Algorithms for Flow Trades at NASDAQ around its Close

by

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Abstract

For many investors, such as mutual fund managers, the closing price of a stock is an important benchmark. The closing price for stocks traded at NASDAQ is determined through an auction, like at many other stock exchanges. Each day and for each stock traded at NASDAQ, the intertemporal order imbalance of the auction is announced beginning ten minutes before the close. We introduce a mathematical framework that takes the order imbalance announcements into account, and then derive an optimal trading algorithm for flow trades, whose benchmark is the closing price. Under suitable assumptions, we find explicit formulas for the optimal trading strategy and that it is not beneficial for the investor to trade after the imbalance announcement. However, in addition to participating in the auction, the investor trades before the imbalance announcement to benefit from prices which do not reflect the later impact of the investor's own auction order. Using real historical data, we simulate the performance of the proposed algorithm and find a small, but persistent out-of-sample improvement and a reduction in average trading costs.

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Chapter 1

Introduction

Closing prices of stocks are important and often serve as reference points for investors to determine their performance. Closing prices are particularly relevant for managers of mutual funds. In mutual funds, flow trades correspond to inflows or outflows of cash when clients decide to buy or sell shares of the fund. Regardless at which specific time the transactions are taking place on any trading day, the mutual fund managers, or an institutional flow trader, will receive from or pay to the client the closing price. Hence, such traders use the closing price as their benchmark: they aim to achieve a price that is as close as possible to the closing price and, if possible, more favourable than the closing price.

Algorithmic trading, the implementation of mathematical and computational algorithms to conduct trading decisions and asset management, is becoming a frequently used tool in the public equity market by institutional investors targeting various trading benchmarks. Some of the standard trading benchmarks include arrival price, VWAP (volume weighted average price), TWAP (time weighted average price), and closing price. The trading algorithm typically aims to achieve optimal trading strategies by minimizing a combination of expected slippage (reflecting average costs) and variance of slippage (reflecting risk), where slippage is the difference between actually paid prices for the order and the benchmark price. The mathematical studies for algorithmic trading started with seminal papers by Bertsimas and Lo [3], who set up a discrete-time model to minimize expected slippage, and by Almgren and Chriss [1], who focused on the trading strategy targeting the arrival price benchmark including risk considerations. In the past several years, a vast literature on trading algorithms targeting different benchmarks has been developed. An overview can be found in the recent books by Cartea et al. [4], Guéant [8], as well as Lehalle and Laruelle [9]. Though the trading strategies for many benchmarks have been well studied, less attention has been paid to the closing price benchmark. Frei and Westray [6] presented a stochastic control formulation targeting a closing price benchmark for stocks traded at the Hong Kong Exchange. At many stock exchanges, the closing price is determining through a closing auction. However, the closing price at the Hong Kong Exchange, instead of a closing auction, is selected as the median of five prices taken over the last minute of trading. The trading algorithm with a closing price benchmark through the closing auction has not yet been widely discussed and mathematically modelled. At first sight, one could think that the only question then is about the percentage that one should place in the auction. However, various exchanges disclose information on the projected order imbalances before the auction closes. This affects the prices before the close so that the question becomes about how to trade before the closing auction as well.

The topic of this thesis focuses on trading around the close in the stocks at NASDAQ. The closing auction mechanism at NASDAQ is as follows. While the regular trading takes place from 9:30 AM to 4:00 PM Eastern Standard Time (ETS), the closing auction takes place from 3:50 PM to 4:00 PM at NASDAQ. At NASDAQ [10] and [11], a closing auction consists of three types of orders; namely, Market-on-Close (MOC), Limit-on-Close (LOC), and Imbalance-Only (IO) orders. MOC and LOC orders must be received before 3:50 PM. MOC orders are executed immediately by matching with the corresponding best buy/sell orders in the limit order book. On the other hand, LOC orders are executed at a given price as they go into the limit order book. An IO order is a type of limit order that is used to provide liquidity and offset the imbalance during the closing auction. The initial imbalance announcement occurs at 3:50 PM, after which, one can only submit IO orders. After 3:50 PM, NASDAQ publishes imbalance information every five seconds until 4:00 PM. All types of orders are accepted by NASDAQ for closing cross, a process to determine the closing price, at 4:00 PM. In the cross process, the closing book and the NASDAQ continuous book are brought together to create the NASDAQ Official Closing Price. Bacidore et al. [2] illustrated and summarized the market behaviour during the closing auction at NASDAQ and NYSE (New York Stock Exchange). The closing auction mechanism at NYSE is similar to that of NASDAQ. At NYSE, restrictions in submitting orders to the closing auction begin at 3:45 PM on each trading day; however, traders may submit MOC and LOC orders during the closing auction if there exists a significant amount of imbalance volume, known as the Regulatory Imbalance[1].

[1] NYSE Rule 123C(1)(d); see http://wallstreet.cch.com/nyse/rules/

Moreover, instead of Imbalance-Only (IO) orders, NYSE offers Closing Offset (CO) orders, which serve a similar purpose as the IO orders at NASDAQ. The major difference between the closing auctions at NASDAQ and NYSE is that the floor brokers have an advantage over other market participants at NYSE. From 2:00 PM to 3:45 PM, the floor brokers are able to view the close book every 15 seconds. The information includes the MOC, LOC, and CO orders, as well as any imbalance. At 3:55 PM, the Closing D-quotes, which are orders floor brokers use that can add or create an imbalance anytime until ten seconds before the market close, are included in the calculation of the imbalance. Due to the multiple layers of additional complexity at NYSE, we choose to study the closing auction at NASDAQ to analyze the underlying financial mechanism.

Since the trader will typically begin trading in the open market sometime before the start of the closing auction, one faces a degree of uncertainty in the quest in attaining the closing price. As a flow trader, the objective is to minimize the average and deviations of the slippage, relative to the closing price benchmark. A trader is guaranteed to receive the closing price if the total volume of the order is placed in the closing auction so that the slippage is zero. In this case, the risk is zero and average cost equal exactly the benchmark cost. However, a trader may perform even better overall by taking some risk and achieving a slippage on average by participating in the continuous trading as well. Such behaviour is mainly due to two reasons. Firstly, the trader can benefit from the impact of one's own order in the closing auction. In particular, the imbalance volumes revealed at the imbalance announcement have an influence on the stock prices. As such, by investing prior to the initial imbalance announcement, the trader could execute orders at more attractive prices. If the order placed in the closing auction is sufficiently large, one can have a negative impact on the closing price when the orders are executed at 4:00 PM; thus, trading in the open market before the prices are affected by the large order submitted to the closing auction can reduce the implementation cost. Secondly, the imbalance announcement may suggest a drift of stock prices if the revealed information goes predominantly in one direction (buy/sell). Hence, the trader placing buy (sell) orders may be able to gain from a lower (higher) price from orders before the closing auction if a buy (sell) imbalance is forecasted. While we include and discuss this second factor in our main results for the optimal strategy, we put more emphasis on the study and implementation of the first impact because it is a crucial feature of the closing price benchmark that the trader's own orders submitted to the auction affect prices before the auction through the imbalance announcement.

Based on the observations discussed in Bacidore et al. [2], they suggested that it is optimal to not trade, or only trade with a small order, continuously after the initial imbalance announcement. After the imbalance is announced, the impact of one's own participation in the closing auction is reflected in the stock prices. By trading in the open market during this time, the trader will receive an unfavourable price due to the imbalance announcement. Thus, it is preferable to trade before the initial imbalance announcement and not after. This statement will be proved mathematically as a part of the main results of this thesis. The main results are expressed in the form of explicit formulas for the optimal strategy. In the empirical part, its implementation to real historical data yields a small but persistent improvement in average costs.

This thesis focuses on presenting an optimal trading strategy for flow traders at NASDAQ. In Chapter 2, we first introduce a discrete-time framework and then derive the optimal trading strategy through the Karush-Kuhn-Tucker conditions. In Chapter 3, we present a continuous-time variant by solving the corresponding Euler-Lagrange equation. In Chapter 4, we estimate the model parameters based on historical data and test the out-of-sample performance on real data from 15 stocks traded at NASDAQ. Chapter 5 provides proofs of the mathematical derivation of the models. Chapter 6 concludes, and the appendix contains auxiliary calculations.

Chapter 2

Discrete-Time Model

2.1 Problem Formulation

Consider a market order with volume of v_i at time i for $i \in \{1, 2, \dots, T-1\}$ where time T corresponds to 4:00 PM EST, the close of the market. Let τ be the time when the initial imbalance is published, which corresponds to 3:50 PM EST, at NASDAQ. Let v_T be the volume of orders submitted to the closing auction. Suppose the order imbalance is cleared immediately and there are no orders in the closing auction after 3:50 PM. If the market impact of our order is only temporary, then our investment decision at time t will only affect the price at time t but not the subsequent stock prices at time $t+1, t+2, \dots, T-1$. Moreover, our order placed in the closing auction, v_T, does not only affect the closing price, P_T, but is also accounted throughout prices from 3:50 PM ET (time τ) to 4:00 PM ET (time T). Then the prices of the stock are given by:

$$P_t = \tilde{P}_t + \beta v_t \quad \text{for} \quad t \in \{1, \dots, \tau - 1, \tau + 1 \dots, T-1\},$$

$$
\begin{aligned}
P_\tau &= \tilde{P}_\tau + \beta v_\tau, \\
P_T &= \tilde{P}_T,
\end{aligned}
$$

for

$$
\begin{aligned}
\tilde{P}_t &= \tilde{P}_{t-1} + Z_t \quad \text{for} \quad t \in \{1, \ldots, \tau - 1, \tau + 1 \ldots, T - 1\}, \\
\tilde{P}_\tau &= \tilde{P}_{\tau-1} + Z_\tau + \alpha N, \\
\tilde{P}_T &= \tilde{P}_{T-1} + \tilde{Z}_i,
\end{aligned}
$$

where β is a non-negative scalar that measures the influence to the stock prices due to the investor's orders in the open market, and α is a non-negative scalar that reflects the impact of the auction imbalance announcement on the stock prices. Z_t is an independent and identically distributed random process and $\tilde{Z}_T$ is a random variable independent from Z_t. For Z_t, we denote its mean by μ_Z and its variance by σ_Z^2, and for $\tilde{Z}_T$, we write $\mu_{\tilde{Z}}$ for its mean and $\sigma_{\tilde{Z}}^2$ for its variance. Moreover, the imbalance N can be expressed as:

$$
N = \tilde{N} + v_T,
$$

where $\tilde{N}$ is the imbalance caused by other market participants. W is the total orders given in advance, which can be written as:

$$
W = \sum_{i=1}^{T} v_i.
$$

Consider any risk aversion, $\lambda > 0$. Flow traders' benchmark is the closing price, P_T; thus, the objective is to minimize:

$$E\left[\sum_{t=1}^{T} v_t P_t - W P_T\right] + \lambda VAR\left[\sum_{t=1}^{T} v_t P_t - W P_T\right]. \tag{2.1}$$

2.2 Optimal Strategy under Drift Condition

In this section, we first analyze the optimal strategy when we impose additional conditions on the drift of stock prices as well as the amount of predetermined total order volume (W). In particular, the assumptions we impose on the drift are:

$$\mu_Z \leq 0, \quad \mu_{\tilde{Z}} \leq 0,$$

which ensures it is not optimal to trade after the initial imbalance announcement. In other words, we assume random drivers reflected in stock prices, Z and $\tilde{Z}$, have non-positive drift.

If the imbalance announcement related to the orders of the other traders has a clear positive direction that outweighs the impact of our trader's order, then it may be optimal for our trader to trade before the imbalance announcement without participating in the closing auction. To avoid such situation, we assume our trader has at least a certain amount of predetermined total order volume (W).

By applying the Karush-Kuhn-Tucker conditions, we derive a set of explicit optimal investment strategies. The detailed proof is shown in section 5.1.1.

Furthermore, we later examine a generalized strategy in section 2.3 when the conditions mentioned above are removed.

Proposition 1. *Suppose that there are no orders in the closing auction after the initial imbalance announcement and the imbalance is cleared immediately. Assuming the investor is a flow trader and his/her participation has temporary market impacts. Suppose the investor has sufficient capital, in particular:*

$$W \geq \frac{\alpha}{2((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2) + \alpha}\mu_{\tilde{N}}. \tag{2.2}$$

If the random drivers reflected in stock prices, Z and $\tilde{Z}$, have non-positive drift, then we have the following:

1. *It is not optimal to trade after the initial imbalance announcement; that is, $v_k = 0$ for $k \in \{\tau, \ldots, T-1\}$.*

2. *Suppose the investor's orders in both the open market and the closing auction have an influence on the stock prices. We denote:*

$$m_t := (T-t)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha,$$
$$p_t := \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_2}{x_1^2 - 1}\right)x_1^t + \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_1}{x_2^2 - 1}\right)x_2^t,$$
$$q_t := \frac{x_1^t}{x_1^2 - 1} + \frac{x_2^t}{x_2^2 - 1},$$

where

$$x_1 := 1 + \frac{\lambda\sigma_Z^2}{2\beta} + \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)},$$

$$x_2 := 1 + \frac{\lambda\sigma_Z^2}{2\beta} - \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)}.$$

Let t^* be the smallest number such that:

$$\left((T - t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) > 0.$$

If there exist such $t^* \in \{1, \ldots, \tau - 2\}$, then the investment strategy in the open market is given by:

$$\begin{aligned}
v_s &= 0 \quad \textit{for } s \in \{0, \ldots, t^* - 1\}, \\
v_{t^*} &= \frac{\left((T - t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}, \\
v_i &= p_{i+1-t^*} v_{t^*} - \frac{\mu_Z}{2\beta} q_{i+1-t^*} \quad \textit{for } i \in \{t^* + 1, \ldots, \tau - 1\},
\end{aligned}$$

and the investment in the closing auction is:

$$v_T = W - \left(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\right) v_{t^*} + \frac{\mu_Z}{2\beta}\sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*}.$$

Otherwise, the optimal strategy is:

$$\begin{aligned}
v_t &= 0 \quad \textit{for } t \in \{1, \ldots, \tau - 2\}, \\
v_{\tau-1} &= \max\left(\frac{(T - \tau + 1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\left(\beta + (T - \tau + 1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right)}, 0\right), \\
v_T &= W - v_{\tau-1}.
\end{aligned}$$

3. *If the investor's orders have no influence on the stock prices in the open market, then the investments in the continuous trading can only occur at the beginning and the moment before the initial imbalance announcement. In particular, we have:*

$$
\begin{aligned}
v_t &= 0 \quad \textit{for } t \in \{1, \dots, \tau - 2\}, \\
v_{\tau-1} &= \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\big((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)}, 0 \right), \\
v_T &= W - v_{\tau-1}.
\end{aligned}
$$

4. *If the investor's orders have no influence on the stock prices in the closing auction, then it is optimal to invest only in the closing auction. That is,* $v_T = W$.

Remark: If the condition 2.2 is not met, we can still give explicit formulas for the optimal strategy, but they become more complicated. In particular, if investor's orders have an impact on the stock prices, we denote by t^* the smallest number such that:

$$
\left((T - t^*) + \frac{1}{\beta} \sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*} \right) \mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta_{t^*} > 0
$$

where

$$
\delta_{t^*} := \max\left(\left((T - t^*) + \frac{1}{\beta} \sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*} \right) \mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) \right.
$$

$$-\frac{2(\beta+m_{t^*}+\sum_{i=t^*+1}^{\tau-1}m_ip_{i+1-t^*})}{1+\sum_{i=t^*+1}^{\tau-1}p_{i+1-t^*}}\Bigg(W+\frac{\mu_Z}{2\beta}\sum_{i=t^*+1}^{\tau-1}q_{i+1-t^*}\Bigg),0\Bigg).$$

If there exist such $t^* \in \{1,\dots,\tau-2\}$, then the investment strategy in the open market is given by:

$$\begin{aligned}
v_s &= 0 \quad \text{for } s \in \{0,\dots,t^*-1\},\\
v_{t^*} &= \frac{\big((T-t^*)+\frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1}m_iq_{i+1-t^*}\big)\mu_Z+\mu_{\tilde{Z}}+\alpha(\mu_{\tilde{N}}+W)-\delta_{t^*}}{2(\beta+m_{t^*}+\sum_{i=t^*+1}^{\tau-1}m_ip_{i+1-t^*})},\\
v_i &= p_{i+1-t^*}v_{t^*}-\frac{\mu_Z}{2\beta}q_{i+1-t^*} \quad \text{for } i \in \{t^*+1,\dots,\tau-1\},
\end{aligned}$$

and the investment in the closing auction is:

$$v_T = \begin{cases} W-\Bigg(1+\sum_{i=t^*+1}^{\tau-1}p_{i+1-t^*}\Bigg)v_{t^*}+\frac{\mu_Z}{2\beta}\sum_{i=t^*+1}^{\tau-1}q_{i+1-t^*} & \text{if } \delta_{t^*}=0,\\ 0 & \text{if } \delta_{t^*}>0. \end{cases}$$

Otherwise, we denote:

$$\begin{aligned}
\delta = \max\Bigg(&(T-\tau+1)\mu_Z+\mu_{\tilde{Z}}+\alpha(\mu_{\tilde{N}}+W)\\
&-2W(\beta+(T-\tau+1)\lambda\sigma_Z^2+\lambda\sigma_{\tilde{Z}}^2+\lambda\alpha^2\sigma_{\tilde{N}}^2+\alpha),0\Bigg),
\end{aligned}$$

and the optimal strategy is

$$\begin{aligned}
v_t &= 0 \quad \text{for } t \in \{1,\dots,\tau-2\}\\
v_{\tau-1} &= \max\Bigg(\frac{(T-\tau+1)\mu_Z+\mu_{\tilde{Z}}+\alpha(\mu_{\tilde{N}}+W)-\delta}{2\big(\beta+(T-\tau+1)\lambda\sigma_Z^2+\lambda\sigma_{\tilde{Z}}^2+\lambda\alpha^2\sigma_{\tilde{N}}^2+\alpha\big)},0\Bigg),\\
v_T &= W-v_{\tau-1}.
\end{aligned}$$

If the investor's orders have no influence on the stock prices, then the investment in the open market will only occur at the moment before the initial imbalance announcement. In particular, we denote:

$$\begin{aligned}\delta = \max\Big(& (T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) \\ & - 2W\big((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big), 0\Big),\end{aligned}$$

and have:

$$\begin{aligned} v_t &= 0 \quad \text{for } t \in \{1, \dots, \tau-2\} \\ v_{\tau-1} &= \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\big((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)}, 0\right), \\ v_T &= W - v_{\tau-1}. \end{aligned}$$

Corollary 1. *Suppose $\mu_Z \leq 0$. As the investor's influence on the stock price in the open market (β) converges to 0, the optimal strategy will converge to*

$$\begin{aligned} v_t &= 0 \quad \textit{for } t \in \{1, \dots, \tau-2\} \\ v_{\tau-1} &= \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\big((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)}, 0\right), \\ v_T &= W - v_{\tau-1}. \end{aligned}$$

The proof of corollary 1 is available in section 5.1.2. The finding suggests that item 2 converges to item 3 in proposition 1, as β converges to 0. In other words, when the stock prices in the open market have a non-positive drift, if a flow trader's investment decision has extremely small influence on the stock prices in the continuous trading, then the optimal strategy converges to the strategy where there is zero effect on the open market (β=0). In addition, we will observe an analogy with its continuous model counterpart later in section 3.3.

2.3 General Optimal Strategy

Suppose we remove the constraints imposed on the drifts, $\mu_Z \leq 0$ and $\mu_{\tilde{Z}} \leq 0$, and the condition on the amount of capital, W. To address this question, we introduce a generalized strategy in this section. Unlike proposition 1, it is difficult to express the optimal strategy in the form of an explicit formula; instead, it will be presented in the form of an algorithm. As a trade-off of a generalized strategy, the execution can be time-consuming due to the computational iteration discussed in this section. The proof of the general strategy can be found in section 5.1.3.

We dissect the strategy into two cases. In particular, the case where the traders have influence on the stock prices in the open market ($\beta > 0$), and the case where the traders do not affect the open market ($\beta = 0$). In the first case where $\beta > 0$, we organize the various scenarios into three categories. We define Strategy A to be the strategy when one does not invest after the initial imbalance announcement. Strategy B is the optimal investment strategy when one would invest both before and after the initial imbalance announcement. Lastly, Strategy C is the strategy when one only invest after the initial imbalance announcement.

Case 1: $\beta > 0$

For all three strategies, we denote:

$$x_1 := 1 + \frac{\lambda\sigma_Z^2}{2\beta} + \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)},$$
$$x_2 := 1 + \frac{\lambda\sigma_Z^2}{2\beta} - \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)},$$

such that:

$$m_i := \begin{cases} (T-i)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha & \text{for } i \in \{1, \ldots, \tau - 1\}, \\ (T-i)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 & \text{for } i \in \{\tau, \ldots, T-1\}, \end{cases}$$

and for $i \in \{1, \ldots, \tau - 1\}$,

$$p_i := \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_2}{x_1^2 - 1}\right)x_1^i + \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_1}{x_2^2 - 1}\right)x_2^i,$$
$$q_i := \frac{x_1^i}{x_1^2 - 1} + \frac{x_2^i}{x_2^2 - 1}.$$

Strategy A: *$v_k = 0$ for $k \in \{\tau, \ldots, T-1\}$.*

Strategy A consider the case when the investment occur only prior to the initial imbalance announcement. The structure follows directly from the remark after proposition 1, which is exactly the optimal strategy under the drift conditions. However, if $\mu_Z > 0$, then the resulting formula may not hold due to the violation of our constraint, $v_i \geq 0$. In particular, applying the strategy directly could result in heavy investment in the earlier period and short selling ($v_i < 0$) in the later period. To avoid this issue, we select the optimal strategy with computational iteration.

Let $t^* \in \{1, \ldots, \tau - 2\}$ and $\bar{t} \in \{t^* + 1, \ldots, \tau - 1\}$ be the starting and ending time of investment, respectively. Moreover, let t^* be the some integer such that $v_{t^*} > 0$ and $v_{\bar{t}} > 0$, for some $\bar{t} \in \{t^* + 1, \ldots, \tau - 1\}$. We consider the auxiliary term:

$$\delta_{t^*}^{\bar{t}} := \max\Bigg(\Bigg(\Bigg((T - t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\bar{t}} m_i q_{i+1-t^*}\Bigg)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \frac{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\bar{t}} m_i p_{i+1-t^*})}{1 + \sum_{i=t^*+1}^{\bar{t}} p_{i+1-t^*}}\Bigg(W + \frac{\mu_Z}{2\beta}\sum_{i=t^*+1}^{\bar{t}} q_{i+1-t^*}\Bigg), 0\Bigg).$$

If there exist t^* and $\bar{t}$ as defined above, then the structure of the optimal investment strategy in the continuous trading, for $i \in \{1, \ldots, T-1\}$, is given by:

$$v_i = \begin{cases} \frac{\left((T-t^*)+\frac{1}{\beta}\sum_{i=t^*+1}^{\bar{t}} m_i q_{i+1-t^*}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}}+W) - \delta_{t^*}^{\bar{t}}}{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\bar{t}} m_i p_{i+1-t^*})} \text{ if } i = t^*, \\ p_{i+1-t^*} v_{t^*} - \frac{\mu_Z}{2\beta} q_{i+1-t^*} \text{ if } i \in \{t^*+1, \ldots, \bar{t}\}, \\ 0 \text{ if otherwise.} \end{cases}$$

The investment in the closing auction is $v_T = W - \sum_{i=1}^{T-1} v_i$.

If $\mu_Z \leq 0$, or $p_{i+1-t^*} v_{t^*} - \frac{\mu_Z}{2\beta} q_{i+1-t^*}$ is non-decreasing over i for all $t^* \in \{1, \ldots, \tau - 2\}$, then we have $\bar{t} = \tau - 1$ and t^* is the smallest integer such that:

$$\Bigg((T - t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\Bigg)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta_{t^*}^{\tau-1} > 0.$$

Otherwise, one would need to iterate through every combination of t^* and $\bar{t}$

such that $v_{t^*}, v_{\bar{t}} > 0$. For each pair of t^* and $\bar{t}$, we compute the strategy given above and compare the objective value, which is equivalent to:

$$\beta \sum_{t=1}^{T-1} v_t^2 + \alpha \sum_{t=1}^{\tau-1} v_t \sum_{t=1}^{T-1} v_t - \mu_Z \sum_{t=1}^{T-1} (T-t) v_t - \mu_{\tilde{Z}} \sum_{t=1}^{T-1} v_t - \alpha(\mu_{\tilde{N}} + W) \sum_{t=1}^{\tau-1} v_t$$
$$+ \lambda\sigma_Z^2 \sum_{t=1}^{T-1} \left(\sum_{i=0}^{t-1} v_i \right)^2 + \lambda\sigma_{\tilde{Z}}^2 \left(\sum_{t=1}^{T-1} v_t \right)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 \left(\sum_{t=1}^{\tau-1} v_t \right)^2, \tag{2.3}$$

as shown in Step 1 of section 5.1.1. The optimal t^* and $\bar{t}$ are given by the combination that leads to the lowest objective value.

If t^* and $\bar{t}$ defined above do not exist, then investment in continuous trading can only occur once at time $\bar{t} \in \{t^* + 1, \dots, \tau - 1\}$. We denote:

$$\delta^{\bar{t}} = \max\Big((T-\bar{t})\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)$$
$$- 2W(\beta + (T-\bar{t})\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha), 0 \Big),$$

and the optimal strategy in the open market is:

$$v_i = \begin{cases} \max\left(\frac{(T-\bar{t})\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta^{\bar{t}}}{2\left(\beta + (T-\bar{t})\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right)}, 0 \right) \text{ if } i = \bar{t}, \\ 0 \text{ if otherwise,} \end{cases}$$

and the investment in the closing auction is simply $v_T = W - v_{\bar{t}}$. Similarly, if $\mu_Z \leq 0$, then we have $\bar{t} = \tau - 1$. Otherwise, after iteratively examining the objective value in eq. (2.3) given by above strategy for all $\bar{t} \in \{1, \dots, \tau - 1\}$, the optimal strategy is given by the $\bar{t}$ yielding the smallest objective value.

Strategy B:

Strategy B is the strategy where one choose to invest during both periods, before and after the initial imbalance announcement at time τ. The beginning and ending time of investment for both period could vary drastically depending on the input parameters. We define $t^* \in \{1, \ldots, \tau-1\}$ to be the starting time and $\bar{t} \in \{t^*, \ldots, \tau-1\}$ to be the ending time of investment for the time horizon prior to time τ. Similarly, we let $k^* \in \{\tau, \ldots, T-1\}$ to be the starting time and $\bar{k} \in \{k^*, \ldots, T-1\}$ to be the ending time of investment after time τ. In other words, we define $t^* \in \{1, \ldots, \tau-1\}$ and $k^* \in \{\tau, \ldots, T-1\}$ to be the integers such that $v_{t^*}, v_{\bar{t}} > 0$ and $v_{k^*}, v_{\bar{k}} > 0$, respectively, for some $\bar{t} \in \{t^*, \ldots, \tau-1\}$ and $\bar{k} \in \{k^*, \ldots, \tau-1\}$.

We denote:

$$r_i := \frac{\beta + \lambda\sigma_Z^2 - \beta x_2}{\beta(x_1^2 - 1)} x_1^i + \frac{\beta + \lambda\sigma_Z^2 - \beta x_1}{\beta(x_2^2 - 1)} x_2^i,$$

and

$$\tilde{p}_i := \begin{cases} \frac{\beta+\lambda\sigma_Z^2}{\beta} p_{\tau-i} - p_{\tau-i-1} \text{ if } i \in \{1, \ldots, \tau-2\}, \\ \frac{\lambda\sigma_Z^2}{\beta} \text{ if } i = \tau - 1. \end{cases}$$

$$\tilde{q}_i := \begin{cases} \frac{\beta+\lambda\sigma_Z^2}{\beta} q_{\tau-i} - q_{\tau-i-1} \text{ if } i \in \{1, \ldots, \tau-2\}, \\ 1 \text{ if } i = \tau - 1. \end{cases}$$

We suppose t^*, k^*, $\bar{t}$, and $\bar{k}$ defined above exist. We further denote:

$$a_1(t^*, k^*, \bar{t}, \bar{k}) := \begin{cases} \beta + m_{t^*} + \sum_{i=t^*+1}^{\bar{t}} m_i p_{i+1-t^*} + \tilde{p}_{t^*} \sum_{i=k^*+1}^{\bar{k}} \big(m_i + \frac{\alpha}{2}\big) q_{i-k^*+1}, \\ \beta + m_{t^*} + \sum_{i=t^*+1}^{\bar{t}} m_i p_{i+1-t^*} \quad \text{if } k^* = \bar{k}, \\ \beta + m_{\bar{t}} + \frac{\lambda \sigma_Z^2}{\beta} \sum_{i=k^*+1}^{\bar{k}} \big(m_i + \frac{\alpha}{2}\big) q_{i-k^*+1} \quad \text{if } t^* = \bar{t}, \\ \beta + m_{\bar{t}} \quad \text{if } t^* = \bar{t} \text{ and } k^* = \bar{k}. \end{cases}$$

$$a_2(t^*, k^*, \bar{t}, \bar{k}) := \begin{cases} \big(m_{k^*} + \frac{\alpha}{2}\big)\big(1 + \sum_{i=t^*+1}^{\bar{t}} p_{i+1-t^*}\big) + \tilde{p}_{t^*} \sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1}, \\ 1 + \sum_{i=t^*+1}^{\bar{t}} p_{i+1-t^*} \quad \text{if } k^* = \bar{k}, \\ m_{k^*} + \frac{\alpha}{2} + \frac{\lambda \sigma_Z^2}{\beta} \sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1} \quad \text{if } t^* = \bar{t}, \\ m_{\bar{k}} + \frac{\alpha}{2} \quad \text{if } t^* = \bar{t} \text{ and } k^* = \bar{k}. \end{cases}$$

$$b_1(t^*, k^*, \bar{k}) := \begin{cases} m_{k^*} + \frac{\alpha}{2} + \sum_{i=k^*+1}^{\bar{k}} \big(m_i + \frac{\alpha}{2}\big) r_{i-k^*+1} \quad \text{if } k^* < \bar{k}, \\ m_{\bar{k}} \quad \text{if } k^* = \bar{k}. \end{cases}$$

$$b_2(t^*, k^*, \bar{k}) := \begin{cases} \beta + m_{k^*} + \sum_{i=k^*+1}^{\bar{k}} m_i r_{i-k^*+1} \quad \text{if } k^* < \bar{k}, \\ \beta + m_{\bar{k}} \quad \text{if } k^* = \bar{k}. \end{cases}$$

$$s_1(t^*, k^*, \bar{t}, \bar{k}) := \begin{cases} \sum_{i=t^*+1}^{\bar{t}} m_i q_{i+1-t^*} + \tilde{q}_{t^*} \sum_{i=k^*+1}^{\bar{k}} \big(m_i + \frac{\alpha}{2}\big) q_{i-k^*+1}, \\ \sum_{i=t^*+1}^{\bar{t}} m_i q_{i+1-t^*} \quad \text{if } k^* = \bar{k}, \\ \sum_{i=k^*+1}^{\bar{k}} \big(m_i + \frac{\alpha}{2}\big) q_{i-k^*+1} \quad \text{if } t^* = \bar{t}, \\ 0 \quad \text{if } t^* = \bar{t} \text{ and } k^* = \bar{k}. \end{cases}$$

$$s_2(t^*, k^*, \bar{t}, \bar{k}) := \begin{cases} \big(m_{k^*} + \frac{\alpha}{2}\big) \sum_{i=t^*+1}^{\bar{t}} q_{i+1-t^*} + \tilde{q}_{t^*} \sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1}, \\ \sum_{i=t^*+1}^{\bar{t}} q_{i+1-t^*} \quad \text{if } k^* = \bar{k}, \\ \sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1} \quad \text{if } t^* = \bar{t}, \\ 0 \quad \text{if } t^* = \bar{t} \text{ and } k^* = \bar{k}. \end{cases}$$

Moreover:

$$A(t^*,k^*,\bar{t},\bar{k}) := \begin{cases} 1+\sum_{i=t^*+1}^{\bar{t}} p_{i+1-t^*} + \tilde{p}_{t^*}\sum_{k^*+1}^{\bar{k}} q_{i-k^*+1}, \\ 1+\sum_{i=t^*+1}^{\bar{t}} p_{i+1-t^*} \quad \text{if } k^*=\bar{k}, \\ 1+\frac{\lambda\sigma_Z^2}{\beta}\sum_{k^*+1}^{\bar{k}} q_{i-k^*+1} \quad \text{if } t^*=\bar{t}, \\ 1 \quad \text{if } t^*=\bar{t} \text{ and } k^*=\bar{k}. \end{cases}$$

$$B(t^*,k^*,\bar{t},\bar{k}) := \begin{cases} 1+\sum_{i=k^*+1}^{\bar{k}} r_{i-k^*+1} \quad \forall t^*<\bar{t}, \\ 1 \quad \text{if } k^*=\bar{k}, \quad \forall t^*<\bar{t}, \\ 0 \quad \text{if } t^*=\bar{t}. \end{cases}$$

$$C(t^*,k^*,\bar{t},\bar{k}) := \begin{cases} \sum_{i=t^*+1}^{\bar{t}} q_i + \tilde{q}_{t^*}\sum_{i=k^*+1}^{\bar{k}} q_{i-k^*+1}, \\ \sum_{i=t^*+1}^{\bar{t}} q_{i+1-t^*} \quad \text{if } k^*=\bar{k}, \\ \sum_{i=k^*+1}^{\bar{k}} q_{i-k^*+1} \quad \text{if } t^*=\bar{t}, \\ 0 \quad \text{if } t^*=\bar{t} \text{ and } k^*=\bar{k}. \end{cases}$$

Furthermore, we let:

$$\begin{aligned} v_1^{num}(t^*,k^*,\bar{t},\bar{k}) :=& \Bigg(b_1(t^*,k^*,\bar{t},\bar{k})\left(T-k^*+\frac{s_2(t^*,k^*,\bar{t},\bar{k})}{\beta}\right) \\ &- b_2(t^*,k^*,\bar{t},\bar{k})\left(T-t^*+\frac{s_1(t^*,k^*,\bar{t},\bar{k})}{\beta}\right)\Bigg)\mu_Z \\ &+ \big(b_1(t^*,k^*,\bar{t},\bar{k}) - b_2(t^*,k^*,\bar{t},\bar{k})\big)\mu_{\tilde{Z}} - b_2(t^*,k^*,\bar{t},\bar{k})\alpha(\mu_{\tilde{N}}+W), \\ v_1^{den}(t^*,k^*,\bar{t},\bar{k}) :=& 2\big(b_1(t^*,k^*,\bar{t},\bar{k})a_2(t^*,k^*,\bar{t},\bar{k}) - b_2(t^*,k^*,\bar{t},\bar{k})a_1(t^*,k^*,\bar{t},\bar{k})\big), \end{aligned}$$

and

$$
\begin{aligned}
v_2^{num}(t^*,k^*,\bar{t},\bar{k}) :=& \Bigg(a_1(t^*,k^*,\bar{t},\bar{k})\Big(T-k^*+\frac{s_2(t^*,k^*,\bar{t},\bar{k})}{\beta}\Big) \\
&- a_2(t^*,k^*,\bar{t},\bar{k})\Big(T-t^*+\frac{s_1(t^*,k^*,\bar{t},\bar{k})}{\beta}\Big)\Bigg)\mu_Z \\
&+ \big(a_1(t^*,k^*,\bar{t},\bar{k})-a_2(t^*,k^*,\bar{t},\bar{k})\big)\mu_{\tilde{Z}} - a_2(t^*,k^*,\bar{t},\bar{k})\alpha(\mu_{\tilde{N}}+W), \\
v_2^{den}(t^*,k^*,\bar{t},\bar{k}) :=& 2\big(a_1(t^*,k^*,\bar{t},\bar{k})b_2(t^*,k^*,\bar{t},\bar{k}) - a_2(t^*,k^*,\bar{t},\bar{k})b_1(t^*,k^*,\bar{t},\bar{k})\big),
\end{aligned}
$$

such that:

$$
X(t^*,k^*,\bar{t},\bar{k}) := \frac{v_1^{num}(t^*,k^*,\bar{t},\bar{k})}{v_1^{den}(t^*,k^*,\bar{t},\bar{k})}, \quad Y(t^*,k^*,\bar{t},\bar{k}) := \frac{v_2^{num}(t^*,k^*,\bar{t},\bar{k})}{v_2^{den}(t^*,k^*,\bar{t},\bar{k})}.
$$

We find that, for the auxiliary term:

$$
\delta(t^*,k^*,\bar{t},\bar{k}) = \max\left(\frac{2\big(b_1a_2-b_2a_1\big)}{A(b_1-b_2)-B(a_1-a_2)}\left(AX+BY-\frac{\mu_Z}{2\beta}C-W\right),0\right),
$$

the optimal strategy in continuous trading takes the form of:

$$
v_i = \begin{cases}
X - \frac{(b_1-b_2)}{2(b_1a_2-b_2a_1)}\delta \text{ if } i=t^*, \\
p_{i+1-t^*}v_{t^*} - \frac{\mu_Z}{2\beta}q_{i+1-t^*} \text{ if } i\in\{t^*+1,\dots,\bar{t}\}, \\
Y + \frac{(a_1-a_2)}{2(b_1a_2-b_2a_1)}\delta \text{ if } i=k^*, \\
\tilde{p}_{t^*}q_{i-k^*+1}v_{t^*} + r_{i-k^*+1}v_{k^*} - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}q_{i-k^*+1} \text{ if } i\in\{k^*+1,\dots,\bar{k}\}, \\
0 \text{ if otherwise,}
\end{cases}
$$

for $i\in\{1,\dots,T-1\}$. The investment in the closing auction is therefore given by $v_T = W - \sum_{i=1}^{T-1} v_i$.

