University of Alberta

Compound Operators and Infinite Dimensional Dynamical Systems

by

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Your file Votre référence
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To my parents and my family in China.

Without their selfless support, this would not have been possible.

ABSTRACT

This thesis studies the qualitative theory of linear and nonlinear infinite dimensional dynamical systems with applications mainly to parabolic partial differential equations. The objective of the study is to examine through linearization the local and global behaviour, including existence and nonexistence, of invariant structures such as equilibria and periodic solutions.

In the linear theory, the dimension of the asymptotically stable solution subspace of a linear differential equation is studied. This gives new insights into the behaviour of linear and nonlinear dynamical systems.

The nonlinear results include such topics as a generalization to infinite dimensional differential equations of a classical stability condition of Poincaré. The main idea is that a periodic orbit is stable if the system diminishes nearby 2-dimensional areas. Similar considerations give conditions for the existence as well as the stability of a periodic solution. If the system diminishes areas globally rather than locally, it is shown that nontrivial periodic solutions can not exist; this is a generalization of the well-known 2-dimensional Bendixson condition for the nonexistence of periodic solutions.

Examples of applications to concrete differential equations are given throughout and the thesis concludes with an application of the Bendixson condition to an epidemiological model.

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Introduction

This thesis extends the theory and applications of compound matrices to differential equations in Banach spaces. The research developed provides new tools for the study of global behaviour of infinite dimensional dynamical systems.

Most of the developments on compound operators have occurred in the context of linear and multilinear algebra. Results on Hilbert spaces are discussed in [1, 7, 8, 94, 104]. A classical body of work dealing with algebraic aspects of multiplicative compound operators in $\mathbb{R}^n$, usually called compound matrices, can be found in [3, 33, 47, 67, 109]. One of the best-known results is the Binet-Cauchy identity. Good historical summaries are [75–78, 91]. In contrast, the literature on additive compound matrices is quite sparse. In the final chapter of the lecture notes [110] Wielandt discusses algebraic and spectral properties of both multiplicative and additive compound matrices. The same approach is taken in the book of Marshall and Olkin [67]. Fiedler [31] presents algebraic aspects of additive and multiplicative compounds in a coordinate-free setting and has applications to stochastic matrices.

Apart from the Abel-Jacobi-Liouville equation, which is also the formula for the determinant of an $n \times n$ matrix solution of a linear differential equation, applications of compound matrices to differential equations begin in the 1970s. Other cases are considered for special equations by Mikusinski [73], Nehari [82] and less directly by Hartman [41], Corollary 3.1, Chapter IV. However, the first treatment in full generality is due to Schwarz [95] in his study on the total positivity of fundamental matrices to general linear systems. London [61] derives a large number of interesting properties of additive compounds based on the relationship between a linear ordinary differential equation and its compound differential equations, and shows how properties of compound matrices may be used to greatly simplify many classical spectral inequalities.

Muldowney [76–78] systematically develops the theory of compound differential equations in his study of linear and nonlinear problems of differential equations in R^n, such as dichotomy and stability theory, orbital stability of periodic solution of nonlinear autonomous systems and establishes a new approach to higher dimensional criteria for the non-existence of periodic orbits analogous to the conditions of Bendixson and Dulac for planar systems. Li and Muldowney [52–54, 56, 58] further explore problems such as lower and upper estimates for the Hausdorff dimension of the global attractor of dissipative dynamical systems, higher dimensional generalization of the criteria of Bendixson and Dulac for the nonexistence of periodic solutions, the existence and stability of equilibria and periodic orbits of nonlinear autonomous systems on invariant manifolds and global stability problems. They solve a long-standing open problem on the global stability of the endemic equilibrium of the SEIRS models with general nonlinear incidence rate in epidemiology.

Temam [104] reviews the definitions of the exterior product of Hilbert spaces and of two multilinear operators, which are the additive and multiplicative compound operators of this thesis. Emphasis is placed on investigation of bounds on the growth of exterior products of solutions of differential equations in terms of Lyapunov exponents rather than specific representations for the compound operators. Central to the discussion in Chapter V of [104] is the behaviour of distance, area, and, more generally, k-volumes which are generated locally by the semigroup $\{S(t)\}_{t\geq 0}$ of a dynamical system. This book gives an exposition of the work of P. Constantin showing that the evolution of oriented k-dimensional volumes is related to the k-th multiplicative compound operator of the differential of the map $S(t)$. Applications to an upper estimation of the exponential decay rate of the volume and the Lyapunov exponents to partial differential equations are also discussed in Chapter VI. The concept of evolution of k-volumes in the dynamics is also emphasized in this thesis.

In Chapter 1, the exterior product of vector spaces and additive and multiplicative compound operators are defined and compound differential equations associated with linear differential equations in a Banach space X are discussed with examples.

In the linear theory, Chapter 2 investigates the codimension of the asymptotically stable solution subspace of a linear differential equation, which gives new insights into the qualitative theory of linear and nonlinear dynamical systems. The objective of the study is to examine, through linearization, the behaviour of nonlinear dynamical systems in the neighbourhood of particular invariant structures such as equilibria and periodic orbits.

Chapter 3 establishes a generalization of the 2-dimensional Poincaré's stability criterion to differential equations in Banach spaces. It is well understood that for periodic orbits of evolutionary differential equations, the moduli of Floquet multipliers determine their stability characteristics. However, the problem of directly estimating the Floquet multipliers is inherently difficult. It is shown that the orbital stability of periodic solutions is equivalent to the stability of a related compound linear system. This permits the use of simpler techniques such as Lyapunov functions in the estimation of the multipliers. Applications to reaction diffusion equations give new insight on the effect of diffusion terms.

Chapter 4 discusses the structure of omega limits of differential equations in Banach spaces. A criterion for the existence of periodic orbits for differential equations in Banach spaces is developed. The finite dimensional motivation is the Poincaré-Bendixson theory. The characterization of the stability of steady state solutions in terms of stability of linearizations and orbital stability of periodic orbits in terms of stability of second compound differential equations, discussed in Chapter 2 and Chapter 3 respectively, are special cases of this chapter.

In Chapter 5, an infinite dimensional analogue of the Bendixson criterion for the nonexistence of periodic orbits is established. The generalized Bendixson criterion states that, if some measure of 2-dimensional surface area tends to zero with time, then there are no closed curves that are left invariant by the dynamics. In particular, there are no nontrivial periodic orbits, homoclinic loops or heteroclinic loops. In this chapter, the Bendixson conditions for general nonlinear differential equations are developed in terms of stability of associated compound differential equations.

In Chapter 6, the results of this thesis are applied to a diffusive SIR model. A SIR model is an epidemiological model that is used to study the spread of an infectious disease, such as measles, mumps and rubella. In this chapter, the Bendixson criterion of Chapter 5 is used to establish the nonexistence of periodic orbits under certain conditions.

Appendix A gives a brief synopsis of relevant properties of Lozinskiĭ measures for linear operators. These arise when norms are used as Lyapunov functions in stability analysis of linear systems.

Appendix B is a summary on sectorial operators and some notations which will be used throughout this thesis.

Chapter 1

Compound Operators and Compound Equations

This chapter contains essential definitions and results that will be used extensively throughout the thesis. A theory of compound operators and compound differential equations in a Banach space X is developed.

The definitions and properties of multiplicative and additive compound operators in $\mathbb{R}^n$, usually called compound matrices, can be found in [3, 31, 33, 47, 67, 75–78, 91, 109, 110]. Applications of compound matrices to differential equations are discussed in [41, 52–54, 56, 58, 61, 73, 76–78, 82, 95]. Results on Hilbert spaces are discussed in [1, 7, 8, 94, 104]. In particular, when X is a Hilbert space, Temam [104] relates the evolution of oriented k-dimensional volumes to the kth multiplicative compound operator of the differential of the semigroup $\{S(t)\}_{t\geq 0}$ of a dynamical system.

This chapter introduces the exterior product of vector spaces and additive and multiplicative compound operators. Compound differential equations associated with linear differential equations in a Banach space X are discussed with examples.

1.1 Exterior Products

Let X, Y be vector spaces over $\mathbb{R}$. Let $\mathscr{L}(X,Y)$ denote the space of all linear functionals from X to Y and $X^* = \mathscr{L}(X,\mathbb{R})$ denote the (algebraic) dual space of X. Define a nondegenerate bilinear map $\langle\cdot,\cdot\rangle \ : \ X^* \times X \to \mathbb{R}$ by

$$\langle v, u\rangle = v(u) \tag{1.1}$$

where $v \in X^*$ and $u \in X$.

Definition 1.1. For every $k \in \mathbb{N}$ and elements $u^1, \cdots, u^k \in X$, define the *kth exterior product* $u^1 \wedge \cdots \wedge u^k$ by

$$u^1 \wedge \cdots \wedge u^k \ : \ (X^*)^k \longrightarrow \mathbb{R}$$

$$u^1 \wedge \cdots \wedge u^k(v_1, \cdots, v_k) := \det[v_i(u^j)], \quad v_i \in X^*, \ i = 1, \cdots, k. \qquad (1.2)$$

Definition 1.2. For each $k \in \mathbb{N}$, the *kth exterior power* of X, $\bigwedge^k X$, is the vector space of all finite linear combinations of elements $u^1 \wedge u^2 \wedge \cdots \wedge u^k$ where $u_i \in X, \ i = 1, \cdots, k$, whose vector addition and scalar multiplication are defined by

$$\begin{aligned} &(u^1 \wedge u^2 \wedge \cdots \wedge u^k + w^1 \wedge w^2 \wedge \cdots \wedge w^k)(v_1, v_2, \cdots, v_k) \\ = {}& u^1 \wedge u^2 \wedge \cdots \wedge u^k(v_1, v_2, \cdots, v_k) + w^1 \wedge w^2 \wedge \cdots \wedge w^k(v_1, v_2, \cdots, v_k), \\ &\alpha(u^1 \wedge u^2 \wedge \cdots \wedge u^k) = (\alpha u^1) \wedge u^2 \wedge \cdots \wedge u^k = \cdots = u^1 \wedge u^2 \wedge \cdots \wedge (\alpha u^k), \end{aligned}$$

where $\alpha \in \mathbb{R}, u^i, w^i \in X, v_i \in X^*, \quad i = 1, \cdots, k$.

Remark 1.1. Equation (1.2) also defines $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k X^*$ as a multilinear map from X^k to $\mathbb{R}$ and a nondegenerate bilinear map $\langle \cdot, \cdot \rangle : \bigwedge^k X^* \times \bigwedge^k X \to \mathbb{R}$ by

$$\langle v_1 \wedge \cdots \wedge v_k, u^1 \wedge \cdots \wedge u^k \rangle = u^1 \wedge \cdots \wedge u^k(v_1, \cdots, v_k).$$

In this thesis, both notations will be used.

Proposition 1.1. *The vectors $u^1, \cdots, u^k$ are linearly dependent in X if and only if*

$$u^1 \wedge u^2 \wedge \cdots \wedge u^k = 0,$$

that is, $u^1 \wedge \cdots \wedge u^k(v_1, \cdots, v_k) = 0$ for all $v_1, \cdots, v_k \in X^*$.

Proof. It is straightforward that $u^1 \wedge \cdots \wedge u^k = 0$ if $u^1, \cdots, u^k$ are linearly dependent in X. Mathematical induction is used to prove that $u^1 \wedge u^2 \wedge \cdots \wedge u^k = 0$ implies the linear dependence of $u^1, \cdots, u^k$. First, it is true if $k = 1$. Assume that $u^1 \wedge \cdots \wedge u^k = 0$ implies that $u^1, \cdots, u^k$ is linearly dependent in X. Next, suppose that $u^1 \wedge \cdots \wedge u^k \wedge u^{k+1} = 0$. Then for every $v_1, \cdots, v_{k+1} \in X^*$,

$$u^1 \wedge \cdots \wedge u^{k+1}(v_1, \cdots, v_{k+1}) = 0.$$

Choose v_1 such that $v_1(u^1) \neq 0$. Let $\alpha_i = -\frac{v_1(u^i)}{v_1(u^1)}, 2 \leq i \leq k+1$. Then

$$\begin{aligned} 0 &= u^1 \wedge \cdots \wedge u^{k+1}(v_1, \cdots, v_{k+1}) \\ &= \det \begin{bmatrix} v_1(u^1) & 0 & \cdots & 0 \\ v_2(u^1) & v_2(u^2 + \alpha_2 u^1) & \cdots & v_2(u^{k+1} + \alpha_{k+1} u^1) \\ \vdots & \vdots & \ddots & \vdots \\ v_{k+1}(u^1) & v_{k+1}(u^2 + \alpha_2 u^1) & \cdots & v_{k+1}(u^{k+1} + \alpha_{k+1} u^1) \end{bmatrix} \\ &= v_1(u^1)(u^2 + \alpha_2 u^1) \wedge \cdots \wedge (u^{k+1} + \alpha_{k+1} u^1)(v_2, \cdots, v_{k+1}), \end{aligned}$$

which implies that

$$(u^2 + \alpha_2 u^1) \wedge \cdots \wedge (u^{k+1} + \alpha_{k+1} u^1)(v_2, \cdots, v_{k+1}) = 0$$

for all $v_2, \cdots, v_{k+1}$ since $v_1(u^1) \neq 0$. Therefore

$$(u^2 + \alpha_2 u^1) \wedge \cdots \wedge (u^{k+1} + \alpha_{k+1} u^1) = 0,$$

and the induction hypothesis implies that $u^2 + \alpha_2 u^1, \cdots, u^{k+1} + \alpha_{k+1} u^1$ are linearly dependent. Hence $u^1, \cdots, u^{k+1}$ are linearly dependent. Mathematical induction implies that $u^1, \cdots, u^k$ are linearly dependent in X if $u^1 \wedge \cdots \wedge u^k = 0$. ∎

Let X be a normed space and X' denote the continuous dual of X, consisting of the continuous linear functionals on X. In general different topologies and norms can be imposed on $\bigwedge^k X$ depending on the application. For $w \in \bigwedge^k X$, define

$$\|w\|_{\bigwedge^k X} := \sup_{v_i} w(v_1, \cdots, v_k) = \sup_{v_i} \langle v_1 \wedge \cdots \wedge v_k, w \rangle . \qquad (1.3)$$

Here the supremum is taken over $v_i \in X', \|v_i\|_{X'} \leq 1, i = 1, \cdots, k$. For simplicity, the symbol $\|\cdot\|$ will be used instead of $\|\cdot\|_{\bigwedge^k X}$ except when the relationship with the norm in X is to be emphasized. Then for $w, w_1, w_2 \in \bigwedge^k X, a \in \mathbb{R}$,

(*i*) $\|w\| \geq 0$;

(*ii*) $\|w\| = 0$ if and only if for every $v_i \in X'$, $w(v_1, v_2, \cdots, v_k) = 0$ if and only if $w = 0$;

(*iii*) $\|aw\| = |a|\|w\|$;

(*iv*) $$\|w_1 + w_2\| = \sup_{v_i}(w_1(v_1, v_2, \cdots, v_k) + w_2(v_1, v_2, \cdots, v_k)) \leq \sup_{v_i} w_1(v_1, v_2, \cdots, v_k) + \sup_{v_i} w(v_1, v_2, \cdots, v_k) \leq \|w_1\| + \|w_2\|.$$

Therefore, $\|\cdot\|$ is a norm on $\bigwedge^k X$. The symbol $\hat{\bigwedge}^k X$ is used to denote the completion of $\bigwedge^k X$ in this norm.

Proposition 1.2. *If X is a normed space, then for every $u^i \in X$, $i = 1, \cdots, k$, the norm on $\bigwedge^k X$ defined by* (1.3) *satisfies*

$$\|u^1 \wedge \cdots \wedge u^k\| \leq 2^{\frac{k(k-1)}{2}} \|u^1\| \cdots \|u^k\|. \tag{1.4}$$

Proof. The proof is by mathematical induction . First, equality is satisfied in (1.4) when $k = 1$. Assume that (1.4) holds for k, a positive integer. Then if $u^i \in X, v_i \in X'$, $i = 1, \cdots, k+1$ and $v_1 \wedge \cdots \wedge \hat{v}_l \wedge \cdots \wedge v_{k+1}$ denotes the exterior product of the k vectors $\{v_i : i = 1, \cdots, k+1, i \neq l\}$, then

$$\begin{aligned}
& \langle v_1 \wedge \cdots \wedge v_{k+1}, u^1 \wedge \cdots \wedge u^{k+1} \rangle \\
= & \sum_{l=1}^{k+1} (-1)^{l+1} \langle v_l, u^1 \rangle \langle v_1 \wedge \cdots \wedge \hat{v}_l \wedge \cdots \wedge v_{k+1}, u^2 \wedge \cdots \wedge u^{k+1} \rangle \\
\leq & (k+1) \cdot \|u^1\| \cdot 2^{\frac{k(k-1)}{2}} \|u^2\| \cdots \|u^{k+1}\| \\
\leq & 2^k \cdot 2^{\frac{k(k-1)}{2}} \|u^1\| \|u^2\| \cdots \|u^{k+1}\| \\
= & 2^{\frac{k(k+1)}{2}} \|u^1\| \|u^2\| \cdots \|u^{k+1}\|.
\end{aligned}$$

Thus

$$\|u^1 \wedge \cdots \wedge u^{k+1}\| \leq 2^{\frac{k(k+1)}{2}} \|u^1\| \cdots \|u^{k+1}\|,$$

and mathematical induction implies that (1.4) holds for all k. ∎

1.1.1 Vector Space: Hamel Basis

Every real vector space X has a Hamel basis $H \subset X$ such that each $u \in X$ is a finite real linear combination of elements $h \in H$, a sum of the form

$$u = \sum_{h \in H} u_h h,$$

where all but a finite number of $u_h = 0$. With each $h \in H$ associate a linear functional $\langle h, \cdot \rangle : X \to \mathbb{R}$ defined for k in H by

$$\langle h, k \rangle = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{if } k \neq h \end{cases}$$

and, for any $u \in X$, by linearity so that $\langle h, u \rangle = u_h$. Thus

$$u = \sum_{h \in H} \langle h, u \rangle \, h. \tag{1.5}$$

Finally, for any $u \in X$, a linear functional $\langle u, \cdot \rangle : X \to \mathbb{R}$ is defined by linearity $\langle u, \cdot \rangle = \sum_{h \in H} \langle h, u \rangle \langle h, \cdot \rangle$. It follows that, when H is a Hamel basis of X, $u^1 \wedge \cdots \wedge u^k$ can be expressed as a finite sum of the form

$$u^1 \wedge \cdots \wedge u^k = \sum_{h^i \in H,\ i=1,\cdots,k} \left\langle h^1, u^1 \right\rangle \left\langle h^2, u^2 \right\rangle \cdots \left\langle h^k, u^k \right\rangle h^1 \wedge \cdots \wedge h^k. \tag{1.6}$$

However, if $k > 1$, $\left\{ h^1 \wedge \cdots \wedge h^k : h^i \in H,\ i = 1, \cdots, k \right\}$ is not a Hamel basis for $\bigwedge^k X$. It is not a linearly independent set as

$$h^1 \wedge \cdots \wedge h^k = \operatorname{sgn}\sigma \; h^{i_1} \wedge \cdots \wedge h^{i_k} \tag{1.7}$$

if $\sigma = (i_1, \cdots, i_k)$ is any permutation of $(1, \cdots, k)$. A subset H_k obtained by including just one element $h^1 \wedge \cdots \wedge h^k$ of each set of $k!$ elements related by (1.7) is a Hamel basis for $\bigwedge^k X$ and, from (1.6),

$$u^1 \wedge \cdots \wedge u^k = \sum_{h^1 \wedge \cdots \wedge h^k \in H_k} \left\langle h^1 \wedge \cdots \wedge h^k, u^1 \wedge \cdots \wedge u^k \right\rangle h^1 \wedge \cdots \wedge h^k,$$

where $\left\langle h^1 \wedge \cdots \wedge h^k, u^1 \wedge \cdots \wedge u^k \right\rangle = \det \langle h^i, u^j \rangle$.

1.1.2 Normed Vector Space: Schauder Basis

Suppose that $S = \{e^i : i = 1, 2, \cdots\}$ is a Schauder (countable) basis of a real normed vector space X, then

$$u = \sum_i u_i e^i,$$

where $u_i = \langle e^i, u \rangle$ is the ith coordinate of u with respect to this basis. Here $\langle e^i, \cdot \rangle$ is defined as for (1.5) but now the sum (1.5) may contain infinitely many non-zero terms when X is not finite dimensional. Analogously to (1.7), the set

$$S_k = \left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \le i_1 < \cdots < i_k\right\}$$

is a basis for $\bigwedge^k X$ under the norm $\|\cdot\|$ defined in (1.3). No particular order is imposed on S_k. By Definition 1.2, if $u^j \in X,\ \ j = 1, \cdots, k,\ u^j = \sum\limits_i u_i^j e^i,\ u_i^j = \langle e^i, u^j\rangle$, then

$$u^1 \wedge \cdots \wedge u^k = \sum_{(i)} u_{(i)} e^{i_1} \wedge \cdots \wedge e^{i_k},$$

where

$$\begin{aligned} u_{(i)} &= \left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, u^1 \wedge \cdots \wedge u^k \right\rangle \\ &= \det\left[\langle e^{i_r}, u^j\rangle\right], \qquad r, j = 1, \cdots, k \\ &= \det\left[u_{i_r}^j\right], \qquad r, j = 1, \cdots, k \\ &= u_{i_1 \cdots i_k}^{1 \cdots k}. \end{aligned}$$

Thus the $(i) = (i_1 \cdots i_k)$-th coordinate $u_{(i)} = u_{i_1 \cdots i_k}^{1 \cdots k}$ of $u^1 \wedge \cdots \wedge u^k$, $1 \le i_1 < \cdots < i_k$, with respect to the basis $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}$ in $\bigwedge^k X$ is the $k \times k$ minor determined by the rows $i_1, \cdots, i_k$ of the matrix $\left[u_i^j\right]$.

If X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then $\bigwedge^k X$ is also a Hilbert space with inner product

$$\left\langle u^1 \wedge \cdots \wedge u^k, v^1 \wedge \cdots \wedge v^k \right\rangle = \det\left[\langle u^i, v^j \rangle\right]. \tag{1.8}$$

The space $\hat{\bigwedge}^k X$ is the completion of $\bigwedge^k X$ under the associated norm $\langle w_1, w_2\rangle^{\frac{1}{2}}$. If X is separable and $\{e^i\}$ is an orthonormal basis, then $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : i_1 < \cdots < i_k\}$ forms an orthonormal basis of $\hat{\bigwedge}^k X$. It follows from

$$\left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, e^{j_1} \wedge \cdots \wedge e^{j_k} \right\rangle = \delta_{(j)}^{(i)}$$

that

$$\begin{aligned} \|u^1 \wedge \cdots \wedge u^k\|^2 &= \left\langle u^1 \wedge \cdots \wedge u^k, u^1 \wedge \cdots \wedge u^k \right\rangle \\ &= \sum_{i_1 < \cdots < i_k} \left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, u^1 \wedge \cdots \wedge u^k \right\rangle^2 \\ &= \sum_{i_1 < \cdots < i_k} \left(u_{i_1 \cdots i_k}^{1 \cdots k}\right)^2. \end{aligned}$$

Remark 1.2. A good reference on the exterior product of a Hilbert space is Temam [104], Chapter V.

