Comparison of different scenarios.

| case: | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---|---|---|---|---|---|---|
| $N$ | 3.24 | 3.25 | 12.9 | 4.34 | 4.72 | 18.7 |
| $\tau_m \times 10^2$ | 22.3 | 14.2 | 3.3 | 22.6 | 14.8 | 1.8 |

Table 4.1 illustrates the number of triggerings ($N$) and MIET ($\tau_m$) for different scenarios. Note that the valus of $\tau_m$ are in msec. Comparing different cases, it is clear that both $\phi_1$ and $\phi_2$ improve transmission rate, however, when $k_2$ is non-zero, the number of samples

and $\tau_m$ improve more significantly. This implies the effectiveness of parameter $\phi_2$ compared to the $\phi_1$. This example suggests that when the trajectories of open-loop ETC system are either converging to the origin or staying bounded, since $\varepsilon$ remains bounded, an appropriate choice of $\bar{\delta}$ and $\delta_k$ in $\phi_2$ avoid unnecessary samplings effectively.

We remark that, suggested by Table 4.1, the value of $\tau_m$ is much greater than $\hat{\tau}$; an important feature of our design compared to time-regularization method. In fact, contrary to our approach, time-regularization TC often result in periodic samplings ($\hat{\tau}$ in the case of our design) whenever state is near origin.

## 4.5 Summary

This chapter introduces a framework for TC design. Although the proposed structure is originally stated for $\mathcal{L}_p$ performance, the approach can be applied to different ETC system problems. Our design introduces several design variables, used for different purposes and we have shown that by proper selection of these variables, several existing TC proposed in the recent literature can be extracted. Also the triggering instants are shown to be uniformly isolated in presence of exogenous disturbance or sensor noise.

Our main contribution is the proposed ETM based on two dynamic variables $\phi_1$ and $\phi_2$. Indeed, $\phi_1$ has a role similar to the one introduced in [28] and is intended to enlarge inter-event times. $\phi_2$, on the other hand, is also used to extend the inter-event times, but has the critical roles of (i) enabling us to analytically predict the increase of inter-event times for a desired period of time, and, more importantly, (ii) excluding Zeno behaviour.

# Chapter 5

# Robust Analysis of Affine Nonlinear Systems Under Network Constraints

## 5.1 Problem Setup

In this chapter we consider the following nonlinear system

$$\begin{cases} \dot{x} = \xi_x(x) + \xi_u(x)u + \xi_w(x)w, \\ z = \eta_x(x) + \eta_u(x)u + \eta_w(x)w. \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ the control input, $w \in \mathcal{L}_2^q$ the exogenous disturbance and $z \in \mathbb{R}^p$ the measured output. Also $\xi_x, \xi_u, \xi_w, \eta_x, \eta_u, \eta_w$ are smooth mappings. We assume that system (5.1) starts off the initial condition $x_0$ at time $t_0 = 0$, i.e., $x_0 = x(0)$. The control signal is sampled at the triggering instants $t_\ell$, $\ell \in \mathbb{Z}_{\geq 0}$ and is held constant between samples using a zero-order hold device. For simplicity, we assuming full state information and consider a state feedback law. This assumption limits the generality of the results but it is analytically convenience. Therefore, using the smooth state feedback law $u = \alpha(x)$, the actuator signal is

$$u(t) = \alpha(x(t_\ell)),\ t \in [t_\ell, t_{\ell+1}). \tag{5.2}$$

In contrast to time-triggered scheme, the updating control instants are by no means specified a *priori*. Instead, the system makes autonomous decisions using a triggering module through continuous monitoring of system's state. This triggering condition, however, has to be designed in accordance to the desired design requirements. Also, to simplify the analysis, we assume that the control task is executed without delay and data dropouts immediately following the update by the triggering module. Interested readers are referred to [9], [16] to see how to deal with these practical considerations. Indeed, as shown by these references, there is a tradeoff between the maximum tolerable delay/maximum allowable number of successive data dropouts and the size of broadcast intervals $(t_\ell, t_{\ell+1})$.

To make use of the theory of continuous time signals, we state the sampled signal $u$ in terms of the actuation error, defined as the difference between the current and the last executed control signals, *i.e.*, $\varepsilon(t) := \alpha(x(t_\ell)) - \alpha(x(t))$, $t \in [t_\ell, t_{\ell+1})$. Then the actuator signal is $u(t) = \alpha(x(t)) + \varepsilon(t)$ and $\varepsilon$ can be treated as an exogenous signal applied to the system. Therefore, (5.1) reduces to

$$\mathcal{P} : \begin{cases} \dot{x} = f_0(x, \alpha) + \xi_u(x)\varepsilon + \xi_w(x)w, \\ z = h_0(x, \alpha) + \eta_u(x)\varepsilon + \eta_w(x)w, \end{cases} \tag{5.3}$$

where $f_0(x, \alpha) = \xi_x(x) + \xi_u(x)\alpha$, $h_0(x, \alpha) = \eta_x(x) + \eta_u(x)\alpha$. For the sake of brevity, throughout the rest of this paper we adopt the notation $f(x, \alpha, w, \varepsilon) = f_0(x, \alpha) + \xi_u(x)\varepsilon + \xi_w(x)w$ and $h(x, \alpha, w, \varepsilon) = h_0(x, \alpha) + \eta_u(x)\varepsilon + \eta_w(x)w$. We assume that the function $\alpha$ satisfies $f_0(0, \alpha(0)) = 0$ and $h_0(0, \alpha(0)) = 0$ implying that the feedback control law $u = \alpha(x)$ renders the origin $x = 0$ of following unperturbed closed loop system

$$\dot{x} = f_0(x, \alpha), \ z = h_0(x, \alpha) \tag{5.4}$$

stable. Critical to any event-triggered control design is the uniform isolation of triggering instants, known as the event separation principle [59]. Therefore, for a set of triggering instants to be admissible, its elements must be isolated according to the following definition (see [66]):

**Definition 5.1 (Uniform isolation of triggering instants)** *The sequence of triggering instants $\{t_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ is said to be uniformly isolated if and only if there exists $\tau_m \in \mathbb{R}_{>0}$ so that for any $j, \ell \in \mathbb{Z}_{\geq 0}$, $j \neq \ell$ we have $|t_\ell - t_j| > \tau_m$.*

The minimum inter-event time is then defined as the largest possible $\tau_m$ in Definition 5.1. The desired robustness criteria is provided in the next definition, [5].

**Definition 5.2 (Robust $\mathcal{L}_2$ gain property)** *The system (5.1) is said to be finite gain $\mathcal{L}_2$-stable with $\mathcal{L}_2$ gain $\leq \gamma$ for some positive $\gamma$ provided that there exist $\mu > 0$ and bias term $\bar{\mu} \geq 0$ such that the quadratic cost function*

$$J(w, z) := \|z|_{[0,t]}\|^2 - \gamma^2 \|w|_{[0,t]}\|^2 \tag{5.5}$$

*satisfes $J(w, z) \leq \mu(x_0) + \bar{\mu}$ for any $x_0 \in \mathbb{R}^n$, any $t \in \mathbb{R}_{\geq 0}$ and any $w \in \mathcal{L}_2^q$.*

In this paper we first aim to provide sufficient conditions for a given controller to be ISS stabilizing w.r.t. actuation error and disturbance. Then we exploit the generalized versions of these conditions to design an event-triggered controller for the system $\mathcal{P}$ defined in (5.3) following both emulation and joint design approaches, where the resulting event-based system is restricted to

- meet the desired robust $\mathcal{L}_2$ gain property (5.5),

while also

- ensuring the uniform isolation of sampling instants in presence of arbitrary disturbances.

## 5.2 Sufficient condition for input-to-state stability

The majority of the literature on robust analysis of event-triggered systems takes the input-to-state stability as a primary assumption. Checking the existence of an ISS stabilizing controller for systems of type (5.3) is not difficult. Indeed, as shown in [72] for the input-affine structure, global asymptotic stabilizability in absence of inputs implies global input-to-state stabilizability. In the realm of event-triggered control, this is interpreted as if the unperturbed model (5.4) can be stabilized in the sense of Lyapunov, then there exists an state feedback $\alpha(x)$ which renders the system $\mathcal{P}$ in (5.3) actuation error-to-state and disturbance-to-state stable. However, finding an ISS stabilizing $\alpha(x)$ is non-trivial and challenging in general. For example, consider $\mathcal{P}$ with $\xi_x = 0$, $\xi_u = 1$ and $\xi_w = x^2$. It is obvious that while $\mathcal{P}$ is not ISS w.r.t. $\varepsilon$ and $w$ for any linear choice of $\alpha(x) = -c_1 x$, the ISS property holds when nonlinearities of higher order than $\xi_w$ are added to the control law, *e.g.*, $\alpha(x) = -c_1 x - c_2 x^3$. This simple example motivates the necessity of finding sufficient condition(s) to filter a proposed controller to be ISS stabilizing. In other words, while the result of [72] serves as a necessary condition for the existence of an ISS stabilizing controller for $\mathcal{P}$, sufficient condition(s) are still required to check whether a given controller is ISS stabilizing or not. To derive results the following assumptions are required:

(**A1**) There exist $\gamma_u, \gamma_w \in \mathbb{R}_{>0}$ such that

(i) $R_u(x) := \gamma_u^2 I - \eta_u^{\mathsf{T}}(x)\eta_u(x) > 0$,

(ii) $R_w(x) := \gamma_w^2 I - \eta_w^{\mathsf{T}}(x)\eta_w(x) > 0$.

(**A2**) There exist a Lyapunov function $V$, class-$\mathcal{K}_\infty$ functions $\underline{\beta}, \overline{\beta}$ and a locally Lipschitz function $\alpha$ such that[1]

$$\underline{\beta}(x) \leq V(x) \leq \overline{\beta}(x) \tag{5.6}$$

and the following NMI hold for all $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{pmatrix} V_x^\mathsf{T} f_0(x,\alpha) + \|h_0(x,\alpha)\|^2 & * & * & * \\ \frac{1}{2}\xi_u^\mathsf{T}(x)V_x + \eta_u^\mathsf{T}(x)h_0(x,\alpha) & -R_u & * & * \\ \frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha) & 0 & -R_w & * \\ \phi(x) & 0 & 0 & -I \end{pmatrix} < 0. \tag{5.7}$$

According to the next theorem, any controller $\alpha$ that is obtained from (5.7) captures the ISS property for $\mathcal{P}$.

**Theorem 5.1** *Suppose that assumptions (**A1**), (**A2**) hold for some $\phi \in \mathcal{K}_\infty$ and $\eta_w^\mathsf{T}\eta_u = 0$. Then $\alpha$ renders the system $\mathcal{P}$ defined in (5.3) ISS w.r.t. actuation error and input disturbance.*

**Proof.** For the sake of brevity, we shall adopt the notation:

$$\hat{\alpha}_2 := R_u^{-1}(x)(\frac{1}{2}\xi_u^\mathsf{T}(x)V_x + \eta_u^\mathsf{T}(x)h_0(x,\alpha)),$$
$$\hat{w} := R_w^{-1}(x)(\frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha)).$$

Using Schur complement argument, we have that (5.7) holds if and only if

$$V_x^\mathsf{T} f_0(x,\alpha) + \|h_0(x,\alpha)\|^2 + \|\hat{w}\|_{R_w}^2 + \|\hat{\alpha}_2\|_{R_u}^2 + \phi^2(x) \leq 0 \tag{5.8}$$

for all $x \in \mathbb{R}^n$. Now from $\eta_w^\mathsf{T}\eta_u = 0$ it can be checked that

$$\|\eta_w^\mathsf{T}\eta_u\varepsilon\|_{R_w^{-1}}^2 + 2\langle\hat{w}, \eta_w^\mathsf{T}\eta_u\varepsilon\rangle + \varepsilon^\mathsf{T}\eta_u^\mathsf{T}\eta_u\varepsilon \leq \gamma_u^2\|\varepsilon\|^2.$$

Hence by adding the last two inequalities and using the completion of squares we conclude that

$$\begin{aligned} V_x^\mathsf{T} f(x,\alpha,w,\varepsilon) + \|h(x,\alpha,w,\varepsilon)\|^2 - \gamma_w^2\|w\|^2 + \|\varepsilon - \hat{\alpha}_2\|_{R_u} \\ + \|w - \hat{w} - R_w^{-1}\eta_w^\mathsf{T}\eta_u\varepsilon\|_{R_w}^2 \leq \gamma_u^2\|\varepsilon\|^2 - \phi^2(x). \end{aligned} \tag{5.9}$$

As a result, we have $V_x^\mathsf{T} f(x,\alpha,w,\varepsilon) \leq -\phi^2(x) + \gamma_u^2\|\varepsilon\|^2 + \gamma_w^2\|w\|^2$ which implies $V$ is an ISS Lyapunov function. Thus, $\mathcal{P}$ is ISS w.r.t. actutor error and exogenous disturbance, [5]. ■

[1] Condition (5.6) implies that $V$ is positive definite and radially unbounded.

Several observations are in order. As suggested by (5.9), the set of control laws $\alpha$ obtained from Theorem 5.1 are not only input-to-state stabilizing but also guarantees $\mathcal{L}_2$ gain property (5.5). Condition $\eta_w^\mathsf{T}\eta_u = 0$ and assumption (**A1**) are borrowed from [73] and is intended to simplify the analysis. Relaxing these assumptions, however, requires more involved mathematical manipulations and possibly additional conditions and hence is left as a follow-up work. Condition $\eta_w^\mathsf{T}\eta_u = 0$ implies that there is no coupling between exogenous and control inputs, however, it is not essential for our main results on the event-triggered control design and will be relaxed in the next section. In addition, assumption (**A1**) states that the control and disturbance weight matrices are norm bounded. When $\|\eta_u(x)\|_\infty$, $\|\eta_w(x)\|_\infty$ are bounded for all $x \in \mathbb{R}^n$, where the $\infty$-norm of a matrix is defined as the maximum absolute row sum of the matrix, assumption (**A1**) can always be satisfied by choosing $\gamma_u$, $\gamma_w$ sufficiently large. A linear counterpart of this assumption can also be found in [74]. We also have the following connection between the result of Theorem 5.1 and [72]. As shown in Section 5.3.1, global asymptotic stability of unperturbed system (5.4) is guaranteed under assumptions (**A1**), (**A2**). Thus, [72] suggests the existence of an ISS stabilizing controller and Theorem 5.1 provides sufficient conditions to characterize such a controller.

### 5.2.1 Robustness with respect to sensor measurement error

As defined in (5.3), our system is expressed in terms of the actuation error. An alternative is to employ the sensor measurement error, $e(t) := x(t_\ell) - x(t)$, $t \in [t_\ell, t_{\ell+1})$, to design the event-triggered rule. Using the sensor error has the advantage that no additional processing time is required to calculate $\varepsilon(t)$. However, doing so presents a fundamental problem: indeed, it was shown in reference [75] that the ISS result of [72] fails to hold when the input in the ISS condition is with sensor noise/error. Indeed, this reference shows via a counter example, that ISS is used w.r.t. the sensor errors there is the possibility of having a finite scape time. Additionally, to the best of our knowledge, there exist no general sufficient condition for input-to-state stability of general nonlinear structures w.r.t. sensor noise. Thus, the use of the input-to-state stability w.r.t. sensor error in [9] and related works should be seen as a primitive assumption, but a more elaborate solution is clearly imperative. It is worth remarking that when $\alpha$ is designed to be a globally Lipschitz function of its arguments, then Theorem 5.1 ensures the ISS property w.r.t. sensor noise as well.

## 5.3 Event-triggered design for robust stability

In this section, we study the $\mathcal{L}_2$ gain stability of system (5.3) under event-triggered communications following two separate scenarios, namely, emulation and joint design. In the emulation approach, we begin by designing a continuous-time controller that stabilize the system in satisfies an $\mathcal{L}_2$ gain performance condition. The event condition is designed afterwards to retain stability and with minimal deterioration of the $\mathcal{L}_2$ gain performance. This is referred to as *emulation* approach. The emulation approach is effective and relatively simple in the sense that breaks down the complexity of original event-based control design into two simple stages. However, as suggested by [30], since the control law is originally designed for the network-free problem, the performance level in presence of a network may not necessarily be optimal. This motivates the more challenging *joint* design approach, where controller and triggering conditions are designed simultaneously.

### 5.3.1 Joint design method

As mentioned earlier, in the joint design approach the event condition is designed together with the control law to achieve a desired $\mathcal{L}_2$ gain bound. Since the controller is designed directly based on commuincation constraint requirements, the resulting event-based system is expected to enjoy an improved performance compared to emulation approach. To proceed with the event-based control design, we need the following generalized versions of assumptions (**A1**), (**A2**), in which the positive definiteness assumption on $R_u(x)$ is relaxed whenever $\Sigma_u(x) := \eta_u^\mathsf{T}(x)\eta_u(x) > 0$.

(**A3**) There exist $\gamma_u, \gamma_w \in \mathbb{R}_{>0}$ such that

(i) $R_u(x) > 0 \bigvee \Sigma_u(x) > 0$,

(ii) $R_w(x) > 0$.

(**A4**) There exist a Lyapunov function $V$, class-$\mathcal{K}_\infty$ functions $\underline{\beta}, \overline{\beta}$, a locally Lipschitz $\alpha$ and some invertible $\phi$ with $\phi(0) = 0$ and locally Lipschitz inverse $\phi^{-1}$ such that (5.6) holds and

- when $\Sigma_u(x) > 0$ the following holds for all $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{pmatrix} V_x^\mathsf{T} f_0(x,\alpha) + \|h_0(x,\alpha)\|^2 & * & * \\ \frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha) & -R_w & * \\ \phi(x) & 0 & -I \end{pmatrix} < 0, \tag{5.10}$$

- otherwise (5.7) holds for all $x \in \mathbb{R}^n \setminus \{0\}$.

We now state the main result of this section.

**Theorem 5.2 (Event-triggered robustness)** *Suppose that assumptions (**A3**), (**A4**) hold. The triggering rule can be designed for system $\mathcal{P}$ so that*

- *the triggering times are uniformly isolated for any bounded $x_0$ and $w$,*

*and the resulting event-based system is:*

- *$\mathcal{L}_2$-stable with $\mathcal{L}_2$ gain $\leq \gamma_d$ for some $\gamma_d > \gamma_w$,*
- *asymptotically stable.*

*In detail, once (5.7) or (5.10) is solved for $V$, $\alpha$, $\gamma_u$, $\gamma_w$, $\phi$, the triggering condition is given by*

$$\rho(\varepsilon) + \|\eta_w^\intercal \eta_u \varepsilon\|^2_{R_w^{-1}} + 2\langle \hat{w}, \eta_w^\intercal \eta_u \varepsilon\rangle \leq \Phi(x,t), \tag{5.11}$$

*where*

$$\rho(\varepsilon) = \begin{cases} \|\varepsilon - \hat{\alpha}_1\|^2_{\Sigma_u} - \|\hat{\alpha}_1\|^2_{\Sigma_u}, & \text{when } \Sigma_u(x) > 0, \\ \gamma_u^2 \|\varepsilon\|^2, & \text{otherwise,} \end{cases}$$

$$\Phi(x,t) = \varsigma \phi^2(x) + \begin{cases} \delta_1, & t_\ell < t \leq t_\ell + \hat{\tau}, \\ \delta_2(t), & t_\ell + \hat{\tau} < t \leq t_{\ell+1}, \end{cases} \tag{5.12}$$

*and $\delta_2(t) < \min\{\delta_3 e^{-\delta_4 t}, (1-\varsigma)\phi^2(x(t))\}$, for some $\delta_1, \delta_3, \delta_4, \hat{\tau} \in \mathbb{R}_{>0}$, $0 < \varsigma < 1$ and*

$$\hat{\alpha}_1 := \Sigma_u^{-1}(x)(\frac{1}{2}\xi_u^\intercal(x) V_x + \eta_u^\intercal(x) h_0(x,\alpha)).$$

The role of constant term $\delta_1$ in (5.12) is to avoid the possible accumulation of sampling instants as discussed in details in [59]. Indeed, in this reference this is carried out at the price of obtaining a practical stability performance for the resulting system. Here, we follow the approach of [66] to avoid such a weak conclusion by restricting $\delta_1$ to be effective only for periods of length $\hat{\tau}$. Therefore, $\hat{\tau}$ has to be designed carefully so that $\mathcal{L}_2$ gain stability of system $\mathcal{P}$ under triggering condition (5.11) is not violated during the interval $[t_\ell, t_\ell + \hat{\tau})$. In this sense, parameter $\hat{\tau}$ has the role of dwell-time concept in time-regularization approach but in a generalized way since the actuator is allowed to trigger during the interval $[t_\ell, t_\ell + \hat{\tau})$, hence better system performance may be attained. On the other hand, $\delta_2$ is a time varying parameter intended to enlarge the broadcast intervals without violating the stability goals defined in Section 5.1. As shown in [66, 76, 77], another role of $\delta_2$ is to turn practical stability results obtained under constant triggering threshold into asymptotic stability. To prove Theorem 5.2 we need the following boundedness result, whose proof is provided in the sequel.

**Lemma 5.1** *The trajectores $x$ of system $\mathcal{P}$ are bounded by some functions of $\|x_0\|$ and $|w|_\infty$.*

**Proof of Theorem 5.2.** We first provide a methodology for designing $\hat{\tau}$. For other parameters $\delta_1, \delta_3, \delta_4$ any positive choices are admissible. The design of $\hat{\tau}$ consists of finding the maximum possible $\hat{\tau}$ such that the $\mathcal{L}_2$ stability of $\mathcal{P}$ with the desired performance level is not violated over $[t_\ell, t_\ell + \hat{\tau})$. Under the assumption on smoothness of the mappings in (5.1), Lemma 5.1 implies the existence of some $a, b$ functions of $\|x_0\|$, $|w|_\infty$ so that the LHS of (5.11) can be upper bounded by $a\|\varepsilon\|^2 + b\|\varepsilon\|\|x\|$. We continue by finding an upper bound on the norm of actuation error. Since $\|\dot{\varepsilon}(t)\| \leq \lambda_k \|\dot{x}(t)\|$, from (5.3) and smoothness of $\xi_x, \xi_u, \xi_w, \alpha$, we conclude that there exists $l_1, l_2, l_3 \in \mathbb{R}^+$ such that $\|\dot{\varepsilon}\| \leq l_1\|x\| + l_2\|\varepsilon\| + l_3\|w\|$. It is then not difficult to show that

$$\|\varepsilon|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2 \leq L_1(\hat{\tau})\|x|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2 + L_2(\hat{\tau})\|w|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2$$

for some $L_1, L_2 \in \mathcal{K}_\infty$. Using Lemma 5.1 it holds that $l_1, l_2, l_3, L_1, L_2$ are in general functions of $\|x_0\|$ and $|w|_\infty$. Therefore, if one choose the maximum possible $\hat{\tau}$ so that

$$(a + bc)L_1(\hat{\tau}) + \frac{b}{c} < \lambda_{\phi^{-1}}^{-2} \bigwedge (a + bc)L_2(\hat{\tau}) < \gamma_d^2 - \gamma_w^2$$

for some $c \in \mathbb{R}^+$, it is not difficult to verify that integral from $t_\ell$ to $t_\ell + \hat{\tau}$ of LHS of (5.11) is upper bounded by $\|\phi|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2 + (\gamma_d^2 - \gamma_w^2)\|w|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2$ and hence in view of assumption (**A4**), it can be checked that

$$V(x(t_\ell+\hat{\tau})) - V(x(t_\ell)) \leq \gamma_d^2\|w|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2 - \|z|_{[t_\ell, t_\ell+\hat{\tau}]}\|^2 \tag{5.13}$$

Part one: Without loss of generality we will assume $\tau_m \leq \hat{\tau}$ since otherwise one can simply choose $\tau_m = \hat{\tau}$. The previous discussion leads us to the fact that the inter sampling intervals obtained from triggering condition (5.11) are lower bounded by those obtained from condition $a\|\varepsilon\|^2 + b\|\varepsilon\|\|x\| - \lambda_{\phi^{-1}}^{-2}\|x\|^2 = \delta_1$. Similarly, the lower bound on the triggering instants of this condition is in turn obtained from $\|\varepsilon\| = \sigma_1\|x\| + \sigma_2$ where

$$\sigma_1 = \frac{1}{2a}\Big(\Big(\frac{4a\lambda_{\phi^{-1}}^{-2} - b^2}{1 + c^{-1}}\Big)^{\frac{1}{2}} - b\Big),\ \sigma_2 = \Big(\frac{\delta_1}{a(1+c)}\Big)^{\frac{1}{2}},$$

where $c > b^2\lambda_{\phi^{-1}}^2/4a$ is an arbitrary constant. Now let $\chi := \|\varepsilon\|/\bar{\delta}$, we can apply Lemma 5.1 to conclude there exists some $\bar{L}$ such that $\dot{\chi} \leq \bar{L}/\bar{\delta}$. Since $\chi = 1$ at the triggering instants, we obtain $\hat{\tau} \geq \bar{\delta}/\bar{L}$.

Part two: To prove $\mathcal{L}_2$ stability, we first need to extend (5.13) to $t \in [t_\ell + \hat{\tau}, t_{\ell+1})$. In this interval, by applying Schur complement to (5.10) we obtain

$$V_x^\mathsf{T} f_0(x, \alpha) + \|h_0(x, \alpha)\|^2 + \|\hat{w}\|^2_{R_w} + \phi^2(x) \leq 0 \tag{5.14}$$

for all $x \in \mathbb{R}^n$, which by adding to triggering condition (5.11) gives

$$\begin{aligned} &V_x^\mathsf{T} f_0(x, \alpha) + \|h_0(x, \alpha)\|^2 + \|\varepsilon - \hat{\alpha}_1\|^2_{\Sigma_u} - \|\hat{\alpha}_1\|^2_{\Sigma_u} + \|\hat{w}\|^2_{R_w} \\ &+ \|\eta_w^\mathsf{T} \eta_u \varepsilon\|^2_{R_w^{-1}} + 2\langle \hat{w}, \eta_w^\mathsf{T} \eta_u \varepsilon \rangle \leq -(1 - \varsigma)\phi^2(x) + \delta_2(t). \end{aligned}$$

Then by completion of squares it is straight forward to conclude

$$V_x^\mathsf{T} f(x, \alpha, w, \varepsilon) + \|h(x, \alpha, w, \varepsilon)\|^2 - \gamma_w^2 \|w\|^2 + \|w - \hat{w} - R_w^{-1} \eta_w^\mathsf{T} \eta_u \varepsilon\|^2_{R_w} \leq 0. \tag{5.15}$$

Therefore, we get

$$\dot{V}(x(t)) \leq \gamma_d^2 \|w(t)\|^2 - \|z(t)\|^2 \tag{5.16}$$

and consequently

$$V(x(t_{\ell+1})) - V(x(t_\ell + \hat{\tau})) \leq \gamma_d^2 \|w|_{[t_\ell + \hat{\tau}, t_{\ell+1}]}\|^2 - \|z|_{[t_\ell + \hat{\tau}, t_{\ell+1}]}\|^2$$

adding which to (5.13) gives

$$V(x(t_{\ell+1})) - V(x(t_\ell)) \leq \gamma_d^2 \|w|_{[t_\ell, t_{\ell+1}]}\|^2 - \|z|_{[t_\ell, t_{\ell+1}]}\|^2.$$

Thus by applying this procedure to the triggering intervals until time $t$, it is obvious that the system $\mathcal{P}$ is $\mathcal{L}_2$ stable with a gain of $\leq \gamma_d$.

Part three: Using Schur complement, (5.10) reduces to (5.14) for all $t \in \mathbb{R}_{>0}$. Let us denote the LHS of (5.14) by $\Lambda$. Then (5.14) is translated as $\Lambda \leq 0$. Then if instead of $\Lambda \leq 0$ we start from the trivial inequality $\Lambda \leq \Lambda$, it is easy to check that instead of (5.16) we will end up with

$$\dot{V}(x(t)) \leq \gamma_d^2 \|w(t)\|^2 - \|z(t)\|^2 + \Lambda. \tag{5.17}$$

Now setting $w = 0$ we have $V_x^\mathsf{T} f(x, \alpha, 0, \varepsilon) \leq -\|h(x, \alpha, 0, \varepsilon)\|^2 + \Lambda$. Since for all $x \in \mathbb{R}^n \setminus \{0\}$ we have $\Lambda < 0$ from (5.10), $V_x^\mathsf{T} f(x, \alpha, 0, \varepsilon) = 0$ holds if and only if $x = 0$. ■

**Sketch of proof of lemma 5.1.** While (5.15) is only valid for the interval $t \in [t_\ell + \hat{\tau}, t_{\ell+1})$, we can apply similar process to conclude that for any $t \in \mathbb{R}^+$ we have

$$\begin{aligned} &V_x^\mathsf{T} f(x, \alpha, w, \varepsilon) + \|h(x, \alpha, w, \varepsilon)\|^2 - \gamma_w^2 \|w\|^2 \\ &+ \|w - \hat{w} - R_w^{-1} \eta_w^\mathsf{T} \eta_u \varepsilon\|_{R_w} < -(1 - \varsigma)\phi^2(x) + \bar{\delta}, \end{aligned}$$

where $\bar{\delta} := \max\{|\delta_1|, |\delta_3|\}$. Therefore, assuming $\delta_1, \delta_3$ to be selected as the functions of $\|x_0\|$, we conclude that $\dot{V}(x) \leq \bar{\delta}(\|x_0\|) + \gamma_d^2 |w|_\infty^2 - (1-\varsigma)\phi^2(x)$. Hence $\dot{V} \leq 0$ for $x \in \mathbb{R}^n$ such that $\phi^2(x) > \frac{\gamma_d^2}{1-\varsigma}|w|_\infty^2 + \frac{1}{1-\varsigma}\bar{\delta}(\|x_0\|)$. The rest of the proof is straight forward and left to the readers. ■

### 5.3.2 Discussion on Theorem 5.2

We conclude several observations without proof.

- **Conditions of Theorem 5.2**: If conditions (5.7), (5.10) are satisfied as non-strict inequalities, then Theorem 5.2 holds if we assume the system $\mathcal{P}$ in (5.3) to be zero-state detectable, *i.e.,* for $w = 0$, and all $x \in \mathbb{R}^n$, $h(x, \alpha, 0, \varepsilon) = 0$ implies $x(t) \to 0$ as $t \to \infty$.

- **Triggering condition**: To simplify our presentation and avoid unnecessary complexity, condition (5.11) is stated in static framework due to the triggering parameters all being static. A more complex dynamic structure for the event design does not affect our main findings. In essence, a generalization to the dynamic case is not difficult and can be attained following the method discussed in [66].

- **Computational costs**: First, to check the triggering condition (5.11), $V$, $\alpha$, $\gamma_u$, $\gamma_w$ and $\phi$ are obtained by solving (5.7) or (5.10) offline. Thus, the triggering condition can be checked on-line with less computational effort. Second, the local Lipschitz-continuity of $\phi^{-1}$ can be easily expressed in terms of linear matrix inequalities on $\phi$. Third, when the functions $\xi_x(x)$ and $\eta_x(x)$ are also affine in the state $x$, *i.e.,* $\xi_x(x) = A(x)x$, $\eta_x(x) = C(x)x$ and state feedback controller is assumed to be of the form $u = K(x)x$, the resulting NMIs (5.7), (5.10) can enjoy computational advantages. Indeed, in the numerical example we show that for affine functions $A(x)x$ and $C(x)x$, even the exact solution of NMIs (5.7), (5.10) can be obtained after some careful manipulations. We refer the interested readers to [78] for a detailed discussions on numerical solutions of NMIs using different algorithms such as finite element and finite difference methods.

### 5.3.3 Emulation design method

The majority of the work on event-triggered systems, up to date, follows the emulation approach which consists of first designing a controller for a continuous-time system based on a desired criteria for stability performance and then designing a triggering condition to recover similar performance under event-based implementations. To analyze the $\mathcal{L}_2$ gain performance of the network-free version of system (5.3):

$$\mathcal{P}_0 : \begin{cases} \dot{x} = f_0(x, \alpha) + \xi_w(x)w, \\ z = h_0(x, \alpha) + \eta_w(x)w. \end{cases} \tag{5.18}$$

One may think of applying Theorem 5.2 since faster sampling (in the limit, continuous sampling) does not deteriorate the disturbance rejection performance level of the system. However, it is reasonable to expect that the bound obtained from Theorem 5.2 is conservative when applied for network-free analysis. Therefore, we recall the following theorem from [78].

**Theorem 5.3 (Network-free robustness)** *Let assumption (**A3**)-(ii) holds. Then the closed loop system (5.18) is finite gain $\mathcal{L}_2$-stable with $\mathcal{L}_2$ gain $\leq \gamma_w$ if there exists a Lyapunov function $V$, control law $\alpha$ and positive $\gamma_w$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$*

$$\begin{pmatrix} V_x^\mathsf{T} f_0(x, \alpha) + \|h_0(x, \alpha)\|^2 & * \\ \frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x, \alpha) & -R_w(x) \end{pmatrix} < 0. \tag{5.19}$$

**Remark 5.1** *Similar to the discussion carried out in Section 5.3.2, if we assume $\mathcal{P}_0$ to be zero-state detectable, then (5.19) can be stated as a non-strict inequality. In this case, one may wonder whether (5.19), which implies $\mathcal{L}_2$ stability of $\mathcal{P}$, together with zero-state detectability implies ISS w.r.t. $w$. However, while this is a well-known result for linear systems, the ISS condition for nonlinear system is a consequence of the $\mathcal{L}_2$ stability and input-output-to-state stability (IOSS), which is a generalization of zero-state detectability, [72].*

Theorem 5.3 provides sufficient conditions for the *Hamiltonian* function

$$H(x, p, \alpha, w) = p^\mathsf{T}\dot{x} + \dot{J}(\alpha, w) \tag{5.20}$$

to be less than zero for all $x \in \mathbb{R}^n \setminus \{0\}$. An interesting aspect of the network-free model (5.18) is that we can state the sufficient conditions for $H < 0$ only in terms of $V$. The results here constitute a mild extention of the seminal papers [73, 78–80]. Similar to above mentioned works, our approach is framed in the context of differential games. Differential games played a major role in the solution of the $H_\infty$ problem in the 1990s, starting with the fundamental works [61, 81]. In this context, the problem of designing a controller to minimize the upper bound of the $\mathcal{L}_2$ gain of a nonlinear system can be interpreted as a zero-sum, two-player differential game with quadratic cost (5.5). In this game the minimizing (respectively, maximizing) player controls the control law $u$ (respectively, disturbance $w$). Then the problem of upper bounding the $\mathcal{L}_2$ gain of $\mathcal{P}_0$ reduces to finding the best strategy for each players. The above observations are then evidenced in the following theorem.

