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DEVELOPMENT OF A VARIABLE DENSITY HEIGHT-GROWTH-MODEL THROUGH DEFINING MULTIDIMENSIONAL HEIGHT GROWTH SPACES

BY

© **CHRIS JANUSZ CIESZEWSKI**

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **DEVELOPMENT OF A VARIABLE DENSITY HEIGHT-GROWTH--MODEL THROUGH DEFINING MULTIDIMENSIONAL HEIGHT GROWTH SPACES** submitted by **CHRIS JANUSZ CIESZEWSKI** in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY**.

Prof. V.J. Lieffers (supervisor)

Prof. J.A. Beck

Prof. W. Krawcewicz

Prof. A.W. Anderson

Prof. P.M. Woodard

Prof. H.E. Burkhart

Date: July 27, 1994

Abstract

Height models, that are general and flexible are needed. The main concept of height modeling here is to look upon height models as multidimensional spaces, cross sections of which can represent different types of the traditional height models: such as the SI based vs. the intercept based models; or the total age based vs. the breast height age based models. A complete height space is considered to be, at minimum, five-dimensional with individual dimensions described by the following five components: 1) predicted height; 2) prediction age; 3) reference-height (SI); 4) base- age (age of SI); and 5) base-height (the height at which the prediction age and the base-age start their counts). The predicted height is always the dependent variable and all the other components are the independent variables. Every step of increase in the model dimensionality involves a considerable increase in the model's algebraic complexity, and the five dimensional model is at the stage beyond which it cannot be further advanced through any algebraic reformulations. Thus, to include algebraically a variable density or a crowding component into this model, one variable has to be eliminated. However, the most advanced height space described in this thesis is defined by an infinite-dimensional, dynamic, variable density height growth model in which the high dimensionality is achieved through iterations of an annual increment prediction equation with dynamically changing variable density or a crowding component.

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Notation and Conventions

The studies presented in this thesis discuss various height models with particular attention to their theoretical foundations. To facilitate understanding, height model components are described below by names which will be used throughout the thesis:

- **prediction-age** (t) is the age for which a height is predicted;
- **prediction-height at age** t ($H(t)$) is the height at age t predicted by a model;
- **reference-height** (h_x) is the height used as site-index **(SI)**, but at a variable base-age (x);
- **base-age** (x) is a variable age used to define the reference-height as a height at this age; it is denoted as Age_{si} when it is constant and used to define SI;
- **base-height** (h_b) is the height at which the age begins to count, e.g, base-height for total age is 0.0m (germination height), for stump age is 0.3m (stump height), for breast height age is 1.3m (breast height), and for an age base-height 2m, is 2m; and
- **age-base-height-rm,** (t_r) is the age counted from the time when the tree reached a height of r meters, e.g., age-base-height-1.3m ($t_{1.3}$) is the breast height age, while age-base-height-0.3m ($t_{0.3}$) is the stump age.

A height model is a mathematical formula relating the above components. These components are either variables, if they are explicit and adjustable, e.g., x; or they are constants if they are implicit or can represent only one preset value, e.g., Age_{si}. Furthermore, the variables can be either measurable, e.g., h_x, or they can be arbitrary, e.g., t. Finally, a variable is either an input variable, e.g., h_x, or an output variable, e.g., $H(t)$.

In addition to the above concepts the following symbols will be used in this thesis:

- **PSP** are permanent sample plots;
- **ADA** stands for algebraic difference approach;
- **MC** stands for model conditioning;
- ln is the natural logarithm;
- e is base of the natural logarithm;
- a, b... A, B... are model coefficients of the models different than those proposed in this thesis; and
- α, β... are coefficients of models proposed in this thesis.

All coefficients of the proposed models represent signs of the intended values of the coefficients. Signs of the coefficients of the models cited from other publications are usually consistent with the original publications. The terms growth, any height or SI, in accord with the above, will imply values above base-height, unless stated otherwise.

Solvability will be understood within practical ramifications of forestry applications, i.e., only closed form solutions or very quickly converging series will be acceptable. If a root of an equations does not have a closed form solution, or a series solution that converges very quickly the equation will be considered unsolvable.

Chapter 1

Background and Problem Definition

1.1 Introduction

Height growth, particularly that of dominant and co-dominant trees (or a fixed number of the largest trees per unit area (TH)) is generally the most stable directly measured stand growth statistic. For most tree species it is independent of the number of trees per unit area within a large range of stand densities and it is often used as a measure of site productivity (Monserud 1984, Alemdag 1988).

Since unbiased and accurate height prediction is essential in almost every aspect of forest management, it is important to develop unbiased and accurate height models. Borders *et al.* (1984) list desirable height model characteristics as: height at age zero equal zero; height at base-age equals SI; curves that are unaffected by a choice of base-age; and increasing asymptotes (if any) with increasing site indexes.

In some situations, even some models meeting the above criteria might not be very useful. A model may not be useful if it is not flexible enough to describe a certain type of growth polymorphism, if stand age cannot be estimated with acceptable accuracy, or if the height growth varies with density. These limitations may apply to a wide range of situations in inventory as well as in estimation of height growth in juvenile stands.

Traditional height-age functions can be difficult to implement when aerial inventory is applied and one can precisely measure current heights and density in even age stands, but cannot estimate age of trees. In juvenile stands in particular, any errors in age estimation will have a great influence on height predictions, because even small errors will be proportionally large in comparison to the small ages. For this reason, researchers have long been searching for alternative solutions to this problem of juvenile height modeling and have proposed a variety of models based on growth.

Problems with height predictions may occur even when the stand age is known, but the stand had undergone a period of slow height growth due to external factors, such as crowding, insect or disease damage, severe thinning or changes in site conditions (site devastation, meliorations, climate changes, acid rain, etc.)

Lodgepole pine (*Pinus concorta* var. *latifolia* Engelm.) (lP) is particularly sensitive to stand density and its height growth is highly affected by it; although, it is not known precisely at what densities its height growth becomes reduced. Smithers (1961) suggests that even in fairly open stands (500 trees/ha at age 90 years) the height growth of lodgepole pine may be affected.

In a simplified approach to modeling height of mature, fire origin, dense lodgepole pine, SI may be assumed to be a measure of lP height growth, which implicitly includes soil fertility, moisture, elevation, and density affects on height growth. This simplified approach cannot be used for situations of dynamically changing stand densities or crowding in managed stand growth simulation with arbitrary density levels. In predicting height of second growth lodgepole pine stands, additional difficulty results from the density influence on the height growth of this species. In situations of varying stand conditions, and generally for managed stands, one needs a height model that will explicitly address density influence on lodgepole pine height growth, and can dynamically simulate the height growth responses to changing stand conditions. Models based on difference or differential equations simulating the height growth in one year increments are potentially useful for such simulations.

Height models have usually been developed on either cross-sectional or stem analysis data (Biging 1985). Stem analysis data are preferred since they provide continuous record of tree growth. Stem analysis data also allows [illegible] interpolation, or approximation of SI with reasonable accuracy (Newnham 1988). The mathematical models used for height modelling can be linear or nonlinear. Linear models, which are less flexible, usually require more terms (or parameters) for satisfactory fit to the data. This may lead to model over-parametrization and unreasonable predictions outside of the range of the data on which the model was calibrated. Linear models with many parameters are likely to become atypical in shape and difficult to support biologically. Nonlinear models are more flexible, more likely to be biologically sound, and usually behave well outside of the data range (Pienaar and Turnbull 1973).

1.2 Literature Overview

1.2.1 *SI* Height Models for Lodgepole Pine In Alberta

The earliest height model for Alberta lodgepole pine is Kirby (1975) which proposes the following linear form: $\frac{H}{SI_{70}} = a + bt + ct^2$ where: SI_{70} is SI base-age 70 years stump age (base-age is an implicit constant equal 70); t is a stump age with implicit base height 0.3m.

Johnstone (1976) published variable density yield tables for lP in Alberta. The tables were based on a system of equations derived from stepwise regressions on permanent sample plot data. The input variables consisted of productivity index (PI), basal area of 100 largest trees per acre (BA_{100}). The linear dominant height growth model is $H(t, DI, PI) = a + bt + \frac{c}{t} + \frac{d}{t^{.25}} + fPI + gDI + iDI^{.25} + jtDI + \frac{kt}{PI}$ where: 1) $[H(t, DI, PI)]$ is total height of dominant trees at stump at age t in a stand with a specified DI and PI; 2) $[DI]$ is development index, calculated from the BA_{100} and t (Johnstone 1976); 3) $[PI]$ productivity index, calculated from TH, stand age and density as a ratio of the TH observed (TH_O) to the TH expected (TH_E). Johnstone (1976) calculated TH_E from a separate model as a function of stand age and density.

Most height models in forestry are based on a SI concept and therefore this model is not typical for a height model and cannot be directly compared with other height models.

In 1982, Dempster and Assoc. (Anon. 1985) developed, on stem analysis data, a polymorphic site-height model for Alberta lodgepole pine $H(t, SI) = SI\frac{1+e^{a+b\ln 50+c\ln SI}}{1+e^{a+b\ln Age+c\ln SI}}$ with 50 as an explicit constant of base-age.

This model gave good predictions as tested on stem analyses and PSP data, while it could not be reformulated for SI, to estimate SI from heights. For this purpose a separate multi-regression model was developed, $SI(h_x, x) = a + bh_x - c\ln^2(x) + dx\ln(x) + f\frac{h_x}{x} - gh_x\ln(h_x)$.. The prediction error of the SI model, was within a range of measurement errors. Although this model[1] gave a reasonably good accuracy for SI predictions, it proved to be incompatible with it's height model for the extremes, i.e., good and poor sites. To solve this problem, numerical searches were used to find compatible SI estimates numerically. Zakrzewski[2] formulated[3] an anamorphic height model for Alberta lP based on the 1822 Hossfeld's model $H(t) = \left(\frac{t}{a+bt}\right)^2$ (Borowski 1979) as $H(t, SI) = SI\left(\frac{t}{A+Bt}\right)^2$ with the SI model $SI(h_x, x) = h_x\left(\frac{A}{x} + B\right)^2$. He calibrated the model on average growth of eleven trees selected from a large data set that reached the greatest height at 50 years of breast height age—an approach developed for height modeling uneven aged Norway spruce stands in Poland (Zakrzewski 1983, 1986).

[1] Two separate models were formulated to use with a stump age and with a breast height age.
[2] Vojtek Zakrzewski, PhD, post-doctoral fellow with CFS–NoFC from 1985 to 1990.
[3] Personal communication

The most recent height model for lodgepole pine in Alberta is a polymorphic SI height model based on a modified half-saturation function (Cieszewski and Bella 1989): $H(t) = \frac{A}{1+\frac{B}{t}}$. To create a set of anamorphic curves, the authors solved the model for A as a function of a known height at a known age, and reformulated the base model (using algebraic differences approach) by substituting the solution for A in place of A in the original equation. To account for IP height growth polimorphism the authors modified B and reformulated the model to a variable base-age SI form: $H(t, h_x, x) = \frac{0.5\left(h_x+\delta+\sqrt{(h_x-\delta)^2+80\beta h_x x^{-1-\alpha}}\right)}{1+40\beta\left[\left(h_x-\delta+\sqrt{(h_x-\delta)^2+80\beta h_x x^{-1-\alpha}}\right)t^{1+\alpha}\right]^{-1}}$, where h_x can be considered as a SI at the base-age x and any age can be assign to x.

1.2.2 Other Nonlinear SI Height Models

Various other nonlinear models are in use. Some of the well known basic models are: Hossfald from 1822 (Borowski 1979; Zakrzewski 1983) $H(t) = \left(\frac{t}{a+bt}\right)^2$, Logistic (Robertson 1923), $H(t) = A/(1 + e^{-bt})$, Schumacher (1939), $H(t) = Ae^{-b/t}$, Gompertz (Medawar 1940), $H(t) = Ae^{-be^{-ct}}$, Chapman-Richards (CR) (Richards 1959), $H(t) = A(1 - e^{-at})^b$, modified Weibull (Yang *at al.* 1978), $H(t) = A\left(1 - e^{-at^b}\right)$, and Bailey (1980) $H(t) = A\left(1 - e^{-at^b}\right)^c$.

There are also many modifications of these base models providing either anamorphic or polymorphic height curves such as the modification of the Chapman Richard's function by, Ek (1971) and Payandeh (1974) $H(t, SI) = aSI^b(1 - e^{-ct})^{d+SI^f}$ and Biging (1985) $H(t, SI) = aSI^b(1 - e^{-ct})^d$, and the modification of the logistic function by Monserud (1984) $H(t, SI) = \frac{aSI^b}{1+e^{c+d\ln(t)+f\ln(SI)}}$. Curves generated by the last equation do not go through the appropriate heights at base-age. For this reason, Dempster and Assoc. in 1982 constrained it to pass through those heights using Burkhart and Tennant (1977) technique for the Chapman Richard's model.

Some of the modified models became so complicated during the process of their modifications toward a better fit, that they became unsolvable for SI as a function of a height and age. Part of the problem is that these models require a prior knowledge of the SI. The SI usually is calculated from a height at any age. To do this, separate models had to be developed for SI as a function of the height at any age, while SI is just a height at a specified fixed age. Models for height and SI that are derived separately (model for height at age 50 separately from model of height at ages 49 or 51), may prove incompatible and cannot easily be used for further modifications such as including adjustable measurement components.

In general, most models, that are considered for growth modeling in forestry can be classified as either exponential function based models, or fractional functions based models. The fractional function based models are easier modifiable for the purpose of inclussion of any of their implicit solutions. In this study mostly fractional function based models will be used for further modifications and new derivations.

1.2.3 Intercept Height Models for Juvenile Stands

Since the traditional use of base-heights in definition of prediction-ages and base-ages has not been very useful for juvenile height modeling and site-indexing, Wakeley proposed the development of a new type of height models based on growth intercepts, i.e., a fixed number of height growth internodes measured close to breast height, instead of a SI (Wakeley 1954, Zahner 1954, Warrack and Fraser 1955, Wakeley and Marrero 1958, Ferree em et al. 1958, Day *et al.* 1960, Brown and Stires 1981, Carmean 1975). Ferree[4] suggested essentially the same measure in 1951 by Wylie (1951) independently developed a similar method and used it to approximate a 100 year SI of Douglas-fir through multiplying an average length of yearly height increments above breast height by six (Smith and Ker 1956).

[4] Ferree, M.J. Comparison of close spaced and wide spaced plantations. Unpublished report presented at the 1952 summer meeting of New York Section, Society of American Foresters. College of Forestry, State University of New York, Syracuse, N. Y. 1952.

In general, the intercept models are based on measurements of a few (usually five) height increments starting at various heights below or above breast height, usually close to breast height and more often above breast height rather than below breast height. For some species, e.g., red pine, height increments at breast height represent the maximum height increment, or are very close to it (Bull 1931, McCormack 1956, Ferree *et al.* 1958). Wakeley and Marrero (1958) tested different lengths of intercepts from three to seven years, starting with a last internode below breast height and concluded that the five years intercept was optimal. Ferree *et al.* (1958) applied the five year intercept, starting at first internode above breast height, of dominant and codominant red pine trees on a wide range of sites. They then constructed *SI* curves, using breast height age *SI*, following the procedure described by Bruce and Schumacher (1950), and compared the curves with those developed by Bull (1931).

Ferree *et al.* (1958) concluded that the intercept method gave reliable height predictions for 15 to 20 years. They also stated that even though the intercept method was not quite reliable for increment extrapolation on sites affected by a "site disease" (Stone *et al.* 1954), it was more reliable for that purpose than the traditional height SI models in other situations following growth disturbances or changes in tree *SI* (Heiberg and White 1956). When using the intercept method, such anomalies can be easily spotted and evaluated. Ferree *et al.* (1958) also conducted preliminary studies which suggested that the intercept method should be suitable for modeling the growth of such other species as white pine, Norway spruce and Scotch pine.

Day *et al.* (1960), similarly to Ferree *et al.* (1958) and Richards *et al.* (1962), studied the 5-year growth intercept, beginning with the first internode above breast height. Their data were collected from at least 20-year-old, healthy stands, on uniformly undisturbed sites. Only dominant and codominant red pine trees were measured. They found the 5-year intercept to be highly correlated with traditional SI, and thus concluded that for the sake of measurement convenience this method is superior to the traditional height over age relationships. On the other hand, as Husch (1956), Ferree *et al.* (1958), Wakeley and Marrero (1958), Alban (1972) and Brown and Stires (1981) found that the age at which a tree reaches breast height is not correlated with site; providing a strong argument against using total age in modeling of height-age relationships for site evaluation purposes in juvenile stands.

Beck (1971) applied the intercept method to stands less than fifteen years old and found that the predictions were at least as accurate as estimates by polymorphic *SI* curves. He found that the predictions from 5-year intercepts were only slightly less accurate than the predictions from 3-year intercepts. Warrack and Fraser (1955) found that both the 3- and 5-year length intercepts above breast height are significantly correlated with the 100 year *SI* of Douglas-fir. Smith and Ker (1956) and Schallau and Miller (1966) found no significant differences between the predictions of SI from 1- to 5-year intercepts. Similarly, Oliver (1972) tested 1- to 6-year intercepts above breast height and found that no significant improvements could be obtained through using more than 4-year intercepts. Wilde (1964, 1965) found for red pine that the ratio of total height to the 5-year intercept was a useful indicator of soil and site conditions for estimating the soil potential productivity.

Brown and Stires (1981), modified the growth intercept method, by measuring the length of 5-year growth intercepts starting two years above breast height. This method was tested on white pine trees from mixed species stands on different soils. They found that this approach was more reliable than the one used by Wakeley and Marrero (1958). They also explored the inclusion of some soil and topographic factors into a modified intercept method. They found that the growth intercept method can be significantly improved by inclusion of information on slope position and/or total soil depth.

Alban (1972 and 1979) further explored the intercept method by applying it at 8, 10 and 15 feet above the ground. He discovered, that such modifications significantly improve predictions of height. Both Alban (1972 and 1979) and Brown and Stires (1981) experimented with growth intercept modifications by starting the measurement of the growth intercept at different internodes above breast height from first to fifth. Hägglund (1976) similarly as Alban (1979) found that the intercept method measuring the growth of 5-year intercept at 2.5 meters gave better results for white pine, red pine, Scots pine, Norway spruce, and Sitka spruce than when measuring the intercept at breast height. Blyth (1974) got similar results for measurements at 3 meters. The growth intercept system developed by Alban (1972) for the Lake States worked well also for other locations of red pine stands (Alban 1985). Brown and Stires (1981) tested 3, 5 and 10 years intercepts at various starting heights. They found no significant improvement in moving the starting height of the intercept measurements above the two years above breast height level.

Gunter (1968) further extended the research on the growth intercept modeling by applying 5-year growth intercept to stands released from suppression. This author measured the intercepts beginning one year after release of these stands by intermediate cuttings.

1.3 Problem Definition

Height modeling has been evolving over time. First, curves were hand drawn as single lines through clusters of plotted height measurements (Grochowski 1973). Hand drawing was later replaced with mathematical equations having parameters determined through statistical procedures. The first attempt to express growth in the form of mathematical models took place in the 18th century and the first half of the 19th century, and they were those of Spath in 1797, Hossfeld in 1822, and Smalian in 1837 (Borowski 1979). Parameters for the mathematical models were either approximated or estimated through linear, and only recently, nonlinear regressions. With these equations becoming more and more complex, containing new added variables such as *SI* and base-age, the generated curves, or rather multi-dimensional height spaces, made gains on biological soundness (polymorphism, increasing asymptotes with site productivity). At the same time with increased sophistication in analysis used to determine model parameters, the model simulations become more exacting.

The changes in the approaches to height modeling have usually progressed toward increasing model functionality. Historically, the model functionality was attributed primarily to model algebraization, generalization, and lack of bias. At present, virtually all height models are in algebraic forms, and the main feature increasing height model functionality is probably model generalization and increases in biological soundness. The generalization may be improved by replacing the model's explicit and/or implicit constants by explicit variables. This means, automatically, expanding model dimensionality.

The earliest efforts in height modeling concentrated on two dimensional models (height over age). Both hand drawn curves (fig. 1.1a), and the earliest equations that were capable of consistently generating more intricate shapes (fig. 1.1b), approximated two dimensional relations. To enhance the applications of these two dimensional relations they could, at times, be developed separately for different sites or even individually for different stands (fig. 1.1c).

For some applications, generic curves were anamorphically adjusted for individual stands by simple means of manual multiplication of a curve by a ratio of observed to predicted height at an arbitrary age so that the new generated curve would pass through this known height and age. In a geometrical sense the collections of curves developed separately for different sites or stands could be classified today as a discrete collection of two dimensional polymorphic non-disjoint (Clutter *et al.* 1983) height curves.

This collection would usually be in forms of graphs, or tables that were developed for a discrete collection of sites, or stands. They represented a four dimensional height space in which the dimensions were: reference-height (discrete); age of reference-height (continuous); prediction age (continuous); and prediction height (continuous). The reference-height was discretized because not all possible height at any age could be matched with existing curves—only a finite number of models can be developed to contribute to any collection of models.

Even though compromising model correctness, the method of adjusting a single generic model to specific situations/stands by scaling it up or down, could be considered an algebraic improvement over multiple models, because it reduces the number of models involved in the prediction system. This method also extends the discrete reference-height to a continuous reference-height through an explicit but simple multiplication. As it requires less work in preparation and applications, this method is more functional, and in principle, similar to current systems of *SI* height curves (Fig. 1.2

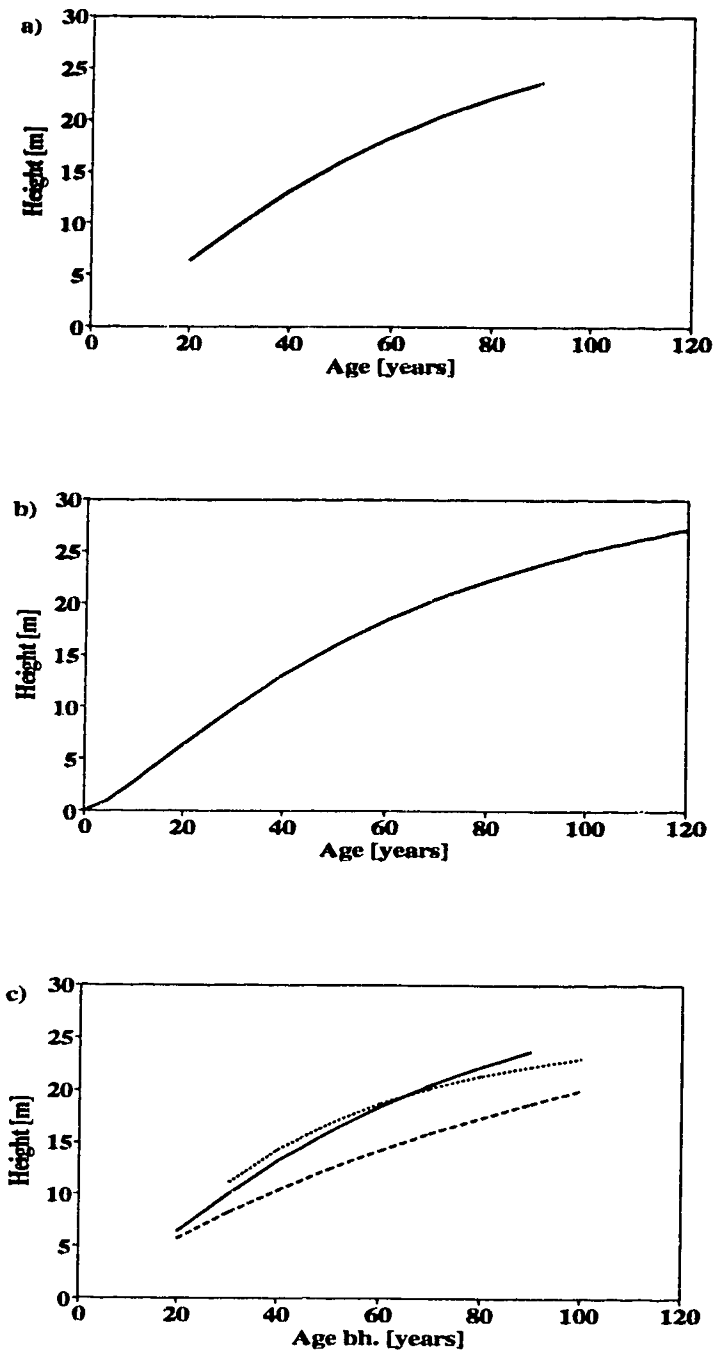

Figure 1.1: Simple two-dimensional height over age curves: a) monotonically decreasing slope; b) sigmoidal growth; c) multiple curves for different stands.

and 1.3). "The Kalman Filter Approach to Localizing Height-Age Equations" (Walters *et al.* 1991) could be considered as a modern application of such method.

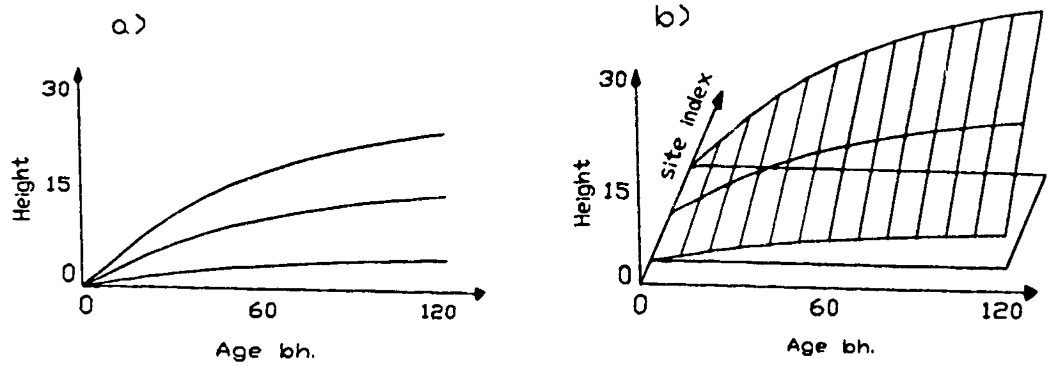

Figure 1.2: Three-dimensional anamorphic SI height curves: a) discrete; and b) continuous.

