University of Alberta

CREDIT RISK MODELS WITH LÉVY PROCESSES

by

Ling Luo ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**

in

Mathematical Finance

Department of Mathematical and Statistical Sciences

Edmonton, Alberta
Fall 2006

Your file Votre référence
ISBN: 978-0-494-23072-5
Our file Notre référence
ISBN: 978-0-494-23072-5

Abstract

In the current literature, there are two distinct families of credit risk models: *firm value model* and *intensity-based model.* The first one makes explicit assumptions about the dynamics of firm value and default happens if the firm value is less than some threshold level. The second one treats default as a totally unexpected event, the stochastic structure is modeled by some intensity process. Is there any connection between these two classes of models?

Before answering the question, we introduce a class of stochastic processes named after the French scientist Paul Pierre Lévy: Lévy processes. In part of the dissertation, we analyze properties of Lévy processes, which includes Lévy densities, multivariate Lévy processes and equivalent martingale measures for Lévy processes. We model the firm value process as an exponential Lévy process. Starting with the firm value models, we obtain the instantaneous default probability in form of the Lévy measure. In this case, default may be caused by either the random jumps or the Brownian motion. The information of the instantaneous default probability is not enough to obtain the survival probability. Thus we provide PIDE (partial integro-differential equation) representations of the survival probability and the corresponding bond prices. Since default is a totally unexpected event in the intensity-based model, it need more assumptions to achieve the point. Under the assumption that the default is only caused by the random jumps in the firm value process, the default intensity is a decreasing function of the nature logarithm ratio of pre firm value to the default threshold, which is not given exogenously. And in this case, the survival

probability has a closed-form expression as in the intensity-based model. It shows the connection between two default risk models. Several examples are shown to justify out setup.

The results of the equivalent martingale measures and the instantaneous default intensities based on the Lévy processes can be extended to additive processes with local characteristics.

Acknowledgements

It is a pleasure to thank the many people who made this thesis possible.

First of all, I would like to express my deep gratitude to my supervisor, Dr. Jacques F. Carrière, whose guidance, patience, encouragement and support were crucial for the successful completion of this thesis. Thankyou to my examiners, for managing to read the whole thing so thoroughly.

I would like to thank the Department of Mathematical and Statistical Sciences, for providing me the opportunity to study and for giving me financial support, and to all faculty members in this department and the School of Business who taught me courses. In particular, I would like to acknowledge the help of Dr. K. C. Carrière. My gratitude extends to the many people who have taught me in school. I would like to acknowledge the assistance of the department staffs for their kindness and help.

I would like to thank all my friends from University of Alberta, for the emotional support, entertainment and caring they provided during the many years I stayed here. I am especially grateful to Lingling Ma and Tingting Shu, my best friends in Edmonton.

Finally, and most importantly, I wish to thank my parents, Rijian Luo and Hongzhu Ling, for their support and encouragement. To them I dedicate this thesis.

This thesis is dedicated to

My Parents

Contents

List of Figures

1
Introduction

1.1 Credit risk models

There are a lot of risks in financial markets such as interest rate risk, volatility risk, currency risk and so on. Credit risk is an important consideration in most financial transactions. It refers to any kind of credit-linked event, for instance, changes in the credit rating, variations of credit spread and the default event. It is gauged by quality ratings assigned by commercial rating companies such as Moody's Investor Service and Standard & Poor's Corporation. The credit risk specified in this thesis is the credit default risk, the possibility that a contractual counterparty does not fulfill its obligation stated in the financial contract, that is the possibility of loss due to the default event.

In the current literature, there are two ways to model the time of default and thus we have two classes of the credit risk models: *firm-value model* and *intensity-based model*. The first one is pioneered by Black and Scholes (1973), Merton (1974), followed by Black and Cox (1976), Longstaff and Schwartz (1995). They define that the default occurs when the firm-value process V reaches a certain boundary D which maybe deterministic or stochastic,

$$\tau = \inf\{t \in \mathcal{A}_T : V_t \leq D_t\}, \tag{1.1}$$

where $\mathcal{A}_T$ is the set of possible default times with maturity T. We denote $\inf \emptyset = \infty$. Let

$\mathcal{F}_t$ be the information filtration which includes all the information up to and including t, then the default time τ is a stopping time with respect to $\mathcal{F}$. In Merton's setup, $\mathcal{A}_T = \{T\}$. If $\mathcal{A}_T$ is infinite, particularly $\mathcal{A}_T = (0, T]$, then it is called the *first-passage model*. In the latter case, the survival probability under the risk-neutral measure Q is

$$Q(\tau > T|\mathcal{F}_t) = I_{\{\tau>t\}} Q\left(\inf_{t<s\leq T}(V_s - D_s) > 0|\mathcal{F}_t\right), \quad t \in [0, T]. \tag{1.2}$$

If $\mathcal{F}_t$ is generated by Brownian motion, it is continuous and then τ is predictable. Thus the instantaneous default probability is zero, in other words, the term structure of credit spreads should always start at zero. It is a property that is not observed in reality. Zhou [56] proposed a jump-diffusion model of V, which makes τ partly unpredictable due to the sudden drop of the firm-value and then the instant credit spread is not zero anymore but the intensity does not exist. Another advantage of this model is that the firm-value at default is not equal to the threshold D, it could be any number between 0 and D. It shows us a way to model the recovery rate at default.

The second class of models focuses directly on describing the evolution of the probability of default in the next instant without defining the exact default event, and the time of default or other credit events are treated as an exogenous random variable, for instance, the jump time of a Lévy jump process X defined in Definition 1.2.1. Thus the intensity h_t is introduced to define the instantaneous likelihood of default, which is the (stochastic) hazard function in reliability analysis. In the intensity-based model, default time can be defined as the first jump time of X:

$$\tau = \inf\{t \in (0, T] : \triangle X_t \neq 0\}, \tag{1.3}$$

it is a totally inaccessible stopping time with intensity h with respect to $\mathcal{F}$. These models were developed by Duffie & Singleton (1999), Jarrow & Turnbull (1995), Lando (1998), Madan & Unal (1998, 2000). Jarrow & Turnbull considered the case of a constant default intensity h, that is X is a Poisson process. Madan & Unal generalized the result to the case when h is a continuous process adapted to a Brownian filtration. If h_t is the default intensity under the risk neutral measure Q, then the survival probability is

$$Q(\tau > T|\mathcal{F}_t) = I_{\{\tau>t\}} E^Q[e^{-\int_t^T h_s ds}|\mathcal{F}_t], \quad t \in [0, T]. \tag{1.4}$$

These two classes of models have different concerns. In firm-value models, default is an endogenous event while in intensity-based models it is exogenous. Are they consistent or can they be unified under some certain conditions? As discussed in Duffie & Lando [16], the default intensity does not exit when the firm value follows a diffusion process or a jump-diffusion process. The default intensity exists only when the firm value follows a pure jump process. Giesecke [27] concludes that the key to unify both approaches lies in the probabilistic properties of the default event. In other words, the default must at least be unpredictable in the structural model. The jumps in the jump-diffusion process are not sufficient, unless the firm-value is modeled through a pure jump process. This ideal is equivalent to identify the two definitions (1.1) and (1.3) with respect to $(\mathcal{F}_t)_{t\geq 0}$. Chen & Panjer [10] show that the forward default intensity exist in both diffusion and jump-diffusion structural models. The probability of default is a function of the forward intensity thus the structural model can link to a forward-intensity model. Another feasible way is to drop the assumption of perfect information commonly made in structural models. With imperfect information on assets and/or default threshold, investors are uncertain about the default time. In the case of existence of the intensity, let $(\mathcal{G}_t)_{t\geq 0}$ represent the imperfect information, then this idea is equivalent to identify the two survival probabilities (1.2) and (1.4) with respect to $\mathcal{G}_t$. Refer to Duffie & Lando [16], Giesecke [27] for example.

Our method to connect two approaches is based on the first idea. We model the firm-value process as an exponential jump-diffusion process: $V_t = V_0 e^{X_t}$, typically, X is a Lévy process. With perfect information in the first-passage model, assume that the default is caused by sudden drops of the firm-value only, then (1.1) and (1.3) can be equalized as

$$\tau = \inf\{t \in (0,T] : V_t \leq D_t\} = \inf\{t \in (0,T] : \triangle X_t \in \mathcal{B}_t\},$$

where $\mathcal{B}_t$ is a subset of $\mathcal{R} \setminus \{0\}$. And then the survival probabilities (1.2) and (1.4) are identical. While in the general case, the default is caused by either the random jumps or the movement of Brownian motion,

$$\tau = \inf\{t \in (0,T] : V_t \leq D_t\} \leq \inf\{t \in (0,T] : \triangle X_t \in \mathcal{B}_t\},$$

and

$$E^Q[e^{-\int_t^T h_s ds}|\mathcal{F}_t] \geq Q\left(\inf_{t<s\leq T}(V_s - D_s) > 0|\mathcal{F}_t\right), \quad t \in [0,T].$$

In this case, even though the default intensity does not exit when the firm value follows a jump-diffusion process, we define the instantaneous default probability as (4.5) or (4.6) and we will show that the instantaneous default probability does exits. Please note that in the intensity-based model, the instantaneous default probability is called the default intensity.

1.2 Stochastic processes in mathematical finance

Economists model the prices of financial assets as stochastic processes evolving with time. Lévy processes are stochastic processes with stationary independent increments, infinitely divisible distribution and they are continuous in probability. Brownian motions and Poisson processes are two special cases. Brownian motion, a Lévy process with continuous sample paths, plays a very important role in modeling financial markets since the work of Bachelier in 1900. He modeled the stock as a Brownian motion. However it may have a negative stock price. A more appropriate model, geometric Brownian motion was suggested by Samuelson (1965) with the stochastic differential equation: $dS_t = S_t(\mu dt + \sigma dW_t)$, μ and σ are called the drift (mean return) and the volatility of the stock S respectively. Black and Scholes (1973) and Merton (1973) used this model to calculate the price of European options and received the Nobel Prize for Economics in 1997. To meet the fact that the log returns of most financial assets are not normally distributed but skewed and fat-tailed, a more flexible model is needed. Lévy processes fulfill that role. The first example of Levy process used in option pricing is Merton (1974), where the stock returns are generated by a mixture of Brownian motion and Poisson process. Mandelbrot used alpha-stable processes earlier, in 1963, but for statistical description of cotton future returns.

Let $X = \{X_t\}_{t\geq 0}$ be an $\mathcal{R}^d$-valued stochastic process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration $\mathcal{F}_t$ satisfies the usual conditions.

Definition 1.2.1 ***(Lévy process)*** *An $\mathcal{R}^d$-valued stochastic process $X = \{X_t\}_{t\geq 0}$ with $X_0 = 0$ almost surely is called a Lévy process if for $0 \leq s < t < \infty$,*

1. *X has independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$;*
2. *X is time homogeneous: $X_t - X_s$ has the same distribution as X_{t-s};*

3. *X is continuous in probability:* $\lim_{u\to 0} P(X_{t+u} - X_t > \epsilon) = 0$ *for any* $\epsilon > 0$;

4. *as a function of time t,* X_t *is right-continuous with left limits almost surely.*

The Fourier transform on characteristic function of the Lévy process X is defined as $\Psi(z,t) = E[e^{i\langle z, X_t\rangle}]$.

Theorem 1.2.1 ***(Lévy-Khintchine formula for Lévy process)***
The characteristic function of a Lévy process X is

$$\Psi(z,t) = \exp\left\{t\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathcal{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x))\nu(dx) + i\langle z,\gamma\rangle\right]\right\}, \tag{1.5}$$

where A is a symmetric nonnegative-definite $d\times d$ matrix, ν is a measure on $\mathcal{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathcal{R}^d}(\|x\|^2 \wedge 1)\nu(dx) < \infty$, $\gamma \in \mathcal{R}^d$. We call (A,ν,γ) the generating triplet, A the Gaussian covariance matrix, ν the Lévy measure of X. The representation by this generating triplet is unique.

X is a Brownian motion when $\nu = 0$. If $A = 0$, then we say that X is purely non-Gaussian. The Gaussian part and jump part of X are independent. If ν satisfies $\int_{\|x\|\le 1}\|x\|\nu(dx) < \infty$, then

$$\Psi(z,t) = \exp\left\{t\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathcal{R}^d}(e^{i\langle z,x\rangle} - 1)\nu(dx) + i\langle z,\gamma_0\rangle\right]\right\},$$

$\gamma_0 = \gamma - \int_{\|x\|\le 1} x\nu(dx)$ is called the *drift of the Lévy process X*. (It is different from the *drift* used in finance.) If $\int_{\|x\|>1}\|x\|\nu(dx) < \infty$, then

$$\Psi(z,t) = \exp\left\{t\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathcal{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu(dx) + i\langle z,\gamma_1\rangle\right]\right\},$$

$\gamma_1 = \gamma + \int_{\|x\|>1} x\nu(dx)$ is called the *center of the Lévy process X*. The three parameters γ, γ_0 and γ_1 have the following relationship if they exist,

$$E[X_1] = \gamma_1 = \gamma_0 + \int_{\mathcal{R}^d} x\nu(dx) = \gamma + \int_{\|x\|>1} x\nu(dx).$$

Thus we say that X_t is a martingale if and only if $\int_{\|x\|>1}\|x\|\nu(dx) < \infty$ and $\gamma_1 = 0$. (See Proposition 3.18 in Cont & Tankov [12].) For the Gaussian covariance matrix A, there always exists a real $d \times m$ matrix σ such that $A = \sigma\sigma^T$, $m \geq d$. Such σ is not unique, but the Cholesky decomposition of A is unique.

A distribution is *infinitely divisible* if its characteristic function can be represented by n^{th} power of a characteristic function for every integer n. The distribution of Lévy process is infinitely divisible. The characteristic function of an infinitely divisible distribution has the form of (1.5) with $t = 1$. Conversely, the characteristic function $\Psi(z, 1)$ implies the existence of an infinitely divisible distribution. A Lévy process is *α-stable* if it has the self-similarity property: for all $b > 0$, X_{bt} and $b^{1/\alpha}X_t$ are identical in law, where $\alpha \in (0, 2]$ is the index of stability. Brownian motion without drift is the only stable process with $\alpha = 2$. A Lévy process on $\mathcal{R}$ with increasing sample path almost surely is called a *subordinator*. For a subordinator $X_t \in \mathcal{R}$, $A = 0, \nu(-\infty, 0) = 0$ and $\int_{(0,1]} x\nu(dx) < \infty$. A Lévy process on $\mathcal{R}$ with no positive jumps is called a *spectrally negative Lévy process*. Its Lévy measure has support in $(-\infty, 0)$ with $\nu(0, \infty) = 0$.

Since the Gaussian part and jump part of X are independent, any Lévy process can be decomposed as following.

Theorem 1.2.2 ***(Lévy decomposition, see Theorem 42 in Protter [48])***
A Lévy process with generating triplet (A, ν, γ) has a decomposition

$$X_t = \underbrace{\sigma W_t + \int_{\|x\|\leq 1} \int_0^t x[N(ds, dx) - \nu(dx)ds]}_{\text{martingale with bounded jumps}} + \underbrace{\int_{\|x\|>1} \int_0^t xN(ds, dx) + \gamma t}_{\text{Lévy process with paths of finite variation}}, \quad (1.6)$$

W_t is an m-dimensional standard Brownian motion and σ is a $d \times m$ matrix with $\sigma\sigma^T = A$, $N(ds, dx)$ is the Poisson random measure of X while $\tilde{N}(ds, dx) := N(ds, dx) - \nu(dx)ds$ is the compensated measure. W is independent of the two jump processes and the two jump processes are independent, that is, they do not jump at the same time.

Besides γt, Brownian motion is the only Lévy process with continuous sample paths. Note that $N(ds, dx) = 1$ if X_t jumps at time ds and the associated jump size dx is recorded. Otherwise $N(ds, dx) = 0$. (See Definition 2.18 in Cont & Tankov [12].) In fact, the stochastic integral $\int_{x\in A} \int_0^t xN(ds, dx) = \sum_{0<s\leq t} \triangle X_s I_A(\triangle X_s)$. If γ_0 and γ_1 exist, X can

be rewritten as

$$X_t = \gamma_0 t + \sigma W_t + \sum_{0<s\leq t} \triangle X_s = \gamma_1 t + \sigma W_t + \int_{\mathcal{R}^d} \int_0^t x\tilde{N}(ds, dx).$$

Lévy processes are mostly used to model the return of financial assets in option pricing as $dS_t = S_{t-}dX_t$. Another generalization of the market model is the stochastic volatility model. In pricing problems, the equivalent martingale measure (EMM) should be determined. It is not unique for Lévy processes. The choice depends on a transform to the Lévy measure and the Brownian motion part of the Lévy process.

1.3 Contribution of this thesis

The first part, Chapter 2, deals with the Lévy process. Section 2.1 is about its basic properties. We show the construction of the Lévy process, the Lévy measure and the Lévy density through a limit procedure. Section 2.2 gives examples of popular Lévy processes with their probability distribution functions and their Lévy densities. In Section 2.3, we investigate the structure of multi-variate Lévy processes. The methods used to obtain multi-variate Lévy process are linear transformation, subordination, transformation of jump sizes and Lévy copula. The generating triplet, particularly the multi-variate Lévy measure after transformation can be expressed as a term of original triplet and other parameters. Time copulas are also mentioned as a complement of copulas. Additive processes are introduced in Section 2.4. They can be easily constructed with Lévy processes and they are similar to Lévy processes in some properties.

Chapter 3 focuses on the equivalent martingale measures for Lévy processes. At first, the main theorem of the measure transform is provided. Examples in general case are shown in Section 3.2, which include the numeraire portfolio, the (Föllmer-Schweizer) minimal martingale measure and the minimal entropy martingale measure. The three particular equivalent martingale measures for Poisson-diffusion processes are treated as special cases. At last, we extend the results to additive processes with local characteristics.

Chapter 4 analyzes the credit risk models. The basic factors in credit models are provided in Section 4.1, including the default time, the recovery amount, the default probability and the bond price as well as the dependence structure of default. Section 4.2 discusses the stopping times. Section 4.3 sets up the firm-value process and obtains the instantaneous default probability. The survival probabilities are described in the forms of partial (integro) differential equations in two cases: the one that the default time is totally inaccessible and the general case. The dependent structure of default events and the dependent structure of default intensities are discussed. Section 4.4 is about the defaultable bond price with different recovery schemes as well as the instant credit spread. Section 4.5 generalizes the instantaneous transition matrix of credit rating. Section 4.6 summarizes the results and extend them to additive processes. The last section shows that the instantaneous default probabilities are different under different risk-neutral measures.

Examples of the first-passage model with simple Lévy processes are given in Chapter 5. The instantaneous default probabilities, the survival probabilities, the hazard rates and the corresponding PIDEs are shown in each example. Simulations are also provided in the Poisson-diffusion firm-value models.

2

Lévy processes

2.1 Lévy measure and Lévy density

Consider a one-dimensional Lévy process X, let Λ be a subset of the real line bounded away from 0 and let N_t^Λ be the number of jumps of X in $(0,t]$ with the jump size in Λ:

$$N_t^\Lambda = \sum_{0<s\le t} I_\Lambda(\triangle X_s), \quad \triangle X_s = X_s - X_{s-}. \tag{2.1}$$

In fact, N_t^Λ is a Poisson process. Let $\nu(\Lambda) = \mathrm{E}(N_1^\Lambda)$ be the parameter of the Poisson process, then $\mathrm{E}(N_t^\Lambda) = t\nu(\Lambda)$ and $\nu(\Lambda) < \infty$.

Theorem 2.1.1 ***(Lévy measure)*** *The set function $\Lambda \to N_t^\Lambda(\omega)$ defines a σ-finite measure on $\mathcal{R} \setminus \{0\}$ for each fixed (t,ω). The set function $\nu(\Lambda) = E(N_1^\Lambda)$ also defines a σ-finite measure on $\mathcal{R} \setminus \{0\}$. And this measure ν is called the* ***Lévy measure*** *of the Lévy process X.*

The Lévy measure of the Lévy process X on Λ is in fact the expected number of jumps of X which belong to Λ in any time interval with length 1. It measures the frequency of jumps.

Proposition 2.1.1 *Let ν be the Lévy measure of the Lévy process X, Λ is a Borel set in $\mathcal{R}$ and N_t^Λ is defined by (2.1), then*

- $\nu(\Lambda)$ *is finite if* Λ *is bounded away from 0;*
- *If* $\Lambda_1 \cap \Lambda_2 = \emptyset$, $N_t^{\Lambda_1}$ *is independent of* $N_t^{\Lambda_2}$*;*
- $\nu(\Lambda_1 \cup \Lambda_2) = \nu(\Lambda_1) + \nu(\Lambda_2) - \nu(\Lambda_1 \cap \Lambda_2)$*;*
- *If* $\Lambda = \cup_{i=1}^n \Lambda_i$ *and* $\Lambda_i \cap \Lambda_j = \emptyset$ *for* $i \neq j$*, then* $N_t^{\Lambda} = \sum_{i=1}^n N_t^{\Lambda_i}$ *and* $\nu(\Lambda) = \sum_{i=1}^n \nu(\Lambda_i)$*.*

The Lévy measure may be infinite when 0 is in the closure of Λ . In such case, the Lévy process has infinite activities near 0. Sato [50] divides Lévy processes into three types: a Lévy process $\{X_t\}$ generated by (A, ν, γ) is

- of type A if $A = 0$ and $\nu(\mathcal{R}^d) < \infty$ (or $\int_{|x|\leq 1} \nu(dx) < \infty$);
- of type B if $A = 0$, $\nu(\mathcal{R}^d) = \infty$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$;
- of type C if $A \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$.

The type A process has a finite number of jumps in every finite time interval, therefore it's a compound Poisson process with drift. The number of jumps in every finite time interval of a type B process is countable. The sample functions of these two processes are of bounded variation in every finite time interval. Type C processes are Lévy processes with infinite variation in any time interval. Surbordinators are of finite variation in every finite time interval, which do not belong to type C.

In one-dimensional case, define $\nu(\mathcal{R}) = \mathrm{E}[\sum_{0<s\leq 1} I_{\{\triangle X_s \neq 0\}}]$, which measures the expected number of jumps in any time interval of length 1. The Lévy measure of Brownian motions is zero on any subsets of $\mathcal{R}$ because it is continuous. Consider a pure-jump Lévy process X, if $\nu(\mathcal{R}) = \lambda < \infty$, type A process, then for any set $\Lambda \subseteq \mathcal{R}$, the Lévy measure is $\nu(\Lambda) = \lambda P(\triangle X \in \Lambda)$. If $\nu(\mathcal{R}) = \infty$, type B or type C process, by Lévy decomposition, this measure can be divided into two parts: $\nu = \nu_1 + \nu_2$, where ν_1 measures the part with bounded jumps and ν_2 measures the part with jumps of size at least 1. Then $\nu_1(\mathcal{R}) = \nu_1(\{x : |x| \leq 1\}) = \lim_{\epsilon\to 0} \int_{\epsilon<|x|\leq 1} v(dx) = \infty$ and $\nu_2(\mathcal{R}) = \nu(\{x : |x| > 1\}) = \int_{|x|>1} \nu(dx) < \infty$.

Lévy process has independent and stationary increments, then

$$X_t = \sum_{i=1}^{n} J_i, \quad \text{where } J_i = X_{\frac{it}{n}} - X_{\frac{(i-1)t}{n}}, i = 1, \cdots, n$$

J_i's have independent identical distributions. As a consequence, the Lévy measure can be approached by the limitation:

$$\nu(\Lambda) = E[\sum_{0<s\leq 1} I_\Lambda(\triangle X_s)] = \lim_{n\to\infty} \sum_{i=1}^{n} P(X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \in \Lambda) = \lim_{n\to\infty} nP(J \in \Lambda).$$

Theorem 2.1.2 ***(Lévy density)***

- *If $\nu(\mathcal{R}) = \lambda < \infty$, that is X is a compound Poisson process with drift. Let J be the jump size of X, then the Lévy density is a combination of these two types at $x \neq 0$*

$$\begin{cases} \nu(dx) = \lambda F_J(dx) & \text{if } J \text{ is continuously distributed with CDF } F_J; \\ \nu(\{x_i\}) = \lambda P(J = x_i) & \text{if } J \text{ is discretely distributed.} \end{cases}$$

- *If $\nu(\mathcal{R}) = \infty$, then the Lévy density is $\nu(dx) = \lim_{n\to\infty} nF(dx; \frac{1}{n})$, where $F(dx;t)$ is the CDF of the increment $X_{t+s} - X_s$, $s, t \geq 0$, the same as the CDF of X_t.*

Consider a bivariate compound Poisson process. Let $Y_t^j = \sum_{i=1}^{N_t^j} J_i^j$, $j = 1, 2$ where N_t^j follow equation (2.1) and the jump sizes are nonzero random numbers. The intensities of the Poisson processes are $\lambda_j = \nu_x(\Lambda_j)$, ν_x is the Lévy measure of X. Y_1 and Y_2 are independent if $\Lambda_1 \cap \Lambda_2 = \emptyset$. In other words, if two processes never jump at the same time, they are independent. More specifically, denote $\Omega = \mathcal{R} \setminus \{0\}$, then for $B_1, B_2 \subseteq \Omega$ the joint Lévy measures in the independent case are

$$\nu(B_1, \{0\}) = \nu_1(B_1), \quad \nu(\{0\}, B_2) = \nu_2(B_2), \quad \nu(B_1, B_2) = 0. \tag{2.2}$$

ν_j are the Lévy measures of Y^j and the measure at point $\{0\}$ measures the event that the corresponding process does not jump. If $\Lambda_1 \cap \Lambda_2 \neq \emptyset$, the joint Lévy measure is

$$\nu(B_1, B_2) = \nu_x(\Lambda_1 \cap \Lambda_2) P(J_1 \in B_1, J_2 \in B_2), \tag{2.3}$$

which is fully determined by $\lambda := \nu_x(\Lambda_1 \cap \Lambda_2)$ and the joint distribution of jump sizes. λ is the intensity of the common jumps of Y_1 and Y_2. Moreover

$$\begin{aligned} \nu(B_1, \{0\}) &= \nu_1(B_1) - \nu(B_1, \Omega) = \lambda_1 P(J_1 \in B_1) - \lambda P(J_1 \in B_1), \\ \nu(\{0\}, B_2) &= \nu_2(B_2) - \nu(\Omega, B_2) = \lambda_2 P(J_2 \in B_2) - \lambda P(J_2 \in B_2), \end{aligned}$$

where $\lambda_j - \lambda$ is the intensity of the specific jump parts of Y_j. The total intensity of the 2-dimensional compound Poisson processes is

$$\begin{aligned}\nu(\mathcal{R}^2) &= \nu(\Omega,\Omega)+\nu(\Omega,\{0\})+\nu(\{0\},\Omega)\\ &= \lambda+(\lambda_1-\lambda)+(\lambda_2-\lambda)=\lambda_1+\lambda_2-\lambda.\end{aligned}$$

If $\lambda_1=\lambda_2=\lambda$ and J_1, J_2 are completely positive (or negative) dependent, then we say Y_1 and Y_2 are completely positive (or negative) dependent.

Let $\Lambda_j \subseteq \Omega$, $j=1,\cdots,d$, the d-dimensional joint Lévy measure may be obtained by limit as

$$\begin{aligned}\nu(\Lambda_1,\cdots,\Lambda_d) &= E\left[\sum_{0<s\le 1} I_{\Lambda_1}(\triangle Y_s^1)\times\cdots\times I_{\Lambda_d}(\triangle Y_s^d)\right]\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} P\left(Y^1_{\frac{i}{n}}-Y^1_{\frac{i-1}{n}}\in\Lambda_1,\cdots,Y^d_{\frac{i}{n}}-Y^d_{\frac{i-1}{n}}\in\Lambda_d\right)\\ &= \lim_{n\to\infty} nP(J_1\in\Lambda_1,\cdots,J_d\in\Lambda_d).\end{aligned}$$

Theorem 2.1.3 ***(Joint Lévy density on nonzero points)***
Let X be a d-dimensional Lévy process and $\vec{x}=(x_1,\cdots,x_d)\neq 0$.

- *If there exists any two independent components of X, then $\nu(\Omega,\cdots,\Omega)=0$;*
- *If $\nu(\Omega,\cdots,\Omega)=\lambda<\infty$, then the joint Lévy density is a combination of these two types*

$$\begin{cases}\nu(d\vec{x}) &= \lambda F_J(\vec{x})\\ & \textit{if the jump sizes } J_j \textit{ are continuous distributed with joint CDF } F_J;\\ \nu(\vec{x}) &= \lambda P(J_1=x_1,\ldots,J_d=x_d)\\ & \textit{if the jump sizes } J_j \textit{ are discrete distributed.}\end{cases}$$

- *If $\nu(\Omega,\cdots,\Omega)=\infty$, then the Lévy density is $\nu(\vec{x})=\lim_{n\to\infty} nF(d\vec{x};\frac{1}{n})$, where $F(d\vec{x};t)$ is the joint CDF of the increment $X_{t+s}-X_s$, $s,t\ge 0$, the same as the joint CDF of X_t.*

Let ν^d be a Lévy measure on $\mathcal{R}^d$ and $\Lambda_i\subseteq\Omega$, $i=1,\cdots d$. For $k<d$,

$$\begin{aligned}\nu^d(\Lambda_1,\cdots,\Lambda_k,\mathcal{R},\cdots,\mathcal{R}) &= \nu^k(\Lambda_1,\cdots,\Lambda_k);\\ \nu^d(\Lambda_1,\cdots,\Lambda_{d-1},\{0\}) &= \nu^{d-1}(\Lambda_1,\cdots,\Lambda_{d-1})-\nu^d(\Lambda_1,\cdots,\Lambda_{d-1},\Omega),\end{aligned}$$

if the measures are not infinity. Others may be induced by the equations above. For example,

$$\begin{aligned} & \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \{0\}, \{0\}) \\ = \; & \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \mathcal{R}, \{0\}) - \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \Omega, \{0\}) \\ = \; & \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \mathcal{R}, \mathcal{R}) - \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \mathcal{R}, \Omega) - \nu^d(\Lambda_1, \cdots, \Lambda_{d-2}, \Omega, \{0\}), \end{aligned}$$

The zero value of multivariate Lévy measure dose not mean the total independence of all the components, while if any two components are independent this measure is zero. Let $B_1, B_2 \subseteq \mathcal{R}$, the 2-dimensional independent Lévy measure, equation (2.2) can be summarized as

$$\nu(B_1, B_2) = \nu_1(B_1) I_{\{0\} \in B_2} + \nu_2(B_2) I_{\{0\} \in B_1}.$$

2.2 Examples of Lévy processes

In these section, we list some popular Lévy processes and their probability densities as well as Lévy densities. (Refer to Schoutens [53] for more examples and details.) Denote $u(x)dx = \nu(dx)$ if $\nu(\cdot)$ is absolutely continuous.

1. The Brownian motion: The probability density of X_t is

$$f(x; ut, \sigma^2 t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{\frac{(x-ut)^2}{2\sigma^2 t}},$$

and the Lévy density is

$$u(x) = \lim_{n \to \infty} n f(x; \frac{u}{n}, \frac{\sigma^2}{n}) = 0.$$

2. The Compound Poisson process: As we discussed before, let $f(x)$ be the density of the jump size and λ is the arrival rate, the Lévy density is

$$u(x) = \lambda f(x).$$

3. The Poisson process: It's the special case of compound poisson process with jump size is equal to 1. Then the Lévy density is

$$u(x) = \lambda I_{\{x=1\}}.$$

Poisson type process has finite activities, and the following Lévy processes are of infinite activities, especially for small jumps as $\int_{\|x\|\leq 1} \nu(dx) = \infty$.

4. The Gamma Process: The Gamma process X_t follows a Gamma distribution, hence it only takes positive value and it's a subordinator. The density is

$$f(x; at, b) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-xb}, \quad a, b, x > 0,$$

where the gamma function $\Gamma(a) = \int_0^\infty y^{a-1} e^{-y} dy$. And the Lévy density is

$$u(x) = \lim_{n\to\infty} n f(x; \frac{a}{n}, b) = a x^{-1} e^{-bx} I_{\{x>0\}}.$$

5. The Inverse Gaussian Process: Let T be the first time a standard Brownian motion with drift $b > 0$ reached the positive level a, $T = \inf\{s \geq 0 : W_s + bs \geq a\}$, then this random time T follows the so-called Inverse Gaussian distribution. And the process $T_t = \inf\{s \geq 0 : W_s + bs \geq at\}$ is a subordinator with density

$$f(x; at, b) = \frac{at}{\sqrt{2\pi}} e^{bat} x^{-3/2} \exp\left\{-\frac{1}{2}[(at)^2 x^{-1} + b^2 x]\right\}, \quad a, b, x > 0.$$

The Lévy density is

$$u(x) = \lim_{n\to\infty} n f(x; \frac{a}{n}, b) = \frac{a}{\sqrt{2\pi}} x^{-3/2} e^{-\frac{1}{2}b^2 x} I_{\{x>0\}}.$$

6. The Tempered Stable Process: The process X_t follows a Tempered Stable distribution with the following series representation of the density: for $x > 0$

$$f(x; k, at, b) = \frac{\exp\{bat - \frac{1}{2}b^{1/k}x\}}{2\pi(at)^{1/k}} \sum_{i=1}^{\infty} (-1)^{i-1} \sin(ik\pi) \frac{\Gamma(ik+1)}{i!} 2^{ik+1} \left(\frac{x}{(at)^{1/k}}\right)^{-ik-1},$$

where the parameters $a > 0, b \geq 0$ and $0 < k < 1$. This process is also a subordinator. And the Lévy density is

$$u(x) = \lim_{n\to\infty} n f(x; k, \frac{a}{n}, b) = a \exp\left\{-\frac{1}{2} b^{1/k} x\right\} 2^k x^{-k-1} \frac{\sin(\pi k)\Gamma(k+1)}{\pi} I_{\{x>0\}}.$$

Because $\Gamma(1+k)\Gamma(1-k) = \pi k / \sin(\pi k)$, the Lévy density is

$$u(x) = \frac{ak}{\Gamma(1-k)} \exp\left\{-\frac{1}{2} b^{1/k} x\right\} 2^k x^{-k-1} I_{\{x>0\}}.$$

7. The Normal Inverse Gaussian Process: The density of X_t is

$$f(x;\alpha,\beta,\delta t) = \frac{\alpha\delta t}{\pi}\exp\{\delta t\sqrt{\alpha^2-\beta^2}+\beta x\}\frac{K_1(\alpha\sqrt{(\delta t)^2+x^2})}{\sqrt{(\delta t)^2+x^2}}, \quad \alpha,\delta>0, -\alpha<\beta<\alpha,$$

where $K_1(\cdot)$ is the modified Bessel function of the third kind, and

$$K_1(x) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{1}{2}x(y+y^{-1})\right\}dy, \quad x>0.$$

The Lévy density is

$$u(x) = \lim_{n\to\infty} nf(x;\alpha,\beta,\frac{\delta}{n}) = \frac{\alpha\delta}{\pi}e^{\beta x}|x|^{-1}K_1(\alpha|x|).$$

8. The Meixner Process: The density of X_t is

$$f(x;\alpha,\beta,\delta t) = \frac{(2\cos\frac{\beta}{2})^{2\delta t}}{2\alpha\pi\Gamma(2\delta t)}e^{\beta x/\alpha}|\Gamma(\delta t+\frac{ix}{\alpha})|^2, \quad \alpha,\delta>0, -\pi<\beta<\pi.$$

The Lévy density is

$$u(x) = \lim_{n\to\infty} nf(x;\alpha,\beta,\frac{\delta}{n}) = \frac{2\delta}{2\alpha\pi}e^{\beta x/\alpha}|\Gamma(\frac{ix}{\alpha})|^2.$$

The gamma function satisfies the following equations: $\Gamma(x) = -\frac{\Gamma(1+x)}{x}$, $\Gamma(1+ix) = (ix)!$ and $|(ix)!|^2 = \pi x/\sinh(\pi x)$, so the Lévy density can be written as

$$u(x) = \frac{\delta\exp(\beta x/\alpha)}{x\sinh(\pi x/\alpha)}.$$

2.3 Constructing dependent Lévy processes

A Lévy process is uniquely determined by its generating triplet (A,ν,γ), the covariance matrix A measures the dependence structure of Brownian motion while the Lévy measure gives the dependence structure of its jump part. To construct a multi-variate Lévy process, instead we may define the generating triplet with a symmetric nonnegative-definite matrix A and a Lévy measure ν ($\nu(\{0\})=0$) which verifies

$$\int_{\mathcal{R}^d}(\|x\|^2\wedge 1)\nu(dx)<\infty \Longleftrightarrow \int_{\|x\|\le 1}\|x\|^2\nu(dx)<\infty \text{ and } \int_{\|x\|>1}\nu(dx)<\infty.$$

Any transformation of the Lévy measure, respecting the singularity at $\{0\}$ and the integrability conditions, will lead to a new homogeneous jump process. Esscher transformation is an example, where the new Lévy measure is $\nu^*(dx) = e^{\langle\theta,x\rangle}\nu(dx)$, θ is a real vector in $\mathcal{R}^d$ such that $\int_{\|x\|>1} e^{\langle\theta,x\rangle}\nu(dx) < \infty$.

The methods used to construct dependent multi-variate Lévy processes from known ones are listed in Chapter 4, Cont & Tankov [12]. Such methods include linear transformation, subordination, tilting and tempering of the Lévy measure. Lévy copulas are introduced in Chapter 5, Cont & Tankov [12] for building multivariate models. In this thesis, we will show the details of these transformations and focus on the representation of Lévy triplets after these transformations.

2.3.1 Linear transformation of independent Lévy processes

Lévy processes have independent stationary increments, so a linear transformation of Lévy processes will also have these properties. A linear transformation of independent Lévy processes will construct dependent Lévy processes. Particularly, for any correlated Brownian motions (or Poisson processes) $Y_t \in \mathcal{R}^m$, there exist independent Brownian motions (or Poisson processes) $X_t \in \mathcal{R}^d$ and a real-valued $m \times d$ matrix C such that Y is a linear transformation of X as $Y = CX$. But not all correlated Lévy processes can be represented as a linear combination of independent components.

Theorem 2.3.1 (Theorem 4.1 in Cont & Tankov [12])
Let X be a $\mathcal{R}^d$-valued Lévy process with generating triplet (A, ν, γ) and let C be a real-valued $m \times d$ matrix, then $Y = CX$ is an m-dimensional Lévy process with generating triplet (A^Y, ν^Y, γ^Y) given by

$$A^Y = CAC^T, \text{ a nonnegative-definite } m \times m \text{ matrix;} \tag{2.4}$$

$$\nu^Y(\{0\}) = 0, \text{ and } \nu^Y(B) = \nu(\{x : Cx \in B\}), \forall B \in \mathcal{B}(\mathcal{R}^m); \tag{2.5}$$

$$\gamma^Y = C\gamma + \int_{\mathcal{R}^m} y \left(I_{\{\|y\|\le 1\}}(y) - I_{\{y=Cx:\|x\|\le 1\}}(y)\right) \nu^Y(dy). \tag{2.6}$$

Proof: The Lévy measure of Y on some Borel set B has the same value as the Lévy measure of X on the set $\{x : Cx \in B\}$. The nonnegative measure ν^Y on $\mathcal{R}^m$ defined by Equation (2.5) is a Lévy measure satisfies

$$\int_{\mathcal{R}^m} (\|y\|^2 \wedge 1)\nu^Y(dx) = \int_{\mathcal{R}^d} (\|Cx\|^2 \wedge 1)\nu(dx) \le \int_{\mathcal{R}^d} (\|C\|^2\|x\|^2 \wedge 1)\nu(dx) < \infty$$

since $||C|| < \infty$ and $\int_{\mathcal{R}^d}(||x||^2 \wedge 1)\nu(dx) < \infty$. The characteristic exponent of CX is

$$-\frac{1}{2}\langle z, CAC^T z\rangle + \int_{\mathcal{R}^d}(e^{i\langle z,Cx\rangle} - 1 - i\langle z, Cx\rangle I_{\{||x||\leq 1\}}(x))\nu(dx) + i\langle z, C\gamma\rangle.$$

On the other hand, the characteristic exponent of Y is

$$-\frac{1}{2}\langle z, A^Y z\rangle + \int_{\mathcal{R}^m}(e^{i\langle z,y\rangle} - 1 - i\langle z, y\rangle I_{\{||y||\leq 1\}}(y))\nu^Y(dy) + i\langle z, \gamma^Y\rangle.$$

By comparing these expressions, we have (2.4) and (2.6). The prove of the existence of the integral in (2.6) is given in details in Cont & Tankov [12].

□

The *quadratic covariation process* $[X, Y]$ of two semimartingales X and Y is defined in Definition 8.4, Cont & Tankov [12] as

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s-}dY_s - \int_0^t Y_{s-}dX_s,$$

which can be rewritten as the limitation in probability

$$[X, Y]_t = \lim_{n\to\infty} \text{ in p. } \sum_{k=0}^{n}(X_{\frac{t(k+1)}{n}} - X_{\frac{tk}{n}})(Y_{\frac{t(k+1)}{n}} - Y_{\frac{tk}{n}}).$$

The process $[X, X]_t$ is called the *quadratic variation process* of X, it is positive and increasing in t. The quadratic covariation of independent semimartingales is 0.

Proposition 2.3.1 *A Lévy process $Y_t \in \mathcal{R}^m$ is a linear transform of independent Lévy processes if and only if its quadratic covariation matrix can be represented as CS_tC^T, where C is a real-valued $m \times d$ ($m \leq d$) matrix and S_t is a $d \times d$ diagonal matrix whose entries are subordinators. Let $X_t \in \mathcal{R}^d$ be the independent Lévy processes, then $Y = CX$ and $S_t = diag\{[X^1, X^1]_t, \ldots, [X^d, X^d]_t\}$.*

It is easy to prove. If $Y_t = CX_t$, let Σ_t denote the quadratic covariation matrix of Y with $(\Sigma_t)_{ij} = [Y^i, Y^j]_t$, $i, j = 1, \ldots m$, then

$$\Sigma_t = C\text{diag}\{[X^1, X^1]_t, \ldots, [X^d, X^d]_t\}C^T = CS_tC^T.$$

For example, the quadratic covariation matrix of an m-dimensional Brownian motion Y with generating triplet $(A, 0, 0)$ can be rewritten as At, that is $S_t = \text{diag}\{t, \ldots, t\}$ and

$CC^T = A$. Thus Y_t can be represented as $Y_t = CW_t$ where W_t is a d-dimensional independent standard Brownian motion. If $d = m$, then C is the Cholesky decomposition (a lower triangle matrix).