The challenge is now to determine the optimal starting time, t^* and k^*, and the ending time $\bar{t}$ and $\bar{k}$. If $\mu_Z \leq 0$ or the following two equations are non-decreasing,

$$p_{i+1-t^*}v_{t^*} - \frac{\mu_Z}{2\beta}q_{i+1-t^*},$$

$$\tilde{p}_{t^*}q_{i-k^*+1}v_{t^*} + r_{i-k^*+1}v_{k^*} - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}q_{i-k^*+1},$$

then it is clear that $\bar{t} = \tau - 1$ and $\bar{k} = T - 1$. In this case, $t^* \in \{1, \ldots, \tau - 1\}$ and $k^* \in \{\tau, \ldots, T-1\}$ are smallest integers such that $v_{t^*} > 0$ and $v_{k^*} > 0$.

Though one cannot easily derive the optimal value $t^*, k^*, \bar{t}$, and $\bar{k}$ explicitly, they can be found iteratively. In the case where the above two equations are increasing, one can iterate every combination of $t^*, k^*, \bar{t}$, and $\bar{k}$ such that $v_{t^*}, v_{\bar{t}}, v_{k^*}, v_{\bar{k}} > 0$. The optimal strategy is given by the set of combination that yields the lowest objective value shown in eq. (2.3). Depending on the over all time horizon and the time increment of the investment, the overall iteration procedure can be very time-consuming.

<u>***Strategy C***: $v_t = 0$ *for* $t \in \{1, \dots, \tau - 1\}$.</u>

The overall presentation of Strategy C is similar to that of Strategy A. Suppose $k^* \in \{\tau, \dots, T-2\}$ and $\bar{k} \in \{t^*+1, \dots, T-1\}$ are the starting and ending time of investment, such that $v_{k^*} > 0$ and $v_{\bar{k}} > 0$, respectively. We consider:

$$\delta_{k^*}^{\bar{k}} := \max\Bigg(\bigg(\Big((T-k^*) + \frac{1}{\beta}\sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1}\Big)\mu_Z + \mu_{\tilde{Z}} \\ - \frac{2(\beta + m_{k^*} + \sum_{i=k^*+1}^{\bar{k}} m_i p_{i-k^*+1})}{1 + \sum_{i=k^*+1}^{T-1} p_{i-k^*+1}}\Big(W + \frac{\mu_Z}{2\beta}\sum_{i=k^*+1}^{\bar{k}} q_{i-k^*+1}\Big), 0\Bigg).$$

If such k^* and $\bar{k}$ exist, then the optimal strategy in the open market is given by:

$$v_i = \begin{cases} \frac{\left((T-k^*)+\frac{1}{\beta}\sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1}\right)\mu_Z + \mu_{\tilde{Z}} - \delta_{k^*}^{\bar{k}}}{2(\beta + m_{k^*} + \sum_{i=k^*+1}^{\bar{k}} m_i p_{i-k^*+1})} \text{ if } i = k^* \\ p_{i-k^*+1} v_{k^*} - \frac{\mu_Z}{2\beta} q_{i-k^*+1} \text{ if } i \in \{k^*+1, \dots, \bar{k}\} \\ 0 \text{ if otherwise} \end{cases}.$$

The investment in the closing auction is again $v_T = W - \sum_{i=1}^{T-1} v_i$.

If $\mu_Z \le 0$, or $p_{i-k^*+1} v_{k^*} - \frac{\mu_Z}{2\beta} q_{i-k^*+1}$ is non-decreasing over i for all $k^* \in \{1, \dots, T-2\}$, then we have $\bar{k} = T-1$ and k^* is the smallest integer such that:

$$\big((T-k^*) + \frac{1}{\beta}\sum_{i=k^*+1}^{\bar{k}} m_i q_{i-k^*+1}\big)\mu_Z + \mu_{\tilde{Z}} - \delta_{k^*}^{\bar{k}} > 0.$$

If not, then the optimal strategy can be found by iterating through all combinations of k^* and $\bar{k}$ such that $v_{k^*}, v_{\bar{k}} > 0$. The pair of k^* and $\bar{k}$ that suggests the lowest objective value shown in eq. (2.3) gives the optimal strategy.

If k^* and $\bar{k}$ defined above do not exist, then investment in continuous trading can only occur once at time $\bar{k} \in \{k^* + 1, \ldots, T-1\}$. We denote:

$$\delta^{\bar{k}} = \max\left((T-\bar{k})\mu_Z + \mu_{\tilde{Z}} - 2W(\beta + (T-\bar{t})\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2), 0\right),$$

and the optimal strategy in the open market is:

$$v_i = \begin{cases} \max\left(\frac{(T-\bar{k})\mu_Z + \mu_{\tilde{Z}} - \delta^{\bar{k}}}{2\left(\beta + (T-\bar{t})\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2\right)}, 0\right) \text{ if } i = \bar{k}, \\ 0 \text{ if otherwise,} \end{cases}$$

and the investment in the closing auction is simply $v_T = W - v_{\bar{k}}$. Similarly, if $\mu_Z \leq 0$, then we have $\bar{t} = T - 1$. Otherwise, after iteratively examining the objective value in eq. (2.3) given by above strategy for all $\bar{k} \in \{1, \ldots, T-1\}$, the optimal strategy is given by the $\bar{k}$ yielding the smallest objective value.

Overall, the optimal strategy for case 1, is the one among the resulting strategy of Strategies A, B, and C, that suggests the lowest target value.

Case 2: $\beta = 0$**.**

In the investor has no influence in the continuous trading at all, then the investment can only occur at three particular time periods. Specifically, at the very beginning of the investment time horizon (time t^*), the period before and the period after the initial imbalance announcement at time τ. We denote by $\bar{t}$ and $\bar{k}$ the starting time of the investment before and after the initial imbalance, respectively. As such, the strategy for the continuous market is given by:

$$
\begin{aligned}
v_{t^*} &= \max\left(\frac{\mu_Z}{2\lambda\sigma_Z^2}, 0\right), \\
v_{\bar{t}} &= \max\left(\frac{(T-\bar{t})\mu_Z + 2\alpha(\mu_{\tilde{N}} + W)}{2\lambda\big((T-\bar{t})\sigma_Z^2 + \alpha^2\sigma_{\tilde{N}}^2\big) + \alpha} - v_{t^*}, 0\right), \\
v_{\bar{k}} &= \max\left(\frac{(T-\bar{k})\mu_Z + \mu_{\tilde{Z}} - \delta_{\bar{t}}^{\bar{k}}}{2m_{\bar{k}}} - (v_1 + v_{\bar{t}})\left(1 + \frac{\alpha}{2m_{\bar{k}}}\right), 0\right),
\end{aligned}
$$

where $\delta_{\bar{t}}^{\bar{k}} = \max\big(\mu_{\tilde{Z}} - \alpha(v_{t^*} + v_{\bar{t}}) - 2Wm_{\bar{k}}, 0\big)$.

If $v_{\bar{k}} = 0$, then the strategy is:

$$
\begin{aligned}
v_{t^*} &= \max\left(\frac{\mu_Z}{2\lambda\sigma_Z^2}, 0\right), \\
v_{\bar{t}} &= \max\left(\frac{(T-\bar{t})\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta_{\bar{t}}}{2m_{\bar{t}}} - v_{t^*}, 0\right),
\end{aligned}
$$

with $\delta_{\bar{t}} = \max\big((T-\bar{t})\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2Wm_{\bar{t}}, 0\big)$.

If both $v_{\bar{t}} = 0$ and $v_{\bar{k}} = 0$, then the strategy is simply:

$$v_{t^*} = \max\left(\frac{(T-t^*)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2m_1}, 0\right),$$

with $\delta = \max\big((T-t^*)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2Wm_1, 0\big)$.

If $\mu_Z \leq 0$, then we have $\bar{t} = \tau - 1$ and $\bar{k} = T - 1$. Otherwise, the optimal strategy can by found iteratively by locating the set of t^*, $\bar{t}$, and $\bar{k}$ that yields the lowest objective value depicted in eq. (2.3).

Chapter 3

Continuous-Time Model

3.1 Problem Formulation

The orders in the open market can be executed at a high frequency, which means the time increment between each transaction is nearly zero. To incorporate the continuous-time structure, we adjust the model accordingly. Suppose we trade at the rate of v_s in the open market time s for $s \in [0, T)$ where time T corresponds to 4:00 PM EST, the close of the market. We denote by τ the time when the initial imbalance is published at 3:50 PM EST. Assume the order imbalance is cleared immediately and there is no orders in the closing auction after 3:50 PM EST. Suppose the flow trader's orders have temporary market impact, then the investment decision at time s can only influence the stock price at time s and not the subsequent prices at time t for $t \in (s, T]$. Similarly to the discrete-time model, the order placed in the closing auction, v_T, is accounted for in all stock prices after the initial imbalance announcement (during the time interval $(\tau, T]$), and does not only affect the closing price. Furthermore, we model the movement of stock prices without the trader's involvement with an arithmetic Brownian motion. An arithmetic Brownian motion satisfies:

$$dY_t = \sigma dW_t + \mu dt$$

which is equivalent to:

$$Y_t = Y_0 + \sigma W_t + \mu t$$

by integration. In addition, the absolute change, dY_t, is normally distributed.

We define β and α to be non-negative scalars to measure the effect on the stock prices due to the investor's orders in the open market and the auction imbalance announcement, respectively. $\tilde{Z}$ is a random variable to capture the stock movement at time T. We denote by $\mu_{\tilde{Z}}$ the mean of $\tilde{Z}$ and by $\sigma^2_{\tilde{Z}}$ its variance. Moreover, the imbalance process N is expressed as:

$$N = \tilde{N} + W - \int_0^T v_t dt.$$

W is the total amount orders given in advance, which can be written as:

$$W = \int_0^T v_t dt + v_T.$$

As such, the prices of the stock are given by:

$$P_t = \tilde{P}_t + \beta v_t \quad \text{for} \quad t \in [0, T),$$

$$P_T = \tilde{P}_T,$$

where

$$\tilde{P}_t = \tilde{P}_0 + \mu t + \sigma W_t \quad \text{for} \quad t \in [0, \tau),$$
$$\tilde{P}_s = \tilde{P}_0 + \mu s + \sigma W_s + \alpha N \quad \text{for} \quad s \in [\tau, T),$$
$$\tilde{P}_T = \tilde{P}_0 + \mu T + \sigma W_T + \alpha N + \tilde{Z}.$$

For any risk aversion parameter, $\lambda > 0$, the objective function for a flow trader is presented as:

$$\begin{aligned} \min \quad & E\Big[\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T\Big] \\ & + \lambda VAR\Big[\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T\Big] \\ s.t. \quad & v_t \geq 0 \quad \forall t \in [0, T), \quad W - \int_0^T v_t dt \geq 0. \end{aligned}$$

Additionally, we define the cumulative order up to time t as:

$$X_t^v = \int_0^t v_s ds.$$

3.2 Excursion: the Euler-Lagrange Equation

The Euler-Lagrange equation is a fundamental mathematical tool in the field of Calculus of Variation. In this chapter, this concept plays a key role in deriving the results. The methodology gives a differential equation such that one can optimize equations of the form:

$$J = \int_a^b F(x, y, y')dx.$$

Gelfand and Fomin [7] give a detailed presentation of the Euler-Lagrange equation and its various forms in the context of Calculus of Variation. In addition, Weinstock [12] shows various application of the Euler-Lagrange equation. In this section, we give a brief description on the application of the Euler-Lagrange equation used in the proof of proposition 2 in section 5.2.1; further detail can be found in Chapter 7 of [7].

We consider the problem of the form $\min_y J = \Phi(I_1, \ldots, I_N)$ where

$$I_k = \int_a^b F_k(x, y, y')dx \quad \text{for } k \in \{1, \ldots, N\},$$

and the function Φ is continuously differentiable. For some Lagrange multipliers, $\lambda_1, \ldots, \lambda_N$, the objective function can be expressed as:

$$\min_y J = \min_y \min_{I_k} \max_{\lambda_k} \left(\Phi + \sum_{k=1}^{N} \lambda_k \left(I_k - \int_a^b F_k(x, y, y')dx \right) \right).$$

Thus, for all k, we have:

$$\frac{\partial \Phi}{\partial I_k} + \lambda_k = 0.$$

We consider:

$$\psi = \sum_{k=1}^{N} \frac{\partial \Phi}{\partial I_k} F_k(x, y, y').$$

The methodology suggests that the optimal solution can be determined by solving the Euler-Lagrange equation:

$$0 = \frac{d}{dt}\frac{\partial \psi}{\partial y'} - \frac{\partial \psi}{\partial y}.$$

3.3 Optimal Strategy under Drift Condition

In this section, we propose a set of optimal investment strategies for flow traders when assuming a continuous-time model. Similarly to the discrete time model in section 3.3, we first examine a model with additional conditions on the drifts of stock prices and the amount of predetermined total order volume (W). We recall that the assumptions we imposed on the drift are:

$$\mu_Z \leq 0, \quad \mu_{\tilde{Z}} \leq 0,$$

As an analogy with the discrete-time model, above assumption suggests no investment after the initial imbalance announcement. Furthermore, we similarly impose an assumption on the capital W. If this condition is not satisfied, then the trader may not invest in the closing auction. Moreover, similarly to section 2.2, we further impose the condition that $\mu \leq 0$ to avoid complicated presentation in this section, due to computational iteration, which is further discussed in section 3.4. Hence, the drift conditions become, $\mu \leq 0$ and $\mu_{\tilde{Z}} \leq 0$.

In comparison with the discrete-time model, the mathematical derivation shown in section 5.2.1 is structurally simpler with the help of the Euler-

Lagrange equation mentioned in section 3.2. Furthermore, later in section 3.4, we present a more generalized strategy when the conditions mentioned previously are removed.

Proposition 2. *Suppose there are no orders in the closing auction after the initial imbalance announcement and the imbalance is cleared immediately. Assuming the investor is a flow trader and his/her participation has temporary market impacts. Suppose the investor has sufficient capital, in particular:*

$$W \geq 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)c(0) - \frac{\mu - \mu e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}}{2\lambda\sigma^2}. \tag{3.1}$$

If the random drivers reflected in stock prices have non-positive drift, $\mu \leq 0$ and $\mu_{\tilde{Z}} \leq 0$, then we have the following:

1. *It is not optimal to trade on a stock after the initial imbalance announcement; that is, $v_t = 0$ for $t \in (\tau, T)$.*
2. *Suppose the investor's orders in the open market influence the stock prices. That is, $\beta > 0$. We denote:*

$$m_1 = T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W),$$
$$m_2 = \frac{\mu}{\lambda\sigma^2}\left(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right),$$

and

$$c_{num}(t) = \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)\left(m_1 - m_2\left(1 - e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}\right)\right) - \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\left(m_1\right.$$
$$\left. - m_2\left(1 - e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\right) - \mu\left(\tau\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - t\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\right.$$

$$
\begin{aligned}
&- \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau} - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\Bigg),\\
c_{den}(t) =& 2\sqrt{\beta\lambda\sigma^2}\Bigg(\sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\Bigg)\\
&+ 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\Bigg(\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\Bigg),
\end{aligned}
$$

such that:

$$
c(t) = \frac{c_{num}(t)}{c_{den}(t)}.
$$

Let t^* be the smallest number such that:

$$
2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c(t^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} > 0.
$$

If there exist such $t^* \in [0, \tau)$, then the rate of trading in the open market at time t is given by:

$$
\begin{aligned}
v_s &= 0 \quad \textit{for } s \in [0, t^*),\\
v_t &= \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c(t^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right),
\end{aligned}
$$

and the cumulative order at time t is:

$$
\begin{aligned}
X_s^v &= 0 \quad \textit{for } s \in [0, t^*),\\
X_t^v &= 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c(t^*) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right).
\end{aligned}
$$

Furthermore, the investment in the closing auction is:

$$
v_T = W - X_\tau^v.
$$

Otherwise, the investment only occur in the closing auction:

$$v_T = W.$$

3. *If the investor's orders have no influence on the stock prices in the open market ($\beta = 0$), then the investments in the open market can only occur at the beginning and the moment before the closing auction, which is denoted as $\tilde{\tau} = \tau - \epsilon$ for some small $\epsilon > 0$. In particular, we have:*

$$V_{\tilde{\tau}} = \max\left(\frac{\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)}, 0\right),$$
$$V_T = W - V_T,$$

where $\delta = \max\Big(\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} - 2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)W, 0\Big).$

In (3.1), the term $\frac{\mu - \mu e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}}{2\lambda\sigma^2}$ is non-negative if $\mu \leq 0$. Therefore, under the assumption $\mu \leq 0$, the condition (3.1) is satisfied if $W \geq 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)c(0)$, which holds if W is big relative to $\frac{\lambda\sigma^2}{\beta}$.

Remark: Suppose the condition (3.1) does not hold. For simplicity, we denote $a_i = e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}i}$, and we consider:

$$\delta = \max\Bigg(\frac{2\sinh(a_\tau)c(t^*) - \frac{\mu}{2\lambda\sigma^2}(1 - e^{-a_\tau}) - W}{\sinh(2a_\tau) - \sinh(2a_{t^*})}\big(2\sqrt{\beta\lambda\sigma^2}(\sinh(2a_\tau) - \sinh(2a_{t^*}))$$
$$+ 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)(\sinh^2(a_\tau) - \sinh^2(a_{t^*}))\big), 0\Bigg).$$

We define:

$$\tilde{c}_{num}(t) = c_{num}(t) - \delta(\sinh(2a_\tau) - \sinh(2a_{t^*})),$$

such that

$$\tilde{c}(t) = \frac{\tilde{c}_{num}(t)}{c_{den}(t)}.$$

If there exists $t^* \in [0, \tau)$, then the rate of trading in the open market at time t is given by:

$$v_s = 0 \quad \text{for } s \in [0, t^*),$$
$$v_t = \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\tilde{c}(t^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right),$$

and the cumulative order at time t is:

$$X_s^v = 0 \quad \text{for } s \in [0, t^*),$$
$$X_t^v = 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\tilde{c}(t^*) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right).$$

We find that as the influence of a flow trader's investment decision on the stock prices diminishes approximately to 0, the optimal strategy is identical to the strategy where the trader has no influence. We recall a similar finding in the discrete-time model shown in corollary 1. The detailed proof can be found in section 5.2.2, and we propose:

Corollary 2. *Suppose $\mu \leq 0$. As β converges to 0, the optimal strategy shown in item 2 of proposition 2 converges to item 3. In particular,*

$$V_{\tilde{\tau}} = \max\left(\frac{\mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)}, 0\right),$$
$$V_T = W - V_T,$$

where $\delta = \max\left(\mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} - 2\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)W, 0\right).$

3.4 General Optimal Strategy

In this section, we discuss the generalized strategy if we omit the conditions we imposed in proposition 2. Similarly to the discrete-time model, we will dissect the overall strategy into two cases. The first case is when a flow trader's investment decision has influence on the stock prices during the continuous trading ($\beta > 0$). The second case is the opposite where the trader has no effect at all ($\beta = 0$). The proof of the optimal strategy is shown in section 5.2.3. As we have in section 2.3, we further separate case 1 into three strategies. We let Strategy A be the strategy when one only invest before the initial imbalance announcement. Strategy B is the optimal investment strategies when the trader invest both before and after the initial imbalance announcement. Strategy C is the strategy when the trader only invest after the initial imbalance announcement.

***Case 1**: $\beta > 0$*

For simplicity, we denote:

$$a(i) := \sqrt{\frac{\lambda\sigma^2}{\beta}}i.$$

in all three strategies.

***Strategy A**: $v_k = 0$ for $k \in [\tau, T)$.*

Strategy A considers the case when the investment occurs only prior to the initial imbalance announcement. The strategy is given by the remark of proposition 2. Similar to the discrete-time model, if $\mu > 0$, then the resulting formula may not hold due to the violation of our constraint, $v_i \geq 0$. Specifically, implementing the strategy without any modification could result in excessive amount of investment in the earlier period and then short selling ($v_i < 0$) afterward. In particular, one can select the optimal strategy with computational iteration.

For some small $\epsilon > 0$, let $t^* \in [0, \tau)$ and $\tau^* \in [t^* + \epsilon, \tau)$ be the starting and ending time of investment, respectively, such that $v_{t^*} > 0$ and $v_{\bar{t}} > 0$. We recall from proposition 2 that:

$$m_1 = T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W),$$
$$m_2 = \frac{\mu}{\lambda\sigma^2}\big(\lambda\sigma^2_{\tilde{Z}} + \lambda\alpha^2\sigma^2_{\tilde{N}} + \alpha\big),$$

and

$$\begin{aligned} c_{num}(t, \bar{t}) =& \sinh\big(a(\bar{t})\big)\big(m_1 - m_2\big(1 - \mathrm{e}^{a(\bar{t})}\big)\big) - \sinh\big(a(t)\big)\big(m_1 - m_2\big(1 - \mathrm{e}^{a(t)}\big)\big) \\ & - \mu\left(\bar{t}\sinh\big(a(\bar{t})\big) - t\sinh\big(a(t)\big) - \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2a(\bar{t})} - \mathrm{e}^{-2a(t)}\right)\right), \\ c_{den}(t, \bar{t}) =& 2\sqrt{\beta\lambda\sigma^2}\big(\sinh\big(2a(\bar{t})\big) - \sinh\big(2a(t)\big)\big) \\ & + 4\big(\lambda\sigma^2_{\tilde{Z}} + \lambda\alpha^2\sigma^2_{\tilde{N}} + \alpha\big)\left(\sinh^2\big(a(\bar{t})\big) - \sinh^2\big(a(t)\big)\right), \end{aligned}$$

such that $c(t, \bar{t}) = \frac{c_{num}(t,\bar{t})}{c_{den}(t,\bar{t})}$. We consider the auxiliary term:

$$\begin{aligned} \delta(t, \bar{t}) =& \max\Bigg(\frac{2\sinh(a(\tau^*))c(t^*, \tau^*) - \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-a(\tau^*)}\big) - W}{\sinh(2a(\tau^*) - \sinh(2a(t^*))}\big(2\sqrt{\beta\lambda\sigma^2}(\sinh(2a(\tau^*)) \\ & - \sinh(2a(t^*))) + 4\big(\lambda\sigma^2_{\tilde{Z}} + \lambda\alpha^2\sigma^2_{\tilde{N}} + \alpha\big)(\sinh^2(a(\tau^*)) - \sinh^2(a(t^*)))), 0\Bigg). \end{aligned}$$

We define:

$$\tilde{c}(t,\bar{t}) = \frac{c_{num}(t,\bar{t}) - \delta(t,\bar{t})\big(\sinh(2a(\bar{t})) - \sinh(2a(\bar{t}))\big)}{c_{den}(t,\bar{t})}.$$

If there exist $t^* \in [0,\tau)$ and $\tau^* \in [t^*+\epsilon,\tau)$, then the rate of trading in the open market at time t is given by:

$$v_t = \begin{cases} \sqrt{\frac{\lambda\sigma^2}{\beta}}\big(2\cosh\big(a(t)\big)\tilde{c}(t^*,\tau^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-a(t)}\big) \text{ if } t \in [t^*,\tau^*], \\ 0 \text{ if otherwise,} \end{cases}$$

and the cumulative order at time t is given by:

$$X_t^v = \begin{cases} 2\sinh\big((a(t)\big)\tilde{c}(t,\bar{t}) + \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-(a(t)}\big) \text{ if } t \in [t^*,\tau^*], \\ 0 \text{ if otherwise.} \end{cases}$$

The investment in the closing auction is, therefore, $v_T = W - X_T^v$.

If $\mu \leq 0$, then $\tau^* = \tau - \epsilon$, and the strategy is identical to the one given in the remark of proposition 2. If $\mu > 0$, then one may need to iterate through every pair of $t^* \in [0,\tau)$ and $\tau^* \in [t^*+\epsilon,\tau)$ and check for the objective value:

$$\begin{aligned} &\beta\int_0^T v_t^2 dt - \mu\int_0^T (T-t)v_t dt - \mu_{\tilde{Z}}\int_0^T v_t dt - \alpha\int_0^\tau v_t\Big(\mu_{\tilde{N}} + W - \int_0^T v_t dt\Big)dt \\ &+ \lambda\sigma^2\int_0^T (X_t^v)^2 dt + \lambda\sigma_{\tilde{Z}}^2\Big(\int_0^T v_t dt\Big)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2\Big(\int_0^\tau v_t dt\Big)^2, \end{aligned} \tag{3.2}$$

which is derived in section 5.2.1. The optimal strategy is given by the set of t^* and τ^* that yields the lowest target value.

In the case that t^* and τ^* do not exist, the investment will only occur in the closing auction, which means $v_T = W$.

Strategy B

Strategy B is the strategy where one choose to invest during both periods before and after the initial imbalance announcement. Consider some small $\epsilon > 0$. We denote by $t^* \in [0, \tau)$ the starting time and $\tau^* \in [t^* + \epsilon, \tau)$ the ending time of investment for the time horizon prior to time τ. Similarly, we let $k^* \in [\tau, T)$ be the starting time and $T^* \in [k^* + \epsilon, T)$ be the ending time of investment after time τ. We define t^*, τ^*, k^*, T^* to satisfy $v_{t^*}, v_{\tau^*} > 0$ and $v_{k^*}, v_{T^*} > 0$.

We consider our denotation for $K_i^j(t^*, \tau^*, k^*, T^*)$ shown in appendix A.2. We denote:

$$\begin{aligned}
A_1(t^*, \tau^*, k^*, T^*) =& \beta K_1^4 + \alpha K_1^1 K_1^2 + \lambda\sigma^2 K_1^5 + \lambda\sigma_{\tilde{Z}}^2 \left(K_1^2\right)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 \left(K_1^1\right)^2, \\
A_2(t^*, \tau^*, k^*, T^*) =& \beta K_2^4 + \lambda\sigma^2 K_2^5 + \lambda\sigma_{\tilde{Z}}^2 \left(K_2^2\right)^2, \\
A_3(t^*, \tau^*, k^*, T^*) =& \beta K_3^4 + \alpha K_1^1 K_2^2 + \lambda\sigma^2 K_3^5 + 2\lambda\sigma_{\tilde{Z}}^2 K_1^2 K_2^2, \\
A_4(t^*, \tau^*, k^*, T^*) =& \beta K_4^4 - (T\mu + \mu_{\tilde{Z}}) K_1^2 + \mu K_1^3 - \alpha(\mu_{\tilde{N}} + W) K_1^1 \\
& + \alpha\left(K_1^1 K_3^2 + K_2^1 K_1^2\right) + \lambda\sigma^2 K_4^5 + 2\lambda\sigma_{\tilde{Z}}^2 K_1^2 K_3^2 + 2\lambda\alpha^2\sigma_{\tilde{N}}^2 K_1^1 K_2^1, \\
A_5(t^*, \tau^*, k^*, T^*) =& \beta K_5^4 - (T\mu + \mu_{\tilde{Z}}) K_2^2 + \mu K_2^3 + \alpha K_2^1 K_2^2 + \lambda\sigma^2 K_5^5 + 2\lambda\sigma_{\tilde{Z}}^2 K_2^2 K_3^2.
\end{aligned}$$

We further denote:

$$\begin{aligned}
D_1(t^*, \tau^*, k^*, T^*) &= \frac{A_3 K_2^2 - 2A_2 K_1^2}{4A_1 A_2 - A_3^2}, \\
D_2(t^*, \tau^*, k^*, T^*) &= \frac{A_3 K_1^2 - 2A_1 K_2^2}{4A_1 A_2 - A_3^2}
\end{aligned}$$

and

$$c_A(t^*,\tau^*,k^*,T^*) = \frac{A_3A_5 - 2A_2A_4}{4A_1A_2 - A_3^2},$$
$$c_B(t^*,\tau^*,k^*,T^*) = \frac{A_3A_4 - 2A_1A_5}{4A_1A_2 - A_3^2}.$$

Moreover, we consider the auxiliary term:

$$\delta = \max\left(\frac{W - c_A\big(e^{2a(\tau^*)} - 1\big)\mathrm{e}^{-a(T^*)} - c_B\big(\mathrm{e}^{a(T^*)} - \mathrm{e}^{2a(\tau^*)-a(T^*)}\big) - \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-a(T^*)}\big)}{D_1\big(e^{2a(\tau^*)} - 1\big)\mathrm{e}^{-a(T^*)} + D_2\big(\mathrm{e}^{a(T^*)} - \mathrm{e}^{2a(\tau^*)-a(T^*)}\big)}, 0\right).$$

If we have:

$$4A_1A_2 - A_3^2 \leq 0,$$

then the optimal strategy will simply be $v_t = 0$ for all $t \in [0, T)$ and $v_T = W$. Otherwise, the optimal rate of trading at time t is given by:

$$v_t = \begin{cases} \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(\big(c_A + D_1\delta\big)\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in [t^*, \tau^*], \\ \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(\big(c_B + D_2\delta\big)\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\right. & \\ \left. -\big(c_A + D_1\delta\big)\big(\mathrm{e}^{2a} - 1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in [k^*, T^*], \end{cases}$$

and the cumulative order at time t is given by:

$$X_t^v = \begin{cases} \big(c_A + D_1\delta\big)\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in [t^*, \tau^*], \\ \big(c_A + D_1\delta\big)\big(e^{2a} - 1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \big(c_B + D_2\delta\big)\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - \mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \\ +\frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in [k^*, T^*]. \end{cases}$$

The investment in the closing auction is, therefore, $v_T = W - X_T^v$.

As we have discussed previously, one can find the optimal t^*, τ^*, k^*, and T^* iteratively by selecting the combination that yields the lowest objective value shown in eq. (3.2). The overall iteration procedure can be extremely time-consuming, especially, in the continuous-time framework.

In addition, if t^*, τ^*, k^*, and T^* as defined previously does not exist, then it is optimal to only invest in the closing auction. Namely, $v_T = W$.

***Strategy C**: $v_t = 0$ for $t \in \{1, \ldots, \tau - 1\}$.*

Strategy C considers the scenario where the investor only invest after time τ. The overall structure and proof of this strategy are nearly identical to that of Strategy A introduced previously in this section. We denote that:

$$m_1 = T\mu + \mu_{\tilde{Z}}, \quad m_2 = \frac{\mu}{\sigma^2}\sigma^2_{\tilde{Z}},$$

and

$$\begin{aligned} c_{num}(k, \bar{k}) =& \sinh\big(a(\bar{k})\big)\big(m_1 - m_2\big(1 - \mathrm{e}^{a(\bar{k})}\big)\big) - \sinh\big(a(k)\big)\big(m_1 - m_2\big(1 - \mathrm{e}^{a(k)}\big)\big) \\ & - \mu\Bigg(\bar{k}\sinh\big(a(\bar{k})\big) - k\sinh\big(a(k)\big) - \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\Big(\mathrm{e}^{-2a(\bar{k})} - \mathrm{e}^{-2a(k)}\Big)\Bigg), \\ c_{den}(k, \bar{k}) =& 2\sqrt{\beta\lambda\sigma^2}\big(\sinh\big(2a(\bar{k})\big) - \sinh\big(2a(k)\big)\big) + 4\lambda\sigma^2_{\tilde{Z}}\big(\sinh^2\big(a(\bar{k})\big) - \sinh^2\big(a(k)\big)\big), \end{aligned}$$

such that $c(k, \bar{k}) = \frac{c_{num}(k,\bar{k})}{c_{den}(k,\bar{k})}$. We consider the auxiliary term:

$$\begin{aligned} \delta(k, \bar{k}) = \max\Bigg(& \frac{2\sinh(a(T^*))c(k^*, T^*) - \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-a(T^*)}\big) - W}{\sinh(2a(T^*) - \sinh(2a(k^*))}\big(2\sqrt{\beta\lambda\sigma^2}(\sinh(2a(T^*)) \\ & - \sinh(2a(k^*))) + 4\lambda\sigma^2_{\tilde{Z}}(\sinh^2(a(T^*)) - \sinh^2(a(k^*)))\big), 0\Bigg). \end{aligned}$$

We consider:

$$\tilde{c}(k,\bar{k}) = \frac{c_{num}(k,\bar{k}) - \delta(k,\bar{k})\big(\sinh(2a(\bar{k})) - \sinh(2a(\bar{k}))\big)}{c_{den}(k,\bar{k})}.$$

If there exist $k^* \in [\tau, T)$ and $T^* \in [k^* + \epsilon, T)$, then the rate of trading in the open market at time k is given by:

$$v_t = \begin{cases} \sqrt{\frac{\lambda\sigma^2}{\beta}}\big(2\cosh\big(a(k)\big)\tilde{c}(k^*, T^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-a(k)}\big) \text{ if } k \in [k^*, T^*], \\ 0 \text{ if otherwise,} \end{cases}$$

and the cumulative order at time k is given by:

$$X_t^v = \begin{cases} 2\sinh\big((a(k)\big)\tilde{c}(k,\bar{k}) + \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-(a(k)}\big) \text{ if } k \in [k^*, T^*], \\ 0 \text{ if otherwise.} \end{cases}$$

The investment in the closing auction is $v_T = W - X_T^v$. The search for the optimal pair of k^* and T^* is the same as seen previously. The optimal strategy will yield the lowest value of eq. (3.2). If k^* and T^* do not exist, then $v_T = W$.

Case 2: $\beta = 0$

In this case, the flow trader has no effect on in the open market. We find that it is not optimal to invest continuously, but rather, only invest at certain point of time. Namely, we find that it optimal to invest once before the initial imbalance announcement (time τ) and/or once before the market end (time T). If the stock prices yield a positive drift (μ), then one should invest at the very beginning as well (time t^*). In particular, for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$, let the time of investment before and after time τ to be $\tilde{\tau} := \tau - \epsilon_1$ and $\tilde{T} := T - \epsilon_2$, respectively.

We denote:

$$\begin{aligned}
c_1 :=& \mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2\big(\alpha + \lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}} + \alpha^2\sigma^2_{\tilde{N}}\big)\big)V_0, \\
c_2 :=& \mu(T - \tilde{T}) + \mu_{\tilde{Z}} - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)V_0.
\end{aligned}$$

Moreover, we let:

$$m := 4\lambda\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \sigma^2_{\tilde{Z}} + \alpha^2\sigma^2_{\tilde{N}}\big)\big)\big(\sigma^2(T + \tilde{T}) + \sigma^2_{\tilde{Z}}\big) - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)^2.$$

In addition, we denote:

$$\begin{aligned}
\delta_{num} :=& m\big(2\lambda\big(\sigma^2(T + \tilde{T}) + \sigma^2_{\tilde{Z}}\big)(W - V_0) - c_2\big) - \big(2\lambda\sigma^2\tilde{T} - \alpha\big)\big(2\lambda\big(\sigma^2(T + \tilde{T}) + \sigma^2_{\tilde{Z}}\big)c_1 \\
& - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)c_2\big)
\end{aligned}$$

such that $\delta = \max\left(\frac{\delta_{num}}{\left(2\lambda\sigma^2\tilde{T} - \alpha\right)\left(\alpha - \sigma^2\tilde{T}\right)}, 0\right)$. The optimal strategy is given by:

$$V_{t^*} = \max\left(\frac{\mu}{2\lambda\sigma^2}, 0\right),$$

$$
\begin{aligned}
V_{\tilde{\tau}} =& \max\left(\frac{2\lambda\big(\sigma^2(T+\tilde{T})+\sigma^2_{\tilde{Z}}\big)c_1-\big(\alpha+2\lambda\big(\sigma^2 T+\sigma^2_{\tilde{Z}}\big)\big)c_2+(\alpha-\sigma^2\tilde{T})\delta}{m},0\right),\\
V_{\tilde{T}} =& \max\left(\frac{c_2-\big(\alpha+2\lambda\big(\sigma^2 T+\sigma^2_{\tilde{Z}}\big)\big)V_{\tilde{\tau}}-\delta}{2\lambda\big(\sigma^2(T+\tilde{T})+\sigma^2_{\tilde{Z}}\big)},0\right),\\
V_T =& W-V_{t^*}-V_{\tilde{\tau}}-V_{\tilde{T}}.
\end{aligned}
$$

If $V_{\tilde{T}}=0$, then the optimal strategy follows from item 3 of proposition 2. In particular, we have:

$$
\begin{aligned}
V_{t^*} &= \max\left(\frac{\mu}{2\lambda\sigma^2},0\right),\\
V_{\tilde{\tau}} &= \max\left(\frac{\mu(T-\tilde{\tau})+\mu_{\tilde{Z}}+\alpha(\mu_{\tilde{N}}+W)-2\big(\alpha+\lambda\big(\sigma^2 T+\alpha^2\sigma^2_{\tilde{N}}+\sigma^2_{\tilde{Z}}\big)\big)V_0-\delta}{2\big(\alpha+\lambda\big(\sigma^2(T+\tilde{\tau})+\alpha^2\sigma^2_{\tilde{N}}+\sigma^2_{\tilde{Z}}\big)\big)},0\right),\\
V_T &= W-V_{t^*}-_{\tilde{\tau}},
\end{aligned}
$$

where

$$
\delta=\max\left(\mu(T-\tilde{\tau})+\mu_{\tilde{Z}}+\alpha\mu_{\tilde{N}}+2\lambda\sigma^2\tilde{\tau}V_0-2\big(\alpha+\lambda\big(\sigma^2(T+\tilde{\tau})+\alpha^2\sigma^2_{\tilde{N}}+\sigma^2_{\tilde{Z}}\big)\big)W,0\right).
$$

The optimal set of $t^*,\epsilon_1,\epsilon_2$ should give a strategy that produce the smallest objective value in eq. (3.2); it can be done iteratively.

Chapter 4

Data Analysis

In this chapter, we use real intraday stock prices and imbalance volumes during the closing auction to estimate input parameters, and then test the performance of the optimal trading strategies. We assume our trader is only submitting buy orders. We choose the time increment of investment to be 1 second; as such, we implement the discrete time strategy introduced in section 2.3 in our simulations. The simulation uses the data from the previous $t-1$ days to estimate the parameters, which are used to test the performance of the strategy on day t. Lastly, we compute the sample value of the objective (2.1) suggested by the optimal strategy for each selected stock and examine if the optimal strategy outperforms the case where the entire purchase is done in the closing auction. The overall investment horizon consists of the last half hour before market close on each trading day, which means each simulation begins at 15:30:00 EST.