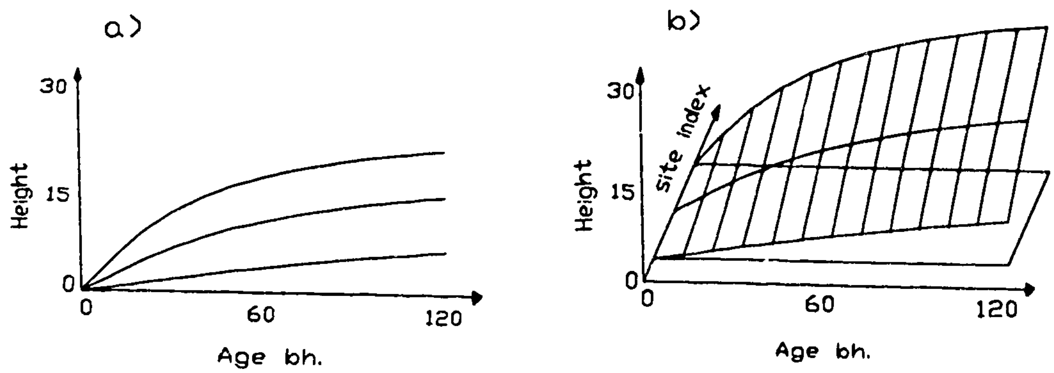

Figure 1.3: Three-dimensional polymorphic SI height curves: a) discrete; and b) continuous.

Newer approaches to height modelling involve almost exclusively three (Fig. 1.2 and 1.3) and four dimensions (Fig. 1.4), by adding to basic height over age models additional explicit variables of SI (third dimension) and SI base-age (fourth dimension). An early algebraic inclusion of a site variable (SI) to simple anamorphic models was followed by increased model complexity necessary to describe height growth polymorphism and other desirable model characteristics, i.e., passing through the origin, increasing asymptotes with site productivity, and height–SI equality at base-age.

Total age, stump age, and breast height age, are all variations of one measurable variable. The different names refer to different fixed base-heights; at which measurements of age begin to count, e.g., 0.0, 0.3, 1.3 m. In that respect, these ages at different fixed base-heights, are similar to site indexes at different fixed base-ages, such as 25, 50, 100 years. The main similarities here are that both age and SI are input variables, they both describe growth features, and their numerical interpretation functionally depends on corresponding specific constants that constitute for each of these variables a fixed base; the measurement variables cannot be interpreted without their corresponding measurement constants. For example, 20 years at base-height 0.0 m (total age) could be equivalent to 10 years at base-height 1.3 m (breast height age), and SI 20 m at base-age 100 years could be equivalent to SI 10 m at base-age 25 years.

Just like age and SI, the two constants base-age and base-height, are analogous. The fixed base-height defines age measurements in terms of height at which the age is measured, e.g., 0.0, 0.3, 1.3 m. This is functionally comparable with the SI fixed base-age, e.g., 25, 50, 100 years, that defines the age at which the height is used for potential growth, or productivity determination. Both of

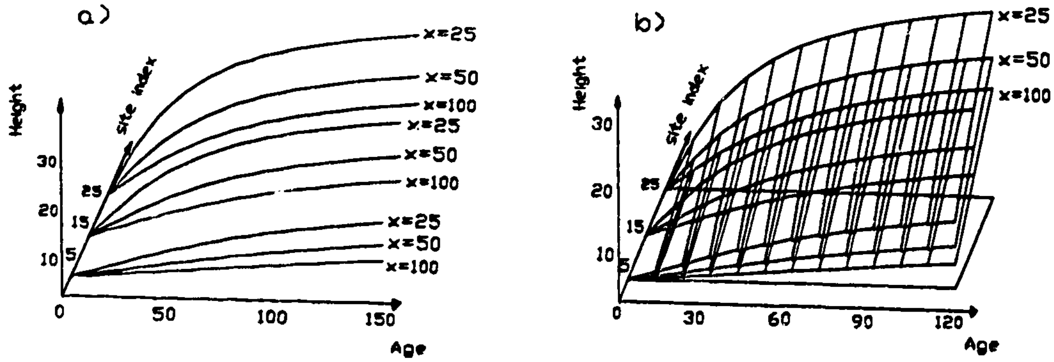

Figure 1.4: Four-dimensional variable age SI height curves: a) discrete over site-index and base-age; b) continuous over SI and discrete over base-age.

these constants, base-age and base-height, contribute to description of the growth and the growth features.

Enabling the measurement constants, i.e., base-age and base-height, to be variable would generalize the height model into an all variable measurement components model with broadened applications. Some variable SI height models derived using the ADA (Bailey and Clutter 1974) are already available in the forestry literature. Unfortunately, their selection is rather limited and the available technology for deriving such models has serious shortcomings. Their major limitation to model formulation is that the resulting models are limited to only one parameter response to different sites. For example, the models derived through the ADA cannot be both polymorphic and have variable asymptotes.

A height model with all measurement components, including the SI variable–base-age and the variable–base-height for the age definition, being described by variables rather than constants, could be a solution to most of the height prediction problems associated with periodic growth irregularity due to both inter and intraspecific competition from density, or growth disruptions, and age ambiguity, e.g., juvenile height modeling. This would permit the use of incomplete growth data following periods of irregularity and growth suppression. Such a model would flexibly use height increments from any number of years, similarly as height intercept models use height intercepts.

In summary, an important direction in the evolution of the modern height modeling is toward increased model generalization and flexibility. The first is expressed through increasing the dimensionality of mathematical forms of the growth models, while the second is expressed through deriving models capable of simulating complex forms of growth polymorphism and variable asymptotes. The main objective of this project is to advance height modeling along this same frontier, by developing the following new technologies:

- a general methodology for derivation of four-dimensional variable-age-SI height models with the adjustable measurement component of SI base-age, and with such desirable growth characteristics as growth polymorphism and variable asymptotes;
- a general height model, that would extend a four dimensional height model to five dimensions by applying modifying functions on its coefficients;
- a five-dimensional application of the above model with a variable base-height, making the model effectively age-free;
- an infinite-dimensional application of the above model with variable density components.

This new multidimensional height growth model will be formulated in a dynamic form allowing predictions of any point (x_2, y_2) from any other point (x_1, y_1). Such a form of an equation is often described as a difference equation, and it can be used as an integral model for the implementation of the variable base-height, and as an increment model for density height growth modeling; this form of an equation can be used as a yearly increment model. The last property of the model is very important because stand density and/or crowding are results of dynamic processes that change from one year to another and the height growth model with adjustable measurement components of *SI* base-age and density should be formulated in a differential (or yearly increment difference) form so that it can be used in yearly iterations. Since the first model should be integral and the second differential, they will be formulated as two separate models.

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Chapter 2

Four–Dimensional Height Spaces with Two Adjustable Measurement Components—A New Methodology for Derivation of Variable Base-Age *SI* Height Models

2.1 Introduction

[1]Height growth, particularly that of dominant and codominant trees (or a fixed number of the largest trees per unit area), is one of the most stable, directly measured stand growth statistics. It is strongly related to site productivity, and for most tree species, is independent of number of trees per unit area over a wide range of densities (Monserud 1984; Alrmdag 1988). To facilitate productivity comparisons on different sites, and for planning stand management activities, one estimates, from an available age and height, *SI*—an average height of crown class(es), or of an "*n*" number of tallest or largest trees at a fixed reference age of 25, 50, 75, or 100 years.

Traditional height models use *SI* as a fixed base-age height input variable. For example, to estimate heights of a 30-year-old stand at say age 20, one usually calculates the stand *SI*, likely at age 50, from the height at age 30, and then uses this height (at age 50) to calculate height at age 20. This procedure is rather awkward. One should formulate height models that use directly any age-height measurements as input, instead of a fixed base-age *SI*. Furthermore, it is suggested to use directly any age-height measurements in all other height model applications.

Earlier approaches with the potential of deriving models using directly height at any age instead of a fixed age *SI*, have been used to force height predictions at base-age to be equal *SI*, e.g., MC (Burkhart and Tennant 1977), and for fitting purposes, e.g., the ADA (ADA) (Bailey and Clutter 1974; Clutter *et al.* 1983; Clutter *et al.* 1984; Borders *et al.* 1984). A common characteristic of the earlier techniques is that when modifying the source model, they use a solution for a model coefficient; while the present approach uses a solution for a model variable for the same purpose. This gives more flexibility than the earlier techniques. It will be shown by examples that the present approach is useful in any modelling situation that involves self-referencing functions, i.e., functions that use a point on a curve as a parameter determining which curve should be generated

[1] A version of this chapter has been submitted for publication in Forest Science.

(Northway 1985).

This chapter starts with a brief review of a customary approach to a fixed-age *SI* height model formulation, and of the earlier techniques capable of deriving *SI*-free height models. Then a new approach is presented and its advantages discussed.

2.2 Background

In modeling practices used to describe height-age data, a researcher needs to choose an appropriate mathematical model. There are many models available in the literature. Ricker (1979) reviews the salient literature on basic growth models while Cieszewski and Bella (1989) list the more renowned mathematical models used to model height in forestry.

From the chosen base model, one may also want to derive a new model. Generally, height models can be derived from linear or nonlinear base equations with or without asymptotes. Although asymptotic growth is sometimes questioned in the forestry literature (Bredenkamp and Gregoire 1988, Smith 1984) and elsewhere (Knight 1968, Ricker 1979, Schnute 1981), nonlinear asymptotic models are intuitively appealing. They can be formulated to calculate the asymptotes through an implicit expression, or to include an estimable proportionality constant that determines the asymptotic value. Such an asymptotic model with an estimable proportionality constant can be derived, for example, from the Chapman-Richards function (Richards 1959) which has been often applied to growth models in its basic and various modified forms (Carmean 1971, 1972; Monserud and Ek, 1976; Krumland and Wensel 1977; Biging 1985). Since its introduction to the North American forestry literature (Pienaar and Turnbull 1973), it has become the most commonly used function (Bredenkamp and Gregoire 1988). Its basic form has zero-intercept, and three coefficients, i.e., $H(t) = \alpha(1 - e^{-\beta t})^{\gamma}$ where $H(t)$ is tree height at age t, α is the asymptote, and β and γ are shape coefficients.

If the coefficient α is replaced by $\alpha' SI$, the result is an anamorphic height/SI model (eq. (2.1), Table 2.1) with proportionality constant α' determining the values of the asymptotes, where $H(t, SI)$ is tree height at age t for a given SI. This model has been applied to many species, e.g., by Hegyi (1981), Lundgren and Dolid (1970), Monserud and Ek (1976) and others. Four important characteristics of models like eq. (2.1) are:

(1) a fixed base-age SI;

(2) a proportionality constant (α') to adjust asymptotes;

(3) $SI \neq$ height at the base-age (Table 2.2, col. 6-8 and Fig. 2.2), or α' is a function of other coefficients (Table 2.3, col. 6-8, and 2 vs. 3);

(4) redundant coefficients.

At the same time, without any loss of model flexibility, eq. (2.1) and many other height and SI models can be formulated in such a way that they:

(1) use any age-height measurements directly instead of the fixed base-age SI;

(2) may have variable asymptotes without proportionality constants;

(3) predict appropriate heights at base-age;

(4) do not contain redundant coefficients.

2.3 Models Using Any Age-Height Measurements

The first step in formulating a height-SI model is to choose a base model with desirable functional characteristics as noted above. Such a model may have the following form:

$$H(t) = f(t, \rho_1 ... \rho_{n-1}, \rho_n) \quad (2.23)$$

Table 2.1: Examples of height/SI models and base equations found in the forestry literature, and those newly formulated (*). $H(t, SI)$ denotes height as a function of age (t) and a fixed base-age SI; α,β,... regression coefficients.

Eq. No.	Author	Date	Equation
(2.1)	Hegyi	1981	
	Lundgren and Dolid	1970	
	Monserud and Ek	1976	$H(t,SI) = \alpha' SI\left(1 - e^{-\beta t}\right)^{\gamma}$
(2.2)	Schumacher	1939	$\ln H(t,SI) = \ln SI + \beta \ln SI(1/50 - 1/t)$
(2.3)	Cieszewski and Bella	*	$\ln H(t,SI) = \ln SI + \alpha' - (\beta' - A_{si} \ln SI)/t$
(2.4)	Cieszewski and Bella	*	$\ln H(t,SI) = \alpha' + \alpha'' \ln SI + (\beta'' \ln SI - \beta')/t$
(2.5)	Biging	1985	$H(t,SI) = \alpha SI^{\delta}(1 - e^{-\beta t})^{\gamma}$
(2.6)	Graney and Bower	1971	$H(t,SI) = \alpha SIt + \beta t + \gamma t^2 + \delta t^{0.5}$
(2.7)	Curtis *et al.*	1974b	$H(t,SI) = SI/\left(\alpha + \frac{\beta}{SI} + \gamma t^{\epsilon} + \frac{\delta t^{\epsilon}}{SI}\right)$
(2.8)	Wiley (after King 1966)	1978	$H(t,SI) = \frac{t^2}{\alpha + \beta/SI + (\gamma + \delta/SI)t + (\epsilon + \mu/SI)t^2}$
(2.9)	Johnson and Worthington	1963	$SI(t,H_t) = H_t\left(\alpha + \frac{\beta}{t}\right)$
(2.10)	Curtis *et al.*	1974a	$H(t,SI) = (SIt^2)/\left(\alpha + \beta t + \gamma t^2\right)$
(2.11)	Kirby	1975	$H(t,SI_{70}) = SI_{70}(\alpha + \beta t + \gamma t^2)$
(2.12)	Zakrzewski	1986	$H(t,SI) = SI(t/(\alpha + \beta t^{\gamma}))^{\delta}$

Table 2.2: Coefficients α', β, and γ (eq. (6)) for major Lake States tree species (Lundgren and Dolid 1970), computed α' ($x = 50$), and heights (in m) at age 50 ($H(50, SI)$) for three SI classes, i.e., 10, 15, and 20 m.

SPECIES	$(1 - e^{-\beta x})^{-\gamma}$	α'	β	γ	$H(50,10)$	$H(50,15)$	$H(50,20)$
Red pine	1.9074	1.890	0.0198	1.3892	9.91	14.86	19.82
Jack pine	1.6365	1.633	0.0223	1.2419	9.98	14.97	19.96
White pine	1.9724	1.966	0.0240	1.8942	9.97	14.95	19.93
Balsam fir	1.4400	1.437	0.0227	0.9381	9.93	14.97	19.96
White Spruce	1.6611	1.637	0.0254	1.5403	[illegible]	14.78	19.71
Black Spruce	1.7517	1.762	0.0201	1.2307	10.03	15.09	20.12
Tamarack	1.5495	1.547	0.0225	1.1129	9.92	14.98	19.97
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

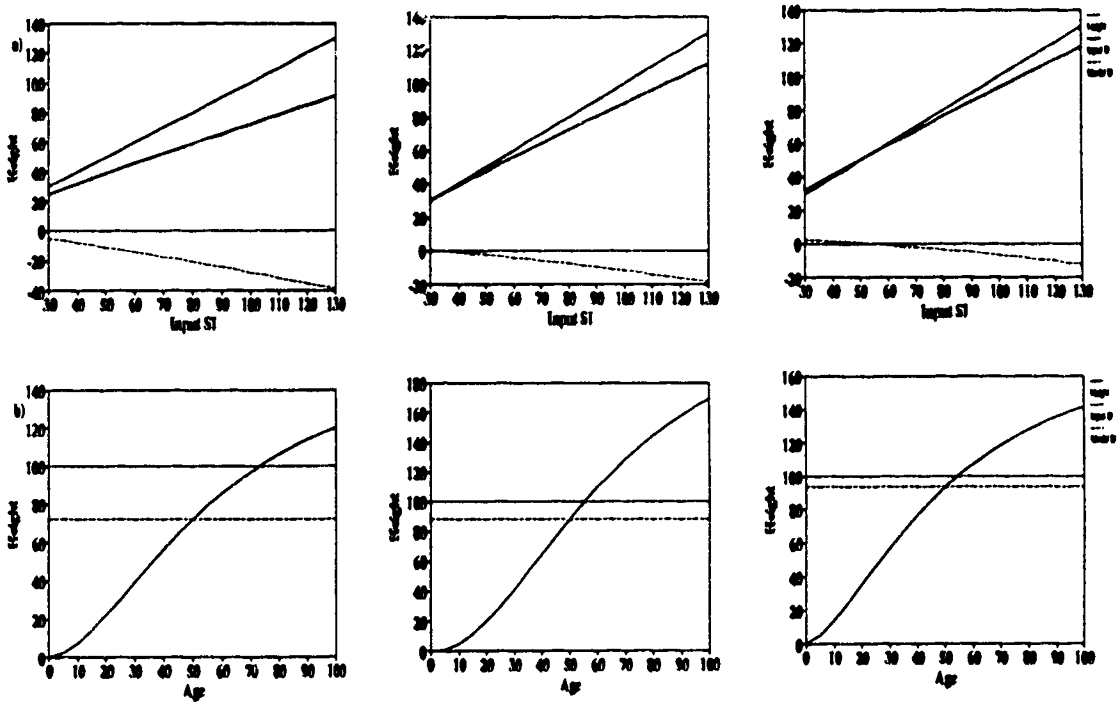

Figure 2.1: Differences between SI and $H(50, t_{SI})$ in eq. (2.5) for all three sets of parameters (Biging 1985) ($\beta = 0.024$, $\delta = 0.89$); a) SI, $H(50, t_{SI})$, and $SI - H(50, t_{SI})$ as a function of SI (in feet); b) $H(t, 100)$, $H(50, 100)$ and $SI = 100$ (in feet).

where $\rho_1 ... \rho_n$ are regression coefficients. The base model (2.23) can then be expanded into a family of anamorphic or polymorphic curves with additional independent variable(s) changing the curves for different sites. Usually, this variable is a fixed base-age SI, but techniques exist that can result in models using any age-height measurements. These techniques may be based on solving model (2.23) for a coefficient (ADA and MC), a coefficient and a variable (Cieszewski and Bella 1989), or just a variable (the new proposed technique).

2.3.1 Model Coefficient Oriented Techniques

The ADA to model derivation (Bailey and Clutter 1974; Clutter *et al.* 1983; Clutter *et al.* 1984; Borders *et al.* 1984; Ramirez *et al.* 1987) consists of solving eq. (2.23) for one coefficient as a function of a known value of the equation and other coefficients, and replacing that coefficient by its algebraic form solution. It can be applied to any base model to reformulate it into either an anamorphic or polymorphic height model.

For anamorphic curves, eq. (2.23) can be rewritten as $H(t) = \rho_n f(t, \rho_1 ... \rho_{n-1})$ so that for any height h_x base-age x, $h_x = \rho_n f(x, \rho_1 ... \rho_{n-1})$. The coefficient ρ_n (asymptote) is replaced by a function of h_x and x. This reduces the number of base model coefficients by one ($\rho_n = h_x / f(x, \rho_1 ... \rho_{n-1})$) and the family of anamorphic height curves becomes: $H(t, x, h_x) = h_x \frac{f(t, \rho_1 ... \rho_{n-1})}{f(x, \rho_1 ... \rho_{n-1})}$ where $H(t, x, h_x)$ defines height as a function of the prediction age t, and any other tree age x with corresponding height h_x. For example, $\ln H(t) = \alpha - \beta / t$ (Schumacher 1939) has the solution for the proportionality

Table 2.3: Coefficients α', β, and γ (eq. (6)) for major British Columbia tree species (Hegyi 1981), computed α' ($x = 50$), and heights (in m) at age 50 ($H(50, SI)$) for three SI classes, i.e., 10, 15, and 20 m.

SPECIES	$(1-e^{-\beta x})^{-\gamma}$	α'	β	γ	$H(50,10)$	$H(50,15)$	$H(50,20)$
Balsam (coast)	1.9306	1.9306	0.0236	1.7918	10.00	15.00	20.00
Balsam (interior)	2.3154	2.3154	0.0155	1.3597	10.00	15.00	20.00
Broadleaf maple	1.0457	1.0457	0.0988	6.2294	10.00	15.00	20.00
Red alder	1.1302	1.1302	0.0421	0.9422	10.00	15.00	20.00
Common paper birch	1.5137	1.5137	0.0175	0.7687	10.00	15.00	20.00
Yellow cedar	1.6317	1.6317	0.0263	1.5662	10.00	15.00	20.00
Larch	1.6672	1.6672	0.0215	1.2243	10.00	15.00	20.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

constant $\alpha = \ln h_x + \beta/x$; so the anamorphic height model is $\ln H(t, x, h_x) = \ln h_x + \beta(1/x - 1/t)$ (Bailey and Clutter 1974, and Borders *et al* 1984).

For polymorphic curves, the right hand side (RHS) of a solution for a different (than the proportionality constant) coefficient ρ_m, is used in place of ρ_m. Thus if: $\rho_m = g(x, h_x, \rho_1 \dots \rho_{m-1})$ where ρ_m could be an exponential coefficient of eq. (2.23), and $H(t, x, h_x) = f\Big(t, \rho_1 \dots \rho_{m-1}, g(x, h_x, \rho_1 \dots \rho_{m-1})\Big)$. For example, the solution for the slope coefficient for the Schumacher's (1939) model is $\beta = x(\alpha - \ln h_x)$, so the polymorphic height model (with a single asymptote) is $\ln H(t, x, h_x) = \alpha + (\ln h_x - \alpha)x/t$ (Bailey and Clutter 1974, and Borders *et al* 1984).

Although MC is used only to assure predictions of appropriate heights at the base-age, it can also sometimes result in models using any age-height measurements. It is similar to ADA in that it uses a solution for a coefficient to replace the coefficient; while it is different from ADA in that it is applied to models already containing SI as a variable. Not all models can use any age-height measurements directly through MC, e.g., models containing SI in more than one entry. An example of using MC for deriving SI-free models is a conditioning of eq. (2.1) over α'. This results in a two coefficient model (2.13) that does not contain the proportionality constant but governs asymptotes by an implicit expression $h_x(1 - e^{-\beta x})^{-\gamma}$, where x and h_x refer to any age-height measurements.

2.3.2 Model Coefficient/Variable Oriented Approach

Cieszewski and Bella (1989) derived an SI-free model from the half-saturation function (Tait *et al.* 1988). Their approach consists of:

- 1) identification of a suitable base-height model with biologically interpretable coefficients;
- 2) formulation of biologically meaningful hypotheses of growth in terms of functional relations between model coefficients and site productivity;
- 3) applying MC to the model;
- 4) replacing SI with an algebraic form of a solution for SI as a function of the model coefficients and any tree age-height measurements[2] and
- 5) revision of the model.

At steps 2) and 3) this approach uses SI temporarily in functional relations with model coefficients (Cieszewski and Bella 1989). At step 3) MC does not result in a model that uses any

[2] This should not be considered MC because the model has already been conditioned in 3) and, by definition, it goes through an appropriate height at SI base-age.

age-height directly, and the fixed-age SI is not ruled out until after step 4). An example of this approach applied to the Schumacher (1939) function is, at step 2), formulation of a hypothesis that the slope is proportional to $\ln SI$, i.e., $\beta \equiv \beta \ln SI$, so that $\ln H(t) = \alpha - \frac{\beta \ln SI}{t}$ Then, at step 3) MC gives $\alpha = \ln SI + \frac{\beta \ln SI}{50}$ and eq. (2.2). At step 4) the RHS of the solution for $\ln SI = \frac{\ln h_x}{1+\beta/50-\beta/x}$ is substituted in place of $\ln SI$, and after revision at step 5) the new model (2.14) uses directly any age and height instead of SI, is polymorphic and has variable asymptotes.

2.3.3 The New, Model Variable Oriented Approach

This new approach is a generalization of that described above by Cieszewski and Bella (1989). It is a variable oriented approach that instead of using solutions for any model coefficients, relies exclusively on the solution for a model variable. The new approach is more flexible and gives a better insight into model derivation by letting the researcher decide about the functional relations in the analyzed model. For example, one can test if the model asymptotes are proportional, linearly related, or curvilinearly related to SI, and still have polymorphic height models that use any age-height measurements directly instead of a fixed base-age SI.

To put the new approach in an algebraic form, one defines ρ_n as a function of SI and any number k of new coefficients, viz., $\rho_n \equiv g_n(SI, \rho_{n_1} ... \rho_{n_k})$. Then the base model (2.23) can be changed into:

$$H(t, SI) = f\Big(t, \rho_1 ... \rho_{m-1}, g_m(SI, \rho_{m_1} ... \rho_{m_k}) ... g_n(SI, \rho_{n_1} ... \rho_{n_l})\Big) \tag{2.24}$$

where $H(t, SI)$ is a function of t, SI, and $m + k + l - 1$ coefficients.

If an equation $h_x = f\Big(x, \rho_1 ... \rho_{m-1}, g_m(SI, \rho_{m_1} ... \rho_{m_k}) ... g_n(SI, \rho_{n_1} ... \rho_{n_l})\Big)$ can be solved for SI, then the RHS of the solution for SI:

$$SI = u(x, h_x, \rho_1 ... \rho_{n_l}) \tag{2.25}$$

can be substituted in eq. (2.24) in place of SI so the height model changes to:

$$H(t, x, h_x) = f\Big(t, \rho_1 ... \rho_m, u(x, h_x, \rho_1 ... \rho_{n_l})\Big) \tag{2.26}$$

After reformulations and elimination of redundant coefficients it becomes:

$$H(t, x, h_x) = f(t, x, h_x, \rho_1 ... \rho_w) \tag{2.27}$$

where $w \leq m + k + l - 1$; in other words, the final SI-free model (2.27) has smaller or equal number of coefficients than the initial fixed base-age SI model (2.24).