Here is an example of correlated Lévy processes which cannot be written as a linear transformation of independent Lévy processes: $Y_t^k = \sum_{i=1}^{N_t} J_i^k$, $k = 1, 2$ where N_t is a Poisson process, J^1 and J^2 are independent for each jump. These two compound Poisson processes jump at the same time but with independent jump sizes.

2.3.2 Lévy process as a time change of Brownian motion

Monroe [46] proposed that a process is equivalent to a time change of Brownian motion with sufficient large σ-field if and only if it is a local semimartingale. The definition of the *local semimartingale* in [46] is following: a process $(X_t, \mathcal{F}_t)$ is a local semimartingale if $X_t = M_t + A_t$ where M_t is a local martingale and A_t is of pathwise bounded variation. Lévy processes are semimartingales, hence any Lévy process is equivalent to a time change of Brownian motion. The term of *time change* is defined in Chapter V §1, Revuz & Yor [49] and in Definition 2, Kallsen & Shiryaev [31].

Definition 2.3.1 ***(Time Change)***

- *A time change T is a family of $\mathcal{F}$-stopping times $\{T_t\}_{t \geq 0}$ such that the map $t \to T_t$ is increasing, right continuous and nonnegative almost surely. It is called finite if $T_t < \infty$ for any $t \geq 0$ almost surely.*
- *Let $\widehat{\mathcal{F}}_t := \mathcal{F}_{T_t}$, which defines the time changed filtration, if X is an $\mathcal{F}_t$-progressively measurable stochastic process then $\widehat{X}_t := X_{T_t}$ is an $\widehat{\mathcal{F}}_t$-adapted process and the process $\widehat{X}$ is called the time change process of X.*
- *The inverse time change $\{\widehat{T}_t\}_{t \geq 0}$ is defined as $\widehat{T}_t = \inf\{s \geq 0 : T_s > t\}$, which is an increasing and right continuous family of $\widehat{\mathcal{F}}$-stopping times..*

If T_t is continuous, strictly increasing and $T_\infty = \infty$, then $\widehat{T}_t$ is continuous, strictly increasing and $\widehat{T}_\infty = \infty$ and it is also true the other way. The processes T and $\widehat{T}$ play totally symmetric roles. For any $\widehat{\mathcal{F}}_t$-progressively measurable stochastic process $\widehat{X}$, the stochastic process $X := \widehat{X}_{\widehat{T}}$ is $\mathcal{F}_t$-progressively measurable. If $T_\infty \neq \infty$ (or $\widehat{T}_\infty \neq \infty$), the same holds but $\widehat{X}$ (or X) is only defined on $t \leq T_\infty$ (or $t \leq \widehat{T}_\infty$).

In Monroe's result, $X = W$, a Brownian motion and the time change T may depend on W. For example, the discrete time process $Y_n = \sum_{i=1}^{n} J_i$, J_i's are independent identical distributed with $P(J = 1) = p$ and $P(J = 0) = 1 - p$, $p \in (0, 1)$. That is J follows a Bernoulli distribution and Y_n follow a Binomial distribution for each $n = 1, 2, \ldots$ Then $Y_n = W_{T_n} + np$ and the time change will be

$$\begin{aligned} T_1 &= \inf\{t > 0, W_t = -p \text{ or } 1 - p\}, \\ T_2 &= \inf\{t > T_1, W_t - W_{T_1} = -p \text{ or } 1 - p\}, \\ \ldots & \quad \ldots \\ T_n &= \inf\{t > T_{n-1}, W_t - W_{T_{n-1}} = -p \text{ or } 1 - p\}. \end{aligned}$$

The time change T is a sequence of stopping times of the Brownian motion W. In this paper, we only consider the case that X and T are independent and both are Lévy processes. The increasing Lévy process T is a subordinator and such time change is called *subordination*. Here are some well-known examples: when $W \in \mathcal{R}$ is a Brownian motion and $\alpha, \beta \in \mathcal{R}$ are constants, then $Y_t = W_{T_t} + \beta T_t + \alpha t$ is an univariate Lévy process. If T_t is a Inverse Gaussian process, then X_t follows a Normal Inverse Gaussian process; if T_t is a Generalized Inverse Gaussian process, then X_t follows a Generalized Hyperbolic processes; if T_t is a Gamma process and $\alpha = 0$, then X_t follows a Variance-gamma process.

The characteristic function and the generating triplet of the subordinated process are shown in Theorem 4.2 in Cont & Tankov [12] as following. Let $\psi(\cdot)$ be the characteristic exponent of X, thus $\mathrm{E}\left[e^{i\langle z, X_t\rangle}\right] = e^{t\psi(z)}$. Let $l(\cdot)$ be the Laplace exponent of T, thus $\mathrm{E}\left[e^{zT_t}\right] = e^{tl(z)}$. $\gamma_0 \geq 0$ is the drift of the subordiator T.

Theorem 2.3.2 ***(Subordination of a Lévy process)***

Fix a probability space $(\Omega, \mathcal{F}, P)$. Let $X_t \in \mathcal{R}^d$ be a Lévy process with characteristic exponent $\psi(\cdot)$ and triplet (A, ν, γ) and let $T_t \in \mathcal{R}$ be a subordinator with Laplace exponent $l(\cdot)$ and triplet $(0, \nu_T, \gamma_0)$, γ_0 is the drift. X and T are stochastically independent. Then the new process $(Y_t)_{t\geq 0}$ defined by the subordination: for each $\omega \in \Omega$, $Y(t, \omega) =$

$X(T(t,\omega),\omega)$ is a Lévy process on $\mathcal{R}^d$ with characteristic function

$$E\left[e^{i\langle z,Y_t\rangle}\right] = e^{tl(\Psi(z))}.$$

Its generating triplet (A^Y, ν^Y, γ^Y) is given by

$$A^Y = \gamma_0 A, \text{ a nonnegative-definite } d\times d \text{ matrix;} \tag{2.7}$$

$$\nu^Y(B) = \gamma_0\nu(B) + \int_0^\infty p_s^X(B)\nu_T(ds),\ \forall B \in \mathcal{B}(\mathcal{R}^d); \tag{2.8}$$

$$\gamma^Y = \gamma_0\gamma + \int_0^\infty \nu_T(ds)\int_{\|x\|\le 1} x p_s^X(dx), \tag{2.9}$$

where p_t^X is the probability distribution of X_t.

Proof: Since X and T are Lévy processes, $Y_t = X_{T_t}$ should also be a Lévy process. (See the proof of Theorem 4.2 in [12] for more details.) The characteristic function of Y can be derived by conditioning on the process T as

$$\begin{aligned} \mathrm{E}\left[e^{i\langle z,Y_t\rangle}\right] &= \mathrm{E}\left\{\mathrm{E}\left[e^{i\langle z,X_{T_t}\rangle}\middle| T_s, s\le t\right]\right\} \\ &= \mathrm{E}\left[e^{\Psi(z)T_t}\right] = \mathrm{E}\left[e^{tl(\Psi(z))}\right]. \end{aligned}$$

$T_t = \gamma_0 t$ is a simple example, where $\mathrm{E}[e^{i\langle z,Y_t\rangle}] = \mathrm{E}[e^{i\langle z,X_{\gamma_0 t}\rangle}] = \mathrm{E}[e^{i\langle z,X_1\rangle\gamma_0 t}]$ and $A^Y = \gamma_0 A$, $\nu^Y(dx) = \gamma_0\nu(dx)$ and $\gamma^Y = \gamma_0\gamma$. Since the subordinator $T_t = \gamma_0 t + \sum_{0<s\le t}\Delta T_t$, the law of the instant jump of Y is

$$\Delta Y_t = X_{T_t} - X_{T_{t-}} \sim^{\mathcal{L}} \Delta X_{\gamma_0 t} + X_{\Delta T_t},$$

where $\Delta X_{\gamma_0 t}$ describes X's own movement at transformed time $\gamma_0 t$ and $X_{\Delta T_t}$ represents the movement of Y with respect to the jump of T. Conditioning on $\Delta T_t = s$, by assuming that X_s has the distribution p_s^X, the Lévy measure corresponding to $\Delta X_{\gamma_0 t}$ is $\gamma_0\nu(dx)$ and the Lévy measure corresponding to the second part is $\int_0^\infty p_s^X(dx)\nu_T(ds)$. Then we obtain Equation (2.8). Since

$$\begin{aligned} \psi(z) &= -\frac{1}{2}\langle z, Az\rangle + \int_{\mathcal{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x))\nu(dy) + i\langle z,\gamma\rangle, \\ l(\mu) &= \int_0^\infty (e^{\mu s} - 1)\nu_T(ds) + \mu\gamma_0. \end{aligned}$$

Then their composition $l(\psi(z))$ is:

$$\begin{aligned}
&\int_0^\infty (e^{\psi(z)s}-1)\nu_T(ds)+\psi(z)\gamma_0 \\
=\;& \int_0^\infty \left(\mathrm{E}[e^{i\langle z,X_s\rangle}]-1\right)\nu_T(ds)+\gamma_0\left[-\frac{1}{2}\langle z,Az\rangle+i\langle z,\gamma\rangle\right. \\
&\left.+\int_{\mathcal{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x)\right)\nu(dx)\right] \\
=\;& \int_0^\infty\int_{\mathcal{R}^d}(e^{i\langle z,x\rangle}-1)p_s^X(dx)\nu_T(ds)-\frac{1}{2}\langle z,\gamma_0 Az\rangle+i\langle z,\gamma_0\gamma\rangle \\
&+\int_{\mathcal{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x)\right)\gamma_0\nu(dx) \\
=\;& \int_{\mathcal{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x)\right)\left[\gamma_0\nu(dx)+\int_0^\infty p_s^X(dx)\nu_T(ds))\right] \\
&-\frac{1}{2}\langle z,\gamma_0 Az\rangle+i\langle z,\gamma_0\gamma\rangle+\int_0^\infty\int_{\mathcal{R}^d} i\langle z,x\rangle I_{\{\|x\|\le 1\}}(x)p_s^X(dx)\nu_T(ds)) \\
=\;& -\frac{1}{2}\langle z,\gamma_0 Az\rangle+\int_{\mathcal{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,X\rangle I_{\{\|x\|\le 1\}}(x)\right)\nu^Y(dx) \\
&+i\left\langle z,\gamma_0\gamma+\int_0^\infty\int_{\|x\|\le 1} x p_s^X(dx)\nu_T(ds)\right\rangle.
\end{aligned}$$

It is the characteristic exponent of the new Lévy process Y, thus $A^Y = \gamma_0 A$ and $\gamma^Y=\gamma_0\gamma+\int_0^\infty\int_{\|x\|\le 1} x p_s^X(dx)\nu_T(ds)$.

□

When X is a Brownian motion, the characteristic function and the Lévy measure of $Y_t = W_{T_t}+\beta T_t+\alpha t$ will be

$$\mathrm{E}\left[e^{izY_t}\right]=e^{iz\alpha t}\mathrm{E}\left[e^{iz\beta T_t-\frac{1}{2}z^2T_t}\right],$$
$$\nu_Y(B)=\int_0^\infty\int_B\frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-\beta s)^2}{2s}}dy\,\nu_T(ds).$$

Based on the expression, not all the Lévy process can be represented as a subordinated Brownian motion with independent time change.

The way to construct dependent Lévy processes by the subordination of Brownian motion lies on the fact that they follow a multi-variate normal distribution given the subordinators.

2.3.3 Transformation of jump sizes

As mentioned before, any transformation of the Lévy measure respecting the integrability conditions will lead to a new homogeneous jump process. Tilting and tempering of the Lévy measure are such transformation as $\tilde{\nu}(dx) = f(x)\nu(dx)$. Here are two examples in section 4.2.3, Cont & Tankov [12]. One is the Esscher transformation $f(x) = \exp\{\theta x\}$, which is also called the *exponential titling*. The other is an asymmetric version transformation as

$$f(x) = I_{\{x>0\}}e^{-\lambda_+ x} + I_{\{x<0\}}e^{-\lambda_-|x|}, \qquad \lambda_+, \lambda_- > 0.$$

This function defines a Lévy process whose large jumps are *tempered*, that is the tails of the Lévy measure are exponential damped.

Denote $\nu(dx) = u(x)dx$ if it is absolutely continuous, now we present another transformation of the Lévy measure as $\tilde{\nu}(dg(x)) = \nu(dx)$. Consider a one-dimensional Lévy process with decomposition

$$X_t = \gamma t + \int_{0<|x|\le 1}\int_0^t x[N(ds,dx) - \nu(dx)ds] + \int_{|x|>1}\int_0^t xN(ds,dx),$$

the new jump process Y_t associated with function $g(\cdot) : \mathcal{R} \to \mathcal{R}$ is defined as

$$\gamma^Y t + \int_{0<|g(x)|\le 1}\int_0^t g(x)[N(ds,dx) - \nu(dx)ds] + \int_{|g(x)|>1}\int_0^t g(x)N(ds,dx), \quad (2.10)$$

$\gamma^Y \in \mathcal{R}$. Y is a Lévy process if

$$\int_{\mathcal{R}} \left(|g(x)|^2 \wedge 1\right) \nu(dx) < \infty \Longleftrightarrow \int_{\mathcal{R}} \left(|y|^2 \wedge 1\right) \nu^Y(dy) < \infty,$$

where the Lévy measure of Y is $\nu^Y(B) = \nu(\hat{B})$, $\hat{B} = \{x : g(x) \in B\}$.

The Lévy measure of Y can be obtained in this way. Consider the simple case that $g(\cdot)$ is a strictly monotone and differentiable function and X_t has the probability density function $f(x;t)$, then the probability density function of Y_t is

$$f^Y(y;t) = \frac{f(g^{-1}(y);t)}{|g'(g^{-1}(y))|},$$

where g^{-1} and g' are the inverse function and the first derivative of g respectively. The Lévy density of Y exists by the following limitation, for $y \neq 0$,

$$u^Y(y) = \lim_{t\to 0}\frac{1}{t}f^Y(y;t) = \lim_{t\to 0}\frac{1}{t}\frac{f(g^{-1}(y);t)}{|g'(g^{-1}(y))|} = \frac{u(g^{-1}(y))}{|g'(g^{-1}(y))|}. \quad (2.11)$$

For other functions of g without constant part (i.e. $g'(\cdot) \neq 0$), u^Y has the similar form as above. For example, if g is a discontinuous strictly monotone function, equation (2.11) still hold in the range of $g(\cdot)$. More generally, for any real function g with non-zero derivatives, there exists a series of functions $a_i(y)$, $b_i(y)$ with $-\infty \leq a_i < b_i \leq \infty$ and $g(a_i(y)) = y$, $g(b_i(y)) = y$, $i = 1, \ldots, k$ such that

$$\{x : g(x) < y\} = \bigcup_{i=1}^{k} \{x : a_i(y) < x < b_i(y)\}.$$

The probability function, the probability density function and the Lévy density function of Y are

$$\begin{aligned} P(g(x) < y) &= \sum_{i=1}^{k} P(a_i(y) < x < b_i(y)), \\ f^Y(y;t) &= \sum_{i=1}^{k} [b_i'(y) f(b_i(y);t) - a_i'(y) f(a_i(y);t)], \\ u^Y(y) &= \sum_{i=1}^{k} [b_i'(y) u(b_i(y)) - a_i'(y) u(a_i(y))]. \end{aligned} \tag{2.12}$$

In the case that a or b take infinity value, their derivative would be set to 0. Equation (2.11) is the special case of (2.12). If g is strictly increasing, then $k = 1$ with $a_1(y) = -\infty$ and $b_1(y) = g^{-1}(y)$. If g is strictly decreasing, then $k = 1$ with $a_1(y) = g^{-1}(y)$ and $b_1(y) = \infty$.

If g has constant parts, for instance, g is a simple function as $g(x) = \sum_i c_i I_{\Lambda_i}(x)$, where c_i's are nonzero constants, Λ_i's are disjoint Borel sets of $\mathcal{R}$. Then $Y_t = \sum_i \int_{\Lambda_i} \int_0^t c_i N(ds, dx) = c_i \sum_i N_t(\Lambda_i)$ is a Poisson point process, and the Lévy measure of Y at non-zero point y is

$$\nu^Y(\{y\}) = \sum_i \nu(\Lambda_i) I_{\{y=c_i\}}. \tag{2.13}$$

The Lévy measure of Y corresponding to a general function g with step part $\sum_i c_i I_{\Lambda_i}$ will have the form of the mixture of (2.12) and (2.13) as

$$\nu^Y(B) = \int_B u^Y(y) dy + \sum_i \nu(\Lambda_i) I_{\{c_i \in B\}}, \tag{2.14}$$

for any Borel set B in $\mathcal{R}$.

Denote $g^{-1}(B) = \{x : g(x) \in B\}$, then the Lévy measure of Y^i defined as Equation (2.10) are $\nu^i(B_i) = \nu(g_i^{-1}(B_i))$, $i = 1, 2$. The joint Lévy measure of Y^1 and Y^2 is

$\nu(B_1, B_2) = \nu(g_1^{-1}(B_1) \bigcap g_2^{-1}(B_2))$. If both g_1 and g_2 are strictly increasing, then the jumps of Y^1 and Y^2 are completely positively correlated. If one is strictly decreasing and the other is strictly increasing, then the jumps of Y^1 and Y^2 are completely negatively correlated.

2.3.4 Probability copula and Lévy copula

As we know, a multivariate probability distribution can be expressed by the marginal distributions and a copula. Copulas capture all the information of the dependence structure of random variables irrespective of their distributions. The Lévy measure has similar properties as probability measures. Specifically, a multivariate Lévy measure can be decomposed into marginal measures and a copula-type function. Here are the definition and the main theorem of the probability copula.

Definition 2.3.2 ***(Copula)*** *A d-dimensional ($d \geq 2$) copula is a function $C : [0,1]^d \rightarrow [0,1]$ satisfying:*

- *(Boundary condition) for $i = 1, \ldots, d$, $C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) = 0$. And the function $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \rightarrow C(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)$ is a $d-1$ copula. $C(1, \ldots, 1, x_i, 1, \ldots 1) = C_i(x_i) = x_i$ is called the margin.*
- *(Monotonicity)*

$$\sum_{V \in R} sgn(V) C(V) \geq 0$$

for all rectangles R of the form $R = \prod_{i=1}^{d} [a_i, b_i]$, $a_i \leq b_i$. Here the sum is over all vertices $V = (\varepsilon_1, \ldots, \varepsilon_d)$ of the rectangle, where $\varepsilon_i = a_i$ or b_i, and

$$sgn(V) = \begin{cases} -1 & \text{if } \varepsilon_i = a_i \text{ for an odd number of } i\text{'s;} \\ & \text{if } \varepsilon_i = a_i \text{ for an even number of } i\text{'s.} \end{cases}$$

Theorem 2.3.3 ***(Sklar)*** *Let H be a d-dimensional CDF with margins $F_1, \ldots, F_d$, then there exists an m-dimensional copula C such that for all $x \in \mathcal{R}^d$,*

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)),$$

Moreover, if all F_i are continuous, then C is unique; otherwise, C is uniquely determined on $RanF_1 \times \ldots \times RanF_d$, where $RanF_i = \{F_i(x) \in [0,1] : x \in \mathcal{R}\}$. Conversely, if C is an *m-copula and $F_1, \dots, F_d$ are 1-dimensional CDF, then the function H is a d-dimensional CDF with margins $F_1, \dots, F_d$.*

The bivariate joint probability function and survival function can be represented as

$$\begin{aligned} P(x_1 \leq a, x_2 \leq b) &= C(P(x_1 \leq a), P(x_2 \leq b)), \\ P(x_1 > a, x_2 > b) &= C^S(P(x_1 > a), P(x_2 > b)). \end{aligned}$$

C and C^S are the probability copula and survival copula of X_1 and X_2 respectively and and $C^S(u, v) = C(u, v) + 1 - u - v$.

When a Lévy process is constructed by subordinating a multivariate Brownian motion, it will follow a multivariate normal distribution given the subordinators. So its probability copula is the mixture of the Brownian copula and the dependence of subordinators. For example, in the two-variate case define $X_t^i = W_{T_t^i}^i$, $i = 1, 2$, let $\Phi(\cdot)$ be the CDF of univariate standard normal distribution and let H_t^i be the CDF of the subordinator T_t^i, $i = 1, 2$ and let $\widehat{H_t}$ be the joint CDF of T_t^1, T_t^2. Then

$$\begin{aligned} F_{i,t}(a_i) := P(X_t^i \leq a_i) &= \int_0^\infty \Phi\left(\frac{a_i}{\sqrt{s}}\right) H_t^i(ds), \\ P(X_t^1 \leq a_1, X_t^2 \leq a_2) &= \int_0^\infty \int_0^\infty P(W_u^1 \leq a_1, W_v^2 \leq a_2) \widehat{H}_t(du, dv) \\ &= \int_0^\infty \int_0^\infty C^N\left(\Phi\left(\frac{a_1}{\sqrt{u}}\right), \Phi\left(\frac{a_2}{\sqrt{v}}\right)\right) \widehat{H}_t(du, dv), \\ C_t(x_1, x_2) &= \int_0^\infty \int_0^\infty C^N\left(\Phi\left(\frac{F_{1,t}^{-1}(x_1)}{\sqrt{u}}\right), \Phi\left(\frac{F_{2,t}^{-1}(x_2)}{\sqrt{v}}\right)\right) \widehat{H}_t(du, dv) \\ &= \int_0^\infty \int_0^\infty C^N(\hat{x}_1, \hat{x}_2) \widehat{H}_t(du, dv), \end{aligned} \tag{2.15}$$

where C^N is the normal copula, C_t is the probability copula of X^1, X^2 and

$$\begin{aligned} x_1 &= F_{1,t}(\sqrt{u}\,\Phi^{-1}(\hat{x}_1)) = \int_0^\infty \Phi\left(\sqrt{\frac{u}{s}}\Phi^{-1}(\hat{x}_1)\right) H_t^1(ds) \\ x_2 &= F_{2,t}(\sqrt{v}\,\Phi^{-1}(\hat{x}_2)) = \int_0^\infty \Phi\left(\sqrt{\frac{v}{s}}\Phi^{-1}(\hat{x}_2)\right) H_t^2(ds). \end{aligned}$$

Generally, the probability copulas of Lévy processes depend on the time t, for example, the one defined by equation (2.15). For the stable processes, their probability copulas are constant.

By using the probability copula and marginal of J instead of the joint probability function, for $x_1, x_2 < 0$ equation (2.3) is equivalent to

$$\begin{aligned}\nu((-\infty, x_1], (-\infty, x_2]) &= \lambda C(P(J_1 \le x_1), P(J_2 \le x_2)) \\ &= \lambda C\left(\frac{\nu_1((-\infty, x_1])}{\lambda_1}, \frac{\nu_2((-\infty, x_2])}{\lambda_2}\right),\end{aligned}$$

for $x_1, x_2 > 0$, use the survival copula C^S instead of probability copula,

$$\begin{aligned}\nu([x_1, \infty), [x_2, \infty)) &= \lambda C^S(P(J_1 \ge x_1), P(J_2 \ge x_2)) \\ &= \lambda C^S\left(\frac{\nu_1([x_1, \infty))}{\lambda_1}, \frac{\nu_2([x_1, \infty))}{\lambda_2}\right).\end{aligned}$$

Now the Lévy measures and probability copula are connected for the compound Poisson process. The Lévy measure of compound Poisson process is finite, we may define a Lévy copula of such process F as following, for lower tail integrals

$$F(\mu_1, \mu_2) = \lambda C\left(\frac{\mu_1}{\lambda_1}, \frac{\mu_2}{\lambda_2}\right), \quad 0 \le \mu_1 \le \lambda_1, 0 \le \mu_2 \le \lambda_2;$$

let F^U be the copula measuring the upper tails, then

$$F^U(\mu_1, \mu_2) = \lambda C^S\left(\frac{\mu_1}{\lambda_1}, \frac{\mu_2}{\lambda_2}\right), \quad 0 \le \mu_1 \le \lambda_1, 0 \le \mu_2 \le \lambda_2.$$

And $F(\lambda_1, \lambda_2) = F^U(\lambda_1, \lambda_2) = \lambda$. But the relation $F^U(u, v) = F(u, v) + \lambda - u - v$ does not hold any more except for the case that $\lambda = \lambda_1 = \lambda_2$. At this point, we may see that the function defined above does not cover the case that two processes do not jump at the same time, it only measures the dependent structure of jumps.

If two compound Poisson processes are completely positive (or negative) dependent, it implies that they have the same driven Poisson process and the jump sizes are also completely positive (or negative) dependent, that is $\lambda_1 = \lambda_2 = \lambda$ and $C(u, v) = \min(u, v)$ (or $C(u, v) = \max(0, u + v - 1)$). Then for $\mu_1, \mu_2 \in [0, \lambda]$, the bivariate Lévy copula is $F(\mu_1, \mu_2) =$

$$\begin{cases} \lambda \min(\frac{\mu_1}{\lambda}, \frac{\mu_2}{\lambda}) = \min(\mu_1, \mu_2) & \text{for perfect positive dependence;} \\ \lambda \max(0, \frac{\mu_1}{\lambda} + \frac{\mu_2}{\lambda} - 1) = \max(0, \mu_1 + \mu_2 - \lambda) & \text{for perfect negative dependence.} \end{cases}$$

Even if the jumps are independent, the two compound Poisson processes are still dependent as long as they have a common driven Poisson process. And the copula will be $F(\mu_1, \mu_2) = \frac{\lambda}{\lambda_1 \lambda_2} \mu_1 \mu_2$.

In the two variate case, given the function F and their margins, the parameters λ, λ_1, λ_2 and the marginal and joint distributions of jump sizes can be easily identified. Our setup is slightly different from Cont & Tankov [12] (Page 151).

Definition 2.3.3 ***(Lévy copula of dependent compound Poisson processes)***
Consider a d-dimensional ($d \geq 2$) compound Poisson process. Let λ_i be the Poisson intensity of i^{th} process and let $\lambda > 0$ be the joint intensity of all processes, its Lévy copula of the associated joint jumps is a function $F : [0, \lambda_1] \times \ldots \times [0, \lambda_d] \to [0, \lambda]$ satisfying:

- *(Boundary condition) $F(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_d) = \lambda$;*
  for all $i = 1, \ldots, d$, $F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) = 0$;
  $$\frac{F(x_1, \ldots, x_i, \lambda_{i+1}, \ldots, \lambda_d)}{\lambda} = \frac{\hat{F}(x_1, \ldots, x_i)}{\hat{F}(\lambda_1, \ldots, \lambda_i)},$$
  where $\hat{F}(x_1, \ldots, x_i)$ is a similar function as F defined on $[0, \lambda_1] \times \ldots \times [0, \lambda_i] \to [0, \hat{F}(\lambda_1, \ldots, \lambda_i)]$. And $\tilde{F}(x_i) = x_i$ for a similar function $\tilde{F} : [0, \lambda_i] \to [0, \lambda_i]$.
- *(Monotonicity) The same as definition 2.3.2.*
- *$F(x_1, \ldots, x_d) = \lambda C(\frac{x_1}{\lambda_1}, \ldots, \frac{x_d}{\lambda_d})$, where C is a probability copula if x_i are lower tail integrals and C is a survival copula if x_i are upper tail integrals.*

Tankov [55] concerned the dependence of Lévy processes with only positive jumps (subordinators) because of the non-integrability at 0. In his later book with Cont [12], they redefine the Lévy copula for a general case that involves positive and negative jumps. See Definition 5.13, 5.14, 5.15 and Theorem 5.7. The original version of S-copula (Definition 3.1 in Tankov [55]) associated with subordinators is following.

Definition 2.3.4 ***(Lévy copula of subordinator with infinite activity)***
A d-dimensional ($d \geq 2$) S-copula is a function $F : [0, \infty]^d \to [0, \infty]$ satisfying:

- *(Boundary condition) $F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) = 0$ for all i;*
  $F(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty) = F_i(x_i) = x_i$ for all i, which is called the margins.
- *(Monotonicity) The same as definition 2.3.2.*

Tankov [55] also proved a theorem, similar as Sklar, about the relationship of the Lévy measure and S-copula.

Theorem 2.3.4 *Let U be the tail integral of a d-dimensional Lévy process with positive components having tail integrals $U_1, \dots, U_d$. $U(x_1, \dots, x_d) = \nu([x_1, \infty], \dots [x_d, \infty])$ and $U_i(x) = \nu_i([x, \infty])$ for all $i = 1, \dots, d$. Then there exists a d-dimensional S-copula F such that for all vectors $(x_1, \dots, x_d)$ in $\mathcal{R}^d_+$,*

$$U(x_1, \dots, x_d) = F(U_1(x_1), \dots, U_d(x_d)).$$

If $U_1, \dots, U_d$ are continuous then this S-copula is unique, otherwise it is unique on $RanU_1 \times \dots \times RanU_d$. Conversely, if F is a d-dimensional S-copula and $U_1, \dots, U_d$ are tail integrals of one-dimensional subordinators, then the function U defined above is the tail integral of a d-dimensional Lévy process with positive components having tail integrals $U_1, \dots, U_d$.

If two subordinators are independent, then for $x_1, x_2 > 0$

$$\nu([x_1, \infty], \{0\}) = \nu_1([x_1, \infty]), \quad \nu(\{0\}, [x_2, \infty]) = \nu_2([x_2, \infty]);$$
$$\nu([x_1, \infty], [x_2, \infty]) = 0.$$

Thus $\nu([x_1, \infty], [0, \infty]) = \nu_1([x_1, \infty])$ and then the independent copula is $F(u, v) = uI_{\{v=\infty\}} + vI_{\{u=\infty\}}$.

Since the inverse of the Lévy measure is not unique and we define $\nu^{-1}(u) = \sup\{x : \nu(x) = u\}$. Roughly, when n is large enough

$$u = \nu(x) = nf(x; \frac{1}{n}) \Longrightarrow \nu^{-1}(u) = f^{-1}(\frac{u}{n}; \frac{1}{n}) = x.$$

For a positive stable process the limitation approach yields the following formula:

$$\begin{aligned} F(u_1, \dots, u_d) &= \int_{[\nu_d^{-1}(u_d), \infty] \times \dots \times [\nu_1^{-1}(u_1), \infty]} u(x_1, \dots, x_d) dx_1 \dots dx_d \\ &= \lim_{n \to \infty} n \int_{[f_d^{-1}(\frac{u_d}{n}; \frac{1}{n}), \infty] \times \dots \times [f_1^{-1}(\frac{u_1}{n}; \frac{1}{n}), \infty]} f(x_1, \dots, x_d; \frac{1}{n}) dx_1 \dots dx_d \\ &= \lim_{n \to \infty} nC^S_{\frac{1}{n}}(\frac{u_1}{n}, \dots, \frac{u_d}{n}) = \lim_{n \to \infty} nC^S(\frac{u_1}{n}, \dots, \frac{u_d}{n}). \end{aligned}$$

C^S is the survival copula of the positive stable process and it does not depend on time which ensures the last equality. F is the S-copula of the positive stable process. Tankov

[55] also gave this formula in Proposition 4.4. Since only subordinator is considered here, the survival copula C^S only represents positive dependence.

Here are some examples. First the perfect positive dependent S-copula, where the probability copula is $C(u_1, \ldots, u_m) = \min(u_1, \ldots, u_m)$. And the S-copula is

$$F(u_1, \ldots, u_m) = \lim_{n\to\infty} n \min(\frac{u_1}{n}, \ldots, \frac{u_m}{n}) = \min(u_1, \ldots, u_m),$$

which has the same form as the probability copula. Another example is to use Cook-Johnson copula as the survival copula

$$C^S(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta},$$

and the associate S-copula is obtained by limit

$$F(u, v) = \lim_{n\to\infty} n \left(\left(\frac{u}{n}\right)^{-\theta} + \left(\frac{v}{u}\right)^{-\theta} - 1 \right)^{-1/\theta} = (u^{-\theta} + v^{-\theta})^{-1/\theta}.$$

The last one is to use the Cuadras-Augé copula as the probability copula, then the corresponding survival copula is

$$C^S(u, v) = (1 - u)(1 - v) \min((1 - u)^{-\alpha}, (1 - v)^{-\beta}) + u + v - 1,$$

where $0 \leq \alpha, \beta \leq 1$. If u and v are bounded, the associate S-copula is

$$F(u, v) = \begin{cases} \alpha u & \text{if } (1 - \frac{u}{n})^{\alpha} \geq (1 - \frac{v}{n})^{\beta}, \text{ i.e. } \alpha u \leq \beta v \\ \beta v & \text{otherwise} \end{cases} = \min(\alpha u, \beta v),$$

which is the compound Poisson case where the jump sizes for the common jumps are perfectly positively dependent. And $\alpha = \lambda/\lambda_1$, $\beta = \lambda/\lambda_2$, λ_1, λ_2 are intensities of two dependent Poisson processes with common intensity λ and $0 \leq u \leq \lambda_1, 0 \leq v \leq \lambda_2$.

2.3.5 Time copula of Lévy processes

The probability copula described above considers the dependence at time t for all processes. Given a Markov process X_t, its joint distribution of different times is

$$F_{s,t}(x_1, x_2) = P(X_s < x_1, X_t < x_2) =: C_{s,t}(P(X_s < x_1), P(X_t < x_2)),$$

$s < t$. $C_{s,t}$ is called the *time copula* for the process X_t. It was introduced by Darsow, Nguyen and Olsen [15] to illustrate the conditions of Markov process. (See Theorem 3.2, 3.3.) Këllezi and Webber [32] found the expression for the time copula of Brownian motion and subordinated Brownian motion.

First, let's see the time copula of one-dimensional Brownian motion W_t. (See section 2.1 in Këllezi and Webber [32].) The joint distribution of W_s and W_t, $s < t$ is

$$\begin{aligned} P(W_s < a_1, W_t < a_2) &= P(W_s < a_1, W_t - W_s < a_2 - W_s) \\ &= \int_{-\infty}^{a_1} \Phi\left(\frac{a_2 - x}{\sqrt{t-s}}\right) d\Phi\left(\frac{x}{\sqrt{s}}\right) \\ &= \int_0^{\Phi(a_1/\sqrt{s})} \Phi\left(\frac{a_2 - \sqrt{s}\Phi^{-1}(y_s)}{\sqrt{t-s}}\right) dy_s, \\ &=: C_{s,t}^N\left(\Phi\left(\frac{a_1}{\sqrt{s}}\right), \Phi\left(\frac{a_2}{\sqrt{t}}\right)\right), \end{aligned}$$

$C_{s,t}^N$ is the time copula of Brownian motion, which has the following expression,

$$C_{s,t}^N(x_1, x_2) = \int_0^{x_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}(x_2) - \sqrt{s}\Phi^{-1}(y_s)}{\sqrt{t-s}}\right) dy_s.$$

Consider the Gaussian Process $X_t = W_{T_t}$. When T_t is deterministic function, the time copula is

$$C_{s,t}(x_1, x_2) = \int_0^{x_1} \Phi\left(\frac{\sqrt{T_t}\Phi^{-1}(x_2) - \sqrt{T_s}\Phi^{-1}(y_s)}{\sqrt{T_t - T_s}}\right) dy_s.$$

When T_t is a stochastic subordinator, we follow a similar way as Equation (2.15). (Also see section 2.2.3 in Këllezi and Webber [32].) Let H_t be the probability function of T_t, let $\widehat{H}_{s,t}$ be the joint probability function of T_s and T_t, $s < t$, then

$$\begin{aligned} F_t(a) = P(X_t \le a) &= \int_0^\infty \Phi\left(\frac{a}{\sqrt{z}}\right) H_t(dz), \\ P(X_s \le a_1, X_t \le a_2) &= \int_0^\infty \int_0^\infty P(W_u \le a_1, W_v \le a_2) \widehat{H}_{s,t}(du, dv) \\ &= \int_0^\infty \int_0^\infty C_{s,t}^N\left(\Phi\left(\frac{a_1}{\sqrt{u}}\right), \Phi\left(\frac{a_2}{\sqrt{v}}\right)\right) \widehat{H}_{s,t}(du, dv), \\ C_{s,t}(x_1, x_2) &= \int_0^\infty \int_0^\infty C_{s,t}^N\left(\Phi\left(\frac{F_s^{-1}(x_1)}{\sqrt{u}}\right), \Phi\left(\frac{F_t^{-1}(x_2)}{\sqrt{v}}\right)\right) \widehat{H}_{s,t}(du, dv) \\ &= \int_0^\infty \int_0^\infty C_{s,t}^N(\hat{x}_1, \hat{x}_2) \widehat{H}_{s,t}(du, dv), \end{aligned}$$

where $C_{s,t}$ is the time copula of X and

$$\begin{aligned} x_1 &= F_s(\sqrt{u}\,\Phi^{-1}(\hat{x}_1)) = \int_0^\infty \Phi\left(\sqrt{\frac{u}{z}}\Phi^{-1}(\hat{x}_1)\right) H_s(dz) \\ x_2 &= F_t(\sqrt{v}\,\Phi^{-1}(\hat{x}_2)) = \int_0^\infty \Phi\left(\sqrt{\frac{v}{z}}\Phi^{-1}(\hat{x}_2)\right) H_t(dz). \end{aligned}$$

2.4 Additive processes

Additive processes are stochastic processes satisfying all conditions except condition 2 in Definition 1.2.1. That is, we relax the assumption of time homogeneity in the law. Their properties are given in Theorem 14.1 in Cont & Tankov [12].

Theorem 2.4.1 ***(Sato)*** *Let $(X_t)_{t\geq 0}$ be an additive process on $\mathcal{R}^d$. Then X_t has infinitely divisible distribution for all t. The law of $(X_t)_{t\geq 0}$ is uniquely determined by its spot characteristics $(A_t, V_t, \Gamma_t)_{t\geq 0}$:*

$$\begin{aligned}\Psi(z,t) &= E\left[e^{i\langle z,X_t\rangle}\right]\\ &= \exp\left\{-\frac{1}{2}\langle z,A_t z\rangle + \int_{\mathcal{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle I_{\{\|x\|\leq 1\}}(x)\right)V_t(dx) + i\langle z,\Gamma_t\rangle\right\}.\end{aligned}$$

The spot characteristics $(A_t, V_t, \Gamma_t)_{t\geq 0}$ satisfy the following conditions

- *For all t, A_t is a nonnegative definite $d\times d$ matrix, V_t is a positive measure on $\mathcal{R}^d$ with $V_t(\{0\}) = 0$ and $\int_{\mathcal{R}^d}(\|x\|^2\wedge 1)V_t(dx) < \infty$.*
- *Positiveness: $A_0 = 0$, $V_0 = 0$, $\Gamma_0 = 0$ and for all s, t such that $s \leq t$, $A_t - A_s$ is a nonnegative-definite matrix and $V_t(B) - V_s(B) \geq 0$ for all measurable sets $B \in \mathcal{B}(\mathcal{R}^d)$.*
- *Continuity: if $s \to t$ then $A_s \to A_t$, $\Gamma_s \to \Gamma_t$ and $V_s(B) \to V_t(B)$ for all $B \in \mathcal{B}(\mathcal{R}^d)$ such that $B \subset \{x : \|x\| \geq \epsilon\}$ for some $\epsilon > 0$.*

Conversely, for family of triplets $(A_t, V_t, \Gamma_t)_{t\geq 0}$ satisfying the three conditions above, there exists an additive process $(X_t)_{t\geq 0}$ with the spot characteristics $(A_t, V_t, \Gamma_t)_{t\geq 0}$.

A triplet $(a_t, \nu_t, \gamma_t)_{t\in[0,T]}, T < \infty$ is called the *local characteristics* of the additive process $(X)_{t\in[0,T]}$ if $A_t = \int_0^t a_s ds$, $V_t(B) = \int_0^t \nu_s(B)ds$ and $\Gamma_t = \int_0^t \gamma_s ds$ for each $t \in [0,T]$. a_t is a nonnegative definite $d\times d$ matrix, a family $(\nu_t)_{0\leq t\leq T}$ of Lévy measure satisfies $\int_0^T\int_{\mathcal{R}^d}(1\wedge\|x\|^2)\nu_t(dx)dt < \infty$ and $\gamma_t \in \mathcal{R}^d$ is a deterministic function. (See Section 14.1 in Cont & Tankov [12].) An additive process admits the local characteristics if all the elements in the spot characteristics are absolutely continuous in t. Similar to equation (1.6), an additive process with local characteristics can be decomposed as

$$X_t = \int_0^t \sigma_s dW_s + \int_{\|x\|\leq 1}\int_0^t x\tilde{N}(ds,dx) + \int_{\|x\|>1}\int_0^t xN(ds,dx) + \int_0^t \gamma_s ds,$$

for some m-dimensional standard Brownian motion W_t and $d \times m$ ($m \geq d$) matrix σ_s with $a_s = \sigma_s \sigma_s^T$, and $\tilde{N}(ds, dx) = N(ds, dx) - \nu_s(dx)ds$ is the compensated Poisson random measure of the additive process X.

Additive processes can be easily constructed by Lévy processes with deterministic volatilities. For example, let $\sigma(t) : [0, \infty) \to \mathcal{R}$ be a measurable function such that $\int_0^t \sigma_s^2 ds < \infty$ for all $t > 0$ and then $X_t = \int_0^t \sigma_s dW_s$ is an additive process with spot characteristics $(\int_0^t \sigma_s^2 ds, 0, 0)_{t \geq 0}$. Let L be a purely non-Gaussian Lévy process in $\mathcal{R}$ with Lévy triplet $(0, \nu, \gamma)$, and $X_t = \int_0^t \sigma_s dW_s + \int_0^t \theta_s dL_s$ with θ_t deterministic and bounded. Then the new process X is an additive process with spot characteristics $(\int_0^t \sigma_s^2 ds, \int_0^t \nu_s^X ds, \int_0^t \gamma_s^X ds)_{t \geq 0}$, where

$$\nu_t^X(B) = \begin{cases} \nu(\hat{B}) & \text{if } \theta_t \neq 0 \\ 0 & \text{if } \theta_t = 0 \end{cases}, \quad \hat{B} = \{x : \theta_t x \in B\}$$

with

$$\int_0^t \int_{\mathcal{R}} (|x|^2 \wedge 1) \nu_s^X(dx) ds = \int_0^t \int_{\mathcal{R}} (|\theta_s x|^2 \wedge 1) \nu(dx) ds < \infty,$$

and γ_t^X satisfies

$$\int_{\mathcal{R}} (e^{iz\theta_t x} - 1 - iz\theta_t x I_{\{|x| \leq 1\}}) \nu(dx) + iz\theta_t \gamma = \int_{\mathcal{R}} (e^{izy} - 1 - izy I_{\{|y| \leq 1\}}) \nu_t^X(dy) + iz\gamma_t^X,$$

which is obtained from the characteristic function of X

$$\Psi(z, t) = \exp\left\{ \int_0^t (-\frac{1}{2}\theta_s^2 + \int_{\mathcal{R}} (e^{iz\theta_s x} - 1 - iz\theta_s x I_{\{|x| \leq 1\}}) \nu(dx) + iz\theta_s \gamma) ds \right\}.$$

The triplet $(\sigma_t^2, \nu_t^X, \gamma_t^X)$ is called the local characteristics of the additive process X.