**Theorem 5.4** *Let assumption (**A1**)-(ii) holds. The closed loop system (5.18) is finite gain $\mathcal{L}_2$-stable with $\mathcal{L}_2$ gain $\leq \gamma_w$ if there exists a Lyapunov function $V$ and positive $\gamma_w$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$*

$$H(x, V_x, \alpha^*, w^*) < 0 \tag{5.21}$$

*where $\alpha^*$, $w^*$ are the best strategies defined as*

$$\begin{pmatrix} \alpha^* \\ w^* \end{pmatrix} = \begin{cases} -\frac{1}{2}M^{-1}N, & M^{-1} \text{ exists}, \\ -\frac{1}{2}M^{\dagger}N + U^{\mathsf{T}} col(0, q), & \text{otherwise}, \end{cases} \tag{5.22}$$

$$M = \begin{pmatrix} \eta_u^{\mathsf{T}}\eta_u & \eta_u^{\mathsf{T}}\eta_w \\ * & \eta_w^{\mathsf{T}}\eta_w - \gamma_w^2 I \end{pmatrix}, \quad N = \begin{pmatrix} \xi_u^{\mathsf{T}}V_x + 2\eta_u^{\mathsf{T}}\eta_x \\ \xi_w^{\mathsf{T}}V_x + 2\eta_w^{\mathsf{T}}\eta_x \end{pmatrix}$$

*and $U^{\mathsf{T}}\Sigma U$ is the singular value decomposition of $M$ and $q$ is an arbitrary vector in $\mathbb{R}^{m+q-r}$ with $r = rank(M)$.*

**Proof.** Let the Hamiltonian function $H$ defined in (5.20) be associated with the differential game, where the derivatives are along the trajectories of the system (5.18). Note that when $\eta_u^{\mathsf{T}}\eta_w = 0$, the function $H$ coincides with the one defined in [80]. The best play corresponds to the saddle point $(\alpha^*, w^*)$ satisfying $H(x, p, \alpha^*, w) \leq H(x, p, \alpha^*, w^*) \leq H(x, p, \alpha, w^*)$ and obained through $(H_\alpha, H_w)(\alpha^*, w^*) = (0, 0)$. Consider

$$H(x, p, \alpha, w) = \langle p, \xi_x \rangle + \|\eta_x\|^2 + \begin{pmatrix} \alpha \\ w \end{pmatrix}^{\mathsf{T}} M \begin{pmatrix} \alpha \\ w \end{pmatrix} + \begin{pmatrix} \alpha \\ w \end{pmatrix}^{\mathsf{T}} N,$$

then we can find $\alpha^*, w^*$ from $2M\text{col}(\alpha^*, w^*) + N = 0$ and thus (5.22) is obtained. Consequently, (5.21) can be obtained through

$$H(x, p, \alpha^*, w^*) = \langle p, \xi_x \rangle + \|\eta_x\|^2 - \Pi(M, N)$$

where

$$\Pi(M, N) = \begin{cases} \frac{1}{4}N^{\mathsf{T}}M^{-1}N, & M^{-1} \text{ exists}, \\ \frac{1}{4}N^{\mathsf{T}}M^{\dagger}N, & \text{otherwise}. \end{cases}$$

■

Clearly, when $M$ is invertible, $\alpha^*$ and $w^*$ are uniquely obtained from (5.22). Also note that the off-diagonal terms in the matrix $M$ are the penalty terms introduced to compensate for the coupling of $\alpha$ and $w$. When such a coupling doesn't exist, *i.e.*, $\eta_u^{\mathsf{T}}\eta_w = 0$, the saddle points reduce to

$$w^* = -R_w^{-1}(x)(\frac{1}{2}\xi_w^{\mathsf{T}}V_x + \eta_w^{\mathsf{T}}\eta_x), \tag{5.23}$$

$$\alpha^* = \begin{cases} -\Sigma_u^{-1}(\frac{1}{2}\xi_u^{\mathsf{T}}V_x + \eta_u^{\mathsf{T}}\eta_x), & \Sigma_u(x) > 0, \\ -\Sigma_u^{\dagger}(\frac{1}{2}\xi_u^{\mathsf{T}}V_x + \eta_u^{\mathsf{T}}\eta_x) + U^{\mathsf{T}}\text{col}(0, q), & \text{otherwise}, \end{cases} \tag{5.24}$$

where $\bar{U}^\mathsf{T}\bar{\Sigma}\bar{U}$ is a singular value decomposition of $\Sigma_u$ and $\bar{q}$ is an arbitrary vector in $\mathbb{R}^{m-s}$ with $s = \text{rank}(\Sigma_u)$. We remark that (5.23), (5.24) coincides with the results of [80] when $\Sigma_u(x) > 0$. However, the discussion above is more general than [80] since it includes non-invertible $\Sigma_u$, and the couplings of $\alpha$, and $w$. Moreover, the obtained $w^*$, $\alpha^*$ in (5.23), (5.24) cover those proposed in [24], which is the closest work to the result of this section.

In the sequel, we make the following assumption:

(**A5**) There exist $\bar{\gamma}_w \in \mathbb{R}_{>0}$ and an invertible function $\psi$ with $\psi(0) = 0$ and locally Lipschitz inverse $\psi^{-1}$ such that

(i) $\eta_w \eta_w^\mathsf{T} < \bar{\gamma}_w^2 I$,

(ii) $\|\frac{1}{2}\xi_w(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha)\| \leq \psi(x)$.

We denote that by restricting $\mathcal{P}_0$ to be a LTI control system and $V$ to be a quadratic Lyapunov function, assumption (**A5**)-(ii) readily holds for a linear choice of function $\psi$. Moreover, assumption (**A5**)-(i) holds when $\eta_w$ is norm bounded. Additionally, for symmetric $\eta_w$ this assumption automatically stems from assumption (**A3**)-(ii).

**Theorem 5.5** *Let assumptions (**A3**), (**A5**) hold and system $\mathcal{P}_0$ has a $\mathcal{L}_2$ gain $\leq \gamma_w$ according to Theorem 5.3. Then the triggering rule can be designed for system $\mathcal{P}$ so that the consequences of Theorem 5.2 hold.*

**Sketch of proof.** We first claim that using assumptions (**A3**)-(ii), (**A5**)-(i) one can show that given $\gamma_d > \gamma_w$, $\delta = \gamma_d^2 - \gamma_w^2$ there exists $\epsilon \in \mathbb{R}_{>0}$ such that

$$\delta I > \epsilon(\gamma_w^2 I - \eta_w^\mathsf{T}\eta_w)(\gamma_d^2 I - \eta_w^\mathsf{T}\eta_w). \tag{5.25}$$

To prove (5.25) it suffices to expand the right-hand side and use the fact that $\eta_w^\mathsf{T}\eta_w\eta_w^\mathsf{T}\eta_w < \bar{\gamma}_w^2\eta_w^\mathsf{T}\eta_w < \bar{\gamma}_w^2\gamma_w^2 I$ where for the first and second ieqnualities we utilized assumptions (**A5**)-(i) and (**A3**)-(ii), respectively. The rest of the proof is easy and hence is left to the readers. Using (5.25) and also assumptions (**A3**)-(ii) we can write $(\gamma_d^2 I - \eta_w^\mathsf{T}\eta_w)(\gamma_w^2 I - \eta_w^\mathsf{T}\eta_w)^{-1} = I + \delta(\gamma_w^2 I - \eta_w^\mathsf{T}\eta_w)^{-1} > I + \epsilon(\gamma_d^2 I - \eta_w^\mathsf{T}\eta_w)$ and hence $(\gamma_w^2 I - \eta_w^\mathsf{T}\eta_w)^{-1} > (\gamma_d^2 I - \eta_w^\mathsf{T}\eta_w)^{-1} + \epsilon I$. Note that $\gamma_d^2 I - \eta_w^\mathsf{T}\eta_w$ is invertible due to assumption (**A3**)-(ii) and the fact that $\gamma_d > \gamma_w$. Therefore, pre- and post-multiplying the last inequality by $\frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha)$, in view of Schur complement, (5.19) and assumption (**A5**)-(iii) we conclude that for all $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{pmatrix} V_x^\mathsf{T} f_0(x,\alpha) + \|h_0(x,\alpha)\|^2 & * \\ \frac{1}{2}\xi_w^\mathsf{T}(x)V_x + \eta_w^\mathsf{T}(x)h_0(x,\alpha) & -\gamma_d^2 I + \eta_w^\mathsf{T}\eta_w \end{pmatrix} + \epsilon\psi^2(x) < 0.$$

We can now construct the triggering rule as follows. Let $\phi$ be:

$$\phi(x) = \begin{cases} \epsilon\psi(x), & \text{when } \Sigma_u(x) > 0, \\ \sqrt{\sigma\epsilon}\psi(x), & \text{otherwise,} \end{cases} \tag{5.26}$$

for some $\sigma \in (0,1)$. Thus, we can choose $\gamma_u$ arbitrarily when $\Sigma_u(x) > 0$ and such that

$$\|\hat{\alpha}_2\|^2_{R_u} \leq (1-\sigma)\psi^2(x), \tag{5.27}$$

otherwise, and adopt the triggering rule (5.11) to guarantee the conditions of Theorem 5.2. The rest of the proof is similar to that of Theorem 5.2, hence left to the interested readers.

■

**Remark 5.2** *We denote that the joint design method has the advantage of offering an optimized solution. There is a simple interpretation of this result: in the emulation method the event-based control design is restricted to*

- *conditions (5.26) and (5.27) on $\phi$ and $\hat{\alpha}_2$,*
- *the initial design of control law $\alpha$ in (5.19).*

*These limitations, however, do not exist in the joint design method. The interested readers are referred to [30] for further discussion of the differences of the two approaches for the linear case.*

## 5.4 Numerical example

In the following two example, we show that under certain affine structure for $\xi_x(x)$, $\eta_x(x)$, the exact solution $V$ for NMIs (5.7), (5.10) can be obtained. It is worth mentioning that the affine characterizations of $\xi_x(x)$, $\eta_x(x)$ enhance the application of the numerical techniques to solve the corresponding NMIs, see [78] for more details. Consider the following model

$$\begin{cases} \dot{x} = e^x u - e^x w_1 + e^x w_2, \\ z = \epsilon x + \epsilon w_1, \end{cases} \tag{5.28}$$

for some $\epsilon \in (0,1)$. With the control law $u = -x$, (5.28) is a two-input single-output process with the $\mathcal{L}_2$ gain $\epsilon$, [78], where $w = \mathrm{col}(w_1, w_2)$, $\xi_w = (-e^x, e^x)$ and $\eta_w = (\epsilon, 0)$.

### 5.4.1 Joint design

Since $\eta_u^\mathsf{T}\eta_u = 0$, we pick some $\gamma_u \in \mathbb{R}_{>0}$, $\gamma_w > \epsilon$ and hence assumption (**A3**) holds. Moreover, choosing $V_x = P(x)x$ in (5.7), the exact solution to (5.7) is obtained from $V(x) =$

$\int_0^x V_x dt = x^2 \int_0^1 tP(tx)dt$ where $\underline{P}(x) \leq P(x) \leq \overline{P}(x)$. To compute the bounds $\underline{P}(x)$ and $\overline{P}(x)$ we choose $\phi(x) = \theta P(x)e^x x$ for some $\theta \in \mathbb{R}_{>0}$. Pre- and post-multiplying (5.7) by $\mathrm{diag}(x, 1, 1, 1, 1)$ and applying the Schur complement, this inequality reduces to

$$\frac{\gamma_w^2-\epsilon^2}{4\gamma_w^2}\Big(4\theta^2\gamma_w^2+1+\frac{\gamma_w^2}{\gamma_w^2-\epsilon^2}+\frac{\gamma_w^2}{\gamma_u^2}\Big)P^2e^{2x}-\gamma_w^2Pe^x+\epsilon^2\gamma_w^2 < 0$$

which is a quadratic algebraic inequality in terms of $Pe^x$. Solving this inequality, we obtain

$$\frac{2\gamma_w^2(1-k\sqrt{1-\lambda})e^{-x}}{\frac{\lambda_1\gamma_w^2}{\epsilon^2}k^2+k+1+k(\frac{\gamma_w}{\gamma_u})^2} \leq P(x) \leq \frac{2\gamma_w^2(1+k\sqrt{1-\lambda})e^{-x}}{\frac{\lambda_1\gamma_w^2}{\epsilon^2}k^2+k+1+k(\frac{\gamma_w}{\gamma_u})^2}$$

where $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 = \frac{4\theta^2\epsilon^2}{k}$, $\lambda_2 = \frac{\epsilon^2}{\gamma_u^2 k}$ and $k = 1 - (\frac{\epsilon}{\gamma_w})^2$. Note that we have a limitation on choosing $\theta, \gamma_w, \gamma_u$ such that $\lambda < 1$. Also the Lipschitz continuity of $\phi^{-1}$ follows from the lower bound on $P(x)$ and the resulting Lipschitz coefficient is $(\frac{\frac{\lambda_1\gamma_w^2}{\epsilon^2}k^2+k+1+k(\frac{\gamma_w}{\gamma_u})^2}{2\gamma_w^2(1-k\sqrt{1-\lambda})\theta})$.

### 5.4.2 Emulation design

To find an exact solution to inequality (5.19) for some $V(x)$, we use similar procedures as above to obtain $V(x) = x^2 \int_0^1 tP(tx)dt$ where

$$\frac{2\epsilon^2e^{-x}}{1+k} \leq P(x) \leq 2\gamma_w^2e^{-x}. \tag{5.29}$$

Therefore, using some careful manipulations, it is not difficult to conclude that taking some $P_0$ so that (5.29) holds for $P(x) = P_0e^{-x}$, we can choose $\psi(x) = \bar{\theta}x$, $\bar{\theta} = \sqrt{\frac{k^2-(\frac{P_0}{2\gamma_w^2}(1+k)-1)^2}{k(1-k)}}\gamma_w$.

## 5.5 Simulation Results

Previous section examines the differences of the emulation and joint methods in designing event condition, where different bounds on the admissible Lyapunov function and also function $\phi$ are obtained. In this section, we study several aspects of our proposed event-triggered control design through numerical simulations. The findings of this section are independent of the particular design scheme, thereby simulations are only provided for the joint design method. In the rest of this section, we adopt the following notations:

| | |
|---|---|
| Minimum inter event time | $\tau_{\text{miet}}$ |
| Average intersampling | $\tau_{\text{avg}}$ |
| Total number of triggerings | N |

*a)* We first consider the following scenario for $w$:

$$w(t) = \begin{cases} \begin{pmatrix} 0.05\sin(2\pi t) \\ 0.05\cos(2\pi t) \end{pmatrix}, & 2 \leq t \leq 8 \\ 0, & \text{otherwise.} \end{cases}$$

The following plots are then obtained based on 10 seconds of simulation and initial condition $x(0) = 1$. The disturbance is applied over the interval $[2, 8]$ and the corresponding measure output signal $z$ is depicted in Fig. 5.1.

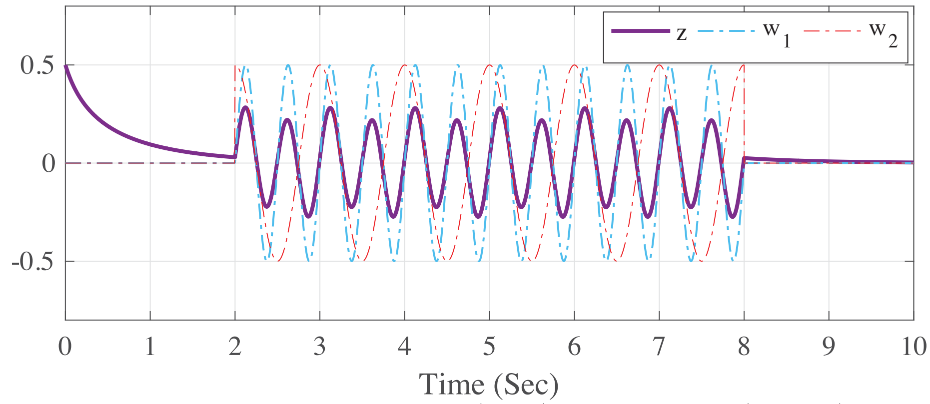



Figure 5.1: Measured output (solid), disturbances (dashed).

The evolution of measurement error signal and triggering threshold (Fig. 5.2) determines the execution instants of the control task (Fig. 5.3).

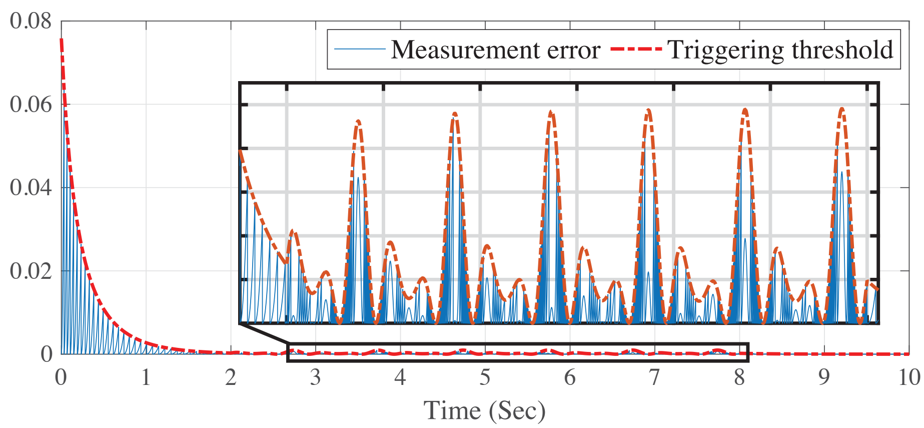



Figure 5.2: Satisfaction of triggering rule.

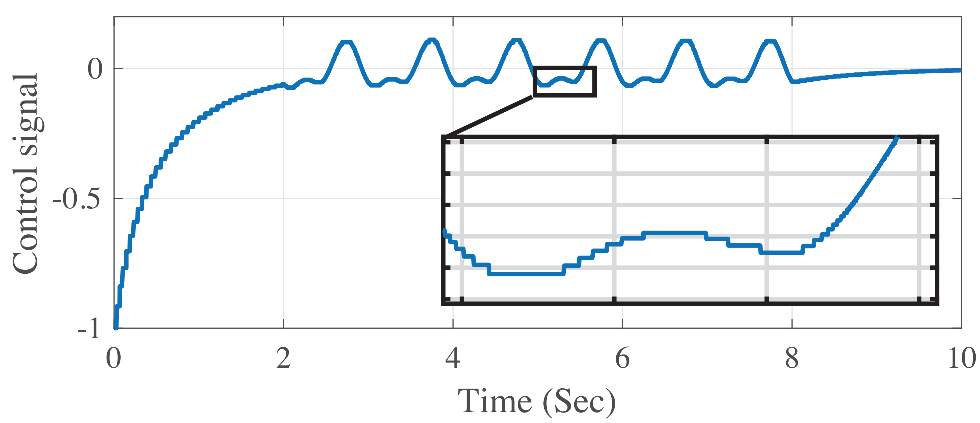



Figure 5.3: Actuator signal.

Next plot (Fig. 5.4) reveals why treating disturbances is challenging and generally a non-trivial task in an event-based setting. Indeed, when states are close to the origin, arbitrary disturbances may drive the measurement error to meet the event condition sooner, thereby decrease the inter-event times.

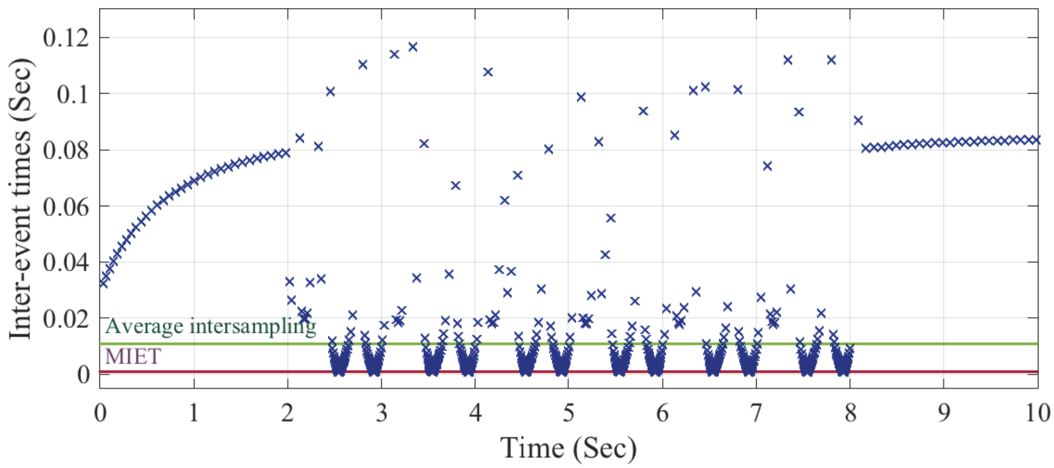



Figure 5.4: Inter-event intervals.

The above simulations are provided neglecting the effect of $\delta_2$ term, *i.e.*, $\delta_2(t) = 0$. The role of $\delta_2(t)$ is studied in the following table, where it clearly improve the sampling rate and inter-event times.

Table 5.1: Effect of decaying function $\delta_2$.

| | $\tau_{\text{miet}}$ | $\tau_{\text{avg}}$ | N |
|---|---|---|---|
| $\delta_2 = 0$ | 1.6512 | 0.0151 | 606 |
| $\delta_2 = e^{-0.5t}$ | 1.6747 | 0.3789 | 21 |

The results of Table 5.1 are obtaind based on 100 initial conditions, uniformly distributed in the inerval $[-1, 1]$, and then average the results. Furthermore, $\hat{\tau} = 0.001$ seconds. Since $\hat{\tau}$ is a guaranteed periodic sampling time, the ratio of average sampling over $\hat{\tau}$ is a good index indicating how far the event-based samplings are from the time-triggered, *i.e.*, periodic one. In fact, as this ratio get close to 1, the event-based system degenerates to a periodic sampled-data system. This undesirable isssue, which contradicts the main goal of an event-based design which is reducing the communication traffic, is the case *e.g.*, for static time-regularization methods, see [48] and the example therein. In our work, however, this ratio is 15 for $\delta = 0$ and 379 for $\delta = e^{-0.5t}$.

*b)* The next set of plots are obtained for $w$ be a random singnal in $(0, 1)$ and based on 10 seconds of simulation and the initial condition $x(0) = 1$. In these plots the effect of time decaying term $\delta_2$ is included by setting $\delta_3 = 1$ and $\delta_4 = -2$. The resulting measured output is shown in (Fig. 5.5)

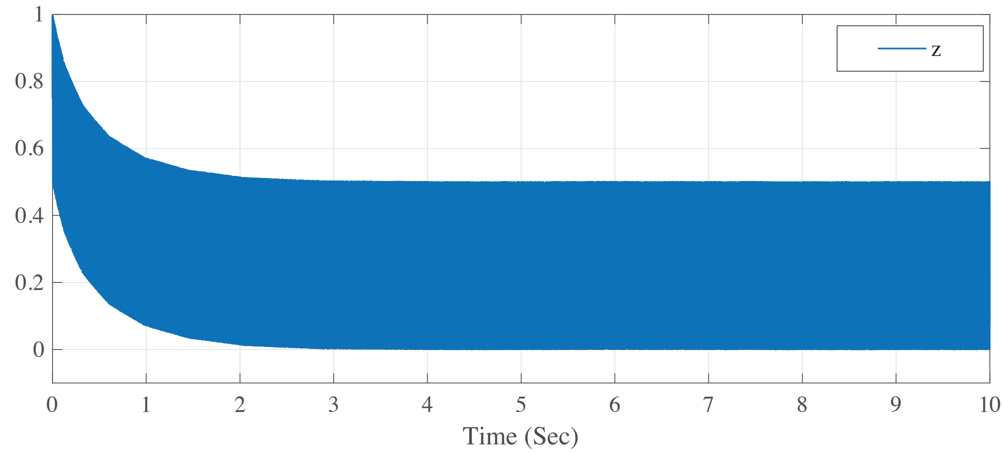



Figure 5.5: Measure output under random disturbances.

The evolution of measurement error signal and triggering threshold (Fig. 5.6) determines the execution instants of the control task (Fig. 5.7).

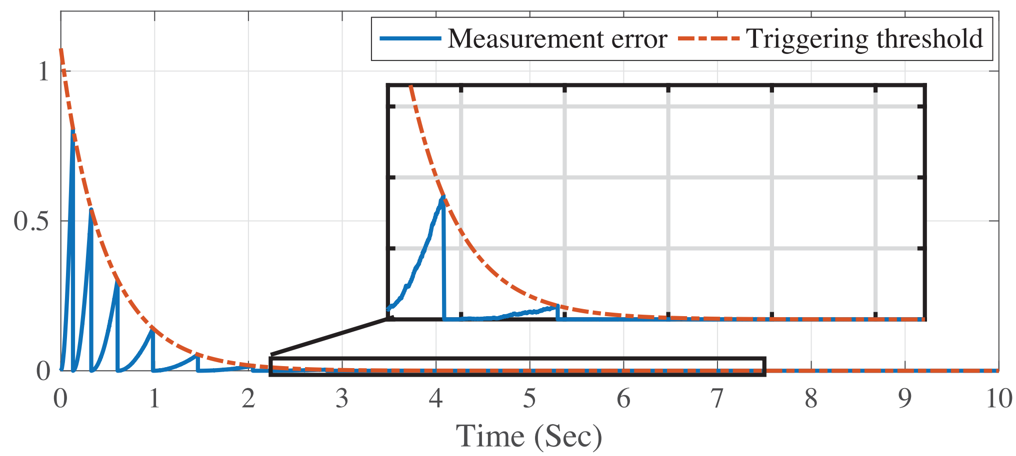



Figure 5.6: Satisfaction of triggering rule.

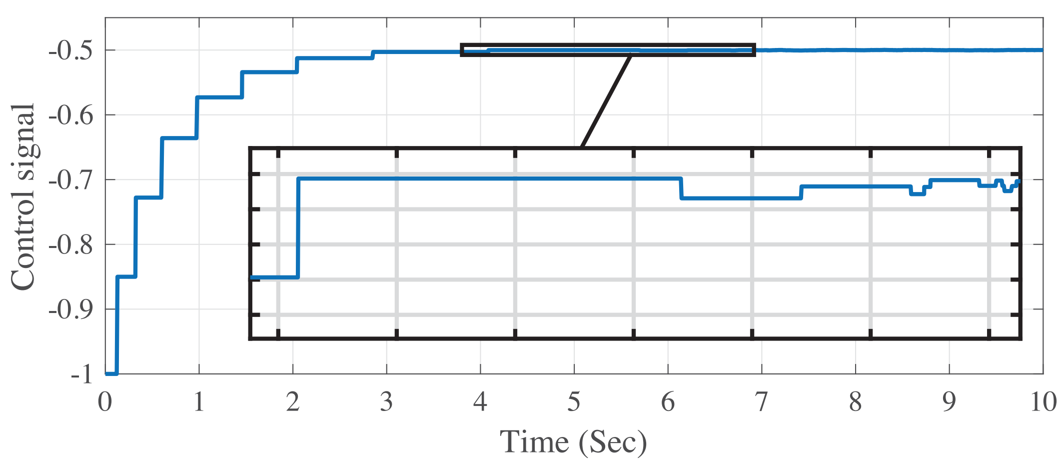



Figure 5.7: Actuator signal.

In accordance with the result of Fig. 5.4, next plot shows that exogenous disturbance may lead to triggering instants get arbitrary close to each other. However, the decaying function $\delta_2(t)$ effectively opposes this advert behaviour until it is vanished at $t \approx 7$, according to Fig. 5.8.

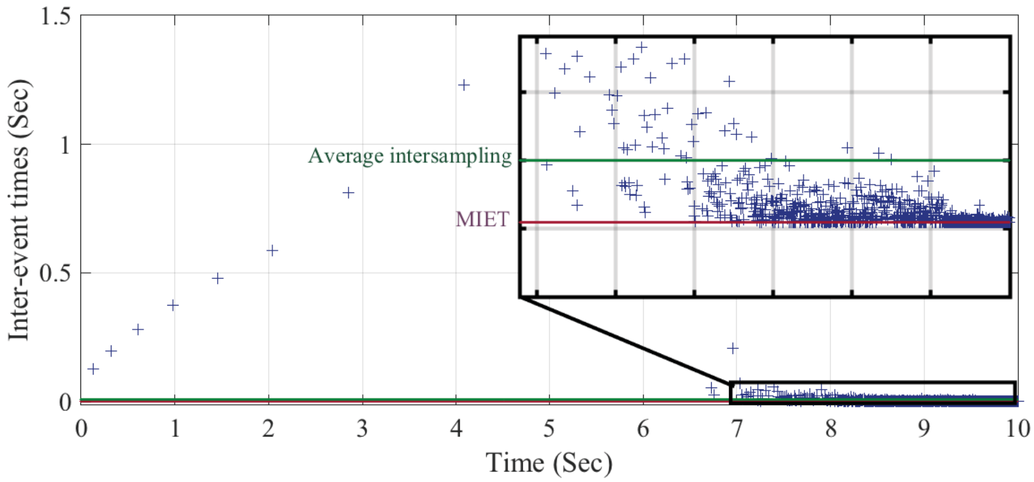



Figure 5.8: Inter-event intervals.

Similar to the previous case, the effect of $\delta_2$ is shown in the following table.

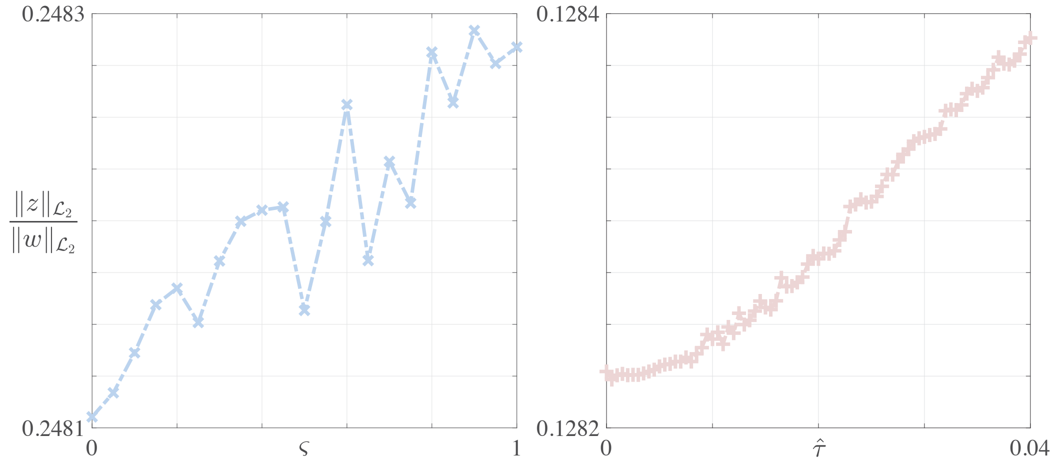
Table 5.2: Effect of decaying function $\delta_2$.

| | $\tau_{\text{miet}}$ | $\tau_{\text{avg}}$ | N |
|---|---|---|---|
| $\delta_2 = 0$ | 1.6545 | 0.0073 | 844 |
| $\delta_2 = e^{-0.5t}$ | 1.8159 | 0.7923 | 12 |

The results of table 5.2 are obtaind based on 100 initial conditions, uniformly distributed in the inerval $[-1, 1]$, and then average the results. Since $\hat{\tau} = 0.001$ seconds, the ratio of $\frac{\tau_{\text{avg}}}{\hat{\tau}}$ is obtained to be 7 for $\delta = 0$ and 792 for $\delta = e^{-0.5t}$, confirming that effectiveness of proposed method compared to periodic sampling.

*c)* Next, we will study the trade-offs between the $\mathcal{L}_2$-gain of the system and triggering parameters, namely the threshold coefficient $\varsigma$ and parameter $\hat{\tau}$. These trade-offs are discussed in the seminal works [20, 46, 48], where it is shown that the guaranteed $\mathcal{L}_2$-gain has an inverse relationship with $\varsigma$, $\hat{\tau}$. This subject will be studied here under the following structure for applied disturbance:

$$w_i(t) = a_i e^{q_i t} \sin(2\pi p_i t).$$




For the left figure, we set $a_1 = 25$, $q_1 = -0.1$, $p_1 = 1$, $\phi_1 = 0$, $a_2 = 2$, $q_2 = -0.1$, $p_2 = 1$, $\phi_2 = \frac{\pi}{2}$ and for the right one, $a_1 = 2$, $q_1 = 0$, $p_1 = 4.5$, $\phi_1 = 0$, $a_2 = 2$, $q_2 = 0$, $p_2 = 2$, $\phi_2 = \frac{\pi}{2}$.

*d)* Finally, we turn our attention into the interesting scenario of efficiently attentive triggering mechanisms, which is recently introduced in [82]. Roughly speaking, when the transmission intervals is non-decreasing over time, the corresponding triggering condition is said to be efficiently attentive since it produces fewer samples as states get close to the equilibrium. This feature is studied under the proposed triggering condition in the present paper, where for the sake of simplicity, we assume no disturbance to be applied to the event-based system. Then, the inter-event intervals are studied for different value of $\delta_3$. Suggested by Fig. 5.9, $\delta_3 = 2.58$ is a critical value to decide whether the proposed triggering condition is efficiently attentive or not. Obviously, when $\delta_3 = 0$ the inter-event times decreasewith time, a fact that reveals another role of time decying function $\delta_2(t)$, which is to turn a pre-designed triggering rule into a one with efficiently attentive property.

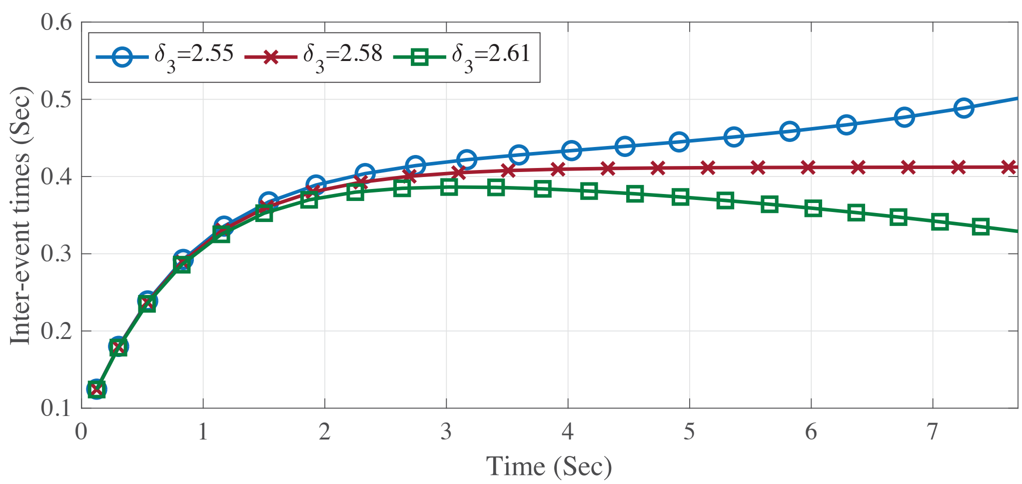



Figure 5.9: Comparison of different scenarios for $\delta_3$.

## 5.6 Summary

We have provided a sufficient condition in terms of NMI for the input-to-state stability of the event-triggered control system with respect to actuator error and disturbance. Additionally, we have designed an event-based state feedback controller, without directly using an ISS assumption, which satisfies the desired $\mathcal{L}_2$ gain performance level, following both emulation and joint design methods. The proposed event condition is shown to successfully rule out the accumulation of triggering instants. Future works will study the more practical scenarios such as output feedback case, decentralized triggering strategies [76], and will include practical issues such as limited bandwidth [82, 83]. Also of interest is the study of existence of an explicit optimal solution under event-based communication.

# Chapter 6

# Event-Triggered Controller Design

## 6.1 Nonlinear Lipschitz Event-Based Modeling

In this chapter[1] we consider a system consisting of a nonlinear Lipschitz plant connected to a dynamic controller through a communication network. We assume the controller receives (respectively, sends) information from sensor (respectively, to the actuator) at discrete instants $t_y^i$ (respectively, $t_u^j$) $i, j \in \mathbb{N}_0$ through the network with $t_y^0 = 0$ (respectively, $t_u^0 = 0$). Thus, the sensor measurements and the controller's output are independently monitored using ETMs (to be described later) that update and send signals through the network, as required. Therefore, the event instants $t_y^i, t_u^j$ are in general asynchronous. The discrete signals $u_s$, $y_s$ are held constant between events using ZOH devices, *i.e.,* for any $i, j \in \mathbb{N}_0$

$$y_s(t) = y(t_y^i), \quad t \in [t_y^i, t_y^{i+1}), \tag{6.1}$$

$$u_s(t) = u(t_u^j), \quad t \in [t_u^j, t_u^{j+1}). \tag{6.2}$$

For the sake of simplicity we neglect the effects of transmission delays in the network as they can be addressed following the method in [9] and neglect also measurement and transmission noises (see [48, 60]). Note that the above mentioned references focus on checking whether a pre-designed static or dynamic controller stabilize the ETC system or not, as opposed to the design problem studied here.