For example, using the new proposed technique for the Schumacher (1939) function, one does not have to compromise using $\alpha = \ln h_x + \beta/x$ for anamorphic curves, or $\beta = (\alpha - \ln h_x)x$ for polymorphic curves with a single asymptote, as ADA would suggest (Bailey and Clutter 1974). Instead, one can define $\alpha = \ln SI + \alpha'$ where α' is a new coefficient, and $\beta = \beta' - A_{si} \ln SI$ where β is another new coefficient independent of α'. This leads to eq. (2.3)—a polymorphic height model with variable asymptotes. In this model, the amount of variation in the asymptotes is unaffected by the amount of variation in the slopes, and vice versa. One can go further and define the model asymptote and slope as linear functions of $\ln SI$, i.e., $\alpha = \alpha' + \alpha'' \ln SI$ and $\beta = \beta' - \beta'' \ln SI$. This results in eq. (2.4). Solving eq. (2.4) for SI gives $\ln SI = (\ln h_x - \alpha' + \beta'/x)/(\alpha'' + \beta''/x)$ Thus, the new height model using any age-height measurements directly, is eq. (2.15), which can be equivalent to eq. (2.3) if $\alpha'' = 1$ and $\beta'' = A_{si}$. One may then test the significance of any of the model coefficients on real data.

All base models, or fixed-age SI/height models listed in Table 2.1 could be reformulated into height models using direct age-height (x and h_x) measurements (Table 2.4) instead of a fixed base-age SI. The new models have up to two coefficients fewer than their original models (Table 2.1), while retaining identical flexibility as their respective source models.

Table 2.4: Height models using directly any height-age measurements (instead of a fixed base-age SI) derived in this study from models in Table 1, and the eliminated coefficients (α,β,... here are not consistent with α,β,... in Table 2.1.

Eq. No.	Source Eq. No.	Eliminated Coefficient(s) No.	Eliminated Coefficient(s) List	Derived Equation
(2.13)	(2.1);(2.5)	1;2	α';α,δ	$H(t,x,h_x) = h_x\left(\frac{1-e^{-\beta t}}{1-e^{-\beta x}}\right)^{\gamma}$
(2.14)	(2.2)	0	n/a	$\ln H(t,x,h_x) = \ln h_x \frac{1+\beta/50-\beta/t}{1+\beta/50-\beta/x}$
(2.15)	(2.4)	1	α''	$\ln H(t,x,h_x) = \frac{\ln h_x - \alpha + \beta/x}{1+\gamma/x}\left(1+\frac{\gamma}{t}\right) + \alpha - \frac{\beta}{t}$
(2.16)	(2.6)	2	α,β	$H(t,x,h_x) = t\left(\frac{h_x}{x} + \gamma(t-x) + \delta\left(t^{-0.5} - x^{-0.5}\right)\right)$
(2.17)	(2.7)	1	α	$H(t,x,h_x) = \frac{0.5h_x\left(1+\gamma x^c+\sqrt{(1+\gamma x^c)^2+4(\beta+\delta x^c)/h_x}\right)}{1+\gamma t^c+2(\beta+\delta t^c)/\left(h_x\left(1+\gamma x^c+\sqrt{(1+\gamma x^c)^2+4(\beta+\delta x^c)/h_x}\right)\right)}$
(2.18)	(2.8)	1	β	$H(t,x,h_x) = \frac{t}{\alpha/t+\gamma+\epsilon t+(x/h_x-\alpha/x-\gamma-\epsilon x)(1/t+\delta+\mu t)/(1/x+\delta+\mu x)}$
(2.19)	(2.9)	1	α	$H(t,x,h_x) = h_x\frac{1+\beta/x}{1+\beta/t}$
(2.20)	(2.10)	1	α	$H(t,x,h_x) = h_x\frac{1+\beta x+\gamma x^2}{1+\beta t+\gamma t^2}\left(\frac{t}{x}\right)^2$
(2.21)	(2.11)	1	α	$H(t,x,h_x) = h_x\frac{1+\beta t+\gamma t^2}{1+\beta x+\gamma x^2}$
(2.22)	(2.12)	1	α	$H(t,x,h_x) = h_x\left(\frac{t(1+\beta x^{\gamma})}{x(1+\beta t^{\gamma})}\right)^{\delta}$

2.4 Discussion

Fixed base-age SI, although traditionally used in height models, is often an unnecessary model limitation. Given a model for a family of height curves, usually any point (x, h_x) on a height curve can unequivocally define that curve.

As x can assume any value, h_x should not be considered a fixed-age SI. Equation (2.27) represents an anamorphic or polymorphic height model that uses directly any age and height measurements, and compared to the base model (2.24), has usually fewer coefficients. Furthermore, eq. (2.27) defines a tree height (H_t) at age (t) as a function of another age (x) and height (h_x) at that age (x). At $t = 25, 50, 75, 100$, H_t can be interpreted as SI; the model can generate SI curves: (i) as a function of height in specified age classes (if x is fixed and h_x varies, Fig. 2.4a), or (ii) as a function of age in height classes (if h_x is fixed and x varies, Fig. 2.4b). If $x = 50$, h_x can be interpreted as SI; the model will generate height curves: (i) as a function of age in SI classes (if SI is fixed and t varies, Fig. 2.4c), or (ii) as a function of SI in age classes (if t is fixed and SI varies, Fig. 2.4d). Because SI is just a height at a base-age, no model reformulation is necessary to calculate SI from height; SI and any other heights are calculated from one common equation.

The ADA was applied successfully by Bailey and Clutter (1974); Clutter *et al.* (1983); Clutter *et al.* (1984); Borders *et al.* (1984); Ramirez *et al.* (1987); to various forestry modelling problems. Bailey and Clutter (1974) used it to modify Schumacher's (1939) model to predict appropriate heights at the base-age, while Clutter *et al.* (1983), Clutter *et al.* (1984) and Borders *et al.* (1984) additionally applied it to model fitting.

The above uses notwithstanding, no one has taken full advantage of the height models derived by ADA in their post-fitting applications. Instead of presenting only one equation that could estimate any height from any other height—be it a forward or back-in-time estimation—once the fitting was done, all authors suggested the customary use of separate equations for SI and height estimations. For example, Clutter *et al.* (1983; p. 50) present: $H(t, SI) = SI\left(\frac{1-e^{-\beta t}}{1-e^{-\beta t_{si}}}\right)^{\gamma}$ and state "...For prediction of SI from height and age the above equation is algebraically rearranged to ..." followed by: $SI(H,t) = H\left(\frac{1-e^{-\beta t_{si}}}{1-e^{-\beta t}}\right)^{\gamma}$ Another example is in Clutter *et al.* (1984; p. 27), where the authors actually recalculate their height model coefficients to absorb the SI base-age to make it permanently

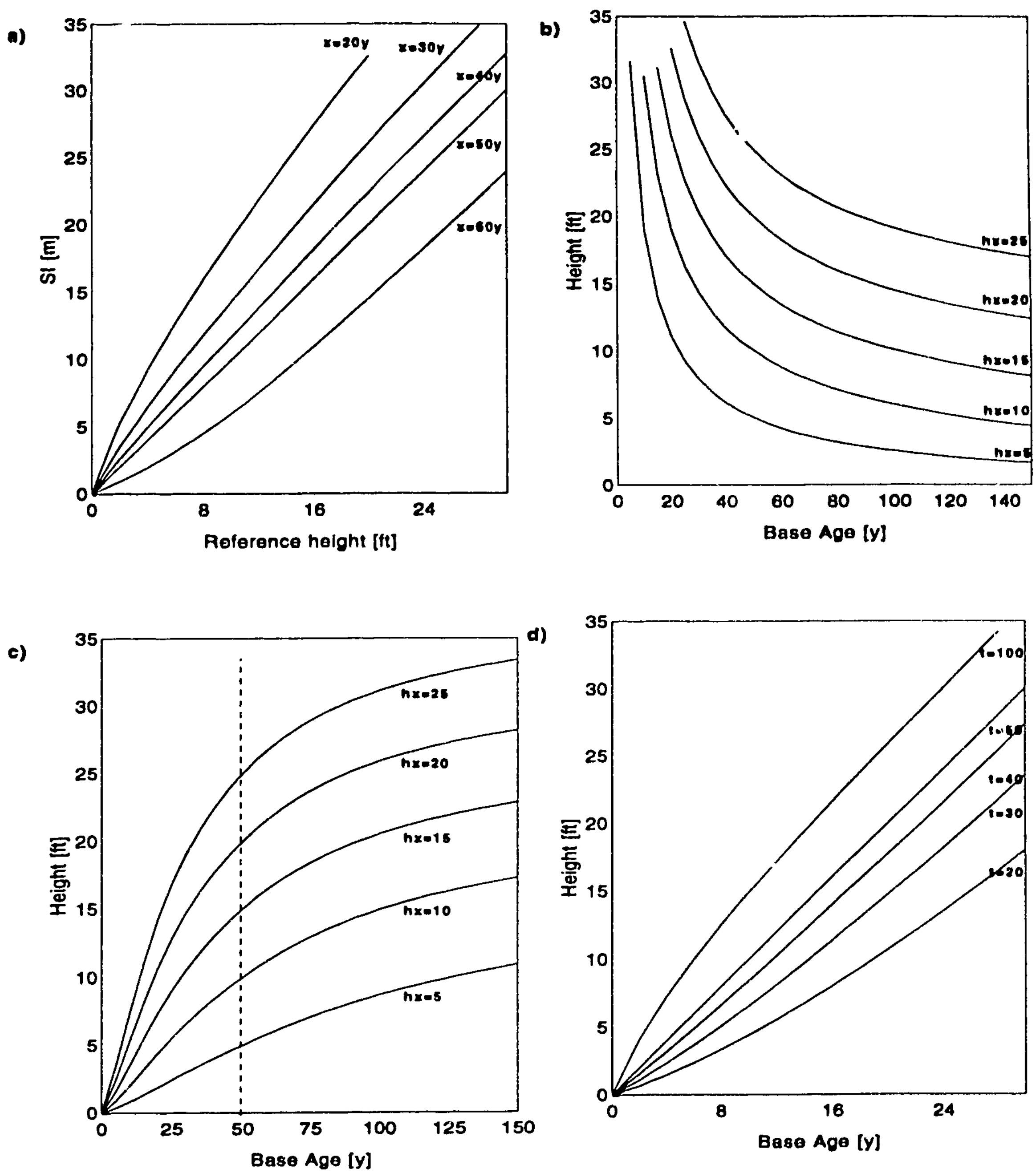

Figure 2.2: Curves generated by a height model using any direct height-age measurements: a) *SI* as a function of height by specified age classes; b) *SI* as a function of age by height classes; c) height as a function of age by *SI* classes; d) height as a function of *SI* by age classes.

unchangeable, and force the user to use two different equations for SI and for height estimations.

The present approach for height model formulation leads to base-age invariant anamorphic, or simple or complex polymorphic height models that at the same time can have varying asymptotes with site productivity. In addition, the models can also have other desirable properties such as: zero height at age zero, height at base-age equal SI, biological interpretations, and easily estimable coefficients.

The approach allows for the development of complex models incorporating several growth hypotheses and/or empirical correlations between productivity potential and different model coefficients. The number of altered coefficients is limited only by the formulated model's solvability for SI. In a limiting case, when applied with one varying coefficient, the method might be equivalent to ADA or to MC, e.g., eq. (2.1), although in general it is not.

2.4.1 Advantages of The New Technique Compared to The Traditional Techniques

MC is somewhat ambiguous about which coefficient should be replaced, e.g., eq. (2.6) conditioned over α or β may result in the two coefficient SI-free model (2.16), but if conditioned over γ or δ it may not. Biging (1985, footnote 5) did not condition eq. (2.5) over α, and in consequence, he did not notice that eq. (2.5) had as many as two redundant coefficients estimated in regression analysis. Moreover, had he used MC to condition eq. (2.5) over α, he might still have not noticed that the resulting model was SI-free, as one generally does not associate MC with such models.

MC is mainly associated with changing the models' predictions rather than the models' structures, and even models that potentially are able to use directly any age-height measurements are usually changed into fixed base-age models. For example, Graney and Bower (1971) conditioned their model: "...Inserting the condition that height at age 50 is SI..." and arrived at the four coefficient fixed base-age model (2.6) (altered in this study into the two coefficient SI-free model (2.16)). Such models as eq. (2.6) to (2.12), normally would not even be considered for this technique because they predict appropriate heights at their respective base-ages.

Using ADA one can predetermine the future model's behaviour by an appropriate choice of only one coefficient to be replaced. Choosing a scaler coefficient results in an anamorphic model with varying (if any) asymptotes with productivity levels (SI). Choosing an exponential coefficient results in a polymorphic model with a fixed (if any) asymptote for all productivity levels, e.g., Bailey and Clutter (1974). Thus, one has to sacrifice polymorphism for varying asymptotes or varying asymptotes for polymorphism. Meanwhile, a researcher may wish to develop an asymptotic polymorphic height model with variable asymptotes.

The approach presented by Cieszewski and Bella (1989) replaces one coefficient (step 3) with a solution for the coefficient. Then it replaces the model variable (SI) (step 4) with a solution for that variable. Hence, it combines both the coefficient and the variable oriented techniques. As site productivity can be associated with more than one arbitrary coefficient in the initial stage of model formulation, this approach provides more control over the final model's properties than the coefficient oriented techniques. It can lead to a polymorphic model with asymptotes changing as required. However, the coefficient oriented part (step 3) of this approach somewhat limits its intrinsic flexibility. The solution for a model asymptotic coefficient predetermines the behaviour of the future model's asymptotes in accordance with the present model's structure. Again, the outcome may not suit the researcher's purposes, or the data trends.

This drawback is common to all coefficient oriented techniques, e.g., for Schumacher's (1939) function, i.e., $\ln H(t) = \alpha - \beta/t$, ADA inflexibly forces $\alpha = \ln h_x + \beta/x$ for anamorphic curves, where β is the slope coefficient; or $\beta = (\alpha - \ln h_x)x$, for polymorphic curves, where a is a logarithm of the asymptote. Thus, the functional relations that govern asymptotes of anamorphic curves or slopes of polymorphic curves depend on one original model coefficient and the base model's assumptions, and are therefore inflexible; in a sense, using ADA, one attempts to prove that the base model is applicable to the data rather than to find the true trend.

Using ADA Bailey and Clutter (1974) derived, from the Schumacher's (1939) function, a polymorphic height model with one common asymptote for all SI values and commented: "...Whether or not the latter property has biological significance is not important...." (Bailey and Clutter 1974, p. 157). In contrast, one can use the new variable oriented approach to derive a polymorphic height model that not only has asymptotes varying with different SI values, but additionally, the degree of the asymptotic variation is determined by the data rather than by the limited flexibility and an arbitrary shape of the base model.

2.4.2 A Reference Variable In Height Models

Incompatibility between input SI and height predictions at base-age (Fig. 2.2) causes height model inaccuracy and inconsistency near the base-age. This may be a minor problem with small differences, but a major one with large differences. More importantly, this incompatibility precludes the development of base-age invariant height models.

The present approach will help to prevent confusing the adjustable parameter, SI, with a model coefficient. The direct use of age-height measurements is more reasonable than the use of SI. Thus, if the height curves pass through appropriate heights at reference age, the coefficient α' in eq. (2.1) is redundant. If desired for any reason its value can be calculated for any base-age x (Table 2.3, col. 2). Any deviation from the calculated value of α' would indicate that the height curves miss the appropriate heights at reference age. For example, in the British Columbia height curves (Hegyi 1981), all of the calculated α'-s (Table 2.3; col. 2) are identical to the published values of α' (col. 3). Thus, all heights computed for the SI base-age for three different SI classes (col. 6–8) are identical with the SI class values. In other words, the curves go through appropriate heights at the reference age and therefore, α' is redundant. In the Lake States curves (Lundgren and Dolid 1970), the values of computed α' (Table 2.2; col. 2) are different from the published α' values (col. 3). Consequently, the heights computed for the SI base-age for the three different SI classes in Table 2.2 are incorrect; these height curves do not go through appropriate heights at the base-age. Other models may have similar disparity problems. For example, eq. (2.5) will not generate curves that unconditionally go through appropriate heights at reference age (Fig. 2.2 and 2.4.2) unless coefficients $\alpha = \alpha'$ (Table 2.3, col. 2), and $\delta = 1$. By setting $H(50, SI) = SI$ I may be able to calculate, for any α and any $\delta \neq 1$, one unique SI value for which a height curve may in fact go through an appropriate height at reference age as $SI = \left[\alpha\left(1 - e^{-\beta t_{SI}}\right)^{\gamma}\right]^{\frac{1}{1-\delta}}$. Using eq. (2.5) as an example and reformulating it to eq. (2.13) results in curves with identical shapes as eq. (2.5), but with two fewer coefficients. In addition, the curves generated by eq. (2.13) pass unconditionally through appropriate heights at any reference age that can be arbitrarily chosen by the model user.

2.4.3 Model Fitting

Estimation of coefficients, techniques used in regression analysis, data requirements, and possible other problems associated with the use of models presented here are independent from their method of derivation. Fitting these models, one may use any coefficient estimation technique suitable to one's specific analysis, as one would do in fitting any traditional height-SI model. However, the present models offer more flexibility in data requirements and handling. In this respect, the models are similar to those derived by ADA they can use any base-age for reference points, and therefore can be base-age invariant. Their coefficients may be estimated on any height over age data, whether from stem analysis or from permanent sample plots (PSP-s). As Borders *et al.* (1984) state, such a model can be fitted to no overlapping measurement periods, i.e., fitting $H(t_n) = f(t_n, t_{n-1}, h_{n-1})$ on $(t_1, h_1; t_2, h_2)$, $(t_2, h_2; t_3, h_3)$, $(t_3, h_3; t_4, h_4)$,... data.

Although rigorous statistical treatments may be used to fit SI-free height models—the implications of which are beyond the scope of this chapter—it is easy to see that a flexible base-age, while retaining the model's sophistication, lends further flexibility to model analysis. If one treated the SI-free height models as fixed base-age height/SI models to help dynamically estimate average SI values between subsequent iterations of a nonlinear regression (Northway 1985), one can simply set x to a constant. Otherwise one may follow the proponents of ADA, or try other approaches like

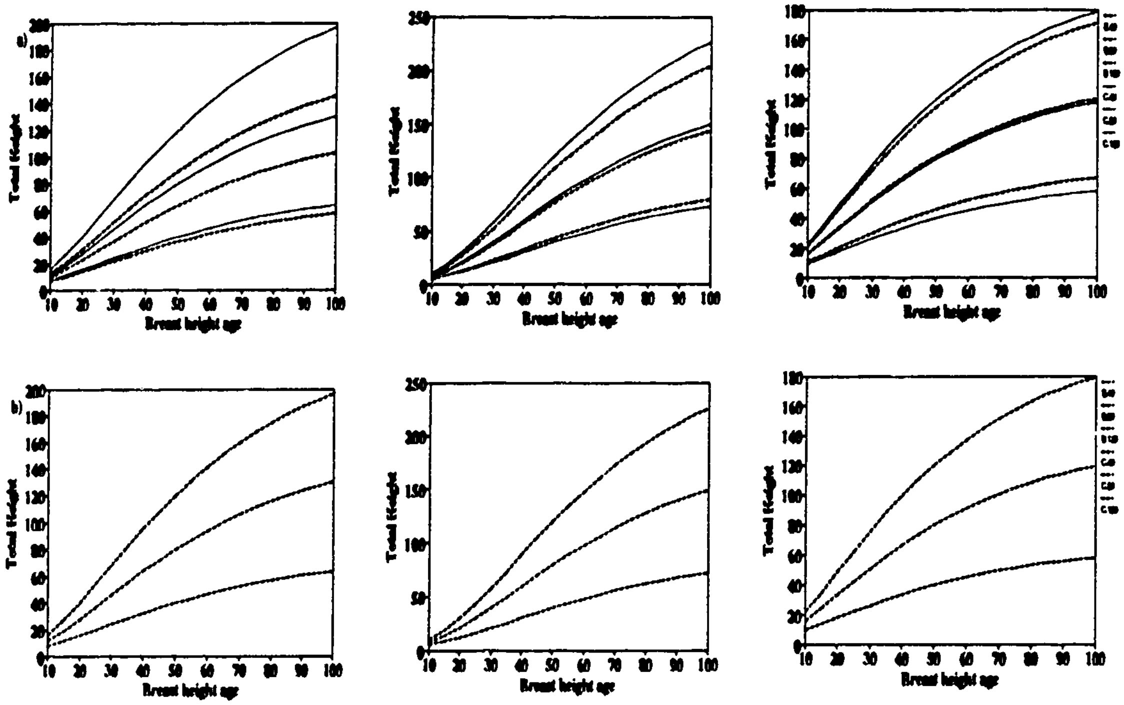

Figure 2.3: Height curves (eq. (2.5) and (2.13)) for all three sets of parameters (Biging 1985)($\beta = 0.024$, $\delta = 0.89$): a) eq. (2.5) and (2.13) using SI 40, 80, and 120'; b) eq. (2.5) using adjusted SI (Fig. 2.2) and eq. (2.13) as in a).

using heights at random ages as reference points, heights at maximum ages, or else heights at all decadel ages (a multi-SI estimation).

2.4.4 Numerical Considerations

Concise models, even if apparently more complex, should be favoured over those with more numerous coefficients, as inclusion of coefficients that cause height deviations at base-age violate the principle of parsimony (William of Occam 14th century[3], Larimore and Mehra 1985). Models with more coefficients may lead to difficulties in nonlinear regression fitting because of the dimensionality problem, and often strong correlation between coefficients.

If the proposed method is used in a modeling situation with only one independent variable, the parameter h_x can be treated as an estimable (through a regression analysis) model coefficient and x as an arbitrary constant (Schnute 1981). One important advantage of coefficient h_x over any other coefficient to be replaced is that h_x corresponds to actual data and therefore has smaller error. In fact, in preliminary regression analysis of complex models, h_x can be set arbitrarily to a value obtained from averaging data points (observations of the dependent variable) at the x value (the independent variable). Since h_x refers directly to data, its estimation by regression, if required, is trivial (initial value is easy to estimate and has a narrow range). This is unlike the estimation of

[3]William of Occam, 14th century, Occam's Razor being, *entia non sunt multiplicanda praeter necessitatem, i.e., parsimony.* — things ought not be multiplied except out of necessity, i.e., parsimony.

asymptote coefficients, e.g., α' in eq. (2.1), which may often tend towards very high values. From a computational point of view, models containing an asymptotic proportionality constant are inferior because of possible difficulties with estimation (e.g., floating point overflow for very high asymptotes). Some researchers using such models have found poor convergence in nonlinear regression fitting because of a lack of clearly defined asymptotic trends in the data (Biging 1985, Goudie 1984[4], Anon. 1985, Bredenkamp and Gregoire 1988, Schnute 1981). In addition, computational problems may occur because of the limited resolution of computers. Thus, for example, due to an inconsistency in computer computations, i.e., $(A/C)B \neq A(B/C)$, the three coefficient eq. (2.1) is in fact a limiting case of the two coefficient eq. (2.13). While $(1-e^{-\beta t})/(1-e^{-\beta x})$ can usually be computed easily, $1/(1-e^{-\beta x})^\gamma$ may not. Accordingly, $h_x\left(\frac{1-e^{-\beta t}}{1-e^{-\beta x}}\right)^\gamma$, i.e., the RHS of eq. (2.13), can be computed safer than $1/\left(1-e^{-\beta x}\right)^\gamma SI\left(1-e^{-\beta t}\right)^\gamma$, or $\hat{\alpha} SI\left(1-e^{-\alpha t}\right)^\gamma$, i.e., RHS of eq. (2.1).

2.5 Summary and Conclusion

A fixed-age SI may be useful in many situations, but height modeling is seldom one of them. The direct use of age-height measurements for this purpose is more advantageous. It eliminates one unnecessary step in height estimation, i.e., the calculation of SI, and simplifies model calibration. In regression analysis, a height model formulated to use directly any age and height measurements, despite the more complex appearance, contain fewer coefficients that are more robust and statistically stable.

The main benefits of using the present approach (compare Tables 2.1 and 2.4) are

- a direct use of age-height measurements and concomitant height-age base-age invariance;
- absence of an explicit coefficient describing an asymptote (though it may be calculated);
- usually reduced number of coefficients;
- associations of numerous model coefficients with site productivity (SI), e.g., variable asymptotes in addition to polymorphism; and
- consistency of mathematical models with biological theory.

In all, instead of the conventional coefficient oriented approaches such as ADA or MC, I recommend the present variable oriented approach to model derivation in five steps:

1) identify a base model with easily interpretable coefficients (eq. (2.23));
2) modify the base model using, as an independent parameter, a value of the dependent variable at a fixed reference point, i.e., SI, (eq. (2.24));
3) solve the modified model for the independent parameter (e.g., SI) as a function of direct measurements and model coefficients (eq. (2.25));
4) substitute the solution obtained in place of the independent parameter (eq. (2.26));
5) revise the model and eliminate redundant coefficients (eq. (2.27)).

[4] James W. Goudie. 1984. Height and SI curves for lodgepole pine and white spruce and interim managed stand yield tables for lodgepole pine in British Columbia. Internal report, FY-1983-84, Submitted to Research Branch, British Columbia Ministry of Forests.

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Chapter 3

Five–Dimensional Height Spaces with Three Adjustable Measurement Components—An Ageless Height Model, and A Static Density Height Model

3.1 A General Model—An Introduction and Framework

The evolution of height models has been strongly stimulated by a general increase in statistical literacy and a rapid development of computer technologies. Nevertheless, new model advancements have always depended on an approach to model derivation and associated with it, limitations of algebraic solutions. Until the year 1989, the advancement of the height models to the fourth dimension of the variable-base-age SI, height models were somewhat limited. They were either polymorphic, but with only one common asymptote for all sites, or had variable asymptotes but anamorphic slopes, because all such efforts were based on the ADA (Bailey and Clutter 1974, Clutter *et al.* 1983, Borders *et al.* 1984, Ramirez *et al.* 1987). Such model derivation was limited to solving equations for only one model coefficient. In addition, the only polymorphic variable-base-age SI height model with variable asymptotes (Cieszewski and Bella 1989, 1991) was developed through a process that was not readily applicable to other base models.