1.1.3 Function Space

Let X be a vector space of real-valued functions on a set Ω. Then a linear functional $x \in X^*$ may be associated with $x \in \Omega$ by

$$\langle x, u \rangle = u(x)$$

if $u \in X$. For $u^1, \cdots, u^k \in X$ and $x_1, \cdots, x_k \in \Omega$,

$$\langle x_1 \wedge \cdots \wedge x_k, u^1 \wedge \cdots \wedge u^k \rangle = u^1 \wedge \cdots \wedge u^k(x_1, \cdots, x_k) = \det \left[u^j(x_i)\right].$$

Norms on $\bigwedge^k X$ may be defined by

$$\|w\|_\infty := \text{ess sup}\left\{|w(x_1, \cdots, x_k)| : x_i \in \Omega\right\} \tag{1.9}$$

or, if $1 \leq p < \infty$ and Ω is a measurable set by

$$\|w\|_{L^p} := \frac{1}{k!}\left[\int_{\Omega^k} |w(x_1, \cdots, x_k)|^p \, dx_1 \cdots dx_k\right]^{\frac{1}{p}}; \tag{1.10}$$

when these expressions are finite. If Ω is finite or countable, the integral (1.10) may be replaced by a discrete sum. For example, when $X = \mathbb{R}^n$ and $\Omega = \{1, 2, \cdots, n\}$, for $u^i = (u^i_1, \cdots, u^i_n), i = 1, \cdots, k$,

$$\langle 1 \wedge \cdots \wedge k, u^1 \wedge \cdots \wedge u^k \rangle = u^1 \wedge \cdots \wedge u^k(1, \cdots, k) = \det \left[u^j_i\right]$$

and the l_p norm of $u^1 \wedge \cdots \wedge u^k$ on $\bigwedge^k \mathbb{R}^n$ is

$$\|u^1 \wedge \cdots \wedge u^k\|_{l_p} = \frac{1}{k!}\left(\sum_{i_1, \cdots, i_k} |u^{1\cdots k}_{i_1 \cdots i_k}|^p\right)^{\frac{1}{p}} = \left(\sum_{1 \leq i_1 < \cdots < i_k} |u^{1\cdots k}_{i_1 \cdots i_k}|^p\right)^{\frac{1}{p}}$$

where $u^{1\cdots k}_{i_1 \cdots i_k}$ is the $k \times k$ minor of the $n \times k$ matrix $(u^1, \cdots, u^k)$ determined by the rows $i_1, \cdots, i_k$.

1.1.4 $L^2(\Omega)$

The choice of v plays an important role in the application of exterior products. The following example on $X = L^2(\Omega)$, where Ω is a measurable set in $\mathbb{R}^n$, shows different representations of $u^1 \wedge u^2$ associated with different choices of v.

Case 1: A pointwise representation of exterior products

For $u \in L^2(\Omega), x \in \Omega$, let x again denote the linear functional $u \mapsto \langle x, u \rangle := u(x)$. Define an inner product

$$\langle u, v \rangle = \int_\Omega u(x)\, v(x)\, dx.$$

For $u^1, u^2 \in X$ and $x_1, x_2 \in \Omega$,

$$u^1 \wedge u^2(x_1, x_2) = \det\left[u^i(x_j)\right].$$

Then, from (1.8),

$$\left\langle u^1 \wedge u^2, v^1 \wedge v^2 \right\rangle = \det\begin{bmatrix} \langle u^1, v^1 \rangle & \langle u^1, v^2 \rangle \\ \langle u^2, v^1 \rangle & \langle u^2, v^2 \rangle \end{bmatrix} = \det\left[\int_\Omega u^i v^j\right] \tag{1.11}$$

is an inner product on $\bigwedge^2 L^2(\Omega)$. Since $u^1 \wedge u^2(x_1, x_2) = \det[u^i(x_j)]$,

$$\left(u^1 \wedge u^2, v^1 \wedge v^2\right) = \int_{\Omega^2} \det\left[u^i(x_j)\right] \det\left[v^i(x_j)\right] dx_1 dx_2 \tag{1.12}$$

is also an inner product on $\bigwedge^k L^2(\Omega)$. In fact, the inner products in (1.11) and (1.12) are equivalent since

$$\left\langle u^1 \wedge u^2, v^1 \wedge v^2 \right\rangle = \frac{1}{2!}\left(u^1 \wedge u^2, v^1 \wedge v^2\right). \tag{1.13}$$

This can be seen by applying the Binet-Cauchy identity for square matrices $[u], [v], \det[u]\det[v] = \det[uv]$, to the integrand in (1.12).

Case 2: A basis representation of exterior products

Suppose that $\{e^i\}$ is an orthonormal basis of $L^2(\Omega)$. Then $u = \sum_i \langle e^i, u \rangle e^i$. Define another linear map $e^i : u \mapsto \langle e^i, u \rangle$. For $u^1, u^2 \in X$, $u^1 \wedge u^2$ is a bilinear map from $\{e^i\} \times \{e^i\}$ to $\mathbb{R}$ and

$$u^1 \wedge u^2(e^{i_1}, e^{i_2}) = \det\begin{bmatrix} \langle e^{i_1}, u^1 \rangle & \langle e^{i_1}, u^2 \rangle \\ \langle e^{i_2}, u^1 \rangle & \langle e^{i_2}, u^2 \rangle \end{bmatrix}.$$

The set $\{e^{i_1} \wedge e^{i_2} : i_1 < i_2\}$ forms an orthonormal basis of $\bigwedge^2 L^2(\Omega)$ and if $u^j = \sum_i \langle e^i, u^j \rangle e^i, j = 1, 2$, then

$$u^1 \wedge u^2 = \sum_{(i)} u_{(i)} e^{i_1} \wedge e^{i_2}$$

where

$$\begin{aligned} u_{(i)} &= \langle e^{i_1} \wedge e^{i_2}, u^1 \wedge u^2 \rangle \\ &= \det \left[\langle e^{i_r}, u^p \rangle \right], \quad r, j = 1, 2 \\ &= \det \left[u^p_{i_r} \right], \quad r, j = 1, 2 \\ &= u^{12}_{i_1 i_2} \end{aligned} \tag{1.14}$$

is the $(i) = (i_1 i_2)$-th coordinate of $u^1 \wedge u^2$, $i_1 < i_2$, with respect to the basis $\{e^{i_1} \wedge e^{i_2}\}$ in $\bigwedge^2 L^2(\Omega)$, which is the 2×2 minor determined by the rows i_1, i_2 of the matrix $\left[u^j_i\right]$ (see Section 1.1.2). It follows from (1.13) that

$$\|u^1 \wedge u^2\|^2 = \sum_{i_1 < i_2} \left(u^{12}_{i_1 i_2}\right)^2 = \frac{1}{2!} \int_{\Omega^2} \det \left[u^i(x_j)\right]^2 dx_1 dx_2,$$

where $u^{12}_{i_1 i_2}$ is defined by (1.14) for any basis orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$ on $L^2(\Omega)$.

1.1.5 Interpretation

In keeping with the tradition of representing a vector u as a directed line segment, the exterior product $u^1 \wedge \cdots \wedge u^k \in \bigwedge^k X$ can be interpreted as the oriented parallelepiped determined by the ordered set of vectors $\{u^1, \cdots, u^k\}$ (see Figure 1.1). Just as $\|u\|$ can be considered a measure of the length of u

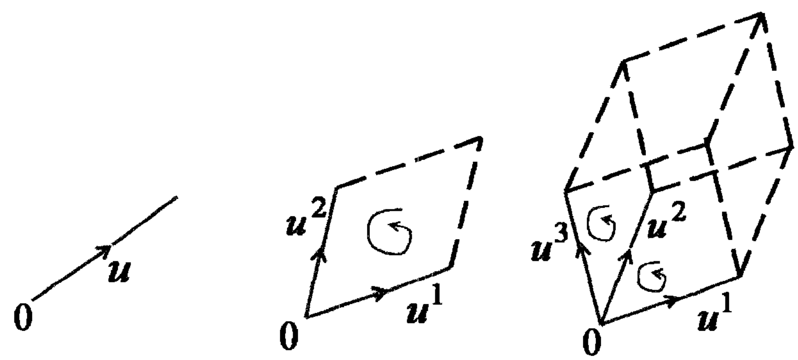

Figure 1.1: Exterior product $u^1 \wedge \cdots \wedge u^k$

when it is represented as directed line segment, a norm $\|u^1 \wedge \cdots \wedge u^k\|$ may be considered a measure of the k-volume of the oriented parallelepiped. If u is referred to a basis $\{e^i\}$, then $u = \sum_i u_i e^i$ and the coordinate $u_i = \langle e^i, u \rangle$ may be considered the projection of the directed line segment onto the coordinate axis e^i. With respect to the basis $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}$ of $\bigwedge^k X$, $u^1 \wedge \cdots \wedge u^k = \sum_{(i)} u_{(i)} e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $u_{(i)} = \langle e^{i_1} \wedge \cdots \wedge e^{i_k}, u^1 \wedge \cdots \wedge u^k \rangle = u^{1 \cdots k}_{i_1 \cdots i_k}$ may be considered the projection of $u^1 \wedge \cdots \wedge u^k$ onto the k-dimensional subspace of X spanned by $e^{i_1}, \cdots, e^{i_k}$. For example, when $X = \mathbb{R}^n$ with the standard orthonormal basis $e^1, \cdots, e^n$, then $\bigwedge^k \mathbb{R}^n \simeq \mathbb{R}^{\binom{k}{n}}$ and the $\binom{n}{k}$ components of $u^1 \wedge \cdots \wedge u^k$ with respect to the k-dimensional coordinate subspace *span* $\{e^{i_1}, \cdots, e^{i_k}\}$ are the determinants $u^{1 \cdots k}_{i_1 \cdots i_k}$.

1.2 Compound Operators

The definitions of compound operators on vector spaces are generalized from compound matrices on $\mathbb{R}^n$, (see [3, 33, 47, 67, 109]) and the two multilinear operators on Hilbert spaces considered in Temam [104], Chapter V.

Definition 1.3. Let X and Y be vector spaces and $A : X \to Y$ be a linear operator. The operator $A^{(k)} : \bigwedge^k X \to \bigwedge^k Y$ defined by

$$A^{(k)} \left(u^1 \wedge \cdots \wedge u^k\right) := Au^1 \wedge \cdots \wedge Au^k, \quad u^i \in X \tag{1.15}$$

and extended by linearity to $\bigwedge^k X$ is called the *kth multiplicative compound* (or *exterior power*) of A.

Definition 1.4. Let X and Y be vector spaces, $X \subset Y$ and $A : X \to Y$ be a linear operator. The operator $A^{[k]} : \bigwedge^k X \to \bigwedge^k Y$ defined by

$$A^{[k]} \left(u^1 \wedge \cdots \wedge u^k\right) := \sum_{j=1}^{k} u^1 \wedge \cdots \wedge Au^j \wedge \cdots \wedge u^k, \quad u^i \in X \tag{1.16}$$

and extended by linearity to $\bigwedge^k X$ is called the *kth additive compound* of A.

Remark 1.3. In [104], Temam uses $\bigwedge^k A$ and A_k to denote what are, in this thesis, called the kth multiplicative compound operator and kth additive compound operator of A.

The multiplicative and additive compound operators have the following properties.

Proposition 1.3.

(*i*) *If* $A : X \to Y$ *and* $B : Y \to Z$ *are linear operators, then* $(AB)^{(k)}$ *satisfies the Binet-Cauchy identity*

$$(AB)^{(k)} = A^{(k)}B^{(k)}.$$

(*ii*) *If* $X \subset Y$ *and* $A, B : X \to Y$ *are linear operators, then* $(A+B)^{[k]}$ *satisfies the additive identity*

$$(A+B)^{[k]} = A^{[k]} + B^{[k]}.$$

(*iii*) *If* $X \subset Y$ *and* $A : X \to Y$ *is a linear operator, then* $A^{[k]} = \frac{d}{dt}(I+tA)^{(k)}\Big|_{t=0} = \lim_{h\to 0}\left[(I+hA)^{(k)} - I^{(k)}\right]\Big/ h$.

(*iv*) *If* $\lambda_1, \lambda_2, \cdots$ *and* $e^1, e^2, \cdots$ *are eigenvalues and eigenvectors of* A*, then* $e^{i_1} \wedge \cdots \wedge e^{i_k}, 1 \le i_1 < i_2 < \cdots < i_k$ *are eigenvectors of* $A^{(k)}$ *and* $A^{[k]}$ *with corresponding eigenvalues* $\lambda_{i_1} \cdots \lambda_{i_k}$ *and* $\lambda_{i_1} + \cdots + \lambda_{i_k}$*, respectively.*

Proof. (*i*) Let $u^i \in X,\ i = 1, \cdots, k$. Then

$$\begin{aligned}(AB)^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right) &= ABu^1 \wedge \cdots \wedge ABu^k \\ &= A^{(k)}(Bu^1 \wedge \cdots \wedge Bu^k) \\ &= A^{(k)}B^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right)\end{aligned}$$

and this equality can be extended by linearity to $\bigwedge^k X$. Thus the Binet-Cauchy identity is proved.

(*ii*) Let $u^i \in X,\ i = 1, \cdots, k$. Then

$$\begin{aligned}&(A+B)^{[k]}\left(u^1 \wedge \cdots \wedge u^k\right) \\ =\ &\sum_{j=1}^{k} u^1 \wedge \cdots \wedge (A+B)u^j \wedge \cdots \wedge u^k \\ =\ &\sum_{j=1}^{k} u^1 \wedge \cdots \wedge Au^j \wedge \cdots \wedge u^k + \sum_{j=1}^{k} u^1 \wedge \cdots \wedge Bu^j \wedge \cdots \wedge u^k \\ =\ &\left(A^{[k]} + B^{[k]}\right)\left(u^1 \wedge \cdots \wedge u^k\right)\end{aligned}$$

and this equality can be extended by linearity to $\bigwedge^k X$. Thus the additive identity is proved.

(*iii*) Let $u^i \in X,\ i = 1, \cdots, k$. Then

$$\begin{aligned}
& \left[(I+hA)^{(k)} - I^{(k)}\right]\left(u^1 \wedge \cdots \wedge u^k\right) \\
= \;& (I+hA)^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right) - u^1 \wedge \cdots \wedge u^k \\
= \;& (1+hA)u^1 \wedge \cdots \wedge (1+hA)u^k - u^1 \wedge \cdots \wedge u^k \\
= \;& (hAu^1) \wedge u^2 \cdots \wedge u^k + \cdots + u^1 \wedge u^2 \cdots \wedge (hAu^k) + O(h^2) \\
= \;& hA^{[k]}\left(u^1 \wedge u^2 \cdots \wedge u^k\right) + o(h^2)
\end{aligned}$$

and thus

$$\lim_{h\to 0}\left[(I+hA)^{(k)} - I^{(k)}\right]\left(u^1 \wedge u^2 \cdots \wedge u^k\right)\Big/ h = A^{[k]}\left(u^1 \wedge u^2 \cdots \wedge u^k\right).$$

This above equality can be extended by linearity to $\bigwedge^k X$. Therefore

$$A^{[k]} = \frac{d}{dt}(I+tA)^{(k)}\bigg|_{t=0} = \lim_{h\to 0}\left[(I+hA)^{(k)} - I^{(k)}\right]\Big/ h\ .$$

(*iv*) The eigenvalues and eigenfunctions properties follow from

$$\begin{aligned}
A^{(k)}(e^{i_1} \wedge \cdots \wedge e^{i_k}) &= Ae^{i_1} \wedge \cdots \wedge Ae^{i_k} \\
&= \lambda_{i_1} e^{i_1} \wedge \cdots \wedge \lambda_{i_k} e^{i_k} \\
&= \lambda_{i_1} \cdots \lambda_{i_k} (e^{i_1} \wedge \cdots \wedge e^{i_k})
\end{aligned}$$

and

$$\begin{aligned}
A^{[k]}(e^{i_1} \wedge \cdots \wedge e^{i_k}) &= \sum_{j=1}^{k} e^{i_1} \wedge \cdots \wedge Ae^{i_j} \wedge \cdots \wedge e^{i_k} \\
&= \sum_{j=1}^{k} e^{i_1} \wedge \cdots \wedge \lambda_{i_j} e^{i_j} \wedge \cdots \wedge e^{i_k} \\
&= (\lambda_{i_1} + \cdots + \lambda_{i_k})(e^{i_1} \wedge \cdots \wedge e^{i_k}).
\end{aligned}$$

■

1.2.1 Representation of Compound Operators

Let X and Y be real normed vector spaces with Schauder bases $\{e^j\}$ and $\{f^i\}$ respectively. If $A : u \mapsto v = Lu$ is a bounded linear function from X to Y, then

$$\begin{aligned}
u &= \sum_j u_j e^j, \quad u_j = \langle e^j, u\rangle\,, \\
Lu &= \sum_i v_i f^i, \quad v_i = \langle f^i, Lu\rangle
\end{aligned}$$

implies that

$$v_i = \sum_j a_i^j u_j, \text{ where } a_i^j = \langle f^i, Le^j \rangle . \tag{1.17}$$

Thus the relationship between the sequences of basis coefficients or coordinates, $\widehat{u} = (u_i)$ and $\widehat{v} = (v_j)$ of u and $v = Lu$ respectively, arrayed as column vectors, satisfy (1.17) which may be written

$$\widehat{v} = A\widehat{u}$$

where $A = \left[a_i^j\right]$ is a matrix whose rows are indexed by i and columns by j.

By Definition 1.2, if $u^r \in X$, $r = 1, \cdots, k$, $u^r = \sum_j u_j^r e^j$, $u_j^r = \langle e^j, u^r \rangle$, implies

$$u^1 \wedge \cdots \wedge u^k = \sum_{(j)} u_{(j)} e^{j_1} \wedge \cdots \wedge e^{j_k}$$

where

$$u_{(j)} = \left\langle e^{j_1} \wedge \cdots \wedge e^{j_k}, u^1 \wedge \cdots \wedge u^k \right\rangle = u^{1\cdots k}_{j_1 \cdots j_k} \tag{1.18}$$

is the $(j) = (j_1 \cdots j_k)$-th coordinate of $u^1 \wedge \cdots \wedge u^k$, $j_1 < \cdots < j_k$, with respect to the basis $\{e^{j_1} \wedge \cdots \wedge e^{j_k}\}$ in $\bigwedge^k X$ (see Section 1.1.2).

From (1.17), $v^r = Lu^r = \sum_i v_i^r f^i$, $v_j^r = \langle f^i, Lu^r \rangle$ implies, since

$$L^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right) = \left(Lu^1\right) \wedge \cdots \wedge \left(Lu^k\right)$$

the $(i) = (i_1 \cdots i_k)$-th coordinate $v_{(i)}$ of $L^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right)$, $i_1 < \cdots < i_k$, with respect to the basis $\{f^{i_1} \wedge \cdots \wedge f^{i_k}\}$ in $\bigwedge^k Y$ is given by

$$L^{(k)}\left(u^1 \wedge \cdots \wedge u^k\right) = \sum_{(i)} v_{(i)} f^{i_1} \wedge \cdots \wedge f^{i_k}$$

where, from (1.17) and (1.18) with $(i) = (i_1 \cdots i_k)$, $(j) = (j_1 \cdots j_k)$,

$$\begin{aligned} v_{(i)} &= v^{1\cdots k}_{i_1 \cdots i_k} = \sum_{(j)} b_{(i)}^{(j)} u^{1\cdots k}_{j_1 \cdots j_k}, \\ b_{(i)}^{(j)} &= \left\langle f^{i_1} \wedge \cdots \wedge f^{i_k}, L^{(k)}\left(e^{j_1} \wedge \cdots \wedge e^{j_k}\right)\right\rangle \\ &= \left\langle f^{i_1} \wedge \cdots \wedge f^{i_k}, \left(Le^{j_1}\right) \wedge \cdots \wedge \left(Le^{j_k}\right)\right\rangle \\ &= \det\left[\left\langle f^{i_r}, Le^{j_s}\right\rangle\right], \ r, s = 1, \cdots, k \\ &= a^{j_1 \cdots j_k}_{i_1 \cdots i_k} \end{aligned}$$

where $a_{i_1\cdots i_k}^{j_1\cdots j_k}$ denotes the $k\times k$ minor of the matrix $A=\left[a_i^j\right]$ determined by the rows $i_1,\cdots,i_k$ and the columns $j_1,\cdots,j_k$. Denote by

$$A^{(k)}=\left[b_{(i)}^{(j)}\right]=\left[a_{i_1\cdots i_k}^{j_1\cdots j_k}\right], \tag{1.19}$$

the matrix of the linear function $L^{(k)}:\bigwedge^k X\to\bigwedge^k Y$ referred to the bases $\{e^{j_1}\wedge\cdots\wedge e^{j_k}\}$, $\{f^{i_1}\wedge\cdots\wedge f^{i_k}\}$.

Similarly, if $X\subset Y$, the $(i)=(i_1\cdots i_k)$-th coordinate $w_{(i)}$ of $L^{[k]}\left(u^1\wedge\cdots\wedge u^k\right)$, $i_1<\cdots<i_k$,, with respect to the basis $\{e^{i_1}\wedge\cdots\wedge e^{i_k}\}$ in $\bigwedge^k Y$ is given by

$$L^{[k]}\left(u^1\wedge\cdots\wedge u^k\right)=\sum_{(i)}w_{(i)}e^{i_1}\wedge\cdots\wedge e^{i_k}$$

where, from (1.17) and (1.18) with $(i)=(i_1\cdots i_k)$, $(j)=(j_1\cdots j_k)$,

$$\begin{aligned}
w_{(i)} &= \sum_{(j)}c_{(i)}^{(j)}u_{j_1\cdots j_k}^{1\cdots k},\\
c_{(i)}^{(j)} &= \left\langle e^{i_1}\wedge\cdots\wedge e^{i_k}, L^{[k]}\left(e^{j_1}\wedge\cdots\wedge e^{j_k}\right)\right\rangle\\
&= \left\langle e^{i_1}\wedge\cdots\wedge e^{i_k}, \sum_{s=1}^{k}e^{j_1}\wedge\cdots\wedge Le^{j_s}\wedge\cdots\wedge e^{j_k}\right\rangle\\
&= \sum_{s=1}^{k}\left\langle e^{i_1}\wedge\cdots\wedge e^{i_k}, e^{j_1}\wedge\cdots\wedge Le^{j_s}\wedge\cdots\wedge e^{j_k}\right\rangle.
\end{aligned}$$

Let

$$\delta_{i_1\cdots\widehat{i_t}\cdots i_k}^{j_1\cdots\widehat{j_s}\cdots j_k}=\begin{cases}1, & \text{if } i_1\cdots\widehat{i_t}\cdots i_k=j_1\cdots\widehat{j_s}\cdots j_k\\ 0, & \text{if } i_1\cdots\widehat{i_t}\cdots i_k\neq j_1\cdots\widehat{j_s}\cdots j_k\end{cases}$$

where $i_1\cdots\widehat{i_t}\cdots i_k$ denotes the $k-1$ numbers $i_1\cdots i_{t-1}i_{t+1}\cdots i_k$. Since

$$\begin{aligned}
&\left\langle e^{i_1}\wedge\cdots\wedge e^{i_k}, e^{j_1}\wedge\cdots\wedge Le^{j_s}\wedge\cdots\wedge e^{j_k}\right\rangle\\
&=\sum_t(-1)^{s+t}\left\langle e^{i_t},Le^{j_s}\right\rangle\left\langle e^{i_1}\wedge\cdots\wedge\widehat{e^{i_t}}\wedge\cdots\wedge e^{i_k}, e^{j_1}\wedge\cdots\wedge\widehat{e^{j_s}}\wedge\cdots\wedge e^{j_k}\right\rangle\\
&=\sum_t(-1)^{s+t}a_{i_t}^{j_s}\delta_{i_1\cdots\widehat{i_t}\cdots i_k}^{j_1\cdots\widehat{j_s}\cdots j_k},
\end{aligned}$$

the matrix of $L^{[k]}$ is $A^{[k]} = \left[c_{(i)}^{(j)}\right]$, where

$$c_{(i)}^{(j)} = \begin{cases} a_{i_1}^{i_1} + \cdots + a_{i_k}^{i_k}, & \text{if } (i) = (j); \\ (-1)^{s+t} a_{i_t}^{j_s}, & \text{if exactly one entry } i_t \text{ of } (i) \text{ does not occur in} \\ & (j) \text{ and } j_s \text{ does not occur in } (i); \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases} \tag{1.20}$$

Example 1.1. When $X = \mathbb{R}^3$, $\bigwedge^k \mathbb{R}^3 = \operatorname{span}\{u^1 \wedge \cdots \wedge u^k : u^i \in \mathbb{R}^3\} \simeq \mathbb{R}^{\binom{3}{k}}$. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator, which has a matrix representation $A = \left[a_i^j\right]_{3\times 3}$. Then

$$A^{(1)} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix} = A, \quad A^{(2)} = \begin{bmatrix} a_{12}^{12} & a_{12}^{13} & a_{12}^{23} \\ a_{13}^{12} & a_{13}^{13} & a_{13}^{23} \\ a_{23}^{12} & a_{23}^{13} & a_{23}^{23} \end{bmatrix}, \quad A^{(3)} = a_{123}^{123} = \det A,$$

$$A^{[1]} = A, \quad A^{[2]} = \begin{bmatrix} a_1^1 + a_2^2 & a_2^3 & -a_1^3 \\ a_3^2 & a_1^1 + a_3^3 & a_1^2 \\ -a_3^1 & a_2^1 & a_2^2 + a_3^3 \end{bmatrix}, \quad A^{[3]} = a_1^1 + a_2^2 + a_3^3 = \operatorname{tr} A.$$

Example 1.2. Let $X = C(\Omega, \mathbb{R})$ where Ω is a measurable set in $\mathbb{R}^n$ and $p \in C(\Omega \times \Omega \to \mathbb{R})$. Consider an integral operator $P : X \to X$,

$$(Pu)(x) = \int_\Omega p(x, y)\, u(y)\, dy. \tag{1.21}$$

Let $(Qu)(x) = \int_\Omega q(x, y)\, u(y)\, dy$ and the multiplication of P and Q is defined by

$$(PQu)(x) = \int_\Omega pq(x, y)\, u(y)\, dy,$$

where $pq(x, y) = \int_\Omega p(x, s)\, q(s, y)\, ds$. Now the operator $P^{(k)} : \bigwedge^k X \to \bigwedge^k X$,

$$\left(P^{(k)} w\right)(x_1, \cdots, x_k) = \frac{1}{k!} \int_{\Omega^k} \det\left[p(x_i, y_j)\right] w(y_1, \cdots, y_k)\, dy_1 \cdots dy_k,$$

satisfies the Binet-Cauchy identity $(PQ)^{(k)} = P^{(k)} Q^{(k)}$; and the operator $P^{[k]} : \bigwedge^k X \to \bigwedge^k X$,

$$\begin{aligned} & \left(P^{[k]} w\right)(x_1, \cdots, x_k) \\ = & \int_\Omega \left[p(x_1, y)\, w(y, x_2, \cdots, x_k) + \cdots + p(x_k, y)\, w(x_1, \cdots, x_{k-1}, y)\right] dy, \end{aligned}$$

satisfies the additive identity $(P + Q)^{[k]} = P^{[k]} + Q^{[k]}$.