We now defined the system to be used throughout the rest of this chapter. We consider the nonlinear plant

$$\dot{x} = Ax + \phi_1(x) + B_u u_s + B_w w \tag{6.3}$$

[1]The results of this chapter have been submitted for publication in the article: M. Ghodrat and H. J. Marquez, "On the Event-Triggered Controller Design", Submitted to *IEEE Trans. Autom. Control*, October 2018.

where $x \in \mathbb{R}^{n_x}$, $u_s \in \mathbb{R}^{n_u}$, $w \in \mathbb{R}^{n_w}$ represent the plant's state, control input and exogenous disturbance. $A$, $B_u$, $B_w$ are constant matrices of appropriate dimensions. The nonlinearity $\phi_1$ satisfies Lipschitz property

$$\|\phi_i(x) - \phi_i(\check{x})\| \leq c_{\phi_i}\|x - \check{x}\| \tag{6.4}$$

for $i = 1$ and some positive constant $c_{\phi_1}$ and all $x, \check{x} \in \mathbb{R}^{n_x}$. Moreover, $\phi_1(0) = 0$, so that $x = 0$ an equilibrium point of the zero-input system. We also assume the state $x$ is driven from initial condition $x(0) = x_0$ in an open subset of $\mathbb{R}^{n_x}$ containing the origin. The plant's output $y \in \mathbb{R}^{n_y}$ is given by

$$y = C_y x + \phi_3(x) \tag{6.5}$$

where $C_y$ is a constant matrix and the nonlinearity $\phi_3$ satisfy property (6.4). Since the state is not available for measurement except in the special case $C_y = \mathbb{I}_{n_x}$, we use an observer to reconstruct the state and implement the output feedback law using the following observer-state feedback formulation:

$$\dot{\hat{x}} = A\hat{x} + \phi_1(\hat{x}) + B_u u_s + L(y_s - C_y\hat{x} - \phi_3(\hat{x})) \tag{6.6}$$

represents the obsever, where $L$ is the observer gain matrix to be designed so that $\hat{x}$ converges to $x$. The control law is then

$$u = K\hat{x} \tag{6.7}$$

for some matrix gain $K \in \mathbb{R}^{n_u \times n_x}$ to be designed. We will also assume the following:

(A1) The pairs $(A, B_u)$ and $(A, C_y)$ are respectively, controllable and observable.

**Remark 6.1** *Assumption (A1) is made for convenience but can be relaxed to stabilizable and detectable, respectively.*

**Remark 6.2** *In model (6.3), matrix $A$ contains the linear part of plant's dynamics and the nonlinearity $\phi_1$ represents the nonlinearities of order two or higher. Thus, in the study of asymptotic stability the linear terms dominate and a necessary condition for local closed-loop asymptotic stability is to design matrix $K$ such that the eigenvalues of $A + B_uK$ are in the left half plane.*

### 6.1.1 Performance Criterion

We will establish our control design using the standard $\mathcal{L}_2$ input-output formalism, [5, 68]. Let $z \in \mathbb{R}^{n_z}$ to be given by

$$z = C_z x + \phi_2(x) \tag{6.8}$$

where $C_z$ is a constant matrix of appropriate dimension and $\phi_2$ satisfy the Lipschitz property (6.4) for $i = 2$. Our design methodology is based on the following finite gain $\mathcal{L}_2$-stability performance index:

$$J^{\gamma}_{\langle r,s\rangle} = \int_r^s (\gamma^2 \|w(\tau)\|^2 - \|z(\tau)\|^2) d\tau. \tag{6.9}$$

The closed-loop system model is

$$\begin{cases} \dot{x} = Ax + \phi_1(x) + B_u u_s + B_w w \\ \dot{\hat{x}} = A\hat{x} + \phi_1(\hat{x}) + B_u u_s + L(y_s - C_y \hat{x} - \phi_3(\hat{x})) \\ z = C_z x + \phi_2(x), \ y = C_y x + \phi_3(x), \ u = K\hat{x}. \end{cases} \tag{6.10}$$

We have:

**Definition 6.1** *( [68]) The closed-loop system (6.10) has the disturbance attenuation index of $J^{\gamma}_{\langle 0,T\rangle}$ provided that there exist finite constants $\gamma \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+_0$ (called bias term) and positive semi-definite continuous function $\alpha$ such that for any $T \in \mathbb{R}^+_0$, any perturbation $w \in \mathcal{L}_2^{n_w}$ and any $x_0 \in \mathbb{R}^{n_x}$*

$$J^{\gamma}_{\langle 0,T\rangle} + \alpha(x_0) + \beta \geq 0. \tag{6.11}$$

**Definition 6.2** *The equilibrium point $x = 0$ of unperturbed system (6.10), obtained by setting $w = 0$, is globally exponentially stable (GES) with a convergence rate $\bar{\sigma}_2$ if there exists $r \in \mathbb{R}^+$ such that*

$$\left\| \Big(x(t), r(x(t) - \hat{x}(t))\Big) \right\| \leq \bar{\sigma}_1 e^{-\bar{\sigma}_2 t} \left\| \Big(x_0, r(x_0 - \hat{x}_0)\Big) \right\| \tag{6.12}$$

*for some $\bar{\sigma}_i \in \mathbb{R}^+$, $i \in \{1, 2\}$ and any $x_0, \hat{x}_0 \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$.*

Our main interest is to obtain a systematic method to jointly design the gain matrices $K$, $L$ and the TCs, so that the following conditions are met:

(i) For a desired $\gamma_d \in \mathbb{R}^+$, the disturbance attenuation index of the resulting ETC system is given by $J^{\gamma_d}_{\langle 0,\infty\rangle}$.

(ii) The unperturbed model (6.10) has a GES equilibrium point at $x = 0$ with a convergence rate $\bar{\sigma}_2$.

## 6.2 State-Feedback Controller

We begin with the full information case, *i.e.* assuming that $C_y = \mathbb{I}_{n_x}$ in (6.5), used primarily to present the core ideas behind our design approach, without the complications of the observer-based case. Let the static control law be:

$$u = Kx. \tag{6.13}$$

Since the state is measured, the observer is unnecessary and we can use a single ETM at the plant's output. Therefore, the event instants will be denoted $t^i := t_u^i = t_y^i$ for any $i \in \mathbb{N}_0$. We can define the state *inter-events error*, denoted by $e \in \mathbb{R}^{n_x}$, as follows:

$$e(t) = x(t^i) - x(t), \quad t \in [t^i, t^{i+1}), \tag{6.14}$$

and we obtain the following closed-loop system:

$$\begin{cases} \dot{x} = \bar{A}_\circ x + \phi_1(x) + B_u K e + B_w w, \\ z = C_z x + \phi_2(x), \end{cases} \tag{6.15}$$

with $e$, and $w$ as the inputs and $\bar{A}_\circ := A + B_u K$.

### 6.2.1 Event-Triggered Mechanism

In this section, the general structure of our ETM will be proposed by generalizing the method of the Chapter 3, where the proposed condition is intended to guarantee the local $\mathcal{L}_2$ stabilty performance. Here, however, as we are interested in $\mathcal{L}_2$ stability in a global sense.

We will assume without loss of generality that $t^0 = 0$. Adopting the notation $\xi := \mathrm{col}(x, e)$, the event times are implicitly defined as

$$t^{i+1} = \inf\{t \in \mathbb{R} : t > t^i \wedge \xi(t)^\mathsf{T} \mathcal{X} \xi(t) - \Delta(t) = 0\} \tag{6.16}$$

for $i \in \mathbb{N}_0$, where $t^i$ denotes the most recent triggering instant. The matrix $\mathcal{X} := (-\mathcal{P}_1\ \mathcal{P}_2; \star\ \mathcal{P}_3)$ for some symmetric $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in \mathbb{R}^{n_x \times n_x}$ is restricted to be designed according to the following assumption:

(A2) $\|\mathcal{P}_2\| \neq 0$, $\lambda_{n_x}(\mathcal{P}_1) \in \mathbb{R}^+$, $\lambda_1(\mathcal{P}_3) \in \mathbb{R}_0^+$.

Also, the function $\Delta$ has the following structure

$$\Delta(t) = \eta e^{-\zeta t} + (\delta - \eta e^{-\zeta t})\chi_\mathcal{T}(t - t^i), \quad t \in [t^i, t^{i+1}) \tag{6.17}$$

for some $\eta, \delta, \zeta \in \mathbb{R}^+$ and $\mathcal{T} := [0, \hat{\tau}]$ for some $\hat{\tau} \in \mathbb{R}^+$. This suggests that $\Delta$ is constant over $t \in [t^i, t^i + \hat{\tau}]$, $i \in \mathbb{N}_0$ and exponentially decreasing elsewhere.

We remark that that (6.16) is a static rule with time varying threshold $\Delta$, which from a practical perspective, is constructed to ensure that the event-based implementation is free of accumulation points, a property known as the *event-separation property*, introduced in [59]:

**Definition 6.3** *The system (6.15) has the robust semi-global event-separation property if there exists $\epsilon \in \mathbb{R}^+$ so that for any compact set $\mathcal{B} \subset \mathbb{R}^{n_x}$, $\inf\{\tau_m : x_0 \in \mathcal{B}, |w|_\infty \leq \epsilon\} > 0$ where $\tau_m = \inf\{t^{i+1} - t^i : i \in \mathbb{N}_0\}$ is the MIET.*

Using the terminology in [59], (6.16) is known as a *mixed* triggering condition suggesting that the triggerings occur whenever measurement error exceeds some mixed threshold of state and $\Delta$, and is aimed to ensure the Zeno-freeness property for the ETC system in the presence of exogenous disturbances. In such a case, since the measurement error is also driven by the disturbance term, if $\Delta = 0$, when the state's norm is near zero the external disturbance may result in a sudden growth of $\|e\|$, possibly leading to redundant events and Zeno behaviour. However, with the addition of the function $\Delta$, the admissible $e$ is lower bounded by $\delta$ for $t - t^i \in \mathcal{T}$. This ensures that the events are uniformly isolated and hence zero behaviour is avoided (see [59] for a more detailed discussion). This observation is summarized in the next theorem.

**Theorem 6.1** *Under the execution rule (6.16) and assumption (A2), the closed-loop system (6.15) has the robust semi-global event-separation property.*

**Proof.** We assume $t^{i+1} \leq t^i + \hat{\tau}$, otherwise $\tau_m = \hat{\tau}$ and hence the proof is immediate. Therefore, (6.16) suggests that the lower bound on MIET can be obtained by assuming the triggerings occur whenever $-x^\top \mathcal{P}_1 x + x^\top \mathcal{P}_2 e + e^\top \mathcal{P}_2^\top x + e^\top \mathcal{P}_3 e = \delta$. Our goal is then to find the lower bound on MIET by introducing new TCs, restricted than (6.16). We start with the condition $-\alpha_1 \|x\|^2 + \alpha_2 \|e\| \|x\| + \alpha_3 \|e\|^2 = \delta$ where $\alpha_1 = \lambda_{n_x}(\mathcal{P}_1)$, $\alpha_2 = 2\|\mathcal{P}_2\|$, $\alpha_3 = \lambda_1(\mathcal{P}_3)$. It is not difficult to see that this condition is more restrictive than (6.16). Next using the completion of squares we define another condition, even more restrictive than the latter condition, as $\|e\| = a\|x\| + b$, where

$$a = \frac{1}{2\alpha_3}\Big(\Big(\frac{4\alpha_1\alpha_3 + \alpha_2^2}{1 + \alpha^{-1}}\Big)^{\frac{1}{2}} - \alpha_2\Big), \; b = \Big(\frac{\delta}{\alpha_3(1+\alpha)}\Big)^{\frac{1}{2}}$$

and $\alpha > \alpha_2^2/(4\alpha_1\alpha_3)$ is an arbitrary parameter. Note that the above choice of $a, b$ is well-defined under assumption (A2). In the rest we will show that the MIET obtained from the latest condition, is bounded away from zero, and hence the same property holds under TC (6.16). To this end, let us define $\rho = \|e\|/(a\|x\|+b)$. Thus we have

$$\dot{\rho} \leq \frac{\|\dot{e}\|}{a\|x\|+b} + \frac{a\|e\|\|\dot{x}\|}{(a\|x\|+b)^2} \leq \Big(1 + \frac{a\|e\|}{a\|x\|+b}\Big)\frac{\|\dot{x}\|}{a\|x\|+b}$$

Now in view of (6.15) we conclude

$$\frac{d}{dt}\|x(t)\| \leq a_1\|x(t)\|+a_2\|e(t)\|+a_3\|w(t)\| \tag{6.18}$$

where $a_i$, $i \in \{1,2,3\}$ are as defined in Proposition 6.1. Thus defining $\kappa = \max\{\frac{a_1}{a}, \frac{a_3\epsilon}{b}\}$ where $|w|_\infty \leq \epsilon$ we can write

$$\dot{\rho} \leq (1+a\rho)\frac{a_1\|x\|+a_2\|e\|+a_3\|w\|}{a\|x\|+b} \leq (1+a\rho)(\kappa + a_2\rho)$$

solving which for $\varsigma \geq t^i$ with $\rho(t^i) = 0$ and $\rho(\varsigma) = 1$ yields

$$\varsigma(\rho) = t^i + \begin{cases} \frac{1}{a_2-\kappa a}\ln(\frac{\kappa+a_2\rho}{\kappa+\kappa a\rho}), & \kappa \neq \frac{a_2}{a}, \\ \frac{a\rho}{a_2(1+a\rho)}, & \kappa = \frac{a_2}{a}. \end{cases} \tag{6.19}$$

Since $\rho \geq 1$ at $t^i$, $i \in \mathbb{N}_0$, we have $\tau_m \geq \varsigma(1) - t^i > 0$. ■

**Remark 6.3** *Assumption (A2) is not essential in the proof of Theorem 6.1 and can be substituted by* $\|\mathcal{P}_2\|=0$, $\lambda_{n_x}(\mathcal{P}_1), \lambda_1(\mathcal{P}_3) \in \mathbb{R}^+$. *The proof, however, is not difficult and left to the interested readers.*

**Remark 6.4** *The concept of dwell-time in time-regularized works [20, 48, 60] has been generalized here to the set* $\mathcal{T}$. *Note that while in the above mentioned articles triggering is forbidden when* $t - t^i \in \mathcal{T}$. *We do not impose such restriction here and hence better performance in terms of* $\mathcal{L}_2$*-gain or GES convergence rate is expected.*

The exponentially decaying term in (6.17) is introduced to enlarge the inter-event times without violating the desired system performance. The amount of enlargement can be estimated explicitly for a given period of time and is shown in the next theorem whose proof is similar to that of Theorem 4.2 and also Section 3.3 and hence is omitted. As an application, one can design $\eta$, $\zeta$, $\delta$, $\mathcal{T}$ to overcome possible delays in communication channels.

**Theorem 6.2** *Given any* $T^\circ \in \mathbb{R}^+$ *and any* $\tau^\circ \in [0, \tau^*_{\max}]$, *where* $\tau^*_{\max} = \varsigma(1) - t^i$ *and* $\varsigma$ *is defined in (6.19), function* $\Delta$ *can be designed so that* $\min\{t^{i+1} - t^i : i \in \mathbb{N}_0, t^{i+1} \leq T^\circ\} \geq \tau^\circ$.

It is shown in Chapters 3, 4 via simulation that compared to dynamic triggering approach introduced in [28], the parameters $\eta$, $\zeta$ in (6.17) can reduce the transmission traffic more effectively, especially when GES is addressed.

### 6.2.2 Event-Based Controller Design

In this section we design an event-based $H_\infty$ controller following the joint design approach, where the control gain matrix $K$ and triggering parameters $\mathcal{X}$, $\Delta$ are simultaneously designed. Moreover, our design incorporates the desired GES convergence rate as a design parameter. Our method is different from the emulation approach used in the majority of literature where the controller is first designed in the abcense of communication network, and then the triggering scheme is constructed to satisfy performance of the original design.

We start with with some preliminary results. The first one gives an upper bound on the measurement error in terms of state $x$ and disturbance $w$.

**Proposition 6.1** *Let* $a_1 = \|\bar{A}_\circ\| + c_{\phi_1}$, $a_2 = \|B_u K\|$, $a_3 = \|B_w K\|$,

$$\Lambda_{b-a} := 2\int_a^b \int_a^s e^{(2a_2+\sigma)(s-\tau)} d\tau ds$$

*for some* $\sigma \in \mathbb{R}^+$, $s \in [0, \bar{\tau})$ *and* $\bar{\tau} \in \mathbb{R}^+$. *Then for any* $i \in \mathbb{N}_0$, $\bar{\sigma} \leq \sigma$

$$\int_{t^i}^t e^{\bar{\sigma}s} \|e(s)\|^2 ds \leq \Lambda_{t-t^i} \int_{t^i}^t e^{\bar{\sigma}s} (a_1^2 \|x(s)\|^2 + a_3^2 \|w(s)\|^2) ds.$$

**Proof.** From the definition of measurement error and since $d\|e(t)\|/dt \leq \|\dot{e}(t)\|$, we conclude from (6.18) that $\frac{d}{dt}\|e(t)\| \leq a_1 \|x(t)\| + a_2 \|e(t)\| + a_3 \|w(t)\|$, which by applying comparison lemma gives

$$\|e(t)\| \leq a_1 \int_{t^i}^t e^{a_2(t-\tau)} \|x(\tau)\| d\tau + a_3 \int_{t^i}^t e^{a_2(t-\tau)} \|w(\tau)\| d\tau.$$

Define $f(a, b, c; x) := \int_a^b e^{c(b-s)} \|x(s)\| ds$, it is easy to verify

$$\|e(t)\|^2 \leq 2(a_1 f(t^i, t, a_2; x))^2 + 2(a_3 f(t^i, t, a_2; w))^2.$$

Multiplying this inequality by $e^{\bar{\sigma}t}$ for some $\bar{\sigma} \in \mathbb{R}_0^+$, the Cauchy-Schwartz inequality can be applied to obtain

$$e^{\bar{\sigma}t} \|e(t)\|^2 \leq 2f(t^i, t, 2a_2 + \bar{\sigma}; 1) \int_{t^i}^t e^{\bar{\sigma}\tau} \left( a_1^2 \|x\|^2 + a_3^2 \|w\|^2 \right) d\tau,$$

integrating which gives the result noting that $2\int_a^b f(a, s, 2a_2 + \bar{\sigma}; 1) ds \leq 2\int_a^b f(a, s, 2a_2 + \sigma; 1) ds =: \Lambda_{b-a}$. ■

As discussed in Section 6.2.1, set $\mathcal{T}$ is responsible to guarantee the separation of event times and Proposition 6.1 plays a key role in designing this set. To clarify this connection, we first apply Proposition 6.1 for $\bar{\sigma} = 0$ to conclude

$$\int_{t^i}^t \|e(s)\|^2 ds \leq \Lambda_{t-t^i} \int_{t^i}^t (a_1^2 \|x(s)\|^2 + a_3^2 \|w(s)\|^2) ds \tag{6.20}$$

which due to definition of $\Lambda_{t-t^i}$ implies that coefficients in the right hand side of (6.20) grow with $t$. However, this growth is allowed until the stability criteria given in Section 6.1.1 is violated. To show this, we need the next proposition.

**Proposition 6.2** *Let $W(x) = x^\top Px$ and*

$$\mathcal{Q} = \bar{A}_\circ^\top P + P\bar{A}_\circ + \gamma^{-2}PB_wB_w^\top P + C_z^\top C_z,$$
$$\mathcal{M} = \mathcal{Q} + \bar{c}\mathbb{I}_{n_x} + \frac{1}{\epsilon_1}P^2 + \frac{1}{\epsilon_2 - 1}C_z^\top C_z$$

*and $\bar{c} = \sum_{i=1}^{2}\epsilon_i c_{\phi_i}^2$ for some $P \succ 0$, $\gamma, \epsilon_1 \in \mathbb{R}^+$, $\epsilon_2 > 1$. Then for any $t \in \mathbb{R}_0^+$*

$$\dot{W} \leq \gamma^2\|w\|^2 - \|z\|^2 + \xi^\top \begin{pmatrix} \mathcal{M} & PB_uK \\ \star & 0 \end{pmatrix} \xi. \tag{6.21}$$

**Proof.** It is rather easy to check $\dot{W} \leq x^\top\mathcal{Q}x + \gamma^2\|w\|^2 - \|C_zx\|^2 + \langle\nabla W, B_uKe + \phi_1(x)\rangle$. Now defining $\bar{\xi} := \mathrm{col}(x, e, \phi_1, \phi_2)$ and since $z = C_zx + \phi_2(x)$, we conclude

$$\dot{W} \leq \gamma^2\|w\|^2 - \|z\|^2 + \bar{\xi}^\top \begin{pmatrix} \mathcal{Q} & PB_uK & P & C_z^\top \\ \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \mathbb{I}_{n_z} \end{pmatrix} \bar{\xi}. \tag{6.22}$$

Writing (6.4) as $\bar{\xi}^\top\mathrm{diag}(-c_{\phi_i}^2\mathbb{I}_{n_x}, 0, r_{1i}\mathbb{I}_{n_x}, r_{2i}\mathbb{I}_{n_z})\bar{\xi} \leq 0$ for $i \in \{1,2\}$ and $r_{11} = r_{22} = 1$, $r_{12} = r_{21} = 0$, (6.22) holds if

$$\dot{W} \leq \gamma^2\|w\|^2 - \|z\|^2 + \bar{\xi}^\top \begin{pmatrix} \mathcal{Q} + \bar{c}\mathbb{I}_{n_x} & PB_uK & P & C_z^\top \\ \star & 0 & 0 & 0 \\ \star & \star & -\epsilon_1\mathbb{I}_{n_x} & 0 \\ \star & \star & \star & (1-\epsilon_2)\mathbb{I}_{n_z} \end{pmatrix} \bar{\xi}.$$

The desired result is then obtained by applying Lemma 2.2 to the last term in the right hand side of above inequality. ■

Critical to the design of $\mathcal{T}$ is the following inequality which is obtained integrating (6.21) and applying (6.20)

$$W(x(t)) - W(x(t^i)) \leq J_{\langle t^i, t\rangle}^{\gamma_\circ} + \int_{t^i}^{t} \xi(\tau)^\top \mathcal{O}\xi(\tau)d\tau, \tag{6.23}$$

where $\gamma_\circ^2 = \gamma^2 + \epsilon_\circ a_3^2\Lambda_{t-t^i}$ and

$$\mathcal{O} = \begin{pmatrix} \mathcal{M} + \epsilon_\circ a_1^2\Lambda_{t-t^i}\mathbb{I}_{n_x} & PB_uK \\ \star & -\epsilon_\circ\mathbb{I}_{n_x} \end{pmatrix}$$

for some $\epsilon_\circ \in \mathbb{R}_0^+$ and $t \geq t^i$. Then, in the view of Definition 6.1, $\Lambda_{t-t^i}$ is allowed to grow until (i) the negative definiteness of $\mathcal{O}$, and (ii) the desired index gain level $\gamma_d$, are violated.

The above restrictions impose an upper bound on $t - t_i$ that will be used in Section 6.2.2 to design set $\mathcal{T}$.

In the sequel, we provide the design methodology in two steps, which due to the joint design nature of our approach, are dependent, *i.e.,* the design condition for controller gain depends on triggering parameters and vice versa.

### $H_\infty$ State Feedback Controller Design

The criterion to design $K$ is to ensure the existence of some $P \succ 0$ so that

$$\mathcal{M} + \sigma P + (\mu + \varepsilon)\mathbb{I}_{n_x} \preceq 0, \tag{6.24}$$

where $\sigma \in \mathbb{R}^+$ is the desired convergence rate of unperturbed model and $\mu, \varepsilon \in \mathbb{R}^+$ are triggering parameters and will be defined in the next subsection. To design $K$ based on (6.24), we follow the ideas given in [84]. Considering the notation

$$\bar{A}_{\sigma+j\omega} := A + B_u K + (\sigma + j\omega)\mathbb{I}_{n_x} = \bar{A}_\circ + (\sigma + j\omega)\mathbb{I}_{n_x}, \tag{6.25}$$

suppose that assumption (A1) holds and $K$ is designed such that $\bar{A}_\circ$ is stable and

$$\min_{\omega \in \mathbb{R}_0^+} \mathcal{S}_{n_x}(\bar{A}_{\frac{\sigma}{2}-j\omega}) > \psi, \tag{6.26}$$

where $\psi = c_{\phi_1} + \frac{\|B_w\|}{\gamma}(\sqrt{\mu+\varepsilon} + \|C_z\| + c_{\phi_2})$, some $\gamma \in \mathbb{R}^+$. While condition (6.26) is in joint design form, it can also be used in *emulation* control, by neglecting $\mu, \varepsilon$ terms. Indeed, if $\mu = \varepsilon = 0$, then (6.26) guarantees finite gain $\mathcal{L}_2$-stability (with $\mathcal{L}_2$-gain $\leq \gamma$) of the network-free system obtained by setting $e = 0$ in (6.15).

In the sequel, the intuition behind (6.26) is clarified through several lemmas. We begin with the following claim, stated without proof.

**Claim 6.1** *Choosing $\epsilon_1$, $\epsilon_2$ in Proposition 6.2 as follows:*

$$\epsilon_1 = \frac{\gamma\sqrt{(c_{\phi_2} + \|C_z\|)^2 + \mu + \varepsilon}}{c_{\phi_1}\|B_w\|}, \quad \epsilon_2 = 1 + \frac{\|C_z\|}{c_{\phi_2}} \tag{6.27}$$

*implies that*

$$\hat{\psi}^2 = \left(\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1}\right)\left(\frac{\epsilon_2\|C_z\|^2}{\epsilon_2 - 1} + \bar{c} + \mu + \varepsilon\right). \tag{6.28}$$

*where $\hat{\psi} = c_{\phi_1} + \frac{\|B_w\|}{\gamma}\sqrt{\mu + \varepsilon + (\|C_z\| + c_{\phi_2})^2}$. Indeed, this particular choice of $\epsilon_1, \epsilon_2$ minimize the right hand side of (6.28).*

**Lemma 6.1** *Let $K$ be designed under (6.26) and $\epsilon_1, \epsilon_2$ are as in (6.27). Then there exists some $\sigma \in \mathbb{R}^+$ so that matrix*

$$\Gamma = \begin{pmatrix} \bar{A}_{\frac{\sigma}{2}} & (\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1})\mathbb{I}_{n_x} \\ -\frac{\epsilon_2 C_z^\mathsf{T} C_z}{\epsilon_2 - 1} - (\bar{c} + \mu + \varepsilon)\mathbb{I}_{n_x} & -\bar{A}_{\frac{\sigma}{2}}^\mathsf{T} \end{pmatrix} \tag{6.29}$$

*has no eigenvalue with zero real part.*

**Proof.** From (6.26) we conclude $(\bar{A}_{\frac{\sigma}{2}-j\omega})^\mathsf{H}(\bar{A}_{\frac{\sigma}{2}-j\omega}) > \psi^2\mathbb{I}_{n_x} > \hat{\psi}^2\mathbb{I}_{n_x}$ and hence in light of (6.28) we have

$$(\bar{A}_{\frac{\sigma}{2}-j\omega})^\mathsf{H}(\bar{A}_{\frac{\sigma}{2}-j\omega}) > \left(\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1}\right)\left(\frac{\epsilon_2 C_z^\mathsf{T} C_z}{\epsilon_2 - 1} + (\bar{c} + \mu + \varepsilon)\mathbb{I}_{n_x}\right)$$

To complete the proof, we shall need the following useful lemma whose proof is along similar lines as the proof of [84, Theorem 2].

**Lemma 6.2** *The eigenvalues $\lambda$ of matrix $\Gamma$ are given by*

$$\det\left\{(\lambda\mathbb{I}_{n_x} + \bar{A}_{\frac{\sigma}{2}}^\mathsf{T})(\lambda\mathbb{I}_{n_x} - \bar{A}_{\frac{\sigma}{2}}) - \left(\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1}\right)\left(\frac{\epsilon_2 C_z^\mathsf{T} C_z}{\epsilon_2 - 1} + (\bar{c} + \mu + \varepsilon)\mathbb{I}_{n_x}\right)\right\} = 0.$$

Substituting $\lambda = j\omega I$ in Lemma 6.2 and noting that $(j\omega\mathbb{I}_{n_x} - \bar{A}_{\frac{\sigma}{2}})^\mathsf{H} = -(j\omega\mathbb{I}_{n_x} + \bar{A}_{\frac{\sigma}{2}}^\mathsf{T})$, Claim 6.1 confirms that $\Gamma$ has no eigenvalue with zero real part. ■

Here we recall a well-known result from [85].

**Lemma 6.3** *For a stable matrix $R$ and some $S \succeq 0$, if*

$$\begin{pmatrix} R & S \\ T & -R^\mathsf{T} \end{pmatrix} \tag{6.30}$$

*has no eigenvalue on imaginary axis, then there exists a positive definite solution $X$ to the algebraic Riccati equation*

$$R^\mathsf{T} X + XR + XSX - T = 0. \tag{6.31}$$

Applying Lemma 6.3 to matrix $\Gamma$ defined in (6.29), we conclude the existence of some $P \succ 0$ solution to (6.24).

### Design of Triggering Parameters

We start with designing $\mathcal{T}$, which as discussed, is restricted to ensure $\mathcal{O} \preceq 0$ and $\gamma_\circ \leq \gamma_d$. By applying Schur complement, the former reads as

$$\mathcal{M} + \epsilon_\circ \Lambda_{t-t^i} a_1^2 \mathbb{I}_{n_x} + \frac{1}{\epsilon_\circ} P B_u K K^\mathsf{T} B_u^\mathsf{T} P \preceq 0$$

for some $\epsilon_\circ \in \mathbb{R}_0^+$. Thus, choosing $\tau_1$ such that $\Lambda_{\tau_1} = \frac{1}{4}a_1^{-2}\|PB_uK\|^{-2}\varepsilon^2$, one can pick $\epsilon_\circ = 2\varepsilon^{-1}\|PB_uK\|^2$ to conclude

$$\epsilon_\circ \Lambda_{t-t^i} a_1^2 \mathbb{I}_{n_x} + \tfrac{1}{\epsilon_\circ} PB_uKK^\mathsf{T}B_u^\mathsf{T}P \preceq \varepsilon \mathbb{I}_{n_x} \tag{6.32}$$

for $t - t^i \in [0, \tau_1]$ and consequently by using Lemma 2.2 and (6.24) we have

$$\xi^\mathsf{T}\mathcal{O}\xi \leq -\mu\|x\|^2 - \sigma x^\mathsf{T}Px.$$

Moreover, choosing $\tau_2$ such that $\Lambda_{\tau_2} = \epsilon_\circ^{-1}a_3^{-2}(\gamma_d^2 - \gamma^2)$, we have $\gamma_\circ \leq \gamma_d$ for $t - t^i \in [0, \tau_2]$. Then $\mathcal{T}$ can be designed as

$$\mathcal{T} = [0, \hat{\tau}], \;\; \hat{\tau} = \min\{\tau_1, \tau_2\} \tag{6.33}$$

**Remark 6.5** *Definition of $\Lambda_{t-t^i}$ implies that $\tau_1$, $\tau_2$ do not depend on particular choice of $t^i$. Thus, $\hat{\tau}$ is positive and independent of triggering index $i$ and hence is defined globally.*

Definition of $\mathcal{T}$ implies that $\Lambda_{\hat{\tau}} \leq \Lambda_{\tau_1}$, $\Lambda_{\hat{\tau}} \leq \Lambda_{\tau_2}$ and hence $J^{\gamma_\circ}_{\langle t^i, t\rangle} \leq J^{\gamma_d}_{\langle t^i, t\rangle}$ and

$$\xi^\mathsf{T}(t)\mathcal{O}\xi(t) \leq -\mu\|x(t)\|^2 - \sigma x^\mathsf{T}(t)Px(t)$$

for $t \in [t^i, t^i + \hat{\tau})$. Hence (6.23) reduces to

$$W(x(t)) - W(x(t^i)) \leq J^{\gamma_d}_{\langle t^i, t\rangle} - \mu \int_{t^i}^{t} \|x(\tau)\|^2 d\tau \tag{6.34}$$

for $t \in [t^i, t^i + \hat{\tau})$. Next, we design matrix $\mathcal{X}$ as

$$\mathcal{X} = \begin{pmatrix} -\mu\mathbb{I}_{n_x} & PB_uK \\ \star & 0 \end{pmatrix}, \tag{6.35}$$

which according to the structure of $\mathcal{X}$ implies $\mathcal{P}_1 = \mu\mathbb{I}_{n_x}$, $\mathcal{P}_2 = PB_uK$ and $\mathcal{P}_3 = 0$. Suggested by Remark 6.3, the event-separation property holds under the above choice. Finally, we remark that there is no restriction on the rest of parameters $\eta$, $\zeta$, $\delta$. However, as stated in Theorem 6.2, they can be chosen properly to enlarge the inter-event times.

### 6.2.3 Main Result

To show finite gain $\mathcal{L}_2$-stability of system (6.15), we need to extend (6.34) to the interval $[t^i, t^{i+1})$, any $i \in \mathbb{N}_0$.

**Theorem 6.3** *Under assumption (A1), let $K$ to be designed such that $\bar{A}_\circ$ is stable and (6.26) holds for some $\gamma \in \mathbb{R}^+$. Also assume triggering parameters $\mathcal{T}$ and $\mathcal{X}$ are defined as in (6.33) and (6.35), respectively. The closed-loop system (6.15) is then finite gain $\mathcal{L}_2$-stable and has $\mathcal{L}_2$-gain $\leq \gamma_d$ for some $\gamma_d > \gamma$.*

**Proof.** First, define $\epsilon_\circ$ for any $i \in \mathbb{N}_0$ as

$$\epsilon_\circ = \begin{cases} 2\varepsilon^{-1}\|PB_uK\|^2, & t \in [t^i, t^i + \hat{\tau}), \\ 0, & t \in [t^i + \hat{\tau}, t^{i+1}), \end{cases} \tag{6.36}$$

which, as shown before, when applied to (6.23) for $t \in [t^i, t^i + \hat{\tau})$ results in (6.34). When $t \in [t^i + \hat{\tau}, t^{i+1})$ we have $\epsilon_\circ = 0$ and hence (6.23) reduces to

$$W(x(t)) - W(x(t^i)) \leq J^{\gamma_d}_{\langle t^i, t\rangle} - \varepsilon \int_{t^i}^{t} \|x(\tau)\|^2 d\tau + \int_{t^i}^{t} \eta e^{-\zeta\tau} d\tau$$

in view of (6.24) and TC (6.16). Combining this inequality with (6.34) and apply the procedure to the triggering intervals from 0 to $t \in \mathbb{R}_0^+$, we get $J^{\gamma_d}_{\langle 0,t\rangle} + W(x_0) + \eta \geq 0$. ■

**Theorem 6.4** *Under the conditions of Theorem 6.3 and taking $\zeta > \sigma$, the closed-loop system (6.15) is GES at equilibrium point $x = 0$ with a convergence rate $\sigma$.*

**Proof.** Setting $w = 0$, it can be inferred from (6.21), (6.24) that for any $t \in \mathbb{R}_0^+$

$$\dot{W} + \sigma W \leq \xi^\mathsf{T} \begin{pmatrix} -(\mu + \varepsilon)\mathbb{I}_{n_x} & PB_uK \\ \star & 0 \end{pmatrix} \xi. \tag{6.37}$$

Solving this inequality from $t^i$ to $t \in [t^i, t^i + \hat{\tau}]$ and apply Proposition 6.1 with $w = 0$, we can write

$$W(x(t))e^{\sigma t} \leq W(x(t^i))e^{\sigma t^i} + \int_{t^i}^{t} e^{\sigma\tau} \xi(\tau)^\mathsf{T} \begin{pmatrix} -(\mu + \varepsilon)\mathbb{I}_{n_x} + \epsilon_\circ a_1^2 \Lambda_{t-t^i}\mathbb{I}_{n_x} & PB_uK \\ \star & \epsilon_\circ \mathbb{I}_{n_x} \end{pmatrix} \xi(\tau)$$

which by using Schur complement and (6.32) reduces to $W(x(t))e^{\sigma t} \leq W(x(t^i))e^{\sigma t^i}$. To Solve (6.37) from $t^i + \hat{\tau}$ to $t \in [t^i + \hat{\tau}, t^{i+1})$, we can apply TC (6.16) and obtain $\dot{W} + \sigma W \leq \eta e^{-\zeta t}$. The solution is then

$$W(x(t))e^{\sigma t} \leq W(x(t^i + \hat{\tau}))e^{\sigma(t^i + \hat{\tau})} + \eta \int_{t^i + \hat{\tau}}^{t} e^{(\sigma - \zeta)\tau} d\tau.$$

Therefore for the interval $[t^i, t^{i+1})$ we obtain

$$W(x(t^{i+1}))e^{\sigma t^{i+1}} \leq W(x(t^i))e^{\sigma t^i} + \eta \int_{t^i}^{t^{i+1}} e^{(\sigma - \zeta)\tau} d\tau.$$

Breaking the interval $[0, t]$ into $\bigcup_{i<N-1}[t^i, t^{i+1}) \cup [t^N, t)$ where $N$ is the most recent triggering instant until $t$, it is rather easy to conclude $W(x(t)) \leq W(x_0)e^{-\sigma t} + \frac{\eta}{\zeta - \sigma}(e^{-\sigma t} - e^{-\zeta t})$. Thus, choosing $\eta = \bar{\eta} W(x_0)$ for some $\bar{\eta} \in \mathbb{R}^+$, Definition 6.2 holds for $r = 0$, $\bar{\sigma}_1 = (1 + \frac{\bar{\eta}}{\zeta - \sigma})\frac{\lambda_1(P)}{\lambda_{n_x}(P)}$, $\bar{\sigma}_2 = \sigma$. ■

### 6.2.4 Admissible Set of Eigenvalues

While criterion (6.26) is stated in terms of the smallest singular value, the relation with the eigenvalues may be of greater importance from design viewpoint. Amongst the earliest attempts to fill this gap, [84, Theorem 5] provides very good insight by exploiting the Bauer-Fike theorem to relate the perturbation of eigenvalues of a diagonalizable matrix in terms of the condition number of the matrix of the eigenvector matrix. It is worth remarking that the Bauer-Fike theorem is a weak form of Gershgorin theorem that locates the eigenvalues in the circles centered in the diagonal elements; hence offers a less-tight radius, [86]. Our proposed method, however, relies on another famous result from computational linear algebra, due to Fan and Hoffman [87].