Generalization of models has always been an attractive idea that in a simplistic version could be paraphrased as developing fewer models that can describe more situations. Diverse examples of model generalization exist in the forestry literature. One example is an application of a simple general height over age growth function and the localizing of its parameters using Kalman filters (Walters *et al.* 1991). In this example, the authors avoid model complexity by using one simple regional equation combined with local stand data.

Zeide (1978) proposed a height model generalization based on using two reference-heights at two base-ages instead of one reference-height at one base-age. The additional reference point in height measurements added to the model flexibility and made it possible for this model to actually simulate growth trends from several disparate equations. Follow up applications included implementation of this system into a stand growth simulation model (Arney 1985) and development of general multi-species height equations (Milner 1987, Hoyer and Chawes 1980, Hoyer and Swanzy 1986). A potential two point principle equation is also the Schnute's (1981) function and its modified form (Cieszewski and Bella 1991).

3.1.1 Development of Model Dimensionality and its Flexibility

The generalized technology described in Chapter 2 provides a new possibility of deriving the variable-base-age *SI* height models from many different base models. The resulting equations contain a variable measurement component of base-age instead of a fixed one, that in many models was implicit. After such a change the resulting models contain two explicit variable measurement components: reference-height and variable-base-age; and one arbitrary input variable: prediction-age.

Since each step to increase a model's dimensionality involves considerable algebraic complication, the variable-base-age *SI* height models are usually very complex and cannot be further modified through any substitutions of their implicit solutions. Figures 1.1– 1.4 in Section 1.3 represent a process of increasing model dimensionality through three general distinct steps, in the evolution of the height models, i.e., 1) generating two dimensional models of height over age; 2) extending the models to three dimensions over fixed base-age *SI*; and 3) adding the fourth dimension of variable base-age. At each step of increasing a height model dimensionality and/or its sophistication/flexibility, e.g., extending a model from anamorphic to polymorphic, the window of applicable algebraic solutions in the newly created equations is drastically reduced. The described process can be illustrated by mathematical forms that are associated with different steps of the model modification towards increasing model dimensionality and flexibility using as a starting point a very simple half-saturation function:

$$H(t) = \frac{A}{1 + \beta/t} \tag{3.1}$$

where: t is the age; $H(t)$ is the height as a function of age; A is the asymptote; and β is a half-saturation coefficient.

While eq. (3.1) can generate a simple monotonically increasing curve, e.g., Fig. [3.1], this equation has to be slightly modified to generate a single biologically sound two-dimensional height curve, e.g., Fig. [3.2], that has an inflection point, i.e., a curve that has a sigmoidal shape, e.g., Fig. [3.2]. The modified half-saturation function generating sigmoidal curves will then have the following form:

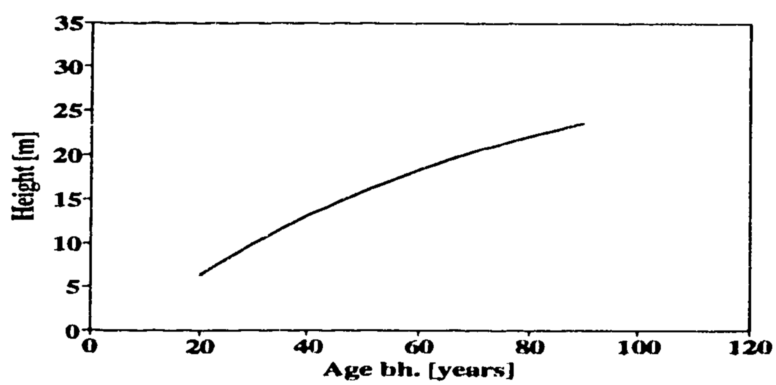

Figure 3.1: Simple monotonic two dimensional height as a function of age.

$$H(t) = \frac{A}{1 + \beta/t^{\alpha}} \tag{3.2}$$

where: A, β, and α are estimable model coefficients; t is prediction age; and H is the predicted height.

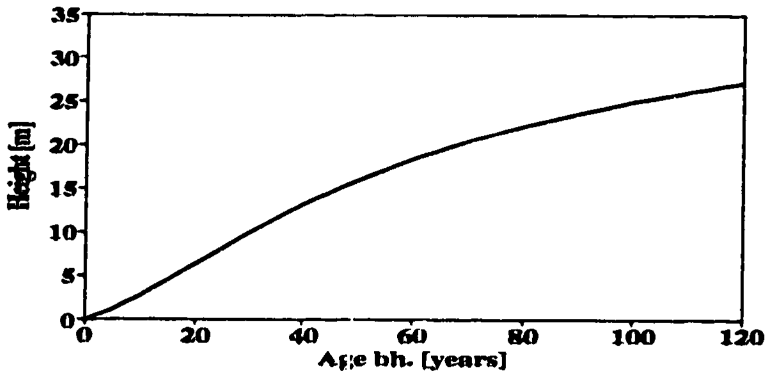

Figure 3.2: An inflected two dimensional height growth as a functions of age.

This equation is a simple form of describing a basic, biologically sound, sigmoidal height over age (Fig. 1.2 and 1.3), of a single tree or stand, and it can be referred to as a base model. Base models are used as a root for derivation of more complex three-dimensional height models describing families of growth curves across ranges of other variables such as site expressed by SI (Fig. 3.2, Section 1.3). There are few different ways to derive a three dimensional SI height model from the base model (3.2). Each way may result in a different model with specific properties. The simplest anamorphic model (Fig. [3.3]) can be derived from this base model by applying to it the ADA and solving the equation for A as a function of SI (SI) and its base-age, e.g., 50, and then replacing the parameter A by this solution in place of A. The resulting model will have the following form:

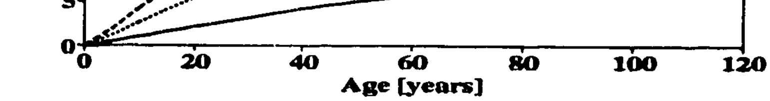

Figure 3.3: A three-dimensional, anamorphic SI height space (multiple asymptotes).

$$H(t) = SI\frac{1+\beta/50^{\alpha}}{1+\beta/t^{\alpha}} \tag{3.3}$$

Since anamorphic models are not considered to be satisfactory for most species and applications, the model should be modified to be flexible enough to simulate growth polymorphism. The ADA can lead to a polymorphic model derived from eq. (3.2) if the equation is solved for β instead of A, and as before, the solution is substituted for β. Now the resulting model would be polymorphic, but have only one asymptote common for all sites (Fig. [3.4]):

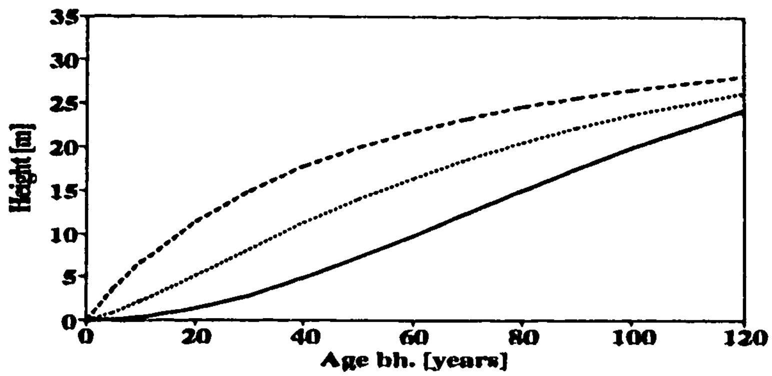

Figure 3.4: A three-dimensional, polymorphic SI height space with a single asymptote common for all SI values.

$$H(t) = \frac{A}{1 + \frac{A/SI - 1}{t^{\alpha}/50^{\alpha}}} \tag{3.4}$$

A simple fixed-base-age SI height model, that is polymorphic and has variable asymptotes (Fig. [3.5]) can be derived from eq. (3.2) following Cieszewski and Bella (1989), while the resulting equation would have a somewhat more complex form:

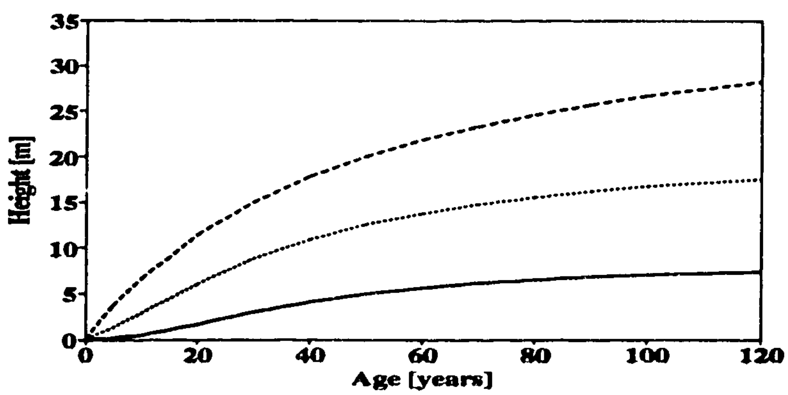

Figure 3.5: A three-dimensional, polymorphic SI height space with multiple asymptotes varying with SI.

$$H(t) = \frac{SI + \beta/50^{\alpha}}{1 + \beta/(SIt^{\alpha})} \tag{3.5}$$

Equation (3.5) describes relatively simple, though biologically sound, polymorphic growth patterns (Fig. [3.5]), according to a hypothesis that the half-saturation parameter β would be inversely proportional to SI ($\beta \propto SI^{-1}$). While seemingly still relatively simple, this equations defines just about the limit to a complexity of modifications that can be performed on a height model to increase its flexibility in simulating different patterns of multidimensional growth. To achieve more complex height growth polymorphism would require further modification of this equation or more flexible initial assumptions about the growth polymorphism (Cieszewski and Bella 1989). Unfortunately, any further increase in complexity of these equations may not be very useful for increasing the model dimensionality to four dimensions (Fig. [3.6]) and deriving variable-base-age height-growth model, as it could preclude the equation's solvability for SI (see Chapter 2). Thus, at this point one faces a dilemma whether to compromise the polymorphism or the dimensionality.

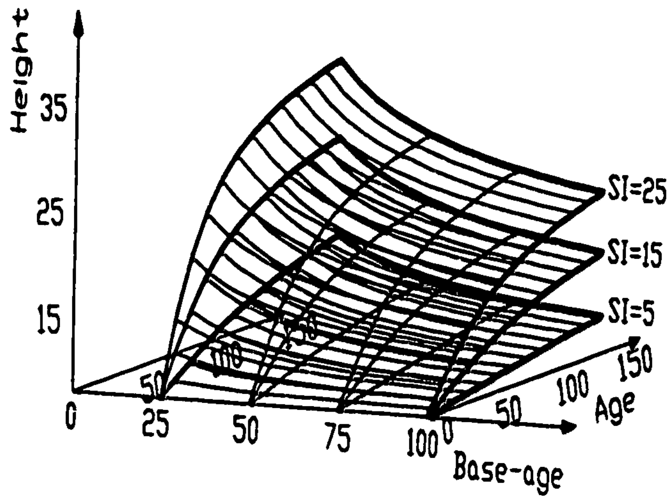

Figure 3.6: A four-dimensional, variable-base-age SI, height-growth space.

If a decision is made in favour of increasing the complexity of the growth polymorphism, an additional parameter could be added to model (3.5) to make the polymorphic height growth hypothesis more flexible. Assuming that $B = \beta SI^{-1+\gamma}$, the model will be either polymorphic according to original assumption ($\gamma = 0$), or anamorphic ($\gamma = 1$), or else ($0 \neq \gamma \neq 1$) polymorphic in a different fashion than originally assumed. With the extra parameter, model (3.5) would become more general, though, it would not have a closed solution for SI and could not be reformulated to variable-base-age SI height model:

$$H(t, SI) = \frac{SI + \beta SI^{\gamma}/Age_{SI}^{\alpha}}{1 + \beta SI^{\gamma-1}/t^{\alpha}} \tag{3.6}$$

If a decision is made in favour of increasing the model dimensionality, then the model has to remain in the form of eq. (3.5). Even in its present relatively simple form, this equation is solvable for SI only under certain assumptions and within a limited range of the model parameter values. Any further modification can easily make the model unsuitable for further advancement, i.e., a derivation of a variable-base-age SI height model (Fig. [3.6]). Since the main objective of this chapter is to demonstrate a development of a five-dimensional height model, the further development here will be based on eq. (3.5).

In biological terms, eq. (3.5) means that the potential growth of a single tree can be defined by a simple half saturation function (3.1) and that, comparable to potential height growth, trees in stands with usual conditions will demonstrate an extra height growth reduction over time ($t^{1+\alpha}$, where $\alpha \gg 0$) as the stand canopy closes and intra specific competition increases (Cieszewski and Bella 1989). In the same terms the model means also that SI differences, that arise between tree growth due to sites, occur in two ways. The upper limit of the height growth is proportional to the SI, and the initial growth rate increases[1] with SI in such a way that the age at which the tree will reach half of its maximum height is inversely proportional to SI.

The complexity of any fixed-base-age SI height model has to increase considerably after reformulation to a variable-base-age form that represents a four-dimensional height space. After deriving equations describing the four dimensional height spaces with polymorphic growth patterns and variable asymptotes (Fig. [3.6]) any further solutions for implicit or explicit constants are practically impossible due to the complexity of these equations. For example, eq. (3.5) describes height growth in terms of SI which allowed for easier interpretation of the underlying biological principles governing the relation between the height growth polymorphism and site productivity. A height model would be convenient if formulated to use height at any age. Solving Equation (3.5) for SI as a function of height h_x at age x results in:

$$SI(h_x, x) = 0.5\left[(h_x - \beta/50^\alpha) + \sqrt{(h_x - \beta/50^\alpha)^2 + 4\beta h_x/x^\alpha}\right] \tag{3.7}$$

where only positive roots are considered.

Substituting right side of eq. (3.7) in place of SI in eq. (3.5) gives the Variable-age-SI height model as:

$$H(t, h_x, x) = \frac{h_x + \beta/50^\alpha + \sqrt{(h_x - \beta/50^\alpha)^2 + 4\beta h_x/x^\alpha}}{2 + \frac{4\beta/t^\alpha}{h_x - \beta/50^\alpha + \sqrt{(h_x - \beta/50^\alpha)^2 + 4\beta h_x/x^\alpha}}} \tag{3.8}$$

where:

α and β are estimable coefficients;

h_x is a tree reference-height;

x is the tree base-age, or the age of h_x; and

t is the prediction-age for the computed height.

Using eq. (3.5) to derive a variable-base-age SI height model results in the model (3.8) that cannot be further modified through using a closed form solution to any of its elements.

3.1.2 Expanding Into the Fifth Dimension

The variable-base-age SI height models do not contain any explicit fixed measurement components. However, just like all other height over age height models, the variable-base-age SI height models also contain various implicit fixed components, other than the base-age. Some of those components are: a base-height (defining the beginning of the age count), density, elevation, average temperature, pollution level, etc. These implicit model components and their effects on height predictions depend on the species growth characteristics and a sensitivity of its height growth to each of these elements.

Specific values of model implicit components should usually depend on data used to calibrate the height models. These values would likely affect the shape of the curves distinctly for each component, be it base-height (Fig. [3.7]) or density (Fig. [3.12]). Depending on which of those components is the most important (either base-height or a density measure) in a model application, it might be advantageous to choose one of the implicit fixed components and provide for it by an inclusion of an explicit variable measurement component.

Since the variable-base-age SI height models cannot be further modified through algebraic reformulations, other possibilities have to be explored. Shapes and patterns resulting from diverse values

[1] It increases until the half saturation point, then decreases past the time.

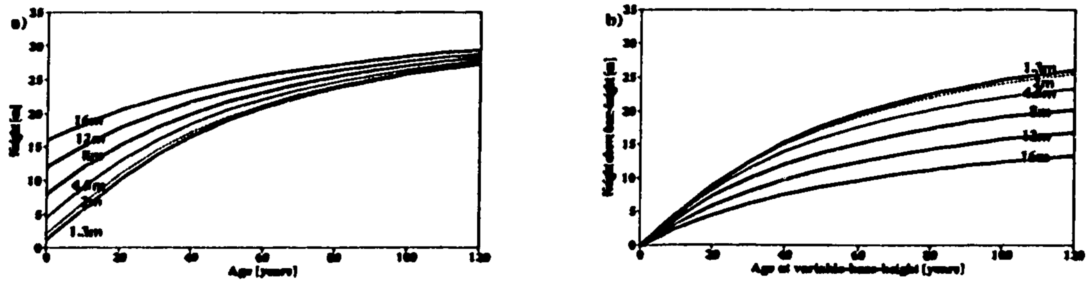

Figure 3.7: Shapes of height trends on medium site for varying base-heights: a) starting at base-height; b) shifted to origin.

of the implicit components (Fig. [3.7]) can be modeled individually using different models. Given an adequately flexible equation, the above changes in height patterns could be expressed by a single model with different sets of coefficients. It is reasonable to assume that if the same equation with different coefficient values could describe different height patterns, resulting from changes in values of an implicit fixed measurement component, the changes in the coefficient values would be in some sort of functional relation with the changes in the measurement components, i.e., the coefficient values would be changing consistently and continuously over a range of the measurement component values. Consequently, the model can be modified to replace an implicit fixed growth component, e.g., base-height or density, with an explicit variable measurement component, i.e., base-height or density, by replacing this model's coefficients with appropriate modifying functions of that measurement component, i.e., base-height or the density.

The new technology for deriving variable-base-age *SI* height models (Chapter 2) can be applied to most basic equations to derive biologically sound polymorphic variable-base-age *SI* height models with variable asymptotes. These models can in turn be modified by imposing modifying functions on their coefficients, without changing their basic structure, as the variable-base-age *SI* height models, just like all other models, generate shape patterns that depend on the model coefficients. Using model (3.8) for that purpose would give a new expanded model with three adjustable measurement components of reference-height and its age, and base-height, or density/crowding measure, that would have the following form:

$$H(t,h_x,x) = \frac{h_x + \frac{f_2(h_b)}{50^{f_1(h_b)}} + \sqrt{(h_x - \frac{f_2(h_b)}{50^{f_1(h_b)}})^2 + 4f_2(h_b)h_x/x^{f_1(h_b)}}}{2 + \frac{4f_2(h_b)/t^{f_1(h_b)}}{h_x - \frac{f_2(h_b)}{50^{f_1(h_b)}} + \sqrt{(h_x - \frac{f_2(h_b)}{50^{f_1(h_b)}})^2 + 4f_2(h_b)h_x)/x^{f_1(h_b)}}}} \qquad (3.9)$$

Applying the technology from Chapter 2 to the base model (3.2) would result in a model more complex than model (3.3) with three estimable parameters. This would add to the model flexibility but it could make a further application of this model more difficult while not contributing to the demonstration purposes. However, this option will exist should model (3.8) prove to be insufficiently flexible.

This study describes an example of a modification of a variable-base-age *SI* height model that replaces implicit fixed input components of base-height by an explicit variable measurement component. The relevant model modifications will be performed on a variable-base-age *SI* height model (3.8) (see Chapter 2). The example is broadly applicable to virtually any height model, and it deals with the implicit input component of the base-height, i.e., the height defining the beginning of an age count. This base-height usually is equal to 1.3 m above ground and is referred to as the breast height, while the age beginning at this height is the breast height age.

Another example is mentioned in this section while treated in detail in the next section. This second example might not be broadly applicable for many models or species, but it is very important for lodgepole pine and it deals with the implicit fixed density input in height models developed for lodgepole pine—a species of exceptional sensitivity to stand density. Both of the examples follow a similar conceptual process of model development mentioned above, i.e., developing height models with coefficients expanded to modifying functions of the variable measurement components.

3.2 Adjustable SI, Base-age and Base-height Components — The Ageless Height Model

3.2.1 The Definition of Prediction-Age for Different Base-heights

Most height models are developed as time series whereby height is a function of time. For simplification, any seasonal variation in growth is ignored and different measures of time are in use. Most of the time measures are referred to as ages of different base-height, and they describe the time needed by a tree or stand to reach a specified height since the time when the tree, or stand, has been "base-height" high. If the base-height is variable, i.e., it can be arbitrarily changed by the user, the model can be effectively considered age-free, or ageless. An example of such ageless growth model is Von Bertallanfy's (1938) (Turnbull and Pienarr 1973) differential growth function: $\partial V = nV^m - kV$ where: ∂ is a slope of V; n is anabolic constant; m photosynthesis exponent; and k is the catabolic constant.

In most of the integral height models, the ages of different base-heights are given adjective names added to the word "age" that describe a common definition of the the base-heights used for the age definition. There are three prevalent ages and associated with these heights: 1) Total age, at base-height 0 usually equivalent to the length of time since the germination of the tree or stand; 2) stump age, defined by stump height 0.3 m; and 3) breast height age, defined by breast height 1.3 m. Other ages can also be used to denote either a prediction time or a time definition of a height reference measure, i.e., an age of the reference-height.

The most traditional age used for development of height models was the total age, and great many models have been developed using this age (Frothingham 1914, Gevorkiantz 1930, 1957, Vimmerstedt 1962, Lundgren and Dolid 1970; Carmean 1972, Payandeh 1974a, 1974b, Monserud and Ek 1976, Burkhart and Tennant 1977, Bruce 1981, Borders *et al.* 1984, Bailey and Clutter 1974). However, since there is a great amount of variation in height growth in an early life stanza of trees and stands before they reach 1.3 m of height, due to a variety of causes from weed competition to animal grazing; some authors (Husch 1956, McCormack 1956) argued that one should use the breast height age for height modeling. Following these, many modern height models have been formulated as functions of breast height age using either a fixed age SI (Barrett 1978, Monserud 1984, 1985, Curtis *et al.* 1974, and Newnham 1988), or a variable-base-age SI (Clutter *et al.* 1983, Cieszewski and Bella 1989).

3.2.2 The Definition of Base-Age for Different Base-heights

The age consideration and associated errors are especially important in juvenile height modeling. Any errors resulting from early growth irregularities have a relatively high impact on growth predictions for juvenile stands due to relatively low ages in those stands. The considered errors in this situation are most important if they are contained in the age of reference-height, or SI, used to describe a site productivity. Thus, the traditional height models, using total age and a fixed age SI, are generally considered unsuitable for juvenile height modeling (Wakeley and Marrero 1958, Ferree *et al.* 1958, Day *et al.* 1960, Brown and Stires 1981, Carmean 1975).

The fixed age SI is also unsuitable for application in juvenile stand height modeling. Measuring of reference-heights in close spacing presents many practical difficulties. On the other hand, counting only several internodes above the breast height should be easy whenever the internodes are regular (Baldwin 1931, Brewster 1918, Pearson 1918). Exceptions from that could apply to southern pines that are multinodal (Shaw 1914) and produce multiple annual internodes and branch whorls

(Reed 1939). Despite those exceptions, most researchers agree that for juvenile height modeling the definition of SI and the base-height used for measuring the SI height and its age should be altered to give more flexibility towards being able to define growth potential using a variable time length growth measured at heights convenient to access. On these premises, a whole class of intercept based height models have been developed for different juvenile height growth applications (See Sub-section 1.2.3).

The intercept height models, using height growth intercepts starting at breast height, are principally not much different from the fixed age SI height models with a relatively small base-age, e.g., 5 years. They also use a reference-height above breast height at a fixed base of breast height age. In fact, they could be considered as special cases of the variable-base-age SI height models with the base-age set to the number of internodes measured above breast height used to compose the growth intercept.

Other intercept height models, using intercepts starting further up the stem above breast height, are quite unique in their class. However, the uniqueness can be viewed as limited to different base-heights used to define the age of the reference-height. It is somewhat erroneous that such models use inconsistently two different base-heights for definition of the two ages used in the height models, i.e., the prediction-age and the base-age, even though the base-age is usually implicit. Also, it can likely be error prone that such models use inconsistently the base-height in definition of the beginning of the reference-height or intercept; as this beginning is usually defined in terms of a number of internodes above breast height. Thus, the height at which the intercept is measured will vary between trees with the usual variation in growth.

Summarizing the above, one can see that many efforts in height modeling concentrate on exploring the possibility of substituting for site-indexing different lengths of height growth measured at different heights. Both an operational forester and a researcher could use it whenever they run into situations where it is best to measure growth intercepts on various heights on different trees. To achieve such model flexibility, one needs to extend a variable-base-age SI height model to a fifth dimension with a variable base-height.

3.2.3 Implementation and Interpretation of The Variable Base-Height

Equation (3.9) is a generalized representation of five-dimensional height space above the base-height. To make this equation applicable for variable base-height implementation, the modifying functions have to describe functional relations between the coefficients changes and the base-height, while the base-height has to be extracted from the element of the variable growth patterns in a form of an intercept. With these changes the equation will take the following form:

$$H(t, h_{x_b}, x, h_b) = \frac{h_{x_b} + \frac{f_2(h_b)}{50^{f_1(h_b)}} + \sqrt{\left(h_{x_b} - \frac{f_2(h_b)}{50^{f_1(h_b)}}\right)^2 + \frac{4 f_2(h_b)(h_{x_b})}{x^{f_1(h_b)}}}}{2 + \frac{4 f_2(h_b) t^{-f_1(h_b)}}{h_{x_b} - \frac{f_2(h_b)}{50^{f_1(h_b)}} + \sqrt{\left(h_{x_b} - \frac{f_2(h_b)}{50^{f_1(h_b)}}\right)^2 + \frac{4 f_2(h_b)(h_{x_b})}{x^{f_1(h_b)}}}}} + h_b \tag{3.10}$$

where:

t is the prediction age;

h_b is variable base-height;

x is the base-age of the reference-height (SI) h_x measured from ground up; and

h_{x_b} is a difference between total reference-height at base-age x and the base-height h_b, i.e., $h_{x_b} = h_x - h_b$.

