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A LAW OF THE ITERATED LOGARITHM FOR STOCHASTIC PROCESSES DEFINED BY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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Consider the following random ordinary differential equation:

$$\dot{X}^{\varepsilon}(\tau) = F\big(X^{\varepsilon}(\tau), \tau/\varepsilon, \omega\big) \quad \text{subject to } X^{\varepsilon}(0) = x_0,$$

where $\{F(x,t,\omega),\ t \geq 0\}$ are stochastic processes indexed by x in $\Re^d$, and the dependence on x is sufficiently regular to ensure that the equation has a unique solution $X^{\varepsilon}(\tau,\omega)$ over the interval $0 \leq \tau \leq 1$ for each $\varepsilon > 0$. Under rather general conditions one can associate with the preceding equation a nonrandom averaged equation:

$$\dot{x}^{0}(\tau) = \overline{F}\big(x^{0}(\tau)\big) \quad \text{subject to } x^{0}(0) = x_0,$$

such that $\lim_{\varepsilon\to 0}\sup_{0\leq\tau\leq 1} E\,|X^{\varepsilon}(\tau) - x^{0}(\tau)| = 0$. In this article we show that as $\varepsilon \to 0$ the random function $(X^{\varepsilon}(\cdot) - x^{0}(\cdot))/\sqrt{2\varepsilon\log\log\varepsilon^{-1}}$ almost surely converges to and clusters throughout a compact set K of $C[0,1]$.

1. Introduction. Consider the random ordinary differential equation in $\Re^d$,

$$\dot{X}^{\varepsilon}(\tau) = F\big(X^{\varepsilon}(\tau), \tau/\varepsilon, \omega\big) \quad \text{subject to } X^{\varepsilon}(0) = x_0, \tag{1.1}$$

where $\varepsilon > 0$ is a "small" parameter and $\{F(x,t,\omega),\ t \geq 0\}$ is an $\Re^d$-valued "mixing" stochastic process, for each $x \in \Re^d$, regular enough to ensure that (1.1) has a unique solution $X^{\varepsilon}(\tau,\omega)$ over the interval $0 \leq \tau \leq 1$ for each $\varepsilon > 0$ [precise conditions on $\{F(x,t,\omega),\ t \geq 0\}$ will be formulated in Section 2]. Taking $\varepsilon \to 0$ in (1.1) corresponds to "accelerating" the right-hand side of the differential equation, and the limiting behavior (if any) of the $X^{\varepsilon}(\tau,\omega)$ as $\varepsilon \to 0$ is very relevant to many problems in diverse areas of physics and engineering. If $\{F(x,t),\ t \geq 0\}$ is weakly stationary as well as mixing for each x and if one defines $\overline{F}(x) \triangleq EF(x,0)$, then it seems reasonable to expect that the solution of the nonrandom "averaged" ordinary differential equation

$$\dot{x}^{0}(\tau) = \overline{F}\big(x^{0}(\tau)\big) \quad \text{subject to } x^{0}(0) = x_0, \tag{1.2}$$

will approximate $X^{\varepsilon}(\cdot)$ for small values of $\varepsilon > 0$. In fact, under rather broad regularity conditions on (1.1), it has been established that the following hold:

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(a) $\lim_{\varepsilon\to 0}\sup_{0\le\tau\le 1} E|X^\varepsilon(\tau)-x^0(\tau)| = 0$; (b) if one defines $Y^\varepsilon(\tau,\omega)\triangleq\varepsilon^{-1/2}(X^\varepsilon(\tau,\omega)-x^0(\tau))$, then the family of processes $\{Y^\varepsilon(\tau),\ 0\le\tau\le 1\}$ converges weakly to a certain limiting Gauss–Markov process as $\varepsilon\to 0$ (see [10], Theorems 1.1 and 3.1). Moreover, Freidlin [9, Theorem 2.1] develops a large-deviations principle for the "weak law of large numbers" in (a).

Motivated by the preceding results, we consider in this article the question of a law of the iterated logarithm for $X^\varepsilon(\cdot)$. Our goal then is to prove that as $\varepsilon\to 0$, $Y^\varepsilon(\cdot,\omega)/\sqrt{2\log\log\varepsilon^{-1}}$ almost surely converges to and clusters throughout a compact set K of $C[0,1]$, the space of continuous functions from $[0,1]$ into $\Re^d$. Previously Tomkins [23] and Lai and Wei [14] have studied laws of the iterated logarithm for weighted sums of the form $\sum_{i=1}^n f(i/n)\xi_i$ and $\sum_{i=1}^n a_{in}\xi_i$, where $f(\cdot)$ is a continuous function on $[0,1]$ and $\{\xi_i\}$ is an i.i.d. sequence. Using (1.1) and (1.2) to write

$$\frac{Y^\varepsilon(\tau,\omega)}{\sqrt{2\log\log\varepsilon^{-1}}} = \frac{1}{\sqrt{2\varepsilon^{-1}\log\log\varepsilon^{-1}}} \times\left[\int_0^{\tau\varepsilon^{-1}}\left\{F\big(X^\varepsilon(\varepsilon s),s\big)-\overline{F}\big(x^0(\varepsilon s)\big)\right\}ds\right], \tag{1.3}$$

one can see that there are certain similarities between the law of the iterated logarithm of interest here and the laws developed in [23] and [14]. Indeed, our approach is suggested to some extent by that adopted in [14], although there are also substantial differences as well. In particular, the proofs in [23] and [14] are based on rather direct use of exponential inequalities. Because our problem involves an underlying process which is mixing (rather than independent) and the interest is in a functional (rather than classical) law of the iterated logarithm, we find it more effective first to obtain rate bounds on the Prokhorov distance arising from the functional central limit theorem associated with the $C[0,1]$-convergence of $X^\varepsilon(\cdot)$ to $x^0(\cdot)$. These bounds are then used, along with an approximation theorem of Berkes and Philipp [4, Theorem 1] and a theorem of Kuelbs [12, Theorem 4.3] (which relates proximity in Prokhorov distance from a Gaussian measure to iterated logarithm behavior) to obtain eventually the desired result.

At this point it might not be inappropriate to point out an approach to functional laws of the iterated logarithm due to Stroock [22], who uses a large deviations principle of Schilder (in place of the Kolmogorov exponential inequalities) to prove the celebrated functional law of the iterated logarithm for Brownian motion due to Strassen [21]. Continuing this approach, Baldi [3] uses the Wentzell–Freidlin theory for stochastic differential equations with "small" noise (extended by Azencott [2] and Priouret [20]) to obtain a functional law of the iterated logarithm for diffusions with a suitably scaled driving Brownian motion (see (2.6) and Theorem 2.2 in Baldi [3]). This certainly suggests the possibility of similarly using the large-deviations theorem of Freidlin [9] [for (1.1)] for the averaging principle. However, the hypotheses associated with this theorem are quite strong and entail uniform boundedness of $F(x,t,\omega)$ [with respect to (x,t,ω)] as well as a rather nonstandard mixing condition (see condition $\mathcal{F}$

in [9], page 130). Moreover, it is not entirely clear how to adapt the general approach of Stroock to our problem. For these reasons we have eschewed large deviations and prefer to use a central limit theorem with rate bound instead.

In Section 2 we list the regularity conditions which will always be assumed in connection with (1.1); Section 3 contains the statement and proof of the law of the iterated logarithm, while Section 4 is a collection of results used for the technical developments in the article.

2. Conditions and notation. Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space on which is defined a system of $\Re^d$-valued processes $\{F(x, s, \omega),\ s \geq 0\}$ indexed by $x \in \Re^d$ and jointly measurable in (s, ω) on $[0, \infty] \times \Omega$ for each x. The following five conditions will be assumed throughout this article.

(C0) For each ω outside some P-null set $\Lambda_1 \in \mathcal{F}$, one has $\int_0^t |F(0, s, \omega)|\, ds < \infty$, for all $0 \leq t < \infty$ [henceforth, for any integer $m \geq 1$ and $x = (x_1, \dots, x_{md}) \in \Re^{md}$, we write $|x| \triangleq \max_{1 \leq i \leq md} |x_i|$].

(C1) There exist a P-null set $\Lambda_2 \in \mathcal{F}$ and a constant $\overline{N} > 0$ such that $x \to F(x, t)$ is twice continuously differentiable for each $t \geq 0, \omega \notin \Lambda_2$ and

$$\max\left\{\left|\frac{\partial F_i}{\partial x_j}(x, t, \omega)\right|, \left|\frac{\partial^2 F_i}{\partial x_j\, \partial x_k}(x, t, \omega)\right|\right\} \leq \overline{N},$$

$$\text{for all } 1 \leq i, j, k \leq d,\ x \in \Re^d,\ t \geq 0,\ \omega \notin \Lambda_2.$$

Henceforth we often use $N \triangleq \overline{N} \cdot d$.