4.1 Data Description

The data used in the simulation come from two sources. Namely, the imbalance information provided by NASDAQ and the intraday stock prices extracted from the Bloomberg terminal.[2] The entire data set consists of a time horizon from November 01, 2016 to January 26, 2017.

The imbalance information shows the imbalance volume at the moment of announcement. The closing auction begins usually at 15:50:00 EST and the information is updated every five seconds. The imbalance information is manually saved through the NASDAQ Net Order Imbalance Indicator on a daily basis. Due to unavoidable technical issues, the information at 15:50:00 EST is missing on certain days. Thus, the information at 15:50:05 EST is used as the initial imbalance information instead.

The Bloomberg terminal provides the intraday stock prices along with their corresponding traded volume. Since the smallest time measurement Bloomberg provides is in seconds, we sometimes have multiple entries for the same point of time. In addition, the order of multiple data at a particular time does not necessarily correspond to the order of transaction. Hence, in order to better reflect the true volatility of the changes in stock prices, we additionally compute the volume weighted average prices (VWAP) per second and choose these prices for the corresponding second. In this chapter, we will refer to these VWAP computed stock prices simply as stock prices.

[2] Data obtained with permission of NASDAQ OMX Group, Inc. and Bloomberg L.P.

In the execution of the simulations, we divide our data into two sets; namely, a training set and a test set. The training set is used to estimate the parameters for the optimal strategy, which is used to test the performance with the stock prices from the test set. The training set will change with an increasing window, which is explained in more detail in section 4.3. In our simulations, we test the performance of our strategy from December 16, 2016 to January 26, 2017. This suggests the smallest training set consists of data from November 01 to December 15, 2016.

4.2 Stock Selection

Among over 100 stocks that comprises the NASDAQ 100 index, we test the performance of a total of 15 stocks. Specifically, we consider three different sets with each containing five stocks. Set 1 consists of the top 5 stocks with the highest dollar-volume, Set 2 is comprised by the five stocks that best fit our assumption of the model, and lastly, Set 3 is randomly selected from the remaining pool of stocks. The stocks in Sets 1 and 2 are selected based on the information between November 01 and December 15 of 2016.

Set 1: The Most Liquid

The dollar-volume liquidity is crucial to institutional investors as they often submit large trading orders. One can easily enter and exit the positions of a highly liquid stock. A measurement of liquidity of stocks is the dollar-volume. From the daily summary of the NASDAQ 100 stocks, one can calculate the dollar-volume for each stock, which is the product of the volume of trade and closing stock price. By examining the average of the dollar-volume from November 01 to December 15, the top five most liquid stocks are given by:

$$\text{Set 1} = \{\text{AAPL,AMZN,FB,MSFT,GOOGL}\}$$

The companies in Set 1 are: Apple Inc., Amazon.com Inc., Facebook Inc., Microsoft Corporation, and Alphabet Inc.

Set 2: The Most Suitable

We recall that one of the assumptions in the theoretical framework is that the imbalance is cleared immediately and will not reappear after the clearance. Though this assumption stays true on certain days, there exist occurrences where the imbalance is not cleared for a long period of time and may reappear after the initial clearance. For certain stocks, there exist times that the order imbalance is never fully cleared. As such, we select the top 5 stocks that, on average, take the shortest time to clear the imbalance order and have the least number of occurrences where the imbalance volume reappears after the initial clearance. Excluding any stock from Set 1, Set 2 is given by:

$$\text{Set 2} = \{\text{ORLY,ESRX,EA,HSIC,DISCA}\}$$

The companies in Set 2 are: O'Reilly Automotive Inc, Express Scripts, Electronic Arts, Henry Schein, and Discovery Communications Inc.

Set 3: The Random

The last set of stocks is determined randomly from NASDAQ 100, excluding stocks from Set 1 and 2. In our simulation, we have:

$$\text{Set 3} = \{\text{SBUX,ULTA,FOX,PYPL,BMRN}\}$$

The companies in Set 3 are: Starbucks Corporation, Ulta Beauty, 21st Century Fox, PayPal, and BioMarin Pharmaceutical.

4.3 Parameter Estimations

We recall that the parameters one needs to determine in order to compute the optimal strategy are: μ_Z, $\mu_{\tilde{Z}}$, $\mu_{\tilde{N}}$, σ_Z^2, $\sigma_{\tilde{Z}}^2$, $\sigma_{\tilde{N}}^2$, α, β, W, and λ. We first examine the parameters that can be determined without any data analysis:

- μ_Z and $\mu_{\tilde{Z}}$ are the drift of the random drivers in the stock prices. As stock prices behave differently in different days, it is difficult to determine a clear direction of drifts on average. Moreover, since we are considering only a relatively short time horizon of 30min, it is sensible to suppose that drifts are close to zero. As such, we assume $\mu_Z = \mu_{\tilde{Z}} = 0$, which suggests the optimal strategy will follow the result of proposition 1.

- $\mu_{\tilde{N}}$ is the average imbalance volume to be cleared within the first five seconds since the initial imbalance announcement. By examining the historical imbalance data, there are few occurrences where the side (buy/sell) of the imbalance remain unchanged for a long period of time. As there is no clear indication on sign of the imbalance volume, it is reasonable to assume $\mu_{\tilde{N}} = 0$ on average.

Remark: For particular stocks, one could observe a tendency of imbalance direction. In the case of AMZN, one can obtain a better performance by assuming a positive $\mu_{\tilde{N}}$. Specifically, one can estimate $\mu_{\tilde{N}}$ as the average value of the cleared imbalance volume within the first five seconds, throughout the training set.

- β is a parameter that quantifies the effect of a trader's action to the stock prices during the continuous trading. For the testing, we choose $\beta = 10^{-7}$, in line with section 3.4 of Almgren and Chriss [1].
- W is the pre-determined shares of a stock to be traded; we assume it is 100,000 in our simulation.
- λ measures the risk aversion of the investor. A smaller (larger) value suggests the investor has a greater (lower) risk tolerance. In our simulation, we examine the performance with $\lambda = 10^{-4}$ and with $\lambda = 10^{-5}$.

We use the historical data to estimate the remaining parameters. We estimate the parameters on an increasing basis. In other words, we always use the information from day 1 to day $t-1$ to estimate parameters in order to test the performance of the strategy on day t. For example, to test the strategy on Dec. 16, we estimate parameters with imbalance data from Nov. 01 to Dec. 15, but to test the strategy on Jan. 02, we would estimate parameters using data in the period Nov. 1 – Dec. 30. With an increasing window, we determine the parameters in the following manner:

- α is a parameter that quantifies how the imbalance volume affects stock prices. In other words, it can be viewed as the effect on the changes of stock prices by the changes of the imbalance volume. We assume our trader has the same influence as other traders in the market in general. One can estimate such parameter with the ordinary least squares (OLS) method. Under the Gauss-Markov assumptions, the estimated parameter is the best linear unbiased estimator. The training set serves as the sample data set of the regression. The linear regression model is given by:

$$y_i = \alpha x_i + \epsilon_i$$

  where x_i is the change in imbalance on day i, y_i is the change in price on day i, and ϵ_i is the estimation residual. Wooldridge [13] gives an overview of the OLS approach.

- $\sigma^2_{\tilde{N}}$ is the variance of the imbalance volume cleared within the first five seconds. For each day in the training set, we calculate the changes of the imbalance volume (the cleared volume) in the first five seconds. $\sigma^2_{\tilde{N}}$ is determined as the variance over the entire training set.

- σ^2_Z captures the volatility of stock prices without any effect from the imbalance announcement. For each day, we compute the variance of the changes in stock prices from the beginning time of investment (15:30:00 EST) to the moment before the initial imbalance announcement. We exclude the changes of stock prices for the last 10 minutes of trading to avoid any effect that may be due to the closing auction. In the end, σ^2_Z is the average value of such daily variance.

- $\sigma^2_{\tilde{Z}}$ is the variance of the changes of stock price at the last moment before the market close. It is simply calculated as such variance throughout the training set.

As we are adapting an increasing window, the set of parameters is different each day. For illustration purpose, Table 4.1 shows the set of estimated parameters using information from Nov. 01 to Dec. 30, 2016. During this period, one can

	$\alpha(\times 10^{-6})$	$\sigma^2_{\tilde{N}}(\times 10^{9})$	$\sigma^2_{Z}(\times 10^{-8})$	$\sigma^2_{\tilde{Z}}(\times 10^{-8})$
AAPL	0.13	58.05	0.45	2.23
AMZN	5.56	0.63	2.31	3.25
FB	0.20	16.37	0.66	1.37
MSFT	0.07	49.11	0.81	2.84
GOOGL	6.90	0.62	4.24	3.21
ORLY	1.40	0.16	3.58	1.60
ESRX	0.16	12.60	1.57	3.44
EA	0.36	2.41	1.70	1.32
HSIC	1.50	0.36	3.37	1.55
DISCA	0.19	1.15	4.50	2.56
SBUX	0.17	6.09	1.03	1.12
ULTA	2.77	0.06	6.30	2.40
FOX	0.19	1.31	3.41	5.94
PYPL	0.14	19.65	2.09	1.57
BMRN	4.03	0.26	12.69	5.43

Table 4.1: Estimated Parameters for Jan. 03, 2017 based on Data from Nov. 01 to Dec. 30, 2016.

see that the investor has the most market impact during the closing auction in GOOGL and least impact in MSFT. Moreover, AAPL has the most volatility in the initial imbalance clearance and BMRN has the most fluctuation in stock prices.

4.4 Algorithmic Simulation

Using the estimated parameters as described in section 4.3, one can apply the algorithm proposed in section 2.3 to determine the optimal strategy. In this section, we examine two cases where the trader has a relatively low or high risk tolerance; that is, $\lambda = 10^{-4}$ and $\lambda = 10^{-5}$, respectively. Having a low risk tolerance suggests the investor will tend to invest less in the earlier time of the investment horizon.

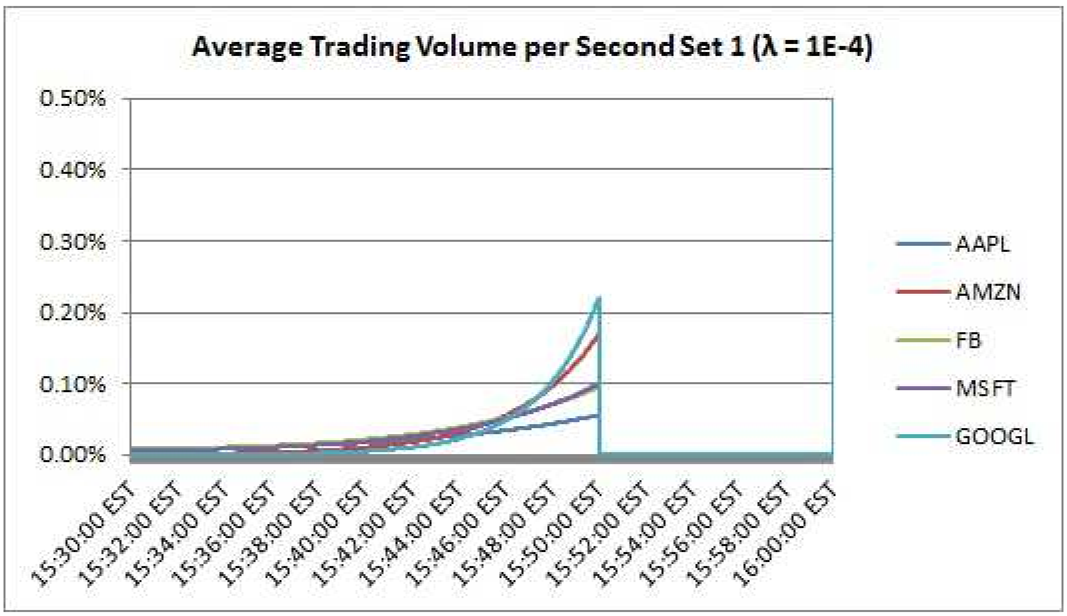

Figure 4.1: Average Trading Volume per Second for Set 1 (Higher λ)

Figures 4.1 − 4.6 show the average progression of trading volume as percentage of the total volume, W, throughout the testing period for all 15 stocks. By comparing Figures 4.1 − 4.3 ($\lambda = 10^{-4}$) with Figures 4.4 − 4.6 ($\lambda = 10^{-5}$), the trading volumes at the earlier period on Figures 4.1 − 4.3 are smaller; however, there is a greater increase in the trading volume when the time is closer to

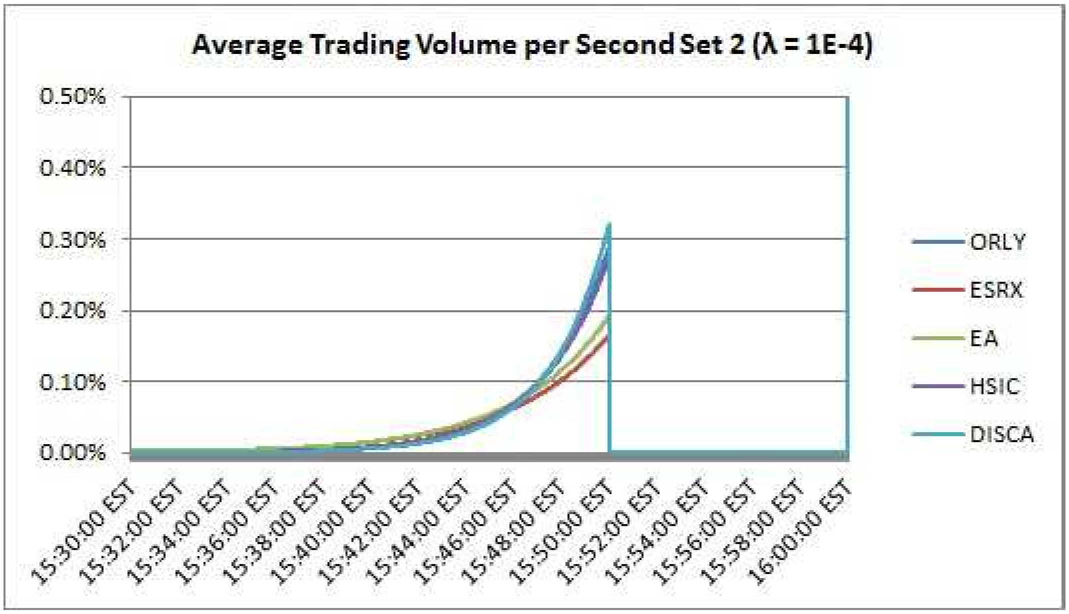

Figure 4.2: Average Trading Volume per Second for Set 2 (Higher λ)

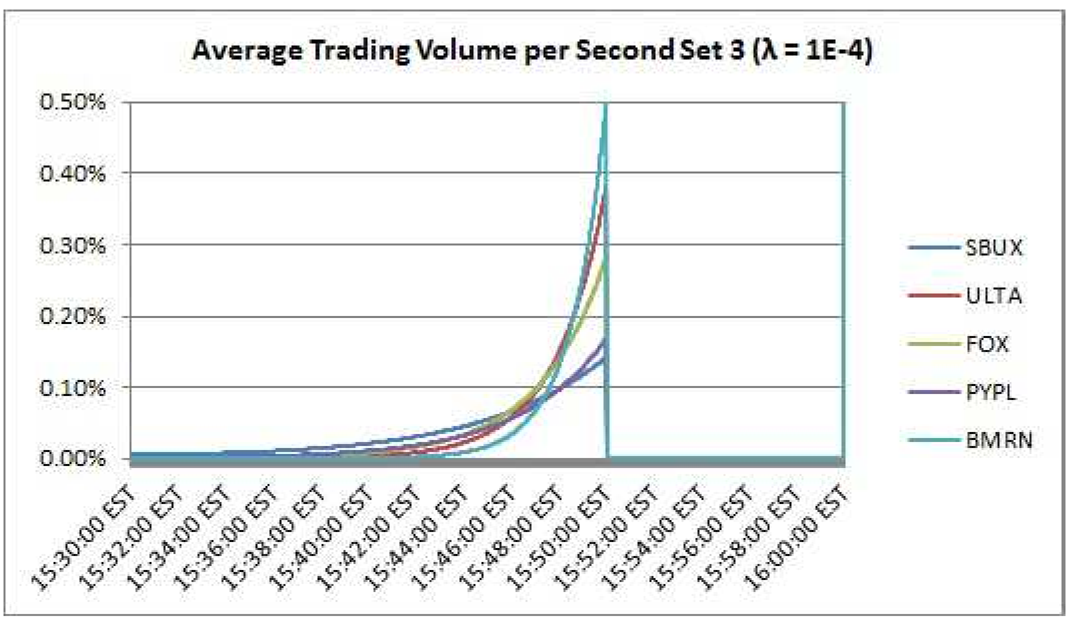

Figure 4.3: Average Trading Volume per Second for Set 3 (Higher λ)

the initial imbalance announcement at 15:50:00 EST. On the other hand, the trading volumes on Figure 4.4 − 4.6 are greater in the earlier period, but lower at the later period. These phenomena are consistent with our intuition and illustrate that a relatively risk-averse investor would place the orders with a larger marginal increase whereas a less risk-averse investor tends to trade with a less marginal increase of volume throughout the trading period so to achieve a lower average price impact by accepting higher deviations compared to the benchmark (closing price). In the case of BMRN, one can observe through Figure 4.3, that there is almost no investment in the earlier period and a larger volume of orders being placed near 15:50:00 EST. Since the trader has a relatively low risk tolerance, the orders will only be placed at a later period, due to the high volatility of BMRN's stock prices, which is reflected in Table 4.1.

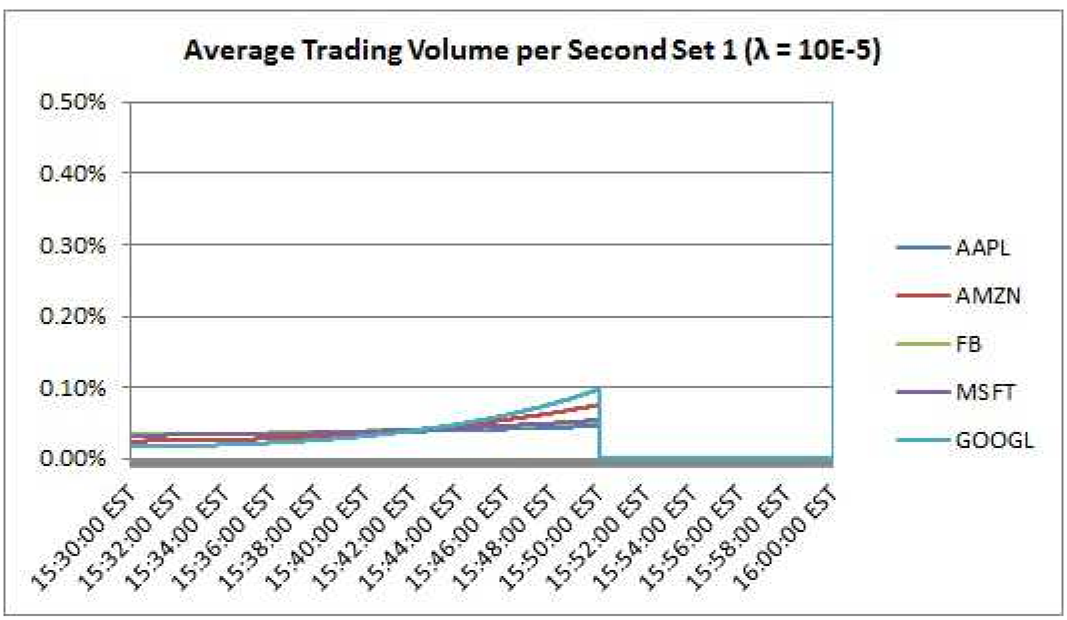

Figure 4.4: Average Trading Volume per Second for Set 1 (Lower λ)

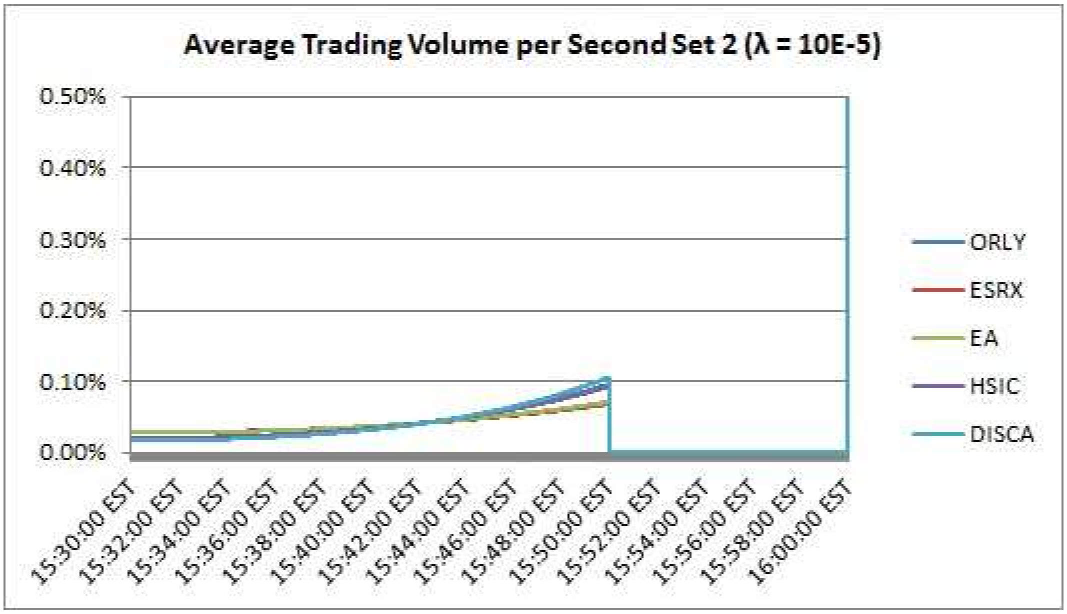

Figure 4.5: Average Trading Volume per Second for Set 2 (Lower λ)

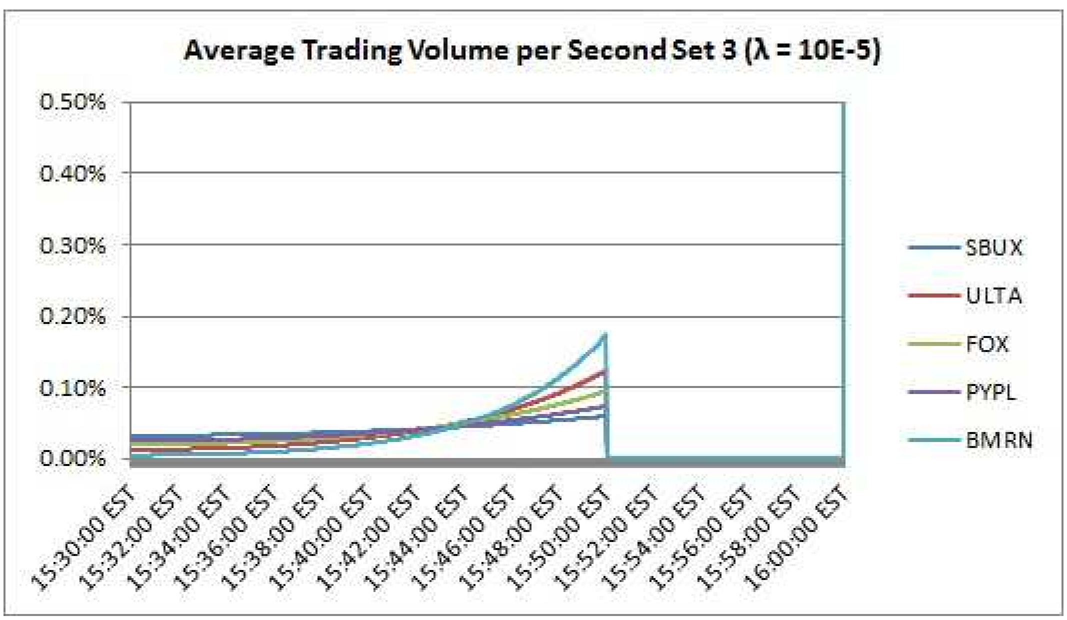

Figure 4.6: Average Trading Volume per Second for Set 3 (Lower λ)

A less risk-averse investor would tend to trade more orders in the open market than a more risk-averse investor; this is reflected through Figures 4.7 − 4.12, which show the cumulative order for each stock. Figures 4.7 − 4.9 show that our trader would invest between 25% to 50% of the total volume in the open market, depending on the selected stock. For a ten times smaller risk aversion parameter, Figures 4.10 − 4.12 suggest the trader would buy generally more in the continuous trading. For most stocks in this analysis, the trader tend to increase the investment in the open market from around 30% to approximately 50%.

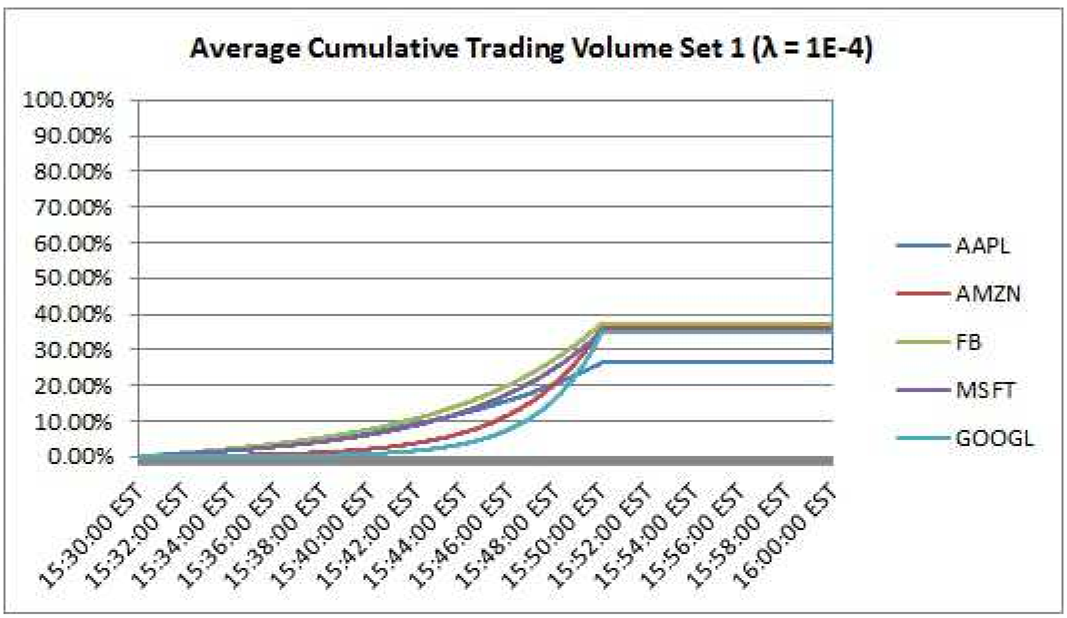

Figure 4.7: Average Cumulative Trading Volume for Set 1 (Higher λ)

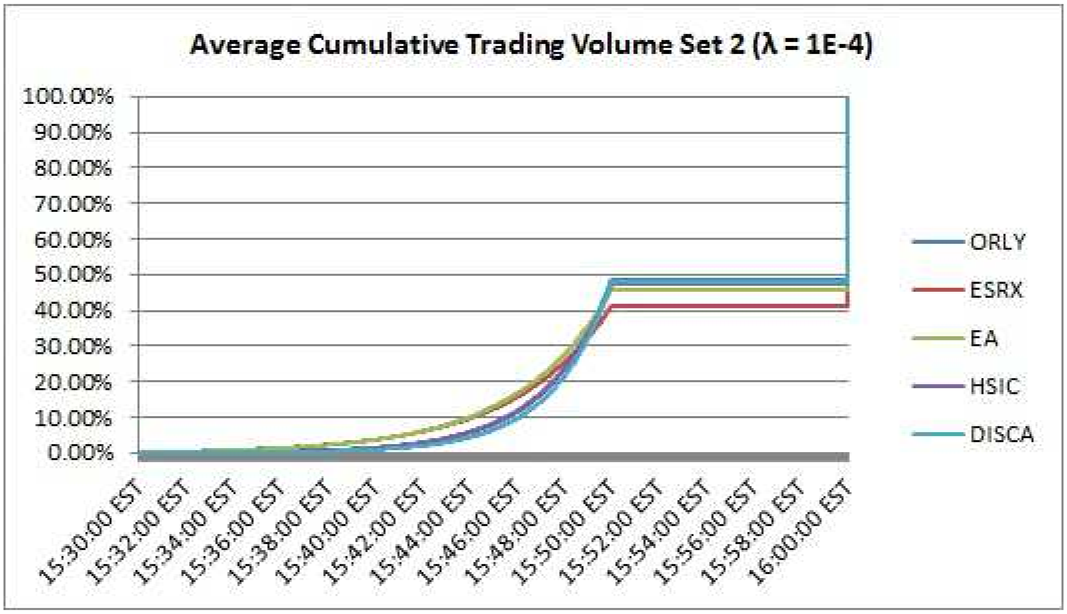

Figure 4.8: Average Cumulative Trading Volume for Set 2 (Higher λ)

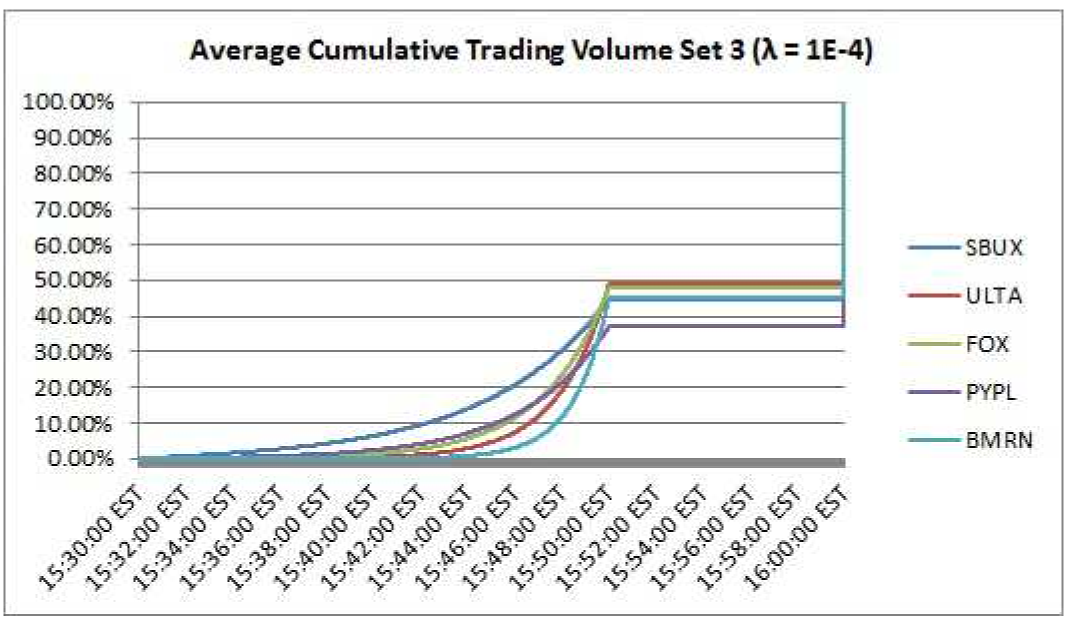

Figure 4.9: Average Cumulative Trading Volume for Set 3 (Higher λ)

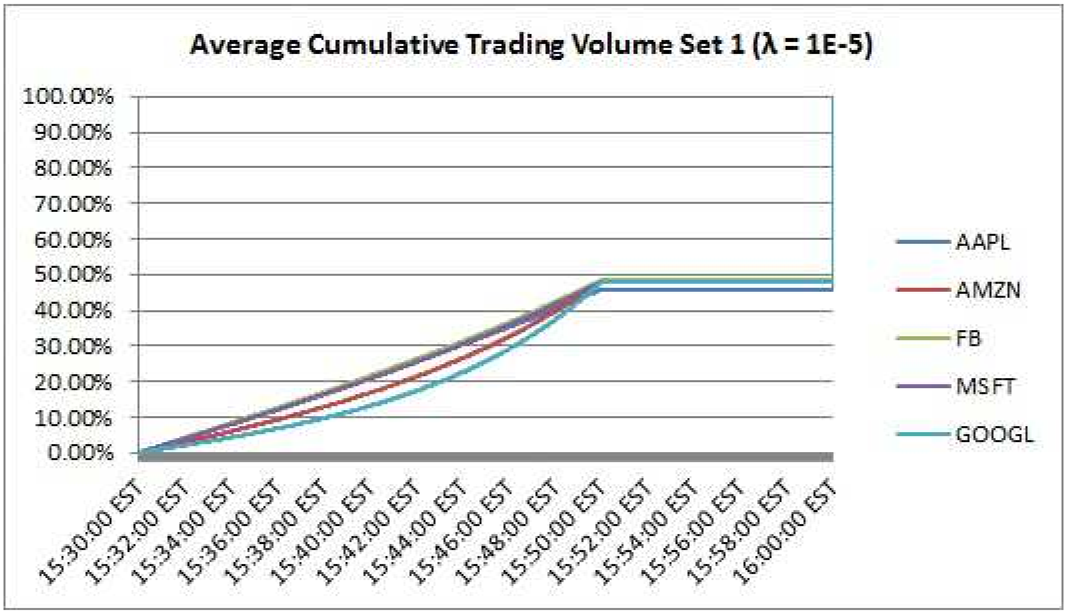

Figure 4.10: Average Cumulative Trading Volume for Set 1 (Lower λ)

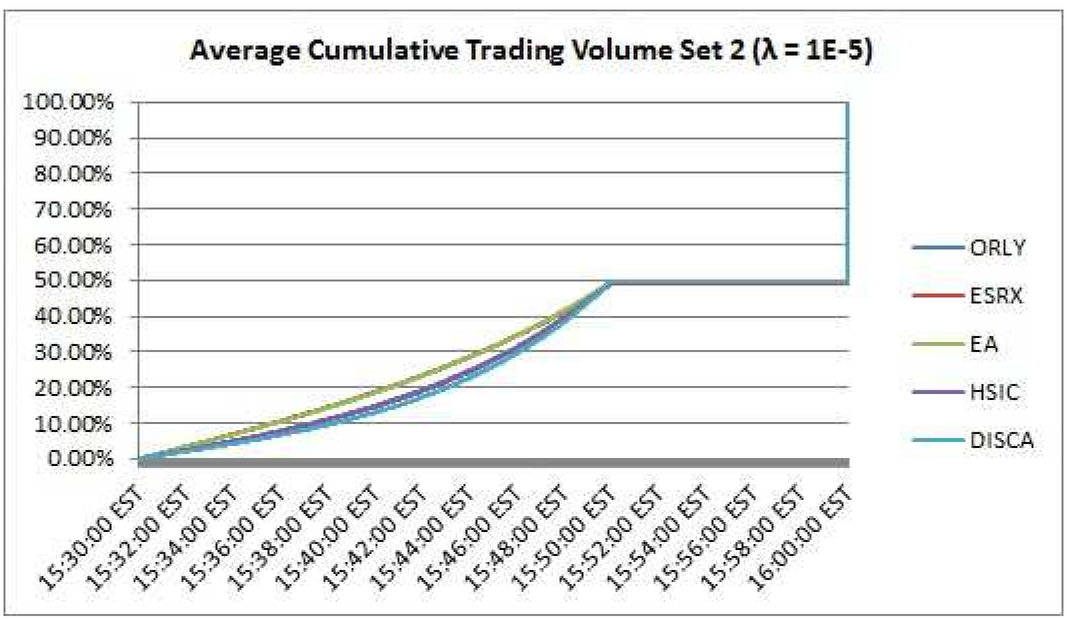

Figure 4.11: Average Cumulative Trading Volume for Set 2 (Lower λ)

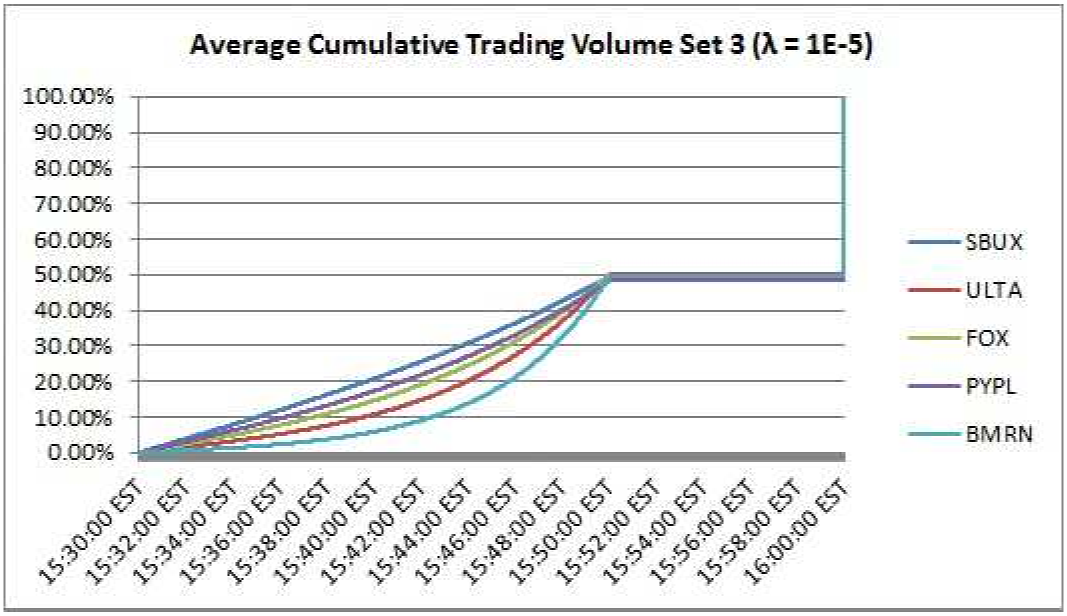

Figure 4.12: Average Cumulative Trading Volume for Set 3 (Lower λ)

We examine the performance of the optimal strategy against the performance of investing entirely in the closing auction, based on two aspects. Namely, the size of the difference in the implementation shortfall and its stability; the two aspects correspond to the two terms in the objective function (2.1). For each day of our testing period, we determine the optimal strategy for each stock and calculate the value for

$$\sum_{t=1}^{T} v_t P_t - W P_T. \tag{4.1}$$

In the testing of the strategy, we adjust the stock prices with corresponding market impact of our orders, as presented in section 2.1. By computing the mean of eq. (4.1) over the entire testing period, we obtain the expected cost reductions for all 15 stocks. Additionally, with the inclusion of the variance, we examine the risk-adjusted expected cost reductions. Table 4.2 shows such objective values under two different values of λ. With a lower risk tolerance,

	Average ($\lambda = 10^{-4}$) amount in \$	Average ($\lambda = 10^{-4}$) change in bps	Risk-adjusted ($\lambda = 10^{-4}$)	Average ($\lambda = 10^{-5}$) amount in \$	Average ($\lambda = 10^{-5}$) change in bps	Risk-adjusted ($\lambda = 10^{-5}$)
AAPL	-2,144	-1.82	-174	-4,101	-3.47	-3,487
AMZN	-40,698	-5.15	87,025	-52,940	-6.69	-24,169
FB	-1,037	-0.84	3,571	-1,640	-1.33	-886
MSFT	-1,157	-1.82	22	-1,942	-3.05	-1,691
GOOGL	-58,040	-7.06	41,205	-75,539	-9.17	-53,668
ORLY	-9,720	-3.48	9,163	-9,358	-3.35	-6,910
ESRX	-1,132	-1.66	1349	-1,610	-2.36	-1,195
EA	-3,464	-4.30	-1,554	-3,666	-4.55	-3,433
HSIC	-13,503	-8.64	-7,978	-14,489	-9.26	-13,668
DISCA	-1,601	-5.75	-1291	-1,636	-5.87	-1,601
SBUX	-1,482	-2.59	-696	-1,667	-2.92	-1,552
ULTA	-339,716	-9.21	-10,613	-24,929	-9.46	-22,985
FOX	-2,077	-7.13	-1,611	-2,293	-7.87	-2,228
PYPL	-764	-1.92	-128	-1,407	-3.50	-1,290
BMRN	-31,740	-36.91	-25,077	-34,300	-39.89	-33,420

Table 4.2: Out-of-Sample Objective Values across the Different Stocks for $\lambda = 10^{-4}$ and $\lambda = 10^{-5}$.

the risk-adjusted value can be positive for certain stocks, such as GOOGL, as illustrated in the fourth column. However, these positive values do not necessarily suggest the underperformance of the strategy, but rather, reflect a notion of fluctuation in the cost reduction. A risk-averse investor would value these fluctuation more than the average value in cost reduction. In fact, the second and third columns of Table 4.2 verify that the optimal strategy would provide an attractive cost reduction on average for each stock. As expected, if the trader has a higher risk tolerance, the risk-adjusted values are negative for all stocks, which can be seen on the last column. Overall, if the trader is less risk-averse, then the simulated objective values for all sets of stocks are negative, which suggests, generally, the proposed strategy yields a positive and stable performance.