**Theorem 6.5** *For any $A \in \mathbb{C}^{n_x \times n_x}$, $j \in \{1, \ldots, n_x\}$ we have*

$$\lambda_j(\operatorname{Re} A) \leq \mathcal{S}_j(A). \tag{6.38}$$

Theorem 6.5 is stated for complex matrices. Moreover, for $j = n_x$, (6.38) gives the relation between smallest eigenvalue of $\operatorname{Re} A$ and smallest singular value of $A$ and will be used later. We start with the following lemma.

**Lemma 6.4** *To satisfy (6.26), it suffices to choose*

$$\min_{\omega \in \mathbb{R}_0^+} \mathcal{S}_{n_x}(\bar{A}_{-j\omega}) > \psi + \tfrac{\sigma}{2}. \tag{6.39}$$

**Proof.** We first recall the following well-known property from linear algebra

$$\mathcal{S}_{n_x}(A) - \mathcal{S}_1(B) \leq \mathcal{S}_{n_x}(A + B).$$

Choosing $A = \bar{A}_{-j\omega}$, $B = \frac{\sigma}{2}\mathbb{I}_{n_x}$ and noting that $\mathcal{S}_j(\alpha I) = |\alpha|$, for any $\alpha \in \mathbb{R}$, $j \in \{1, \ldots, n_x\}$, (6.39) reads as $\mathcal{S}_{n_x}(A) - \mathcal{S}_1(B) \geq \psi$ and hence (6.26) is immediate. ■

In our next Lemma we state (6.39) as a necessary condition on the eigenvalues of $\bar{A}_\circ$. Our conjecture is that the eigenvalues of the matrix $\bar{A}_\circ$ should be placed to the left of $-\psi - \frac{\sigma}{2}$ in the complex plane. To gain familiarity with the result, we first provide a simple interpretation of how this conjecture is obtained for $n_x = 2$. Consider two extreme scenarios: First, assume $\bar{A}_\circ$ has pure imaginary eigenvalues, *i.e.*, $\bar{A}_\circ = (0\ \alpha; -\alpha\ 0)$ and hence $\bar{A}_{-j\omega} = (j\omega\ \alpha; -\alpha\ j\omega)$ where $\omega \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}$. The eigenvalues of $(\bar{A}_{-j\omega})^{\mathsf{H}}(\bar{A}_{-j\omega})$ are $\omega^2 + \alpha^2 \pm 2\alpha\omega$. Therefore, when $\alpha \geq 0$ (respectively, when $\alpha < 0$) to ensure $\mathcal{S}_{n_x}(\bar{A}_{-j\omega}) > \psi + \frac{\sigma}{2}$, $\alpha$ needs to satisfy $\alpha > \psi + \frac{\sigma}{2} + \omega$ (respectively, $\alpha < -\psi - \frac{\sigma}{2} - \omega$). Thus to satisfy (6.39), the imaginary values between $-j(\psi + \frac{\sigma}{2} + \omega)$ and $j(\psi + \frac{\sigma}{2} + \omega)$ must to

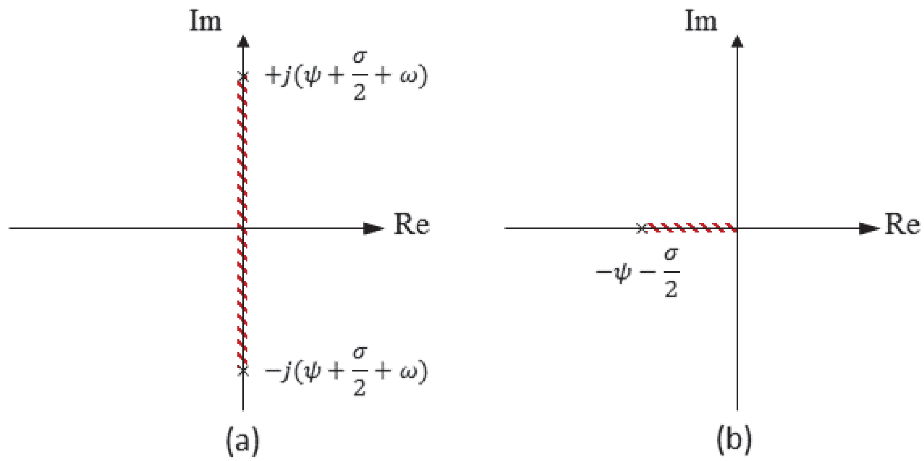



Figure 6.1: Admissible region for eigenvalues of matrix $\bar{A}_\circ$ when $\bar{A}_\circ$: (a) has pure imaginary eigenvalues and (b) is symmetric.

be excluded from the eigenvalues of matrix $\bar{A}_\circ$ (see Figure 6.1-(a)). Second, assume that $\bar{A}_\circ$ is symmetric and hence has pure real eigenvalues, *i.e.,* $\bar{A}_\circ = (c_1\ c_2; \star\ c_3)$ and hence $\bar{A}_{-j\omega} = (c_1 - j\omega\ c_2; \star\ c_3 - j\omega)$ for some $q_i \in \mathbb{R}$, $i = 1, 2, 3$ and $\omega \in \mathbb{R}_0^+$. The eigenvalues of $(\bar{A}_{-j\omega})^{\mathsf{H}}(\bar{A}_{-j\omega})$ are $\frac{1}{2}(2\omega^2 + c_1^2 + c_3^2 + 2c_2^2 \pm \sqrt{(c_1^2 - c_3^2)^2 + 4c_2^2(c_1 + c_3)^2})$. Since $\alpha$ is non-negative, we have $\mathcal{S}_{n_x}(\bar{A}_{j\omega}) > \mathcal{S}_{n_x}(\bar{A}_\circ) = |\lambda_{n_x}(\bar{A}_\circ)|$, where the last equality follows from the fact that $\bar{A}_\circ$ is symmetric. Thus if $\lambda_{n_x}(\bar{A}_\circ) < -\psi - \frac{\sigma}{2}$ and we have $|\lambda_{n_x}(\bar{A}_\circ)| > \psi + \frac{\sigma}{2}$ and consequently (6.39) holds (see Figure 6.1-(b)). The above observations lead us to the conjecture that to satisfy (6.39), points to the right hand of $-\psi - \frac{\sigma}{2}$ must to be excluded from the spectrum of $\bar{A}_\circ$.

Note that while these cases represent extreme conditions, they serve to provide a necessary condition that (6.39) must satisfy. Next lemma brings clarity to our conjecture and is starting point for designing $K$.

**Lemma 6.5** *To satisfy (6.39),* $\bar{A}_\circ$ *can not have any eigenvalue on the right hand side of* $-\psi - \frac{\sigma}{2}$.

**Proof.** Let us assume $\bar{A}_\circ$ has an eigenvalue at $-\psi_1 + j\omega_1$ for some $\psi_1 < \psi + \frac{\sigma}{2}$, $\omega_1 \in \mathbb{R}_0^+$. This implies that $\det(\bar{A}_\circ + \psi_1 \mathbb{I}_{n_x} - j\omega_1 \mathbb{I}_{n_x}) = 0$ and hence $-\psi_1$ is an eigenvalue of matrix $\bar{A}_{-j\omega_1}$. Thus we conclude that $|\lambda_{n_x}(\bar{A}_{-j\omega_1})| \leq \psi_1$. Now we use a useful relation from linear algebra, $\mathcal{S}_{n_x}(A) \leq |\lambda_j(A)|$ for any $j \in \{1, \ldots n_x\}$. Applying this inequality for $j = n_x$ we come to the following contradiction $\psi + \frac{\sigma}{2} < \mathcal{S}_{n_x}(\bar{A}_{-j\omega_1}) \leq |\lambda_{n_x}(\bar{A}_{-j\omega_1})| \leq \psi_1$, where the first inequality is obtained from (6.39) for $w = w_1$. ■

Next, we provide a sufficient condition to show that bound $-\psi - \frac{\sigma}{2}$ for choosing the eigenvalues of $\bar{A}_\circ$ is almost sharp.

**Lemma 6.6** *Condition (6.26) holds if* $\lambda_1(\mathrm{Re}\,\bar{A}_\circ) < -\psi - \frac{\sigma}{2}$.

**Proof.** We start with a simple property

$$\mathcal{S}_{n_x}(AB) \geq \mathcal{S}_{n_x}(A)\mathcal{S}_{n_x}(B)$$

which by choosing $A = \bar{A}_{-j\omega}$ and $B = -\mathbb{I}_{n_x}$ reduces to $\mathcal{S}_{n_x}(\bar{A}_{-j\omega}) \geq \mathcal{S}_{n_x}(-\bar{A}_{-j\omega})$ since $\mathcal{S}_{n_x}(-\mathbb{I}_{n_x}) = 1$. Now because $\mathrm{Re}(-\bar{A}_{-j\omega}) = \mathrm{Re}(-\bar{A}_\circ)$ and $\lambda_{n_x}(\mathrm{Re}(-\bar{A}_\circ)) = -\lambda_1(\mathrm{Re}\,\bar{A}_\circ)$, Theorem 6.5 suggests that $\mathcal{S}_{n_x}(-\bar{A}_{-j\omega}) \geq -\lambda_1(\mathrm{Re}\,\bar{A}_\circ)$. Therefore, choosing $\lambda_1(\mathrm{Re}\,\bar{A}_\circ) < -\psi - \frac{\sigma}{2}$ ensures $\mathcal{S}_{n_x}(\bar{A}_{-j\omega}) > \psi + \frac{\sigma}{2}$ which by applying Lemma 6.4 gives the desired result.
■

It is worth noting that there exist certain situations where for any matrix $K$, the largest eigenvalue of $\mathrm{Re}\,\bar{A}_\circ$ can not be pushed to the left side of $-\psi - \frac{\sigma}{2}$, restrict the application of Lemma 6.6. The authors of [84], however, provide another method to design matrix $K$, summarized in the next lemma.

**Lemma 6.7** *If $K$ is chosen such that for any $j = 1, \dots n_x$,*

$$\mathrm{Re}\{-\lambda_j(\bar{A}_\circ)\} > \mathcal{C}(X)(\psi + \frac{\sigma}{2}) \tag{6.40}$$

*where $\bar{A}_\circ = X\Upsilon X^{-1}$ and $\mathcal{C}(X)$ denotes the condition number of $X$, then (6.39) holds.*

Note that since the choice of $X$ is non-unique, it is much more reasonable to define the condition number of $X$ as

$$\mathcal{C}(X) = \inf\{\|X\|\|X^{-1}\|: X\Upsilon X^{-1} = A\}. \tag{6.41}$$

When $X$ is fixed in (6.41), the condition number can be obtained from the ratio of its largest and smallest singular values, *i.e.*, $\mathcal{C}(X) = \mathcal{S}_1/\mathcal{S}_{n_x}$ and suggests that $\mathcal{C}(X) \geq 1$. Due to difficulty in calculating condition number, the method of Lemma 6.7 often reduces to a trial and error procedure. We refer the interested reader to [84, Example 1] for more details. The following example shows that our proposed method may relax the possible conservatism associated with Lemma 6.7.

**Example 6.1** *Let*

$$\bar{A}_\circ = \begin{pmatrix} -1 & -\theta \\ 0 & -1-\theta^2 \end{pmatrix}$$

*for some parameter $\theta \in \mathbb{R}^+$. The matrix $\mathrm{Re}\,\bar{A}_\circ$ is calculated as $\frac{1}{2}(\bar{A}_\circ + \bar{A}_\circ^\mathsf{T})$, thus $\lambda_1(\mathrm{Re}\,\bar{A}_\circ) = -1 - \frac{\theta^2}{2} + \frac{\theta^2}{2}\sqrt{1+\frac{4}{\theta^2}}$. Therefore, to satisfy (6.39) using the method of Lemma 6.6, the following has to be satisfied*

$$1 + \tfrac{\theta^2}{2} - \tfrac{\theta^2}{2}\sqrt{1+\tfrac{4}{\theta^2}} > \psi. \tag{6.42}$$

*However, following the approach of Lemma 6.7, the matrix $\bar{A}_\circ$ has the eigenvalues at $-1$, $-1-\theta^2$ with the corresponding eigenvectors $(1,0)$, $(1,\theta)$. The condition number of $X$ as defined in (6.41) is $\frac{1}{\theta}+\frac{\theta}{2}$. Thus (6.40) reduces to*

$$1 > (\tfrac{1}{\theta}+\tfrac{\theta}{2})\psi. \tag{6.43}$$

*Therefore, while (6.42) has a solution for $\theta$ for $\psi \in [\frac{\sqrt{2}}{2}, 1)$, (6.43) does not (since $\frac{1}{\theta}+\frac{\theta}{2} \geq \sqrt{2}$).*

**Remark 6.6** *Our proposed method provides a sharp bound for designing controller gain $K$. Indeed, suggested by [84, Theorem 3], if $K$ is chosen such that (6.26) does not hold, then there exists some $E \in \mathbb{R}^{n_x \times n_x}$ so that the function $\phi_1(x) = Ex$ has Lipschitz constant $c_{\phi_1}$ and (6.15) is unstable.*

## 6.3 Output-Based Controller

We consider the more important scenario of output-based control. We use the dynamic observer (6.6) to reconstruct the state vector $\hat{x}$, needed to implement the control law

$$u = K\hat{x}.$$

To formalize the problem, let us define the output measurement error $e_y$ and actuator error $e_u$ as

$$\begin{aligned} e_y(t) &= y_s(t) - y(t) = y(t_y^i) - y(t),\ t \in [t_y^i, t_y^{i+1}), \\ e_u(t) &= u_s(t) - u(t) = u(t_u^j) - u(t),\ t \in [t_u^j, t_u^{j+1}), \end{aligned}$$

for any $i, j \in \mathbb{N}_0$. Contrary to the state feedback case, we assume asynchronous triggering instants at the sensor-to-controller and controller-to-actuator channels which necessitates the design of two independent ETMs.

**Remark 6.7** *Stating the TC in terms of $e_{\hat{x}} := \hat{x}(t_u^j) - \hat{x}(t)$ rather than actuation error $e_u = Ke_{\hat{x}}$ has certain practical advantages, as pointed out in [9]. Note that the use of the actuation error requires processing time to compute $e_u$ and decide next execution instant, which can only be ignored when the shared resource is transmission bandwidth rather than processing time. Note that the result of this section are still valid if the TC is stated in terms of $e_{\hat{x}}$.*

Define $\tilde{x} := x - \hat{x}$ and $\hat{A}_\circ := A - LC_y$, we can use (6.3) and (6.6) to write

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} \bar{A}_\circ & -B_u K \\ 0 & \hat{A}_\circ \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} B_u & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} e_u \\ e_y \end{pmatrix} + \begin{pmatrix} B_w \\ B_w \end{pmatrix} w + \begin{pmatrix} \phi_1(x) \\ \tilde{\phi}_1(x,\hat{x}) - L\tilde{\phi}_3(x,\hat{x}) \end{pmatrix}, \tag{6.44}$$

where $\tilde{\phi}_j(x,\hat{x}) = \phi_j(x) - \phi_j(\hat{x})$, $j \in \{1,3\}$. Treating $(e_u, e_y)$ and $w$ as the exogenous inputs, (6.44) describes the overall closed-loop model.

We propose two triggering policies at sensor-to-controller and controller-to-actuator channels, inspired by (6.16). Assuming the first event to ocurr at $t_y^0 = t_u^0 = 0$, and adopting the notations $\xi_y := \mathrm{col}(y, e_y)$, $\xi_u := \mathrm{col}(u, e_u)$, the execution times are implicitly defined as:

$$t_y^{i+1} = \inf\{t \in \mathbb{R} : t > t_y^i \wedge \xi_y^\mathsf{T}(t)\mathcal{X}_y\xi_y(t) - \Delta_y(t) = 0\} \tag{6.45a}$$

$$t_u^{j+1} = \inf\{t \in \mathbb{R} : t > t_u^j \wedge \xi_u^\mathsf{T}(t)\mathcal{X}_u\xi_u(t) - \Delta_u(t) = 0\} \tag{6.45b}$$

for $i, j \in \mathbb{N}_0$, where $t_y^i$, $t_u^j$ are the most recent execution instants. Matrices $\mathcal{X}_y = (-\mathcal{P}_1^y\ \mathcal{P}_2^y; \star\ \mathcal{P}_3^y)$ and $\mathcal{X}_u = (-\mathcal{P}_1^u\ \mathcal{P}_2^u; \star\ \mathcal{P}_3^u)$ for some symmetric $\mathcal{P}_l^y \in \mathbb{R}^{n_y \times n_y}$, $\mathcal{P}_l^u \in \mathbb{R}^{n_u \times n_u}$, $l \in \{1,2,3\}$ are designed according to the next assumption:

(A3) $\|\mathcal{P}_2^y\| \neq 0, \lambda_{n_y}(\mathcal{P}_1^y) \in \mathbb{R}^+, \lambda_1(\mathcal{P}_3^y) \in \mathbb{R}_0^+,\ \|\mathcal{P}_2^u\| \neq 0, \lambda_{n_u}(\mathcal{P}_1^u) \in \mathbb{R}^+, \lambda_1(\mathcal{P}_3^u) \in \mathbb{R}_0^+$.

The functions $\Delta_y$, $\Delta_u$ are also given by

$$\Delta_y(t) = \eta_y e^{-\zeta_y t} + (\delta_y - \eta_y e^{-\zeta_y t})\chi_{\mathcal{T}_y}(t - t_y^i),\ t \in [t_y^i, t_y^{i+1}),$$
$$\Delta_u(t) = \eta_u e^{-\zeta_u t} + (\delta_u - \eta_u e^{-\zeta_u t})\chi_{\mathcal{T}_u}(t - t_u^j),\ t \in [t_u^j, t_u^{j+1})$$

for some $\eta_y, \delta_y, \zeta_y, \eta_u, \delta_u, \zeta_u \in \mathbb{R}^+$ and $\mathcal{T}_y := [0, \hat{\tau}_y]$, $\mathcal{T}_u := [0, \hat{\tau}_u]$ for some $\hat{\tau}_y, \hat{\tau}_u \in \mathbb{R}^+$.

Proving the non-existence of accumulation point for the triggering instants $t_y^i$, $t_u^j$ requires boundedness of state trajectories of (6.44), and hence is postponed to Section 6.3.2.

### 6.3.1 Event-Based Output-Based Controller Design

Next we introduce a design methodology for jointly designing matrix gains $K$ and $L$ and corresponding triggering parameters $\mathcal{X}_y$, $\Delta_y$ and $\mathcal{X}_u$, $\Delta_u$. The method is similar to the state feedback case where the desired GES convergence rate of closed-loop system is taken as design parameter.

We start with some preliminary results. Similar to Proposition 6.1, $H_\infty$-synthesis of system (6.10) relies on finding an upper bound for the errors $e_y$, $e_u$ in terms of state $x$, $\tilde{x}$ and disturbance $w$. However, as we will see, the output feedback assumption imposes a technical difficulty originates from the asynchronous triggerings at sensor-to-controller and controller-to-actuator channels.

From the definition of $e_u$ we have

$$\|\dot{e}_u\| = \|K\dot{\hat{x}}\|. \tag{6.46}$$

For $e_y$ to be differentiable we need the following assumption:

(A4) The function $\phi_3$ is continuously differentiable.

Under assumption (A4) we have $\dot{e}_y = -\dot{y} = -C_y\dot{x} - \dot{\phi}_3(x)$ which in the view of the following observation

$$\begin{aligned}\|\dot{\phi}_3(x)\| &= \left\| \lim_{dt\to 0^+} \frac{\phi_3(x(t+dt)) - \phi_3(x(t))}{dt} \right\| \\ &\leq c_{\phi_3} \lim_{dt\to 0^+} \frac{\|x(t+dt) - x(t)\|}{dt} = c_{\phi_3}\|\dot{x}\|\end{aligned}$$

gives $\|\dot{e}_y\| \leq (\|C_y\| + c_{\phi_3})\|\dot{x}\|$. Therefore, from (6.44), (6.46) we conclude that

$$\dot{\overline{\|e_y\|}} \leq b_1\|x\| + b_2\|\tilde{x}\| + b_3\|e_u\| + b_4\|w\|, t \in [t_y^i, t_y^{i+1}) \tag{6.47a}$$

$$\dot{\overline{\|e_u\|}} \leq \bar{b}_1\|x\| + \bar{b}_2\|\tilde{x}\| + \bar{b}_3\|e_y\| + \bar{b}_4\|e_u\|, t \in [t_u^j, t_u^{j+1}) \tag{6.47b}$$

for any $i, j \in \mathbb{N}_0$, where $\bar{b}_1 = \|K\bar{A}_\circ\| + \|K\|c_{\phi_1}$, $\bar{b}_2 = \bar{b}_1 + \|KLC_y\| + \|KL\|c_{\phi_3}$, $\bar{b}_3 = \|KL\|$, $\bar{b}_4 = \|KB_u\|$ and

$$\frac{b_1}{\|\bar{A}_\circ\| + c_{\phi_1}} = \frac{b_2}{\|B_uK\|} = \frac{b_3}{\|B_u\|} = \frac{b_4}{\|B_w\|} = \|C_y\| + c_{\phi_3}.$$

As claimed, the technical difficulty of simultaneously solving (6.47a), (6.47b) arises from asynchronous triggerings. This, however, is addressed in the next proposition.

**Proposition 6.3** *Given $H_1, H_2 \in \mathbb{R}^+$ so that*

$$b_3^2\bar{b}_3^2 H_1 H_2 < 1, \tag{6.48}$$

*let $\hat{\tau}_y$ and $\hat{\tau}_u$ be the solutions to $h_1(\hat{\tau}_y) = H_1$ and $h_2(\hat{\tau}_u) = H_2$, where*

$$h_i(s) = r_i \int_r^{r+s} \int_r^{\tau_2} e^{\nu_i(\tau_2 - \tau_1)} d\tau_1 d\tau_2$$

*for $r_1 = 4$, $r_2 = 3$, $\nu_1 = \sigma_\circ$, $\nu_2 = 2\bar{b}_4 + \sigma_\circ$ for some $\sigma_\circ \in \mathbb{R}^+$. Then there exist constants $k_i \in \mathbb{R}^+$, $i \in \{1, \cdots, 5\}$ so that for any $t \in \mathbb{R}^+$, $\bar{\sigma}_\circ \leq \sigma_\circ$, $\alpha_j \in \mathbb{R}^+$, $j = \{1, 2\}$, the following* *holds*

$$\int_0^t e^{\bar{\sigma}_\circ \tau}\Big(\alpha_1\|e_y(\tau)\|^2+\alpha_2\|e_u(\tau)\|^2\Big)d\tau \leq$$
$$\int_0^t e^{\bar{\sigma}_\circ \tau}\Big(k_1\|x(\tau)\|^2+k_2\|\tilde{x}(\tau)\|^2+k_3 r_y\|y(\tau)\|^2$$
$$+k_4 r_u\|K\hat{x}(\tau)\|^2+k_3\eta_y e^{-\zeta_y\tau}+k_4\eta_u e^{-\zeta_u\tau}+k_5\|w\|^2\Big)d\tau$$

*where* $k_1=\alpha_1\beta_1+\alpha_2\beta_3,\ k_2=\alpha_1\beta_2+\alpha_2\beta_4,\ k_3=\beta(\alpha_1+\alpha_2H_2\bar{b}_4^2),\ k_4=\beta(\alpha_2+\alpha_1H_1\bar{b}_3^2)$, $k_5=k_3H_1b_4^2,\ \beta=(1-H_1H_2b_3^2\bar{b}_4^2)^{-1},\ \beta_1=\beta H_1(b_1^2+H_2b_3^2\bar{b}_1^2),\ \beta_2=\beta H_1(b_2^2+H_2b_3^2\bar{b}_2^2)$, $\beta_3=\beta H_2(\bar{b}_1^2+H_1\bar{b}_3^2b_1^2),\ \beta_4=\beta H_2(\bar{b}_2^2+H_1\bar{b}_3^2b_2^2)$.

**Proof.** Solving (6.47a) for $\|e_y\|$ gives

$$\|e_y(t)\|\leq \int_{t_y^i}^t \Big\{b_1\|x(\tau)\|+b_2\|\tilde{x}(\tau)\|+b_3\|e_u(\tau)\|+b_4\|w(\tau)\|\Big\}d\tau.$$

Therefore we have

$$\|e_y(t)\|^2\leq 4\Big\{\Big(b_1\int_{t_y^i}^t\|x\|d\tau\Big)^2+\Big(b_2\int_{t_y^i}^t\|\tilde{x}\|d\tau\Big)^2$$
$$+\Big(b_3\int_{t_y^i}^t\|e_u\|d\tau\Big)^2+\Big(b_4\int_{t_y^i}^t\|w\|d\tau\Big)^2\Big\}, \tag{6.49}$$

which by using Cauchy-Schwartz inequality reduces to

$$\|e_y(t)\|^2\leq 4\int_{t_y^i}^t e^{-\bar{\sigma}_\circ\tau}d\tau\int_{t_y^i}^t e^{\bar{\sigma}_\circ\tau}\Big\{b_1^2\|x(\tau)\|^2$$
$$+b_2^2\|\tilde{x}(\tau)\|^2+b_3^2\|e_u(\tau)\|^2+b_4^2\|w(\tau)\|^2\Big\}d\tau$$

and since $4\int_r^{r+s}\int_r^{\tau_2}e^{\bar{\sigma}_\circ(\tau_2-\tau_1)}d\tau_1 d\tau_2\leq h_1(s)$, we have

$$\int_{t_y^i}^t e^{\bar{\sigma}_\circ\tau}\|e_y(\tau)\|^2d\tau\leq h_1(t-t_y^i)\int_{t_y^i}^t e^{\bar{\sigma}_\circ\tau}\Big\{b_1^2\|x(\tau)\|^2$$
$$+b_2^2\|\tilde{x}(\tau)\|^2+b_3^2\|e_u(\tau)\|^2+b_4^2\|w(\tau)\|^2\Big\}d\tau. \tag{6.50}$$

Similarly, solving (6.47b) for $\|e_u\|$ gives

$$\|e_u(t)\|\leq\int_{t_u^j}^t e^{\bar{b}_4(t-\tau)}\Big\{\bar{b}_1\|x(\tau)\|+\bar{b}_2\|\tilde{x}(\tau)\|+\bar{b}_3\|\hat{e}_y\tau)\|\Big\}d\tau$$

which using Cauchy-Schwartz inequality reduces to

$$\|e_u(t)\|^2\leq 3\int_{t_u^j}^t e^{2\bar{b}_4(t-\tau)}e^{-\bar{\sigma}_\circ\tau}d\tau\int_{t_u^j}^t e^{\bar{\sigma}_\circ\tau}\Big\{\bar{b}_1^2\|x(\tau)\|^2$$
$$+\bar{b}_2^2\|\tilde{x}(\tau)\|^2+\bar{b}_3^2\|\hat{e}_y\tau)\|^2\Big\}d\tau.$$

Since $3\int_r^{r+s}\int_r^{\tau_2} e^{(2\bar{b}_4+\bar{\sigma}_\circ)(\tau_2-\tau_1)}d\tau_1 d\tau_2 \leq h_2(s)$, we have

$$\int_{t_u^j}^{t} e^{\bar{\sigma}_\circ\tau}\|e_u(\tau)\|^2 d\tau \leq h_2(t-t_u^j)\int_{t_y^i}^{t} e^{\bar{\sigma}_\circ\tau}\Big\{\bar{b}_1^2\|x(\tau)\|^2 + \bar{b}_2^2\|\tilde{x}(\tau)\|^2 + \bar{b}_3^2\|e_y(\tau)\|^2\Big\}d\tau. \tag{6.51}$$

Now consider

$$\int_0^t e^{\bar{\sigma}_\circ\tau}\|e_y(\tau)\|^2 d\tau = \sum_i \Bigg\{\underbrace{\int_{t_y^i}^{t_y^i+\hat{\tau}_y}}_{1} + \underbrace{\int_{t_y^i+\hat{\tau}_y}^{t_y^{i+1}}}_{2} e^{\bar{\sigma}_\circ\tau}\|e_y(\tau)\|^2 d\tau\Bigg\},$$

$$\int_0^t e^{\bar{\sigma}_\circ\tau}\|e_u(\tau)\|^2 d\tau = \sum_j \Bigg\{\underbrace{\int_{t_u^j}^{t_u^j+\hat{\tau}_u}}_{1} + \underbrace{\int_{t_u^j+\hat{\tau}_u}^{t_u^{j+1}}}_{2} e^{\bar{\sigma}_\circ\tau}\|e_u(\tau)\|^2 d\tau\Bigg\}.$$

For the terms in 1, we can apply (6.50) and (6.51) with $h_1(\hat{\tau}_y) \leq H_1$ and $h_2(\hat{\tau}_u) \leq H_2$. For the terms in 2, however, we may apply TC (6.45). Therefore, we have

$$\int_0^t e^{\bar{\sigma}_\circ\tau}\|e_y(\tau)\|^2 d\tau \leq \int_0^t e^{\bar{\sigma}_\circ\tau}\Big(H_1\Big\{b_1^2\|x(\tau)\|^2 + b_2^2\|\tilde{x}(\tau)\|^2 + b_3^2\|e_u(\tau)\|^2 + b_4^2\|w(\tau)\|^2\Big\} + r_y\|y\|^2 + \eta_y e^{-\zeta_y\tau}\Big)d\tau,$$

$$\int_0^t e^{\bar{\sigma}_\circ\tau}\|e_u(\tau)\|^2 d\tau \leq \int_0^t e^{\bar{\sigma}_\circ\tau}\Big(H_2\Big\{\bar{b}_1^2\|x(\tau)\|^2 + \bar{b}_2^2\|\tilde{x}(\tau)\|^2 + \bar{b}_3^2\|e_y(\tau)\|^2\Big\} + r_u\|k\hat{x}\|^2 + \eta_u e^{-\zeta_u\tau}\Big)d\tau,$$

which can be written in matrix form as follows

$$\begin{pmatrix} 1 & -H_1 b_3^2 \\ -H_2\bar{b}_3^2 & 1 \end{pmatrix}\begin{pmatrix} \int_0^t e^{\bar{\sigma}_\circ\tau}\|e_y(\tau)\|^2 d\tau \\ \int_0^t e^{\bar{\sigma}_\circ\tau}\|e_u(\tau)\|^2 d\tau \end{pmatrix} \leq \begin{pmatrix} H_1 b_1^2 & H_1 b_2^2 \\ H_2\bar{b}_1^2 & H_2\bar{b}_2^2 \end{pmatrix}\begin{pmatrix} \int_0^t e^{\bar{\sigma}_\circ\tau}\|x(\tau)\|^2 d\tau \\ \int_0^t e^{\bar{\sigma}_\circ\tau}\|\tilde{x}(\tau)\|^2 d\tau \end{pmatrix} + \begin{pmatrix} \int_0^t e^{\bar{\sigma}_\circ\tau}(r_y\|y(\tau)\|^2 + \eta_y e^{-\zeta_y\tau} + H_1 b_4^2\|w(\tau)\|^2)d\tau \\ \int_0^t e^{\bar{\sigma}_\circ\tau}(r_u\|K\hat{x}(\tau)\|^2 + \eta_u e^{-\zeta_u\tau})d\tau \end{pmatrix}. \tag{6.52}$$

For $\beta := (1 - b_3^2\bar{b}_3^2 H_1 H_2)^{-1}$, which is well-defined due to condition (6.48), we have

$$\int_0^t e^{\bar{\sigma}_\circ\tau}\Big(\alpha_1\|e_y(\tau)\|^2 + \alpha_2\|e_u(\tau)\|^2\Big)d\tau = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix} \beta & \beta H_1 b_3^2 \\ \beta H_2\bar{b}_3^2 & \beta \end{pmatrix} \times \text{RHS of (6.52)}$$

which gives the desired result. ■

Note that the upper bound of the output measurement error $e_y$ (respectively, actuation error $e_u$) can be obtained by setting $\alpha_2 = 0$ (respectively, $\alpha_1 = 0$) in Proposition 6.3 for $\bar{\sigma}_\circ = 0$.

**Remark 6.8** *design of $\hat{\tau}_y$, $\hat{\tau}_u$ are coupled according to condition (6.48) and hence unique solutions does not exist.*

**Remark 6.9** *Proposition 6.3, which can be viewed as the output-based counterpart of Proposition 6.1, introduces several penalty terms for asynchronous triggerings at the sensor-to-controller and controller-to-actuator channels.*

The following proposition is the output-based counterpart of Proposition 6.2 and serve to clarify the design ideas in the next section.