(C2) There are σ-algebras $\{\mathcal{F}_s^t,\ 0 \leq s \leq t \leq \infty\}$ in Ω such that $F(x, t, \omega)$ is $\mathcal{F}_t^t$-measurable in ω, for each $x \in \Re^d,\ t \geq 0$, where the following hold:

(i) $\mathcal{F}_s^t \subset \mathcal{F}_u^v \subset \mathcal{F}$, for all $0 \leq u \leq s \leq t \leq v$;

(ii) The $\mathcal{F}_s^t$ are strong mixing in the sense of Rosenblatt. That is, if $\alpha(\tau)$ is defined for all $\tau \geq 0$ by $\alpha(\tau) \triangleq \sup |P(AB) - P(A)P(B)|$ (the supremum being taken over all $A \in \mathcal{F}_0^t,\ B \in \mathcal{F}_{\tau+t}^{\infty}$ and $t \geq 0$), then $\alpha(\tau) \to 0$ as $\tau \to \infty$.

(C3) For each $x \in \Re^d$, the process $\{F(x, t),\ t \geq 0\}$ is weakly stationary [i.e., $EF(x, s) = EF(x, 0)$ and $E\{F_i(x, s)F_j(x, t)\} = E\{F_i(x, 0)F_j(x, t - s)\}$, for all $0 \leq s \leq t,\ 1 \leq i, j \leq d$]. Moreover, for all $p \geq 1$, one has

$$M_p \triangleq \sup_{t \geq 0} \left\|F(0, t)\right\|_p < \infty,$$

where, for any d-dimensional random vector $X = (X_1, \dots, X_d)$ and $1 \leq r < \infty$, we write $\|X\|_r \triangleq E^{1/r}(|X|^r)$.

(C4) The mixing coefficient $\alpha(\cdot)$ defined in (C2) satisfies

$$\alpha(\tau) \leq \eta \exp(-\tau^{\kappa}) \quad \text{at each } \tau \geq 1, \text{ for some constants } \kappa > 0,\ \eta > 0.$$

In view of (C0) and (C1) the random ordinary differential equation

$$\dot{x}(\tau) = F\big(x(\tau), \tau/\varepsilon, \omega\big) \quad \text{subject to } x(0) = x_0, \tag{2.1}$$

has a unique solution $X^\varepsilon(\tau,\omega)$ defined for all $\tau \in [0,1]$, $\omega \notin \Lambda_1 \cup \Lambda_2$, $\varepsilon > 0$. Moreover, if one defines an "averaged" right-hand side for (2.1) by $\overline{F}(x) \triangleq EF(x,0)$, then the averaged differential equation

$$\dot{x}(\tau) = \overline{F}\big(x(\tau)\big) \quad \text{subject to } x(0) = x_0, \tag{2.2}$$

has a unique solution $x^0(\tau)$, $0 \le \tau \le 1$. In the above and throughout this article, the initial condition x_0 is held fixed as we study the a.s. asymptotics of $X^\varepsilon(\cdot,\omega)$ with $\varepsilon \to 0$. For later reference, define the following:

$$D \triangleq \sup_{0\le\tau\le 1} \left|x^0(\tau)\right|, \qquad N \triangleq \overline{N}d, \qquad M \triangleq M_1 \quad \text{[see (C3)]}; \tag{2.3}$$

$$\widetilde{F}(x,t) \triangleq F(x,t) - EF(x,t) = F(x,t) - \overline{F}(x), \tag{2.4}$$

for each x and t. Moreover, for each x define the $d \times d$ symmetric positive semi-definite matrix

$$A(x) \triangleq \int_0^\infty E\Big\{\big(\widetilde{F}(x,0)\big)\big(\widetilde{F}(x,t)\big)^T\Big\} + E\Big\{\big(\widetilde{F}(x,t)\big)\big(\widetilde{F}(x,0)\big)^T\Big\}\,dt, \tag{2.5}$$

and let $\{\widehat{B}^0(\tau,\widehat{\omega}),\ 0 \le \tau \le 1\}$ be an $\Re^d$-valued standard Brownian motion defined on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$. We now define the Gauss–Markov processes $\{\widehat{W}^0(\tau,\widehat{\omega}),\ 0 \le \tau \le 1\}$ and $\{\widehat{Y}^0(\tau,\widehat{\omega}),\ 0 \le \tau \le 1\}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ by

$$d\widehat{W}^0(\tau) \triangleq A^{1/2}\big(x^0(\tau)\big)\,d\widehat{B}^0(\tau) \qquad \text{subject to } \widehat{W}^0(0) = 0, \tag{2.6}$$

$$d\widehat{Y}^0(\tau) \triangleq \frac{\partial \overline{F}}{\partial x}\big(x^0(\tau)\big)\widehat{Y}^0(\tau)\,d\tau + d\widehat{W}^0(\tau) \quad \text{subject to } \widehat{Y}^0(0) = 0, \tag{2.7}$$

where $A(x) \equiv (A^{1/2}(x))(A^{1/2}(x))^T$. Finally, we define on the original probability space $(\Omega, \mathcal{F}, \mathcal{P})$ the process $\{Y^\varepsilon(\tau,\omega),\ 0 \le \tau \le 1\}$, for each $\varepsilon > 0$, by

$$Y^\varepsilon(\tau,\omega) \triangleq \varepsilon^{-1/2}\big(X^\varepsilon(\tau,\omega) - x^0(\tau)\big), \qquad 0 \le \tau \le 1. \tag{2.8}$$

Theorem 3.1 of Khas'minskii [10] implies that, under conditions which certainly include (C0)–(C4), the family of processes $\{Y^\varepsilon(\cdot)\}$ converges weakly to the limiting process $\{\widehat{Y}^0(\cdot)\}$ as $\varepsilon \to 0$. In this note we establish a corresponding functional law of the iterated logarithm (LIL) which is suggested by this "central limit theorem." Namely, it will be shown that, for a.a. ω, the set of all $C[0,1]$-accumulation points of $\{Y^\varepsilon(\cdot,\omega)(2\log\log\varepsilon^{-1})^{-1/2},\ \varepsilon > 0\}$ (taken as $\varepsilon \to 0$) is equal to the unit ball of the reproducing kernel Hilbert space generated by the covariance function of the Gauss–Markov process $\{\widehat{Y}^0(\cdot)\}$ in (2.7).

REMARK 2.1. We let $C[0,1]$ denote the Banach space of all functions x from $T \triangleq \{1,2,\dots,d\} \times [0,1]$ into $\Re$ such that $\tau \to x(i,\tau)$ is continuous on $[0,1]$, for all $1 \le i \le d$, with norm defined by

$$\|x\|_c \triangleq \max_{\substack{1\le i\le d\\ 0\le\tau\le 1}} |x(i,\tau)|.$$

The theory of reproducing kernel Hilbert spaces (RKHS) generated by a symmetric positive semidefinite function from $T \times T$ into $\Re$ has been amply developed elsewhere (see, e.g., Aronszajn [1], LePage [15] and Kuelbs [12]). For our purpose we need only the following observation, which follows from Kuelbs [12], Lemma 2.1 (vi)). If μ is some zero-mean Gaussian measure on $B \triangleq C[0,1]$ [i.e., each $f \in B^*$ is a zero-mean Gaussian random variable with variance $\int_B f^2(x)\,d\mu(x)$] and $R((i,\sigma),(j,\tau)) \triangleq E^{\mu}[x(i,\sigma)x(j,\tau)]$, $(i,\sigma),(j,\tau) \in T$, is the covariance function of μ, then the RKHS generated by R coincides with the Hilbert space H_μ in Kuelbs [12, Lemma 2.1]. (A similar statement can be made if $B \triangleq \Re^k$, for some $k \ge 1$.) If H_W and H_Y are the reproducing kernel Hilbert spaces corresponding to $\{\widehat{W}^0(\cdot)\}$ and $\{\widehat{Y}^0(\cdot)\}$ [see (2.6) and (2.7)], then their unit balls can be shown to be

$$K_W = \{\phi \in AC_0[0,1],\ I_W(\phi) \le 1\} \quad \text{and} \quad K_Y = \{\phi \in AC_0[0,1],\ I_Y(\phi) \le 1\},$$

respectively, where

$$AC_0[0,1] \triangleq \{\psi \in C[0,1]\colon \psi(0) = 0,\ \psi_i(\cdot) \text{ is absolutely continuous}\},$$

$$I_W(\phi) \triangleq \inf_u \left\{ \frac{1}{2}\int_0^1 u^T(s)u(s)\,ds\colon u \in L_2[0,1],\ \dot\phi(t) = A^{1/2}(x^0(t))u(t) \right\},$$

$$I_Y(\phi) \triangleq \inf_u \left\{ \frac{1}{2}\int_0^1 u^T(s)u(s)\,ds\colon u \in L_2[0,1],\right.$$
$$\left. \dot\phi(t) = \frac{\partial \overline{F}}{\partial x}(x^0(t))\phi(t) + A^{1/2}(x^0(t))u(t) \right\}.$$

We mention these explicit representations of K_W and K_Y for interest only. These forms will not be used in any of our proofs.

REMARK 2.2. We shall use the following notation due to Kuelbs [12]: If $\phi(\cdot)$ is some function from $(0,1]$ into a metric space $(\mathcal{S},\rho)$, then set $\rho(x,A) \triangleq \inf_{y\in A} \cdot \rho(x,y)$ for any set $A \subset \mathcal{S}$, and let $\mathcal{C}\{\phi(\varepsilon),\ \varepsilon > 0\}$ denote the set of all possible limit points (if any) of $\phi(\varepsilon)$ as $\varepsilon \to 0$. Finally, we write $\{\phi(\varepsilon),\ \varepsilon > 0\} \twoheadrightarrow A$ to indicate (i) $\rho(\phi(\varepsilon),A) \to 0$ as $\varepsilon \to 0$ and (ii) $\mathcal{C}\{\phi(\varepsilon),\ \varepsilon > 0\} = A$.