Let's go back to the equation (2.1), which constructs a homogeneous Poisson process. A similar construction can be made with an additive process X as $N_t^\Lambda = \sum_{0 < s \leq t} I_\Lambda(\triangle X_s)$. The set function $\Lambda \to N_t^\Lambda(\omega)$ defines a σ-finite measure on $\mathcal{R} \setminus \{0\}$ for each fixed (t, ω). The intensity (instantaneous jump probability) of N_t^Λ exists if X admits local characteristics (a_t, ν_t, γ_t), and it is equal to $\nu_t(\Lambda)$. If θ_t is deterministic in the example above, N_t^Λ is an inhomogeneous poisson process with deterministic intensity. If θ_s is random, then N_t^Λ is a Cox process with stochastic intensity.

3

Equivalent martingale measures for Lévy processes

3.1 Main theorem

Consider a financial market with d risky assets traded up to a horizon $T < \infty$ and one riskless asset $S_t^0 = e^{rt}$ with constant interest rate r. In the physical world, the prices of the risky assets $(S^1, \ldots, S^d)$ are given by a multi-variate exp-Lévy process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and define $S = (S^0, S^1, \ldots, S^d)$.

For a predictable S-integrable process $\theta = (\theta^0, \theta^1, \ldots, \theta^d)$, the value process of the self-financing trading strategy θ with initial capital 1 is given by $V_t = 1 + \int_0^t \theta_u^T dS_u$. If the value process $\{V_t\}_{t\geq 0}$ is strictly positive, it is called a *numeraire asset.* Let $\mathcal{N}$ denote the set of all numeraire assets. Any asset process S (without dividends) discounted by a numeraire $V \in \mathcal{N}$ should be a (local) martingale under some measure Q which is equivalent to the physical probability measure P. And such measure Q associated with V is called the *Equivalent (Local) Martingale Measure (EMM)* for the process S. Generally, given a pair of the EMM and numeraire (P^1, N^1) and another numeraire N^2, an EMM P^2 associated with N^2 can be defined by the Radon-Nikodym density process: $dP_t^2/dP_t^1 = N_t^2/N_t^1$, $t \in [0, T]$. Becherer [5] provides two dual approaches for EMMs:

1. Fix $V \in \mathcal{N}$ and choose $Q \sim P$ such that S/V become (local) Q-martingales.
2. Fix $Q \sim P$ and choose $V \in \mathcal{N}$ such that S/V become (local) Q-martingales.

The so-called risk-neutral measure is the EMM with $V = S^0$, the riskless asset, which is the most used EMM in pricing. It belongs to the first approach. The risk-neutral measure is unique when the market is complete, otherwise the choice of the optimal martingale should be made according to different criteria. Such optimal EMMs include the minimal martingale measure ([1], [22], [54]) and the minimal entropy martingale measure ([21], [23]). In the second approach, a certain probability measure Q is fixed first. The physical measure P is a natural choice and the corresponding portfolio process of the numeraire V is called the *numeraire portfolio* ([5], [33], [37]).

Long [37] proved the existence of a numeraire portfolio in the case of finite Ω and discrete time and in the case where the asset prices follow a regular multi-dimensional diffusion model. Korn, Oertel and Schäl [33] studied it for a multi-dimensional Poisson-diffusion process with univariate Poisson process. In these cases, the numeraire portfolio turns out to be growth-optimal. The minimal martingale measure for (continuous) semimartingales was introduced by Föllmer and Schweizer [22] and it is in general only a signed measure for discontinuous processes (see Schweizer [54]). Arai [1] discussed the case when the asset prices follow jump-diffusion processes. Fujiwara and Miyahara [23] investigated the minimal entropy martingale measures for the geometric Lévy processes.

In this section, we focus on the change of measure, specifically, the EMM for the Lévy/additive processes. The three martingale measures: the numeraire portfolio, the minimal martingale measure and the minimal entropy martingale measure are used as examples to illustrate the key ideas of measure transformation. The results for the numeraire portfolio are new.

Sato [50] shows the conditions of equivalent Lévy measures in Theorem 7.2 and Theorem 7.4, also see Proposition 9.8 in Cont & Tankov [12] for the one-dimension case. We modify it as following:

Theorem 3.1.1 ***(Equivalent measures for Lévy processes)*** *Let (X_t, P^1) and (X_t, P^2) be two Lévy processes on $\mathcal{R}^d$ with generating triplets (A_1, ν_1, b_1) and (A_2, ν_2, b_2). Then P_t^1 and P_t^2 are equivalent for all $t > 0$ if and only if the following three conditions are satisfied:*

1. *$A_1 = A_2$, denote $A := A_1 = A_2$;*
2. *The Lévy measures are equivalent with $\int_{\mathcal{R}^d}(\sqrt{\nu_1} - \sqrt{\nu_2})^2(dx) < \infty$;*
3. *$b_2 - b_1 - \int_{\|x\|\leq 1} x(\nu_2 - \nu_1)(dx) = A\eta$ for some $\eta \in \mathcal{R}^d$.*

Under the P^1 measure, let W_t^1 be a d-dimensional standard Brownian motion and σ be the unique Cholesky decomposition of A, then X has a decomposition as in (1.6)

$$X_t = b_1 t + \sigma W_t^1 + \int_{\|x\|\leq 1}\int_0^t x(N(ds,dx) - \nu_1(dx)ds) + \int_{\|x\|>1}\int_0^t xN(ds,dx).$$

Under the P^2 measure, the Lévy decomposition with the same dimension of the Brownian motion is

$$X_t = b_2 t + \sigma W_t^2 + \int_{\|x\|\leq 1}\int_0^t x(N(ds,dx) - \nu_2(dx)ds) + \int_{\|x\|>1}\int_0^t xN(ds,dx),$$

then $W_t^2 = W_t^1 - \sigma^T\eta t$ is the standard Brownian motion under P^2. Denote the function $\phi(x) = \ln[\nu_2(dx)/\nu_1(dx)] : \mathcal{R}^d \to \mathcal{R}$, when P^1 and P^2 are equivalent, the Radon-Nikodym density process is

$$Z_t = \frac{dP^2|\mathcal{F}_t}{dP^1|\mathcal{F}_t} = \frac{dP_t^2}{dP_t^1} = e^{U_t} \quad \textit{with}$$

$$U_t = \eta^T\sigma W_t^1 - \frac{t}{2}\|\sigma^T\eta\|^2 + \int_{\mathcal{R}^d}\int_0^t \phi(x)N(ds,dx) - t\int_{\mathcal{R}^d}(e^{\phi(x)} - 1)\nu_1(dx).$$

Then U_t is a Lévy process on $\mathcal{R}$ with generating triplet (A_U, ν_U, b_U) given by

$$\begin{aligned}
A_U &= \|\sigma^T\eta\|^2, \\
\nu_U(B) &= \int_{\mathcal{R}^d} I_{\{\phi(x)\in B\}}\nu_1(dx) \quad \textit{for } B \in \mathcal{B}(\mathcal{R}), \\
b_U &= -\frac{1}{2}\|\sigma^T\eta\|^2 - \int_{\mathcal{R}^d}(e^{\phi(x)} - 1 - \phi(x)I_{\{\|\phi(x)\|\leq 1\}})\nu_1(dx) \\
&= -\frac{1}{2}A_U - \int_{\mathcal{R}}(e^y - 1 - yI_{\{|y|\leq 1\}})\nu_U(dy).
\end{aligned}$$

And the density process $Z_t = e^{U_t}$ satisfies the stochastic differential equation

$$dZ_t/Z_{t-} = \eta^T\sigma dW_t^1 + \int_{\mathcal{R}^d}(e^{\phi(x)} - 1)[N(dt,dx) - \nu_1(dx)ds],$$

which is a positive P^1-martingale with initial value 1.

Given a Lévy processes (X_t, P^1) with generating triplet (A, ν_1, b_1), one may define an equivalent probability measure P^2 by defining the function $\phi(\cdot)$ and the parameter η such that condition 2 and 3 in Theorem 3.1.1 are satisfied. A simple example is the *Esscher transform*, where $dP_t^2/dP_t^1 = e^{\langle\theta, X_t\rangle}/E[e^{\langle\theta, X_t\rangle}]$. In this case, $\phi(x) = \langle\theta, x\rangle$ for some $\theta \in \mathcal{R}^d$ with condition $\int_{\|x\|\geq 1} e^{\langle\theta, x\rangle}\nu_1(dx) = \int_{\|x\|\geq 1} \nu_2(dx) < \infty$. And $b_2 = b_1 + \int_{\|x\|\leq 1} x(e^{\langle\theta, x\rangle} - 1)\nu_1(dx) + A\eta$, where $\eta = \theta$. Thus the generating triplet of X under P^2 is $(A, e^{\langle\theta, x\rangle}\nu_1(dx), b_2)$. The function $\phi(\cdot)$ is the key to the construction of the equivalent measures P^2, which transforms the Lévy measure. The parameter η is used to adjust the drift (or γ in its generating triplet) of the Lévy process, which also transforms the Brownian motion. The covariance matrix A remains unchanged after the measure transformation.

If we want P^2 to be a martingale measure, that is X_t is a martingale under P^2, we must have $\int_{\|x\|>1} \|x\|\nu_2(dx) < \infty$ and $E^2[X_t] = 0$ yields $b_2 = -\int_{\|x\|>1} x\nu_2(dx)$. Thus η will solve the equation (see condition 3 in Theorem 3.1.1)

$$b_1 + A\eta + \int_{\|x\|\leq 1} x(e^{\phi(x)} - 1)\nu_1(dx) + \int_{\|x\|>1} xe^{\phi(x)}\nu_1(dx) = 0. \tag{3.1}$$

3.2 Equivalent martingale measures for exp-Lévy processes

Assume that under the physical measure P, the asset prices satisfy:

$$dS_t^i = S_{t-}^i(rdt + dX_t^i), \quad S_0^i > 0,\, 0 < t \leq T < \infty,\, i = 1, \ldots, d. \tag{3.2}$$

Then the discounted asset processes $\tilde{S}_t^i = S_t^i/S_t^0$ satisfy: $d\tilde{S}_t^i = \tilde{S}_{t-}^i dX_t^i$. Here $X = (X^1, \ldots, X^d)^T$ is a Lévy process with generating triplet (A, ν, γ). By assuming that the assets have finite expectation under the physical measure and the risk-neutral measure, that is γ_1, the center of the Lévy process X, exists under both measures, we will **use the centering triplet (σ, ν, γ_1) instead of its generating triplet** because it is more informative.

Theorem 1.2.2 shows that under measure P, X has a decompensation

$$X_t = \gamma_1 t + \sigma W_t + \int_{\mathcal{R}^d}\int_0^t x[N(ds, dx) - \nu(dx)ds]. \tag{3.3}$$

Under the risk-neutral measure Q, the discounted asset process $\tilde{S}$ is a martingale. $\tilde{S}$ is a stochastic exponent of X, thus X is also a Q-martingale and the risk-neutral measure Q for the asset S is the EMM for the process X. Let $W_t^Q = W_t - \sigma^T \eta t$ be the standard Brownian motion and $\nu^Q(\cdot) = e^{\phi(x)}\nu(dx)$ be the Lévy measure under Q satisfying condition 2 in Theorem 3.1.1 and $\int_{\|x\|>1} \|x\| \nu^Q(dx) < \infty$, then the Q-martingale X will have the **centering triplet** $(\sigma, \nu^Q, 0)$ with the decompensation $X_t = \sigma W_t^Q + \int_{\mathcal{R}^d} \int_0^t x[N(ds, dx) - \nu^Q(dx)ds]$. Compared to equation (3.3), we obtain

$$\gamma_1 + \sigma\sigma^T \eta + \int_{\mathcal{R}^d} x(e^{\phi(x)} - 1)\nu(dx) = 0, \tag{3.4}$$

which is the same as equation (3.1). Here $\sigma^T \eta$ represents the market price of the diffusion risk while $e^{\phi(x)} - 1$ is the risk premium associated with the jump risk.

For each $i = 1, \dots, d$, the solution to (3.2) is

$$S_t^i = S_0^i \exp\left\{rt + X_t - \frac{1}{2}\|\sigma^i\|^2 t\right\} \prod_{0<s\le t} (1 + \triangle X_s^i) e^{-\triangle X_s^i},$$

where σ^i is the i^{th} row of the matrix σ. To keep the asset processes positive, we must assume that for each i the jump sizes of X^i are always greater than -1.

If $\int_{\|x\|\le 1} \|x\| \nu(dx) < \infty$, the drift of X exists as $\gamma_0 = \gamma_1 - \int_{\mathcal{R}^d} x\nu(dx)$. Thus

$$S_t^i = S_0^i \exp\left\{rt + \gamma_0 t + \sigma W_t - \frac{1}{2}\|\sigma^i\|^2 t\right\} \prod_{0<s\le t} (1 + \triangle X_s^i),$$

and equation (3.4) yields (recall that $\gamma_1 = \gamma_0 + \int_{\mathcal{R}^d} x\nu(dx)$)

$$\gamma_0 + \sigma\sigma^T \eta + \int_{\mathcal{R}^d} x e^{\phi(x)} \nu(dx) = 0. \tag{3.5}$$

3.2.1 Numeraire portfolio

The value process $V_t = 1 + \int_0^t \theta_\mu^T dS_\mu$ defined at the beginning of the section satisfies

$$dV_t = \theta_t^T dS_t = \sum_{i=0}^{d} \theta_t^i dS_t^i = V_{t-} \sum_{i=0}^{d} \pi_t^i (dS_t^i / S_{t-}^i) = V_{t-}(rdt + \sum_{i=1}^{d} \pi_t^i dX_t^i),$$

where $\theta_t = (\theta_t^0, \dots, \theta_t^d)^T$ is called the *trading strategy* and $\pi_t = (\pi_t^0, \pi_t^1, \dots, \pi_t^d)^T$, $\pi_t^i = \theta_t^i S_{t-}^i / V_{t-}$ is called the *portfolio process*. Since $\pi_t^0 = 1 - \sum_{i=1}^d \pi_t^i$ for each t, we denote $\pi_t = (\pi_t^1, \dots, \pi_t^d)^T$ from now on. Long [37] defined the numeraire portfolio as:

Definition 3.2.1 *(Numeraire portfolio) A numeraire portfolio π_t is a self-financing portfolio such that its value process $\{V_t\}_{t\geq 0}$ is strictly positive and the physical probability measure P is an EMM with respect to the numeraire V_t. That is, S_t^i/V_t are positive P-martingales for $i = 0, \dots, d$.*

Just as in Korn, Oertel and Schäl[33], we may focus on the case where the portfolio process is described by a constant vector $\pi \in \mathcal{R}^d$.

Theorem 3.2.1 *(Numeraire portfolio for exp-Lévy processes)*
Suppose that the asset process is modeled by a d-dimensional exp-Lévy process as (3.2), the numeraire portfolio $\pi \in \mathcal{R}^d$ exists if

1. $1 + \langle \pi, x\rangle > 0$ *and* $\vec{1} + x > 0$ *for any x, the possible jump size of X;*
2. $\int_{\mathcal{R}^d} \left(1 - \frac{1}{\sqrt{1+\langle\pi, x\rangle}}\right)^2 \nu(dx) < \infty$;
3. $\int_{\mathcal{R}^d} \frac{\|x\|^2}{1+\langle\pi, x\rangle} \nu(dx) < \infty$.

π is the solution to the following equation

$$\gamma_1 - \sigma\sigma^T\pi - \int_{\mathcal{R}^d} \frac{x\langle\pi,\, x\rangle}{1 + \langle\pi,\, x\rangle}\nu(dx) = 0. \tag{3.6}$$

The corresponding numeraire V defines a risk-neutral measure Q^* with Radon-Nikodym derivative $dQ^*/dP = S_T^0/V_T$. The density process $Z_t = e^{rt}/V_t$ is

$$\exp\left\{-\langle\pi, \sigma W_t\rangle - \frac{1}{2}\|\sigma^T\pi\|^2 t - \int_{\mathcal{R}^d}\int_0^t \ln(1 + \langle\pi, x\rangle) N(ds, dx) + t\int_{\mathcal{R}^d} \frac{\langle\pi, x\rangle}{1 + \langle\pi, x\rangle}\nu(dx)\right\}.$$

Moreover, the Lévy process X is a martingale under Q^* with centering triplet $(\sigma, \nu^*, 0)$, $\nu^*(dx) = \nu(dx)/(1 + \langle\pi, x\rangle)$. $W_t^* = W_t + \sigma^T\pi t$ is the standard Brownian motion.

Proof: Suppose that the numeraire portfolio π exists, then S_t/V_t must be positive martingales under P. Denote π as the constant portfolio process,

$$\begin{aligned}
V_t &= \exp\left\{rt - \frac{1}{2}\|\sigma^T\pi\|^2 t + \langle\pi, X_t\rangle\right\} \prod_{0<s\leq t} (1 + \langle\pi, \Delta X_s\rangle)e^{-\langle\pi, \Delta X_s\rangle}, \\
S_t^i/V_t &= S_0^i \exp\left\{\frac{1}{2}\|\sigma^T\pi\|^2 t - \frac{1}{2}\|\sigma^i\|^2 t - \langle\pi, X_t\rangle + X_t^i\right\} \times \\
&\prod_{0<s\leq t} \frac{1 + \Delta X_s^i}{1 + \langle\pi, \Delta X_s\rangle} \exp\left\{\langle\pi,\, \Delta X_s\rangle - \Delta X_s^i\right\}, \quad i = 1, \dots, d.
\end{aligned}$$

Condition 1 guarantees that S and V are positive. Condition 2 is equivalent to condition 2 in Theorem 3.1.1 as

$$\int_{\mathcal{R}^d} (\sqrt{\nu} - \sqrt{\nu^*})^2(dx) = \int_{\mathcal{R}^d} \left(1 - \frac{1}{\sqrt{1 + \langle \pi, x \rangle}}\right)^2 \nu(dx) < \infty.$$

Condition 3 guarantees the existence of the solution to (3.6) and X_t is a Q^*-martingale since $\int_{\|x\|>1} \|x\| \nu^*(dx) < \infty$. For $0 \le u < t$,

$$\begin{aligned}
&\frac{V_u}{S_u^0} E\left[\frac{S_t^0}{V_t} | \mathcal{F}_u\right] \\
= \; & E_u\left[\exp\left\{\frac{1}{2}\|\sigma^T\pi\|^2(t-u) - \langle \pi, X_t - X_u\rangle - \sum_{u<s\le t} [\ln(1 + \langle \pi, \triangle X_s\rangle) - \langle \pi, \triangle X_s\rangle]\right\}\right] \\
= \; & E_u\left[\exp\left\{-\int_{\mathcal{R}^d}\int_u^t \ln(1+\langle \pi, x\rangle) N(ds, dx) + (t-u)\int_{\mathcal{R}^d} \frac{\langle \pi, x\rangle}{1+\langle \pi, x\rangle}\nu(dx)\right\}\right] \times \\
& \exp\left\{(t-u)\left[\|\sigma^T\pi\|^2 - \langle \pi, \mu\rangle + \int_{\mathcal{R}^d} \langle \pi, x\rangle \nu(dx) - \int_{\mathcal{R}^d} \frac{\langle \pi, x\rangle}{1+\langle \pi, x\rangle}\nu(dx)\right]\right\}.
\end{aligned}$$

If π satisfies equation (3.6), the expectation is equal to S_u^0/V_u and S_t^0/V_t is

$$\exp\left\{-\langle \pi, \sigma W_t\rangle - \frac{1}{2}\|\sigma^T\pi\|^2 t - \int_{\mathcal{R}^d}\int_0^t \ln(1+\langle \pi, x\rangle) N(ds, dx) + t\int_{\mathcal{R}^d} \frac{\langle \pi, x\rangle}{1+\langle \pi, x\rangle}\nu(dx)\right\}. \tag{3.7}$$

Similarly, for each $i = 1, \ldots, d$, $E[S_t^i/V_t|\mathcal{F}_u]$ is

$$\begin{aligned}
&\frac{S_u^i}{V_u} \exp\left\{(t-u)\left[\frac{1}{2}(\|\sigma^i - \sigma^T\pi\|^2 + \|\sigma^T\pi\|^2 - \|\sigma^i\|^2) - \langle \pi, \mu\rangle + \mu^i\right]\right\} \times \\
&\exp\left\{(t-u)\int_{\mathcal{R}^d} (\langle \pi, x\rangle - x^i)\nu(dx) - (t-u)\int_{\mathcal{R}^d} \frac{\langle \pi, x\rangle - x^i}{1+\langle \pi, x\rangle}\nu(dx)\right\},
\end{aligned}$$

which is S_u^i/V_u based on equation (3.6).

So S/V are positive P-martingales. So S_T^0/V_T is the Radon-Nikodym derivative of the risk-neutral measure Q^* to the physical probability measure P. By comparing equation (**??**) to the density process in Theorem 3.1.1, we find that $\eta = -\pi$ and $\phi(x) = -\ln(1 + \langle \pi, x\rangle)$, that is $\nu^*(dx) = \nu(dx)/(1 + \langle \pi, x\rangle)$.

□

If π_t is not set to be constant at first, equation (3.6) turns to be

$$\gamma_1 - \sigma\sigma^T\pi_t - \int_{\mathcal{R}^d} \frac{x\langle \pi_t, x\rangle}{1+\langle \pi_t, x\rangle}\nu(dx) = 0 \quad \text{for each } t \in [0, T].$$

The solution π_t will not depend on the time t. So the restriction that the portfolio process is constant will not affect the results.

It is well known that the numeraire portfolio coincides to the growth optimal portfolio $\tilde{\pi}$ that maximizes the expectation of the logarithm:

$$E\left[\ln \tilde{V}_T\right] = \sup_{V\in\mathcal{N}} E\left[\ln V_T\right],$$

where $d\tilde{V}_t/\tilde{V}_{t-} = rdt + \sum_{i=1}^{d} \tilde{\pi} dX_t^i$. Under the conditions listed in Theorem 3.2.1,

$$\ln V_T = \langle \pi, X_T\rangle - \frac{1}{2}\|\sigma^T\pi\|^2 T + \sum_{0<s\leq T} \left[\ln(1+\langle \pi, \triangle X_s\rangle) - \langle \pi, \triangle X_s\rangle\right],$$

with the expectation under the physical measure P:

$$E[\ln V_T] = T\left\{\langle \pi, \gamma_1\rangle - \frac{1}{2}\|\sigma^T\pi\|^2 + \int_{\mathcal{R}^d} \left[\ln(1+\langle \pi, x\rangle) - \langle \pi, x\rangle\right]\nu(dx)\right\}.$$

To achieve its maximum, let us take the derivative with respect to the portfolio process π and equalize it to be 0:

$$\gamma_1 - \sigma\sigma^T\pi - \int_{\mathcal{R}^d} \frac{x\langle \pi,\, x\rangle}{1+\langle \pi,\, x\rangle}\nu(dx) = 0.$$

The growth optimal portfolio process $\tilde{\pi}$ satisfies the same equation as (3.6) if it exists, which confirms that the numeraire portfolio coincides to the growth optimal portfolio. The maximal value is

$$E\left[\ln \tilde{V}_T\right] = T\left\{\frac{1}{2}\|\sigma^T\tilde{\pi}\|^2 + \int_{\mathcal{R}^d}\left[\ln(1+\langle\tilde{\pi}, x\rangle) - \frac{\langle\tilde{\pi}, x\rangle}{1+\langle\tilde{\pi}, x\rangle}\right]\nu(dx)\right\}.$$

3.2.2 Minimal martingale measure

The asset processes in (3.2) are semimartingales. Thus for each $i = 1, \ldots, d$, the discounted prices can be decomposed as $\tilde{S}_t^i = \int_0^t \tilde{S}_{s-}^i dX_s^i = \tilde{S}_0^i + M_t^i + A_t^i$:

$$\begin{aligned} M_t^i &= \int_0^t \tilde{S}_{s-}^i \sigma^i dW_s + \int_0^t \tilde{S}_{s-}^i \int_{\mathcal{R}} x[N^i(ds, dx) - \nu^i(dx)ds], \\ A_t^i &= \int_0^t \tilde{S}_{s-}^i \gamma_1^i ds. \end{aligned}$$

ν^i is the Lévy measure of X^i and γ_1^i is the center of X^i. Thus M^i is a P-martingale and A^i is a predictable process with finite variation.

Definition 3.2.2 ***(Minimal martingale measure)*** *An equivalent local martingale measure $\hat{Q}$ is the minimal martingale measure if any square-integrable P-martingale X which is P-orthogonal to M^i for $i = 1, \ldots, d$ remains a local martingale under $\hat{Q}$. X is P-orthogonal to M^i if the quadratic variation process $[X, M^i]$ is a local P-martingale.*

The density process of the minimal martingale measure follows $dZ_t/Z_{t-} = -\alpha_t dM_t$, where $M = (M^1, \ldots, M^d)^T$ is a d-dimensional P-martingale and $\alpha_t = (\alpha_t^1, \ldots, \alpha_t^d) \in \mathcal{R}^d$ is a predictable process which satisfies

$$dA_t = (\tilde{S}_{t-} \circ \gamma_1)dt = (\alpha_t d\langle M, M\rangle_t)^T,$$

$A = (A^1, \ldots, A^d)^T$ and $\tilde{S} \circ \gamma_1 = (\tilde{S}^1\gamma_1^1, \ldots, \tilde{S}^d\gamma_1^d)^T$. Denote ν^{ij} the 2-variate Lévy measure of L^i and L^j, then $\langle M, M\rangle_t$ is a $d \times d$ positive definite matrix with entry

$$\begin{aligned}\langle M^i, M^j\rangle_t &= \int_0^t \tilde{S}_{s-}^i \tilde{S}_{s-}^j \left[\sigma^i(\sigma^j)^T + \int_{\mathcal{R}^2} xy\nu^{ij}(dxdy)\right] ds, \quad i \neq j,\ i,j = 1, \ldots, d \\ \langle M^i, M^i\rangle_t &= \int_0^t (\tilde{S}_{s-}^i)^2 \left[\sigma^i(\sigma^i)^T + \int_{\mathcal{R}} x^2\nu^i(dx)\right] ds.\end{aligned}$$

Let Θ be a $d \times d$ positive definite matrix with entries $\Theta^{ij} = \int_{\mathcal{R}^2} xy\nu^{ij}(dxdy)$, $i \neq j$ and $\Theta^{ii} = \int_{\mathcal{R}} x^2\nu^i(dx)$. Or $\Theta = \int_{\mathcal{R}^d} xx^T\nu(dx)$. The matrix $\sigma\sigma^T + \Theta$ is the variance matrix of X. If Θ is bounded, the predictable process α_t solves the equation

$$\gamma_1 = (\sigma\sigma^T + \Theta)(\tilde{S}_{t-} \circ \alpha_t)^T,$$

and then $(\tilde{S}_{t-} \circ \alpha_t)^T = (\sigma\sigma^T + \Theta)^{-1}\gamma_1$. So the density process is

$$dZ_t/Z_{t-} = -\alpha_t dM_t = -\gamma_1^T(\sigma\sigma^T + \Theta)^{-1}\left[\sigma dW_t + \int_{\mathcal{R}^d} x\tilde{N}(dt, dx)\right]$$

provided that $1 - \langle(\sigma\sigma^T + \Theta)^{-1}\gamma_1, \triangle X_t\rangle > 0$.

Compared to the density process in Theorem 3.1.1, one finds that $\hat{\eta} = -(\sigma\sigma^T + \Theta)^{-1}\gamma_1$ and $\hat{\phi}(x) = \ln(1 + \langle\hat{\eta}, x\rangle)$. If the condition

$$\int_{\mathcal{R}^d} (\sqrt{\nu} - \sqrt{\hat{\nu}})^2(dx) = \int_{\mathcal{R}^d} (1 - \sqrt{1 + \langle\hat{\eta}, x\rangle})^2\nu(dx) < \infty$$

is satisfied, there exists a minimal martingale measure $\hat{Q}$.

Theorem 3.2.2 ***(Minimal martingale measure for exp-Lévy processes)***
Suppose that the asset process is modeled by a d-dimensional exp-Lévy process as (3.2), the density process of the minimal martingale measure $\hat{Q}$ is given by

$$Z_t = \exp\left\{\langle \hat{\eta}, \sigma W_t\rangle - \frac{1}{2}\|\sigma^T\hat{\eta}\|^2 t + \int_{\mathcal{R}^d}\int_0^t \ln(1+\langle\hat{\eta}, x\rangle)N(ds,dx) - t\int_{\mathcal{R}^d}\langle\hat{\eta}, x\rangle\nu(dx)\right\},$$

$\hat{\eta} = -(\sigma\sigma^T + \Theta)^{-1}\gamma_1$, *under the following assumptions:*

1. $1 + \langle\hat{\eta}, x\rangle > 0$ *and* $\vec{1} + x > 0$ *for any* x, *the possible jump size of* X;
2. $\int_{\mathcal{R}^d}(\sqrt{1+\langle\hat{\eta}, x\rangle} - 1)^2\nu(dx) < \infty$;
3. *the matrix* $\Theta = \int_{\mathcal{R}^d} xx^T\nu(dx)$ *is bounded;*
4. $\int_{\|x\|>1}\|x\|(1+\langle\hat{\eta}, x\rangle)\nu(dx) < \infty$.

The Lévy process X *is a martingale under* $\hat{Q}$ *with centering triplet* $(\sigma, \hat{\nu}, 0)$, $\hat{\nu}(dx) = (1+\langle\hat{\eta}, x\rangle)\nu(dx)$. $\hat{W}_t = W_t - \sigma^T\hat{\eta}t$ *is the standard Brownian motion.*

The first assumption implies that Z and S are strictly positive. The second assumption assures that the measure $\hat{Q}$ is equivalent to the original probability measure P. The third assumption is to guarantee the existence of Θ^{-1}, it also guarantees the square integrability of Z. Z is square integrable if

$$\int_{\mathcal{R}^d}\left(e^{2\ln(1+\langle\hat{\eta},x\rangle)} - 1\right)\nu(dx) < \infty, \quad \text{that is} \quad \int_{\mathcal{R}^d}\left(2\langle\hat{\eta}, x\rangle + \langle\hat{\eta}, x\rangle^2\right)\nu(dx) < \infty.$$

Since $\int_{\mathcal{R}^d}\langle\hat{\eta}, x\rangle\nu(dx) = \langle\hat{\eta}, \gamma_1\rangle$ and $\langle\hat{\eta}, x\rangle^2 \le \|x\|^2\|\hat{\eta}\|^2$, both are finite. X is a $\hat{Q}$-martingale by the last assumption.

3.2.3 Minimal entropy martingale measure

Definition 3.2.3 ***(Minimal entropy martingale measure)*** *Denote the set of EMM by* $\mathcal{M}$, *an EMM* $\bar{Q}$ *is a minimal entropy martingale measure if it minimized the so-called relative entropy with respect to the original measure* P:

$$\frac{d\bar{Q}}{dP}\ln\frac{d\bar{Q}}{dP} = \min_{Q\in\mathcal{M}} E\left[\frac{dQ}{dP}\ln\frac{dQ}{dP}\right].$$

The inside of the expectation is the *relative entropy* which is often used as a measure of proximity of two equivalent probability measures. Fujiwara & Miyahara [23] derived the result in the one-dimension case with a geometric Lévy model defined as $S_t = S_0 \exp\{X_t\}$. (Also see Proposition 10.7 in Cont & Tankov [12].) Here we extend it to a multi-dimensional stochastic exp-Lévy model. In their result, one may check that for stochastic exp-Lévy process, $\bar{\phi}(\cdot)$ is a function of the parameter $\bar{\eta}$ as $\bar{\phi}(x) = \langle \bar{\eta}, x\rangle$, an Esscher transform. In comparison, the functions of the numeraire portfolio and the minimal martingale measure are $\phi^*(x) = -\ln(1 - \langle \eta^*, x\rangle)$ and $\hat{\phi}(x) = \ln(1 + \langle \hat{\eta}, x\rangle)$ respectively. Theorem 3.1.1 and equation (3.4) give:

Theorem 3.2.3 ***(Minimal entropy martingale measure for exp-Lévy processes)*** *Suppose that the asset process is modeled by a d-dimensional exp-Lévy process as (3.2) with $\vec{1} + x > 0$ for any x, the possible jump size of X, if there exists a solution $\bar{\eta} \in \mathcal{R}^d$ to the equation:*

$$\gamma_1 + \sigma\sigma^T\eta + \int_{\mathcal{R}^d} x(e^{\langle \eta, x\rangle} - 1)\nu(dx) = 0, \tag{3.8}$$

and $\int_{\|x\|>1} \|x\| e^{\langle \bar{\eta}, x\rangle}\nu(dx) < \infty$, then the density process of the minimal entropy martingale measure $\bar{Q}$ is

$$\begin{aligned}\bar{Z}_t &= \frac{\exp\{\langle \bar{\eta}, X_t\rangle\}}{E[\exp\{\langle \bar{\eta}, X_t\rangle\}]} \\ &= \exp\left\{\langle \bar{\eta}, X_t\rangle - \frac{1}{2}\|\sigma^T\bar{\eta}\|^2 t - \langle \bar{\eta}, \gamma_1\rangle t - t\int_{\mathcal{R}^d} (e^{\langle \bar{\eta}, x\rangle} - 1 - \langle \bar{\eta}, x\rangle)\nu(dx)\right\}\end{aligned}$$

Under $\bar{Q}$, the Lévy process X is a martingale with centering triplet $(\sigma, \bar{\nu}, 0)$, $\bar{\nu}(dx) = e^{\langle \bar{\eta}, x\rangle}\nu(dx)$. $\bar{W}_t = W_t - \sigma^T\bar{\eta}t$ is the standard Brownian motion.

3.2.4 Pure diffusion model: $\nu(\cdot) = 0$

In the simple case where the asset process is pure-diffusion, that is $\nu(\cdot) = 0$ and $\sigma\sigma^T$ is not singular, the problem of the existence of EMMs is reduced to solve the equation (3.4) without ν^* and ν as

$$\gamma_1 + \sigma\sigma^T\eta = 0 \Longrightarrow \eta = -(\sigma\sigma^T)^{-1}\gamma_1.$$

Such η is unique even if the market is incomplete, which means that the three EMMs: the numeraire portfolio, the minimal martingale measure and the minimal entropy martingale measure are identical. Now, let $\mu = r\vec{1} + \gamma_1$ be the mean of the return process of S and $W_t^Q = W_t + \beta t$, $\beta \in \mathcal{R}^d$ be the standard Brownian motion under the risk-neutral measure Q, then β satisfies the equation $\mu - r\vec{1} = \sigma\beta$. The solution is not unique when $m > d$ and $\beta^Q = -\sigma^T\eta = \sigma(\sigma\sigma^T)^{-1}(\mu - r\vec{1})$ is just a special root where the randomness W_t is considered globally. When $m = d$, the financial market is complete and then there is only one equivalent martingale measure, that is only one $\beta \in \mathcal{R}^d$, specifically $\beta = \beta^Q = \sigma^{-1}(\mu - r\vec{1})$ is called the *market price of risk*. In this case, all the risk-neutral measures are identical.

3.3 Special case: exponential Poisson-diffusion processes

Here, we consider (3.2) where for each $i = 1, \ldots, d$

$$X_t^i = \gamma_0^i t + \sum_{k=1}^{m} \sigma^{ik} W_t^k + \sum_{l=1}^{n} \sum_{h=1}^{N_t^l} J_h^{il}, \quad 0 \le t \le T.$$

$\gamma_0 = (\gamma_0^1, \ldots, \gamma_0^d)^T$ is the drift of the Lévy process X. $N = (N^1, \ldots, N^n)^T$ is an n-variate independent Poisson process with intensity $\lambda = (\lambda^1, \ldots, \lambda^n)^T$, the jump sizes $J^{i1}, \ldots, J^{in}$ are independent for each $i = 1, \ldots, d$ and the joint distribution of $J^{1l}, \ldots, J^{dl}$ is given by F^l for each $l = 1, \ldots, n$. Denote the average jump size by $\mu^l = \int_{\mathcal{R}^d} x dF^l(x) \in \mathcal{R}^d$ for each l. W and σ are the same as before. W, N and $\mathcal{J} = (J^{il})_{1 \le i \le d,\, 1 \le l \le n}$ are mutually independent. The joint Lévy measure for each compound Poisson process is $\nu^l(dx) = \lambda^l dF^l(x)$, $l = 1, \ldots, n$. We also require $1 + \triangle X_s^i > 0$ for each i and $s \in [0, T]$ to ensure the asset price is positive.

Suppose there exists a risk-neutral measure Q, then Theorem 3.1.1 shows that under Q, $W_t^Q = W_t - \sigma^T\eta t$ is the standard Brownian motion and the new drift for X is $\gamma_0^Q = \gamma_0 + \sigma\sigma^T\eta$. For each $l = 1, \ldots, n$, the Lévy measure is $\nu^{lQ}(dx) = e^{\phi(x;\eta)}\nu^l(dx)$, N^l is a Poisson process with intensity $\lambda^{lQ} = \lambda^l \int_{\mathcal{R}^d} e^{\phi(x;\eta)} dF^l(x)$ and the joint distribution of the jump sizes associated with N^l is $dF^{lQ}(x) = e^{\phi(x;\eta)} dF^l(x) / \int_{\mathcal{R}^d} e^{\phi(x;\eta)} dF^l(x)$. Since X is a Q-martingale, $\sum_{l=1}^n \int_{\mathcal{R}^d} \|x\| \nu^{lQ}(dx) < \infty$ and $\gamma_0^Q + \sum_{l=1}^n \int_{\mathcal{R}^d} x\nu^{lQ}(dx) = 0$ and η satisfies

$$\gamma_0 + \sigma\sigma^T\eta + \sum_{l=1}^{n} \lambda^l \int_{\mathcal{R}^d} x e^{\phi(x;\eta)} dF^l(x) = 0, \tag{3.9}$$

same as equation (3.5). For constant jump sizes $\mathcal{J}$, we get $\gamma_0 + \sigma\sigma^T\eta + \mathcal{J}\lambda^* = 0$. The financial market is complete if there is only one equivalent martingale measure, that is the equation only admits one solution (η, λ^*). When $\sigma = 0$, the solution is unique if $d = n$ and J is not singular. The unique solution $\lambda^* = -J^{-1}\gamma_0$ is the new Poisson intensity under Q, which must be positive.

3.3.1 Numeraire portfolio: $\phi^*(x) = -\ln(1-\langle\eta^*,\, x\rangle) = -\ln(1+\langle\pi,\, x\rangle)$

The conditions in Theorem 3.2.1 can be modified as

- $1 + \langle\pi, x\rangle > 0$ and $\vec{1} + x > 0$ for any x, the possible jump size of X;
- $\int_{\mathcal{R}^d} \frac{\|x\|}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty$ for each l.

Then the portfolio process π solves a similar equation as (3.6) or (3.9)

$$\gamma_0 - \sigma\sigma^T\pi + \sum_{l=1}^{n} \lambda^l \int_{\mathcal{R}^d} \frac{x}{1+\langle\pi,\, x\rangle} dF^l(x) = 0.$$

Note that $\int_{\mathcal{R}^d} \frac{\|x\|}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty$ shows $\sum_{l=1}^{n} \lambda^l \int_{\|x\|>1} \frac{\|x\|}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty$, then X_t is a martingale under Q^*. Also

$$\left.\begin{array}{l} \int_{\mathcal{R}^d} \frac{\langle\pi,\, x\rangle}{1+\langle\pi,\, x\rangle} dF^l(x) \le \|\pi\| \int_{\mathcal{R}^d} \frac{\|x\|}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty \\ \int_{\mathcal{R}^d} \frac{1}{1+\langle\pi,\, x\rangle} dF^l(x) = 1 - \int_{\mathcal{R}^d} \frac{\langle\pi,\, x\rangle}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty \end{array}\right\} \Longrightarrow$$

$$\int_{\mathcal{R}^d} (\sqrt{\nu^l} - \sqrt{\nu^{l*}})^2(dx) = \lambda^l \int_{\mathcal{R}^d} \frac{2+\langle\pi,\, x\rangle - 2\sqrt{1+\langle\pi,\, x\rangle}}{1+\langle\pi,\, x\rangle} dF^l(x) < \infty.$$

By the measure transformation, the sum of the intensities varies by

$$\sum_{l=1}^{n}(\lambda^{l*} - \lambda^l) = \sum_{l=1}^{n} \lambda^l \int_{\mathcal{R}^d} \left(\frac{1}{1+\langle\pi,\, x\rangle} - 1\right) dF^l(x) = \langle\pi, \gamma_0\rangle - \|\sigma^T\pi\|^2 = \langle\pi, \gamma_0^*\rangle.$$

3.3.2 Minimal martingale measure: $\hat{\phi}(x) = \ln(1 + \langle\hat{\eta},\, x\rangle)$

Theorem 3.2.2 or equation (3.9) shows $\hat{\eta} = -(\sigma\sigma^T + \Theta)^{-1}(\gamma_0 + \sum_{l=1}^{n} \mu^l)$. Where $\Theta^{ij} = \sum_{l=1}^{n} \lambda^l \int_{\mathcal{R}^2} xy dF^l_{ij}(x, y)$ for $i \neq j$ and $\Theta^{ii} = \sum_{l=1}^{n} \lambda^l \int_{\mathcal{R}^2} x^2 dF^l_i(x)$, $i, j = 1, \ldots, d$. F^l_{ij} is the joint distribution of the jump sizes of i^{th} and j^{th} assets with respect to N^l. Then the conditions of existence are:

- $1 + \langle \hat{\eta}, x \rangle > 0$ and $\vec{1} + x > 0$ for any x, the possible jump size of X;
- the matrix $\int_{\mathcal{R}^d} x x^T dF^l(x)$ is bounded for each l;
- $\int_{\|x\|>1} \|x\|(1 + \langle \hat{\eta}, x \rangle) dF^l(x) < \infty$ for each l.