Though the second and fifth columns of Table 4.2 show that the average cost reduction are negative for all selected stocks, it does not necessarily suggest the optimal strategy will outperform everyday for all stocks. In particular, FB yields one of the lowest cost reductions among the selected stocks. This is to be expected as FB is one of a few stocks that deviate materially from the theoretical model assumptions. In particular, throughout the training and testing period, FB had ten days where the imbalance volume took a long time to clear, and there exist three days where the imbalance volume was never cleared. Moreover, the initial imbalance volume clearance was relatively small on multiple occasions. In this section, we present detailed information of FB as an illustration. For $\lambda = 10^{-5}$, Table 4.3 shows the implementation costs of the optimal strategy as well as investing only in the closing auction. The fourth column shows the cost reduction in dollar, given by eq. (4.1), whereas the last column measures it in percentage. One can observe that the investor is incurring losses from January 04 to January 10, as well as January 20 to January 26. The highest loss occurred on January 06, where the strategy underperformed by 0.13%. However, the losses incurred on such days are completely covered by the gains from other days; thus, resulting in a decent implementation cost reduction on average, as shown in Table,4.2. Although the trader may experience temporary losses from time to time, the proposed optimal strategy outperforms in general, nonetheless.

FB	Cost: Optimal Strategy	Cost: Only C.A.	Difference ($)	Difference (%)
2016-12-16	$11,985,009	$11,989,018	-$4,009	-0.03%
2016-12-19	$11,920,018	$11,926,009	-$5,991	-0.05%
2016-12-20	$11,907,823	$11,911,006	-$3,183	-0.03%
2016-12-21	$11,899,183	$11,906,000	-$6,817	-0.06%
2016-12-22	$11,728,834	$11,742,006	-$13,172	-0.11%
2016-12-23	$11,725,144	$11,729,007	-$3,863	-0.03%
2016-12-27	$11,804,965	$11,803,010	$1,955	0.02%
2016-12-28	$11,695,062	$11,694,017	$1,046	0.01%
2016-12-29	$11,634,884	$11,637,009	-$2,125	-0.02%
2016-12-30	$11,507,593	$11,507,053	$541	0.00%
2017-01-03	$11,671,078	$11,687,981	-$16,903	-0.14%
2017-01-04	$11,879,775	$11,871,003	$8,772	0.07%
2017-01-05	$12,071,586	$12,069,013	$2,573	0.02%
2017-01-06	$12,358,789	$12,343,012	$15,777	0.13%
2017-01-09	$12,500,051	$12,492,020	$8,030	0.06%
2017-01-10	$12,447,244	$12,437,040	$10,204	0.08%
2017-01-11	$12,602,509	$12,611,032	-$8,523	-0.07%
2017-01-12	$12,660,938	$12,664,031	-$3,093	-0.02%
2017-01-13	$12,845,679	$12,836,028	$9,652	0.08%
2017-01-17	$12,776,859	$12,789,026	-$12,166	-0.10%
2017-01-18	$12,788,408	$12,793,991	-$5,582	-0.04%
2017-01-19	$12,755,764	$12,756,970	-$1,206	-0.01%
2017-01-20	$12,709,382	$12,705,986	$3,396	0.03%
2017-01-23	$12,904,073	$12,894,944	$9,129	0.07%
2017-01-24	$12,943,840	$12,939,011	$4,829	0.04%
2017-01-25	$13,152,468	$13,150,011	$2,458	0.02%
2017-01-26	$13,282,190	$13,280,014	$2,176	0.02%
2017-01-27	$13,211,519	$13,220,019	-$8,500	-0.06%
2017-01-30	$13,079,683	$13,100,017	-$20,334	-0.16%
2017-01-31	$13,019,756	$13,034,034	-$14,278	-0.11%

Table 4.3: Implementation Costs in Facebook Inc. of the Optimal Strategy vs. Investing only in Closing Auction

Overall, the performance of the proposed strategy shows cost reduction for all sets of our selected stocks. The sixth column of Table 4.2 summarizes the average cost reduction for each stock in basis point (bps) when $\lambda = 10^{-5}$. One can see that BMRN performed exceptionally well with an average cost reduction of almost 40 bps. Excluding BMRN, the average value of cost reduction for all 15 stocks is 5.20 bps. Set 1 and Set 2 each have two stocks performed above the average while Set 3 has three stocks performed above the average. We recall that Set 2 consists of five stocks that satisfy the model assumptions the best. However, Table 4.2 suggests that Set 2 does not necessarily always outperform the average performance. Though stocks in Set 2 satisfy the theoretical assumptions well, they are rather small-capitalized and less liquid stocks, hence their stock prices and impact of imbalance announcement are more difficult to model and estimate.

Chapter 5

Proofs of Results

5.1 Proofs for the Discrete-Time Model

5.1.1 Proof of Proposition 1

Suppose the order imbalance is cleared immediately and there are no orders in the closing auction afterward. We assume the market impact of our order is only temporary. We will first restructure our objective function for a flow trader. Then, we will construct a Lagrange function and examine its the first order condition with respect to the investment strategy at each point of time. By using the Karush-Kuhn-Tucker conditions, we show that it is not optimal to trade after the initial imbalance announcement using contradiction. As such, we derive the explicit optimal investment strategy for the period before the initial imbalance announcement. We will show various strategies base on the existence of the investor's market influence. In particular, we analyze the structure in the following four cases:

Investor's Market Impact	Exist in Closing Auction	None in Open Market
Exist in Open Market	$\beta > 0$, $\alpha > 0$	$\beta = 0$, $\alpha > 0$
None in Open Market	$\beta > 0$, $\alpha = 0$	$\beta = 0$, $\alpha = 0$

Step 1: Preparation.

We begin the proof with rewriting our objective function. We recall that the stock prices are:

$$P_t = \tilde{P}_t + \beta v_t \quad \text{for} \quad t \in \{1, \dots, \tau - 1, \tau + 1 \dots, T - 1\}$$
$$P_\tau = \tilde{P}_\tau + \beta v_\tau$$
$$P_T = \tilde{P}_T$$

where

$$\tilde{P}_t = \tilde{P}_{t-1} + Z_t \quad \text{for} \quad t \in \{1, \dots, \tau - 1, \tau + 1 \dots, T - 1\}$$
$$\tilde{P}_\tau = \tilde{P}_{\tau-1} + Z_\tau + \alpha N$$
$$\tilde{P}_T = \tilde{P}_{T-1} + \tilde{Z}_i,$$

and

$$N = \tilde{N} + \sum_{i=\tau}^{T} w_i = \tilde{N} + v_T,$$

which can be expressed as:

$$P_t = P_0 + \sum_{i=1}^{t} Z_i + \beta v_t \quad \text{for} \quad t \in \{1, \dots, \tau - 1\}$$
$$P_k = P_0 + \sum_{i=1}^{k} Z_i + \beta v_k + \alpha(\tilde{N} + v_T) \quad k \in \{\tau \dots, T - 1\}$$
$$P_T = P_0 + \sum_{i=1}^{T-1} Z_i + \tilde{Z} + \alpha(\tilde{N} + v_T).$$

For flow traders, we recall that the objective function is:

$$
\begin{aligned}
\min \quad & E\Bigg[\sum_{t=1}^{T} v_t P_t - W P_T\Bigg] + \lambda VAR\Bigg[\sum_{t=1}^{T} v_t P_t - W P_T\Bigg] \\
s.t. \quad & W = \sum_{t=1}^{T} v_i \\
& W - \sum_{t=1}^{T-1} v_t \geq 0, \quad v_t \geq 0 \quad \text{for all } t \in \{1, \dots, T-1\}.
\end{aligned}
$$

One can show:

$$
\begin{aligned}
& \sum_{t=1}^{T} v_t P_t - W P_T \\
&= \sum_{t=1}^{T-1} v_t P_t - \Bigg(\sum_{i=1}^{T-1} v_i\Bigg) P_T \\
&= \sum_{t=1}^{\tau-1} v_t \Bigg(P_0 + \sum_{i=1}^{t} Z_i + \beta v_t\Bigg) + \sum_{t=\tau}^{T-1} v_t \Bigg(P_0 + \sum_{i=1}^{t} Z_i + \beta v_t + \alpha\Bigg(\tilde{N} + W - \sum_{t=1}^{T-1} v_t\Bigg)\Bigg) \\
&\quad - \Bigg(\sum_{t=1}^{T-1} v_i\Bigg)\Bigg(P_0 + \sum_{t=1}^{T-1} Z_t + \tilde{Z} + \alpha\Bigg(\tilde{N} + W - \sum_{t=1}^{T-1} v_t\Bigg)\Bigg) \\
&= \beta \sum_{t=1}^{T-1} v_t^2 + \sum_{t=1}^{T-1} \Bigg(v_t \sum_{i=1}^{t} Z_i\Bigg) - \sum_{t=1}^{T-1} v_t \sum_{t=1}^{T-1} Z_t - \alpha \sum_{t=1}^{\tau-1} \Bigg(\tilde{N} + W - \sum_{t=1}^{T-1} v_t\Bigg) v_t - \tilde{Z} \sum_{t=1}^{T-1} v_t.
\end{aligned}
$$

In particular, we show:

$$
\begin{aligned}
\sum_{t=1}^{T-1} \Bigg(v_t \sum_{i=1}^{t} Z_i\Bigg) &= \big(v_1 Z_1 + v_2(Z_1 + Z_2) + \cdots + v_{T-1}(Z_1 + \cdots + Z_{T-1})\big) \\
&= Z_1(v_1 + \cdots + v_{T-1}) + Z_2(v_2 + \cdots + v_{T-1}) + \cdots + Z_{T-1} v_{T-1},
\end{aligned}
$$

which suggests:

$$
\begin{aligned}
\sum_{t=1}^{T-1} \Bigg(v_t \sum_{i=1}^{t} Z_i\Bigg) - \sum_{t=1}^{T-1} v_t \sum_{t=1}^{T-1} Z_t &= -\big(Z_2 v_1 + Z_3(v_1 + v_2) + \cdots + Z_{T-1}(v_1 + \cdots + v_{T-2})\big) \\
&= -\sum_{t=2}^{T-1} \Bigg(\sum_{i=1}^{t-1} v_i\Bigg) Z_t.
\end{aligned}
$$

Hence, we have:

$$\beta\sum_{t=1}^{T-1}v_t^2+\sum_{t=1}^{T-1}\Big(v_t\sum_{i=1}^{t}Z_i\Big)-\sum_{t=1}^{T-1}v_t\sum_{t=1}^{T-1}Z_t-\alpha\sum_{t=1}^{\tau-1}\Big(\tilde{N}+W-\sum_{t=1}^{T-1}v_t\Big)v_t-\tilde{Z}\sum_{t=1}^{T-1}v_t$$
$$=\Big(\beta\sum_{t=1}^{T-1}v_t^2-\alpha W\sum_{t=1}^{\tau-1}v_t+\alpha\sum_{t=1}^{\tau-1}v_t\sum_{t=1}^{T-1}v_t\Big)-\sum_{t=2}^{T-1}\Big(\sum_{i=1}^{t-1}v_i\Big)Z_t-\Big(\sum_{t=1}^{T-1}v_t\Big)\tilde{Z}-\Big(\alpha\sum_{t=1}^{\tau-1}v_t\Big)\tilde{N}.$$

We will then examine the expected value and the variance of the above equation, but before we do so, we denote $v_0=0$ and observe:

$$\begin{aligned}E\Big(\sum_{t=2}^{T-1}\Big(\sum_{i=1}^{t-1}v_i\Big)Z_t\Big)&=E\big(v_1Z_2+(v_1+v_2)Z_3+\cdots+(v_1+\cdots+v_{T-2})Z_{T-1}\big)\\&=\mu_Z\big(v_1+(v_1+v_2)+\cdots+(v_1+\cdots+v_{T-2})\big)\\&=\mu_Z\big((T-1)v_1+(T-2)v_2+\cdots+v_{T-2}\big)\\&=\mu_Z\sum_{t=1}^{T-1}(T-t)v_t,\end{aligned}$$

and

$$\begin{aligned}VAR\Big(\sum_{t=2}^{T-1}\Big(\sum_{i=1}^{t-1}v_i\Big)Z_t\Big)&=VAR\big(v_1Z_2+(v_1+v_2)Z_3+\cdots+(v_1+\cdots+v_{T-2})Z_{T-1}\big)\\&=\sigma_Z^2\big(v_1^2+(v_1+v_2)^2+\cdots+(v_1+\cdots+v_{T-2})^2\big)\\&=\sigma_Z^2\sum_{t=1}^{T-1}\Big(\sum_{i=0}^{t-1}v_i\Big)^2.\end{aligned}$$

Now, the expected value term can be formulated as:

$$E\Big[\sum_{t=1}^{T}v_tP_t-WP_T\Big]$$
$$=\beta\sum_{t=1}^{T-1}v_t^2-\alpha W\sum_{t=1}^{\tau-1}v_t+\alpha\sum_{t=1}^{\tau-1}v_t\sum_{t=1}^{T-1}v_t-\mu_Z\sum_{t=1}^{T-1}(T-t)v_t-\mu_{\tilde{Z}}\sum_{t=1}^{T-1}v_t-\alpha\mu_{\tilde{N}}\sum_{t=1}^{\tau-1}v_t$$

$$= \beta \sum_{t=1}^{T-1} v_t^2 + \alpha \sum_{t=1}^{\tau-1} v_t \sum_{t=1}^{T-1} v_t - \mu_Z \sum_{t=1}^{T-1} (T-t) v_t - \mu_{\tilde{Z}} \sum_{t=1}^{T-1} v_t - \alpha(\mu_{\tilde{N}} + W) \sum_{t=1}^{\tau-1} v_t$$

On the other hand, we have the following for the variance term of the objective function:

$$VAR\Big[\sum_{t=1}^{T} v_t P_t - W P_T\Big] = \sigma_Z^2 \sum_{t=1}^{T-1} \Big(\sum_{i=0}^{t-1} v_i\Big)^2 + \sigma_{\tilde{Z}}^2 \Big(\sum_{t=1}^{T-1} v_t\Big)^2 + \alpha^2 \sigma_{\tilde{N}}^2 \Big(\sum_{t=1}^{\tau-1} v_t\Big)^2$$

Step 2: Lagrange Systems of Equations.

Our objective function is now transformed into:

$$\begin{aligned} \min \quad & \beta \sum_{t=1}^{T-1} v_t^2 + \alpha \sum_{t=1}^{\tau-1} v_t \sum_{t=1}^{T-1} v_t - \mu_Z \sum_{t=1}^{T-1} (T-t) v_t - \mu_{\tilde{Z}} \sum_{t=1}^{T-1} v_t - \alpha(\mu_{\tilde{N}} + W) \sum_{t=1}^{\tau-1} v_t \\ & + \lambda \sigma_Z^2 \sum_{t=1}^{T-1} \Big(\sum_{i=0}^{t-1} v_i\Big)^2 + \lambda \sigma_{\tilde{Z}}^2 \Big(\sum_{t=1}^{T-1} v_t\Big)^2 + \lambda \alpha^2 \sigma_{\tilde{N}}^2 \Big(\sum_{t=1}^{\tau-1} v_t\Big)^2 \\ s.t. \quad & W - \sum_{t=1}^{T-1} v_t \geq 0, \quad v_t \geq 0 \quad \text{for all } t \in \{1, \dots, T-1\}. \end{aligned}$$

From this system of equations, we construct a Lagrange function L. For some $\delta \geq 0$, we have:

$$\begin{aligned} L =& \beta \sum_{t=1}^{T-1} v_t^2 + \alpha \sum_{t=1}^{\tau-1} v_t \sum_{t=1}^{T-1} v_t - \mu_Z \sum_{t=1}^{T-1} (T-t) v_t - \mu_{\tilde{Z}} \sum_{t=1}^{T-1} v_t - \alpha(\mu_{\tilde{N}} + W) \sum_{t=1}^{\tau-1} v_t \\ & + \lambda \sigma_Z^2 \sum_{t=1}^{T-1} \Big(\sum_{i=0}^{t-1} v_i\Big)^2 + \lambda \sigma_{\tilde{Z}}^2 \Big(\sum_{t=1}^{T-1} v_t\Big)^2 + \lambda \alpha^2 \sigma_{\tilde{N}}^2 \Big(\sum_{t=1}^{\tau-1} v_t\Big)^2 + \delta\Big(\sum_{t=1}^{T-1} v_t - W\Big) \end{aligned}$$

We differentiate L with respect to v_t for all $t \in \{1, \dots, T-1\}$ and δ. Note that for all t, we have:

$$\frac{\partial}{\partial v_t} \lambda \sigma_Z^2 \sum_{t=1}^{T-1} \Big(\sum_{i=0}^{t-1} v_i\Big)^2 = \frac{\partial}{\partial v_t} \lambda \sigma_Z^2 (v_1^2 + (v_1 + v_2)^2 + \cdots + (v_1 + \cdots + v_{T-1})^2)$$

$$= 2\lambda\sigma_Z^2\Bigg((T-t)\sum_{i=0}^{t-1} v_i + \sum_{i=t}^{T-1}(T-i)v_i\Bigg)$$

Using the above equation, we are able to find the following partial derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial v_t} &= 2\beta v_t + \alpha\sum_{i=1}^{\tau-1} v_i + \alpha\sum_{i=1}^{T-1} v_i - \mu_Z(T-t) - \mu_{\tilde{Z}} - \alpha(\mu_{\tilde{N}} + W) \\
&\quad + 2\lambda\sigma_Z^2\Bigg((T-t)\sum_{i=0}^{t-1} v_i + \sum_{i=t}^{T-1}(T-i)v_i\Bigg) + 2\lambda\sigma_{\tilde{Z}}^2\sum_{i=1}^{T-1} v_i + 2\lambda\alpha^2\sigma_{\tilde{N}}^2\sum_{i=1}^{\tau-1} v_i + \delta \\
&= \beta v_t + \left(\lambda\alpha^2\sigma_{\tilde{N}}^2 + \frac{\alpha}{2}\right)\sum_{i=1}^{\tau-1} v_i + \left(\lambda\sigma_{\tilde{Z}}^2 + \frac{\alpha}{2}\right)\sum_{i=1}^{T-1} v_i + \lambda\sigma_Z^2(T-t)\sum_{i=0}^{t-1} v_i \\
&\quad + \lambda\sigma_Z^2\sum_{i=t}^{T-1}(T-i)v_i - c_t \qquad \text{for } t \in \{1,\dots,\tau-1\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial v_k} &= 2\beta v_k + \alpha\sum_{i=1}^{\tau-1} v_i - \mu_Z(T-k) - \mu_{\tilde{Z}} + 2\lambda\sigma_Z^2\Bigg((T-k)\sum_{i=0}^{k-1} v_i + \sum_{i=k}^{T-1}(T-i)v_i\Bigg) \\
&\quad + 2\lambda\sigma_{\tilde{Z}}^2\sum_{i=1}^{T-1} v_i + \delta \\
&= \beta v_k + \frac{\alpha}{2}\sum_{i=1}^{\tau-1} v_i + \lambda\sigma_{\tilde{Z}}^2\sum_{i=1}^{T-1} v_i + \lambda\sigma_Z^2(T-k)\sum_{i=0}^{k-1} v_i + \lambda\sigma_Z^2\sum_{i=k}^{T-1}(T-i)v_i - c_k \\
&\qquad \text{for } k \in \{\tau,\dots,T-1\},
\end{aligned}$$

$$\frac{\partial L}{\partial \delta} = \sum_{i=1}^{T-1} v_i - W,$$

where

$$\begin{aligned}
c_t &:= \frac{1}{2}\big((T-t)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big) \qquad \text{for } t \in \{1,\dots,\tau-1\}, \\
c_k &:= \frac{1}{2}\big((T-k)\mu_Z + \mu_{\tilde{Z}} - \delta\big) \qquad \text{for } k \in \{\tau,\dots,T-1\}.
\end{aligned}$$

To minimize the objective function, the following Karush-Kuhn-Tucker conditions must hold:

$$\begin{aligned}
v_t \frac{\partial L}{\partial v_t} &= 0; & v_t &\geq 0; & \frac{\partial L}{\partial v_t} &\geq 0 \\
v_k \frac{\partial L}{\partial v_k} &= 0; & v_k &\geq 0; & \frac{\partial L}{\partial v_k} &\geq 0 \\
\delta \frac{\partial L}{\partial \delta} &= 0; & \delta &\geq 0; & \frac{\partial L}{\partial \delta} &\leq 0.
\end{aligned}$$

Step 3: Not Optimal to trade after time τ.

We will show here that it is not optimal to trade after the initial imbalance announcement by contradiction. Based on our assumptions, $\mu_Z, \mu_{\tilde{Z}} \leq 0$, which suggest $(T-t)\mu_Z \leq -\mu_{\tilde{Z}}$. Thus, we know:

$$c_k = \frac{1}{2}\big((T-k)\mu_Z + \mu_{\tilde{Z}} - \delta\big) \leq 0 \qquad \text{for } k \in \{\tau, \ldots, T-1\}.$$

Suppose there exist $k \in \{\tau, \ldots, T-1\}$ such that $v_k > 0$, then by the Karush-Kuhn-Tucker conditions, we have $\frac{\partial L}{\partial v_k} = 0$. For any k, we have

$$0 = \beta v_k + \frac{\alpha}{2}\sum_{i=1}^{\tau-1} v_i + \lambda\sigma_{\tilde{Z}}^2 \sum_{i=1}^{T-1} v_i + \lambda\sigma_Z^2 (T-k)\sum_{i=0}^{k-1} v_i + \lambda\sigma_Z^2 \sum_{i=k}^{T-1}(T-i)v_i - c_k.$$

Since $\beta, \alpha \geq 0$ and $\lambda > 0$, it is clear that each term of the above equation is non-negative since $v_i \geq 0$ for all i. The RHS of above equality is strictly positive since $\lambda > 0$, unless $v_i = 0$ for all $i \in \{1, \ldots, T-1\}$, which will imply $v_k = 0$. Now, we have a contradiction; as such, we conclude that $v_k = 0$ for all $k \in \{\tau, \ldots, T-1\}$.

Step 4: Optimal Investment Strategy - Case $\beta > 0$ and $\alpha > 0$.

Step 4a: Scenario $v_1 > 0$.

We will now examine the set of optimal v_t for $t \in \{1, \ldots, \tau - 1\}$. We first consider the case where the investor's decision have impact on stock prices in both the open market and the closing auction. Namely, $\beta > 0$ and $\alpha > 0$.

To show the explicit expression of the set of optimal strategies for $t \in \{1, \ldots, \tau - 1\}$, we suppose that $v_t > 0$ for any t, which implies $\frac{\partial L}{\partial v_t} = 0$ by the Karush-Kuhn-Tucker conditions. In other words, we solve for the following set of equations:

$$c_t = \beta v_t + \left(\lambda \sigma_{\tilde{Z}}^2 + \lambda \alpha^2 \sigma_{\tilde{N}}^2 + \alpha\right) \sum_{i=1}^{\tau-1} v_i + \lambda \sigma_Z^2 (T - t) \sum_{i=0}^{t-1} v_i + \lambda \sigma_Z^2 \sum_{i=t}^{\tau-1} (T - i) v_i.$$

We denote:

$$m_i = (T - i)\lambda \sigma_Z^2 + \lambda \sigma_{\tilde{Z}}^2 + \lambda \alpha^2 \sigma_{\tilde{N}}^2 + \alpha$$

for $i \in \{1, \ldots, \tau - 1\}$. Note that for any $i \in \{1, \ldots, \tau - 2\}$, we have:

$$c_i - c_{i+1} = \frac{1}{2}\mu_Z, \qquad m_i - m_{i+1} = \lambda \sigma_Z^2.$$

We observe that the system of equations can be expressed as:

$$\left[\begin{array}{ccccc|c} \beta+m_1 & m_2 & m_3 & \dots & m_{\tau-1} & c_1 \\ m_2 & \beta+m_2 & m_3 & \dots & m_{\tau-1} & c_2 \\ m_3 & m_3 & \beta+m_3 & \dots & m_{\tau-1} & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \dots & \beta+m_{\tau-1} & c_{\tau-1} \end{array}\right].$$

We subtract row $i-1$ from row i for $i \in \{2, \dots, \tau-1\}$, then the above matrix is transformed into:

$$\left[\begin{array}{ccccc|c} \beta+m_1 & m_2 & m_3 & \dots & m_{\tau-1} & c_1 \\ m_2-m_1-\beta & \beta & 0 & \dots & 0 & c_2-c_1 \\ m_3-m_2 & m_3-m_2-\beta & \beta & \dots & 0 & c_3-c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{\tau-1}-m_{\tau-2} & m_{\tau-1}-m_{\tau-2} & m_{\tau-1}-m_{\tau-2} & \dots & \beta & c_{\tau-1}-c_{\tau-2} \end{array}\right],$$

which is equivalent to:

$$\left[\begin{array}{ccccc|c} \beta+m_1 & m_2 & m_3 & \dots & m_{\tau-1} & c_1 \\ \beta+\lambda\sigma_Z^2 & -\beta & 0 & \dots & 0 & \frac{1}{2}\mu_Z \\ \lambda\sigma_Z^2 & \beta+\lambda\sigma_Z^2 & -\beta & \dots & 0 & \frac{1}{2}\mu_Z \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & -\beta & \frac{1}{2}\mu_Z \end{array}\right].$$

Moreover, we can further transform this matrix into:

$$\left[\begin{array}{ccccccc|c} \beta + m_1 & m_2 & m_3 & m_4 & \dots & m_{\tau-2} & m_{\tau-1} & c_1 \\ \beta + \lambda\sigma_Z^2 & -\beta & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}\mu_Z \\ -\beta & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 \end{array}\right].$$

We can see that:

$$v_2 = \left(1 + \frac{\lambda\sigma_Z^2}{\beta}\right) v_1 - \frac{1}{2}\frac{\mu_Z}{\beta}.$$

By examining the above matrix, for $i \in \{3, \dots, \tau - 1\}$, we have a recursive function:

$$v_i = \left(2 + \frac{\lambda\sigma_Z^2}{\beta}\right) v_{i-1} - v_{i-2}.$$

We denote:

$$b := 2 + \frac{\lambda\sigma_Z^2}{\beta}$$

and note that b is strictly positive. Specifically, we have $b > 2$ because λ and σ_Z^2 are strictly positive. The solution to the above recursive function depends on the roots of the characteristic equation:

$$x^2 - bx + 1.$$

By applying the quadratic formula, the roots are given by:

$$x_1 := \frac{b + \sqrt{b^2 - 4}}{2} \quad \text{and} \quad x_2 := \frac{b - \sqrt{b^2 - 4}}{2},$$

where $x_1 \neq x_2$ and $x_1, x_2 > 0$ since $b > 2$. The recursive function can now be expressed as:

$$v_i = Ax_1^i + Bx_2^i$$

for some $A, B \in \mathbb{R}$ and $i \in \{3, \ldots, \tau - 1\}$. For v_1 and v_2 we have:

$$v_1 = Ax_1 + Bx_2 \quad \text{and} \quad v_2 = Ax_1^2 + Bx_2^2.$$

One can easily show that:

$$A = \frac{(b-1-x_2)v_1 - \frac{\mu_Z}{2\beta}}{x_1^2 - x_1x_2} \quad \text{and} \quad B = \frac{(b-1-x_1)v_1 - \frac{\mu_Z}{2\beta}}{x_2^2 - x_1x_2}.$$

Thus, the recursive function is equivalent to:

$$\begin{aligned} v_i &= \left(\frac{(b-1-x_2)v_1 - \frac{\mu_Z}{2\beta}}{x_1^2 - x_1x_2}\right)x_1^i + \left(\frac{(b-1-x_1)v_1 - \frac{\mu_Z}{2\beta}}{x_2^2 - x_1x_2}\right)x_2^i \\ &= \left(\left(\frac{b-1-x_2}{x_1^2 - x_1x_2}\right)x_1^i + \left(\frac{b-1-x_1}{x_2^2 - x_1x_2}\right)x_2^i\right)v_1 - \frac{\mu_Z}{2\beta}\left(\frac{x_1^i}{x_1^2 - x_1x_2} + \frac{x_2^i}{x_2^2 - x_1x_2}\right). \end{aligned}$$

We observe that:

$$x_1x_2 = \left(\frac{b + \sqrt{b^2-4}}{2}\right)\left(\frac{b - \sqrt{b^2-4}}{2}\right) = \frac{b^2 - (b^2-4)}{4} = 1.$$

We denote:

$$p_i := \left(\frac{b-1-x_2}{x_1^2 - 1}\right)x_1^i + \left(\frac{b-1-x_1}{x_2^2 - 1}\right)x_2^i$$

and

$$q_i := \frac{x_1^i}{x_1^2 - 1} + \frac{x_2^i}{x_2^2 - 1}.$$

Moreover, since $b > 2$, we obtain the following:

$$
\begin{aligned}
b - 1 - x_1 = b - 1 - \frac{b + \sqrt{b^2 - 4}}{2} &= \frac{b}{2} - 1 - \sqrt{\frac{b^2}{4} - 1} < 0, \\
b - 1 - x_2 = b - 1 - \frac{b - \sqrt{b^2 - 4}}{2} &= \frac{b}{2} - 1 + \sqrt{\frac{b^2}{4} - 1} > 0, \\
x_1^2 - 1 = \frac{(b + \sqrt{b^2 - 4})^2}{4} - 1 > 0&, \\
x_2^2 - 1 = \frac{(b - \sqrt{b^2 - 4})^2}{4} - 1 < 0&.
\end{aligned}
$$

Hence, we know that $p_i > 0$ for all i. As such, we now have:

$$
\begin{aligned}
c_1 &= (\beta + m_1)v_1 + \sum_{i=2}^{\tau-1} m_i v_i \\
&= (\beta + m_1)v_1 + \sum_{i=2}^{\tau-1} m_i \left(p_i v_1 - \frac{\mu_Z}{2\beta} q_i \right) \\
&= \left(\beta + m_1 + \sum_{i=2}^{\tau-1} m_i p_i \right) v_1 - \frac{\mu_Z}{2\beta} \sum_{i=2}^{\tau-1} m_i q_i,
\end{aligned}
$$

which leads to:

$$
\begin{aligned}
v_1 &= \frac{c_1 + \frac{\mu_Z}{2\beta} \sum_{i=2}^{\tau-1} m_i q_i}{\beta + m_1 + \sum_{i=2}^{\tau-1} m_i p_i} \\
&= \frac{\left((T-1) + \frac{1}{\beta} \sum_{i=2}^{\tau-1} m_i q_i\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2(\beta + m_1 + \sum_{i=2}^{\tau-1} m_i p_i)} \qquad (5.1)
\end{aligned}
$$

where

$$
\begin{aligned}
m_i &= (T - 1 - i)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha, \\
p_i &= \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_2}{x_1^2 - 1} \right) x_1^i + \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_1}{x_2^2 - 1} \right) x_2^i,
\end{aligned}
$$

$$q_i = \frac{x_1^i}{x_1^2 - 1} + \frac{x_2^i}{x_2^2 - 1},$$

with

$$x_1 = 1 + \frac{\lambda\sigma_Z^2}{2\beta} + \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)}$$

and

$$x_2 = 1 + \frac{\lambda\sigma_Z^2}{2\beta} - \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)}.$$

Subsequently, the investment at time $i \in \{2, \dots, \tau - 1\}$ can be determined by:

$$v_i = p_i v_1 - \frac{\mu_Z}{2\beta} q_i.$$

Step 4b: Scenario $v_1 = 0$.

Now, we observe from eq. (5.1) that $v_1 > 0$ only if the following holds:

$$W > \frac{\delta - \big((T-1) + \frac{1}{\beta}\sum_{i=2}^{\tau-1} m_i q_i\big)\mu_Z - \mu_{\tilde{Z}} - \alpha\mu_{\tilde{N}}}{\alpha}.$$

Otherwise, we must have $v_1 = 0$ to satisfy the Karush-Kuhn-Tucker Condition.

If we have $v_1 = 0$, then our system of equations can be written as:

$$\left[\begin{array}{cccc|c} \beta + m_2 & m_3 & \dots & m_{\tau-1} & c_2 \\ \beta + \lambda\sigma_Z^2 & -\beta & \dots & 0 & \frac{1}{2}\mu_Z \\ \lambda\sigma_Z^2 & \beta + \lambda\sigma_Z^2 & \dots & 0 & \frac{1}{2}\mu_Z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & -\beta & \frac{1}{2}\mu_Z \end{array}\right].$$

One can repeat the identical procedure and have:

$$v_2 = \max\left(\frac{\big((T-2)+\frac{1}{\beta}\sum_{i=3}^{\tau-1} m_i q_{i-1}\big)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}}+W)-\delta}{2(\beta+m_2+\sum_{i=3}^{\tau-1} m_i p_{i-1})}, 0\right)$$

Similarly, if $v_2 > 0$, then the investment at time $i \in \{3, \dots, \tau-1\}$ can be determined by:

$$v_i = p_{i-1}v_2 - \frac{\mu_Z}{2\beta}q_{i-1}.$$

If $v_2 = 0$, then the identical procedure is repeatable until time $\tau - 3$.