REMARK 2.3. In many applications the process $\{F(x,t,\omega),\ t \ge 0\}$ is typically of the form $F(x,t,\omega) \triangleq F_1(x,\xi(t,\omega))$, where $(x,y) \to F_1(x,y)\colon \Re^d \times \Re^q \to \Re^d$ is sufficiently regular for (C1) to hold, and $\{\xi(t,\omega),\ t \ge 1\}$ is a stationary process obtained from a given stationary "discrete-time" process $\{\tilde\xi_k(\omega),\ k = 0,1,2,\ldots\}$ by $\xi(t,\omega) \triangleq \tilde\xi_k(\omega)$, for all $t \in [k,k+1)$. [Two common cases are (i)

$F(x,t,\omega) \triangleq a(x) + \xi(t,\omega)$ and (ii) $F(x,t,\omega) \triangleq b(x) \cdot \xi(t,\omega)$, where, in the case of (ii), $\xi(t,\omega)$ must be uniformly bounded in (t,ω) and typically arises from a finite state Markov chain $\{\widetilde{\xi}_k\}$ in the manner indicated previously.] Bradley [5] gives several examples of processes $\{\widetilde{\xi}_k,\ k = 0, 1, 2, \dots\}$ which are strong mixing with exponentially decreasing mixing rates and thus are sufficient to ensure that (C4) holds (note in particular Theorems 4.2 and 5.1 and the examples in Section 6 of [5], as well as the related Theorem 3.1 of Pham and Tran [19] and Theorem 1 of Mokkadem [17]).

REMARK 2.4. From (1.3) one sees that the parameter $\varepsilon > 0$ determines not only the range of integration $[0, \tau\varepsilon^{-1}]$ but also affects the mixing process appearing after the integral sign. This causes some technical complications, and our approach to resolving this issue uses the full strength of the moment bounds and mixing rates in (C3) and (C4). These conditions are invoked only to conclude the proof of Proposition 3.2 (to follow). For all other proofs and arguments much less restrictive mixing rates and moment bounds suffice.

3. A law of the iterated logarithm for $\{Y^{\varepsilon}(\cdot), \varepsilon > 0\}$. The main result of this article is the following.

THEOREM 3.1. *Under conditions* (C0)–(C4) *of Section* 2 *one has*

$$\left\{ \frac{Y^{\varepsilon}(\cdot)}{\sqrt{2 \log\log \varepsilon^{-1}}},\ \varepsilon > 0 \right\} \twoheadrightarrow K_Y \quad a.s.,$$

where K_Y is a compact subset of $C[0,1]$, namely, the unit ball of the reproducing kernel Hilbert space generated by the covariance function of the Gauss–Markov process $\{\widehat{Y}^0(\cdot)\}$ defined in (2.7), *and $\{Y^{\varepsilon}(\cdot)\}$ is defined by* (2.8).

PROOF. Define G: $C[0,1] \to C[0,1]$, by $G(\psi) \triangleq \phi$, where

$$\phi(\tau) = \psi(\tau) + \int_0^{\tau} \frac{\partial \overline{F}}{\partial x}\big(x^0(s)\big)\phi(s)\,ds \quad \text{for } 0 \le \tau \le 1. \tag{3.1}$$

From standard theory of linear integral equations (see [6], Theorem 3.3.2), corresponding to each $\psi(\cdot) \in C[0,1]$ there is a unique $\phi(\cdot) \in C[0,1]$ which solves (3.1), so $G(\cdot)$ is well defined on $C[0,1]$. Clearly $G(\cdot)$ is one-to-one and onto, and both $G(\cdot)$ and $G^{-1}(\cdot)$ are continuous and linear. Now, by Proposition 3.2 and 3.3,

$$\left\{ \frac{W_1^{\varepsilon}(\cdot)}{\sqrt{2 \log\log \varepsilon^{-1}}},\ \varepsilon > 0 \right\} \twoheadrightarrow K_W \quad \text{a.s.}, \tag{3.2}$$

where K_W is defined to be the unit ball in the RKHS generated by the covariance function of the Gauss–Markov process $\{\widehat{W}^0(\cdot)\}$ defined in (2.6), and $\{W_1^{\varepsilon}(\tau),\ 0 \le \tau \le 1\}$ is defined by

$$W_1^{\varepsilon}(\tau) \triangleq \varepsilon^{-1/2} \int_0^{\tau} \widetilde{F}\big(x^0(s), s/\varepsilon\big)\,ds, \tag{3.3}$$

with $\widetilde{F}$ given by (2.4). Since $\mu \triangleq \mathcal{L}(\widehat{W}^0)$ is easily seen to be a Gaussian measure on $B = C[0,1]$, it follows from Remark 2.1 and Kuelbs [12, Lemma 2.1(iv)] that K_W is a compact subset of $C[0,1]$. Set

$$K_Y \triangleq G[K_W]; \tag{3.4}$$

by the continuity of G, the set K_Y is a compact subset of $C[0,1]$. Defining $Z_1^\varepsilon(\cdot)$ by $Z_1^\varepsilon \triangleq G(W_1^\varepsilon)$, one sees from (3.4) and (3.2) that

$$\left\{ \frac{Z_1^\varepsilon(\cdot)}{\sqrt{2\log\log\varepsilon^{-1}}},\ \varepsilon > 0 \right\} \twoheadrightarrow K_Y \quad \text{a.s.} \tag{3.5}$$

Moreover, from the definition of $\{\widehat{Y}^0(\cdot)\}$ in (2.7) it follows that

$$\widehat{Y}^0 = G(\widehat{W}^0). \tag{3.6}$$

If $(H_Y, \|\cdot\|_Y)$ and $(H_W, \|\cdot\|_W)$ are the RKHS's generated by the covariance function of the Gauss–Markov processes $\{\widehat{Y}^0(\cdot)\}$ and $\{\widehat{W}^0(\cdot)\}$, respectively, then it is easy to see from (3.6), the characterization of RKHS in Kuelbs [12, Lemma 2.1] and the fact that $G(\cdot)$ and $G^{-1}(\cdot)$ are continuous linear one-to-one functions from $C[0,1]$, onto $C[0,1]$, that $H_Y = G[H_W]$ and $\|x\|_W = \|G(x)\|_Y$, for all $x \in H_W$. Now it follows at once that the set K_Y in (3.4) is the unit ball of $(H_Y, \|\cdot\|_Y)$. Theorem 3.1 follows from (3.5) and Lemma 3.5. □

3.1. *A law of the iterated logarithm for* $\{W_1^\varepsilon(\cdot),\ \varepsilon > 0\}$.

PROPOSITION 3.2. *Under* (C0)–(C4) *of Section* 2 *we have that, for a.a.* ω,

$$\lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\| \frac{W_1^\varepsilon(\cdot)}{\sqrt{2\log\log\varepsilon^{-1}}} - K_W \right\|_C = 0, \tag{3.7}$$

where $\{W_1^\varepsilon(\tau),\ 0 \le \tau \le 1\}$ *is defined in* (3.3) *and* K_W *is a compact subset of* $C[0,1]$, *namely, the unit ball of the* RKHS *generated by the covariance function of the Gauss–Markov process* $\{\widehat{W}^0(\tau),\ 0 \le \tau \le 1\}$ *in* (2.6).

PROOF. Fix any $\sigma \in [1/2, 1)$ and put $\varepsilon_r \triangleq \exp(-r^\sigma)$, $r = 2, 3, \ldots$. By Lemma 3.4 there exist $c_1 > 0$ and $\rho > 0$ such that the Prokhorov distance between the laws of $\{W_1^{\varepsilon_r}(\cdot)\}$ and $\{\widehat{W}^0(\cdot)\}$ satisfies

$$\Pi\big(\mathcal{L}(W_1^{\varepsilon_r}), \mathcal{L}(\widehat{W}^0)\big) \le c_1 \exp(-\rho r^\sigma), \qquad r = 2, 3, \ldots .$$

Now $\mu \triangleq \mathcal{L}(\widehat{W}^0)$ is easily seen to be a mean-zero Gaussian measure on $B \triangleq C[0,1]$. Hence, by Theorem 4.5(i), one obtains

$$\lim_{r\to\infty} \left\| \frac{W_1^\varepsilon(\cdot)}{\sqrt{2\log\log\varepsilon_r^{-1}}} - \frac{K_W}{\sqrt{\sigma}} \right\|_C = 0 \quad \text{for a.a. } \omega, \tag{3.8}$$

where K_W is as specified in the statement of the lemma. Now we prove

$$P\Big\{\lim_{r\to\infty}\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|W_1^{\varepsilon}-W_1^{\varepsilon_r}\|_c=0\Big\}=1 \tag{3.9}$$

by adapting an idea of Lai and Wei [14, page 328] for i.i.d. random variables to the case of $C[0,1]$-valued mixing random functions. In order to do this we define

$$S_T(\tau)\triangleq\begin{cases}\displaystyle\int_0^{\tau T}\widetilde{F}(x^0(s/T),s)\,ds, & 0\le\tau\le 1,\ T>0,\\ 0, & 0\le\tau\le 1,\ T=0.\end{cases} \tag{3.10}$$