By the second assumption, the inverse of Θ exists and the density process is square-integrable. The two martingale measures are equivalent since $(\sqrt{\nu^l} - \sqrt{\hat{\nu}^l})^2 = 2 + \langle \hat{\eta}, x \rangle - 2\sqrt{1 + \langle \hat{\eta}, x \rangle}$ and

$$\int_{\mathcal{R}^d} (\sqrt{\nu^l} - \sqrt{\hat{\nu}^l})^2 (dx) \leq \lambda^l \int_{\mathcal{R}^d} (2 + \langle \hat{\eta}, x \rangle) dF^l(x) = \lambda^l (2 + \langle \hat{\eta}, \mu^l \rangle) < \infty.$$

Under the minimal martingale measure $\hat{Q}$, N^l is a Poisson process with intensity

$$\hat{\lambda}^l = \lambda^l \int_{\mathcal{R}^d} (1 + \langle \hat{\eta}, x \rangle) dF^l(x) = \lambda^l (1 + \langle \hat{\eta}, \mu^l \rangle).$$

3.3.3 Minimal entropy martingale measure: $\bar{\phi}(x) = \langle \bar{\eta}, x \rangle$

Both equation (3.8) and equation (3.9) show that $\bar{\eta} \in \mathcal{R}^d$ is the solution to

$$\gamma_0 + \sigma\sigma^T \bar{\eta} + \sum_{l=1}^n \lambda^l \int_{\mathcal{R}^d} x e^{\langle \bar{\eta}, x \rangle} dF^l(x) = 0,$$

$\vec{1} + x > 0$ for any x, the possible jump size of X and $\int_{\|x\|>1} \|x\| e^{\langle \bar{\eta}, x \rangle} dF^l(x) < \infty$. If $\bar{\eta}$ exists, $M^l(\bar{\eta}) = E[e^{\langle \bar{\eta}, X_1 \rangle}] < \infty$ for each $l = 1, \ldots, n$, then

$$\int_{\|x\|>1} e^{\langle \bar{\eta}, x \rangle} \nu(dx) = \sum_{l=1}^n \lambda^l \int_{\|x\|>1} e^{\langle \bar{\eta}, x \rangle} dF^l(x) < \sum_{l=1}^n \lambda^l M^l(\bar{\eta}) < \infty,$$

and the following condition is satisfied automatically

$$\begin{aligned} \int_{\mathcal{R}^d} (\sqrt{\nu^l} - \sqrt{\bar{\nu}^l})^2 (dx) &= \lambda^l \int_{\mathcal{R}^d} \left(1 + e^{\langle \bar{\eta}, x \rangle} - 2e^{\frac{\langle \bar{\eta}, x \rangle}{2}}\right) dF^l(x) \\ &= \lambda^l \left[1 + M^l(\bar{\eta}) - 2M^l(\frac{\bar{\eta}}{2})\right] < \infty. \end{aligned}$$

Under the minimal entropy martingale measure $\bar{Q}$, the intensity of N^l is

$$\bar{\lambda}^l = \lambda^l \int_{\mathcal{R}^d} e^{\langle \bar{\eta}, x \rangle} dF^l(x) = \lambda^l M^l(\bar{\eta}).$$

3.4 Extension: equivalent martingale measures for additive processes

Theorem 3.1.1 and equation (3.1) can be extended to additive processes with local characteristics.

Theorem 3.4.1 ***(Equivalent measures for additive processes)***

Let (X_t, P^1) and (X_t, P^2) be two additive processes on $\mathcal{R}^d$ with local characteristics (a_t^1, ν_t^1, b_t^1) and (a_t^2, ν_t^2, b_t^2), $0 \leq t \leq T < \infty$. Then P_t^1 and P_t^2 are equivalent for all $t \in [0, T]$ if and only if the following three conditions are satisfied:

1. $a_t^1 = a_t^2$, *denote* $a_t := a_t^1 = a_t^2$;
2. ν_t^1 *and* ν_t^2 *are equivalent with* $\int_{\mathcal{R}^d} (\sqrt{\nu_t^1} - \sqrt{\nu_t^2})^2 (dx) < \infty$;
3. $b_t^2 - b_t^1 - \int_{\|x\| \leq 1} x(\nu_t^2 - \nu_t^1)(dx) = a_t \eta_t$ *for some* $\eta_t \in \mathcal{R}^d$.

Under the P^1 measure, let W_t^1 be a d-dimensional standard Brownian motion and σ_t be the Cholesky decomposition of a_t, then X has a decomposition as

$$\int_0^t b_s^1 ds + \int_0^t \sigma_s dW_s^1 + \int_{\|x\| \leq 1} \int_0^t x(N(ds, dx) - \nu_s^1(dx) ds) + \int_{\|x\| > 1} \int_0^t x N(ds, dx).$$

Under the P^2 measure, the Lévy decompositions with the same dimension of the Brownian motion is

$$\int_0^t b_s^2 ds + \int_0^t \sigma_s dW_s^2 + \int_{\|x\| \leq 1} \int_0^t x(N(ds, dx) - \nu_s^2(dx) ds) + \int_{\|x\| > 1} \int_0^t x N(ds, dx),$$

then $W_t^2 = W_t^1 - \int_0^t \sigma_s^T \eta_s ds$ is the standard Brownian motion under P^2. Denote $\phi(t, x) = \ln[\nu_t^2(dx)/\nu_t^1(dx)] : [0, T] \times \mathcal{R}^d \to \mathcal{R}$, when P^1 and P^2 are equivalent, the Radon-Nikodym density process is

$$\begin{aligned} Z_t &= \frac{dP^2|\mathcal{F}_t}{dP^1|\mathcal{F}_t} = \frac{dP_t^2}{dP_t^1} = e^{U_t} \quad \textit{with} \\ U_t &= \int_0^t \eta_s^T \sigma_s dW_s^1 - \frac{1}{2} \int_0^t \|\sigma_s^T \eta_s\|^2 ds \\ &\quad + \int_{\mathcal{R}^d} \int_0^t \phi(s, x) N(ds, dx) - \int_0^t \int_{\mathcal{R}^d} (e^{\phi(s,x)} - 1) \nu_s^1(dx) ds. \end{aligned}$$

U_t is an additive process on $\mathcal{R}$ with local characteristics $(a_U(t), \nu_U(t), b_U(t))$ given by

$$\begin{aligned} a_U(t) &= \|\sigma_t^T \eta_t\|^2, \\ \nu_U(t, B) &= \int_{\mathcal{R}^d} I_{\{\phi(t,x)\in B\}} \nu_t^1(dx) \quad \textit{for } B \in \mathcal{B}(\mathcal{R}), \\ b_U(t) &= -\frac{1}{2}\|\sigma_t^T \eta_t\|^2 - \int_{\mathcal{R}^d} (e^{\phi(t,x)} - 1 - \phi(t,x) I_{\{\|\phi(t,x)\|\le 1\}}) \nu_t^1(dx) \\ &= -\frac{1}{2} a_U(t) - \int_{\mathcal{R}} (e^y - 1 - y I_{\{|y|\le 1\}}) \nu_U(t, dy). \end{aligned}$$

Moreover, if P^2 is a martingale measure for X_t, then ν_t^2 must satisfy $\int_{\|x\|>1} \|x\| \nu_t^2(dx) < \infty$. Thus $b_t^2 = -\int_{\|x\|>1} x \nu_t^2(dx)$ and η_t will solve the equation

$$b_t^1 + a_t \eta_t + \int_{\|x\|\le 1} x(e^{\phi(t,x)} - 1)\nu_t^1(dx) + \int_{\|x\|>1} x e^{\phi(t,x)} \nu_t^1(dx) = 0.$$

Here we only give a sketch of the proof of the first part (equivalent part). If the first part is true, then the second part (martingale part) comes from the fact that X_t is a martingale under P^2 with $E^2[X_t] = 0$ for all $t \in [0, T]$.

Proof: Additive processes share a common property with Lévy processes: independent increments. Under P^1, fix $T \in (0, \infty)$ and we may construct an additive process as

$$X_t = \sum_{i=1}^{n} L^i_{t_i - t_{i-1}} I_{\{t \ge t_i\}}, \quad t \in [0, T] \tag{3.10}$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ is a subdivision of $[0, T]$ and L_t^i are d-dimensional Lévy processes with generating triplet (a^i, ν^i, b^i), $i = 1, \ldots, n$. The spot characteristics of X is $(A_t^1, V_t^1, \Gamma_t^1)$, where $A_t^1 = \sum_{i=1}^n a^i (t_i - t_{i-1}) I_{\{t\ge t_i\}}$, $V_t^1(dx) = \sum_{i=1}^n \nu^i(dx)(t_i - t_{i-1}) I_{\{t \ge t_i\}}$ and $\Gamma_t^1 = \sum_{i=1}^n b^i (t_i - t_{i-1}) I_{\{t\ge t_i\}}$.

Since any two Lévy processes L_t^i and L_t^j ($i \ne j$ and $i, j = 1, \ldots, n$) in (3.10) work on disjoint time periods, we may assume that L^i and L^j are independent for any $i \ne j$. Theorem 3.1.1 shows that for $((L_t^1, \ldots, L_t^n), P^1)$ there exists an equivalent measure P^2 (or there exist functions $\phi^i(x) : \mathcal{R}^d \to \mathcal{R}$ and $\eta^i \in \mathcal{R}^d$, $i = 1, \ldots, n$) for $t \in [0, T]$. For each i, under the measure P^2, L_t^i is a Lévy process with triplet $(a^i, e^{\phi^i(x)}\nu^i(dx), b^{i*})$ if $\int_{\mathcal{R}^d} (e^{\phi^i(x)/2} - 1)^2 \nu^i(dx) < \infty$ and $b^{i*} = b^i + \int_{\|x\|\le 1} x(e^{\phi^i(x)} - 1)\nu^i(dx) + a^i \eta^i$. Thus X_t is an additive process under P^2 with spot characteristics $(A_t^2, V_t^2, \Gamma_t^2)$, where

$$\begin{aligned} A_t^2 &= \sum_{i=1}^n a^i(t_i - t_{i-1}) I_{\{t \geq t_i\}} = A_t^1 =: A_t, \\ V_t^2(dx) &= \sum_{i=1}^n e^{\phi^i(x)} \nu^i(dx)(t_i - t_{i-1}) I_{\{t \geq t_i\}}, \\ \Gamma_t^2 &= \sum_{i=1}^n b^{i*}(t_i - t_{i-1}) I_{\{t \geq t_i\}} = \Gamma_t^1 + \int_{\|x\| \leq 1} x(V_t^2 - V_t^1)(dx) + A_t \eta^i. \end{aligned} \tag{3.11}$$

For each L^i, the Radon-Nikodym derivative on $t \in [t_i, t_{i+1})$ is

$$\frac{dP^2 | \mathcal{F}_t}{dP^1 | \mathcal{F}_t} = e^{U_t^i},$$

where U_t^i is a Lévy process on $\mathcal{R}$ with triplet (a_U^i, ν_U^i, b_U^i) given by

$$\begin{aligned} a_U^i &= (\eta^i)^T a^i \eta^i, \\ \nu_U^i(B) &= \int_{\mathcal{R}^d} I_{\{\phi^i(x) \in B\}} \nu^i(dx) \quad \text{for } B \in \mathcal{B}(\mathcal{R}), \\ b_U^i &= -\frac{1}{2}(\eta^i)^T a^i \eta^i - \int_{\mathcal{R}^d} (e^{\phi^i(x)} - 1 - \phi^i(x) I_{\{\|\phi^i(x)\| \leq 1\}}) \nu^i(dx) \\ &= -\frac{1}{2} a_U^i - \int_{\mathcal{R}} (e^y - 1 - y I_{\{|y| \leq 1\}}) \nu_U^i(dy) \end{aligned} \tag{3.12}$$

So the density process Z_t associated with X_t satisfies

$$\ln Z_t = \sum_{i=1}^n U_t^i I_{\{t \geq t_i\}},$$

which is an additive process with spot characteristics $(A_U(t), V_U(t), \Gamma_U(t))$ given by $A_U(t) = \sum_{i=1}^n a_U^i (t_i - t_{i-1}) I_{\{t \geq t_i\}}$, $V_U(t, dx) = \sum_{i=1}^n \nu_U^i(dx)(t_i - t_{i-1}) I_{\{t \geq t_i\}}$ and $\Gamma_U(t) = \sum_{i=1}^n b_U^i (t_i - t_{i-1}) I_{\{t \geq t_i\}}$.

Now let $n \to \infty$, A^i will be the derivative (from right) of $dA_t/dt =: a_t$ at t_i. Denote $dV_t^1/dt = \nu_t$ and $d\Gamma_t^1/dt = b_t$. Then X_t in (3.10) is an additive process with local characteristics (a_t, ν_t, b_t) under P^1 when $n \to \infty$. There exists an equivalent measure P^2 with density process e^{U_t}, U_t is an additive process with the local characteristics $(a_U(t), \nu_U(t), b_U(t))$ which have almost the same form as (a_U^i, ν_U^i, b_U^i) in (3.12) and the parameters $(a^i, \nu^i, b^i, \eta^i)$ are replaced by $(a_t, \nu_t, b_t, \eta_t)$ respectively. The function $\phi^i(x)$ becomes $\phi(t, x)$ and satisfies $\int_{\mathcal{R}^d} (e^{\phi(t,x)/2} - 1)^2 \nu_t(dx) < \infty$ for each $t \in (0, T]$.

Compared to the spot characteristics of X_t in (3.11) under the new measure P^2, the local characteristics will be $(a_t, e^{\phi(t,x)}\nu_t(dx), b_t^*)$ with $b_t^* = b_t + \int_{\|x\|\leq 1} x(e^{\phi(t,x)} - 1)\nu_t(dx) + a_t\eta_t$.

□

In financial market, if the asset process is modeled as $dS_t = S_{t-}(r_t dt + dX_t)$ where r_t is the deterministic interest rate and X is an additive process with local characteristics (a_t, ν_t, γ_t), then the numeraire portfolio may be obtained by $\phi^*(t,x) = -\ln(1 - \langle\eta_t^*, x\rangle)$ and the portfolio process is $\pi_t = -\eta_t^*$. The minimal martingale measure and the minimal entropy martingale measure may be determined by $\hat{\phi}(t,x) = \ln(1 + \langle\hat{\eta}_t, x\rangle)$ and $\bar{\phi}(t,x) = \langle\bar{\eta}_t, x\rangle$ respectively. The functions $\eta_t^*, \hat{\eta}_t, \bar{\eta}_t$ and ϕ will satisfy Theorem 3.4.1. Discussions about the minimal martingale measure and the minimal entropy martingale measure for exp-additive processes can also be found in Henderson & Hobson [28].

4

Credit models with Lévy processes

4.1 Basic factors in credit models

Consider a financial market with filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, Q)$, Q is the risk-neutral measure. For a defaultable zero-coupon bond with face value F and maturity T, let r_t be the short rate and τ is default time, the bondholders will receive C_τ at default and F at maturity if there is no default before or at T. Then the price of such defaultable bond before default is

$$B(t,T) = E^Q\left[Fe^{-\int_t^T r_s ds} I_{\{\tau>T\}} + C_\tau e^{-\int_t^\tau r_s\, ds} I_{\{\tau\leq T\}} \,|\, \mathcal{F}_t\right], \tau > t.$$

The price depends on the default time τ, the recovery amount C_τ and the risk-neutral measure Q. Equation (1.1) defines the default time in the firm value models and equation (1.3) defines the default time in the intensity-based models.

Let $P(t,T) = E^Q[e^{-\int_t^T r_s ds}|\mathcal{F}_t]$ be the price of a zero-coupon default-free bond at $t < T$ with face value 1 and maturity T. Then the defaultable bond price before default is

$$B(t,T) = FP(t,T) - E^Q\left[e^{-\int_t^T r_s ds}(F - C_\tau e^{\int_\tau^T r_s ds}) I_{\{\tau\leq T\}} \,|\, \mathcal{F}_t\right].$$

The credit spread, which is defined as the difference between the yield on the defaultable bond and the yield on the equivalent default-free bond, is

$$cs(t,T) = -\frac{\ln B(t,T) - \ln(FP(t,T))}{T-t}, \quad \tau > t.$$

And the instantaneous credit spread is defined as

$$cs_t = \frac{E^Q[dB(t,T)|\mathcal{F}_{t-}]}{B(t-,T)dt} - \frac{E^Q[dP(t,T)|\mathcal{F}_{t-}]}{P(t-,T)dt}, \quad \tau > t.$$

4.1.1 Recovery scheme

We define the *recovery rate* as the ratio of debt recovered once a default event happens. There are three recovery schemes in the current literature.

- *Recovery of Face Value* (RFV): at default, bondholders will receive a fraction ω_1 of face value of the bond $C_\tau = \omega_{1,\tau}F$;
- *Recovery of Treasury Value* (RTV): at default, bondholders will receive a fraction ω_2 of the price of default-free bond with same maturity and face value, which is equivalent as receiving the same fraction of face value at maturity $C_\tau = \omega_{2,\tau}FP(\tau,T)$. In this case, $B(t,T)$ can simply rewritten as $F\left(P(t,T) - E^Q[e^{-\int_t^T r_s ds}(1-\omega_{2,\tau})I_{\{\tau \leq T\}}|\mathcal{F}_t]\right)$;
- *Recovery of Market Value* (RMV): at default, bondholders will receive a fraction ω_3 of the price of pre-default market value of this defaultable bond $C_\tau = \omega_{3,\tau}B(\tau-,T)$.

Duffie & Singleton [18] shows that under the RMV scheme in the intensity-based model, the price of a risky bond can be generally written as

$$B(t,T) = FE^Q\left[e^{-\int_t^T (r_s + (1-\hat{\omega}_s)h_s)ds}|\mathcal{F}_t\right],$$

where F is the face value, r_t is the interest rate, h_t is the default intensity and $\hat{\omega}_t$ is the expected recovery rate given all the information up to, but not including, time t if default happens immediately. The instantaneous credit spread is $cs_t = h_t(1-\hat{\omega}_t)$.

Let $B^0(t,T)$ be the defaultable bond price with zero recovery, that is

$$B^0(t,T) = E^Q[e^{-\int_t^T r_s ds}FI_{\{\tau > T\}}|\mathcal{F}_t].$$

If the recovery rate ω is a random variable which is independent of other random variables or processes, the bond price under the RTV scheme is

$$\begin{aligned} B(t,T) &= F\left(P(t,T) - E^Q[e^{-\int_t^T r_s ds}(1-\omega)I_{\{\tau\le T\}}|\mathcal{F}_t]\right) \\ &= FP(t,T) - (1-E^Q[\omega])E^Q[e^{-\int_t^T r_s ds}F(1-I_{\{\tau>T\}})|\mathcal{F}_t] \\ &= E^Q[\omega]FP(t,T) + (1-E^Q[\omega])B^0(t,T), \end{aligned}$$

where Since $\omega \in [0,1]$ and then $E^Q[\omega] \in [0,1]$, $B(t,T)$ can be rewritten as the expected value of the random variable X, where

$$X = \begin{cases} FP(t,T) & \text{with } p = E^Q[\omega]; \\ B^0(t,T) & \text{with } 1-p. \end{cases}$$

4.1.2 Bond price in the firm value models

Merton used the geometric Brownian motion as the firm value process: $dV_t = V_t(rdt + \sigma dW_t)$, $V_0 > 0$, $r > 0$ is the constant interest rate and $\sigma > 0$ is the volatility. The defaultable bond only defaults at maturity if $V_T < F$ and its value at maturity is $B(T,T) = \min(F, V_T) = F - (F - V_T)^+$. Then

$$\begin{aligned} B(t,T) &= V_t\Phi(-d_t^1) + Fe^{-r(T-t)}\Phi(d_t^2); \\ d_t^1 &= \frac{\ln V_t - \ln F + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \\ d_t^2 &= d_t^1 - \sigma\sqrt{T-t}. \end{aligned}$$

Here $\Phi(x)$ is the CDF of a standard normal distribution. The default probability is $Q(\tau = T|V_t) = \Phi(-d_t^2)$. Let $l_t = Fe^{-r(T-t)}/V_t$ be the leverage at time t, the credit spread is

$$\begin{aligned} cs(t,T) &= -\frac{1}{T-t}\ln\left(\frac{1}{l_t}\Phi(h_t^1) + \Phi(h_t^2)\right); \\ h_t^1 &= \frac{\ln l_t}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}; \\ h_t^2 &= -\frac{\ln l_t}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}. \end{aligned}$$

It increases with the leverage and the volatility. While the instantaneous credit spread at maturity is

$$\lim_{t\to T} cs(t,T) = \begin{cases} 0 & \text{if } l_T < 1, \text{ that is } V_T > F; \\ \infty & \text{if } l_T \ge 1, \text{ that is } V_T \le F. \end{cases}$$

This is the consequence of the fact that the firm value process is continuous.

In the first-Passage Model, let F be the default boundary, Musiela & Rutkowski [47] shows that

$$Q(\tau < T \,|\tau > t) = \Phi\left(\frac{\ln x_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right) + e^{\frac{2\mu x_t}{\sigma^2}}\Phi\left(\frac{\ln x_t + \mu(T-t)}{\sigma\sqrt{T-t}}\right),$$

where $x_t = F/V_t$, and $\mu = r - \frac{1}{2}\sigma^2$.

Zhou [56] used a jump-diffusion model instead of the geometric Brownian motion in the firm value model, where

$$dV_t = V_t\left[(r - \lambda v)dt + \sigma dW_t + (J-1)dN_t\right],$$

N_t is a Poisson process with parameter λ and v is the expected value of jump size $J - 1$. N_t is independent of the Brownian Motion W_t. Now, the default may be caused by the movement of Brownian motion or the Poisson jumps. The first one is predictable while the latter one is a surprise.

By assuming that the jump size follows a log-normal distribution with $\ln J \sim N(\mu_\pi, \sigma_\pi^2)$, under Merton's setup the default probability is

$$Q(\tau = T) = Q(V_T < F) = \sum_{i=0}^{\infty}\frac{e^{-\lambda T}(-\lambda T)^i}{i!}\times\Phi\left(\frac{\ln F - \ln V_0 - (r - \frac{1}{2}\sigma^2\lambda v)T - i\mu_\pi}{\sqrt{\sigma^2 T + \sigma_\pi^2 i}}\right).$$

And in the first-passage model, the bond price and the survival probability can be approached by PDEs with numerical solutions.

4.1.3 Dependence of default

In the firm value model, the correlation between the dynamics of firm's assets determines the default dependence. Consider two firms that follow $dV_t^i = V_t^i(rdt + \sigma_i dW_t^i)$ with $V_0^i, \sigma^i > 0$ and $dW_t^1 dW_t^2 = \rho dt$. The joint distribution of default times is a bivariate standard normal distribution with correlation coefficient ρ in Merton's model, and it is a bivariate inverse Gaussian distribution with correlation ρ in the first-passage model. For some deterministic D_t, Zhou [57] shows a closed-form joint default probability.

In the intensity-based model, there are several ways to model the dependency. (See Duffie & Singleton [19] for more details.) One is to introduce the correlation between the intensity processes through time and assume the jump processes X in equation (1.3) are independent given the paths of intensities. For example, let $\lambda_t^i = h_t^i + a_i h_t$ be the default intensity of firm i, where h and h^i are mutually independent processes and $a_i > 0$. Then the join survival probability are,

$$\begin{aligned} Q(\tau^i > T_i) &= E^Q\left[e^{-\int_0^t \lambda_s^i ds}\right] = E^Q\left[e^{-\int_0^t h_s^i ds}\right] E^Q\left[e^{-\int_0^t a_i h_s ds}\right] \\ Q(\tau^1 > T_1, \tau^2 > T_2) &= E^Q\left[e^{-\int_0^t (\lambda_s^1 + \lambda_s^2) ds}\right] \\ &= E^Q\left[e^{-\int_0^t h_s^1 ds}\right] E^Q\left[e^{-\int_0^t h_s^2 ds}\right] E^Q\left[e^{-\int_0^t (a_1+a_2) h_s ds}\right]. \end{aligned}$$

Jarrow & Yu [29] consider the counterparty risk: a firm's default will increase the default probability of another firm. Specifically, the default intensity can be described by $\lambda_t^1 = a_1 + a_2 I_{\{t \geq \tau^2\}}$, $a_1, a_2 > 0$.

One may consider the dependence between two jump processes. Giesecke [25] gave a simple exponential model for bivariate correlation. Let the jump process in (1.3) be a Poisson process. The defaults are driven by firm-specific (N^i) and common shock (N) events. N and N^i are Poisson processes with intensities λ and λ^i respectively and the three poisson process are mutually independent. Then the default times are $\tau^i = \inf\{t \geq 0 : N_t^i + N_t > 0\}$ and

$$Q(\tau^i > t) = Q(N_t^i + N_t = 0) = e^{-(\lambda^i + \lambda)t},$$

The join survival function is

$$\begin{aligned} Q(\tau^1 > t_1, \tau^2 > t_2) &= Q(N_{t_1}^1 = 0, N_{t_2}^2 = 0, N_{\max\{t_1,t_2\}} = 0) \\ &= \exp\{-\lambda^1 t_1 - \lambda^2 t_2 - \lambda \max\{t_1, t_2\}\} \\ &= Q(\tau^1 > t_1) Q(\tau^2 > t_2) \min\{e^{\lambda t_1}, e^{\lambda t_2}\}. \end{aligned} \quad (4.1)$$

Given $N_t = 0$, the default events are mutually independent up to time t. Such construction only admits positive correlation.

Another way to model the dependence structure is to use the copula method

$$Q(\tau^1 > t_1, \tau^2 > t_2, \cdots, \tau^n > t_n) = C(Q(\tau^1 > t_1), Q(\tau^2 > t_2), \cdots, Q(\tau^n > t_n)).$$

The joint survival probability in (4.1) implies that the survival copula

$$C(\mu_1, \mu_2) = \mu_1 \mu_2 \min(\mu_1^{-\theta_1}, \mu_2^{-\theta_2}), \quad \theta_i = \lambda/(\lambda + \lambda^i).$$

4.2 Default time as a stopping time

Let h_t be the intensity (stochastic hazard rate) of a totally inaccessible stopping time τ, then the survival function $S(t) = Q(\tau > t)$, the probability density function $f(t) = -dS(t)/dt$ and the hazard function $\lambda(t) = f(t)/S(t)$ are

$$\begin{aligned} S(t) &= E^Q[I_{\{\tau>t\}}] = E^Q[e^{-\int_0^t h_s ds}], \\ f(t) &= E^Q[h_t e^{-\int_0^t h_s ds}] = E^Q[h_t I_{\{\tau>t\}}], \\ \lambda(t) &= \frac{E^Q[h_t I_{\{\tau>t\}}]}{E^Q[I_{\{\tau>t\}}]} = E^Q[h_t|\tau > t]. \end{aligned}$$

In fact, the hazard function $\lambda(t)$ is the forward default rate at time t assessed at time 0. Duffie & Singleton [19] summarize that the intensity h_t is the arrival rate of default at t, conditioning on all information available at t and the forward default rate (hazard rate) $\lambda(t)$ is the mean arrival rate of default at t, conditioning only on survival to t.

On $\{\tau > t\}$, let $S(t,T) = Q(\tau > T|\mathcal{F}_t)$, $T \geq t$, be the conditional survival probability, define

$$\begin{aligned} f(t,T) &= -\frac{\partial S(t,T)}{\partial T}, \\ \lambda(t,T) &= \frac{f(t,T)}{S(t,T)}. \end{aligned}$$

Here $f(t,T)$ is the probability density function at time T assessed at time t and $\lambda(t,T)$ is called the forward hazard function at time T assessed at time t. Similar to the relationship between the forward interest rate and the short rate, the instantaneous default probability is the limitation of the forward default rate

$$h_t = \lim_{T\to t} \lambda(t,T) = \lim_{T\to t} f(t,T).$$

Thus the survival probability of τ can be represented as

$$S(t,T) = E^Q[e^{-\int_t^T h_s ds}|\mathcal{F}_t] = e^{-\int_t^T \lambda(t,s)ds},$$

and the forward hazard rate is

$$\begin{aligned} \lambda(t,T) &= -\frac{\partial S(t,T)}{S(t,T)\partial T} = \frac{E^Q[h_T e^{-\int_t^T h_s ds}|\mathcal{F}_t]}{S(t,T)} \\ &= \frac{E^Q[h_T I_{\{\tau>T\}}|\mathcal{F}_t]}{E^Q[I_{\{\tau>T\}}|\mathcal{F}_t]} = E^Q[h_T|\mathcal{F}_t, \tau > T], \end{aligned} \tag{4.2}$$

which is the expectation of the intensity h_T conditioning on all the information available at time t and survival to T. While the forward interest rate is the expectation of the short rate given all the information available at time t under the forward measure with maturity T. We must note that equation (4.2) is not true if the intensity does not exist.

Consider the First-passage model, define the default time as

$$\tau = \inf\{t > 0 : V_t \leq D_t\}, \tag{4.3}$$

where V_t follows a jump-diffusion process, the threshold D is predictable and is left to be empirically determined. Both of them are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, Q)$. Then τ is a stopping time with respect to $\mathcal{F}_t$, that is $\{\tau \leq t\} \in \mathcal{F}_t$. Given $\{\tau \geq t\}$, we have

$$\{\tau = t\} = \{V_t \leq D_t \leq V_{t-}\} = \{V_t \leq D_t < V_{t-}\} \cup \{V_t \leq D_t = V_{t-}\}.$$

Here are some important definitions and theorem from Métivier [45].

Definition 4.2.1 ***(Definition 4.5, 4.9, 7.1, 7.4 in [45])***

- *The **graph** of the stopping time τ: $[\tau] := \{(t, \omega) \in \mathcal{R}^+ \times \Omega : t = \tau(\omega) < \infty\}$.*
- *A stopping time τ is called **predictable** when $[\tau]$ is a predictable subset of $\mathcal{R}^+ \times \Omega$.*
- *A stopping time τ is called **totally inaccessible** when, for every admissible measure μ on $\mathcal{F}$ and every predictable stopping time S, the following holds: $\mu([\tau] \cap [S]) = 0$.*
- *A stopping time τ, the graph $[\tau]$ of which is included in the union of denumerably many graphs of predictable stopping times, is called **accessible**.*

A stopping time τ is *totally inaccessible* if, for every predictable stopping time S, $Q\{\omega : \tau(\omega) = S(\omega) < \infty\} = 0$. A totally inaccessible stopping time is non-predictable, but there exist unpredictable stopping times which are accessible. In the later case, the intensity does not exit. Refer to Example 7.5 in Métivier [45].

Theorem 4.2.1 ***(Decomposition theorem for stopping times, theorem 7.3 in [45])***
For every stopping time τ, there exists one and (up to $Q-$negligibility) only one pair (τ_e, τ_u) of stopping times with the properties:

- $[\tau] = [\tau_e] \cup [\tau_u]$, $[\tau_e] \cap [\tau_u] = \emptyset$;
- T_u *is totally inaccessible;*
- T_e *is accessible, that is, there exists a sequence* $(\tau_n)_{n\geq 1}$ *of predictable stopping times such that* $[\tau_e] \subset \cup_n[\tau_n]$.

For a stopping time τ to have an associated intensity, it must be totally inaccessible. The default time defined in equation (4.3) is the first time that the $\mathcal{F}_t$-adapted process $\{\ln V_t - \ln D_t\}_{t\geq 0}$ hits the Borel set $(-\infty, 0]$. If $\ln V_t$ has a diffusion part, the default time is not totally inaccessible and does not have an associated intensity. Thus equation (4.2) does not hold anymore and in fact

$$\lambda(t,T) > E^Q[h_T|\mathcal{F}_t, \tau > T].$$

Chen & Panjer [10] divided the forward hazard rate into two parts: jump part which is $E^Q[h_T|\mathcal{F}_t, \tau > T]$ here and diffusion part which is $\lambda(t,T) - E^Q[h_T|\mathcal{F}_t, \tau > T]$.

Based on the decomposition theorem above, let us define the following stopping times:

$$\begin{aligned}
\tau_0 &= \inf\{t > 0 : V_t \leq D_t < V_{t-}, V_s > D_s \text{ for } s < t\} \\
&= \inf\{t > 0 : \Delta \ln V_t \leq \ln D_t - \ln V_{t-} < 0, V_s > D_s \text{ for } s < t\}; \\
\tau^* &= \inf\{t > 0 : V_t \leq D_t = V_{t-}, V_s > D_s \text{ for } s < t-\} \\
&= \inf\{t > 0 : \Delta \ln V_t \leq 0, D_t = V_{t-}, V_s > D_s \text{ for } s < t-\}.
\end{aligned}$$

τ_0 is the default time that the default is caused by a jump and τ^* is the default time that the default is caused by diffusion. The definition of τ^* is similar to Definition 4.1 in [10]. Clearly, τ_0 is *totally inaccessible* while τ^* is neither *totally inaccessible* nor *predictable* because the event $\{\Delta \ln V_t \leq 0\}$ is unpredictable and there exists a predictable stopping time $S = \inf\{t > 0 : D_t = V_{t-}, V_s > D_s \text{ for } s < t-\}$ such that $Q\{\omega : \tau^*(\omega) = S(\omega) < \infty\} > 0$. The graphs of the stopping times are

$$\begin{aligned}
[\tau] &= \{(t,\omega) \in \mathcal{R}^+ \times \Omega : V_t \leq D_t \leq V_{t-}, V_s > D_s \text{ for } s < t-\}, \\
[\tau_0] &= \{(t,\omega) \in \mathcal{R}^+ \times \Omega : V_t \leq D_t < V_{t-}, V_s > D_s \text{ for } s < t-\}, \\
[\tau^*] &= \{(t,\omega) \in \mathcal{R}^+ \times \Omega : V_t \leq D_t = V_{t-}, V_s > D_s \text{ for } s < t-\}, \\
[S] &= \{(t,\omega) \in \mathcal{R}^+ \times \Omega : V_{t-} = D_t, V_s > D_s \text{ for } s < t-\}.
\end{aligned}$$

Thus $[\tau] = [\tau_0] \cup [\tau^*]$, $[\tau_0] \cap [\tau^*] = \emptyset$ and $[\tau^*] \subset [S]$. Here τ^* is *accessible*, τ_0 and τ^* are mutually independent. Moreover, since we define $\inf \emptyset = \infty$, $\tau_0 = \infty$ when $\tau = \tau^* < \infty$ and $\tau^* = \infty$ when $\tau = \tau_0 < \infty$, thus the default time can be expresses as $\tau = \tau_0 \wedge \tau^*$. Notice that

$$[S] \setminus [\tau^*] = \{(t, \omega) \in \mathcal{R}^+ \times \Omega : V_t > D_t, V_{t-} = D_t, V_s > D_s \text{ for } s < t-\},$$

which is the case that even though $V_{t-} = D_t$, $\tau > t$ due to the positive jump of V at time t.

4.3 First-passage model with Lévy processes

Let the firm value be modeled as an exponential Lévy process,

$$V_t = V_0 \exp\{\sigma_V W_t + L_t\}, \qquad \int_{|x|>1} e^x \nu(dx) < \infty \tag{4.4}$$

where W is a one-dimensional standard Brownian motion, $L \in \mathcal{R}$ is a Lévy process with generating triplet $(0, \nu, \gamma_V)$ and $\sigma_V > 0$ is the volatility. The inequality of the Lévy measure ensures that the firm value has finite expectation. Let r be the constant risk-free interest rate, then the discounted firm value is a Q-martingale and we have $\frac{1}{2}\sigma_V^2 + \ln E(e^{L_1}) = r$. The generating triplet of the Lévy process $\ln V_t - \ln V_0$ is $(\sigma_V^2, \nu, \gamma_V)$. The firm value follows the SDE:

$$\begin{aligned} dV_t/V_{t-} &= \frac{\sigma_V^2}{2} dt + \sigma_V dW_t + dL_t + (e^{\Delta L_t} - 1 - \Delta L_t) \\ &= r dt + \sigma_V dW_t + \int_R (e^x - 1)\,[N(ds, dx) - \nu(dx) ds]. \end{aligned}$$

4.3.1 Instantaneous default probability in Exp-Lévy model

On $\{\tau > t\}$, the (stochastic) hazard rate (also known as the intensity) in reliability analysis is defined as

$$h_t = \lim_{s \downarrow 0} \frac{P(t < \tau \le t + s\,|\mathcal{F}_t)}{s}.$$

It is well known that the intensity does not exist when the firm value follows a jump diffusion process. One may refer to section 4 in Chen [9] for the existence of instantaneous default intensity. As he discussed, default occurs with probability 1 due to diffusion when the firm value is about to cross the default boundary. When the firm value is far away from the default boundary, default is caused by jump only with some finite intensity. In the latter case, there exists a positive number δ such that $\ln V_t - \ln D_t > \delta$ and the finite intensity is

$$h_t = \lim_{s \downarrow 0} \frac{P(V_{t+s} \leq D_{t+s} | \mathcal{F}_t)}{s}. \tag{4.5}$$

In the following part of this thesis, we only consider the case that the firm value is far away from the default boundary. Under such assumption, the instantaneous default probability is finite as shown in the following theorem.

Theorem 4.3.1 *If the firm value process follows an exp-Lévy process as equation (4.4), the default time is defined as equation (4.3) and the default boundary is either deterministic or follows a geometric Brownian motion. Assume that the distance* $\ln V_t - \ln D_t$ *is bounded by* $\delta > 0$ *from below for all* $t < \tau$. *Given all the information up to time t, then on* $\{\tau > t\}$ *the instantaneous default probability is*

$$h_t = \nu((-\infty, \ln D_t - \ln V_t]),$$

where $\nu(\cdot)$ *is the Lévy measure of* $\ln V_t - \ln V_0$ *and* h_t *is bounded by* $\nu((-\infty, -\delta]) < \infty$.

For two exp-Lévy firm value processes with default threshold level D_t^1 *and* D_t^2, D_t^i, $i = 1, 2$, *are deterministic or follow geometric Brownian motion, denote* $\nu^J(\cdot, \cdot)$ *be the joint Lévy measure of* $\ln V_t^1 - \ln V_0^1$ *and* $\ln V_t^2 - \ln V_0^2$. *If the distances* $\ln V_t^i - \ln D_t^i$ *are bounded by* $\delta^i > 0$, *then the joint instantaneous default probability is*

$$h_t^J = \nu^J((-\infty, \ln D_t^1 - \ln V_t^1], (-\infty, \ln D_t^2 - \ln V_t^2]),$$

which is bounded by $\nu^J((-\infty, -\delta^1], (-\infty, \delta^2])$. *These instantaneous default probabilities are adapted to the filtration* $\mathcal{F}_t$.

Proof: Let $F(\cdot; t)$ be the CDF of the Lévy process $\ln V_t - \ln V_0$, followed by equation (4.5),

$$\begin{aligned} h_t &= \lim_{s \downarrow 0} \frac{P(\ln V_{t+s} - \ln V_t \leq \ln D_{t+s} - \ln V_t | \mathcal{F}_t)}{s} \\ &= \lim_{s \downarrow 0} \frac{E[F(\ln D_{t+s} - \ln V_t; s) | \mathcal{F}_t]}{s} \end{aligned}$$

If the boundary D_t is a continuous function of t, since V_t is a Markov process, then

$$h_t = \lim_{s \downarrow 0} \frac{F(\ln D_{t+s} - \ln V_t; s)}{s} = \lim_{s \downarrow 0} \frac{\int_{-\infty}^{\ln D_t - \ln V_t} F(dx; s)}{s}.$$

Theorem 2.1.2 shows that the Lévy density $\nu(dx) = \lim_{s\downarrow 0} F(dx; s)/s$, thus

$$h_t = \int_{-\infty}^{\ln D_t - \ln V_t} \nu(dx) = \nu((-\infty, \ln D_t - \ln V_t]).$$

If D_t is a geometric Brownian motion, assume that $\ln D_{t+s} - \ln D_t$ follows a normal distribution with mean $a(t)s$ and standard deviation $b(t)\sqrt{s}$, where $a(t), b(t)$ are deterministic functions, then

$$\begin{aligned} h_t &= \lim_{s\downarrow 0} \frac{E[F(\ln D_{t+s} - \ln V_t; s)|V_t, D_t]}{s} \\ &= \lim_{s\downarrow 0} \int_{\mathcal{R}} \frac{F(\ln D_t + (a(t)s + xb(t)\sqrt{s}) - \ln V_t; s)}{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \end{aligned}$$

First observe that the CDF $F(\ln D_t + (a(t)s + xb(t)\sqrt{s}) - \ln V_t; s) \leq 1$ and

$$\lim_{s\downarrow 0} \frac{F(\ln D_t + (a(t)s + xb(t)\sqrt{s}) - \ln V_t; s)}{s} = \nu((-\infty, \ln D_t - \ln V_t]),$$

which is bounded by $\nu((-\infty, -\delta])$. Thus there exists some constant B such that

$$\left| \frac{F(\ln D_t + (a(t)s + xb(t)\sqrt{s}) - \ln V_t; s)}{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right| \leq B \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and the upper bound is integrable. By dominated convergence theorem, we have

$$\begin{aligned} h_t &= \int_{\mathcal{R}} \nu((-\infty, \ln D_t - \ln V_t]) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \nu((-\infty, \ln D_t - \ln V_t]). \end{aligned}$$

Similarly, in the two-variate case, let $F^J(\cdot, \cdot; t)$ be the joint CDF of $\ln V_t^1 - \ln V_0^1$ and $\ln V_t^2 - \ln V_0^2$, we may prove that the joint instantaneous default probability is

$$\begin{aligned} h_t^J &= \lim_{s\downarrow 0} \frac{P(\ln V_{t+s}^i \leq \ln D_{t+s}^i | \mathcal{F}_t, \ln V_t^i > \ln D_t^i)}{s}, \quad i = 1, 2 \\ &= \nu^J((-\infty, \ln D_t^1 - \ln V_{t-}^1], (-\infty, \ln D_t^2 - \ln V_{t-}^2]). \end{aligned}$$

Since D_t and V_t are $\mathcal{F}_t$−measurable, thus the instantaneous default probabilities h_t and h_t^J are adapted to the filtration $\mathcal{F}_t$.

The results need to be verified for other types of predictable default boundary processes.