To generalize our observation, we denote t^* to be the smallest integer such that:

$$\left((T-t^*)+\frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}}+W)-\delta > 0.$$

If there exists such $t^* \in \{2, \dots, \tau-3\}$, then:

$$\begin{aligned} v_s &= 0 \quad \text{for } s \in \{1, \dots, t^*-1\}, \\ v_{t^*} &= \frac{\big((T-t^*)+\frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\big)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}}+W)-\delta}{2(\beta+m_{t^*}+\sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}, \\ v_i &= p_{i+1-t^*}v_{t^*} - \frac{\mu_Z}{2\beta}q_{i+1-t^*} \quad \text{for } i \in \{t^*+1, \dots, \tau-1\}. \end{aligned} \tag{5.2}$$

We will now analyze the auxiliary term δ. Suppose $\delta > 0$, then we must have $0 = W - \sum_{i=1}^{\tau-1} v_i$. In particular, we have:

$$\begin{aligned}
0 &= W - \sum_{i=1}^{\tau-1} v_i \\
&= W - \Bigg(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\Bigg) v_{t^*} + \frac{\mu_Z}{2\beta} \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*} \\
&= W - \Bigg(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\Bigg) \frac{\big((T-t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\big)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})} \\
&\quad + \frac{\mu_Z}{2\beta} \sum_{i=t^*+1}^{\tau-1} q_i + \frac{1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}}{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}\delta,
\end{aligned}$$

which is equivalent to:

$$\begin{aligned}
\delta = &\Bigg((T-t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\Bigg)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) \\
&- \frac{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}{1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}} \Bigg(W + \frac{\mu_Z}{2\beta}\sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*}\Bigg).
\end{aligned}$$

Suppose $W \geq \frac{\alpha}{2m_{\tau-1} - \alpha}\mu_{\tilde{N}}$, which implies:

$$\begin{aligned}
\mu_{\tilde{N}} &\leq \Bigg(\frac{2m_{\tau-1}(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})}{\alpha(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})} - 1\Bigg) W \\
&< \Bigg(\frac{2(\beta + \lambda\sigma_Z^2 + m_{\tau-1} + m_{\tau-1}\sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})}{\alpha(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})} - 1\Bigg) W \\
&= \Bigg(\frac{2(\beta + m_{\tau-2} + m_{\tau-1}\sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})}{\alpha(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*})} - 1\Bigg) W \\
&\leq \Bigg(\frac{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}{1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}} - \alpha\Bigg) \frac{W}{\alpha}.
\end{aligned}$$

We see that:

$$\alpha(\mu_{\tilde{N}} + W) - \frac{2(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*})}{1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}} W \leq 0.$$

Since $\mu_Z \leq 0$ and $\mu_{\tilde{Z}} \leq 0$, we get $\delta \leq 0$, which is a contradiction. Therefore, $\delta = 0$ must hold. As such, the order placed in the closing auction is:

$$v_T = W - \left(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\right) v_{t^*} + \frac{\mu_Z}{2\beta} \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*}.$$

<u>*Step 4c: Scenario $v_i = 0$ for $i \in \{1, \ldots, \tau - 3\}$.*</u>

Suppose for all $i \in \{1, \ldots, \tau - 3\}$, $v_i = 0$. In this case, we our system of equations is reduced to:

$$\left[\begin{array}{cc|c} \beta + m_{\tau-2} & m_{\tau-1} & c_{\tau-2} \\ \beta + \lambda\sigma_Z^2 & -\beta & \frac{1}{2}\mu_Z \end{array}\right].$$

By solving this matrix, we find:

$$\begin{aligned} v_{\tau-2} &= \frac{\left((T - \tau + 2) + \frac{m_{\tau-1}}{\beta}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\left(\beta + m_{\tau-2} + m_{\tau-1}\left(1 + \frac{\lambda\sigma_Z^2}{\beta}\right)\right)} \\ v_{\tau-1} &= \left(1 + \frac{\lambda\sigma_Z^2}{\beta}\right) v_{\tau-2} - \frac{\mu_Z}{2\beta}. \end{aligned} \tag{5.3}$$

We note that if $v_{\tau-2} > 0$, then $v_{\tau-1} > 0$. In this case, the optimal strategy is given as above equalities. Moreover, similar as in the previous cases, if $\delta > 0$, we have

$$\delta = \left((T-\tau+2) + \frac{m_{\tau-1}}{\beta} \right) \mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) \\ - \frac{2\left(\beta + m_{\tau-2} + m_{\tau-1}\left(1 + \frac{\lambda\sigma_Z^2}{\beta}\right)\right)}{1 + \frac{\lambda\sigma_Z^2}{\beta}} \left(W + \frac{\mu_Z}{2\beta} \right).$$

Due to our assumption that $W \geq \frac{\alpha}{2m_{\tau-1}-\alpha}\mu_{\tilde{N}}$, one can show that $\delta = 0$, which means :

$$v_T = W - v_{\tau-2} - v_{\tau-1}.$$

<u>*Step 4d: Scenario* $v_i = 0$ *for* $i \in \{1, \ldots, \tau-2\}$.</u>

Suppose $v_{\tau-2} = 0$, in this case, we simply have

$$v_{\tau-1} = \frac{c_{\tau-1}}{\beta + m_{\tau-1}} = \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\big(\beta + (T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)}, 0 \right), \tag{5.4}$$

and

$$\delta = (T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) \\ - 2W(\beta + (T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha).$$

Similarly, by our assumption:

$$W \geq \frac{\alpha}{2m_{\tau-1} - \alpha}\mu_{\tilde{N}},$$

we have:

$$\mu_{\tilde{N}} \leq \left(\frac{2m_{\tau-1} - \alpha}{\alpha}\right)W < \left(\frac{2(\beta + m_{\tau-1})}{\alpha} - 1\right)W,$$

which implies $\delta \leq 0$, a contradiction. Thus, $\delta = 0$, and we have :

$$v_T = W - v_{\tau-1}.$$

<u>*Step 5: Optimal Investment Strategy - Case $\beta > 0$ and $\alpha = 0$.*</u>

Now, we consider the case where the stock prices is insensitive to the investor's closing auction order, or $\alpha = 0$. As such, we note that $v_t < 0$ for all t from eq. (5.1), eq. (5.2), eq. (5.3), and eq. (5.4), due to the non-positive drift of random drivers Z and $\tilde{Z}$. This is a contradiction from our initial assumption with $v_1 > 0$. In order to satisfy the Karush-Kuhn-Tucker condition, we have $v_t = 0$. As such, we conclude that:

$$W = v_T.$$

<u>*Step 6: Optimal Investment Strategy - Case $\beta = 0$ and $\alpha > 0$.*</u>

We now examine the case where the investor's orders in a open market have no impact on the stock prices but the order in the closing auction can influence the stock prices; in other words, $\beta = 0$ and $\alpha > 0$.

Once again, to show the explicit expression of optimal strategy, we suppose $v_t > 0$ for all $t \in \{1, \ldots, \tau - 1\}$. In turn, the Karush-Kuhn-Tucker conditions suggest:

$$0 = \frac{\partial L}{\partial v_t} = \left(\lambda\sigma^2_{\tilde{Z}} + \lambda\alpha^2\sigma^2_{\tilde{N}} + \alpha\right)\sum_{i=1}^{\tau-1} v_i + \lambda\sigma^2_Z(T-t)\sum_{i=0}^{t-1} v_i + \lambda\sigma^2_Z\sum_{i=t}^{\tau-1}(T-i)v_i - c_t,$$

which can be expressed as:

$$\left[\begin{array}{ccccc|c} m_1 & m_2 & m_3 & \dots & m_{\tau-1} & c_1 \\ m_2 & m_2 & m_3 & \dots & m_{\tau-1} & c_2 \\ m_3 & m_3 & m_3 & \dots & m_{\tau-1} & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \dots & m_{\tau-1} & c_{\tau-1} \end{array}\right].$$

If we subtract row i from row $i+1$ of the matrix for $i \in \{1, \dots, \tau-2\}$, we have:

$$\left[\begin{array}{cccccc|c} m_1 - m_2 & 0 & 0 & \dots & 0 & 0 & c_1 - c_2 \\ m_2 - m_3 & m_2 - m_3 & 0 & \dots & 0 & 0 & c_2 - c_3 \\ m_3 - m_4 & m_3 - m_4 & m_3 - m_4 & \dots & 0 & 0 & c_3 - c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{\tau-2} - m_{\tau-1} & m_{\tau-2} - m_{\tau-1} & m_{\tau-2} - m_{\tau-1} & \dots & m_{\tau-2} - m_{\tau-1} & 0 & c_{\tau-2} - c_{\tau-1} \\ m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \dots & m_{\tau-1} & m_{\tau-1} & c_{\tau-1} \end{array}\right],$$

which is equivalent to:

$$\left[\begin{array}{cccccc|c} \lambda\sigma_Z^2 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}\mu_Z \\ \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & 0 & \dots & 0 & 0 & \frac{1}{2}\mu_Z \\ \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & 0 & 0 & \frac{1}{2}\mu_Z \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & \lambda\sigma_Z^2 & 0 & \frac{1}{2}\mu_Z \\ m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \dots & m_{\tau-1} & m_{\tau-1} & c_{\tau-1} \end{array}\right].$$

We see that:

$$v_1 = \max\left(\frac{\mu_Z}{2\lambda\sigma_Z^2}, 0\right) = 0.$$

Moreover, it is clear that $v_t = 0$ for $t \in \{1, \dots, \tau-2\}$. Our system of equations is now reduced to:

$$m_{\tau-1}(v_1 + v_{\tau-1}) = c_{\tau-1}.$$

Suppose the investor's order in the closing auction has market effect, which means $\alpha > 0$, the investment at time $\tau - 1$ is given by:

$$v_{\tau-1} = \frac{c_{\tau-1}}{m_{\tau-1}} - v_1 = \max\left(\frac{1}{2}\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{(T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha} - v_1, 0\right).$$

Suppose that $\delta > 0$, then by the Karush-Kuhn-Tucker condition, we have

$$W - v_{\tau-1} - v_1 = 0,$$

This suggests:

$$\delta = (T - \tau + 1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2W((T - \tau + 1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha),$$

which is non-positive due to our assumptions. As such, the order placed in the closing auction is given by:

$$v_T = W - v_{\tau-1}.$$

Step 7: Optimal Investment Strategy - Case $\beta = 0$ and $\alpha = 0$.

Lastly, we examine the case when the investor's orders have no impact on the market price at all, which means $\beta = 0$ and $\alpha = 0$. One can see that $c_{\tau-1} \leq 0$, which implies $v_{\tau-1} \leq 0$. This is a contradiction, so $v_{\tau-1} = 0$. Since $v_i = 0$ for all $i \in \{1, \ldots, T-1\}$, we conclude that $v_T = W$, which means that the investor will only participate in the closing auction.

In addition to the previous result where $\beta > 0$ and $\alpha = 0$, we again have an optimal strategy to not participate in the open market since $v_i = 0$ for all i. We can conclude that, in general, if the investor's order in the closing auction has no effect on the stock prices, then he/she will only invest during the closing auction to minimize the implementation cost.

5.1.2 Proof of Corollary 1

We first denote $a := \frac{2\beta}{\lambda\sigma_Z^2}$. We note that a converges to 0 as β converges to 0. Furthermore, we have:

$$x_1 = 1 + \frac{1}{a}(1 + \sqrt{1+2a}),$$
$$x_2 = 1 + \frac{1}{a}(1 - \sqrt{1+2a}).$$

We recall that $x_1 x_2 = 1$ and:

$$p_t = \left(\frac{\frac{2}{a} + 1 - x_2}{x_1^2 - 1}\right)x_1^t + \left(\frac{\frac{2}{a} + 1 - x_1}{x_2^2 - 1}\right)x_2^t,$$
$$q_t = \frac{x_1^t}{x_1^2 - 1} + \frac{x_2^t}{x_2^2 - 1}.$$

Since β is non-negative, we will examine the right limit only. It is clear that $\lim_{a\to 0^+} x_1 = \infty$ since $\lim_{a\to 0^+} \frac{1}{a} = \infty$. For x_2, we have:

$$\lim_{a\to 0^+} x_2 = \lim_{a\to 0^+} 1 + \frac{1}{a}(1 - \sqrt{1+2a}) = 1 + \lim_{a\to 0^+} \frac{1 - \sqrt{1+2a}}{a}$$

By L'Hospital's Rule, we know:

$$\lim_{a\to 0^+} \frac{1 - \sqrt{1+2a}}{a} = \lim_{a\to 0^+} \frac{-(1+2a)^{-\frac{1}{2}}}{1} = -1;$$

thus, $\lim_{a\to 0^+} x_2 = 0$. Moreover, we have:

$$\lim_{a\to 0^+} \frac{x_2^t}{x_2^2 - 1} = \frac{0}{-1} = 0,$$

and by applying L'Hospital's Rule, we see that:

$$\lim_{a\to 0^+} \frac{x_1^t}{x_1^2-1} = \lim_{a\to 0^+} \frac{t x_1^{t-1}}{2x_1} = \lim_{a\to 0^+} \frac{t}{2} x_1^{t-2} = \frac{t}{2} \lim_{a\to 0^+} x_1^{t-2} = \begin{cases} 0, & \text{if } t = 1 \\ 1, & \text{if } t = 2 \\ \infty, & \text{if } t \geq 3 \end{cases} \cdot$$

Also, we observe:

$$\begin{aligned} \lim_{a\to 0^+} \frac{1+\frac{2}{a}-x_2}{x_1^2-1} x_1^t &= \lim_{a\to 0^+} \frac{x_1^t + \frac{2x_1^t}{a} - x_1^{t-1}}{x_1^2-1} \\ &= \lim_{a\to 0^+} \frac{x_1^t}{x_1^2-1} + \lim_{a\to 0^+} \frac{\frac{2}{a}x_1^t}{x_1^2-1} + \lim_{a\to 0^+} \frac{x_1^{t-1}}{x_1^2-1}. \end{aligned}$$

We note that $\lim_{a\to 0^+} \frac{2x_1^t}{a} = \infty$ for $t \geq 2$; as such, we find:

$$\lim_{a\to 0^+} \frac{1+\frac{2}{a}-x_2}{x_1^2-1} x_1^t = \infty \quad \text{for } t \geq 2.$$

On the other hand, we have:

$$\begin{aligned} \lim_{a\to 0^+} \frac{1+\frac{2}{a}-x_1}{x_2^2-1} x_2^t &= \lim_{a\to 0^+} \frac{x_2^t + \frac{2x_2^t}{a} - x_2^{t-1}}{x_2^2-1} \\ &= \lim_{a\to 0^+} \frac{x_2^t}{x_2^2-1} + \lim_{a\to 0^+} \frac{\frac{2}{a}x_2^t}{x_2^2-1} + \lim_{a\to 0^+} \frac{x_2^{t-1}}{x_2^2-1}. \end{aligned}$$

We analyze the numerator of the second term. By using Taylor expansion at 0, we have $\sqrt{1+2a} \approx 1 + a - \frac{a^2}{2}$. Thus, we can show:

$$\begin{aligned} \lim_{a\to 0^+} \frac{2x_2^t}{a} &= \lim_{a\to 0^+} \frac{2(1+\frac{1}{a}(1-\sqrt{1+2a}))^t}{a} \\ &= \lim_{a\to 0^+} \frac{2\big(1+\frac{1}{a}\big(1-1-a+\frac{a^2}{2}\big)\big)^t}{a} \end{aligned}$$

$$= \lim_{a \to 0^+} \frac{2a^t}{2^t a} = \lim_{a \to 0^+} \left(\frac{a}{2}\right)^{t-1} = 0 \quad \text{for } t \geq 2.$$

Therefore:

$$\lim_{a \to 0^+} \frac{\frac{2}{a} x_2^t}{x_2^2 - 1} = 0 \quad \text{for } t \geq 2,$$

which implies that:

$$\lim_{a \to 0^+} \frac{1 + \frac{2}{a} - x_1}{x_2^2 - 1} x_2^t = 0 + 0 + 0 = 0 \quad \text{for } t \geq 2.$$

As a result, we discover that:

$$\lim_{a \to 0^+} p_t = \lim_{a \to 0^+} \frac{1 + \frac{2}{a} - x_2}{x_1^2 - 1} x_1^t + \lim_{a \to 0^+} \frac{1 + \frac{2}{a} - x_1}{x_2^2 - 1} x_2^t = \infty \quad \text{for } t \geq 2$$

and

$$\lim_{a \to 0^+} q_t = \lim_{a \to 0^+} \frac{x_1^t}{x_1^2 - 1} + \lim_{a \to 0^+} \frac{x_2^t}{x_2^2 - 1} = \begin{cases} 0, & \text{if } t = 1 \\ 1, & \text{if } t = 2 \\ \infty, & \text{if } t \geq 3 \end{cases} \cdot$$

Furthermore, we note that:

$$\begin{aligned} \lim_{\beta \to 0^+} \frac{q_t}{\beta} &= \frac{2}{\lambda \sigma_Z^2} \lim_{a \to 0^+} \frac{q_t}{a} \\ &= \frac{2}{\lambda \sigma_Z^2} \left(\lim_{a \to 0^+} \frac{\frac{1}{a} x_1^t}{x_1^2 - 1} + \lim_{a \to 0^+} \frac{\frac{1}{a} x_2^t}{x_2^2 - 1} \right) \\ &= \infty \quad \text{for } t \geq 2 \end{aligned}$$

In addition, one can observe that:

$$\begin{aligned} p_t &= \left(\frac{1+\frac{2}{a}-x_2}{x_1^2-1}\right)x_1^t + \left(\frac{1+\frac{2}{a}-x_1}{x_2^2-1}\right)x_2^t \\ &= \frac{x_1^t}{x_1^2-1} + \frac{\frac{2}{a}x_1^t}{x_1^2-1} - \frac{x_1^{t-1}}{x_1^2-1} + \frac{x_2^t}{x_2^2-1} + \frac{\frac{2}{a}x_2^t}{x_2^2-1} - \frac{x_2^{t-1}}{x_2^2-1} \\ &= q_t + \frac{2}{a}q_t - q_{t-1}. \end{aligned}$$

Now, we will analyze the strategy itself. Suppose there exist $t^* \in \{1, \dots, \tau-2\}$, such that:

$$\left((T-t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\right)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) > 0.$$

We know that:

$$\lim_{\beta\to 0^+} \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*} = \infty.$$

Suppose $\mu_Z < 0$, which suggests:

$$\lim_{\beta\to 0^+}\left((T-t^*) + \frac{1}{\beta}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}\right)\mu_Z = -\infty,$$

Hence, such t^* does not exist. In this case, we have $v_t = 0$ at the limit as β converges to 0 for $t \in \{1, \dots, \tau-2\}$.

Suppose $\mu_Z = 0$ and we have

$$\mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) > 0.$$

In particular, we have $t^* = 1$ and recall that $\lim_{a\to 0^+} \sum_{i=2}^{\tau-1} m_i p_i = \infty$; thus, we get:

$$\lim_{a\to 0^+} v_1 = \lim_{a\to 0^+} \frac{\mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\left(a\frac{\lambda\sigma_Z^2}{2} + m_1 + \sum_{i=2}^{\tau-1} m_i p_i\right)} = 0$$

Since $v_1 = 0$ at the limit, we will now analyze v_2 at the limit; by our construction, we have:

$$\lim_{a\to 0^+} v_2 = \lim_{a\to 0^+} \frac{\mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2(a\frac{\lambda\sigma_Z^2}{2} + m_2 + \sum_{i=3}^{\tau-1} m_i p_{i-1})}$$

By the same argument, one can show that $\lim_{a\to 0^+} v_2 = 0$. To generalize, for any $t \in \{1, \ldots, \tau - 2\}$,

$$\lim_{a\to 0^+} v_t = \lim_{a\to 0^+} \frac{\mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2(a\frac{\lambda\sigma_Z^2}{2} + m_t + \sum_{i=t+1}^{\tau-1} m_i p_{i+1-t})}$$

suggests $v_t = 0$ at limit as β converges to 0.

In general, for any $\mu_Z \leq 0$, we have $\lim_{a\to 0^+} v_t = 0$ where $t \in \{1, \ldots, \tau - 2\}$. To analyze the strategy right before the initial imbalance announcement, we obtain:

$$\begin{aligned}\lim_{\beta\to 0^+} v_{\tau-1} &= \lim_{\beta\to 0^+} \frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\left(\beta + (T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right)} \\ &= \frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\left((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right)}\end{aligned}$$

Since $v_{\tau-1}$ is non-negative, we have:

$$\lim_{\beta \to 0^+} v_{\tau-1} = \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W)}{2\big((T-\tau+1)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)}, 0\right).$$

In turn, we have $v_T = W - v_{\tau-1}$.

5.1.3 Proof of the Discrete-Time General Strategy

In this section, we present the mathematical derivation of the structure of the generalized optimal strategy shown in section 2.3. Suppose we remove the constraint imposed on the drifts, $\mu_Z, \mu_{\tilde{Z}} \leq 0$, in proposition 1. We recall that $t^* \in \{1, \ldots, \tau-1\}$ and $k^* \in \{\tau, \ldots, T-1\}$ are the integers such that $v_{t^*}, v_{\bar{t}} > 0$ and $v_{k^*}, v_{\bar{k}} > 0$, respectively, for some $\bar{t} \in \{t^*, \ldots, \tau-1\}$ and $\bar{k} \in \{k^*, \ldots, \tau-1\}$. With loss of generality, we assume $\bar{t} = \tau-1$ and $\bar{k} = T-1$ for simplicity of presentation in this section. The mathematical procedure is identical if otherwise.

Recall that, in section 5.1.1, we have:

$$m_t = (T-t)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha \quad \text{for } t \in \{1, \ldots, \tau-1\};$$

we now denote:

$$m_k := (T-k)\lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2 \quad \text{for } k \in \{\tau, \ldots, T-1\}.$$

Let $\gamma := \lambda\alpha^2\sigma^2_{\tilde{N}} + \frac{\alpha}{2}$ and we observe that:

$$m_{\tau-1} - m_\tau = \lambda\sigma^2_Z + \gamma + \frac{\alpha}{2}.$$

Moreover, we recall that:

$$\begin{aligned} c_t &= \frac{1}{2}\big((T-t)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big) \quad \text{for } t \in \{1, \dots, \tau-1\}, \\ c_k &= \frac{1}{2}\big((T-k)\mu_Z + \mu_{\tilde{Z}} - \delta\big) \quad \text{for } k \in \{\tau, \dots, T-1\}. \end{aligned}$$

Case 1: $\beta > 0$.

Strategy A*:* $v_k = 0$ *for* $k \in \{\tau, \dots, T-1\}$.

In this case, the strategy and its mathematical derivation are identical to the optimal strategy shown in section 5.1.1. The more generalized variant is derived in the remark of proposition 1.

Strategy B

If $t^* = \tau - 1$ ($t^* = \bar{t}$) and $k^* = T - 1$ ($k^* = \bar{k}$), we can arrive to a conclusion without any computation. Otherwise, the set of equations in the Lagrange system presented at the end of Step 2 of section 5.1.1 can be expressed as the following matrix:

$$\left[\begin{array}{cccccccc|c}
\beta+m_{t^*} & m_{t^*+1} & m_{t^*+2} & \dots & m_{\tau-1} & m_{k^*}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & c_{t^*} \\
m_{t^*+1} & \beta+m_{t^*+1} & m_{t^*+2} & \dots & m_{\tau-1} & m_{k^*}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & c_{t^*+1} \\
m_{t^*+2} & m_{t^*+2} & \beta+m_{t^*+2} & \dots & m_{\tau-1} & m_{k^*}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & c_{t^*+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \dots & \beta+m_{\tau-1} & m_{k^*}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & c_{\tau-1} \\
m_{k^*}+\frac{\alpha}{2} & m_{k^*}+\frac{\alpha}{2} & m_{k^*}+\frac{\alpha}{2} & \dots & m_{k^*}+\frac{\alpha}{2} & \beta+m_{k^*} & \dots & m_{T-1} & c_{k^*} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{T-1}+\frac{\alpha}{2} & m_{T-1}+\frac{\alpha}{2} & m_{T-1}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & m_{T-1} & \dots & \beta+m_{T-1} & c_{T-1}
\end{array}\right].$$

We subtract row $i-1$ from row i for $i \in \{t^*+1, \dots \tau-1, k^*, \dots, T-1\}$ and then multiply each row by -1, the above matrix is transformed into:

$$\left[\begin{array}{cccccccc|c}
\beta+m_{t^*} & m_{t^*+1} & \dots & m_{\tau-1} & m_{k^*}+\frac{\alpha}{2} & m_{k^*+1}+\frac{\alpha}{2} & \dots & m_{T-1}+\frac{\alpha}{2} & c_{t^*} \\
\beta+\lambda\sigma_Z^2 & -\beta & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu_Z}{2} \\
\lambda\sigma_Z^2 & \beta+\lambda\sigma_Z^2 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu_Z}{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & -\beta & 0 & 0 & \dots & 0 & \frac{\mu_Z}{2} \\
\zeta_1 & \zeta_1 & \dots & \beta+\zeta_1 & \frac{\alpha}{2}-\beta & \frac{\alpha}{2} & \dots & \frac{\alpha}{2} & \omega_1 \\
\lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & \lambda\sigma_Z^2 & \beta+\lambda\sigma_Z^2 & -\beta & \dots & 0 & \frac{\mu_Z}{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \lambda\sigma_Z^2 & \dots & -\beta & \frac{\mu_Z}{2}
\end{array}\right].$$

where we denote $\zeta_1 := (k^*-\tau+1)\lambda\sigma_Z^2+\gamma$ and $\omega_1 := \frac{(k^*-\tau+1)\mu_Z}{2}+\alpha(\mu_{\tilde{N}}+W)$. Once more, we subtract row $i-1$ from row i and get:

$$\left[\begin{array}{ccccccc|c}
\beta + m_{t^*} & \dots & m_{\tau-2} & m_{\tau-1} & m_{k^*} + \frac{\alpha}{2} & \dots & m_{T-1} + \frac{\alpha}{2} & c_{t^*} \\
\beta + \lambda\sigma_Z^2 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu_Z}{2} \\
-\beta & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\beta & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 & \dots & 0 & 0 \\
\zeta_2 & \dots & \zeta_2 - \beta & 2\beta + \zeta_1 & \frac{\alpha}{2} - \beta & \dots & \frac{\alpha}{2} & \omega_2 \\
-\zeta_2 & \dots & -\zeta_2 & -\beta - \zeta_2 & 2\beta + \lambda\sigma_Z^2 - \frac{\alpha}{2} & \dots & -\frac{\alpha}{2} & -\omega_2 \\
0 & \dots & 0 & 0 & -\beta & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & -\beta & 0
\end{array}\right],$$

which is equivalent to:

$$\left[\begin{array}{cccccccc|c}
\beta + m_{t^*} & \dots & m_{\tau-2} & m_{\tau-1} & m_{k^*} + \frac{\alpha}{2} & m_{k^*+1} + \frac{\alpha}{2} & \dots & m_{T-1} + \frac{\alpha}{2} & c_{t^*} \\
\beta + \lambda\sigma_Z^2 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\mu_Z}{2} \\
-\beta & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\beta & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 & 0 & \dots & 0 & 0 \\
\zeta_2 & \dots & \zeta_2 - \beta & 2\beta + \zeta_1 & \frac{\alpha}{2} - \beta & \frac{\alpha}{2} & \dots & \frac{\alpha}{2} & \omega_2 \\
0 & \dots & -\beta & \beta + \lambda\sigma_Z^2 & \beta + \lambda\sigma_Z^2 & -\beta & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & -\beta & 2\beta + \lambda\sigma_Z^2 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & \dots & -\beta & 0
\end{array}\right],$$

where we denote $\zeta_2 := (k^* - \tau)\lambda\sigma_Z^2 + \gamma$ and $\omega_2 := \frac{(k^*-\tau)\mu_Z}{2} + \alpha(\mu_{\tilde{N}} + W)$. If $t^* < \tau - 1$, then one can see that the top-left $(k^* - 1) \times (k^* - 1)$ submatrix is:

$$\left[\begin{array}{ccccccc|c} \beta + m_{t^*} & m_{t^*+1} & m_{t^*+2} & m_{t^*+3} & \dots & m_{\tau-2} & m_{\tau-1} & c_1 \\ \beta + \lambda\sigma_Z^2 & -\beta & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}\mu_Z \\ -\beta & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 \end{array}\right].$$

We recall that:

$$\begin{aligned} p_i &= \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_2}{x_1^2 - 1}\right)x_1^i + \left(\frac{\frac{\lambda\sigma_Z^2}{\beta} + 1 - x_1}{x_2^2 - 1}\right)x_2^i, \\ q_i &= \frac{x_1^i}{x_1^2 - 1} + \frac{x_2^i}{x_2^2 - 1}, \end{aligned}$$

with

$$x_1 = 1 + \frac{\lambda\sigma_Z^2}{2\beta} + \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)}$$

and

$$x_2 = 1 + \frac{\lambda\sigma_Z^2}{2\beta} - \sqrt{\frac{\lambda\sigma_Z^2}{\beta}\left(1 + \frac{\lambda\sigma_Z^2}{4\beta}\right)}.$$

As we have shown in Step 4a of section 5.1.1, given that $t^* < \tau - 1$, the optimal strategy for time $i \in \{t^* + 1, \dots, \tau - 1\}$ can be determined by:

$$v_i = p_{i+1-t^*}v_{t^*} - \frac{\mu_Z}{2\beta}q_{i+1-t^*}.$$

For $i \in \{1, \ldots, \tau - 2\}$, we denote:

$$\begin{aligned}
\tilde{p}_i &:= \frac{\beta + \lambda\sigma_Z^2}{\beta} p_{\tau-i} - p_{\tau-i-1}, \\
\tilde{q}_i &:= \frac{\beta + \lambda\sigma_Z^2}{\beta} q_{\tau-i} - q_{\tau-i-1}.
\end{aligned}$$

If $t^* = \tau - 1$ and $k^* < T - 1$, then we denote $\tilde{p}_{\tau-1} := \frac{\lambda\sigma_Z^2}{\beta}$ and $\tilde{q}_{\tau-1} := 1$. In this case, we have:

$$v_{k^*+1} = \frac{\beta + \lambda\sigma_Z^2}{\beta} v_{k^*} + \tilde{p}_{\tau-1} v_{\tau-1} - \frac{\mu_Z}{2\beta} \tilde{q}_{\tau-1}.$$

Otherwise, by analyzing row $k^* + 1$ of the above matrix, we have:

$$\begin{aligned}
v_{k^*+1} =& \frac{\beta + \lambda\sigma_Z^2}{\beta} (v_{\tau-1} + v_{k^*}) - v_{\tau-2} \\
=& \frac{\beta + \lambda\sigma_Z^2}{\beta} v_{k^*} + \left(\frac{\beta + \lambda\sigma_Z^2}{\beta} p_{\tau-t^*} - p_{\tau-t^*-1} \right) v_{t^*} - \frac{\mu_Z}{2\beta} \left(\frac{\beta + \lambda\sigma_Z^2}{\beta} q_{\tau-t^*} - q_{\tau-t^*-1} \right) \\
=& \frac{\beta + \lambda\sigma_Z^2}{\beta} v_{k^*} + \tilde{p}_{t^*} v_{t^*} - \frac{\mu_Z}{2\beta} \tilde{q}_{t^*}.
\end{aligned}$$

Now, if $k^* < T - 1$, then we can examine the bottom-right $(T - k^* - 1) \times (T - k^* + 1)$ section of the above matrix:

$$\left[\begin{array}{ccccccc|c}
-\beta & 2\beta + \lambda\sigma_Z^2 & -\beta & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -\beta & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 2\beta + \lambda\sigma_Z^2 & -\beta & 0
\end{array}\right].$$

For $i \in \{k^* + 2, \ldots, T-1\}$, we have a recursive function:

$$v_i = \left(2 + \frac{\lambda \sigma_Z^2}{\beta}\right) v_{i-1} - v_{i-2},$$

which can be expressed as

$$v_i = A x_1^{i-k^*+1} + B x_2^{i-k^*+1}$$

for some $A, B \in \mathbb{R}$ and $i \in \{k^* + 2, \ldots, T-1\}$, as we have shown in Step 4a section 5.1.3. Moreover, we have:

$$v_{k^*} = A x_1 + B x_2$$

and

$$v_{k^*+1} = A x_1^2 + B x_2^2.$$

Since $x_1 x_2 = 1$, we can express A and B as:

$$A = \frac{v_{k^*+1} - x_2 v_{k^*}}{x_1^2 - 1}$$
$$B = \frac{v_{k^*+1} - x_1 v_{k^*}}{x_2^2 - 1}.$$

By the structure of v_{k^*+1}, we have:

$$A = \frac{\left(\frac{\beta + \lambda \sigma_Z^2}{\beta} - x_2\right) v_{k^*} + \tilde{p}_{t^*} v_1 - \frac{\mu_Z}{2\beta} \tilde{q}_{t^*}}{x_1^2 - 1}$$
$$B = \frac{\left(\frac{\beta + \lambda \sigma_Z^2}{\beta} - x_1\right) v_{k^*} + \tilde{p}_{t^*} v_1 - \frac{\mu_Z}{2\beta} \tilde{q}_{t^*}}{x_2^2 - 1}.$$

As such, for $i \in \{k^*+1, \dots, T-1\}$, the recursive function is:

$$
\begin{aligned}
v_i &= \frac{\left(\frac{\beta+\lambda\sigma_Z^2}{\beta} - x_2\right)v_{k^*} + \tilde{p}_{t^*}v_1 - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}}{x_1^2-1}x_1^{i-k^*+1} + \frac{\left(\frac{\beta+\lambda\sigma_Z^2}{\beta} - x_1\right)v_{k^*} + \tilde{p}_{t^*}v_1 - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}}{x_2^2-1}x_2^{i-k^*+1} \\
&= \left(\frac{\frac{\beta+\lambda\sigma_Z^2}{\beta} - x_2}{x_1^2-1}x_1^{i-k^*+1} + \frac{\frac{\beta+\lambda\sigma_Z^2}{\beta} - x_1}{x_2^2-1}x_2^{i-k^*+1}\right)v_{k^*} + \tilde{p}_{t^*}\left(\frac{x_1^{i-k^*+1}}{x_1^2-1} + \frac{x_2^{i-k^*+1}}{x_2^2-1}\right)v_1 \\
&\quad - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}\left(\frac{x_1^{i-k^*+1}}{x_1^2-1} + \frac{x_2^{i-k^*+1}}{x_2^2-1}\right).
\end{aligned}
$$

We denote $r_i := \frac{\beta+\lambda\sigma_Z^2-\beta x_2}{\beta(x_1^2-1)}x_1^i + \frac{\beta+\lambda\sigma_Z^2-\beta x_1}{\beta(x_2^2-1)}x_2^i$ such that:

$$
v_i = \tilde{p}_{t^*}q_{i-k^*+1}v_{t^*} + r_{i-k^*+1}v_{k^*} - \frac{\mu_Z}{2\beta}\tilde{q}_{t^*}q_{i-k^*+1} \quad \text{for } i \in \{k^*+1, \dots, T-1\}.
$$

We now examine the first row of the above full matrix, we get:

$$
\begin{aligned}
c_{t^*} &= (\beta + m_{t^*})v_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i v_i + \left(m_{k^*} + \frac{\alpha}{2}\right)v_{k^*} + \sum_{i=k^*+1}^{T-1}\left(m_i + \frac{\alpha}{2}\right)v_i \\
&= \left(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*} + \tilde{p}_{t^*}\sum_{i=k^*+1}^{T-1}\left(m_i + \frac{\alpha}{2}\right)q_{i-k^*+1}\right)v_{t^*} \\
&\quad + \left(m_{k^*} + \frac{\alpha}{2} + \sum_{i=k^*+1}^{T-1}\left(m_i + \frac{\alpha}{2}\right)r_{i-k^*+1}\right)v_{k^*} \\
&\quad - \frac{\mu_Z}{2\beta}\left(\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*} + \tilde{q}_{t^*}\sum_{i=k^*+1}^{T-1}\left(m_i + \frac{\alpha}{2}\right)q_{i-k^*+1}\right).
\end{aligned}
$$

Similarly, row k^* gives us:

$$
\begin{aligned}
c_{k^*} &= \left(m_{k^*} + \frac{\alpha}{2}\right)v_{t^*} + \left(m_{k^*} + \frac{\alpha}{2}\right)\sum_{i=t^*+1}^{\tau-1} v_i + (\beta + m_{k^*})v_{k^*} + \sum_{i=k^*+1}^{T-1} m_i v_i \\
&= \left(\left(m_{k^*} + \frac{\alpha}{2}\right)\left(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\right) + \tilde{p}_{t^*}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}\right)v_{t^*} \\
&\quad + \left(\beta + m_{k^*} + \sum_{i=k^*+1}^{T-1} m_i r_{i-k^*+1}\right)v_{k^*}
\end{aligned}
$$

$$-\frac{\mu_Z}{2\beta}\Bigg(\Big(m_{k^*}+\frac{\alpha}{2}\Big)\sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*}+\tilde{q}_{t^*}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}\Bigg).$$

We denote:

$$a_1^{t^*,k^*}=\begin{cases}\beta+m_{t^*}+\sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*}+\tilde{p}_{t^*}\sum_{i=k^*+1}^{T-1}\big(m_i+\frac{\alpha}{2}\big)q_{i-k^*+1},\\ \beta+m_{t^*}+\sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*} \quad \text{if } k^*=T-1,\\ \beta+m_{\tau-1}+\frac{\lambda\sigma_Z^2}{\beta}\sum_{i=k^*+1}^{T-1}\big(m_i+\frac{\alpha}{2}\big)q_{i-k^*+1} \quad \text{if } t^*=\tau-1,\\ \beta+m_{\tau-1} \quad \text{if } t^*=\tau-1 \text{ and } k^*=T-1.\end{cases}$$

$$a_2^{t^*,k^*}=\begin{cases}\big(m_{k^*}+\frac{\alpha}{2}\big)\big(1+\sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\big)+\tilde{p}_{t^*}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1},\\ 1+\sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*} \quad \text{if } k^*=T-1,\\ m_{k^*}+\frac{\alpha}{2}+\frac{\lambda\sigma_Z^2}{\beta}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1} \quad \text{if } t^*=\tau-1,\\ m_{T-1}+\frac{\alpha}{2} \quad \text{if } t^*=\tau-1 \text{ and } k^*=T-1.\end{cases}$$

$$b_1^{t^*,k^*}=\begin{cases}m_{k^*}+\frac{\alpha}{2}+\sum_{i=k^*+1}^{T-1}\big(m_i+\frac{\alpha}{2}\big)r_{i-k^*+1} \quad \text{if } k^*<T-1,\\ m_{T-1} \quad \text{if } k^*=T-1.\end{cases}$$

$$b_2^{t^*,k^*}=\begin{cases}\beta+m_{k^*}+\sum_{i=k^*+1}^{T-1} m_i r_{i-k^*+1} \quad \text{if } k^*<T-1,\\ \beta+m_{T-1} \quad \text{if } k^*=T-1.\end{cases}$$

and

$$s_1^{t^*,k^*}=\begin{cases}\sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*}+\tilde{q}_{t^*}\sum_{i=k^*+1}^{T-1}\big(m_i+\frac{\alpha}{2}\big)q_{i-k^*+1},\\ \sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*} \quad \text{if } k^*=T-1,\\ \sum_{i=k^*+1}^{T-1}\big(m_i+\frac{\alpha}{2}\big)q_{i-k^*+1} \quad \text{if } t^*=\tau-1,\\ 0 \quad \text{if } t^*=\tau-1 \text{ and } k^*=T-1.\end{cases}$$

$$s_2^{t^*,k^*} = \begin{cases} \left(m_{k^*} + \frac{\alpha}{2}\right) \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*} + \tilde{q}_{t^*} \sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}, \\ \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*} \quad \text{if } k^* = T-1, \\ \sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1} \quad \text{if } t^* = \tau - 1, \\ 0 \quad \text{if } t^* = \tau-1 \text{ and } k^* = T-1. \end{cases}$$