**Proposition 6.4** *Define $\bar{\gamma}_\circ^2 = \gamma^2+\theta\tilde{\gamma}^2$ for some $\gamma, \tilde{\gamma} \in \mathbb{R}^+$ and let $V(x,\tilde{x}) = W(x)+\theta\tilde{W}(\tilde{x})$ be the candidate Lyapunov function where $W(x) = x^\top Px$, $\tilde{W}(\tilde{x}) = \tilde{x}^\top \tilde{P}\tilde{x}$ for some $P, \tilde{P} \succ 0$ and $\theta \in \mathbb{R}^+$. Also define $\mathcal{Q}$ as in Proposition 6.2 and*

$$\begin{aligned}
\tilde{\mathcal{Q}} &= \hat{A}_\circ^\top \tilde{P} + \tilde{P}\hat{A}_\circ + \tilde{\gamma}^{-2}\tilde{P}B_wB_w^\top\tilde{P},\\
\mathcal{M}_y &= \mathcal{Q} + \bar{c}_y\mathbb{I}_{n_x} + \frac{P^2}{\epsilon_1^y} + \frac{C_z^\top C_z}{\epsilon_2^y - 1},\\
\mathcal{M}_u &= \tilde{\mathcal{Q}} + \bar{c}_u\mathbb{I}_{n_x} + \frac{\tilde{P}^2}{\epsilon_1^u} + \frac{\tilde{P}LL^\top\tilde{P}}{\epsilon_3^u}
\end{aligned}$$

*for $\bar{c}_u = \epsilon_1^u c_{\phi_1}^2 + \epsilon_3^u c_{\phi_3}^2$, $\bar{c}_y = \epsilon_1^y c_{\phi_1}^2 + \epsilon_2^y c_{\phi_2}^2$ and some $\epsilon_1^u, \epsilon_3^u, \epsilon_1^y, \epsilon_2^y \in \mathbb{R}^+$. Then for any $t \in \mathbb{R}_0^+$*

$$V(x,\tilde{x}) - V(x_0,\tilde{x}_0) \leq J_{\langle 0,t\rangle}^{\bar{\gamma}_\circ} + \int_0^t \hat{\xi}(\tau)^\top \bar{\mathcal{O}}\hat{\xi}(\tau)d\tau, \tag{6.53}$$

*where $\hat{\xi} = col(x, \tilde{x}, e_y, e_u)$ and*

$$\bar{\mathcal{O}} := \begin{pmatrix} \mathcal{M}_y & -PB_uK & 0 & PB_u \\ \star & \theta\mathcal{M}_u & \theta\tilde{P}L & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{pmatrix}.$$

**Proof.** Similar to the proof of Proposition 6.2, we have

$$\begin{aligned}
\dot{W} \leq{}& \gamma^2\|w\|^2 - \|z\|^2 - \langle\nabla W, B_uK\tilde{x}\rangle \\
&+ \begin{pmatrix} x \\ e_u \\ \phi_1 \\ \phi_2 \end{pmatrix}^\top \begin{pmatrix} \mathcal{Q} + \bar{c}_y\mathbb{I}_{n_x} & PB_u & P & C_z^\top \\ \star & 0 & 0 & 0 \\ \star & \star & -\epsilon_1^y\mathbb{I}_{n_x} & 0 \\ \star & \star & \star & (1-\epsilon_2^y)\mathbb{I}_{n_z} \end{pmatrix} \begin{pmatrix} x \\ e_u \\ \phi_1 \\ \phi_2 \end{pmatrix}
\end{aligned}$$

for some $\epsilon_1^y \in \mathbb{R}^+$, $\epsilon_2^y > 1$. From Schur complement we conclude

$$\dot{W} \leq \gamma^2\|w\|^2 - \|z\|^2 - \langle\nabla W, B_uK\tilde{x}\rangle + \bar{\xi}_y^\top \begin{pmatrix} \mathcal{M}_y & PB_u \\ \star & 0 \end{pmatrix}\bar{\xi}_u, \tag{6.54}$$

where $\bar{\xi}_u := \mathrm{col}(x, e_u)$. Similarly, for $\tilde{x}$ subspace we can write

$$\dot{\tilde{W}} \leq \tilde{\gamma}^2 \|w\|^2 + \begin{pmatrix} \tilde{x} \\ e_y \\ \tilde{\phi}_1 \\ \tilde{\phi}_3 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \tilde{\mathcal{Q}} + \bar{c}_u \mathbb{I}_{n_x} & \tilde{P}L & \tilde{P} & \tilde{P}L \\ \star & 0 & 0 & 0 \\ \star & \star & -\epsilon_1^u \mathbb{I}_{n_x} & 0 \\ \star & \star & \star & -\epsilon_3^u \mathbb{I}_{n_y} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ e_y \\ \tilde{\phi}_1 \\ \tilde{\phi}_3 \end{pmatrix}.$$

Using Schur complement and defining $\bar{\xi}_y := \mathrm{col}(\tilde{x}, e_y)$, we have

$$\dot{\tilde{W}} \leq \tilde{\gamma}^2 \|w\|^2 + \bar{\xi}_y^{\mathsf{T}} \begin{pmatrix} \mathcal{M}_u & \tilde{P}L \\ \star & 0 \end{pmatrix} \bar{\xi}_y \tag{6.55}$$

The desired result is then obtained by merging (6.54), (6.55). ∎

### $H_\infty$ Observer-Based Controller Design

Inspired by state feedback case, the controller and observer gains $K$, $L$ are designed to ensure the existence of some $P \succ 0$, $\alpha \in \mathbb{R}^+$ so that

$$\mathcal{M}_y + \frac{1}{\alpha} P B_u B_u^{\mathsf{T}} P + \sigma_y P + \mu_y \mathbb{I}_{n_x} + \varepsilon_y \mathbb{I}_{n_x} \preceq 0, \tag{6.56}$$

and some $\tilde{P} \succ 0$ such that

$$\mathcal{M}_u + \sigma_u \tilde{P} + \mu_u \mathbb{I}_{n_x} + \varepsilon_u \mathbb{I}_{n_x} \preceq 0 \tag{6.57}$$

where $\sigma_y, \sigma_u \in \mathbb{R}^+$ will be used to determine the desired convergence rate of unperturbed model and $\mu_y, \mu_u, \varepsilon_y, \varepsilon_u \in \mathbb{R}^+$ are triggering parameters and will be specified in the next subsection and $\alpha \in \mathbb{R}^+$ is . Note that (6.56), (6.57) are required later to prove the stability of system (6.44). Let us adopt the following notation

$$\hat{A}_{\sigma+j\omega} := A - LC_y + (\sigma + j\omega)\mathbb{I}_{n_x} = \hat{A}_\circ + (\sigma + j\omega)\mathbb{I}_{n_x}.$$

Under assumption (A1), the controller and observer gains $K$ and $L$ are designed such that $\bar{A}_\circ$ and $\hat{A}_\circ$ are stable and

$$\min_{\omega \in \mathbb{R}_0^+} \mathcal{S}_{n_x}\left(\bar{A}_{\frac{\sigma_y}{2} - j\omega}\right) > \psi_y, \tag{6.58}$$

$$\min_{\omega \in \mathbb{R}_0^+} \mathcal{S}_{n_x}\left(\hat{A}_{\frac{\sigma_u}{2} - j\omega}\right) > \psi_u, \tag{6.59}$$

where $\psi_y = c_{\phi_1} + (\frac{\|B_w\|}{\gamma} + \frac{\|B_u\|}{\sqrt{\alpha}})(\sqrt{\mu_y + \varepsilon_y} + \|C_z\| + c_{\phi_2})$ and $\psi_u = c_{\phi_1} + c_{\phi_3}\|L\| + \frac{\|B_w\|}{\tilde{\gamma}}\sqrt{\mu_u + \varepsilon_u}$, some $\gamma, \tilde{\gamma} \in \mathbb{R}^+$.

**Remark 6.10** *Compared to the state feedback design (6.26), $\frac{\|B_u\|}{\sqrt{\alpha}}$ in (6.58) is introduced as the penalty for output feedback.*

**Remark 6.11** *The design of $L$ is non-trivial due to the presence of $\|L\|$ in the right hand side of (6.59). However, when $c_{\phi_3}$ is small enough, one can readily guarantee the existance of a solution for $L$. In limit, when $\phi_3 = 0$ and henec $y = C_y x$, $L$ can be designed similar to design of $K$ in (6.58), (6.26) which is discussed in details in Section 6.2.4. To cover a wider range of nonlinearities, one can maximize the admissible $c_{\phi_3}$ denoted by $\bar{c}_{\phi_3}$ as the maximum possible $c_{\phi_3}$ such that (6.59) has a solution for some $L$.*

Similar to Claim 6.1, it can be verified that choosing

$$\begin{aligned} \epsilon_1^y &= \frac{\sqrt{(c_{\phi_2}+\|C_z\|)^2+\mu_y+\varepsilon_y}}{c_{\phi_1}\sqrt{\frac{\|B_w\|^2}{\gamma^2}+\frac{\|B_u\|^2}{\alpha}}}, \quad \epsilon_2^y = 1 + \frac{\|C_z\|}{c_{\phi_2}}, \\ \epsilon_1^u &= \frac{1}{c_{\phi_1}}\sqrt{\frac{\epsilon_3^u c_{\phi_3}^2+\mu_u+\varepsilon_u}{\frac{\|B_w\|^2}{\tilde{\gamma}^2}+\frac{\|L\|^2}{\epsilon_3^u}}}, \quad \epsilon_3^u = \frac{\tilde{\gamma}\|L\|\sqrt{\mu_u+\varepsilon_u}}{c_{\phi_3}\|B_w\|}, \end{aligned} \tag{6.60}$$

implies that

$$\hat{\psi}_y^2 = \Big(\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1^y} + \frac{\|B_u\|^2}{\alpha}\Big)\Big(\frac{\epsilon_2^y\|C_z\|^2}{\epsilon_2^y-1} + \bar{c}_y + \mu_y + \varepsilon_y\Big), \tag{6.61}$$

$$\psi_u^2 = \Big(\frac{\|B_w\|^2}{\tilde{\gamma}^2} + \frac{1}{\epsilon_1^u} + \frac{\|L\|^2}{\epsilon_3^u}\Big)\Big(\bar{c}_u + \mu_u + \varepsilon_u\Big). \tag{6.62}$$

where $\hat{\psi}_y = c_{\phi_1} + (\frac{\|B_w\|}{\gamma} + \frac{\|B_u\|}{\sqrt{\alpha}})\sqrt{\mu_y + \varepsilon_y + (\|C_z\| + c_{\phi_2})^2}$. The following lemma then clarifies the idea behind (6.58), (6.59).

**Lemma 6.8** *Let $K$ and $L$ be designed under (6.58), (6.59). Under assumption (6.60), there exist some $\sigma_y, \sigma_u \in \mathbb{R}^+$ such that the following matrices have no eigenvalue on imaginary axis.*

$$\Gamma_1 = \begin{pmatrix} \bar{A}_{\frac{\sigma_y}{2}} & (\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1^y} + \frac{\|B_u\|^2}{\alpha})\mathbb{I}_{n_x} \\ -\frac{\epsilon_2^y C_z^\mathsf{T} C_z}{\epsilon_2^y - 1} - (\bar{c}_y + \mu_y + \varepsilon_y)\mathbb{I}_{n_x} & -\bar{A}_{\frac{\sigma_y}{2}}^\mathsf{T} \end{pmatrix}$$

$$\Gamma_2 = \begin{pmatrix} \hat{A}_{\frac{\sigma_u}{2}} & (\frac{\|B_w\|^2}{\tilde{\gamma}^2} + \frac{1}{\epsilon_1^u} + \frac{\|L\|^2}{\epsilon_3^u})\mathbb{I}_{n_x} \\ -(\bar{c}_u + \mu_u + \varepsilon_u)\mathbb{I}_{n_x} & -\hat{A}_{\frac{\sigma_u}{2}}^\mathsf{T} \end{pmatrix}.$$

**Proof.** From (6.58) we conclude $(\bar{A}_{\frac{\sigma_y}{2}-j\omega})^\mathsf{H}(\bar{A}_{\frac{\sigma_y}{2}-j\omega}) > \psi_y^2\mathbb{I}_{n_x} > \hat{\psi}_y^2\mathbb{I}_{n_x}$. Next, similar to Lemma 6.2, it can be shown that the eigenvalues of matrix $\Gamma_1$, $\Gamma_2$ are obtained from

$$\det\Big\{(\lambda\mathbb{I}_{n_x} + \bar{A}_{\frac{\sigma_y}{2}}^\mathsf{T})(\lambda\mathbb{I}_{n_x} - \bar{A}_{\frac{\sigma_y}{2}}) - \Big(\frac{\|B_w\|^2}{\gamma^2} + \frac{1}{\epsilon_1^y} + \frac{\|B_u\|^2}{\alpha}\Big)\Big(\frac{\epsilon_2^y C_z^\mathsf{T} C_z}{\epsilon_2^y-1} + (\bar{c}_y + \mu_y + \varepsilon_y)\mathbb{I}_{n_x}\Big)\Big\} = 0,$$

$$\det\Big\{(\lambda\mathbb{I}_{n_x} + \hat{A}_{\frac{\sigma_u}{2}}^\mathsf{T})(\lambda\mathbb{I}_{n_x} - \hat{A}_{\frac{\sigma_u}{2}}) - \Big(\frac{\|B_w\|^2}{\tilde{\gamma}^2} + \frac{1}{\epsilon_1^u} + \frac{\|L\|^2}{\epsilon_3^u}\Big)(\bar{c}_u + \mu_u + \varepsilon_u)\mathbb{I}_{n_x}\Big\} = 0.$$

Therefore, in light of (6.61), (6.62) the result follows. ■

Applying Lemma 6.3 to the matrices $\Gamma_1$, $\Gamma_2$, it is not difficult to verify the existence of $P, \tilde{P} \succ 0$ solutions to (6.56), (6.57).

### Design of Triggering Parameters

First define

$$\mathcal{O}_{\mathcal{M}} := \begin{pmatrix} \mathcal{M}_y & -PB_uK \\ \star & \theta\mathcal{M}_u \end{pmatrix}, \quad \mathcal{O}_K := \begin{pmatrix} K^\intercal K & -K^\intercal K \\ \star & K^\intercal K \end{pmatrix}.$$

The admissible range of parameters are restricted to ensure the stability of (6.44). To show this, we apply Young's inequality to (6.53) to get

$$\begin{aligned} V(x,\tilde{x}) - V(x_0,\tilde{x}_0) \le J^{\bar{\gamma}_\circ}_{\langle 0,t\rangle} + \int_0^t \Big\{ \tilde{\xi}(\tau)^\intercal \mathcal{O}_{\mathcal{M}} \tilde{\xi}(\tau) + \alpha\|e_u\|^2 \\ + \frac{1}{\alpha} x^\intercal PB_uB_u^\intercal Px + \frac{1}{\hat{\alpha}}\theta\tilde{x}^\intercal \tilde{P}LL^\intercal\tilde{P}\tilde{x} + \theta\hat{\alpha}\|e_y\|^2 \Big\} d\tau, \end{aligned}$$

where $\alpha, \hat{\alpha} \in \mathbb{R}^+$, $V(x,\tilde{x})$ as defined in Proposition 6.4 and $\tilde{\xi} := \mathrm{col}(x,\tilde{x})$. Then applying Proposition 6.3 for $\alpha_1 = \hat{\alpha}\theta$, $\alpha_2 = \alpha$, $\bar{\sigma}_\circ = 0$, we obtain

$$\begin{aligned} V(x,\tilde{x}) - V(x_0,\tilde{x}_0) \le J^{\sqrt{\bar{\gamma}_\circ^2 + k_5}}_{\langle 0,t\rangle} + \int_0^t \Big\{ \tilde{\xi}^\intercal (\mathcal{O}_{\mathcal{M}} + k_4 r_u \mathcal{O}_K)\tilde{\xi} \\ + x^\intercal \Big( \frac{1}{\alpha} PB_uB_u^\intercal P + (k_1 + k_3 r_y(\|C_y\| + c_{\phi_3})^2)\mathbb{I}_{n_x} \Big) x \\ + \tilde{x}^\intercal \Big( \frac{1}{\hat{\alpha}}\theta\tilde{P}LL^\intercal\tilde{P} + k_2\mathbb{I}_{n_x} \Big)\tilde{x} + k_3\eta_y e^{-\zeta_y\tau} + k_4\eta_u e^{-\zeta_u\tau} \Big\} d\tau, \end{aligned} \tag{6.63}$$

where we used the fact that $\|y\| \le (\|C_y\| + c_{\phi_3})\|x\|$. The design parameters are restricted to the following conditions:

$$k_5 \le \gamma_d^2 - \bar{\gamma}_\circ^2, \tag{6.64a}$$

$$k_1 + \kappa(\|C_y\| + c_{\phi_3})^2 \le \varepsilon_y, \tag{6.64b}$$

$$\tilde{P}LL^\intercal\tilde{P} + \hat{\alpha}\vartheta\mathbb{I}_{n_x} \preceq \hat{\alpha}\varepsilon_u\mathbb{I}_{n_x}, \tag{6.64c}$$

$$k_2 \le \theta\vartheta, \tag{6.64d}$$

$$k_3 r_y \le \kappa, \tag{6.64e}$$

$$\begin{pmatrix} -\mu_y\mathbb{I}_{n_x} + k_4 r_u K^\intercal K & -PB_uK - k_4 r_u K^\intercal K \\ \star & -\theta\mu_u\mathbb{I}_{n_x} + k_4 r_u K^\intercal K \end{pmatrix} \preceq 0. \tag{6.64f}$$

Under conditions (6.64a-e) and in the light of (6.56), (6.57) we have the stability for subsystem $x$, *i.e.*,

$$\mathcal{M}_y + \frac{1}{\alpha} PB_uB_u^\intercal P + k_1\mathbb{I}_{n_x} + k_3 r_y(\|C_y\| + c_{\phi_3})^2\mathbb{I}_{n_x} \preceq 0, \tag{6.65}$$

and the subsystem $\tilde{x}$, *i.e.*,

$$\mathcal{M}_u + \frac{1}{\hat{\alpha}}\tilde{P}LL^\intercal\tilde{P} + \frac{k_2}{\theta}\mathbb{I}_{n_x} \preceq 0 \tag{6.66}$$

and also the guaranteed $\mathcal{L}_2$ performance level $\gamma_d$.

Due to dependence on $k_i$, $i \in \{1, \dots 5\}$, $H_1$ and $H_2$ have to be designed not only based on (6.48) but also such that (6.64) is satisfied. Therefore, we define

$$\mathcal{T}_y \times \mathcal{T}_u := \left\{ (\tau_1, \tau_2) : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} h_1(\tau_1) \\ h_2(\tau_2) \end{pmatrix}, (6.48), (6.64\text{a-d}) \text{ hold} \right\}. \tag{6.67}$$

Note that there is a tradeoff in the selection of $(\hat{\tau}_y, \hat{\tau}_u)$ from $\mathcal{T}_y \times \mathcal{T}_u$ in the way that the larger one is chosen, the smaller the other will be. Once $H_1, H_2$ and consequently $k_i$, $i \in \{1, \dots, 5\}$ get fixed, $\theta$, $r_u$, $r_y$, $\mu_u$, $\mu_y$ has to be designed to satisfy (6.64e,f). In addition, matrices $\mathcal{X}_y$, $\mathcal{X}_u$ are given by

$$\mathcal{X}_y = \text{diag}(-r_y \mathbb{I}_{n_y}, \mathbb{I}_{n_y}), \quad \mathcal{X}_u = \text{diag}(-r_u \mathbb{I}_{n_u}, \mathbb{I}_{n_u}). \tag{6.68}$$

Similar to the state feedback case, the rest of parameters, $\eta_y, \delta_y, \zeta_y, \eta_u, \delta_u, \zeta_u$, can be properly designed to improve inter-event times with no restriction on them.

**Remark 6.12** *Since $k_i$, $i \in \{1, 3, 5\}$ are linear functions of $H_1$, $H_2$, inequalities in (6.64) are well-defined provided that $H_1$, $H_2$, $r_y$ and $r_u$ chosed small enough. However, suggested by the multiplicative terms $k_y r_u$, $k_3 r_y$, the design seems to be a compromise between choosing $H_1$, $H_2$ from one hand and $r_y$, $r_u$ from the other hand.*

**Remark 6.13** *Compared to state feedback case, design of $\mathcal{T}_y, \mathcal{T}_u$ involves more complex steps due to (i) output feedback assumption and (ii) asynchronous triggerings.*

### 6.3.2 Isolation of Triggering Instants

To show the successful implementation of control task under the proposed ETM, the event-separation property is required to be guaranteed. We first provide the following extension of event-separation property definition for the system (6.44).

**Definition 6.4** *Let $\tau_m^y = \inf\{t_y^{i+1} - t_y^i : i \in \mathbb{N}_0\}$ and $\hat{\tau}_m^u = \inf\{t_u^{j+1} - t_u^j : j \in \mathbb{N}_0\}$ be the MIETs for the sensor-to-controller and controller-to-actuator channels. System (6.44) has the robust semi-global event-separation property if there exists $\epsilon \in \mathbb{R}^+$ so that for any compact sets $\mathcal{B}, \tilde{\mathcal{B}} \subset \mathbb{R}^{n_x}$, $\inf\{\tau_m^y, \tau_m^u : x_0 \in \mathcal{B}, \tilde{x}_0 \in \tilde{\mathcal{B}}, |w|_\infty \leq \epsilon\} > 0$.*

The event-separation property for system (6.44) relies on the following boundedness property for trajectories $x$, $\tilde{x}$.

**Lemma 6.9** *Suppose that under assumption (A1), matrices $K, L$ are designed such that $\bar{A}_\circ, \hat{A}_\circ$ are stable and (6.58), (6.59) hold for some $\gamma, \tilde{\gamma} \in \mathbb{R}^+$. Then under the TC (6.45)* *with $\mathcal{X}_y, \mathcal{X}_u$ and $\mathcal{T}_y, \mathcal{T}_u$ as defined in (6.68) and (6.67), respectively, and some $\zeta_y, \zeta_u > \sigma_\circ$, the trajectories $(x, \tilde{x})$ of the system (6.44) starting from $\mathcal{B}\times\tilde{\mathcal{B}}$ will remain in $\mathcal{B}'\times\tilde{\mathcal{B}}'$ provided that $|w|_\infty \leq \epsilon$, where*

$$\mathcal{B}' = \{x \in \mathbb{R}^{n_x} : \|x\| \leq \varrho(\lambda_{n_x}(P))^{-\frac{1}{2}}\}$$
$$\tilde{\mathcal{B}}' = \{\tilde{x} \in \mathbb{R}^{n_x} : \|\tilde{x}\| \leq \theta^{-\frac{1}{2}}\varrho(\lambda_{n_x}(\tilde{P}))^{-\frac{1}{2}}\}$$

*and* $\varrho = \frac{\gamma_d^2\epsilon^2}{\sigma_\circ} + \frac{k_3\eta_y}{\zeta_y - \sigma_\circ} + \frac{k_4\eta_u}{\zeta_u - \sigma_\circ} + \sup_{(x_0,\tilde{x}_0)\in\mathcal{B}\times\tilde{\mathcal{B}}} V(x_0, \tilde{x}_0).$

**Proof.** Using similar argument as in the proof of Theorem 6.8, we conclude that for $|w|_\infty \leq \epsilon$

$$\begin{aligned} V(x,\tilde{x}) \leq{}& V(x_0,\tilde{x}_0)e^{-\sigma_\circ t} + \frac{\gamma_d^2\epsilon^2}{\sigma_\circ}(1 - e^{-\sigma_\circ t}) \\ &+ \frac{k_3\eta_y}{\zeta_y - \sigma_\circ}(e^{-\sigma_\circ t} - e^{-\zeta_y t}) + \frac{k_4\eta_u}{\zeta_u - \sigma_\circ}(e^{-\sigma_\circ t} - e^{-\zeta_u t}). \end{aligned}$$

The fact that $\lambda_{n_x}(P)\|x\|^2 + \theta\lambda_{n_x}(\tilde{P})\|\tilde{x}\|^2 \leq V(x,\tilde{x})$ implies $\|x\| \leq \lambda_{n_x}(P)^{-\frac{1}{2}}V(x,\tilde{x})$ and $\|\tilde{x}\| \leq \theta^{-\frac{1}{2}}\lambda_{n_x}(\tilde{P})^{-\frac{1}{2}}V(x,\tilde{x})$, hence completes the proof. ■

A decentralized version of event separation property is given in the following theorem. Compared to the centralized case originally stated in [59], the main technical issue here is the boundedness of $x$ ,$\tilde{x}$, proven in Lemma 6.9.

**Theorem 6.6** *Under the execution rule (6.45) and assumption (A2), the closed-loop system (6.44) has the robust semi-global event-separation property.*

**Proof.** We first prove the result for the sensor-to-controller channel. We will assume $t_y^{i+1} \leq t_y^i + \hat{\tau}_y$, otherwise $\tau_m^y = \hat{\tau}_y$ and hence the proof is immediate. Using similar process as in the proof of Theorem 6.1, we conclude from TC (6.45) that a lower bound on MIET can be obtained through the condition $\|e_y\| \leq a_y\|y\| + b_y$ where $a_y$, $b_y$ are obtained by substituting $\alpha_1$, $\alpha_2$, $\alpha_3$, $\delta$ in (6.18) by $\alpha_1^y = \lambda_{n_y}(\mathcal{P}_1^y)$, $\alpha_2^y = 2\|\mathcal{P}_2^y\|$, $\alpha_3^y = \lambda_1(\mathcal{P}_3^y)$, $\delta_y$, respectively. Defining $\rho_y = \|e_y\|/(a_y\|y\| + b_y)$ we have

$$\dot{\rho}_y \leq \frac{\|\dot{e}_y\|}{a_y\|y\| + b_y} + \frac{a_y\|e_y\|\|\dot{y}\|}{(a_y\|y\| + b_y)^2} \leq (1 + a_y\rho_y)\frac{\|\dot{y}\|}{a_y\|y\| + b_y}$$

Therefore, from (6.47a) we have

$$\dot{\rho}_y \leq (1 + a_y\rho_y)\Big(\frac{b_1\|x\| + b_2\|\tilde{x}\| + b_3\|e_u\| + b_4\|w\|}{a_y\|y\| + b_y}\Big).$$

Since the second bracket is bounded in view of Lemma 6.9, we conclude that $\dot{\rho}_y \leq \frac{c_y}{b_y}(1 + a_y\rho_y)$, where $c_y = (b_1 + b_3a_u\|K\|)(\varrho\lambda_{n_x}(P)^{-\frac{1}{2}}) + (b_2 + b_3a_u\|K\|)(\theta^{-\frac{1}{2}}\varrho\lambda_{n_x}(\tilde{P})^{-\frac{1}{2}}) + b_3b_u + b_4\epsilon$

and $|w|_\infty \leq \epsilon$. Solving this inequality from $\rho_y(t_y^i) = 0$ to $\rho_y(\varsigma_y) = 1$, at which the modified TC is satisfied, we conclude $\varsigma_y = t_y^i + \frac{b_y}{c_y}\ln(1 + a_y)$ and hence $\tau_m^y \geq \frac{b_y}{c_y}\ln(1 + a_y)$.

Now we use similar arguments to prove the result for the controller-to-actuator channel. We assume $t_u^j \leq t_u^j + \hat{\tau}_u$. Therefore, the TC $\|e_u\| \leq a_u\|K\hat{x}\| + b_u$ gives the lower bound on MIET where $a_u$, $b_u$ are obtained by substituting $\alpha_1$, $\alpha_2$, $\alpha_3$, $\delta$ in (6.18) by $\alpha_1^u = \lambda_{n_u}(\mathcal{P}_1^u)$, $\alpha_2^u = 2\|\mathcal{P}_2^u\|$, $\alpha_3^u = \lambda_1(\mathcal{P}_3^u)$, $\delta_u$, respectively. Thus by defining $\rho_u = \|e_u\|/(a_u\|K\hat{x}\| + b_u)$ we have

$$\dot{\rho}_u \leq \frac{\|\dot{e}_u\|}{a_u\|K\hat{x}\| + b_u} + \frac{a_u\|e_u\|\|K\dot{\hat{x}}\|}{(a_u\|K\hat{x}\| + b_u)^2} \leq (1 + a_u\rho_u)\frac{\|K\dot{\hat{x}}\|}{a_u\|K\hat{x}\| + b_u}$$

which using (6.47b) gives

$$\dot{\rho}_u \leq (1 + a_u\rho_u)\Big(\frac{\bar{b}_1\|x\| + \bar{b}_2\|\tilde{x}\| + \bar{b}_3\|e_y\| + \bar{b}_4\|e_u\|}{a_u\|K\hat{x}\| + b_u}\Big).$$

Applying Lemma 6.9, we conclude that $\dot{\rho}_u \leq (1 + a_u\rho_u)(\bar{b}_4 + \frac{c_u}{b_u})$, where $c_u = (\bar{b}_1 + \bar{b}_3 a_y(\|C_y\| + c_{\phi_3}))(\varrho\lambda_{n_x}(P)^{-\frac{1}{2}}) + \bar{b}_2(\theta^{-\frac{1}{2}}\varrho\lambda_{n_x}(\tilde{P})^{-\frac{1}{2}}) + \bar{b}_3 b_y$. Using similar process as in the proof of first statement, we solve this inequality from $\rho_u(t_u^j) = 0$ to $\rho_u(\varsigma_u) = 1$ to conclude $\tau_m^u \geq (\bar{b}_4 + \frac{c_u}{b_u})^{-1}\ln(1 + a_u)$. ■

### 6.3.3 Discussion: Separation Principle

So far, the matrices $K$, $L$ were designed in (6.58), (6.59) to stabilize the subsystems $x$, $\tilde{x}$. However, there remains to establish overall system stability (6.44) obtained by combining these subsystems. In the absence of a network, overall stability of the combination follows by the well-known separation principle. However, the same does not hold in the presence of the ETMs, [21]. This can be seen from (6.66), (6.65) which are insufficient to prove the stability of (6.44) and hence the additional condition (6.64f) is required. We now show that fast sampling at the controller-to-actuator channel is the key to solving this issue and guarantee the independency of $K$ and $L$ design in an event-based scenario. Indeed, from (6.64f) it can be inferred that when $\theta$ is chosen large enough so that

$$\begin{pmatrix} -\mu_y\mathbb{I}_{n_x} & -PB_uK \\ \star & -\theta\mu_u\mathbb{I}_{n_x} \end{pmatrix} \prec 0, \tag{6.69}$$

(6.64f) holds for sufficiently small $r_u$, *i.e.,* fast sampling at the controller-to-actuator channel. On the other hand, due to presence of the term $r_y$ in (6.65), the effect of fast sampling at the sensor-to-controller channel is to reduce the conservatism in designing matrix $K$. Note that choosing $K$ such that the eigenvalues of $A + B_uK$ are pushed further negative, from the definition of $\mathcal{M}_y$ it can be concluded that (6.65) can be solved for larger $r_y$.

### 6.3.4 Main Result

**Theorem 6.7** *Suppose that assumption (A1) holds and let $K, L$ to be designed such that $\bar{A}_\circ, \hat{A}_\circ$ are stable and (6.58), (6.59) hold for some $\gamma, \tilde{\gamma} \in \mathbb{R}^+$. Also assume triggering parameters $\mathcal{X}_y, \mathcal{X}_u$ and $\mathcal{T}_y, \mathcal{T}_u$ are defined as in (6.68) and (6.67), respectively. Then the closed-loop system (6.44) is finite gain $\mathcal{L}_2$-stable and has $\mathcal{L}_2$-gain $\leq \gamma_d$ for some $\gamma_d > \bar{\gamma}_\circ := \sqrt{\gamma^2 + \theta\tilde{\gamma}^2}$, $\theta \in \mathbb{R}^+$.*

**Proof.** From (6.56), (6.57), (6.66) and (6.64) one can easily show that the integral in the right hand side of (6.63) is upper bounded by $-\sigma_y x^\top P x - \sigma_u \tilde{x}^\top \tilde{P} \tilde{x} + k_3\eta_y + k_4\eta_u$. Thus we conclude $J^{\gamma_d}_{\langle 0,t\rangle} + V(x_0, \tilde{x}_0) + k_3\eta_y + k_4\eta_u \geq 0$. ■

**Theorem 6.8** *Under the conditions of Theorem 6.7, the closed-loop system (6.44) is GES at equilibrium point $x = 0$ with convergence rate $\sigma_\circ = \min\{\sigma_y, \sigma_u\}$.*

**Proof.** Define

$$\bar{\Theta}_3 = \begin{pmatrix} -(\mu_y + \varepsilon_y)\mathbb{I}_{n_x} & -PB_uK \\ \star & -\theta(\mu_u + \varepsilon_u)\mathbb{I}_{n_x} + \theta\frac{\tilde{P}LL^\top\tilde{P}}{\hat{\alpha}} \end{pmatrix},$$

similar to the proof of Theorem 6.7, from (6.54), (6.55), (6.56), (6.57) we conclude that for $w = 0$ and any $t \in \mathbb{R}_0^+$

$$\dot{V}(x, \tilde{x}) \leq -\sigma_\circ V(x, \tilde{x}) + \tilde{\xi}^\top \bar{\Theta}_3 \tilde{\xi} + \alpha_1\|e_y\|^2 + \alpha_2\|e_u\|^2,$$

which can be solved as $V(x, \tilde{x})e^{\sigma_\circ t} - V(x_0, \tilde{x}_0) \leq \int_0^t e^{\sigma_\circ \tau}\{\tilde{\xi}^\top \bar{\Theta}_3 \tilde{\xi} + \alpha_1\|e_y\|^2 + \alpha_2\|e_u\|^2\}d\tau$. Therefore, by applying (6.66), (6.64) and Proposition 6.3 for $\bar{\sigma}_\circ = \sigma_\circ$ we conclude

$$V(x, \tilde{x})e^{\sigma_\circ t} - V(x_0, \tilde{x}_0) \leq \int_0^t e^{\sigma_\circ \tau}(k_3\eta_y e^{-\zeta_y \tau} + k_4\eta_u e^{-\zeta_u \tau}),$$

and finally

$$V(x, \tilde{x}) \leq V(x_0, \tilde{x}_0)e^{-\sigma_\circ t} + \frac{k_3\eta_y}{\zeta_y - \sigma_\circ}(e^{-\sigma_\circ t} - e^{-\zeta_y t}) + \frac{k_4\eta_u}{\zeta_u - \sigma_\circ}(e^{-\sigma_\circ t} - e^{-\zeta_u t}).$$

It is then easy to check (6.12) holds for $r = \sqrt{\theta}$, $\bar{\sigma}_2 = \sigma_\circ$ and

$$\bar{\sigma}_1 = \frac{\min\{\lambda_1(P), \lambda_1(\tilde{P})\}}{\max\{\lambda_{n_x}(P), \lambda_{n_x}(\tilde{P})\}}\left(1 + \frac{k_3\bar{\eta}_y}{\zeta_y - \sigma_\circ} + \frac{k_4\bar{\eta}_u}{\zeta_u - \sigma_\circ}\right)$$

choosing $\eta_y = \bar{\eta}_y V(x_0, \tilde{x}_0)$, $\eta_u = \bar{\eta}_u V(x_0, \tilde{x}_0)$, $\bar{\eta}_y, \bar{\eta}_u \in \mathbb{R}^+$. ■

## 6.4 Application: Event-Based Lyapunov's Indirect Method

In this section, we take the advantage of the results of previous sections to design local $H_\infty$ controller for general nonlinear ETC systems. To simplify our presentation we restrict attention to the state feedback case, premising that the more complicated output-based control structure will not change the resulting outcomes. Therefore, consider the nonlinear plant

$$\dot{x} = f(x) + B_u u + B_w w$$

with the performance output $z$ given by

$$z = h(x).$$

The functions $f$ and $h$ are continuously differentiable and locally Lipschitz-continuous. According to the classical Lyapunov indirect method if the continuous-time control law

$$u = Kx$$

stabilizes the linearized plant, then the original nonlinear model is also stable, at least locally. It it, however, unclear whether or not this important classical result still holds when the control law is implemented in event-triggered form. To the best of our knowledge, this problem has not been studied so far. Let us define

$$\mathcal{A} := \frac{\partial f(x)}{\partial x}\Big|_{x=0},\ \mathcal{C}_z := \frac{\partial h(x)}{\partial x}\Big|_{x=0},$$

the resulting ETC system is described by

$$\begin{cases} \dot{x} = (\mathcal{A} + B_u K)x + \varphi_1(x) + B_u K e + B_w w, \\ z = \mathcal{C}_z x + \varphi_2(x). \end{cases} \tag{6.70}$$

where

$$\varphi_1(x) = f(x) - \mathcal{A}x,\ \varphi_2(x) = h(x) - \mathcal{C}_z x.$$

The nonlinear functions $\varphi_1, \varphi_2$ are locally Lipschitz-continuous. Given some $c_{\varphi_1}, c_{\varphi_2}$, system (6.70) is equivalent to (6.15) provided that the trajectories of (6.70) remains in

$$\mathcal{D} = \{x \in \mathbb{R}^{n_x} : \|x\| \leq x_d\}$$

where $x_d = \inf\{x, \breve{x} \in \mathbb{R}^{n_x} : \|\varphi_i(x) - \varphi_i(\breve{x})\| \leq c_{\varphi_i}\|x - \breve{x}\|,\ i = 1, 2\}$. To see that this is the case, let us first redefine $\bar{c}$ in Proposition 6.2 by replacing $c_{\phi_i}$ by $c_{\varphi_i}$, $i \in \{1, 2\}$. Then we conclude from the TC (6.16), Proposition 6.2 and (6.24) that for any $t \in \mathbb{R}_0^+$

$$\dot{W} + \sigma W \leq \gamma_d^2 \|w\|^2 + \Delta,$$

solving which gives

$$\lambda_{n_x}(P)\|x\|^2 \leq \lambda_1(P)\|x_0\|^2 + \sup_{0\leq\tau\leq t}\Big\{\frac{\gamma_d^2}{\sigma}\|w(\tau)\|^2 + \frac{\|\Delta(\tau)\|}{\sigma}\Big\}.$$

Thus choosing $r_0, \epsilon$ such that

$$\sqrt{\tfrac{\lambda_1(P)}{\lambda_{n_x}(P)}}r_0 + \tfrac{\gamma_d\epsilon + \sqrt{|\Delta|_\infty}}{\sqrt{\sigma\lambda_{n_x}(P)}} \leq x_d,$$

it can be inferred that $x(t) \in \mathcal{D}$ for all $t \in \mathbb{R}_0^+$ and any $w \in \mathcal{L}_2^{n_w}$ with $|w|_\infty \leq \epsilon$, and any initial conditions with $\|x_0\| \leq r_0$. Hence the $H_\infty$ ETC design discussed in Section 6.2 can be applied to linearized model (6.70). The above results are summarized in the following theorem.