The above model can simulate changes in height patterns above any base-height. Below the base-height the model is assumed to be constant and cannot describe any patterns; model (3.9) cannot be used to estimate height below the base-height. This is similar with all other height models—just like any breast height age based height model cannot estimate heights or ages below 1.3 m. The interpretation of ages at variable base-height is similar to the interpretation of stump age and the breast height age, i.e., an age base-height h_b is a number of years since the tree, or stand, had reached the height h_b. For example, when a tree is 1.3 m high its breast height age is zero, and when it is 5 m high its base-height 5 m age is zero as well. Consequently, a tree that has 15 internodes starting at and above 2 m would be 15 years old base-height 2 m and the model could simulate the tree's height growth pattern between the 2 m height and the top of the tree. If the 15 internodes had total length of 8 m, the tree could be described as reference-height 10 m at base-age 15 years, base-height 2 m. This could be equivalent to reference-height 10 m at breast height age 17 years if it had taken the tree 2 years to grow from 1.3 m to 2 m, or it could be equivalent to reference-height 10 m at total age 25 years if it took the tree a total of 10 years to grow from the ground level to 2 m height.

It was implied above that the measure of the tree reference-height would not be affected by the changes in base-height. A tree of reference-height 15 m should always be reference-height 15 m regardless what base-height is used to describe the growth of this tree. However, this would not be practical if such *SI* were applied directly for height prediction. If a tree growth is disturbed during life, the total height over age relation may be drastically changed and it may no longer reflect actual site potential. For this reason it is an important advantage of model (3.9) that it actually applies only the portion of the reference-height that is above the base-height for defining the future growth. For example, a *SI* 10 m at base-age 5 years and base-height 8 m means that only the last 2 m of growth during last 5 years will be defining the future growth pattern. Even though a total growth of the tree or stand during last 25 years could be only 3 m, the fact that it grew 2 m in the last 5 years would take precedence over the disturbance.

As any variable base-age *SI* height model, model (3.9) uses any height at any age as *SI*. At the same time the age of the reference-height can be measured from any base-height and therefore, the model gives virtually unrestricted flexibility in what part of a tree/stand growth is used as input, i.e., the model is effectively ageless in reference input and it can predict future height from any time length height increment measured at any height.

3.2.4 Data

Model (3.9) can be calibrated on the same data as any other height over age model, disregarding the implicit conditions that the data may represent, e.g., an average density. Moreover, the model could also be calibrated on many of the data that is normally rejected in a process of data screening due to such problems as early growth suppression, broken tops, and different periodic growth disturbances from insects and diseases. When computing statistical analyses, one should be aware that the number of observations for any age class will be usually decreasing with age for all continuous time series. Consequently, the model may have different statistical properties for different age classes. Broad cross sectional representation of time series from different age periods may help to remedy this problem if different site classes are well represented.

Advanced statistical treatment of the model and investigation of its properties and implications from different fitting techniques were beyond the scope of this study. Also, no attempt was made to actually calibrate this model on actual data from Alberta and test its performance in comparison to other models. The sole intent of this study was to present the theoretical potential of the model application for prediction of height, with limited information on the growth, potential reference such as a periodic growth increment with any height at which it occurred, without actual age information.

The purpose of this research was to demonstrate the algebraic capability of the ageless height model to simulate periodic growth patterns using only height increments measured at different base-heights. The model (3.9) was test calibrated on artificial data generated by a polymorphic variable-base-age *SI* height model for lodgepole pine in Alberta (Cieszewski and Bella 1989).

The data consisted of predictions of height for 32 ages (0.01, 5, 10, 20,..., 290, and 300 years), seven base-heights, and five site classes of 5, 10, 15, 20, and 25 m *SI* at base-age of 50 years breast height (Fig. [3.8] a–c). The total number of observations was $32 \times 7 \times 5 = 1120$. There were seven time series of 32 observations for each site class and five series per each base-height. The ages at different base-heights were calculated as breast height age of a predicted height minus breast height age at the new base-height.

One might notice that some of the 300 years ages at higher base-heights and lower sites would imply unreasonably high total ages to be biologically justified because lodgepole pine lives only about 400 years from germination. On a low site (5 m SI base-age 50 years breast height), it takes 260 years for a 1.3 m high tree to reach 16 m height (Fig. [3.8]a). By this time the tree is over 270 years of its physiological age. After adding another 300 years of growth above the 16 m (Fig. [3.8]f) the tree would have to be 570 years old, which biologically is impossible. However, this fact does not preclude experimental questions on the model application and its calibration on such data for theoretical purposes.

While in this study the biological significance of the model was not important for a real application of an ageless model, since the total age of similar predictions might be constrained to a maximum total age of 400, or less. In using real data this problem would not occur, as no such data would exist, therefore it is of no practical concern.

For the purpose of the model calibration, all of the generated height values and ages at different base-heights have been shifted to the origin of the coordinate system. There was no advantage in calibration of the model with its intercepts above zero because any changes in coefficients would affect only shapes of the part of the curves above the intercepts. It was more effective to calibrate eq. (3.9) rather than eq. (3.10) as a simpler equation; model (3.10) can be formulated by just adding the intercept term of base-height to the calibrated curvature equation (3.9).

After shifting all the height series to the origin (Fig. [3.9]) new SI values had to be calculated appropriately for each base-height by subtracting the base-height from the original SI values used to generate the series. This was needed because the new series had reflected height as a function of ages at changed base-heights, and had different slopes then the original series (Fig. [3.9]). The higher the base-height of an age, the lower the SI that the height will reach on a given site, which was defined by the original site indexes used to generate the series; increase in the base-height reduces growth potential as a functions of age and its new SI.

The final data for five sites and seven base-heights included: 1) base-heights; 2) base-height ages; 3) heights above base-height; and 4) SI above base-height at 50 years base-height.

The original site indexes at 50 years breast height were not needed in the data set even though all the sites were represented, because this representation became implicit to the new explicit site indexes at 50 years of different base-heights. The new site indexes above base-height represented an extended range of values between 0.88 m, for the lowest site and greatest base-height (16 m), to 25 m for the highest site and smallest base-height (1.3 m). The highest SI at base-height 16 m was 14.3 m above the base-height (16 m), i.e., total height of 30.3 m, and it was reached at 72 years breast height.

3.2.5 Model Development and Calibration

Individual Parameter Fitting

Since eq. (3.9) describes a very complex nonlinear system that is difficult to fit and may have many local minimums, initially a simpler model (3.8) was fitted to the above described data individually for each set of heights computed for one base-height. This method, known in forest biometrics as a parameter prediction method, greatly simplified the process, giving point parameter estimates for discrete values of the base-heights. Estimates obtained through the parameter prediction method are particularly reliable for the type of data sets as used in this study, that were generated from another model and typically do not contain any appreciable variation.

Estimates of the parameters α and β in model (3.8) were to be used for modeling the modifying functions $f_1(h_b)$ and $f_2(h_b)$ in model (3.9), as functions of the independent variable base-height (h_b). Given the high regularity of the pattern changes in simulated data, the obtained point estimates at different values of the base-height, in model (3.9), can be used to define functional forms of the modifying functions to be imposed on the original model coefficients.

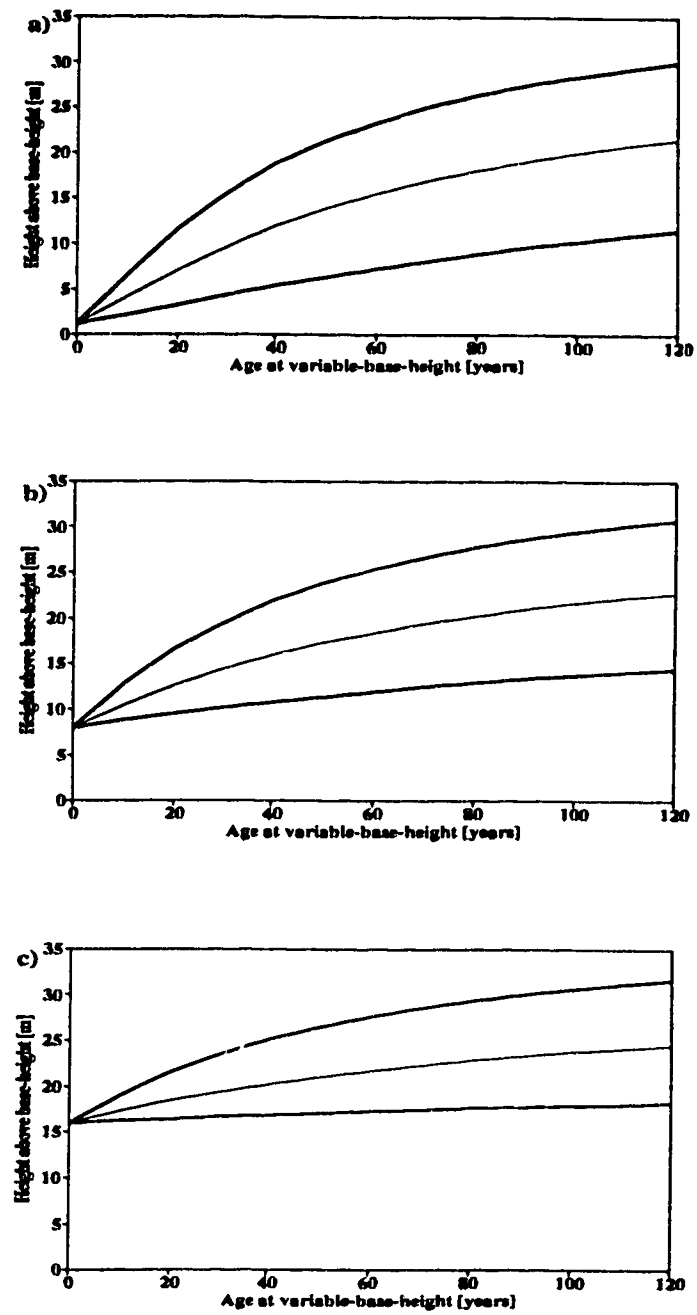

Figure 3.8: Heights for 3 sites (Cieszewski and Bella 1989), for ages 0 to 300 years base-heights: a) 1.3, b) 8, and c) 16m.

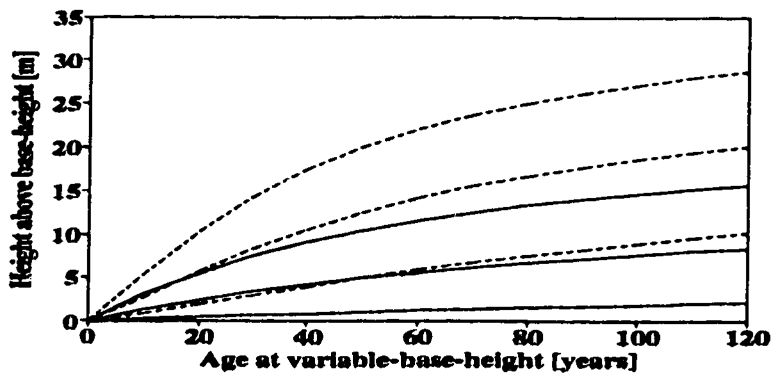

Figure 3.9: Height patterns above the base-heights, shifted to origin of the system of coordinates to easier illustrate differences in slopes of curves. Broken lines are for base-height 1.3m and solid lines are for base-height 16m.

Formulation of The Modifying Functions

The values of both parameters α and β, obtained from the above fitting, were decreasing with increasing values of base-height (Fig. [3.10]a,b). This would suggest nonlinear functional relations between the parameter values and the base-height values. The degree of the decline in the parameter values, with the increasing base-heights, was different for α than it was for β and therefore two separate models had to be found for the parameter modifying functions.

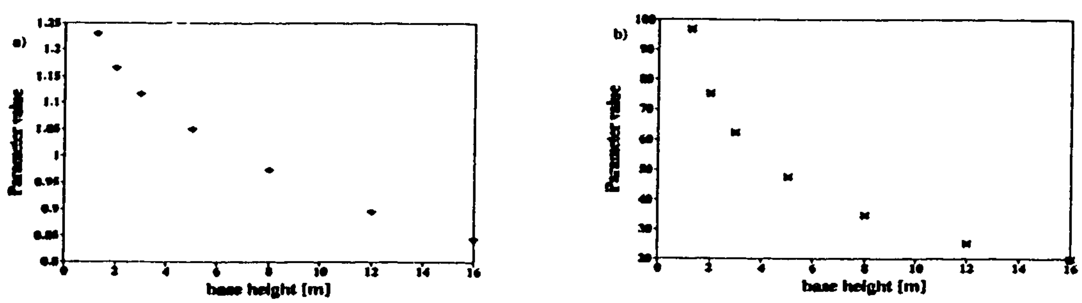

Figure 3.10: Values of the modifying functions of the variable-base-height height-growth model, at different base-heights: a) $f_1(h_{b_x})$ for α; b) $f_2(h_{b_x})$ for β.

After exploratory analysis of reciprocal trends, log and exponential functions, fractional functions, and combinations of all the above, two relatively simple equations were developed that very closely described the changes in the model coefficients with changing base-heights

$$f_1(h_b) = \frac{\alpha'}{1 + \alpha''\sqrt{h_b}} \tag{3.11}$$

$$f_2(h_b) = \frac{\beta'}{1 + h_b^{\beta''}} \tag{3.12}$$

where:

$f_1(h_b)$ and $f_2(h_b)$ denote the modifying functions of model (3.9) that can be used in place of parameters of model (3.8) calibrated for different base-height ages;

α', α'', β', and β'' are parameters of the parameter modifying functions in model (3.9);

α' and β' are upper limits of the parameters α and β in model (3.8) that can be reached when the base-height is zero, i.e., the model describes height as a function of a total age;

h_b is base-height that can assume any positive value.

The modifying functions (3.11) and (3.12) give good fit to the parameter values that are changing with base-heights (Fig. [3.11]), and have well defined and stable coefficients (Table [3.1]). The deviations in predictions of the coefficient values were within a normal range of the "coefficient standard errors" in estimation of height models (Cieszewski and Bella 1989). The α parameter's standard error of the estimate was some 30% lower than the standard error of estimate of the model calibrated on stem analysis data. The β parameter's standard error was 300% lower than the standard error of β estimate for the model calibrated on stem analysis data (see Cieszewski and Bella 1989, Table 2.4.)

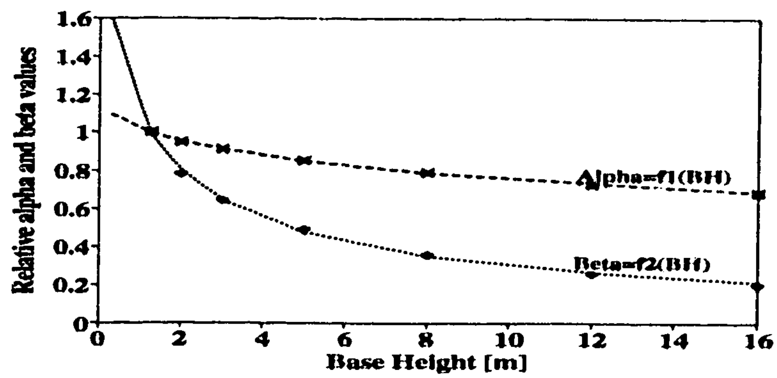

Figure 3.11: Observed and predicted relative values of the ageless height model parameters as functions of the base-heights: a) $f_1(h_{b_x})$ for α; and b) $f_2(h_{b_x})$ for β.

Table 3.1: The variable-base-height model coefficients (α', α'', β' and β''), and standard errors of predictions for the two modifying functions.

Coefficient	Estim.	SE	t-ratio	Res.SD
α'	1.4879	0.0095576	155.68	
α''	.18947	0.0041626	45.517	0.005184
β'	213.35	2.6160	81.558	
β''	.80161	0.014585	54.961	1.287556

The Final Form of The Ageless Height Model

After applying to eq. (3.10) the functional forms of eq. (3.11) and (3.12) and the coefficient values listed in Table [3.1], the ageless lodgepole pine height models for Alberta will have the following form:

$$H(t,h_x,x) = \frac{h_{x_b} + \frac{\beta'/\left(1+h_b^{\beta''}\right)}{50^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}} + \sqrt{\left(h_{x_b} - \frac{\beta'/\left(1+h_b^{\beta''}\right)}{50^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}}\right)^2 + \frac{4\beta' h_{x_b}}{\left(1+h_b^{\beta''}\right)x^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}}}}{2 + \frac{4\beta' t^{-\alpha'/(1+\alpha''\sqrt{h_b})}\left(1+h_b^{\beta''}\right)^{-1}}{h_{x_b} - \frac{\beta'/\left(1+h_b^{\beta''}\right)}{50^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}} + \sqrt{\left(h_{x_b} - \frac{\beta'/\left(1+h_b^{\beta''}\right)}{50^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}}\right)^2 + \frac{4\beta' h_{x_b}}{\left(1+h_b^{\beta''}\right)x^{\frac{\alpha'}{1+\alpha''\sqrt{h_b}}}}}}} + h_b \qquad (3.13)$$

where α', α'', β', and β'' are the model estimable coefficients and h_{x_b} is the reference-height above the base-height ($h_x - h_b$).

Global Fitting

When calibrating the modifying functions (3.11) and (3.12) separately from the complete model (3.10), using the parameter prediction approach, one may expect some statistical inconsistencies resulting from the fact that the coefficients will have elliptical confidence intervals. Problems associated with these inconsistencies may be easily avoided by refitting the whole model (3.13) directly to height growth data, or even to the coefficient values of α and β. The current functional forms of the modifying functions would most likely stay the same, while the coefficient values of α', α'', β', and β'' would provide excellent initial coefficient values for the nonlinear regression on this complex model.

Any problems with the coefficient elliptical confidence intervals would be more likely to have significant implications with real growth data, and these would likely depend on the amount of variation in height growth. When calibrating one model on simulated values obtained from another model, such statistical inconsistencies are not likely to have much influence on the model. Furthermore, the purpose of this study is to demonstrate the conceptual development of an ageless height model, which has been achieved, with the present model (3.13).

3.3 Adjustable SI, Base-age and Density Measure—A Static Density Height Model

Similarly as the implicit height measurement component of base-height can be changed from a constant to a variable, any other implicit height measurement component (density, elevation, etc.), inherent in a height model, can also be changed to a variable. An example of this is an implicit density or a crowding measure. When changing values, a crowding measure would affect the model predictions resulting in different height shapes (Fig. [3.12]).

The implicit fixed density component is usually some average density represented by the sample data, and therefore an equation calibrated on two separate samples of height growth data from two different density levels would produce two different height patterns (Fig. [3.12]). This variation in growth patterns resulting from using different values of an implicit fixed density component is similar in principle to changes that occur as a result of changing the base-height—an equation calibrated on two similar data sets, one using total age, and the other using breast height age, will produce two different height patterns (Fig. [3.8]). This means that a variable density height model could follow a similar general development framework as any other five dimensional height model whereby the coefficients of model (3.8) are altered by modifying functions of an explicit variable. Thus, given that a static density or crowding measure is described by a crowding index CI, the conceptual static variable density height model could have the following form:

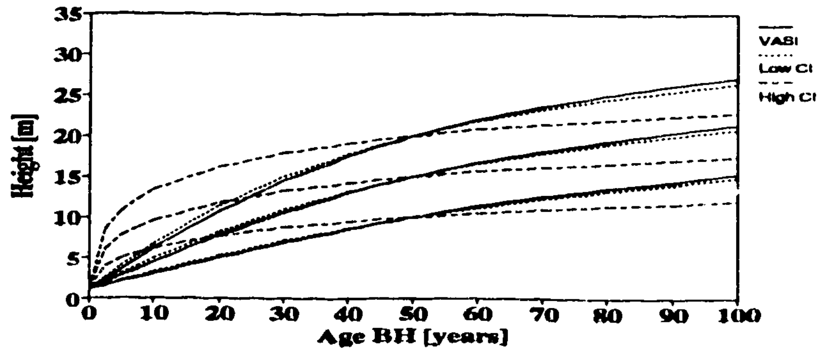

Figure 3.12: Height shapes corresponding to different crowding levels.

$$H(t,h_x,x) = \frac{h_x + \frac{g_2(CI)}{50g_1(CI)} + \sqrt{(h_x - \frac{g_2(CI)}{50g_1(CI)})^2 + 4g_2(CI)h_x/x^{g_1(CI)}}}{2 + \frac{4g_2(CI)/t^{g_1(CI)}}{h_x - \frac{g_2(CI)}{50g_1(CI)} + \sqrt{(h_x - \frac{g_2(CI)}{50g_1(CI)})^2 + 4g_2(CI)h_x/x^{g_1(CI)}}}} \quad (3.14)$$

$g_1(CI)$ and $g_2(CI)$ are coefficient modifying functions of the variable CI; and all other symbols are the same as in previous equations.

In the presented form, model (3.14) can represent only static variable density scenarios in which density, or at least some sort of crowding measure, does not change over time. Construction of a suitable, relatively stable over time crowding index is a challenge in itself. Furthermore, eq. (3.14) represents only a basic conceptual form of a variable density height model. The complete development of this model would require exploratory analysis towards identification of the actual algebraic forms of the modifying functions $g_1(CI)$ and $g_2(CI)$, and extensive analysis towards calibration of the model.

3.4 Discussion

General Comments

Model (3.13) represents five-dimensional height space (Fig. [3.13]), where the dimensions are:

- prediction height Ht;
- reference-height h_x;
- base-height h_b;
- prediction age t_{h_b} base-height h_b; and
- reference age x_{h_b} base-height h_b.

The presence of the base-height, h_b, in this model introduces an unprecedented flexibility in the model's applications. The model can be considered practically age free, because the definition of age is so flexible that any time length of the tree growth/increment starting at any time and finishing at any other time can be interpreted as age of a given base-height.

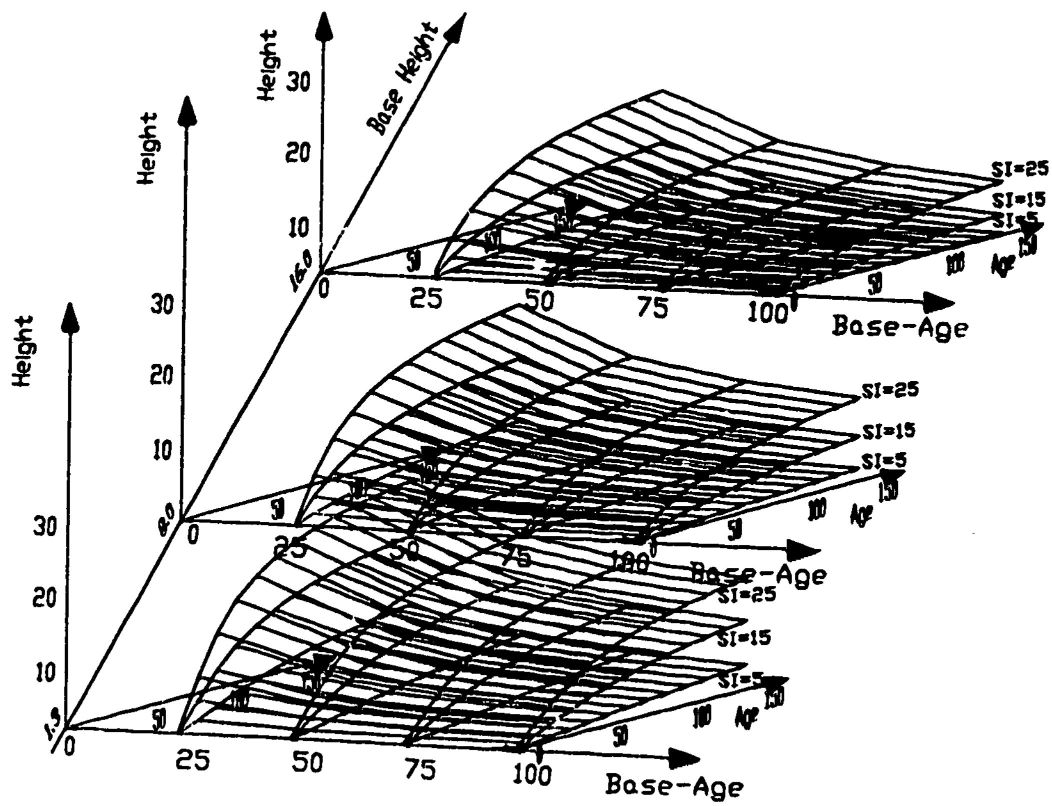

Figure 3.13: A five-dimensional, variable-base-height and variable-base-age SI, height-growth space.

Advantages in Operational Applications

In applications to remeasured plots, the model can be used to model future growth of trees or stands based on subsequent measurements without knowledge of the actual age of the trees or stands. In juvenile stands, the model can be used as an intercept model of any base, i.e., any number of internodes starting at any height can be used as SI, driving the model's future predictions. In fact, model (3.13) constitutes a general platform, unifying all existing intercept models in terms of the flexibility of the length of growth intercept and its location on the stem. It can also likely improve on some existing models by making them more precise, because the intercept models that define a beginning of intercept with an internode above or below a breast height, will actually use a growth intercept located on a different height for virtually every tree.

Depending on the chosen base-height, the proposed model simulates different height patterns for the same lengths of the active reference-heights (Fig. [3.14]). These growth patterns have diminishing slopes with increasing base-height. They are unique for each base-height that is corresponding to the specific growth characteristics associated with the given base-height defining the age. The variable base-height model (3.13) resolves the arguments, with respect to age and SI, on modeling strategies for height predictions, e.g., Husch (1956) and McCormack (1956).