□

h_t increases with the ratio D_t/V_t (or it decreases with the distance of $\ln V_t - \ln D_t$). If $\lim_{\delta\downarrow 0}\nu((-\infty,-\delta]) = \infty$, the instantaneous default probability tends to be infinity when the firm value V_t approaches to the threshold D_t but default may not occur immediately if the movement of the Brownian motion or the jump is upward at t. Similarly, if $\lim_{\delta\downarrow 0}\nu((-\infty,-\delta]) < \infty$. When V_t approaches to D_t, even though the instantaneous default probability is less than infinity, default happens immediately when the movement of the Lévy process is downward. If the Lévy measure is undefined on the negative real line, that is $\nu((-\infty,0)) = 0$, the instantaneous default probability is zero, then $\tau = \tau^*$ and the default is caused by the Brownian motion only.

Empirically, the instantaneous default probability h_t will match the short term credit spread with zero recovery and the forward hazard rate $\lambda(t,T) = -\partial \ln S(t,T)/\partial T$ will match the long term credit spread with zero recovery.

Proposition 4.3.1 *h_t is the intensity of the totally inaccessible stopping time τ_0 on $\{\tau > t\}$, a subset of $\{\tau_0 > t\}$, and the intensity of τ_0 is 0 on $\{\tau_0 > t > \tau\}$.*

Proof: Since $\tau \le \tau_0$, given $\{\tau > t\}$, we have $\{\tau_0 > t\}$. The instantaneous default probability

$$\begin{aligned} h_t &= \lim_{s\downarrow 0}\frac{P(\tau > t, V_{t+s} \le D_{t+s}\,|\mathcal{F}_t)}{s} \\ &= \lim_{s\downarrow 0}\frac{P(\tau_0 > t, V_{t+s} \le D_{t+s}, V_t > D_t\,|\mathcal{F}_t)}{s} \\ &= \lim_{s\downarrow 0}\frac{P(t < \tau_0 \le t+s\,|\mathcal{F}_t)}{s}, \end{aligned}$$

which is the intensity of τ_0 on $\{\tau > t\}$. On $\{\tau_0 > t > \tau\}$, the intensity of τ_0 is 0 because the default time $\tau = \tau^*$ and $\tau_0 = \infty$.

□

Here is an example that uses the structure approach in the intensity-based model. Madan & Unal [41] define default as occurring when the outgoing cash flow triggers a short-fall of equity. Equity is composed of non-financial assets S (modeled as a geometric Brownian motion) and financial assets less liability in the amount $g(r)$, a function of the short rate.

And the unforseen cash flow has the arrival rate λ and the loss distribution is exponential with mean loss rate μ_L. This intensity model can be represented as

$$\tau = \inf\{t > 0 : \Delta L_t \geq S_t + g(r_t)\}, \quad h_t = \lambda \exp\left\{-\frac{S_t + g(r_t)}{\mu_L}\right\}.$$

Here L_t is the cumulative value of outgoing cash flows and the default intensity has a similar form as ours. Taking a local linear approximation in $\ln S$ and r around their initial values, the intensity becomes

$$\hat{h}_t = h_0 \left[1 - \frac{1}{\mu_L}\left(e^{\ln S_0}(\ln S_t - \ln S_0) + g'(r_0)(r_t - r_0)\right)\right] = a - b \ln S_t + c r_t.$$

$g'(r_0)$ is call the *duration* of the net financial asset $g(r)$ at time 0, $b, c > 0$ and

$$a = \lambda \exp\left\{-\frac{S_0 + g(r_0)}{\mu_L}\right\} + b \ln S_0 - c r_0.$$

$\hat{h}$ is a decreasing function of the asset value S and it increases with r_t.

Given no default before (but not including) t, that is $\tau \geq t$, we define another instantaneous default probability as

$$h_t^* = \lim_{s \downarrow 0} \frac{P(t - s < \tau \leq t \,|\mathcal{F}_{t-s})}{s}. \tag{4.6}$$

Under the same assumptions in Theorem 4.3.1,

$$\begin{aligned} h_t^* &= \lim_{s \downarrow 0} \frac{P(\ln V_t - \ln V_{t-s} \leq \ln D_t - \ln V_{t-s} | \ln V_{t-s}, \ln D_{t-s})}{s} \\ &= \nu((-\infty, \ln D_t - \ln V_{t-}]). \end{aligned}$$

h_t^* is $\mathcal{F}_{t-}$−measurable (predictable). There is no big difference between h_t and h_t^* except that one is $\mathcal{F}_t$−measurable and the other is predictable. We choose to use h_t^* in the following sections due to its predictability. In Chapter 5, we will use h_t for simplicity.

4.3.2 Intensity-based model: $\tau = \tau_0$

If the default time is restricted to be τ_0, that is $\tau^* = \infty$, then this structure model is an intensity-based model. Similar to the example above. The default time is

$$\tau = \tau_0 = \inf\{t > 0 : \Delta L_t + R_t \leq 0\}, \tag{4.7}$$

where $R_t = \ln V_{t-} - \ln D_t$ is the log ratio of the (pre-)firm value to the threshold level, which can also be considered as a measure of the distance to default. Since $V_{t-} > D_t$ before default, R_t is a positive predictable process. There are lots of choices to model the positive predictable process R_t. One may model R as a diffusion process, which means the default level D will jump after the jump of the firm value if such jump in V does not cause default and both jumps will be canceled out as $\ln D_{t+} - \ln D_t = \ln V_t - \ln V_{t-}$. Here the threshold process D_t is left continuous with right limits. Then the survival probability can be obtained by $Q(\tau > t) = E^Q[e^{-\int_0^t h_s^* ds}]$, where the default intensity is $h_s^* = \nu((-\infty, -R_s])$.

This method can be treated as the case that the hazard rate depends on the firm value in the intensity-based model. Here is an example, Madan & Unal [40] take as a sufficient statistic on the well being of a firm its equity value measured in units of the money market account. The hazard rate is modeled as a function of this relative price $s_t = S_t/B_t$, where S_t is the stock price, B_t is the value of money market account and $ds_t = \theta s_t dW_t$, $\theta > 0$. Thus s_t is a martingale and the hazard rate is

$$h_t^* = \frac{c}{(\ln s_t - \ln \delta)^2}, \qquad c, \delta > 0. \tag{4.8}$$

If $\delta < s_0$, h^* is a decreasing function of s and default happens immediately as s_t approaches δ. If $\delta > s_0$, h^* increases with s and default happens immediately as s_t approaches δ. The exact location of δ is left to be empirical determined.

In equation (4.7), let $R_t = (\ln s_t - \ln \delta)^2$ for $s_t \neq \delta$, a strictly positive predictable process and let $\nu(dx) = cx^{-2} I_{\{x<0\}} dx$ with

$$\int_{\mathcal{R}} (|x|^2 \wedge 1)\nu(dx) = \int_{-1}^{0} x^2 (cx^{-2}) dx + \int_{-\infty}^{-1} cx^{-2} dx = 2c < \infty.$$

Since $ds_t = \theta s_t dW_t$, R_t will be equal to $\left(\ln s_0 - \ln \delta - \frac{1}{2}\theta^2 t + \theta W_t\right)^2$, which satisfies the following SDE

$$dR_t = (1 - \sqrt{R_t})\theta^2 dt + 2\sqrt{R_t}\theta dW_t.$$

And the default intensity is matched to equation (4.8):

$$\nu((-\infty, -R_t]) = cR_t^{-1} = \frac{c}{(\ln s_t - \ln \delta)^2},$$

a decreasing function of R.

Assume that R follows a Markov process as

$$dR_t = \alpha(t, R_t)dt + \beta(t, R_t)dW_t,$$

then by following Madan and Unal's idea in [40]: $e^{-\int_0^t h_s^* ds} Q(\tau > T|\mathcal{F}_t, \tau > t)$ is a Q-martingale. It's not true if default is caused by either Brownian motion or random jumps and in such case we consider the Q-martingale $E^Q[I_{\{\tau>T\}}|\mathcal{F}_t]$. Let the function $\psi(t, R_t) = Q(\tau > T|\mathcal{F}_t, \tau > t)$ and apply the Itô's formula on $e^{-\int_0^t h^*(R_s)ds}\psi(t, R_t)$, we obtain the following PDE

$$\begin{cases} \psi_t + \alpha(t,R)\psi_R + \frac{1}{2}\beta(t,R)^2\psi_{RR} = h^*(R)\psi, \quad t \in [0,T], R > 0 \\ \psi(T,R) = 1 \end{cases} \tag{4.9}$$

where ψ_t, ψ_R are the first derivative of ψ to t and R and ψ_{RR} is the second derivative of ψ to R. The absorbing boundary condition $\psi(t, 0) = 0$ can be omitted since R is strictly positive. Its Feynman-Kac solution is the survival probability $E^Q[e^{-\int_t^T h^*(R_s)ds}|\mathcal{F}_t]$.

As in Zhou [56], we assume that the jump component of the firm value is purely firm-specific and is uncorrelated with the market. Then in this setup, the random jumps are the direct reason of default. The systematic risk appears in the Gaussian component and so in the stochastic structure of default intensity, which only affects the likelihood of default but cannot lead to the default event directly. Generally, when the whole economy goes down, the default intensity will increase and then the likelihood of default will increase. But default only happens on some companies because of the different firm-specific risk (jump risk).

4.3.3 Survival probability in general case: $\tau = \tau_0 \wedge \tau^*$

In the general case where default is caused by either Brownian motion or random jumps, that is $\tau = \tau_0 \wedge \tau^*$. We rewrite it as

$$\tau = \inf\{t > 0 : Y_t := \ln V_t - \ln D_t \le 0\}.$$

The intensity of τ_0 (or the instantaneous default probability) is $\nu((-\infty, -Y_{t-}])$. For certain threshold that $\ln D_t - \ln D_0$ is a Lévy process, in fact a Brownian motion, then $Y_t - Y_0$ is also a Lévy process with the same Lévy measure as $\ln V_t - \ln V_0$. The default probability is

$$Q(\tau \leq t) = Q\left(\inf_{0\leq s\leq t} \frac{V_s}{D_s} \leq 1\right) = Q\left(\inf_{0\leq s\leq t} (Y_s - Y_0) \leq -Y_0\right),$$

define $I_t = \inf_{0\leq s\leq t}(Y_s - Y_0)$, the infimum process of the Lévy process $Y_t - Y_0$. The probability $Q(I_t \leq Y_0)$ does not have a general closed form expression, but it can be generated from the Fourier transform of I which satisfies (see Theorem 5.3 in Sato [50])

$$q\int_0^\infty e^{-qt}E^Q[e^{izI_t}]dt = \exp\left\{\int_0^\infty t^{-1}e^{-qt}dt\int_{-\infty}^0 (e^{izx}-1)du_t(x)\right\},\quad q > 0$$

$u_t(x)$ is the CDF of $Y_t - Y_0$. It may also be generated from Baxter & Donsker's result. Denote $p(a,t) = Q(I_t > -a)$, $a > 0$, then the default probability is $1 - p(Y_0, t)$. The double Laplace transform of $p(a,t)$ can be expressed in terms of $\psi(\xi) = \ln E^Q[e^{-i\xi(Y_1-Y_0)}]$, the characteristic exponent of $-(Y_t - Y_0)$.

Theorem 4.3.2 *(Theorem 1 in Baxter & Donsker [4]) For all positive u and λ,*

- *If $\psi(\xi)$ is real,*
$$u\int_0^\infty\int_0^\infty e^{-ut-\lambda a}d_a p(a,t)dt = \exp\left\{\frac{1}{2\pi}\int_u^\infty\int_{-\infty}^\infty \frac{\lambda}{\lambda^2+\xi^2}\frac{\psi(\xi)}{s(s-\psi(\xi))}d\xi ds\right\}.$$
- *If $\psi(\xi)$ is complex, and for some $\delta > 0$, $\int_{-\delta}^{\delta}|\psi(\xi)/\xi|d\xi < \infty$,*
$$u\int_0^\infty\int_0^\infty e^{-ut-\lambda a}d_a p(a,t)dt = \exp\left\{\frac{1}{2\pi}\int_u^\infty\int_{-\infty}^\infty \frac{\lambda}{\xi(\xi-i\lambda)}\frac{\psi(\xi)}{s(s-\psi(\xi))}d\xi ds\right\}.$$

The numeric solution of the default probability may be obtained by inverting the Fourier or Laplace transforms. Another option is to use PIDE (partial-integro differential equation) approach to obtain the survival probability.

Proposition 4.3.2 *Consider a first passage time model where $\tau = \inf\{t > 0 : X_t \leq a\}$, where X is a Lévy process with generating triplet (σ^2, ν, γ), $\sigma > 0$ and $a < 0$. Fix $T > t$ and let $Y_t = X_t - a$, the survival probability $Q(\tau > T|Y_t = y)$ on $\{\tau > t\}$ is given by $\psi(t,y)$ on $\{y > 0\}$, where $\psi(t,y) = 0$ for $t \in [0,T]$, $y \leq 0$ and for $t \in [0,T]$, $y > 0$*

$$\psi : [0,T]\times(0,\infty) \to (0,1]$$

verifies

$$\begin{aligned} h^*(y)\psi &= \psi_t + \gamma\psi_y + \frac{1}{2}\sigma^2\psi_{yy} + \\ &\int_{\mathcal{R}}[(\psi(t,y+x) - \psi(t,y))I_{\{x+y>0\}} - x\psi_y(t,y)I_{\{|x|\leq 1\}}]\nu(dx) \end{aligned} \quad (4.10)$$

on $[0, T) \times \mathcal{R}^+$ *with the terminal condition*

$$\forall y > 0, \quad \psi(T, y) = 1. \tag{4.11}$$

Here $h^*(y) = \int_{\{x+y \le 0\}} \nu(dx)$ *and* $h^*(Y_{t-})$ *is the instantaneous default probability of* τ *on* $\{\tau \ge t\}$. *If* $\psi(t, y)$ *is* $C^{1,2}$ *(or* $C^{1,1}$ *in the case of* $\sigma = 0$*), then* ψ *is a solution to (4.10) and (4.11) with the boundary* $\psi(t, y) = 0$ *for* $t \in [0, T]$, $y \le 0$.

Proof: Since $\{Y_t > 0\} \supseteq \{\tau > t\}$, given $\tau > t$ then $y > 0$ and

$$\begin{aligned} Q(\tau > T | Y_t = y) &= Q(\inf_{t \le s \le T} X_s > a | X_t = y + a) = Q(\inf_{t \le s \le T} Y_s > 0 | Y_t = y) \\ &= Q(\inf_{t \le s \le T} (Y_s - Y_t) > -y) = Q(\inf_{0 \le s \le T-t} Y_s > -y). \end{aligned}$$

The last equality is the consequence of the property of independent stationary increments of X. Fix the maturity T, define

$$S(t, Y_t) = E^Q[I_{\{\tau > T\}} | \mathcal{F}_t] = I_{\{\tau > t\}} E^Q[I_{\{\tau > T\}} | Y_t, \tau > t] = \psi(t, Y_t) I_{\{\tau > t\}}, \tag{4.12}$$

and

$$\psi(t, Y_t) = \begin{cases} Q(\tau > T | Y_t, \tau > t) & \text{here } Y_t > 0 \text{ since } \tau > t, \\ 0 & \text{if } Y_t \le 0. \end{cases}$$

Then $S(t, Y_t)$ is a Q-martingale satisfying $E^Q[dS(t, Y_t) | \mathcal{F}_{t-}] = 0$ with $dS(t, Y_t)$ is equal to

$$\psi(t, Y_{t-}) dI_{\{\tau > t\}} + I_{\{\tau > t-\}} d\psi(t, Y_t) + (I_{\{\tau > t\}} - I_{\{\tau > t-\}})(\psi(t, Y_t) - \psi(t, Y_{t-})).$$

Here $E^Q[dI_{\{\tau > t\}} | \mathcal{F}_{t-}] = -I_{\{\tau > t-\}} h^*(Y_{t-}) dt$, $h^*(Y_{t-}) = \nu((-\infty, -Y_{t-}])$ is the instantaneous default probability. If $\psi(t, y)$ is C^2 continuous on y such that ψ, ψ_y and ψ_{yy} are bounded by a constant, the martingale-drift decomposition of functions of a Lévy process (Proposition 8.16 in Cont & Tankov [12]) shows that $E^Q[d\psi(t, Y_t) | \mathcal{F}_{t-}]$ is equal to

$$\begin{aligned} &\Big(\int_{\mathcal{R}} [\psi(t, Y_{t-} + x) - \psi(t, Y_{t-}) - x\psi_y(t, Y_{t-}) I_{\{|x| \le 1\}}] \nu(dx) + \\ &\psi_t + \gamma \psi_y + \frac{1}{2} \sigma^2 \psi_{yy} \Big) \, dt =: (\psi_t(t, Y_{t-}) + \mathcal{L}\psi(t, Y_{t-})) dt, \end{aligned}$$

here $\mathcal{L}$ is called the integro-differential operator of Y. And the conditional expectation of the last term in $dS(t, Y_t)$ is

$$\begin{aligned}
& E^Q\left[(I_{\{\tau>t\}} - I_{\{\tau>t-\}})(\psi(t,Y_t) - \psi(t,Y_{t-}))|\mathcal{F}_{t-}\right] \\
= & E^Q\left[(I_{\{Y_t\leq 0,\tau>t-\}} - I_{\{\tau>t-\}})(\psi(t,Y_t) - \psi(t,Y_{t-}))|\mathcal{F}_{t-}\right] \\
= & -h^*(Y_{t-})I_{\{\tau>t-\}} \times (0 - \psi(t,Y_{t-})) \\
= & h^*(Y_{t-})I_{\{\tau>t-\}}\psi(t,Y_{t-}).
\end{aligned}$$

Thus $E^Q[dS(t,Y_t)|\mathcal{F}_{t-}]$ is equal to

$$-I_{\{\tau>t-\}}h^*(Y_{t-})dt + I_{\{\tau>t-\}}(\psi_t(t,Y_{t-}) + \mathcal{L}\psi(t,Y_{t-}))dt + h^*(Y_{t-})I_{\{\tau>t-\}}\psi(t,Y_{t-})dt,$$

then we have $I_{\{\tau>t-\}}(\psi_t(t,Y_{t-}) + \mathcal{L}\psi(t,Y_{t-})) = 0$. That is, given $\{\tau \geq t\}$, $E^Q[d\psi(t,Y_t)|\mathcal{F}_{t-}] = 0$, in other words, $\psi(t,Y_t)$ is a Q-martingale on $\{\tau \geq t\}$ and $\psi(t,y)$satisfies the PIDE

$$\begin{cases} \psi_t(t,y) + \mathcal{L}\psi(t,y) = 0, & t \in [0,T), y > 0 \\ \psi(t,y) = 0, & t \in [0,T], y \leq 0. \end{cases}$$

To obtain equation (4.10), we may arrange $\mathcal{L}\psi(t,y)$ as

$$\begin{aligned}
& \gamma\psi_y + \frac{1}{2}\sigma^2\psi_{yy} + \int_{\mathcal{R}} [\psi(t,y+x) - \psi(t,y) - x\psi_y(t,y)I_{\{|x|\leq 1\}}]\nu(dx) \\
= & \gamma\psi_y + \frac{1}{2}\sigma^2\psi_{yy} + + \int_{\{x+y\leq 0\}} (0 - \psi(t,y))\nu(dx) \\
& + \int_{\mathcal{R}} [(\psi(t,y+x) - \psi(t,y))I_{\{x+y>0\}} - x\psi_y(t,y)I_{\{|x|\leq 1\}}]\nu(dx) \\
= & \gamma\psi_y + \frac{1}{2}\sigma^2\psi_{yy} - h^*(y)\psi(t,y) \\
& + \int_{\mathcal{R}} [(\psi(t,y+x) - \psi(t,y))I_{\{x+y>0\}} - x\psi_y(t,y)I_{\{|x|\leq 1\}}]\nu(dx).
\end{aligned}$$

When $\tau > T$, $Q(I_{\{\tau>T\}}|\mathcal{F}_T) = 1$, thus $\psi(T,y) = 1$ for $y > 0$, the terminal condition (4.11). Proposition 12.6 in Cont & Tankov [12] shows that the survival probability is the (unique) Feynman-Kac solution to (4.10) and (4.11) if $\psi(t,y)$ is $C^{1,2}$ continuous on $[0,T] \times (0,\infty)$.

□

Remark 4.3.1 *As shown in this proof, $\psi(t, Y_t)$ is a Q-martingale on $\{\tau \geq t\}$. While when we look at the behavior of $\psi(t, Y_t)$ itself,*

$$d\psi(t, Y_t) = \psi_t dt + \frac{1}{2}\sigma^2 \psi_{yy} dt + \psi_y dY_t + \psi(t, Y_{t-} + \triangle Y_t) - \psi(t, Y_{t-}) - \psi_y \triangle Y_t.$$

Given $\{\triangle Y_t + Y_{t-} > 0\}$, that is $\{\tau > t\}$, ψ represents the survival probability and

$$E^Q[d\psi(t, Y_t)|\mathcal{F}_{t-}] = \psi_t(t, Y_{t-}) + \mathcal{L}\psi(t, Y_{t-}) + h^*(Y_{t-})\psi(t, Y_{t-}) = h^*(Y_{t-})\psi(t, Y_{t-}).$$

It implies that $\psi(t, Y_t)$ is a submartingle on $\{\tau > t\}$ under measure Q.

Remark 4.3.2 *The survival probability is similar to the barrier option, particularly the down-and-out option, discussed in Cont & Voltchkova [13], where the terminal payoff is 1 if the barrier is not crossed. Proposition 5 in [13] summarizes the results on the continuity of the barrier options. The smooth property of ψ in y is not required in the case of pure jump processes with $\sigma = 0$ and $\gamma - \int_{\{|x|\leq 1\}} x\nu(dx) = 0$, where the PIDE (4.10) is $h^*(y)\psi = \psi_t + \int_{\mathcal{R}}(\psi(t, y+x) - \psi(t, y))I_{\{x+y>0\}}\nu(dx)$.*

Consider the case that $\int_{\{|x|\leq 1\}} |x|\nu(dx) < \infty$, let $\mu = \gamma - \int_{\{|x|\leq 1\}} x\nu(dx)$, then $\psi(t, y) \in C^{1,2}([0, T] \times \mathcal{R}^+)$ will satisfies the PIDE

$$\begin{cases} \psi_t + \mu\psi_y + \frac{1}{2}\sigma^2\psi_{yy} + \int_{\{x+y>0\}}[\psi(t, y+x) - \psi(t, y)]\nu(dx) = h^*(y)\psi(t, y) \\ \psi(T, y) = 1. \end{cases} \tag{4.13}$$

Generally, this PIDE dose not have a closed-form solution and numeric methods are required. For a discuss of such PIDEs, refer to Cont & Voltchkova[13]. Here are some special cases. The first one is that Y is a Brownian motion with mean μ and standard deviation σ, then $\tau = \tau^*$ and $\tau_0 = \infty$ with zero instantaneous default probability and the PIDE is reduced to the following PDE: $\psi(t, y) \in C^{1,2}([0, T] \times \mathcal{R}^+)$

$$\begin{cases} \psi_t + \mu\psi_y + \frac{1}{2}\sigma^2\psi_{yy} = 0 \\ \psi(T, y) = 1. \end{cases}$$

Its solution is the survival probability in the first-passage model when the firm value follows a geometric Brownian motion:

$$\psi(t, y) = \Phi\left(\frac{\mu(T-t) + y}{\sigma\sqrt{T-t}}\right) - e^{\frac{-2\mu y}{\sigma^2}}\Phi\left(\frac{\mu(T-t) - y}{\sigma\sqrt{T-t}}\right).$$

Here $\Phi(\cdot)$ is the CDF of a standard normal distribution. If y is not restricted to be positive, that is $\psi \in C^{1,2}([0,T] \times \mathcal{R})$, then the (Feynman-Kac) solution to the PDE

$$\begin{cases} \psi_t + \mu\psi_y + \frac{1}{2}\sigma^2\psi_{yy} = 0 \\ \psi(T,y) = I_{\{y>0\}} \end{cases}$$

is the survival probability in Merton's Model, where

$$\psi(t,y) = \Phi\left(\frac{\mu(T-t)+y}{\sigma\sqrt{T-t}}\right) = E^Q[I_{\{Y_T>0\}} \,|\, Y_t = y].$$

Define $\tau_y = \inf\{T > t : Y_{T-t} \leq -y\}$, consider the Laplace transform of τ_y, $l : (0,\infty) \to (0,1)$,

$$l(y) = E[e^{-a(\tau_y - t)}] = \int_t^\infty e^{-a(u-t)} f(u;t,y)du, \qquad a, y > 0.$$

Here $f(u;t,y), u > t$ is the PDF of τ_y. Then the Laplace transform of the survival function $\psi(t,T,y) = P(\tau > T | \tau > t, Y_t = y)$ is

$$\begin{aligned}
\int_t^\infty e^{-a(u-t)}\psi(t,u,y)du &= \int_t^\infty e^{-a(u-t)} \int_u^\infty f(s;t,y)dsdu \\
&= \int_t^\infty f(s;t,y) \int_t^s e^{-a(u-t)}duds \\
&= \frac{1}{a}\int_t^\infty (1 - e^{-a(s-t)})f(s;t,y)ds = \frac{1-l(y)}{a}.
\end{aligned}$$

Let L^{-1} be the operator of Laplace inverse, the survival probability $\psi(t,y) = \psi(t,T,y)$ and its derivatives are

$$\begin{aligned}
\psi(t,y) &= L^{-1}\left\{\frac{1-l(y)}{a}\right\}; \\
\psi_t(t,y) &= f(T;t,y) = L^{-1}\{l(y)\}; \\
\psi_y(t,y) &= L^{-1}\left\{\frac{-l'(y)}{a}\right\}; \\
\psi_{yy}(t,y) &= L^{-1}\left\{\frac{-l''(y)}{a}\right\}.
\end{aligned}$$

Since the Laplace inverse has the rule of linearity, if $l(y)$ is twice continuous differentiable at y, the PIDE of ψ in equation (4.10) will be

$$\begin{aligned}
h^*(y)\frac{1-l(y)}{a} &= l(y) + \gamma\frac{-l'}{a} + \frac{\sigma^2}{2}\frac{-l''}{a} \\
&+ \int_{\mathcal{R}} \left[\left(\frac{1}{a}[1-l(x+y)] - \frac{1}{a}[1-l(y)]\right) I_{\{x+y>0\}} - x\frac{-l'}{a}I_{\{|x|\leq 1\}}\right]\nu(dx).
\end{aligned}$$

That is

$$al(y) = h^*(y)[1 - l(y)] + \gamma l' + \frac{\sigma^2}{2} l''$$
$$+ \int_{\mathcal{R}} \left[(l(x+y) - l(y)) I_{\{x+y>0\}} - x l' I_{\{|x|\le 1\}} \right] \nu(dx). \tag{4.14}$$

It is the integro-differential equation of the function $l(y)$ and it does not involve the time. The terminal condition (4.11) is automatically satisfied because

$$\lim_{t\to T} \psi(t,y) = \lim_{a\to\infty} L^{-1}\left\{\frac{1-l(y)}{a}\right\} = \lim_{a\to\infty} a\left\{\frac{1-l(y)}{a}\right\} = 1, \quad \text{as} \lim_{a\to\infty} l(y) = 0.$$

It is the consequence of *the Initial and Final-Value Theorems of Laplace transforms* stated as following: let $L\{f(t)\} = \hat{f}(a)$, $t \ge 0$, then

$$\lim_{t\to 0^+} f(t) = \lim_{a\to\infty} a\hat{f}(a) \quad \text{and} \quad \lim_{t\to\infty} f(t) = \lim_{a\to 0} a\hat{f}(a).$$

On $y \le 0$, defined $l(y) = 1$ and then $l(y)$ satisfies

$$\begin{cases} al = \gamma l' + \frac{1}{2}\sigma^2 l'' + \int_{\mathcal{R}} [l(x+y) - l(y) - x l' I_{\{|x|\le 1\}}]\nu(dx), & y > 0 \\ l(y) = 1, \quad y \le 0 \end{cases}$$

the right side of the equation is the integro-differential operator generator of the Lévy process Y. Kuo & Wang [34] solved this equation in the case of the double exponential jump diffusion process. It is shown in section 5.2.1. Without the boundary condition, the general solution to $al = \mathcal{L}l(y)$ has the form of

$$l(y) = \sum_{i=1}^{n} A_i e^{\beta_i y},$$

where β_i are the solutions to $\ln E[e^{\beta Y_1}] = a$ and A_i are constant.

4.3.4 Dependence of default when $\tau = \tau_0$

Giesecke [26] discussed the default correlation in intensity-based model as following. A natural way is to introduce correlation between firm's intensity processes through time, while defaults are independent given the intensity paths. Another way is to allow common jumps. In our model, the default correlation can also be built in these two ways. Basically, the dependence of two Lévy processes (jump part vs. jump part, Brownian motion vs. Brownian motion) determines the type of default correlation.

The dependence of jump parts in two Lévy processes controls the common jumps. For two firm value processes with log ratios R^i, their marginal and joint survival probabilities are

$$\begin{aligned} P(\tau_i > t_i) &= E\left[\exp\left\{-\int_0^{t_i} h_s^i ds\right\}\right], \quad i = 1, 2; \\ P(\tau_1 > t_1, \tau_2 > t_2) &= E\left[\exp\left\{-\int_0^{t_1} h_s^1 ds - \int_0^{t_2} h_s^2 ds + \int_0^{\min\{t_1,t_2\}} h_s^J ds\right\}\right], \end{aligned}$$

where $h_t^i = \nu^i((-\infty, R_t^i])$ and the joint default intensity $h_t^J = \nu^J((-\infty, R_t^1], (-\infty, R_t^2])$.

If two jump parts are independent, then defaults are independent given the intensity paths. And the default correlation relies on the dependence of intensity processes h^1 and h^2, which is drawn from the correlation of R_t^1 and R_t^2. More specifically, the resource of the second type of default correlation is the dependence of the Gaussian parts in two Lévy processes.

4.4 Bond price and credit spread in general case

4.4.1 Recovery of treasury value: deterministic interest rate

Now, consider the RTV recovery schema. If the recovery rate at default is a random variable with mean μ_ω, given no default before t ($\tau \geq t$) the price of a defaultable bond with maturity T and face value 1 is

$$\begin{aligned} B(t,T) &= e^{-\int_t^T r_s ds} E^Q[I_{\{\tau > T\}} + \omega_\tau I_{\{\tau \leq T\}} \,|\, \mathcal{F}_t, \tau \geq t] \\ &= e^{-\int_t^T r_s ds}[\omega_t I_{\{\tau = t\}} + \mu_\omega I_{\{\tau > t\}} + (1-\mu_\omega) S(t, Y_t)]. \end{aligned}$$

Denote $\bar{B}(t, Y_t) = e^{\int_t^T r_s ds} B(t,T)$, then on $\{\tau \geq t\}$

$$d\bar{B}(t, Y_t) = (1-\mu_\omega) dS(t, Y_t) + (\omega_t - \mu_\omega) I_{\{\tau = t\}}.$$

It is well known that the discounted bond price $e^{-\int_0^T r_s ds} \bar{B}(t, Y_t)$ is a Q-martingale and so is $\bar{B}(t, Y_t)$. On the other hand,

$$E^Q[d\bar{B}(t, Y_t) | \mathcal{F}_{t-}] = (1-\mu_\omega) E^Q[dS(t, Y_t) | \mathcal{F}_{t-}] + (E^Q[\omega_t | \mathcal{F}_{t-}] - \mu_\omega) h^*(Y_{t-}) dt = 0,$$

which also confirms that $\bar{B}(t, Y_t)$ is a Q-martingale. On $\{\tau > t\}$, the instantaneous credit spread is

$$\begin{aligned} cs_t &= \frac{E^Q[(1-\mu_\omega)d\psi(t,Y_t)|\mathcal{F}_{t-}]}{\bar{B}(t,Y_{t-})dt} \\ &= \frac{(1-\mu_\omega)h^*(Y_{t-})\psi(t,Y_{t-})}{\mu_\omega + (1-\mu_\omega)\psi(t,Y_{t-})} \\ &= h^*(Y_{t-})\left(1 - \frac{\mu_\omega}{\bar{B}(t,Y_{t-})}\right). \end{aligned}$$

If Y_t is a jump-diffusion process, the bond price $B(t,T)$ on $\{\tau > t\}$ is also a jump-diffusion process. If Y_t is a diffusion process, then $h^*(y) = 0$ and the bond price $B(t,T)$ on $\{\tau > t\}$ is a diffusion process and only jumps at default. $\omega_t e^{-\int_\tau^T r_s ds}$ is the bond price at default, while the bond price just before default is

$$B(\tau-,T) = \lim_{y\downarrow 0} e^{-\int_{\tau-}^T r_s ds}\bar{B}(\tau-,y) = \mu_\omega e^{-\int_\tau^T r_s ds}.$$

It is the amount that the bondholders expected to receive if default happens in the next moment. $(\omega_\tau - \mu_\omega)e^{-\int_\tau^T r_s ds}$ is the difference between the real amount and the expectation. Thus, it is possible that the bond price will jump up upon default if $\omega_\tau - \mu_\omega > 0$ or remain at the same level if $\omega_\tau - \mu_\omega = 0$ or jump down if $\omega_\tau - \mu_\omega < 0$.

Now consider the case that the recovery rate is a function of Y at default, that is

$$\begin{aligned} \bar{B}(t,Y_t) &= E^Q[I_{\{\tau>T\}} + \omega(Y_\tau)I_{\{\tau\le T\}}|\mathcal{F}_t, \tau \ge t] \\ &= \psi(t,Y_t) + E^Q[\omega(Y_\tau)I_{\{\tau\le T\}}|Y_t, \tau \ge t]. \end{aligned}$$

On $\{\tau \ge t\}$, apply Itô's formula to $\bar{B}(t, Y_t)$,

$$\begin{aligned} d\bar{B}(t,Y_t) &= (\bar{B}(t,Y_t) - \bar{B}(t,Y_{t-}))I_{\{\tau=t\}} + [\bar{B}_t dt + \frac{1}{2}\sigma^2\bar{B}_{yy}dt + \bar{B}_y dY_t \\ &\quad + \bar{B}(t, Y_{t-} + \Delta Y_t) - \bar{B}(t,Y_{t-}) - \bar{B}_y\Delta Y_t]I_{\{\Delta Y_t + Y_{t-} > 0\}}. \end{aligned} \tag{4.15}$$

$\bar{B}(t, Y_t)$ is a Q-martingale on $\{\tau \ge t\}$ and then $\bar{B}(t,y) : [0,T] \times \mathcal{R}^+ \to (0,1]$ satisfies the following PIDE on $[0,T) \times \mathcal{R}^+$

$$\begin{aligned} &(\mu_\omega(y) - \bar{B}(t,y))h^*(y) + \bar{B}_t + \gamma\bar{B}_y + \frac{1}{2}\sigma^2\bar{B}_{yy} + \\ &\int_{\mathcal{R}}[(\bar{B}(t,y+x) - \bar{B}(t,y))I_{\{x+y>0\}} - x\bar{B}_y I_{\{|x|\le 1\}}]\nu(dx) = 0 \end{aligned}$$

with terminal condition

$$\bar{B}(T,y) = 1, \quad y > 0,$$

and boundary condition

$$\bar{B}(t,y) = \omega(y), \quad t \in (0,T], y \leq 0.$$

Here $\mu_\omega(y), y > 0$ is the expected recovery rate under Q

$$\mu_\omega(y) = E^Q[\omega(x+y)|x+y \leq 0] = \frac{\int_{\{x+y\leq 0\}} \omega(x+y)\nu(dx)}{\int_{\{x+y\leq 0\}} \nu(dx)}. \tag{4.16}$$

The PIDE of the survival probability (4.10) is a special case where the recovery rate $\omega(\cdot) = 0$.

On $\{\tau > t\}$, the instantaneous credit spread is

$$\begin{aligned} cs_t &= \frac{E^Q[I_{\{\tau>t\}} d\bar{B}(t,Y_t)|\mathcal{F}_{t-}]}{\bar{B}(t,Y_{t-})dt} \\ &= h^*(Y_{t-})\left(1 - \frac{\mu_\omega(Y_{t-})}{\bar{B}(t,Y_{t-})}\right) \\ &= \int_{\{x+Y_{t-}\leq 0\}} \left(1 - \frac{\omega(x+Y_{t-})}{\bar{B}(t,Y_{t-})}\right)\nu(dx). \end{aligned} \tag{4.17}$$

At maturity, $\bar{B}(t,Y_{T-}) = 1$ on $\{\tau > T\}$ and the instantaneous credit spread has the same form as the one in Duffie & Singleton's model,

$$cs_T = h^*(Y_{T-})(1 - \mu_\omega(Y_{T-})).$$

Chen & Kou [11] (Theorem 2) state a similar result with a double exponential jump diffusion process, which is shown as an example in section 5.2.1 with $h^*(y) = \lambda p e^{-y\eta_1}$ in equation (5.7) and the recovery rate function $\omega(y) = ce^y$, here $c \in [0,1]$ is constant and $y = \ln(V/F)$. The mean recovery rate is

$$\mu_\omega(y) = \frac{\int_{\{x+y\leq 0\}} ce^{x+y}\lambda p\eta_1 e^{\eta_1 x}dx}{\int_{\{x+y\leq 0\}} \lambda p\eta_1 e^{\eta_1 x}dx} = c\frac{\eta_1}{\eta_1+1}.$$

If $\nu(\cdot) = 0$, we cannot model the recovery rate as a function of Y because there would be no surprise of both the default event and the recovery rate when Y is a continuous process. And then the bond price is continuous all the time, even at default. It does not make sense.

4.4.2 Recovery of market value: deterministic interest rate

Assume that the recovery rate is a function of Y as $B(\tau,T) = \omega^*(Y_\tau)B(\tau-,T)$, that is $\bar{B}(\tau,T) = \omega^*(Y_\tau)\bar{B}(\tau-,T)$. Similarly as (4.15), on $\{\tau \geq t\}$

$$\begin{aligned} d\bar{B}(t,Y_t) &= (\omega^*(Y_t)-1)\bar{B}(t,Y_{t-})I_{\{\tau=t\}} + [\bar{B}_t dt + \frac{1}{2}\sigma^2\bar{B}_{yy}dt + \bar{B}_y dY_t \\ &\quad + \bar{B}(t,Y_{t-}+\triangle Y_t) - \bar{B}(t,Y_{t-}) - \bar{B}_y\triangle Y_t]I_{\{\triangle Y_t+Y_{t-}>0\}}. \end{aligned}$$

$\bar{B}(t,y)$ will satisfy the following PIDE on $[0,T)\times\mathcal{R}^+$,

$$\begin{aligned} &(\mu^*_\omega(y)-1)h^*(y)\bar{B}(t,y) + \bar{B}_t + \gamma\bar{B}_y + \frac{1}{2}\sigma^2\bar{B}_{yy} + \\ &\int_{\mathcal{R}}[(\bar{B}(t,y+x)-\bar{B}(t,y))I_{\{x+y>0\}} - x\bar{B}_y I_{\{|x|\leq 1\}}]\nu(dx) = 0 \end{aligned} \tag{4.18}$$

with terminal condition $\bar{B}(T,y) = 1$ on $y > 0$, where $\mu^*_\omega(y)$ the the expected recovery rate as in equation (4.16). The instantaneous credit spread on $\{\tau > t\}$ is

$$cs_t = h^*(Y_{t-})(1-\mu^*_\omega(Y_{t-})) = \int_{\{x+Y_{t-}\leq 0\}}(1-\omega^*(x+Y_{t-}))\nu(dx),$$

which has the similar form as equation (4.17), the credit spread in the RTV scheme.

The problem here is that there is no way to set the boundary condition as in the RTV model since the recovery amount does not dependent on t and y only, it also depends on the pre-default bond value. Thus, the solution is not unique.

4.4.3 Stochastic interest rate

We may extend the results to time homogeneous stochastic short rate:

$$dr_t = a(r_t)dt + b(r_t)dW_t^r,$$

here W_t^r is the standard Brownian motion with $dW_t dW_t^r = \rho dt$. The default bond price at time t will be a function of Y_t, r_t and t. Denote

$$\begin{aligned} LB(t,Y_t,r_t) &= B_t + \gamma B_y + a(r)B_r + \frac{1}{2}[\sigma^2 B_{yy} + b(r)^2 B_{rr} + 2\rho\sigma b(r)B_{yr}] \\ &\quad + \int_{\mathcal{R}}[(B(t,y+x,r) - B(t,y,r))I_{\{x+y>0\}} - xB_y I_{\{|x|\leq 1\}}]\nu(dx). \end{aligned}$$

Consider the case that recovery rate is a function of Y. The PIDE of the bond price in the RTV model is

$$\begin{cases} LB(t,y,r) = (r + h^*(y))B(t,y,r) - \mu_\omega(y)h^*(y)P(t,r), & t \in [0,T), y > 0 \\ B(T,y,r) = 1, \quad y > 0 \\ B(t,y,r) = \omega(y)P(t,T), \quad t \in (0,T], y \leq 0. \end{cases}$$

Where $P(t,r_t) = E^Q[e^{-\int_t^T r_s ds}|\mathcal{F}_t]$ is the Treasury value at time t with maturity T and face value 1. The PIDE of the bond price in the RMV model

$$\begin{cases} LB(t,y,r) = [r + (1 - \mu_\omega^*(y))h^*(y)]B(t,y,r), & t \in [0,T), y > 0 \\ B(T,y,r) = 1, \quad y > 0. \end{cases}$$

Let $C(\tau, Y_\tau, r_\tau)$ be the recovery amount at default and define its expectation just before default as $\mu_C(\tau, Y_{\tau-}, r_{\tau-}) = E^Q[C(\tau, Y_\tau, r_\tau)|\mathcal{F}_{\tau-}]$, then the bond price can be expressed as

$$B(t,T) = E^Q[e^{-\int_t^T r_s ds}I_{\{\tau > T\}} + e^{-\int_t^\tau r_s ds}C(\tau, Y_\tau, r_\tau)I_{\{\tau \leq T\}} \mid \mathcal{F}_t, \tau \geq t].$$

Thus the defaultable bond price before default $B(t,y)$ will satisfy the generalized PIDE

$$\begin{cases} LB(t,y,r) = (r + h^*(y))B(t,y,r) - h^*(y)\mu_C(t,y,r), & t \in [0,T), y > 0 \\ B(T,y,r) = 1, \quad y > 0 \\ B(t,y,r) = C(t,y,r), \quad t \in (0,T], y \leq 0. \end{cases} \tag{4.19}$$

$\mu_C(t,y,r) = \mu_\omega(y)P(t,r)$ in the RTV model and $\mu_C(t,y,r) = \mu_\omega^*(y)B(t,y,r)$ in the RMV model. The instantaneous credit spread is

$$cs_t = h^*(Y_{t-})\left(1 - \frac{\mu_C(t, Y_{t-}, r_t)}{B(t, Y_{t-})}\right), \quad \tau > t.$$

4.5 Instantaneous transition matrix of credit rating

$Y_t = \ln V_t - \ln D_t$ is the logarithmal of the ratio of firm value to the default level, which measures the distance to default. When we consider the class of credit rating with state $\{1, 2, \cdots, K\}$, where state 1 is the highest credit rating class, state 2 is second highest and K is the default state. Let $D_1 > D_2 > \cdots > D_{K-1} = 0$, be the boundaries of states, that is a firm is in state 1 if its state variable $Y_t > D_1$, it is in state 2 if $D_2 < Y_t \leq D_1$ and so on. Default (state K) occurs when $Y_t \leq D_{K-1} = 0$.