**Theorem 6.9** *There exist some neighborhoods $\mathcal{D}_0 \subset \mathbb{R}^{n_x}$, $\mathcal{D}_w \subset \mathcal{L}_2^{n_w}$ of origin such that the closed-loop system (6.70) is finite gain $\mathcal{L}_2$-stable and has $\mathcal{L}_2$-gain $\leq \gamma_d$ for all $w \in \mathcal{D}_w$, all $x_0 \in \mathcal{D}_0$.*

## 6.5 Numerical Examples

In this section, we illustrate the proposed procedures for designing matrices $K$ and $L$. Note that conditions (6.58), (6.59) are of limited use from a design perspective, as they are not originally stated in terms of the eigenvalues of stability matrices. To circumvent this limitation, two different strategies are provided in Section 6.2.4 and are summerized in Lemmas 6.6, 6.7. In our first example, we follow the method of Lemma 6.6 to design matrix gains $K$, $L$. In the second example, however, we provide a model for which the design can not be performed based on the result of Lemma 6.6 and hence Lemma 6.7 is exploited to for design.

### 6.5.1 Example 1

The first example is chosen in consistent with the statement of Lemma 6.6, *i.e.*, the largest eigenvalue of $\mathrm{Re}\,\bar{A}_\circ$ can be assigned negative by appropriate choice of $K$. Consider the system model (6.10) with

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_y = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

Note that due to the symmetry of matrix $A$ and similarity of $B_u$, $C_y^\mathsf{T}$, the desgin procdure for $K$, $L$ are similar. Taking

$$K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}, \quad L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^\mathsf{T} \tag{6.71}$$

implies that assumption (A1) holds. It is then easy to check

$$\textstyle\prod_{i=1}^{2} \lambda_i(\operatorname{Re}\bar{A}_\circ) = -k_2 - (1+\frac{k_1}{2})^2,$$
$$\textstyle\sum_{i=1}^{2} \lambda_i(\operatorname{Re}\bar{A}_\circ) = -1 + k_2,$$

solving which for $\lambda_1(\operatorname{Re}\bar{A}_\circ)$ gives

$$\lambda_1(\operatorname{Re}\bar{A}_\circ) = \tfrac{1}{2}(k_2 - 1) + \tfrac{1}{2}\Big((k_2+1)^2 + 4(1+\tfrac{k_1}{2})^2\Big)^{\frac{1}{2}}.$$

Then choosing $k_1, k_2$ such that $k_2 < -(1+\frac{k_1}{2})^2$ results in $\lambda_1(\operatorname{Re}\bar{A}_\circ) \in (-1, 0]$ and hence Lemma 6.6 can be used to design $K$, *i.e.*, we can assign $\psi_y + \frac{\sigma_y}{2} \in [0, 1)$. Also since $\bar{A}_\circ^\intercal = \hat{A}_\circ|_{L=-K^\intercal}$ and $C_y^\intercal = B_u$, we have $\operatorname{Re}\hat{A}_\circ = \operatorname{Re}\bar{A}_\circ$. Similarly, matirx $L$ can be designed appropriately to ensure $\psi_u + \frac{\sigma_u}{2} \in [0, 1)$.

### 6.5.2 Example 2

We now consider the case where the the largest eigenvalue of $\operatorname{Re}\bar{A}_\circ$ can not be set negative and hence Lemma 6.6 is useless to design $K$, $L$. Therefore, the design is performed using the result of Lemma 6.7. Consider the system model (6.10) with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad B_u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_y = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

which is the generalization of the model proposed in [84] to the output feedback case. It is easy to check that assumption (A1) holds. Moreover, choosing $K$, $L$ as in (6.71), we have $\det(\operatorname{Re}\bar{A}_\circ), \det(\operatorname{Re}\hat{A}_\circ) \leq 0$ and hence $\lambda_1(\operatorname{Re}\bar{A}_\circ), \lambda_1(\operatorname{Re}\hat{A}_\circ) \geq 0$. This implies that Lemma 6.6 is useless to design $K, L$. Thus, we design these matices according to Lemma 6.7. In [84], it is shown that the design can be performed for matrix $L$ as $L = \operatorname{col}(69.5523, 11.5679)$. Also the resulting closed-loop eigenvalues of $\hat{A}_\circ$ are placed at $-6.2839 \pm j5.3911$. Now since $\bar{A}_\circ^\intercal = \hat{A}_\circ|_{L=-K^\intercal}$ it can be inferred that $K = (-69.5523, -11.5679)$ and hence the same closed-loop eigenvalues as $\bar{A}_\circ$. The maximum $\psi_y + \frac{\sigma_y}{2}$ (respectively, $\psi_u + \frac{\sigma_u}{2}$) for which the stable controller (respectively, observer) is guaranteed can be calculated as 0.49. We refer the interested readers to [84] for more details.

### 6.5.3 Range of Parameters and Admissible Nonlinearities

In previous examples, it is shown that by proper choices of matrices $K, L$ one can ensure $\psi_y + \frac{\sigma_y}{2} \in [0, \varpi)$, where $\varpi = 1$ (respectively, $\varpi = 0.49$) in the first (respectively, second) example. Therefore, from definition of $\psi_y$ we obtain $c_{\phi_1} + (\frac{\|B_w\|}{\gamma} + \frac{\|B_u\|}{\sqrt{\alpha}})(\sqrt{\mu_y + \varepsilon_y} + \|C_z\| + c_{\phi_2}) + \frac{\sigma_y}{2} \leq \varpi$. This introduce an upper bound on the admissible range of $\gamma, \mu_y, \varepsilon_y, \sigma_y$ and Lipschitz coefficients $c_{\phi_1}, c_{\phi_2}$. Similarly, from $\psi_u + \frac{\sigma_u}{2} \in [0, \varpi)$, the parameters $\tilde{\gamma}, \mu_u, \varepsilon_u, \sigma_u$ and Lipschitz coefficients $c_{\phi_1}, c_{\phi_3}$ are restricted to satisfy $c_{\phi_1} + c_{\phi_3}\|L\| + \frac{\|B_w\|}{\tilde{\gamma}}\sqrt{\mu_u + \varepsilon_u} + \frac{\sigma_u}{2} \leq \varpi$.

## 6.6 Summary

We have proposed a design methodology to simultaneously synthesize the feedback law and the TC for nonlinear Lipschitz systems. The study covered both state and output-based controller designs. Several triggering and performance variables are introduced as the design parameters which has to chosen properly to meet the desired configuration of the closed-loop poles. In addition, it is shown for the first time that for the output feedback case, under the fast sampling at the controller-to-actuator channel, the separation principle holds. Moreover, the results are shown to be a platform for the local stability of general nonlinear systems.

# Chapter 7

# Dissipativity Properties of Nonlinear Systems Under Network Constraints

## 7.1 Problem Statement

In this chapter[1] we consider two subsystems $\Sigma_i$ $(i = 1, 2)$ of the following form

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i), \\ y_i = h_i(x_i), \end{cases} \tag{7.1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{m_{\bar{i}}}$. The functions $f_i$ and $h_i$ are locally Lipschitz with respect to their arguments and satisfy $f_i(0,0) = 0$ and $h_i(0) = 0$ so that $x_i = 0$ is an equilibrium point of the unforced subsystem $i$. The following notation is used throughout this chapter

$$\bar{i} = \begin{cases} 1, & i = 2, \\ 2, & i = 1. \end{cases}$$

**Definition 7.1 (Dissipativity property)** *A state space system $\Sigma_i$ is said to be dissipative with respect to the supply rate $s_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{\bar{i}}} \to \mathbb{R}$ if there exists $\eta_i(x_{i0}) \geq 0$ such that for all $x_{i0} := x_i(t_0) \in \mathbb{R}^{n_i}$, all $u_i \in \mathcal{L}_2^{m_i}$, and all $t_1 \geq t_0$*

$$\int_{t_0}^{t_1} s_i(u_i(t), y_i(t))dt + \eta_i(x_{i0}) \geq 0. \tag{7.2}$$

With the assistance of a differentiable function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$, called storage function, condition (7.2) can be re-stated as follows: $\Sigma_i$ is *dissipative* with respect to $s_i$, provided that for all $x_{i0} \in \mathbb{R}^{n_i}$, all $u_i \in \mathcal{L}_2^{m_i}$, and all $t_1 \geq t_0$

$$\dot{V}_i(x_i(t)) \leq s_i(u_i(t), y_i(t)).$$

[1]The results of this chapter have been submitted for publication in the article: M. Ghodrat and S. H. Mousavi and A. H. J. de Ruiter and H. J. Marquez, "Dissipativity Properties of Nonlinear Systems Under Network Constraints", Submitted to *IEEE Trans. Autom. Control*, November 2018.

**Assumption 7.1** *Each subsystem $\Sigma_i$ $(i = 1, 2)$ is dissipative with respect to the supply rate*

$$s_i(u_i, y_i) = y_i^\top Q_i y_i + y_i^\top S_i u_i + u_i^\top R_i u_i, \tag{7.3}$$

*with differentiable storage function $V_i$, where $Q_i, S_i, R_i, i = 1, 2$ are matrices of appropriate dimensions and $Q_i^\top = Q_i$, $R_i^\top = R_i$.*

For the feedback interconnections of the two subsystems which is obtained through the relations $u_1 = r_1 - y_2$, $u_2 = r_2 + y_1$, dissipativity follows immediately from Assumption 7.1. However, this manuscript focuses on a more realistic scenario where the interconnection between systems takes place through a communication network. More specifically, we assume that data exchange between subsystems $\Sigma_i$, $i = 1, 2$ is carried out through a digital network with limited communication rate. Being stated in the context of sampled-data theory, this situation has received a lot of attention up to date and has a rich body of literature, see [4]. In this chapter we relax the periodic sampling assumption and focus on aperiodic sampling in the form of an ETM while maintaining the properties of the original design.

### 7.1.1 Event-Based Architecture

We will consider two independent triggering modules $\Gamma_i$ for the subsystems $\Sigma_i$. $\Gamma_1$ (resp. $\Gamma_2$) exploits its local available information to schedule the data transmission from $\Sigma_1$ to $\Sigma_2$ (resp. $\Sigma_2$ to $\Sigma_1$). In such structure, the interconnections between subsystems $\Sigma_i$, $i = 1, 2$ are expressed as follows:

$$u_1 = r_1 - \hat{y}_2, \quad u_2 = r_2 + \hat{y}_1, \tag{7.4}$$

where $r_i \in \mathbb{R}^{q_i}$ is the exogenous input to the system $\Sigma_i$, $i = 1, 2$ and $\hat{y}_i$, $i = 1, 2$ is the intermittent information exchanged between two subsystems defined by

$$\hat{y}_i(t) := y(t^i_{j_i}), \quad \forall\, t \in [t^i_{j_i}, t^i_{j_i+1}),$$

for $j_i \in \mathbb{N}_0$, and $i = 1, 2$. In the above definition, $\{t^1_{j_1} : j_1 \in \mathbb{N}_0\}$ and $\{t^2_{j_2} : j_2 \in \mathbb{N}_0\}$ are the *asynchronous* transmission instants at the $\Sigma_1$ and $\Sigma_2$ sides, respectively. The general system structure is depicted in Fig. 7.1, where the solid and dashed lines represent continuous and intermittent information flow of information, respectively. Denote, $x := (x_1^\top, x_2^\top)^\top \in \mathbb{R}^n$, $r := (r_1^\top, r_2^\top)^\top \in \mathbb{R}^m$ and $y := (y_1^\top, y_2^\top)^\top \in \mathbb{R}^p$ as the state, input and output, respectively, with $n = n_1 + n_2$ and $m = p = m_1 + m_2$. Then, In order to study the input-output

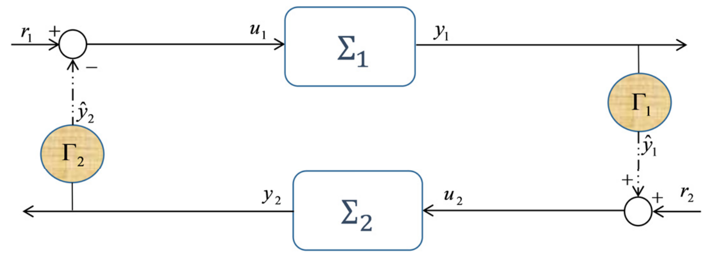



Figure 7.1: General structure of feedback system $\Sigma$

properties, the overall interconnected system is described as follows:

$$\Sigma : \begin{cases} \dot{x} = \begin{pmatrix} f_1(x_1, r_1 - \hat{y}_2) \\ f_2(x_2, r_2 + \hat{y}_1) \end{pmatrix}, \\ y = \begin{pmatrix} h_1(x_1) \\ h_2(x_2) \end{pmatrix}. \end{cases}$$

As mentioned earlier, the problem of event-triggered dissipative has been recently studied in [11,26,27,51,52]. However, there are two main issues with the proposed mechanisms: i) The results in the aforementioned references are limited to the cases that inputs $r_i$, $i = 1, 2$ are continuously differentiable signals with bounded derivative; a restrictive assumption when dealing with arbitrary unknown disturbances. ii) More importantly, the event separation in the presence of arbitrary inputs $r_i$, $i = 1, 2$ is not fully guaranteed. In fact, when states approach the origin, arbitrary exogenous inputs may result in an unlimited number of events.

In this chapter, our goal is to design triggering policies such that the dissipativity property of the proposed feedback system is guaranteed, while the separation property for triggering instants of each module holds, assuming only disturbances in $\mathcal{L}_2$. In this regard, define the following sampling error for each subsystems $\Sigma_i$, $i = 1, 2$

$$e_i(t) = \hat{y}_i(t) - y_i(t). \tag{7.5}$$

**Remark 7.1** *While the connections between subsystems $\Sigma_1$ and $\Sigma_2$ are interrupted in between the sampling instants and they operate in an open-loop fashion, taking $e_i$, $i = 1, 2$ as external inputs one can treat $\Sigma$ as a closed-loop control system. This enables us to apply the existing input/output theories to the system $\Sigma$.*

Assuming the initial sampling instants at both sides occur simultaneously at $t_0^i = 0$, and denoting $t_{j_i}^i$ as the most recent triggering instants for the triggering modules $\Gamma_i$, $i = 1, 2$, the upcoming sampling instants are decided through

$$t_{j_i+1}^i = \inf\{t > t_{j_i}^i : \|e_i(t)\|^2 \geq \sigma_i \|y_i(t)\|^2 + \Delta_i(t)\}, \tag{7.6}$$

where the parameters $\sigma_i \in \mathbb{R}^+$ and the functions $\Delta_i(\cdot)$, $i = 1, 2$ have the following structure

$$\Delta_i(t) = \begin{cases} \delta_i, & t^i_{j_i} \leq t \leq t^i_{j_i} + \tau_i, \\ \psi_i(t), & t^i_{j_i} + \tau_i \leq t \leq t^i_{j_i+1}, \end{cases}$$

where $\tau_i, \delta_i \in \mathbb{R}^+$ are constant parameters and $\psi_i \in \mathcal{L}_\infty$ are bounded functions which can be designed based on three different scenarios $\psi_i \in \mathcal{Z}_0 \cap \mathcal{L}_\infty$, $\psi_i \in \mathcal{L}_1 \cap \mathcal{L}_\infty$ or $\psi_i \in \mathcal{L}_\infty$, each leads to a different dissipativity property for the ETC system. In the sequel, it will be shown that the positiveness of $\tau_i$, $\delta_i$ are incorporated to guarantee the separation of triggering instants. Moreover, as suggested in Chapter 4, proper choices of functions $\psi_i$ provide the flexibility to enlarge the inter-event times. It is worth remarking that an alternative to the TC proposed in (7.6), is the time-regularization approach recently proposed in [20, 48], where triggering is not permitted until some dwell-time has passed since the last transmitted signal. Condition (7.6) is free of this limitation and hence may offer better performance when compared to time-regularization method.

It will be shown in Section 7.3 that the boundedness of the trajectoriesof subsystems $\Sigma_i$, $i = 1, 2$ is key to guarantee Zeno-free behavior for the system $\Sigma$. Generally speaking, the dissipativity property (7.2) does not imply boundedness of states and hence an auxiliary assumption is required. The missing element in the input-output theory is the link between boundedness of input-output signals and that of state trajectories. This is entirely analogous to the detectability-type properties that ensure ultimate boundedness of states by some function of inputs and outputs norms [88]. Thus, our conjecture is to exploit a detectability assumption in one of its forms, in order to admit the boundedness property of states.

## 7.2 Motivation

It is not difficult to verify that finite-gain $\mathcal{L}_2$ stability is a special case of the dissipativity property (7.3), with a specific choice of matrices $Q_i$, $S_i$ and $R_i$, [5]. The $\mathcal{L}_2$ stability analysis of ETC systems has seen much attention in recent years, including [16–18, 20, 49] to mention just a few. All these works are built on two main assumptions: first, $\mathcal{L}_2$ stability of network-free system and second, ISS of ETC model. Therefore, in order to address the dissipativity problem, it is reasonable to generalize these assumptions. To generalize the first assumption on the $\mathcal{L}_2$ stability of a network-free setup, it suffices to assume that each subsystem $\Sigma_i$, $i = 1, 2$ is dissipative. Moreover, a general counterpart of ISS assumption is the following notion of input-output to state stability (IOSS). An equivalent definition can be found in [89].

**Definition 7.2 (Input-output to state stability)** *The system $\Sigma_i$ has IOSS property provided that there exists a class $\mathcal{KL}$ function $\xi_i$ and class $\mathcal{K}$ functions $\gamma_i, \nu_i$ so that*

$$\|x_i(t)\| \leq \xi_i(\|x_{i0}\|, t) + \gamma_i(\|u_i(t)\|_\infty) + \nu_i(\|y_i(t)\|_\infty)$$

*for every intial state $x_{i0} \in \mathbb{R}^{n_i}$.*

The above definition suggests that when inputs and outputs are zero, states converge to zero. So, it is strongly connected to detectability of $\Sigma$, [88]. This property serves to show the boundedness of trajectories of $\Sigma$. Note that a similar but stronger concept, defined as strongly finite-time detectability, is used in [90] for the same purpose.

Depending on how the parameters in the TCs (7.6) are selected, these generalized assumptions render a dissipativity property for the event-based setup, either in classical form or weak form. Indeed, we will show that there exists a trade-off in the parameter selection such that improved inter-event behaviour comes at the expense of a weaker form of dissipativity for feedback system $\Sigma$. The above statement also explain why the dissipativity results given in [51, 52] have been stated in its classical form by sacrificing the separation of triggering instants.

The following variation of the dissipativity property will be required. Note that a similar definition was introduced in [63].

**Definition 7.3 (Weakly quasi-dissipativity property)** *System $\Sigma$ is said to be weakly quasi-dissipative with respect to supply rate $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if there exist some $\alpha, \beta \in \mathbb{R}_0^+$ such that for all $x_0 \in \mathbb{R}^n$, all $r \in \mathbb{R}^m$, and all $t_1 \geq t_0$*

$$\int_{t_0}^{t_1} s(r(t), y(t))dt + \alpha(t_1 - t_0) + \beta \geq 0.$$

Whenever $\alpha = 0$ (resp. $\beta = 0$), the system $\Sigma$ is called weakly dissipative (resp. quasi-dissipative). Thus the main deviation of weak dissipativity from the regular concept is that it allows the storage function to include finite power sources. This is analogous to the concept of a bias term in theory of $\mathcal{L}_2$ stability [68].

## 7.3 Boundedness Properties

We shall need the following notion of state boundedness.

**Definition 7.4 (Bounded-input bounded-state stability)** *System $\Sigma$ is bounded-input bounded-state stable (BIBS) if there exist $\varpi_1, \varpi_2 \in \mathcal{N}$ such that for any $x_0 \in \mathbb{R}^n$ and any*

$r \in \mathcal{L}_2^m$

$$\sup_{t \geq t_0} \|x(t, x_0, r)\| \leq \max\{\varpi_1(\|x_0\|), \varpi_2(|r|_\infty)\}. \tag{7.7}$$

As stated earlier, the IOSS property is the key toward showing the state boundedness in the ETC system. However, Definition 7.2 is not practically applicable for checking this property for a given system. As a solution, the following definition characterizes this property in terms of Lyapunov functions.

**Definition 7.5 (Characterization of IOSS property)** *An IOSS-Lyapunov function for system $\Sigma_i$ is any continuously differentiable function $W_i$, satisfying*

$$\underline{\varphi}_i(x_i) \leq W_i(x_i) \leq \bar{\varphi}_i(x_i), \tag{7.8}$$

$$\frac{d}{dt} W_i(x_i(t)) \leq -\alpha_i(x_i(t)) + \kappa_i(u_i(t)) + \phi_i(y_i(t)), \tag{7.9}$$

*for some class $\mathcal{K}_\infty$ functions $\underline{\varphi}_i$, $\bar{\varphi}_i$, $\alpha_i$, $\kappa_i$, $\phi_i$.*

**Assumption 7.2** *Each subsystem $\Sigma_i$ $(i = 1, 2)$ has IOSS property.*

The following lemma characterizes the BIBS property for the proposed ETC system.

**Lemma 7.1** *Suppose that:*

*(i) Assumption 7.2 holds with respect to IOSS-Lyapunov functions $W_i$, $i = 1, 2$. Moreover, functions $\bar{\varphi}_i$, $\alpha_i$, $\kappa_i$, $\phi_i$ in (7.8), (7.9) satisfy $\bar{\varphi}_i(x_i) \leq \bar{d}_i \|x_i\|^2$, $\underline{\varphi}_i(x_i) \geq \underline{d}_i \|x_i\|^2$, $\alpha_i(x_i) \geq \alpha_i^* \|x_i\|^2$, $\kappa_i(u_i) \leq \bar{\kappa}_i \|u_i\|^2$, $\phi_i(y_i) \leq \bar{\phi}_i \|y_i\|^2$ for some $\bar{d}_i$, $\alpha_i^*$, $\bar{\kappa}_i$, $\bar{\phi}_i \in \mathbb{R}^+$;*

*(ii) Assumption 7.1 holds with respect to storage functions $V_i$ $i = 1, 2$, satisfying $\underline{v}_i \|x_i\|^2 \leq V_i(x_i) \leq \bar{v}_i \|x_i\|^2$ for some $\underline{v}_i, \bar{v}_i \in \mathbb{R}^+$;*

*(iii) There exists some $\varrho \in \mathbb{R}^+$ such that matrix $\widehat{Q}$ defined as*

$$\widehat{Q} = \begin{bmatrix} Q_1 + \varrho R_2 & \frac{1}{2}(-S_1 + \varrho S_2^\top) \\ \frac{1}{2}(-S_1^\top + \varrho S_2) & R_1 + \varrho Q_2 \end{bmatrix}. \tag{7.10}$$

*is negative definite*

*Then, feedback system $\Sigma$ has BIBS property.*

**Proof.** According to Assumption 7.1, there exist storage functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$, $i = 1, 2$ such that

$$\dot{V}_i(x_i) \leq y_i^\top Q_i y_i + y_i^\top S_i u_i + u_i^\top R_i u_i. \tag{7.11}$$

Consider the storage function $V(x) = V_1(x_1) + \varrho V_2(x_2)$ for some $\varrho \in \mathbb{R}^+$, it follows from (7.11), (7.4) and (7.5) that

$$\dot{V}(x) \leq y^\mathsf{T}\widehat{Q}y + y^\mathsf{T}\widehat{S}r + r^\mathsf{T}\widehat{R}r + \Phi(e,r,y)$$

where $\widehat{Q}$ is defined in (7.10),

$$\widehat{S} = \begin{bmatrix} S_1 & 2\varrho R_2 \\ -2R_1 & \varrho S_2 \end{bmatrix}, \quad \widehat{R} = \begin{bmatrix} R_1 & 0 \\ 0 & \varrho R_2 \end{bmatrix}$$

and $\Phi(e,r,y) = e^\mathsf{T}(\Phi_e e + \Phi_r r + \Phi_y y)$ with

$$\Phi_e = \begin{bmatrix} \varrho R_2 & 0 \\ 0 & R_1 \end{bmatrix},\ \Phi_r = \begin{bmatrix} 0 & 2\varrho R_2 \\ -2R_1 & 0 \end{bmatrix},\ \Phi_y = \begin{bmatrix} 2\varrho R_2 & \varrho S_2 \\ -S_1 & 2R_1 \end{bmatrix}.$$

Using the fact that

$$e^\mathsf{T}\Phi_r r \leq \frac{\|e\|^2}{\lambda_r} + \lambda_r\|\Phi_r r\|^2,\ e^\mathsf{T}\Phi_y y \leq \frac{\|e\|^2}{\lambda_y} + \lambda_y\|\Phi_y y\|^2,$$

for some $\lambda_r, \lambda_y \in \mathbb{R}^+$, we conclude that

$$\dot{V}(x) \leq y^\mathsf{T}\bar{Q}y + y^\mathsf{T}\bar{S}r + r^\mathsf{T}\bar{R}r + \sum_{i=1}^{2}\bar{\varrho}_i\|e_i\|^2, \tag{7.12}$$

where $\bar{\varrho}_1 = \varrho\|R_2\| + \frac{1}{\lambda_r} + \frac{1}{\lambda_y}$, $\bar{\varrho}_2 = \|R_1\| + \frac{1}{\lambda_r} + \frac{1}{\lambda_y}$ and

$$\bar{Q} = \widehat{Q} + \lambda_y\Phi_y^\mathsf{T}\Phi_y,\ \bar{S} = \widehat{S},\ \bar{R} = \widehat{R} + \lambda_r\Phi_r^\mathsf{T}\Phi_r.$$

Now, take $U(x) = V(x) + \rho W(x)$ for some $\rho \in \mathbb{R}^+$ where $W(x) = W_1(x_1) + W_2(x_2)$. According to inequality (7.9) and condition (i) we have

$$\dot{W}(x) \leq -\alpha^*\|x\|^2 + \kappa\|u\|^2 + \phi\|y\|^2 \tag{7.13}$$

with $\alpha^* = \min\{\alpha_1^*, \alpha_2^*\}$, $\kappa = \max\{\bar{\kappa}_1, \bar{\kappa}_2\}$ and $\phi = \max\{\bar{\phi}_1, \bar{\phi}_2\}$. Using Young's inequality it is obtained that

$$\begin{aligned} \|u\|^2 =& \sum_{i=1}^{2}\|u_i\|^2 = \|r_1 - y_2 - e_2\|^2 + \|r_2 + y_1 + e_1\|^2 \\ &\leq y^\mathsf{T}Q_u y + y^\mathsf{T}S_u r + r^\mathsf{T}R_u r + (1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4})\|e_1\|^2 + (1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2})\|e_2\|^2 \end{aligned} \tag{7.14}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}^+$ and

$$Q_u = \begin{bmatrix} 1+\epsilon_4 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix},\ S_u = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},\ R_u = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_3 \end{bmatrix}.$$

In view of (7.12), (7.13) and (7.14) we can write

$$\dot{U}(x) \leq y^\top \tilde{Q} y + y^\top \tilde{S} r + r^\top \tilde{R} r + \sum_{i=1}^{2} \varrho_i \|e_i\|^2 - \rho\alpha^* \|x\|^2 \tag{7.15}$$

where $\varrho_1 = \bar{\varrho}_1 + \rho\kappa(1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4})$, $\varrho_2 = \bar{\varrho}_2 + \rho\kappa(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2})$ and

$$\tilde{Q} = \bar{Q} + \rho\phi I + \rho\kappa Q_u,\ \tilde{S} = \bar{S} + \rho\kappa S_u,\ \tilde{R} = \bar{R} + \rho\kappa R_u.$$

Now since $\widehat{Q}$ is negative definite in the light of condition (iii), there exists some $\rho$, $\lambda_y$ such that $\tilde{Q}$ is also negative definite. Hence, applying [90, Proposition 1] we see that there exist some positive $\eta$, $\mu$ such that

$$y^\top \tilde{Q} y + y^\top \tilde{S} r + r^\top \tilde{R} r \leq -\eta \|y\|^2 + \mu \|r\|^2. \tag{7.16}$$

Exploiting conditions (i), (ii) there exists some $\theta \in \mathbb{R}^+$ such that $\theta U(x) \leq \rho\alpha^* \|x\|^2$. Using the last inequality alongside with (7.15), (7.16), and TCs (7.6) results in

$$\dot{U}(x) \leq -\theta U(x) - y^\top (\eta I - Q_e) y + \mu r^\top r + \sum_{i=1}^{2} \varrho_i \Delta_i$$

for $\rho^* < \rho$ and

$$Q_e = \begin{bmatrix} \varrho_1 \sigma_1 & 0 \\ 0 & \varrho_2 \sigma_2 \end{bmatrix}.$$

Choosing $\sigma_i$ such that

$$\sigma_i < \frac{\eta}{\varrho_i}, \quad (i = 1, 2) \tag{7.17}$$

we have $\eta I > Q_e$, and hence

$$\dot{U}(x) \leq -\theta U(x) + \mu |r|_\infty^2 + \sum_{i=1}^{2} \varrho_i \Delta_i.$$

Multiplying both sides of above inequality with $e^{\theta t}$ and integrating from 0 to $t$ yields

$$U(x(t)) e^{\theta t} - U(x_0) \leq \int_0^t e^{\theta\tau} \Big\{ \mu |r|_\infty^2 + \sum_{i=1}^{2} \varrho_i |\Delta_i|_\infty \Big\} d\tau.$$

Note that $|\Delta_i|_\infty$ is well-defined due to boundedness of $\delta_i$, $\psi_i(t)$. It is then easy to verify that

$$U(x(t)) \leq U(x_0) + \frac{\mu}{\theta} |r|_\infty^2 + \sum_{i=1}^{2} \frac{\varrho_i}{\theta} |\Delta_i|_\infty. \tag{7.18}$$

It is then straight forward to show that BIBS property (7.7) holds. ■

**Remark 7.2** *The BIBS property of system $\Sigma$ can also be shown if condition (i) of Lemma 7.1 is substituted by an IOSS condition on the overall feedback system $\Sigma$. The details are, however, left to the interested readers.*

**Remark 7.3** *In [90], a similar condition to (7.10) is used to guarantee boundedness of interconnected systems. Condition (7.10) is a special version of the graph separation condition (originally defined in [91]) and is used to prove similar boundedness results. Moreover, as shown in [90], following condition (7.10) system $\Sigma$ has a uniform finite power gain.*

We continue with the following lemma that gives an upper bound on the linear combination of measurement errors.

**Lemma 7.2** *Define*

$$A = \Big\{(z_1, z_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : z_1 z_2 < 1\Big\} \tag{7.19}$$

*and let compact sets $\mathcal{D}_i \in \mathbb{R}^{n_i}$ and $H_1, H_2, \epsilon \in \mathbb{R}^+$given such that $x_0 \in \mathcal{D}_1 \times \mathcal{D}_2$, $(H_1, H_2) \in A$, $|r|_\infty \leq \epsilon$. Also let $\vartheta \in \mathbb{R}^+$ and denote $\tau_i \in \mathbb{R}^+$, $i = 1, 2$ as the solutions to*

$$\vartheta \int_l^{l+\tau_i} \int_l^{\pi_2} d\pi_1 d\pi_2 = H_i \quad (i = 1, 2). \tag{7.20}$$

*Then, for any $k_1, k_2$, $t \in \mathbb{R}^+$ we have*

$$\int_0^t \sum_{i=1}^{2} k_i \|e_i(\tau)\|^2 d\tau \leq \int_0^t \Big(y^\mathsf{T}(\tau) Q_c y(\tau) + r^\mathsf{T}(\tau) R_c r(\tau) + \sum_{i=1}^{2} \Big\{c_i \|x_i(\tau)\|^2 + \psi_i(\tau)\Big\}\Big) d\tau,$$

*where*

$$c_i = \frac{(k_i + k_{\bar{i}} H_{\bar{i}}) H_i}{1 - H_i H_{\bar{i}}}, \qquad (i = 1, 2),$$

$c_3 = c_1$, $c_4 = c_2$, $c_5 = c_2 + \frac{\sigma_1 c_1}{H_1}$, $c_6 = c_1 + \frac{\sigma_2 c_2}{H_2}$ *and*

$$Q_c = \begin{bmatrix} c_5 & 0 \\ 0 & c_6 \end{bmatrix}, R_c = \begin{bmatrix} c_3 & 0 \\ 0 & c_4 \end{bmatrix}.$$

**Proof.** Define $X_0 = \max_{x_0 \in \mathcal{D}_1 \times \mathcal{D}_2} \|x_0\|$. Since $x_0 \in \mathcal{D}_1 \times \mathcal{D}_2$ and $|r|_\infty \leq \epsilon$, it can be inferred from Lemma 7.1 that the state trajectories $x$ remain in the compact set

$$\mathcal{D}_x = \Big\{\xi_x \in \mathbb{R}^n : \|\xi_x\| \leq \max\{\varpi_1(X_0), \varpi_2(\epsilon)\}\Big\}. \tag{7.21}$$

Moreover, using this fact and interconnection equations (7.4), we can write $\|u\|=\|(u_1,u_2)\|\leq \|r\|+\|\hat{y}\|$, and thus conclude that the signal $u$ does not leave the compact set

$$\mathcal{D}_u=\Big\{\xi_u\in\mathbb{R}^m : \|\xi_u\|\leq \epsilon+\max_{\xi_x\in\mathcal{D}_x}\|h(\xi_x)\|\Big\}. \tag{7.22}$$

Denote the Lipschitz coefficients of functions $f_i$, $h_i$ on the compact sets $\mathcal{D}_x,\mathcal{D}_u$ by $\lambda_{f_i}$, $\lambda_{h_i}$. From (7.5) and (7.1) it is concluded that

$$\begin{aligned}\|\dot{e}_i\|=\|\dot{y}_i\| &= \lim_{dt\to 0^+}\frac{1}{dt}\|h_i(x_i(t+dt))-h_i(x_i(t))\|\\ &\leq \lambda_{h_i}\|f_i(x_i,u_i)\|.\end{aligned}$$

Exploiting the interconnection equations (7.4) we can write

$$\|\dot{e}_i\|\leq \bar{\lambda}_i(\|x_i\|+\|r_i\|+\|y_{\bar{i}}\|+\|e_{\bar{i}}\|), \tag{7.23}$$

where $\bar{\lambda}_i=\lambda_{f_i}\lambda_{h_i}$, $i=1,2$. Using the fact $d\|e_i\|/dt\leq\|\dot{e}_i\|$ and the inequality (7.23) we have

$$\|e_i(t)\|\leq\int_{t^i_{j_i}}^t \bar{\lambda}_i(\|x_i\|+\|r_i\|+\|y_{\bar{i}}\|+\|e_{\bar{i}}\|)d\tau.$$

By applying Young's inequality it is obtained that

$$\begin{aligned}\|e_i(t)\|^2\leq 4\bar{\lambda}_i^2\Big\{&\Big(\int_{t^i_{j_i}}^t\|x_i(\tau)\|d\tau\Big)^2+\Big(\int_{t^i_{j_i}}^t\|r_i(\tau)\|d\tau\Big)^2\\ &+\Big(\int_{t^i_{j_i}}^t\|y_{\bar{i}}(\tau)\|d\tau\Big)^2+\Big(\int_{t^i_{j_i}}^t\|e_{\bar{i}}(\tau)\|d\tau\Big)^2\Big\}.\end{aligned}$$

Finally, we can apply C-S inequality to get

$$\begin{aligned}\|e_i(t)\|^2\leq 4\bar{\lambda}_i^2\int_{t^i_{j_i}}^t d\tau\int_{t^i_{j_i}}^t\Big\{&\|x_i(\tau)\|^2+\|r_i(\tau)\|^2\\ &+\|y_{\bar{i}}(\tau)\|^2+\|e_{\bar{i}}(\tau)\|^2\Big\}d\tau.\end{aligned} \tag{7.24}$$

Breaking down the integral term of $\int_0^t\|e_i(\tau)\|^2d\tau$ as

$$\int_0^t\|e_i(\tau)\|^2d\tau=\sum_{j_i}\Big\{\int_{t^i_{j_i}}^{t^i_{j_i}+\tau_i}+\int_{t^i_{j_i}+\tau_i}^{t^i_{j_i+1}}\|e_i(\tau)\|^2d\tau\Big\},$$

where for the most recent triggering index $N_i$ until time $t$, we define $t^i_{N_i+1}=t$, $i=1,2$. We can upper bound $\|e_i\|^2$ using TC (7.24) for $[t^i_{j_i},t^i_{j_i}+\tau_i)$ (resp. using (7.6) for $[t^i_{j_i}+\tau_i,t^i_{j_{i+1}}))$

and apply (7.20) with $\vartheta = 4(\bar{\lambda}_i)^2$ to write the result in the following matrix form

$$\begin{bmatrix} 1 & -H_1 \\ -H_2 & 1 \end{bmatrix} \begin{bmatrix} \int_0^t \|e_1(\tau)\|^2 d\tau \\ \int_0^t \|e_2(\tau)\|^2 d\tau \end{bmatrix} \leq$$
$$\begin{bmatrix} H_1 \int_0^t (\|x_1(\tau)\|^2 + \|r_1(\tau)\|^2 + \|y_2(\tau)\|^2) d\tau \\ H_2 \int_0^t (\|x_2(\tau)\|^2 + \|r_2(\tau)\|^2 + \|y_1(\tau)\|^2) d\tau \end{bmatrix}$$
$$+ \begin{bmatrix} \int_0^t (\sigma_1 \|y_1(\tau)\|^2 + \psi_1(\tau)) d\tau \\ \int_0^t (\sigma_2 \|y_2(\tau)\|^2 + \psi_2(\tau)) d\tau \end{bmatrix}. \tag{7.25}$$

This completes the proof noting that

$$\int_0^t \sum_{i=1}^{2} k_i \|e_i(\tau)\|^2 d\tau \leq \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} 1 & -H_1 \\ -H_2 & 1 \end{bmatrix}^{-1} \times \text{RHS of (7.25)}.$$

■

## 7.4 Main Results

As the first part of our main results, Theorem 7.1 shows the Zeno-free behaviour for the proposed TC (7.6). Before that we need the following definition.