Relevance to Traditional Height Models

Model (3.13) can be used as a traditional height model using a total age (Frothingham 1914, Gevorkiantz 1957, Gevorkiantz and Zon 1930, Vimmerstedt 1962, Lundgren and Dolid 1970, Carmean 1972, Payandeh 1974a and 1974b, Monserud and Ek 1976, Burkhart and Tennant 1977, Bruce 1981,

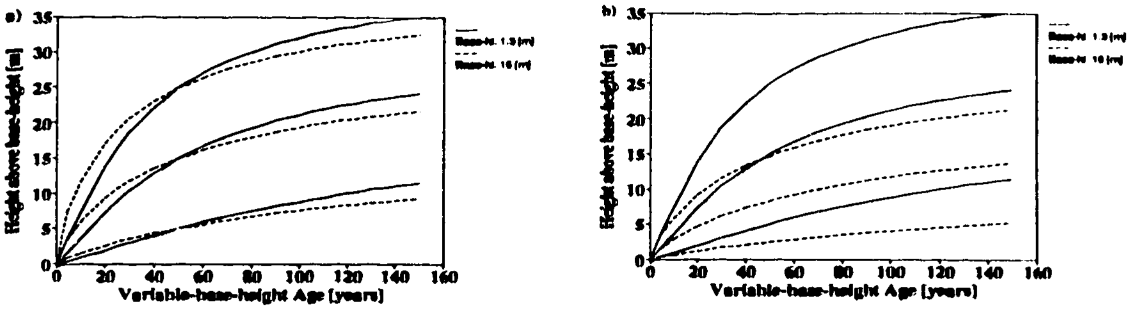

Figure 3.14: Shifted to origin height patterns for three reference-heights and two base-heights, for: a) base-age 50 years of either base-height; b) base-age 5 years of either base-height.

Borders *et al.* 1984, Bailey and Clutter 1974), if $h_b = 0$; or as the newer type height model using breast height age (Barrett 1978, Monserud 1984, Curtis *et al.* 1974, Newnham 1988) if $h_b = 1.3$; or as a stump age if $h_b = 0.3$; or even as any type of intercept model (Wakeley and Marrero 1958, Ferree *et al.* 1958, Day *et al.* 1960, Brown and Stires 1981, Carmean 1975) if h_b equals the exact height at which the internodes start being measured. In principle, the model (3.13) unifies all of the above outlined types of models into a common general model of five-dimensional height space. Different crossections of this space can represent any one of these models individually, as a special case corresponding to the model particular limited parameter values.

In juvenile stands where estimation of a fixed age SI is not possible due to close spacing (Wakeley and Marrero 1958, Day *et al.* 1960) the model can be used with measurements of variable number of internodes, whenever they are regular (Baldwin 1931, Brewster 1918, Pearson 1918), and these in addition can be taken at a different height for every tree. Any height of the beginning of the measured internodes, whether below or above breast height, can be assigned to h_b. At the same time, x can be any number of internodes, while any upper height limit of the measured internodes can be assigned to h_x, or any summed length of the yearly height increments can be assigned to h_{x_b}.

Advantages in Data Handling

In a traditional process of data screening for development of the traditional height models, all data from trees showing early growth suppression, leader damage, or some periodic growth suppression/disturbances from disease or animal grazing, or missing breast height, or total ages, have to be omitted. In traditional height modeling with limited data at times, some risky decisions may have to be made whether to retain some of the tree measurements of questionable value, e.g., those showing periodic growth disturbances, because a rigorous screening would not leave sufficient amount and/or ranges of data. If the former is the limiting factor, one may be forced to sacrifice some data quality for data quantity. While data collection cost can easily exceed budgets, a too tolerant data screening may seriously bias the model fit.

The traditional height models, whether based on site indexes or intercepts, do not offer much flexibility in data handling and selection of best growth intervals. The model presented in this study offers a new possibility in data screening and selection of best parts of height growth. Taken to an extreme, the model could be calibrated on data that should normally be rejected due to any number of reasons, by simply taking the healthy unsuppressed growth sections of the tree and their respective variable-base-height ages. In a more moderate approach, one could use somewhat more rigorous screening criteria that rejects all disturbed growth sections while retaining the "good" growth sections, even if both of these sections are part of the same tree. Using the variable-base-height model there is no need to give up data from trees with missing breast height or total ages, or with signs of early suppression, or any other periodic irregularities, as long as certain parts of the tree can qualify for contribution to the height modeling.

Computer Implementation

Despite the complex appearance of model (3.13) the implementation of this model is relatively simple. In a computer application written in FORTRAN, a basic code for this model could consist of only a few lines:

```
F1 = 1.4879/(1+0.18947*SQRT(Hb))
F2 = 213.35/(1+Hb**0.80161)*20
hR = hx-Hb+SQRT((hx-Hb-F2/50**F1)**2+4*F2*(hx-Hb)/x**F1)
H  = (hR+F2/50**F1)/(2+4*F2/t**F1/(hR-F2/50**F1))
```

where:

F1, F2, and hR are intermediate variables;

Hb is the base-height, or a height at the beginning of intercept;

hx is the reference-height, or a height of the end of intercept;

x is the base-age–base-height Hb, or a time period of intercept;

t is the prediction-age–base-height Hb, or the number of years for growth prediction above Hb;

A similar code can be written in any spreadsheet using only one column for the actual height prediction as a function of prediction-age and just three cells for the intermediate computations for given reference-height, base-age, and base-height.

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Chapter 4

Infinite-Dimensional Height Spaces with Three Adjustable Measurement Components—A Dynamic Variable-Density Height Growth Model

4.1 Introduction

[1]For many timber species, height of dominant and codominant trees (of a certain number of the largest trees per unit area, like Top Height - TH) is assumed to be independent of stand density over a wide range and is often used as a measure of site productivity. In a lP stand, however, density has a strong influence on all growth characteristics, although it is not known at what density this influence begins; height growth of lP may be reduced even in stands with 500 trees/ha at 90 years (Smithers 1961). This growth reduction can be dramatic in dense stands, although the height pattern in these stands is similar to that of open stands on less productive sites (R. Dempster, Personal communication, Edmonton, Alberta Jan. 1989). Thus, the height-based SI in lP stands is confounded by stand density.

Height models are often developed from stem analysis data (Biging 1985) because they provide a continuous record of tree height growth. Such data allow for either interpolation, or approximation of SI with reasonable accuracy (Newnham 1988). The underlying mathematical models can be either linear, or nonlinear. Nonlinear models are flexible, usually behave reasonably outside the data range, and are more likely to be biologically meaningful (Pienaar and Turnbull 1973).

Previous attempts to model density effect on lP height produced empirical, linear, static (integral form) models (Alexander *et al.* 1967, Johnstone 1976) with intrinsic problems. Some of these problems arise from the fact that the underlying height growth–density relationship is a dynamic process that changes during the life of the stand, although it may be negligible in very young stands before crown closure. Density effects increase in importance in young and intermediate aged stands as crowding intensifies until it reaches certain stability in mature stands. Although, traditionally, SI models have been used to describe height growth processes, such models are not suitable to account for density influence.

Height-growth prediction is essential in growth and yield estimations, and should be based on an unbiased and accurate height-growth model. A model of height growth response of lP to different densities, sites, and silvicultural treatments should be based on repeated measurements of permanent

[1]A version of this chapter has been published. Cieszewski & Bella 1993. Can. J. For. Res. 23: 2499–2506.

sample plots (PSP) representing various stand densities, ages and sites. An important feature of such a model, which was lacking in previous attempts (Alexander *et al.* 1967, Johnstone 1976), would be the ability to mimic diminishing effects of density on height growth below certain density thresholds.

The main objective here was to develop a relatively simple, theoretically based, difference model of IP height growth as affected by density.

This chapter presents a new approach to modeling such relationships in IP stands, based on an extension of Czarnowski's(1961) stand dynamics theory of height growth-density relationships. The model predicts annual height growth from the current year's height, age, and density. Simulating the dynamic interactions between height growth and density over time, that would require a mortality model estimating changes of densities with height growth and site conditions, was beyond the scope of this study.

4.2 Model Derivation

4.2.1 Base Model Selection

A new IP height model derived from a generic base equation by Cieszewski and Bella (1989) was selected for further modification to investigate density influence on IP height growth. This model, a dynamic equation, predicts a height growth at any age t, using as a reference a height h_x at any age x and thus, it is a variable age SI height growth model. Other characteristics of this model are that the curves always pass through SI at any base-age, the model is polymorphic with variable asymptotes, and it contains only two coefficients - the age exponent α and the half saturation coefficient β that is an approximate age when tree height reaches half of its potential maximum.

The age exponent α reduces height growth when it is positive, or it enhances height growth when it is negative. For $\alpha > 0$, larger α indicates slower height growth.

Decreases in β lowers future growth predictions, while an increase in β would enhance it. Including a reference point in the half-saturation function with an added age exponent (Cieszewski and Bella 1989) is, in a sense, a scaling of the curves up or down to pass through such a point. To illustrate this, one can construct two curves with two different β values, then scale down the higher curve with a smaller β so that it crosses the lower curve with a larger β at a reference point. Then, past the reference point the scaled down curve with smaller β should fall below the second curve with higher β.

4.2.2 Adding Crowding Effects

To model density-related height growth relationships one needs a suitable measure of crowding, or crowding index (CI), that unlike common measures of density, such as number of trees per ha, is independent of time and tree size. This index then needs to be incorporated into the base model.

Selecting a Crowding Index

In this study, the main criteria in choosing a crowding measure were: (i) simplicity of input requirements and algebraic expression; and if possible (ii) reasonable stability of the measure over time in fully stocked stands.

Potentially useful crowding measures (CM) can be adopted from the literature, e.g., crown competition factor - CCF (Krajicek *et al.* 1961), or derived from size (growth) and number of trees (NT) self-limiting relationships, viz., Yoda's *et al.* (1963) −3/2 power law, Reineke's (1933) density-quadratic mean diameter relationship, and Czarnowski's (1947) height-related stand dynamics theory, e.g., $CM_1 = \ln NTSize^{slope}$ $CM_2 = Constant - \ln Size^{slope} - \ln NT$ $CM_3 = \ln NT/\left(Constant - \ln size^{slope}\right)$.

The height growth related Czarnowski's (1947, 1961, 1978) stand dynamics theory with regard to height growth states that, in fully stocked stands, the product of squared stand height and number of trees per unit area remains constant during the stand's life. This product, after replacing stand height with TH, was chosen as the most convenient measure of crowding because of its algebraic simplicity, moderate input requirements, direct relation to height, and relative stability over time (Czarnowski 1947, 1961, 1978):

$$CI = TH^2 \times NT \times 10^{-4} \tag{4.1}$$

where 10^{-4} converts hectares into square meters.

Table 4.1: Regression statistics of future over current crowding measures over time in 70-year and older stands (n=268).

Bases of measure	Regression Statistics		
	Slope	SE	t-ratio
Height (CI)	1.0078	0.0077	1.0
Volume	1.0521	0.0067	7.7
Diameter	1.0473	0.0066	7.2
Crown (CCF)	1.0334	0.0064	5.2

To compare the stability and utility of eq. (4.1) with those of other crowding measures, i.e, volume and diameter based measures derived from the self-limiting relationships and CCF (Krajicek *et al.* 1961), simple linear regressions without constants were fitted on fully stocked stands, future CIs over current CIs (Tables 4.1 and 4.2). To ensure that the selected stands were at or near their maximum crowding levels, a subset of PSP data containing stands that were at least 70 years old (268 observations) was taken. These comparisons showed that eq. (4.1) provided the most stable crowding measure, based on the slope and its *t*-ratio (Table 4.1). This test had to be performed on stands that have most likely reached their maximum crowding levels. Lacking clear criteria when younger stands had reached this maximum crowding, only age criteria could be used to avoid including understocked stands in these tests.

The top height based CI used here is relatively stable in highly crowded stands (Czarnowski 1947, 1961, 1978; see also Table 4.1), although it may vary between stands, and it may be increasing or decreasing at any particulary time due to irregular mortality. In young plantations and spaced or thinned stands, CI will be increasing until the stand reaches its maximum crowding level, because these stands clearly must be below their maximum crowding level. CI maximum value for any particular stand may also be influenced by initial stand conditions and/or treatments. Testing CI stability in young natural stands would be inconclusive because of the uncertainty about the stand's position in relation to its maximum achievable crowding. However, one can expect that, on average, CI would be increasing in such stands, because even a small number of understocked stands in the sample might influence the results. Thus young stands were excluded from these CI stability tests.

Incorporating Crowding Index Into The Model

After selecting the crowding index, it was incorporated into the base-height growth model (Cieszewski and Bella 1989). This was done by replacing both α and β with suitable modifying functions to simulate the changes in height growth at different crowding levels.

In general, with increased crowding, i.e., CI, growth should decrease. To describe such growth response, α should also increase with CI. Thus, the modifying function of this measure, which replaces α in the half-saturation function with added age exponent (Cieszewski and Bella 1989), would likely have a positive first derivative.

Because increased crowding generally reduces growth, it may be considered in this context as a growth-delaying factor. To allow for a growth delay in the half-saturation function, β would have to increase with crowding. This density influence on height growth in the half-saturation function can be included by substituting β with a modifying function of CI with a positive slope.

In the SI height growth model (Cieszewski and Bella 1989), the functional interpretation of how crowding influences the model's predictions is changed by SI as a height-age reference point. Crowding and site quality have a direct but opposite influence on IP height growth; growth declines with increased crowding but increases with increased SI. This means that if crowding were used as the only variable to describe site in a height growth model—as, for example, in the half-saturation function with added age exponent—then a function of crowding with a positive slope should be used in place of α and β to predict greater height growth. Because SI is already in the SI height growth model, (modifying the asymptote and the half-saturation coefficient β), the interpretation of the modifying functions is reversed. Stands with higher crowding would indicate a better site than stands of the same height and age at lower crowding. In other words, to reach the same height at higher crowding, despite the density-related height growth suppression, the stand would have to grow on a better site. Likewise, the biological interpretation of crowding effect on height growth is reversed for the SI height growth model when compared to the half-saturation function. Since the coefficients in the SI height growth model have a different effect on height growth predictions due to SI, the modifying functions of CI that replace α and β would likely have negative slopes.

When both coefficients are replaced by decreasing functions of CI, and if actual crowding is reduced—for example by thinning—the values of α and β will increase, resulting in higher future growth predictions. When the decreasing functions of CI describing α and β are denoted as $f_1(CI)$ and $f_2(CI)$, the new density—SI–height growth model will have the following form:

$$H_t(t,SI) = \frac{SI + 20 f_2(CI)/t_{SI}^{1+f_1(CI)}}{1 + 20 f_2(CI)/\left(SI t^{1+f_1(CI)}\right)} \tag{4.2}$$

where all symbols are as previously defined.

Final forms of the modifying functions $f_1(CI)$ and $f_2(CI)$ in eq. (4.2) are not predefined because of the difficulty in anticipating the shapes necessary to have a suitable affect on the function's performance throughout the range of CI. Because the relationships between crowding and the coefficients α and β are unknown, the modifying functions should be flexible enough to assume a wide variety of shapes—from simple to complex—with robust conversions between different shape patterns. Thus, f_1 and f_2 were initially defined by the modified Schnute's (1981, 1990) function (Cieszewski and Bella 1992):

$$f(CI) = \begin{cases} \left[y_1^b + (y_2^b - y_1^b)\frac{1-e^{-a|CI-\tau_1|^c}}{1-e^{-a|\tau_2-\tau_1|^c}}\right]^{1/b} \\ \text{for: } a \neq 0,\ b \neq 0 \\ y_1 \left(\frac{y_2}{y_1}\right)^{\frac{1-e^{-a|CI-\tau_1|^c}}{1-e^{-a|\tau_2-\tau_1|^c}}} \\ \text{for: } a \neq 0,\ b = 0 \\ \left[y_1^b + (y_2^b - y_1^b)\left|\frac{CI-\tau_1}{\tau_2-\tau_1}\right|^c\right]^{1/b} \\ \text{for: } a = 0,\ b \neq 0 \\ y_1 \left(\frac{y_2}{y_1}\right)^{\left|\frac{CI-\tau_1}{\tau_2-\tau_1}\right|^c} \\ \text{for: } a = 0,\ b = 0 \end{cases} \tag{4.3}$$

where: a, b, c, y_1, and y_2 are estimable coefficients; τ_1 and τ_2 are arbitrary constants that can be set to subsequently lowest and highest value of CI observed in the data; and $|x|$ defines an absolute value of x.

This function can assume a variety of shapes, from linear nonasymptotic, to nonlinear asymptotic, and has been obtained from, and thus has similar properties to, Schnute (1981) function. The submodels, whether simple linear or complex nonlinear equations, can be identified during regression analysis through the significance of the final model coefficients $t_{ratio} = \frac{coef}{SE_{coef}}$ and a replacement of the model by a simpler form can be tested with an F-test, as in Schnute (1981).

4.2.3 Height Dynamic Equation Derivation

As stated before, a dynamic difference equation is needed to simulate height growth of trees at different crowding levels. Such an equation can be derived from eq. (4.2) if crowding is assumed constant for the growth trajectory. This assumption is reasonable either for fully stocked stands, or for short time periods during which crowding is practically constant. In this analysis, crowding value during any one year was assumed to be constant, even though it may be increasing in young, fast growing, understocked stands with virtually no mortality, as well as in stands that have recently undergone a considerable reduction in density. It may be decreasing in stands at the time of undergoing a drastic reductions in density.

If crowding is assumed constant, f_1 and f_2 can also be treated as a constants even though the values of the modifying functions will vary between different stands. Therefore, the solution for SI, which is required to derive a dynamic difference equation from eq. (4.2), can be derived as from the *SI* height growth model (Cieszewski and Bella 1989), and will have the following form:

$$SI(h_t, t) = 0.5\left[h_t - \delta + \sqrt{(h_t-\delta)^2 + \zeta h_t/t^{\iota}}\right] \tag{4.4}$$

where: h_t is an observed top height at breast height age t; $\delta = 20f_2(CI)/t_{SI}^{1+f_1(CI)}$; $\zeta = 80f_2(CI)$; and $\iota = 1 + f_1(CI)$.

To ensure that eq. (4.4) is biologically sound and will generate only positive SI values, only the positive roots were considered. Once the solution for SI is available, the right hand side of eq. (refeq:root) can be substituted for SI in eq. (4.2), and—after appropriate reformulation—the dynamic difference equation that predicts next year height from current year height, breast height age, and stand crowding can be written as:

$$H_{t+1}(h_t, t, CI) = \frac{h_t + \delta + \sqrt{(h_t-\delta)^2 + \zeta h_t/t^{\iota}}}{2 + \zeta/(t+1)^{\iota}\left[h_t - \delta + \sqrt{(h_t-\delta)^2 + \zeta h_t/t^{\iota}}\right]^{-1}} \tag{4.5}$$

where: $H_{t+1}(h_t, t, CI)$ is height at age $t+1$, predicted as a function of current height (h_t), breast height age (t), and CI.

If any understocked stands are included in the calibration data for model [4.5], the calibration should be performed on one-year increments because eq. (4.5) has been derived using an assumption of constant crowding. The modifying functions, f_1 and f_2, are expected to be identified from eq. (4.3) as simpler nonlinear, or even linear functions with a reduced number of estimable coefficients (Schnute 1981). The calibration of eq. (4.5) on one-year increments does not preclude the future use of the model for the simulation of dynamic systems in multiple one-year iterations, with a mortality model predicting changes in crowding.

4.3 Data Sources and Processing

In this analysis, two kinds of long-term growth data were used from a total of 402 PSPs (Table 4.2). First, data from essentially pure, natural, fully stocked stands that had at least 80% of IP by basal area. This data was collected by the Alberta Forest Service (AFS) and the Canadian Forest Service (CFS). The plots were located in the foothills of western Alberta, Sections B.19a and B.19c of the Boreal Forest Region (Rowe 1972), between Rocky Mountain House in the south, and Grand Prairie in the north. At plot examination, stand age varied between 19 and 152 years, and *SI* between 4.6 and 29 m (base-age 50 years breast height age; Cieszewski and Bella 1989) (Table 4.2). The AFS data contained up to two, and the CFSdata contained three remeasurements in intervals varying from five to 20 years.

The second data set came from CFS spacing trials in two locations; Gregg Burn (south of Hinton) and Teepee Pole Creek (northwest of Sundre). Both experiments had the same spacing design; plots at establishment had 100 trees each, and were replicated twice on three different sites of low, medium, and high productivity.

The Gregg Burn stands, originating after fire in 1956, were spaced to five density levels at age 7. This data set provided two to three remeasurements (Table 4.2), depending on early height growth, i.e., the time trees reached breast height. The Teepee Pole stands, also of fire origin, were spaced at age 25. These plots all had four remeasurements.

The measurements in the natural stands included diameter tally by species, and up to 30 heights per plot. In the spaced stands, all trees were measured for diameter and height, and their health status was noted. Stand densities and crowding levels were computed using all trees of all species that grew on plots counting the other species as IP.

The data processing included computations of the number of trees per hectare, top height, quadratic mean diameter, and various crowding measures computed from densities that were based on all trees on a plot, approximation of SI, and annual height growth of top height by height curve slope estimation at measurement times. Tree heights not measured were estimated from height-diameter curves for each plot. To approximate yearly top height increments at the times of initial measurements, individual nonlinear height over age curves were fitted to the top height values for each plot. Annual top height increments were computed from the individual curves at the points of measurement, as yearly increments following initial measurement of each period between two measurements. Hence, each plot contributed as many observations in the final data set as there were remeasurements on that plot. These annual increments were then added to the actual height measurements, thus approximating the following year height growth. Actual height measurements and the estimated heights in the following years were used to calibrate a yearly increment height growth model. In this model, the actual height measurements from each plot at $t_1 \dots t_{T-1}$—where t is stand age and τ is number of all measurements on the plot—are used as the independent variable h_t in eq. (4.5), and the estimated next year heights at $t_1 + 1 \dots t_{T-1} + 1$, are the dependent variable $H_{t+1}(h_t, t, CI)$ in the same equation.

4.4 Model Calibration and Testing

In fitting this annual increment density height growth model, two parallel procedures were used; (i) trend initiation through average guide points; and (ii) trend detection through a progressive expansion. Both procedures used maximum likelihood procedure for nonlinear regression estimation (White 1978) of height models predicting annual height growth increments explicitly by subtracting heights at age t from heights at age $t + 1$ with eq. (4.5), i.e., $dH(t, h_t, CI) = H_{t+1} - h_t$ (the error terms in this model were assumed to be normally distributed).

In the first procedure, a SI height growth model (Cieszewski and Bella 1989) was fitted to separate subsamples of data representing 30 plots with the highest, medium, and lowest CI values. The coefficients so obtained provided three approximate values of α and β to predefine the modifying functions f_1 (from αs) and f_2 (from βs) through the three classes of CIs for the SI height growth model. These coefficient values were then used in an interactive spreadsheet simulation to define the most suitable modifying functions f_1 and f_2. The coefficients of these functions were to then be used as initial points for the final fit of eq. (4.5) with the main data set. Threshold values of these coefficients, allowing the reductions of the function to a simpler form, can be recognised from their t-ratios, then with the help of F-test, one can replace the generic model by a simpler function (Schnute 1981). Although this fitting began with coefficients that defined eq. (4.3) as asymptotic nonlinear modifying functions, after convergence the final coefficients a, b, and c were close enough (judging from their relative standard errors) to values of 0, 1, and 1 respectively, so that they could be fixed to these values without appreciable loss in accuracy. Thus, the two modifying functions, with $a = 0$, $b = 1$, and $c = 1$, turned out to be linear: $f(CI) = y_1 + (y_2 - y_1)CI$ with $\tau_1 = 0$, and $\tau_2 = 1$.

The second procedure—progressive expansion—was started by fitting the variable age SI height growth model (Cieszewski and Bella 1989) to all PSP data (Table 4.3), using coefficients reported by Cieszewski and Bella (1989) as initial values. The next step used the results of this first fitting

Table 4.2: Summary statistics for natural and spaced stands, with SI base-age 50 years breast height (estimated from Cieszewski and Bella 1989). For the spacing trials below 50 years, SI values were extrapolated using the same model.

Characteristic	Statistics	Source			
		Natural stands		Spaced stands	
		AFS	CFS	CFS	C/S
				Teepee Pole	Gregg Burn
Plots	No.	268	74	30	30
Observ.	No.	521	222	120	83
Tree Age at breast-height	Avg	71.5	79.0	28.0	17.0
	SD	22.8	27.2	7.1	5.4
	Min	19.0	23.0	18.0	9.0
	Max	152.0	143.0	38.0	24.0
Top H (m)	Avg	19.1	17.9	9.4	6.1
	SD	3.7	4.2	2.2	2.3
	Min	8.8	5.9	4.5	2.1
	Max	28.5	27.2	14.2	11.0
Number of trees/ha	Avg	2747.9	2457.7	2743.2	25[illegible]3.0
	SD	1841.5	1566.2	2386.9	2298.5
	Min	415.2	580.7	308.9	324.3
	Max	12451.9	9834.8	7907.0	7907.0
Competition Index	Avg	85.8	66.2	24.0	9.3
	SD	29.7	21.6	22.0	10.7
	Min	13.4	19.6	1.7	0.2
	Max	202.1	139.2	99.2	50.2
Site-index (m)	Avg	15.0	13.1	12.9	11.9
	SD	3.3	2.76	1.8	2.7
	Min	5.7	5.2	8.5	4.6
	Max	29.0	18.4	16.8	16.8

as initial coefficient values in the initial fitting of eq. (4.2), which also included CI. The final fit in this procedure produced identical linear modifying functions as the previous procedure. In symbolic form these linear functions are: $f_1(CI) = \alpha' + \alpha''CI$ and $f_2(CI) = \beta' + \beta''CI$ where $\alpha' = y_1$ and $\alpha'' = y_2 - y_1$ in the modifying function describe α; and $\beta' = y_1$ and $\beta'' = y_2 - y_1$ in the modifying function describe β (see Table 4.3 for coefficient values).

To test whether the addition of modifying functions f_1 and/or f_2 was needed, three additional regression analyses were performed on model [4.5] with: i) no modifying functions; ii) only one modifying function describing changes in α; iii) only one modifying function describing changes in β; and iv) the complete model with both modifying functions describing α and β. To evaluate the resulting models, in addition to usual fit statistics such as standard errors and log of likelihood function, four model selection tests (see Judge *et al.* 1985, Judge *et al.* 1988, and White 1978) were applied. These were: i) Akaike (1969) Final Prediction Error (FPE) also known as Amemiya Prediction Criterion (PC); ii) Akaike (1973) Information Criterion (AIC); iii) Schwarz (1978) Criterion (SC); and iv) the log likelihood ratio test using for comparisons χ^2 at 5% level. All tests indicated that using two modifying functions produced a better model than one modifying function, or no modifying function (Table 4.3), while the first three tests indicated that using one modifying function produced a better model then no modifying functions (the likelihood ratio test indicated that for α but not for β). All test indicated that between the two models with only one modifying function, either for α or β, the

Table 4.3: Model coefficients (α', α'', β', and β''), autocorrellation coef. (ρ), error (SE), log likelihood function (LLF), final prediction error (FPE), Akaike (1973) information criterion (AIC), Schwarz (1978) Criterion (SC), and the log likelihood ratio test (LLR) (Judge *et al.* 1988) with corresponding χ^2 values at 5% level, for exploratory runs of the linear form modifying functions.