Then, for any integer $p\ge 3$,

$$\begin{aligned}&E\Big[\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|W_1^{\varepsilon}-W_q^{\varepsilon_r}\|_C^{2p}\Big]\\ &\quad\le 2^{2p}\Big\{E\Big[\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|(\varepsilon^{1/2}-\varepsilon_r^{1/2})S_{\varepsilon^{-1}}\|_C^{2p}\Big]\\ &\qquad+E\Big[\varepsilon_r^p\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|S_{\varepsilon^{-1}}-S_{\varepsilon_r^{-1}}\|_C^{2p}\Big]\Big\}.\end{aligned} \tag{3.11}$$

Now, from Lemma 3.8, one sees that

$$\begin{aligned}E\Big[\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|(\varepsilon^{1/2}-\varepsilon_r^{1/2})S_{\varepsilon^{-1}}\|_c^{2p}\Big]&\le c_1\big(\varepsilon_r^{1/2}-\varepsilon_{r+1}^{1/2}\big)^{2p}\big(\varepsilon_{r+1}^{-1}\big)^p\\ &\le c_1\Big\{\big(\varepsilon_r-\varepsilon_{r+1}\big)\big(\varepsilon_{r+1}^{-1}\big)\Big\}^p,\end{aligned} \tag{3.12}$$

for some constant c_1. Again, from Lemma 3.8,

$$\begin{aligned}E\Big[\varepsilon_r^p\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|S_{\varepsilon^{-1}}-S_{\varepsilon_r^{-1}}\|_C^{2p}\Big]&\le c_1\varepsilon_r^p\big(\varepsilon_{r+1}^{-1}\big)\big(\varepsilon_{r+1}^{-1}-\varepsilon_r^{-1}\big)^{p-1}\\ &\le c_1 e\Big\{\big(\varepsilon_r-\varepsilon_{r+1}\big)\big(\varepsilon_{r+1}^{-1}\big)\Big\}^{p-1},\end{aligned} \tag{3.13}$$

where the second inequality follows because $(\varepsilon_r/\varepsilon_{r+1})\le e$ (since $\sigma<1$). However, by the mean value theorem and the fact that $\sigma<1$, one sees that $(\varepsilon_r-\varepsilon_{r+1})\varepsilon_r^{-1}\le e\sigma r^{\sigma-1}$. Thus, choosing an integer $p>(2-\sigma)/(1-\sigma)$, we see from (3.11)–(3.13) and the monotone convergence theorem that

$$E\Big[\sum_{r\ge 2}\max_{\varepsilon_{r+1}\le\varepsilon\le\varepsilon_r}\|W_1^{\varepsilon}-W_1^{\varepsilon_r}\|_C^{2p}\Big]<\infty,$$

which is enough to establish (3.9). Equation (3.7) follows easily from (3.8), (3.9) and the fact that one can choose $\sigma<1$ arbitrarily near to 1 in the definition of ε_r. □

PROPOSITION 3.3. *Under* (C0)–(C4) *of Section* 2, *we have for a.a.* ω *that*

$$K_W = C\left\{ \frac{W_1^\varepsilon(\cdot)}{\sqrt{2\log\log\varepsilon^{-1}}},\ \varepsilon > 0 \right\}, \tag{3.14}$$

where $\{W_1^\varepsilon(\cdot)\}$ *and* K_W *are as in Proposition* 3.2.

PROOF. From (C0), (C1) and (C3), one easily sees that $\varepsilon \to W_1^\varepsilon(\cdot)$: $(0,\infty) \to C[0,1]$ is continuous for a.a. ω. In view of (3.7) and the fact that K_W is a compact subset of $C[0,1]$, hence totally bounded, it follows that

$$\left\{ \frac{W_1^\varepsilon(\cdot)}{\sqrt{2\log\log\varepsilon^{-1}}},\ 1 \geq \varepsilon > 0 \right\} \tag{3.15}$$

is a relatively compact subset of $C[0,1]$ for a.a. ω. Now, we will use this compactness along with Lemma 4.1 to prove (3.14). Fix any $0 < \tau_1 < \tau_2 < \cdots < \tau_m = 1$, some $1 < \sigma \leq 2$ and define $\varepsilon_r \triangleq \exp(-r^\sigma)$, for all $r = 2, 3, \ldots$. Also choose an integer r_0 such that $r_0 \geq (10^8 m)^{4/9}$, $(r_0 - 1)^{\sigma-1} \geq 2\log r_0$ and $r_0 \geq (2/\tau_1)^{1/2}$. To lighten the notation define

$$U_r(\omega) \triangleq \left[\left(W_1^{\varepsilon_r}(\tau_1,\omega)\right)^T, \left(W_1^{\varepsilon_r}(\tau_2,\omega)\right)^T, \ldots, \left(W_1^{\varepsilon_r}(\tau_m,\omega)\right)^T \right]^T,$$
$$\widehat{U}^0(\widehat{\omega}) \triangleq \left[\left(\widehat{W}^0(\tau_1,\omega)\right)^T, \left(\widehat{W}^0(\tau_2,\omega)\right)^T, \ldots, \left(\widehat{W}^0(\tau_m,\omega)\right)^T \right]^T,$$

where $\{\widehat{W}^0(\tau),\ 0 \leq \tau \leq 1\}$ is defined by (2.6). Also, let $\mathcal{G}_r \triangleq \sigma\{U_1, \ldots, U_r\}$. Now $|e^{ix} - e^{iy}| \leq |x - y|$, $x, y \in \Re$, so, for any $u \in \Re^{md}$, $r \geq r_0$, one has

$$\begin{aligned} &E\left| E\left\{ \exp\left(iu^T U_r\right) \mid \mathcal{G}_{r-1} \right\} - E\left\{ \exp\left(iu^T U_r\right) \right\} \right| \\ &\quad \leq 2E\left| u^T (V_r - U_r) \right| \\ &\qquad + E\left| E\left\{ \exp\left(iu^T V_r\right) \mid \mathcal{G}_{r-1} \right\} - E\left\{ \exp\left(iu^T V_r\right) \right\} \right|, \end{aligned} \tag{3.16}$$

where V_r is the md-dimensional random vector whose $((i-1)d + j)$-th scalar component is defined for $i = 1, \ldots, m$, $j = 1, \ldots, d$, by

$$V_r^{((i-1)d+j)} \triangleq \varepsilon_r^{1/2} \int_{\tau_i r^2 \varepsilon_{r-1}^{-1}}^{\tau_i \varepsilon_r^{-1}} \widetilde{F}_j\left(x^0(\varepsilon_r s), s\right) ds, \tag{3.17}$$

for all $r \geq r_0$, and $\widetilde{F}$ is given by (2.4). (From the definition of r_0 one sees that $r^2 \varepsilon_{r-1}^{-1} < \varepsilon_r^{-1}$ for all $r \geq r_0$.) Now by (C2)–(C4) of Section 2 and by Lemma 4.7(a), there exists a number $c_1 > 0$, independent of r, such that [setting $k \triangleq (i-1)d+j$]

$$
\begin{aligned}
E\left|U_r^k - V_r^k\right| &\leq \varepsilon_r^{1/2} E^{1/2}\left|\int_0^{\tau_i r^2 \varepsilon_{r-1}^{-1}} \widetilde{F}_j\left(x^0(\varepsilon_r s), s\right) ds\right|^2 \\
&\leq c_1\left(\varepsilon_r \tau_i r^2 \varepsilon_{r-1}^{-1}\right)^{1/2} \\
&\leq c_1 r \exp\left[-\tfrac{1}{2}(r-1)^{(\sigma-1)}\right].
\end{aligned}
\tag{3.18}
$$

Using (C2) and (C4) of Section 2, the fact that $\mathcal{G}_{r-1} \subset \mathcal{F}_0^{\varepsilon_{r-1}^{-1}}$ and Lemma 4.2, there is a constant $c_2 > 0$ such that

$$
\begin{aligned}
&E\left|E\left\{\exp\left(iu^T V_r\right) \mid \mathcal{G}_{r-1}\right\} - E\left\{\exp\left(iu^T V_r\right)\right\}\right| \\
&\qquad \leq 2\pi\alpha\left(\left(\tau_1 r^2 - 1\right)\varepsilon_{r-1}^{-1}\right) \leq c_2 \varepsilon_{r-1},
\end{aligned}
\tag{3.19}
$$

for all $r \geq r_0$, $u \in \Re^{md}$. However, by Lemma 3.4 and the definition of Prokhorov distance, there are constants $c_3 > 0$ and $\rho > 0$ such that

$$
\Pi_\infty^{md}\left(\mathcal{L}(U_r),\ \mathcal{L}(\widehat{U}^0)\right) \leq c_3 \varepsilon_r^\rho \quad \text{for all } r \geq r_0. \tag{3.20}
$$

[Here $\Pi_\infty^{md}(P, Q)$ denotes the Prokhorov distance between probability measures P and Q defined on the Borel σ-algebra of the md-dimensional vector space $(\Re^{md}, |\cdot|)$.] Thus, by the Strassen–Dudley theorem (see Theorem 4.4), there exists some $2md$-dimensional random vector $(\widetilde{U}_r^T, \widetilde{Y}_r^T)^T$ on a probability space $(\widetilde{\Omega}_r, \widetilde{\mathcal{F}}_r, \widetilde{P}_r)$ such that

$$
\mathcal{L}(U_r) = \mathcal{L}(\widetilde{U}_r), \qquad \mathcal{L}(\widetilde{Y}_r) = \mathcal{L}(\widehat{U}^0) \quad \text{and} \quad \widetilde{P}_r\left(|\widetilde{U}_r - \widetilde{Y}_r| > c_3\varepsilon_r^\rho\right) \leq c_3 \varepsilon_r^\rho \quad \text{for each } r \geq r_0.
$$

