Topological Recursion for Transalgebraic Spectral Curves and the TR/QC Connection

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematical Physics

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Abstract

The topological recursion is a construction in algebraic geometry that takes in the data of a so-called spectral curve, $\mathcal{S} = (\Sigma, x, y)$ where Σ is a Riemann surface and $x, y : \Sigma \to \mathbb{C}_\infty$ are meromorphic, and recursively constructs correlators which, in applications, are then interpreted as generating functions. In many of these applications, for example the r-spin Hurwitz case $\mathcal{S} = \left(\mathbb{C}, x(z) = z\mathrm{e}^{-z^r}, y(z) = \mathrm{e}^{z^r}\right)$, x has essential singularities when the underlying Riemann surface is compactified. Previously, these essential singularities have been ignored and the topological recursion considered on the non-compact surface. Here we argue that it is more natural to include the essential singularities as ramification points and give the corresponding definition for topological recursion; that is, a topological recursion for *transalgebraic* spectral curves rather than *algebraic* spectral curves. We use this definition to shed light on the TR/QC connection, Hurwitz theory, the Gromov-Witten invariants of $\mathbb{CP}^1$, and mirror curves.

Acknowledgements

The thesis presented here would not have been possible without the support and input of numerous individuals beyond that of myself. I would therefore like to thank Prof Gaëtan Borot, for allowing me to audit his helpful course on Airy structures and topological recursion, Dr Reinier Kramer, for recognising that our construction had an interesting enumerative interpretation, my family, for providing me with support throughout my life and degree, and the University of Alberta, for giving me the opportunity to complete a thesis based master's program at a quality research-focussed institution. Most of all, I would like to thank my supervisor Prof Vincent Bouchard, for his invaluable advice and mentorship throughout my degree and nascent academic career.

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Chapter 1

Introduction

The intersection between mathematics and physics has long been a fruitful area for researchers to examine. In the past, research in this area has been focused upon providing new physical insights using mathematics. However, as the mathematics used in physical theories has become increasingly sophisticated research has been conducted in the opposite vein, using physical intuition to arrive at new mathematical insight. It is this rich vein that is mined in the following work. In particular, we will focus on a construction in algebraic geometry known as topological recursion [Eynard and Orantin, 2007c, Bouchard and Eynard, 2013] and its relation to formal WKB solutions of certain Schrödinger-like equations [Gukov and Sulkowski, 2011, Norbury, 2015, Bouchard and Eynard, 2017].

The topological recursion (TR) was first developed in the context of matrix theory and QFT [Eynard, 2004], wherein it was viewed as a special solution to certain loop equations, but it was quickly realised it could be generalised to a wide field of applications [Eynard and Orantin, 2007c, Bouchard and Eynard, 2013, Borot et al., 2018, Bouchard et al., 2008, Bouchard and Mariño, 2008]. It is a recursive formalism that starts with the data of a so-called spectral curve, which is an algebraic curve $P(x, y) = 0$ along with additional structure on the first homology class. We then parametrise the two variables x and y as meromorphic functions on the corresponding Riemann surface, denoted Σ, by choosing an atlas on Σ and considering the coordinate expression of projection onto x and y. With this the topological recursion then constructs an infinite tower of symmetric n-differentials denoted $\omega_{g,n}$ that take in points on Σ^n. The interest in these $\omega_{g,n}$ comes from the fact that they often act as generating functions encoding interesting information in their expansion coefficients. The most celebrated example of this is for the simple spectral curve $P(x, y) = x - y^2$ where the corresponding $\omega_{g,n}$ encode the famous Witten-Kontsevich numbers that were central to Kontsevich's Fields Medal winning proof of Witten's conjecture [Witten, 1991, Kontsevich, 1992, Eynard and Orantin, 2007c]. But this is only one such example, and the topological recursion profits from deep connections with such diverse topics as BPS structures [Iwaki and Kidwai, 2022, 2021], Givental formalsim and Frobenius manifolds [Dunin-Barkowski et al., 2014], Hurwitz theory [Bouchard and Mariño, 2008, Eynard et al., 2009, Borot et al., 2011], Gromov-Witten invariants of the complex projective line [Dunin-Barkowski et al., 2017], Weil-Petterson volume of moduli spaces [Eynard and Orantin, 2007b, Eynard, 2011], and beyond [Eynard and Orantin, 2009].

One important connection that topological recursion has to other fields, and the connection that will be of primary importance to this thesis, is an intimate relation to WKB theory, called the quantum curve/topological recursion connection or QC/TR connection for short. To outline this connection we define the so-called wave function from our tower of meromorphic forms $\omega_{g,n}$ in a natural way

$$\psi(z) = \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g+n-2}}{n!}\int_b^z\cdots\int_b^z\left(\omega_{g,n} - \delta_{g,0}\delta_{n,2}x^*x_*\omega_{0,2}\right)\right], \tag{1.1}$$

where b is a base point and the $x^*x_*\omega_{0,2}$[1] term can just be thought of as a 'correction'. Then, if we quantise a spectral curve $P(x, y)$ by sending x and y to operators $\hat{x}$ and $\hat{y}$ such that the canonical commutation relation $[\hat{y}, \hat{x}] = \hbar$ holds there is often a so-called quantum curve $\hat{P}(\hat{x}, \hat{y}; \hbar)$ with $\hat{P}(x, y; 0) = P(x, y)$ and

$$\hat{P}(\hat{x}, \hat{y}; \hbar)\Psi(z) = 0. \tag{1.2}$$

This is a rather remarkable result and has been shown to hold in a wide variety of cases [Bouchard and Eynard, 2017, Eynard and Garcia-Failde, 2019, Eynard et al., 2021].

Our focus then is on extending the numerous results in this area, but we will particularly focus on the methods of Bouchard and Eynard [2017]. Critical to the argument in Bouchard and Eynard [2017] was that the functions x and y lived on a compact genus zero Riemann surface and were meromorphic. This precludes important curves such as the r-spin Hurwitz curve $P(x, y) = y - \mathrm{e}^{(xy)^r}$ whose corresponding parametrisation can only be defined on a non-compact Riemann surface [Bouchard and Mariño, 2008, Mulase et al., 2013]. To attack this case we redefine the topological recursion so it can deal with essential singularities of x and then consider the curve on a compact Riemann surface where the function x has essential singularities. Dealing with the resulting spectral curves necessarily involves dealing with transalgebraic [Pérez-Marco, 2019a] rather than algebraic geometry; we therefore denote these spectral curves with essential singularities as transalgebraic spectral curves. Importantly, this approach allows us to get quantum curves in the r-spin Hurwitz case, but they are, shockingly, not the same quantum curves as those obtained working directly with the r-spin Hurwitz numbers [Mulase

[1] $\omega_{0,2}$ is symmetric, so it does not matter which of its two variables one takes the pullback of the pushforward in.

et al., 2013].

In particular, this journey leads us to realise that our new topological recursion calculates not the regular r-spin Hurwitz numbers (which are the ones calculated when one neglects the essential singularity), but the r-atlantes Hurwitz numbers and their corresponding quantum curve [Alexandrov et al., 2016]. Hitherto, it was thought atlantes Hurwitz could not be calculated via topological recursion despite satisfying loop equations as their spectral curves should be the same as regular r-spin Hurwitz numbers, and it was know regular r-spin Hurwitz numbers were calculated by topological recursion [Dunin-Barkowski et al., 2019]; our results conclusively resolve this conundrum.

Furthermore, there are other curves of importance that have essential singularities. For example, the curve $P(x, y) = x - 2\cosh(y)$, which encodes information about Gromov-Witten invariants of $\mathbb{CP}^1$ [Norbury and Scott, 2014, Dunin-Barkowski et al., 2014, Zhou, 2012], is considered; to do this we need to define topological recursion when $\mathrm{d}x$ has infinitely many zeros, which is another case that hasn't been considered previously. Relatedly, there are a wide class of curves [Bouchard et al., 2008, Liu, 2012], related to string theory and the Gromov-Witten theory of toric Calabi-Yau threefolds, called mirror curves, that take the form $P(\mathrm{e}^x, \mathrm{e}^y) = 0$ where our method yields further insight into their quantum curves.

Outline

We begin, in the first chapter, to set the stage, with an examination of the well-established Eynard-Bouchard topological recursion. This necessitates, perforce, a discussion on spectral curves and their geometry, the initial data inputted into the recursive formalism of TR, along with an introduction to the key players and objects that have a role in the formalism. With this established, we move onto the definition of the topological recursion and outline the myriad properties enjoyed by this remarkable formalism. Finally, the chapter is concluded with an original reformulation of the topological recursion that will be of ineluctable utility for our purposes.

With the stage set, the players well-rehearsed, and the audience's interest perked, we begin with main production in the second chapter. Following the outline for the prior chapter, we first examine how the transition to a transalgebraic setting affects spectral curves and their geometry. Here we will also briefly examine transalgebraic geometry in general, a field that, excitingly, is still in its incipient stages. Next, we present our definition of transalgebraic topological recursion based on limits of sequences of algebraic spectral curves; the well-definedness of this is established, multiple properties are discussed, and a formula for calculating the limiting topological recursion in a wide variety of cases is presented. Briefly, an example is considered, where we get to see the formula for the limiting topological recursion in action calculating a correlator in the q-orbifold r-atlantes Hurwitz case. Finally, we conclude the chapter by further generalising the topological recursion to the case where x has a ramification locus of countably infinite cardinality.

In the final chapter we demonstrate how our new topological recursion may be used to construct quantum curves. Just as the first chapter required a prefatory discussion on spectral curves, here we must begin with a prefatory discussion on quantum curves and their relation to spectral curves. With this newfound knowledge on quantum curves we are able to describe in some detail the connection between topological recursion and quantum curves and present some new results in this area. Finally, the denouement comes and we use our previous results to construct quantum curves for the r-atlantes Hurwitz case, the case of the Gromov-Witten invariants of the projective line, and provide some conjectural statements on mirror curves, all exclusively from TR based considerations, rather than considerations based on the individual theories.

Chapter 2

Algebraic Topological Recursion

2.A The Geometry of Algebraic Spectral Curves

Only a brief perusal of the literature will lead one to as many different definitions of spectral curves as papers one reads [Eynard and Orantin, 2007c, Borot et al., 2018, Andersen et al., 2017]. For us, we will be mainly interested in two equivalent viewpoints on spectral curves; one that focuses on analytic properties of the objects and another that focus on the algebro-geometric properties. First, we must define the notion of a Torelli marking.

Definition 2.A.1. Given a connected algebraic curve Σ of genus g, a *Torelli marking* is a choice of a symplectic basis of cycles, *i.e.*, a canonical basis of $H_1(\Sigma)$, $\mathcal{A}_1, \ldots, \mathcal{A}_g, \mathcal{B}_1, \ldots, \mathcal{B}_g$ with the obvious symplectic pairing

$$\mathcal{A}_i \cap \mathcal{A}_j = \mathcal{B}_i \cap \mathcal{B}_j = 0, \quad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}. \tag{2.1}$$

With this definition we can define the algebro-geometric notion of a spectral curve.

Definition 2.A.2. A *spectral curve* $\mathcal{S}$ is a Torelli marked connected algebraic curve $\Sigma = \overline{\{P(x,y) = 0\}}$.

However, although this is a geometrically nice way to view a spectral curve, when we wish to feed it into the topological recursion, ultimately a recursive analytic formula, we don't want the abstract algebraic curve but a parametrisation of the curve. To get to this promised land, take the defining polynomial equation for our algebraic curve $P(x,y)$ and note that any point on the algebraic curve is given by the pair $(x,y) \in \mathbb{C}^2_\infty$. Then, in a slight abuse of notation we denote the projection onto x and y as meromorphic functions $x, y : \Sigma = \overline{\{P(x,y) = 0\}} \to \mathbb{C}_\infty$, respectively, and, given an atlas on Σ, x and y will then define meromorphic functions between compact Riemann surfaces. This brings us to the following definition of a spectral curve.

Definition 2.A.3. A *spectral curve* is a triple $\mathcal{S} = (\Sigma, x, y)$ where Σ is a compact connected Torelli marked Riemann surface, and x and y are two meromorphic functions on Σ that generate the function field on Σ, *i.e.*, $K(\Sigma) = \mathbb{C}(x,y)$.

This definition is equivalent to the previous one, as given two meromorphic functions on Σ, they must satisfy identically a polynomial relation $P(x,y) = 0$ that we may take to be irreducible and non-trivial as $K(\Sigma) = \mathbb{C}(x,y)$. In this later definition, the polynomial equation P is hidden, so it is more natural to not think of Σ as an algebraic curve, but rather just a compact Riemann surface (the two, of course, being equivalent). In either definition, the genus of the spectral curve is taken to be the genus of the underlying algebraic curve/Riemann surface.

Remark 2.A.4. Both these definitions bring the geometry of the spectral curves to the forefront. This geometry plays a key role in the so-called quantum curves and therefore the TR/QC connection, and therefore this thesis. For the purposes of TR, the definition of a spectral curve can be generalised to a fairly large degree, but a lot of the geometry, and therefore the relevance to this thesis, is lost. For details on this, see Borot et al. [2018].

Now that we have our spectral curves we proceed by defining a number of key supporting characters that will appear throughout our story. First, let us return to the defining polynomial equation of a spectral curve

$$P(x,y) = \sum_{i=0}^{d} p_{d-i}(x) y^i = \sum_{(i,j)\in\mathbb{Z}^2_{\geq 0}} \alpha_{i,j} x^i y^j, \tag{2.2}$$

where the p_i are polynomials in one variable and the $\alpha_{i,j}$ are complex valued coefficients. We denote by $A = \{(i,j)|\alpha_{i,j} \neq 0\} \subset \mathbb{Z}^2_{\geq 0}$ the set of points (i,j) where the corresponding coefficient $\alpha_{i,j}$ is non-zero. This allows us to make the following definition.

Definition 2.A.5. The *Newton polygon* of P is the convex hull of A, which we will denote by Δ.

With this important definition under our belt we can define two sets of numbers that will play critical roles when we attempt to construct quantum curves

$$\alpha_m = \inf\{a|(a,m) \in \Delta\}, \qquad \beta_m = \sup\{a|(a,m) \in \Delta\}, \tag{2.3}$$

and using these, we can calculate the number of interior points (considered as a subset of $\mathbb{N}^2$) I of Δ quite easily

$$I = \sum_{i=1}^{d-1} (\lceil \beta_i \rceil - \lfloor \alpha_i \rfloor), \tag{2.4}$$

where $\lfloor \cdot \rfloor$ is the floor function and $\rceil \cdot \lceil$ is the ceiling function. There is a classical result due to Baker [Baker, 1895], know as Baker's formula, that says $g \leq I$ where g is the genus of our algebraic curve Σ. Next, if we define the meromorphic functions on Σ

$$P_m(x,y) = \sum_{i=1}^{m} p_{m-1-i}(x) y^i, \tag{2.5}$$

we get the following result from Beelen [2009].

Lemma 2.A.6. *For $m = 2, \ldots, d$ we have*

$$\mathrm{div}(P_m) \geq \alpha_{d-m+1} \mathrm{div}_0(x) - \beta_{d-m+1} \mathrm{div}_\infty(x), \tag{2.6}$$

where, for a meromorphic function f on Σ, $\mathrm{div}(f)$ denotes the divisors of f, $\mathrm{div}_0(f)$ denotes the divisors of zeros of f, and $\mathrm{div}_\infty(f)$ denotes the divisors of poles of f.

We now turn our attention away from the polynomial P and define two differentials from the data of our Riemann surface and Torelli marking that are ubiquitous in the classical theory of Riemann surfaces and, as we will see, in the topological recursion.

Definition 2.A.7. The *canonical bilinear differential of the second kind*[1] B is the unique meromorphic 2-differential on Σ^2 satisfying:

• it is symmetric, *i.e.*, $B(z_1, z_2) = B(z_2, z_1)$;

[1] In the physics literature, this is often referred to as the *Bergman kernel*. As the Bergman kernel already refers to an entirely unrelated notion, this terminology is avoided in recent mathematics publications and here.

• its only pole is on the diagonal and near this pole it has the following expansion[2]

$$B(z_1, z_2) \overset{z_1 \to z_2}{\sim} \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \mathcal{O}(1);$$

• it is normalised on A-cycles

$$\int_{\mathcal{A}_i} B(\cdot, z) = 0, \qquad \forall i.$$

Definition 2.A.8. Fix $a, b \in \Sigma$. The *canonical differential of the third kind* S_a^b is the unique meromorphic 1-form on Σ satisfying:

• it is holomorphic on $\Sigma \setminus \{a, b\}$;

• it has a simple pole at a, b with residues $+1, -1$ respectively;

• it is normalised on A-cycles

$$\int_{\mathcal{A}_i} S_a^b = 0, \qquad \forall i.$$

By integrating along the unique homological chain $[b] - [a]$ that doesn't intersect our homology basis we can observe the following relation between B and S_a^b

$$S_a^b(z) = \int_a^b B(\cdot, z). \tag{2.7}$$

Whenever we write this integral as simply being from a to b we actually mean the homological chain described above. For an example of these two objects we can look at curves of genus zero where $\Sigma \cong \mathbb{C}_\infty$ and we have the following simple

[2]We will always presume that we have an atlas on Σ and write things in coordinate charts in this atlas; we abuse notation by not distinguishing between points and their values in coordinate charts.

expressions

$$S_a^b(z_1) = \frac{1}{z_1 - a} - \frac{1}{z_1 - b} = \int_a^b \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} = \int_a^b \omega_{0,2}(z_1, \cdot), \tag{2.8}$$

where z_i is an affine coordinate on $\mathbb{C}_\infty$.

As the next step in our journey through the geometry of spectral curves we briefly discus the properties of $x : \Sigma \to \mathbb{C}_\infty$. As a meromorphic function between compact Riemann surfaces, x automatically induces a branched covering between Σ and $\mathbb{C}_\infty$ which we will also denote by x. Given that x is meromorphic on Σ for each point z_0 on Σ there exists a unique integer l such that

$$x(z) \overset{z \to z_0}{\sim} \sum_{n=l}^{\infty} c_n (z - z_0)^n,$$

where c_l is assumed to be non-zero. Then we may define the *order* and *multiplicity* of x at z_0 as $\mathrm{Ord}_x(z_0) = l$ and $\mathrm{Mult}_x(z_0) = \max\{1, |l|\}$, respectively. We may also define the ramification locus of x, $R = \{a \in \Sigma | \mathrm{Mult}_x(a) \geq 2\}$, as the set of points on Σ where x fails to be a proper covering map.

Let us now zoom in on the local geometry about a ramification point $a \in R$. In an open neighbourhood U_a near a, $x|_{U_a}$ is a fully ramified Galois covering of degree $d_a = \mathrm{Mult}_x(a)$. Thus there are d_a local biholomorphic involutions $\sigma_1, \ldots, \sigma_{d_a}$ on U_i such that $x|_{U_a} \circ \sigma_i \equiv x|_{U_a}$ and $\sigma_i(a) = a$. Furthermore, there exists a local coordinate ζ defined by

$$\zeta^d = \begin{cases} x - x(a), & x(a) \neq \infty, \\ x^{-1}, & x(a) = \infty, \end{cases}$$

and in this local coordinate the σ_i just amount to multiplications by d_a'th roots of unity. We denote by $\boldsymbol{\sigma}_a$ the group of these local deck transformations σ_i and for $z \in U_a$ we denote by $\boldsymbol{\sigma}_a \cdot z = \{\sigma_i(z) | i = 1, \dots, d_a\}$ the orbit of z where $\boldsymbol{\sigma}_a$ acts on U_a in the obvious way and z is any coordinate chart on U_a. We introduce for later the notation $\boldsymbol{\sigma}'_a = \boldsymbol{\sigma}_a \setminus \{\mathrm{id}_{U_a}\}$, which is the local deck transformation group of x about a without the identity map.

We can, to some extent, extend our discussion globally. Let $d = \deg(x)$ be the degree of x as a covering map, *i.e.*, the number of preimages of a generic point (non-ramification point) in Σ under x. Then we denote by $\boldsymbol{\sigma} = \{\sigma : \Sigma \to \Sigma | x \circ \sigma = x\}$ the group of all deck transformations of x; it is important to note these are not, in general, well-defined functions on the whole of Σ[3] and will require branch cuts between ramification points to become so. If we define the *pushforward* or *trace* of a 1-form $\eta : \Sigma \to \mathbb{C}_\infty$ under x as

$$x_*\eta : \mathbb{C}_\infty \to \mathbb{C}_\infty, \; z \mapsto \sum_{w \in x^{-1}(\{z\})} \eta(w), \tag{2.9}$$

we can then take the pullback of $x_*\eta$ to get the symmetrisation of η under the deck transformation group of x

$$\begin{aligned} x^* x_* \eta : &\Sigma \to \mathbb{C}_\infty, \\ & z \mapsto \sum_{w \in x^{-1}(x(\{z\}))} \eta(w) = \sum_{w \in \boldsymbol{\sigma} \cdot z} \eta(w). \end{aligned} \tag{2.10}$$

It is rather important to observe that such a construction is automatically a well-defined 1-form of x, *i.e.*, there exists a 1-form $\omega : \mathbb{C}_\infty \to \mathbb{C}_\infty$ (depending on both x and η) such that

$$x^* x_* \eta(z) = \omega(x(z)). \tag{2.11}$$

[3]Many authors require the maps to be well-defined without branch cuts for it to be considered a deck transformation. For us, this would highly inconvenient.

Finally, as the last meander in our journey through the properties of algebraic spectral curves, we define two important classes of 'nice' curves.

Definition 2.A.9. Given a spectral curve $\mathcal{S} = (\Sigma, x, y)$ define the 1-form $\omega_{0,1} = y\mathrm{d}x$. For a ramification point $a \in R$ put $r_a = \mathrm{Mult}_x(a)$ and take the local coordinate $t_a^{r_a} = x - x(a)$ if $x(a) \neq \infty$ and $t_a^{-r_a} = x$ if $x(a) = \infty$. Then, near a, $\omega_{0,1}$ has the expansion

$$\omega_{0,1}(t) = \sum_{l=r+1} \tau_l^a t_a^{l-1} \mathrm{d}t_a.$$

Let $s_a = \min\{l | \tau_l^a \neq 0 \wedge l \nmid r_a\}$ be the smallest integer such that τ_l^a is non-zero and l does not divide r_a.

The spectral curve is then called *admissible* if for every $a \in R$ we have that the point $(x(a), y(a))$ is non-singular, $s_a \leq r_a + 1$, and one of the following two conditions is satisfied

• $r_a = \pm 1 \bmod s_a$;

• $s_a \leq -1$.