**Definition 7.6 (Event-separation property)** *Let $\tau_m^i = \inf\{t_{j_i+1}^i - t_{j_i}^i : j_i \in \mathbb{N}_0\}$ be the MIETs for the triggering modules $\Gamma_i$, $i = 1, 2$. The event-separation property holds for the system $\Sigma$ provided that for any $\epsilon \in \mathbb{R}^+$ and any compact sets $\mathcal{D}_i \in \mathbb{R}^{n_i}$, $i = 1, 2$, we have*

$$\inf\{\tau_m^i : x_0 \in (\mathcal{D}_1 \times \mathcal{D}_2), |r|_\infty \leq \epsilon, i = 1, 2\} > 0.$$

**Theorem 7.1** *Under the TC (7.6) the event-separation property holds for system $\Sigma$.*

**Proof.** We assume $t_{j_i+1}^i < t_{j_i}^i + \tau_i$, $i = 1, 2$ at least for some $j_i \in \mathbb{N}_0$, since otherwise we can choose $\tau_m^i = \tau_i$, $i = 1, 2$ and the proof is immediate from the positiveness of $\tau_i$. The rest of proof relies on modifying TC (7.6) to obtain a more conservative one so that TC would be reached sooner. Then, MIET for this new condition is readily a lower bound for the inter-event times of (7.6). In this regard, let us choose

$$\|e_i\| \geq a_i \|y_i\| + b_i, \quad (i = 1, 2), \tag{7.26}$$

as the modified TC where

$$a_i = \left(\frac{\sigma_i}{1 + \varepsilon_i}\right)^{\frac{1}{2}}, \quad b_i = \left(\frac{\delta_i}{1 + \varepsilon_i^{-1}}\right)^{\frac{1}{2}},$$

for some $\varepsilon_i \in \mathbb{R}^+$. Note that one can easily check that (7.26) is more conservative than (7.6) through verfying the following inequality

$$(A+B)^2 \leq (1+\kappa)A^2 + (1+\kappa^{-1})B^2,$$

and setting $A = a_i\|y_i\|$, $B = b_i$ and $\kappa = \varepsilon_i$. Now defining $\chi_i = \|e_i\|/b_i$ we have

$$\dot{\chi}_i \leq \frac{\|\dot{e}_i\|}{b_i}$$

Using the fact $\|\dot{y}_i\|= \|\dot{e}_i\|$ and inequality (7.23), it is concluded that

$$\dot{\chi}_i \leq \frac{\bar{\lambda}_i}{b_i}\Big(\|x_i\|+\|r_i\|+\|y_{\bar{i}}\|+\|e_{\bar{i}}\|\Big).$$

Since the second brackets is bounded due to Lemma 7.1, there exists some positive $L_1$, $L_2$ such that

$$\chi_i \leq \frac{\bar{\lambda}_i L_i}{b_i} t, \quad (i = 1, 2). \tag{7.27}$$

Based on the definition of $\chi_i$ and condition (7.26), the triggering in the modified rule occurs when $\chi_i = 1$. Thus Solving (7.27) for $\chi_i(0) = 0$ and $\chi_i(\bar{\tau}_i) = 1$ it is straight forward to conclude

$$\bar{\tau}_i = \frac{b_i}{\bar{\lambda}_i L_i} > 0 \quad (i = 1, 2).$$

Since $\tau_i \geq \bar{\tau}_i$, the proof is complete. ■ As the second part of our main result, Theorem 7.2 states the dissipativity properties of system $\Sigma$.

**Theorem 7.2** *Under conditions (i)-(iii) of Lemma 7.1 and if*

$$c_i < \rho\alpha^* \quad (i = 1, 2), \tag{7.28}$$

*then, system $\Sigma$ has the following dissipativity properties with respect to the supply rate*

$$\tilde{s}(r, y) = y^\mathsf{T}(\tilde{Q} + Q_c)y + y^\mathsf{T}\tilde{S}r + r^\mathsf{T}(\tilde{R} + R_c)r. \tag{7.29}$$

**Proof.** Integrating (7.15) from 0 to $t$ and applying Lemma 7.2, it is obtained that

$$U(x(t)) - U(x_0) \leq \int_0^t \Big\{y^\mathsf{T}(\tilde{Q} + Q_c)y + y^\mathsf{T}\tilde{S}r + r^\mathsf{T}(\tilde{R} + R_c)r$$
$$+ \underbrace{c_1\|x_1\|^2 + c_2\|x_2\|^2 - \rho\alpha^*\|x\|^2}_{(a)} + \underbrace{\psi_1(\tau) + \psi_2(\tau)}_{(b)}\Big\}d\tau$$

Suggested by (7.28), the integral term (a) is upper bounded by 0. Moreover, we have the following three scenarios For the integral term (b):

Table 7.1: Different dissipativity properties

| Admissible space for $\psi_i(t)$ | Dissipativity (Diss.) of system $\Sigma$ weakly diss. | quasi-diss. | (classical) diss. |
|---|---|---|---|
| $\mathcal{Z}_0 \cap \mathcal{L}_\infty$ | ✓ | ✓ | ✓ |
| $(\mathcal{L}_1 \backslash \mathcal{Z}_0) \cap \mathcal{L}_\infty$ | ✓ | ✓ | × |
| $\mathcal{L}_\infty \backslash \mathcal{L}_1$ | × | ✓ | × |

1) $\psi_i(t) \in \mathcal{Z}_0$ for $i = 1, 2$: we have $(b) = 0$.

2) $\psi_i(t) \in \mathcal{L}_1 \backslash \mathcal{Z}_0$ for $i = 1, 2$: there exists some $\beta \in \mathbb{R}^+$ such that $(b) \leq \beta$.

3) $\psi_i(t) \in \mathcal{L}_\infty \backslash \mathcal{L}_1$ for $i = 1, 2$: there exists some $\alpha \in \mathbb{R}^+$ such that $(b) \leq \alpha t$.

In view of Definition 7.3, the above formulations yields the results summarized in Table 7.1. This completes the proof. ■

### 7.4.1 Design Algorithm for Triggering Conditions Parameters

Sections 7.3 - 7.4 introduce several constraints in the design of TC. In this section we proposed an algorithm to complete the design and select the parameters in (7.6).

i. Choose $\sigma_i$, $i = 1, 2$ based on (7.17).

ii. Select $\epsilon > 0$, the upper-bound of the norm of the disturbance, and $X_0 > 0$, the upper-bound of the initial state, respectively. Define compact sets $\mathcal{D}_x$, $\mathcal{D}_u$ according to (7.21), (7.22). Then, calculate $\lambda_{f_i}$, $\lambda_{h_i}$ as the Lipschitz coefficients of functions $f_i$, $h_i$ on these compact sets.

iii. For a desired pair of $(H_1, H_2) \in A$, where $A$ is defined in (7.19), and for $\vartheta = 4(\lambda_{f_i}\lambda_{h_i})^2$, calculate $\tau_i$, $i = 1, 2$ according to (7.20).

iv. Choose a positive constant $\delta_i$, $i = 1, 2$ and $\psi_i$, $i = 1, 2$ according to desired dissipativity property in Table 7.1.

Under the above algorithm, system $\Sigma$ is dissipative with respect to supply rate $\tilde{s}$ defined in (7.29), for the initial conditions $\|x_0\| \leq X_0$ and exogenous disturbance $|r|_\infty \leq \epsilon$.

## 7.5 Case Study

Let $f_i, h_i$ in (7.1) have the linear structures $f_i(x_i, u_i) = A_i x_i + u_i$, $h_i(x_i) = x_i$ for $i = 1, 2$, which under interconnection (7.4) gives

$$\begin{cases} \dot{x}_1 = A_1 x_1 - \hat{x}_2 + w_1, \\ \dot{x}_2 = A_2 x_2 + \hat{x}_1 + w_2, \\ y_i = x_i, \ (i = 1, 2). \end{cases}$$

We assume the following TCs proposed in [51]

$$t^i_{j_i+1} = \inf\{t > t^i_{j_i} : \|e_i(t)\| \geq \sigma_i \|y_i(t)\|\}, \tag{7.30}$$

for $i = 1, 2$. Choosing $w_i$, $i = 1, 2$ as follows

$$\begin{aligned} w_1(t) &= -A_1 x_1(t) + \hat{x}_2(t) - x_1(0), \\ w_2(t) &= -A_2 x_2(t) - \hat{x}_1(t) - x_2(0) \end{aligned}$$

results in the following dynamics

$$\dot{x}_i = -x_i(0), \quad (i = 1, 2)$$

between sampling instants. Thus using a process similar to [59, Theorem IV.1], it can be shown that

$$x_i(t) = x_i(0)(1 - t), \ e_i(t) = x_i(0)(t - t^i_{j_i}), \quad t \in [t^i_{j_i}, t^i_{j_{i+1}}).$$

Thus applying TCs (7.30), the sampling instants are analytically obtained from

$$t^i_{j_i} = 1 - \left(\frac{1}{1 + \sigma_i}\right)^{j_i}, \ (i = 1, 2).$$

This proves the existence of accumulation points at $t = 1$ for both $\Gamma_1$, $\Gamma_2$ sides. However, under our proposed TCs (7.6), Theorem 7.1 suggests the existence of some positive $L_i$, $\varepsilon_i$, $i = 1, 2$ such that the inter-event times are guaranteed to be separated at least by

$$\bar{\tau}_i = \frac{1}{\max\{1, \|A_i\|\} L_i} \left(\frac{\delta_i}{1 + \varepsilon_i^{-1}}\right)^{\frac{1}{2}}.$$

## 7.6 Summary

This chapter proposed a TC design structure to meet different types of dissipativity property for the event-triggered interconnected subsystems. It is shown that by proper choices of triggering parameters, the resulting closed-loop system may enjoy weak-, quasi- or the classical dissipativity property. Moreover, the proposed TC is showed to be favorable from implementation perspective by showing the sampling times to be uniformly isolated. Finally, the obtained results are validated through a compelling example.

# Chapter 8

# An Integral Based Event-Triggered Control Scheme of Distributed Network Systems

## 8.1 Problem Definition

[1] In [9], using an ISS Lyapunov function, an ETM is designed for nonlinear systems:

$$\dot{x} = f(x, u) \tag{8.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and inputs. Assuming state feedback controller

$$u = k(x) \tag{8.2}$$

and defining measurements error as $e(t) := x(t_i) - x(t)$ for $t \in [t_i, t_{i+1})$ where $t_i$'s are triggering instants, one can rewrite (8.1) as

$$\dot{x} = f(x, k(x + e)) \tag{8.3}$$

Assume also that an ISS Lyapunov function $V$ exists so that nonlinear system $\dot{x} = f(x, k(x + e))$ is ISS with respect to measurements error $e$, *i.e.*, there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \le \dot{V}(x) \le \overline{\alpha}(|x|), \tag{8.4}$$

$$\dot{V}(x) \le -\alpha(|x|) + \gamma(|e|). \tag{8.5}$$

[1] An early part of the results of this chapter has been published in the article: S. H. Mousavi and M. Ghodrat and H. J. Marquez, "Integral-Based Event Triggered Control Scheme for a General Class of Non-Linear Systems", *IET Control Theory & Appl.*, vol. 9, no. 13, pp. 1982-1988, 2015.

Then the following TC

$$\gamma(|e|) \leq \sigma\alpha(|x|), \quad \sigma \in (0,1) \tag{8.6}$$

guarantee that $\dot{V}(x) < 0$ and hence the system is asymptotically stable in presence of measurements error, i.e.,

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq (\sigma - 1)\alpha(|x|) \tag{8.7}$$

In this chapter we look for a less conservative TC that can increase the inter-execution times. To improve the inter-event times, we introduce in integral event-based scheme which allow the Lyapunov function to have positive derivative until the asymptotic stability of overall system does not violate. We will show that the new law achieves stability while significantly reducing the amount of information sent between plant and controller. To this end, we integrate (8.5) over the interval $[t_i, t)$:

$$V(t) - V(t_i) \leq -\int_{t_i}^{t} \alpha(|x|)d\tau + \int_{t_i}^{t} \gamma(|e|)d\tau \tag{8.8}$$

and define the integral-based TC as follows:

$$\int_{t_i}^{t} \gamma(|e|)d\tau \leq \sigma \int_{t_i}^{t} \alpha(|x|)d\tau \qquad\qquad t \geq t_i \tag{8.9}$$

where $0 < \sigma < 1$ is an arbitrary coefficient and next execution time ($t_{i+1} \in \mathbb{T}$) is the time when above inequality is violated; *i.e.*

$$\int_{t_i}^{t_{i+1}} \gamma(|e|)d\tau = \sigma \int_{t_i}^{t_{i+1}} \alpha(|x|)d\tau. \tag{8.10}$$

In the following theorem, we show that the TC (8.9) preserves asymptotic stability of the closed-loop system, while a positive MIET is guaranteed for the scheme.

**Theorem 8.1** *Consider the continuous-time nonlinear system (8.1) with the pre-defined stable state feedback law (8.2) and assume that the following conditions, introduced in [9], hold:*

1. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz continuous on compacts.*

2. $k : \mathbb{R}^n \to \mathbb{R}^m$ *is Lipschitz continuous on compacts.*

3. *There exists an ISS Lyapunov function for the closed-loop system, satisfying (8.4) and (8.5) with* $\alpha^{-1}$ *and* $\gamma$ *Lipschitz continuous on compacts.*

*Assume now that instead of continuous information flow from plant to the controller, the control law updates based on an event-based scheme with integral-based TC (8.9). If* $0 < \sigma < 1$*, then we have the following properties for the ETC system:*

*(A) The origin is an asymptotically stable equilibrium point.*

*(B) For any compact set* $S \subset \mathbb{R}^n$*, containing the origin, there exists a lower bound* $\tau_{min} \in \mathbb{R}^+$ *such that for any initial condition in* $S$ *we have*

$$t_{i+1} - t_i \geq \tau_{min} \qquad \forall\ t_i, t_{i+1} \in \mathbb{T} \tag{8.11}$$

*where partition* $\mathbb{T}$*, defined in Definition 2.4, is the sequence of the triggering instants.*

**Proof.**

(A) Substituting (8.9) in (8.8) we have

$$V(t) - V(t_i) \leq (\sigma - 1) \int_{t_i}^{t} \alpha(|x|) d\tau \tag{8.12}$$

and so, for $\sigma < 1$:

$$V(t) < V(t_i) \qquad \forall t \in [t_i, t_{i+1}), \tag{8.13}$$

and asymptotic stability follows from [58, Lemma 1].

(B) To show the existence of positive MIET $\tau_{min}$, we introduce an auxiliary system with the same dynamic as (8.3):

$$\dot{\zeta} = f(\zeta, k(\zeta + e')) \tag{8.14}$$

but with the TC proposed in [9]:

$$\gamma(|e'|) \leq \sigma\alpha(|\zeta|). \tag{8.15}$$

Assume now that both systems update their control law at time instant $t_i$ and also have the same state values at this time, *i.e.*:

$$x(t_i) = \zeta(t_i). \tag{8.16}$$

Denote the next execution times of system (8.14) by $t'_{i+1}$; *i.e.*

$$\gamma(|e'(t'_{i+1})|) = \sigma\alpha(|\zeta(t'_{i+1})|) \tag{8.17}$$

and

$$\gamma(|e'(t)|) < \sigma\alpha(|\zeta(t)|) \qquad \forall t \in [t_i, t'_{i+1}). \tag{8.18}$$

Based on (8.16) we have

$$e'(t) = e(t) \qquad \forall t \in [t_i, t'_{i+1}). \tag{8.19}$$

Integrating (8.18) from $t_i$ to $t'_{i+1}$ and using (8.10), we can easily see that $t_{i+1} > t'_{i+1}$. Since the auxiliary system has lower bound for its execution time, [9], so does the ETC system with integral-based TC (8.9).

■

In order to make a fair comparison between the proposed and traditional TC schemes, in this section we consider two ETC systems with the same dynamic but different triggering strategies and compare the resulting inter-event times. The following theorem is then stated. The proof, however, can be find in [58].

**Theorem 8.2** *Consider the event-triggered nonlinear system (8.3) implemented using the integral-based TC (8.9), and let partition $\mathbb{T} = \{t_i : i \in \mathbb{N}_0\}$ denote the triggering instants. Consider also the system (8.14) with the same dynamic but implemented using the classical TC (8.15) and let the partition $\mathbb{T}' = \left\{t'_j : j \in \mathbb{N}_0\right\}$ represent the triggering instants. Assuming that $\alpha$ is Lipschitz continuous on compacts and that the conditions of Theorem 1 are satisfied, then the following properties hold:*

*(A) Zero Triggering-Time State Difference: If $x(t_m) = \zeta(t'_n)$ for some $t_m \in \mathbb{T}$ and $t'_n \in \mathbb{T}'$, then*
$t_{m+1} - t_m > t'_{n+1} - t'_n$.

*(B) Non-Zero Triggering-Time State Difference: For every $t'_n \in \mathbb{T}'$, there exists $\epsilon > 0$ such that if*

$$|x(t_m) - \zeta(t'_n)| < \epsilon \qquad \forall t_m \in \mathbb{T} \tag{8.20}$$

*then*

$$t_{m+1} - t_m > t'_{n+1} - t'_n \tag{8.21}$$

## 8.2 Integral Based Event Triggered Cooperative Control of Distributed Network Systems

In this section[2] we consider a system of $n$ agents operating in $\mathbb{R}$. Let $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ denote the state and control input for agent $i$, respectively, with the following single-

[2]The results of this section have been published in the article: M. Ghodrat and H. J. Marquez, "An Integral Based Event Triggered Control Scheme of Distributed Network Systems", *Proc. Eur. Control Conf. (ECC)*, pp. 1724-1729, 2015.

integrator dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, n. \tag{8.22}$$

Assume agent $i$ communicates with a limited number of agents in the network. Our problem is to design a decentralized control law together with a TC (centralized and decentralized) to insure that the agents achieve average consensus, i.e., they converge to the agreement point $\frac{1}{n}\sum_i x_i(0)$ which is the average of initial states of the agents. In this section we consider the problem of designing an event-based controller for a group of agents. Our goal is to ensure the consensus of agents both using centralized and decentralized approaches

### 8.2.1 Centralized Approach

In the centralized approach, each agent needs information from other agents in the network to decide the next triggering instant based on a global (same) execution rule. To this end, each agent needs to communicate with all other agents in the network. This condition, however, is relaxed in decentralized approach. Moreover, in a centralized control scheme all agents' actuators update simultaneously which is perhaps too conservative compared to the decentralized approach. Let us define the measurements error $e$ as the difference between the current and the last sampling value of state, i.e.,

$$e(t) := x(t_k) - x(t), \quad t \in [t_k, t_{k+1}), \tag{8.23}$$

where $\{t_k : k \in \mathbb{N}_0\}$ is the sequence of control task execution times and $x = (x_1, \ldots, x_n)^\mathsf{T}$. It is known that using the control law $u(t) = (u_1, \ldots, u_n)^\mathsf{T} = -Lx(t)$, average consensus is achieved for a connected network. In the presence of measurements error, the closed-loop dynamics for the systems is given by

$$\dot{x}(t) = -Lx(t_i) = -L(x(t) + e(t)). \tag{8.24}$$

Now defining the vector $\delta(t)$ as the deviation of agent's positions from the average, $a = \frac{1}{n}\sum_i x_i(t)$, we can write $x(t) = a\mathbf{1}_n + \delta(t)$. We avoid using time dependence representation for $a$ since it is easy to verify that $\dot{a} = \frac{1}{n}\sum_i \dot{x}_i(t) = 0$, i.e., the average position remains constant over time [92][3]. Then it is possible to find the dynamics for position disagreements vector $\delta(t)$ as

$$\dot{\delta} = -L(\delta + e), \tag{8.25}$$

where we use the fact that $\mathbf{1}_n$ is an eigenvector of Laplacian matrix corresponding to the eigenvalue 0. Now we are at the point to find the TC using the Lyapunov function $V(\delta(t)) =$

[3]This property relies on the structure of matrix $L$ and hence can be easily extended to ETC systems.

$\frac{1}{2}\delta^{\mathsf{T}}(t)\delta(t)$. Note that to ensure the average consensus of all agents, we only need to show that $V(\delta(t)) \to 0$ as $t \to \infty$. Using the same method as in [93], an upper bound for dynamics of $V(\delta)$ can be found as

$$\dot{V}(\delta) = -\delta^{\mathsf{T}} L\delta - \delta^{\mathsf{T}} Le \leq -\lambda_2(L)|\delta|^2 + |\delta|\|L\||e|. \tag{8.26}$$

Enforcing control task to execute when

$$|e| - \sigma\frac{\lambda_2(L)|\delta|}{\|L\|} = 0, \tag{8.27}$$

where $\sigma \in (0,1)$, $\dot{V}(\delta)$ becomes negative definite, i.e., $\dot{V}(\delta) \leq -(1-\sigma)\lambda_2(L)|\delta|^2$. In the following theorem we propose an integral based execution rule which, compared to (8.27), improves inter-event times.

**Theorem 8.3** *Consider a network of agents having single integrator dynamics $\dot{x}(t) = u(t)$, where the state feedback control law $u(t) = -Lx(t)$ is designed to ensure the average consensus of the network. Suppose that $t_k$ is the most recent triggering instant of the agents. Then if the actuator value for each agent updates at time instant $t_{k+1} > t_k$ when the following execution rule is satisfied*

$$\int_{t_k}^{t_{k+1}} |e(\tau)||\delta(\tau)| d\tau - \frac{\sigma\lambda_2(L)}{\|L\|}\int_{t_k}^{t_{k+1}} |\delta(\tau)|^2 d\tau = 0, \tag{8.28}$$

*where $\sigma \in (0,1)$, and ZOH module is used to keep the last transmitted control value at the intervals between triggering times, the following hold:*

*(I) The network achieves average consensus.*

*(II) The time intervals between consecutive triggering instants are bounded below by $\frac{1}{\|L\|}\ln(1+\frac{\sigma\lambda_2(L)}{\|L\|})$.*

**Proof.**

(I) Integrate (8.26) from $t_k$ to $t \in [t_k, t_{k+1})$, we get

$$V(\delta(t)) - V(\delta(t_k)) \leq -\lambda_2(L)\int_{t_k}^{t} |\delta(\tau)|^2 d\tau + \|L\|\int_{t_k}^{t} |\delta(\tau)||e(\tau)| d\tau. \tag{8.29}$$

Now, considering the integral TC

$$\int_{t_k}^{t} |e(\tau)||\delta(\tau)| d\tau \leq \frac{\sigma\lambda_2(L)}{\|L\|}\int_{t_k}^{t} |\delta(\tau)|^2 d\tau, \tag{8.30}$$

we can rewrite (8.29) as

$$V(\delta(t)) - V(\delta(t_k)) \leq -(1-\sigma)\lambda_2(L)\int_{t_k}^{t} |\delta(\tau)|^2 d\tau. \tag{8.31}$$

Since $\sigma \in (0, 1)$, we have

$$V(\delta(t)) < V(\delta(t_k)), \quad t \in [t_k, t_{k+1}[. \tag{8.32}$$

Therefore we conclude that the value of Lyapunov function at the next triggering instant, $V(\delta(t_{k+1}))$, is strictly less than its value at the most recent triggering instant, $V(\delta(t_k))$. As a consequence, the discrete time sequence of $\{V(\delta(t_k)) : k \in \mathbb{N}_0\}$ is monotonically decreasing, bounded below and consequently convergent. In fact, we show that the convergence point is 0. It is important to note that $V(\delta(t))$ can be increasing at some points in the interval $[t_k, t_{k+1})$, while not deteriorating the convergence of Lyapunov function to 0. To explain this point, consider inequality (8.32). It is obvious that even if the Lyapunov function increases in the interval $[t_k, t_{k+1})$, its value can not exceed $V(\delta(t_k))$. Therefore we can write

$$0 \le V(\delta(t)) < V(\delta(t_k)), \quad t \in [t_k, t_{k+1}). \tag{8.33}$$

Now if $\{t_i : i \in \mathbb{N}_0\}$ constitutes a partition, i.e., the number of triggering instants tend to infinity, from sandwich rule we get $V(\delta(t)) \to 0$ as $t_k \to \infty$. However, if execution of control task stops after finite number of triggering instants, then one can integrate (8.26) from 0 to $t_k$ and apply TC (8.28) on inter-event periods $[t_{k'}, t_{k'+1})$ for $k' = 0, \ldots, k-1$ to get

$$V(\delta(t_k)) - V(\delta(0)) \le -(1-\sigma)\lambda_2(L) \int_0^{t_k} |\delta(\tau)|^2 d\tau. \tag{8.34}$$

From (8.31), (8.34) and positive definiteness of $V(\delta(t))$, we obtain $2\int_0^t V(\delta(\tau))d\tau = \int_0^t |\delta(\tau)|^2 d\tau \le \frac{V(\delta(0))}{(1-\sigma)\lambda_2(L)}$. Since $\int_0^t |\delta(\tau)|^2 d\tau$ is a nondecreasing function of $t$ which is bounded from above and $\frac{d^2}{dt^2}\int_0^t |\delta(\tau)|^2 d\tau = 2\dot{V}(\delta(t))$ is finite, from Barbalat's Lemma we have $\lim\limits_{t\to\infty} \frac{d}{dt}\int_0^t |\delta(\tau)|^2 d\tau = \lim\limits_{t\to\infty} |\delta(t)|^2 = 0$ and hence $\delta(t)$ converges to 0.

(II) Let us define new variable

$$\varrho := \frac{\int_{t_k}^t |e(\tau)||\delta(\tau)| d\tau}{\int_{t_k}^t |\delta(\tau)|^2 d\tau} \tag{8.35}$$

which has the following dynamics

$$\dot{\varrho} = \frac{d}{dt}\frac{\int_{t_k}^t |e(\tau)||\delta(\tau)| d\tau}{\int_{t_k}^t |\delta(\tau)|^2 d\tau} = \frac{|e(t)||\delta(t)| \int_{t_i}^t |\delta(\tau)|^2 d\tau}{(\int_{t_k}^t |\delta(\tau)|^2 d\tau)^2} - \frac{|\delta(t)|^2 \int_{t_k}^t |e(\tau)||\delta(\tau)| d\tau}{(\int_{t_k}^t |\delta(\tau)|^2 d\tau)^2}. \tag{8.36}$$

Since the second part in the last inequality is always negative we can write

$$\dot{\varrho} \le \frac{|e(t)||\delta(t)| \int_{t_k}^t |\delta(\tau)|^2 d\tau}{(\int_{t_k}^t |\delta(\tau)|^2 d\tau)^2} \le \frac{|e(t)||\delta(t)|}{(\int_{t_k}^t |\delta(\tau)|^2 d\tau)}. \tag{8.37}$$

Integrating (8.24) from $t_k$ to $t$ and using $\dot{e}(t) = -\dot{x}(t)$ we get

$$e(t) = \int_{t_k}^{t} L(x(\tau) + e(\tau))d\tau = \int_{t_k}^{t} L(\delta(\tau) + e(\tau))d\tau,$$

and thus

$$|e(t)| \leq \int_{t_k}^{t} |L(\delta(\tau) + e(\tau))|d\tau \leq \|L\| \int_{t_k}^{t} (|\delta(\tau)| + |e(\tau)|)d\tau. \tag{8.38}$$

Multiplying the last inequality by $\min_{\tau \in [t_k, t]} |\delta(\tau)|$, we get

$$\min_{\tau \in [t_k, t]} |\delta(\tau)||e(t)| \leq \|L\| \int_{t_k}^{t} (|\delta(\tau)|^2 + |e(\tau)||\delta(\tau)|)d\tau. \tag{8.39}$$

Now consider the following TC

$$|\delta(t)||e(t)| \leq \|L\| \int_{t_k}^{t} (|\delta(\tau)|^2 + |e(\tau)||\delta(\tau)|)d\tau \tag{8.40}$$

which is more restrictive than the one given in (8.39) and hence gives a lower bound for inter sampling times. Then we are able to find an upper bound for $\dot{\varrho}$ as

$$\dot{\varrho} \leq \|L\|\left(\frac{\int_{t_k}^{t} (|\delta(\tau)|^2)d\tau + \int_{t_k}^{t} (|e(\tau)||\delta(\tau)|)d\tau}{(\int_{t_k}^{t} |\delta(\tau)|^2 d\tau)}\right). \tag{8.41}$$

Then we have

$$\dot{\varrho} \leq \|L\|(1 + \varrho), \quad \varrho(t_k) = 0, \tag{8.42}$$

which shows that the trajectory of $\varrho$ over $[t_k, t_{k+1})$ is bounded above by $\varphi$ which has the following dynamics

$$\dot{\varphi} = \|L\|(1 + \varphi), \quad \varphi(t_k) = 0. \tag{8.43}$$

Thus, the inter sampling times are lower bounded by the solution $\bar{\tau}$ of $\varphi(\bar{\tau}) = \frac{\sigma\lambda_2(L)}{\|L\|}$, i.e., $\bar{\tau} = \frac{1}{\|L\|} \ln(1 + \frac{\sigma\lambda_2(L)}{\|L\|})$. ■

**Remark 8.1** *The main reason that this method, compared to the one proposed in [93], results in larger inter sampling intervals is that condition (8.28) allows increase of Lyapunov function in some interval within two consecutive triggering instants, see, e.g., [94].*

### 8.2.2 Decentralized Approach

In this section we consider decentralized approach. Here we assume that each agent in the network has some communication limits in the sense that it can only exchange information with its neighbors in the network. The importance of the decentralized approach becomes more apparent as the number of agents in the network increases. To design a decentralized ETC law for each agent, we propose two methods both using a decentralized control law but one with a centralized TC and the other one with a decentralized one.

### Semi-Decentralized Triggering

In this approach the control law for each agent is decentralized but the TC is designed in a semi-decentralized fashion. We assume that in order to check the TC, each agent needs not only information from its neighbors in the network but also some global information about the whole network. We now make the following assumption.

**Assumption 8.1** *Each agent updates the control signal both at its own event instants as well as the triggering instants of its neighbors.*

To formulate the problem in this case, we first introduce the following notation from [93]. The set of triggering instants for agent $i$ is denoted by $\{t_k^i : k \in \mathbb{N}_0\}$. The measurements error and control law for agent $i$ are then defined as $e_i(t) = x_i(t_k^i) - x_i(t)$ and

$$u_i(t) = -\sum_{j \in N_i} (x_i(t_k^i) - x_j(t_{k'}^j)) = -\sum_{j \in N_i} [(x_i(t) - x_j(t)) + (e_i(t) - e_j(t))] \tag{8.44}$$

for $t \in [t_k^i, t_{k+1}^i)$, where $t_{k'}^j := \max\{t_K^j : t_k^i - t_K^j \geq 0, K \in \mathbb{N}_0\}$ is the most recent triggering instant for agent $j$. Then under Assumption 8.1, the set of actuator updating times for agent $i$ is $\{t_k^i\} \cup \{\sum_{j \in N_i} t_{k'}^j\}$, $k, k' \in \mathbb{N}_0$. Similar to the centralized approach, we may find an upper bound for dynamics of Lyapunov function $V(\delta) = \frac{1}{2}\delta^\mathsf{T}\delta$ as

$$\dot{V}(\delta) = -\delta^\mathsf{T} L\delta - \delta^\mathsf{T} Le \leq -\lambda_2(L)\sum_i \delta_i^2 - \sum_i \sum_{j \in N_i} \delta_i(e_i - e_j). \tag{8.45}$$

Then we conclude

$$\begin{aligned}\dot{V}(\delta) &\leq -\lambda_2(L)\sum_i \delta_i^2 - \sum_i \sum_{j \in N_i} |\delta_i||(e_i - e_j)| \\ &\leq -\lambda_2(L)\sum_i (\delta_i^2 - \left|\tfrac{\delta_i}{\lambda_2(L)}\right| \sum_{j \in N_i} (|e_i| + |e_j|)).\end{aligned}$$

Now using the following execution rule for control task

$$\sum_{j \in N_i} (|e_i| + |e_j|) - \lambda_2(L)\sigma_i|\delta_i| = 0, \tag{8.46}$$

where $\sigma_i \in (0, 1)$, it can be shown (see [93]) that the time derivative of the Lyapunov function becomes negative definite, i.e., $\dot{V}(\delta) \leq -\lambda_2(L)\sum_i (1 - \sigma_i)\delta_i^2$. In the following theorem, however, we relax this execution condition to achieve larger sampling times for the network.