Now let $g(u)$, $u \in \Re^{md}$, be the characteristic function of $\widehat{U}_0$. Then

$$
\begin{aligned}
&\left|E\left\{\exp\left(iu^T U_r\right)\right\} - g(u)\right| \\
&\quad \leq \widetilde{E}_r\left[\left|\exp\left(iu^T \widetilde{U}_r\right) - \exp\left(iu^T \widetilde{Y}_r\right)\right|; \left\{\left|\widetilde{U}_r - \widetilde{Y}_r\right| > c_3 \varepsilon_r^\rho\right\}\right] \\
&\qquad + \widetilde{E}_r\left[md|u| \cdot \left|\widetilde{U}_r - \widetilde{Y}_r\right|; \left|\widetilde{U}_r - \widetilde{Y}_r\right| \leq c_3 \varepsilon_r^\rho\}\right] \\
&\quad \leq 2c_3 \varepsilon_r^\rho + md\left|u\right| c_3 \varepsilon_r^\rho,
\end{aligned}
\tag{3.21}
$$

for $r \geq r_0$, where $\widetilde{E}_r$ denotes integration with respect to $\widetilde{P}_r$. In view of (3.16), (3.18), (3.19) and (3.21), there is a $c_4 > 0$, independent of r, such that

$$
\begin{aligned}
&E\left|E\left\{\exp\left(iu^T U_r\right) \mid \mathcal{G}_{r-1}\right\} - g(u)\right| \\
&\qquad \leq c_4 r^{13/4} \exp\left(-\beta(r-1)^{(\sigma-1)}\right) \triangleq \lambda_r,
\end{aligned}
\tag{3.22}
$$

for all $r \geq r_0$, $|u| \leq T_r \triangleq r^{9/4}$, where $\beta \triangleq \min(\rho, 1/2)$. Moreover, one can easily bound the Gaussian distributed random vector $\widehat{U}^0$, to find some constant c_5 such that

$$\widehat{P}\Big(|\widehat{U}^0| > \tfrac{1}{4}T_r\Big) \leq \exp\big(-c_5 T_r^2\big) \triangleq \delta_r \tag{3.23}$$

(see Berkes and Philipp [4], page 43, and observe that $\operatorname{cov}(\widehat{U}_0)$ need not be nonsingular for this estimate to be valid, as follows from the comment on page 1015 of Kuelbs and Philipp [13]). If one defines $\alpha_r \triangleq 16mdT_r^{-1}\log T_r + 4\lambda_r^{1/2}T_r^{md} + \delta_r$, then it follows that $\sum\alpha_r < \infty$; hence, from (3.23), (3.22) and Theorem 4.3, there exist on some $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ the md-dimensional vector processes $\{\overline{U}_r,\ r \geq r_0\}$ and $\{\overline{U}_r^{(0)},\ r \geq r_0\}$ such that the following hold:

(a) $\{\overline{U}_r,\ r \geq r_0\} =_{\mathcal{D}} \{U_r,\ r \geq r_0\}$;
(b) $\{\overline{U}_r^0,\ r \geq r_0\}$ is i.i.d. with $\mathcal{L}(\overline{U}_r^0) = \mathcal{L}(\widehat{U}^0)$, for $r \geq r_0$;
(c) $\lim_{r\to\infty} |\overline{U}_r - \overline{U}_r^0| = 0$, a.s. $[\overline{P}]$.

By virtue of (b), (c) and Theorem 4.5(ii), it is easy to see that

$$\left\{\frac{\overline{U}_r}{\sqrt{2\log r}},\ r \geq r_0\right\} \twoheadrightarrow K_W[\tau_1 \cdots \tau_m] \quad \text{a.s. } [\overline{P}], \tag{3.24}$$

where K_w is as in Proposition 3.2, and the notation $K_W[\tau_l \cdots \tau_m]$ is defined in the statement of Lemma 4.1. Now $\log\log\varepsilon_r^{-1} = \sigma\log r$ and [see (a)] (3.24) continues to hold a.s. $[P]$ when $\overline{U}_r$ is replaced by U_r. Thus, by Lemma 4.1 and a.s. relative compactness of (3.15), one sees that

$$\frac{K_W}{\sqrt{\sigma}} = \mathcal{C}\left\{\frac{W_1^{\varepsilon_r}}{\sqrt{2\log\log\varepsilon_r^{-1}}}\right\} \subset \mathcal{C}\left\{\frac{W_1^{\varepsilon}}{\sqrt{2\log\log\varepsilon^{-1}}},\ \varepsilon > 0\right\},$$

for a.a. ω. The arbitrary choice of $1 < \sigma$ gives one half of the set equality in (3.14); the opposite half is an immediate consequence of Proposition 3.2. □

3.2. *Central limit theorem with rate of convergence for* $\{W_1^{\varepsilon}(\cdot), \varepsilon > 0\}$ (*from* [11], *Lemma* A6.1).

LEMMA 3.4. *Under* (C0)–(C4), *there are constants* $c_1 > 0$, $\rho > 0$ *and* $\varepsilon_0 \in (0, 1]$ *such that*

$$\Pi\Big(\mathcal{L}(W_1^{\varepsilon}), \mathcal{L}(\widehat{W}^0)\Big) \leq c_1\varepsilon^{\rho}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Here $\mathcal{L}(W_1^{\varepsilon})$ and $\mathcal{L}(\widehat{W}_0)$ are the respective probability measures generated in $C[0,1]$ by the processes $\{W_1^{\varepsilon}(\cdot)\}$ and $\{\widehat{W}^0(\cdot)\}$ in (3.3) and (2.6), and $\Pi(P, Q)$ denotes the Prokhorov distance between probability measures P and Q in $C[0,1]$.

3.3. *An a.s. approximation of Y^ε by $G(W_1^\varepsilon)$.*

LEMMA 3.5. *Assume* (C0)–(C4) *of Section* 2. *If* $Z_1^\varepsilon \triangleq G(W_1^\varepsilon)$, *where* $\{W_1^\varepsilon(\cdot)\}$ *is defined by* (3.3) *and* $G(\cdot)$ *is defined by* (3.1), *then*

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \|Z_1^\varepsilon - Y^\varepsilon\|_C = 0 \quad a.s., \tag{3.25}$$

where $\{Y^\varepsilon(\cdot)\}$ *is given by* (2.8).

PROOF. Put $U_1^\varepsilon(\tau) \triangleq Y^\varepsilon(\tau) - Z_1^\varepsilon(\tau)$. From (3.1), (2.8), (3.3), (2.2) and (2.1), one sees that

$$\begin{aligned} U_1^\varepsilon(\tau) = &\int_0^\tau \big(\Psi(s,\varepsilon) - \Xi(s,\varepsilon)\big)\, ds + \int_0^\tau \Xi(s,\varepsilon)\, ds + I_1^\varepsilon(\tau) \\ &+ \int_0^\tau \frac{\partial F}{\partial x}\Big(x^0(s), \frac{s}{\varepsilon}\Big) U_1^\varepsilon(s)\, ds, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} \Psi(s,\varepsilon) \triangleq \varepsilon^{-1/2}\bigg[&F\Big(X^\varepsilon(s), \frac{s}{\varepsilon}\Big) - F\Big(x^0(s), \frac{s}{\varepsilon}\Big) \\ &- \varepsilon^{1/2}\frac{\partial F}{\partial x}\Big(x^0(s), \frac{s}{\varepsilon}\Big) Y^\varepsilon(s)\bigg], \end{aligned} \tag{3.27}$$

$$\begin{aligned} \Xi(s,\varepsilon) \triangleq \varepsilon^{-1/2}\bigg[&F\Big(x^0(s) + \varepsilon^{1/2} Z_1^\varepsilon(s), \frac{s}{\varepsilon}\Big) - F\big(x^0(s), s/\varepsilon\big) \\ &- \varepsilon^{1/2}\frac{\partial F}{\partial x}\Big(x^0(s), \frac{s}{\varepsilon}\Big) Z_1^\varepsilon(s)\bigg], \end{aligned} \tag{3.28}$$

$$I_1^\varepsilon(\tau) \triangleq \int_0^\tau \bigg[\frac{\partial F}{\partial x}\Big(x^0(s), \frac{s}{\varepsilon}\Big) - \frac{\partial \overline{F}}{\partial x}\big(x^0(s)\big)\bigg] Z_1^\varepsilon(s)\, ds. \tag{3.29}$$

Now, by the mean value theorem and (2.8),

$$\Psi(s,\varepsilon) - \Xi(s,\varepsilon) = \bigg[\frac{\partial F}{\partial x}\Big(\sigma^\varepsilon(s), \frac{s}{\varepsilon}\Big) - \frac{\partial F}{\partial x}\Big(x^0(s), \frac{s}{\varepsilon}\Big)\bigg] U_1^\varepsilon(s)$$