From now on we will always assume we are dealing with only admissible spectral curves. The topological recursion is not well-defined for spectral curves that are not admissible; for a discussion of why this is so, see Borot et al. [2018], Bouchard and Eynard [2013].

Definition 2.A.10. Given a spectral curve $\overline{\{P(x,y) = 0\}}$ we say it is *regular* if the following two conditions hold:

• the Newton polygon Δ has no interior points;

• if the origin is on our curve, *i.e.* $P(0,0) = 0$, then as an affine curve it must be smooth at this point.

It is important to note that, by Baker's formula, all regular spectral curves are genus zero[4].

2.B The Eynard-Bouchard Topological Recursion

The goal of the Eynard-Bouchard Topological Recursion is to construct an infinite tower of symmetric n-differentials $\omega_{g,n}$, called correlators (the terminology comes from matrix models [Eynard, 2004]), from the initial data of a spectral curve; the hope being that these correlators will be generating functions for numbers interesting to the given problem. In this spirit, we take as given a spectral curve $\mathcal{S} = (\Sigma, x, y)$ and assuming we have a set of symmetric meromorphic n-differentials on Σ, $\{\omega_{g,n}\}_{g,n=0}^{\infty}$ we define the following two combinatorial combinations of these $\omega_{g,n}$

Definition 2.B.1. Let $A, B \subset \Sigma$ be two sets of points with cardinality i and $n-1$ respectively. Then define

$$\begin{aligned} \Omega^i_{g,n}(A|B) &= \sum_{A_1,\ldots,A_j \vdash A} \sideset{}{'}\sum_{\substack{g_1+\cdots+g_j=g+j-i \\ B_1 \sqcup\cdots\sqcup B_j = B}} \bigotimes_{k=1}^{j} \omega_{g_k, \sharp A_k + \sharp B_k}(A_k, B_k), \\ \mathcal{E}^i_{g,n}(A|B) &= \sum_{A_1,\ldots,A_j \vdash A} \sum_{\substack{g_1+\cdots+g_j=g+j-i \\ B_1 \sqcup\cdots\sqcup B_j = B}} \bigotimes_{k=1}^{j} \omega_{g_k, \sharp A_k + \sharp B_k}(A_k, B_k), \end{aligned} \tag{2.1}$$

where $\vdash$ means set partition, $\sqcup$ is the disjoint union (note: using this notation we allow the B_k to be empty), for a set S $\sharp S$ denotes the cardinality of the set, and the prime over the sum means we exclude terms with factors of $\omega_{0,1}$.

As a final step before we define the topolgical recursion itself, we must define the recursion kernel that will appear in the recursive formula.

[4] It is possible to classify all regular spectral curves. See Bouchard and Eynard [2017].

Definition 2.B.2. Let $A \subset \Sigma$ be a set of cardinality $i-1$ and let $z_1, t \in \Sigma$. Then we define the so-called *Eynard-Bouchard Recursion Kernel*

$$K_i(z_1, t, A) = S^t_*(z_1) \prod_{z \in A} \frac{1}{\omega_{0,1}(t) - \omega_{0,1}(z)}, \tag{2.2}$$

where $S^t_*(z_1)$ was defined in Definition 2.A.8, $\omega_{0,1} = y\mathrm{d}x$, and the base point $*$ is arbitrary; it can be seen that the topological recursion does not depend on this choice [Bouchard and Eynard, 2013].

We now come to the main definition of this chapter, the Eynard-Bouchard topological recursion itself.

Definition 2.B.3. Set $\omega_{0,1} = y\mathrm{d}x$ and let $\omega_{0,2}$ be the canonical bilinear differential of the second kind. Then define recursively, for $2g+n-2 \geq 1$

$$\begin{aligned} \omega_{g,n}(z_1, B) &= \sum_{a \in R} \operatorname*{Res}_{t=a} \sum_{i=2}^{\mathrm{Mult}_x(a)} \sum_{\substack{A \subset \boldsymbol{\sigma}_a \cdot t \\ |A|=i, t \in A}} K_i(z_1, t, A \setminus \{t\}) \Omega^i_{g,n}(A|B), \qquad n \geq 1, \\ \omega_{g,0} &= \frac{1}{2-2g} \sum_{a \in R} \operatorname*{Res}_{t=a} \left(\int_*^t \omega_{0,1} \right) \omega_{g,1}(t), \qquad g \geq 2, \end{aligned} \tag{2.3}$$

where the base point $*$ is arbitrary (other than the fact it can't be a pole of $\omega_{0,1}$). The definition is recursive on the negative of the "Euler characteristic" $-\chi = 2g+n-2$ (the integer g is often referred to as the 'genus', although it has nothing to do with the genus of the spectral curve). The correlators with $2g+n-2 \geq 1$ are called *stable*.

Remark 2.B.4. The 0-differentials (complex numbers) $\omega_{0,0}$ and $\omega_{1,0}$ can be defined in a natural way. However, this definition is rather involved and strays too far from the topic of this thesis to be included. See Eynard and Orantin [2007c] for details. Furthermore, if $\omega_{0,1}$ is not exact in an open set near a ramification point a, then the definition of $\omega_{g,0}$ will obviously have to be amended.

Remark 2.B.5. When taken as the initial data for topological recursion, a spectral curve is often given as a quadruple $\mathcal{S} = (\Sigma, x, \omega_{0,1}, \omega_{0,2})$ where Σ is a compact connected Riemann surface, $x : \Sigma \to \mathbb{C}_\infty$ is meromorphic, $\omega_{0,1}$ is a meromorphic 1-form, and $\omega_{0,2}$ is a meromorphic bidifferential of the second kind with its only pole a double pole on the diagonal and normalised so this double pole term has a coefficient of unity. To get back to our original definition, we set $y = \omega_{0,1}/\mathrm{d}x$ and take the Torelli marking on Σ that gives $\omega_{0,2}$ as the canonical bilinear differential of the second kind.

It should now be clear where the 'topological' in topological recursion comes from; the recursion is done on the negative of the Euler characteristic $-\chi_{g,n} = 2g + n - 2$, which is an invariant of topological surfaces. After defining the topological recursion the first step in our study is to examine some of the extraordinary properties enjoyed by the correlators $\omega_{g,n}$. This is the content of the next theorem.

Theorem 2.B.6. *For $2g + n - 2 \geq 1$ the $\omega_{g,n}$ constructed from the topological recursion satisfy the following properties [Eynard and Orantin, 2007c, Bouchard and Eynard, 2013, Borot et al., 2018].*

- Symmetry: *the $\omega_{g,n}$ are symmetric in all of their n variables.*
- Pole structure: *the $\omega_{g,n}$ have poles only at the ramification points of x.*
- Residueless: *for $k = 0, 1$, $2g + n - 2 \geq 0$, and every ramification point $a \in R$ we have*

$$\operatorname*{Res}_{z=a} x(z)^{\pm k} \omega_{g,n}(z, z_2, \ldots, z_n) = 0,$$

where the plus and the minus correspond to ramification points that are zeros and poles of $\mathrm{d}x$*, respectively.*

• String Equations: *for* $k = 0, 1$ *and* $2g + n - 2 \geq -1$ *the following relation holds* [5]

$$\sum_{a \in R} \operatorname*{Res}_{z=a} x(z)^k y(z) \omega_{g,n+1}(z, z_1, \dots, z_n) = \sum_{j=1}^{n} \mathrm{d}_{z_j} \frac{x(z_j)^k \omega_{g,n}(z_1, \dots, z_n)}{\mathrm{d}x(z_j)}.$$

• Homogeneity: *under the rescaling* $\omega_{0,1} \to f\omega_{0,1}$ *we have* $\omega_{g,n} \to f^{2-2g-n}\omega_{g,n}$.

• Modularity: *under the group of Torelli marking changes the* $\omega_{g,n}$ *are quasi-modular forms.*

• Normalised: *for* $2g + n - 2 \geq 0$ *the* $\omega_{g,n}$ *are normalised on* $\mathcal{A}$*-cycles, i.e.,*

$$\int_{\mathcal{A}_i} \omega_{g,n}(\cdot, z_2, \dots, z_n) = 0, \qquad \forall i.$$

• Diagrammatic Representation: *The recursive definition of the correlators* $\omega_{g,n}$ *can be represented using Feynman-like* [6] *graphs [Eynard and Orantin, 2007c, Bouchard et al., 2013].*

• Deformations: *the* $\omega_{g,n}$ *satisfy variational deformation equations [Eynard and Orantin, 2007c].*

• Limits: *under reasonable restrictions, the* $\omega_{g,n}$ *are well-defined when taking limits of spectral curves; see Eynard and Orantin [2007c] for singular limits and Bouchard and Eynard [2013] for the case when ramification points collide.*

• Loop Equations: *the* $\omega_{g,n}$ *can be viewed as 'nice' solutions of certain abstract loop equations [Borot et al., 2018]. In the case when the* $\omega_{g,n}$ *are the correlator in a matrix model these loop equations are simply the ones of the underlying QFT*

[5] In general, when such relations are stated, it is implicitly assumed that the only ramification points that are to be included in R are those at which the $\omega_{g,n}$ have poles.

[6] It is important to note that these are emphatically *not* actual Feynman graphs.

[Eynard, 2004].

• Dilaton Equations: *when $\omega_{0,1}$ is locally exact near ramification points the following relation holds for an arbitrary base point $*$ and set of n points $A \subset \Sigma$*

$$(2-2g-n)\omega_{g,n}(A) = \sum_{a\in R} \operatorname*{Res}_{t=a} \left(\int_*^t \omega_{0,1} \right) \omega_{g,n+1}(t, A).$$

On the above list, we intentionally omitted what is thought to be perhaps the deepest and most mysterious property of these $\omega_{g,n}$, which is the so-called *symplectic invariance*, *i.e.*, that the $\omega_{g,0}$ should somehow remain unchanged under the transformations of $(x, y) \to (\tilde{x}, \tilde{y})$ that preserve the natural symplectic form $\mathrm{d}x \wedge \mathrm{d}y = \pm \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ up to sign. Unfortunately, it is not know whether this is true in general and if so, why. See Eynard and Orantin [2007c,a] for a proof of this valid in some generality. It is, however, known that the proof in Eynard and Orantin [2007c,a] fails in some cases; see Bouchard et al. [2013] for details.

2.C Rewriting the Topological Recursion

Eventually, we wish to study topological recursion not for algebraic spectral curves, but for certain transalgebraic curves. Practically, this will involve allowing the function x to have essential singularities; our task will then be to define the contributions at these essential singularities. The observant reader might instantly spot one of the chief issues with this: the local deck transformation group of x near the essential singularity will, if it can be defined in a sensible manner, certainly be an infinite group due to the arbitrarily non-injective behavior of an analytic function near an essential singularity. To avoid this problem, we present a rewriting of the topological recursion that trades out the sum over the deck transformation group of x, for a sum over ramification points, coinciding points, and deck transformations of y.[7] This approach is accomplished by rewriting the symmetrisation over the non-trivial sheets as integration over a contour integral before using the compactness of the Riemann surface Σ to pick out the residues on the other side of the contour integral.

Before we present the rewriting, we need some notation. Let $C = \{t, t_1, \ldots, t_i\} \subset \Sigma$ and $C' = C \setminus \{t\}$ be sets of $i+1$ and i points, respectively. For an i-differential η that has vanishing residue when any two of its arguments coincide we define

$$\operatorname*{Res}_{C=t} \eta(t_1, \ldots, t_i) = \operatorname*{Res}_{C'=t} \eta(t_1, \ldots, t_i) = \operatorname*{Res}_{t_1=t} \cdots \operatorname*{Res}_{t_i=t} \eta(t_1, \ldots, t_i).$$

This notation makes sense precisely because η has vanishing residue at points where its arguments coincide so the order in which we take the residues is irrelevant. For our purposes, we note this condition is clearly satisfied if η is taken to be symmetric. Similarly, we use the related, but common, notation that

$$\operatorname*{Res}_{t=C'} = \sum_{t_0 \in C'} \operatorname*{Res}_{t=t_0}.$$

For a set $C \subset \Sigma$ we denote by t_C one fixed point in this set. Lastly, as we will have to take many residues at once, we define the conveniently compact notation

$$\operatorname*{Res}_{\substack{t_l=a_l \\ l=1,\ldots,n}} = \operatorname*{Res}_{t_1=a_1} \cdots \operatorname*{Res}_{t_n=a_n},$$

along with the obvious generalisation to the previous two notations. With all this notation safely in our memory vaults, we may proceed to the theorem of this section.

Theorem 2.C.1. *Let $Y(t) = y^{-1}(y(\{t\}))$ and $B \subset \Sigma$ a set of $n-1$ points. Then*

[7] The author was made aware of the possibility of this approach by Prof Vincent Bouchard from notes by Dr Nitin K Chidambaram. The actual result presented here is, however, original.

the correlators of the topological recursion satisfy the alternative recursive formula

$$\omega_{g,n}(z_1,B) = \operatorname*{Res}_{t=R} \sum_{i=2}^{\deg(x)} S_*^t(z_1) \sum_{C_1,\ldots,C_j \vdash \{t_1,\ldots,t_{i-1}\}} \frac{(-1)^{1-\delta_{j,i-1}}}{j!} \operatorname*{Res}_{\substack{t_{C_l}=R,B,Y(t)\\ l=1,\ldots,j}} \operatorname*{Res}_{\substack{C_l=t_{C_l}\\ l=1,\ldots,j}} \left(\prod_{l=1}^{j} \frac{1}{x(t)-x(t_{C_l})} \prod_{t_0 \in C_l \setminus \{t_{C_l}\}} \frac{1}{x(t_0)-x(t_{C_l})} \right) \frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(y(t)-y(t_l))}, \quad (2.1)$$

where, as before, R is the ramification locus of x, S_*^t is defined in Definition 2.A.8, and $\vdash$ denotes a set partition. Note that we have committed a notational peccadillo by writing that the C_l partition the dummy integration variables t_i outside the integrand when the t_i are, of course, only defined inside the integrand (i.e., under the residue). This is of no fundamental importance as we really only need a partition of the set $\{1,\ldots,i-1\}$; it is just more convenient to immediately attach these indices to the coordinates.

Proof. We first perform the obvious rewriting[8]

$$\begin{aligned} &\sum_{\substack{A\subset \boldsymbol{\sigma}\cdot t\\ |A|=i, t\in A}} K_i(z_1,t,A\setminus\{t\})\Omega^i_{g,n}(A|B) \\ &= S_*^t(z_1) \sum_{\{\sigma_1,\ldots,\sigma_{i-1}\}\subset \boldsymbol{\sigma}'} \operatorname*{Res}_{\substack{t_l=\sigma_l(t)\\ l=1,\ldots,i-1}} \frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(x(t_l)-x(t))(y(t)-y(t_l))}. \end{aligned} \quad (2.2)$$

Now we notice that, in our new writing, the summand is actually well-defined when two or more of the σ_l coincide, it will just result in higher order poles at $t_l=\sigma_l(t)$. Therefore, we may add and subtract all terms where two or more σ_l coincide. This gives us two main terms: the original sum plus the added terms where two or more σ_l coincide; the subtracted terms where two or more σ_l coincide. We first examine

[8]It is a theorem in Bouchard and Eynard [2013] that one may replace $\boldsymbol{\sigma}_a$ with $\boldsymbol{\sigma}$ in the sum in topological recursion, *i.e.*, one may replace the sum over the *local* deck transformations of x about a ramification point a with the sum over all the *global* sheets.

the first term

$$\begin{aligned}\sum_{\sigma_1,\ldots,\sigma_{i-1}\in\boldsymbol{\sigma}'}&\frac{1}{(i-1)!}\operatorname*{Res}_{\substack{t_l=\sigma_l(t)\\ l=1,\ldots,i-1}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(x(t_l)-x(t))(y(t)-y(t_l))}\\ &=\frac{1}{(i-1)!}\operatorname*{Res}_{\substack{t_l=R,B,Y(t)\\ l=1,\ldots,i-1}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(x(t)-x(t_l))(y(t)-y(t_l))}.\end{aligned}\tag{2.3}$$

All we have done here is used the fact that Σ is a compact Riemann surface so the sum of all the residues of any differential must be zero. That we only pick up residues at the listed points is because the $\omega_{g,n}$ only have poles at coinciding points and ramification points.

We now wish to apply the same logic to the terms with coinciding points. To this end, we want to know what happens when $j \leq i-1$ of the same t_l are specialised to the same sheet. Thus, for illustration, we examine the following expression

$$\begin{aligned}&\sum_{\sigma\in\boldsymbol{\sigma}'}\operatorname*{Res}_{\substack{t_l=\sigma(t)\\ l=1,\ldots,j}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(x(t_l)-x(t))(y(t)-y(t_l))}\\ =&\sum_{\sigma\in\boldsymbol{\sigma}'}\operatorname*{Res}_{\substack{t_l=\sigma(t)\\ l=1,\ldots,j-1}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{j-1},\sigma(t),t_{j+1},\ldots,t_{i-1}|B)}{\mathrm{d}x(t)(y(t)-y(\sigma(t)))\prod_{\substack{l=1\\ l\neq j}}^{i-1}(x(t_l)-x(\sigma(t)))(y(t)-y(t_l))}\\ =&\sum_{\sigma\in\boldsymbol{\sigma}'}\operatorname*{Res}_{t_j=\sigma(t)}\operatorname*{Res}_{\substack{t_l=t_j\\ l=1,\ldots,j-1}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{(x(t_j)-x(t))(y(t)-y(t_j))\prod_{\substack{l=1\\ l\neq j}}^{i-1}(x(t_l)-x(t_j))(y(t)-y(t_l))}\\ =&\operatorname*{Res}_{t_j=R,B,Y(t)}\operatorname*{Res}_{\substack{t_l=t_j\\ l=1,\ldots,j-1}}\frac{\Omega^i_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{(x(t)-x(t_j))(y(t)-y(t_j))\prod_{\substack{l=1\\ l\neq j}}^{i-1}(x(t_l)-x(t_j))(y(t)-y(t_l))}.\end{aligned}\tag{2.4}$$

With the above calculation in mind, the subtracted terms with the coinciding deck transformations may be written as

$$
\begin{aligned}
&-\sum_{j=1}^{i-2}\frac{1}{j!}\sum_{\sigma_1,\dots,\sigma_j\in\boldsymbol{\sigma}'}\sum_{C_1,\dots,C_j\vdash\{t_1,\dots,t_{i-1}\}}\operatorname*{Res}_{\substack{C_l=\sigma_l(t)\\ l=1,\dots,j}}\frac{\Omega^i_{g,n}(t,t_1,\dots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(x(t_l)-x(t))(y(t)-y(t_l))}\\
&=-\sum_{j=1}^{i-2}\frac{1}{j!}\sum_{C_1,\dots,C_j\vdash\{t_1,\dots,t_{i-1}\}}\operatorname*{Res}_{\substack{t_{C_l}=R,B,Y(t)\\ l=1,\dots,j}}\operatorname*{Res}_{\substack{C_l=t_{C_l}\\ l=1,\dots,j}}\\
&\qquad\left(\prod_{l=1}^{j}\frac{1}{x(t)-x(t_{C_l})}\prod_{t_0\in C_l\setminus\{t_{C_l}\}}\frac{1}{x(t_0)-x(t_{C_l})}\right)\frac{\Omega^i_{g,n}(t,t_1,\dots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(y(t)-y(t_l))},
\end{aligned}
\tag{2.5}
$$

and the desired result then follows. $\square$

Chapter 3

Transalgebraic Topological Recursion

3.A The Geometry of Transalgebraic Spectral Curves

Now we wish to generalise the topological recursion to a more inclusive notion of spectral curves so as to analyse new cases using the powerful properties of the correlation functions $\omega_{g,n}$. For example, an important use of topological recursion is its application to Hurwitz theory [Bouchard and Mariño, 2008, Dunin-Barkowski et al., 2019]. Here the spectral curves for what are known as the q-orbifold r-spin Hurwitz numbers are[1]

$$\mathcal{S} = \left(\mathbb{C}, x(z) = z\mathrm{e}^{-z^{qr}}, y(z) = z^{q-1}\mathrm{e}^{z^{qr}}\right). \tag{3.1}$$

[1]Usually in the literature this curve is written slightly differently with $y(z) = z^q$; $\omega_{0,1}$ is then taken to be $y\mathrm{d}\log(x)$ rather than $y\mathrm{d}x$. This is, of course, entirely equivalent to what is presented here.

Indeed, this is not a spectral curve in the sense of Definition 2.A.3 and the key missing component is the fact that $\mathbb{C}$ is not a compact Riemann surface[2]. After seeing the definition of the topological recursion, the astute reader may note that we actually need, in a sense, very little of the spectral curve structure to define it. All that is needed, strictly speaking, is for there to be a set of points R, an open disk around each of these points, and a germ of x, $\omega_{0,1}$, and $\omega_{0,2}$ on these disks. This is the approach of Borot et al. [2018]. For instance, in our example, the q-orbifold r-spin Hurwitz case, in the traditional approach in the literature [Dunin-Barkowski et al., 2019] one merely ignores the essential singularity at infinity and considers and simply computes the topological recursion for the ramification points $\mathrm{d}x(z) = 0$ with $z \in \mathbb{C}$.

This dropping of the essential singularity does, at first glance, solve the issue. However, the more one loses geometric properties of the spectral curve and drifts towards the more geometrically nebulous union-of-disks approach, the more the topological recursion itself loses its geometric properties for the simple reason that there is no underlying geometry. In particular, establishing the TR/QC connection becomes increasingly difficult.

To regain these properties, the aim of this chapter is to define the topological recursion at essential singularities; in the aforementioned example this would allow us to work on a compact Riemann surface, namely $\mathbb{C}_\infty \cong \mathbb{CP}^1$, and, therefore, gain back the lost properties of the correlators. In summary, rather than restrict the domains of x and y in (3.1), we treat x and y as *transalgebraic* functions on the compactified Riemann surface. To do so, we must chart the waters of transalgebraic geometry.

Unfortunately, the study of transalgebraic geometry of this type is only in its

[2]For genus zero curves the Torelli marking is irrelevant as the first homology class is trivial.

nascent stages and the waters are therefore relatively uncharted. Even more unfortunately, a digression into this new field would bring us too far away from our topic, so we will have to take a low road of sorts. But before we venture down our low road, we make a brief bibliographic divertissement for those interested in seeing what waters have been charted in this geometry: for preliminary study and motivation of these surfaces see Marco [1995]; particularly relevant to us will be the the convergence of surfaces [Biswas and Pérez-Marco, 2010a] and defining the class of transalgebraic functions on a compact Riemann surface [Pérez-Marco, 2019a]; as far as the author knows, a complete bibliography on the study of this geometry is (in chronological order) Marco [1995], Biswas and Pérez-Marco [2010a,b, 2015a,b], Pérez-Marco [2019b,a], Biswas and Pérez-Marco [2019], Pérez-Marco [2020], Biswas and Biswas [2020].