The instantaneous transition matrix can be obtained in the similar way as the instantaneous default probability, the instantaneous transition probability from state i to state j ($i \neq j, i, j = 1, 2, \cdots, K-1$) is

$$\lambda_{ij}(Y_{t-}) = \int_{D_j < Y_{t-}+y \leq D_{j-1}} \nu(dy) = \nu((D_j - Y_{t-}, D_{j-1} - Y_{t-}]), \quad Y_{t-} \in (D_i, D_{i-1}].$$

$\lambda_{ij} \in [0, \infty)$ because the interval $(D_j - Y_{t-}, D_{j-1} - Y_{t-}]$ is bounded away from 0. And the instantaneous transition probability from i to absorption state K is

$$\lambda_{iK} = \nu((-\infty, -Y_{t-}]), \quad Y_{t-} \in (D_i, D_{i-1}].$$

When $\nu((-\infty, 0)) = 0$, $\lambda_{ij} = 0$ if $i > j$. When $\nu((0, \infty)) = 0$, $\lambda_{ij} = 0$ if $i < j$. The instantaneous transition matrix is

$$A_X(t) = \begin{pmatrix} -\lambda_1(X_t) & \lambda_{12}(X_t) & \lambda_{13}(X_t) & \cdots & \lambda_{1K}(X_t) \\ \lambda_{21}(X_t) & -\lambda_2(X_t) & \lambda_{23}(X_t) & \cdots & \lambda_{2K}(X_t) \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \lambda_{K-1,1}(X_t) & \lambda_{K-1,2}(X_t) & \cdots & -\lambda_{K-1}(X_t) & \lambda_{K-1,K}(X_t) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$\lambda_i = \sum_{j=1, j\neq i}^{K} \lambda_{ij}$, $i = 1, 2, \cdots, K-1$.

4.6 Summary and extension

We model the firm value as an exponential Lévy process (equation (4.4)) and the default threshold is predictable. Starting with the first-passage firm value model, the instantaneous default probability h_t^* is the Lévy measure of X on the interval $(-\infty, -Y_{t-}]$, where Y_{t-} is the log ratio of the (pre) firm value and threshold and it measures the distance to default. We decompose the default time into two parts: a totally inaccessible stopping time with intensity h_t^* and a predictable stopping time.

By assuming the default is only caused by jumps, we discover that it is an intensity-based model. In the general case, assume that Y is also a Lévy process, the PIDE of the defaultable bond price (4.19) is derived. The PIDE for the survival probability (4.10) is a special case of (4.19) with zero interest rate and zero recovery amount. The solution to (4.10) with terminal condition (4.11) is not closed form except for some special cases.

The results can be generated to additive process with local characteristics. For example, let $(\sigma_t^2, \nu_t, \gamma_t)$ be the local characteristics of $Y_t = \ln V_t - \ln D_t$ with

$$\int_0^T \int_{|x|>1} e^x \nu_t(dx)dt < \infty,$$

T is the maturity od the default bond. The default intensity is $h_t^* = \nu_t((-\infty, -Y_{t-}]) =: h^*(t, Y_{t-})$ and and define $LB(t, Y_t, r_t)$ be

$$\begin{aligned} & B_t + \gamma_t B_Y + a(r)B_r + \frac{1}{2}[\sigma_t^2 B_{YY} + b(r)^2 B_{rr} + 2\rho\sigma_t b(r) B_{Yr}] \\ + & \int_{\mathcal{R}} [(B(t, y+x, r) - B(t, y, r))I_{\{x>-y\}} - xB_Y(t, y, r)I_{\{|x|\le 1\}}]\nu_t(dx). \end{aligned}$$

Then the PIDE (4.19) can be modified as

$$\begin{cases} LB(t, y, r) = (r + h^*(t, y))B(t, y, r) - h^*(t, y)\mu_C(t, y, r), & t \in [0, T], y > 0 \\ B(T, y, r) = 1, \quad y > 0. \end{cases}$$

4.7 Risk-neutral measure vs. physical measure

In the intensity-based model, the default intensity under the physical measure h^{*P} is different from the default intensity under the risk-neutral measure h^{*Q} not only in current levels but also in dynamics. Generally, $h_t^{*Q} \ge h_t^{*P}$ due to the that investors are risk-aversion and their difference reflects the premium for default-timing risk. Duffie & Singleton [19] show two approached to parameterize h^{*Q} from h^{*P}. One is to infer information about h^{*Q} from the market prices of defautlable bonds. The other is to parameterize the transformation between h^{*Q} and h^{*P} explicitly. In our model, their relation lies on the equivalent martingale transformation of the firm values.

The EMM transformation for Lévy processes has been studied in Chapter 3. Assume that under P, the firm value follow an exponential Lévy process

$$\ln V_t = \ln V_0 + \gamma_V^P t + \sigma_V W_t^P + \int_{|x|\le 1} \int_0^t x[N(ds, dx) - \nu^P(dx)ds] + \int_{|x|>1} \int_0^t xN(ds, dx),$$

with $\int_{|x|>1} e^x \nu^P(dx) < \infty$. After the martingale measure transformation

$$\ln V_t = \ln V_0 + \gamma_V^Q t + \sigma_V W_t^Q + \int_{|x|\le 1} \int_0^t x[N(ds, dx) - \nu^Q(dx)ds] + \int_{|x|>1} \int_0^t xN(ds, dx),$$

with $\int_{|x|>1} e^x \nu^Q(dx) < \infty$ and γ_V^Q satisfies

$$r = \gamma_V^Q + \frac{1}{2}\sigma_V^2 + \int_{\mathcal{R}} (e^x - 1 - xI_{\{|x|\le 1\}})\nu^Q(dx).$$

The key points of the transformation are

$$\begin{aligned} \nu^Q(dx) &= e^{\phi(e^x-1)}\nu^P(dx), \quad \int_{\mathcal{R}} (e^{\frac{1}{2}\phi(e^x-1)} - 1)^2 \nu^P(dx) < \infty \\ W_t^Q &= W_t^P + \sigma_V^{-1}\left[\gamma_V^P - \gamma_V^Q - \int_{|x|\le 1} x(\nu^P - \nu^Q)(dx)\right] t. \end{aligned}$$

Here we use $\phi(e^x - 1)$ since the firm value process is an exponential Lévy process, while we use $\phi(x)$ in Chapter 3 where the asset processes are stochastic exponential Lévy processes. An exponential Lévy process e^{L_t} satisfies the SDE

$$de^{L_t} = e^{L_{t-}}\left[dL_t + \frac{1}{2}d[L^c, L^c]_t + (e^{\Delta L_t} - 1 - \Delta L_t)\right] =: e^{L_{t-}} d\hat{L}_t,$$

where $\hat{L}_t$ is a Lévy process and e^{L_t} is a stochastic exponential of $\hat{L}_t$. The Lévy measure of L and $\hat{L}$ satisfies

$$\nu_L(A) = \nu_{\hat{L}}(\hat{A}), \quad \hat{A} = \{e^z - 1 : z \in A\} = \{x : \ln(1+x) \in A\}.$$

See Proposition 8.22 in Cont & Tankov [12] for the details of the relation between ordinary and stochastic exponential.

Under measure P and Q, the firm values (the threshold) are at the same level but with different dynamics. Then the values of $Y_t = \ln V_t - \ln D_t$ are the same under both measures. The instantaneous default probabilities are

$$\begin{aligned} h^{*P}(Y_{t-}) &= \int_{-\infty}^{-Y_{t-}} \nu^P(dz), \\ h^{*Q}(Y_{t-}) &= \int_{-\infty}^{-Y_{t-}} \nu^Q(dz) = \int_{-\infty}^{-Y_{t-}} e^{\phi(e^z-1)}\nu^P(dz). \end{aligned}$$

Their current levels and dynamics differ from each other except the case of $\nu^P = \nu^Q$, where they have the same value with different dynamics. Their difference is

$$h^{*Q}(y) - h^{*P}(y) = \int_{-\infty}^{-y} (e^{\phi(e^z-1)} - 1)\nu^P(dz), \quad y > 0. \tag{4.20}$$

In one-dimension case, recall that $\phi(x) = -\ln(1+\pi x)$ in the numeraire portfolio, π the equation (3.6) based on the stochastic exponential form as

$$\gamma_1 - \sigma^2\pi - \int_{\mathcal{R}} \frac{x^2\pi}{1+\pi x}\nu(dx) = 0.$$

γ_1 is the instantaneous return on assets minus the risk-free interest rate. $\gamma_1 > 0$ in the risk-aversion world, then $\pi > 0$. The difference in (4.20) is

$$\int_{-\infty}^{-y} (e^{\phi(e^z-1)} - 1)\nu^P(dz) = \int_{-\infty}^{-y} \left(\frac{1}{1+\pi(e^z-1)} - 1\right)\nu^P(dz) > 0,$$

since $e^z - 1 \in (-1, 0)$ as $z < -y < 0$.

$\phi(x) = \ln(1+\hat{\eta}x)$ in the (Föllmer-Schweizer) minimal martingale measure and $\hat{\eta} = -(\sigma^2 + \int_{\mathcal{R}} x^2\nu(dx))^{-1}\gamma_1 < 0$. The difference in (4.20) is

$$\int_{-\infty}^{-y} (e^{\phi(e^z-1)} - 1)\nu^P(dz) = \int_{-\infty}^{-y} \hat{\eta}(e^z - 1)\nu^P(dz) > 0.$$

$\phi(x) = \bar{\eta}x$ in the minimal entropy martingale measure and $\bar{\eta}$ satisfies equation (3.8)

$$\gamma_1 + \sigma^2\bar{\eta} + \int_{\mathcal{R}} x(e^{\bar{\eta}x} - 1)\nu(dx) = 0.$$

If $\bar{\eta} > 0$, then $x(e^{\bar{\eta}x} - 1) > 0$ for all x and the equation above does not hold. Thus $\bar{\eta}$ must be negative and the difference in (4.20) is

$$\int_{-\infty}^{-y} (e^{\phi(e^z-1)} - 1)\nu^P(dz) = \int_{-\infty}^{-y} (e^{\bar{\eta}(e^z-1)} - 1)\nu^P(dz) > 0.$$

For the three risk-neutral measures, the instantaneous default probability under the risk-neutral measure is higher than the instantaneous default probability under the physical measure in the risk-aversion world. If investors are risk-favor, $\gamma_1 < 0$ and $h_t^{*Q} - h_t^{*P} < 0$. If investors are risk-neutral, $\gamma_1 = 0$ and there is no difference between h_t^{*Q} and h_t^{*P}.

5

Examples of first-passage models with Lévy processes

Define $\tau := \inf\{t > 0 : X_t \geq b\}$ and $\inf\emptyset = \infty$, where $b > 0$ and X is a one-dimensional Lévy process with generating triplet (σ^2, γ, ν) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_{t\geq 0}, P)$. Then τ is a stopping time with respect to $\mathcal{F}$. If $\sigma \neq 0$, the stopping time τ does not have an associated intensity but its instantaneous default probability does exist as

$$h_t = \lim_{s\downarrow 0} \frac{P(t < \tau \leq t + s|\mathcal{F}_t)}{s} = \lim_{s\downarrow 0} \frac{P(X_{t+s} - X_t \geq b - X_t|\mathcal{F}_t)}{s}, \quad \tau > t.$$

While $\sigma = 0$ does not guarantee the existence of the intensity because the drift of X may also cause the default. The formula used here is slightly different from the one used in Chapter 4, equation (4.6). We choose the $\mathcal{F}_t$-adapted version just for simplifying the expression.

The default time τ with constant barrier is also called the time to ruin in ruin problem for Lévy-typed risk processes. A lot of works have been done in ruin probabilities, one may refer to Asmussen [2] for more information.

Before analyzing the default event in the jump diffusion processes, we deals with the first-passage time of jump processes. Examples are the jump processes with nondecreasing sample paths with/without drifts and the jump processes whose sample paths are not monotone. The instantaneous default probabilities, the survival probabilities, the hazard functions and the PIDEs of survival probabilities are analyzed in each case.

5.1 Jump processes with $\sigma = 0$

5.1.1 Hazard rate and intensity when τ is totally inaccessible

Since the stopping time is defined as the first time that the Lévy process approaches to a positive number b, $\{X_t < b\} \supseteq \{\tau > t\} = \{X_s < b, s \in [0,t]\}$ and $\{X_t < b\} = \{\tau > t\}$ if and only if X has a nondecreasing sample path. Given $\{\tau > t\}$, the instantaneous arrival probability is

$$h_t = \lim_{s \downarrow 0} \frac{P(X_{t+s} - X_s \geq b - X_t | X_t)}{s} = \nu([b - X_t, \infty)) =: h(X_t), \quad X_t < b.$$

It is an increasing function of X_t with the condition $X_t < b$ and it is adapted to $\mathcal{F}_t$.

If $\gamma + \int_{|x|>1} x\nu(dx) \leq 0$ in addition to $\sigma = 0$, $h(X_t)$ is the intensity of τ, then

$$I_{\{\tau \leq t\}} - \int_0^t h(X_s) I_{\{\tau > s\}} ds$$

is a P-martingale. Then the probability density function and the hazard function are

$$\begin{aligned} f(t) &= \frac{dE[I_{\{\tau \leq t\}}]}{dt} = E[h(X_t) I_{\{\tau > t\}}], \\ \lambda(t) &= \frac{f(t)}{P(\tau > t)} = \frac{E[h(X_t) I_{\{\tau > t\}}]}{E[I_{\{\tau > t\}}]} = E[h(X_t) | \tau > t]. \end{aligned}$$

If X_t has nondecreasing sample path, the survival probability is $S(t) = P(X_t < b)$ and $f(t) = E[h(X_t) I_{\{X_t < b\}}]$, $\lambda(t) = E[h(X_t) | X_t < b]$ respectively.

Let $E_n = \{X_{t_i} < b,\ t_i = it/n,\ i = 1 \ldots n\}$, then $E_1 \supseteq E_2 \supseteq \ldots$ and $\lim_{n \to \infty} E_n = \{\tau > t\} = \cap_{n=1}^{\infty} E_n$. For $n = 1, 2, \ldots$, the probability density function of τ

$$f(t) \leq \ldots \leq E[h(X_t) I_{\{E_{n+1}\}}] \leq E[h(X_t) I_{\{E_n\}}] \leq \ldots \leq E[h(X_t) I_{\{X_t < b\}}].$$

Since h_t is an increasing function of X_t with the condition $X_t \leq b_-$, where b_- is the possible value of X_t that is most closed to b from below. For instance, $b_- = \lfloor b \rfloor$ for non-integer b and $b_- = b - 1$ for integer b when X_t is a Poisson process. Thus $h(X_t) \leq h(b_-)$.

If $h(b_-) < \infty$, for each n

$$\begin{aligned} f(t) &= E[h(X_t)I_{\{\tau>t\}}] = E[h(X_t)I_{\{E_n\}}] - E[h(X_t)I_{\{E_n,\tau\le t\}}], \\ f(t) &\le E[h(X_t)I_{\{E_n\}}] := R_n, \\ f(t) &\ge E[h(X_t)I_{\{E_n\}}] - E[h(b_-)I_{\{E_n,\tau\le t\}}] \\ &= E[h(X_t)I_{\{E_n\}}] - h(b_-)[P(E_n) - P(\tau > t)] =: L_n. \end{aligned}$$

Since

$$L_{n+1} - L_n = E[(h(X_t) - h(b_-))(I_{\{E_{n+1}\}} - I_{\{E_n\}})] \ge 0,$$

as a conclusion, for $n = 1, 2, \ldots$

$$L_1 \le L_2 \le \ldots \le L_n \le \ldots \le f(x) \le \ldots \le R_n \le \ldots \le R_2 \le R_1,$$

and $f(x) = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$. $f(t) = L_n = R_n$ for any n if X_t has a nondecreasing sample path.

On the other hand for each n, the hazard function satisfies

$$\frac{E[h(X_t)I_{E_n}]}{P(\tau > t)} - h(b_-)\left[\frac{P(E_n)}{P(\tau > t)} - 1\right] \le h(t) \le \frac{E[h(X_t)I_{E_n}]}{P(\tau > t)}. \tag{5.1}$$

Define $h_t^n = E[h(X_t)|E_n]$, then

$$h_t^n = \frac{E[h(X_t)I_{E_n}]}{P(E_n)} \le \frac{E[h(X_t)I_{E_n}]}{P(\tau > t)}, \quad h_t^n \le \frac{E[h(b_-)I_{E_n}]}{P(E_n)} = h(b_-) \text{ and}$$
$$\frac{E[h(X_t)I_{E_n}]}{P(\tau > t)} - h(b_-)\left[\frac{P(E_n)}{P(\tau > t)} - 1\right] \le \frac{E[h(X_t)I_{E_n}]}{P(\tau > t)} - h_t^n\left[\frac{P(E_n)}{P(\tau > t)} - 1\right] = h_t^n.$$

Thus h_t^n and h_t are in the same range as in (5.1). In most cases that the joint distribution of X_t and its running maximum is unknown, we may use h_t^n to estimate $\lambda(t) = E[h(X_t)|\tau > t] = \lim_{n\to\infty} h_t^n$. The absolute error is much less than $h(b_-)[\frac{P(E_n)}{P(\tau>t)} - 1]$ provided that $h(b_-) < \infty$. When n increases, the value becomes smaller and approaches to 0 as n goes to infinity.

For each example with explicit form of the survival probability, we will show its survival function, probability density function, the instantaneous arrival probability and check whether the conditional expectation $E[h_t|\tau > t]$ is equal to the hazard function. (They are equal if h_t is the intensity.) Moreover, let $Y_t = b - X_t$, we will calculate the following functions on $y > 0$

$$\begin{aligned}
\psi(t,y) &= P(\tau > T | Y_t = y), \\
\lambda(T-t,y) &= -\frac{\partial \ln \psi(t,y)}{\partial T}, \\
h(y) &= \lim_{T \to t} \lambda(T-t,y).
\end{aligned}$$

And we will check the PIDE (4.13) of $\psi(t,y)$

$$\begin{cases} \psi_t - \mu\psi_y + \frac{1}{2}\sigma^2\psi_{yy} + \int_{\{x+y>0\}}[\psi(t,y+x) - \psi(t,y)]\nu(dx) = h(y)\psi(t,y), \\ \psi(T,y) = 1. \end{cases}$$

5.1.2 Compound Poisson process with positive jumps

1. Poisson process

Let N_t be a Poisson process with intensity λ and the threshold level $b = n$, a positive integer. Obviously, $\tau = \inf\{t > 0 : N_t \geq n\}$ is the n^{th} jump time of N_t, which follows a gamma distribution with shape n and rate λ, denoted by Gamma (n, λ). Since the sample path of N_t is nondecreasing, $\{\tau > t\} = \{N_t < n\}$. The survival function and the density function are

$$\begin{aligned}
P(\tau > t) &= \sum_{i=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} = P(N_t < n), \\
f(t) &= \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{\Gamma(n)} = \lambda P(N_t = n-1).
\end{aligned}$$

On $\{\tau > t\}$, the intensity can be obtained by analyzing the instantaneous behavior of N_t as

$$\begin{aligned}
h_t &= \lim_{s \downarrow 0} \frac{P(N_{t+s} = n | N_t)}{s} \\
&= \lim_{s \downarrow 0} \frac{P(N_{t+s} - N_t = 1, N_t = n-1 | N_t)}{s} \\
&= \lim_{s \downarrow 0} \frac{\lambda s I_{\{N_t = n-1\}}}{s} = \lambda I_{\{N_t = n-1\}}.
\end{aligned}$$

Its conditional expectation is

$$E[h_t | \tau > t] = \lambda P(N_t = n-1)/P(N_t < n) = f(t)/P(\tau > t),$$

which is exactly the hazard function.

Here $Y_t = n - N_t$, and the survival probability $P(\tau > T|Y_t = y)$ is

$$\psi(t,y) = \sum_{i=0}^{y-1} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^i}{i!} = \sum_{i=0}^{y-1} P(N_{T-t} = i).$$

Then $\psi \in C^1([0,T])$, $y = 1, 2, \ldots$ The forward hazard rate $\lambda(T-t, y) = -\partial \ln \psi(t,y)/\partial T$ is

$$\lambda(T-t,y) = \frac{\lambda^y (T-t)^{y-1} e^{-\lambda(T-t)}}{(y-1)!} / \psi(t,y) = \frac{\lambda P(N_{T-t} = y-1)}{\psi(t,y)}.$$

The instantaneous default intensity may be rewritten as a function of y as

$$h(y) = \lim_{T \to t} \lambda(T-t,y) = \lambda P(N_0 = y-1) = \lambda I_{\{y=1\}}.$$

Consider the PIDE representation of ψ, the Lévy density of Y is $\nu(dx) = \lambda I_{\{x=-1\}} dx$ and then

$$\begin{aligned}
&\psi_t + \int_{\{x>-y\}} [\psi(t,y+x) - \psi(t,y)] \nu(dx) \\
&= \lambda(T-t,y)\psi(t,y) + \lambda[\psi(t,y-1) - \psi(t,y)] I_{\{y-1>0\}} \\
&= \lambda P(N_{T-t} = y-1) I_{\{y>0\}} - \lambda P(N_{T-t} = y-1) I_{\{y>1\}} \\
&= \lambda P(N_{T-t} = y-1) I_{\{y=1\}} = \lambda P(N_{T-t} = 0) \\
&= \lambda I_{\{y=1\}} \psi(t,1) = h(y)\psi(t,y).
\end{aligned}$$

The terminal condition is $\psi(T,y) = 1$, $y > 0$.

2. Compound Poisson process with exponential jumps

Now consider a compound Poisson process as

$$X_t = \sum_{i=1}^{N_t} J_i, \tag{5.2}$$

J_i's have independent and identical exponential distribution with parameter $\eta > 0$. Given $N_t = k$, the distribution of X_t is Gamma (k, η), thus the survival function and the probability density function of τ are

$$\begin{aligned}
P(\tau > t) &= P(X_t < b) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} G(b; i, \eta), \\
f(t) &= \lambda \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \times \frac{e^{-\eta b}(\eta b)^i}{i!}.
\end{aligned} \tag{5.3}$$

Where $G(b; i, \eta), b > 0$ is the CDF of gamma distribution with shape $i > 0$ and rate $\eta > 0$, which satisfies

$$G(b; i, \eta) - G(b; i+1, \eta) = \frac{e^{-\eta b}(\eta b)^i}{i!} \quad \text{and} \quad G(b; 0, \eta) = 1.$$

Moreover, let $g(b; i, \eta)$ be the associated density function. On $\{\tau > t\}$,

$$\begin{aligned} h_t &= \lim_{s \downarrow 0} \frac{P(X_{t+s} - X_t \geq b - X_t | X_t)}{s}, \quad X_t < b \\ &= \lim_{s \downarrow 0} \frac{P(N_{t+s} - N_t = 1, J \geq b - X_t | X_t)}{s} \\ &= \lim_{s \downarrow 0} \frac{(\lambda s) \times P(J \geq b - X_t)}{s} = \lambda e^{-\eta(b - X_t)}, \end{aligned}$$

with the conditional expectation

$$E[h_t | \tau > t] = \lambda e^{-\eta b} E[e^{\eta X_t} I_{\{X_t < b\}}] / P(X_t < b).$$

And

$$E[e^{\eta X_t} I_{\{X_t < b\}}] = e^{\eta \times 0} P(N_t = 0) + \int_0^b e^{\eta x} \sum_{i=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \times \frac{e^{-\eta x} \eta^i x^{i-1}}{\Gamma(i)} dx$$

$$= e^{-\lambda t} + \sum_{i=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \times \frac{\eta^i \int_0^b x^{i-1} dx}{(i-1)!} = \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \times \frac{\eta^i b^i}{i!},$$

thus the expected intensity conditioned on $\{\tau > t\}$ is the hazard function as $E[h_t | \tau > t] = f(t)/P(\tau > t)$. We will not analyze the forward hazard function and the PIDE representation of the survival function here since it is the special case of the next example where the drift $\mu = 0$.

In these two examples, the events occur at one of the random jump times of Poisson process. The default time can be represented as

$$\tau = \sum_{i=1}^{\infty} \tau_i I_{\{A_i\}},$$

where τ_i is the i^{th} jump time of N_t with density $f_i(t) = g(t; i, \lambda)$ and $A_i = \{\tau = \tau_i\} = \{X_{\tau_i} \geq b, X_s < b, s \in [0, \tau_i)\}$. For instance, $A_i = \{X_{\tau_i} \geq b, X_{\tau_{i-1}} < b\}$ when the sample path of X is nondecreasing. In this examples, the CDF of X_{τ_i} is $G(\cdot; i, \eta)$ and the i^{th} jump size $X_{\tau_i} - X_{\tau_{i-1}}$ follows an exponential distribution with parameter η, which is independent of $X_{\tau_{i-1}}$. That is X_{τ_i} and $X_{\tau_{i-1}}$ do not dependent on the behavior of τ_i. Thus the probability of A_i can be calculated directly as

$$\begin{aligned}
P(A_1) &= P(X_{\tau_1} \geq b, X_{\tau_0} = X_0 < b) = e^{-\eta b}, \\
P(A_i) &= P(X_{\tau_{i-1}} < b, X_{\tau_i} \geq b) = P(X_{\tau_{i-1}} < b, X_{\tau_i} - X_{\tau_{i-1}} \geq b - X_{\tau_{i-1}}) \\
&= \int_0^b g(x; i-1, \eta) \int_{b-x}^{\infty} g(y; 1, \eta) dy dx \\
&= \int_0^b \frac{e^{-\eta x} \eta^{i-1} x^{i-2}}{\Gamma(i-1)} e^{-\eta(b-x)} dx = \frac{e^{-\eta b} (\eta b)^{i-1}}{(i-1)!}.
\end{aligned}$$

And the density function of τ, equation (5.3) can be obtained as

$$\begin{aligned}
f(t) &= \sum_{i=1}^{\infty} f_i(t) \times P(A_i) = \sum_{i=1}^{\infty} \frac{e^{-\lambda t} \lambda^i t^{i-1}}{\Gamma(i)} \times \frac{e^{-\eta b} (\eta b)^{i-1}}{(i-1)!} \\
&= \lambda \sum_{j=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^j}{j!} \times \frac{e^{-\eta b} (\eta b)^j}{j!}.
\end{aligned}$$

It is true if the events A_i do not dependent on the random jump times τ_i, the density function of the first passage time will be $f(t) = \sum_{i=1}^{\infty} g(t; i, \lambda) P(A_i)$. Otherwise, the survival probability is $P(\tau > t) = \sum_{i=1}^{\infty} P(\tau_i > t, A_i)$.

3. Compound Poisson process with exponential jumps and positive drift

Now consider the case that a drift μt is added to X_t. If $u < 0$, the occurrence of the event $\{X_t + \mu t \geq b\}$ will be caused by the random jumps only with intensity $\lambda e^{-\eta(b - X_t - \mu t)}$ on $\{\tau > t\}$. But $X_t + \mu t$ does not have nondecreasing sample path and then $P(\tau > t) \neq P(X_t + \mu t < b)$. In this case, the survival function can be obtained as

$$P(\tau > t) = E\left[\exp\left\{-\int_0^t \lambda e^{-\eta(b - X_s - \mu s)} ds\right\}\right], \mu < 0.$$

If $\mu > 0$, the occurrence of the event $\{X_t + \mu t \geq b\}$ will be caused not only by the random jumps but also by the nature increase of the drift. Thus, the first passage time does not have an associate intensity because of the increasing drift.

$\forall t \in [0, b/\mu)$, the survival function and density function of the first passage time are

$$\begin{aligned} P(\tau > t) &= P(X_t + \mu t < b) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} G(b - \mu t; i, \eta), \\ f(t) &= \underbrace{\lambda \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \frac{e^{-\eta(b-\mu t)}(\eta(b-\mu t))^i}{i!}}_{\textit{caused by random jumps}} + \underbrace{\mu \sum_{i=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} g(b - \mu t; i, \eta)}_{\textit{caused by the increase of drift}}. \\ &:= \mathrm{I}_t + \mathrm{II}_t \end{aligned}$$

The second part II_t can be rewritten as ($\mu \times$ the PDF of $X_t + \mu t$ at b). In this case, the event $A_i = \{X_{\tau_{i-1}} + \mu\tau_{i-1} < b, X_{\tau_i} + \mu\tau_i \geq b\}$ depends on τ_i, thus the density function cannot be represented as $\sum_{i=1}^{\infty} g(t; i, \lambda) P(A_i)$. The instantaneous arrival intensity is

$$\begin{aligned} h_t &= \lim_{s \downarrow 0} \frac{P(X_{t+s} + \mu(t+s) \geq b | X_t)}{s} \\ &= \lim_{s \downarrow 0} \frac{\lambda s P(J \geq b - X_t - \mu(t+s) | X_t) + (1 - \lambda s) I_{\{X_t + \mu(t+s) \geq b\}}}{s} \\ &= \lambda e^{-\eta(b - X_t - \mu t)} + \lim_{s \downarrow 0} \frac{I_{\{X_t + \mu(t+s) \geq b\}}}{s}. \end{aligned}$$

Given $\{\tau > t\} = \{X_t + \mu t < b\}$, there exists a constant $\epsilon > 0$ such that $\epsilon = b - (X_t + \mu t)$. Then for all $s < \epsilon/\mu$, $I_{\{X_t + \mu(t+s) \geq b\}} = I_{\{\mu s \geq \epsilon\}} = 0$. Thus $h_t = \lambda e^{-\eta(b - X_t - \mu t)}$ which shows the motion of the random jumps. And the survival function $S(t) \neq$ (in fact $<$)$E[e^{-\int_0^t h_s ds}]$. But in the sense of the conditional expectation, $E[h_t | \tau > t]$ is equal to

$$\begin{aligned} &\left(E[\lambda e^{-\eta(b - X_t - \mu t)} I_{\{X_t + \mu t < b\}}] + \lim_{s \downarrow 0} \frac{P(b > X_t + \mu t \geq b - \mu s)}{s} \right) / P(X_t + \mu t < b) \\ = &\left(\mathrm{I}_t + \lim_{s \downarrow 0} \sum_{i=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \frac{G(b - \mu t; i, \eta) - G(b - \mu t - \mu s; i, \eta)}{s} \right) / P(\tau > t) \\ = &\left(\mathrm{I}_t + \sum_{i=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \mu g(b - \mu t; i, \eta) \right) / P(\tau > t) = f(t)/P(\tau > t). \end{aligned}$$

It is the hazard function when $t \in [0, b/\mu)$.

Here $Y_t = b - (\mu t + \sum_{i=0}^{N_t} J_i)$, $\mu > 0$, $J_i \sim \exp(\eta)$ and the Lévy density of Y is $\nu(dx) = I_{\{x<0\}} \lambda \eta e^{\eta x} dx$. Given $Y_t = y > 0$, the survival probability $P(\tau > T | Y - t = y)$, $T \in [0, b/\mu)$ is

$$\psi(t, y) = \sum_{i=0}^{\infty} P(N_{T-t} = i) G(y - \mu(T - t); i, \eta),$$

which is zero if $y \le \mu(T-t)$. Then $\psi \in C^{1,1}([0,T] \times (\mu(T-t), \infty))$. The forward hazard function is

$$\begin{aligned}\lambda(T-t,y) &= \left\{\lambda\sum_{i=0}^{\infty}P(N_{T-t}=i)\frac{e^{-\eta(y-\mu(T-t))}[\eta(y-\mu(T-t))]^i}{i!}\right.\\ &\left.+\mu\sum_{i=1}^{\infty}P(N_{T-t}=i)g(y-\mu(T-t);i,\eta)\right\}/\psi(t,y),\end{aligned}$$

and the instantaneous default intensity

$$h(y) = \lim_{T\to t}\lambda(T-t,y) = \lambda e^{-\eta y}.$$

Now consider the PIDE representation of ψ, first $\psi_t + (-\mu)\psi_y$ is equal to

$$\begin{aligned}&\lambda(T-t,y)\psi(t,y) - \mu\sum_{i=1}^{\infty}P(N_{T-t}=i)g(y-\mu(T-t);i,\eta)\\ =\ &\lambda\sum_{i=0}^{\infty}P(N_{T-t}=i)\frac{e^{-\eta(y-\mu(T-t))}[\eta(y-\mu(T-t))]^i}{i!}.\end{aligned}$$

Second,

$$\begin{aligned}&\int_{x+y>0}[\psi(t,x+y)-\psi(t,y)]\nu(dx)\\ =\ &\int_{x+y>\mu(T-t)}\psi(t,x+y)\nu(dx) - \int_{x+y>0}\psi(t,y)\nu(dx)\\ =\ &\lambda\sum_{i=0}^{\infty}P(N_{T-t}=i)\int_{\mu(T-t)-y}^{0}\eta e^{\eta x}\int_0^{x+y-\mu(T-t)}g(z;i,\eta)dzdx\\ &-\lambda\psi(t,y)(1-e^{\eta(-y)}),\end{aligned}$$

the double integration is

$$\begin{aligned}&\int_0^{y-\mu(T-t)}\int_{\mu(T-t)-y+z}^{0}\eta e^{\eta x}dxg(z;i,\eta)dz\\ =\ &\int_0^{y-\mu(T-t)}\left(1-e^{\eta(\mu(T-t)-y+z)}\right)g(z;i,\eta)dz\\ =\ &G(y-\mu(T-t);i,\eta) - e^{-\eta(y-\mu(T-t))}\frac{\eta^i(y-\mu(T-t))^i}{i!}\end{aligned}$$

Thus the left side of the PIDE is

$$\begin{aligned}
&\psi_t - \mu\psi_y + \int_{x+y>0} [\psi(t, x+y) - \psi(t, y)]\nu(dx) \\
= \;& \lambda \sum_{i=0}^{\infty} P(N_{T-t} = i) G(y - \mu(T-t); i, \eta) - \lambda\psi(t, y)(1 - e^{-\eta y}) \\
= \;& \lambda e^{-\eta y}\psi(t, y) = h(y)\psi(t, y).
\end{aligned}$$

The terminal condition is $\psi(T, y) = 1, y > \mu(T-t)$.

5.1.3 Coin tossing problem

Consider the process $\{X_t\}_{t\geq 0}$ with the form (5.2), where J_i's are independent identically distributed Bernoulli variables with distribution $P(J = -1) = P(J = 1) = 1/2$. Its sample path is not monotone and $\{\tau > t\} \subseteq \{X_t < n\}$. Such process is referred as *coin tossing at random times* in Baxter & Donsker [4]. It has been shown that for each positive integer $n = 1, 2, \ldots$

$$P(\sup_{0\leq s\leq t} X_s < n) = 1 - n\int_0^t e^{-\lambda s}\frac{I_n(\lambda s)}{s} ds,$$

here $I_n(x)$ is the modified Bessel function of the first kind with the expression

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{k!\Gamma(n+k+1)} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!(n+k)!}.$$

Define $\tau = \inf\{t > 0 : X_t \geq n\}$, $n = 1, 2, \ldots$, then for each n, the survival function, the density function and the hazard function are

$$\begin{aligned}
P(\tau > t) &= P(\sup_{0\leq s\leq t} X_s < n) = 1 - n\int_0^t e^{-\lambda s}\frac{I_n(\lambda s)}{s} ds, \\
f(t) &= \frac{n}{t} e^{-\lambda t} I_n(\lambda t), \\
\lambda(t) &= f(t)/P(\tau > t).
\end{aligned}$$

On $\{\tau > t\}$, the instantaneous arrival intensity is

$$\begin{aligned}
h_t &= \lim_{s\downarrow 0} \frac{P(X_{t+s} = n|X_t)}{s} \\
&= \lim_{s\downarrow 0} \frac{P(X_{t+s} - X_t = 1|X_t = n-1)}{s} \\
&= \frac{\lambda}{2} I_{\{X_t = n-1\}}.
\end{aligned}$$

And $f(t) = E[h_t I_{\{\tau>t\}}] = \frac{\lambda}{2}E[I_{\{X_t=n-1,\tau>t\}}]$. Assume that there are $2k+n-1$ jumps in $[0,t)$, let X_i be the value of X_t at i^{th} jump of N_t and $M(k,n)$ be the number of paths such that $X_0 = 0, X_{2k+n-1} = n-1, X_i < n$ for $i = 1, 2, \ldots, 2k+n-2$, $k = 0, 1, \ldots$. Then $M(k,n) = n(2k+n-1)!/(k!(n+k)!)$ (see appendix A for proof). While the total number of paths of $2k+n-1$ jumps is 2^{2k+n-1}. Thus the density function is

$$
\begin{aligned}
f(t) &= \frac{\lambda}{2}E[I_{\{X_t=n-1,\tau>t\}}] \\
&= \frac{\lambda}{2}\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{2k+n-1}}{(2k+n-1)!}\times\frac{M(k,n)}{2^{2k+n-1}} \\
&= \frac{\lambda}{2}\sum_{k=0}^{\infty}\frac{ne^{-\lambda t}(\lambda t/2)^{2k+n-1}}{k!(n+k)!} \\
&= \sum_{k=0}^{\infty}\frac{ne^{-\lambda t}(\lambda t/2)^{2k+n}}{tk!(n+k)!} = \frac{n}{t}e^{-\lambda t}I_n(\lambda t).
\end{aligned}
$$

The other way to figure out the density function is based on the decomposition $\tau = \sum_{k=0}^{\infty}\tau_i I_{\{A_k\}}$, where

$$
\begin{aligned}
A_k &= \{X_0 = 0, X_{2k+n} = n, X_i < n \text{ for } i = 1, \ldots, 2k+n-1\} \\
&= \{X_0 = 0, X_{2k+n-1} = n-1, X_{2k+n} = n, X_i < n \text{ for } i = 1, \ldots, 2k+n-2\}.
\end{aligned}
$$

The events do not depended on τ_i, then $f(t) = \sum_{k=0}^{\infty} g(t; 2k+n, \lambda)P(A_k)$ with $P(A_k) = M(k,n)/2^{2k+n}$ and the density is

$$
\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} g(t; 2k+n, \lambda)\times\frac{M(k,n)}{2^{2k+n}} \\
&= \sum_{k=0}^{\infty}\frac{\lambda e^{-\lambda t}(\lambda t)^{2k+n-1}}{(2k+n-1)!}\times\frac{M(k,n)}{2^{2k+n}} \\
&= \sum_{k=0}^{\infty}\frac{ne^{-\lambda t}(\lambda t/2)^{2k+n}}{tk!(n+k)!}.
\end{aligned}
$$

Here $Y_t = n - \sum_{i=0}^{N_t} J_i$, $P(J=1) = P(J=-1) = 1/2$ and the Lévy density of Y is $\nu(dx) = \frac{\lambda}{2}I_{\{x=\pm1\}}$. Given $Y_t = y > 0$, the survival probability is

$$
\psi(t,y) = 1 - y\int_0^{T-t} e^{-\lambda s}\frac{I_y(\lambda s)}{s}ds.
$$

Then $\psi \in C^1([0,T])$, $y = 1, 2, \ldots$ The forward hazard function is

$$\lambda(T-t,y) = ye^{-\lambda(T-t)}\frac{I_y(\lambda(T-t))}{T-t}/\psi(t,y),$$

and the instantaneous default intensity

$$h(y) = \lim_{T\to t}\lambda(T-t,y) = y\sum_{k=0}^{\infty}\frac{(\lambda/2)^{2k+y}(T-t)^{2k+y-1}}{k!(y+k)!} = \frac{\lambda}{2}I_{\{y=1\}}.$$

Thus $h(y)\psi(t,y) = \frac{\lambda}{2}I_{\{y=1\}}\psi(t,y)$ and

$$\begin{aligned}
&\psi_t + \int_{x+y>0}(\psi(x+y)-\psi(y))\nu(dx) \\
= \;& \lambda(T-t,y)\psi(t,y) + \frac{\lambda}{2}\left[I_{\{y+1>0\}}(\psi(t,y+1)-\psi(t,y))\right. \\
&\left.+I_{\{y-1>0\}}(\psi(t,y-1)-\psi(t,y))\right],
\end{aligned}$$

then for $t \in [0,T)$, PIDE (4.13) is equivalent to

$$\begin{cases} \lambda(T-t,y)\psi(t,y) + \frac{\lambda}{2}[\psi(t,y+1)+\psi(t,y-1)-2\psi(t,y)] = 0 & \text{if } y \geq 2, \\ \lambda(T-t,y)\psi(t,y) + \frac{\lambda}{2}[\psi(t,y+1)-2\psi(t,y)] = 0 & \text{if } y = 1, \end{cases} \tag{5.4}$$

with terminal condition $\psi(T,y) = 1$, $y \geq 1$. We will prove it in appendix B.