Example 1.3. Let X be a vector space of real-valued functions on a set Ω. Consider a linear operator $A : X \to X$ defined by

$$\langle x, Au \rangle = a(x)u(x), \quad x \in \Omega. \tag{1.22}$$

Then, by Definition 1.3 and Definition 1.4, pointwise representations of the kth multiplicative and additive compounds of A satisfy

$$\left\langle x_1 \wedge \cdots \wedge x_k, A^{(k)}w \right\rangle = \left(\prod_{i=1}^{k} a(x_i) \right) w\left(x_1, \cdots, x_k\right)$$

and
$$\left\langle x_1 \wedge \cdots \wedge x_k, A^{[k]}w \right\rangle = \sum_{i=1}^{k} a(x_i) w\left(x_1, \cdots, x_k\right),$$

where $x_1, \cdots, x_k \in \Omega$ and $w \in \bigwedge^k X$.

More generally, Example 1.3 can be included in Example 1.2 if singular kernels $p(x, y)$ are permitted in (1.21).

In the case that X is a normed space with a basis $\{e^i\}$, this example falls under the discussion of bases at the beginning of this section. The basis representation of A is $a^i_j = \langle e^j, ae^i \rangle = a^j_i$ and thus the basis representations of $A^{(k)}$ and $A^{[k]}$ are given by (1.19) and (1.20).

1.2.2 The Laplacian Operator

Let $X = C^2(\Omega, \mathbb{R})$ and $Y = C(\Omega, \mathbb{R})$ where Ω is a bounded domain of $\mathbb{R}^m$. Then $X \subset Y$. By Definition 1.4, the kth multiplicative and additive compounds of the Laplacian $\Delta : X \to Y$, $\Delta^{(k)}$ and $\Delta^{[k]} : \bigwedge^k X \to \bigwedge^k Y$, are defined on wedges by

$$\Delta^{(k)}(u^1 \wedge \cdots \wedge u^k) = \left(\Delta u^1\right) \wedge \cdots \wedge \left(\Delta u^k\right),$$
$$\Delta^{[k]}\left(u^1 \wedge \cdots \wedge u^k\right) = \left(\Delta u^1\right) \wedge \cdots \wedge u^k + \cdots + u^1 \wedge \cdots \wedge \left(\Delta u^k\right)$$

and extended by linearity. For $u \in X$ and $x \in \Omega$, the pointwise representation of Δu is

$$\langle x, \Delta u \rangle = \Delta u(x),$$

the Laplacian of u with respect to x. For $w \in \bigwedge^k X$ and $x_1, \cdots, x_k \in \Omega$, pointwise representations of $\Delta^{(k)}w$ and $\Delta^{[k]}w$ are

$$\left\langle x_1 \wedge \cdots \wedge x_k, \Delta^{(k)}w \right\rangle = \left(\prod_{i=1}^{k} \Delta_i \right) w\left(x_1, \cdots, x_k\right),$$

and $$\left\langle x_1 \wedge \cdots \wedge x_k, \Delta^{[k]} w \right\rangle = \sum_{i=1}^{k} \Delta_i w\left(x_1, \cdots, x_k\right),$$

where $\Delta_i w = \langle x_i, \Delta w\rangle$ is the Laplacian of $w(x_1, \cdots, x_k)$ with respect to x_i.

Remark 1.4. It is emphasized that $\Delta^{(k)}$ and $\Delta^{[k]}$ are not considered as operators on $w(x_1, \cdots, x_k)$ in $C^2\left(\Omega^k, \mathbb{R}\right)$, but rather in $\bigwedge^k C^2\left(\Omega, \mathbb{R}\right)$ and, in particular, $w(x_1, \cdots, x_k)$ is antisymmetric in $(x_1, \cdots, x_k)$.

In the more general context, $X = H^2(\Omega) = \{u \in L^2(\Omega) : \nabla u, \Delta u \in L^2(\Omega)\}$ is associated with some boundary conditions on a smooth boundary $\partial\Omega$ and $Y = L^2(\Omega)$; $X \subset Y$. Let $\{e^i : i = 1, 2, \cdots\}$ be an orthonormal basis of X composed of eigenfunctions of the Laplacian Δ. Then the set of vectors $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k\}$ is an orthonormal basis of $\bigwedge^k X$ composed of eigenfunctions of $\Delta^{(k)}$ and $\Delta^{[k]}$. Since $\langle e^i, \Delta e^j\rangle = \lambda_j \delta_i^j$,

$$\left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, \Delta^{(k)}(e^{j_1} \wedge \cdots \wedge e^{j_k})\right\rangle = \left(\lambda_{j_1} \cdots \lambda_{j_k}\right) \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k}$$

and $$\left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, \Delta^{[k]}(e^{j_1} \wedge \cdots \wedge e^{j_k})\right\rangle = \left(\lambda_{j_1} + \cdots + \lambda_{j_k}\right) \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k},$$

it follows that if $u \in X$, then a basis representation of Δu is

$$\left\langle e^i, \Delta u\right\rangle = \lambda_i u,$$

and if $w \in \bigwedge^k X$, then basis representations of $\Delta^{(k)} w$ and $\Delta^{[k]} w : \bigwedge^k X \to \bigwedge^k Y$ are

$$\left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, \Delta^{(k)} w\right\rangle = (\lambda_{i_1} \cdots \lambda_{i_k}) w,$$

and $$\left\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, \Delta^{[k]} w\right\rangle = (\lambda_{i_1} + \cdots + \lambda_{i_k}) w.$$

1.3 Compound Differential Equations

This section deals with linear differential equations in a Banach space X.

Definition 1.5. A two-parameter family of linear operators $\{T(t, s) : 0 \leq s \leq t\} \subseteq \mathscr{L}(X, X)$ is said to be an *evolution operator* if

(*i*) $T(s, s) = I$,

(*ii*) $T(t, s)T(s, r) = T(t, r)$ for $0 \leq r \leq s \leq t$,

(*iii*) $t \mapsto T(t, s)$ is continuous for $t \geq s$.

Let

$$\lim_{h\to 0}\frac{T(t+h,t)x-x}{h}=A(t)x, \tag{1.23}$$

the domain $\mathcal{D}(A(t))$ of $A(t)$ being the set of all $x\in X$ for which the limit defined above exists. The map $A(t):\mathcal{D}(A(t))\subseteq X\to X$ is called the generator of $T(t,s)$. Assume that $\mathcal{D}(A(t))=\mathcal{D}(A)$, which is independent of t and dense in X.

If there exists a unique solution of the initial value problem

$$\begin{aligned}\frac{du}{dt}&=A(t)u,\\ u(s)&=u_s\in\mathcal{D}(A),\end{aligned} \tag{1.24}$$

then there is an evolution operator given by the relation $u(t)=T(t,s)u_s$. The operators $A(t)$ are usually unbounded. Detailed discussions of these topics may be found in [34, 42, 87, 111, 113].

Now assume that $u(t)=T(t,s)u_s, 0\le s\le t$ solves the initial value problem (1.24). The kth multiplicative compound $T^{(k)}(t,s)$ of $T(t,s)$ satisfies, from Definition 1.5,

(*i*) $T^{(k)}(s,s)=I_X^{(k)}=I_{\bigwedge^k X}$;

(*ii*) $T^{(k)}(t,s)T^{(k)}(s,r)=T^{(k)}(t,r)$, $0\le r\le s\le t$, from (ii) of Definition 1.5 and the Binet-Cauchy identity;

(*iii*) $t\mapsto T^{(k)}(t,s)$ is continuous for $t\ge s$.

Thus $T^{(k)}(t,s)$ is also an evolution operator. Let $u^1(t),\cdots,u^k(t)$ be k solutions of (1.24) with initial conditions $u_s^1,\cdots,u_s^k\in\mathcal{D}(A)$. Since

$$T^{(k)}(t,s)\,u_s^1\wedge\cdots\wedge u_s^k=\left(T(t,s)u_s^1\right)\wedge\cdots\wedge\left(T(t,s)u_s^k\right)=u^1(t)\wedge\cdots\wedge u^k(t)$$

and

$$\begin{aligned}\frac{d}{dt}\left[u^1(t)\wedge\cdots\wedge u^k(t)\right]&=\sum_i u^1(t)\wedge\cdots\wedge\frac{d}{dt}u^i(t)\wedge\cdots\wedge u^k(t)\\ &=\sum_i u^1(t)\wedge\cdots\wedge A(t)\,u^i(t)\wedge\cdots\wedge u^k(t)\\ &=A^{[k]}(t)\left[u^1(t)\wedge\cdots\wedge u^k(t)\right]\end{aligned}$$

it follows that $w(t) = u^1(t) \wedge \cdots \wedge u^k(t)$ is a solution of

$$\begin{aligned} \frac{dw}{dt} &= A^{[k]}(t)\, w, \ t \geq s, \\ w(s) &= u_s^1 \wedge \cdots \wedge u_s^k. \end{aligned}$$

This equation is called the kth compound differential equation of (1.24).

For example, if $A(t) = A$ is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ and the evolution operator $T(t,s) = e^{-(t-s)A}$ (see [87], page 20). The evolution operator of the kth compound differential equation of (1.24) is

$$T^{(k)}(t,s) = \left(e^{-(t-s)A}\right)^{(k)} = e^{-(t-s)A^{[k]}}.$$

Example 1.4. Consider a linear differential equation in $\mathbb{R}^n$

$$\frac{dy}{dt} = A(t)y, \tag{1.25}$$

where $A(t)$ is a continuous real or complex matrix-valued function of t. Suppose that $y^1(t), \cdots, y^k(t)$ are k solutions of (1.25). Then $w(t) = y^1(t) \wedge \cdots \wedge y^k(t)$, is a solution of the *kth compound differential equation* of (1.24)

$$\frac{dw}{dt} = A^{[k]}(t)w, \tag{1.26}$$

For the special case when $k = n$, it is well known that $w(t)$ is the solution of the Abel-Liouville-Jacobi equation

$$\frac{dw}{dt} = \mathrm{tr}A(t)w.$$

If $Y(t)$ is a fundamental matrix of (1.25), then $Y^{(k)}(t)$ is a fundamental matrix of (1.26). Here $T(t,s) = Y(t)Y^{-1}(s)$ and $T^{(k)}(t,s) = Y^{(k)}(t)\,(Y^{-1})^{(k)}(s)$. For example, if A is a constant matrix, then $Y(t) = e^{tA}$ and $Y^{(k)}(t) = \left(e^{tA}\right)^{(k)} = e^{tA^{[k]}}$ is a fundamental matrix of (1.26). For a detailed exposition and applications in the finite dimensional situation, see [76–78].

Example 1.5. Consider a reaction diffusion equation

$$\frac{du}{dt} = (\Delta + A(t))u. \tag{1.27}$$

Suppose that (1.27) generates a semigroup on $H_0^1(0,1) = \{\phi \in L^2(0,1) : \phi' \in L^2(0,1), \phi(0) = \phi(1) = 0\}$. It is assume that the linear operator $A(t)$ has the pointwise representation

$$\langle x, A(t)u \rangle = a(t,x)u(x). \tag{1.28}$$

This example illustrates how different choices of families of linear functionals v used to examine the exterior products of solutions of (1.27) yield different representations of the differential equation and its compound equations. Choosing first as v the pointwise evaluation functionals gives partial differential equations representing (1.27) and its compounds. Then, using eigenfunctions of the Laplacian a basis and the spectral projections as the linear operators v, equation (1.27) and its compounds are represented by infinite systems of coupled ordinary differential equations.

Case 1: A pointwise representation of the kth compound differential equation of (1.27)

For $u \in H_0^1(0,1), x \in (0,1)$, let x also denote a linear functional $u \mapsto \langle x, u \rangle = u(x)$. Then (1.27) can be written as

$$\begin{aligned} &u_t = u_{xx} + a(t,x)u, \quad 0 < x < 1, \quad t > 0, \\ &u(t,0) = u(t,1) = 0, \quad t \geq 0. \end{aligned} \tag{1.29}$$

For $w(t) \in \bigwedge^k H_0^1(0,1)$, a pointwise representation of w is

$$\langle x_1 \wedge \cdots \wedge x_k, w(t) \rangle = w(t, x_1, \cdots, x_k).$$

A pointwise representation of the kth compound operator $(\Delta + A(t))^{[k]} = \Delta^{[k]} + A(t)^{[k]} : \bigwedge^k H_0^1(0,1) \to \bigwedge^k L^2(0,1)$ follows from Example 1.3 and Section 1.2.2 and thus, a pointwise representation of the kth compound differential equation of (1.27) in $\bigwedge^k H_0^1(0,1)$ is

$$w_t = \sum_{i=1}^{k} w_{x_i x_i} + \sum_{i=1}^{k} a(t, x_i) w. \tag{1.30}$$

If $u^1(t), \cdots, u^k(t)$ are solutions of (1.29), then

$$\langle x_1 \wedge \cdots \wedge x_k, u^1(t) \wedge \cdots \wedge u^k(t) \rangle = \det[u^j(t, x_i)].$$

It can be checked by differentiating the above determinant that $u^1(t) \wedge \cdots \wedge u^k(t)$ satisfies the kth compound differential equation (1.30) and Dirichlet boundary conditions of u^i implies that

$$\langle x_1 \wedge \cdots \wedge x_k, u^1(t) \wedge \cdots \wedge u^k(t) \rangle \big|_{x_i=0,1} = 0, \qquad i = 1, \cdots, k.$$

Again, (1.30) is not considered as an equation for $w(t, x_1, \cdots, x_k)$ in $H_0^1\left((0,1)^k\right)$, but rather in $\bigwedge^k H_0^1(0,1)$.

Case 2: A basis representation of the kth compound differential equation of (1.27)

Let $\{e^j(x) = \sqrt{2}\sin j\pi x : j = 1, 2, \cdots\}$ be an orthonormal basis of $H_0^1(0,1)$. Define another linear map $e^j : u \mapsto \langle e^j, u\rangle$. Then $u(t) = \sum_j u_j(t) e^j$ and $u_j = \langle e^j, u\rangle$ satisfies

$$\frac{du_j}{dt} = [\lambda_j + a_j^j(t)]u_i + \sum_{i \neq j} a_j^i(t) u_j, \tag{1.31}$$

where $\lambda_j = -(j\pi)^2$ and $a_j^i = \langle e^i, ae^j\rangle = a_i^j$. For $w(t) \in \bigwedge^k H_0^1(0,1)$, a basis representation of w is

$$\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, w(t) \rangle := w_{(i)}(t) = w_{(i_1 \cdots i_k)}(t).$$

A basis representation of the kth compound operator $(\Delta + A(t))^{[k]} = \Delta^{[k]} + A(t)^{[k]}$ follows from Example 1.3 and Section 1.2.2 and thus, a basis representation of the kth compound differential equation of (1.27) is

$$\begin{aligned} \frac{dw_{(i_1 \cdots i_k)}}{dt} &= \left(\lambda_{i_1} + \cdots + \lambda_{i_k} + a_{i_1}^{i_1}(t) + \cdots + a_{i_k}^{i_k}(t)\right) w_{(i_1 \cdots i_k)} \\ &\quad + \sum_{j \notin (i_1 \cdots i_k)} \left(a_{i_1}^j(t) w_{(j i_2 \cdots i_k)} + \cdots + a_{i_k}^j(t) w_{(i_1 \cdots i_{k-1} j)}\right). \end{aligned} \tag{1.32}$$

If $u^1(t), \cdots, u^k(t)$ are solutions of (1.29), then

$$\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, u^1(t) \wedge \cdots \wedge u^k(t) \rangle = \det\left[\langle e^{i_r}, u^p(t)\rangle\right] = u_{i_1 \cdots i_k}^{1 \cdots k}(t).$$

It can be checked by differentiating the above determinants that $u^1(t) \wedge \cdots \wedge u^k(t)$ satisfies the kth compound differential equation (1.32).

Chapter 2

Dimension Problems

This chapter addresses the problem of estimation of the codimension of the asymptotically stable solution subspace of a linear differential equation. This question arises not only for its intrinsic interest as a singular linear boundary value problem but also for its importance in investigations of the dimension of stable manifolds of equilibria and periodic orbits of nonlinear equations by linearization techniques.

The approach is motivated by results on the dimension of the asymptotically stable solution subspace of a differential equation

$$\frac{du}{dt} = A(t)u, \qquad u(\cdot) \in \mathbb{R}^n. \tag{2.1}$$

When $n = 2$, Milloux [74] shows that if $0 < a(t)$ is a nondecreasing function, then the scalar equation $u'' + a(t)u = 0$ has a nontrivial solution $u = u_0(t)$ such that

$$\lim_{t\to\infty} u_0(t) = 0 \quad \text{if and only if} \quad \lim_{t\to\infty} a(t) = \infty.$$

Higher dimensional equations have been studied by Hartman [40], Coppel [19], Macki and Muldowney [65] and Muldowney [76–78]. Hartman [40] ([41], page 501) shows that if $0 \le \lim_{t\to\infty} \|u(t)\| < \infty$ exists for all solutions u, and $\|\cdot\|$ is the Euclidean norm, then there exists a nontrivial solution such that $\lim_{t\to\infty} \|u_0(t)\| = 0$ if and only if $\lim_{t\to\infty} \int_0^t \operatorname{Re} \operatorname{tr} A = -\infty$. Hartman's proof relies heavily on properties of the norm. Coppel [19], page 60, shows that the result holds for any norm. Macki and Muldowney [65], by an entirely different approach, weaken the restriction that $\|u(t)\|$ tends to a limit to a stability requirement on $u(t)$. Muldowney [76–78] extends these results by means of a sequence of compound matrices $A^{[k]}(t), k = 1, \cdots, n$ of $A(t)$ such that, under the same stability assumptions as in [65], the given system has an $(n - k + 1)$-dimensional asymptotically stable solution subspace if and only if all nontrivial solutions of the system

$$\frac{dw}{dt} = A^{[k]}(t)w$$

tend to zero.

The applications of compound matrices to differential equations are further developed in [51, 56, 58, 76–78]. In [78] a general result on the existence of an asymptotically stable subspace in a linear space of $\mathbb{R}^n$-valued functions is obtained. This chapter extends that result to general vector spaces. Nonlinear applications include estimation of the codimension of stable manifolds in Section 2.2. It also provides an important step in the proof of Theorem 4, Section 4.2.1, on the existence of stable periodic solutions.

2.1 Asymptotically stable subspaces

This section extends a result of Muldowney [78] on the existence of an asymptotically stable subspace in a set of vector space valued functions on $[0, \infty)$.

Let X be a vector space and X^* be its (algebraic) dual space. Let $\mathcal{U}$ be a vector space of maps $t \mapsto u(t)$ from $[0, \infty)$ to X and $\mathcal{V}$ be a family of maps $t \mapsto v(t)$ from $[0, \infty)$ to X^*.

Condition L**:** The pair $\{\mathcal{U}, \mathcal{V}\}$ satisfies Condition L if for each $u \in \mathcal{U}$, the following two assumptions hold:

(*i*) $\limsup_{t\to\infty} |\langle v, u\rangle (t)| < \infty$ for every $v \in \mathcal{V}$;

(*ii*) $\liminf_{t\to\infty} |\langle v, u\rangle (t)| = 0$ for every $v \in \mathcal{V}$ implies that $\lim_{t\to\infty} \langle v, u\rangle (t) = 0$ for every $v \in \mathcal{V}$.

Here $\langle v, u\rangle (t) = \langle v(t), u(t)\rangle$.

Remark 2.1. For example, suppose that X is a normed space and that $\mathcal{U}$ is a linear space of functions $t \mapsto u(t)$ from $[0, \infty)$ to X that satisfy in some sense a linear differential equation of the form

$$\frac{du}{dt} = A(t)\,u, \qquad u(\cdot) \in X. \tag{2.2}$$

If $\mathcal{V}$ is the set of all functions $t \mapsto v(t)$ from $[0, \infty)$ to X', the continuous dual space of X, such that $\|v(t)\| = 1$, then the pair $\{\mathcal{U}, \mathcal{V}\}$ satisfies Condition L provided that (i) and (ii) hold:

(i) $\|u(t)\|$ is bounded, $0 \le t < \infty$;

(ii) $\liminf_{t\to\infty} \|u(t)\| = 0$ implies $\lim_{t\to\infty} \|u(t)\| = 0$.

The above conditions (i) and (ii) are satisfied if $\lim_{t\to\infty} \|u(t)\| < \infty$ exists for all $u \in \mathcal{U}$. Conditions (i) and (ii) are also satisfied if there exists a constant C such that

$$\|u(t)\| \le C\,\|u(s)\|, \; 0 \le s \le t < \infty. \tag{2.3}$$

Thus the solution space of a linear differential equation satisfies Condition L if the equation is uniformly stable. The condition (2.3) can be weakened to

$$\|u(t)\| \le C_u\,\|u(s)\|, \; 0 \le s \le t < \infty,$$

where the constant C_u depends on the solution u.

Let

$$\mathcal{U}_0 = \left\{ u \in \mathcal{U} : \lim_{t\to\infty} \; \langle v, u\rangle(t) = 0 \text{ for every } v \in \mathcal{V} \right\}. \tag{2.4}$$

In the following proposition codim $\mathcal{U}_0 < k$ means that any subspace of $\mathcal{U}$ that has dimension k or greater must intersect $\mathcal{U}_0$ nontrivially.