To begin our trek down this low road of sorts we follow Pérez-Marco [2019a] in defining the notion of an exponential singularity, and then define the precisely what a transalgebraic function is.

Definition 3.A.1. A point $s_0 \in \Sigma$ is said to be an *exponential singularity* of a function f if there exists an open neighbourhood $U \subset \Sigma$ of s_0 such that $f : U \setminus \{s_0\} \to \mathbb{C}^\times$ is a well-defined holomorphic function, but x does not extend to a meromorphic function on all of U. The *exponential order* of f at $s_0 \in S$ is defined as

$$\mathrm{Erd}_f(s_0) = \inf\{d \in \mathbb{R}_{\geq 0} | \limsup_{z \to s_0} |z - s_0|^d \log |f(z)| < \infty\}, \tag{3.2}$$

and $\mathrm{Erd}_f(s_0)$ is taken to be infinity if the infimum does not exist (*i.e.*, the set of such d is empty).

Definition 3.A.2. We define the *transalgebraic functions* with at most $n \in \mathbb{Z}_{\geq 0}$

zeros, poles, and exponential singularities as

$$\mathcal{T}_n(\Sigma) =$$
$$\{f \in K(\Sigma \backslash S) | \big(\sharp S \le n\big) \wedge \big(f(z) \in \mathbb{C}^\times\ \forall z \in \Sigma \backslash S\big) \wedge \big(\mathrm{Erd}_f(s_0) < \infty\ \forall s_0 \in S\big)\}. \tag{3.3}$$

In words, these are all the holomorphic functions on Σ without n points, that have no zeros or poles (*i.e.*, zeros and poles are only possible at the n removed points), and finite exponential order at the n removed points. Then, we define the *class of transalgebraic functions* as the union

$$\mathcal{T}(\Sigma) = \bigcup_{n \in \mathbb{Z}_{\ge 0}} \mathcal{T}_n(\Sigma) \tag{3.4}$$

It is a natural question to ask why we insist that there are finitely many zeros and poles near each s_0 in Definition 3.A.1. Of course, by the great Picard theorem, all points in $\mathbb{C}_\infty$ are obtained infinitely often as one approaches an essential singularity save for possibly two points and we may put those two points (if they exist) at zero and infinity via a change of coordinates. The following proposition tells us that if we didn't have two such points, sometimes called *Picard points* in the literature, then s_0 would be a cluster point of ramification points. Although we will deal with such things in the section following the next, for now we wish to exclude them.

Proposition 3.A.3. *Let $\pi : B_\epsilon(0) \setminus \{0\} \to \mathbb{C}_\infty$ be a branched covering from the punctured disk of radius $\epsilon > 0$ to $\mathbb{C}_\infty$ with an essential singularity at zero. Assume the only Picard point of π is infinity, and if infinity is a Picard point ϵ is small enough that π never takes the value infinity. Then π has a ramification point in $B_\epsilon(0) \setminus \{0\}$.*

Proof. Proceed by contradiction and assume $\pi : B_\epsilon(0) \setminus \{0\} \to C$ is an honest covering map where $C = \pi(B_\epsilon(0) \setminus \{0\})$ is either $\mathbb{C}$ or $\mathbb{C}_\infty$. As C is simply connected the monodromy group of π is trivial. Ergo, there exists a right inverse map (non-unique) $\pi^{-1} : C \to B_\epsilon(0) \setminus \{0\}$ so that $\pi \circ \pi^{-1} = \mathrm{id}_{B_\epsilon(0)\setminus\{0\}}$. If we define the image of π^{-1} to be B then the map $\pi' : C \to B$ with $\pi' \equiv \pi^{-1}$ is biholomorphic and, in particular, a homeomorphism so B is simply connected; therefore, ${\pi'}^{-1} : B \to C$ is the universal cover of C so there exists a covering map $\phi : B \to B_\epsilon(0) \setminus \{0\}$ such that $\pi \circ \phi = {\pi'}^{-1}$. As ${\pi'}^{-1}$ has the inverse π', it is bijective so ϕ must be injective. As ϕ is a covering map, it must be surjective so it is in fact a homeomorphism. Thus, $B_\epsilon(0) \setminus \{0\}$ is simply connected, an obvious contradiction. □

Thus, if we only have one (or zero) Picard points then, for any $\epsilon > 0$ sufficiently small, we get a ramification point in the disk of radius ϵ about the essential singularity so that the essential singularity must be an accumulation point of ramification points.

Noting that algebraic functions automatically have a discrete ramification locus we see transalgebraic functions as, in some sense, a natural generalisation of algebraic functions where instead of considering functions of finite order we consider functions of finite exponential order. This, however, tells us little about how fearsome transalgebraic functions encountered in the wild may be. May our low road end in some unruly jungle with no order? The following theorem from Pérez-Marco [2019a] tells us that these functions are quite shockingly tame by giving us a relatively simple explicit expression for them.

Theorem 3.A.4.

$$\mathcal{T}(\Sigma) = \{M_0 \exp(M_1) | M_0, M_1 \in K(\Sigma)\}. \tag{3.5}$$

From this we easily see that for s_0 a pole of M_1 we have the simple formula $\mathrm{Erd}_{M_0 \exp(M_1)}(s_0) = -\mathrm{Ord}_{M_1}(s_0)$. With this knowledge, we are now fully prepared to generalise the notion of a spectral curve to allow for transalgebraic functions.

Definition 3.A.5. A *spectral curve* is a triple $\mathcal{S} = (\Sigma, x, y)$ where Σ is a Torelli marked connected compact Riemann surface, x and y are in $\mathcal{T}(\Sigma)$, $xy \in K(\Sigma)$, and $y(z) \neq z_0$ for every $z \in x^{-1}(x(z_0))$ and $z_0 \in \Sigma \setminus R$ where R is the set of all ramification points (in the sense of the previous chapter) and essential singularities of x. The spectral curve is called *algebraic* if $x \in K(\Sigma)$ and *transalgebraic* otherwise.

Recalling that the topological recursion always produces meromorphic differentials, we see the requirement that $xy \in K(\Sigma)$ is so that $\omega_{0,1} = y\mathrm{d}x$ is meromorphic and, as we will see in the subsequent section, the topological recursion for transalgebraic spectral curves will still give meromorphic differentials on Σ. More practically, the proof of Theorem 3.B.2 requires this fact. Note that the above definition reduces to Definition 2.A.3 in the case where M_1 is constant, *i.e.*, $x \in K(\Sigma)$.

Now, as in the algebraic case, we must examine the structure of x as a branched covering from Σ to $\mathbb{C}_\infty$.[3] First, for any function $x \in \mathcal{T}(\Sigma) \setminus K(\Sigma)$ we define $\deg(x) = \infty$ and for any point z with $\mathrm{Erd}_x(z) > 0$ we define $\mathrm{Mult}_x(z) = +\infty$. Then, the definition of ramification points as a point with multiplicity greater than or equal to two extends in the obvious way to include exponential singularities as ramification points. Following Biswas and Pérez-Marco [2015a] we naturally split the ramification locus R of x into two sets by defining *infinite ramification points* as those with infinite multiplicity and *finite ramification points* as those with finite multiplicity; we denote the set of these as R_∞ and R_0, respectively, so that $R = R_\infty \sqcup R_0$.

[3]The concerned reader may wonder whether x even is a branched covering; see Biswas and Pérez-Marco [2015a] for a verification of this and details.

The local geometry around elements of R_0 remains unchanged from the algebraic case so to complete a local study of the transalgebraic geometry we therefore only have to study the local geometry around a generic infinite ramification point $a \in R_\infty$. Let $x = M_0 \exp(M_1)$, denote by m_0 the order of M_0 at a and denote by m_1 the multiplicity of M_1. Of course, the choice of M_0, M_1 is not unique; we can add to M_1 a holomorphic function h on Σ if we then multiply M_0 by $\exp(-h)$. It is important to note, however, that this non-uniqueness will not affect the values of m_0 and m_1 so these are, perforce, defined independently of the choice of M_0 and M_1. We can then define the natural local coordinate of x near a as a coordinate ζ such that near a

$$x(\zeta) = \zeta^{m_0} \mathrm{e}^{\zeta^{-m_1}}. \tag{3.6}$$

The existence of such a coordinate is assured as follows. If $m_0 = 0$, existence is clear as near a $\log(x)$ is a well defined meromorphic function of order $-m_1$ at a (fixing a branch of the logarithm) so there exists a local coordinate such that $\zeta^{-m_1} = \log(x(\zeta))$. If $m_0 \neq 0$ we may take z as the coordinate such that near a $M_0(z) = z^{m_0}$. The coordinate ζ, if it exists, can then be defined through the relation (up to branch choices)

$$M_1(z) - \zeta^{-m_1} - m_0 \log\left(\zeta/z\right) = 0, \tag{3.7}$$

where we are then able to sub in the ansatz $\zeta = \sum_{n \geq 1} a_n z^n$ with $a_1 \neq 0$ and recursively solve for the coefficients. This explicitly constructs ζ in terms of z and thereby guarantees the existence of such a ζ.

However, unlike in the case of finite ramification points we see that there are infinitely many different choices of ζ corresponding to the branch choices for the

m_1th root and the logarithm. It follows that our local deck transformation group about a will be of infinite order. Unlike the finite case, even in a local coordinate ζ, the local deck transformations have no simple expression in terms of elementary functions (except when $m_0 = 0$). In principle, it is possible to write them in terms of the different branches of the Lambert W function, but this approach appears to be mostly useless for us.

As an alternative, one can derive series expansions by subbing in an ansatz of the form

$$\sum_{n\geq 0} s_n \zeta^{nm_1+1}, \tag{3.8}$$

where s_0 is a m_1th root of unity and s_1 is $\log(s_0^{m_0})$ where different choices of the branch of the logarithm will yield different local deck transformations. It is a rather annoying feature of the geometry that the radius of convergence of these series will depend on the choice of logarithm; there is no open set on which all such expansions converge.[4]

As we trek along our low road it is hopefully becoming impressed upon the reader that the rewriting of the topological recursion at the end of the first chapter so as to remove the deck transformation group of x will be, in the transalgebraic case, a key simplification. There is, however, something quite nice we can say about the local deck transformation group of x at a and that is the asymptotic behaviour as the chosen branch of the logarithm becomes 'large' in some sense. To define this precisely recall that each local deck transformation is uniquely defined by the first

[4]This is obvious in the case $m_0 = 0$ where all local deck transformations take the form

$$\left(\zeta^{-m_1} + 2\pi \mathrm{i} k\right)^{-1/m_1}$$

for some integer k and choice of the m_1th root as the radius of convergence is clearly $|2\pi\mathrm{i}k|^{-1/m_1}$. That this holds in general can easily be seen through subbing in series expansions. For each choice of m_1th root of unity one will get two series with radius of convergence $|2\pi\mathrm{i}k|^{-1/m_1}$ for each $k \in \mathbb{Z}_{>0}$.

and second coefficient in its expansion about a (3.8). The first, which we denoted s_0, was just a m_1th root of unity. If we let θ be a primitive m_1th root of unity then we can index the choice of s_0 by $m = 0, 1, \dots, m_1 - 1$ where $s_0 = \theta^m$. The second coefficient was $\log(s_0^{m_0}) = \log(\theta^{mm_0})$. If we fix a choice of $\log$ with a branch cut chosen along an irrational angle in the complex plane (in particular, the cut must not 'cut out' any integer power of θ) then the choices of s_1 are in a one-to-one correspondence with the integers where $s_1 = \log(\theta^{mm_0}) - 2\pi\mathrm{i}k$ for $k \in \mathbb{Z}$. We denote the local deck transformation with first coefficient θ^m and second coefficient $\log(\theta^{mm_0}) + 2\pi\mathrm{i}k$ as $\sigma_a^{k,m}$. The following lemma, then, characterises the large k behaviour of these deck transformations.

Lemma 3.A.6.

$$\begin{aligned}
\sigma_a^{m,k}(\zeta) &= \theta^m (2\pi\mathrm{i}k)^{-1/m_1}\left(1 + \mathcal{O}\left(\frac{\log(|k|)}{k}\right)\right), \qquad |k| \to \infty, \\
\frac{\mathrm{d}\sigma_a^{m,k}}{\mathrm{d}\zeta}(\zeta) &= \theta^m \zeta^{-m_1-1}(2\pi\mathrm{i}k)^{-1/m_1-1}\left(1 + \mathcal{O}\left(\frac{\log(|k|)}{k}\right)\right), \qquad |k| \to \infty,
\end{aligned} \tag{3.9}$$

Proof. Taking the equation for $\sigma_a^{m,k}$ (we denote by Log the principle branch of the logarithm)

$$\zeta^{-m_1} + 2\pi\mathrm{i}k - \mathrm{Log}\left(\left(\frac{\sigma_a^{m,k}(\zeta)}{\theta^m\zeta}\right)^{m_0}\right) - \log(\theta^{mm_0}) = \sigma_a^{m,k}(\zeta)^{-m_1},$$

we can rearrange to find

$$\sigma_a^{m,k}(\zeta) = \theta\zeta k^{-1/m_1}\left(-\frac{1}{k}\mathrm{Log}\left(\left(\frac{\sigma_a^{m,k}(\zeta)}{\theta^m\zeta}\right)^{m_0}\right) - \frac{1}{k}\log(\theta^{mm_0}) + \frac{1}{k} + 2\pi\mathrm{i}\zeta^{m_1}\right)^{-1/m_1}.$$

Then, clearly, as $|k| \to \infty$ we have the leading order result $\sigma_a^{m,k}(\zeta) = \mathcal{O}(k^{-1/p_\infty})$;

subbing this in to the logarithm yields the estimate for the NLO term. Taking derivatives and using identical arguments gives the second line of the lemma. $\square$

Using the above result, it is easy to see that for a 1-form η that is holomorphic at all essential singularities of x we have that $x^*x_*\eta$ is well-defined in the sense that the sum over deck transformations is convergent. However, for the topological recursion, we want to look at forms with poles at the essential singularities. In this case the sum in $x^*x_*\eta$ is not absolutely convergent,[5] which is yet another reason to rewrite the topological recursion. Using the previous lemma, we can make the work of checking the $y(z) \neq z_0$ condition in Definition 3.A.5 more straightforward. This is the content of our next lemma.

Lemma 3.A.7. *Given a ramification point $a \in R$ and a non-trivial deck transformation $\sigma \in \boldsymbol{\sigma}'_a$ of x with infinite order (i.e. σ composed with itself n times will never yield the identity for any n) the only possible zero of $(xy)(\sigma(z)) - (xy)(z)$ is a or xy is a constant.*

Proof. Assume there exists $z_0 \neq a$ near a such that $(xy)(\sigma(z_0)) = (xy)(z_0)$. As σ has infinite order and z_0 is not a ramification point, $\sigma^{\circ n}(z_0) \neq z_0$ for every $n \in \mathbb{Z}_{\geq 1}$, but $(xy)(\sigma^{\circ n}(z_0)) = (xy)(z_0)$. The set $S = \{\sigma^{\circ n}(z) | n \in \mathbb{Z}_{\geq 0}\}$ is therefore an infinite set and has an accumulation point at a. As xy is continuous we see $(xy)(z) = (xy)(a)$ for all $z \in S$. If a is a pole of xy then clearly $xy \notin K(\Sigma)$, whereas if a is not a pole of xy we have $xy \equiv (xy)(a)$. $\square$

Recalling our discussion about deck transformations, and that they are fixed by the first two coefficients in their series, a finite order deck transformation can occur

[5]Curiously, if we define summation to be over the index m first, then sum over the sign of k if $k \neq 0$, and then finally sum from $k = 1, \ldots, \infty$ (at some point adding in the $k = 0$ term) it is straightforward to see the sum from $k = 1$ to $k = \infty$ *is absolutely convergent* if the 1-form η has, at each $a \in R_\infty$, a pole of order no more than $\mathrm{Erd}_x(a)$.

only if $m_0 = 0$. In this way, the $y(z) \neq z_0$ condition in Definition 3.A.5, although initially, perhaps, daunting, is in applications usually rather obvious and not worth mentioning.

Next, we would like to naturally extend notions like regularity, admissibility, and the topological recursion from algebraic spectral curves to transalgebraic spectral curves. A natural way to accomplish this is to take a sequence of algebraic spectral curves $S_N = (\Sigma, x_N, y_N)$ such that $x_N \to x$ and $y_N \to y$ where $S = (\Sigma, x, y)$ is a transalgebraic spectral curve. Explicitly, if we write $x = M_0 \exp(M_1), y = M_2/x$ (clearly, by Definition 3.A.5 $M_2 \in K(\Sigma)$) we will consider the approximations, fixing $\tau \in \mathbb{C}$,

$$x_N = M_0 \left(1 + (\tau - 1)\frac{M_1}{N}\right)^{-N} \left(1 + \tau\frac{M_1}{N}\right)^{N}, \; y_N = M_2/x_N, \tag{3.10}$$

which converge compactly to x and y, respectively, away from the poles of M_1. At this point, the reader is probably wondering about the introduction of τ; why would we consider a 1-parameter family of sequences rather than the clearly simpler option of choosing, say, $\tau = 0$? The answer is twofold. First, we will see in Theorem 3.B.2 that the limiting correlators do not depend on the choice of τ, which is, philosophically speaking, good evidence that the our definition of the transalgebraic topological recursion is indeed the correct one for the limiting curve and not an artefact of the particular sequence chosen. Second, there is a practical consideration involved. When we eventually construct quantum curves we will see that we get different a quantum curve for each choice of τ. However, in both the r-spin Hurwitz case and the case of the Gromov-Witten invariants of the projective line, we will see that this dependence can be naturally transformed away.

For these curves S_N we divide the ramification points of x_N, denoted collectively as R^N, into two sets of ramification points $R^N_\infty = \{M_1 = -N/\tau\} \cup \{M_1 = N/(1-\tau)\} \cup \{M_1 = \infty\}$ consisting of the ramification points colliding at essential singularities of x_N and $R^N_0 = R^N \setminus R^N_\infty$ consisting of those ramification points not colliding at essential singularities of x_N. The notion of admissibility of the ramification points in R_0 is clear; it should be the same as in Definition 2.A.9. At the elements of R_∞ we quite obviously need a new definition based on the notion of admissibility for the elements of R^N_∞.

We distinguish two distinct cases for a ramification point $a \in R_\infty$: first, when $M_2 = xy$ has a pole at a; second, when $M_2 = xy$ does not have a pole at a. In the first case, for finite N, it is easy to see via pole counting arguments that the ramification point at a will not contribute to the topological recursion. For sufficiently large N, M_2 will have multiplicity one at the solutions of $M_1 = N$ and it is then obvious that S^N is admissible at these points. Ergo, for this case, the admissibility condition should be nothing beyond requiring xy has a pole at a. There appears to be significant challenges in defining the topological recursion in the second case and as it appears in no applications of signifiant interest, it is not done here. However, from the admissibility requirement at the pole at a for finite N, it is somewhat straightforward to see that in the limit we should have the requirement $\pm 1 \bmod s_a = m_{0,a} \bmod m_{1,a}$ with $m_{0,a} = \mathrm{Ord}_{M_0}(a)$ and $m_{1,a} = \mathrm{Mult}_{M_1}(a)$, $m_{0,a} = \mathrm{Ord}_{M_0}(a)$, which naturally reduces to Definition 2.A.9 when $m_1 = 0$. Given this caveat, for us, admissibility at infinite ramification points will be defined as follows.

Definition 3.A.8. Given an infinite ramification point $a \in R_\infty$ we say a spectral curve $\mathcal{S} = (\Sigma, x, y)$ is *admissible* at a if the meromorphic function xy has a pole at a.

Next, we turn our attention to the notion of regularity. As was mentioned previously, in Bouchard and Eynard [2017] the authors classified all algebraic regular spectral curves $P(x, y) = 0$ as follows.

- Linear in x, *i.e.*, $P(x, y) = xE_1(y) - E_2(y)$ where E_1, E_2 are polynomials.
- Has Newton polygon Δ given by the convex hull of $\{(0, 0), (0, 2), (2, 0)\}$.
- Is obtained from one of the two previous cases via a transformation $(x, y) \to (x^a y^b, x^c y^d)$ with $ad - bc = 1$ and rescaling by powers of x and y to get an irreducible polynomial equation.

In the case of transalgebraic spectral curves we must replace the polynomial P by an entire function in two variables. The Taylor expansion of P will therefore have infinitely many terms; clearly, in the case when Δ is given by the convex hull of $\{(0, 0), (0, 2), (2, 0)\}$ (up to transformation and rescaling) P can have at most six terms so we can eliminate this option in the transalgebraic case. Thus, up to rescaling, a transalgebraic regular spectral curve should take the form

$$x^a y^b E_1(x^c y^d) - E_2(x^c y^d) = 0,$$

where $ad - cb = 1$, E_1, E_2 are entire functions of finite order[6], and at least one of E_1, E_2 is transcendental[7]. However, we want a curve that results in functions $x, y \in \mathcal{T}(\mathbb{C}_\infty)$ (in particular, no exponentials of exponentials allowed) with $xy \in K(\mathbb{C}_\infty)$. As such, the only combination of x and y that should appear with arbitrarily high powers should be xy. Thus, the argument of our entire functions E_1, E_2 must be xy

[6]The order of an entire function E is defined as the infimum over all positive numbers $d \in \mathbb{R}_{>0}$ such that $E(z) = \mathcal{O}\left(\exp(|z|^d)\right),\ z \to \infty$. If no such positive integers d exist, then the order of E is defined to be infinite. Clearly, if $E \in \mathcal{T}(\mathbb{C}_\infty)$, then the order is given by $\mathrm{Erd}_E(\infty)$.

[7]An entire function is transcendental if it is not a polynomial; equivalently, it is transcendental if it has non-zero order.

so we get $c = d = 1$ and $a = b + 1$. Therefore, up to rescaling to get an irreducible entire equation in x and y, we have

$$x(xy)^b E_1(xy) - E_2(xy) = 0.$$

We then absorb the factor of $(xy)^b$ into E_1 to get $x = E_1(xy)/E_2(xy)$. Now, x and xy can not share any nontrivial symmetries. Since x should be a function of xy this means xy should have no non-trivial symmetry. Ergo, we can choose a global coordinate z on $\mathbb{C}_\infty$ such that $xy = z$. This motivates the following definition.