[for some $\sigma^\varepsilon(s)$ on the line joining $X^\varepsilon(s)$ and $x^0(s)+\sqrt{\varepsilon} Z_1^\varepsilon(s)$], so, by (3.26) and (C1),

$$|U_1^\varepsilon(\tau)| \le N \int_0^\tau |U_1^\varepsilon(s)|\, ds + \int_0^1 |\Xi(s,\varepsilon)|\, ds + \|I_1^\varepsilon\|_C, \tag{3.30}$$

for $0 \le \tau \le 1$, and by the Gronwall inequality one obtains

$$\|U_1^\varepsilon\|_C \le e^N \bigg\{ \int_0^1 |\Xi(s,\varepsilon)|\, ds + \|I_1^\varepsilon\|_C \bigg\}. \tag{3.31}$$

Now we show that both of the terms in braces on the right-hand side of (3.31) go to zero a.s. as $\varepsilon \to 0$. From (3.28), (C1) and the mean value theorem, one sees that $|\Xi(s,\varepsilon)| \le (\frac{1}{2})\varepsilon^{1/2} dN \|Z_1^\varepsilon\|_C^2$, and applying the Gronwall inequality to (3.1) gives $\|Z_1^\varepsilon\|_C \le e^N \|W_1^\varepsilon\|_C$, whence, by (3.3),

$$\text{(3.32)} \quad \int_0^1 |\Xi(s,\varepsilon)|\,ds \le \left(\frac{1}{2}\right) dNe^{2N}\left\{\varepsilon^{3/4} \max_{0\le t\le\varepsilon^{-1}} \left|\int_0^t \widetilde{F}\big(x^0(s\varepsilon),s\big)\,ds\right|\right\}^2;$$

so, in view of Lemma 3.6 and (3.32),

$$\text{(3.33)} \qquad \lim_{\varepsilon\to 0} \int_0^1 |\Xi(s,\varepsilon)|\,ds = 0 \quad \text{a.s.}$$

As for the second term in braces on the right-hand side of (3.31), if we let $K(\tau,u)$, $0 \le \tau, u \le 1$, be the resolvent kernel (see [6], page 139) for the integral equation in (3.1), then, by definition of $Z_1^\varepsilon \triangleq G(W_1^\varepsilon)$, we get $Z_1^\varepsilon(s) = W_1^\varepsilon(s) + \int_0^s K(s,u) W_1^\varepsilon(u)\,du$, and hence, by (3.29),

$$\text{(3.34)} \qquad I_1^\varepsilon(\tau) = A^\varepsilon(\tau) + B^\varepsilon(\tau),$$

where one easily sees from (3.3) that

$$\text{(3.35)} \qquad \begin{aligned} A^\varepsilon(\tau) &= \int_0^\tau \frac{\partial \widetilde{F}}{\partial x}\big(x^0(s), s/\varepsilon\big) W_1^\varepsilon(s)\,ds \\ &= \varepsilon^{3/2} \int_0^{\tau/\varepsilon} \int_0^s \widetilde{\Phi}^\varepsilon(s)\Phi^\varepsilon(u)\,du\,ds, \end{aligned}$$

$$\text{(3.36)} \qquad \begin{aligned} B^\varepsilon(\tau) &= \int_0^\tau \frac{\partial \widetilde{F}}{\partial x}\big(x^0(s), s/\varepsilon\big) \int_0^s K(s,u) W_1^\varepsilon(u)\,du\,ds \\ &= \varepsilon^{5/2} \int_0^{\tau/\varepsilon} \widetilde{\Phi}^\varepsilon(s) \int_0^s K(\varepsilon s, \varepsilon u) \int_0^u \Phi^\varepsilon(w)\,dw\,du\,ds \end{aligned}$$

for all $0 \le \tau \le 1$. Here we have used the notation

$$\Phi^\varepsilon(s) \triangleq \widetilde{F}\big(x^0(\varepsilon s), s\big) \quad \text{and} \quad \widetilde{\Phi}^\varepsilon(s) \triangleq \frac{\partial \widetilde{F}\big(x^0(\varepsilon s), s\big)}{\partial x},$$

for $0 \le s \le \varepsilon^{-1}$ with $\widetilde{F}$ defined in (2.4). For ease of notation define $T_\varepsilon \triangleq \varepsilon^{-1}$, $n_\varepsilon \triangleq [T_\varepsilon]$ (the largest integer less than or equal to T_ε) and

$$\text{(3.37)} \qquad G_T(s) \triangleq \int_0^s \widetilde{\Phi}^{1/T}(s) \cdot \Phi^{1/T}(v)\,dv,$$

for all $0 \le s \le T$, $T > 0$. From (3.35) it is easily seen that

$$\text{(3.38)} \qquad \begin{aligned} \|A^\varepsilon\|_C &\le \left(\frac{1+n_\varepsilon}{T_\varepsilon}\right)^{3/2} \left\|A^{1/(1+n_\varepsilon)}\right\|_C \\ &\quad + T_\varepsilon^{-3/2} \int_0^{T_\varepsilon} \left|G_{T_\varepsilon}(s) - G_{n_\varepsilon+1}(s)\right|\,ds. \end{aligned}$$

Moreover, since $x^0(s)$ has a Lipschitz constant of $M+ND$ in s, while $\widetilde{F}(x,s)$ and $\partial\widetilde{F}(x,s)/\partial x_j$ have Lipschitz constants of $2N$ in x, it is not difficult to see from (3.37) that

$$\left|G_{T_\varepsilon}(s) - G_{n_\varepsilon+1}(s)\right| \le 2N(M+ND)\left\{T_\varepsilon^{-1} d \max_{0\le t\le T_\varepsilon}\left|\int_0^t \widetilde{F}\big(x^0(u/T_\varepsilon),u\big)\,du\right| + N\right\},$$

for all $0 \le s \le T_\varepsilon$. It is then clear from Lemma 3.6 that the second term on the right-hand side of (3.38) goes to zero a.s. as $\varepsilon \to 0$. As for the first term on the right-hand side of (3.38), we see from Lemma 3.7 that there is some constant $c_1 > 0$ such that $E\|A^{1/n}\|_C^4 \le c_1 n^{-2}$ so $\|A^{1/n}\|_C \to 0$ a.s. as $n \to \infty$, and it follows that $\|A^\varepsilon\|_C \to 0$ a.s. as $\varepsilon \to 0$. It remains to show that $\|B^\varepsilon\|_C \to 0$ a.s. as $\varepsilon \to 0$, where $B^\varepsilon(\cdot)$ is given by (3.35). For all $0 \le s \le T$, $T > 0$, define

$$H_T(s) \triangleq \int_0^s K(s/T,\tau/T)\int_0^\tau \Phi^{1/T}(u)\,du\,d\tau, \tag{3.39}$$

$$J_T(s) \triangleq \widetilde{\Phi}^{1/T}(s) H_T(s). \tag{3.40}$$

Then one easily sees from (3.35) that

$$\begin{aligned}\|B^\varepsilon\|_C \le{}& \left(\frac{1+n_\varepsilon}{T_\varepsilon}\right)^{5/2}\|B^{1/(1+n_\varepsilon)}\|_C \\ &+ T_\varepsilon^{-5/2}\int_0^{T_\varepsilon}\left|J_{T_\varepsilon}(s) - J_{n_\varepsilon+1}(s)\right|ds,\end{aligned} \tag{3.41}$$

where, as before, $T_\varepsilon \triangleq \varepsilon^{-1}$ and $n_\varepsilon \triangleq [T_\varepsilon]$. Now a tedious calculation involving the mean value theorem and the uniform boundedness of $K(\cdot,\cdot)$ on the unit square shows that

$$\begin{aligned}&\left|H_{T_\varepsilon}(s) - H_{n_\varepsilon+1}(s)\right| \\ &\quad\le \frac{c_2 s^3}{T_\varepsilon^2} + \frac{c_3 s^2}{T_\varepsilon^{5/4}(n_\varepsilon+1)^{3/4}}\max_{0\le t\le 1+n_\varepsilon}\left|\int_0^t \Phi^{1/(n_\varepsilon+1)}(u)\,du\right|,\end{aligned} \tag{3.42}$$

for constants $c_2, c_3 > 0$; from (3.39) and the uniform boundedness of $K(\cdot,\cdot)$ one sees that

$$\left|H_{T_\varepsilon}(s)\right| \le c_4 s \max_{0\le t\le T_\varepsilon}\left|\int_0^t \Phi^{1/T_\varepsilon}(s)\,ds\right|. \tag{3.43}$$

Putting together (3.40), (3.42), (3.43) and (C1) and using Lemma 3.6, it is easily seen that the second term on the right-hand side of (3.41) goes to zero a.s. as $\varepsilon \to 0$. As for the first term on the right-hand side of (3.41), we can see from Lemma 3.7 that $E\|B^{1/n}\|_C^4 \le c_5 n^{-2}$ for some $c_5 > 0$, whence $\|B^{1/n}\|_C \to 0$ a.s. as $n \to \infty$; from (3.41) we then get $\|B^\varepsilon\|_C \to 0$ a.s. as $\varepsilon \to 0$. Lemma 3.5 follows by (3.34). □

LEMMA 3.6. *Under* (C0)–(C4) *of Section* 2, *if* $n_T \triangleq [T]$, *then*

$$\lim_{T\to\infty} n_T^{-3/4} \max_{0\le\tau\le T}\left|\int_0^\tau \widetilde{F}\left(x^0\left(\frac{s}{T}\right), s\right) ds\right| = 0 \quad a.s., \tag{3.44}$$

where $\widetilde{F}(\cdot)$ *is defined by* (2.4).