Proposition 2.1. *Let $\mathcal{U}$ be a vector space of maps $t \mapsto u(t)$ from $[0, \infty)$ to X and $\mathcal{V}$ be a family of maps $t \mapsto v(t)$ from $[0, \infty)$ to X^*. Suppose that the pair $\{\mathcal{U}, \mathcal{V}\}$ satisfies Condition L. Then*

$$\textit{codim}\; \mathcal{U}_0 < k$$

if and only if

$$\lim_{t\to\infty} \langle v_1 \wedge \cdots \wedge v_k, u^1 \wedge \cdots \wedge u^k\rangle(t) = 0 \tag{2.5}$$

is satisfied for all $u^1, \cdots, u^k \in \mathcal{U}$ and $v_1, \cdots, v_k \in \mathcal{V}$.

Proof. Suppose that codim $\mathcal{U}_0 < k$. Then for every k linearly independent functions $u^1, \cdots, u^k \in \mathcal{U}$, there exists a nontrivial linear combination $u_0 = c_1 u^1 + \cdots + c_k u^k$ such that

$$\lim_{t\to\infty} \langle v, u_0\rangle(t) = 0 \quad \text{for all } v \in \mathcal{V}. \tag{2.6}$$

Without loss of generality, it is assumed that $c_1 \neq 0$. Then, it follows from Condition $L(i)$ and (2.6) that for every $v_1, \cdots, v_k \in \mathcal{V}$,

$$\begin{aligned} &\lim_{t\to\infty} \left\langle v_1 \wedge \cdots \wedge v_k, u^1 \wedge \cdots \wedge u^k\right\rangle(t) \\ = \;& \frac{1}{c_1} \lim_{t\to\infty} \left\langle v_1 \wedge \cdots \wedge v_k, u_0^1 \wedge \cdots \wedge u^k\right\rangle(t) \\ = \;& 0. \end{aligned} \tag{2.7}$$

Conversely, suppose that (2.5) is satisfied. If $u^1, \cdots, u^k \in \mathcal{U}$ are linearly independent, it is to be proved that span$\{u^1, \cdots, u^k\}$ intersects $\mathcal{U}_0$ nontrivially. The proof is by mathematical induction on k. It is evidently true when $k = 1$ since, in that case,

$$\left\langle v_1, u^1\right\rangle(t) \underset{t\to\infty}{\longrightarrow} 0$$

for all $v_1 \in \mathcal{V}$ and hence $u^1 \in \mathcal{U}_0$. Suppose that the proposition is true if $1 \leq k < h$. Let $v_1, \cdots, v_h \in \mathcal{V}$. From Condition $L(i)$ any sequence $t_n \underset{n\to\infty}{\longrightarrow} \infty$ has a subsequence, which will also be denoted t_n, such that

$$\lim_{n\to\infty} \left\langle v_i, u^j\right\rangle(t_n) = \xi_i^j \tag{2.8}$$

exists $i, j = 1, \cdots, h$. Let $\xi_{12\cdots k}^{12\cdots k}$ denote det $\left[\xi_i^j\right]$, $i, j = 1, \cdots, k$. Now (2.5), $k = h$, implies $\xi_{12\cdots h}^{12\cdots h} = 0$. It may be assumed that t_n and $v_1, \cdots, v_{h-1} \in \mathcal{V}$ can be chosen so that $\xi_{12\cdots h-1}^{12\cdots h-1} \neq 0$ since otherwise $\lim_{t\to\infty} \left\langle v_1 \wedge \cdots \wedge v_{h-1}, u^1 \wedge \cdots \wedge u^{h-1}\right\rangle(t) = 0$ for all $v_1, \cdots, v_{h-1} \in \mathcal{V}$ and the induction hypothesis implies that there is a nontrivial $u_0 = \sum_{j=1}^{h-1} c_j u^j \in \mathcal{U}_0$. Thus, with

$$\xi_{12\cdots h-1}^{12\cdots h-1} \neq 0, \qquad \xi_{12\cdots h}^{12\cdots h} = 0,$$

expanding the determinant $\xi_{12\cdots h}^{12\cdots h}$ along the last row gives

$$0 = \xi_{12\cdots h}^{12\cdots h} = \sum_{j=1}^{h} c_j \xi_h^j \tag{2.9}$$

where c_j is the cofactor of ξ_h^j, $j = 1, \cdots, h$. In particular, $c_h = \xi_{12\cdots h-1}^{12\cdots h-1} \neq 0$. Then (2.9) implies

$$0 = \sum_{j=1}^{h} c_j \xi_h^j = \lim_{n\to\infty} \sum_{j=1}^{h} c_j \left\langle v_h, u^j \right\rangle (t_n)$$

and thus $u_0 = \sum_{j=1}^{h} c_j u^j$ satisfies

$$\lim_{n\to\infty} \langle v_h, u_0 \rangle (t_n) = 0. \tag{2.10}$$

Furthermore, for any choice of $v_h \in \mathcal{V}$, a subsequence of t_n, also denoted t_n, may be chosen so that (2.8) is still satisfied and the preceding analysis shows that each choice of v_h satisfies (2.10) with the same constants c_j, which do not depend on v_h, for some sequence $t_n \underset{n\to\infty}{\longrightarrow} \infty$. Therefore $\liminf_{t\to\infty} \langle v, u_0 \rangle (t) = 0$ for all $v \in \mathcal{V}$ which, by Condition $L(ii)$, implies

$$\lim_{t\to\infty} \langle v, u_0 \rangle (t) = 0$$

and so

$$u_0 = \sum_{j=1}^{h} c_j u^j \in \mathcal{U}_0.$$

The proposition is thus true for $k = h$ also and hence for all k by mathematical induction. ■

2.1.1 Function Space

Let X be a vector space of real-valued functions on a finite, countable or uncountable set Ω. For $x \in \Omega$, let x also denote the linear functional $u \mapsto \langle x, u \rangle := u(x)$, $u \in X$. The pointwise representation of the exterior products discussed in Section 1.1.3 is $u^1 \wedge u^2 \wedge \cdots \wedge u^k \; : \; \bigwedge^k X \longrightarrow \mathbb{R}$,

$$\left\langle x_1 \wedge \cdots \wedge x_k, u^1 \wedge \cdots \wedge u^k \right\rangle = u^1 \wedge \cdots \wedge u^k(x_1, \cdots, x_k) = \det \left[u^i(x_j) \right].$$

Let $\mathcal{U}$ be a linear space of maps $t \mapsto u(t, \cdot)$ and $\mathcal{V}$ be the set of constant maps $\{t \mapsto v(t) = x : x \in X^*\}$.

The following corollary is a particular case of Proposition 2.1.

Corollary 2.2. *Suppose that, for every $u \in \mathcal{U}$, the following two conditions hold:*

(*i*) $\limsup_{t\to\infty} |u(t,x)| < \infty$ *for all* $x \in \Omega$;

(*ii*) $\liminf_{t\to\infty} |u(t,x)| = 0$ *for all* $x \in \Omega$ *implies* $\lim_{t\to\infty} u(t,x) = 0$ *for all* $x \in \Omega$.

Then

$$\text{codim}\left\{u \in \mathcal{U} : \lim_{t\to\infty} |u(t,x)| = 0 \quad \text{for all } x \in \Omega\right\} < k \tag{2.11}$$

if and only if, for each $u^1, \cdots, u^k \in \mathcal{U}$ *and all* $x_1, \cdots, x_k \in \Omega$,

$$\lim_{t\to\infty} \det\left[u^i(t,x_j)\right] = 0.$$

In particular, conditions (i) and (ii) are satisfied if, for every $u \in \mathcal{U}$, $\lim_{t\to\infty} |u(t,x)| < \infty$ *exists for all* $x \in \Omega$.

2.1.2 Normed Vector Space

Let X be a normed vector space, $\mathcal{U}$ a linear space of functions $t \mapsto u(t)$ from $[0,\infty)$ to X and $\mathcal{V}$ the set of functions $t \mapsto v(t)$ from $[0,\infty)$ to X', $\|v(t)\|_{X'} = 1$. Proposition 2.1 has the following corollary.

Corollary 2.3. *Suppose that for every* $u \in \mathcal{U}$, *the following two conditions are satisfied:*

(*i*) $\limsup_{t\to\infty} \|u(t)\|_X < \infty$;

(*ii*) $\liminf_{t\to\infty} \|u(t)\|_X = 0$ *implies* $\lim_{t\to\infty} \|u(t)\|_X = 0$.

Then

$$\text{codim}\left\{u \in \mathcal{U} : \lim_{t\to\infty} \|u(t)\|_X = 0\right\} < k$$

if and only if, for all $u^1, \cdots, u^k \in \mathcal{U}$,

$$\lim_{t\to\infty} \left\|u^1 \wedge \cdots \wedge u^k(t)\right\|_{\bigwedge^k X} = 0.$$

In particular, conditions (i) and (ii) are satisfied if, for every $u \in \mathcal{U}$, $\lim_{t\to\infty} \|u(t)\|_X < \infty$ *exists.*

Consider X, a vector space of real-valued functions on Ω. The following table gives examples of the norms $\|u(t)\|_X$ and $\left\|u^1 \wedge \cdots \wedge u^k(t)\right\|_{\wedge^k X}$ in

$\lVert u(t) \rVert_X$	$\lVert u^1 \wedge \cdots \wedge u^k(t) \rVert_{\wedge^k X}$
$\left(\int_\Omega \lvert u(t,x) \rvert^p \, dx \right)^{\frac{1}{p}}$	$\left(\frac{1}{k!} \int_{\Omega^k} \left\lvert \det \left[u^j(t, x_i) \right] \right\rvert^p dx_1 \cdots dx_k \right)^{\frac{1}{p}}$
$\sup_{x \in \Omega} \lvert u(t,x) \rvert$	$\sup_{x_1, \cdots, x_k \in \Omega} \lvert \det [u^j(t, x_i)] \rvert$
$\left(\sum_i \lvert u_i(t) \rvert^p \right)^{\frac{1}{p}}$	$\left(\sum_{(i)} \lvert u^{1 \cdots k}_{i_1 \cdots i_k}(t) \rvert^p \right)^{\frac{1}{p}}$
$\sup_i \lvert u_i(t) \rvert$	$\sup_{(i)} \lvert u^{1 \cdots k}_{i_1 \cdots i_k}(t) \rvert$

Table 2.1: Norms of kth exterior products

Corollary 2.3 for a set $\mathcal{U}$ of functions $t \mapsto u(t)$ in this space. In the first two examples Ω is a measurable set in $\mathbb{R}^n$ and in the last two examples Ω is a subset of the set $\mathbb{N}$ of natural numbers. In each case, it is assumed that X is the space of functions for which the expression defining the norm is finite. Notation in the last two examples is defined in Section 1.1.2.

The examples may be related, for example, through an orthonormal basis $\{e^i\}$ with respect to an inner product $\langle u, v \rangle = \int_\Omega uv$ and

$$\begin{aligned} u(t,x) &\sim \sum_i u_i(t) \, e^i(x), \\ u_i(t) &= \int_\Omega u(t,x) \, e^i(x) \, dx \end{aligned}$$

and, then

$$u^1 \wedge \cdots \wedge u^k (t, x_1, \cdots, x_k) \sim \sum_{(i)} u^{1 \cdots k}_{i_1 \cdots i_k}(t) \, e^{i_1} \wedge \cdots \wedge e^{i_k}(x_1, \cdots, x_k).$$

Remark 2.2. When X is a Hilbert space and $p = 2$, the second and fourth lines in the table above give the same condition in Corollary 2.3.

2.1.3 Differential Equations

In Remark 2.1, **Condition** L is discussed in the context of the solution space $\mathcal{U}$ of a linear differential equation. This discussion is continued with applications of Proposition 2.1 to estimation of the codimension of the asymptotically stable subspace $\mathcal{U}_0$.

Theorem 2.4. *Suppose that X is a normed space and that the differential equation*

$$\frac{du}{dt} = A(t)\, u$$

has a solution space $\mathcal{U}$ that satisfies, if $u \in \mathcal{U}$,
(*i*) $\limsup_{t\to\infty} \|u(t)\|_X < \infty$;
(*ii*) $\liminf_{t\to\infty} \|u(t)\|_X = 0$ *implies* $\lim_{t\to\infty} \|u(t)\|_X = 0$.
Then

$$\mathit{codim} \left\{ u \in \mathcal{U} : \lim_{t\to\infty} \|u(t)\|_X = 0 \right\} < k$$

if and only if all solutions $w = u^1 \wedge \cdots \wedge u^k$, $u^i \in \mathcal{U}$, of the kth compound equation

$$\frac{dw}{dt} = A^{[k]}(t)\, w$$

satisfy

$$\lim_{t\to\infty} \|w(t)\|_{\wedge^k X} = 0.$$

Remark 2.3. In particular conditions (*i*) and (*ii*) of Theorem 2.4 are satisfied if, for every $u \in \mathcal{U}$, $\lim_{t\to\infty} \|u(t)\|_X < \infty$ exists or if the equation $\frac{du}{dt} = A(t)\, u$ is uniformly stable. The finite dimensional results discussed in the introduction to this chapter are all special cases of Theorem 2.4.

Consider a scalar reaction diffusion equation

$$\frac{du}{dt} = (\Delta + A(t))u, \quad u(\cdot) \in H_0^1(0,1), \tag{2.12}$$

where the linear operator $A(t)$ has the pointwise representation

$$\langle x, A(t)u \rangle = a(t,x)u(x).$$

In Example 1.5 of Section 1.3, two representations of the kth compound differential equation of (2.12) are considered.

Case 1: The pointwise representation of the kth compound differential equation of (2.12) on $\bigwedge^k H_0^1(0,1)$ is

$$w_t = \sum_{i=1}^{k} w_{x_i x_i} + \sum_{i=1}^{k} a(t, x_i) w. \tag{2.13}$$

Case 2: The basis representation of the kth compound differential equation of (2.12) on $\bigwedge^k H_0^1(0,1)$ is

$$\begin{aligned} \frac{dw_{(i_1 \cdots i_k)}}{dt} &= \left(\lambda_{i_1} + \cdots + \lambda_{i_k} + a_{i_1 i_1}(t) + \cdots + a_{i_k i_k}(t)\right) w_{(i_1 \cdots i_k)} \\ &\quad + \sum_{j \notin (i_1 \cdots i_k)} \left(a_{i_1 j}(t) w_{(j i_2 \cdots i_k)} + \cdots + a_{i_k j}(t) w_{(i_1 \cdots i_{k-1} j)}\right) \end{aligned} \tag{2.14}$$

where $\langle e^{i_1} \wedge \cdots \wedge e^{i_k}, w(t)\rangle := w_{(i)}(t) = w_{(i_1 \cdots i_k)}(t)$, $e^j(x) = \sqrt{2} \sin j\pi x$, $\lambda_j = -(j\pi)^2$ and $a_{ij} = a_{ji} = \langle e^i, ae^j\rangle$.

To compare different choices of the functionals in X', Theorem 2.4 implies the following two corollaries. Corollary 2.5 discusses the problem in $L^2(0,1)$, $\|u\|_{L^2}^2 = \int_0^1 u(t,x)^2\, dx$, while Corollary 2.6 considers the l_∞ norm, $\|u\|_\infty = \sup_i |u_i(t)|$.

Corollary 2.5. *Suppose that there exists a constant M such that*

$$\int_s^t \left[\lambda_1 + \max_x a(\tau, x)\right] d\tau \le M, \quad 0 \le s \le t < \infty, \tag{2.15}$$

where the maximum is taken over $0 \le x \le 1$ and $\lambda_j = -(j\pi)^2$, $j = 1, 2, \cdots$. Then

$$\text{codim}\left\{u(\cdot) \text{ satisfies (2.12) and } \lim_{t\to\infty} \|u(t)\|_{L^2} = 0\right\} < k$$

if

$$\int_0^\infty \left[\lambda_1 + \cdots + \lambda_k + k \max_x a(t,x)\right] dt = -\infty. \tag{2.16}$$

Corollary 2.6. *Suppose that there exists a constant M such that*

$$\int_s^t \sup_i \left\{\lambda_i + a_{ii}(\tau) + \sum_{j\neq i} |a_{ij}(\tau)|\right\} d\tau \le M, \quad 0 \le s \le t < \infty, \tag{2.17}$$

where $a_{ij} = a_{ji} = \langle e^i, ae^j \rangle$ *and* $\lambda_j = -(j\pi)^2,\ j = 1, 2, \cdots$ *. Then*

$$\text{codim}\left\{u(\cdot) \text{ satisfies } (2.12) \text{ and } \lim_{t\to\infty} \|u(t)\|_\infty = 0\right\} < k$$

if

$$\int_0^\infty \sup_{(i)} \left\{ \sum_{s=1}^{k} (\lambda_{i_s} + a_{i_s i_s}(\tau)) + \sum_{j\notin(i)} (|a_{i_1 j}(\tau)| + \cdots + |a_{i_k j}(\tau)|) \right\} d\tau = -\infty \tag{2.18}$$

where $(i) = (i_1, \cdots, i_k)$, $i_1 < i_2 < \cdots < i_k$.

Remark 2.4. In Corollary 2.5, condition (2.15) implies that the reaction diffusion equation (2.12) is uniformly stable in $L^2(0,1)$ by considering $\|u\|_{L^2}$ as a Lyapunov function. The asymptotic stability of the kth compound differential equation (2.13) in $\bigwedge^k L^2(0,1)$ is obtained from condition (2.16) by considering $\|w\|_{\bigwedge^k L^2}$. Similar arguments using conditions (2.17) and (2.18) with $\|u\|_\infty$ and $\|w\|_{\bigwedge^k l_\infty}$ prove Corollary 2.6 respectively.

2.2 Dimension Problems for Steady State Solutions

This section derives results on the stability of steady state solutions of differential equations. For an autonomous ordinary differential equation

$$\frac{du}{dt} = f(u), \qquad u(\cdot) \in \mathbb{R}^n,$$

a necessary and sufficient condition for an equilibrium $u = u^*$ to be stable hyperbolic is that the linear variational equation at $u = u^*$,

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(u^*)v,$$

is uniformly asymptotically stable. General results on the stability of steady state solutions of differential equation in Banach space can be found in Smoller [103], Theorem 11.20, page 120 and Henry [42], Theorem 5.1.1, page 98.

Let A be a sectorial operator in a Banach space X and f be continuously differentiable from X^α into X where $0 \le \alpha < 1$. Let $\mathcal{D}(A)$ denote the domain of A. Consider an autonomous differential equation

$$\frac{du}{dt} + Au = f(u), \qquad u(\cdot) \in X. \tag{2.19}$$

Equation (2.19) has been discussed in Henry [42]. Definitions and properties of sectorial operators and the space X^α are listed in Appendix B.

Definition 2.1. A *solution* of the initial value problem

$$\frac{du}{dt} + Au = f(u),$$
$$u(0) = u_0,$$

on $(0,T)$ is a continuous function $u : [0,T) \to X$ such that $u(0) = u_0$ and on $(0,T)$, $(t, u(t)) \in \mathbb{R} \times X^\alpha, u(t) \in \mathcal{D}(A), \frac{du}{dt}(t)$ exists and the differential equation (3.1) is satisfied on $(0,T)$.

Lemma 2.7. *Assume that A is a sectorial operator, $0 \le \alpha < 1$, and that $f : X^\alpha \to X$ is locally Lipschitz continuous in u. Then for any $u_0 \in X^\alpha$, there exists $T = T(u_0) > 0$ such that (3.1) has a unique solution u on $(0,T)$ with initial value $u(0) = u_0$.*

Definition 2.2. A solution is a *steady state solution* or (an *equilibrium*) if $u = u^* \in \mathcal{D}(A)$ and $Au^* = f(u^*)$. The steady state solution u^* is *stable hyperbolic* if the spectrum of $A - \frac{\partial f}{\partial u}(u^*)$ lies in $\{\text{Re}\lambda > \beta\}$ for some $\beta > 0$.

Theorem 2.8. *A steady state solution u^* is stable hyperbolic if and only if the linear variational equation at $u(t) = u^*$,*

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u^*)v, \tag{2.20}$$

is uniformly asymptotically stable.

Remark 2.5. Definition 2.2 is equivalent to Definition 11.19 in Smoller [103], page 120. Theorem 11.20 in Smoller [103], page 120, shows that the hyperbolic stability of u^* is equivalent to the uniform asymptotic stability of (2.20). Theorem 5.1.1 in Henry [42] proves that if $f = f(t,u)$, then $u = u^*$ is stable hyperbolic if

$$\frac{dv}{dt} + Av = Bv$$

is uniformly asymptotically stable where B is a bounded linear operator from X^α to X such that

$$f(t, u^* + z) = f(t, u^*) + Bz + g(t, z)$$

and $\|g(t, z)\| = o(\|z\|_\alpha)$ as $\|z\|_\alpha \to 0$.

The effect of adding diffusion to the dynamics of an ordinary differential equation in $\mathbb{R}^n$ has been studied by [16, 17, 57, 62, 63, 81, 83, 85, 103, 105]. Turing [105] is the first to demonstrate that different diffusion coefficients can cause a stable equilibrium of the ordinary differential equation to cease to be stable for the reaction diffusion equation. The diffusion-driven instability has become an important mechanism for the occurrence of interesting patterns in many model systems. Turing's idea has been explored by many authors; see [17, 81, 83, 85]. With the help of compound matrices, for a stable matrix A with real entries, Wang and Li [57] derive sufficient and necessary conditions for $A - D$ to be stable for any nonnegative diagonal matrix D. These conditions can be used to study the stability and instability of constant steady state solutions to reaction diffusion equations.

In the following, the stability of steady state solutions of scalar reaction diffusion equations is studied. Theorem 2.4 gives an estimate of the codimension of the asymptotically stable solution subspace, which is related to the stability of a compound differential equation. Rather than use Theorem 2.4 directly, estimates on the spectrum of an operator A from the spectrum of its compound $A^{[k]}$ are derived.

Consider a nonlinear scalar reaction diffusion equation

$$\begin{aligned} &u_t = u_{xx} + f(x, u), \quad a < x < b, \quad t > 0, \\ &u(t, a) = u(t, b) = 0, \quad t > 0. \end{aligned} \tag{2.21}$$

Assume that (2.21) generates a semiflow on $H_0^1(a, b)$ and that $u(t, x)$ exists for all t. Suppose that $u = u^*(x)$ is a steady state solution of (2.21). The linear variational equation at $u = u^*(x)$ is

$$\begin{aligned} &v_t = v_{xx} + q(x)v, \quad a < x < b, \quad t > 0, \\ &v(t, a) = v(t, b) = 0, \quad t > 0, \end{aligned} \tag{2.22}$$

where

$$q(x) := f_u(x, u^*(x)). \tag{2.23}$$

Assume that (2.22) also generates a semiflow on $H_0^1(a,b)$ and that $v(t,x)$ exists for all t. Theorem 2.8 states that the steady state solution $u = u^*(x)$ of (2.21) is stable hyperbolic if the linear variational equation (2.22) is uniformly asymptotically stable and thus, if the eigenvalues of the following eigenvalue problem

$$\begin{aligned} \lambda\phi &= \phi'' + q(x)\phi, \\ \phi(a) &= \phi(b) = 0, \end{aligned} \tag{2.24}$$

satisfy $\lambda < 0$, the steady state solution $u = u^*(x)$ is stable hyperbolic. The eigenvalue problem (2.24) is a Sturm-Liouville boundary value problem; see Hartman [41], page 337-344 for details.