Definition 3.A.9. A transalgebraic spectral curve is regular if and only if it is genus zero and $xy \in \mathrm{Aut}(\mathbb{CP}^1 \cong \mathbb{C}_\infty)$ is a Möbius transformation.

Thus, a regular transalgebraic spectral curve is of the form $x = x(xy)$ (our abuse of notation in denoting by x both the projection onto the coordinate x and the coordinate itself is a bit awkward here) for $x \in \mathcal{T}(\mathbb{C}_\infty) \cap K(\mathbb{C})$ where infinity is placed at the pole of xy. It is then clear if we introduce a sequence of algebraic curves as $x_N = x_N(x_N y_N)$ with the sequences x_N, y_N defined as before, this will be a sequence of regular algebraic spectral curves. That we have such a sequence of algebraic regular spectral curves will be of near paramount importance when we construct quantum curves from transalgebraic spectral curves.

3.B The Topological Recursion for Transalgebraic Spectral Curves

Now that we have finished our walk along the low road of the previous section, we can present the definition of the topological recursion for transalgebraic spectral curves straight away, and then spend the rest of the section arguing and demonstrating why it works.

Definition 3.B.1. Given a transalgebraic admissible spectral curve $\mathcal{S} = (\Sigma, x, y)$ with $x = M_0 \exp(M_1)$ and $y = M_2/x$ with $M_0, M_1, M_2 \in K(\Sigma)$ fix $\tau \in \mathbb{C}$ and define the sequence of spectral curves $\mathcal{S} = (\Sigma, x_N, y_N)$ where

$$x_N = M_0 \left(1 + (\tau - 1)\frac{M_1}{N}\right)^{-N} \left(1 + \tau\frac{M_1}{N}\right)^{N}, \; y_N = M_2/x_N.$$

Then, if $\omega_{g,n}^N$ are the correlators constructed from the spectral curve $\mathcal{S}_N$ we define the correlators of the spectral curve $\mathcal{S}$ as the $N \to \infty$ limit. This topological recursion shall be referred to as the *transalgebraic topological recursion* in contrast to the topological recursion of Definition 2.B.3, which will be referred to as the *algebraic topological recursion*.

Obviously, this defines nothing if the limit depends on τ or does not yield well-defined meromorphic correlators. Allaying these fears is the content of the next theorem, which will unfortunately take some work to prove.

Theorem 3.B.2. *Let $\mathcal{S} = (\Sigma, x, y)$ be a transalgebraic spectral curve. Then the $\omega_{g,n}$ constructed from Definition 3.B.1 are well defined meromorphic n-differentials and do not depend on the choice of τ.*

Proof. Our strategy will be to first prove the that $\omega_{g,n}$ are well-defined for $\tau = 0$, and then show that the limit is independent of τ.

Proceeding inductively in the $\tau = 0$ case on $-\chi_{g,n} = 2g+n-2$, and noting that the induction beginning (corresponding to $\omega_{0,1}$ and $\omega_{0,2}$) the result holds trivially, we fast-forward directly to the induction step. For finite N use Theorem 2.C.1 to write

$$\omega^N_{g,n}(z_1, B) = \operatorname*{Res}_{t=R_N} \sum_{i=2}^{\deg(x_N)} S^t_*(z_1) \sum_{C_1,\ldots,C_j \vdash \{t_1,\ldots,t_{i-1}\}} \frac{(-1)^{1-\delta_{j,i-1}}}{j!} \operatorname*{Res}_{\substack{t_{C_l}=R^N,B,\mathcal{Y}(t)\\ l=1,\ldots,j}} \operatorname*{Res}_{\substack{C_l=t_{C_l}\\ l=1,\ldots,j}}$$

$$\left(\prod_{l=1}^{j} \frac{x_N(t)}{x_N(t)-x_N(t_{C_l})} \prod_{t_0\in C_l\setminus\{t_{C_l}\}} \frac{x_N(t_{C_l})}{x_N(t_0)-x_N(t_{C_l})}\right) \frac{\Omega^{i,N}_{g,n}(t,t_1,\ldots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(M_2(t)-M_2(t_l))}, \tag{3.1}$$

where $\mathcal{Y}(t) = (xy)^{-1}\big((xy)(\{t\})\big)$. This isn't a copy and paste of the result of Theorem 2.C.1 so a couple of remarks are in order so it is clear how we get here:

- in the denominator of the integrand in the original TR we rewrote $y_N(t) - y_N(\sigma(t)) = (M_2(t) - M_2(\sigma(t)))/x_N(t)$ so we ended up with the $M_2 = xy = x_N y_N$ in the denominator and the $x_N(t)$ in the numerator;

- as $x_N(t) = x_N(\sigma(t))$ for every deck transformation, we can choose which deck transformation we take the argument of x_N to be at; in particular, we take j of them to just be t and the other $i-1-j$ to be precisely those deck transformations that gives us t_{C_l};

- when we flipped the contour we then had to pick out residues at $\mathcal{Y}(t)$ rather than $Y(t)$.

The residues at $t_{C_l} = R^N_\infty$ actually vanish for sufficiently large N as the following argument shows. For the points in R^N_∞ that satisfy $M_1 = N$ we have that $x_N(t_{C_l})$ has a pole of order N. $x_N(t_{C_l})$ appears in the denominator one time with no corresponding $x_N(t_{C_l})$ in the numerator to give the overall integrand in the variable t_{C_l} a zero of order N. We then claim the rest of the integrand has, at worst, a pole of uniformly bounded order in N. At these ramification points $\omega_{0,1}^N$ has a simple pole; it is known that in this case the $\omega_{g,n}^N$ have poles of order no more than $2g$ [Do and Norbury, 2018, 2016] and so the $\Omega_{g,n}^{i,N}$ certainly won't have a pole of unbounded order in N. Similarly, M_2 is meromorphic and constant in N. For the $x_N(t_{C_l})/(x_N(t_0) - x_N(t_{C_l}))$ the x_N appears in both the denominator and the numerator. Finally, we need to examine the residues at $C_l = t_{C_l}$. This will be a residue of a pole of order no more than three (two from a potential $\omega_{0,2}$ contribution plus one for the difference of the x_N in the denominator). Thus, this residue may be replaced by multiplication by $(t_0 - t_{C_l})^3$ and twice differentiating by t_0, for each $t_0 \in C_l \setminus \{t_{C_l}\}$, before taking the limit as $t_{C_l} \to t_0$. By the quotient rule for differentiation we will have the same total power of derivatives of x_N in the numerator and denominator, just in different combinations and orders of differentiation. Thus, at the residues at points where $M_1 = N$ we may drop the residue in t_{C_l}.

Now we examine the residues in t_{C_l} where the point at which the residue is taken satisfies $M_1 = \infty$. Here, when we take the residue at $C_l = t_{C_l}$, as discussed previously, this ultimately corresponds to derivatives. However, the pole counting is now a little more subtle so we do it explicitly. In particular, observe[8]

$$\begin{aligned}\frac{x_N(t_{C_l})\omega_{0,2}(t_0,t_{C_l})(t_0-t_{C_l})^3}{x_N(t_0)-x_N(t_{C_l})} &= \frac{x_N(t_{C_l})}{x'_N(t_{C_l})} - (t_0-t_{C_l})\frac{x_N(t_{C_l})x''_N(t_{C_l})}{x'_N(t_{C_l})^2} \\ &+ (t_0-t_{C_l})^2\left(\frac{x_N(t_{C_l})x''(t_{C_l})^2}{4x'_N(t_{C_l})^3} - \frac{x_N(t_{C_l})x'''_N(t_{C_l})}{x'_N(t_{C_l})^2}\right) + \mathcal{O}\left((t_0-t_{C_l})^3\right). \quad (3.2)\end{aligned}$$

In the constant term and the $t_0 - t_{C_l}$ term there is no pole at t_{C_l} equalling a pole of M_1. However, the $(t_0 - t_{C_l})^2$ has a simple pole here. On the other hand, the $\omega_{g,n}^N$ are

[8]We are technically using t to denote points on Σ and not as a coordinate. Thus the derivatives in the following should be understood to mean derivatives with respect to a chosen coordinate. This abuse of notation does not matter, as the particular coordinate that is chosen is irrelevant to the argument.

regular at these points (this is clear from pole-counting the normal, non-rewritten topological recursion, Definition 2.B.3) and $M_2(t_0)$, which has at least a simple pole by admissibility, appears in the denominator. Thus, in terms with an $\omega_{0,2}(t_0, t_{C_l})$ we don't have contributions from these points. For terms without this factor, the pole at $t_0 = t_{C_l}$ is simple. Ergo, observing

$$\frac{x_N(t_{C_l})(t_0 - t_{C_l})}{x_N(t_0) - x_N(t_{C_l})} = \frac{x_N(t_{C_l})}{x'_N(t_{C_l})} + \mathcal{O}\left(t_0 - t_{C_l}\right), \tag{3.3}$$

we see the exact same argument still holds. In summary, we may replace the residues in each t_{C_l} at all the points in R_N with just those at R_N^0.

First we note this argument shows that the integrand in t is well defined in the limit. We may simply commute the limit in $N \to \infty$ with the residues (integrals) in the $t_0 \in C_l$ and t_{C_l} using dominated convergence before using the induction assumption that $\omega_{g,n}^N \to \omega_{g,n}$. However, we of course want the integral, not just the integrand, to be well-defined in the limit. To this end, we note the contributions from the residues at $t = R_0^N$ go to the contributions at $t = R_0$ in the limit simply by pulling the limit in N inside each integral using dominated convergence, as before, and applying the induction assumption. This simplistic argument will not, however, work for the residues at $t = R_\infty$ as these points are not isolated singularities in the limit.

To deduce that the actual integral must be well-defined in the limit, we will pushforward to work in the M_1 plane where all elements of R_N^∞ fall at either $M_1 = N, \infty$. To move the action to the origin, let $w = 1/M_1$ be a coordinate in the M_1-plane. For a deck transformation σ of M_2, *i.e.*, $\sigma(t) \in \mathcal{Y}(t)$, we define a corresponding transformation ν through $\nu(w) = \nu(1/M_1(t)) = 1/M_1(\sigma(t))$. One should note that the value of individual ν may depend on where we are in the t-plane which initially may seem to render them totally ill-defined; however, we sum over ν at every step and so avoid this issue. In performing this sum, as from now on we will suppress this detail, it is advisable to realise that the sum over ν must include both the sum over σ (from all elements of $\mathcal{Y}(t)$) and partial inverse of M_1 (from the fact we pushforward in M_1). Let $\nu_1, \ldots, \nu_r$ be all such non-trivial ν and note, for general N, $\nu_p(w = 1/N) \neq 1/N$ for all $p = 1, \ldots, r$. We claim the integrand of the topological recursion, pushed forward under M_1 and then written in the coordinate w takes the following form (where $\exp_N(z) = (1 - z/N)^{-N}$)

$$\frac{\mathcal{N}_N^d(w|z_1, B)\exp_N(w^{-1})^d + \mathcal{N}_N^{d-1}(w|z_1, B)\exp_N(w^{-1})^{d-1} + \cdots + \mathcal{N}_N^0(w|z_1, B)}{\mathcal{D}_N^d(w|z_1, B)\exp_N(w^{-1})^d + \mathcal{D}_N^{d-1}(w|z_1, B)\exp_N(w^{-1})^{d-1} + \cdots + \mathcal{D}_0^N(w|z_1, B)}, \tag{3.4}$$

where each of the $\mathcal{N}_N^k$ and $\mathcal{D}_N^k$ are meromorphic functions[9] with the order of their zeros and poles at of $w = 1/N$ bounded uniformly in N and it is assumed $\mathcal{D}_N^d(w|B)$ is not identically zero. To get this expansion first note we can write every derivative of x_N as $\exp_N(M_1) = \exp_N(1/w)$ times a sequence of meromorphic functions with the order of their poles and zeros bounded uniformly in N. Then, when we take the residues at $t_0 = t_{C_l}$ we get expansions like (3.2) and (3.3) which we may put over a common denominator. Next, when we take the residues at $t_{C_l} = R_N^0, B, \mathcal{Y}(t)$ we can, if necessary (such as the residue at $t_{C_l} = t$), do similar expansions. However, the important thing to note is we always, for each term, get the same leading power of x_N in the numerator and denominator. Thus, when we pushforward and put everything over a common denominator, we get an expansion of the form (3.4). Now, such an expansion is self-evidently non-unique, but by the above construction we can take the coefficients $\mathcal{N}_N^k$ and $\mathcal{D}_N^k$ to be well-defined in the limit, although

[9]Due to the presence of $S_*^t(z_1)$ in the integrand, this isn't strictly true for non-zero genus. However, all we want to do in the end is integrate, so the reader is encouraged to forgive our slight misuse of terminology.

they may have essential singularities due to the presence of $\exp(1/\nu_p(w))$ factors and therefore not be meromorphic in the limit. In particular, we will be interested in the ratio $\mathcal{N}_N^d(w|B)/\mathcal{D}_N^d(w|B)$, so the non-uniqueness will not matter, as this ratio is unique.

We now need to verify two important properties of $\mathcal{N}_N^d(w|B)/\mathcal{D}_N^d(w|B)$. First, we claim it has no pole at $w = 0$. Second, we claim that it does not have essential singularity in the limit and its only potential pole near $w = 0$ is the one at $w = 1/N$. Note the only place an essential singularity could come from are the $\exp_N(1/\nu_p(w))$.

The proof of these claims is a bit more involved. Our strategy will be to examine the individual terms that we put over a common denominator before pushing forward to the w-plane; by examining the ratio of the coefficient of the highest power of x_N in the numerator, to the one in the denominator, we can deduce the behaviour of the coefficients in the fraction put over a common denominator (note that, before pushing forward, x_N is $\exp_N(M_1)$ times a meromorphic function M_0, so looking at leading powers of x_N is the same as looking at leading powers of $\exp_N(M_1)$). The lack of poles at $t = R_\infty$ will give us the lack of a pole at $w = 0$, and the lack of $\exp_N(M_1(\sigma(t)))$ factors in the leading order, will give us the no essential singularities result. To this end we first examine the contributions

$$\operatorname*{Res}_{\substack{t_{C_l}=R_N^0,B \\ l=1,\dots,j}} \operatorname*{Res}_{\substack{C_l=t_{C_l} \\ l=1,\dots,j}} \left(\prod_{l=1}^{j} \frac{x_N(t)}{x_N(t)-x_N(t_{C_l})} \prod_{t_0\in C_l\setminus\{t_{C_l}\}} \frac{x_N(t_{C_l})}{x_N(t_0)-x_N(t_{C_l})} \right) \frac{\Omega_{g,n}^{i,N}(t,t_1,\dots,t_{i-1}|B)}{\prod_{l=1}^{i-1}(M_2(t)-M_2(t_l))}. \tag{3.5}$$

Here, the residues in the t_{C_l} are taken at points that do not depend on t. Thus, due to the pole of M_2 at the poles of M_1 guaranteed by admissibility, these residues will all contribute some sort of zero to the ratio of leading order coefficients in powers of x_N. Obviously, these will never contribute $\exp_N(M_1(\sigma(t)))$ factors.

More involved are the contributions from the residues at $\mathcal{Y}(t)$. Here, we divide these into three sub-cases: the first, where we take the residues at $\sigma(t) \in \mathcal{Y}'(t)$ that do not preserve M_1, and it is these that correspond to the $\nu_1, \ldots, \nu_r$ where the potential dangers of essential singularities in the limit of $\mathcal{N}_N^d(w|B)/\mathcal{D}_N^d(w|B)$ lurk; the second, where we take the residues at $\sigma(t) \in \mathcal{Y}'(t)$ that preserve M_1, *i.e.*, $M_1 \circ \sigma = M_1$; the third, where we take the residue at t itself. Starting with the first case take an element $\sigma(t) \in \mathcal{Y}'(t)$, fix an l, and note

$$\operatorname*{Res}_{t_{C_l}=\sigma(t)} \operatorname*{Res}_{C_l=t_{C_l}} \left(\frac{x_N(t)}{x_N(t) - x_N(t_{C_l})} \prod_{t_0 \in C_l \setminus \{t_{C_l}\}} \frac{x_N(t_{C_l})}{x_N(t_0) - x_N(t_{C_l})} \right) \frac{\Omega_{g,n}^{i,N}(t, t_1, \ldots, t_{i-1}|B)}{\prod_{t_0 \in C_l}(M_2(t) - M_2(t_0))}. \tag{3.6}$$

After performing the residues $C_l = t_{C_l}$ we will be left with an expression of the form

$$\operatorname*{Res}_{t_{C_l}=\sigma(t)} \frac{x_N(t)}{x_N(t) - x_N(t_{C_l})} \frac{f_N(t, t_{C_l}, \{t_1, \ldots, t_{i-1}\} \setminus C_l|B)}{(M_2(t) - M_2(t_{C_l}))^{\sharp C_l}}, \tag{3.7}$$

where f_N is a differential in all its variables except t. Furthermore, f_N is meromorphic in t and t_{C_l} and remains so in the limit (note in (3.2) and (3.3) the derivatives in x_N appear in the same power in the numerator and denominator so we may cancel out the factor of $\exp_N(M_1)$), and there is no pole in t or t_{C_l} at the poles of M_1 (we established this before to show the residues at $t_{C_l} = R_\infty$ do not contribute). Furthermore, there is no way f_N has a pole at $t_{C_l} = \sigma(t)$.

Now, clearly, we have a pole of order no more than $\sharp C_l$. We can therefore calculate the residue with the formula

$$\lim_{t_{C_l}\to\sigma(t)} \frac{(-1)^{\sharp C_l}}{\sharp C_l!} \frac{\mathrm{d}^{\sharp C_l - 1}}{\mathrm{d}t_{C_l}^{\sharp C_l - 1}} \frac{x_N(t) f_N(t, t_{C_l}, \{t_1, \dots, t_{i-1}\} \setminus C_l | B)}{x_N(t) - x_N(t_{C_l})} \left(\frac{t_{C_l}\sigma(t)}{M_2(t_{C_l}) - M_2(t)} \right)^{\sharp C_l}. \tag{3.8}$$

If we take the derivatives of the $1/(x_N(t) - x_N(t_{C_l}))$ we will end up with subleading terms in powers of x_N; these do not concern our analysis. All we care about when we take derivatives of f, is that derivatives can't create poles. Finally, we have the expansion

$$\left(\frac{t_{C_l}\sigma(t)}{M_2(t_{C_l}) - M_2(\sigma(t))} \right)^{\sharp C_l} = \frac{1}{M_2'(\sigma(t))^{\sharp C_l}} \sum_{k=0}^{\infty} H_k(\sigma(t))(t_{C_l} - \sigma(t))^k, \tag{3.9}$$

where H_k has a pole of order at most k at elements of R_∞ (poles of M_1). The pre-factor has a zero of at least order $\sharp C_l$ (as M_2 has a pole at all elements of R_∞), and the only H_k that can contribute are those with $k \leq \sharp C_l$. Thus, for the leading terms in x_N, we will never get poles. Furthermore, from the above discussion, it is clear the $\exp_N(1/\nu_p(w))$ will never enter the leading order power in $\exp_N(1/w)$.

Now, we move on to the deck transformations that preserve M_1; let $\sigma(t)$ be such a deck transformation and examine the expression

$$\lim_{t_{C_l}\to\sigma(t)} \frac{(-1)^{\sharp C_l}}{\sharp C_l!} \frac{\mathrm{d}^{\sharp C_l - 1}}{\mathrm{d}t_{C_l}^{\sharp C_l - 1}} \frac{x_N(t) f_N(t, t_{C_l}, \{t_1, \dots, t_{i-1}\} \setminus C_l | B)}{x_N(t) - x_N(t_{C_l})} \left(\frac{t_{C_l}\sigma(t)}{M_2(t_{C_l}) - M_2(t)} \right)^{\sharp C_l}, \tag{3.10}$$

which is the same as the prior case as nothing changes prior to the examination of this expression. The thing that does change upon analysing this expression is that when we take derivatives of the $x_N(t)/(x_N(t) - x_N(t_{C_l}))$ factor we don't end up with only subleading terms as $x_N(\sigma(t))$ now has a factor of $\exp(M_1(t))$. Instead, when we take k derivatives of this factor we get a pole of order at most k in the ratio of the coefficients of the leading powers of x_N. Thus, we still can't get a pole as we have the factor of $M_2'(\sigma(t))^{-\sharp C_l}$, as before.

Finally, we examine the case where we take the residue at $t_{C_l} = t$

$$\operatorname*{Res}_{t_{C_l}=t} \frac{x_N(t)}{x_N(t) - x_N(t_{C_l})} \frac{f_N(t, t_{C_l}, \{t_1, \dots, t_{i-1}\} \setminus C_l | B)}{(M_2(t) - M_2(t_{C_l}))^{\sharp C_l}}. \tag{3.11}$$

Here, we may have a pole of order at most $\sharp C_l + 3$. In particular, we get a pole of order one from the $s_N(t)/(x_N(t) - x_N(t_{C_l}))$ factor, a pole of order $\sharp C_l$ from the difference of the M_2 in the denominator, and a potential double pole in f_N at $t_{C_l} = t$ due to possible presence of an $\omega_{0,2}(t, t_{C_l})$. Here, however, the $M_2'(t)^{-\sharp C_l}$ has at least a zero of order $2\sharp C_l$. Using identical arguments with the expansions of the individual factors it is then clear that this case will not create an undesired pole in $\mathcal{N}_N^d(w|z_1, B)/\mathcal{D}_N^d(w|z_1, B)$.