If $t^* = \tau - 1$, then we define $\sum_{i=t^*+1}^{\tau-1} m_i v_i = 0$. In addition, if $k^* = T-1$ and $t^* < \tau - 1$, then one can easily show that:

$$c_{t^*} = \left(\beta + m_{t^*} + \sum_{i=t^*+1}^{\tau-1} m_i p_{i+1-t^*}\right) v_{t^*} + m_{T-1} v_{T-1} - \frac{\mu_Z}{2\beta} \sum_{i=t^*+1}^{\tau-1} m_i q_{i+1-t^*},$$
$$c_{T-1} = \left(1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*}\right) v_{t^*} + (\beta + m_{T-1}) v_{T-1} - \frac{\mu_Z}{2\beta} \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*},$$

In general, the system of equations becomes:

$$a_1^{t^*,k^*} v_{t^*} + b_1^{t^*,k^*} v_{k^*} = c_{t^*} + \frac{\mu_Z}{2\beta} s_1^{t^*,k^*}$$
$$a_2^{t^*,k^*} v_{t^*} + b_2^{t^*,k^*} v_{k^*} = c_{k^*} + \frac{\mu_Z}{2\beta} s_2^{t^*,k^*}.$$

By solving this system, we have:

$$v_{t^*} = \frac{v_{t^*}^{num} - \left(b_1^{t^*,k^*} - b_2^{t^*,k^*}\right)\delta}{v_{t^*}^{den}}$$
$$v_{k^*} = \frac{v_{k^*}^{num} - \left(a_1^{t^*,k^*} - a_2^{t^*,k^*}\right)\delta}{v_{k^*}^{den}},$$

where we denote:

$$v_{t^*}^{num} := \left(b_1^{t^*,k^*}\left(T - k^* + \frac{s_2^{t^*,k^*}}{\beta}\right) - b_2^{t^*,k^*}\left(T - t^* + \frac{s_1^{t^*,k^*}}{\beta}\right)\right)\mu_Z$$

$$
\begin{aligned}
&+ \big(b_1^{t^*,k^*} - b_2^{t^*,k^*}\big)\mu_{\tilde{Z}} - b_2^{t^*,k^*}\alpha(\mu_{\tilde{N}} + W) \\
v_{t^*}^{den} :=& 2\big(b_1^{t^*,k^*} a_2^{t^*,k^*} - b_2^{t^*,k^*} a_1^{t^*,k^*}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
v_{k^*}^{num} :=& \left(a_1^{t^*,k^*} \left(T - k^* + \frac{s_2^{t^*,k^*}}{\beta} \right) - a_2^{t^*,k^*} \left(T - t^* + \frac{s_1^{t^*,k^*}}{\beta} \right) \right) \mu_Z \\
&+ \big(a_1^{t^*,k^*} - a_2^{t^*,k^*}\big)\mu_{\tilde{Z}} - a_2^{t^*,k^*}\alpha(\mu_{\tilde{N}} + W) \\
v_{k^*}^{den} :=& 2\big(a_1^{t^*,k^*} b_2^{t^*,k^*} - a_2^{t^*,k^*} b_1^{t^*,k^*}\big)
\end{aligned}
$$

Moreover, we denote:

$$
X_{t^*}^{k^*} := \frac{v_{t^*}^{num}}{v_{t^*}^{den}}, \quad Y_{t^*}^{k^*} := \frac{v_{k^*}^{num}}{v_{k^*}^{den}}.
$$

We further denote:

$$
A_{t^*}^{k^*} := \begin{cases}
1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*} + \tilde{p}_{t^*} \sum_{k^*+1}^{T-1} q_{i-k^*+1}, \\
1 + \sum_{i=t^*+1}^{\tau-1} p_{i+1-t^*} \quad \text{if } k^* = T-1, \\
1 + \frac{\lambda \sigma_Z^2}{\beta} \sum_{k^*+1}^{T-1} q_{i-k^*+1} \quad \text{if } t^* = \tau - 1, \\
1 \quad \text{if } t^* = \tau - 1 \text{ and } k^* = T - 1.
\end{cases}
$$

$$
B_{t^*}^{k^*} := \begin{cases}
1 + \sum_{i=k^*+1}^{T-1} r_{i-k^*+1} \quad \forall t^* < \tau - 1, \\
1 \quad \text{if } k^* = T-1, \quad \forall t^* < \tau - 1, \\
0 \quad \text{if } t^* = \tau - 1.
\end{cases}
$$

$$C_{t^*}^{k^*} := \begin{cases} \sum_{i=t^*+1}^{\tau-1} q_i + \tilde{q}_{t^*} \sum_{i=k^*+1}^{T-1} q_{i-k^*+1}, \\ \sum_{i=t^*+1}^{\tau-1} q_{i+1-t^*} \quad \text{if } k^* = T-1, \\ \sum_{i=k^*+1}^{T-1} q_{i-k^*+1} \quad \text{if } t^* = \tau - 1, \\ 0 \quad \text{if } t^* = \tau - 1 \text{ and } k^* = T-1. \end{cases}$$

If $\delta > 0$, then we must have:

$$\begin{aligned} 0 &= W - \sum_{i=t^*}^{T-1} v_i \\ &= W - A_{t^*}^{k^*} v_{t^*} - B_{t^*}^{k^*} v_{k^*} + \frac{\mu_Z}{2\beta} C_{t^*}^{k^*} \\ &= W - A_{t^*}^{k^*} X_{t^*}^{k^*} - B_{t^*}^{k^*} Y_{t^*}^{k^*} + A_{t^*}^{k^*} \frac{(b_1^{t^*,k^*} - b_2^{t^*,k^*})\delta}{2(b_1^{t^*,k^*} a_2^{t^*,k^*} - b_2^{t^*,k^*} a_1^{t^*,k^*})} \\ &\quad + B_{t^*}^{k^*} \frac{(a_1^{t^*,k^*} - a_2^{t^*,k^*})\delta}{2(a_1^{t^*,k^*} b_2^{t^*,k^*} - a_2^{t^*,k^*} b_1^{t^*,k^*})} + \frac{\mu_Z}{2\beta} C_{t^*}^{k^*}. \end{aligned}$$

As such, we have:

$$\delta_{t^*}^{k^*} = \max\left\{ \frac{2\left(b_1^{t^*,k^*} a_2^{t^*,k^*} - b_2^{t^*,k^*} a_1^{t^*,k^*}\right)}{A_{t^*}^{k^*}(b_1^{t^*,k^*} - b_2^{t^*,k^*}) - B_{t^*}^{k^*}(a_1^{t^*,k^*} - a_2^{t^*,k^*})} \left(A_{t^*}^{k^*} X_{t^*}^{k^*} + B_{t^*}^{k^*} Y_{t^*}^{k^*} - \frac{\mu_Z}{2\beta} C_{t^*}^{k^*} - W \right), 0 \right\}.$$

Summarizing various cases, for $t \in \{1, \dots, \tau - 1\}$ and $k \in \{\tau, \dots, T-1\}$, we arrive to:

$$\begin{aligned} v_s &= 0 \quad \text{for } s \in \{1, \dots, t^* - 1\} \quad \text{if } t^* > 1, \\ v_{t^*} &= X_{t^*}^{k^*} - \frac{\left(b_1^{t^*,k^*} - b_2^{t^*,k^*}\right)}{2\left(b_1^{t^*,k^*} a_2^{t^*,k^*} - b_2^{t^*,k^*} a_1^{t^*,k^*}\right)} \delta_{t^*}^{k^*}, \\ v_t &= p_{t+1-t^*} v_{t^*} - \frac{\mu_Z}{2\beta} q_{t+1-t^*} \quad \text{for } t \in \{t^* + 1, \dots, \tau - 1\}, \end{aligned}$$

$$
\begin{aligned}
v_{\bar{s}} &= 0 \quad \text{for } \bar{s} \in \{\tau, \dots, k^* - 1\} \quad \text{if } k^* > \tau, \\
v_{k^*} &= Y_{t^*}^{k^*} + \frac{\left(a_1^{t^*,k^*} - a_2^{t^*,k^*}\right)}{2\left(b_1^{t^*,k^*} a_2^{t^*,k^*} - b_2^{t^*,k^*} a_1^{t^*,k^*}\right)} \delta_{t^*}^{k^*}, \\
v_k &= \tilde{p}_{t^*} q_{k-k^*+1} v_{t^*} + r_{k-k^*+1} v_{k^*} - \frac{\mu_Z}{2\beta} \tilde{q}_{t^*} q_{k-k^*+1} \\
&\qquad\qquad\qquad\qquad \text{for } k \in \{k^* + 1, \dots, T - 1\}, \\
v_T &= W - \sum_{i=1}^{T-1} v_i.
\end{aligned}
$$

<u>***Strategy C***: *$v_t = 0$ for $t \in \{1, \dots, \tau - 1\}$.*</u>

Suppose $v_t = 0$ for $t \in \{1, \dots, \tau - 1\}$. We denote $k^* \in \{\tau, \dots, T - 2\}$ to be the smallest integer such that:

$$
\left((T - k^*) + \frac{1}{\beta} \sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}\right) \mu_Z + \mu_{\tilde{Z}} + \delta > 0.
$$

for some $\delta \geq 0$. Moreover, the system of equation is now:

$$\left[\begin{array}{ccccc|c} \beta + m_{k^*} & m_{k^*+1} & m_{k^*+2} & \cdots & m_{T-1} & c_{k^*} \\ m_{k^*+1} & \beta + m_{k^*+1} & m_{k^*+2} & \cdots & m_{T-1} & c_{k^*+1} \\ m_{k^*+2} & m_{k^*+2} & \beta + m_{k^*+2} & \cdots & m_{T-1} & c_{k^*+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{T-1} & m_{T-1} & m_{T-1} & \cdots & \beta + m_{T-1} & c_{T-1} \end{array}\right].$$

We can see that the matrix appears nearly identical to the original matrix from Step 4 of section 5.1.1, so by the exact same procedure, we can show that the optimal strategy is given by:

$$\begin{aligned} v_s &= 0 \quad \text{for } s \in \{1, \ldots, k^* - 1\}, \\ v_{k^*} &= \frac{\left((T - k^*) + \frac{1}{\beta}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}\right)\mu_Z + \mu_{\tilde{Z}} - \delta_{k*}}{2(\beta + m_{k^*} + \sum_{i=k^*+1}^{T-1} m_i p_{i-k^*+1})}, \\ v_i &= p_{i-k^*+1} v_{k^*} - \frac{\mu_Z}{2\beta} q_{i-k^*+1} \quad \text{for } i \in \{k^* + 1, \ldots, T - 1\}. \end{aligned}$$

where

$$\begin{aligned} \delta_{k^*} := \max\Bigg(&\left(\left((T - k^*) + \frac{1}{\beta}\sum_{i=k^*+1}^{T-1} m_i q_{i-k^*+1}\right)\mu_Z + \mu_{\tilde{Z}}\right. \\ &\left. - \frac{2(\beta + m_{k^*} + \sum_{i=k^*+1}^{T-1} m_i p_{i-k^*+1})}{1 + \sum_{i=k^*+1}^{T-1} p_{i-k^*+1}}\left(W + \frac{\mu_Z}{2\beta}\sum_{i=k^*+1}^{T-1} q_{i-k^*+1}\right), 0\right)\Bigg). \end{aligned}$$

Similarly, if k^* does not exist, then the strategy is:

$$v_{T-1} = \max\left(\frac{\mu_Z + \mu_{\tilde{Z}} - \delta}{2(\beta + \lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2)}, 0\right),$$

for $\delta = \max\left(\mu_Z + \mu_{\tilde{Z}} - 2W(\beta + \lambda\sigma_Z^2 + \lambda\sigma_{\tilde{Z}}^2), 0\right)$.

Case 2: $\beta = 0$**.**

Suppose $\beta = 0$. The matrix of equations will be:

$$\left[\begin{array}{ccccccccc|c}
m_1 & m_2 & m_3 & \cdots & m_{\tau-1} & m_\tau & m_{\tau+1} & \cdots & m_{T-1} & c_1 \\
m_2 & m_2 & m_3 & \cdots & m_{\tau-1} & m_\tau & m_{\tau+1} & \cdots & m_{T-1} & c_2 \\
m_3 & m_3 & m_3 & \cdots & m_{\tau-1} & m_\tau & m_{\tau+1} & \cdots & m_{T-1} & c_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{\tau-1} & m_{\tau-1} & m_{\tau-1} & \cdots & m_{\tau-1} & m_\tau & m_{\tau+1} & \cdots & m_{T-1} & c_{\tau-1} \\
m_\tau + \frac{\alpha}{2} & m_\tau + \frac{\alpha}{2} & m_\tau + \frac{\alpha}{2} & \cdots & m_\tau + \frac{\alpha}{2} & m_\tau & m_{\tau+1} & \cdots & m_{T-1} & c_\tau \\
m_{\tau+1} + \frac{\alpha}{2} & m_{\tau+1} + \frac{\alpha}{2} & m_{\tau+1} + \frac{\alpha}{2} & \cdots & m_{\tau+1} + \frac{\alpha}{2} & m_{\tau+1} + \frac{\alpha}{2} & m_{\tau+1} & \cdots & m_{T-1} & c_{\tau+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & \cdots & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & \cdots & m_{T-1} & c_{T-1}
\end{array}\right],$$

which can be represented as:

$$\left[\begin{array}{ccccccccc|c}
\lambda\sigma_Z^2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{\mu_Z}{2} \\
0 & \lambda\sigma_Z^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \lambda\sigma_Z^2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & \gamma & \gamma & \cdots & \lambda\sigma_Z^2 + \gamma & 0 & 0 & \cdots & 0 & \alpha(\mu_{\tilde{N}} + W) \\
0 & 0 & 0 & \cdots & 0 & \lambda\sigma_Z^2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \lambda\sigma_Z^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & \cdots & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & \cdots & m_{T-1} & c_{T-1}
\end{array}\right].$$

by subtracting row $i+1$ from row i and then subtracting row i from row $i+1$ for $i \in \{1, \ldots, T-1\}$. We can see that $v_t = 0$ for all $t \in \{2, \ldots, \tau - 2\}$ and $k \in \{\tau, \ldots, T-2\}$. We can directly see that:

$$v_1 = \max\left(\frac{\mu_Z}{2\lambda\sigma_Z^2}, 0\right).$$

We first consider the case where $v_T > 0$. In this case, the number of equations is reduced to three:

$$\left[\begin{array}{ccc|c} m_1 & m_{\tau-1} & m_{T-1} & c_1 \\ m_{\tau-1} & m_{\tau-1} & m_{T-1} & c_{\tau-1} \\ m_{T-1} + \frac{\alpha}{2} & m_{T-1} + \frac{\alpha}{2} & m_{T-1} & c_{T-1} \end{array}\right].$$

By subtracting the third row from the second row, we have:

$$\frac{(T-\tau+1)\mu_Z}{2} + \alpha(\mu_{\tilde{N}} + W) = \big((T-\tau+1)\lambda\sigma_Z^2 + \gamma\big)(v_1 + v_\tau),$$

which suggests:

$$v_{\tau-1} = \max\left(\frac{(T-\tau+1)\mu_Z + 2\alpha(\mu_{\tilde{N}} + W)}{2\lambda\big((T-\tau+1)\sigma_Z^2 + \alpha^2\sigma_{\tilde{N}}^2\big) + \alpha} - v_1, 0\right).$$

Moreover, the last row suggests:

$$c_{T-1} = m_{T-1}(v_1 + v_{\tau-1} + v_{T-1}) + \frac{\alpha}{2}(v_1 + v_{\tau-1}),$$

which leads to:

$$v_{T-1} = \max\left(\frac{\mu_Z + \mu_{\tilde{Z}} - \delta}{2m_{T-1}} - (v_1 + v_{\tau-1})\left(1 + \frac{\alpha}{2m_{T-1}}\right), 0\right)$$

where

$$\delta = \max\big(\mu_{\tilde{Z}} - \alpha(v_1 + v_{\tau-1}) - 2Wm_{T-1}, 0\big).$$

In the case where $v_{T-1} = 0$, then the strategy is similar to what has been shown in Step 6 of section 5.1.1. In particular, we have:

$$v_{\tau-1} = \max\left(\frac{(T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2m_{\tau-1}} - v_1, 0\right),$$

with

$$\delta = \max\left((T-\tau+1)\mu_Z + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2Wm_{\tau-1}, 0\right).$$

5.2 Proofs for the Continuous-Time Model

5.2.1 Proof of Proposition 2

Suppose the order imbalance is cleared immediately and there are no orders in the closing auction afterward. We assume the market impact of our order is only temporary. We will first restructure our objective function. By examining the objective function, we show that it is not optimal to trade after the initial imbalance announcement. As such, we derive the explicit optimal investment strategy for the period prior to the initial imbalance announcement by applying the Euler-Lagrange equation. We study the case where the trader's investment decision has some influence on stock prices in the open market ($\beta > 0$) and the case where there is no influence ($\beta = 0$).

Step 1: Preparation.

We recall that the prices of the stock is given by

$$P_t = \tilde{P}_t + \beta v_t \quad \text{for} \quad t \in [0, T)$$
$$P_T = \tilde{P}_T$$

where

$$\tilde{P}_t = \tilde{P}_0 + \mu t + \sigma W_t \quad \text{for} \quad t \in [0, \tau)$$
$$\tilde{P}_k = \tilde{P}_0 + \mu k + \sigma W_k + \alpha N \quad \text{for} \quad k \in [\tau, T)$$
$$\tilde{P}_T = \tilde{P}_0 + \mu T + \sigma W_T + \alpha N + \tilde{Z},$$

and

$$N = \tilde{N} + W - \int_0^T v_t dt.$$

The objective function is:

$$\begin{aligned}
\min \quad & E\Big[\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T\Big] \\
& + \lambda VAR\Big[\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T\Big] \\
s.t. \quad & v_t \geq 0 \quad \forall t \in [0, T), \quad W - \int_0^T v_t dt \geq 0.
\end{aligned}$$

We note that:

$$\begin{aligned}
& \int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T \\
= & \int_0^\tau v_t (\tilde{P}_0 + \mu t + \sigma W_t + \beta v_t) dt + \int_\tau^T v_t \Big(\tilde{P}_0 + \mu t + \sigma W_t + \beta v_t + \alpha \Big(\tilde{N} + W \\
& - \int_0^T v_t dt\Big)\Big) dt - \Big(\tilde{P}_0 + \mu T + \sigma W_T + \alpha \Big(\tilde{N} + W - \int_0^T v_t dt\Big) + \tilde{Z}\Big) \int_0^T v_t dt \\
= & \beta \int_0^T v_t^2 dt - \mu \int_0^T (T - t) v_t dt + \sigma \int_0^T W_t v_t dt - (\sigma W_T + \tilde{Z}) \int_0^T v_t dt \\
& - \alpha \int_0^\tau v_t \Big(\tilde{N} + W - \int_0^T v_t dt\Big) dt.
\end{aligned}$$

Since W_t is a Brownian motion, we have $E(W_t) = 0$ and $Var(W_t) = t$ for all t. Also, we recall $E\big(\int_0^T x_t dt\big) = \int_0^T E(x_t) dt$ under integrability assumptions. Thus, the expectation of the above equation is:

$$\begin{aligned}
& E\Big(\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big) P_T - W P_T\Big) \\
= & \beta \int_0^T v_t^2 dt - \mu \int_0^T (T - t) v_t dt - \mu_{\tilde{Z}} \int_0^T v_t dt - \alpha \int_0^\tau v_t \Big(\mu_{\tilde{N}} + W - \int_0^T v_t dt\Big) dt.
\end{aligned}$$

Similar to what has been shown in Section 2.1 of Frei and Westray [5], we denote:

$$dX_t^v = v_t dt,$$

and the product rule yields the following:

$$\int_0^T W_t v_t dt = \int_0^T W_t dX_t^v = -\int_0^T X_t^v dW_t + YW_T$$

where $Y := X_T^v - X_0^v = \int_0^T v_t dt$. As such, we observe that:

$$\sigma \int_0^T W_t v_t dt - \sigma W_T \int_0^T v_t dt = -\sigma \int_0^T X_t^v dW_t.$$

Moreover, since $\int_0^T X_t^v dW_t$ is a martingale, we know $E(\int_0^T X_t^v dW_t) = 0$. Thus:

$$\begin{aligned} VAR\Big(-\sigma \int_0^T X_t^v dW_t\Big) &= E\Big(\Big(-\sigma \int_0^T X_t^v dW_t\Big)^2\Big) - E\Big(\Big(-\sigma \int_0^T X_t^v dW_t\Big)\Big)^2 \\ &= \sigma^2 E\Big(\int_0^T (X_t^v)^2 dt\Big) - 0 = \sigma^2 \int_0^T (X_t^v)^2 dt. \end{aligned}$$

Hence, the variance term in the objective function is given by:

$$\begin{aligned} &VAR\Big(\int_0^T v_t P_t dt + \Big(W - \int_0^T v_t dt\Big)P_T - WP_T\Big) \\ =&\sigma^2 \int_0^T (X_t^v)^2 dt + \sigma_{\tilde{Z}}^2 \Big(\int_0^T v_t dt\Big)^2 + \alpha^2 \sigma_{\tilde{N}}^2 \Big(\int_0^\tau v_t dt\Big)^2. \end{aligned}$$

Therefore, the objective function is reformulated into:

$$\begin{aligned} \min \quad & \beta \int_0^T v_t^2 dt - \mu \int_0^T (T-t) v_t dt - \mu_{\tilde{Z}} \int_0^T v_t dt - \alpha \int_0^\tau v_t \Big(\mu_{\tilde{N}} + W - \int_0^T v_t dt\Big) dt \\ & + \lambda \sigma^2 \int_0^T (X_t^v)^2 dt + \lambda \sigma_{\tilde{Z}}^2 \Big(\int_0^T v_t dt\Big)^2 + \lambda \alpha^2 \sigma_{\tilde{N}}^2 \Big(\int_0^\tau v_t dt\Big)^2 \end{aligned}$$

$$s.t. \quad v_t \geq 0 \quad \forall t \in [0, T), \quad W - \int_0^T v_t dt \geq 0.$$

We define Φ to be the Lagrange function such that, for some $\delta \geq 0$:

$$\begin{aligned}\Phi =& \beta \int_0^T v_t^2 dt - \int_0^T \big((T-t)\mu + \mu_{\tilde{Z}}\big) v_t dt - \alpha \int_0^\tau v_t \Big(\mu_{\tilde{N}} + W - \int_0^T v_t dt\Big) dt \\ &+ \lambda\sigma^2 \int_0^T (X_t^v)^2 dt + \lambda\sigma_{\tilde{Z}}^2 \Big(\int_0^T v_t dt\Big)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 \Big(\int_0^\tau v_t dt\Big)^2 + \delta\Big(\int_0^T v_t dt - W\Big).\end{aligned}$$

<u>*Step 2: Not Optimal to trade after time τ.*</u>

We can rewrite above objective into two portions; namely, before and after the initial imbalance announcement (time τ):

$$\begin{aligned}\Phi =& \beta \int_0^\tau v_t^2 dt + \beta \int_\tau^T v_t^2 dt - \int_0^\tau \big((T-t)\mu + \mu_{\tilde{Z}}\big) v_t dt - \int_\tau^T \big((T-t)\mu + \mu_{\tilde{Z}}\big) v_t dt \\ &- \alpha \int_0^\tau v_t \Big(\mu_{\tilde{N}} + W - \int_0^\tau v_t dt\Big) dt - \alpha \int_0^\tau v_t \Big(\mu_{\tilde{N}} + W - \int_\tau^T v_t dt\Big) dt \\ &+ \lambda\sigma^2 \int_0^\tau (X_t^v)^2 dt + \lambda\sigma^2 \int_\tau^T (X_t^v)^2 dt + \lambda\sigma_{\tilde{Z}}^2 \Big(\int_0^\tau v_t dt\Big)^2 + \lambda\sigma_{\tilde{Z}}^2 \Big(\int_\tau^T v_t dt\Big)^2 \\ &+ \lambda\alpha^2\sigma_{\tilde{N}}^2 \Big(\int_0^\tau v_t dt\Big)^2 + \delta\Big(\int_0^\tau v_t dt + \int_\tau^T v_t dt - W\Big).\end{aligned}$$

At time τ, the prior orders, v_t for $t < \tau$, are determined; thus, one can only optimize the objective function over v_t for $t \geq \tau$. As such, the target function of the minimization problem after time τ is:

$$\begin{aligned}\min \quad & \beta \int_\tau^T v_t^2 dt - \int_\tau^T \big((T-t)\mu + \mu_{\tilde{Z}}\big) v_t dt + \Big(\alpha \int_0^\tau v_t dt + \delta\Big) \int_\tau^T v_t dt \\ &+ \lambda\sigma^2 \int_\tau^T (X_t^v)^2 dt + \lambda\sigma_{\tilde{Z}}^2 \Big(\int_\tau^T v_t dt\Big)^2.\end{aligned}$$

We now analyze each term of the target function given that $v_t \geq 0$ for all t:

- Since $\beta \geq 0$, we must have $v_t = 0$ to minimize $\beta \int_\tau^T v_t^2 dt$.
- We assume $\mu \leq 0$ and $\mu_{\tilde{Z}} \leq 0$; thus, $-\big((T-t)\mu + \mu_{\tilde{Z}}\big) \geq 0$. In order to minimize $\int_\tau^T -\big((T-t)\mu + \mu_{\tilde{Z}}\big) v_t dt$, we have $v_t = 0$.
- We have $\alpha, \delta \geq 0$ and we know $\int_0^\tau v_t dt \geq 0$ since $v_t \geq 0$ for $t \in [0, \tau)$. Hence, $v_t = 0$ will minimize $\int_\tau^T v_t dt$, thus, minimizing $\big(\alpha \int_0^\tau v_t dt + \delta\big) \int_\tau^T v_t dt$.
- We have $\lambda > 0$ and $\sigma > 0$ and we note that $X_t^v = \int_0^t v_s ds$. As such, $v_s = 0$ will minimize $\lambda\sigma^2 \int_\tau^T (X_t^v)^2 dt$.
- Since $\sigma_{\tilde{Z}}^2 > 0$, $v_t = 0$ minimizes $\lambda\sigma_{\tilde{Z}}^2 \big(\int_\tau^T v_t dt\big)^2$.

Since each term of the above target function is minimized by $v_t = 0$ for $t \in [\tau, T)$, we can conclude that it is not optimal to trade after time τ.

<u>*Step 3: Euler-Lagrange Equation.*</u>

Since $v_t = 0$ for $t \in [\tau, T)$, we have:

$$\Phi = \int_0^\tau \beta v_t^2 - \big((T-t)\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big) v_t + \lambda\sigma^2 (X_t^v)^2 - \frac{\delta W}{T} dt$$
$$+ \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\bigg(\int_0^\tau v_t dt\bigg)^2$$

Now, we will apply the Euler-Lagrange equation to determine v_t for $t \in [0, \tau)$; see Section 2.3.1 for descriptions. Suppose $v_t > 0$ for all $t \in [0, \tau)$. We denote:

$$u(t) := X_t^v \qquad \text{and} \qquad u'(t) = v_t.$$

Moreover, let:

$$I_1 := \int_0^T L_1(t, u, u')dt, \quad I_2 = \int_0^T L_2(t, u, u')dt$$

with

$$L_1(t, u, u') := \beta u'^2 - \big((T-t)\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)u' + \lambda\sigma^2 u^2 - \frac{\delta W}{T}$$
$$L_2(t, u, u') := u'$$

We rewrite the Lagrange equation as:

$$\Phi(t, u, u') = I_1 + \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)I_2^2$$

We consider:

$$\frac{\partial\Phi}{\partial I_1}L_1(t, u, u') = \beta u'^2 - \big((T-t)\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)u' + \lambda\sigma^2 u^2 - \frac{\delta W}{T}$$
$$\frac{\partial\Phi}{\partial I_2}L_2(t, u, u') = 2(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)I_2 u'$$

Furthermore:

$$\psi := \frac{\partial\Phi}{\partial I_1}L_1(t, u, u') + \frac{\partial\Phi}{\partial I_2}L_2(t, u, u')$$

$$
\begin{aligned}
=&\beta u'^2 - \big((T-t)\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)I_2 - \delta\big)u' \\
&+ \lambda\sigma^2 u^2 - \frac{\delta W}{T}.
\end{aligned}
$$

The Euler-Lagrange equation suggests:

$$
0 = \frac{d}{dt}\frac{\partial\psi}{\partial u'} - \frac{\partial\psi}{\partial u}.
$$

We compute that:

$$
\begin{aligned}
\frac{\partial\psi}{\partial u'} &= 2\beta u' - (T-t)\mu - \mu_{\tilde{Z}} - \alpha(\mu_{\tilde{N}} + W) + 2(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)I_2 + \delta \\
\frac{\partial\psi}{\partial u} &= 2\lambda\sigma^2 u.
\end{aligned}
$$

The Euler-Lagrange equation can be written as:

$$
2\lambda\sigma^2 u = \frac{d}{dt}\Big[2\beta u' - (T-t)\mu - \mu_{\tilde{Z}} - \alpha(\mu_{\tilde{N}} + W) + 2(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)I_2 + \delta\Big],
$$

which is equivalent to $2\beta\frac{d^2}{dt^2}u - 2\lambda\sigma^2 u = -\mu$.

Step 4: Solve for optimal strategy.

Case 1: $\beta > 0$.

By solving the above non-homogeneous ODE, we have

$$
u(t) = c_1 e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + c_2 e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \frac{\mu}{2\lambda\sigma^2}
$$

for some $c_1, c_2 \in \mathbb{R}$. We recall that $u(0) = 0$, which implies $c_2 = -\big(c_1 + \frac{\mu}{2\lambda\sigma^2}\big)$. Hence, we have:

$$\begin{aligned} u(t) &= c_1\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) \\ &= 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1 + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) \end{aligned}$$

Moreover, the rate of trading is:

$$v_t = u'(t) = \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1 + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right).$$

Now, we will determine the optimal value of c_1 by differentiate the objective function:

$$\begin{aligned} \Phi =& \beta\int_0^\tau u'(t)^2dt - \big(T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)\int_0^\tau u'(t)dt + \mu\int_0^\tau tu'(t)dt \\ &+ \lambda\sigma^2\int_0^\tau u^2(t)dt + \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\left(\int_0^\tau u'(t)dt\right)^2 - \delta W \end{aligned}$$

with respect to c_1. Before we do so, there are some preparation steps. We compute that:

$$\begin{aligned} u'(t)^2 =& \frac{4\lambda\sigma^2}{\beta}\cosh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1^2 + \frac{2\mu}{\beta}\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}c_1 + \frac{\mu^2}{4\beta\lambda\sigma^2}e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} \\ u(t)^2 =& 4\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1^2 + \frac{2\mu}{\lambda\sigma^2}\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\left(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)c_1 \\ &+ \frac{\mu^2}{4\lambda^2\sigma^4}\left(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2 \end{aligned}$$

We denote $a := \sqrt{\frac{\lambda\sigma^2}{\beta}}\tau$ and compute the following integrals:

$$\int_0^\tau t\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)dt = \frac{\beta}{\lambda\sigma^2}\big(a\sinh(a) - \cosh(a) + 1\big),$$

$$\int_0^\tau te^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{\beta}{\lambda\sigma^2}\big(1 - e^{-a}(1+a)\big),$$

$$\int_0^\tau \cosh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)dt = \frac{1}{4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a + \sinh(2a)\big),$$

$$\int_0^\tau \cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{1}{4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a - \mathrm{e}^{-2a} + 1\big),$$

$$\int_0^\tau e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1 - \mathrm{e}^{-2a}\big),$$

$$\int_0^\tau \sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)dt = \frac{1}{4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a) - 2a\big),$$

and

$$\int_0^\tau \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\left(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)dt = -\frac{1}{4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a + \mathrm{e}^{-2a} - 4\cosh(a) + 3\big)$$

$$\int_0^\tau \left(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2 dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2a} - \frac{\mathrm{e}^{-4a} + 3}{4} + a\right)$$

Using above integrals, we compute:

$$\int_0^\tau tu'(t)dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\left[2\big(a\sinh(a) - \cosh(a) + 1\big)c_1 + \frac{\mu}{2\lambda\sigma^2}\big(1 - e^{-a}(1+a)\big)\right],$$

$$\begin{aligned}\int_0^\tau u'(t)^2 dt = &\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(2a + \sinh(2a)\big)c_1^2 + \frac{\mu}{2\beta}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a - \mathrm{e}^{-2a} + 1\big)c_1\\ &+ \frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1 - \mathrm{e}^{-2b}\big),\end{aligned}$$

$$
\begin{aligned}
\int_0^\tau u(t)^2 dt =& \sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a) - 2a\big)c_1^2 - \frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a + \mathrm{e}^{-2a} - 4\cosh(a) + 3\big)c_1 \\
&+ \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2a} - \frac{\mathrm{e}^{-4b}+3}{4} + a\right)
\end{aligned}
$$

Moreover, we note that:

$$
\begin{aligned}
\int_0^\tau u'(t)dt = u(\tau) =& 2\sinh(a)c_1 + \frac{\mu}{2\lambda\sigma^2}\big(1 - \mathrm{e}^{-a}\big), \\
\left(\int_0^\tau u'(t)dt\right)^2 = u^2(\tau) =& 4\sinh^2(a)c_1^2 + \frac{2\mu}{\lambda\sigma^2}\sinh(a)\big(1 - \mathrm{e}^{-2a}\big)c_1 + \frac{\mu^2}{4\lambda^2\sigma^4}\big(1 - \mathrm{e}^{-a}\big)^2.
\end{aligned}
$$

We denote the following constants:

$$
\begin{aligned}
K_1 &:= 2\sinh(a) \\
K_2 &:= 4\sinh^2(a) \\
K_3 &:= \frac{2\mu}{\lambda\sigma^2}\sinh(a)\big(1 - \mathrm{e}^{-2a}\big) \\
K_4 &:= \sqrt{\frac{\beta}{\lambda\sigma^2}}2\big(a\sinh(a) - \cosh(a) + 1\big) \\
K_5 &:= \sqrt{\frac{\lambda\sigma^2}{\beta}}\big(2a + \sinh(2a)\big) \\
K_6 &:= \frac{\mu}{2\beta}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a - \mathrm{e}^{-2a} + 1\big) \\
K_7 &:= \sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a) - 2a\big) \\
K_8 &:= -\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a + \mathrm{e}^{-2a} - 4\cosh(a) + 3\big)
\end{aligned}
$$

As such, we have:

$$
\Phi = \beta(K_5 c_1^2 + K_6 c_1) - \big(T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)K_1 c_1 + \mu K_4 c_1
$$

$$
\begin{aligned}
&+ \lambda\sigma^2(K_7c_1^2 + K_8c_1) + \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)(K_2c_1^2 + K_3c_1) + constants\\
=&\big(\beta K_5 + \lambda\sigma^2K_7 + \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)K_2\big)c_1^2 + constants\\
&+ \big(\beta K_6 - \big(T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)K_1 + \mu K_4 + \lambda\sigma^2K_8\\
&+ \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)K_3\big)c_1.
\end{aligned}
$$

We denote:

$$
\begin{aligned}
m_1 &= T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W),\\
m_2 &= \frac{\mu}{\lambda\sigma^2}\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big).
\end{aligned}
$$

To find the optimal c_1, we obtain:

$$
\begin{aligned}
0 = \frac{\partial\Phi}{\partial c_1} =&2\big(\beta K_5 + \lambda\sigma^2K_7 + \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)K_2\big)c_1\\
&+ \big(\beta K_6 + \big(T\mu + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta\big)K_1 + K_4 + \lambda\sigma^2K_8\\
&+ \big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)K_3\big)\\
=&\big(2\sqrt{\beta\lambda\sigma^2}\sinh(2a) + 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\sinh^2(a)\big)c_1\\
&+ \frac{\mu}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a\sinh(a) - \mathrm{e}^{-2a} + 1\big) + \sinh(a)\big(m_2\big(1 - \mathrm{e}^{-2a}\big) - m_1\big)\\
&+ \delta\sinh(2a).
\end{aligned}
$$

Therefore, we find:

$$
c_1^* = \frac{\sinh(a)\big(m_1 - m_2\big(1 - \mathrm{e}^{-2a}\big)\big) - \frac{\mu}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a\sinh(a) - \mathrm{e}^{-2a} + 1\big) - \delta\sinh(2a)}{2\sqrt{\beta\lambda\sigma^2}\sinh(2a) + 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\sinh^2(a)},
$$

and the optimal rate of trading is:

$$v_t = u'(t) = \max\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\left(2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1^* + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right), 0\right).$$

Now, we denote by t^* the smallest integer such that $v_{t^*} > 0$. We can find c_1^* similarly by analyzing the integrals shown previously over the interval $[t^*, \tau)$ instead of $[0, \tau)$. We denote:

$$\begin{aligned}
c_{num}(t) =& \sinh(a_\tau)\big(m_1 - m_2\big(1 - \mathrm{e}^{-2a_\tau}\big)\big) - \sinh(a_t)\big(m_1 - m_2\big(1 - \mathrm{e}^{-2a_t}\big)\big) \\
& - \mu\left(\tau\sinh(a_\tau) - t\sinh(a_t) - \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2a_\tau} - \mathrm{e}^{-2b_t}\big)\right), \\
c_{den}(t) =& 2\sqrt{\beta\lambda\sigma^2}\big(\sinh(2a_\tau) - \sinh(2a_t)\big) \\
& + 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)\big(\sinh^2(a_\tau) - \sinh^2(a_t)\big),
\end{aligned}$$

where

$$a_t := \sqrt{\frac{\lambda\sigma^2}{\beta}}t \quad \text{for } t \in [0, \tau],$$

such that:

$$c(t) = \frac{c_{num}(t)}{c_{den}(t)}.$$

As such, we have:

$$c_1^*(t^*) = c(t^*) - \frac{\delta(\sinh(2a_\tau) - \sinh(2a_{t^*}))}{c_{den}(t^*)}.$$

Hence, the rate of trading at time t is:

$$v_t^* = \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(2\cosh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1^*(t^*) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right),$$

and the cumulative order at time t is:

$$u^*(t) = 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_1^*(t^*) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right).$$

We will now analyze the structure of δ. Suppose that $\delta > 0$, then we must have:

$$\begin{aligned}
0 =& W - \int_{t^*}^{\tau} v_t dt \\
=& W - u^*(\tau) \\
=& W - 2\sinh(a_\tau)c_1^*(t^*) + \frac{\mu}{2\lambda\sigma^2}(1 - \mathrm{e}^{-a_\tau}) \\
=& W - 2\sinh(a_\tau)c(t^*) + \frac{\mu}{2\lambda\sigma^2}(1 - \mathrm{e}^{-a_\tau}) + \frac{\delta(\sinh(2a_\tau) - \sinh(2a_{t^*}))}{c_{den}(t^*)}.
\end{aligned}$$

Thus:

$$\delta = \frac{2\sinh(a_\tau)c(t^*) - \frac{\mu}{2\lambda\sigma^2}(1 - \mathrm{e}^{-a_\tau}) - W}{\sinh(2a_\tau) - \sinh(2a_{t^*})} c_{den}(t^*).$$

We note that:

$$\sinh(2a_\tau) - \sinh(2a_{t^*}) > 0,$$

and

$$2\sqrt{\beta\lambda\sigma^2}(\sinh(2a_\tau) - \sinh(2a_{t^*})) + 4\big(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\big)(\sinh^2(a_\tau) - \sinh^2(a_{t^*})) > 0.$$

By our assumption, $2\sinh(a_\tau)c(0) - \frac{\mu(1-e^{a_\tau})}{2\lambda\sigma^2} \leq W$, we can see that $\delta \leq 0$, which is a contradiction as $\delta \geq 0$ must hold. Therefore, we have $\delta = 0$, which implies $c_1^* = c$. Moreover, we have:

$$v_T = W - u^*(\tau) > 0.$$

If such t^* does not exist, then we can conclude that there is no investment in the continuous trading. That is, $v_T = W$.