Let $L\phi = \phi'' + q(x)\phi$. Then L is a self-adjoint operator on $H_0^1(a,b)$. The standard analysis in [2, 20, 21, 103] on the eigenvalue problem of a self-adjoint second order differential equation implies that the principal eigenvalue of (2.24) is

$$\lambda_1 = \sup_{\phi \in H_0^1} \int_a^b \left[-(\phi')^2 + q\phi^2 \right] \Big/ \int_a^b \phi^2$$

where the supremum is taken over the functions $\phi \in H_0^1(a,b)$ with $\int_a^b \phi^2 \neq 0$ and the maximizing function ϕ_1 is the corresponding eigenfunction. In particular, if $\mu_1 \geq \mu_2 \geq \cdots$ and $e_1, e_2, \cdots$ are the eigenvalues and orthonormal eigenfunctions of the Laplacian in $H_0^1(a,b)$,

$$\mu_1 = \sup_{\phi \in H_0^1} \int_a^b -(\phi')^2 \Big/ \int_a^b \phi^2 = -\int_a^b (e_1')^2 .$$

Therefore

$$\mu_1 + \int_a^b q\,(e_1)^2 \leq \lambda_1 \leq \mu_1 + \max_{a<x<b} q\,(x) .$$

Since L is self-adjoint, the kth additive compound operator $L^{[k]}$ of L is also self-adjoint on $\bigwedge^k H_0^1(a,b)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the eigenvalues of L, counting multiplicities, and $\phi_1, \phi_2, \cdots$ be the corresponding eigenfunctions. Then $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_k} : 1 \leq i_1 < \cdots < i_k\}$ forms a basis of $\bigwedge^k H_0^1(a,b)$ and the principal eigenvalue of $L^{[k]}$ is $\lambda_1 + \cdots + \lambda_k$ satisfying

$$\lambda_1 + \cdots + \lambda_k = \sup_{w \in \bigwedge^k H_0^1} \int_{(a,b)^k} \sum_{i=1}^k \left[-(w_{x_i})^2 + q\,(x_i)\,w^2 \right] \Big/ \int_{(a,b)^k} w^2 \quad ,$$

where the supremum is taken over the functions $w \in \bigwedge^k H_0^1(a,b)$ with $\int_{(a,b)^k} w^2 \neq 0$. Therefore

$$\begin{aligned}&\mu_1 + \cdots + \mu_k + \int_{(a,b)^k} \sum_{i=1}^{k} q(x_i)\,(e_1 \wedge \cdots \wedge e_k)^2 \Big/ \int_{(a,b)^k} (e_1 \wedge \cdots \wedge e_k)^2 \\ \leq\ & \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k + k \max_x q(x).\end{aligned}$$

It follows from

$$\begin{aligned}\delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} &= \langle e_{i_1} \wedge \cdots \wedge e_{i_r}, e_{j_1} \wedge \cdots \wedge e_{j_r} \rangle \\ &= \frac{1}{r!} \int_{(a,b)^r} (e_{i_1} \wedge \cdots \wedge e_{i_r})\ (e_{j_1} \wedge \cdots \wedge e_{j_r}),\end{aligned}$$

where $r \geq 1$, that

$$\int_{(a,b)^k} (e_1 \wedge \cdots \wedge e_k)^2 = k! \tag{2.25}$$

and

$$\int_{(a,b)^k} \sum_{i=1}^{k} q(x_i)\,(e_1 \wedge \cdots \wedge e_k)^2 = k! \sum_{i=1}^{k} \int_a^b q(e_i)^2$$

which implies the following proposition.

Proposition 2.9. *Let $\mu_1 \geq \mu_2 \geq \cdots$ and $e_1, e_2, \cdots$ be the eigenvalues and orthonormal eigenfunctions of the Laplacian in $H_0^1(a,b)$. Then the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ of the eigenvalue problem* (2.24) *satisfy*

$$\begin{aligned}&\mu_1 + \cdots + \mu_k + \sum_{i=1}^{k} \int_a^b q(e_i)^2 \\ \leq\ & \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k + k \max_{a<x<b} q(x).\end{aligned} \tag{2.26}$$

Remark 2.6. The inequalities (2.26) are sharp in the sense that both can be replaced by equality when $q(x)$ is a constant. The above techniques can be applied to a more general eigenvalue problem, for example,

$$\lambda \phi = \Delta \phi + a(x)\phi,$$

with homogeneous boundary conditions on $\partial\Omega$,

$$u(\partial\Omega) = 0, \quad \text{or} \quad \frac{du}{dn} + b(x)u = 0,$$

where Δ is the Laplacian on Ω, a is bounded and b is a piecewise continuous function on $\partial\Omega$. Here Ω is a bounded domain of $\mathbb{R}^n$ and $\partial\Omega$ is smooth.

The estimates obtained in Proposition 2.9 on the eigenvalues for the regular Sturm-Liouville boundary value problem (2.24) may be of independent interest and a particular case is noted separately in the following proposition with $a = 0, b = 1$. In this case, $\mu_j = -(j\pi)^2$, $e_j = \sqrt{2}\sin(j\pi x)$, $j = 1, 2, \cdots$.

Proposition 2.10. *The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ of the Sturm-Liouville boundary value problem*

$$\begin{aligned} \lambda\phi &= \phi'' + q(x)\phi, \\ \phi(0) &= \phi(1) = 0, \end{aligned}$$

satisfy

$$\begin{aligned} &-\frac{1}{6}k(k+1)(2k+1)\pi^2 + 2\sum_{j=1}^{k}\int_0^1 q(x)\sin^2(j\pi x)\,dx \\ \leq\ &\sum_{j=1}^{k}\lambda_j \leq -\frac{1}{6}k(k+1)(2k+1)\pi^2 + k\max_{0\leq x\leq 1} q(x)\,. \end{aligned} \tag{2.27}$$

Stability of the steady state solution $u = u^*$ of the scalar reaction diffusion equation (2.21) follows from Theorem 2.8 and Proposition 2.9.

Corollary 2.11. *If*

$$\mu_1 + \max_{a<x<b} q(x) < 0, \tag{2.28}$$

then $u = u^*(x)$ is stable hyperbolic.

Corollary 2.12. *If*

$$\mu_1 + \cdots + \mu_k + k\max_{a<x<b} q(x) < 0, \tag{2.29}$$

then $u = u^*(x)$ has a stable manifold with codimension at most $k-1$.

The above procedure also applies to a more general reaction diffusion equation $u_t = u_{xx} + f(x, u, u_x)$. However, the eigenvalue problems of its linear variational equation and compound differential equations need to be transformed to the self-adjoint form. An example of $f = f(u, u_x)$ is explained as follows.

Example 2.1. Consider a scalar reaction diffusion equation

$$\begin{aligned} &u_t = u_{xx} + u + \varepsilon u(1 - u^2 - u_x^2), \quad 0 < x < 2\pi, \quad t > 0, \\ &u(t, 0) = u(t, 2\pi), \quad u_x(t, 0) = u_x(t, 2\pi), \quad t > 0. \end{aligned} \tag{2.30}$$

When $\varepsilon \geq 0$, since

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2 dx &= \int_0^{2\pi} u_t u dx \\ &= \int_0^{2\pi} [-u_x^2 + (1+\varepsilon)u^2 - \varepsilon(u^4 + u_x^2 u^2)]dx \\ &\leq (1+\varepsilon)\int_0^{2\pi} u^2\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u_x^2 dx &= -\int_0^{2\pi} u_t u_{xx} \\ &= \int_0^{2\pi} [-u_{xx}^2 + (1+\varepsilon)u_x^2 + \varepsilon(u^3 + uu_x^2)u_{xx}]dx \\ &= \int_0^{2\pi} \left[-u_{xx}^2 + (1+\varepsilon)u_x^2 - \varepsilon\left(3u^2u_x^2 + \frac{u_x^4}{3}\right)\right] dx \\ &\leq (1+\varepsilon)\int_0^{2\pi} u_x^2 dx,\end{aligned}$$

the solution $u(t,x)$ of (2.30) exists in $X = \{u \in H^1(0,2\pi) : u(0) = u(2\pi),\ u_x(0) = u_x(2\pi)\}$ for all $t \geq 0$. Each solution of (2.30) can be written as $u(t,x) = a_0(t) + \sum_{j=1}^{\infty}(a_j(t)\cos(jx) + b_j(t)\sin(jx))$. It can be shown that the subspaces $S_1 = \text{span}\{1\}$ and $S_2 = \text{span}\{\cos(x), \sin(x)\}$ are invariant with respect to (2.30). In fact, if $u = a_0(t) + a_1(t)\cos(x) + b_1(t)\sin(x)$, then

$$\begin{aligned}& a_0'(t) + a_1'(t)\cos(x) + b_1'(t)\sin(x) \\ = \ & -a_1(t)\cos(x) - b_1(t)\sin(x) + a_0(t) + a_1(t)\cos(x) + b_1(t)\sin(x) \\ & +\varepsilon(a_0(t) + a_1(t)\cos(x) + b_1(t)\sin(x)) \\ & \quad \cdot(1 - a_0^2(t) - a_1^2(t) - b_1^2(t) - 2a_0(t)a_1(t)\cos(x) - 2a_0(t)b_1(t)\sin(x)).\end{aligned}$$

Thus, if $u = a_0(t)$, then

$$a_0' = a_0 + \varepsilon a_0(1 - a_0^2);$$

if $u = a_1(t)\cos(x) + b_1(t)\sin(x)$, then

$$\begin{aligned} a_1' &= \varepsilon a_1(1 - a_1^2 - b_1^2), \\ b_1' &= \varepsilon a_1(1 - a_1^2 - b_1^2). \end{aligned} \tag{2.31}$$

When $\varepsilon > 0$, the reaction diffusion equation (2.30) has steady states solutions $u = 0$, $u = \pm\sqrt{\frac{1+\varepsilon}{\varepsilon}}$ and $u = \cos(x + \alpha)$, where $\alpha \in \mathbb{R}$. The stability of these

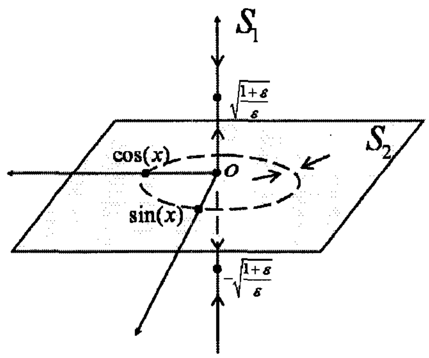

Figure 2.1: Stability of steady state solutions in the invariant subspace S_1 and S_2.

three steady state solutions in the invariant space S_1 and S_2 is illustrated by Figure 2.1.

In the following the stability of these steady states solutions in the whole space X will be studied.

Steady state Solution $u = 0$:

The linear variational equation at $u = 0$ is

$$\begin{aligned} &v_t = v_{xx} + (1+\varepsilon)v, \quad 0 < x < 2\pi, \quad t > 0, \\ &v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \quad t > 0 \end{aligned}$$

and its eigenvalue problem is

$$\begin{aligned} &\lambda\phi = \phi'' + (1+\varepsilon)\phi, \\ &\phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi), \end{aligned} \tag{2.32}$$

for some λ, which has eigenvalues

$$\lambda_1 = 1+\varepsilon > 0, \quad \lambda_2 = \lambda_3 = \varepsilon > 0, \quad \lambda_4 = \lambda_5 = -2^2 + 1 + \varepsilon, \quad \cdots,$$

and corresponding eigenfunctions

$$\phi_1 = 1, \quad \phi_2 = \cos(x), \quad \phi_3 = \sin(x), \quad \phi_4 = \cos(2x), \quad \phi_5 = \sin(2x), \quad \cdots,$$

respectively.

Case 1: $n^2 - 1 < \varepsilon < (n+1)^2 - 1, n \geq 2$

$u = 0$ has a $(2n+1)$-dimensional unstable manifold and a stable manifold with codimension $(2n+1)$.

Case 2: $\varepsilon = n^2 - 1, n \geq 2$

$u = 0$ has a $(2n+1)$-dimensional unstable manifold, a 2-dimensional center manifold, a stable manifold with codimension $(2n+3)$.

Steady state solution $u = \pm\sqrt{\frac{1+\varepsilon}{\varepsilon}}$**:**

The linear variational equation at $u = \pm\sqrt{\frac{1+\varepsilon}{\varepsilon}}$ is

$$\begin{array}{l} v_t = v_{xx} - 2(1+\varepsilon)v, \quad 0 < x < 2\pi, \quad t > 0, \\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \quad t > 0 \end{array}$$

and its eigenvalue problem is

$$\begin{array}{l} \lambda\phi = \phi'' - 2(1+\varepsilon)\phi, \\ \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi), \end{array} \tag{2.33}$$

for some λ, which has eigenvalues

$$\lambda_1 = -2(1+\varepsilon) < 0, \quad \lambda_2 = \lambda_3 = -1 - 2(1+\varepsilon) < 0,$$

$$\lambda_4 = \lambda_5 = -2^2 - 2(1+\varepsilon) < 0, \quad \cdots,$$

and corresponding eigenfunctions

$$\phi_1 = 1, \quad \phi_2 = \cos(x), \quad \phi_3 = \sin(x), \quad \phi_4 = \cos(2x), \quad \phi_5 = \sin(2x), \quad \cdots,$$

respectively. Therefore $u = \pm\sqrt{\frac{1+\varepsilon}{\varepsilon}}$ is stable.

Steady state solution $u = \cos(x+\alpha)$**:**

Only the case $u = \cos(x)$ will be discussed and other cases are similar. The linear variational equation at $u = \cos(x)$ is

$$\begin{array}{l} v_t = v_{xx} + \varepsilon\sin(2x)v_x + (1 - \varepsilon - \varepsilon\cos(2x))v, \quad 0 < x < 2\pi, \quad t > 0, \\ v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \quad t > 0 \end{array} \tag{2.34}$$

and its eigenvalue problem is

$$\begin{array}{l} \lambda\phi = \phi'' + \varepsilon\sin(2x)\phi' + (1 - \varepsilon - \varepsilon\cos(2x))\phi, \\ \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi), \end{array} \tag{2.35}$$

for some λ, which has solutions

$$\phi_1 = \sin(x), \quad \text{when } \lambda_1 = 0,$$

and $$\phi_2 = \cos(x), \quad \text{when } \lambda_2 = -2\varepsilon < 0.$$

Rewrite (2.35) as a self-adjoint second order differential equation

$$L\phi - \lambda p\phi = (p\phi')' + q\phi - \lambda p\phi = 0, \tag{2.36}$$

with periodic boundary condition, where

$$p(x) = \exp\left(-\frac{\varepsilon}{2}\cos(2x)\right) > 0, \quad q(x) = (1 - \varepsilon - \varepsilon\cos(2x))p(x).$$

Then the first principal eigenvalue λ_p of (2.36), and thus of (2.35), is given by

$$\lambda_p = \sup_{\phi \in H^1_{per}} \int_0^{2\pi} \left[-p(\phi')^2 + q\phi^2\right] \Big/ \int_0^{2\pi} p\phi^2 \tag{2.37}$$

where the supremum is taken over $\phi \in H^1_{per}(0, 2\pi) = \{\phi \in L^2(0, 2\pi) : \phi' \in L^2(0, 2\pi), \phi(0) = \phi(2\pi), \ \phi'(0) = \phi'(2\pi)\}$ with $\int_0^{2\pi} p\phi^2 \neq 0$. Substitute $\phi = \frac{1}{\sqrt{p(x)}}$ into the right-hand side of (2.37) to obtain

$$\lambda_p \geq 1 - \varepsilon - \frac{\varepsilon^2}{8}.$$

Moreover,

$$\lambda_p \leq 1.$$

For $\varepsilon > 0$, $1 - \varepsilon - \frac{\varepsilon^2}{8} = 0$ when $\varepsilon = 2(\sqrt{6} - 2) \approx 0.898979$ (see Figure 2.2). If $0 < \varepsilon < 2(\sqrt{6} - 2) \approx 0.898979$, then $\lambda_p > 0$ and thus λ_1, λ_2 and λ_p are three different eigenvalues of (2.36).

If v^1, v^2, v^3, v^4 are solutions of (2.34), then $w(t) = (v^1 \wedge v^2 \wedge v^3 \wedge v^4)(t) \in \bigwedge^4 X$ has a pointwise representation

$$w(t, x_1, x_2, x_3, x_4) = \det\left[v^i(t, x_j)\right],$$

which satisfies the $4th$ compound differential equation of the linear variational equation (2.34)

$$w_t = \sum_{i=1}^{4} w_{x_i x_i} + \varepsilon \sum_{i=1}^{4} \sin(2x_i) w_{x_i} + 4(1 - \varepsilon)w - \varepsilon \sum_{i=1}^{4} \cos(2x_i) w. \tag{2.38}$$

Consider a Lyapunov function

$$V(t) = \int_{(0,2\pi)^4} w^2 = \int_{(0,2\pi)^4} w^2(t, x_1, x_2, x_3, x_4)\, dx_1 dx_2 dx_3 dx_4.$$

Then

$$\begin{aligned}
\frac{d^+V}{dt} &= \int_{(0,2\pi)^4} ww_t \\
&= \int_{(0,2\pi)^4} \left(w \sum_{j=1}^{4} w_{x_j x_j} + 4(1-\varepsilon)w^2 - \varepsilon \sum_{j=1}^{4} \cos(2x_j) w^2 \right) \\
&\quad + \int_{(0,2\pi)^4} \sum_{j=1}^{4} \varepsilon \sin(2x_j) w w_{x_j} \\
&\leq (0 - 1^2 - 1^2 - 2^2 + 4(1-\varepsilon)) \int_{(0,2\pi)^4} w^2 \\
&\quad -2\varepsilon \int_{(0,2\pi)^4} \sum_{j=1}^{4} \cos(2x_j + 2ct) w^2 \\
&\leq (-2 - 4\varepsilon + 2\varepsilon \cdot 4) \int_{(0,2\pi)^4} w^2 = 2(-2+4\varepsilon)V.
\end{aligned}$$

Thus, for any four eigenvalues $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}$ of (2.35), by choosing $w = \phi_{i_1} \wedge \cdots \wedge \phi_{i_4}$ where $\phi_{i_s}, s = 1, \cdots, 4$ are the corresponding eigenfunctions, it follows that

$$\lambda_{i_1} + \lambda_{i_2} + \lambda_{i_3} + \lambda_{i_4} \leq -2 + 4\varepsilon.$$

In particular,

$$\lambda_p + \lambda_1 + \lambda_2 + \lambda_3 \leq -2 + 4\varepsilon.$$

Therefore,

$$\lambda_3 \leq -3 + 7\varepsilon + \frac{\varepsilon^2}{8}.$$

For $\varepsilon > 0$, $-3 + 7\varepsilon + \frac{\varepsilon^2}{8} = 0$ when $\varepsilon = 2(\sqrt{202} - 14) \approx 0.425341$ and $1 - \varepsilon - \frac{\varepsilon^2}{8} = -3 + 7\varepsilon + \frac{\varepsilon^2}{8}$ when $\varepsilon = 2(13\sqrt{2} - 18) \approx 0.769553$ (See Figure 2.2).

Corollary 2.13. *If* $0 < \varepsilon < 2(\sqrt{202} - 14)$, *then* $u = \cos(x)$ *has a 1-dimensional center manifold, a 1-dimensional unstable manifold and a stable manifold with codimension* 2.

Corollary 2.14. *If* $2(\sqrt{202} - 14) \leq \varepsilon < 2(13\sqrt{2} - 18)$, *then* $u = \cos(x)$ *has a stable manifold with codimension at most* 3.

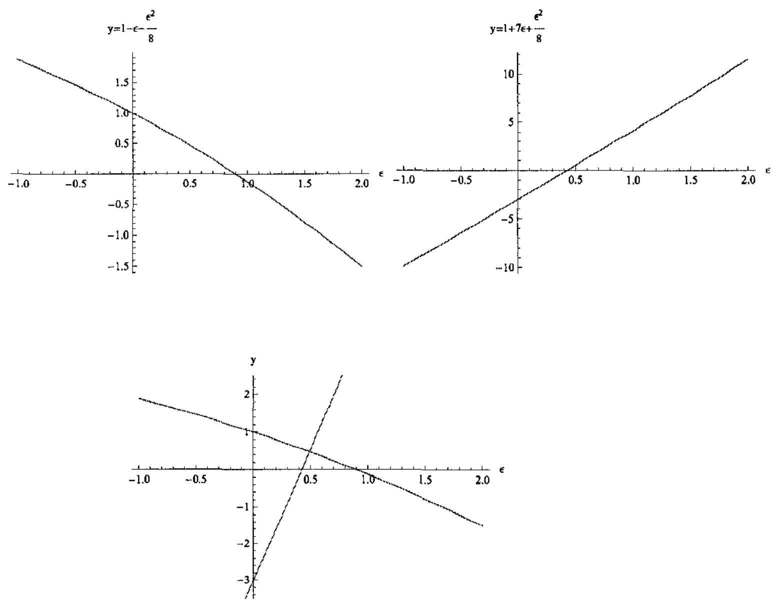

Figure 2.2: Graphs of $y = 1 - \varepsilon - \frac{\varepsilon^2}{8}$ and $y = -3 + 7\varepsilon + \frac{\varepsilon^2}{8}$.

Chapter 3

On Poincaré's Stability Criterion for Periodic Orbits

This chapter deals with the stability of periodic solutions of differential equations. A fundamental result on this topic is that a nonconstant periodic solution is orbitally asymptotically stable with asymptotic phase if, with the exception of a single characteristic multiplier which equals 1, the moduli of all characteristic multipliers of the periodic solution are less than 1. For a 2-dimensional autonomous ordinary differential equation, this is equivalent to Poincaré's stability criterion: a nonconstant ω-periodic solution $u = \phi(t)$ of

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^2,$$

is orbitally asymptotically stable with asymptotic phase if

$$\int_0^\omega \text{div } f(\phi(t))dt < 0.$$

This is the requirement that the Liouville equation

$$\frac{dw}{dt} = \text{div } f(\phi(t))w$$

be uniformly asymptotically stable, which means that 2-dimensional areas near the orbit of $\phi(t)$ decrease exponentially under the flow of the differential equation and, as a consequence, nearby orbits are attracted to the orbit of $\phi(t)$.

A higher dimensional generalization of Poincaré's stability criterion is obtained by Muldowney [78]: a nonconstant ω-periodic solution $u = \phi(t)$

of

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^n,$$

is orbitally asymptotically stable with asymptotic phase if

$$\frac{dw}{dt} = \frac{\partial f}{\partial u}^{[2]}(\phi(t))w$$

is uniformly asymptotically stable. This equation is called the second compound differential equation of the linear variational equation

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(\phi(t))v,$$

which reduces to the Liouville equation when $n = 2$. The uniform asymptotic stability of the second compound differential equation means that, under the flow of the nonlinear equation, 2-dimensional areas near the orbit of $\phi(t)$ diminish exponentially, which is the reason that the characteristic multipliers of the periodic solution $\phi(t)$, with the exception of a single characteristic multiplier which equals one, all have moduli less than 1.

When, with the exception of a single characteristic multiplier which equals 1, the characteristic multipliers have moduli less than 1, the periodic solution is said to be *stable hyperbolic*. It was shown by Muldowney [78] that the periodic solution is stable hyperbolic if and only if the second compound differential equation is uniformly asymptotically stable.

In this chapter the property of the diminishing area being a sufficient condition for a nonconstant periodic solution to be stable hyperbolic is extended to autonomous differential equations in general Banach spaces X with particular applications to reaction diffusion equations. As in the finite dimensional situation, this property is characterized by the asymptotic stability of an associated compound differential equation. This allows the possibility of applying standard asymptotic stability techniques such as Lyapunov theory to the difficult problem of estimation of the characteristic multipliers.

3.1 Orbital Stability of Periodic Solutions

The following notation, definitions, and lemmas are taken from Henry [42], page 53-54, page 197-202 and page 247-261.

Let A be a sectorial operator in a Banach space X and f be continuously differentiable from X^α into X where $0 \le \alpha < 1$. Let $\mathcal{D}(A)$ denote the domain of A. Consider an autonomous differential equation

$$\frac{du}{dt} + Au = f(u), \quad u(\cdot) \in X. \tag{3.1}$$

In the following, let $u(t) = u(t, u_0)$ denote the solution of (3.1) with initial value $u(0) = u_0$. A *nonconstant periodic solution* of the autonomous differential equation (3.1) is $u = \phi(t)$ if there exists a least $\omega > 0$ such that $\phi(t + \omega) = \phi(t)$. Let $\gamma = \{\phi(t), 0 \le t \le \omega\}$ denote the orbit of the ω-periodic solution $\phi(t)$.