With these properties established it is easy to prove the $\omega_{g,n}^N$ well-defined in the limit. First note

$$\begin{aligned}\operatorname*{Res}_{w=1/N} \frac{\mathcal{N}_N^d(w|z_1, B)\exp_N(1/w)^d + \dots}{\mathcal{D}_N^d(w|z_1, B)\exp_N(1/w)^d + \dots} &= \operatorname*{Res}_{w=1/N} \frac{\mathcal{N}_N^d(w|z_1, B)}{\mathcal{D}_N^d(w|z_1, B)} \left(1 + \mathcal{O}(w - 1/N)^N\right) \\ &= \operatorname*{Res}_{w=1/N} \frac{\mathcal{N}_N^d(w|z_1, B)}{\mathcal{D}_N^d(w|z_1, B)}.\end{aligned} \tag{3.12}$$

As we have established $\mathcal{N}_N^d(w|z_1, B)/\mathcal{D}_N^d(w|z_1, B)$ has no pole at $w = 0$ we may change the residue at $w = 1/N$ to a contour integral about a small circle around $w = 0$. Then, using dominated convergence to bring the limit as $N \to \infty$ inside the contour we conclude

$$\lim_{N\to\infty} \operatorname*{Res}_{w=1/N} \frac{\mathcal{N}_N^d(w|z_1,B)\exp_N(1/w)^d + \dots}{\mathcal{D}_N^d(w|z_1,B)\exp_N(1/w)^d + \dots} = \operatorname*{Res}_{w=0} \frac{\mathcal{N}_\infty^d(w|z_1,B)}{\mathcal{D}_\infty^d(w|z_1,B)}. \tag{3.13}$$

Finally, we establish that all choices of τ yield the same result in the $N \to \infty$ limit as the $\tau = 0$ case. To this end we proceed inductively on $-\chi_{g,n} = 2g + n - 2$ with the induction assumption that $\partial_\tau^m \omega_{g,n}^N(z_1,\dots,z_n)|_{\tau=0}$ exists and goes to zero for every value of $m \in \mathbb{Z}_{\geq 1}$ as $N \to \infty$. Before proving this, we pause briefly to note that this result straightforwardly establishes the theorem since we have the expansion for sufficiently small τ and generic choices of $z_1,\dots,z_n \in \Sigma$

$$\omega_{g,n}^N(z_1,\dots,z_n) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} \partial_\tau^m \omega_{g,n}^N(z_1,\dots,z_n)|_{\tau=0}, \tag{3.14}$$

and may take the limit as $N \to \infty$ inside the sum as the $\omega_{g,n}$ may only have singularities at ramification points, wherefore the radius of convergence remains non-zero in the limit.

Ergo, we turn our cerebration to the induction proof first noting that the result is straightforward for $\omega_{0,1}^N$ and entirely trivial for $\omega_{0,2}$ so we may proceed directly to the induction step and assume the result holds for all prior correlators. We first argue that we may commute derivatives in τ with all residues in (3.1). To do this, we transform all residues into contour integrals; even if the point at which the residue is being taken depends on τ, the contour may be taken to be locally constant in τ. Then we may commute the derivatives in τ with the τ-independent contour integrals and, as the derivatives in τ can not create new poles, we may switch all contour integrals back to the same residues.

As discussed previously, the residues at $t_{C_l} = R_\infty^N$ do not contribute, and so we may conclude the derivatives in τ of the integrand in t go to zero as $N \to \infty$. Therefore, as before, we must now check that the same result holds for the integral and not just the integrand. Again, as before, for the residues at $t = R_0^N$ this result is clear as we may merely take the $N \to \infty$ limit inside the contour integral so we concentrate our thoughts on the residues at $t = R_\infty^N$. Here, for generic τ, we will have residues at solutions of $M_1(z) = N/(1-\tau), -N/\tau$. When we set $\tau = 0$, we end up with no poles at $M_1(z) = \infty$, even after taking derivatives, as we can't create poles by taking derivatives. For our purposes, we may therefore neglect the residues at $M_1(z) = -N/\tau$, as they will drop out in the end.

Thus, at these points, we need to take τ derivatives of the analogous expression to (3.4) where the $\mathcal{N}$ and $\mathcal{D}$ coefficients acquire the suitable τ dependence and $\exp_N(w^{-1}) = (1+(\tau-1)/(Nw))^{-N}(1+\tau/(Nw))^N$ is appropriately modified[10]. After taking m τ derivatives, the ratio of the new leading order coefficients won't have a pole at $w = \tau = 0$ by the quotient rule and the fact that the τ derivative can only decrease the order of poles at $w = 0$. We may then conclude that the same argument with the $N \to \infty$ limit holds, but this time the ratio of the leading order coefficients must go to zero, as, recalling the manner in which we put everything over a common denominator, the whole integrand is converging to zero. This proves the theorem.

$\square$

First we state an obvious corollary of the fact that our definition of the transalgebraic topological recursion is based on the limit of the algebraic topological recursion.

Corollary 3.B.3. *The symmetry, pole structure, residueless, homogeneity, nor-*

[10] τ derivatives obviously commute with the pushforward in M_1.

malised, diagrammatic, and modularity properties listed in Theorem 2.B.6 carry over to the case of transalgebraic topological recursion.

Proof. The claimed properties obviously carry over as they hold for each element of our sequence of spectral curves and there is no issue with taking the limit. □

Now, we provide a bound on the order of the poles of the correlators at the infinite ramification points in certain cases.

Proposition 3.B.4. *Given an infinite ramification point $a \in R_\infty$ let m_1 denote the multiplicity of M_1 and $\zeta^{m_1} = M_1$ be a local coordinate on a punctured disk about a (note: $\zeta(a) = \infty$). If locally near a we have that $M_0(\zeta) = \zeta^{d_0} f_0(\zeta^{m_1})$ and $M_2(\zeta) = \zeta^{d_2} f_2(\zeta^{m_1})$ for some integers $d_0, d_2 \in \mathbb{Z}$ and f_0, f_2 meromorphic functions at a, then the $\omega_{g,n}$ have poles of order no greater than m_1 at a provided $m_1 \geq 2$.*

Proof. Let θ be a primitive m_1th root of unity and take the sequence of spectral curves $\mathcal{S}_N$ as before (choose $\tau = 0$ for simplicity, as the result will be τ independent). Note that, under the rescaling $\zeta \to \theta^m \zeta$ we have that $\omega^N_{0,1} \to \theta^{md_0 d_2} \omega^N_{0,1}$. Therefore, by the homogeneity of the topological recursion, the simultaneous coordinate change $\zeta \to \theta^m \zeta$ in all variables is equivalent to the rescaling $\omega^N_{g,n} \to \theta^{(2-2g-n)md_0 d_2} \omega^N_{g,n}$ where $B \subset \Sigma$. Thus, letting B be a set of $n-1$ points, the principle part of $\omega^N_{g,n}(\zeta, B)$ at the solutions of $M_1 = N$ that are near a should read

$$\sum_{l=2}^{2g} \sum_{m=1}^{m_1} \frac{w^N_{m,l}(B) \mathrm{d}\log(\zeta)}{(\zeta - \theta^m N^{1/m_1})^l}. \tag{3.15}$$

If we re-scale the coordinates $\zeta \to \theta^{m'} \zeta$ and use homogeneity we get the following equality

$$\sum_{l=2}^{2g} \sum_{m=1}^{m_1} \frac{w^N_{m,l}(\theta^{m'} B) \mathrm{d}\log(\zeta)}{(\theta^{m'} \zeta - \theta^m N^{1/m_1})^l} = \theta^{(2-2g-n)m' d_0 d_2} \sum_{l=2}^{2g} \sum_{m=1}^{m_1} \frac{w^N_{m,l}(B) \mathrm{d}\log(\zeta)}{(\zeta - \theta^m N^{1/m_1})^l}, \tag{3.16}$$

so we find $w^N_{m,l}(\theta^{m'}B) = \theta^{m'l}w^N_{m-m',l}(B)$ where the subtraction $m - m'$ is taken modulo m_1 so the answer is between one and m_1. Thus, if we put everything on a common denominator we obtain

$$\sum_{l=2}^{2g}\sum_{m=1}^{m_1}\frac{w^N_{m,l}(B)\mathrm{d}\log(\zeta)}{(\zeta-\theta^m N^{1/m_1})^l} = \sum_{l=2}^{2g}\frac{\zeta^{(2-2g-n)d_0d_2-l}W^N_l(\zeta^{m_1};B)\mathrm{d}\log(\zeta)}{(\zeta^{m_1}-N)^l}, \tag{3.17}$$

where the $W^N_l(\zeta^{m_1};B)$ are polynomials in $\zeta^{\pm m_1}$ and the $\zeta^{(2-2g-n)d_0d_2-l}$ pre-factor ensures the correct scaling property. The terms that will survive in the $N\to\infty$ limit of the $W^N_l(\zeta^{m_1};B)$ are the ones with the highest powers of N. However, this term will be the surviving term with the lowest power of ζ^{m_1}, as when we put things over a common denominator to get the RHS of the above expression, the term with the most powers of N will have the fewest powers of ζ. Thus, in each term in the sum over l, the leading order expression in N must have a power of ζ that is strictly less than m_1. This proves the claim. $\square$

The following conjecture is the natural extension of the previous proposition to $m_1 = 1$, given the fact that the $\omega_{g,n}$ are residueless.

Conjecture 3.B.5. *At points where M_1 has simple poles, the $\omega_{g,n}$ are regular for $2g+n-2\geq 1$.*

This conjecture will be proven in Bouchard et al. [2022]. Finally, we give a formula for the topological recursion, in a wide variety of cases, without using a sequence of correlators and taking limits.

Lemma 3.B.6. *When M_1 is a well-defined function of M_2, given a set $B\subset\Sigma$ of cardinality $n-1$, then we may use the following formula to recursively compute the* *correlators of the transalgebraic topological recursion.*

$$\omega_{g.n}(z_1, B) = \operatorname*{Res}_{t=R} \sum_{i=2}^{\deg(x)} S_*^t(z_1) \sum_{C_1,\ldots,C_j \vdash \{t_1,\ldots,t_{i-1}\}} \frac{(-1)^{1-\delta_{j,i-1}}}{j!} \operatorname*{Res}_{\substack{t_{C_l}=R_0,B,\mathcal{Y}(t) \\ l=1,\ldots,j}} \operatorname*{Res}_{\substack{C_l=t_{C_l} \\ l=1,\ldots,j}} \left(\prod_{l=1}^{j} \frac{x(t)}{x(t)-x(t_{C_l})} \prod_{t_0 \in C_l \setminus \{t_{C_l}\}} \frac{x(t)}{x(t_0)-x(t_{C_l})} \right) \frac{\Omega^i_{g,n}(t, t_1, \ldots, t_{i-1}|B)}{\prod_{l=1}^{i-1}((xy)(t)-(xy)(t_l))}, \tag{3.18}$$

where $\mathcal{Y}(t) = M_2^{-1}\big(M_2(\{t\})\big)$ *and the residues at the infinite ramification points* R_∞ *are defined as, letting* $x = M_0 \exp(M_1)$*,*

$$\operatorname*{Res}_{t=R_\infty} \leftrightarrow \frac{1}{(2g-1)!} \lim_{w\to 0^+} \frac{\mathrm{d}^{2g-1}}{\mathrm{d}w^{2g-1}} M_{1*}, \tag{3.19}$$

where the expression on the left is to be interpreted as follows: we take the push-forward under the map M_1 *so are now working in the* M_1 *plane; we define the coordinate* $w = 1/M_1$ *so the infinite ramification points are all located at* $w = 0$*; the formula is then the standard one for a pole of order* $2g$ *at* $w = 0$ *except we take the limit as* $w \to 0$ *along the positive real axis.*

Proof. Adopting the notation of the proof of Theorem 3.B.2 note that, as $\mathcal{N}_N^d(w|z_1, B)/\mathcal{D}_N^d(w|z_1, B)$ has a pole of order at most $2g$ at $w = 1/N$, we will have that $\mathcal{N}_\infty^d(w|z_1, B)/\mathcal{D}_\infty^d(w|z_1, B)$ will have a pole of order no more than $2g$ at $w = 0$. Ergo, we can compute the topological recursion in the limit (note here, by assumption, there are no ν). By definition, this is

$$\begin{aligned}
&\frac{1}{(2g-1)!}\lim_{w\to 0^+}\frac{\mathrm{d}^{2g-1}}{\mathrm{d}w^{2g-1}}\frac{\mathcal{N}^d_\infty(w|z_1,B)\exp(d/w)+\dots}{\mathcal{D}^d_\infty(w|z_1,B)\exp(d/w)+\dots}\\
=&\frac{1}{(2g-1)!}\lim_{w\to 0^+}\frac{\mathrm{d}^{2g-1}}{\mathrm{d}w^{2g-1}}\frac{\mathcal{N}^d_\infty(w|z_1,B)}{\mathcal{D}^d_\infty(w|z_1,B)}\big(1+\mathcal{O}(\exp(-w^{-1}))\big)\\
=&\frac{1}{(2g-1)!}\lim_{w\to 0^+}\frac{\mathrm{d}^{2g-1}}{\mathrm{d}w^{2g-1}}\frac{\mathcal{N}^d_\infty(w|z_1,B)}{\mathcal{D}^d_\infty(w|z_1,B)}=\operatorname*{Res}_{w=0}\frac{\mathcal{N}^d_\infty(w|z_1,B)}{\mathcal{D}^d_\infty(w|z_1,B)}.
\end{aligned}\tag{3.20}$$

It is the belief of the author that this works even when M_1 is not a well-defined function of M_2, but it is tricky to establish due to the possibility of the $\exp(1/\nu(w))$ contributing to the leading order coefficients in the limit. It is therefore clear that this corollary still holds if M_1 is not a well-defined function of M_2, but one can prove the $\exp(1/\nu(w))$ do not contribute to the leading order in the limit. □

We then conjecture, with reasonable evidence, that the principal parts of the correlators at essential singularities actually take an exceedingly nice form.

Conjecture 3.B.7. *The projection of $\omega_{g,n}$ onto its principal parts (principal part being defined through the choice of polarisation induced by $\omega_{0,2}$) at the essential singularities is ($B\subset\Sigma$ is a set of $n-1$ points)*

$$\begin{aligned}
&\sum_{a\in R_\infty}\operatorname*{Res}_{t=a}S^t_*(z_1)\omega_{g,n}(t,B)\\
&\qquad=\delta_{n,1}\frac{(2^{1-2g}-1)B_{2g}}{(2g)!}\sum_{a\in R_\infty}\operatorname*{Res}_{t=a}S^t_*(z_1)\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}M_2(t)}\right)^{2g-1}\log(x(t))\\
&\qquad=\delta_{n,1}\left(\mathrm{d}M_2(z_1)\right)\frac{(2^{1-2g}-1)B_{2g}}{(2g)!}\left(\frac{\mathrm{d}}{\mathrm{d}M_2(z_1)}\right)^{2g}M_1(z_1),
\end{aligned}\tag{3.21}$$

where B_{2g} denotes the $2g$th Bernoulli number. Note that this formula necessarily implies that the only correlators that have poles at essential singularities are those *with $n = 1$ and of these, only finitely many have poles.*

The only explicit evidence for this conjecture provided here is that this agrees with the rescaling arguments presented in Proposition 3.B.4, the fact that if we replace x by x_N in the above formula we reproduce the correct pole-structure for finite N in the $n = 1$ case, and the fact that the $\omega_{0,n}$ will have vanishing principal parts at essential singularities. However, in an upcoming publication [Bouchard et al., 2022], it will be shown that this conjecture is always true for the simplest non-trivial correlators with $-\chi_{g,n} = 2g + n - 2 = 1$ and the conjecture will be proven for all correlators in the r-atlantes Hurwitz case.

If this conjecture is indeed true, it means that throughout this section we have been something like Krylov's inquisitive man sifting through fine details and noticing gnats on the wall while missing the obvious big picture; we have, in short, done something rather complicated to achieve something rather remarkably simple. In such cases, there should be a different lens that brings the big picture more into focus and allows us to see the elephant in the room.

3.B.1 Example: q-Orbifold r-Atlantes Hurwitz Curve

At this point, we pause briefly our elucidation of the general theory of the transalgebraic topological recursion to consider an example. Let z be an affine coordinate on $\mathbb{C}_\infty$, $r, q \in \mathbb{Z}_{\geq 1}$, and consider the spectral curve

$$\mathcal{S} = \left(\mathbb{C}_\infty, x(z) = z\mathrm{e}^{-z^{qr}}, y(z) = z^{q-1}\mathrm{e}^{z^{qr}}\right), \tag{3.22}$$

which we will call the q-orbifold r-atlantes Hurwitz curve. It is the content of Theorem 4.B.8 that this curve, in the case $q = 1$, produces correlators that are generating functions of the r-atlantes Hurwitz numbers (see Alexandrov et al. [2016]

for an explanation of atlantes Hurwitz numbers). For $q > 1$ it can be taken as conjectural that this spectral curve will calculate the q-orbifold r-atlantes Hurwitz numbers.

In any case, here we will use Lemma 3.B.6 to calculate the contribution from the essential singularity at infinity to the correlator $\omega_{1,1}$. Given our spectral curve we have $M_1(z) = -z^{qr}$, $M_2(z) = z^q$, and $\Omega^2_{1,1}(t, t_1|\emptyset) = \omega_{0,2}(t, t_1)$. Choosing our base point $*$ to be infinity and letting θ be a primitive qth root of unity we see that this contribution will be

$$\operatorname*{Res}_{t=\infty} S^t_\infty(z_1)\omega_{1,1}(t) = \operatorname*{Res}_{t=\infty} \frac{\mathrm{d}z_1}{z_1 - t} \operatorname*{Res}_{t_1=R_0,\mathcal{Y}(t)} \frac{t\mathrm{e}^{-t^{qr}}}{t\mathrm{e}^{-t^{qr}} - t_1\mathrm{e}^{-t_1^{qr}}} \frac{\omega_{0,2}(t, t_1)}{t^q - t_1^q}. \tag{3.23}$$

The residues at $t_1 = R_0$ will drop out, as the integrand has no poles here. For the residues at $t_1 = \mathcal{Y}(t)$ we must be careful to distinguish between the trivial and non-trivial sheets of M_2, as the pole structure of the integrand is different in these two cases. First, we look at the non-trivial sheets, where there is only a simple pole

$$\sum_{m=2}^{q-1} \operatorname*{Res}_{t_1=\theta^m t} \frac{t\mathrm{e}^{-t^{qr}}}{t\mathrm{e}^{-t^{qr}} - t_1\mathrm{e}^{-t_1^{qr}}} \frac{\omega_{0,2}(t, t_1)}{t^q - t_1^q} = \sum_{m=2}^{q-1} \frac{1}{1-\theta^m} \frac{\theta^m}{-q} \frac{\mathrm{d}t}{t^2(1-\theta^m)^2}, \tag{3.24}$$

which we see has no pole at $t = \infty$ and so will not contribute to the final result. Next, we examine the residue at $t_1 = t$. The calculation is done on Sagemath as it is long and unenlightening and we just give the result here

$$\operatorname*{Res}_{t_1=t} \frac{t\mathrm{e}^{-t^{qr}}}{t_1\mathrm{e}^{-t_1^{qr}} - t\mathrm{e}^{-t^{qr}}} \frac{\omega_{0,2}(t, t_1)}{t_1^q - t^q} = -\frac{qr(r-1)t^{qr-q-1}\mathrm{d}t}{24} + \mathcal{O}(t^{-2})\mathrm{d}t. \tag{3.25}$$

Then, multiplying by $S^t_\infty(z_1)$ and taking the residue at infinity we obtain

$$\operatorname*{Res}_{t=\infty} S^t_\infty(z_1)\omega_{1,1}(t) = -\frac{r\mathrm{d}z_1^{q(r-1)}}{24}, \tag{3.26}$$

which is in agreement with Conjecture 3.B.7. Note that we did not have to use the re-definition of the residue at $t = R_\infty$ in Lemma 3.B.6 as the integrand is meromorphic. It is straightforward to see this is generic to calculations of $\omega_{1,1}$, but that this will not hold for more complicated correlators.

3.C The Topological Recursion for Curves with Infinite Ramification Loci

It has hitherto been assumed that the ramification locus of x, R, is of finite cardinality. In the case of algebraic curves, this is always the case, however, in the transalgebraic case it is interesting to examine the loosening of this restriction. Here, we define the topological recursion for a broad class of transalgebraic spectral curves[11] where the ramification locus is countable infinite.

To avoid toilsome complications we will impose some restrictions on the behaviour of these spectral curves, although it seems likely the topological recursion is definable for virtually any branched covering x. Indeed, by the Weierstraß factorisation theorem, given a countable subset $R \subset \mathbb{C}$, there exists an entire function with zeros at R of any prescribed order so tackling the general case seems rather broad even for genus zero. We will therefore restrict to genus zero, assume that x is periodic with period p, and that the set of all ramification points may be written as (assuming an essential singularity only at infinity)

$$R = \{a_j + kp | k \in \mathbb{Z}, 1 \leq j \leq r\} \cup \{\infty\}, \tag{3.1}$$

[11] Of course, these won't be spectral curves in the sense of Definition 3.A.5 where the ramification locus is more or less assumed finite.

where $a_1, \ldots, a_r$ are ramification points that generate the ramification locus; we assume such a collection of a_j is chosen to be minimal, *i.e.*, $a_{j_1} \neq a_{j_2} + kp$. These assumptions are broad enough to encompass applications to the Gromov-Witten theory of the projective line and mirror curves, as we will see in Chapter 4, but are still restrictive enough that that the development of the general theory is manageable.

We now argue that these assumptions imply a rather specific form for $\mathrm{d}x$. Letting b_j be the order of $\mathrm{d}x$ at a_j we may conclude that

$$\frac{\mathrm{d}x}{\mathrm{d}z} \prod_{j=1}^{r} (z-a_j)^{-b_j} \prod_{k=1}^{\infty} \left[1 - \left(\frac{z-a_j}{kp}\right)^2\right]^{-b_j}, \tag{3.2}$$

is a never zero entire function. Thus, it is equal to $C \exp(f(z))$ for some $C \in \mathbb{C}$ and entire function f with $f(0) = 0$. We assume, as before, that f is in fact meromorphic on $\mathbb{C}_\infty$; in this case, as $f(0) = 0$ and $f(z+p) \in 2\pi\mathrm{i}\mathbb{Z} - f(z)$ from the periodicity of x, we may then conclude that f is linear so $f(z) = \lambda z$ where $\lambda p \in 2\pi\mathrm{i}\mathbb{Z}$. Putting it all together, and setting $C = 1$ as the normalisation of x is irrelevant for topological recursion (we can always put the normalisation in y), we arrive at

$$\mathrm{d}x(z) = \mathrm{e}^{\lambda z} \prod_{j=1}^{r} (z-a_j)^{b_j} \prod_{k=1}^{\infty} \left[1 - \left(\frac{z-a_j}{kp}\right)^2\right]^{b_j} \mathrm{d}z. \tag{3.3}$$

This leads us to the following definition of admissibility for curves of this form.