PROOF. Define $\widetilde{F}_T(s) \triangleq \widetilde{F}(x^0(s/T), s)$, for all ω, $0 \le s \le T$. Since the functions $\tau \to x^0(\tau)$ and $x \to \widetilde{F}(x, s)$ have Lipschitz constants $(M + ND)$ and $2N$, respectively, it is easy to see that

$$\begin{aligned} n_T^{-3/4} &\max_{0\le\tau\le T}\left|\int_0^\tau \widetilde{F}_T(s)\,ds\right| \\ &\le n_T^{-3/4}\left[\max_{0\le\tau\le 1+n_T}\left|\int_0^\tau \widetilde{F}_{1+n_T}(s)\,ds\right| + N(M+ND)\right]. \end{aligned} \tag{3.45}$$

Now, in view of Lemma 4.7(a) and Theorem 4.6 there exists a constant $c_1 > 0$ such that

$$E\left[\max_{0\le\tau\le n+1}\left|\int_0^\tau \widetilde{F}_{1+n}(s)\,ds\right|^6\right] \le c_1(n+1)^3. \tag{3.46}$$

Since $\sum n^{-9/2}(n+1)^3 < \infty$, we get (3.44) from (3.46) and (3.45). □

The following technical result is proved in [11], Lemmas A5.2 and A5.3.

LEMMA 3.7. *Assume* (C0)–(C4) *of Section* 2, *and let* $A^\varepsilon(\tau)$ *and* $B^\varepsilon(\tau)$, $0 \le \tau \le 1$, *be defined by* (3.35) *and* (3.36). *Then there is a constant* $c > 0$ *such that, for all* $\varepsilon > 0$,

(i) $E\|A^\varepsilon\|_C^4 \le c\,\varepsilon^2$ *and*

(ii) $E\|B^\varepsilon\|_C^4 \le c\,\varepsilon^2$.

3.4. *A required maximal inequality.*

LEMMA 3.8. *Suppose conditions* (C0)–(C4) *of Section* 2 *hold. Corresponding to each integer* $p \ge 3$ *there exists some constant* $\alpha_p > 0$ *such that*

$$E\left[\max_{T\le t\le U}\|S_t - S_T\|_C^{2p}\right] \le \alpha_p U(U-T)^{p-1}, \tag{3.47}$$

for $U > T \ge 0$, *where* $S_t(\cdot)$ *is defined by* (3.10).

PROOF. Suppose that, for each integer $p \ge 3$, there is some constant $c_p > 0$ such that

$$E\left[\|S_u - S_t\|_C^{2p}\right] \le c_p u(u-t)^{p-1}, \tag{3.48}$$

for all $0 \leq t < u$. If we set $h(t,u) \triangleq \{c_p u(u-t)^{p-1}\}^{1/(p-1)}$, $0 \leq t < u$, then it is easily checked that

$$h(t,u) + h(u,v) \leq h(t,v), \tag{3.49}$$

for all $0 \leq t < u < v$. Now a simple but tedious calculation using (C0) and (C1) shows that the function $t \to S_t(\cdot)$ from $[0,\infty)$ into $C[0,1]$ is continuous for a.a. ω. Fix arbitrary $0 \leq T < U$. Then (3.47) follows from (3.48), (3.49) and Theorem 4.6 (with $\gamma \triangleq p-1$, $\nu \triangleq 2p$, $Q_t \triangleq S_t$ and $\mathcal{X} \triangleq C[0,1]$). Thus, in order to finish the proof of Lemma 3.8, it remains to show that corresponding to each integer $p \geq 3$ is some constant $c_p > 0$ such that (3.48) holds for all $0 \leq t < u$. Choose some integer $p \geq 3$, and fix arbitrary numbers $0 < t < u$. Then

$$\begin{aligned} &E\left[\|S_u - S_t\|_C^{2p}\right] \\ &\quad \leq 2^{2p} E\left[\max_{0\leq\tau\leq1}\left|\int_0^{\tau t} \Phi_{tu}(s)\,ds\right|^{2p}\right] \\ &\quad\quad + 2^{2p} E\left[\max_{0\leq\tau\leq1}\left|\int_{\tau t}^{\tau u} \widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p}\right], \end{aligned} \tag{3.50}$$

where $\Phi_{tu}(s) \triangleq \widetilde{F}(x^0(s/u)s) - \widetilde{F}(x^0(s/t),s)$, for all $s \in [0,t]$. Considering the first term on the right-hand side of (3.50), we have by the $2N(M+2ND)$ Lipschitz constant of $v \to \widetilde{F}(x^0(v),s)$, for all $0 \leq v \leq 1$ and all $s \geq 0$, that $|\Phi_{tu}(s)| \leq 2N(M+2ND)(u-t)/u$, for all $s \in [0,t]$. Thus, from Lemma 4.7(b), corresponding to $p \geq 3$ is some constant $b_p > 0$ such that

$$E\left|\int_v^w \Phi_{tu}(s)\,ds\right|^{2p} \leq \{h_1(v,w)\}^p, \tag{3.51}$$

for all $0 \leq v \leq w \leq t$, where $h_1(v,w) \triangleq (b_p)^{1/p}\{(u-t)/u\}^2(w-v)$ (b_p does not depend on t and u). In view of (3.51) and Theorem 4.6, corresponding to $p \geq 3$ is some constant $\beta_p > 0$ (likewise not depending on t and u), where

$$E\left[\max_{0\leq v\leq t}\left|\int_0^v \Phi_{tu}(s)\,ds\right|^{2p}\right] \leq \beta_p\{h_1(0,t)\}^p \leq \beta_p b_p (u-t)^p. \tag{3.52}$$

For the second term on the right-hand-side of (3.50) define the half-overlapping intervals I_i by putting $l \triangleq [u/(u-t)]$ (where $[a]$ is the integer part of a), $t_i \triangleq i\cdot(u-t)$ for $i = 0,1,2,\ldots,l$, $I_i \triangleq [t_{i-1},t_{i+1}]$ for $i = 1,2,\ldots,l-1$, and

$I_l \triangleq [t_{l-1}, u]$. Then

$$
\begin{aligned}
&E\left[\max_{0\le\tau\le 1}\left|\int_{\tau t}^{\tau u}\widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p}\right]\\
(3.53)\qquad &\le E\left[\max_{i=1,\dots,l}\ \max_{v,w\in I_i}\left|\int_{v}^{w}\widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p}\right]\\
&\le 2^{2p}\sum_{i=1}^{l}E\left[\max_{v\in I_i}\left|\int_{t_{i-1}}^{v}\widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p}\right].
\end{aligned}
$$

Now by (C1), (C3), (C4) and Lemma 4.7(a), corresponding to $p \ge 3$ is some constant $\gamma_p > 0$ such that

$$
(3.54)\qquad E\left|\int_{v}^{w}\widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p} \le \gamma_p(w-v)^p,
$$

for all $0 \le v < w \le u$. In view of Theorem 4.6 and (3.54), there is a constant $\alpha_p' > 0$ such that

$$
(3.55)\quad E\left[\max_{v\in I_i}\left|\int_{t_{i-1}}^{v}\widetilde{F}\big(x^0(s/u),s\big)\,ds\right|^{2p}\right] \le \alpha_p'(t_{i+1}-t_i)^p \le 2^p\alpha_p'(u-t)^p.
$$

Since $(u-t)^p l \le u(u-t)^{p-1}$, we get (3.48) from (3.50), (3.52) and (3.53), for all $0 < t < u$. The case of $0 = t < u$ is established in the same way but is even easier since one need be concerned only with the second term on the right-hand side of (3.50). □

4. Useful results. For convenience we collect here the main results required for the proofs of Section 3.

Lemma 4.1 is a simple extension to the vector case of Proposition 2 in [18], page 257. It is used in the proof of Proposition 3.3.

LEMMA 4.1. *Let μ be a zero-mean Gaussian measure on $C[0,1]$ with covariance function $R((i,\sigma),(j,\tau)) \triangleq E^{\mu}[x^i(\sigma)x^j(\tau)]$, $1 \le i,j \le d$, $0 \le \sigma,\tau \le 1$, and let K be the unit ball of the RKHS generated by R. For any $\varphi \in C[0,1]$ and $\{\tau_1,\dots,\tau_m\} \subset (0,1]$ with $0 < \tau_1 < \tau_2 <,\dots,< \tau_m = 1$, set $\varphi[\tau_1,\dots,\tau_m] \triangleq (\varphi^T(\tau_1),\dots,\varphi^T(\tau_m))^T$, and let $K[\tau_1,\dots,\tau_m] \triangleq \{\varphi[\tau_1,\dots,\tau_m]: \varphi \in K\}$. Suppose that $\{\varphi_r(t,\omega)\}$ is a sequence of $C[0,1]$-valued random variables such that $\{\varphi_r(\cdot,\omega)\}$ is a relatively compact subset of $C[0,1]$ for a.a. ω. If, for each such finite collection $\{\tau_1,\dots,\tau_m\} \subset (0,1]$, one has*

$$
\{\varphi_r[\tau_1,\dots,\tau_m]\} \twoheadrightarrow K[\tau_1,\dots,\tau_m] \quad a.s., \quad then \quad \{\varphi_r\} \twoheadrightarrow K \quad a.s.
$$

Lemma 4.2 (due to Dvoretzky [8], Lemma 5.3 on page 528) is also used in the proof of Proposition 3.3 [see (3.19)].