**Theorem 8.4** *Consider a network of agents having single integrator dynamics $\dot{x}(t) = u(t)$, where the state feedback control law $u(t) = -Lx(t)$ is designed to ensure the average consensus of the network. Suppose $t_k^i$ is the most recent triggering instant of agent $i$. If the control task execution is performed when the following execution rule holds at time instant $t_{k+1}^i > t_k^i$ for agent $i$ or one of its neighbors*

$$\int_{t_k^i}^{t_{k+1}^i} |\delta_i(\tau)| \sum_{j\in N_i} (|e_i(\tau)|+|e_j(\tau)|)d\tau - \sigma_i\lambda_2(L)\int_{t_k^i}^{t_{k+1}^i} |\delta_i(\tau)|^2 d\tau = 0, \tag{8.47}$$

*where $\sigma_i \in (0,1)$, and the system uses a ZOH module to keep the last transmitted control value at the intervals between triggering times, the following hold:*

*(I) The network achieves average consensus.*

*(II) The inter sampling time intervals are bounded below by some non-zero constant.*

**Proof.**

(I) Integrate inequality (8.46) from 0 to $t \in [t_k^i, t_{k+1}^i)$ gives

$$V(\delta(t)) - V(\delta(0)) \leq \sum_l [-\lambda_2(L)\int_0^t |\delta_l(\tau)|^2 + \int_0^t |\delta_l(\tau)| \sum_{j\in N_l} (|e_l(\tau)|+|e_j(\tau)|)d\tau. \tag{8.48}$$

One can brake the interval $[0,t]$ into subintervals $\bigcup_{0\leq r\leq k-1}[t_r^i, t_{r+1}^i) \cup [t_k^i, t]$. Under Assumption 8.1, triggering instants are same for $l \in \{i\} \cup N_i$ and since (8.47) holds over any of these subintervals, we conclude that

$$\int_0^t |\delta_l(\tau)| \sum_{j\in N_l} (|e_l(\tau)|+|e_j(\tau)|)d\tau \leq \sigma_l\lambda_2(L)\int_0^t |\delta_l(\tau)|^2 d\tau. \tag{8.49}$$

For $l \notin \{i\} \cup N_i$, however, one can choose another subintervals $\bigcup_{0\leq r\leq k'-1}[t_r^l, t_{r+1}^l) \cup [t_{k'}^l, t]$, where $t_{k'}^l$ is the most recent triggering instant of agent $l$. Now by applying (8.47) on any these subintervals, we conclude that (8.49) holds. Then (8.48) can be rewritten as

$$\begin{aligned} V(\delta(t)) - V(\delta(0)) &\leq -\lambda_2(L)\sum_l (1-\sigma_l)\int_0^t |\delta_l(\tau)|^2 d\tau \\ &\leq -\lambda_2(L)(1-\max_l \sigma_l)\int_0^t |\delta(\tau)|^2 d\tau. \end{aligned} \tag{8.50}$$

From definition of $V(\delta(t))$ we get an upper bound for $V(\delta(t))$ as

$$V(\delta(t)) \leq V(\delta(0)) - 2\lambda_2(L)(1-\max_l \sigma_l)\int_0^t V(\delta(\tau))d\tau \tag{8.51}$$

which is a decreasing function of $t$. Since $V(\delta(t)) \geq 0$, we conclude that $\int_0^t V(\delta(\tau))d\tau \leq \frac{V(\delta(0))}{2\lambda_2(L)(1-\max_l \sigma_l)}$, i.e., $\int_0^t V(\delta(\tau))d\tau$ is bounded from above and because it is a nondecreasing function of $t$ and $\frac{d^2}{dt^2}\int_0^t V(\delta(\tau))d\tau = \dot{V}(\delta(t))$ is finite, from Barbalat's Lemma we conclude that $\lim\limits_{t\to\infty} \frac{d}{dt}\int_0^t V(\delta(\tau))d\tau = \lim\limits_{t\to\infty} V(\delta(t)) = 0$. As a consequence, $\delta(t)$ converges to 0.
(II) The proof outline is similar to Theorem 1. ■

**Remark 8.2** *As we mentioned before, in semi-decentralized triggering scheme each agent needs some global information from the network system. This is actually because of existence of terms $\lambda_2(L)$ and $\delta_i(t)$ in the TC (the later needs the knowledge of $a$).*

### Decentralized Triggering

We now consider the problem of designing a decentralized control law and TC. In contrast to the previous results, each agent now decides the next execution of the control task only based on the information from its neighbors. Using the same expressions for the measurements error $e_i(t)$, control law $u_i(t)$ and triggering time instants $t_k^i$ for agent $i$, and considering the Lyapunov function $V(x) = \frac{1}{2}x^\top Lx$, we get

$$\begin{aligned}\dot{V}(x) = x^\top L\dot{x} &= -(Lx)^\top(Lx) - (Lx)^\top Le \\ &= -\sum_i (Lx)_i^2 - \sum_i \sum_{j\in N_i} (Lx)_i(e_i - e_j) \\ &= -\sum_i (Lx)_i^2 - \sum_i \|N_i\|(Lx)_i e_i + \sum_i \sum_{j\in N_i} (Lx)_i e_j,\end{aligned} \tag{8.52}$$

where $\|N_i\|$ denotes the number of neighbors for agent $i$. Now using the Young's inequality, we can write

$$\begin{aligned}\dot{V}(x) \leq &-\sum_i (Lx)_i^2 + \sum_i \gamma\|N_i\|(Lx)_i^2 \\ &+ \sum_i \frac{1}{2\gamma}\|N_i\|e_i^2 + \sum_i \sum_{j\in N_i} \frac{1}{2\gamma}e_j^2,\end{aligned} \tag{8.53}$$

where $\gamma \in \mathbb{R}^+$ is an arbitrary constant. From the symmetric property of adjacency matrix of the network, the last two terms in (8.53) are the same and hence

$$\dot{V}(x) \leq -\sum_i (1 - \gamma\|N_i\|)(Lx)_i^2 + \sum_i \frac{1}{\gamma}\|N_i\|e_i^2. \tag{8.54}$$

In [95], it has been shown that using the execution rule

$$e_i^2 - \frac{\sigma_i\gamma(1-\gamma\|N_i\|)}{\|N_i\|}(Lx)_i^2 = 0 \tag{8.55}$$

for agent $i$, $\dot{V}(x) < 0$ for $\sigma_i \in (0, 1)$ and $0 < \gamma < \frac{1}{\|N_i\|}$. In a similar way to the previous two sections, we relax this condition by defining an integral based one.

**Theorem 8.5** *Consider a network of agents having single integrator dynamics $\dot{x}(t) = u(t)$, where the state feedback control law $u(t) = -Lx(t)$ is designed to ensure the average consensus of the network. Suppose $t_k^i$ is the most recent triggering time instant of agent $i$. Assume that the control task for this agent executes whenever the following execution rule holds at time $t_{k+1}^i > t_k^i$ for agent $i$ or one of its neighbors*

$$\int_{t_k^i}^{t_{k+1}^i} e_i^2(\tau)d\tau - \frac{\sigma_i\gamma(1-\gamma\|N_i\|)}{\|N_i\|}\int_{t_k^i}^{t_{k+1}^i}(Lx(\tau))_i^2 d\tau = 0, \tag{8.56}$$

*where $\sigma_i \in (0,1)$ and $0 < \gamma < \frac{1}{\|N_i\|}$. Then using a ZOH module to keep the last transmitted control value at the intervals between triggering time instants, the following hold:*

*(I) The network achieves average consensus.*

*(II) The inter sampling times are bounded below by some non-zero constant.*

**Proof.**

(I) Integrate inequality (8.54) from 0 to $t \in [t_k^i, t_{k+1}^i)$, we get

$$\begin{aligned} V(x(t)) - V(x(0)) \leq & -\sum_l (1-\gamma\|N_l\|)\int_0^t (Lx(\tau))_l^2 \\ & + \sum_l \frac{1}{\gamma}\|N_l\|\int_0^t e_l^2(\tau)d\tau. \end{aligned} \tag{8.57}$$

Using Assumption 8.1, for $l \in \{i\} \cup N_i$, we can brake the interval $[0,t]$ into subintervals $\bigcup_{0\leq r\leq k-1}[t_r^i, t_{r+1}^i) \cup [t_k^i, t]$ so that over each one (8.56) holds. The same argument is true for $l \notin \{i\} \cup N_i$ and subintervals $\bigcup_{0\leq r\leq k'-1}[t_r^l, t_{r+1}^l) \cup [t_{k'}^l, t]$, where $t_{k'}^l$ is the most recent triggering instant of agent $l$. Thus we have

$$\int_0^t e_l^2(\tau)d\tau \leq \frac{\sigma_l\gamma(1-\gamma\|N_l\|)}{\|N_l\|}\int_0^t (Lx(\tau))_l^2 d\tau \tag{8.58}$$

for $l = 1, \ldots, n$. Then (8.57) can be rewritten as

$$\begin{aligned} V(x(t)) - V(x(0)) \leq & -\sum_l \bar{\sigma}_l \int_0^t (Lx(\tau))_l^2 d\tau \\ \leq & -\min_l \bar{\sigma}_l \int_0^t |Lx(\tau)|^2 d\tau, \end{aligned} \tag{8.59}$$

where $\bar{\sigma}_l = (1-\sigma_l)(1-\gamma\|N_l\|)$ and $\min_l \bar{\sigma}_l = (1-\max_l \sigma_l)(1-\gamma \max_l \|N_l\|)$. From $0 < \gamma < \frac{1}{\|N_i\|}$, we conclude $1 - \gamma\max_l \|N_l\| \geq 0$. Then since $V(x(t)) \geq 0$, we get $\int_0^t |Lx(\tau)|^2 d\tau \leq \frac{V(x(0))}{(1-\max_l \sigma_l)(1-\gamma\max_l\|N_l\|)}$, meaning that $\int_0^t |Lx(\tau)|^2 d\tau$ is upper bounded by some positive constant and since it is a nondecreasing function of $t$ and $\frac{d^2}{dt^2}\int_0^t |\delta(\tau)|^2 d\tau =$

$2\dot{V}(x(t))$ is finite, from Barbalat's Lemma we get $\lim_{t\to\infty} \frac{d}{dt}\int_0^t |Lx(\tau)|^2 d\tau = \lim_{t\to\infty} |Lx(t)|^2 = 0$. This means that $Lx(t)$ (and hence $V(x(t)) = \frac{1}{2}x(t)^\mathsf{T} Lx(t)$) converges to zero.

(II) The proof outline is similar to Theorem 1. ■

## 8.3 Simulation Results

### 8.3.1 Centralized and Semi-Decentralized Approach

Consider the network of agents given in Fig. 8.1 with the following neighboring sets

$$N_1 = \{2, 3\}, N_2 = \{1, 3\}, N_3 = \{1, 2, 4\}, N_4 = \{3\}. \tag{8.60}$$

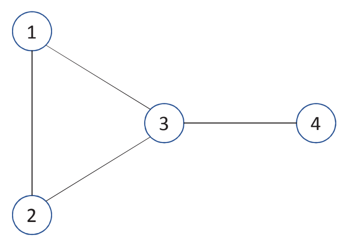



Figure 8.1: Communication graph of network system

Then the corresponding Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{8.61}$$

We have $\lambda_2(L) = 1$ and $\|L\| = 4$. We also assume $\sigma = 0.65$ for centralized case and $\sigma_1 = \sigma_2 = 0.55$ and $\sigma_3 = \sigma_4 = 0.65$ for decentralized case. In Figs. 2, 3, we present the control input of the agents for initial conditions $x_1(0) = -0.4$, $x_2(0) = -0.2$, $x_3(0) = 0$ and $x_4(0) = 0.6$. Fig. 2 is corresponding to the centralized approach and Fig. 3 to the semi-decentralized one.

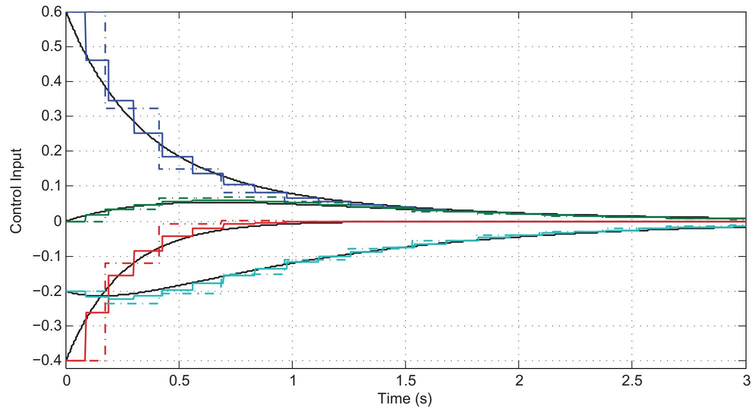



Figure 8.2: Control input of the agents: traditional classic state feedback controller (black), traditional event-based controller (solid) and the proposed event-based controller (dashed).

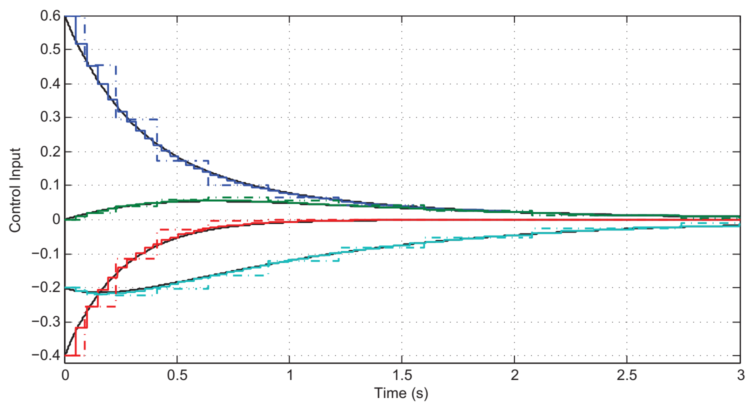



Figure 8.3: Control input of the agents: traditional classic state feedback controller (black), traditional event-based controller (solid) and the proposed event-based controller (dashed).

### 8.3.2 Decentralized Approach

Consider the following network of agents

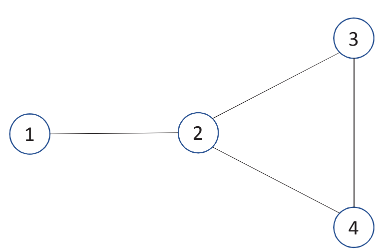



Figure 8.4: Communication graph of network system

with the following Laplacian matrix

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}. \tag{8.62}$$

Also assume $\sigma_1 = \sigma_2 = 0.55$ and $\sigma_3 = \sigma_4 = 0.65$. Fig. 5 shows the control input of the agents.

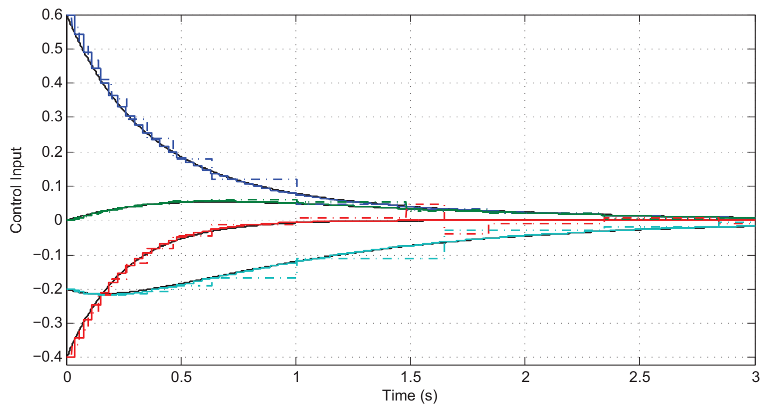



Figure 8.5: Control input of the agents: traditional classic state feedback controller (black), traditional event-based controller (solid) and the proposed event-based controller (dashed).

The simulation results support our claim that, compared to the existing methods, our integral based approach increases the inter sampling times.

## 8.4 Summary

In this chapter we proposed a novel event-triggered condition to solve the stability problem of general nonlinear systems while reducing the frequency of samplings. Our slution involves relaxing the conservative condition on the derivative of Lyapunov function to be negative all the times, which is the core assumption in many related works. In Chapter 4 it is shown that the integral-based type of TCs are indeed a special case of the proposed general TC design structure. The obtained results are then applied to the multi agent consensus problem.

# Chapter 9

# Summary and Conclusions

This thesis investigates the ETC design of nonlinear systems by focusing on the robustness properties of the resulting ETC system with respect to exogenous disturbances. In this regard, while our main focus is on different input-output stability performances, we also visit some of the main practical issues that may affect proper implementation of the ETC task.

The main contributions of this research can be highlighted as follows.

1. In Chapter 3, we address the problem of disturbance rejection of general nonlinear ETC systems while an $\mathcal{L}_2$-type stability performance is preserved. Our study is built on the assumption that the disturbance originates from structural uncertainties and hence its norm can be upper bounded by the norm of some function of states. As a consequence, the validity of the obtained results is only locally proved. It is also shown that by properly modifying the proposed TC, the intersampling intervals can be enlarged for a desired period of time or a desired number of sampling instants. This is, however, obtained at the price of relaxing the stability properties of the design.

2. In Chapter 4, we propose a rather general platform for TC design. Indeed, several dynamic and static parameters are introduced to capture the existing TC in the literature. Additionally, the results of Chapter 3 are extended in two aspects. First, instead of local $\mathcal{L}_2$ stability, the proposed TC is shown to guarantee global $\mathcal{L}_p$ performance for the ETC system. Second, the proposed TC is designed in a way that Zeno-behaviour is excluded in the presence of arbitrary disturbances. The restriction on admissible disturbances made in Chapter 3 is relaxed.

3. Chapter 5 considers the event-triggered analysis of a special class of nonlinear systems. Indeed, it is shown that when the system's inputs are introduced through an affine

structure, several restrictive assumptions in previous chapters can be relaxed. In detail, it is shown that the ETC problem can be solved without using an ISS condition. Both emulation and joint design approaches are studied. Moreover, the input-to-state stability of this class of nonlinear systems can be guaranteed through sufficient NLMI conditions.

4. Chapter 6 addresses one of the aspects of ETC systems that has not seen much attention yet. Indeed, in this chapter, we propose a design methodology to jointly design an ETC law for nonilinear Lipschitz systems under both state and output feedback scenarios. The obtined results is novel in that while most of the existing results in the literature solve the problem by proposing a set of LMIs to be solved, the design here is directly based on assigning the eigenvalues of stability matrices. As an another contribution of this chapter, we show that the output-based controller can be designed following the classical separation principle, *i.e.,* th controller and observer gains are designed independently, provided that the sampling at the controller-to-actuator channel is performed sufficiently fast. As an application, it is shown that the results can effectively serve as an event-based version of Lyapunov's indirect method, where an $H_\infty$ controller for the linearized ETC model renders the nonlinear ETC system locally stable.

5. Dissipativity is known as a powerful tool in unifying different forms of input-output stabilities. Thus to generalize the robustness performances studied in previous chapters, it is natural to study the dissipativity properties of nonlinear systems under event-based communications. In Chapter 7, it is shown that the general framework of TC design that is proposed in Chapter 4 can serve to extract different dissipativity properties for nonlinear ETC systems. Moreover, the proposed TC is proved to guarantee the isolation of triggering instants in presence of arbitrary disturbances.

6. Chapter 8 focuses on the problem of reducing the sampling frequency of ETC systems. In particular, it is shown that by using an integral based Lyapunov approach, we can proposed a less conservative triggering condition and hence improve the average frequency of samplings. The results, are then applied to solve the cooperative control problem of multi agent systems under event-based communications. As shown in details, the integral-based TCs can be extracted from the general TC design structure proposed in Chapter 4.

## 9.1 Directions for Future Work

Our proposed results in this thesis can be pursued in the following areas:

- Network control systems are considered as one of the main applications of the theory of event-based control. While, in this research we have focused on some interesting aspects of ETC systems such as performance in presence of disturbances and inter-event time properties, the validity and/or efficiency of the obtained results, when typical practical issues of a network control system such as time-delays and data packet dropouts are introduced, remains as an open research area.

- In Chapter 6, we have shown that under the proposed ETC design, when the TC at the observer-to-actuator channel is designed with a small enough triggering threshold, the event-triggered observer design reduces to two simpler design steps on the controller and observer gains. This is in accordance with the classical separation principle. However, this interesting result is restricted to our proposed method for designing controller and observer gains and also triggering conditions, an open area of research is then to extend the results to the other design methods.

- We proposed a general platform for designing TC in Chapter 4, which is shown to capture several existing TC designs. Our focus in this comparison was merely based on the structure of the TC with no reference to the relative performance in each design. Therefore, an interesting open area for future research is to extend this result and possibly propose a general TC famework which not only capture different TC structures but also covers the performance of each design.

- In Chapter 5 it is shown that while the notion of game theory is powerful tool in analysis of continuous-time input-affine nonlinear systems, when the communication between plant and controller is performed in an event-based manner this method this tool is of limited use to obtain the best control strategy when an event-based communicaton network is introduced. We have a conjecture that in such case, thebest strategy for minimizing player (control signal) can not be obtained explicitly. The proof or rejection of this conjecture can be pursued in future studies.

- The attention of this research is mostly focused on the robust stability of ECT systems under more famous robustness indices such as $\mathcal{L}_2$-gain and in general $\mathcal{L}_p$-gain performances. However, the (more stronger) incremental form of these performances has not seen much attention yet. Indeed, while the usual gain definition deals with the ratio of the norm of output to the norm of input, the incremental gain considers the ratio of the norm of changes at the output to the one at the input. Therefore, a natural way of generalization of the the obtained results in this thesis is to re-visit the problems when the usual gain performance is replaced with the incremental one.

# Bibliography


[1] K. Astrom and B. Wittenmark, *Computer-Control Systems: Theory and Design, 3rd Edition.* Prentice Hall, Englewood Cliffs, NJ, 1997.

[2] T. Chen and B. Francis, *Optimal sampled-data control systems.* London: Springer, 1995.

[3] D. Nesic, A. R. Teel, and P. Kokotovic, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Syst. Contr. Lett.*, vol. 38, pp. 259–270, 1999.

[4] D. Nešić and A. R. Teel, "Sampled-data control of nonlinear systems: An overview of recent results," *In: Moheimani S.R. (eds) Perspectives in robust control. Lecture Notes in Control and Information Sciences*, vol. 268, pp. 221–239, 2001.

[5] H. J. Marquez, *Nonlinear Control Systems: Analysis and Design.* Wiley, NJ, 2003.

[6] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[7] K. Astrom and B. Bernhardsson, "Comparison of periodic and event based sampling for first order stochastic systems," *Proc. IFAC World Conf.*, pp. 301–306, 1999.

[8] K. E. Arzen, "A simple event-based pid controller," *Proc. IFAC World Conf.*, vol. 18, pp. 423–428, 1999.

[9] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control task," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680 – 1685, 2007.

[10] W. Heemels, M. Donkers, and A. R. Teel, "Periodic event-triggered control for linear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 847–861, 2013.

[11] H. Yu and P. Antsaklis, "Event-triggered real-time scheduling for stabilization of passive and output feedback passive systems," *Proc. Amer. Control Conf.*, pp. 1674–1679, June-July 2011.

[12] R. Postoyan, A. Anta, D. Nešić, and P. Tabuada, "A unifying lyapunov-based framework for the event-triggered control of nonlinear systems," *Proc. IEEE Conf. Dec. Control and Eur. Control Conf.*, pp. 2559 – 2564, 2011.

[13] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, "A framework for the event-triggered stabilization of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 982–996, 2015.

[14] P. Tallapragada and N. Chopra, "On event triggered tracking for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2343–2348, 2013.

[15] R. W. Brockett, "Minimum attention control," *Proc. IEEE Conf. Dec. Control*, vol. 3, pp. 2628–2632, 1997.

[16] X. Wang and M. Lemmon, "Finite-gain $\mathcal{L}_2$ stability in distributed event-triggered networked control systems with data dropouts," *Proc. Eur. Control Conf.*, pp. 1746 – 1751, 2009.

[17] ——, "Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability," *IEEE Trans. Autom. Control*, vol. 54, no. 3, pp. 452–467, 2009.

[18] ——, "Self-triggering under state-independent disturbances," *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1494–1500, 2010.

[19] V. S. Dolk, D. P. Borgers, and W. P. M. H. Heemels, "Dynamic event-triggered control: Tradeoffs between transmission intervals and performance," *Proc. IEEE Conf. Dec. Control*, pp. 2764 – 2769, 2014.

[20] ——, "Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and zeno-freeness," *IEEE Trans. Autom. Control*, vol. 62, no. 1, pp. 34–49, 2017.

[21] W. Heemels, K. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," *Proc. IEEE Conf. Dec. Control*, pp. 3270–3285, December 2012.

[22] M. Donkers and W. Heemels, “Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering,” *IEEE Trans. Autom. Control*, vol. 57, no. 6, pp. 1362–1376, 2012.

[23] M. Lemmon, T. Chantem, X. Hu, and M. Zyskowski, “On self-triggered full information h-infinity controllers,” *Hybrid Systems: Computation and Control*, July 2007.

[24] M. Lemmon, “Networked control systems,” A. Bemporad, M. Heemels, and M. Johansson, Eds. Berlin Heidelberg: Springer-Verlag, 2010, ch. Event-triggered Feedback in Control, Estimation, and Optimization, pp. 293–358.

[25] D. Dimarogonas, “L2 gain stability analysis of event-triggered agreement protocols,” *Proc. IEEE Conf. Dec. Control and Eur. Control Conf.*, pp. 2130–2135, December 2011.

[26] H. Yu and P. Antsaklis, “Event-triggered output feedback control for networked control systems using passivity: Triggering condition and limitations,” *Proc. IEEE Conf. Dec. Control and Eur. Control Conf.*, pp. 199–204, December 2011.

[27] ——, “Event-triggered output feedback control for networked control systems using passivity: Time-varying network induced delays,” *Proc. IEEE Conf. Dec. Control and Eur. Control Conf.*, pp. 205–210, December 2011.

[28] A. Girard, “Dynamic triggering mechanisms for event-triggered control,” *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1992 – 1997, 2015.

[29] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić, “Co-design of output feedback laws and event-triggering conditions for linear systems,” *Proc. IEEE Conf. Dec. Control*, pp. 3560–3565, 2014.

[30] M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nešić, and M. Heemels, “Co-design of output feedback laws and event-triggering conditions for the $\mathcal{L}_2$-stabilization of linear systems,” *Automatica*, vol. 87, pp. 337–344, 2018.

[31] C. Peng and T. Yang, “Event-triggered communication and $\mathcal{H}_\infty$ control co-design for networked control systems,” *Automatica*, vol. 49, no. 5, pp. 1326–1332, 2013.

[32] F. Zhang, M. M. Jr., and N. van de Wouw, “Absolute stabilization of lur’s systems under event-triggered feedback,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 15 301–15 306, 2017.

[33] A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian, "Lq-based event-triggered controller co-design for saturated linear systems," *Automatica*, vol. 74, pp. 47–54, 2016.

[34] P. Tallapragada and N. Chopra, "Event-triggered dynamic output feedback control for LTI systems," *Proc. IEEE Conf. Dec. Control*, pp. 6597–6602, 2012.

[35] ——, "Event-triggered dynamic output feedback control of LTI systems over sensor-controller-actuator networks," *Proc. IEEE Conf. Dec. Control*, p. 4625–4630, 2013.

[36] D. Lehmann and J. Lunze, "Event–based output–feedback control," *Proc. Mediterranean Conf. Cont. Aut.*, p. 982–987, 2011.

[37] L. Etienne, S. D. Gennaro, and J.-P. Barbot, "Event-triggered observers and observer-based controllers for a class of nonlinear systems," *Proc. Amer. Control Conf.*, pp. 4717 – 4722, 2015.

[38] S. Durand, L. Torres, and J. F. Guerrero-Castellanos, "Event-triggered observer-based output-feedback stabilization of linear system with communication delays in the measurements," *Proc. IEEE Eur. Control Conf.*, pp. 666 – 671, 2014.

[39] S. Tarbouriech, A. Seuret, J. M. G. da Silva, and D. Sbarbaro, "Observer-based event-triggered control co-design for linear systems," *IET Control Theory Appl.*, vol. 10, no. 18, pp. 2466 – 2473, 2016.

[40] L. Etienne, S. D. Gennaro, and J.-P. Barbot, "Event triggered observer-based control for linear systems with time varying uncertainties," *Proc. Amer. Control Conf.*, pp. 1531 – 1536, 2015.

[41] S. Tarbouriech, A. Seuret, L. G. Moreira, and J. M. G. da Silva, "Observer-based event-triggered control for linear systems subject to cone-bounded nonlinearities," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 7893–7898, 2017.

[42] L. Shanbin and X. Bugong, "Co-design of event generator and controller for event-triggered control system," *Proc. Chin. Control Conf.*, pp. 175–179, 2011.

[43] X.Wang, H. Yu, and F. Hao, "Observer-based disturbance rejection for linear systems by aperiodical sampling control," *IET Control Theory Appl.*

[44] D. Borgers, V. Dolk, and W. Heemels, "Dynamic event-triggered control with time regularization for linear systems," *Proc. IEEE Conf. Dec. Control*, pp. 1352–1357, 2016.

[45] ——, “Dynamic periodic event triggered control for linear systems,” *Proc. Hybrid Syst., Comput. Control*, pp. 179–186, 2017.

[46] ——, “Riccati-based design of event-triggered controllers for linear systems with delays,” *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 174– 188, 2018.

[47] M. D. D. Benedetto, S. D. Gennaro, and A. D’Innocenzo, “Digital self triggered robust control of nonlinear systems,” *Proc. IEEE Conf. Dec. Control and Eur. Control Conf.*, pp. 1674–1679, 2011.

[48] M. Abdelrahim, J. Daafouz, and D. Nešić, “Robust event-triggered output feedback controllers for nonlinear systems,” *Automatica*, vol. 75, pp. 96–108, 2017.

[49] M. Ghodrat and H. J. Marquez, “On the local input-output stability of event-triggered control systems,” *IEEE Trans. Autom. Control*, pp. 1–1, 2018. [Online]. Available: https://doi.org/10.1109/tac.2018.2809594

[50] U. Tiberi and K. Johansson, “A simple self–triggered sampler for perturbed nonlinear systems,” *Nonlinear Analysis: Hybrid Systems*, vol. 10, pp. 126– 140, 2013.

[51] A. Rahnama, M. Xia, and P. Antsaklis, “A qsr-dissipativity and passivity based analysis of event-triggered networked control systems,” *Proc. IEEE Conf. Dec. Control*, pp. 3072– 3077, 2016.

[52] ——, “Passivity-based design for event-triggered networked control systems,” *IEEE Trans. Autom. Control*, pp. 1–1, 2017. [Online]. Available: https://doi.org/10.1109/tac.2017.2774600

[53] M. Xia, A. Rahnama, S. Wang, and P. Antsaklis, “Control design using passivation for stability and performance,” *IEEE Trans. Autom. Control*, pp. 1–1, 2018. [Online]. Available: https://doi.org/10.1109/tac.2018.2789681

[54] D. Nešić, A. R. Teel, and D. Carnevale, “Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems,” *IEEE Trans. Autom. Control*, vol. 54, no. 3, pp. 619–624, 2009.

[55] F. Forni, S. Galeani, D. Nešić, and L. Zaccarian, “Event-triggered transmission for linear control over communication channels,” *Automatica*, vol. 50, no. 2, pp. 490–498, 2014.

[56] M. Mazo and P. Tabuada, "Decentralized event-triggered control over wireless sensor/actuator networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2456–2461, 2011.

[57] H. Marquez, "A local theory of input-output stability of dynamical systems," *The J. of the Franklin Inst.*, vol. 332B, no. 1, pp. 91 – 106, 1996.

[58] S. H. Mousavi, M. Ghodrat, and H. J. Marquez, "Integral-based event-triggered control scheme for a general class of non-linear systems," *IET Control Theory Appl.*, vol. 9, no. 13, pp. 1982–1988, 2015.

[59] D. P. Borgers and W. P. M. H. Heemels, "Event-separation properties of event-triggered control systems," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2644–2656, 2014.

[60] H. Yu and F. Hao, "Input-to-state stability of integral-based event-triggered control for linear plants," *Automatica*, vol. 85, pp. 248–255, 2017.

[61] T. Basar and P. Bernhard, *$H_\infty$-Optimal Control and Related Minimax Design Problems.* Birkhauser, 1990.

[62] J. C. Willems, "Dissipative dynamical systems part i: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[63] A. L. Fradkov and I. G. Polushin, "Quasidissipative systems with one or several supply rates," *Proc. Eur. Control Conf.*, pp. 1237 – 1242, 1997.

[64] I. G. Polushin, "Stability results for quasi-dissipative systems," *Proc. IEEE Eur. Control Conf.*, 1995.

[65] D. Nešić and D. S. Laila, "On preservation of dissipation inequalities under sampling," *Proc. IEEE Conf. Dec. Control*, pp. 2472– 2477, 2000.

[66] M. Ghodrat and H. J. Marquez, "Event-triggered design with guaranteed minimum inter-event times and $\mathcal{L}_p$ performance," *Submitted to IEEE Trans. Autom. Control.*

[67] X. Wang and M. Lemmon, "On event design in event-triggered feedback systems," *Automatica*, vol. 47, no. 9, pp. 2319–2322, 2011.

[68] M. Vidyasagar, *Nonlinear Systems Analysis, 2nd Ed.* Englewood Cliffs: Prentice-Hall, 1993.

[69] M. R. James and S. Yuliar, “Numerical approximation of the $H_\infty$ norm for nonlinear systems,” *Automatica*, vol. 31, no. 8, pp. 1075 – 1086, 1995.

[70] E. D. Sontag and Y. Wang, “On characterizations of the input-to-state stability property,” *Systems Contr. Lett.*, vol. 24, pp. 351–359, 1995.

[71] A. R. Teel, J. Peuteman, and D. Aeyels, “Semi-global practical asymptotic stability and averaging,” *Syst. Contr. Lett.*, vol. 37, no. 5, pp. 329 – 334, 1999.

[72] E. D. Sontag, “Smooth stabilization implies coprime factorization,” *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435 – 443, 1989.

[73] A. Isidori and A. Astolfi, “Disturbance attenuation and $h_\infty$-control via measurement feedback in nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1283 – 1293, 1992.

[74] J. C. Doyle, K. Glover, P. P. khargonekar, and B. A. Francis, “State-space solutions to standard $H_2$ and $H_\infty$ control problems,” *IEEE Trans. Autom. Control*, vol. 34, no. 8, pp. 831 – 847, 1989.

[75] R. Freeman, “Global internal stabilizability does not imply global external stabilizability for small sensor disturbances,” *IEEE Trans. Autom. Control*, vol. 40, no. 12, pp. 2119 – 2122, 1995.

[76] M. Mazo and M. Cao, “Asynchronous decentralized event-triggered control,” *Automatica*, vol. 50, no. 12, pp. 3197–3203, 2014.

[77] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, “Event-based broadcasting for multi-agent average consensus,” *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[78] W.-M. Lu and J. C. Doyle, “Robustness analysis and synthesis for nonlinear uncertain systems,” *IEEE Trans. Autom. Control*, vol. 42, no. 12, pp. 1654 – 1662, 1997.

[79] A. Isidori, “Dissipation inequalities in nonlinear $h_\infty$ control,” *Proc. IEEE Conf. Dec. Control*, pp. 3265 – 3270, 1992.

[80] ——, “$h_\infty$ control via measurement feedback for affine nonlinear systems,” *Int. J. Robust Nonlinear Control*, vol. 4, pp. 553 – 574, 1994.

[81] J. A. Ball and J. W. Helton, “$h_\infty$ optimal control for nonlinear plants: connection with differential games,” *Proc. IEEE Conf. Dec. Control*, pp. 956 – 962, 1989.

[82] L. Li, X. Wang, and M. Lemmon, "Efficiently attentive event-triggered systems with limited bandwidth," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1491 – 1497, 2017.

[83] P. Tallapragada and J. Cortés, "Event-triggered stabilization of linear systems under bounded bit rates," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1575 – 1589, 2016.

[84] R. Rajamani, "Observers for lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 397 – 401, 1998.

[85] B. A. Francis, *A Course in $H_\infty$ Control Theory, Lecture Notes in Control and Information Sciences.* New York: Springer-Verlag, 1987.

[86] G. H. Golub and C. F. V. Loan, *Matrix Computations.* Johns Hopkins Univ. Press, 3rd edition, 1996.

[87] K. Fan and A. J. Hoffman, "Some metric inequalities in the space of matrices," *Proc. Amer. Math. Soc.*, vol. 6, no. 1, pp. 111–116, 1955.

[88] E. D. Sontag and Y. Wang, "Detectability of nonlinear systems," *Proc. Conf. on Inform. Sci. and Systems*, pp. 1031– 1036, 1996.

[89] ——, "Output-to-state stability and detectability of nonlinear systems," *Syst. Contr. Lett.*, vol. 29, pp. 279– 290, 1997.

[90] I. G. Polushin and H. J. Marquez, "Boundedness properties of nonlinearquasi-dissipative systems," *IEEE Trans. Autom. Control*, vol. 49, no. 12, pp. 2257– 2261, 2004.

[91] A. R. Teel, "On graphs, conic relations, and input-output stability of nonlinear feedback systems," *IEEE Trans. Autom. Control*, vol. 41, no. 5, pp. 702 – 709, 1996.

[92] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[93] D. V. Dimarogonas and K. H. Johansson, "Event-triggered cooperative control," *European Control Conference*, pp. 3015–3020, 2009.

[94] S. H. Mousavi, M. Ghodrat, and H. J. Marquez, "A novel integral-based event triggering control for linear time-invariant systems," *Proc. IEEE Conf. Dec. Control*, 2014.

[95] D. V. Dimarogonas and K. H. Johansson, "Event triggered control for multi-agent systems," *48th IEEE Conference on Decision and Control*, 2009.