Run	Coefficients estimates							Model selection tests			
No.	α'	α''	β'	β''	ρ	SE	LLF	FPE	AIC	SC	LLR vs. χ^2
1	.15	N/A	77.3	N/A	.379	.0671	1212	.00454	-5.39	-5.36	38 > 6.0
2	.27	-.0029	66.6	N/A	.390	.0663	1225	.00441	-5.42	-5.39	12 > 3.8
3	.16	N/A	72.3	.090	.383	.0671	1213	.00453	-5.39	-5.37	36 > 3.8
4	.37	-.0043	93.0	-.36	.330	.0658	1231	.00435	-5.43	-5.41	N/A

former gave a better model.

Data from repeated measurements, such as used in this analysis, are prone to show autocorrelation when used in regression analysis. To find out if indeed a significant autocorrelation was present in the analysis, a simple linear regression was performed on the residuals over their respective lag values, i.e., residuals from predictions for current over preceding measurements (Judge *et al.* 1985): $Residual_T = \gamma + \delta Residual_{T-1}$ where T is a subsequent measurement number, and γ and δ are linear coefficients. This indeed showed a significant positive autocorrelation ($R^2 = 13\%$, see Fig. 4.1a). To account for, or remove, the effect of this autocorrelation, the Pegan's (1974) technique (White 1978) was applied with nonlinear regression analysis to derive new corrected model coefficients. The new model so obtained resulted in residuals that did not show significant autocorrelation (Table 4.4 and Fig. 4.1b), nor were these residuals correlated with either age or CI.

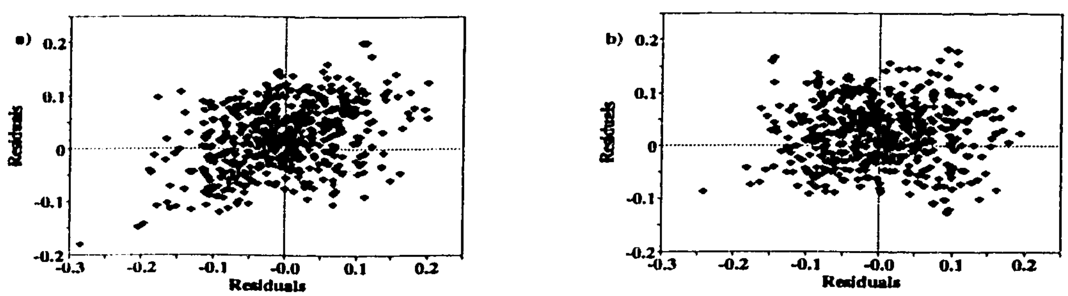

Figure 4.1: Residuals over lag residuals (Judge 1985) from two regressions: a) not corrected for autocorrelation; and b) corrected for autocorrelation (Pegan 1974, White 1978)

In summary, both of the above approaches lead to the same final coefficient estimates (Table 4.4), and there was no inherent advantage of one over the other. Nonlinear regression problems often have more then one minimum, and their presence is easily overlooked. Depending where a "local" minimum is in relation to the "global" minimum, one approach may be better than another, yet may produce a false convergence to a local minimum. As a precaution, one fits nonlinear regressions at least twice from two different initial coefficient values. Using the two above approaches is, in essence, equivalent to such a practice. A common convergence from the two fittings would suggest a global minimum, although it still is not an absolute proof.

Table 4.4: Model coefficients (α', α'', β' and β''), autocorrelation coefficient (ρ), and other summary statistics of the height-growth model fitted to all data $n = 946$.

Coefficient	Estim.	SE	t-ratio
α'	.37389	.055	6.8
α''	-.004254	.0006	-6.5
β'	92.960	10.5	8.9
β''	-.35966	.095	-3.8
ρ	.33002	.038	8.7

Other Regression Statistics

Mean of Dependent Variable	.33

Asymptotic Covariance matrix

β'	109.72				
β''	-.83618	.0089616			
α'	.41474	-.0031920	.0029949		
α''	-.0017435	.00002946	-.00002369	$.42276 \times 10^{-6}$	
ρ	-.23140	.0016884	-.0010496	$.49809 \times 10^{-5}$	.0014399
	β'	β''	α'	α''	ρ

Autocorrelation Measures

DURBIN-WATSON STAT.	2.0936
VON NEUMAN RATIO	2.0959
R^2 ($Res_T = \gamma + \delta Res_{T-1}$)	0.0001

Other Residual Analysis

RESIDUAL SUM	6.3651
RESIDUAL VARIANCE	0.0043
SUM OF ABSOL. ERRORS	49.963
R^2 (ANNUAL INCREMENT REGRESSION)	0.8978

4.5 Application of The Model to Thinning Evaluation

Three main steps are required to evaluate the effect of thinning on lodgepole pine growth: prediction of height growth at current density from current age to rotation, prediction of potential height growth at post-thinning densities in yearly iterations for the same period of time, and finally, comparison of the two results.

4.5.1 Growth at Current Density

When predicting height growth at current density, CI is calculated first. The resulting CI is used with current height and age to predict any future height growth from eq. (4.5). For untreated stands, the CI is assumed to be constant ($NT_1 \times TH_1^2 = NT_2 \times TH_2^2 = ... = NT_n \times TH_n^2$) therefore, the same value of CI can be used, either at any time, or over the whole growth period. In both cases eq. (4.5) is used as a cumulative function.

4.5.2 Post-thinning Growth

Prediction of potential height growth after thinning requires three additional steps:

1) calculation of CI with post thinning NT_t and TH_t;

2) prediction of height (TH_{t+1}) for following year;

3) use of this height to calculate a new CI from NT_t and TH_{t+1};

Steps 1 to 3 should be repeated until the end of the desired period, ensuring that the CI used in eq. (4.5) does not exceed the value before thinning.

4.5.3 Comparisons

Comparison of the above predicted results may require a creation of a spreadsheet. The LOTUS 1-2-3 program shown in Appendix A can be the basis for such a spreadsheet. In the spreadsheets illustrated in Table [A.1] the post-thinning density was set to a value that would result in a near maximum height growth for this stand. The density height growth model developed in this study indicates greatest height-growth reduction in young high-density stands. In older stands, changes in density result in a more modest height-growth response. Nevertheless, it should be noted that the approach used above accounts for only crowding-related mortality based on a simplistic approach to "constant" maximum crowding, that is known and specific to the stand under question. The user is advised to allow for site-specific, competition-independent mortality, based on local information and experience.

Application of a FORTRAN Program

The LOTUS 123 code presented in Appendix A can be used with other spreadsheets like SYMPHONY or Quatro Pro. If neither of these spreadsheets is available, the FORTRAN program, PROGRAM HtDens, listed in Appendix A, can be used for the same purpose on any computer. This interactive program predicts future top height from present height, breast height age, and number of trees per hectare, and it si es height-growth response to thinning using the same assumptions as the LOTUS spreadsheets ıstrated in Table [A.1].

4.6 Adjusting Lodgepole Pine *SI* for Density Related Height Growth Reduction

The new model is sufficiently flexible to estimate density related height and SI reduction in stands where a potential maximum crowding, and the age of reaching this crowding, can be approximated. The basic premise for developing this model was to describe height growth in annual increments so that a height growth curve could be generated in annual steps. The resulting model, a difference equation, predicts lP future height as a function of its present height, age, and CI. In the simplest scenario, with constant crowding, the model can be used as a cumulative function, the same way as the variable age SI height growth model (Cieszewski and Bella 1989) that estimates a height at any age in the future or past, from another height and age. In such a cumulative form, the model is more convenient because it predicts a height value in one step, e.g., SI at age 50, and it can generate complete height growth curves for specific CI values. Different CI values result in different height growth trajectories and different amounts of height reduction (Fig 4.2).

This section presents a practical application of the new density-height growth model (described in detail earlier) in the form of a density-SI diagram, to adjust lP SI for stand density.

4.6.1 Estimating SI Adjustment

Maximum Crowding

For the estimation of SI adjustment, one needs information about the stand's potential maximum crowding and the age of its commencement. This maximum crowding is generally reached after

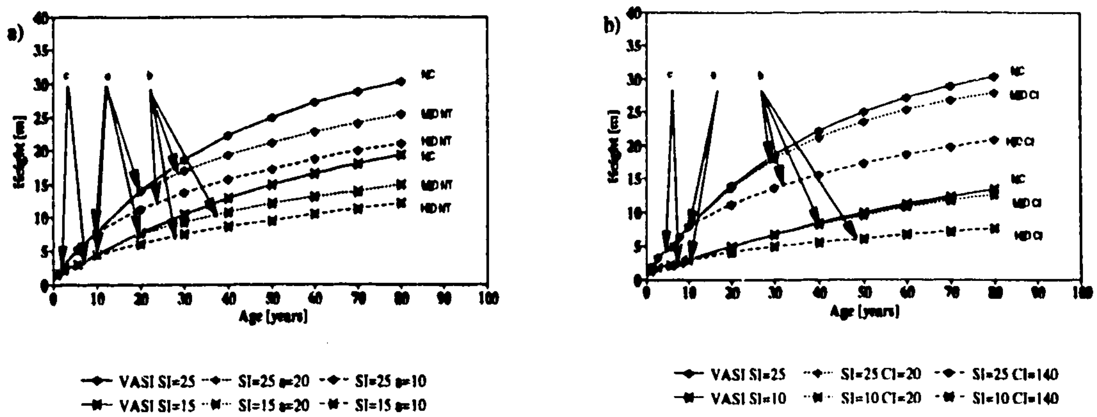

Figure 4.2: Height growth curves for two sites (SI 10 and 25) from the height/SI model (solid lines; Cieszewski and Bella 1991) and that from the new density-height growth model (broken lines) for: a) two initial density levels; and b) two crowding levels (CI 20 and 140). NC, MID and HID denote: no crowding, and medium, and high initial density/crowding; "a", "b", "c" different stages of crowding development.

crown closure in young or intermediate age fully stocked stands. In young stands, crowding is likely to be below maximum and would generally increase with time. A rough estimation of this maximum may be possible from site characteristics, because potential crowding is likely to be related to site productivity, although it may be influenced by initial stand densities.

Estimating SI adjustment would generally require three steps: 1) estimating the height at the age of reaching the maximum CI (Table 4.5) from current stand conditions, 2) estimating SI from previous height and age for this maximum CI and for the unsuppressed growth (Table 4.6), and 3) calculating the difference between these two SI values.

Table 4.5: Estimation of potential SI from suppressed SI, CI, and ages of reaching maximum CI

	Observed SI = 5				Observed SI = 10				Observed SI = 15			
x_m/CI	10	20	30	40	10	20	30	40	10	20	30	40
0	1.9	2.6	3.5	4.3	3.0	5.1	7.0	8.6	4.6	8.1	11.0	13.3
20	1.9	2.7	3.5	4.3	3.1	5.2	7.0	8.6	4.7	8.1	10.9	13.2
40	2.0	2.8	3.5	4.3	3.2	5.2	7.0	8.6	4.8	8.1	10.9	13.1
60	2.1	2.9	3.6	4.3	3.4	5.4	7.1	8.6	5.0	8.3	10.9	13.1
80	2.2	3.0	3.7	4.4	3.6	5.5	7.2	8.7	5.3	8.5	11.0	13.2
100	2.3	3.1	3.8	4.4	3.9	5.8	7.4	8.8	5.7	8.8	11.2	13.3
120	2.4	3.2	3.9	4.5	4.2	6.1	7.6	8.9	6.2	9.2	11.5	13.4
140	2.6	3.4	4.0	4.5	4.6	6.4	7.8	9.0	6.9	9.7	11.8	13.6
160	2.8	3.5	4.1	4.6	5.2	6.9	8.1	9.1	7.8	10.5	12.3	13.8
180	3.1	3.8	4.3	4.7	6.1	7.6	8.6	9.4	9.6	11.8	13.2	14.2

Table 4.6: Potential SI values for three sites of observed SI and four assumed ages of reaching maximum CI.

x_m/CI	Observed SI = 5				Observed SI = 10				Observed SI = 15			
	10	20	30	40	10	20	30	40	10	20	30	40
0	4.7	4.9	5.0	5.0	9.9	10.2	10.2	10.1	15.4	15.7	15.6	15.3
20	5.0	5.0	5.0	5.0	10.3	10.3	10.2	10.1	15.7	15.6	15.4	15.2
40	5.3	5.2	5.1	5.0	10.7	10.4	10.3	10.1	16.1	15.7	15.4	15.2
60	5.7	5.4	5.2	5.1	11.4	10.7	10.4	10.1	16.8	15.9	15.4	15.2
80	6.2	5.6	5.3	5.1	12.1	11.0	10.5	10.2	17.7	16.3	15.6	15.2
100	6.8	5.9	5.5	5.2	13.1	11.5	10.7	10.3	18.9	16.8	15.8	15.3
120	7.4	6.2	5.6	5.3	14.2	12.0	11.0	10.4	20.3	17.5	16.2	15.4
140	8.1	6.5	5.8	5.3	15.5	12.7	11.3	10.5	22.1	18.3	16.6	15.6
160	9.0	6.9	6.0	5.4	17.2	13.5	11.8	10.7	24.5	19.5	17.2	15.9
180	10.3	7.5	6.3	5.5	20.1	14.8	12.4	11.0	28.8	21.6	18.3	16.3

Two Models

The first step may require the use of either the new density height growth model in its cumulative form, or a SI height growth model, e.g., Cieszewski and Bella (1991), depending on whether or not the subject stand has reached maximum crowding. The second step requires the use of both models.

In the third step, one simply subtracts the appropriate suppressed SI from the corresponding potential, unsuppressed SI. Figure 4.3 illustrates the relation between the suppressed and potential SI values for three different sites and four different ages of the commencement of maximum crowding as listed in Table 4.6.

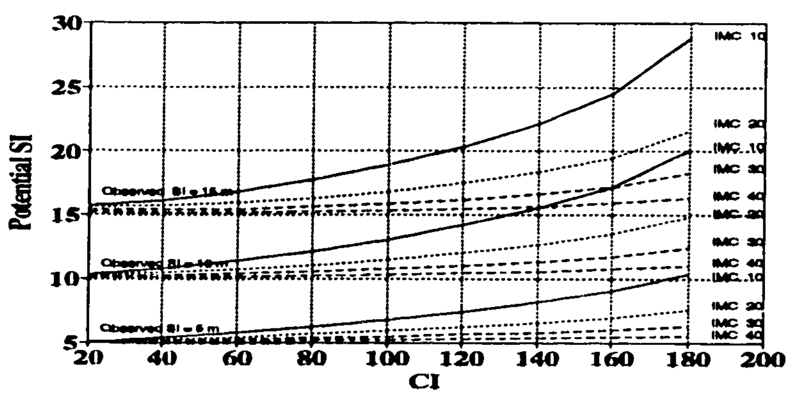

Figure 4.3: Suppressed (5, 10, 15m) and potential (5 to 28m) SI values for different stands as functions of maximum CIs (20 to 180) and onset of these maximum CIs (IMC = 10, 20, 30, and 40 years), i.e., ages at which these CIs were reached by the stands.

Three Stages of Stand Development

Depending on the development stage of the stand (Fig. 4.4), the estimation of SI reduction adjustment for density varies in difficulties. The easiest situation is when it is established that the stand has just reached its maximum crowding (Fig. 4.4 "a"). A certain equilibrium between mortality and growth would be indicative of this; so would an increase in mortality and a reduction in stand diameter and height growth. Repeated observations of the stand, or examination of increment cores and their comparisons with that of similar neighbouring stands, may also provide a good clue about reaching this stage. In that situation, one can estimate the expected suppressed SI from the stand's present age, height and CI, using the new density height growth model. The potential SI for the same stand can be estimated from a height SI model. The SI adjustment is then simply the difference between these two SI values.

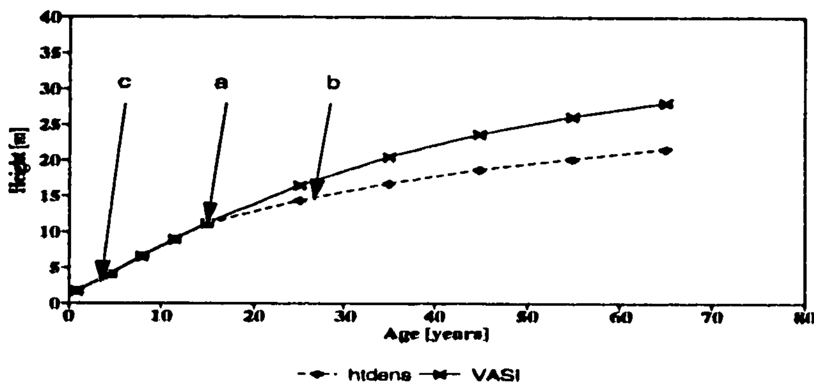

Figure 4.4: Illustration of different stages of stand development relating to height growth suppression from crowding: "*a*" the stand has just reached maximum crowding; "*b*" the stand has been growing at maximum crowding for some time; and "*c*" the stand before reaching maximum crowding.

Estimation of SI reduction for a stand that has already been growing at maximum crowding (Fig. 4.4 "b"), or has not yet reached maximum crowding (Fig. 4.4 "c"), is more difficult. In the former case, one has to estimate the age when the maximum crowding commenced. This may not be easy, although an examination of historical records and/or increment cores might be helpful. Next, one predicts height at this age from the present height, age and CI, using the new density height growth model. From this height and age, one can predict the expected suppressed SI using again the density height growth model, and the potential SI using a height SI model. The SI adjustment again is simply the difference between these two SI values.

The most difficult case to estimate SI adjustment is for a stand before it reaches maximum crowding. Then, one first has to estimate the maximum CI and the age when it will be reached. Again, examination of increment cores from the subject stand and assessment of the neighbouring stands may be helpful. One then predicts height at this age from present height and age using a SI-height growth model. From this height and age, one can then predict the expected suppressed SI using the new density height growth model, and the potential SI using a height SI model. The SI adjustment is again obtained as a difference.

In summary, to estimate density related height growth or SI reduction one needs to know the stand's potential maximum crowding and the age and height when it is reached. Using these, one can compute suppressed SI with the density height growth model and potential SI with a height/SI model, and their difference will be the SI adjustment.

This approach disregards height growth reduction before maximum crowding is reached, as such reduction would likely be relatively small. This reduction, however, may be estimated by yearly simulations that would compute changes in crowding and their effect on height increment.

4.6.2 Practical Application

A diagram (Fig. 4.5) was constructed using the two models (Cieszewski and Bella 1992) to facilitate

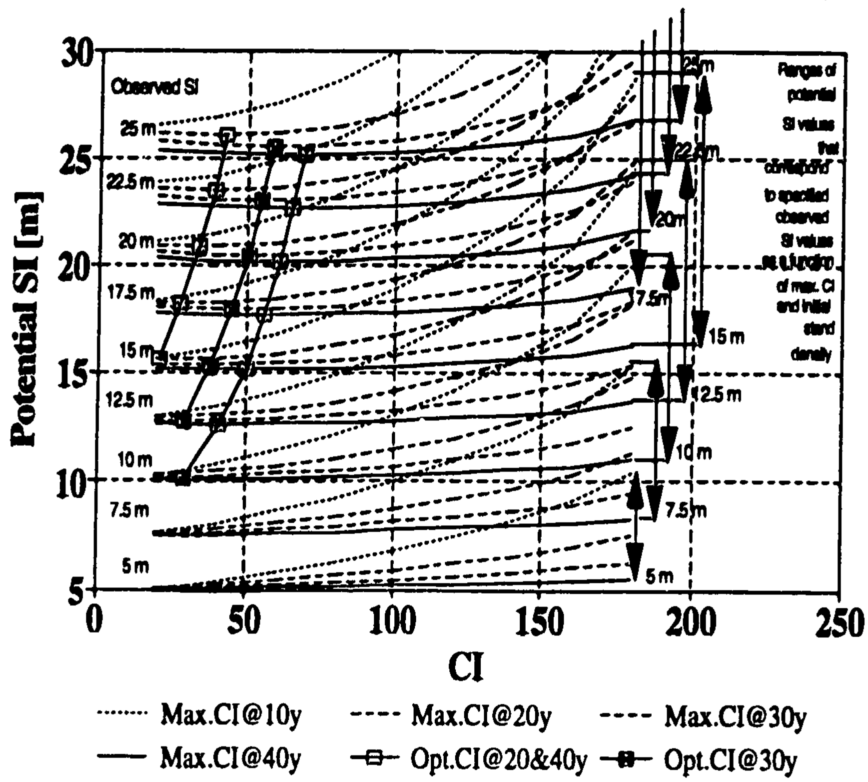

Figure 4.5: Diagram of suppressed and potential SI values (5 to 30m) for stands of different maximum CIs (20–180) and initial densities as indicated by the ages of reaching these CIs (10, 20, 30, and 40 years).

the field application of this approach for estimating density related SI reduction in IP. The diagram requires a prior estimation of the observed suppressed SI in the crowded stand. Then, to use the diagram one first needs to select a subset of four trajectories representing the suppressed SI, e.g., 10 m. Then, one has to identify the trajectory within the subset that corresponds to the age when the stand had reached its maximum CI, say 20 years, which is the second trajectory from the top within that set. This trajectory will define a potential SI as a function of CI for this stand. Then projecting along that trajectory to the desired CI value, say 150, one can read on the vertical axis the corresponding potential unsuppressed SI, in this example about 12.5 m.

Conditions not represented by trajectories can be interpolated. Such conditions may include interpolation between suppressed SI values, i.e., 5, 7.5, 10, m etc.; between the commencement age of maximum CI, i.e., 10, 20, 30, and 40 years; or between interpolated values of both.

For example, to estimate potential unsuppressed SI for a crowded stand with a suppressed SI 11.2 m that reached maximum CI 100 at age 20 years, one first finds on the diagram the sets of trajectories for stands of suppressed SI 10 and 12.5 m, i.e., the third and fourth set from the bottom of the diagram. Then one locates within these sets the trajectories that correspond to the commencement of maximum crowding at 20 years, i.e., the second trajectory from the top in each set. Projecting along these trajectories to the value of CI 100, one reads the potential SI values for each trajectory. For a stand of suppressed SI 10 m that reached its maximum CI 100 at age 20 years, the potential SI is 11.5 m. For a similar stand of suppressed SI 12.5 m, the potential SI is 14.2 m.

Interpolating between 14.2 and 11.5 m gives 12.8 m, which is an approximate potential SI for the stand of suppressed SI 11.2 m and maximum CI 100 commenced at 20 years. By subtraction, one can obtain density related SI reduction of 1.6 m for the stand examined. A similar interpolation can be done with respect to the age of maximum crowding commencement.

Further, one can interpolate between potential SI values for different ages of reaching maximum crowding and suppressed SI values by drawing curves in two steps. First, one needs to draw two curves, one for each of the closest available suppressed SI trajectory subsets. These curves are interpolated between available trajectories that bracket the specified age, e.g., for SI 11.2 m and crowding stabilization age 15 years it would be 10-and 20-year trajectories in the SI 10 and 12.5 m subsets. Second, one has to draw a new interpolated trajectory between the above two just created trajectories.

4.6.3 Equations

For operational use, the base density height growth model (4.5) can be written in the following form:

$$H(t,h_t,CI)-1.3=\frac{h_t+d+\sqrt{(h_t-d)^2+zh_tt^i}}{2+z(t+1)^i\left[h_t-d+\sqrt{(h_t-d)^2+zh_tt^i}\right]^{-1}} \tag{4.6}$$

where: $H(t,h_t,CI)$ is the height at age $t+1$, predicted as a function of current age (t), height (h_t), and CI; $i=0.004543CI-1.33124$; $d=20b/50^i$; $b=91.468352-0.478853CI$; and $z=80b$.

When crowding is stable, e.g., in fully stocked mature stands, eq. (4.6) can also be used as a cumulative function to predict height for a given density level, i.e.,

$$H(Age,h_x,x,CI)-1.3=\frac{h_x+d+\sqrt{(h_x-d)^2+zh_xx^i}}{2+z(Age)^i\left[h_x-d+\sqrt{(h_x-d)^2+zh_xx^i}\right]^{-1}} \tag{4.7}$$

where $H(Age,h_x,x,CI)$ is a height at any age (Age), predicted as a function of any other height above breast height (h_x) at an age (x), and CI; other symbols are the same as in eq. (4.6).

For constant crowding, eq (4.7) predicts identical values to that accumulated by yearly simulations of eq. (4.6). For predicting SI, a simpler equation (Cieszewski and Bella 1989) can also be used:

$$SI(h_x,x,CI)-1.3=0.5\left[h_x-d+\sqrt{(h_x-d)^2+zh_xx^i}\right] \tag{4.8}$$

where $SI(h_x,x,CI)$ is SI at breast height base-age 50 years, predicted as a function of any height above breast height (h_x) at a breast height age x, and a CI; other symbols are the same as in eq. (4.6).

To predict unsuppressed SI, an equation similar to eq. (4.8) can be used:

$$SI_p(h_m,x_m)-1.3=0.5\left[h_m-d_p+\sqrt{(h_m-d_p)^2+z_ph_mx_m^{i_p}}\right] \tag{4.9}$$

where:

$SI_p(h_m,x_m)$ denotes a potential SI at breast height base-age 50 years, predicted as a function of the stand height above breast height (h_m) at breast height age (x_m) at the time when the stand reached maximum crowding level CI; and $i_p=-1.20373434$; $d_p=20b_p\times 50_p^i$; $b_p=97.37473618$; and $z_p=80b_p$.