Definition 3.C.1. A triple $\mathcal{S} = (\mathbb{C}_\infty, x, y)$ is called a spectral curve with infinite ramification locus R if $\mathrm{d}x$ takes the form given in (3.3), and $\omega_{0,1}(z)/\omega(x(z)) = y(z)\mathrm{d}x(z)/\omega(x(z))$ is holomorphic at each ramification point of x and meromorphic on the whole of $\mathbb{C}_\infty$ where $\omega(x)$ is a meromorphic 1-form on $\mathbb{C}_\infty$. We say $\mathcal{S}$ is *acceptable* if $\omega(x) = \mathrm{d}x$ and $\lambda = 0$ in (3.3).

Remark 3.C.2. Note that we now allow $\omega_{0,1}$ itself to not be meromorphic on $\mathbb{C}_\infty$. Even more generally, we could actually choose ω to be locally different around each ramification point (provided any two ramification points ω is different at are not related by addition of the period), but this seems like rather pointless generality.

The definition of admissibility carries over verbatim at the finite ramification points; given Conjecture 3.B.5 we will simply ignore the ramification point at infinity. With these facts in mind, the following definition of the topological recursion is the obvious one.

Definition 3.C.3. Given a spectral curve admissible in the sense of definition 2.A.9 we define the topological recursion to be exactly that of definition 2.B.3 except the sum over R is now infinite. The ramification point at infinity is ignored.

The next lemma checks that we may indeed take Definition 2.B.3 to hold at each ramification point and sum up the infinitely many contributions to obtain a well-defined result. The lemma succeeding the next demonstrates that, for acceptable curves, we may obtain this topological recursion as a limit of the algebraic topological recursion.

Lemma 3.C.4. *The sum over R in Definition 3.C.3 converges absolutely for admissible spectral curves. Furthermore, for $g \geq 0$, $n \geq 1$, and $2g + n - 2 \geq 1$ the n differentials constructed are periodic in all their variables with period p.*

Proof. We prove this by induction on $2g + n - 2$. The statement trivially holds for $\omega_{0,2}$ and $\omega_{0,1}$ so we may fast forward straight to the induction step. First note that by the induction assumption $\omega_{g,n}$ is automatically periodic in all its arguments save the first, provided it converges. Therefore, we must check only that $\omega_{g,n}$ is well-defined, and that it is periodic in its first arguments.

We will start by demonstrating the periodicity property, from which the convergence property will follow naturally. To accomplish this, we will begin with an examination of the local deck transformations to find (perhaps unsurprisingly) that they obey a periodicity-like property. For notational convenience, up to periodicity, we assume we have only one ramification point a (it will be clear that the proof easily generalises). Let θ be a primitive r_a'th root of unity. Fixing $k \in \mathbb{Z}$, every $\sigma^m_{a+kp} \in \boldsymbol{\sigma}_{a+kp}$ has an expansion of the form

$$\sigma^m_{a+kp}(z) = a + kp + \theta^m(z - a - kp) + \mathcal{O}(z - a - kp)^2.$$

We then compute

$$x(\sigma^m_{a+kp}(z + p)) = x(z + p) = x(z),$$

so $\sigma(z) := \sigma^m_{a+kp}(z + p)$ is also a deck transformation. One can then observe, using prime to denote a z derivative

$$\sigma'(a + (k - 1)p) = \theta^m,$$

to conclude that, using the uniqueness of local deck transformations upon fixing their first derivative,

$$\sigma^m_{a+kp}(z+a) = \sigma^m_{a+(k-1)p}(z) = a+(k-1)p+\theta^m(z-a-(k-1)p)+\mathcal{O}(z-a-(k-1)p)^2.$$

Finally, we may use induction to establish the result that for any $k - l$ with $l \in \mathbb{Z}$ there are expansion coefficients s^m_l not depending on k such that

$$\sigma^m_{a+kp}(z) = a + kp + \sum_{l=1}^{\infty} s^m_l (z - a - kp)^l. \tag{3.4}$$

Let us now take ω as in Definition 3.C.1 and define $M_2(z) := \omega_{0,1}(z)/\omega(x(z))$. We wish to show there exists coefficients M_l^m which do not depend on k such that

$$(M_2 \circ \sigma_{a+kp}^m)(z) - M_2(z) = \sum_{l=1}^{\infty} M_l^m (z - a - kp). \tag{3.5}$$

As M_2 is assumed regular at each ramification point it suffices to show the result for the case $M_2(z) = z^j$ for some $j \in \mathbb{Z}_{>0}$; if it is true for each j, then it is true for a general expansion of a holomorphic function. We use the well-known identity for $a, b \in \mathbb{C}$

$$a^j - b^j = (a - b) \prod_{i=1}^{j-1} \left(a - b\mathrm{e}^{2\pi\mathrm{i}/j}\right).$$

Applying this to our situation we find

$$\begin{aligned}\left(\sigma_{a+kp}^m(z)\right)^j - z^j &= \left(\sigma_{a+kp}^m(z) - z\right) \prod_{i=1}^{j-1} \left(\sigma_{a+kp}^m(z) - z\mathrm{e}^{2\pi\mathrm{i}/j}\right) \\ &= \left((\theta^m - 1)(z - a - kp) + \sum_{l=2}^{\infty} s_l^m (z - a - kp)^l\right) \\ &\times \prod_{i=1}^{j-1} \left((\theta^m - \mathrm{e}^{2\pi\mathrm{i}/j})(z - a - kp) + \sum_{l=2}^{\infty} s_l^m (z - a - kp)^l\right),\end{aligned} \tag{3.6}$$

from which the existence of such M_l^m is clear. Next, we note

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}(z_1 - a - kp) \otimes \mathrm{d}(z_2 - a - kp)}{\left((z_1 - a - kp) - (z_2 - a - kp)\right)^2}, \tag{3.7}$$

so any $\omega_{0,2}(\sigma_{a+kp}^{m_1}(z), \sigma_{a+kp}^{m_2}(z))$ clearly has an expansion about $a + kp$ with coefficients independent of k. As x is periodic in p, $\omega(x(z))$ certainly has an expansion about $z = a + kp$ with coefficients that do not depend on k. By the induction assumption and the same reasoning, the $\omega_{g',n'}$ with $2g' + n' - 2 < 2g + n - 2$ also have such expansions. Ergo the entire integrand

$$\sum_{i=2}^{r_{a+kp}} \sum_{\substack{A\subseteq \boldsymbol{\sigma}_{a+kp}\cdot t\\ |A|=i, t\in A}} K_i(z_1, t, A\setminus\{t\})\Omega^i_{g,n}(A|B),$$

clearly has such an expansion in $t = a + kp$. Thus the $\omega_{g,n}$ will have an expansion of the form

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{k=-\infty}^{\infty}\sum_{l=2}^{\infty} \frac{a_l(z_2,\dots,z_n)}{(z_1-a-kp)^l}, \tag{3.8}$$

where all but finitely many of the a_l are zero. This proves that the sum over the ramification points (which is the sum over k) converges and that $\omega_{g,n}$ is periodic in its first argument. Note that the property that the $\omega_{g,n}$ are residueless (and therefore don't have simple poles) is locally guaranteed at each ramification point; the fact we have an infinite sum does nothing to the argument. □

Lemma 3.C.5. *Given an acceptable admissible spectral curve with a countably infinite ramification locus, $\mathcal{S} = (\mathbb{C}, x, y)$, there exists a sequence of spectral curves $\mathcal{S}_N = (\mathbb{C}, x_N, y_N)$ with finite ramification locus (a spectral curve in the sense of Definition 2.A.3 or Definition 3.A.5) such that the $\omega^N_{g,n}$ constructed from $\mathcal{S}_N$ converge to the $\omega_{g,n}$ constructed from $\mathcal{S}$.*

Proof. By admissibility we may assume the form (3.3) for $\mathrm{d}x$. By relabelling the generating set of ramification points $\{a_1,\dots,a_j\}$, we will assume a_1 is a zero of $\mathrm{d}x$. Then define

$$\mathrm{d}x_N(z) = \prod_{j=1}^{r}(z-a_j)^{b_j}\prod_{k=1}^{N}\left[1-\left(\frac{z-a_j}{kp}\right)^2\right]^{b_j}\mathrm{d}z,\ x_N(z) = C+\int_{a_1}^{z}\mathrm{d}x_N, \tag{3.9}$$

where $C\in\mathbb{C}$ is the unique value so that $x_N\to x$. There is no ambiguity in defining the integral of $\mathrm{d}x_N$ as $\mathrm{d}x_N$ clearly has no simple poles and is thus exact (recall we are working in genus zero). Define

$$\omega_{0,1}^N(z) = \frac{\omega_{0,1}(z)\mathrm{d}x_N(z)}{\mathrm{d}x(z)}, \tag{3.10}$$

and note that this defines y_N through $y_N = \omega_{0,1}^N/\mathrm{d}x_N$.

We now make two key observations. First, the finite N ramification locus is a strict subset of the limiting ramification locus. Second, by Bouchard and Eynard [2013] at each finite ramification point of x_N we get locally uniform convergence of the $\omega_{g,n}^N$ constructed from $\mathcal{S}^N$ from the locally uniform convergence of the x_N and $\omega_{0,1}^N$ in small punctured disks about these points. We will use the Weirerstrauss M-test to then commute the limit in N with the sum over ramification points. For simplicity[12] we assume that $p = 1$, and up to periodicity there is only one ramification point a. Then, inductively, we claim for $2g+n-2 \geq 1$ (we make no claim on $\omega_{0,1}^N$ and $\omega_{0,2}^N$)

$$\omega_{g,n}^N(z_1,\dots,z_n) = \sum_{l=2}^{L_{g,n}} \sum_{k=-\infty}^{\infty} \frac{a_{l,k}^N(z_2,\dots,z_n)}{(z_1-a-k)^l}, \tag{3.11}$$

where $L_{g,n}$ are some positive integers and the $a_{l,k}^N(z_2,\dots,z_n)$ are uniformly bounded in k and N, i.e. there exists $M(z_2,\dots,z_n) \in \mathbb{R}_{\geq 0}$ such that $|a_{l,k}^N(z_2,\dots,z_n)| < M(z_2,\dots,z_n)$ (as the sum over l is finite the non-dependence of M on l is clear). As we make no claim on the form of $\omega_{0,1}^N$ and $\omega_{0,2}^N$ the induction beginning is trivial. Thus, we move onto the induction step. The claim follows if we can show the following integral is bounded independently of $k, m, N \in \mathbb{Z}$ where $B = \{z_2,\dots,z_n\}$ (we abuse notation slightly by saying bounded as this is a $n-1$

[12] If the reader is concerned this assumption is too much, in our applications it is the case up to an additional $\mathbb{Z}_2$ symmetry.

differential)

$$\oint_{|t-a-k|=\epsilon} (t-a-k)^m \sum_{i=2}^{r_{a+k}} \sum_{\substack{A\subseteq \boldsymbol{\sigma}_{a+k}\cdot t\\ |A|=i, t\in A}} \left(\frac{1}{\mathrm{d}x_N(t)}\right)^{i-1} \prod_{t_0\in A} \left(M_2^N(t) - M_2^N(t_0)\right)^{-1} \Omega_{g,n}^{i,N}(A|B), \tag{3.12}$$

where $M_2^N = \omega_{0,1}^N/\mathrm{d}x_N$. This follows if we can show the following is bounded

$$\epsilon^m \max_{|t-a-k|=\epsilon} \sum_{i=2}^{r_{a+k}} \sum_{\substack{A\subseteq \boldsymbol{\sigma}_{a+k}\cdot t\\ |A|=i, t\in A}} \left|\frac{\mathrm{d}t}{\mathrm{d}x_N(t)}\right|^{i-1} \prod_{t_0\in A} \left|M_2^N(t) - M_2^N(t_0)\right|^{-1} \left|\frac{\Omega_{g,n}^{i,N}(A|B)}{\mathrm{d}t^{\otimes i}}\right|. \tag{3.13}$$

Obviously, the pre-factor $|t-a-k|^m = \epsilon^m$ is bounded in the desired manner, and by the induction assumption the $\left|\Omega_{g,n}^{i,N}(A|B)\right|$ term is also bounded in the desired manner so all that is left is the two factors from the recursion kernel. The argument that the factor involving the difference $M_2^N(t) - M_2^N(t_0)$ is bounded independently of k is a virtual copy and paste of the argument given in the previous lemma. As both M_2^N and all deck transformations will converge uniformly near $a+k$ we get that this is bounded uniformly in N for free. The remaining factor is

$$\left|\frac{\mathrm{d}t}{\mathrm{d}x_N(t)}\right|^{i-1},$$

and this is the most involved part. We wish to show $\mathrm{d}x_N(t)/\mathrm{d}t$ is bounded below independently of k and N for $t = a+k+\epsilon \mathrm{e}^{\mathrm{i}\theta}$ for some $0 \le \theta < 2\pi$. Explicitly, we have

$$\mathrm{d}x_N(t) = \prod_{l=1}^{N} \left[1 - \frac{(k+\epsilon \mathrm{e}^{\mathrm{i}\theta})^2}{l^2}\right]^b \mathrm{d}t, \tag{3.14}$$

for some $b \in \mathbb{Z}$. Now note

$$\begin{aligned}\left|1 - \frac{(k+\epsilon \mathrm{e}^{\mathrm{i}\theta})^2}{l^2}\right|^2 &= 1 + \frac{k^2}{l^2}\left(-2 - \frac{4\epsilon\cos\theta}{k}\right) \\ &+ \frac{k^4}{l^4}\left(1 + \frac{4\epsilon\cos\theta}{k} + \frac{2\epsilon^2}{k^2}(1+2\cos^2\theta) + \frac{4\epsilon^3\cos\theta}{k^3}\right) + \mathcal{O}\left(\frac{\epsilon^2 k^0}{l^2}\right) \\ &= \frac{(l^2 - (k^2 + \mathcal{O}(k^0 l^0 \epsilon))^2)^2}{l^4} + \mathcal{O}\left(\frac{\epsilon^2 k^0}{l^2}\right). \quad (3.15)\end{aligned}$$

Ergo, we can choose $\epsilon \ll 1$ small enough to get arbitrarily close to having this expression be $(l^2 - k^2)^2/l^4$ independently of k, provided $k \neq l$ so the leading order behaviour does not cancel out. We then note

$$\prod_{l=1}^{k-1}\left(\frac{k^2 - l^2}{l^2}\right)^b = \prod_{l=1}^{k-1}\left(1 + \frac{k}{l}\right)^b = \left(\frac{(k+1)^{\bar{k}}}{(k-1)!}\right)^b, \qquad (3.16)$$

where $x^{\bar{n}} = x(x+1)\cdots(x+n-1)$ is the rising factorial. Next note

$$\prod_{l=k+1}^{N}\left(\frac{k^2 - l^2}{l^2}\right)^b \geq \prod_{l=k+1}^{\infty}\left(\frac{k^2 - l^2}{l^2}\right)^b = \left(\frac{(k!)^2}{(2k)!}\right)^b, \qquad (3.17)$$

putting these two together we see

$$\left(\prod_{l=1}^{k-1}\frac{k^2 - l^2}{l^2}\right)^b\left(\prod_{l=k+1}^{N}\frac{k^2 - l^2}{l^2}\right)^b \geq k^b. \qquad (3.18)$$

Finally, the $k = l$ term is clearly $\mathcal{O}_s(\epsilon/k)^b$ so we get our k-independent lower bound. The desired result then follows. □

Although we define topological recursion for all admissible spectral curves, we only prove the existence of a sequence of spectral curves for the acceptable curves. This is not ideal (it would be better to have a sequence for every curve), but as the definition of the topological recursion remains unchanged locally at each ramification points the key properties, like symmetry or residueless, still hold without requiring the sequence. However, when we construct quantum curves, the results will be rigorous only for acceptable curves.

Chapter 4

Quantum Curves

4.A What is a Quantum Curve

In the context of matrix models, the correlators $\omega_{g,n}$ are related to the expectation values of the traces of the matrices under consideration [Eynard, 2004, Eynard and Orantin, 2007c]. But the trace is only one of the two natural basis-independent objects one can form from a matrix; the other is, of course, the determinant. Quantum curves provide the topological recursion link to this other perspective. In particular, in a matrix model, the expectation values of the determinants satisfy certain differential equations; roughly speaking, the solution of these differential equations is the so-called *wavefunction* and the operator that kills it is the *quantum curve*. It is then intuitively clear that the wavefunction should somehow involve the exponential of the $\omega_{g,n}$ given that the exponential of a trace is a determinant. This is indeed the case, as, explicitly, given a spectral curve $\mathcal{S} = (\Sigma, x, y)$ we define the wavefunction as

$$\psi(x(z)) = \exp\left[\sum_{n=1}^{\infty}\sum_{g=0}^{\infty}\frac{\hbar^{2g+n-2}}{n!}\left(\int^{z}\cdots\int^{z}\omega_{g,n} + \text{corrections}\right)\right], \tag{4.1}$$

which is an exponential of the correlators as claimed. Here $\hbar$ is a formal expansion parameter, there are n integrations in each term, and it is conventional to write ψ as a function of $x(z)$, rather than z, even though it is not globally well-defined as such.[1] The exact nature of the integration will be more carefully defined later.

Given the prior discussion, it should come as no surprise that there should be a Schrödinger-like operator that kills the wavefunction. To gain insight into what form this operator takes, we make the following curious observation. If we define a quantisation of our original (x, y) variables $(x, y) \to (\hat{x}, \hat{y})$ where $\hat{x} = x$ and $\hat{y} = \hbar \mathrm{d}/\mathrm{d}x$ then

$$P(\hat{x}, \hat{y})\psi(x(z)) = P(x, \hbar\frac{\mathrm{d}}{\mathrm{d}x}) \exp\left[\frac{1}{\hbar}\int y\mathrm{d}x + \mathcal{O}(\hbar^0)\right] = P(x, y)\psi(x(z)) + \mathcal{O}(\hbar), \tag{4.2}$$

where our spectral curve is defined by $P(x, y) = 0$.[2] Thus, up to order $\hbar$, our desired Schrödinger-like equation is just the spectral curve evaluated at our canonically quantised variables $(\hat{x}, \hat{y})$! We formalise this 'coincidence' with the following definition.

Definition 4.A.1. Let $\mathcal{S}$ be a meromorphic spectral curve, with x and y satisfying the relation $P(x, y) = 0$. We say that $\hat{P}(\hat{x}, \hat{y}; \hbar)$ is a quantisation of $P(x, y)$ if we have the following expansion for some $m \in \mathbb{N} \cup \{\infty\}$:

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = P(\hat{x}, \hat{y}) + \sum_{i=1}^{m} \hbar^i \hat{P}_i(\hat{x}, \hat{y}),$$

where $P(\hat{x}, \hat{y})$ is taken to be normally ordered (in each term all the $\hat{x}$ are put to the

[1] This convention is the natural one as the way one obtains the expectation values of the traces from the $\omega_{g,n}$ is through formal expansion in x where the expectation values of the traces are read off from the expansion coefficients.

[2] There is, of course, ambiguities in writing down $P(\hat{x}, \hat{y})$ due to ordering, but different orderings are the same up to order $\hbar$, so are irrelevant for the given argument.

left of the $\hat{y}$) and the $\hat{P}_i$ are normal ordered polynomials of degree at most $\deg P - 1$. We say that the quantisation is *simple* if $m < \infty$.

Of course, such a definition will require a formal approach to defining the wavefunction, one that defines both the integrations and the correction terms. Nevertheless, given the previous, along with other evidence [Norbury, 2015], we are lead to the following conjecture, commonly referred to as the Gukov-Sulkowski conjecture in the literature [Gukov and Sulkowski, 2011]. For the sake of correctness, however, it should be noted the name of this conjecture is somewhat of a misnomer; the result has been well-known in the context of matrix models [Mehta, 1990] long before the topological recursion and it was already being considered more generally in Bergère and Eynard [2009] in the context of the then recently discover topological recursion.

Conjecture 4.A.2. *Given a spectral curve and the correlators $\omega_{g,n}$ constructed through the topological recursion, there exists a wavefunction of the form* (4.1) *and a corresponding quantum curve in the sense of Definition 4.A.1.*

4.B The Quantum Curve/Topological Recursion Connection

We saw in the previous section that there is good reason to believe that quantum curves and the topological recursion enjoy an intimate and deep connection. This 'good reason' has been turned into hard proof by a number of authors exploring a myriad of different cases [Bergère and Eynard, 2009, Bouchard and Eynard, 2017, Marchal, 2017, Eynard and Garcia-Failde, 2019, Eynard et al., 2021]. For us, the key results in this area will be those of Bouchard and Eynard [2017], wherein the authors proved Conjecture 4.A.2 for all regular spectral curves. More precisely, they derived quantum curves for the wavefunctions[3] (recall regular spectral curves are genus zero and so integration along a path is unambiguously defined)

$$\psi(x(z);b) = \exp\left[\sum_{n=1}^{\infty}\sum_{g=0}^{\infty}\frac{\hbar^{2g+n-2}}{n!}\left(\int_b^z\cdots\int_b^z \omega_{g,n} - \delta_{g,0}\delta_{n,2}x^*x_*\omega_{0,2}\right)\right], \tag{4.1}$$

where, as $\omega_{0,2}$ is symmetric, it does not matter which variable the pullback of the pushforward is taken in. The integral of $\omega_{0,1}$ may need to be regularised, but this character plays no part in the QC/TR story.

Unfortunately, for the sake of maintaining a reasonable level of brevity, this section will not be self contained and will have to refer to Bouchard and Eynard [2017] with great frequency. Whenever possible, we will refer explicitly to the analogues of what we are doing in Bouchard and Eynard [2017] so confusion will hopefully be avoided. However, in Bouchard and Eynard [2017] everything was indexed starting from the degree of the curve which for us may be infinite;[4] it is therefore an inauspicious, but ultimately unavoidable, fact that we will have to re-index virtually all objects considered in Bouchard and Eynard [2017] where infinite degree curves were not a consideration.

We begin by defining multiple re-indexed objects related to the irreducible equation in Definition 2.A.2.

Definition 4.B.1. Given a spectral curve $P(x,y) = 0$ and letting Δ be the Newton

[3]The authors actually examined a slightly more general case than the one listed here, where the chosen integration divisor could be more complex than integration from a base point b to z.

[4]For us, any curve that does not have a well-defined finite degree is of infinite degree.

polygon of P and d be the degree, define, for $m = 0, \ldots, d$,

$$q_m(x) = \sum_{(i,m)\in A} \alpha_{i,m} x^m, \qquad Q_m(x,y) = \sum_{i=1}^{d-m-1} q_{m+i+1}(x) y^i$$
$$\alpha_m = \inf\{a|(a,m) \in \Delta\}, \qquad \beta_m = \sup\{a|(a,m) \in \Delta\} \tag{4.2}$$

The α_m and β_m correspond directly to Definition 2.3 in Bouchard and Eynard [2017] and were previously defined in Section 2.A. The q_m and Q_m correspond to re-indexed versions of the p_m and P_m of Remark 2.2 and Definition 2.5, respectively, in Bouchard and Eynard [2017].