Case 2: $\beta = 0$.

We recall that the Euler-Lagrange equation suggests:

$$2\beta \frac{d^2}{dt^2} u - 2\lambda\sigma^2 u = -\mu.$$

If $\beta = 0$, then we simply have $u(t) = \frac{\mu}{2\lambda\sigma^2}$; in particular, we have:

$$X_0 = \max\left(\frac{\mu}{2\lambda\sigma^2}, 0\right) = 0.$$

Moreover, we have $v_t = 0$ for $t \in (0, \tau)$ and v_t does not exist for $t = 0, \tau$. We note that the transactions in the continuous trading only occur at time 0 and the moment before time τ; that is, $\tau - \epsilon$ for some small $\epsilon > 0$. We denote $\tilde{\tau} := \tau - \epsilon$. Let $V_0, V_{\tilde{\tau}}$, and V_T be the order volume at time $0, \tau$, and T, respectively. In order to determine $u(\tilde{\tau})$, we rewrite the objective function accordingly. In particular, we have:

$$\begin{aligned}
\min \quad & E\Big[V_0P_0 + V_{\tilde{\tau}}P_{\tilde{\tau}} + \big(W - V_0 - V_{\tilde{\tau}}\big)P_T - WP_T\Big] \\
& + \lambda VAR\Big[V_0P_0 + V_{\tilde{\tau}}P_{\tilde{\tau}} + \big(W - V_0 - V_{\tilde{\tau}}\big)P_T - WP_T\Big] \\
s.t. \quad & V_{\tilde{\tau}} \geq 0, \quad W - V_0 - V_{\tilde{\tau}} \geq 0
\end{aligned}$$

where

$$\begin{aligned}
P_{\tilde{\tau}} &= P_0 + \beta V_0 + \mu\tilde{\tau} + \sigma W_{\tilde{\tau}}, \\
P_T &= P_0 + \beta V_0 + \mu T + \sigma W_T + \alpha\big(\tilde{N} + W - V_0 - V_{\tilde{\tau}}\big) + \tilde{Z}.
\end{aligned}$$

We can rewrite the following:

$$\begin{aligned}
& V_0P_0 + V_{\tilde{\tau}}P_{\tilde{\tau}} + \big(W - V_0 - V_{\tilde{\tau}}\big)P_T - WP_T \\
=& V_0P_0 + V_{\tilde{\tau}}\big(P_0 + \beta V_0 + \mu\tilde{\tau} + \sigma W_{\tilde{\tau}}\big) \\
& - (V_0 + V_{\tilde{\tau}})\big(P_0 + \beta V_0 + \mu T + \sigma W_T + \alpha\big(\tilde{N} + W - V_0 - V_{\tilde{\tau}}\big) + \tilde{Z}\big) \\
=& V_{\tilde{\tau}}\big(\mu\tilde{\tau} + \sigma W_{\tilde{\tau}}\big) - \beta V_0^2 - (V_0 + V_{\tilde{\tau}})\big(\mu T + \sigma W_T + \alpha(\tilde{N} + W) + \tilde{Z}\big) \\
& + \alpha(V_0 + V_{\tilde{\tau}})^2.
\end{aligned}$$

We recall that $E(W_t) = 0$ and $Var(W_t) = t$ for all t. As such, we obtain:

$$\begin{aligned}
& E\Big[V_0P_0 + V_{\tilde{\tau}}P_{\tilde{\tau}} + \big(W - V_0 - V_{\tilde{\tau}}\big)P_T - WP_T\Big] \\
=& \mu\tilde{\tau}V_{\tilde{\tau}} - \beta V_0^2 - (V_0 + V_{\tilde{\tau}})\big(\mu T + \alpha(\mu_{\tilde{N}} + W) + \mu_{\tilde{Z}}\big) + \alpha(V_0 + V_{\tilde{\tau}})^2, \\
& VAR\Big[V_0P_0 + V_{\tilde{\tau}}P_{\tilde{\tau}} + \big(W - V_0 - V_{\tilde{\tau}}\big)P_T - WP_T\Big]
\end{aligned}$$

$$=\sigma^2\tilde{\tau}V_{\tilde{\tau}}^2 + (V_0 + V_{\tilde{\tau}})^2\big(\sigma^2 T + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big).$$

Hence, the objective function is equivalent to: ;

$$\begin{aligned}\min \quad L :=& \mu\tilde{\tau}V_{\tilde{\tau}} - \beta V_0^2 - (V_0 + V_{\tilde{\tau}})\big(\mu T + \alpha(\mu_{\tilde{N}} + W) + \mu_{\tilde{Z}}\big) + \alpha(V_0 + V_{\tilde{\tau}})^2 \\ &+ \lambda\Big[\sigma^2\tilde{\tau}V_{\tilde{\tau}}^2 + (V_0 + V_{\tilde{\tau}})^2\big(\sigma^2 T + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\Big] + \delta\big(V_0 + V_{\tilde{\tau}} - W\big).\end{aligned}$$

for some $\delta \geq 0$. To find the optimal $V_{\tilde{\tau}}$, we analyze:

$$\begin{aligned}0 = \frac{\partial L}{\partial V_{\tilde{\tau}}} =& \mu\tilde{\tau} - \big(\mu T + \alpha(\mu_{\tilde{N}} + W) + \mu_{\tilde{Z}}\big) + 2\alpha(V_0 + V_{\tilde{\tau}}) \\ &+ 2\lambda\sigma^2\tilde{\tau}V_{\tilde{\tau}} + 2\lambda\big(\sigma^2 T + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)(V_0 + V_{\tilde{\tau}}) + \delta,\end{aligned}$$

which suggests:

$$V_{\tilde{\tau}} = \max\left(\frac{\mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2\big(\alpha + \lambda\big(\sigma^2 T + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)V_0 - \delta}{2\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)}, 0\right).$$

Furthermore, we have $V_T = W - V_0 - V_{\tilde{\tau}}$ and $V_0 = X_0$.

We now analyze δ. We consider $V_{\tilde{\tau}} > 0$. Suppose that $\delta > 0$, then we must have:

$$\begin{aligned}0 =& V_0 + V_{\tilde{\tau}} - W \\ 2\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \alpha^2\sigma_{\tilde{N}}^2 + \sigma_{\tilde{Z}}^2\big)\big)W =& \mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} + 2\lambda\sigma^2\tilde{\tau}V_0 - \delta,\end{aligned}$$

which suggests:

$$\delta = \mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} + 2\lambda\sigma^2\tilde{\tau}V_0 - 2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)W.$$

Since $\delta \geq 0$, we have:

$$\delta = \max\left(\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} + 2\lambda\sigma^2\tilde{\tau}V_0 - 2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)W, 0\right).$$

Since $\mu \leq 0$, the strategy is given by:

$$\begin{aligned}
V_{\tilde{\tau}} &= \max\left(\frac{\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - \delta}{2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)}, 0\right),\\
V_T &= W - V_T,
\end{aligned}$$

where $\delta = \max\left(\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} - 2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)W, 0\right)$.

5.2.2 Proof of Corollary 2

We recall that:

$$\begin{aligned}
c_{num}(t) =& \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)\left(m_1 - m_2\left(1 - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}\right)\right) - \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\Bigg(m_1\\
&- m_2\left(1 - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\Bigg) - \mu\Bigg(\tau\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - t\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\\
&- \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau} - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\Bigg),\\
c_{den}(t) =& 2\sqrt{\beta\lambda\sigma^2}\left(\sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\right)\\
&+ 4\big(\lambda\sigma^2_{\tilde{Z}} + \lambda\alpha^2\sigma^2_{\tilde{N}} + \alpha\big)\left(\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\right),
\end{aligned}$$

such that: $c(t) = \frac{c_{num}(t)}{c_{den}(t)}$. If t^* exists, then the optimal cumulative order at time t is:

$$
\begin{aligned}
X_s^v &= 0 \quad \text{for } s \in [0, t^*), \\
X_t^v &= 2\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c(t^*) + \frac{\mu}{2\lambda\sigma^2}\left(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right).
\end{aligned}
$$

It is clear that $\lim_{\beta\to 0} \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} = 0$. Moreover, we compute:

$$
\begin{aligned}
&\lim_{\beta\to 0} \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_{num}(t^*) \\
=&\lim_{\beta\to 0} \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)\left(m_1 - m_2\left(1 - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}\right) - \mu\tau\right) \\
&- \lim_{\beta\to 0} \sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t^*\right)\left(m_1 - m_2\left(1 - \mathrm{e}^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t^*}\right) - \mu t^*\right) \\
=&\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\beta\to 0} c_{num}(\tau) \\
=&4\left(\lambda\sigma_{\tilde{Z}}^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 + \alpha\right)\lim_{\beta\to 0}\left(\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\right) \\
&+ 2\lim_{\beta\to 0}\sqrt{\beta\lambda\sigma^2}\left(\sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) - \sinh\left(2\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\right) = \infty.
\end{aligned}
$$

We now analyze $\lim_{\beta\to 0} \frac{\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_{num}(t)}{c_{den}(t)}$. In particular, we note that the term with the highest order in the numerator is:

$$
\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right),
$$

and the term with the highest order in the denominator is:

$$\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right).$$

Since $t \leq \tau$, we know that:

$$\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right) \leq \sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)$$

Hence, we have:

$$\lim_{\beta\to 0}\frac{\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)}{\sinh^2\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau\right)} = 0,$$

which, in turn, suggests:

$$\lim_{\beta\to 0}\frac{\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_{num}(t)}{c_{den}(t)} = 0.$$

Therefore, we find:

$$\lim_{\beta\to 0} X_t^v = \lim_{\beta\to 0}\frac{\sinh\left(\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)c_{num}(t)}{c_{den}(t)} + \frac{\mu}{2\lambda\sigma^2}\left(1 - \lim_{\beta\to 0}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) = \frac{\mu}{2\lambda\sigma^2}.$$

We apply the same argument in case 2 in section 5.2.1 and obtain the result stated in corollary 2.

5.2.3 Proof of the Continuous-Time General Strategy

In this section, we show the mathematical derivation behind the generalized strategy for the continuous-time model. In particular, we consider the condition $(T-t)\mu \leq -\mu_{\tilde{Z}}$ does not necessarily hold. Moreover, as we did in section 5.1.3, we consider the case for $\beta > 0$ as well as the case with $\beta = 0$. Consider some small $\epsilon > 0$. We recall that $t^* \in [0, \tau)$, $\tau^* \in [t^* + \epsilon, \tau)$, $k^* \in [\tau, T)$, and $T^* \in [k^* + \epsilon, T)$ are some real numbers such that $v_{t^*}, v_{\tau^*} > 0$ and $v_{k^*}, v_{T^*} > 0$. The search for the optimal set of t^*, τ^*, k^*, T^* is discussed in section 3.4 and the mathematical proof remain the same for any combination. For the simplicity of the presentation of the proof, we write τ for τ^* and T for T^*. This is done to ensure the consistency notation-wise with section 5.2.1 when drawing references. For a more generalized presentation, one can simply replace τ and T with τ^* and T^*.

Case 1: $\beta > 0$**.**

Strategy A*: $v_k = 0$ for $k \in \{\tau, \dots, T-1\}$.*

This strategy and its mathematical derivation are identical to the optimal strategy shown in section 5.2.1. The more generalized variant is derived in the remark of proposition 2.

Strategy B

This strategy consider the scenario where investor choose to invest in the time period before and after time τ. The below steps follow immediately after Step

1 in section 5.2.1. We can rewrite:

$$\int_0^\tau v_t dt = \int_0^T v_t \mathbb{I}_{\{t<\tau\}} dt$$

As such, the objective function is:

$$\Phi = \int_0^T \beta v_t^2 - \big((T-t)\mu + \mu_{\tilde{Z}} - \delta\big)v_t - \alpha(\mu_{\tilde{N}} + W)v_t \mathbb{I}_{\{t<\tau\}} + \lambda\sigma^2 (X_t^v)^2 - \frac{\delta W}{T} dt$$
$$+ \lambda\sigma_{\tilde{Z}}^2 \Big(\int_0^T v_t dt\Big)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 \Big(\int_0^T v_t \mathbb{I}_{\{t<\tau\}} dt\Big)^2 + \alpha \int_0^T v_t dt \int_0^T v_t \mathbb{I}_{\{t<\tau\}} dt$$

We denote:

$$u(t) := X_t^v \qquad \text{and} \qquad u'(t) = v_t.$$

Moreover, let:

$$I_1 := \int_0^T L_1(t,u,u')dt, \quad I_2 = \int_0^T L_2(t,u,u')dt, \quad I_3 = \int_0^T L_3(t,u,u')dt$$

with

$$L_1(t,u,u') := \beta u'^2 - \big((T-t)\mu + \mu_{\tilde{Z}} - \delta\big)u' - \alpha(\mu_{\tilde{N}} + W)u'\mathbb{I}_{\{t<\tau\}} + \lambda\sigma^2 u^2 - \frac{\delta W}{T}$$
$$L_2(t,u,u') := u'$$
$$L_3(t,u,u') := u'\mathbb{I}_{\{t<\tau\}}$$

We rewrite the Lagrange equation as:

$$\Phi(t,u,u') = I_1 + \lambda\sigma_{\tilde{Z}}^2 I_2^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2 I_3^2 + \alpha I_2 I_3$$

We consider:

$$\frac{\partial\Phi}{\partial I_1}L_1(t,u,u') = \beta u'^2 - \big((T-t)\mu + \mu_{\tilde{Z}} - \delta\big)u' - \alpha(\mu_{\tilde{N}} + W)u'\mathbb{I}_{\{t<\tau\}} + \lambda\sigma^2 u^2 - \frac{\delta W}{T},$$
$$\frac{\partial\Phi}{\partial I_2}L_2(t,u,u') = \big(2\lambda\sigma_{\tilde{Z}}^2 I_2 + \alpha I_3\big)u',$$
$$\frac{\partial\Phi}{\partial I_3}L_3(t,u,u') = \big(2\lambda\alpha^2\sigma_{\tilde{N}}^2 I_3 + \alpha I_2\big)u'.$$

Furthermore:

$$\begin{aligned}\psi :=& \frac{\partial\Phi}{\partial I_1}L_1(t,u,u') + \frac{\partial\Phi}{\partial I_2}L_2(t,u,u') + \frac{\partial\Phi}{\partial I_3}L_3(t,u,u')\\ =&\beta u'^2 - \Bigg((T-t)\mu + \mu_{\tilde{Z}} - \big(\alpha + 2\lambda\sigma_{\tilde{Z}}^2\big)I_2 - \big(\alpha + 2\lambda\alpha^2\sigma_{\tilde{N}}^2\big)I_3 - \delta\Bigg)u'\\ &- \alpha(\mu_{\tilde{N}} + W)u'\mathbb{I}_{\{t<\tau\}} + \lambda\sigma^2 u^2 - \frac{\delta W}{T}.\end{aligned}$$

The Euler-Lagrange equation suggests:

$$0 = \frac{d}{dt}\frac{\partial\psi}{\partial u'} - \frac{\partial\psi}{\partial u}.$$

We compute that:

$$\begin{aligned}\frac{\partial\psi}{\partial u'} =& 2\beta u' - (T-t)\mu - \mu_{\tilde{Z}} + \big(\alpha + 2\lambda\sigma_{\tilde{Z}}^2\big)I_2\\ &+ \big(\alpha + 2\lambda\alpha^2\sigma_{\tilde{N}}^2\big)I_3 + \delta - \alpha(\mu_{\tilde{N}} + W)\mathbb{I}_{\{t<\tau\}},\\ \frac{\partial\psi}{\partial u} =& 2\lambda\sigma^2 u.\end{aligned}$$

The Euler-Lagrange equation can be written as:

$$2\lambda\sigma^2 u = \frac{d}{dt}\Bigg[2\beta u' - (T-t)\mu - \mu_{\tilde{Z}} + \big(\alpha + 2\lambda\sigma_{\tilde{Z}}^2\big)I_2$$

$$+\left(\alpha+2\lambda\alpha^2\sigma_{\tilde{N}}^2\right)I_3+\delta-\alpha(\mu_{\tilde{N}}+W)\mathbb{I}_{\{t<\tau\}}\Big]$$

$$2\lambda\sigma^2 u = 2\beta u''+\mu-\alpha(\mu_{\tilde{N}}+W)\frac{d}{dt}\mathbb{I}_{\{t<\tau\}}$$

On intervals $[0,\tau)$ and (τ,T), we have

$$2\lambda\sigma^2 u = 2\beta u''+\mu$$

In both cases, we a non-homogeneous ODE:

$$2\beta\frac{d}{dt}u'+\mu = 2\lambda\sigma^2 u$$
$$2\beta\frac{d^2}{dt^2}u-2\lambda\sigma^2 u = -\mu. \tag{5.5}$$

By solving this ODE, one can show that:

$$u(t)=\begin{cases} c_1 \mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+c_2\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2} & \text{for } t\in[0,\tau) \\ \\ c_3 \mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+c_4\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2} & \text{for } t\in[\tau,T) \end{cases}$$

for some $c_1,c_2,c_3,c_4\in\mathbb{R}$. We recall that $u(0)=0$, which implies $c_2 = -\left(c_1+\frac{\mu}{2\lambda\sigma^2}\right)$.

One one hand, we have:

$$\lim_{t\to\tau^-}u(t)=\lim_{t\to\tau^-}\left(c_1\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+c_2\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2}\right)$$
$$=\lim_{t\to\tau^-}\left(c_1\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-\left(c_1+\frac{\mu}{2\lambda\sigma^2}\right)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2}\right)$$

$$
\begin{aligned}
=&c_1 \lim_{t\to\tau^-}\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)-\frac{\mu}{2\lambda\sigma^2}\lim_{t\to\tau^-}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2}\\
=&c_1\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}\right)-\frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+\frac{\mu}{2\lambda\sigma^2}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\lim_{t\to\tau^+}u(t)=&\lim_{t\to\tau^+}\left(c_3\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+c_4\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+\frac{\mu}{2\lambda\sigma^2}\right)\\
=&c_3\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+c_4\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+\frac{\mu}{2\lambda\sigma^2}.
\end{aligned}
$$

Since the function $u(t)$ measures the cumulative order up to time t, it is continuous as τ, which suggests:

$$
\begin{aligned}
c_1\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}\right)-\frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+\frac{\mu}{2\lambda\sigma^2}&=c_3\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+c_4\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+\frac{\mu}{2\lambda\sigma^2}\\
c_1\left(\mathrm{e}^{2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}-1\right)-\frac{\mu}{2\lambda\sigma^2}&=c_3\mathrm{e}^{2\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau}+c_4
\end{aligned}
$$

Hence, for $a:=\sqrt{\frac{\lambda\sigma^2}{\beta}}\tau$, we have:

$$
c_4=c_1\big(\mathrm{e}^{2a}-1\big)-c_3\mathrm{e}^{2a}-\frac{\mu}{2\lambda\sigma^2}.
$$

Now, we can rewrite the function u as:

$$
u(t)=\begin{cases}
c_1\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)+\frac{\mu}{2\lambda\sigma^2}\left(1-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t\in[0,\tau)\\
c_1\big(e^{2a}-1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}+c_3\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-\mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \\
+\frac{\mu}{2\lambda\sigma^2}\left(1-\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t\in[\tau,T).
\end{cases}
$$

Moreover, the rate of trading is:

$$v_t = u'(t) = \begin{cases} \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(c_1\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in [0,\tau) \\ \\ \sqrt{\frac{\lambda\sigma^2}{\beta}}\left(c_3\left(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) - c_1\big(\mathrm{e}^{2a}-1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} \right. \\ \left. +\frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) & \text{for } t \in (\tau, T). \end{cases}$$

We will now analyze the optimal structure for constants c_1 and c_3. We recall the objective Lagrange function:

$$\begin{aligned} \Phi =& \beta\int_0^T u'(t)^2 dt - \int_0^T ((T-t)\mu + \mu_{\tilde{Z}})u'(t)dt - \alpha\int_0^\tau u'(t)\Big(\mu_{\tilde{N}} + W - \int_0^\tau u'(t)dt\Big)dt \\ &+ \lambda\sigma^2\int_0^T u(t)^2 dt + \lambda\sigma_{\tilde{Z}}^2\Big(\int_0^T u'(t)dt\Big)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2\Big(\int_0^\tau u'(t)dt\Big)^2 + \delta\Big(\int_0^T u'(t)dt - W\Big). \end{aligned}$$

Suppose t^* and k^* are the smallest real number such that $v_{t^*} > 0$ and $v_{k^*} > 0$ for $t^* \in [0,\tau)$ and $k^* \in [\tau, T)$, respectively. As such, the objective function becomes:

$$\begin{aligned} \Phi =& \beta\Big(\int_{t^*}^\tau u'(t)^2 dt + \int_{k^*}^T u'(t)^2 dt\Big) - \big(T\mu + \mu_{\tilde{Z}} - \delta\big)\Big(\int_{t^*}^\tau u'(t)dt + \int_{k^*}^T u'(t)dt\Big) \\ &+ \mu\Big(\int_{t^*}^\tau tu'(t)dt + \int_{k^*}^T tu'(t)dt\Big) - \alpha(\mu_{\tilde{N}} + W)\int_{t^*}^\tau u'(t)dt \\ &+ \alpha\int_{t^*}^\tau u'(t)dt\Big(\int_{t^*}^\tau u'(t)dt + \int_{k^*}^T u'(t)dt\Big) + \lambda\sigma^2\Big(\int_{t^*}^\tau u^2(t)dt + \int_{k^*}^T u^2(t)dt\Big) \\ &+ \lambda\sigma_{\tilde{Z}}^2\Big(\int_{t^*}^\tau u'(t)dt + \int_{k^*}^T u'(t)dt\Big)^2 + \lambda\alpha^2\sigma_{\tilde{N}}^2\Big(\int_{t^*}^\tau u'(t)dt\Big)^2 - \delta W. \end{aligned}$$

Using the integrals shown in appendix A.1, we have:

$$\begin{aligned}\Phi =&\beta\big(K_1^4c_1^2+K_2^4c_3^2+K_3^4c_1c_3+K_4^4c_1+K_5^4c_3+K_6^4\big)\\&-(T\mu+\mu_{\tilde{Z}}-\delta)\big(K_1^2c_1+K_2^2c_3+K_3^2\big)+\mu\big(K_1^3c_1+K_2^3c_3+K_3^3\big)\\&-\alpha(\mu_{\tilde{N}}+W)\big(K_1^1c_1+K_1^2\big)+\alpha\big(K_1^1K_1^2c_1^2+K_1^1K_2^2c_1c_3+\big(K_1^1K_3^2+K_2^1K_1^2\big)c_1\\&+K_2^1K_2^2c_3+K_2^1K_3^2\big)+\lambda\sigma^2\big(K_1^5c_1^2+K_2^5c_3^2+K_3^5c_1c_3+K_4^5c_1+K_5^5c_3+K_6^5\big)\\&+\lambda\sigma_{\tilde{Z}}^2\bigg(\big(K_1^2\big)^2c_1^2+\big(K_2^2\big)^2c_3^2+2K_1^2K_2^2c_1c_3+2K_1^2K_3^2c_1+2K_2^2K_3^2c_3+\big(K_3^2\big)^2\bigg)\\&+\lambda\alpha^2\sigma_{\tilde{N}}^2\bigg(\big(K_1^1\big)^2c_1^2+2K_1^1K_2^1c_1+\big(K_2^1\big)^2\bigg)-\delta W.\end{aligned}$$

We now denote:

$$\begin{aligned}A_1(t^*,k^*):=&\beta K_1^4(t^*,k^*)+\alpha K_1^1K_1^2(t^*,k^*)+\lambda\sigma^2K_1^5(t^*,k^*)+\lambda\sigma_{\tilde{Z}}^2\big(K_1^2(t^*,k^*)\big)^2\\&+\lambda\alpha^2\sigma_{\tilde{N}}^2\big(K_1^1(t^*,k^*)\big)^2,\\A_2(t^*,k^*):=&\beta K_2^4(t^*,k^*)+\lambda\sigma^2K_2^5(t^*,k^*)+\lambda\sigma_{\tilde{Z}}^2\big(K_2^2(t^*,k^*)\big)^2,\\A_3(t^*,k^*):=&\beta K_3^4(t^*,k^*)+\alpha K_1^1(t^*,k^*)K_2^2(t^*,k^*)+\lambda\sigma^2K_3^5(t^*,k^*)\\&+2\lambda\sigma_{\tilde{Z}}^2K_1^2(t^*,k^*)K_2^2(t^*,k^*),\end{aligned}$$

$$\begin{aligned}\tilde{A}_4(t^*,k^*):=&\beta K_4^4(t^*,k^*)-(T\mu+\mu_{\tilde{Z}})K_1^2(t^*,k^*)+\mu K_1^3(t^*,k^*)\\&-\alpha(\mu_{\tilde{N}}+W)K_1^1(t^*,k^*)+\alpha\big(K_1^1(t^*,k^*)K_3^2(t^*,k^*)\\&+K_2^1(t^*,k^*)K_1^2(t^*,k^*)\big)+\lambda\sigma^2K_4^5(t^*,k^*)\\&+2\lambda\sigma_{\tilde{Z}}^2K_1^2(t^*,k^*)K_3^2(t^*,k^*)+2\lambda\alpha^2\sigma_{\tilde{N}}^2K_1^1(t^*,k^*)K_2^1(t^*,k^*),\\\tilde{A}_5(t^*,k^*):=&\beta K_5^4(t^*,k^*)-(T\mu+\mu_{\tilde{Z}})K_2^2(t^*,k^*)+\mu K_2^3(t^*,k^*)\\&+\alpha K_2^1(t^*,k^*)K_2^2(t^*,k^*)+\lambda\sigma^2K_5^5(t^*,k^*)\\&+2\lambda\sigma_{\tilde{Z}}^2K_2^2(t^*,k^*)K_3^2(t^*,k^*).\end{aligned}$$

and

$$\begin{aligned} A_4(t^*,k^*) :=& \tilde{A}_4(t^*,k^*) + \delta K_1^2(t^*,k^*),\\ A_5(t^*,k^*) :=& \tilde{A}_5(t^*,k^*) + \delta K_2^2(t^*,k^*). \end{aligned}$$

As such, we express the objective function as:

$$\Phi = A_1(t^*,k^*)c_1^2 + A_2(t^*,k^*)c_3^2 + A_3(t^*,k^*)c_1c_3 + A_4(t^*,k^*)c_1 + A_5(t^*,k^*)c_3 + const.$$

To find the optimal value of c_1 and c_3, we analyze:

$$\begin{aligned} 0 = \frac{\partial \Phi}{\partial c_1} = 2A_1(t^*,k^*)c_1 + A_3(t^*,k^*)c_3 + A_4(t^*,k^*),\\ 0 = \frac{\partial \Phi}{\partial c_3} = 2A_2(t^*,k^*)c_3 + A_3(t^*,k^*)c_1 + A_5(t^*,k^*). \end{aligned}$$

By solving this system of equation, we find:

$$\begin{aligned} c_1^*(t^*,k^*) = \frac{A_3(t^*,k^*)A_5(t^*,k^*) - 2A_2(t^*,k^*)A_4(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)},\\ c_3^*(t^*,k^*) = \frac{A_3(t^*,k^*)A_4(t^*,k^*) - 2A_1(t^*,k^*)A_5(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)}. \end{aligned}$$

By the second partial derivative test, we must have:

$$4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*) > 0$$

to attain a minimum solution. If this condition is not met, that is,

$$4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*) \leq 0,$$

then the optimal solution will simply be $v_t = 0$ for all $t \in [0, T)$ and $v_T = W$.

If the above condition is satisfied, then the optimal rate of trading at time t is:

$$v_t^* = \begin{cases} \sqrt{\frac{\lambda\sigma^2}{\beta}}\Big(c_1^*(t^*,k^*)\Big(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) & \text{for } t \in [0,\tau) \\ \\ \sqrt{\frac{\lambda\sigma^2}{\beta}}\Big(c_3^*(t^*,k^*)\Big(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) \\ -c_1^*(t^*,k^*)\big(\mathrm{e}^{2a}-1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + \frac{\mu}{2\lambda\sigma^2}\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) & \text{for } t \in (\tau, T), \end{cases}$$

and the cumulative order at time t is given by:

$$u^*(t) = \begin{cases} c_1^*(t^*,k^*)\Big(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) + \frac{\mu}{2\lambda\sigma^2}\Big(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) & \text{for } t \in [0,\tau) \\ \\ c_1^*(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + c_3^*(t^*,k^*)\Big(\mathrm{e}^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - \mathrm{e}^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) \\ +\frac{\mu}{2\lambda\sigma^2}\Big(1 - \mathrm{e}^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) & \text{for } t \in (\tau, T). \end{cases}$$

Furthermore, we will now determine δ. We denote:

$$D_1(t^*,k^*) := \frac{A_3(t^*,k^*)K_2^2(t^*,k^*) - 2A_2(t^*,k^*)K_1^2(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)},$$
$$D_2(t^*,k^*) := \frac{A_3(t^*,k^*)K_1^2(t^*,k^*) - 2A_1(t^*,k^*)K_2^2(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)},$$

such that we have:

$$c_1^*(t^*,k^*) = c_A(t^*,k^*) + D_1(t^*,k^*)\delta,$$
$$c_3^*(t^*,k^*) = c_B(t^*,k^*) + D_2(t^*,k^*)\delta,$$

where

$$c_A(t^*,k^*) := \frac{A_3(t^*,k^*)\tilde{A}_5(t^*,k^*) - 2A_2(t^*,k^*)\tilde{A}_4(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)},$$
$$c_B(t^*,k^*) := \frac{A_3(t^*,k^*)\tilde{A}_4(t^*,k^*) - 2A_1(t^*,k^*)\tilde{A}_5(t^*,k^*)}{4A_1(t^*,k^*)A_2(t^*,k^*) - A_3^2(t^*,k^*)}.$$

Recall that we denoted $a := \sqrt{\frac{\lambda\sigma^2}{\beta}}\tau$ and we now denote $b := \sqrt{\frac{\lambda\sigma^2}{\beta}}T$. Suppose $\delta > 0$, then we must have:

$$\begin{aligned} 0 =& W - u^*(T) \\ =& W - c_A(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} - c_B(t^*,k^*)\big(\mathrm{e}^{b} - \mathrm{e}^{2a-b}\big) - \frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-b}\big) \\ &- \Big(D_1(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} + D_2(t^*,k^*)\big(\mathrm{e}^{b}-\mathrm{e}^{2a-b}\big)\Big)\delta, \end{aligned}$$

which suggests:

$$\delta = \frac{W - c_A(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} - c_B(t^*,k^*)\big(\mathrm{e}^{b}-\mathrm{e}^{2a-b}\big) - \frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-b}\big)}{D_1(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} + D_2(t^*,k^*)\big(\mathrm{e}^{b}-\mathrm{e}^{2a-b}\big)}.$$

We recall that $\delta \geq 0$; thus, we have:

$$\delta(t^*,k^*) = \max\left(\frac{W - c_A(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} - c_B(t^*,k^*)\big(\mathrm{e}^{b}-\mathrm{e}^{2a-b}\big) - \frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-b}\big)}{D_1(t^*,k^*)\big(e^{2a}-1\big)\mathrm{e}^{-b} + D_2(t^*,k^*)\big(\mathrm{e}^{b}-\mathrm{e}^{2a-b}\big)}, 0\right).$$

***Strategy C**: $v_t = 0$ for $t \in \{1, \dots, \tau-1\}$.*

Suppose the investor choose to trade only after the initial imbalance announcement. The proof for this strategy can be viewed as a simplified version of section 5.2.1. Suppose $v_t = 0$ for $t \in \{1, \dots, \tau-1\}$, from Step 1 of section 5.2.1, we have the objective function:

$$\Phi = \int_{k^*}^{T} \beta v_t^2 - \big((T-t)\mu + \mu_{\tilde{Z}} + W) - \delta\big)v_t + \lambda\sigma^2 (X_t^v)^2 - \frac{\delta W}{T} dt + \lambda\sigma_{\tilde{Z}}^2 \Big(\int_0^{\tau} v_t dt\Big)^2$$

With identical procedure shown in Steps 3 and 4 of section 5.2.1, and the remark of proposition 2, one can derive the result shown in section 3.4.