At time t, X_t is a discrete random variable, for each nonnegative integer $n = 0, 1, \ldots$

$$\begin{aligned}
P(X_t = n) &= P(X_t = -n) \\
&= \sum_{i=0}^{\infty} P(N_t = 2i+n) \times P(\text{there are } i\ -1\text{'s and } n+i\ 1\text{'s}) \\
&= \sum_{i=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{2i+n}}{(2i+n)!}\binom{2i+n}{i}\left(\frac{1}{2}\right)^{2i+n} \\
&= \sum_{i=0}^{\infty}\frac{e^{-\lambda t}(\lambda t/2)^{2i+n}}{i!(i+n)!}.
\end{aligned}$$

Since $\{\tau > t\} \subseteq \{X_t < n\}$, we may use h_t^1 to estimate the hazard function.

$$\begin{aligned}
h_t^1 &= E[h_t|X_t < n] = \frac{\frac{\lambda}{2}P(X_t = n-1)}{P(X_t < n)} \\
&= \frac{\frac{\lambda}{2}P(X_t = n-1)}{\frac{1}{2}(1+P(X_t=0)) + \sum_{i=1}^{n-1}P(X_t = i)}.
\end{aligned} \tag{5.5}$$

Figure 5.1 shows the difference of the estimated hazard rate and its true value $h_t^1 - \lambda(t)$ in basis points, the parameters used are $\lambda = 0.6$, $n = 1, 2, 3, 4$ and $t \in (0, 1]$. 1 basis points

= 0.01. The difference is small for high value of the level n and it increases with time. Figure 5.2 shows the relative difference of survival probability $S(t)$ and $\exp\{-\int_0^t h_s^1 ds\}$: $1 - \exp\{-\int_0^t h_s^1 ds\}/S(t)$. Both show that it is not bad to use h_t^1 to replace $\lambda(t)$ at $\lambda = 0.6$ and $t \in (0, 1]$.

The following table shows the maximum difference of $h_t^1 - \lambda(t)$ (in basis points) at different level: $\lambda = 0.6, 1, 10$; $t \leq 1$, $t \leq 10$ and $n = 1, 2, 3, 4$. Most maximum difference happens on the end of the time horizon except for those two highlighted.

level n	$t \leq 1, \lambda = 0.6$	$t \leq 10, \lambda = 0.6$	$t \leq 1, \lambda = 1$	$t \leq 10, \lambda = 1$	$t \leq 1, \lambda = 10$
1	0.168439	3.800164	0.913636	**6.506188**	**65.061880**
2	0.063237	2.249781	0.327769	4.477918	44.779180
3	0.005520	1.202774	0.048034	2.922774	29.227740
4	0.000338	0.574709	0.004955	1.786208	17.862080

It is brought to our attention that the density function of the first passage time satisfies

$$f(t; \lambda, n) = 10 f(0.1t; 10\lambda, n).$$

Thus

$$\begin{aligned} S(t; \lambda, n) &= \int_0^t f(s; \lambda, n) ds = \int_0^t 10 f(0.1s; 10\lambda, n) ds \\ &= \int_0^{0.1t} 100 f(0.1s; 10\lambda, n) d(0.1s) = 100 S(0.1t; 10\lambda, n); \\ \lambda(t; \lambda, n) &= \frac{f(t; \lambda, n)}{S(t; \lambda, n)} = \frac{10 f(0.1t; 10\lambda, n)}{100 S(0.1t; 10\lambda, n)} = 0.1\lambda(0.1t; 10\lambda, n). \end{aligned}$$

And

$$\begin{aligned} P(X_t = n; \lambda) &= P(X_{0.1t} = n; 10\lambda); \\ h^1(t; \lambda, n) &= \frac{\lambda}{2} \frac{P(X_t = n-1; \lambda)}{P(X_t < n; \lambda)} = 0.1 \frac{10\lambda}{2} \frac{P(X_{0.1t} = n-1; 10\lambda)}{P(X_{0.1t} < n; 10\lambda)} \\ &= 0.1 h^1(0.1t; 10\lambda, n). \end{aligned}$$

Keeping λt constant, we have $\lambda(t/a; a\lambda, n) = a\lambda(t; \lambda, n)$ and $h^1(t/a; a\lambda, n) = a h^1(t; \lambda, n)$ for any positive value of a. It is good to use h_t^1 to estimate the hazard function only for small value of λ and high value of n.

5.2 Poisson-diffusion processes

Now we turn to the jump-diffusion processes. Let

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} J_i, \qquad \sigma > 0.$$

μ is the drift, σ is the volatility, W is a standard Brownian motion and N is a Poisson process with parameter λ, the jump sizes $J_1, J_2, \ldots$ are independently and identically distributed random variables. Two most used distributions of the jump size J are normal distribution and double exponential distribution. Zhou [56] assumed the normal distribution of J and got some results in valuing defaultable securities. Kou & Wang [34] obtained some properties of the first passage times of the double exponential jump diffusion process. These two distributions of the jump size will be discussed in this section.

In this kind of model, the instantaneous arrival intensity is

$$\begin{aligned} h_t &= \lim_{s\downarrow 0} \frac{P(X_{t+s} \geq b|X_t)}{s}, \qquad \tau > t \\ &= \lim_{s\downarrow 0} \frac{\lambda s P(X_t + \mu s + \sigma W_s^* + J \geq b|X_t) + (1-\lambda s)P(X_t + \mu s + \sigma W_s^* \geq b|X_t)}{s}, \end{aligned}$$

where W^* is a standard Brownian motion and is independent of X. Thus

$$\begin{aligned} h_t &= \lambda P(J > b - X_t|X_t) + \lim_{s\downarrow 0} \frac{P(\sigma W_s^* \geq b - X_t - \mu s|X_t)}{s} \\ &= \lambda P(J > b - X_t|X_t). \end{aligned}$$

While when we look at its conditional expectation,

$$E[h_t|\tau > t] = \frac{E[\lambda P(J > b - X_t|X_t) I_{\{\tau>t\}}]}{P(\tau > t)} + \lim_{s\downarrow 0} \frac{E\left[\Phi\left(\frac{b-X_t-\mu s}{\sigma\sqrt{s}}\right) I_{\{\tau>t\}}\right]}{s} / P(\tau > t),$$

here $\Phi(\cdot)$ is the CDF of a standard normal distribution. It will be the overall hazard function at time t, and we call the first part the hazard function due to jumps and call the second part the hazard function due to diffusion.

5.2.1 Exponential jumps

Fist we list some results from Kou & Wang [34]. Define the first passage time of the jump diffusion process X_t as

$$\tau_b = \inf\{t > 0 : X_t \geq b\}, \qquad b > 0, \tag{5.6}$$

where the jump size follows a double exponential distribution with density

$$f(x) = p\eta_1 e^{-\eta_1 x} I_{\{x \geq 0\}} + q\eta_2 e^{\eta_2 x} I_{\{x<0\}},$$

$\eta_1, \eta_2 > 0, p \in (0, 1]$ are constant and $p + q = 1$. Denote $X_{\tau_b} = \limsup_{t\to\infty} X_t$ on the set $\{\tau_b = \infty\}$. Then the CDF of τ_b is $P(\tau_b \leq t) = P(\sup_{0\leq s\leq t} X_s \geq b)$. For any $\theta \in (-\eta_2, \eta_1)$, let the moment generate function of X_t be

$$\begin{aligned} E[e^{\theta X_t}] &= \exp\{G(\theta)t\}, \quad \text{where} \\ G(x) &= \mu x + \frac{1}{2}\sigma^2 x^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right). \end{aligned}$$

Lemma 5.2.1 ***(See Lemma 2.1 in [34])*** *The equation*

$$G(x) = a \qquad \text{for all } a > 0$$

has exactly four roots: $\beta_{1,a}, \beta_{2,a}, -\beta_{3,a}, -\beta_{4,a}$ *with*

$$0 < \beta_{1,a} < \eta_1 < \beta_{2,a} < \infty, \qquad 0 < \beta_{3,a} < \eta_2 < \beta_{4,a} < \infty.$$

In addition, let the overall drift of X_t *be* $d = \mu + \lambda(\frac{p}{\eta_1} - \frac{q}{\eta_2})$, *then as* $a \to 0$,

$$\beta_{1,a} \to \begin{cases} 0 & \text{if } d \geq 0, \\ \beta_1^* & \text{if } d < 0, \end{cases} \quad \text{and } \beta_{2,a} \to \beta_2^*,$$

where β_1^* *and* β_2^* *are the unique roots of* $G(x) = 0$ *with* $0 < \beta_1^* < \eta_1 < \beta_2^* < \infty$.

The distribution of τ_b is represented explicitly in the form of its Laplace transform as following.

Theorem 5.2.1 ***(See Lemma 2.1 in [34])*** *For any* $a > 0$, *let* $\beta_{1,a}$ *and* $\beta_{2,a}$ *be the only two positive roots of the equation* $G(x) = a$ *with* $0 < \beta_{1,a} < \eta_1 < \beta_{2,a} < \infty$. *Then the Laplace transform of* τ_b *is*

$$E[e^{-a\tau_b}] = \frac{\eta_1 - \beta_{1,a}}{\eta_1}\frac{\beta_{2,a}}{\beta_{2,a} - \beta_{1,a}}e^{-b\beta_{1,a}} + \frac{\beta_{2,a} - \eta_1}{\eta_1}\frac{\beta_{1,a}}{\beta_{2,a} - \beta_{1,a}}e^{-b\beta_{2,a}}.$$

If $d \geq 0$, $\beta_{1,a} \to 0$ as $a \to 0$, then $P(\tau < \infty) = \lim_{a\to 0} E[e^{-a\tau_b}] = 1$. If $d < 0$, $P(\tau < \infty) < 1$. Theorem 5.2.1 holds as long as $\beta_{1,a}$ and $\beta_{2,a}$ exist, which is the case that $\{p > 0, \sigma > 0\}$ or $\{p > 0, \sigma = 0, \mu > 0\}$. If $p = 0$, there is only one positive root and two negative roots for $G(x) = a > 0$. If $p > 0$ and $\sigma = 0$, the equation has three roots: when $\mu > 0$, there are the two positive roots $0 < \beta_{1,a} < \eta_1 < \beta_{2,a} < \infty$ and one negative root $-\beta_{3,a} > -\eta_2$; when $\mu < 0$, there are the two negative roots $0 < -\beta_{4,a} < -\eta_2 < -\beta_{3,a} < \infty$ and one positive root $\beta_{1,a} < \eta_1$.

Now consider the PIDE representation of the Laplace transform of τ in (4.14), here $Y_t = b - X_t$. Then

$$\begin{aligned} l(a,y) &= \frac{\eta_1 - \beta_{1,a}}{\eta_1} \frac{\beta_{2,a}}{\beta_{2,a} - \beta_{1,a}} e^{-y\beta_{1,a}} + \frac{\beta_{2,a} - \eta_1}{\eta_1} \frac{\beta_{1,a}}{\beta_{2,a} - \beta_{1,a}} e^{-y\beta_{2,a}} \\ &= A_1 e^{-y\beta_{1,a}} + A_2 e^{-y\beta_{2,a}} \end{aligned}$$

and

$$\begin{aligned} al(a,y) &= h(y)[1 - l(a,y)] - \mu l_y(a,y) + \frac{\sigma^2}{2} l_{yy}(a,y) \\ &+ \int_{\{x+y>0\}} [l(a, x+y) - l(a,y)]\nu(dx). \end{aligned}$$

The Lévy measure of Y is $\nu(dx) = \lambda f(-x)dx$. And the instantaneous default probability is

$$h(y) = \lambda P(J > y) = \lambda p e^{-y\eta_1}. \tag{5.7}$$

The right side of the PIDE is

$$\begin{aligned} &\left[\mu\beta_{1,a} + \frac{(\sigma\beta_{1,a})^2}{2} - h(y)\right] A_1 e^{-y\beta_{1,a}} + \left[\mu\beta_{2,a} + \frac{(\sigma\beta_{2,a})^2}{2} - h(y)\right] A_2 e^{-y\beta_{2,a}} \\ &+ h(y) + \lambda\left[\int_{x>-y} l(a, x+y) f(-x)dx - \int_{x>-y} f(-x) l(a,y) dx\right]. \end{aligned} \tag{5.8}$$

And

$$\begin{aligned} l(a,y)\int_{x>-y} f(-x)dx &= l(a,y)[q + p(1 - e^{-y\eta_1})] = l(a,y)(1 - pe^{-y\eta_1}), \\ \int_{x>-y} f(-x) l(a, x+y) dx &= \int_0^\infty q\eta_2 e^{-\eta_2 x} l(a, x+y)dx + \int_{-y}^0 p\eta_1 e^{\eta_1 x} l(a, x+y) dx \\ &= q\eta_2\left(\frac{A_1 e^{-y\beta_{1,a}}}{\eta_2 + \beta_{1,a}} + \frac{A_2 e^{-y\beta_{2,a}}}{\eta_2 + \beta_{2,a}}\right) + \\ &\quad p\eta_1\left(\frac{A_1(e^{-y\beta_{1,a}} - e^{-y\eta_1})}{\eta_1 - \beta_{1,a}} + \frac{A_2(e^{-y\beta_{1,a}} - e^{-y\eta_1})}{\eta_1 - \beta_{2,a}}\right). \end{aligned}$$

Then the coefficient of $A_1e^{-y\beta_{1,a}}$ in (5.8) is

$$\begin{aligned}&\mu\beta_{1,a} + \frac{(\sigma\beta_{1,a})^2}{2} - h(y) + \lambda\left[\frac{q\eta_2}{\eta_2+\beta_{1,a}} - \frac{p\eta_1}{\eta_1-\beta_{1,a}} - (1-pe^{-y\eta_1})\right] \\ &= G(\beta_{1,a}) - h(y) + \lambda pe^{-y\eta_1} = a.\end{aligned}$$

Similarly,the coefficient of $A_2e^{-y\beta_{2,a}}$ in (5.8) is also a. And the remaining term is

$$h(y) - \lambda p\eta_1\left(\frac{A_1}{\eta_1-\beta_{1,a}}e^{-y\eta_1} + \frac{A_2}{\eta_1-\beta_{2,a}}e^{-y\eta_1}\right) = h(y) - \lambda pe^{-y\eta_1} = 0.$$

Thus the right side of the PIDE, (5.8) is $a(A_1e^{-y\beta_{1,a}} + A_2e^{-y\beta_{2,a}}) = al(a,y)$, the left side of the PIDE.

The forward hazard function is

$$\lambda(T-t,y) = \frac{f(T;t,y)}{\psi(t,T,y)} = L^{-1}\{l(a,y)\}/L^{-1}\left\{\frac{1-l(a,y)}{a}\right\},$$

and the instantaneous default probability will be

$$h(y) = \lim_{T\to t}\lambda(T-t,y) = \lim_{a\to\infty}L^{-1}\{l(a,y)\} = \lim_{a\to\infty}al(a,y).$$

As $a\to\infty$, the two positive roots of equation $G(x)=a$ are $\beta_{1,a}\to\eta_1$ and $\beta_{2,a}\to\infty$. Let $G(\beta_{1,a})$ replace a, then

$$\begin{aligned}h(y) &= \lim_{a\to\infty}G(\beta_{1,a})l(a,y) \\ &= \lim_{a\to\infty}\left(\mu\beta_{1,a} + \frac{1}{2}\sigma^2\beta_{1,a}^2 + \frac{\lambda q\eta_2}{\eta_2+\beta_{1,a}} - \lambda\right)l(a,y) + \lim_{a\to\infty}\frac{\lambda p\eta_1}{\eta_1-\beta_{1,a}}l(a,y) \\ &= 0 + \lim_{\beta_{1,a}\to\eta_1}\frac{\lambda p\eta_1}{\eta_1-\beta_{1,a}} \times \frac{\eta_1-\beta_{1,a}}{\eta_1}e^{-\beta_{1,a}y} = \lambda pe^{-\eta_1 y}.\end{aligned}$$

Now we may apply the result to our credit model. Let the nature log of the firm value $(\ln V_t)_{t\geq 0}$ be a jump-diffusion process on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, Q)$, Q is the risk-neutral measure.

$$\ln V_t = \ln V_0 - X_t = \ln V_0 - \mu t - \sigma W_t - \sum_{i=1}^{N_t} J_i,$$

then the firm value process follows the SDE

$$dV_t = V_t\left[(-\mu + \frac{1}{2}\sigma^2)dt - \sigma dW_t + (e^{-J_t}-1)dN_t\right]. \tag{5.9}$$

Under Q, the expected instantaneous return on V is the risk-free interest rate r,

$$r = -\mu + \frac{1}{2}\sigma^2 + \lambda E^Q[e^{-J} - 1].$$

Define the default time be $\tau = \inf\{t > 0 : V_t \leq F\}$, then it is equivalent to

$$\tau = \inf\{t > 0 : X_t \geq \ln V_0 - \ln F\},$$

which has the same form as τ_b in equation (5.6) with $b = \ln(V_0/F) > 0$. To simplifies the model, let $p = 1$, that is the jump size J follows an exponential distribution with parameter $\eta = \eta_1$. Then Theorem 5.2.1 shows that the Laplace transform of τ is

$$E^Q[e^{-a\tau}] = \frac{\eta - \beta_{1,a}}{\eta}\frac{\beta_{2,a}}{\beta_{2,a} - \beta_{1,a}}\left(\frac{F}{V_0}\right)^{\beta_{1,a}} + \frac{\beta_{2,a} - \eta}{\eta}\frac{\beta_{1,a}}{\beta_{2,a} - \beta_{1,a}}\left(\frac{F}{V_0}\right)^{\beta_{2,a}},$$

where $\beta_{1,a}$ and $\beta_{2,a}$ are two positive roots of the equation

$$a = G(x) = \mu x + \frac{1}{2}\sigma^2 x^2 + \frac{\lambda x}{\eta - x}, \qquad a > 0.$$

The laplace transform of the CDF of τ is

$$\begin{aligned}\hat{F}(a) &= \int_0^\infty e^{-at}F(t)dt = \int_0^\infty e^{-at}\int_0^t f(s)dsdt \\ &= \frac{1}{a}\int_0^\infty e^{-as}f(s)ds = \frac{1}{a}E^Q[e^{-a\tau}].\end{aligned}$$

As Kou & Wang [34] discussed, numerical inversion of Laplace transform should be used to obtain the distribution of τ, and they decided to use the Gaver-Stehfest algorithm since it is the one that does the inversion on the real line. Here is how the algorithm works. For any bounded real-valued function $f(\cdot)$ defined on $[0, \infty)$ that is continuous at t,

$$\begin{aligned}f(t) &= \lim_{n\to\infty} \tilde{f}_n(t), \qquad \text{where} \\ \tilde{f}_n(t) &= \frac{\ln 2}{t}\frac{(2n)!}{n!(n-1)!}\sum_{k=0}^{n}(-1)^k \binom{n}{k} \hat{f}((n+k)\frac{\ln 2}{t}).\end{aligned}$$

On $\{\tau > t\}$, the instantaneous default probability is

$$h_t = \lambda Q(J \geq b - X_t | X_t) = \lambda e^{-\eta(b - X_t)} = \lambda(F/V_t)^\eta. \tag{5.10}$$

So at the beginning of the time horizon, the instantaneous default probability is not zero any more and it is

$$h_0 = \lambda(F/V_0)^\eta.$$

The hazard function which is considered as a function of time $\lambda(t)$ can be divided into two parts: the one due to random jumps is $E^Q[h_t|\tau > t]$ and the one due to the Brownian motion with drift is $\lambda(t) - E^Q[h_t|\tau > t]$.

As mentioned before, we may use $h_t^n = E^Q[h(X_t)|E_n]$ to estimate $E^Q[h_t|\tau > t]$. And

$$h_t^1 = \lambda e^{-\eta b} E^Q[e^{\eta X_t} I_{\{X_t<b\}}]/Q(X_t < b).$$

$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} J_i$, where the diffusion part $\mu t + \sigma W_t (:= x)$ follows a normal distribution with mean μt and variance $\sigma^2 t$, the jump part $\sum_{i=1}^{N_t} J_i (:= y)$ follows a gamma distribution with rate η and shape k conditioning on $N_t = k$, $k = 1, 2, \ldots$. So

$$Q(X_t < b) = Q(N_t = 0) \int_{\{x<b\}} f(x)dx + \sum_{k=1}^{\infty} Q(N_t = k) \int_{\{x+y<b\}} f(x)g(y)dxdy,$$

where $f(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\{-\frac{(x+\mu t)^2}{2\sigma^2 t}\}$ is the PDF of x and $g(y) = \eta^k y^{k-1} e^{-\eta y}/\Gamma(k)$ is the PDF of y given $N_t = k$. Similarly, the expectation $E^Q[e^{\eta X_t} I_{\{X_t<b\}}]$ is

$$Q(N_t = 0) \int_{\{x<b\}} e^{\eta x} f(x)dx + \sum_{k=1}^{\infty} Q(N_t = k) \int_{\{x+y<b\}} e^{\eta(x+y)} f(x)g(y)dxdy.$$

Figure 5.3 show the overall hazard function in [0,10], estimated hazard function due to random jumps (h_t^1) and their difference which can be considered as the estimated hazard function due to the movement of Brownian motion with drift. The parameters used are $r = 0.06$, $\sigma = 0.4$, $\lambda = 0.6$, $\eta = 10$ and $F/V_0 = 0.5$.

A more reasonable setting is to let the threshold level of default be the discounted value of the face value, $Fe^{-r(T-t)}$. Then the default event $\{V_t \le Fe^{-r(T-t)}\}$ is equivalent to

$$\{X_t \ge \ln V_0 - \ln F + r(T-t)\} = \{X_t + rt \ge \ln V_0 - \ln F + rT\},$$

So the barrier $b = \ln V_0 - \ln F + rT > 0$ and the function $G(x) = (\mu + r)x + \frac{1}{2}\sigma^2 x^2 + \lambda x/(\eta - x)$. The instantaneous default probability is

$$h_t = \lambda e^{-\eta(b - X_t - rt)} = \lambda \left(\frac{F}{V_t}\right)^\eta e^{-\eta r(T-t)} \le \lambda \left(\frac{F}{V_t}\right)^\eta.$$

Compared to the previous case, the default will be less likely to happen because of the lower default boundary.

Figure 5.4 show the overall hazard function, estimated hazard function due to random jumps (h_t^1) and their difference when the threshold level of default is $Fe^{-r(T-t)}$. The parameters are the same as Figure 5.3 and the maturity is $T = 10$.

Figure 5.5 compares the estimated default intensity due to jumps in two cases. In the latter case (discounted face value), the estimated default intensity is lower because its default barrier $Fe^{-r(T-t)}$ is much lower than the other one F but the firm value processes are identical. The estimated default intensities are the same at maturity when two default barriers are both equal to F. In fact, the instantaneous default intensity is $\lambda(F/V_t)^\eta$ in the first case and it is $\lambda e^{-\eta r(T-t)}(F/V_t)^\eta$ in the latter case. Figure 5.6 compares the overall hazard rate in two cases.

We also simulate the firm value process in $[0, 1]$, where there is only one jump at $t = 0.5$. The initial firm value is 1 and the face value of bond is 0.5. The value is showed in Figure 5.7 and there is no default in $[0, 1]$ for both threshold level F and $Fe^{-r(t-t)}$. The associated instantaneous default intensities for both threshold level are compared in Figure 5.8.

Figure 5.9 shows the overall hazard rates with three different initial leverages in the case that the threshold is F. When $l = V_0/F = 6$, it can be considered as a high grade issue and the hazard rate has an upward trend at the beginning and then remains flat. When $l = 2$, the hazard rate increases first and then follows a downward trend and remains flat thereafter. When $l = 1.5$, which is the lowest grade among the three, the hazard rate follows a similar pattern as the one when $i = 2$ but with higher slopes. It increases dramatically at the beginning and reaches its highest at $t = 0.41$. Overall, the hazard rate of a high grade issue is lower than the hazard rate of a low grade issue.

5.2.2 Normal jumps

Zhou [56] assumed that the jumps $-J_i$ in equation (5.9) follows a normal distribution with mean μ_π and standard deviation σ_π. Define the default time be

$$\tau = \inf\{t > 0 : V_t \leq D_t\} = \inf\{t > 0 : \ln V_t - \ln V_0 \leq \ln D_t - \ln V_0\},$$

$V_0 > D_0$. Then the instantaneous default probability is

$$h_t = \lambda Q(-J \leq \ln D_t - \ln V_t | V_t, D_t) = \lambda \Phi\left(-\frac{R_t + \mu_\pi}{\sigma_\pi}\right),$$

where R_t is the log ratio of V_t/D_t and $\Phi(x)$ is the CDF of a standard normal distribution.

Obviously, h increases with λ, while decreases with R and μ_π. When $R > -\mu_\pi$, it will increase with σ_π; when $R = -\mu_\pi$, it will be invariant with σ_π; otherwise h will decrease with σ_π. R_t measures the distance between the firm value and the threshold level. The lower the R_t, the higher the likelihood of default. The jump size J with lower mean will have more chance to being small, so more chance to default. λ and σ_π represent the jump risk. High value of λ implies high frequency of jumps. Figure 5.10, 5.11 and 5.12 show how these four parameters $(\lambda, R, \mu_\pi, \sigma_\pi)$ affect the instantaneous default intensity (due to jumps) based on the normality assumption of J.

Following the assumption in equation (4.7), the default time is defined as $\tau := \inf\{t > 0 : R_t + J_t \triangle N_t \leq 0\}$, $R_t > 0$ before default, the default event will only be caused by unpredictable jumps. And it admits an intensity, h_t. The estimated hazard rate is

$$h_t^1 = E^Q[h_t | R_t > 0] = E^Q[h_t].$$

Let $\lambda(t)$ be the hazard function, then the survival probability can be represented as

$$\begin{aligned} Q(\tau > t) &= \exp\left\{-\int_0^t \lambda(s)ds\right\} = E^Q\left[\exp\left\{-\int_0^t h_s ds\right\}\right] \\ &\geq \exp\left\{E^Q\left[-\int_0^t h_s ds\right]\right\} = \exp\left\{-\int_0^t h_s^1 ds\right\}. \end{aligned}$$

The inequality is an application of *Jensen's Inequality* on the convex function e^x. And it is true for all $t \geq 0$, then $h_t^1 \geq \lambda(t)$ for all $t > 0$ and we say that h_t^1 overestimates the hazard function $\lambda(t)$. The survival probability based on h^1 is less than its true value.

Now consider the dependence of default. Let R^1, R^2 be strictly positive diffusion processes as

$$dR_t^i = R_t^i(\mu_i dt + \sigma_i dW_t^i), \quad R_0^i > 0 \tag{5.11}$$

and define the default times as

$$\tau^i = \inf\{t : R_t^i + J^i \triangle N_t + M^i \triangle N_t^i \leq 0\}.$$

μ_i, σ_i are constants. W_t^1, W_t^2 are standard Brownian motions with $dW_t^1W_t^2 = \rho dt$. N_t, N_t^1, N_t^2 are independent Poisson process with arrival rate λ, λ_1, λ_2 respectively. The jump sizes (J^1, M^1) and (J^2, M^2) are normal random variables with mean μ_π^1, μ_π^2 and standard deviation σ_π^1, σ_π^2 respectively. When $\triangle N_t = 1$, the correlation coefficient of two associate jumps J^1 and J^2 is ρ_π. Other random parts are mutually independent.

The individual default intensities are

$$\begin{aligned} h^1(R_t^1) &= (\lambda+\lambda_1)\Phi\left(-\frac{R_t^1+\mu_\pi^1}{\sigma_\pi^1}\right), \\ h^2(R_t^2) &= (\lambda+\lambda_2)\Phi\left(-\frac{R_t^2+\mu_\pi^2}{\sigma_\pi^2}\right), \end{aligned}$$

with the joint default intensity

$$h(R_t^1, R_t^2) = \lambda\Phi\left(-\frac{R_t^1+\mu_\pi^1}{\sigma_\pi^1}, -\frac{R_t^2+\mu_\pi^2}{\sigma_\pi^2}; \rho_\pi\right) \le \lambda\Phi\left(-\frac{R_t^i+\mu_\pi^i}{\sigma_\pi^i}\right), \quad i = 1, 2.$$

Given the path of R^i, the joint survival probability is

$$\begin{aligned} Q(\tau^1 > t, \tau^2 > s) &= \exp\left\{-\int_0^t h^1(R_u^1)du - \int_0^s h^2(R_u^2)du + \int_0^{\min\{t,s\}} h(R_u^1, R_u^2)du\right\} \\ &= Q(\tau^1 > t)Q(\tau^2 > s)\min\left(e^{\int_0^t h(R_u^1,R_u^2)du}, e^{\int_0^s h(R_u^1,R_u^2)du}\right). \end{aligned}$$

If R_t^i are constants, the joint distribution of the default times follows the *bivariate exponential distribution.* It has been studied in Giesecke [25]. The linear default time correlation coefficient is

$$\frac{h(R^1, R^2)}{h^1(R^1) + h^2(R^2) - h(R^1, R^2)},$$

and Spearman's rank correlation is

$$\frac{3h(R^1, R^2)}{2h^1(R^1) + 2h^2(R^2) - h(R^1, R^2)}.$$

If $\lambda = 0$, then there is no common default for the two firms, while the individual intensities are correlated through W^1 and W^2 as

$$dh_t^1 dh_t^2 = \frac{dh^1}{dR^1}\frac{dh^2}{dR^2}R_t^1R_t^2\sigma_1\sigma_2\rho dt.$$

Figure 5.13 simulating the R^i values when $R_0^i = 1$, $\mu_i = 0.2$, $\sigma_i = 0.3$ and $\rho = 0.5$. Figure 5.14 shows the associated default intensities, where $\lambda = 0.2$, $\lambda_i = 0.4$, $\mu_\pi^i = -0.1$

and $\sigma_{\pi}^{i} = 0.3$. With these parameters, the joint default intensity increases with the correlation coefficient ρ_{π}, which is shown in Figure 5.15. Figure 5.16 simulates the default events. The common Poisson process N_t jumps at time 1.5 and 6, the jump times of N_t^1 are 2.6 2.9 and 8.3, and there are 5 jumps of N_t^2 in $[0, 10]$. Firm 1 defaults at $t = 6$, and firm 2 survives in $[0, 10]$.

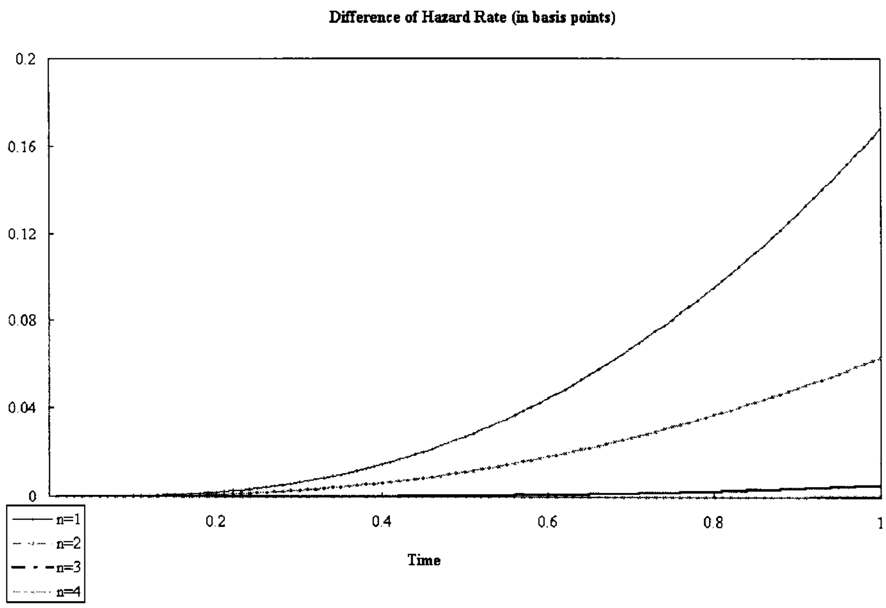

Figure 5.1: Coin tossing: the difference $h_t^1 - \lambda(t)$ increases with n.

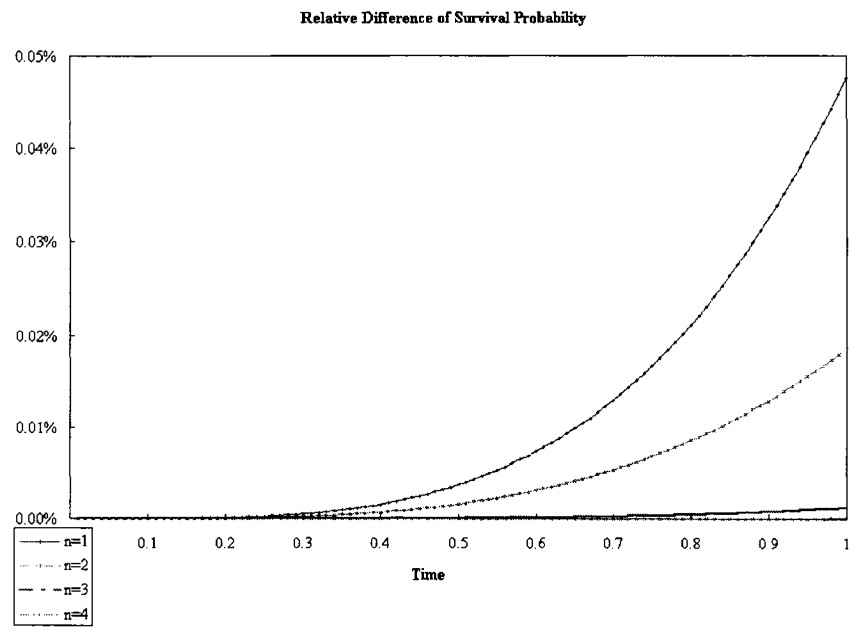

Figure 5.2: Coin tossing: the relative difference of survival probability and $\exp\{-\int_0^t h_s^1 ds\}$ increases with n.

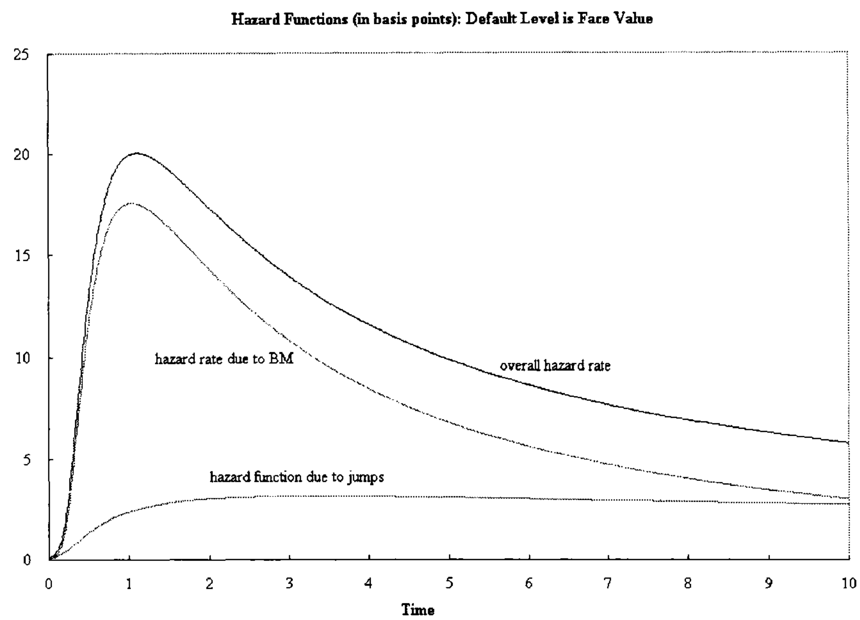

Figure 5.3: Exp-jumps: the overall hazard rate $\lambda(t)$, the estimated hazard function due to jumps h_t^1 and their difference when the threshold level is the face value.

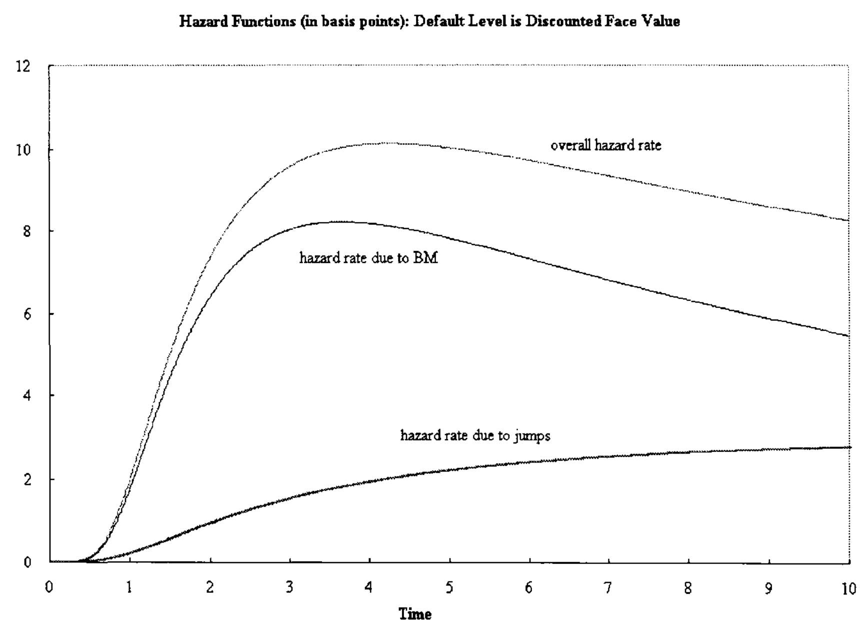

Figure 5.4: Exp-jumps: the overall hazard function $\lambda(t)$, the estimated hazard function due to random jumps h_t^1 and their difference when the threshold level is the discounted face value.

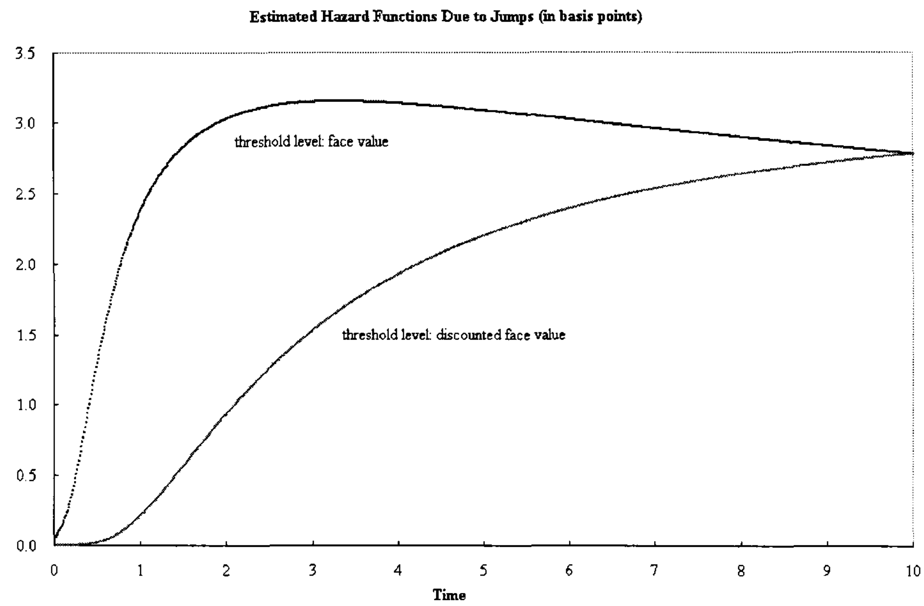

Figure 5.5: Exp-jumps: h_t^1 is higher when the threshold level is the face value than the case that the threshold level is the discounted face value.

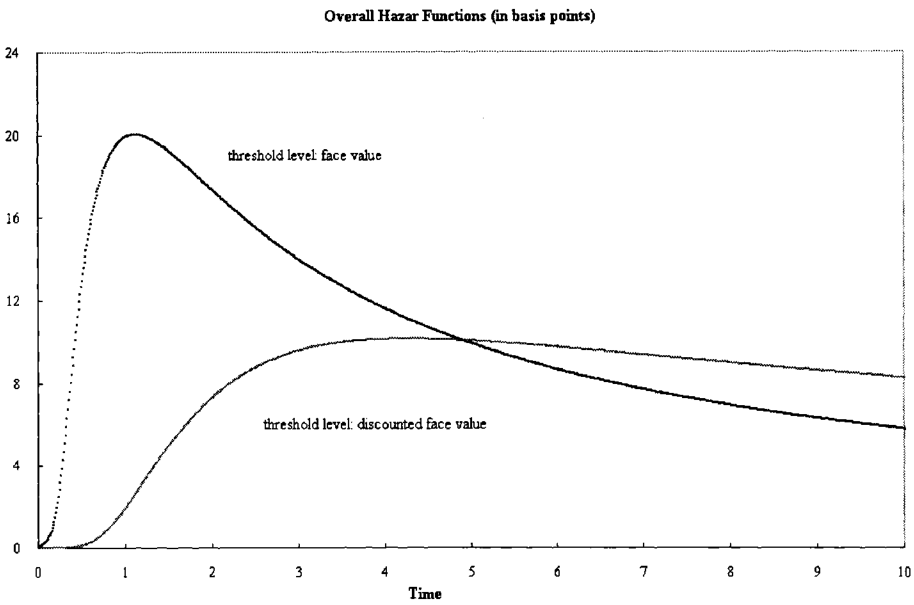

Figure 5.6: Exp-jumps: $\lambda(t)$ when the threshold levels are the face value and the discounted face value.

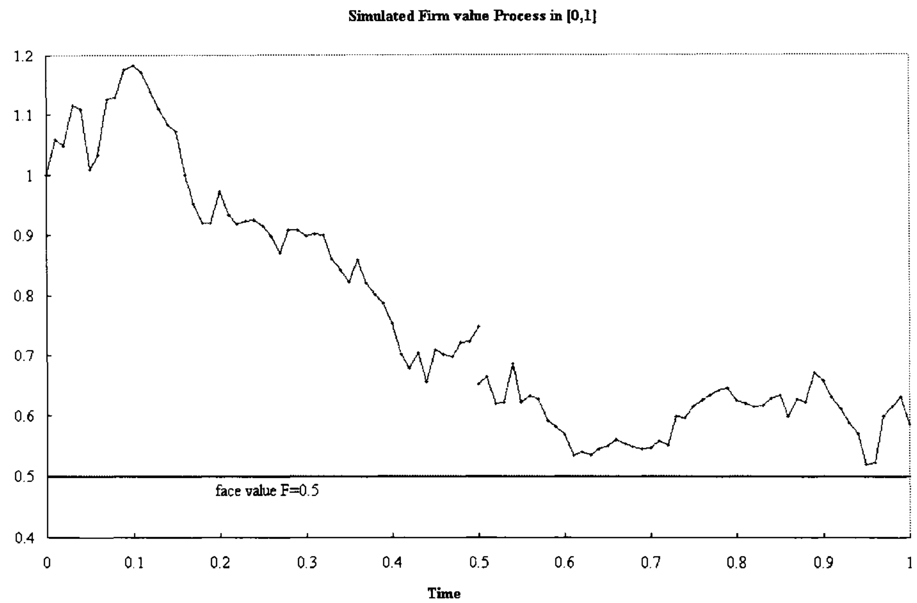

Figure 5.7: Exp-jumps: the simulated firm value process where there is only one jump in $[0, 1]$.