Definition 3.1. An ω-periodic solution $\phi(t)$ of (3.1) is said to be *orbitally stable* if the orbit γ is stable, that is if for any neighbourhood U of γ, there exists a neighbourhood N of γ such that $u_0 \in N$ implies that the solution of (3.1) $u(t, u_0) \in U$ for all $t > 0$. An ω-periodic solution $\phi(t)$ is said to be *orbitally asymptotically stable* if the orbit γ is asymptotically stable, that is, the orbit γ is stable and there exists a δ such that $\mathrm{dist}_{X^\alpha}(u_0, \phi(0)) < \delta$ implies that $\mathrm{dist}_{X^\alpha}(u(t, u_0), \phi(t)) \to 0$ as $t \to \infty$. An ω-periodic solution $\phi(t)$ is said to be *orbitally asymptotically stable with asymptotic phase* if there exist $M, \rho, \beta > 0$ such that

$$\mathrm{dist}_{X^\alpha}\{u_0, \gamma\} = \min_{0 \le t \le p} \|u_0 - \phi(t)\|_\alpha < \rho$$

implies that there exists a real constant $h = h(u_0)$ such that the solution $u(t, u_0)$ of (3.1) satisfies

$$\|u(t, u_0) - \phi(t - h)\|_\alpha < Me^{-\beta t}, \quad \text{for } t \ge 0.$$

An ω-periodic solution $\phi(t)$ of (3.1) is *orbitally unstable* if it is not orbitally stable.

The linear variational equation of (3.1) at the solution $\phi(t)$ is

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(\phi(t))v, \tag{3.2}$$

whose solutions satisfy

$$v(t) = T(t, s)v(s), \quad t \ge s,$$

where $T(t, s)$ is the evolution operator generated by (3.2).

Definition 3.2. The *period map* of (3.2) is

$$U(t) = T(t+\omega, t). \tag{3.3}$$

The nonzero eigenvalues of $U(t)$ are called *characteristic multipliers* of the periodic solution $\phi(t)$ or *characteristic multipliers* of the linear variational equation (3.2).

Remark 3.1. For an ordinary differential equation, this map is called the *monodromy matrix*, whose eigenvalues are also the eigenvalues of the Jacobian matrix of the Poincaré map.

Lemma 3.1. *$U(t+\omega) = U(t)$ for all t. The characteristic multipliers are independent of t, that is, the nonzero eigenvalues of $U(t)$ coincide with those of $U(s)$. In fact, $\sigma(U(t))\backslash\{0\}$ is independent of t. If A has compact resolvent, then $U(t)$ is compact and $\sigma(U(t))\backslash\{0\}$ consists entirely of characteristic multipliers.*

Lemma 3.2. *Suppose that σ_1 is a spectral subset of $\sigma(U(t))$ for all t; the usual case is when σ_1 is a finite collection of isolated eigenvalues or the complement of such a set. Then for each t, the space X may be decomposed as $X = X_1(t) \oplus X_2(t)$, the direct sum of closed subspaces invariant under $U(t)$, $\sigma(U(t)|_{X_1(t)}) = \sigma_1, \sigma(U(t)|_{X_2(t)}) = \sigma(U(t))\backslash\sigma_1$. If $t \geq s, T(t,s)$ maps $X_1(s)$ into $X_1(t)$, and is a one-to-one map onto $X_1(t)$ if $0 \notin \sigma_1$.*

Let $e^{\beta\omega} = \sup\{|\lambda|, \lambda \in \sigma_1\}$. Then for each $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\|T(t,s)u\| \leq M_\varepsilon e^{(\beta+\varepsilon)(t-s)}\|u\|$$

for $t \geq s$ and $u \in X_1(s)$.

Suppose $0 \notin \sigma_1$ and let $e^{\alpha\omega} = \inf\{|\lambda|, \lambda \in \sigma_1\} > 0$. Then $T(t,s)u, u \in X_1(s)$, may be defined also for $t \leq s$, and for small $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\|T(t,s)u\| \leq M_\varepsilon e^{(\alpha-\varepsilon)(t-s)}\|u\|$$

for $t \leq s$ and $u \in X_1(s)$.

Assume that $\phi(t)$ is a nonconstant ω-periodic solution of (3.1). Let S be a C^1 manifold (in X^α) of codimension 1 such that $\phi(0) \in S$ and $\frac{d\phi}{dt}(0)$ is not tangent to S at $\phi(0)$. For $\xi \in S$ near $\phi(t_0)$, define $\Phi(\xi) \in S$ by

$$\Phi(\xi) = u(\omega(\xi), \xi), \tag{3.4}$$

where $\omega(\xi) = \omega + O(\|\xi - \phi(0)\|_\alpha)$ is chosen to ensure $\Phi(\xi) \in S$. Let $\Phi'(\xi) = \frac{d\Phi(\xi)}{d\xi}$.

Definition 3.3. The map Φ defined in (3.4) is called the *Poincaré map.*

It is shown in Henry [42], page 258-260, that Φ is a well-defined C^1 function on S near $\phi(0)$ and

$$\Phi'(\phi(0)) = U(0)$$

where $T(t, s)$ is the evolution operator generated by (3.2). Notice that any fixed point of Φ yields an ω-periodic solution of (3.1).

Without loss of generality, assume that $\phi(0) = 0, \frac{d\phi}{dt}(0) \neq 0$ and S is represented near 0 in the form $x = h(y)$ for $y \in Y$ where Y is the tangent space to S at 0. Thus $\|h(y) - y\|_\alpha = o(\|y\|_\alpha)$ as $y \to 0$ in Y. Now $X^\alpha = \text{span}\{\frac{d\phi}{dt}(0), Y\}$. Let $P_Y : X^\alpha \to Y$ be the projection onto Y where $P_Y\frac{d\phi}{dt}(0) = 0, P_Y y = y$ for $y \in Y$.

If $\{1\}$ is an isolated eigenvalue of $U(0)$ and $X^\alpha = X_1^\alpha \oplus X_2^\alpha$ is the corresponding decomposition with

$$\sigma(U(0)|_{X_1^\alpha}) = \{1\},$$
$$\sigma(U(0)|_{X_2^\alpha}) = \sigma(U(0))\backslash\{1\},$$

then $Y = Y_1 \oplus Y_2$ where $Y_j = P_Y X_j^\alpha, j = 1, 2$ are $\Phi'(0)$-invariant subspaces and

$$\sigma(\Phi'(0)|_{Y_1}) \subseteq \{1\},$$
$$\sigma(\Phi'(0)|_{Y_2}) = \sigma(U(t_0))\backslash\{1\}.$$

In particular, if $\{1\}$ is an isolated simple eigenvalue of $U(0)$, then $Y_1 = \{0\}$ and $1 \notin \sigma(\Phi'(0))$. If $\{1\}$ is an eigenvalue of multiplicity m, then $\text{dim}Y_1 = m - 1$. Suppose that $Y_2 = Y_S \oplus Y_U$ where

$$\sigma(\Phi'(0)|_{Y_S}) = \sigma(U(0)) \cap \{|\lambda| < 1, \lambda \in \sigma(U(0))\},$$
$$\sigma(\Phi'(0)|_{Y_U}) = \sigma(U(0)) \cap \{|\lambda| > 1, \lambda \in \sigma(U(0))\}.$$

Let $e^{\beta\omega} = \sup\{|\lambda|, \lambda \in \sigma(\Phi'(0)|_{Y_S})\}$ and $e^{\alpha\omega} = \inf\{|\lambda|, \lambda \in \sigma(\Phi'(0)|_{Y_U}\}$. Assume that either $\sigma(\Phi'(0)|_{Y_S})$ or $\sigma(\Phi'(0)|_{Y_U})$ is a finite collection of isolated eigenvalues and $\beta < 0, \alpha > 0$. Then Lemma 3.2 implies that there exist constants $M, N > 0$ such that

$$\|T(t,0)u_0\|_\alpha \leq Me^{\frac{\beta}{2}t}\|u_0\|_\alpha, \quad \text{for } t \geq 0 \text{ and } u_0 \in Y_S,$$

and $T(t,s)u, u \in Y_U$, may be defined also for $t \leq s$, and

$$\|T(t,0)u_0\|_\alpha \leq Ne^{\frac{\alpha}{2}t}\|u_0\|_\alpha, \quad \text{for } t \leq 0 \text{ and } u_0 \in Y_U.$$

Definition 3.4. The dimension of Y_S is called the *dimension of the (local) stable manifold* of $\phi(t)$. The dimension of Y_U is called the *dimension of the (local) unstable manifold* of $\phi(t)$. The dimension of span$\{\phi'(0), Y_1\}$ is called the *dimension of the (local) center manifold* of $\phi(t)$. The sum of the dimension of the (local) center manifold and the dimension of the (local) unstable manifold is called the *codimension of the (local) stable manifold* of $\phi(t)$.

Lemma 3.3. *If A has compact resolvent, then the zero solution of* (3.2) *is uniformly asymptotically stable if and only if all nonzero eigenvalues of $U(t)$ have moduli less than 1.*

Lemma 3.4. *Let f be continuously differentiable from X^α into X in a neighbourhood of a nonconstant periodic solution $\phi(t)$ of* (3.1). *Suppose that $t \mapsto \frac{\partial f}{\partial x}(\phi(t)) \in \mathscr{L}(X^\alpha, X)$ is Hölder continuous. Then the linear variational equation*

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(\phi(t))v,$$

has 1 as a characteristic multiplier.

Lemma 3.5. *Suppose that the assumptions in Lemma* 3.4 *are satisfied. If $\sigma(U(t)) \cap \{|\lambda| > 1\}$ is a nonempty spectral set, then $\phi(t)$ is orbitally unstable.*

Lemma 3.6. *Suppose that the assumptions in Lemma* 3.4 *are satisfied. If 1 is an isolated simple characteristic multiplier of the periodic solution $\phi(t)$, and the remainder of the spectrum lies in $\{|\lambda| < e^{-\beta\omega}\}$ for some $\beta > 0$, then $\phi(t)$ is orbitally asymptotically stable with asymptotic phase.*

These theoretical results from Henry [42] furnish a context in which the compound differential equation approach to orbital stability may be developed. The next theorem uses the compound differential equation introduced in Chapter 1 to provide a sufficient condition for the remainder of the spectrum of the period map lying in $\{|\lambda| < e^{-\beta\omega}\}$ for some $\beta > 0$.

Theorem 3.7. *Suppose that $\phi(t)$ is a nonconstant ω-periodic solution of* (3.1). *If the second compound differential equation*

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(\phi(t))w$$

of the linear variational equation (3.2) *is uniformly asymptotically stable in $\bigwedge^2 X^\alpha$, then $\phi(t)$ is orbitally asymptotically stable with asymptotic phase.*

Theorem 3.7 will follow from Theorem 4.12 in Section 4.2. No proof is given here. A particular case when A has compact resolvent is discussed in detail as follows.

Theorem 3.8. *Suppose that A has compact resolvent and $\phi(t)$ is a nonconstant ω-periodic solution of* (3.1). *If the kth compound differential equation*

$$\frac{dw}{dt} + A^{[k]}w = \frac{\partial f}{\partial u}^{[k]}(\phi(t))w \tag{3.5}$$

of the linear variational equation (3.2) *is uniformly asymptotically stable in $\bigwedge^k X^\alpha$, then at most $k-2$ characteristic multipliers of the periodic solution $\phi(t)$ have moduli greater than or equal to 1.*

Proof. Let $T(t,s)$ be the evolution operator generated by the linear variational equation

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(\phi(t))v.$$

Let $U(t) = T(t+\omega, t)$ denote the period map. Since $\phi(t)$ is a nonconstant ω-periodic solution of (3.1), Lemma 3.4 implies that one of the characteristic multipliers of the periodic solution $\phi(t)$ is 1. From Section 1.3, $w(t) = T^{(k)}(t,s)w(s)$ is a solution of (3.5), and thus the period map of (3.5) is $U^{(k)}(t) = T^{(k)}(t+\omega, t)$. Since (3.5) is uniformly asymptotically stable and A has compact resolvent, Lemma 3.3 implies that the nonzero eigenvalues of $U^{(k)}(t)$ have moduli less than 1. Let $\{\lambda_i\}$ be the eigenvalues, counting multiplicities, of $U(t)$. Then all possible products of k eigenvalues of $U(t)$ in the form $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, i_1 < i_2 < \cdots < i_k$, counting multiplicities, are the eigenvalues of $U^{(k)}(t)$. Since one of the characteristic multipliers of $\phi(t)$ is 1, it can be concluded that at most $k-2$ characteristic multipliers of $\phi(t)$ have moduli greater than or equal to 1. ∎

Definition 3.5. A periodic solution is *stable hyperbolic* if, with the exception of a single characteristic multiplier which equals 1, the moduli of the characteristic multipliers of the periodic solution are less than 1.

Corollary 3.9. *Suppose that A has compact resolvent and $\phi(t)$ is a nonconstant periodic solution of* (3.1). *If $\phi(t)$ is stable hyperbolic, then $\phi(t)$ is orbitally asymptotically stable with asymptotic phase.*

Corollary 3.10. *Suppose that A has compact resolvent and $\phi(t)$ is a nonconstant ω-periodic solution of* (3.1). *Then $\phi(t)$ is stable hyperbolic if the second compound differential equation*

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(\phi(t))w \tag{3.6}$$

of the linear variational equation (3.2) *is uniformly asymptotically stable in* $\bigwedge^2 X^\alpha$.

Remark 3.2. Let $U(t)$ be the periodic map of the linear variational equation of (3.1) at the solution $\phi(t)$

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(\phi(t))v. \tag{3.7}$$

Suppose that $\{\lambda_i\}$ are the eigenvalues, counting multiplicities, of $U(t)$. If the corresponding eigenfunctions of $U(t)$ form a basis of X, then the eigenvalues of $U^{(k)}(t)$, counting multiplicities, are given by all possible products of k eigenvalues of $U(t)$ in the form $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, i_1 < i_2 < \cdots < i_k$. In this case, the uniform asymptotic stability of the second compound differential equation (3.5) is equivalent to the hyperbolic stability of the periodic solution $\phi(t)$.

Remark 3.3. The finite linear combinations of expression of the form $v^1 \wedge v^2$ are solutions of the second compound differential equation (3.6), where $v^i, i = 1, 2$ are solutions of the linear variational equation (3.7) of (3.1) at the solution $\phi(t)$. If the second compound differential equation (3.6) is uniformly asymptotically stable in $\bigwedge^2 X^\alpha$, then there exists a $\sigma > 0$, such that

$$\|(v^1 \wedge v^2)(w)\|_{\bigwedge X^\alpha} \leq e^{-\sigma w}\|(v^1 \wedge v^2)(0)\|_{\bigwedge X^\alpha}. \tag{3.8}$$

Hence

$$\|U^{(2)}(0)(v^1 \wedge v^2)(0)\|_{\bigwedge X^\alpha} < \|(v^1 \wedge v^2)(0)\|_{\bigwedge X^\alpha},$$

where $U^{(2)}(t)$ is defined in the proof of Theorem 3.8. Let $\{\lambda_i\}$ be the eigenvalues, counting multiplicities, of $U(0)$. Then all possible products of two eigenvalues of $U(0)$ in the form $\lambda_{i_1}\lambda_{i_2}, i_1 < i_2$, counting multiplicities, are the eigenvalues of $U^{(2)}(0)$. Since one of the characteristic multipliers of $\phi(t)$ is 1, it can be concluded from that (3.8) that all the other characteristic multipliers of $\phi(t)$ have moduli less than 1 and thus $\phi(t)$ is stable hyperbolic.

3.2 Reaction Diffusion Systems

In this section, Corollary 3.10 is applied to study the stability of periodic solutions of reaction diffusion systems.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and $D = \mathrm{diag}(d_1, \cdots, d_n)$, $d_i \geq 0$. Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz continuous in u, uniformly in x. Suppose that $\phi(t, x)$ is an ω-periodic solution of

$$\frac{\partial u}{\partial t} = D\Delta u + f(x, u), \tag{3.9}$$

with Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \tag{3.10}$$

or Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{3.11}$$

or a more general boundary condition of the form

$$Bu|_{\partial\Omega} = 0. \tag{3.12}$$

In the following, the case with Dirichlet boundary condition is discussed and similar arguments can be applied to other boundary conditions. Let

$X = L^2(\Omega, \mathbb{R}^n)$ and $A : \mathcal{D}(A) \subset X \to X$ be the linear unbounded operator $Au = -\Delta u$, where

$$\mathcal{D}(A) = \left\{u \in H^2(\Omega, \mathbb{R}^n),\ u|_{\partial\Omega} = 0\right\}.$$

The initial value problem of (3.9) is well-posed in X^α, $0 \le \alpha < 1$. The linear variational equation of (3.9) at $u = \phi(t, x)$ is

$$\begin{aligned} \frac{\partial v}{\partial t} &= D\Delta v + \frac{\partial f}{\partial u}(x, \phi(t,x))v, \\ v|_{\partial\Omega} &= 0. \end{aligned} \tag{3.13}$$

Let $\lambda_i, i = 1, 2, \cdots$ be the eigenvalues of the Laplace equation

$$\Delta u = -\lambda_i u, \tag{3.14}$$

with Dirichlet boundary condition (3.10) and $e^1, e^2, \cdots$ be its corresponding orthonormal eigenfunctions. It is assumed that $0 < \lambda_1 \le \lambda_2 \le \cdots$.

Remark 3.4. If $\phi(t,x)$ is an ω-periodic solution of (3.9) with Neumann boundary condition, then $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ is assumed. For other boundary conditions, λ_i are the eigenvalues of the Laplace equation (3.14) and $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ is assumed.

Let $v^i = (v_1^i, v_2^i, \cdots, v_n^i)^T \in X^\alpha, i = 1, 2$ be solutions of (3.13) and $f(x,u) = (f_1(x,u), f_2(x,u), \cdots, f_n(x,u))^T$. In the following, two representations of the second compound differential equation of (3.13) are discussed.

Case 1: A pointwise representation of the second compound differential equation of (3.13)

For $u \in X^\alpha$, $x \in \Omega$, $i = 1, \cdots, n$, define a linear functional $(i, x) : u \mapsto \langle (i,x), u\rangle = u_i(x)$. Then (3.13) can be written as

$$\frac{\partial v_j}{\partial t} = d_j \Delta v_j + \sum_{k=1}^{n} a_{jk}(t,x) v_k,$$

where

$$a_{jk}(t,x) = \frac{\partial f_j}{\partial u_k}(x, \phi(t,x)).$$

For $w(t) \in \bigwedge^2 X^\alpha,\ x_1, x_2 \in \Omega,\ i,j = 1,\cdots,n$, a pointwise representation of $w(t)$ is

$$\langle (i,x_1) \wedge (j,x_2), w(t)\rangle := w_{ij}(t,x_1,x_2)$$

and a pointwise representation of the operator $\left(D\Delta + \dfrac{\partial f}{\partial u}\right)^{[2]}$ is

$$\begin{aligned}
&\left\langle (i,x_1) \wedge (j,x_2), \left(D\Delta + \frac{\partial f}{\partial u}\right)^{[2]} w \right\rangle \\
= \;& d_i\Delta_1 w_{ij}(t,x_1,x_2) + d_j\Delta_2 w_{ij} + (a_{ii}(t,x_1) + a_{jj}(t,x_2))w_{ij}(t,x_1,x_2) \\
&+ \sum_{k\neq i} a_{ik}(t,x_1)w_{kj} + \sum_{k\neq j} a_{jk}(t,x_2)w_{ik},
\end{aligned}$$

where $\Delta_1 w_{ij}$ and $\Delta_2 w_{ij}$ are the Laplacian of $w_{ij}(x_1,x_2)$ with respect to x_1 and x_2. Since w_{ij} satisfies

$$w_{ij}(t,x_1,x_2) = -w_{ji}(t,x_2,x_1),$$

there are $k+(k-1)+(k-2)+\cdots+1 = \binom{k+1}{2}$ different types of determinants that will need to be discussed. Thus a pointwise representation of the second compound differential equation of (3.13) in $\bigwedge^2 X^\alpha$ is

$$\begin{aligned}
\frac{\partial w_{ii}}{\partial t} &= d_i(\Delta_1 w_{ii} + \Delta_2 w_{ii}) + (a_{ii}(t,x_1) + a_{ii}(t,x_2))w_{ii} \\
&\quad + \sum_{k\neq i}(a_{ik}(t,x_1)w_{ki} + a_{ik}(t,x_2)w_{ik}), \quad i = 1,2,\cdots n, \\
\frac{\partial w_{ij}}{\partial t} &= d_i\Delta_1 w_{ij} + d_j\Delta_2 w_{ij} + (a_{ii}(t,x_1) + a_{jj}(t,x_2))w_{ij} \\
&\quad + \sum_{k\neq i} a_{ik}(t,x_1)w_{kj} + \sum_{k\neq j} a_{jk}(t,x_2)w_{ik}, \quad 1 \le i < j \le n.
\end{aligned} \tag{3.15}$$

If $v^1(t), v^2(t)$ are solutions of (3.13), then

$$\begin{aligned}
w_{ij}(t,x_1,x_2) &= \langle (i,x_1) \wedge (j,x_2), (v^1 \wedge v^2)(t)\rangle \\
&= \det\begin{bmatrix} v_i^1(t,x_1) & v_i^2(t,x_1) \\ v_j^1(t,x_2) & v_j^2(t,x_2) \end{bmatrix}, \quad x_1, x_2 \in \Omega, \quad i,j = 1,2,\cdots,n.
\end{aligned}$$

It can be checked by differentiating the above determinant that $v^1(t) \wedge v^2(t)$ satisfies the second compound differential equation (3.15) and the Dirichlet boundary conditions on v^i imply that

$$w_{ij}(t,x_1,x_2)\big|_{x_1\in\partial\Omega} = 0, \quad w_{ij}(t,x_1,x_2)\big|_{x_2\in\partial\Omega} = 0.$$

Case 2: A basis representation of the second compound differential equation of (3.13)

Let $\{e^i(x)\}$ be the orthonormal eigenfunctions of the Laplace equations (3.14) with Dirichlet boundary condition (3.10) and $\lambda_i, i = 1, 2, \cdots$ be the corresponding eigenvalues. Then the set of vectors $\{e^i\}$ is an orthonormal basis of X^α and $\{e^i \wedge e^j, 1 \leq i < j\}$ is an orthonormal basis of $\bigwedge^2 X^\alpha$. Define another linear map $(i, e^j) : v = (v_1, v_2, \cdots, v_n) \mapsto \langle (i, e^j), v \rangle = \int_\Omega v^i e^j$. Then

$$v(t,x) = (v_1, v_2, \cdots, v_n) = \left(\sum_j c_{1j}(t) e^j(x), \cdots, \sum_j c_{nj}(t) e^j(x) \right)^T$$

where $c_{ij} := \langle (i, e^j), v \rangle$, $i = 1, 2, \cdots, n, j = 1, 2, \cdots$ satisfies

$$\frac{dc_{ij}}{dt} = [-d_i\lambda_j + a_{ii}(t,j,j)]c_{ij} + \sum_{k \neq j} a_{ii}(t,j,k)c_{ik} + \sum_{l \neq i} \sum_k a_{il}(t,j,k)c_{lk}, \quad (3.16)$$

where $a_{ij}(t,k,l) := \langle e^k, a_{ij} e^l \rangle = \int_\Omega a_{ij}(t,x) e^l(x) e^k(x) dx$, $i, j = 1, 2, \cdots, n$, $k, l = 1, 2, \cdots$. Here $a_{ij}(t,k,l) = a_{ij}(t,l,k)$.