***Case 2:** $\beta = 0$.*

If $\beta = 0$, then following from eq. (5.5), we simply have: $u(t) = \frac{\mu}{2\lambda\sigma^2}$; in particular, we have:

$$X_0 = \max\left(\frac{\mu}{2\lambda\sigma^2}, 0\right).$$

This suggests that we have $v_t = 0$ for $t \in (0,\tau) \cap (\tau, T)$ and v_t does not exist for $t = 0, \tau, T$. Similar to case 2 in section 5.2.1, the transactions in the continuous trading only occur at time 0 and the moment before time τ and T. For some small $\epsilon > 0$, we denote by $\tilde{\tau} := \tau - \epsilon$ and $\tilde{T} := T - \epsilon$ the moment before time τ and T, respectively. Let $V_0, V_{\tilde{\tau}}, V_{\tilde{T}}$, and V_T be the order volume at time $0, \tilde{\tau}, \tilde{T}$, and T, respectively. We have shown $V_0 = X_0$. We now rewrite the objective function accordingly. In particular, we consider:

$$\min \quad E\Big[V_0 P_0 + V_{\tilde{\tau}} P_{\tilde{\tau}} + V_{\tilde{T}} P_{\tilde{T}} + \big(W - V_0 - V_{\tilde{\tau}} - V_{\tilde{T}}\big) P_T - W P_T\Big]$$

$$+ \lambda VAR\Big[V_0P_0 + V_{\tilde\tau}P_{\tilde\tau} + V_{\tilde T}P_{\tilde T} + \big(W - V_0 - V_{\tilde\tau} - V_{\tilde T}\big)P_T - WP_T\Big]$$

$$s.t. \quad V_{\tilde\tau} \geq 0, \quad V_{\tilde T} \geq 0, \quad W - V_0 - V_{\tilde\tau} - V_{\tilde T} \geq 0$$

where

$$P_{\tilde\tau} = P_0 + \beta V_0 + \mu\tilde\tau + \sigma W_{\tilde\tau},$$

$$P_{\tilde T} = P_0 + \beta V_0 + \mu\tilde T + \sigma W_{\tilde T} + \alpha(\tilde N + W - V_0 - V_{\tilde\tau} - V_{\tilde T}),$$

$$P_T = P_0 + \beta V_0 + \mu T + \sigma W_T + \alpha(\tilde N + W - V_0 - V_{\tilde\tau} - V_{\tilde T}) + \tilde Z.$$

We rewrite the following:

$$\begin{aligned}
& V_0P_0 + V_{\tilde\tau}P_{\tilde\tau} + V_{\tilde T}P_{\tilde T} + \big(W - V_0 - V_{\tilde\tau} - V_{\tilde T}\big)P_T - WP_T \\
=& V_0P_0 + V_{\tilde\tau}\big(P_0 + \beta V_0 + \mu\tilde\tau + \sigma W_{\tilde\tau}\big) + V_{\tilde T}\big(P_0 + \beta V_0 + \mu\tilde T + \sigma W_{\tilde T} + \alpha(\tilde N + W - V_0 - V_{\tilde\tau} \\
& - V_{\tilde T})\big) - (V_0 + V_{\tilde\tau} + V_{\tilde T})\big(P_0 + \beta V_0 + \mu T + \sigma W_T + \alpha(\tilde N + W - V_0 - V_{\tilde\tau} - V_{\tilde T}) + \tilde Z\big) \\
=& V_{\tilde\tau}\big(\mu\tilde\tau + \sigma W_{\tilde\tau}\big) + V_{\tilde T}\big(\mu\tilde T + \sigma W_{\tilde T}\big) - \beta V_0^2 - (V_0 + V_{\tilde\tau} + V_{\tilde T})\big(\mu T + \sigma W_T + \tilde Z\big) \\
& - \alpha(V_0 + V_{\tilde\tau})(\tilde N + W - V_0 - V_{\tilde\tau} - V_{\tilde T}).
\end{aligned}$$

We recall that $E(W_t) = 0$ and $VAR(W_t) = t$ for all t. As such, we obtain:

$$\begin{aligned}
& E\Big[V_0P_0 + V_{\tilde\tau}P_{\tilde\tau} + \big(W - V_0 - V_{\tilde\tau}\big)P_T - WP_T\Big] \\
=& \mu\tilde\tau V_{\tilde\tau} + \mu\tilde T V_{\tilde T} - \beta V_0^2 - (V_0 + V_{\tilde\tau} + V_{\tilde T})\big(\mu T + \mu_{\tilde Z}\big) \\
& - \alpha(V_0 + V_{\tilde\tau})(\mu_{\tilde N} + W - V_0 - V_{\tilde\tau} - V_{\tilde T}),
\end{aligned}$$

$$\begin{aligned}
& VAR\Big[V_0P_0 + V_{\tilde\tau}P_{\tilde\tau} + \big(W - V_0 - V_{\tilde\tau}\big)P_T - WP_T\Big] \\
=& \sigma^2\tilde\tau V_{\tilde\tau}^2 + \sigma^2\tilde T V_{\tilde T}^2 + (V_0 + V_{\tilde\tau} + V_{\tilde T})^2\big(\sigma^2 T + \sigma_{\tilde Z}^2\big)
\end{aligned}$$

$$+ \alpha^2(V_0 + V_{\tilde{\tau}})^2\sigma^2_{\tilde{N}}.$$

Thus, for some $\delta \geq 0$, the objective function is equivalent to:

$$\begin{aligned} L :=& \mu\tilde{\tau}V_{\tilde{\tau}} + \mu\tilde{T}V_{\tilde{T}} - \beta V_0^2 - (V_0 + V_{\tilde{\tau}} + V_{\tilde{T}})\big(\mu T + \mu_{\tilde{Z}}\big) \\ & - \alpha(V_0 + V_{\tilde{\tau}})(\mu_{\tilde{N}} + W - V_0 - V_{\tilde{\tau}} - V_{\tilde{T}}) + \delta\big(V_0 + V_{\tilde{\tau}} + V_{\tilde{T}} - W\big) \\ & + \lambda\Big[\sigma^2\tilde{\tau}V_{\tilde{\tau}}^2 + \sigma^2\tilde{T}V_{\tilde{T}}^2 + (V_0 + V_{\tilde{\tau}} + V_{\tilde{T}})^2\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big) + \alpha^2(V_0 + V_{\tilde{\tau}})^2\sigma^2_{\tilde{N}}\Big]. \end{aligned}$$

To minimize L, we consider:

$$0 = \frac{\partial L}{\partial V_{\tilde{\tau}}} \quad \text{and} \quad 0 = \frac{\partial L}{\partial V_{\tilde{T}}}.$$

In particular, we have:

$$\begin{aligned} 0 =& \mu\tilde{\tau} - \big(\mu T + \mu_{\tilde{Z}}\big) - \alpha\big(\mu_{\tilde{N}} + W - 2V_0 - 2V_{\tilde{\tau}} - V_{\tilde{T}}\big) + 2\lambda\sigma^2\tilde{\tau}V_{\tilde{\tau}} \\ & + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)(V_0 + V_{\tilde{\tau}} + V_{\tilde{T}}) + 2\lambda\alpha^2(V_0 + V_{\tilde{\tau}})\sigma^2_{\tilde{N}} + \delta, \\ 0 =& \mu\tilde{T} - \big(\mu T + \mu_{\tilde{Z}}\big) + \alpha(V_0 + V_{\tilde{\tau}}) + 2\lambda\sigma^2\tilde{T}V_{\tilde{T}} + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)(V_0 + V_{\tilde{\tau}} + V_{\tilde{T}}) + \delta. \end{aligned}$$

We denote:

$$\begin{aligned} c_1 =& \mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2\Big(\alpha + \lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}} + \alpha^2\sigma^2_{\tilde{N}}\big)\Big)V_0, \\ c_2 =& \mu(T - \tilde{T}) + \mu_{\tilde{Z}} - \Big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\Big)V_0, \end{aligned}$$

such that:

$$c_1 - \delta = 2\Big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \sigma^2_{\tilde{Z}} + \alpha^2\sigma^2_{\tilde{N}}\big)\Big)V_{\tilde{\tau}} + \Big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\Big)V_{\tilde{T}},$$

$$c_2 - \delta = \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)V_{\tilde{\tau}} + 2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)V_{\tilde{T}}.$$

Moreover, we let:

$$m := 4\lambda\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \sigma^2_{\tilde{Z}} + \alpha^2\sigma^2_{\tilde{N}}\big)\big)\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big) - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)^2.$$

By solving the system of equations, we find:

$$V_{\tilde{\tau}} = \max\left(\frac{2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)c_1 - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)c_2 + (\alpha - \sigma^2\tilde{T})\delta}{m}, 0\right)$$
$$V_{\tilde{T}} = \max\left(\frac{c_2 - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)V_{\tilde{\tau}} - \delta}{2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)}, 0\right)$$

We now analyze δ. We consider $V_{\tilde{\tau}} > 0$ and $V_{\tilde{T}} > 0$. Suppose that $\delta > 0$, then we must have $0 = V_0 + V_{\tilde{\tau}} + V_{\tilde{T}} - W$, which is equivalent to:

$$\delta = \frac{m\big(2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)(W - V_0) - c_2\big) - \big(2\lambda\sigma^2\tilde{T} - \alpha\big)\big(2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)c_1 - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)c_2\big)}{\big(2\lambda\sigma^2\tilde{T} - \alpha\big)\big(\alpha - \sigma^2\tilde{T}\big)}.$$

We let:

$$\delta_{num} := m\big(2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)(W - V_0) - c_2\big) - \big(2\lambda\sigma^2\tilde{T} - \alpha\big)\big(2\lambda\big(\sigma^2(T+\tilde{T}) + \sigma^2_{\tilde{Z}}\big)c_1 - \big(\alpha + 2\lambda\big(\sigma^2 T + \sigma^2_{\tilde{Z}}\big)\big)c_2\big)$$

Since $\delta \geq 0$, we have:

$$\delta = \max\left(\frac{\delta_{num}}{\big(2\lambda\sigma^2\tilde{T} - \alpha\big)\big(\alpha - \sigma^2\tilde{T}\big)}, 0\right).$$

If $V_{\tilde{T}} = 0$, then the strategy follows from case 2 in section 5.2.1; we have:

$$V_{\tilde{\tau}} = \max\left(\frac{\mu(T-\tilde{\tau}) + \mu_{\tilde{Z}} + \alpha(\mu_{\tilde{N}} + W) - 2\big(\alpha + \lambda\big(\sigma^2 T + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)V_0 - \delta}{2\big(\alpha + \lambda\big(\sigma^2(T+\tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)}, 0\right),$$

$$V_T = W - V_0 - V_T,$$

where

$$\delta = \max\left(\mu(T - \tilde{\tau}) + \mu_{\tilde{Z}} + \alpha\mu_{\tilde{N}} + 2\lambda\sigma^2\tilde{\tau}V_0 - 2\big(\alpha + \lambda\big(\sigma^2(T + \tilde{\tau}) + \alpha^2\sigma^2_{\tilde{N}} + \sigma^2_{\tilde{Z}}\big)\big)W, 0\right).$$

Chapter 6

Conclusion

The optimal strategy derived in this thesis gives a trading algorithm for an investor who targets the closing prices of stocks listed at NASDAQ. The investor attempts to minimize a combination of average costs and deviations to the closing price benchmark. In both discrete-time and continuous-time models, we proved formulas for the optimal trading strategies, which depend on parameters from the stock price dynamics as well as the investor's level of risk aversion. Under assumptions on the drift of the underlying stock price dynamics, the formulas for the optimal trading strategies become explicit, and there is no investment after the imbalance announcement. Using historical imbalance volume and intraday stock prices, we performed out-of-sample simulations for the optimal strategy. The strategy tested on 15 NASDAQ stocks shows, persistently across different levels of the investor's risk aversion, an improvement compared to investing in the closing auction only; in particular, our optimal strategy has lower average costs for all 15 stocks.

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Appendix A

Auxiliary Calculations

A.1 Integral Calculation

This appendix contains the detailed integral derivation used for Strategy B in section 5.2.3.

Recall that we denoted $a := \sqrt{\frac{\lambda\sigma^2}{\beta}}\tau$ and we now denote $b := \sqrt{\frac{\lambda\sigma^2}{\beta}}T$. We compute

$$u'(t)^2 = \begin{cases} c_1^2\frac{\lambda\sigma^2}{\beta}\left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2 + c_1\frac{\mu}{\beta}\left(1 + e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right) \\ +\frac{\mu^2}{4\beta\lambda\sigma^2}e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} \quad \text{for } t \in [0,\tau) \\ \\ c_1^2\frac{\lambda\sigma^2}{\beta}(\mathrm{e}^{2a}-1)^2e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - c_1\frac{\mu}{\beta}(\mathrm{e}^{2a}-1)e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} \\ -c_1c_3\frac{2\lambda\sigma^2}{\beta}(\mathrm{e}^{2a}-1)\left(1+e^{2\left(a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)}\right) + c_3^2\frac{\lambda\sigma^2}{\beta}\left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2 \\ +c_3\frac{\mu}{\beta}\left(1+e^{2\left(a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)}\right) + \frac{\mu^2}{4\beta\lambda\sigma^2}e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} \quad \text{for } t \in (\tau, T) \end{cases}$$

and

$$u(t)^2 = \begin{cases} c_1^2\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 + c_1\frac{\mu}{\lambda\sigma^2}\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)\Big(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) \\ +\frac{\mu^2}{4\lambda^2\sigma^4}\Big(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 \quad \text{for } t \in [0,\tau) \\ \\ c_1^2(e^{2a}-1)^2 e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + c_1\frac{\mu}{\lambda\sigma^2}(e^{2a}-1)\Big(e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) \\ +2c_1c_3(e^{2a}-1)\Big(1 - e^{2\big(a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t\big)}\Big) + c_3^2\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 \\ +c_3\frac{\mu}{\lambda\sigma^2}\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} - e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)\Big(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big) \\ +\frac{\mu^2}{4\lambda^2\sigma^4}\Big(1 - e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 \quad \text{for } t \in [\tau, T). \end{cases}$$

In addition, we note that:

$$\begin{aligned} \int_0^\tau u'(t)dt = u(\tau) =& 2\sinh(a)c_1 + \frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-a}\big), \\ \int_0^T u'(t)dt = u(T) =& c_1\big(\mathrm{e}^{2a}-1\big)e^{-b} + c_3\big(\mathrm{e}^{b} - \mathrm{e}^{2a-b}\big) + \frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-b}\big). \end{aligned}$$

We compute the following integrals:

$$\begin{aligned} &\int_0^\tau t\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)dt = \frac{\beta}{\lambda\sigma^2}\big(2a\sinh(a) - 2\cosh(a) + 2\big) \\ &\int_0^T t\Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)dt = 2\mathrm{e}^a\frac{\beta}{\lambda\sigma^2}\big(\cosh(a) + b\sinh(b-a) - \cosh(b-a)\big) \\ &\int_0^T te^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{\beta}{\lambda\sigma^2}\big(1 - e^{-b}(1+b)\big) \\ &\int_0^T e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1-\mathrm{e}^{-2b}\big) \\ &\int_0^\tau \Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a) + 2a\big) \\ &\int_0^T \Big(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t} + e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\Big)^2 dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a}\big(\sinh(2a) + \sinh(2(b-a)) + 2b\big) \end{aligned}$$

$$\int_0^\tau 1+e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1-\mathrm{e}^{-2a}+2a\big)$$

$$\int_0^T 1+e^{2\left(a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2a}-\mathrm{e}^{2(a-b)}+2b\big)$$

$$\int_0^T 1-e^{2\left(a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t\right)}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b)}-\mathrm{e}^{2a}+2b\big)$$

$$\int_0^T e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-e^{-2\sqrt{\frac{\lambda\sigma^2}{\beta}}t}dt = \frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\mathrm{e}^{-2b}\big(\mathrm{e}^{2b}-1\big)^2$$

$$\int_0^\tau \left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a)-2a\big)$$

$$\int_0^T \left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a}\big(\sinh(2a)+\sinh(2(b-a))-2b\big)$$

$$\int_0^\tau \left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\left(1-e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)dt = -\frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a+\mathrm{e}^{-2a}-4\cosh(a)+3\big)$$

$$\int_0^T \left(1-e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)^2dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2b}-\frac{\mathrm{e}^{-4b}+3}{4}+b\right)$$

$$\int_0^T \left(e^{\sqrt{\frac{\lambda\sigma^2}{\beta}}t}-e^{2a-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)\left(1-e^{-\sqrt{\frac{\lambda\sigma^2}{\beta}}t}\right)dt = -\frac{1}{2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2a}+\mathrm{e}^{2(a-b)}-2\mathrm{e}^{2a-b}+2b-2\mathrm{e}^{b}+2\big)$$

Using above integrals, we compute:

$$\int_0^\tau tu'(t)dt = \sqrt{\frac{\beta}{\lambda\sigma^2}}\left[2c_1\big(a\sinh(a)-\cosh(a)+1\big)+\frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-a}(1+a)\big)\right],$$

$$\begin{aligned}\int_0^T tu'(t)dt =& \sqrt{\frac{\beta}{\lambda\sigma^2}}\Big[c_1\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-b}(1+b)-1\big)+2c_3\mathrm{e}^{a}\big(\cosh(a)+b\sinh(b-a)-\cosh(b-a)\big)\\ &+\frac{\mu}{2\lambda\sigma^2}\big(1-\mathrm{e}^{-b}(1+b)\big)\Big],\end{aligned}$$

$$\int_0^\tau u'(t)^2dt = c_1^2\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\sinh(2a)+2a\big)+c_1\frac{\mu}{2\beta}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1-\mathrm{e}^{-2a}+2a\big)+\frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1-\mathrm{e}^{-2a}\big),$$

$$\int_0^T u'(t)^2dt = \frac{1}{2}c_1^2\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(e^{2a}-1\big)^2\big(1-\mathrm{e}^{-2b}\big)-c_1\frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(e^{2a}-1\big)\big(1-\mathrm{e}^{-2b}\big)$$

$$
\begin{aligned}
&- c_1 c_3 \sqrt{\frac{\lambda\sigma^2}{\beta}} \big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{2a}-\mathrm{e}^{2(a-b)}+2b\big) + \frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(1-\mathrm{e}^{-2b}\big) \\
&+ c_3^2 \sqrt{\frac{\lambda\sigma^2}{\beta}} \mathrm{e}^{2a}\big(\sinh(2a)+\sinh(2(b-a))+2b\big) + c_3\frac{\mu}{2\beta}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2a}-\mathrm{e}^{2(a-b)}+2b\big),
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\tau u(t)^2 dt =& c_1^2\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a)-2a\big) - c_1\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2a+\mathrm{e}^{-2a}-4\cosh(a)+3\big) \\
&+ \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2a}-\frac{\mathrm{e}^{-4a}+3}{4}+a\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T u(t)^2 dt =& c_1^2\sqrt{\frac{\beta}{\lambda\sigma^2}}\frac{\big(e^{2a}-1\big)^2\big(1-e^{-2b}\big)}{2} + c_1\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(e^{2a}-1\big)\mathrm{e}^{-2b}\big(\mathrm{e}^{2b}-1\big)^2 \\
&+ c_1 c_3\big(e^{2a}-1\big)\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b)}-\mathrm{e}^{2a}+2b\big) + \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2b}-\frac{\mathrm{e}^{-4b}+3}{4}+b\right) \\
&+ c_3^2\sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a}\big(\sinh(2a)+\sinh(2(b-a))-2b\big) \\
&- c_3\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2a}+\mathrm{e}^{2(a-b)}-2\mathrm{e}^{2a-b}+2b-2\mathrm{e}^{b}+2\big)
\end{aligned}
$$

We recall that t^* and k^* are the smallest real number such that $v_{t^*} > 0$ and $v_{k^*} > 0$ for $t^* \in [0,\tau)$ and $k^* \in [\tau, T)$, respectively. We denote $a^* := \sqrt{\frac{\lambda\sigma^2}{\beta}}t^*$ and $b^* := \sqrt{\frac{\lambda\sigma^2}{\beta}}k^*$. Moreover, we denote the following constants:

$$
\begin{aligned}
K_1^1(t^*,k^*) :=& 2\big(\sinh(a)-\sinh(a^*)\big) \\
K_2^1(t^*,k^*) :=& -\frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a}-\mathrm{e}^{-a^*}\big) \\
K_1^2(t^*,k^*) :=& 2\big(\sinh(a)-\sinh(a^*)\big) + \big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-b}-\mathrm{e}^{-b^*}\big) \\
K_2^2(t^*,k^*) :=& \mathrm{e}^{b}-\mathrm{e}^{b^*}-\mathrm{e}^{2a-b}+\mathrm{e}^{2a-b^*} \\
K_3^2(t^*,k^*) :=& -\frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a}+\mathrm{e}^{-b}-\mathrm{e}^{-a^*}-\mathrm{e}^{-b^*}\big)
\end{aligned}
$$

$$
\begin{aligned}
K_1^3(t^*,k^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}\big(2\big(a\sinh(a)-a^*\sinh(a^*)-\cosh(a)+\cosh(a^*)\big)\\
&+\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-b}(1+b)-\mathrm{e}^{-b^*}(1+b^*)\big)\big)\\
K_2^3(t^*,k^*) :=& 2\sqrt{\frac{\beta}{\lambda\sigma^2}}\mathrm{e}^{a}\big(b\sinh(b-a)-b^*\sinh(b^*-a)-\cosh(b-a)+\cosh(b^*-a)\big)\\
K_3^3(t^*,k^*) :=& -\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(e^{-a}(1+a)+e^{-b}(1+b)-e^{-a^*}(1+a^*)-e^{-b^*}(1+b^*)\big)\\
K_1^4(t^*,k^*) :=& \frac{1}{2}\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a}-1\big)^2\big(\mathrm{e}^{-2b^*}-\mathrm{e}^{-2b}\big)\\
K_2^4(t^*,k^*) :=& \sqrt{\frac{\lambda\sigma^2}{\beta}}\mathrm{e}^{2a}\big(\sinh(2(b-a))-\sinh(2(b^*-a))+2(b-b^*)\big)\\
K_3^4(t^*,k^*) :=& -\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{2(a-b^*)}-\mathrm{e}^{2(a-b)}+2(b-b^*)\big)\\
K_4^4(t^*,k^*) :=& \frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{-2a^*}-\mathrm{e}^{-2a}+2(a-a^*)-\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-2b^*}-\mathrm{e}^{-2b}\big)\big)\\
K_5^4(t^*,k^*) :=& \frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b^*)}-\mathrm{e}^{2(a-b)}+2(b-b^*)\big)\\
K_6^4(t^*,k^*) :=& -\frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2a^*}-\mathrm{e}^{-2a}+\mathrm{e}^{-2b^*}-\mathrm{e}^{-2b}\big)\\
K_1^5(t^*,k^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}\Big(\sinh(2a)-\sinh(2a^*)-2(a-a^*)+\frac{1}{2}\big(e^{2a}-1\big)^2\big(e^{-2b^*}-e^{-2b}\big)\Big)\\
K_2^5(t^*,k^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a}\big(\sinh(2(b-a))+\sinh(2(b^*-a))-2(b-b^*)\big)\\
K_3^5(t^*,k^*) :=& \big(e^{2a}-1\big)\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b)}-\mathrm{e}^{2(a-b^*)}+2(b-b^*)\big)\\
K_4^5(t^*,k^*) :=& \frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\Big(\big(e^{2a}-1\big)\big(\mathrm{e}^{-2b}\big(\mathrm{e}^{2b}-1\big)^2-\mathrm{e}^{-2b^*}\big(\mathrm{e}^{2b^*}-1\big)^2\big)\\
&+\mathrm{e}^{-2a^*}-\mathrm{e}^{-2a}+4(\cosh(a)-\cosh(a^*))-2(a-a^*)\Big)\\
K_5^5(t^*,k^*) :=& \frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b^*)}-\mathrm{e}^{2(a-b)}-2\big(\mathrm{e}^{2a-b^*}-\mathrm{e}^{2a-b}\big)+2\big(\mathrm{e}^{b}-\mathrm{e}^{b^*}\big)-2(b-b^*)\big)\\
K_6^5(t^*,k^*) :=& \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\Big(\mathrm{e}^{-2b}+\mathrm{e}^{-2a}-\mathrm{e}^{-2a^*}-\mathrm{e}^{-2b^*}-\frac{\mathrm{e}^{-4b}+\mathrm{e}^{-4a}-\mathrm{e}^{-4a^*}-\mathrm{e}^{-4b^*}}{4}\\
&+(b+a-b^*-a^*)\Big)
\end{aligned}
$$

We determine the following integrals:

$$\begin{aligned}\int_{t^*}^{\tau} u'(t)dt =& \int_0^{\tau} u'(t)dt - \int_0^{t^*} u'(t)dt \\ =& 2\big(\sinh(a) - \sinh(a^*)\big)c_1 - \frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a} - \mathrm{e}^{-a^*}\big) \\ =& K_1^1 c_1 + K_2^1,\end{aligned}$$

$$\left(\int_{t^*}^{\tau} u'(t)dt\right)^2 = \big(K_1^1\big)^2 c_1^2 + 2K_1^1 K_2^1 c_1 + \big(K_2^1\big)^2,$$

$$\begin{aligned}\int_{t^*}^{\tau} u'(t)dt + \int_{k^*}^{T} u'(t)dt =& \int_0^{T} u'(t)dt - \int_0^{k^*} u'(t)dt + \int_{t^*}^{\tau} u'(t)dt \\ =& \big(\mathrm{e}^{2a} - 1\big)\big(\mathrm{e}^{-b} - \mathrm{e}^{-b^*}\big)c_1 + \big(\mathrm{e}^{b} - \mathrm{e}^{b^*} - \mathrm{e}^{2a-b} + \mathrm{e}^{2a-b^*}\big)c_3 \\ & - \frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-b} - \mathrm{e}^{-b^*}\big) + 2\big(\sinh(a) - \sinh(a^*)\big)c_1 \\ & - \frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a} - \mathrm{e}^{-a^*}\big) \\ =& \Big(2\big(\sinh(a) - \sinh(a^*)\big) + \big(\mathrm{e}^{2a} - 1\big)\big(\mathrm{e}^{-b} - \mathrm{e}^{-b^*}\big)\Big)c_1 \\ & + \big(\mathrm{e}^{b} - \mathrm{e}^{b^*} - \mathrm{e}^{2a-b} + \mathrm{e}^{2a-b^*}\big)c_3 \\ & - \frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a} + \mathrm{e}^{-b} - \mathrm{e}^{-a^*} - \mathrm{e}^{-b^*}\big) \\ =& K_1^2 c_1 + K_2^2 c_3 + K_3^2,\end{aligned}$$

$$\begin{aligned}&\left(\int_{t^*}^{\tau} u'(t)dt + \int_{k^*}^{T} u'(t)dt\right)^2 \\ =& \big(K_1^2\big)^2 c_1^2 + \big(K_2^2\big)^2 c_3^2 + 2K_1^2 K_2^2 c_1 c_3 + 2K_1^2 K_3^2 c_1 \\ & + 2K_2^2 K_3^2 c_3 + \big(K_3^2\big)^2,\end{aligned}$$

$$\begin{aligned}
&\int_{t^*}^{\tau} u'(t)dt\bigg(\int_{t^*}^{\tau} u'(t)dt + \int_{k^*}^{T} u'(t)dt\bigg)\\
=&K_1^1K_1^2c_1^2 + K_1^1K_2^2c_1c_3 + \big(K_1^1K_3^2 + K_2^1K_1^2\big)c_1\\
&+ K_2^1K_2^2c_3 + K_2^1K_3^2,
\end{aligned}$$

$$\begin{aligned}
&\int_{t^*}^{\tau} tu'(t)dt + \int_{k^*}^{T} tu'(t)dt\\
=&\int_0^T tu'(t)dt - \int_0^{k^*} tu'(t)dt + \int_0^{\tau} tu'(t)dt - \int_0^{t^*} tu'(t)dt\\
=&\sqrt{\frac{\beta}{\lambda\sigma^2}}\Big[c_1\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-b}(1+b) - \mathrm{e}^{-b^*}(1+b^*)\big)\\
&+ 2c_3\mathrm{e}^{a}\big(b\sinh(b-a) - b^*\sinh(b^*-a) - \cosh(b-a) + \cosh(b^*-a)\big)\\
&+ 2c_1\big(a\sinh(a) - a^*\sinh(a^*) - \cosh(a) + \cosh(a^*)\big)\\
&- \frac{\mu}{2\lambda\sigma^2}\big(e^{-a}(1+a) - e^{-a^*}(1+a^*) + e^{-b}(1+b) - e^{-b^*}(1+b^*)\big)\\
=&K_1^3c_1 + K_2^3c_3 + K_3^3,
\end{aligned}$$

$$\begin{aligned}
&\int_{t^*}^{\tau} u'(t)^2dt + \int_{k^*}^{T} u'(t)^2dt\\
=&\int_0^T u'(t)^2dt - \int_0^{k^*} u'(t)^2dt + \int_0^{\tau} u'(t)^2dt - \int_0^{t^*} u'(t)^2dt\\
=&\frac{1}{2}c_1^2\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a}-1\big)^2\big(\mathrm{e}^{-2b^*} - \mathrm{e}^{-2b}\big)\\
&- c_1\frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{-2b^*} - \mathrm{e}^{-2b}\big)\\
&- c_1c_3\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a}-1\big)\big(\mathrm{e}^{2(a-b^*)} - \mathrm{e}^{2(a-b)} + 2(b-b^*)\big)\\
&- \frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2b^*} - \mathrm{e}^{-2b}\big)
\end{aligned}$$

$$
\begin{aligned}
&+ c_3^2\sqrt{\frac{\lambda\sigma^2}{\beta}}\mathrm{e}^{2a}\big(\sinh(2(b-a)) - \sinh(2(b^*-a)) + 2(b-b^*)\big)\\
&+ c_3\frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b^*)} - \mathrm{e}^{2(a-b)} + 2(b-b^*)\big)\\
&+ c_1\frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{-2a^*} - \mathrm{e}^{-2a} + 2(a-a^*)\big)\\
&- \frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2a^*} - \mathrm{e}^{-2a}\big)\\
=&K_1^4c_1^2 + K_2^4c_3^2 + K_3^4c_1c_3 + K_4^4c_1 + K_5^4c_3 + K_6^4,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t^*}^{\tau} u(t)^2dt + \int_{k^*}^{T} u(t)^2dt\\
=&\int_0^T u(t)^2dt - \int_0^{k^*} u(t)^2dt + \int_0^{\tau} u(t)^2dt - \int_0^{t^*} u(t)^2dt\\
=&c_1^2\sqrt{\frac{\beta}{\lambda\sigma^2}}\frac{\big(e^{2a}-1\big)^2\big(e^{-2b^*} - e^{-2b}\big)}{2}\\
&+ c_1\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2a}-1\big)\Big(\mathrm{e}^{-2b}\big(\mathrm{e}^{2b}-1\big)^2 - \mathrm{e}^{-2b^*}\big(\mathrm{e}^{2b^*}-1\big)^2\Big)\\
&+ c_1c_3\big(e^{2a}-1\big)\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b)} - \mathrm{e}^{2(a-b^*)} + 2(b-b^*)\big)\\
&+ c_3^2\sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a}\big(\sinh(2(b-a)) + \sinh(2(b^*-a)) - 2(b-b^*)\big)\\
&+ c_3\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a-b^*)} - \mathrm{e}^{2(a-b)} - 2\big(\mathrm{e}^{2a-b^*} - \mathrm{e}^{2a-b}\big) + 2\big(\mathrm{e}^{b} - \mathrm{e}^{b^*}\big) - 2(b-b^*)\big)\\
&+ \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2b} - \mathrm{e}^{-2b^*} - \frac{\mathrm{e}^{-4b} - \mathrm{e}^{-4b^*}}{4} + (b-b^*)\right)\\
&+ c_1^2\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\sinh(2a) - \sinh(2a^*) - 2(a-a^*)\big)\\
&- c_1\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2a} - \mathrm{e}^{-2a^*} - 4(\cosh(a) - \cosh(a^*)) + 2(a-a^*)\big)\\
&+ \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\left(\mathrm{e}^{-2a} - \mathrm{e}^{-2a^*} - \frac{\mathrm{e}^{-4a} - \mathrm{e}^{-4a^*}}{4} + (a-a^*)\right)\\
=&K_1^5c_1^2 + K_2^5c_3^2 + K_3^5c_1c_3 + K_4^5c_1 + K_5^5c_3 + K_6^5.
\end{aligned}
$$

A.2 Continuous-Time Strategy B

This section lists other notations used in the presentation of Strategy B in section 3.4. A variant of the notations listed here was introduced in appendix A.1. We recall that:

$$a_i := \sqrt{\frac{\lambda\sigma^2}{\beta}}i.$$

We denote the following:

$$
\begin{aligned}
K_1^1(t^*,\tau^*,k^*,T^*) :=& 2\big(\sinh(a_{\tau^*}) - \sinh(a_{t^*})\big)\\
K_2^1(t^*,\tau^*,k^*,T^*) :=& -\frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a_{\tau^*}} - \mathrm{e}^{-a_{t^*}}\big)\\
K_1^2(t^*,\tau^*,k^*,T^*) :=& 2\big(\sinh(a_{\tau^*}) - \sinh(a_{t^*})\big) + \big(\mathrm{e}^{2a_{\tau^*}} - 1\big)\big(\mathrm{e}^{-a_{T^*}} - \mathrm{e}^{-a_{k^*}}\big)\\
K_2^2(t^*,\tau^*,k^*,T^*) :=& \mathrm{e}^{a_{T^*}} - \mathrm{e}^{a_{k^*}} - \mathrm{e}^{2a_{\tau^*}-a_{T^*}} + \mathrm{e}^{2a_{\tau^*}-a_{k^*}}\\
K_3^2(t^*,\tau^*,k^*,T^*) :=& -\frac{\mu}{2\lambda\sigma^2}\big(\mathrm{e}^{-a_{\tau^*}} + \mathrm{e}^{-a_{T^*}} - \mathrm{e}^{-a_{t^*}} - \mathrm{e}^{-a_{k^*}}\big)\\
K_1^3(t^*,\tau^*,k^*,T^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}\Big(2\big(a_{\tau^*}\sinh(a_{\tau^*}) - a_{t^*}\sinh(a_{t^*}) - \cosh(a_{\tau^*})\\
&+ \cosh(a_{t^*})\big) + \big(\mathrm{e}^{2a_{\tau^*}} - 1\big)\big(\mathrm{e}^{-a_{T^*}}(1 + a_{T^*}) - \mathrm{e}^{-a_{k^*}}(1 + a_{k^*})\big)\Big)\\
K_2^3(t^*,\tau^*,k^*,T^*) :=& 2\sqrt{\frac{\beta}{\lambda\sigma^2}}\mathrm{e}^{a_{\tau^*}}\big(a_{T^*}\sinh(a_{T^*} - a_{\tau^*}) - a_{k^*}\sinh(a_{k^*} - a_{\tau^*})\\
&- \cosh(a_{T^*} - a_{\tau^*}) + \cosh(a_{k^*} - a_{\tau^*})\big)\\
K_3^3(t^*,\tau^*,k^*,T^*) :=& -\frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(e^{-a_{\tau^*}}(1 + a_{\tau^*}) + e^{-a_{T^*}}(1 + a_{T^*})\\
&- e^{-a_{t^*}}(1 + a_{t^*}) - e^{-a_{k^*}}(1 + a_{k^*})\big)\\
K_1^4(t^*,\tau^*,k^*,T^*) :=& \frac{1}{2}\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a_{\tau^*}} - 1\big)^2\big(\mathrm{e}^{-2a_{k^*}} - \mathrm{e}^{-2a_{T^*}}\big)\\
K_2^4(t^*,\tau^*,k^*,T^*) :=& \sqrt{\frac{\lambda\sigma^2}{\beta}}\mathrm{e}^{2a_{\tau^*}}\big(\sinh(2(a_{T^*} - a_{\tau^*})) - \sinh(2(a_{k^*} - a_{\tau^*})) + 2(a_{T^*} - a_{k^*})\big)
\end{aligned}
$$

$$K_3^4(t^*,\tau^*,k^*,T^*) := -\sqrt{\frac{\lambda\sigma^2}{\beta}}\big(\mathrm{e}^{2a_{\tau^*}}-1\big)\big(\mathrm{e}^{2(a_{\tau^*}-a_{k^*})}-\mathrm{e}^{2(a_{\tau^*}-a_{T^*})}+2(a_{T^*}-a_{k^*})\big)$$

$$\begin{aligned}
K_4^4(t^*,\tau^*,k^*,T^*) :=& \frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{-2a_{t^*}}-\mathrm{e}^{-2a_{\tau^*}}+2(a_{\tau^*}-a_{t^*})\\
&-\big(\mathrm{e}^{2a_{\tau^*}}-1\big)\big(\mathrm{e}^{-2a_{k^*}}-\mathrm{e}^{-2a_{T^*}}\big)\big)\\
K_5^4(t^*,\tau^*,k^*,T^*) :=& \frac{\mu}{2\sqrt{\beta\lambda\sigma^2}}\big(\mathrm{e}^{2(a_{\tau^*}-a_{k^*})}-\mathrm{e}^{2(a_{\tau^*}-a_{T^*})}+2(a_{T^*}-a_{k^*})\big)\\
K_6^4(t^*,\tau^*,k^*,T^*) :=& -\frac{\mu^2}{8\beta\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{-2a_{t^*}}-\mathrm{e}^{-2a_{\tau^*}}+\mathrm{e}^{-2a_{k^*}}-\mathrm{e}^{-2a_{T^*}}\big)\\
K_1^5(t^*,\tau^*,k^*,T^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}\bigg(\sinh(2a_{\tau^*})-\sinh(2a_{t^*})-2(a_{\tau^*}-a_{t^*})\\
&+\frac{1}{2}\big(e^{2a_{\tau^*}}-1\big)^2\big(e^{-2a_{k^*}}-e^{-2a_{T^*}}\big)\bigg)\\
K_2^5(t^*,\tau^*,k^*,T^*) :=& \sqrt{\frac{\beta}{\lambda\sigma^2}}e^{2a_{\tau^*}}\big(\sinh(2(a_{T^*}-a_{\tau^*}))+\sinh(2(a_{k^*}-a_{\tau^*}))\\
&-2(a_{T^*}-a_{k^*})\big)\\
K_3^5(t^*,\tau^*,k^*,T^*) :=& \big(e^{2a_{\tau^*}}-1\big)\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a_{\tau^*}-a_{T^*})}-\mathrm{e}^{2(a_{\tau^*}-a_{k^*})}+2(a_{T^*}-a_{k^*})\big)\\
K_4^5(t^*,\tau^*,k^*,T^*) :=& \frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\bigg(\big(e^{2a_{\tau^*}}-1\big)\big(\mathrm{e}^{-2a_{T^*}}\big(\mathrm{e}^{2a_{T^*}}-1\big)^2-\mathrm{e}^{-2a_{k^*}}\big(\mathrm{e}^{2a_{k^*}}-1\big)^2\big)\\
&+\mathrm{e}^{-2a_{t^*}}-\mathrm{e}^{-2a_{\tau^*}}+4(\cosh(a_{\tau^*})-\cosh(a_{t^*}))-2(a_{\tau^*}-a_{t^*})\bigg)\\
K_5^5(t^*,\tau^*,k^*,T^*) :=& \frac{\mu}{2\lambda\sigma^2}\sqrt{\frac{\beta}{\lambda\sigma^2}}\big(\mathrm{e}^{2(a_{\tau^*}-a_{k^*})}-\mathrm{e}^{2(a_{\tau^*}-a_{T^*})}-2\big(\mathrm{e}^{2a_{\tau^*}-a_{k^*}}-\mathrm{e}^{2a_{\tau^*}-a_{T^*}}\big)\\
&+2\big(\mathrm{e}^{a_{T^*}}-\mathrm{e}^{a_{k^*}}\big)-2(a_{T^*}-a_{k^*})\big)\\
K_6^5(t^*,\tau^*,k^*,T^*) :=& \frac{\mu^2}{4\lambda^2\sigma^4}\sqrt{\frac{\beta}{\lambda\sigma^2}}\bigg(\mathrm{e}^{-2a_{T^*}}+\mathrm{e}^{-2a_{\tau^*}}-\mathrm{e}^{-2a_{t^*}}-\mathrm{e}^{-2a_{k^*}}\\
&-\frac{\mathrm{e}^{-4a_{T^*}}+\mathrm{e}^{-4a_{\tau^*}}-\mathrm{e}^{-4a_{t^*}}-\mathrm{e}^{-4a_{k^*}}}{4}+(a_{T^*}+a_{\tau^*}-a_{k^*}-a_{t^*})\bigg)
\end{aligned}$$