Given that we now have a number of definitions that will, in short order, be critical to our construction of quantum curves it is helpful to pause briefly and consider an example.

Example 4.B.2. Consider the degree two spectral curve

$$P(x,y) = y^2 + (2 - x^2)y + 1 = 0,$$

which has the parametrisation

$$\mathcal{S} = \left(\mathbb{C}_\infty, x(z) = z + 1/z, y(z) = z^2\right).$$

Here the non-zero q_m and Q_m are

$$q_0(x) = 1, \quad q_1(x) = 2 - x^2, \quad q_2(x) = 1, \quad Q_0(x,y) = y = z^2. \tag{4.3}$$

Next note the Newton polygon is $\Delta = \{(0,0),(0,1),(0,2),(1,1),(2,1)\}$, which means this spectral curve is not regular as $(1,1)$ is an interior point. From Δ we also write down the α_m and β_m for illustrative purposes

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \beta_0 = 0, \quad \beta_1 = 2, \quad \beta_2 = 0. \tag{4.4}$$

In this simple case, the infimum and supremum in the definition of the α_m and β_m, respectively, are actually achieved; however, in general, this may not be the case and the α_m and β_m could take on non-integer values.

We now define analogues of the C_k and D_k that appear in Bouchard and Eynard [2017]. Let $x = x(z)$ and $x_i = x(z_i)$ for $z, z_i \in \mathbb{C}$ and $i \in \mathbb{N}$, *i.e.*, we write things in terms of the coordinate on the base.

Definition 4.B.3. Let $b \in \mathbb{C}$ be a pole of $\mathrm{d}x$ where all the $\omega_{g,n}$ are holomorphic and x is meromorphic. Then define

$$E_i = -\lim_{z_1 \to b} \frac{Q_{i-1}(x_1, y_1)}{x_1^{\lfloor \alpha_{d-k} \rfloor + 1}}, \qquad F_i = \hbar \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}} \frac{\mathrm{d}}{\mathrm{d}x}, \tag{4.5}$$

where $\lfloor \cdot \rfloor$ is the floor function.

With these definitions out of the way we are ready to state that, as expected, we can construct quantum curves for transalgebraic regular admissible spectral curves.

Theorem 4.B.4. *Given a transalgebraic regular spectral curve $\mathcal{S} = (\mathbb{C}, x, y)$, a base point b that is a pole of $\mathrm{d}x$ but not an essential singularity of x, and at which the $\omega_{g,n}$ are regular, and $\psi(z; b)$ as in (4.1) we have that,*

$$\left(\frac{q_0(x)}{x^{\lfloor \alpha_0 \rfloor}} + \sum_{i=1}^{d} F_1 F_2 \cdots F_{i-1} \frac{q_i(x)}{x^{\lfloor \alpha_i \rfloor}} F_i + \hbar \sum_{i=1}^{d-1} E_i F_1 F_2 \cdots F_{i-1} \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}} \right) \psi(z; b) = 0. \tag{4.6}$$

Proof. Let $x = M_0 \exp(M_1)$, $y = M_2/x$ with M_0, M_1, M_2 rational and take the sequence of spectral curves (recall regularity implies genus zero)

$$\mathcal{S}^N = \left(\mathbb{C}_\infty, x^N = M_0\left(1+(\tau-1)\frac{M_1}{N}\right)^{-N}\left(1+\tau\frac{M_1}{N}\right)^N, y^N = \frac{M_2}{x^N}\right).$$

As the $\omega_{g,n}^N$ are convergent, if they are regular at b in the limit then they must be regular for large enough N. We then define B_i^N, E_i^N, F_i^N, and $\psi^N(z;b)$ in the natural fashion. We quickly see that $\psi^N(z;b) \to \psi(z;b)$ as the exponential is continuous, the sum is formal, and the $\omega_{g,n}^N$ are well-defined in the limit so we can bring the limit inside the integrals using dominated convergence. The fact that $F_i^N \to F_i$ is clear, as the Newton polygon will converge. Finally, we must deal with the E_i^N. In Bouchard and Eynard [2017] it was argued, using arguments based on an inequality of divisors, that the C_i (which correspond to re-indexed E_i) must be finite as $z_1 \to b$. That an argument will carry over in the limit as $N \to \infty$ is clear as: (i) x is meromorphic near b by assumptions on b; (ii) as x^N will be uniformly convergent away from x infinity it will be uniformly convergent, in particular, near b; (iii) the required inequalities are non-strict. Finally, as already noted, near b, x^N is uniformly convergent so by the Moore-Osgood theorem,

$$\lim_{N\to\infty}\lim_{z_1\to b}\frac{Q_{i-1}^N(x_1,y_1)}{(x^N-x_1^N)(x_1^N)^{\lfloor\alpha_{d-k}^N\rfloor}} = \lim_{z_1\to b}\lim_{N\to\infty}\frac{Q_{i-1}^N(x_1,y_1)}{(x^N-x_1^N)(x_1^N)^{\lfloor\alpha_{d-k}^N\rfloor}}, \tag{4.7}$$

at which point we may just take the limit, concluding that $B_i^N \to B_i$ and the B_i's are not identically infinity. $\square$

Remark 4.B.5. It is important when one constructs the limiting ($d=\infty$) quantum curve to take the right spectral curve. For example, there is no guarantee that the family of equations

$$ye^{(\tau-1)(xy)^r} - e^{\tau(xy)^r} = 0,$$

which all yield the same correlators, will give the same quantum curve. The correct choice(s) are clearly given by considering the sequence of spectral curves. For example, the choice of sequence in the theorem gives the listed equation for each τ in the r-atlantes Hurwitz case.

This theorem therefore gives us a canonical way of creating a quantum curve for a regular transalgebraic spectral curves. In particular, we don't actually have to construct a quantum curve for each finite N and take the limit; the mere existence of such a sequence of curves guarantees we can construct the quantum curve directly from the limiting curve. It is important to note, that when $d = \infty$, the constructed quantum curve is under no obligation to be simple, which is in sharp contrast to the $d < \infty$ case.

As in Bouchard and Eynard [2017], the choice of divisor can be generalised from being the rather trivial $D = [z] - [b]$ in an analogous way to the generalisation presented in Remark 5.15 in Bouchard and Eynard [2017]; the key steps of the proof carry over virtually without modification.

However, choosing one's base point to be a pole of $\mathrm{d}x$ is extraordinarily inconvenient when $\mathrm{d}x$ has no pole; a case that may arise when x has an essential singularity. In Bouchard and Eynard [2017], they considered the case of the base point b being a zero of $q_d(x(b)) = 0$, but only when $d = 2$. Here, we generalise this choice to the case $d > 2$ and then use it to construct quantum curves with this base point. We begin this process with a lemma before proving a theorem analogous to Theorem 4.B.4.

Lemma 4.B.6. *For b a zero of $q_d(x)$ that is not in the ramification locus of x we have*

$$\psi_i(x(b); z; b) = \psi(z; b) \lim_{z_1 \to b} \frac{1}{x(z_1)^{\lfloor \alpha_{d-i} \rfloor}} Q_{d-i-1}(x(z_1), y(z_1)), \tag{4.8}$$

where the ψ_i are defined in Definition 5.9 of Bouchard and Eynard [2017].

Proof. From Bouchard and Eynard [2017] Definition 5.9

$$\psi_i(x(b); z; b) = \psi(z; b) \lim_{z_1 \to b} \left(\frac{1}{x(z_1)^{\lfloor \alpha_{d-i} \rfloor}} \left(q_d(x(z_1)) \xi_i(x(z_1); D) - q_{d-i}(x(z_1)) \right) \right). \tag{4.9}$$

Using the notation of Bouchard and Eynard [2017] and subbing in the definition of the ξ_k (Definition 5.6 in Bouchard and Eynard [2017])

$$q_d(x(z_1))\xi_i(z_1) = (-1)^i q_d(x(z_1)) \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \frac{\hbar^{2g+n}}{n!} \frac{G^i_{g,n+1}(z_1)}{\mathrm{d}x(z_1)^{\otimes i}}, \tag{4.10}$$

where the $G^i_{g,n+1}$ are defined in Definition 5.3 of Bouchard and Eynard [2017]. First we examine the power $\hbar^0$. Here we have, where the $U^i_{0,1}$ are defined in Definition 4.1 of Bouchard and Eynard [2017]

$$(-1)^i q_d(x(z_1)) \frac{G^i_{0,1}(z_1)}{\mathrm{d}x(z_1)^{\otimes i}} = (-1)^i q_d(x(z_1)) \frac{U^i_{0,1}(z_1)}{\mathrm{d}x(z_1)^{\otimes i}} = Q_{d-i-1}(x(z_1), y(z_1)) + q_{d-i}(x(z_1)). \tag{4.11}$$

Note then we have the inequality (Lemma 2.6 in Bouchard and Eynard [2017])

$$\mathrm{div}(Q_{d-i-1}(x, y)) \geq \alpha_{d-i} \mathrm{div}_0(x) - \beta_{d-i} \mathrm{div}_\infty(x). \tag{4.12}$$

So we therefore have that the limit

$$\lim_{z_1 \to b} \frac{1}{x(z_1)^{\lfloor \alpha_{d-i} \rfloor}} Q_{d-i-1}(x(z_1), y(z_1)), \tag{4.13}$$

is finite. This is in agreement with the result in Bouchard and Eynard [2017] for $d = 2$ as, when $d = 2$, we have $Q_0(x, y) = q_d(x)y$. Now we examine the higher order powers of $\hbar$. As b is not in the ramification locus of x we have that each

$G^i_{g,n+1}(z_1)$ is regular at b for $2g+n \geq 1$, furthermore it can't be a zero of $\mathrm{d}x$ so for each i,

$$\frac{G^i_{g,n+1}(z_1)}{\mathrm{d}x(z_1)^{\otimes i}}, \tag{4.14}$$

is regular at b. Ergo, if b is not a zero of x the terms of higher order in $\hbar$ never contribute. Assume then that b is a simple zero of x and we claim that still

$$\lim_{z_1 \to b} \frac{q_d(x(z_1))}{x(z_1)^{\lfloor \alpha_{d-i} \rfloor}} = 0, \tag{4.15}$$

for $i = 1, \ldots, d-1$. As our curve is irreducible there is some $k = 0, \ldots, d-1$ with $q_k(x) =$ const. as we could otherwise cancel out an overall factor of x in $P(x,y)$. Let $i_1 \leq k \leq i_2$ such that the $\alpha_{i_1} = \cdots = \alpha_{i_2} = 0$. Then, as the α_m are the smallest point on the convex hull at the power of y^m, they are strictly increasing for $m \geq i_2$ and strictly decreasing for $m \leq i_1$ by the convexity of the convex hull. Finally, observe that α_0 and α_d will be non-negative integers and note $\alpha_0 \leq \alpha_d$ as, if this inequality didn't hold, $(1, i_1)$ would be an interior point of the Newton polygon. Thus, we have $\alpha_d = \lfloor \alpha_d \rfloor > \lfloor \alpha_m \rfloor$ for all $d > m > 0$. This establishes (4.15) as the order of the zero of $q_d(x)$ in x is α_d.

Ergo, we get that the $\hbar$ corrections vanish and we have the explicit expressions,

$$\psi_i(x(b); z; b) = \psi(z; b) \lim_{z_1 \to b} \left(\frac{1}{x(z_1)^{\lfloor \alpha_{d-i} \rfloor}} Q_{d-i-1}(x(z_1), y(z_1)) \right), \tag{4.16}$$

as claimed. $\square$

This gives an analogous theorem to Theorem 4.B.4 except with this new choice of base point. First, we define the new coefficients G_i and H_i

$$G_i = \lim_{z_1\to b} \frac{1}{x(z_1)^{\lfloor\alpha_i\rfloor}} Q_{i-1}(x(z_1), y(z_1)), \qquad H_i = \hbar \frac{x^{\lfloor\alpha_i\rfloor}}{x^{\lfloor\alpha_{i-1}\rfloor}} \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x - x(b)} \right). \tag{4.17}$$

Then, Theorem 5.11 of Bouchard and Eynard [2017] reduces to

$$\hbar \frac{\mathrm{d}}{\mathrm{d}x} \psi_{i-1}(x; z; b) = \frac{x^{\lfloor\alpha_{d-i}\rfloor}}{x^{\lfloor\alpha_{d-i+1}\rfloor}} \psi_i(x; z; b) - \frac{q_{d-i+1}(x) x^{\lfloor\alpha_{d-1}\rfloor}}{q_d(x) x^{\lfloor\alpha_{d-i+1}\rfloor}} \psi_1(x; z; b) \\ + \hbar \frac{1}{x - x(b)} \left(\psi_{i-1}(x; z; b) - G_{d-i+1} \psi(z; b) \right). \tag{4.18}$$

We can now derive a quantum curve in the manner of Lemma 5.14 of Bouchard and Eynard [2017].

Theorem 4.B.7. *Given a regular spectral curve $\mathcal{S} = (\mathbb{C}, x, y)$, a base point b that is a zero of $q_d(x)$ for $d < \infty$ or, if $d = \infty$, a zero of x, with b not in the ramification locus of x, and $\psi(z; b)$ as in (4.1) we have that*

$$\left(\frac{q_0(x)}{x^{\lfloor\alpha_0\rfloor}} + \sum_{i=1}^{d} H_1 \cdots H_{i-1} \frac{q_i(x)}{x^{\lfloor\alpha_i\rfloor}} F_i + \hbar \sum_{i=1}^{d-1} G_i H_1 \cdots H_{i-1} \frac{x^{\lfloor\alpha_i\rfloor}}{x^{\lfloor\alpha_0\rfloor}(x - x(b))} \right) \psi(z; b) = 0 \tag{4.19}$$

Proof. First assume $d < \infty$. Rewriting (4.18) we have

$$\psi_i(x; z; b) = H_{d-i+1} \psi_{i-1}(x; z; b) + \frac{q_{d-i+1}(x)}{x^{\lfloor\alpha_{d-i+1}\rfloor}} F_{d-i+1} \psi(z; b) \\ + \hbar \frac{x^{\lfloor\alpha_{d-i+1}\rfloor}}{x^{\lfloor\alpha_{d-i}\rfloor}(x - x(b))} G_{d-i+1} \psi(z; b), \tag{4.20}$$

where we used the fact that (Lemma 5.10 in Bouchard and Eynard [2017])

$$\psi_1(x; b) = \frac{q_d(x)}{x^{\lfloor\alpha_{d-1}\rfloor}} \hbar \frac{\mathrm{d}}{\mathrm{d}x} \psi(z; b).$$

We can sub the $i = d-1$ result into the $i = d$ result to obtain

$$\begin{aligned}\psi_d(x;z;b) &= H_1\psi_{d-1}(x;z;b) + \frac{q_1(x)}{x^{\lfloor\alpha_1\rfloor}}F_1\psi(z;b) + \hbar\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}(x-x(b))}G_1\psi(z;b)\\ &= H_1H_2\psi_{d-2}(x;z;b) + H_1\frac{q_2(x)}{x^{\lfloor\alpha_2\rfloor}}F_2\psi(z;b) + \hbar H_1\frac{x^{\lfloor\alpha_2\rfloor}}{x^{\lfloor\alpha_1\rfloor}(x-x(b))}G_2\psi(z;b)\\ &\qquad + \frac{q_1(x)}{x^{\lfloor\alpha_1\rfloor}}F_1\psi(z;b) + \hbar\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}(x-x(b))}G_1\psi(z;b). \end{aligned}\tag{4.21}$$

Applying this iteratively, before finally using the fact that (again Lemma 5.10 in Bouchard and Eynard [2017])

$$\psi_d(x;z;b) = -\frac{q_0(x)}{x^{\lfloor\alpha_0\rfloor}}\psi(z;b),$$

yields the desired result. Taking the limit to get the $d = \infty$ result is completely analogous to the $d = \infty$ case in Theorem 4.B.4. □

4.B.1 Hurwitz Numbers

Here we examine derive the quantum curve for the spectral curve

$$\mathcal{S} = \left(\mathbb{C}_\infty, x(z) = z\mathrm{e}^{-z^r}, y(z) = e^{z^r}\right), \tag{4.22}$$

where $r \in \mathbb{Z}_{\geq 1}$ is a fixed integer referred to as the 'spin'. As in Subsection 3.B.1 we will refer to this curve as the r-atlantes Hurwitz curve; in fact, it is the content of Theorem 4.B.8 that this curve indeed calculates the r-atlantes Hurwitz numbers. First, note that this is a regular spectral curve as $xy = z$ and that the following irreducible equation holds for all $\tau \in \mathbb{C}$

$$P(x,y) = y\mathrm{e}^{(\tau-1)(xy)^r} - \mathrm{e}^{\tau(xy)^r} = \sum_{m=0}^{\infty} \frac{(\tau-1)^m}{m!} x^{rm} y^{rm+1} - \sum_{m=0}^{\infty} \frac{\tau^m}{m!} x^{rm} y^{rm}, \tag{4.23}$$

so it is easily obtained that $\alpha_m = m - 1 + \delta_{m,0}$, $q_m(x) = 0$ if $r \nmid m-1$ and $r \nmid m$, $q_{rm}(x) = -\frac{\tau^m x^{rm}}{m!}$, and $q_{rm+1}(x) = \frac{(\tau-1)^m x^{rm}}{m!}$. Choosing the base point $b = 0$ we easily compute the following coefficients

$$\begin{aligned} H_1 =& \hbar \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x} \right),\ H_i = \hbar \left(x\frac{\mathrm{d}}{\mathrm{d}x} - 1 \right), \\ F_1 =& \hbar \frac{\mathrm{d}}{\mathrm{d}x},\ F_i = \hbar x \frac{\mathrm{d}}{\mathrm{d}x},\ G_i = 0. \end{aligned} \tag{4.24}$$

Then, from Theorem 4.B.7 we get the quantum curve, where $\hat{x} = x$ and $\hat{y} = \hbar\frac{\mathrm{d}}{\mathrm{d}x}$

$$\begin{aligned} &\hat{P}(\hat{x}, \hat{y}; \hbar) = \\ &-1 + \frac{1}{x}\sum_{m=0}^{\infty} \frac{(\tau-1)^m \hbar^{rm+1}}{m!} \left(x\frac{\mathrm{d}}{\mathrm{d}x} - 1 \right)^{rm} x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\sum_{m=1}^{\infty} \frac{\tau^m \hbar^{rm}}{m!} \left(x\frac{\mathrm{d}}{\mathrm{d}x} - 1 \right)^{rm-1} x^2 \frac{\mathrm{d}}{\mathrm{d}x} \\ &\qquad = \frac{\hbar}{x}\mathrm{e}^{(\tau-1)\hbar^r (x\frac{\mathrm{d}}{\mathrm{d}x}-1)^r} x\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{e}^{\tau\hbar^r x^r \frac{\mathrm{d}^r}{\mathrm{d}x^r}}. \end{aligned} \tag{4.25}$$

We can rearrange to obtain a cleaner result

$$\begin{aligned} &\left(\mathrm{e}^{(\tau-1)\hbar^r (x\frac{\mathrm{d}}{\mathrm{d}x}-1)^r} x\hbar\frac{\mathrm{d}}{\mathrm{d}x} - x\mathrm{e}^{\tau\hbar^r x^r \frac{\mathrm{d}^r}{\mathrm{d}x^r}} \right) \psi(x;0) = 0, \\ \Rightarrow &\left(x\hbar\frac{\mathrm{d}}{\mathrm{d}x} - x\mathrm{e}^{-(\tau-1)\hbar^r x^r \frac{\mathrm{d}^r}{\mathrm{d}x^r}} \mathrm{e}^{\tau\hbar^r x^r \frac{\mathrm{d}^r}{\mathrm{d}x^r}} \right) \psi(x;0) = 0, \\ \Rightarrow &\left(\hat{y} - \mathrm{e}^{(\hat{x}\hat{y})^r} \right) \psi(x;0) = 0, \end{aligned} \tag{4.26}$$

where we used the identity

$$\exp\left(a\hbar^r\left(\frac{\mathrm{d}}{\mathrm{d}u}-1\right)^r\right)\exp(bu)=\exp(u)\exp\left(a\hbar^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r\right)\exp((b-1)u), \tag{4.27}$$

with $a = 1-\tau$, $b = 1$, and $u = \log(x)$. It is important to note that we started with a quantum curve that depended on τ and ended up with a result that had no τ dependence.[5] This is because all the quantum curves for different τ were related by multiplication on the left by an invertible operator and multiplying on the left by an invertible operator does not change the solution of the corresponding differential equation.

Next, we can compare with Mulase et al. [2013], who obtained their result not by working with TR, but by working directly with the r-spin Hurwitz numbers to compute the following quantum curve where $\hat{Y} = \hat{x}\hat{y}$,

$$\hat{P}'(\hat{x},\hat{Y};\hbar)=\hat{Y}-\hat{x}^{3/2}\mathrm{e}^{\frac{1}{r+1}\sum_{i=0}^{r}\hat{x}^{-1}\hat{Y}^i\hat{x}\hat{Y}^{r-i}}\hat{x}^{-1/2}. \tag{4.28}$$

Clearly, in general, this is not the same curve. However, we can observe an interesting relation between the two results that was first noticed in a more limited form in [Chotai, 2016]. Defining $u = \log(x)$ assume an operator,

$$\hat{H}=\exp\left(\sum_{n=1}^{r}\hbar^{r-n}h_n\frac{\mathrm{d}^n}{\mathrm{d}u^n}\right), \tag{4.29}$$

such that $\hat{H}\psi = \tilde{\psi}$ where $\tilde{\psi}$ is killed by (4.28). Then, noting that $[\hat{Y},\hat{H}] = 0$ and

[5] It is unclear to the author whether this holds in general, or is unique to the cases considered.