For $w(t) \in \bigwedge^2 X^\alpha$, a basis representation of $w(t)$ is

$$\left\langle (i, e^{i_1}) \wedge (j, e^{i_2}), w(t) \right\rangle := w_{ij}(t, i_1, i_2)$$

and a basis representation of the second compound differential equation of (3.13) in $\bigwedge^2 X^\alpha$ is

$$\begin{aligned}
\frac{dw_{ii}}{dt}(t,i_1,i_2) &= (-d_i\lambda_{i_1} - d_i\lambda_{i_2} + a_{ii}(t,i_1,i_1) + a_{ii}(t,i_2,i_2))w_{ii}(t,i_1,i_2) \\
&\quad + \sum_{k \neq i_1} a_{ii}(t,k,i_1)w_{ii}(t,k,i_2) + \sum_{l \neq i} \sum_k a_{il}(t,k,i_1)w_{li}(t,k,i_2) \\
&\quad + \sum_{k \neq i_1} a_{ii}(t,k,i_2)w_{ii}(t,i_1,k) + \sum_{l \neq i} \sum_k a_{il}(t,k,i_2)w_{il}(t,i_1,k) \\
\frac{dw_{ij}}{dt}(t,i_1,i_2) &= (-d_i\lambda_{i_1} - d_j\lambda_{i_2} + a_{ii}(t,i_1,i_1) + a_{jj}(t,i_2,i_2))w_{ij}(t,i_1,i_2) \\
&\quad + \sum_{k \neq i} a_{ii}(t,k,i_1)w_{ij}(t,k,i_2) + \sum_{l \neq i} \sum_k a_{il}(t,k,i_1)w_{ll}(t,k,i_2) \\
&\quad + \sum_{k \neq i} a_{jj}(t,k,i_2)w_{ij}(t,i_1,k) + \sum_{l \neq i} \sum_k a_{jl}(t,k,i_2)w_{ll}(t,i_1,k).
\end{aligned} \quad (3.17)$$

A direct application of Corollary 3.10 implies the following corollary.

Corollary 3.11. *If the second compound differential equation* (3.15) *or* (3.17) *of the linear variational equation* (3.13) *is uniformly asymptotically stable in* $\bigwedge^2 X^\alpha$, *then* $\phi(t,x)$ *is orbitally asymptotically stable with asymptotic phase in* X^α.

In the following, the two cases when $\Omega = (0, L)$ are discussed in detail.

Scalar Case: $n = 1$ with periodic boundary condition

Suppose that $u = \phi(t,x)$ is a nonconstant ω-periodic solution of

$$\begin{aligned} &u_t = u_{xx} + f(x,u), \quad 0 < x < L, \quad t > 0, \\ &u(t,0) = u(t,L), \quad u_x(t,0) = u_x(t,L) \quad t \geq 0, \end{aligned} \tag{3.18}$$

where $f \in C^2((0,L) \times \mathbb{R} \to \mathbb{R})$. The linear variational equation of (3.18) at $\phi(t,x)$ is

$$\begin{aligned} &v_t = v_{xx} + a(t,x)v, \quad 0 < x < L, \quad t > 0, \\ &v(t,0) = v(t,L), \quad v_x(t,0) = v_x(t,L) \quad t \geq 0, \end{aligned} \tag{3.19}$$

where

$$a(t,x) = \frac{\partial f}{\partial u}(x, \phi(t,x)).$$

If v^1, v^2 are solutions of (3.19), then $w(t) = (v^1 \wedge v^2)(t) \in \bigwedge^2 X$ has a pointwise representation

$$w(t,x_1,x_2) = \langle x_1 \wedge x_2, (v^1 \wedge v^2)(t) \rangle = \det \begin{bmatrix} v^1(t,x_1) & v^2(t,x_1) \\ v^1(t,x_2) & v^2(t,x_2) \end{bmatrix},$$

which satisfies the second compound differential equation of (3.19) defined on $\bigwedge^2 X$

$$w_t = w_{x_1x_1} + w_{x_2x_2} + (a(t,x_1) + a(t,x_2))w. \tag{3.20}$$

The periodic boundary conditions of v^i implies that

$$\begin{aligned} &w(t,0,x_2) = w(t,L,x_2), \quad w_{x_1}(t,0,x_2) = w_{x_1}(t,L,x_2), \\ &w(t,x_1,0) = w(t,x_1,L), \quad w_{x_2}(t,x_1,0) = w_{x_2}(t,x_1,L). \end{aligned}$$

The eigenvalues and orthonormal eigenfunctions of the Laplace equation

$$\begin{aligned} &\Delta u = -\lambda_i u, \\ &u(0) = u(L), \quad u_x(0) = u_x(L) \end{aligned} \tag{3.21}$$

are $\lambda_1 = 0, \lambda_{2n} = \left(\frac{n\pi}{L}\right)^2, \lambda_{2n+1} = \left(\frac{n\pi}{L}\right)^2$, and $e^1(x) = 1, e^{2n}(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi}{L}\right)x, e^{2n+1}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}\right)x, n = 1, 2, \cdots$, respectively. Consider a Lyapunov function

$$V(t) = \frac{1}{2}\int_{(0,L)^2} w^2 = \frac{1}{2}\int_{(0,L)^2} w^2(t, x_1, x_2)\, dx_1 dx_2.$$

Then

$$\begin{aligned}
\frac{d^+V}{dt} &= \int_{(0,L)^2} ww_t \\
&= \int_{(0,L)^2} ww_{x_1x_1} + ww_{x_2x_2} + \int_{(0,L)^2} (a(t,x_1) + a(t,x_2))\, w^2 \\
&= ww_{x_1}|_{x_1=0}^{L} + ww_{x_2}|_{x_2=0}^{L} - \int_{(0,L)^2} \left[(w_{x_1})^2 + (w_{x_2})\ ^2\right] \\
&\quad + \int_{(0,L)^2} (a(t,x_1) + a(t,x_2))\, w^2 \\
&= -\int_{(0,L)^2} \left[(w_{x_1})^2 + (w_{x_2})^2\right] + \int_{(0,L)^2} (a(t,x_1) + a(t,x_2))\, w^2 \\
&\leq \int_{(0,L)^2} (-\lambda_1 - \lambda_2) w^2 + \int_{(0,L)^2} (a(t,x_1) + a(t,x_2))\, w^2 \\
&= \int_{(0,L)^2} \left(-\left(\frac{\pi}{L}\right)^2 + 2 \max_{0<x<L}\{a(t,x)\}\right) w^2.
\end{aligned}$$

Corollary 3.12. *If*

$$\int_0^\omega \max_{0<x<L}\left\{\frac{\partial f}{\partial u}(x, \phi(t,x))\right\} dt < -\frac{\pi^2}{2L^2},$$

then $\phi(t,x)$ is orbitally asymptotically stable with asymptotic phase in $L^2(0,L)$.

Suppose that $v(t,x) = \sum_j c_j(t)e^j(x)$, where $c_j = \langle e^j, v\rangle = \int_0^L ve^j$, $j = 1, 2, \cdots$. Then

$$\frac{dc_j}{dt} = [-\lambda_j + a(t,j,j)]c_j + \sum_{k\neq j} a_j(t,j,k)c_k, \tag{3.22}$$

where $a(t,k,l) := \langle e^k, ae^l\rangle = \int_0^L a(t,x)e^l(x)e^k(x)dx$, $k, l = 1, 2, \cdots$. If v^1, v^2 are solutions of (3.19), then $w(t) = (v^1 \wedge v^2)(t) \in \bigwedge^2 X$ has a basis representation

$$w(t,i_1,i_2) = \langle e^{i_1} \wedge e^{i_2}, w(t)\rangle = \det \begin{bmatrix} c_{i_1}^1 & c_{i_1}^2 \\ c_{i_2}^1 & c_{i_2}^2 \end{bmatrix}, \quad \text{for } 1 \le i_1 < i_2,$$

which satisfies the second compound differential equation of (3.19)

$$\begin{aligned} \frac{dw(t,i_1,i_2)}{dt} &= (-\lambda_{i_1} - \lambda_{i_2} + a(t,i_1,i_1) + a(t,i_2,i_2))w(t,i_1,i_2) \\ &\quad + \sum_{k\neq i_1,i_2} a(t,i_1,k)w(t,k,i_2) + \sum_{k\neq i_1,i_2} a(t,k,i_2)w(t,i_1,k). \end{aligned} \tag{3.23}$$

Consider a Lyapunov function

$$V(t) = \sum_{1\le i_1<i_2} w^2(t,i_1,i_2).$$

Then

$$\begin{aligned} \frac{d^+V}{dt} &\le \sum_{1\le i_1<i_2} \mu_{i_1i_2}(t)w^2(t,i_1,i_2) \\ &\le \mu(t)V, \end{aligned}$$

where

$$\begin{aligned} &\mu_{i_1i_2}(t) = -\lambda_{i_1} - \lambda_{i_2} + a(t,i_1,i_1) + a(t,i_2,i_2) + \sum_{k\neq i_1,i_2} (|a(t,i_1,k)| + |a(t,k,i_2)|), \\ &a(t,x) = \frac{\partial f}{\partial u}(x,\phi(t,x)), \quad a(t,k,l) = \int_0^L a(t,x)e^l(x)e^k(x)dx, \quad k,l = 1,2,\cdots, \\ &\mu(t) = \sup_{1\le i_1<i_2} \{\mu_{i_1i_2}(t)\}. \end{aligned} \tag{3.24}$$

Corollary 3.13. *If*

$$\int_0^\omega \mu < 0,$$

where $\mu(\cdot)$ is given by (3.24), *then $\phi(t,x)$ is orbitally asymptotically stable with asymptotic phase in $L^2(0,L)$.*

Planar Case: $n=2$ with Dirichlet boundary condition

Suppose that $u = \phi(t,x) = (\phi_1(t,x), \phi_2(t,x))$ is a nonconstant ω-periodic solution of

$$\begin{aligned} &u_t = d_1u_{xx} + f(x,u,v), \quad v_t = d_2v_{xx} + g(x,u,v), \quad 0<x<L, \quad t>0, \\ &u(t,0) = u(t,L) = 0, \qquad v(t,0) = v(t,L) = 0, \quad t\ge 0, \end{aligned} \tag{3.25}$$

where $f, g \in C^2((0,L) \times \mathbb{R}^2 \to \mathbb{R}^2)$. Let

$$\begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{21}(t,x) & a_{22}(t,x) \end{bmatrix} := \begin{bmatrix} \frac{\partial f}{\partial u}(x,\phi_1(t,x),\phi_2(t,x)) & \frac{\partial f}{\partial v}(x,\phi_1(t,x),\phi_2(t,x)) \\ \frac{\partial g}{\partial u}(x,\phi_1(t,x),\phi_2(t,x)) & \frac{\partial g}{\partial v}(x,\phi_1(t,x),\phi_2(t,x)) \end{bmatrix}.$$

The previous discussion implies that a pointwise representation of the second compound differential equation defined on $\bigwedge^2 X^\alpha$ is

$$\begin{aligned} \frac{\partial w_{11}}{\partial t} &= d_1(\Delta_1 w_{11} + \Delta_2 w_{11}) + (a_{11}(t,x_1) + a_{11}(t,x_2))w_{11} \\ &\quad + a_{12}(t,x_1)w_{21} + a_{12}(t,x_2)w_{12}, \\ \frac{\partial w_{22}}{\partial t} &= d_2(\Delta_1 w_{22} + \Delta_2 w_{22}) + (a_{22}(t,x_1) + a_{22}(t,x_2))w_{22} \\ &\quad + a_{21}(t,x_1)w_{12} + a_{21}(t,x_2)w_{21}, \\ \frac{\partial w_{12}}{\partial t} &= d_1\Delta_1 w_{12} + d_2\Delta_2 w_{12} + (a_{11}(t,x_1) + a_{22}(t,x_2))w_{12} \\ &\quad + a_{12}(t,x_1)w_{22} + a_{21}(t,x_2)w_{11}. \end{aligned} \tag{3.26}$$

Let

$$\begin{aligned} V_1(t) &= \frac{1}{2}\int_{(0,L)^2} (w_{11})^2 = \frac{1}{2}\int_{(0,L)^2} (w_{11}(t,x_1,x_2))^2 dx_1 dx_2, \\ V_2(t) &= \frac{1}{2}\int_{(0,L)^2} (w_{22})^2 = \frac{1}{2}\int_{(0,L)^2} (w_{22}(t,x_1,x_2))^2 dx_1 dx_2, \\ V_3(t) &= \frac{1}{2}\int_{(0,L)^2} (w_{12})^2 = \frac{1}{2}\int_{(0,L)^2} (w_{12}(t,x_1,x_2))^2 dx_1 dx_2. \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d^+ V_1}{dt} \\ =\ & \int_{(0,L)^2} d_1(\Delta_1 w_{11} + \Delta_2 w_{11})w_{11} + \int_{(0,L)^2} (a_{11}(t,x_1) + a_{11}(t,x_2))(w_{11})^2 \\ &+ \int_{(0,L)^2} [a_{12}(t,x_1)w_{21}w_{11} + a_{12}(t,x_2)w_{12}w_{11}] \\ \leq\ & \int_{(0,L)^2} d_1(-\lambda_1 - \lambda_2 + a_{11}(t,x_1) + a_{11}(t,x_2))(w_{11})^2 \\ &+ \int_{(0,L)^2} [a_{12}(t,x_1)w_{21}w_{11} + a_{12}(t,x_2)w_{12}w_{11}]. \end{aligned}$$

It follows from

$$\begin{aligned}
&\int_{(0,L)^2} a_{12}(t,x_1)w_{21}(t,x_1,x_2)w_{11}(t,x_1,x_2)dx_1dx_2 \\
= &\int_{(0,L)^2} a_{12}(t,x_2)w_{21}(t,x_2,x_1)w_{11}(t,x_2,x_1)dx_1dx_2 \\
= &\int_{(0,L)^2} a_{12}(t,x_2)w_{12}(t,x_1,x_2)w_{11}(t,x_1,x_2)dx_1dx_2
\end{aligned}$$

that

$$\begin{aligned}
\frac{1}{2}\frac{d^+V_1}{dt} \leq &\int_{(0,L)^2} d_1\left[-\lambda_1-\lambda_2+a_{11}(t,x_1)+a_{11}(t,x_2)\right](w_{11})^2 \\
&+2\int_{(0,L)^2} a_{12}(t,x_2)w_{12}w_{11}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{2}\frac{dV_2}{dt} \leq &\int_{(0,L)^2} d_2\left[-\lambda_1-\lambda_2+a_{22}(t,x_1)+a_{22}(t,x_2)\right](w_{22})^2 \\
&+2\int_{(0,L)^2} a_{21}(t,x_1)w_{12}w_{22} \\
\frac{1}{2}\frac{dV_3}{dt} \leq &\int_{(0,L)^2} \left[-\lambda_1(d_1+d_2)+a_{11}(t,x_1)+a_{22}(t,x_2)\right](w_{12})^2 \\
&+\int_{(0,L)^2} \left[a_{12}(t,x_1)w_{22}w_{12}+a_{21}(t,x_2)w_{11}w_{12}\right].
\end{aligned}$$

Consider a Lyapunov function

$$V(t) = \frac{1}{2!}(V_1+V_2+2V_3).$$

It follows from Young's inequality

$$u(x)v(x) \leq \frac{1}{2}\left[\nu(x)u^2(x)+\frac{v^2(x)}{\nu(x)}\right], \quad \text{for every positive function } \nu(x), \tag{3.27}$$

that for any positive functions $\nu_1(x), \nu_2(x)$,

$$\begin{aligned}
\frac{d^+V}{dt} &\leq \int_{(0,L)^2} \left[\mu_1(t)(w_{11})^2+\mu_2(t)(w_{22})^2+2\mu_3(t)(w_{12})^2\right] \\
&\leq 2\mu(t)V,
\end{aligned} \tag{3.28}$$

where

$$
\begin{aligned}
\mu_1(t) &= -(\lambda_1+\lambda_2)\,d_1 + \max_x\left\{2a_{11}(t,x) + \frac{1}{\nu_1(x)}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\},\\
\mu_2(t) &= -(\lambda_1+\lambda_2)\,d_2 + \max_x\left\{2a_{22}(t,x) + \frac{1}{\nu_2(x)}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\},\\
\mu_3(t) &= -\lambda_1(d_1+d_2) + \max_x\left\{a_{11}(t,x) + \frac{\nu_2(x)}{2}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\}\\
&\quad + \max_x\left\{a_{22}(t,x) + \frac{\nu_1(x)}{2}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\},\\
\mu(t) &= \max\{\mu_1(t),\mu_2(t),\mu_3(t)\}
\end{aligned}
\tag{3.29}
$$

and the maximum is taken over $0 < x < L$.

Corollary 3.14. *If*

$$\int_0^\omega \mu < 0, \tag{3.30}$$

where $\mu(\cdot)$ is defined by (3.29), then $\phi(t,x)$ is orbitally asymptotically stable with asymptotic phase in $L^2((0,L),\mathbb{R}^2)$.

Remark 3.5. Corollary 3.14 is used in Section 3.3 to give a stability condition for a periodic solution of a system arising from an ordinary differential equation.

3.3 Effect of Diffusion

In this section the effect of adding diffusion to the dynamics of an ordinary differential equation in $\mathbb{R}^n$ on the stability of a periodic solution is studied. It has been shown by Henry [42], page 201-202, that for 2-dimensional systems, the orbital stability is preserved under certain circumstances. But he proved that, in general, stability is not preserved for general diffusion terms. Leiva [50] extended Henry's sufficient condition on the preservation of orbital stability to systems in $\mathbb{R}^n$. Here the necessary and sufficient condition for hyperbolic stability is formulated in terms of stability of an infinite dimensional system of uncoupled ordinary differential equations, which includes both conditions of [42] and [50] with weaker restrictions on the added diffusion.

The following necessary and sufficient condition for a periodic solution of an ordinary differential equation to be hyperbolic stable is from Muldowney [78]. Let $u = \phi(t)$ be a nonconstant ω-periodic solution of

$$\frac{du}{dt} = f(u), \quad f \in C^1(\mathbb{R}^n, \mathbb{R}^n). \tag{3.31}$$

The linear variational equation at $u = \phi(t)$ is

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(\phi(t))v.$$

Theorem 3.15. *The periodic solution $\phi(t)$ of (3.31) is stable hyperbolic if and only if the second compound differential equation*

$$\frac{dw}{dt} = \frac{\partial f}{\partial u}^{[2]}(\phi(t))w$$

is uniformly asymptotically stable in $\bigwedge^2 \mathbb{R}^n \left(\cong \mathbb{R}^{\binom{n}{2}}\right)$.

Corollary 3.16. *The periodic solution $\phi(t)$ of (3.31) is stable hyperbolic if, for some Lozinskiĭ measure μ,*

$$\int_0^\omega \mu\left(\frac{\partial f}{\partial u}^{[2]}(\phi(t))\right) < 0.$$

Theorem 3.15 is the special case of Theorem 3.7 when $X = \mathbb{R}^n$ and $A = 0$. When $n = 2$, it is the Poncaré stability criterion which asserts that a nonconstant ω-periodic solution $u = \phi(t)$ of

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^2,$$

is orbitally asymptotically stable with asymptotic phase if

$$\int_0^\omega \operatorname{div} f(\phi(t)) < 0.$$

Suppose that a diffusion term is added to (3.31) as follows. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and $D = \operatorname{diag}(d_1, d_2, \cdots, d_n), d_i \geq 0$. Then $u = \phi(t)$ is also a periodic solution of the reaction diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u), \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= 0. \end{aligned} \tag{3.32}$$

Let $X = L^2(\Omega, \mathbb{R}^n)$ and $A : \mathcal{D}(A) \subset X \to X$ be the linear unbounded operator $Au = -\Delta u$, where

$$\mathcal{D}(A) = \left\{ u \in H^2(\Omega, \mathbb{R}^n),\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \right\}.$$

Theorem 1.6.1 in Henry [42], page 39, implies that if $\frac{m}{4} < \alpha \leq 1$, then

$$X^\alpha \subset C(\Omega, \mathbb{R}^n), \text{ and } X^\alpha \subset L^p(\Omega, \mathbb{R}^n),\ p \geq 2.$$

The initial value problem of (3.32) is well-posed in X^α. The linear variational equation of (3.32) at $u = \phi(t)$ is

$$\frac{\partial v}{\partial t} = D\Delta v + \frac{\partial f}{\partial u}(\phi(t))v. \tag{3.33}$$

The eigenvalues and orthonormal eigenfunctions of the Laplace equation

$$\begin{aligned} \Delta u &= -\lambda_i u, \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= 0, \end{aligned} \tag{3.34}$$

are $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $e^0(x), e^1(x), e^2(x), \cdots$, respectively. Let $v(t) \in X^\alpha$ be any solution of (3.33). Then $v(t,x) = \sum_{j=0}^{\infty} v_j(t)e^j(x)$ where $v_j(t) := \langle e^j, v(t)\rangle = \int_\Omega v(t,x)e^j(x)dx$, $j = 0, 1, 2, \cdots$ and (3.33) is equivalent to the following infinite dimensional system of linear equations

$$\frac{dv_j}{dt} = -\lambda_j D v_j + \frac{\partial f}{\partial u}(\phi(t))v_j. \tag{3.35}$$

While the methods of Section 3.1 may be used directly to analyze the characteristic multipliers of the periodic solution $\phi(t)$ with the help of the second compound differential equation of (3.33), a simpler development presented here is based on the finite dimensional theory of compound matrices by Muldowney [78] and the material in Henry [42], page 201-202. To that end, the above infinite system of uncoupled n-dimensional linear differential equations (3.35) is considered.

Lemma 3.17. *The characteristic multipliers of the periodic solution $\phi(t)$ of (3.32) are the characteristic multipliers of the infinite dimensional system (3.35).*

Proof. Let $v(t) \in X^\alpha$ be any solution of (3.33). then $v(t,x) = \sum_{j=0}^{\infty} v_j(t)e^j(x)$, where $v_j(t) = \langle e^j, v(t)\rangle$ satisfies (3.35).

If μ_j is a characteristic multiplier of (3.35), then there exists a nonzero $v_j(t)$ such that

$$v_j(\omega) = \mu_j v_j(0).$$

Since $v_j(t)e^j(x)$ is a solution of (3.33), μ_j is a characteristic multiplier of the periodic solution $\phi(t)$ of (3.32).

If μ is a characteristic multiplier of the periodic solution $\phi(t)$ of (3.32), then there exists a nonzero $v(t,x) = \sum_{j=0}^{\infty} v_j(t)e^j(x)$ such that

$$v(\omega) = \mu v(0),$$

or

$$\sum_{j=0}^{\infty} v_j(\omega)e^j(x) = \mu \sum_{i=0}^{\infty} v_j(0)e^j(x).$$

Thus for all $j = 0, 1, 2, \cdots$,

$$v_j(\omega) = \mu v_j(0).$$

Since $v(t)$ is nonzero, at least one of the v_j is not identically zero and thus μ is a characteristic multiplier of (3.35) for some j.

■

Remark 3.6. The technique developed here can also be applied to other boundary conditions such that $p(t)$ is still a periodic solution for the reaction diffusion equation, for example, periodic boundary condition.

3.3.1 Stability of Periodic Solutions

The following theorem is an analogue for the reaction diffusion equation (3.32) of Poincaré's stability criterion for the ordinary differential equation (3.31).

Theorem 3.18. *Suppose that $\phi(t)$ is a periodic solution of the ordinary differential equation* (3.31). *Then $\phi(t)$ is stable hyperbolic for the diffusive system* (3.32) *if and only if the second compound differential equation*

$$\frac{dw}{dt} = \frac{\partial f}{\partial u}^{[2]}(\phi(t))w, \tag{3.36}$$

and the linear equations

$$\frac{dv_j}{dt} = -\lambda_j D v_j + \frac{\partial f}{\partial u}(\phi(t))v_j, \quad j = 1, 2, \cdots, \tag{3.37}$$

are uniformly asymptotically stable.