To estimate SI reduction for a given density one needs to subtract predicted height at SI base-age for that density from potential open-grown height at the same age: $SI_{loss}=f_4(f_2(x_m,h_x,x,CI),x_m)-f_3(h_x,x,CI)$ where f_2 denotes eq. (4.7) and is equivalent here to h_m, f_3 denotes eq. (4.8), and f_4 denotes eq. (4.9).

4.7 Discussion and Conclusions

The model developed here is a dynamic difference equation that simulates IP height growth over a wide range of density conditions, sites, and ages. Thus, it represents a four dimensional space that has been derived from two-dimensional observations of height over age data at different crowding levels on different sites. It is a polymorphic model with nonlinearly entering coefficients. Its greatest value lies in illustrating the changes of height growth patterns between different crowding levels on different sites.

The model is designed to predict height growth in annual steps, thus providing the flexibility required for use under dynamically changing stand conditions. To take full advantage of this capability, however, requires a complete stand growth simulation model that can predict stand crowding and mortality.

The model is based on the simple half-saturation function, with its asymptotic and half-saturation parameter modified by equations applying Czarnowski's (1947) stand dynamics theory with a simple yet relatively stable crowding measure. This measure is based on the product of top height and number of trees per unit area, which can be easily obtained from sample plot summaries, forestry inventory information, or directly from aerial photographs.

To illustrate the magnitude of differences in height growth that can arise from differences in stand density and crowding, the new model was used to simulate three separate sets of height trajectories for good, medium, and poor growing sites (Figs. 4.6 and 4.7). These show that differences in height

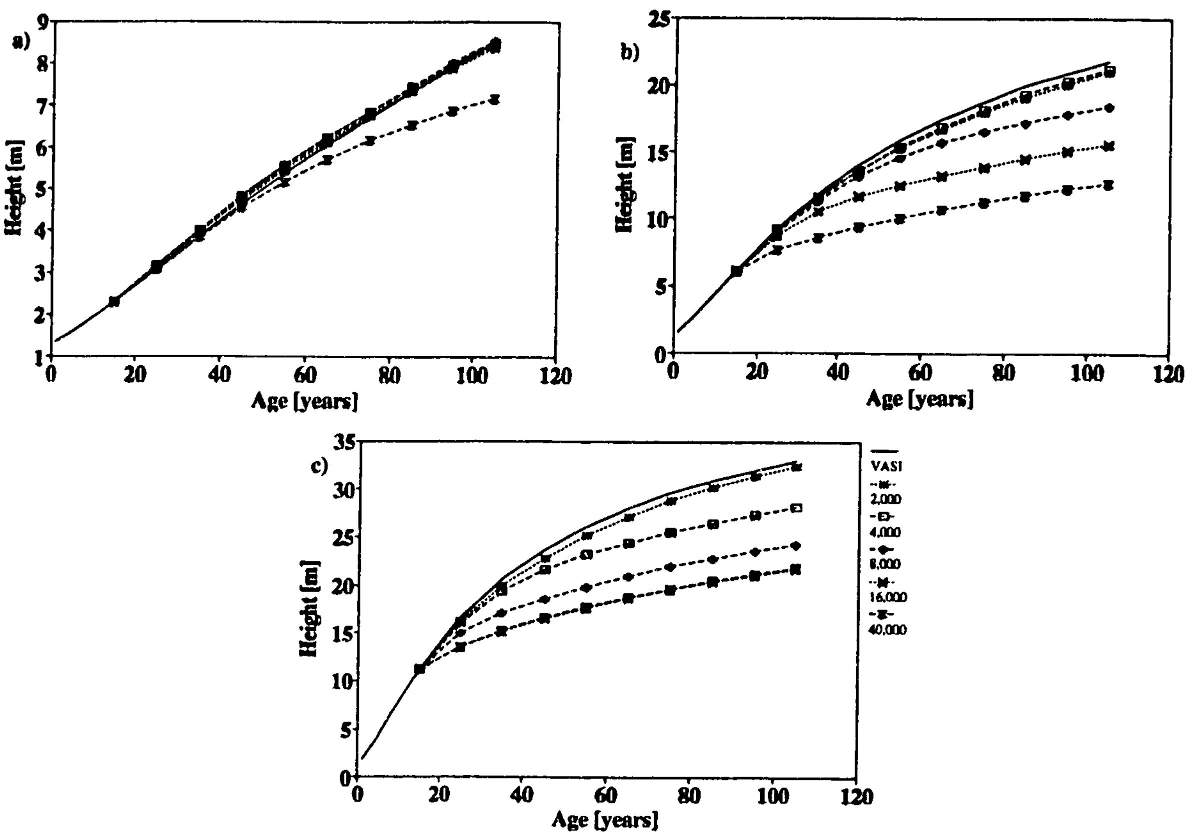

Figure 4.6: Simulated height trajectories for different density levels on (a) poor, (b) medium and (c) good sites.

growth as great as 50% can arise from excessive density, e.g., 40,000 trees/ha at age 60–70 years.

As good sites produce faster tree growth and support larger trees, the reduction in height growth is greater for the same increase in crowding on a good site than on a poor site.

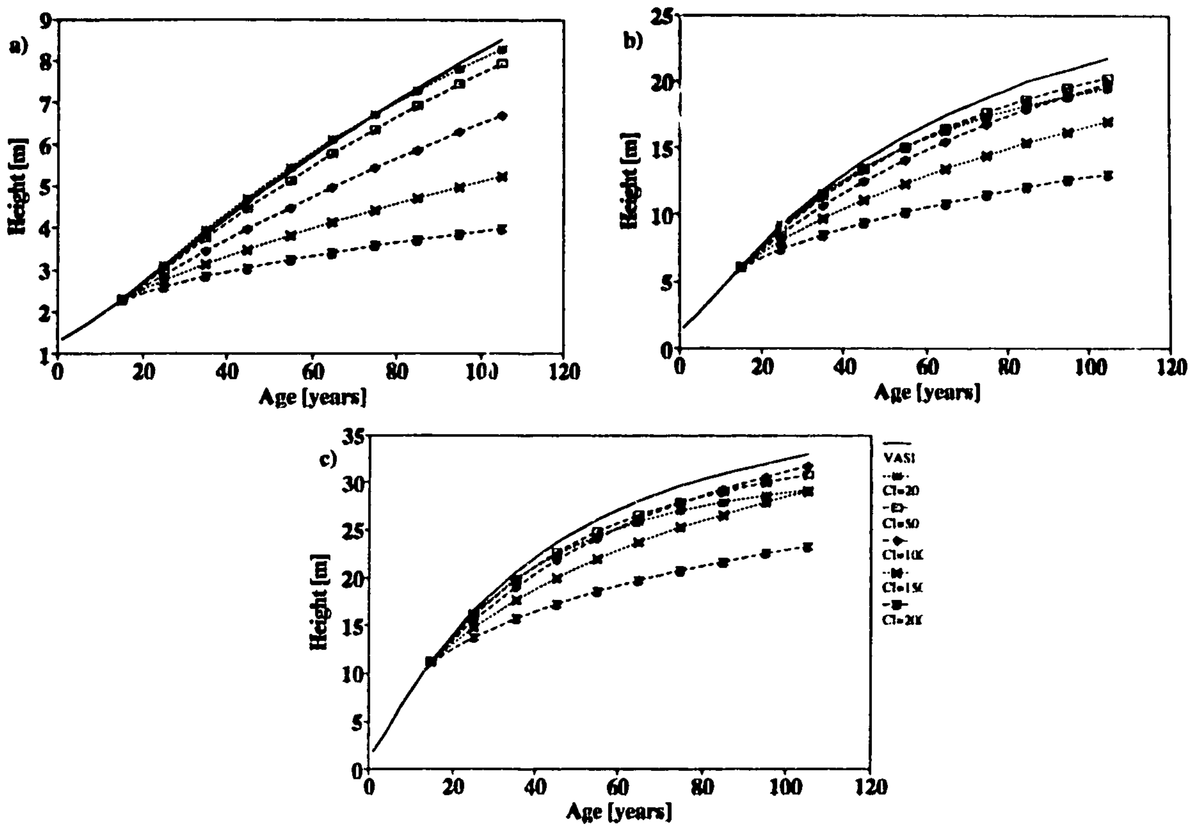

Figure 4.7: Simulated height trajectories for different crowding levels on (a) poor, (b) medium and (c) good sites.

Under open stand conditions, the values of the modifying functions replacing α and β approach values of α and β in the IP SI–height-growth model developed from stem analysis of dominant and codominant trees (Cieszewski and Bella 1989). The α values are closest at about $CI = 38.78$, and β values near $CI = 21.14$.

For decreasing values of the modifying functions, the model predicts diminished growth rates before growth completely ceases. When the value of $CI = -\beta'/\beta'' = 258.47$ the modifying function replacing β takes on a zero value, and the model is reduced to a simple constant of $H_{t+1}(t, h_t) = h_t$. This may be considered as state of stagnation. For crowding values over 258.47, β becomes negative and meaningless, and the model estimates become unreasonable. Therefore, the model should not be applied with crowding values of 258 and over.

One can expect the CI to increase with time in young and open stands until they reach their maximum stocking level. After that CI stabilizes, irregular tree mortality may cause some variation. The value of CI in fully stocked stands is likely to vary between sites. In fact, Czarnowski (1947) used this expression as a measure of site productivity in fully stocked Scots pine (*Pinus sylvestris* L.) stands.

As with SI, CI may combine the effect of site and the influence of density on height growth. While SI in open stands adequately describes site productivity, CI may be a better descriptor in fully stocked stands, where SI is strongly affected by density. Therefore, the most difficult site-indexing problem is in the transitional stage before the stand reaches full stocking, yet is already showing density effects on height growth. Under such conditions, CI can be only a competition measure. For the above reasons, the combined use of SI and CI in a height-growth simulation model provides both site and crowding information. The two major challenges are to find an appropriate algebraic form for such a model, and to calibrate it.

The density–height growth model developed in this study provides a reasonable representation and description of the height-density interrelationship for lP stands with a minimal input of information. This relationship is required for the prediction of reduced height growth trajectories as a function of stand density. The relationship can also be useful in assisting evaluation of the outcome of thinning treatment alternatives in lP and deriving thinning prescriptions.

The diagram (Fig. 4.5) presented here provides a means for estimation of height growth reduction and SI adjustment for lodgepole pine stands. This approach is a first approximation for SI density adjustment. The approach for the next approximation may be through detailed stem analysis information that represents a range of site and crowding conditions.

If the new model is applied to open stand conditions, it gives predictions similar to the lP height/SI model (Cieszewski and Bella 1991). At extreme crowding height growth and SI reduction may exceed 50% (Fig. 4.5).

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Chapter 5

General Discussion and Conclusions

The measurement components in the height growth modeling are the elements that need to be measured on a tree or in a stand to contribute to the identification of the underlying growth trends for this tree or stand. Given the traditionally explicit use of two independent variables (age and SI) in height growth models, the concept of the implicit measurement components may pertain to those elements of the height growth that are necessary to define meanings of such explicit variables as the prediction age and SI. While prediction age and SI can be considered a basic input in any height model, they both are meaningless when dissociated from their respective co-variables: the base-height and base-age.

This thesis contains a series of papers advancing mathematical height modeling towards unravelling the traditionally "hidden" (implicit) height measurement components and implementing them as explicit variables. The achievements of this thesis include a general methodology of derivation of four-dimensional height growth models with explicit variable base-ages (Chapter 2) capable of describing polymorphic height growth with variable asymptotes.

A further achievement of this thesis was in developing a concept of a variable base-height, five-dimensional height model, (Chapter 3). As the base-height is a key element in definition of any age, this new model is essentially age free. Through an adjustment of the base-height, the age can be defined as starting at any arbitrary height. In addition to this property the new model also contains all of the properties of the previous variable base-age four-dimensional height growth models (Chapter 2), thus containing three variable measurement components of the reference-height, base-age, and base-height, as independent variables. The combination of these three variable measurement components can describe any growth increment or growth intercept, that begins or ends at any height and have any length. Hence, this variable base-height and base-age model constitutes a general height growth platform that can unify all SI and intercept height models based either on total age, or breast height age, or any other age.

The final achievement of this thesis is the development of the dynamic variable density height growth model in Chapter 4 that predicts yearly height growth increments and can be iterated over time with a dynamically changing element of density or crowding. After validation, the model should be suitable for implementation into growth prediction systems, and simulations of height growth responses to different thinning regimes. Since the density responses are entering the model in an unrestricted way and can assume any values and any changes over time in these values, the resulting variable density height growth system represents an infinite-dimensional height growth space, similar in dimensionality to spaces defined by systems of simultaneous interacting differential equations with continuous variables. At every step, when an implicit variable is implemented in an explicit form, the model dimensionality increases by an additional dimension and the model mathematical equation dramatically increases in complexity. Yet this infinite-dimensional dynamic variable density height growth model is algebraically relatively simple and contains only four estimable coefficients. The great flexibility of this model is mainly due to its iterative approach and the fact that it is in the form of a difference equation, that is iteratively simulating height growth in one year increments.

In summary, all objectives of this thesis have been achieved through the development of:

- the general methodology for derivation of four-dimensional variable-base-age polymorphic height growth models with variable asymptotes (Chapter 2);
- the methodology of extending the above four dimensional height growth models to the fifth dimension including the variable-base-height (Chapter 3); and
- the methodology of developing the infinite-dimensional, dynamic, variable-density height growth model and its calibration for lodgepole pine data in Alberta (Chapter 4).

The above methodologies represent a journey towards an advancement of height growth modeling that is consistent with a general trend in the history of increasing model flexibility and generality. Individual parts of this journey built on the previous knowledge/chapters and expanded this knowledge to new subsequently higher levels, e.g., the dynamic variable density height growth model (Chapter 4), that uses coefficient modifying functions (Chapter 3) to implement a third explicit measurement component of variable density, can be formulated as a dynamic equation only if it already contains the explicit base-age as an independent variable (Chapter 2). The results of the methodologies presented in this thesis aim towards, and achieve, a means of improving the understanding of the growth dynamics associated with a height growth model development in general. The resulting new models are capable of better reflecting existing height growth patterns, while they require the same or less demanding data for their development. At least in theory, the new models should be:

- easier to use, e.g., direct use of measurable variables such as reference-height;
- easier to calibrate and collect data for, e.g., more flexible screening of the data with possibility of using higher parts of the height growth (ageless model);
- more exacting, e.g., in juvenile height growth modeling a base of an intercept can be defined with arbitrary precision as the base height; and
- more informative, e.g., predicting responses of height growth to different density levels.

The scope of this thesis includes no rigid statistical investigations associated with the developed models and their validations. These should be a next step in the relevant research following this study. The models should be tested and validated on available permanent sample plot and stem analysis data, and in real applications. As for the further research prospective it would be desirable to investigate statistical properties of these multidimensional height growth models and implications of different ways in which they could be calibrated. Given that the implications of different approaches to calibration of even the most popular, three-dimensional, fixed base-age, SI height growth models is unknown to date, it is not likely that much in depth will be known soon about these more complex models. Despite this, one should not hesitate to use these advanced multi-dimensional height growth models in applications in which they can be more useful then the traditional height growth models, while the extent of use may depend on individual judgement and relevant confirmation of the usefulness.

Appendix A

Computer Implementation of The Variable Density Height Growth Model

A.1 An Example of a FORTRAN Program for Computing a Variable Density Height Growth Predictions

```
      PROGRAM HtDens

      IMPLICIT NONE
      DOUBLE PRECISION Est_Ht, hx, pred, t, x1, switch, bh_age,
     &    CI, coef_a, coef_b, gen_a1, gen_a2, gen_b1, gen_b2, calc_cf,
     &    no_trees, thin_trs, orig_CI
      CHARACTER*1 cont, model, new_mdl
      INTEGER i, count
      parameter (gen_a1=0.33121384, gen_a2=-0.004543, gen_b1=91.468352,
     &    gen_b2=-0.478853)

      WRITE (*,'(T25,34A/,2(T25,A/),T25,34A//,A,2(A/))') ('-',i=1,34),
     & '    LP Height Growth Estimator','     using Ht-Density Model',
     & ('-',i=1,34)

1     switch = 0
      WRITE (*,'(/A,A)') ' Select (g)rowth model predictions,',
     &        ' or (t)hinning simulation (g/t)?'
      READ (*,'(A1)') model

10    IF ((model.EQ.'t').OR.(model.EQ.'T')) THEN
         count = 0
         switch = 1
      ENDIF
      WRITE(*,'(/A,A)') '.Enter a known BH AGE and HT, eg., 50 15.0'
      READ (*,*) x1, hx

      WRITE(*,'(/A)') ' Enter number of trees per ha at this age:'
      READ (*,*) no_trees
      CI = hx**2 * no_trees / 10000
```

```
      IF (switch.EQ.1) THEN
         WRITE(*,'(/A)') ' Enter number of trees per ha after thinning:'
         READ (*,*) thin_trs
         orig_CI = CI
         CI = hx**2 * thin_trs / 10000
      ENDIF

      coef_a = gen_a1 + gen_a2 * CI
      coef_b = gen_b1 + gen_b2 * CI

* check for a realistic calculated SI (should be 5 < ht < 30 m at bh age 50)

      pred = Est_Ht(50.0, coef_a, coef_b, x1, hx, switch)
      IF (pred.LT.5.0) THEN
         WRITE (*,*) '     * WARNING: Trees unusually short--future',
     &               ' ht predictions may be unreliable.'
      ENDIF
      IF (pred.GT.30.0) THENø¯h)
         WRITE (*,*) '     * WARNING: Trees unusually tall--future',
     &               ' ht predictions may be unreliable.'
      ENDIF

      IF (switch.EQ.1) THEN
         WRITE(*,'(/A,F8.1,A)') ' After thinning to',
     &             thin_trs, ' trees per ha . . .'

*  do thinning simulation, calculating ht every year and using it as new hx

13       IF (x1.LE.100) THEN
            t = x1 + 1
            pred = Est_Ht(t, coef_a, coef_b, x1, hx, switch)
            count = count + 1
            IF (count.EQ.10) THEN
               count = 0
               WRITE(*,15) 'At bh age',t,', ht will be',pred,' m.'
            ENDIF
            hx = pred
            x1 = t
            CI = hx**2 * thin_trs / 10000
            IF (CI.GT.orig_CI) CI = orig_CI
            coef_a = gen_a1 + gen_a2 * CI
            coef_b = gen_b1 + gen_b2 * CI
            GOTO 13
         ENDIF
      ELSE

*  predict heights every 10 years

         t = x1 + 10.0
14       IF (t.LE.100) THEN
            pred = Est_Ht(t, coef_a, coef_b, x1, hx, switch)
            WRITE(*,15) 'At bh age',t,', ht will be',pred,' m.'
```

```
15          FORMAT (1X, A, F6.1, A, F8.4, A)
            t = t + 10.0
            GOTO 14
         ENDIF
      ENDIF

      WRITE (*,'(/A)') ' Again (y/n)?'
      READ (*,'(A1)') cont
      IF ((cont.NE.'N').AND.(cont.NE.'n')) THEN
         WRITE (*,*) 'Choose new model (y/n)?'
         READ (*,'(A1)') new_mdl
         IF ((new_mdl.EQ.'Y').OR.(new_mdl.EQ.'y')) GOTO 1
         GOTO 10
      ENDIF
      WRITE (*,'(/A)') ' Good-bye.'
      END
      DOUBLE PRECISION FUNCTION Est_Ht(t,coef_a,coef_b,x1,hx,switch)

      IMPLICIT NONE
      DOUBLE PRECISION t,coef_a,coef_b,x1,hx,switch,z,j,d,hxRoot

      z = 80 * coef_b
      j = - 1 - coef_a
      d = 20 * coef_b * 5d1**j

      hxRoot=(hx-1.3) + DSQRT(((hx-1.3)-d)**2 + z*(hx-1.3)*x1**j)
      Est_Ht = ( hxRoot + d) / ( 2 + z*t**j/(hxRoot-d) ) + 1.3

      RETURN
      END
```

A.2 An Example of a Spreadsheet for Computing Variable Density Height Growth Predictions

A basic code for LOTUS 123 spreadsheet for computation of density related height growth can consist of the following listing:

```
A1: "Age
B1: "TH1
C1: "PresNT
D1: "CI
E1: "alpha
F1: "beta
G1: "gama
H1: "R
A2: 10
B2: (F0) 6
C2: (F0) 50000
D2: (F1) @MIN(C2/10000*B2^2,183)
E2: "<== These are the initial conditions
A3 to H3: \=
A4: (F0) "Thin to=>
C4: (F0) 2000
```

Table A.1: Example LOTUS 1-2-3 spreadsheet—a comparison of two lodgepole pine stand densities on a good site: a) no thinning; b) with thinning to 1200 trees/ha.

a)

	A	B	C	D	E	F	G	H	I	J
1	Age	TH1	PresNT	CI	alpha	beta	gama	R	TH2	VASI
2	10	7.6	30980	180.0	<== These are the initial conditions					
3	======	======	======	======	======	======	======	======	======	======
4	Thin	to=>	30980	180.0						
5	1			---------------------						1.8
6	3			| VASI SI 24.00 |						3.1
7	6			| HTDE SI 12.21 |						4.6
8	8			=====================						6.2
9	10	7.62	30980	180.0	-0.513	421.99	14.153	35.974	7.86	7.6
10	20	9.45	20170	180	-0.513	421.99	14.153	35.974	9.59	13.4
11	30	10.63	15929	180	-0.513	421.99	14.153	35.974	10.73	17.9
12	40	11.51	13586	180	-0.513	421.99	14.153	35.974	11.59	21.3
13	50	12.21	12073	180	-0.513	421.99	14.153	35.974	12.27	24.0
14	60	12.79	11003	180	-0.513	421.99	14.153	35.974	12.84	26.1
15	70	13.28	10200	180	-0.513	421.99	14.153	35.974	13.33	27.9
16	80	13.71	9571	180	-0.513	421.99	14.153	35.974	13.75	29.3
17	90	14.09	9064	180	-0.513	421.99	14.153	35.974	14.13	30.4
18	100	14.43	8643	180	-0.513	421.99	14.153	35.974	14.46	31.4

b)

	A	B	C	D	E	F	G	H	I	J
1	Age	TH1	PresNT	CI	alpha	beta	gama	R	TH2	VASI
2	10	7.6	30980	180.0	<== These are the initial conditions					
3	======	======	======	======	======	======	======	======	======	======
4	Thin	to=>	1200	7.0						
5	1			---------------------						1.8
6	3			| VASI SI 24.00 |						3.1
7	6			| HTDE SI 23.07 |						4.6
8	8			=====================						6.2
9	10	7.62	1200	7.0	-1.299	7050.3	10.921	53.836	8.28	7.6
10	20	13.35	1200	21	-1.234	6498.0	13.005	56.131	13.83	13.4
11	30	17.50	1200	37	-1.164	5909.4	15.541	58.931	17.85	17.9
12	40	20.62	1200	51	-1.099	5363.5	18.170	61.692	20.89	21.3
13	50	23.07	1200	64	-1.040	4869.8	20.745	64.294	23.29	24.0
14	60	25.09	1200	76	-0.987	4423.2	23.180	66.718	25.27	26.1
15	70	26.79	1200	86	-0.939	4017.3	25.415	68.957	26.95	27.9
16	80	28.26	1200	96	-0.895	3646.6	27.399	70.997	28.39	29.3
17	90	29.54	1200	105	-0.855	3307.0	29.088	72.618	29.66	30.4
18	100	30.66	1200	113	-0.818	2995.5	30.444	74.400	30.77	31.4

```
D4: (F1) +D6
A5: "Height growth as a function of age and density:
A6: +A2
B6: (F2) +B2
C6: (F0) +C4
D6: (F1) +C6/10000*B6^2
E6: 0.004543*D6-1.33121384
F6: 7317.4681216-38.3082064*D6
G6: +F6*50^E6/4
H6: +B6-1.3+@SQRT((B6-1.3-G6)^2+F6*(B6-1.3)*A6^E6)
A7: +A6+1
B7: (F2) (H6+G6)/(2+F6*A7^E6/(H6-G6))+1.3
C7: (F0) @IF(D7<$D$2,C6,$D$2*10000/B7^2)
D7: (F0) @MIN($D$2,C6/10000*B7^2)
E7: 0.004543*D7-1.33121384
F7: 7317.4681216-38.3082064*D7
G7: +F7*50^E7/4
H7: +B7-1.3+@SQRT((B7-1.3-G7)^2+F7*(B7-1.3)*A7^E7)
.
.
.
```

An example of a basic spreadsheet based on the above code is the spreadsheet below for computation of density related height growth:

```
-----A----B------C------D------E------F------G------H--
1    Age    TH  PresNT   CI   alpha   beta   gama    R
2     10     6   50000 180.0 <== These are the initial conditions
3    ==================================================
4    Thin  to=>   2000   7.2
5    Height growth as a function of age and density:
6     10  6.00    2000    7.2 -1.298 7041.6 10.952 45.973
7     11  6.51    2000      8 -1.292 6993.1 11.124 46.135
8     12  7.00    2000     10 -1.286 6941.8 11.308 46.315
9     13  7.49    2000     11 -1.280 6887.8 11.505 46.511
10    14  7.96    2000     13 -1.273 6831.7 11.712 46.721
11    15  8.43    2000     14 -1.266 6773.6 11.930 46.945
12    16  8.88    2000     16 -1.259 6713.8 12.157 47.181
13    17  9.31    2000     17 -1.252 6652.7 12.393 47.428
14    18  9.74    2000     19 -1.244 6590.3 12.637 47.684
15    19 10.16    2000     21 -1.237 6527.0 12.888 47.950
16    20 10.56    2000     22 -1.229 6463.0 13.147 48.223
.
.
.
```

Modified spreadsheets for computing height predictions for different thinning scenarios with the possible results of a thinning can be compared to those of a no-thinning scenarios are included in Table [A.1]. The two spreadsheets in this Table implement computations for two density scenarios on a good site based on the code listed on the beginning of this section. By examining several of such scenarios one can determine the density required to achieve near-maximum height growth.