LEMMA 4.2. *Let ξ be a complex-valued random variable with $|\xi| \leq 1$, and let $\mathcal{F}$ be the σ-field generated by ξ. Then, for any σ-field $\mathcal{G}$,*

$$E\big|E(\xi \mid \mathcal{G}) - E\xi\big| \leq 2\pi \sup_{\substack{A\in\mathcal{F}\\ B\in\mathcal{G}}} \big|P(AB) - P(A)P(B)\big|.$$

The following approximation theorem is a special case of Berkes and Philipp [4, Theorem 1]. It is used in the proof of Proposition 3.3.

THEOREM 4.3. *Let $\{X_k,\ k \geq 1\}$ be a sequence of $\Re^d$-valued random variables and let $\{\mathcal{F}_k,\ k \geq 1\}$ be a nondecreasing sequence of σ-fields such that X_k is $\mathcal{F}_k$-measurable. Moreover, let G be a probability distribution on $\Re^d$ with characteristic function $g(u)$, $u \in \Re^d$. Suppose that, for each $k \geq 1$, there are some nonnegative numbers λ_k, σ_k, and $T_k \geq 10^8 d$ such that*

$$E\Big|E\Big\{\exp\big(i\langle u,\ X_k\rangle\big)|\mathcal{F}_{k-1}\Big\} - g(u)\Big| \leq \lambda_k$$

$$\text{for all } u \text{ with } |u| \leq T_k \text{ and } G\{u\colon |u| > \tfrac{1}{4}T_k\} \leq \delta_k.$$

Then there is a sequence of $\Re^{2d}$-valued random variables $\{(\widehat{X}_k^T, \widehat{Y}_k^T)^T,\ k \geq 1\}$ on some $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ such that (i) *$\{\widehat{Y}_k,\ k \geq 1\}$ is i.i.d. with marginal distribution G,* (ii) *$\{\widehat{X}_k,\ k \geq 1\} =_{\mathcal{D}} \{X_k,\ k \geq 1\}$ and* (iii) *$\widehat{P}\{|\widehat{X}_k - \widehat{Y}_k| \geq \alpha_k\} \leq \alpha_k$, $k \geq 1$, where $\alpha_1 = 1$ and $\alpha_k \triangleq 16dT_k^{-1}\log T_k + 4\lambda_k^{1/2}T_k^d + \delta_k$, $k \geq 2$.*

The Prokhorov distance between probability measures P and Q on the Borel σ-field of a metric space (S, ρ) is defined by

$$\Pi_S(P, Q) \triangleq \inf\big\{\delta > 0\colon P(A) \leq Q(A^\delta) + \delta,\ \text{for all closed sets } A \subset S\big\},$$

where $A^\delta \triangleq \{x \in S;\ \rho(x, A) < \delta\}$. The following theorem of Strassen and Dudley (see [7], Theorem 1) is required for the proof of Proposition 3.3 [see (3.21)].

THEOREM 4.4. *Let (S, ρ) be a separable metric space with Borel σ-field $\mathcal{S}$. Suppose P_1 and P_2 are probability measures on $\mathcal{S}$ such that $\Pi_S(P_1, P_2) < \alpha$. Then there is a probability measure Q on the Borel sets of $S \times S$ with marginals P_1 and P_2 such that $Q\{(x,y)\colon \rho(x,y) > \alpha\} \leq \alpha$.*

Theorem 4.5 (which is due to Kuelbs [12] Theorem 4.3) is used in the proofs of Proposition 3.2 [see (3.8)] and Proposition 3.3 [see (3.24)].

THEOREM 4.5. *Let $(B, \|\cdot\|)$ be either $(C[0, 1], \|\cdot\|_C)$ or $(\Re^d, |\cdot|)$, for some $d \geq 1$, and let $\{Y_k,\ k \geq 1\}$ be a sequence of B-valued random variables. Suppose μ is a* *mean-zero Gaussian measure on B such that* $\sum_{k=1}^{\infty}\Pi_B(\mathcal{L}(Y_k),\mu)<\infty$, *where* $\mathcal{L}(Y_k)$ *denotes the distribution of* Y_k. *Then the following hold for a.a.* ω:

(i)
$$\lim_{k\to\infty}\left\|\frac{Y_k(\omega)}{(2\log k)^{1/2}}-K\right\|_C=0.$$

(ii) *If, in addition to the preceding conditions, the* Y_k*'s are independent, then, for a.a.* ω,

$$C\left(\frac{Y_k(\omega)}{(2\log k)^{1/2}}: k\geq 1\right)=K,$$

where K is the unit ball of RKHS generated by the covariance function of μ.

The following maximal inequality is a simple generalization of Longnecker and Serfling's maximal inequality (see [16], Theorem 1) to the case of continuous-time processes in a normed vector space. The proof of Theorem 4.6 follows from the inequality

$$\max_{a\leq r\leq s\leq b}\left\|\sum_{k=r}^{s}x_k\right\|^{\nu}\leq 2^{\nu}\max_{a\leq r\leq b}\left\|\sum_{k=a}^{r}x_k\right\|^{\nu},$$

for any $x_a, x_{a+1},\dots,x_b$ in some normed vector space $(\mathcal{X},\|\cdot\|)$, the proof of Theorem 1 in [16] (with norms replacing absolute values) along with discretization and passage to a continuous limit.

THEOREM 4.6. *Let* $0\leq T<U<\infty$ *and suppose that* $\{Q_t,\ T\leq t\leq U\}$ *is a process assuming values in some normed vector space* $\mathcal{X}$ *with norm* $\|\cdot\|$ *such that the following hold:* (i) $t\to Q_t(\omega)$ *is continuous on* $[T,U]$ *for almost all* ω *and* (ii) *there exist constants* $\gamma>1$ *and* $\nu>0$ *such that* $E\|Q_u-Q_t\|^{\nu}\leq[h(t,u)]^{\gamma}$, *for all* $T\leq t\leq u\leq U$, *where* $h(t,u)$ *is a nonnegative function satisfying* $h(t,u)+h(u,v)\leq h(t,v)$, *for all* $T\leq t<u<v\leq U$. *Then there exists a constant* $\widetilde{A}_{\nu,\gamma}$ *depending only on* ν *and* γ *such that*

$$E\Big[\max_{T\leq t\leq u\leq U}\|Q_u-Q_t\|^{\nu}\Big]\leq\widetilde{A}_{\nu,\gamma}[h(T,U)]^{\gamma}.$$

The following moment bounds for strong mixing processes are due to Khas'minskii (see [10], the statement of Lemma 2.1 and the line following equation (2.10) on page 215).

LEMMA 4.7. *For some positive integer k, let* $\Phi_1(t),\Phi_2(t),\dots,\Phi_{2k}(t)$, $t\geq 0$, *be zero-mean stochastic processes on some probability space* $(\Omega,\mathcal{F},P)$. *Suppose each* $\Phi_i(t)$ *is* $\mathcal{F}_t^t$*-measurable, where* $\{\mathcal{F}_s^t,\ 0\leq s\leq t\leq\infty\}$ *satisfies condition* (C2) *of Section 2 with some function* $\alpha(\cdot)$.

(a) *Suppose, for some* $\delta>0$, *the following hold:*

(i) $$M' \triangleq \sup_{\substack{t \geq 0 \\ i = 1,2,\ldots,2k}} \|\Phi_i(t)\|_{(2+\delta)(2k-1)} < \infty;$$

(ii) $$R'_n \triangleq \int_0^\infty \tau^{n-1}\big[\alpha(\tau)\big]^{\delta/(2+\delta)} d\tau < \infty, \qquad n = 1,2,\ldots,k.$$

Then there exists a $c_{2k} > 0$, *depending only on* $k, M', R'_1, \ldots, R'_k$, *such that, for all* $0 \leq t \leq u$,

$$\int_t^u \cdots \int_t^u \Big|E\big\{\Phi_1(s_1)\cdots\Phi_{2k}(s_{2k})\big\}\Big| \, d_{s_1}\cdots d_{s_{2k}} \leq c_{2k}(u-t)^k.$$

(b) *Suppose there is some number* N *such that the following hold:*

(i) $|\Phi_i(t,\omega)| \leq N$ *for all* $t \geq 0$, $i = 1,2,\ldots,2k$, *a.a.* $\omega \in \Omega$;

(ii) $$R_n \triangleq \int_0^\infty \tau^{n-1}\big[\alpha(\tau)\big]\, d\tau < \infty \text{ for all } n = 1,2,\ldots,k.$$

Then there exists a constant c_{2k}, *depending only on* $k, R_1, \ldots, R_k$, *such that, for all* $0 \leq t \leq u$,

$$\int_t^u \cdots \int_t^u \big|E\Phi_1(s_1)\cdots\Phi_{2k}(s_{2k})\big| \, ds_1\cdots ds_{2k} \leq c_{2k}N^{2k}(u-t)^k.$$

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