$\hat{x}^{-1}\hat{Y}\hat{x} = \hat{Y} + \hbar$, so $[\hat{x}^{-1}\hat{H}\hat{x}, \hat{Y}] = 0$,[6]

$$\begin{aligned} 0 = \hat{H}0 &= \hat{H}\big(\hat{Y} - \hat{x}e^{\hat{Y}^r}\big)\psi \\ &= \big(\hat{Y} - \hat{H}\hat{x}e^{\hat{Y}^r}\hat{H}^{-1}\big)\hat{H}\psi \\ &= \big(\hat{Y} - \hat{x}\hat{x}^{-1}\hat{H}\hat{x}e^{\hat{Y}^r}\hat{H}^{-1}\big)\psi' \\ &= \big(\hat{Y} - \hat{x}e^{\hat{Y}^r}\hat{x}^{-1}\hat{H}\hat{x}\hat{H}^{-1}\big)\psi'. \end{aligned} \tag{4.30}$$

Ergo, our operator $\hat{H}$ is the solution to

$$\begin{aligned} \hat{x}e^{\hat{Y}^r}\hat{x}^{-1}\hat{H}\hat{x}\hat{H}^{-1} &= \hat{x}^{3/2}e^{\frac{1}{r+1}\sum_{i=0}^{r}\hat{x}^{-1}\hat{Y}^i\hat{x}\hat{Y}^{r-i}}\hat{x}^{-1/2} \\ \Rightarrow \hat{x}^{-1/2}e^{\hat{Y}^r}e^{\sum_{n=1}^{r}\hbar^{r-n}h_n(\hat{Y}+\hbar)^n}e^{\sum_{n=1}^{r}\hbar^{r-n}h_n\hat{Y}^n}\hat{x}^{1/2} &= e^{\frac{1}{r+1}\sum_{i=0}^{r}(\hat{Y}+\hbar)^i\hat{Y}^{r-i}} \\ \Rightarrow e^{(\hat{Y}+\frac{\hbar}{2})^r}e^{-\sum_{n=1}^{r}\hbar^{r-n}h_n\left((\hat{Y}+\frac{3\hbar}{2})^n-(\hat{Y}+\frac{\hbar}{2})^n\right)} &= e^{\frac{1}{r+1}\sum_{i=0}^{r}(\hat{Y}+\hbar)^i\hat{Y}^{r-i}}. \end{aligned} \tag{4.31}$$

Then, equating the exponentials, absorbing any factor of $2\pi\mathrm{i}$ into h_0, which is

[6]This calculation was originally done incorrectly by the author. The author would like to acknowledge Dr Reinier Kramer for providing the correct calculation.

arbitrary anyway,

$$
\begin{aligned}
&\big(\hat{Y}+\frac{\hbar}{2}\big)^{r}+\sum_{n=1}^{r}\hbar^{r-n}h_{n}\Big(\big(\hat{Y}+\frac{3\hbar}{2}\big)^{n}-\big(\hat{Y}+\frac{\hbar}{2}\big)^{n}\Big)\\
=&\frac{1}{r+1}\sum_{i=0}^{r}(\hat{Y}+\hbar)^{i}\hat{Y}^{r-i}\\
\Rightarrow&\sum_{n=1}^{r}h_{n}\sum_{j=0}^{n}\binom{n}{j}\Big(\big(\frac{3}{2}\big)^{n-j}-\big(\frac{1}{2}\big)^{n-j}\Big)\hbar^{r-j}\hat{Y}^{j}\\
=&-\sum_{j=0}^{r}\binom{r}{j}\big(\frac{\hbar}{2}\big)^{r-j}\hat{Y}^{j}+\frac{1}{r+1}\sum_{i=0}^{r}\sum_{k=0}^{i}\binom{i}{k}\hbar^{i-k}\hat{Y}^{k+r-i}\\
\Rightarrow&\sum_{j=0}^{r}\sum_{n=j}^{r}h_{n}\binom{n}{j}\Big(\big(\frac{3}{2}\big)^{n-j}-\big(\frac{1}{2}\big)^{n-j}\Big)\hbar^{r-j}\hat{Y}^{j}\\
=&\sum_{j=0}^{r}\Big(-\binom{r}{j}\big(\frac{1}{2}\big)^{r-j}+\frac{1}{r+1}\sum_{i=r-j}^{r}\binom{i}{r-j}\Big)\hbar^{r-j}\hat{Y}^{j}\\
\Rightarrow&\sum_{n=j+1}^{r}h_{n}\binom{n}{j}\Big(\big(\frac{3}{2}\big)^{n-j}-\big(\frac{1}{2}\big)^{n-j}\Big)\\
=&-\binom{r}{j}\big(\frac{1}{2}\big)^{r-j}+\frac{1}{r+1}\binom{r+1}{j},\qquad 0\le j\le r\,.
\end{aligned}
\tag{4.32}
$$

Such an $\hat{H}$ therefore exists and is unique. It is easy to see now that $\hat{H}\neq 1$ except in the case when $r=1$ where it can be seen that (4.26) and (4.28) agree.

Although it's certainly clear that there should be some operator that transforms ψ to $\tilde{\psi}$, it is not at all obvious that this operator is the exponential of an operator that is of degree precisely $r-1$; this immediately begs the deeper question of why this is the case. To see this deeper reason we must note a couple of facts: the quantum curve we get, (4.26), is precisely the one obtained for the r-atlantes Hurwitz numbers [Alexandrov et al., 2016]; it is known the topological recursion ignoring the essential singularity computes the regular r-spin Hurwitz numbers [Dunin-Barkowski et al., 2019]; for $r=1$ when there are no contributions from infinity (Conjecture 3.B.5) atlantes and regular Hurwitz numbers coincide [Alexandrov et al., 2016] and the two quantum curves, as already discussed, are the same. Therefore, in the face of such evidence, the answer to the deeper question appears to be that the transalgebraic topological recursion actually computes the r-atlantes Hurwitz numbers. This is the content of the following theorem from the upcoming publication Bouchard et al. [2022].

Theorem 4.B.8. *For the choice of spectral curve* $\mathcal{S} = \left(\mathbb{C}_\infty, x(z) = z\mathrm{e}^{-z^r}, y(z) = \mathrm{e}^{z^r}\right)$, *the transalgebraic topological recursion defined in Definition 3.B.1 computes the r-atlantes Hurwitz numbers.*

The degree of the operator, from this perspective, is no surprise; a degree $r-1$ operator is precisely the degree needed to reduce all contributions from the essential singularity to constants by Corollary 3.B.4 and/or Conjecture 3.B.7.

4.B.2 Gromov-Witten Theory of the Projective Line

It is well-known that topological recursion produces generating functions for the Gromov-Witten invariants of $\mathbb{CP}^1$ from the initial data of the following spectral curve [Zhou, 2012, Norbury and Scott, 2014, Dunin-Barkowski et al., 2014][7]

$$\mathcal{S} = \left(\mathbb{CP}^1 \cong \mathbb{C}_\infty, x(z) = z + \frac{1}{z}, y(z) = \log(z)\right), \tag{4.33}$$

which is a parametrisation of the transalgebraic equation

$$P(x,y) = x - 2\cosh(y) = 0. \tag{4.34}$$

[7] This isn't a spectral curve in the sense of this paper, but the notion of a spectral curve can be generalised to include this case [Borot et al., 2018].

Given our previous results, it seems far more natural to consider a different parametrisation of the above equation, namely

$$\mathcal{S}_\infty = (\mathbb{C}_\infty, x(z) = 2\cosh(z), y(z) = z)\,. \tag{4.35}$$

Initially, one might naively guess that since x and y in the curve (4.35) are just the pullbacks of x and y in the curve (4.33) that the correlators $\omega_{g,n}$ constructed from the initial data of (4.35) will be the pullbacks of the correlators constructed from the initial data of (4.33). This, however, is not the case as $\omega_{0,2}$ is the same for both curves.[8] We will denote by $\omega_{g,n}$ the correlators constructed from (4.33) and $\omega_{g,n}^\infty$ the correlators constructed from (4.35). In the spirit of our previous work, we define the sequence of spectral curves

$$\mathcal{S}_N = \left(\mathbb{C}_\infty, x_N(z) = 2\int_{\pi/2}^{z} w \prod_{k=0}^{N}\left(1+\frac{w^2}{k^2\pi^2}\right)\mathrm{d}w, y_N(z) = y(z) = z\right), \tag{4.36}$$

which corresponds to the polynomial equation

$$\begin{aligned} 0 = P_N(x_N, y) &= x_N - 2\int_{\pi/2}^{y} w \prod_{k=0}^{N}\left(1+\frac{w^2}{k^2\pi^2}\right)\mathrm{d}w \\ &= x_N - \int_{w=\pi/2}^{w=y} \sum_{m=0}^{N}\left(\frac{w^2}{\pi^2}\right)^m \sum_{N\geq k_m>\cdots>k_1\geq 0} \frac{1}{k^2\cdots k^2}\mathrm{d}w^2 \\ &= x_N - \sum_{m=0}^{N}\frac{\pi^2}{m+1}\left(\frac{y^2}{\pi^2}\right)^{m+1} \sum_{N\geq k_m>\cdots>k_1\geq 0} \frac{1}{k_1^2\cdots k_m^2} + C_N, \end{aligned} \tag{4.37}$$

[8]If one did redefine $\omega_{0,2}$ in the curve (4.35) to be the pullback under the exponential map, then it is straightforward to see that all the correlators will be pullbacks.

for a constant C_N whose explicit form is irrelevant but makes $x_N(\pi/2) = 0$. Denote the correlators constructed by this curve as $\omega^N_{g,n}$. To construct the quantum curve, we choose the base point to be $b = \infty$ (a pole of $\mathrm{d}x_N$ at which the $\omega^N_{g,n}$ are holomorphic) and note

$$q_{2m+2}(x_N) = (x_N + C_N)\delta_{m,-1} - \frac{1}{(m+1)\pi^{2m}} \sum_{N \geq k_m > \cdots > k_1 \geq 1} \frac{1}{k_1^2 \cdots k_m^2}, m = -1, \ldots, N, \tag{4.38}$$

with all other q_m being zero; ergo, $\alpha_m = 0$. With this, using Theorem 4.B.4, we see the quantum curve is

$$\hat{P}_N(\hat{x}_N, \hat{y}_N; \hbar) = \hat{x}_N - \sum_{m=0}^{N} \frac{\pi^2}{m+1} \left(\frac{\hat{y}^2}{\pi^2}\right)^{m+1} \sum_{N \geq k_m > \cdots > k_1 \geq 0} \frac{1}{k_1^2 \cdots k_m^2} + C_N, \tag{4.39}$$

where $\hat{x}_N = x_N$ and $\hat{y}_N = \hbar \mathrm{d}/\mathrm{d}x_N$. This kills the wavefunction

$$\psi_N(x_N(z)) = \exp\left[\sum_{n=1}^{\infty} \sum_{g=0}^{\infty} \frac{\hbar^{2g+n-2}}{n!} \left(\int_{\infty}^{z} \cdots \int_{\infty}^{z} \omega^N_{g,n} - \delta_{g,0}\delta_{n,2} x_N^* x_{N*}\omega_{0,2}\right)\right]. \tag{4.40}$$

Taking the $N \to \infty$ limit of the quantum curve is straightforward; clearly, one obtains

$$\hat{P}_\infty(\hat{x}, \hat{y}; \hbar) = \hat{x} - 2\cosh(\hat{y}), \tag{4.41}$$

where $\hat{x} = x = 2\cosh(z)$ and $\hat{y} = \hbar \mathrm{d}/\mathrm{d}x$, which agrees with the result in Marchal [2017]. However, taking the $N \to \infty$ limit of the wavefunction is a bit more subtle as ∞ will be an essential singularity of the limiting $\omega^\infty_{g,n}$. To deal with this we note, as we saw in the proof of Lemma 3.C.4, that the $\omega^\infty_{g,n}$ will have an expansion of the form

$$\omega^\infty_{g,n}(z_1, B) = \sum_{l=2}^{M_{g,n}} W^l_{g,n}(B) \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}z_1}{(z_1 - k\pi)^l}, \tag{4.42}$$

where $M_{g,n}$ is some positive integer and the $W^l_{g,n}(B)$ are symmetric $n-1$-differentials in their variables. For $l \geq 3$ note

$$\int_{-\infty}^{z} \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}z_1}{(z_1 - k\pi)^l} = -\frac{1}{l-1} \sum_{k=-\infty}^{\infty} \frac{1}{(z_1 - k\pi)^{l-1}}, \tag{4.43}$$

where $\int_{-\infty}^{z}$ means we approach ∞ along the negative real axis. We could commute the integral and sum using dominated convergence as, along the negative real axis, we avoid the singularities of the integrand. For $l = 2$, this argument doesn't work as the RHS is no longer convergent. It is easy to adopt the argument

$$\begin{aligned} \int_{-\infty}^{z} \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}z_1}{(z_1 - k\pi)^2} &= -\frac{1}{z} + \int_{-\infty}^{z} \sum_{k=1}^{\infty} \left(\frac{\mathrm{d}z_1}{(z_1 - k\pi)^2} + \frac{\mathrm{d}z_1}{(z_1 + k\pi)^2} \right) \\ &= -\frac{1}{z} - 2 \sum_{k=1}^{\infty} \frac{1}{z^2 + k^2\pi^2}. \end{aligned} \tag{4.44}$$

Thus, we see the wavefunction is well-defined for the limiting curve if we agree to approach infinity along the negative real axis. That the limit of the wavefunction is the wavefunction of the limit is clear from the arguments of Lemma 3.C.5.

As noted, in the limit, we obtain the same quantum curve (up to changing x from $z + z^{-1}$ to $2\cosh(z)$) that was found from topological recursion using other means in Marchal [2017]. Although initially unsurprising, this is in fact a bit miraculous, as this quantum curve was constructed for the wavefunction corresponding to the $\omega_{g,n}$ of the spectral curve (4.33), not the $\omega^{\infty}_{g,n}$ of the spectral curve (4.35).

To examine the relation, define π to be the exponential. As discussed, the naïve guess that the relation $\pi^*\omega_{g,n} = \omega^{\infty}_{g,n}$ holds is incorrect; however, as we know the two quantum curves are the same (up to the change in x) and the relation $\omega^{\infty}_{0,1} = \pi^*\omega_{0,1}$ holds, our wavefunction must be the pullback of their wavefunction under π.[9]

[9]One may wonder what happens with the $\hbar^0$ term, as $\omega_{0,2}$ is not pulledback under π. In fact,

Even more curiously, the author knows of no other known case in the literature where two different sets of correlators produce the same wavefunction. That this is so clearly connected to a change in $\omega_{0,2}$ makes one suspicious that it is related to the properties of the $\omega_{g,n}$ under a change in polarisation[10] [Borot et al., 2018] and somehow the wavefunction is left invariant under certain polarisation changes. Further investigation of this would certainly be of great interest, not only for the Gromov-Witten theory of $\mathbb{CP}^1$, but for the theory of the topological recursion itself.

4.B.3 Mirror Curves

Here we study the so-called mirror curves. For background on mirror curves and their interpretation the reader may refer to Bouchard et al. [2008], Bouchard and Sulkowski [2012], Zhou [2012]. In particular, the specific curve we will study here is what is commonly referred to in the literature as the framed mirror curve of $\mathbb{C}^3$ which, given a parameter $f \in \mathbb{Z}$ called the framing, has the following form

$$P(x, y) = -y^{f+1} + y^f - x = 0, \tag{4.45}$$

where we consider $(x, y) \in (\mathbb{C}^\times)^2$. Initially, transalgebraic geometry may appear to be irrelevant here, but we are considering x and y to be in $\mathbb{C}^\times$ so the natural choice for the 1-form $\omega_{0,1}$ is $\omega_{0,1} = \log(y)\mathrm{d}\log(x)$ [Bouchard et al., 2008]. However, if we make the transformation $y \to \mathrm{e}^{xy}$ we see we restore the natural form $\omega_{0,1} = y\mathrm{d}x$.

the $\hbar^0$ term in the wavefunction has to be fudged from what one would naturally expect when the quantum curve is derived directly from enumerative considerations [Dunin-Barkowski et al., 2017]. This fudge term makes it precisely agree with our results once the wavefunction is pulled back. In fact, the author has verified explicit agreement of the two wavefunctions up to and including the $\mathcal{O}(\hbar)$ term.

[10] The fundamental bilinear differential of the second kind induces a symplectic structure on the space of 1-forms, and changing it therefore induces a change in polarisation on the space of 1-forms [Borot et al., 2018].

This transformation applied to our spectral curve yields

$$P(x,y) = -\mathrm{e}^{(f+1)xy} + \mathrm{e}^{fxy} - x = 0, \tag{4.46}$$

which has the parametrisation

$$\mathcal{S} = \left(\mathbb{C}_\infty, x(z) = \mathrm{e}^{fz}(1-\mathrm{e}^z), y(z) = z/x(z)\right), \tag{4.47}$$

where we now see the transalgebraic geometry brought to the fore. To truncate this curve in the style of Lemma 3.C.5, for $f \neq 0$ (the case $f = 0$ is uninteresting as all the stable correlators should be zero by Conjecture 3.B.5)

$$\begin{aligned}
\mathrm{d}x(z) &= \mathrm{e}^{fz}(f-(f+1)\mathrm{e}^z)\mathrm{d}z \\
&= -f\mathrm{e}^{fz+z/2+\mathrm{Log}(1+f^{-1})/2}\left(\mathrm{e}^{z/2+\mathrm{Log}(1+f^{-1})/2} - \mathrm{e}^{-z/2-\mathrm{Log}(1+f^{-1})/2}\right)\mathrm{d}z \\
&= -\sqrt{f^2+f}\,\mathrm{e}^{(f+1/2)z}\left[z+\mathrm{Log}(1+f^{-1})\right]\prod_{k=1}^{\infty}\left[1+\left(\frac{z+\mathrm{Log}(1+f^{-1})}{2\pi k}\right)^2\right]\mathrm{d}z,
\end{aligned} \tag{4.48}$$

where Log denotes the principal branch of the logarithm. By now, there should be no surprises in our approach. To compute the quantum curve we will consider the sequence of transalgebraic spectral curves

$$\mathcal{S}_N = \left(\mathbb{C}_\infty, x_N(z) = \int_{b_0}^{z}\mathrm{d}x_N, y_N(z) = z/x_N(z)\right), \tag{4.49}$$

for a base point $b_0 \in 2\pi\mathrm{i}\mathbb{Z}$ where

$$\mathrm{d}x_N(z) = -\sqrt{f^2+f}\,\mathrm{e}^{(f+1/2)z}\left[z+\mathrm{Log}(1+f^{-1})\right]\prod_{k=1}^{N}\left[1+\left(\frac{z+\mathrm{Log}(1+f^{-1})}{2\pi k}\right)^2\right]\mathrm{d}z. \tag{4.50}$$

Here we should pause and note that $\omega_{0,1}^N = y_N \mathrm{d}x_N$, by Conjecture 3.B.5 there should not be a contribution from the pole at infinity, and this curve is regular and admissible as $x_N y_N = z$ has a simple pole at the infinite ramification point at $z = \infty$. However, this curve is not quite acceptable as $\omega(x) = \mathrm{d}\log(x)$ rather than $\mathrm{d}x$ so the fact that the correlators of the limit are the limit of the correlators should be taken as conjectural. We choose our base point to be $b = 0$, which is a simple zero of x_N so we may apply the results of Theorem 4.B.7. We could choose b to be any element of $2\pi\mathrm{i}\mathbb{Z}$, but due to the $2\pi\mathrm{i}$ periodicity of the correlators, it doesn't seem like this choice gives a meaningful difference. Given this choice of b, it is easiest to set $b_0 = b = 0$.

Now, computing the quantum curve for each N and then taking the limit would clearly be rather involute. Ergo, our approach will be to argue that all the coefficients H_i^N, F_i^N, G_i^N converge to the right coefficients in the limit. Due to the convergence of the Newton polygon, this is clear for the H_i^N and F_i^N. The G_i^N are easily seen to be zero for both for finite N and in the limit as $\alpha_i^N = i$ and $Q_i^N(x_N, y_N) = \mathcal{O}(x_N^{i+2})$ for all N. Thus, we examine the limiting curve

$$-\frac{1}{x}\sum_{m=1}^{\infty}\left[(f+1)^m - f^m\right]\frac{(xy)^m}{m!} - 1 = 0. \tag{4.51}$$

Then, noting $\alpha_m = m - 1 + \delta_{m,0}$ and $q_m(x) = -\left[(f+1)^m - f^m\right]\frac{x^{m-1}}{m!} - \delta_{m,0}$ we find the coefficients

$$H_1 = \hbar\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right),\ H_i = xH_1,\ F_1 = \hbar\frac{\mathrm{d}}{\mathrm{d}x}\ F_i = xF_1,\ G_i = 0. \tag{4.52}$$

Therefore, we may compute the quantum curve as

$$\left[-1-\frac{1}{x}\sum_{m=1}^{\infty}\frac{\hbar^m}{m!}\left[(f+1)^m-f^m\right]\left(x\frac{\mathrm{d}}{\mathrm{d}x}-1\right)^{m-1}x\frac{\mathrm{d}}{\mathrm{d}x}\right]\psi(z;0)=0. \quad (4.53)$$

Unfortunately, this does not appear to have a nice closed form, or match the results of Zhou [2012]. One can introduce a factor of $\mathrm{e}^{\tau x y}$ in P, but this seems to do little to remedy either problem.[11] As in the Gromov-Witten case, it would be interesting to examine these, for lack of a better word, inconsistencies to yield further insight into the TR/QC connection.

[11]Interestingly, differentiating with respect to τ after quantising allows one to get a closed form for the 'quantum curve', but this equation is no longer irreducible so is not an actual quantum curve as one can't properly take the classical limit

Chapter 5

Conclusion

In trying to extend the results of Bouchard and Eynard [2017] to the scenario where x was not meromorphic on a compact Riemann surface we were motivated to consider sequences of spectral curves converging to the desired curve. These sequences led to a new, natural, definition of the topological recursion at essential singularities, the transalgebraic topological recursion, that takes in the initial data of a transalgebraic spectral curve. This new topological recursion enjoys most of the key properties enjoyed by the original formalism of Eynard and Orantin [2007c].

Armed with this new definition and the technique of constructing sequences of algebraic spectral curves that converge to transalgebraic ones, we were able to rigorously construct quantum curves for the r-atlantes Hurwitz numbers and the Gromov-Witten invariants of $\mathbb{CP}^1$ as well as obtain some conjectural results on quantum mirror curves.

However, there is more to be done. Although our new definition of the topological recursion works for arbitrary genus, all quantum curve related results are done only for genus zero curves; there are, however, interesting higher genus curves where our results should yield new insight [Bouchard et al., 2008, Liu, 2012, Eynard and Garcia-Failde, 2019, Eynard et al., 2021]. Furthermore, it is somewhat unclear how the new topological recursion should be viewed from the perspective of the higher quantum Airy structures of Borot et al. [2018]. Studying this should yield interesting new connections between the topological recursion and certain twisted modules of $\mathcal{W}(\mathfrak{gl}_{\infty+1})$ algebras.

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