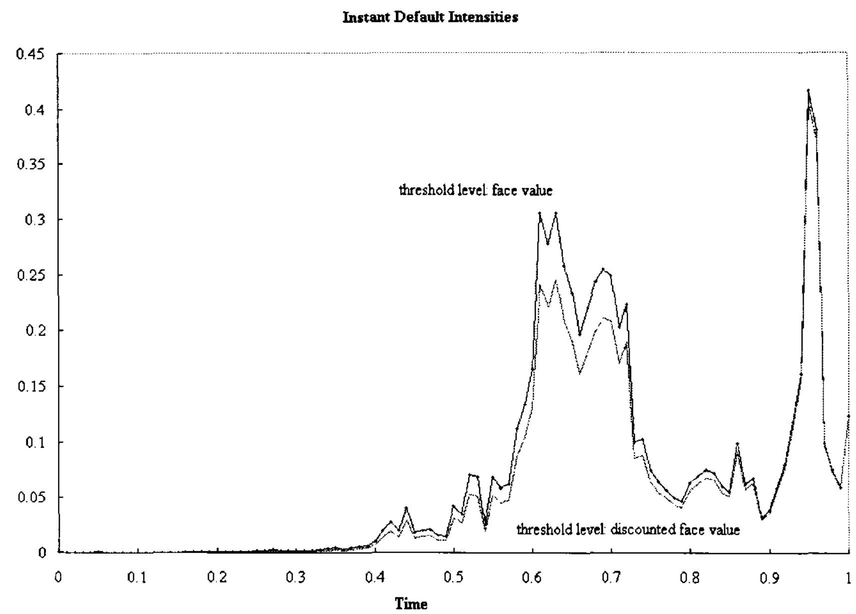

Figure 5.8: Exp-jumps: h_t is higher when the threshold level is the face value than the case that the threshold level is the discounted face value.

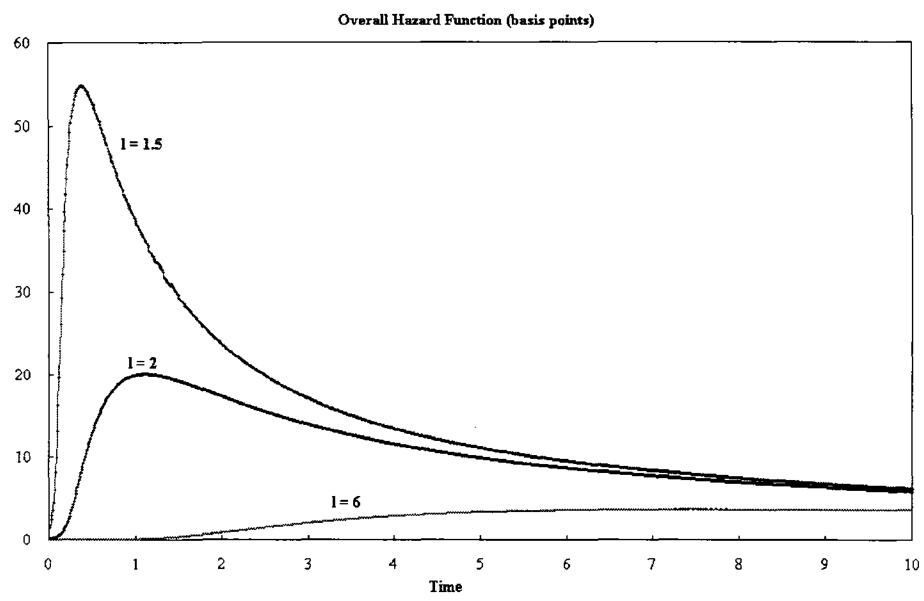

Figure 5.9: Exp-jumps: the overall hazard rate follows an upward trend for a high grade issue; for a low grade issue, the hazard rate is hump shaped for the first few years and keeps flat thereafter. Here $l = V_0/F$ is the initial leverage.

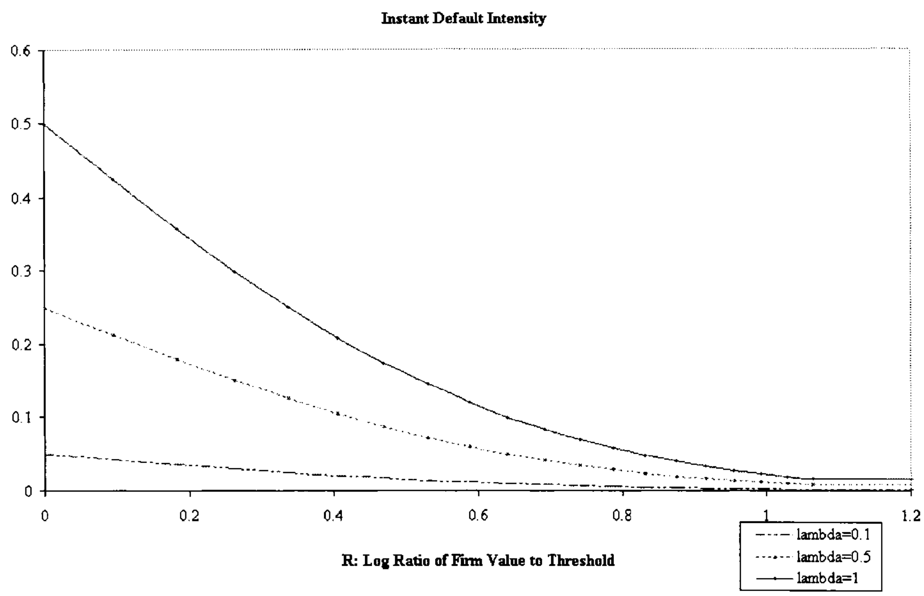

Figure 5.10: Normal-jumps: h_t increases with mean jump rate λ and decreases with R. ($\mu_\pi = 0, \sigma_\pi = 0.5$.)

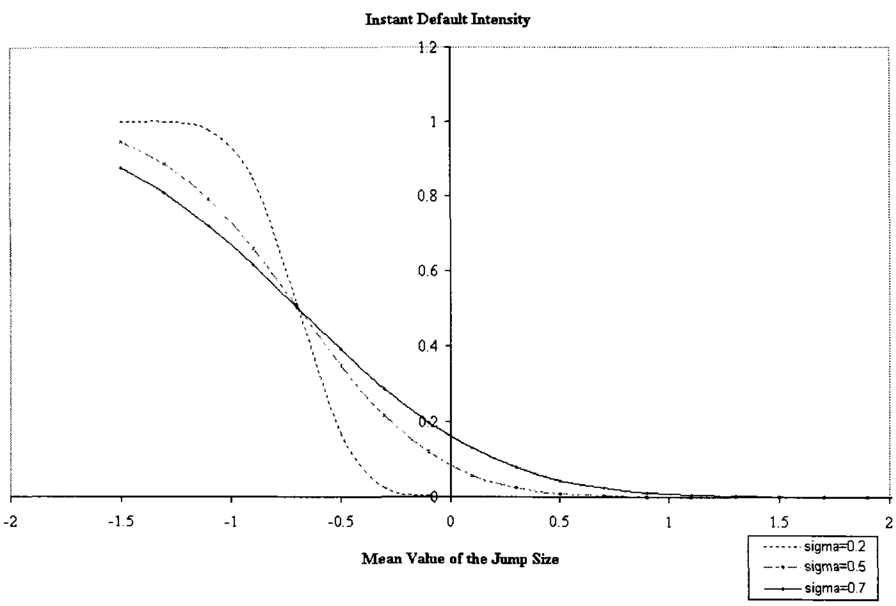

Figure 5.11: Normal-jumps: h_t decreases with μ_π. It increases with σ_π when $\mu_\pi > -R$ and it decreases with σ_π when $\mu_\pi < -R$. ($\lambda = 1, R = \ln 2$.)

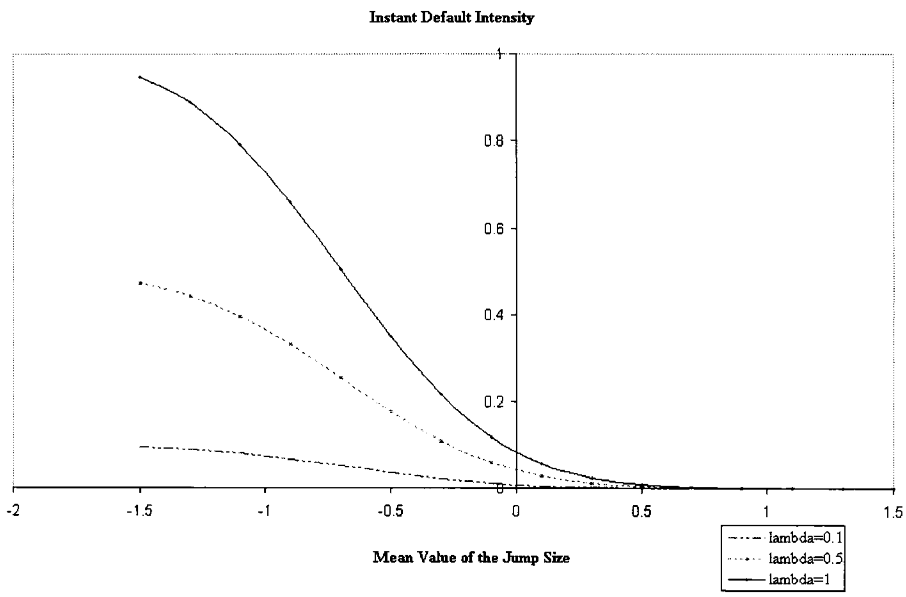

Figure 5.12: Normal-jumps: h_t increases with mean jump rate λ and decreases with μ_π. ($R = \ln 2, \sigma_\pi = 0.5$.)

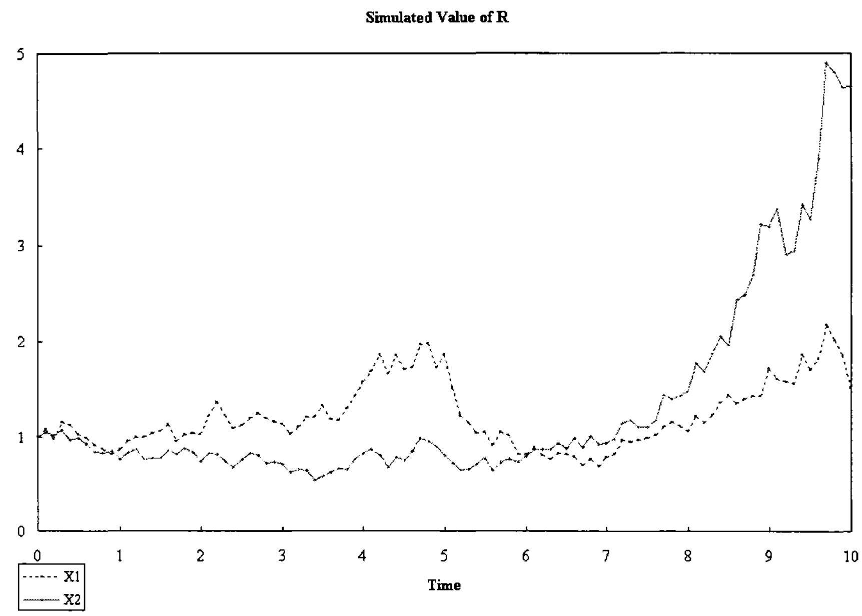

Figure 5.13: Joint case: the simulated R value, $dR_t^i = R_t^i(0.2dt + 0.3dW_t^i)$, $R_0^i = 1$ and $dW_t^1 dW_t^2 = 0.5t$.

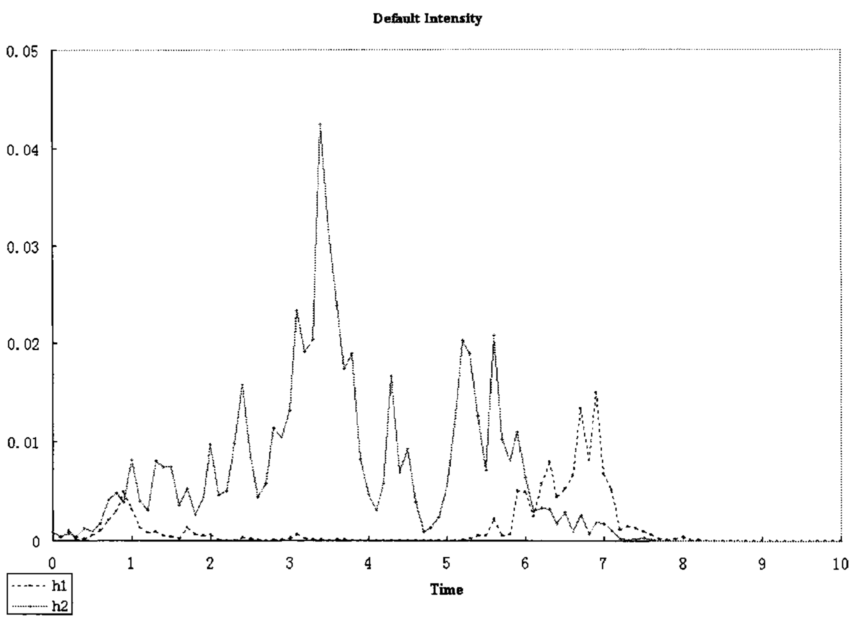

Figure 5.14: Joint case: the default intensities, $h^i = (\lambda + \lambda_i)\Phi(-\frac{R_t^i + \mu_\pi^i}{\sigma_\pi^i})$, a decreasing function of R. Here $\lambda = 0.2$, $\lambda^i = 0.4$, $\mu_\pi^i = -0.1$ and $\sigma_\pi^i = 0.3$.

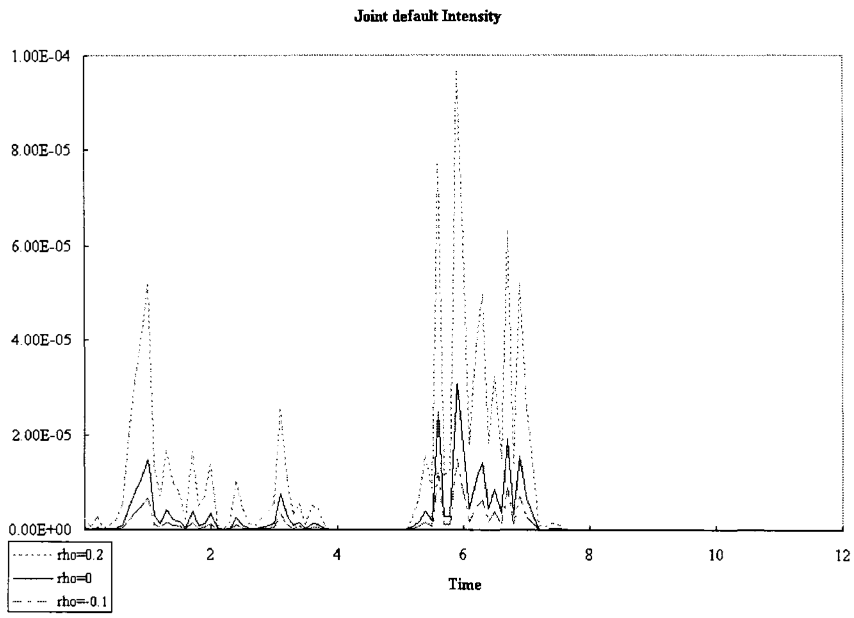

Figure 5.15: Joint case: the joint default intensity, $\lambda\Phi(-\frac{R_t^1+\mu_\pi^1}{\sigma_\pi^1}, -\frac{R_t^2+\mu_\pi^2}{\sigma_\pi^2}; \rho_\pi)$, increases with ρ_π.

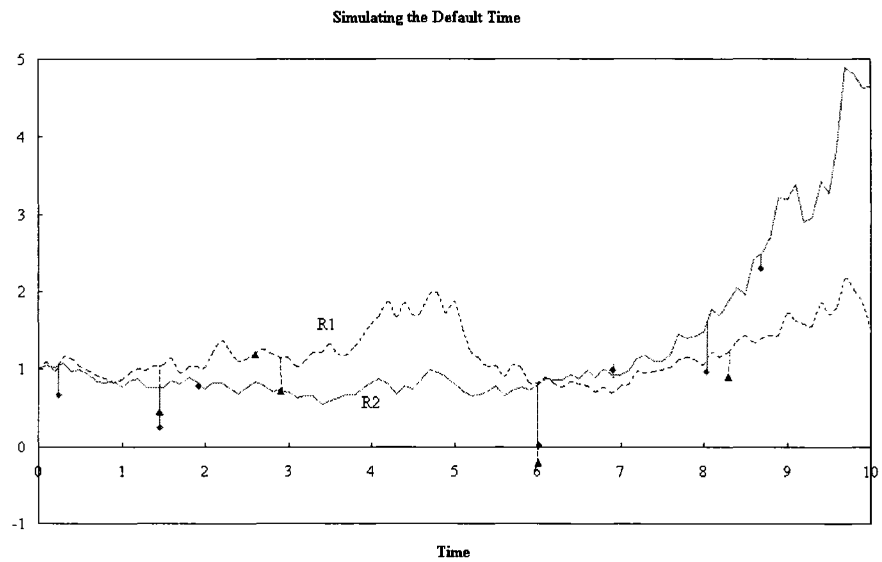

Figure 5.16: Joint case: $\tau^i = \inf\{t : R_t^i + J^i \triangle N_t + M^i \triangle N_t^i \leq 0\}$. $\triangle N_t = 1$ at 1.5, 6; $\triangle N_t^1 = 1$ at 2.6 2.9 and 8.3 Firm 1 defaults at time 6 and firm 2 survives.

Bibliography

[1] ARAI, T. (2004). Minimal Martingale Measures for Jump Diffusion Processes. *Journal of Applied Probability*, **41**, 263-270.

[2] ASMUSSEN, S. (2000). *Ruin Probabilities*. In O. E. Barndorff-Nielsen (ed.): Advanced Series on Statistical Science and Applied Probability, vol 2, World Scientific, Singapore.

[3] BARNDORFF-NIELSEN, O. E. (2000). Probability Densities and Lévy Densities. *Working Paper Series No. 66*, Centre for Analytical Finance, Aarhus School of Business, University of Aarhus.

[4] BAXTER, G. AND DONSKER, M. D. (1957). On the Distribution of the Supremum Functional for Processes with Stationary Independent Increments. *Transactions of the American Mathematical Society*, **85**(1), 73-87.

[5] BECHERER, D. (2001). The Numeraire Portfolio for Unbounded Semimartingales. *Finance and Stochastics*, **5**, 327-341.

[6] BLACK, F. AND COX, J. C. (1976). Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. *Journal of Finance*, **31**, 351-367.

[7] BLACK, F. AND SCHOLES, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81**, 637-654.

[8] CARRIÈRE, J. F. (2002). Copulas. *Working Paper*, University of Alberta.

[9] CHEN, C. J. (2006). The Instantaneous and Forward Default Intensity of Structural Models. *Working Paper*, University of Alberta.

[10] CHEN, C. J. AND PANJER, H. (2006). A Bridge from Ruin Theory to Crdit Risk. *Working Paper*, University of Alberta and University of Waterloo.

[11] CHEN, N. AND KOU, S. (2006). Credit Spreads, Optimal Capital Structure, and Implied Volatility with Endogenous Default and Jump Risk. *Working Paper*, Columbia University.

[12] CONT, R. AND TANKOV, P. (2004). *Financial Modelling with Jump Processes*, Chapman & Hall/CRC.

[13] CONT, R. AND VOLTCHKOVA, E. (2005). Integro-differential Equations for Option Prices in Exponential Lévy Models. *Finance and Stochastics*, **9**, 299-325.

[14] COSSIN, D. AND PIROTTE, H. (2001). *Advanced Credit Risk Analysis*, John Wiley & Sons, Ltd, Chichester/New York/Weinheim/Brisbane/Singapore/Toronto.

[15] DARSOW, W. F., NGUYEN, B. AND OLSEN, E. T. (1992). Copula and Markov Processes. *Illinois Journal of Mathematics*, **36(4)**, 600-642.

[16] DUFFIE, D. AND LANDO, D. (2001). Term Structures of Credit Spreads with Incomplete Accounting Information. *Econometrica*, **69(3)**, 633-664.

[17] DUFFIE, D. AND PAN, J. (1999). Analytical Value at Risk with Jumps and Credit Risk. *Finance and Stochastics*, **5**, 155-180.

[18] DUFFIE, D. AND SINGLETON, K. (1999). Modeling Term Structures of Default Risky Bonds. *Review of Financial Studies*, **12**, 687-720.

[19] DUFFIE, D. AND SINGLETON, K. (2003). *Credit Risk: Pricing, Measurement and Management*, Princeton University Press, New Jersey.

[20] DUFFIE, D. AND ZIEGLER, A. (2003). Liquidation Risk. *Financial Analysts Journal*, May-June 2003, 42-51.

[21] ESCHE, F. AND SCHWEIZER, M. (2005). Minimal Entropy Preserves the Lévy Property: How and Why. *Stochastic Processes and their Applications*, **115**, 299-327.

[22] FÖLLMER, H. AND SCHWEIZER, M. (1991). Hedging of Contingent Claims under Incomplete Information". In M.H.A. Davis and R.J. Elliott (eds.): *Applied Stochastic Analysis*, Stochastics Monographs, **5**, Gordon and Breach, London/New York, 389-414.

[23] FUJIWARA, T. AND MIYAHARA, Y. (2003). The Minimal Entropy Martingale Measures for Geometric Lévy Processes. *Finance and Stochastics*, **7**, 509-531.

[24] GARCIA, J., GINDEREN, H. V. AND GARCIA, R. (2001). On the Pricing of Credit Spread Options: A Two Factor HW-BK Algorithm. *Working Paper*, Artesia BC, Artesia BC and University of California at Berkeley.

[25] GIESECKE, K. (2003). A Simple Exponential Model for Dependent Defaults. *Journal of Fixed Income*, **13(3)**, 74-83.

[26] GIESECKE, K. (2004). Credit Risk Modeling and Valuation: An Introduction. In D. Shimko (ed.): *Credit Risk: Models and Management*, Risk Books, London, second edition, 487-526.

[27] GIESECKE, K.(2005). Default and Information. *Working Paper*, Cornell Univerisity.

[28] HENDERSON, V. AND HOBSON, D. (2003). Coupling and Option Price Comparisons in a Jump Diffusion model. *Stochastics and Stochastics Reports*, **75(3)**, 79-101.

[29] JARROW, R. A. AND YU, F. (2001). Counterparty Risk and the Pricing of Defaultable Securities. *Journal of Fiance*, **56(5)**, 555-576.

[30] JEANBLANC, M. AND RUTKOWSKI, M (1999). Modelling of Default Risk: An Overview. In R. Cont and J. Yong (eds.): *Mathematical Finance: Theory and Practice*, High Education Press, Beijing, 171-269.

[31] KALLSEN, J. AND SHIRYAEV, A. N. (2002). Time Change Representation of Stochastic Integrals. *Theory of Probability and Its Applications*, **46(3)**, 522-528.

[32] KËLLEZI, E. AND WEBBER, N. (2001). Numerical Methods for Lévy Processes: Lattice Methods and the Density, the Subordinator and the Time Copula. *Working paper*, University of Geneva and University of Warwick.

[33] KORN, R., OERTEL, F. AND SCHÄL, M. (2003). The Numeraire Portfolio in Financial Markets Modeled by a Multi-dimensional Jump Diffusion Process. *Decisions in Economics and Finance*, **26**, 153-166.

[34] KOU, S. G. AND WANG, H. (2002). First Passage Times of a Jump Diffusion Process. *Advances in Applied Probability*, **35(2)**, 504-531.

[35] LANDO, D. (1998). On Cox Processes and Credit Risky Securities. *Review of Derivatives Research*, **2**, 99-120.

[36] LI, D. X. (2000). On Default Correlation: A Copula Function Approach. *working paper No. 99-07*, the RiskMetrics Group.

[37] LONG, J. B. (1990). The Numeraire Portfolio. *Journal of Financial Economics*, **26**, 29-69.

[38] LONGSTAFF, F. AND SCHWARTZ, E. (1995). A Simple Approach to Valuing Risky Fixed and Floating Rate Debt. *Journal of Finance*, **50(3)**, 789-819.

[39] MADAN, D. (2000). Pricing the Risks of Default. *Working Paper*, University of Maryland.

[40] MADAN, D. AND UNAL H. (1998). Pricing the Risks of Default. *Review of Derivatives Research*, **2**, 121-160.

[41] MADAN, D. AND UNAL H. (2000). A Two-Factor Hazard Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads. *Journal of Financial and Quantitative Analysis*, **35**, 43-56.

[42] MELLA-BARRAL, P. AND TYCHON, P. (1999). Default Risk in Asset Pricing. *Revue Finance*, **20(1)**.

[43] MERTON, R. C. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, **29**, 449-470.

[44] MERTON, R. C. (1976). Option Pricing when the Underlying Stock Returns are Discontinuous. *Journal of Financial Economics*, **3**, 115-144.

[45] MÉTIVIER, M. (1982). *Semimartingales: A Course on Stochastic Processes*. Walter de Gryter, Berlin.

[46] MONROE, I (1978). Processes that can be Embedded in Brownian Motion. *Annals of Probability*, **6(1)**, 42-56.

[47] MUSIELA, M AND RUTKOWSKI, M. (1997). *Martingale Methods in Financial Modelling*. Springer, Berlin.

[48] PROTTER, P. E. (1990). *Stochastic Integration and Differential Equations*. Springer, Berlin.

[49] REVUZ, D AND YOR, M. (1999). *Continuous Martingales and Brownian Motion.* Springer, Berlin, third edition.

[50] SATO, K. (2001). Basic Results on Lévy Processes. In O. E. Barndorff-Nielsen, T. Mikosch and S. I. Resnick (eds.): *Lévy Processes: Theory and Application*, Birkhäuser, London, 3-28.

[51] SCHONBUCHER, P. J. (1998). Pricing Credit Risk Derivatives. *Working Paper*, Department of Statistics, University of Bonn.

[52] SCHONBUCHER, P. J. (2000). Factor Models for Portfolio Credit Risk. *Working Paper*, Department of Statistics, University of Bonn.

[53] SCHOUTENS, W. (2003). *Lévy Processes in Finance: Pricing Financial Derivatives.* John Wiley & Sons, England.

[54] SCHWEIZER, M. (1995). On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition. *Stochastic Analysis and Applications*, **13**, 573-599.

[55] TANKOV, P. (2003). Dependence Structure of Multivariate Lévy Processes with Applications in Risk Management. *ECOLE POLYTECHNIQUE* R.I. N^0 501.

[56] ZHOU, C. (1997). A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities. *Review of Financial Studies*, **14(2)**, 555-576.

[57] ZHOU, C. (2001). An Analysis of Default Correlation and Multiple Defaults. *Federal Reserve Board*, Washington.

A

$M(k,n)$ in coin tossing problem

We want to prove that

$$M(k,n) = \frac{n(2k+n-1)!}{k!(n+k)!}, \qquad k = 0,1,\cdots; \; n = 1,2,\cdots. \tag{A.1}$$

$M(k,n)$ is the number of pathes such that $X_0 = 0, X_{2k+n-1} = n-1, X_i < n$ for $i = 1,2,\cdots,2k+n-2$, where

$$X_m = \sum_{i=0}^{m} J_i, \qquad P(J=1) = P(J=-1) = \frac{1}{2}.$$

Define $\#(\cdot)$ be the number of paths satisfied the condition inside, then $M(k,n)$ is

$$\#(X_0 = 0, X_{2k+n-1} = n-1) - \sum_{i=0}^{k-1} M(i,n) \times \#(X_{2i+n} = n, X_{2k+n-1} = n-1)$$

$$= \binom{2k+n-1}{k} - \sum_{i=0}^{k-1} M(i,n) \binom{2k-2i-1}{k-i}$$

When $k = 0$, for each $n = 1,2,\cdots$,

$$M(0,n) = \#(X_0 = 0, X_1 = 1, X_2 = 2, \cdots, X_{n-1} = n-1) = 1 = \frac{n(2*0+n-1)!}{0!(n+0)!}.$$

For $k = 1,2,\cdots$ and $n = 1,2,\cdots$, define

$$N(k,n) = \sum_{i=0}^{k-1} M(i,n) \binom{2k-2i-1}{k-i} = \binom{2k+n-1}{k} - M(k,n),$$

and define

$$B(k,n) = \frac{(2k+n-1)!}{(k-1)!(n+k)!}.$$

Then equation (A.1) is equivalent to

$$N(k,n) = B(k,n), \quad k,n = 1,2,\cdots \tag{A.2}$$

We will prove it for each k and n by inductive method.

When $k = 1$, for each $n > 0$,

$$\begin{aligned} N(1,n) &= M(0,n)\binom{2*1-2*0-1}{1-0} = 1 = \frac{(2*1+n-1)!}{(1-1)!(n+1)!} = B(1,n). \\ M(1,n) &= \binom{2*1+n-1}{1} - N(1,n) = (n+1)-1 = n. \end{aligned}$$

When $k = 2$, for each $n > 0$,

$$\begin{aligned} N(2,n) &= M(1,n)\binom{2*2-2*1-1}{2-1} + M(0,n)\binom{2*2-2*0-1}{2-0} \\ &= n*1+1*3 = n+3 = \frac{(2*2+n-1)!}{(2-1)!(n+2)!} = B(2,n). \end{aligned}$$

Next, let's show that $N(l,1) - N(l-1,2) = B(l,1) - B(l-1,2)$ for each $l > 1$ by assuming that (A.2) is true for $k = 1,\cdots,l-n$, $n = 1,2$. First,

$$B(l,1) - B(l-1,2) = \frac{(2l+1-1)!}{(l-1)!(l+1)!} - \frac{(2(l-1)+2-1)!}{(l-1-1)!(l-1+2)!} = \frac{(2l-1)!}{(l)!(l-1)!},$$

and $N(l,1) - N(l-1,2)$ is equal to

$$\begin{aligned} & \sum_{i=0}^{l-1} \frac{1*(2i+1-1)!}{i!(i+1)!}\binom{2l-2i-1}{l-i} - \sum_{i=0}^{l-2} \frac{2*(2i+2-1)!}{i!(i+2)!}\binom{2(l-1)-2i-1}{l-1-i} \\ = & \binom{2l-1}{l} + \sum_{i=1}^{l-1} \frac{(2i)!}{i!(i+1)!}\frac{(2l-2i-1)!}{(l-i)!(l-i-1)!} \\ & - \sum_{i=0}^{l-2} \frac{2*(2i+1)!}{i!(i+2)!}\frac{(2l-2i-3)!}{(l-i-1)!(l-i-2)!} \\ = & \frac{(2l-1)!}{(l)!(l-1)!} + \sum_{j=0}^{l-2} \frac{2(j+1)*(2j+1)!}{(j+1)!(j+2)!}\frac{(2l-2j-3)!}{(l-j-1)!(l-j-2)!} \\ & - \sum_{i=0}^{l-2} \frac{2*(2i+1)!}{i!(i+2)!}\frac{(2l-2i-3)!}{(l-i-1)!(l-i-2)!} \\ = & \frac{(2l-1)!}{(l)!(l-1)!} = B(l,1) - B(l-1,2). \end{aligned}$$

Then, we will show that $N(l,n+1) = N(l,n) + N(l-1,n+2)$ and $B(l,n+1) = B(l,n) + B(l-1,n+2)$ for $n,l > 0$ by assuming that (A.2) is true for $k = 1,\cdots,l-1$, $n = 1,2,3,\cdots$.

$$\begin{aligned}
& N(l,n) + N(l-1,n+2) \\
= & \sum_{i=0}^{l-1} \frac{n(2i+n-1)!}{i!(n+i)!} \binom{2l-2i-1}{l-i} \\
& + \sum_{i=0}^{l-2} \frac{(n+2)(2i+n+2-1)!}{i!(n+2+i)!} \binom{2(l-1)-2i-1}{l-1-i} \\
= & \binom{2l-1}{l} + \sum_{i=1}^{l-1} \frac{n(2i+n-1)!}{i!(n+i)!} \binom{2l-2i-1}{l-i} \\
& + \sum_{j=1}^{l-1} \frac{(n+2)(2j+n-1)!}{(j-1)!(n+1+j)!} \binom{2l-2j-1}{l-j} \\
= & \binom{2l-1}{l} + \sum_{i=1}^{l-1} \frac{(2i+n-1)![n(n+i+1)+(n+2)i])}{i!(n+i+1)!} \binom{2l-2i-1}{l-i} \\
= & \binom{2l-1}{l} + \sum_{i=1}^{l-1} \frac{(2i+n-1)!(n+1)(2i+n)}{i!(n+i+1)!} \binom{2l-2i-1}{l-i} \\
= & \sum_{i=0}^{l-1} \frac{(n+1)(2i+n)!}{i!(n+i+1)!} \binom{2l-2i-1}{l-i} = N(l,n+1).
\end{aligned}$$

And

$$\begin{aligned}
B(l,n) + B(l-1,n+2) &= \frac{(2l+n-1)!}{(l-1)!(n+l)!} + \frac{(2(l-1)+n+2-1)!}{(l-1-1)!(n+2+l-1)!} \\
&= \frac{(n+l+1)*(2l+n-1)! + (l-1)*(2l+n-1)!}{(l-1)!(n+l+1)!} \\
&= \frac{(2l+n)*(2l+n-1)!}{(l-1)!(n+1+l)!} = B(l,n+1).
\end{aligned}$$

Now we have the followings information,

i For each $n = 1,2,\cdots$, $M(0,n) = 1$ and $N(k,n) = B(k,n)$ is true when $k = 1,2$.

ii For each $k = 2,3,\cdots$, $N(k,1) - N(k-1,2) = B(k,1) - B(k-1,2)$ if $N(l,n) = B(l,n)$ holds when $l = 1,\cdots,k-n$, $n = 1,2$.

iii For each $k = 2,3,\cdots$, $n = 1,2,\cdots$, $N(k,n+1) = N(k,n) + N(k-1,n+2)$ if $N(l,n) = B(l,n)$ holds when $l = 1,\cdots,k-1$, $n = 1,2,3,\cdots$.

iv For each $k = 2, 3, \cdots, n = 1, 2, \cdots$, $B(k, n+1) = B(k, n) + B(k-1, n+2)$.

When $k = 3$, clause ii shows that $N(3,1) = N(2,2) + B(3,1) - B(2,2) = B(3,1)$ since $N(2,n) = B(2,n)$ for each $n > 0$ which is given in clause i.

$$\begin{aligned}
N(3,2) = N(3,1) + N(2,3) &= B(3,1) + B(2,3) = B(3,2), \\
N(3,3) = N(3,2) + N(2,4) &= B(3,2) + B(2,4) = B(3,3), \\
\cdots\cdots \quad & \quad \cdots\cdots \\
N(3,n-1) = N(3,n-2) + N(2,n) &= B(3,n-2) + B(2,n) = B(3,n-1), \\
\underbrace{N(3,n) = N(3,n-1) + N(2,n+1)}_{\textit{clause iii}} &= \underbrace{B(3,n-1) + B(2,n+1) = B(3,n)}_{\textit{clause iv}}, \quad n > 1.
\end{aligned}$$

Then $N(3,n) = B(3,n)$ is true for each $n = 1, 2, \cdots$.

When $k = 4$, clause ii shows that $N(4,1) = N(3,2) + B(4,1) - B(3,2) = B(4,1)$ since $N(3,n) = B(3,n)$ for each $n > 0$. And then $N(4,n) = N(4,n-1) + N(3,n+1) = B(4,n-1) + B(3,n+1) = B(4,n)$ for $n > 1$.

By induction, we may conclude that $N(k,n) = B(k,n)$ for each $k, n = 1, 2, \cdots$. Equivalently, we have proved that $M(k,n) = n(2k+n-1)!/(k!(n+k)!)$, for each $k = 0, 1, \cdots, n = 1, 2, \cdots$.

B

Equation (5.4) in coin tossing problem

First we recall the equation and denote, for $t \in [0,T)$,

$$A(t,y) := \begin{cases} \lambda(T-t,y)\psi(t,y) + \frac{\lambda}{2}[\psi(t,y+1)+\psi(t,y-1)-2\psi(t,y)] = 0 & y \geq 2, \\ \lambda(T-t,1)\psi(t,1) + \frac{\lambda}{2}[\psi(t,2)-2\psi(t,1)] = 0 & y = 1, \end{cases}$$

with terminal condition $\psi(T,y) = 1$, $y \geq 1$. Where

$$\psi(t,y) = 1 - y\int_0^{T-t} e^{-\lambda s}\frac{I_y(\lambda s)}{s}ds, \quad I_n(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{2k+n}}{k!(n+k)!}$$

and

$$\lambda(T-t,y) = ye^{-\lambda(T-t)}\frac{I_y(\lambda(T-t))}{T-t}/\psi(t,y).$$

Let $f(t,y) = \lambda(T-t,y)\psi(t,y) = \partial\psi(t,y)/\partial t = -\partial\psi(t,y)/\partial T$, which is the probability density function at time T assessed at time t. Then $f(T,y) = I_{\{y=1\}}\lambda/2$ and

$$f(t,y) = ye^{-\lambda(T-t)}\frac{I_y(\lambda(T-t))}{T-t}, \quad t \in [0,T).$$

The terminal value of $A(t,y)$ is

$$A(T,1) = f(T,1) + \frac{\lambda}{2}[\psi(T,2) - 2\psi(T,1)] = \frac{\lambda}{2} + \frac{\lambda}{2}(1-2) = 0$$

and

$$A(T,y) = f(T,y) + \frac{\lambda}{2}[\psi(T,y+1) + \psi(T,y-1) - 2\psi(T,y)] = 0, \quad y \geq 2.$$

Next we will prove that $\partial A(t,y)/\partial t = 0$ for $t \in [0,T)$ and $y \geq 1$.

When $y \geq 2$, the derivative $\partial A(t,y)/\partial t$ is

$$\begin{aligned}
&\lambda f(t,y) - ye^{-\lambda(T-t)}\frac{\partial}{\partial u}\left(\frac{I_y(\lambda u)}{u}\right)|_{u=T-t} + \frac{\lambda}{2}[f(t,y+1)+f(t,y-1)-2f(t,y)] \\
= \; & \frac{\lambda}{2}[f(t,y+1)+f(t,y-1)] - ye^{-\lambda(T-t)}\frac{\partial}{\partial u}\left(\frac{I_y(\lambda u)}{u}\right)|_{u=T-t} \\
= \; & e^{-\lambda(T-t)}\left\{\frac{\lambda}{2u}[(y+1)I_{y+1}(\lambda u)+(y-1)I_{y-1}(\lambda u)] - y\frac{\partial}{\partial u}\left(\frac{I_y(\lambda u)}{u}\right)\right\}|_{u=T-t}.
\end{aligned}$$

Where

$$y\frac{\partial}{\partial u}\left(\frac{I_y(\lambda u)}{u}\right) = y\sum_{k=0}^{\infty}\frac{(2k+y-1)(\frac{\lambda}{2})^{2k+y}u^{2k+y-2}}{k!(y+k)!} = \frac{y\lambda}{2u}\sum_{k=0}^{\infty}\frac{(2k+y-1)(\frac{\lambda u}{2})^{2k+y-1}}{k!(y+k)!},$$

and

$$\begin{aligned}
& (y+1)I_{y+1}(\lambda u) + (y-1)I_{y-1}(\lambda u) \\
= \; & \sum_{k=0}^{\infty}\frac{(y+1)(\frac{\lambda u}{2})^{2k+y+1}}{k!(y+1+k)!} + \sum_{k=1}^{\infty}\frac{(y-1)(\frac{\lambda u}{2})^{2k+y-1}}{k!(y-1+k)!} + \frac{(y-1)(\frac{\lambda u}{2})^{y-1}}{(y-1)!} \\
= \; & \frac{(\frac{\lambda u}{2})^{y-1}}{(y-2)!} + \sum_{k=0}^{\infty}\frac{(y+1)(\frac{\lambda u}{2})^{2k+y+1}}{k!(y+1+k)!} + \sum_{k=0}^{\infty}\frac{(y-1)(\frac{\lambda u}{2})^{2k+y+1}}{(k+1)!(y+k)!} \\
= \; & \frac{(\frac{\lambda u}{2})^{y-1}}{(y-2)!} + \sum_{k=0}^{\infty}\frac{y(y+2k+1)(\frac{\lambda u}{2})^{2k+y+1}}{(k+1)!(y+1+k)!} \\
= \; & \frac{(\frac{\lambda u}{2})^{y-1}}{(y-2)!} + \sum_{k=0}^{\infty}\frac{y(y+2k-1)(\frac{\lambda u}{2})^{2k+y-1}}{k!(y+k)!} - \frac{y(y-1)(\frac{\lambda u}{2})^{y-1}}{y!} \\
= \; & y\sum_{k=0}^{\infty}\frac{(y+2k-1)(\frac{\lambda u}{2})^{2k+y-1}}{k!(y+k)!}.
\end{aligned}$$

Thus $\partial A(t,y)/\partial t = 0$ for $t \in [0,T)$ and $y \geq 2$. When $y=1$,

$$\begin{aligned}
\frac{dA(t,1)}{dt} &= \lambda f(t,1) - e^{-\lambda(T-t)}\frac{d}{du}\left(\frac{I_1(\lambda u)}{u}\right)|_{u=T-t} + \frac{\lambda}{2}[f(t,2)-2f(t,1)] \\
&= e^{-\lambda(T-t)}\left[\frac{\lambda I_2(\lambda(T-t))}{T-t} - \frac{d}{du}\left(\frac{I_1(\lambda u)}{u}\right)|_{u=T-t}\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{du}\left(\frac{I_1(\lambda u)}{u}\right) &= \sum_{k=0}^{\infty}\frac{(2k)(\frac{\lambda}{2})^{2k+1}u^{2k-1}}{k!(1+k)!} = \sum_{k=1}^{\infty}\frac{2(\frac{\lambda}{2})^{2k+1}u^{2k-1}}{(k-1)!(1+k)!} \\
&= \sum_{k=0}^{\infty}\frac{2(\frac{\lambda}{2})^{2k+3}u^{2k+1}}{k!(2+k)!} = \frac{\lambda I_2(\lambda u)}{u},
\end{aligned}$$

then $dA(t,1)/dt = 0$ for all $t \in [0,T)$.

We conclude that $\partial A(t,y)/\partial t = 0$ for $t \in [0,T)$ and $y \geq 1$, thus $A(t,y)$ is constant in t. And this constant number is equal to its terminal value $A(T,y) = 0$. Equation (5.4) has been proven.