Note that the stability of equation (3.36) is a necessary and sufficient condition for the periodic solution $\phi(t)$ of the ordinary differential equation (3.31) to be stable hyperbolic. In addition, Poincaré's stability criterion here also requires the stability of the infinite system of decoupled n-dimensional linear equations (3.37). The following corollaries give concrete sufficient conditions on D for the preservation of stability in the diffusion system. The concept, "Lozinskiĭ measure", is discussed in Appendix A. Good references on Lozinskiĭ measure are Coppel [19], page 41 and Muldowney [76–78].

Corollary 3.19. *Suppose that the periodic solution $\phi(t)$ of the ordinary differential equation* (3.31) *is stable hyperbolic. If, for some Lozinskiĭ measure μ,*

$$\mu(-D) \le 0 \tag{3.38}$$

and

$$\int_0^{\omega} \mu\left(\frac{\partial f}{\partial u}(\phi(t)) - \lambda_1 D\right) < 0, \tag{3.39}$$

then $u = \phi(t)$ is stable hyperbolic for the diffusive system (3.32).

Remark 3.7. The Lozinskiĭ measure μ in Corollary 3.19 is assumed to be *admissible.* The term "admissible" was introduced by Li and Wang [57]. Precisely, a Lozinskiĭ measure μ is said to be *admissible* if $\mu(-D) \le 0$ for all diagonal $D = \text{diag}(d_1, d_2, \cdots, d_n)$ where $d_i \ge 0, 1 \le i \le n$. The Lozinskiĭ measures μ listed in Table $A.1$ of Appendix A are admissible.

Corollary 3.20. *Suppose that the periodic solution $\phi(t)$ of the ordinary differential equation* (3.31) *is stable hyperbolic. Then there exists $K > 0$ such that*

$$|X(t)X^{-1}(s)| \le K, \quad 0 \le s \le t,$$

where $X(t)$ is a fundamental matrix solution of

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(\phi(t))v.$$

If there exists a constant $d > 0$ such that

$$|D - dI| < \frac{d}{K}, \tag{3.40}$$

then $u = \phi(t)$ is stable hyperbolic for the diffusive system (3.32).

Remark 3.8. Neither of Corollary 3.19 and Corollary 3.20 is implied by the other. This is discussed through an example in Section 3.3.3. For the case $D = dI$, inequality (3.39) is true but inequality (3.40) may not be satisfied.

Remark 3.9. Corollary 3.20 includes as special cases results of Henry [42] and Leiva [50]. When $n = 2$ and $D = \text{diag}(d_1, d_2)$, Henry [42] proved that if $|d_1 - d_2|$ is small, $u = \phi(t)$ is orbitally asymptotically stable with asymptotic phase for the reaction diffusion system (3.32). Leiva [50] generalized this result for $D = dI + \text{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ with ε_i small enough. The results of Henry and Leiva are strengthened slightly in that a concrete bound is given here on ε_i. A particular example obtaining such a bound is discussed in Section 3.3.3.

Proof of Theorem 3.18:

Proof. Theorem 3.15 implies that $\phi(t)$ is stable hyperbolic for the first equation of (3.37) with λ_0 if and only if the second compound differential equation (3.36) is uniformly asymptotically stable. Theorem 7.2 in Hale [37], page 120, implies that the characteristic multipliers of the infinite dimensional linear equations (3.37) with $\lambda_j = \lambda_1, \lambda_2, \cdots$ have moduli less than 1 if and only if it is uniformly asymptotically stable. By Lemma 3.17, the proof is completed. ■

Proof of Corollary 3.19:

Proof. Let $A(t) = \frac{\partial f}{\partial u}(\phi(t))$. A solution of (3.37) satisfies

$$|v_j(t)| \le |v_j(s)| \exp\left(\int_s^t \mu(A - \lambda_j D)\right), \quad s \le t, \quad j = 1, 2, \cdots,$$

which implies that

$$|X_j(t)X_j^{-1}(s)| \leq \exp\left(\int_s^t \mu(A - \lambda_j D)\right), \quad s \leq t, \quad j = 1, 2, \cdots,$$

where $X_j(t)$ is a fundamental matrix solution of (3.37) and $X_j(0) = I$. Thus

$$|X_j(\omega)| \leq \exp\left(\int_0^\omega \mu(A - \lambda_j D)\right).$$

If $\mu(-D) \leq 0$ and $\int_0^\omega \mu(A - \lambda_1 D) < 0$, then for $j = 2, 3, \cdots$,

$$\begin{aligned} \int_0^\omega \mu(A - \lambda_j D) &= \int_0^\omega \mu(A - \lambda_1 D - (\lambda_j - \lambda_1)D) \\ &\leq \int_0^\omega (\mu(A - \lambda_1 D) + (\lambda_j - \lambda_1)\mu(-D)) < 0, \end{aligned}$$

which implies that all of the characteristic multipliers of (3.37) with $\lambda = \lambda_j, j = 1, 2, \cdots$ have moduli less than 1. Therefore, $u = \phi(t)$ is stable hyperbolic for the diffusive system (3.32). ■

Proof of Corollary 3.20:

Proof. Since the periodic solution $\phi(t)$ of the ordinary differential equation (3.31) is stable hyperbolic, Theorem 7.2 in Hale [37], page 120, implies that there exists $K > 0$ such that

$$|X(t)X^{-1}(s)| \leq K, \quad s \leq t,$$

where $X(t)$ is a fundamental matrix solution of

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(\phi(t))v.$$

Let $A(t) = \frac{\partial f}{\partial u}(\phi(t))$. Choose $d > 0$ and consider the following equation

$$\frac{dy}{dt} = -dIy + A(t)y. \tag{3.41}$$

Then the fundamental matrix solution $Y(t)$ of (3.41) satisfies $Y(t) = X(t)\exp(-dt)$ and thus

$$|Y(t)Y^{-1}(s)| \leq K\exp(-d(t-s)), \quad s \leq t.$$

Now consider differential equation

$$\frac{dz}{dt} = (-D + A(t))z. \tag{3.42}$$

Then the solution of (3.42),

$$z(t) = Y(t)Y^{-1}(s)z(s) + \int_s^t Y(t)Y^{-1}(\tau)(dI - D)z(\tau)d\tau, \quad s \leq t,$$

satisfies

$$|z(t)| \leq K\exp(-d(t-s))|z(s)| + \int_s^t K\exp(-d(t-\tau))|D - dI| \cdot |z(\tau)|d\tau.$$

Gronwall's inequality implies

$$|z(t)| \leq K\exp((-d + K|D - dI|)(t-s))|z(s)|, \quad s \leq t.$$

Therefore, if

$$|D - dI| < \frac{d}{K}, \tag{3.43}$$

then (3.42) is uniformly asymptotically stable. Since $\lambda_j > 0, j = 1, 2, \cdots$ and

$$|\lambda_j D - \lambda_j dI| = \lambda_j |D - dI| < \frac{\lambda_j d}{K},$$

the above argument implies that if $|D - dI| < \frac{d}{K}$, then

$$\frac{dv_j}{dt} = (-\lambda_j D + A(t))v_j$$

is uniformly asymptotically stable. Therefore, from Theorem 3.18, $u = \phi(t)$ is stable hyperbolic for the diffusive system (3.32). ■

3.3.2 Instability of Periodic Solutions

Since solutions of the ordinary differential equation (3.31) are also solutions of the reaction diffusion equation (3.32), it follows that an orbitally unstable solution of (3.31) is also an orbitally unstable solution of (3.32). In the following, it is assumed that $u = \phi(t)$ is an orbitally stable solution and conditions under which it is an orbitally unstable solution of (3.32) are explored.

Theorem 3.21. *Suppose that for some Lozinskiĭ measure* μ,

$$\mu(-D) < 0.$$

Then the stable manifold of the solution $u = \phi(t)$ *of the diffusive system* (3.32) *has finite codimension.*

Proof. Since the Jacobian matrix $A(t) = \frac{\partial f}{\partial u}(\phi(t))$ is a continuous and periodic function of t, there exists a $\beta > 0$ such that

$$\int_s^t \mu(A) \leq \beta(t-s), \quad s \leq t. \tag{3.44}$$

If $\lambda > \frac{\beta}{-\mu(-D)}$, then

$$\int_s^t \mu(-\lambda D + A) \leq \lambda\mu(-D)(t-s) + \int_s^t \mu(A) \leq (\lambda\mu(-D) + \beta)(t-s) < 0,$$

which implies that the linear ordinary differential equation

$$\frac{dv}{dt} = (-\lambda D + A(t))v$$

is uniformly asymptotically stable. Thus there exists N such that (3.37) with $\lambda = \lambda_j$ is uniformly asymptotically stable for $j \geq N$. Therefore the stable manifold of the solution $u = \phi(t)$ of the diffusive system (3.32) has finite codimension. ■

Theorem 3.22. *Suppose that* $u = \phi(t)$ *is orbitally asymptotically stable for the ordinary differential equation*

$$\frac{du}{dt} = f(u).$$

Let $A(t) = \frac{\partial f}{\partial u}(\phi(t))$. *Suppose that there is a principal* $m \times m$ *submatrix* A_{11} *of* A *and a Lozinskiĭ measure* μ_1 *on* $\mathbb{R}^m$ *such that*

$$\int_0^\omega -\mu_1(-A_{11}) > 0. \tag{3.45}$$

Then, for any integer $k > 0$, *there exist an* $\varepsilon > 0$ *and a* $n \times n$ *matrix* $D = diag(\varepsilon I_{m\times m}, \frac{1}{\varepsilon} I_{(n-m)\times(n-m)})$ *such that* $u = \phi(t)$ *is orbitally unstable for the diffusive system*

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u), \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= 0, \end{aligned} \tag{3.46}$$

and the dimension of the unstable manifold of $u = \phi(t)$ *is at least* k.

Condition (3.45) in Theorem 3.22 is satisfied if for some i, $\int_0^\omega a_{ii}(t)dt > 0$. Without loss of generality, let $i = 1$.

Corollary 3.23. *If $\int_0^\omega a_{11}(t)dt > 0$, then for any integer $k > 0$, there exist an $\varepsilon > 0$ and a $n \times n$ matrix $D = diag(\varepsilon, \frac{1}{\varepsilon} I_{(n-1)\times(n-1)})$, where $\varepsilon > 0$, such that $u = \phi(t)$ is orbitally unstable for the diffusive system (3.46) and the dimension of the unstable manifold of $u = \phi(t)$ is at least k.*

The proof of Theorem 3.22 is based on an argument in Muldowney [77].

Proof. Consider the linear equation

$$\begin{aligned} \frac{dv_1}{dt} &= (A_{11}(t) - \lambda\varepsilon I_{m\times m})v_1 + A_{12}(t)v_2, \\ \frac{dv_2}{dt} &= A_{21}(t)v_1 + \left(A_{22}(t) - \frac{\lambda}{\varepsilon} I_{(n-m)\times(n-m)}\right) v_2, \end{aligned} \tag{3.47}$$

where $\varepsilon > 0$, $\int_0^\omega -\mu_1(-A_{11}) > 0$ for some Lozinskiĭ measure μ_1 and $v = (v_1, v_2), v_1 \in \mathbb{R}^m, v_2 \in \mathbb{R}^{n-m}$. Let $|\cdot|_1$ and $|\cdot|_2$ be vector norms in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, μ_1 and μ_2 the corresponding Lozinskiĭ measures, respectively. Let $r_1 = |v_1|_1$ and $r_2 = |v_2|_2$. Then

$$\begin{aligned} \frac{d^+r_1}{dt} &\geq -\mu_1(-A_{11} + \lambda\varepsilon I_{m\times m})r_1 - \alpha_{12}r_2 \geq (\alpha_{11} - \lambda\varepsilon)\, r_1 - \alpha_{12}r_2, \\ \frac{d^+r_2}{dt} &\leq \alpha_{21}r_1 + \mu_2\left(A_{22} - \frac{\lambda}{\varepsilon} I_{(n-m)\times(n-m)}\right) r_2 \leq \alpha_{21}r_1 + \left(\alpha_{22} - \frac{\lambda}{\varepsilon}\right) r_2, \end{aligned}$$

where

$$\begin{aligned} \alpha_{11}(t) = -\mu_1(-A_{11}(t)), &\qquad \alpha_{22}(t) = -\mu_2(-A_{22}(t)), \\ \alpha_{12}(t) = |A_{12}(t)|_{21} &= \sup_{x\in\mathbb{R}^{n-k},\ x\neq 0} \frac{|A_{12}(t)x|_1}{|x|_2}, \\ \alpha_{21}(t) = |A_{21}(t)|_{12} &= \sup_{x\in\mathbb{R}^{k},\ x\neq 0} \frac{|A_{21}(t)x|_2}{|x|_1}, \end{aligned}$$

and $\frac{d^+}{dt}$ denotes the right-hand derivative. Since $\int_0^\omega -\mu_1(-A_{11}) > 0$, we can choose $\varepsilon, \delta > 0$ sufficiently small such that

$$\int_0^\omega (\alpha_{11} - \lambda\varepsilon - \delta\alpha_{21}) > 0, \tag{3.48}$$

and

$$\alpha_{22}(t) - \frac{\lambda}{\varepsilon} + \frac{1}{\delta}\alpha_{12}(t) \le \alpha_{11}(t) - \lambda\varepsilon - \delta\alpha_{21}(t), \quad \text{for all } t \ge 0. \tag{3.49}$$

Let $r = r_1 - \delta r_2$. Then

$$\begin{aligned}
\frac{d^+ r}{dt} &\ge (\alpha_{11} - \lambda\varepsilon) r_1 - \alpha_{12} r_2 - \delta\alpha_{21} r_1 - \delta\left(\alpha_{22} - \frac{\lambda}{\varepsilon}\right) r_2 \\
&= (\alpha_{11} - \lambda\varepsilon - \delta\alpha_{21}) r_1 + \left(\frac{\lambda}{\varepsilon} - \alpha_{22} - \frac{1}{\delta}\alpha_{12}\right)\delta r_2 \\
&\ge (\alpha_{11} - \lambda\varepsilon - \delta\alpha_{21}) r,
\end{aligned}$$

which implies

$$r(t) \ge r(s)\exp\left(\int_s^t (\alpha_{11} - \lambda\varepsilon - \delta\alpha_{21})\right), \quad \text{for } t \ge s.$$

Choose $v_1(0)$ and $v_2(0)$ such that $r(0) > 0$. Then

$$r(n\omega) \ge r(0)\exp\left(n\int_0^\omega (\alpha_{11} - \lambda\varepsilon - \delta\alpha_{21})\right) \longrightarrow \infty, \quad \text{as} \quad n \to \infty,$$

and thus

$$v_1(n\omega) \longrightarrow \infty, \quad \text{as} \quad n \to \infty.$$

Therefore (3.47) is unstable and at least one of the characteristic multipliers of (3.47) has modulus greater than 1. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the first k nonzero eigenvalues of the Laplace equation

$$\begin{aligned}
\Delta u &= -\lambda_i u, \\
\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= 0.
\end{aligned}$$

Then there exists an $\varepsilon > 0$ such that the inequalities (3.48) and (3.49) hold for all $\lambda_1, \lambda_2, \cdots, \lambda_k$. The above argument for λ implies that for each $j = 1, 2, \cdots, k$, the linear equation (3.47) has at least one of the characteristic multipliers with modulus greater than 1. Hence, Lemma 3.5 and Lemma 3.17 imply that $u = \phi(t)$ is orbitally unstable for the diffusive system and the dimension of the unstable manifold of $u = \phi(t)$ is at least k. ∎

3.3.3 Planar Reaction Diffusion Systems

The following results include Henry [42], page 201-202, as the special case $n = 2$ of Theorem 3.18 and Corollary 3.23.

Suppose that the planar system

$$\begin{aligned} \frac{du}{dt} &= f(u,v), \\ \frac{dv}{dt} &= g(u,v), \end{aligned} \tag{3.50}$$

has a nonconstant periodic solution $(p(t), q(t))$ with period ω, where $f, g \in C^1(\mathbb{R}^2 \to \mathbb{R})$. In this case the second compound differential equation of the linear variational equation at $(p(t), q(t))$ is the Liouville equation

$$\begin{aligned} \frac{dw}{dt} &= \left(\frac{\partial f}{\partial u}(p,q) + \frac{\partial f}{\partial v}(p,q)\right) w \\ &= (f_u(p,q) + g_v(p,q))w. \end{aligned}$$

Theorem 3.15 implies that $(p(t), q(t))$ is stable hyperbolic for the ordinary differential equation (3.50) if and only if

$$\int_0^{\omega} (f_u(p,q) + g_v(p,q)) < 0. \tag{3.51}$$

Solution $(p(t), q(t))$ is also a periodic solution of the diffusive system

$$\begin{aligned} &u_t = d_1 u_{xx} + f(u,v), \\ &v_t = d_2 v_{xx} + g(u,v), \quad \text{in } (0, 2\pi) \times (0, \infty), \quad d_1, d_2 \geq 0, \end{aligned} \tag{3.52}$$

with Neumann boundary conditions

$$\begin{aligned} &u_x(t,0) = u_x(t, 2\pi) = 0, \\ &v_x(t,0) = v_x(t, 2\pi) = 0, \quad \text{for } t \geq 0, \end{aligned} \tag{3.53}$$

or periodic boundary conditions

$$\begin{aligned} &u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi) \\ &v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \quad \text{for } t \geq 0. \end{aligned} \tag{3.54}$$

Let

$$M_j(t) = \begin{bmatrix} f_u(p,q) - \lambda_j d_1 & f_v(p,q) \\ g_u(p,q) & g_u(p,q) - \lambda_j d_2 \end{bmatrix}, \tag{3.55}$$

where $\lambda_0 = 0, \lambda_j = (j/2)^2$ for Neumann boundary conditions and $\lambda_0 = 0, \lambda_{2j-1} = \lambda_{2j} = j^2$ for periodic boundary conditions, $j = 1, 2, \cdots$. The characteristic multipliers of $(p(t), q(t))$ for the reaction diffusion system (3.52) are the characteristic multipliers of

$$\begin{bmatrix} \dfrac{du}{dt} \\ \dfrac{dv}{dt} \end{bmatrix} = M_j(t) \begin{bmatrix} u \\ v \end{bmatrix}, \quad j = 0, 1, 2, \cdots . \tag{3.56}$$

Let $d_1, d_2 \geq 0$ and $d_1 + d_2 \neq 0$. Corollary 3.24, 3.25 and 3.26 are given in Henry [42]. Corollary 3.27 and 3.28 are new and follow from Corollary 3.23.

Corollary 3.24. *Suppose* $\int_0^\omega (f_u(p,q) + g_v(p,q)) > 0$. *Then* $(p(t), q(t))$ *is orbitally unstable for the diffusive system* (3.52).

Corollary 3.25. *Suppose* $\int_0^\omega (f_u(p,q) + g_v(p,q)) < 0$. *If* $\int_0^\omega g_v(p,q) > 0$, *then* $(p(t), q(t))$ *is orbitally unstable for the diffusive system* (3.52) *with* $d_1 = \varepsilon^{-1}, d_2 = \varepsilon$ *where* ε *is sufficiently small.*

Corollary 3.26. *Suppose* $\int_0^\omega (f_u(p,q) + g_v(p,q)) < 0$. *If* $|d_1 - d_2|$ *is small, then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system* (3.52).

Corollary 3.27. *Suppose* $\int_0^\omega (f_u(p,q) + g_v(p,q)) < 0$. *If*

$$\int_0^\omega \max\{f_u(p,q) - \lambda_1 d_1 + |f_v(p,q)|, g_v(p,q) - \lambda_1 d_2 + |g_u(p,q)|\} < 0, \tag{3.57}$$

or

$$\int_0^\omega \max\{f_u(p,q) - \lambda_1 d_1 + |g_u(p,q)|, g_u(p,q) - \lambda_1 d_2 + |f_v(p,q)|\} < 0, \tag{3.58}$$

or

$$\int_0^\omega \lambda\left(\frac{M_1 + M_1^*}{2}\right) < 0, \tag{3.59}$$

where $\lambda\left(\frac{M_1+M_1^*}{2}\right)$ *denotes the largest eigenvalue of* $\frac{M_1+M_1^*}{2}$, *then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system* (3.52).

The orbital asymptotic stability of $(p(t), q(t))$ can also be discussed by using the result of Corollary 3.14 in Section 3.2. Let

$$\begin{aligned}
\mu_1(t) &= -\lambda_1 d_1 + \max_x \left\{ 2f_u(p,q) + \frac{1}{\nu_1(x)} \left| f_v(p,q) + g_u(p,q) \right| \right\}, \\
\mu_2(t) &= -\lambda_1 d_2 + \max_x \left\{ 2g_v(p,q) + \frac{1}{\nu_2(x)} \left| f_v(p,q) + g_u(p,q) \right| \right\}, \\
\mu_3(t) &= \max_x \left\{ f_u(p,q) + \frac{\nu_2(x)}{2} \left| f_v(p,q) + g_u(p,q) \right| \right\} \\
&\quad + \max_x \left\{ g_v(p,q) + \frac{\nu_1(x)}{2} \left| f_v(p,q) + g_u(p,q) \right| \right\}, \\
\mu(t) &= \sup\{\mu_1(t), \mu_2(t), \mu_3(t)\},
\end{aligned} \tag{3.60}$$

where $\nu_1(x), \nu_2(x)$ are positive functions and the maximum is taken over $0 < x < 2\pi$.

Corollary 3.28. *Suppose that*

$$\int_0^\omega \mu < 0,$$

where $\mu(\cdot)$ *is defined by* (3.60). *Then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system* (3.52).

The next example is an application of Corollary 3.27.

Example 3.1. ([18, 35]) Consider the following predator-prey system with logistic growth for prey u in the absence of predation and Holling type II functional response for predator v:

$$\begin{aligned}
\frac{du}{dt} &= ru\left(1 - \frac{u}{K}\right) - \frac{Mu}{N+u} v := f(u,v), \\
\frac{dv}{dt} &= \left(\frac{Mu}{N+u} - \beta\right) v := g(u,v),
\end{aligned} \tag{3.61}$$

where $M, N, K, r, \beta > 0$. Let $M > \beta$ and $\eta = \frac{N\beta}{M-\beta}$. This system has been studied in Cheng [18]. If $\frac{K-N}{2} < \eta < K$, then the equilibrium (u^*, v^*) is globally asymptotically stable where $u^* = \eta, v^* = \frac{r}{M}(N + u^*)\left(1 - \frac{u^*}{K}\right)$. If $0 < \eta < \frac{K-N}{2}$, then the equilibrium (u^*, v^*) is unstable and a unique periodic solution exists. Let $(p(t), q(t))$ denote the unique periodic solution. Then

$$\gamma = \{(p(t), q(t)),\ 0 \le t \le \omega\} \subseteq \{(u,v),\ 0 < u < K,\ 0 < v\}.$$

The second compound differential equation of the linear variational equation at $(p(t), q(t))$ is the Liouville equation

$$\frac{dw}{dt} = \left(r\left(1 - \frac{2p}{K}\right) - \frac{MNq}{(N+p)^2} + \frac{Mp}{N+p} - \beta \right) w.$$

It has been shown in Cheng [18] that

$$\int_0^\omega \left(\frac{Mp}{(N+p)^2} - \beta \right) = 0, \tag{3.62}$$

and

$$\int_0^\omega \left(r\left(1 - \frac{2p}{K}\right) - \frac{MNq}{(N+p)^2} \right) < 0. \tag{3.63}$$

Thus if $0 < \eta < \frac{K-N}{2}$, then Poincaré's stability criterion implies that the periodic solution $(p(t), q(t))$ of (3.61) is orbitally asymptotically stable with asymptotic phase. Let

$$\begin{aligned} M_1(t) &= \begin{bmatrix} f_u(p,q) - \lambda_1 d_1 & f_v(p,q) \\ g_u(p,q) & g_v(p,q) - \lambda_1 d_2 \end{bmatrix} \\ &= \begin{bmatrix} r\left(1 - \frac{2p}{K}\right) - \frac{MNq}{(N+p)^2} - \lambda_1 d_1 & -\frac{Mp}{N+p} \\ \frac{MNq}{(N+p)^2} & \frac{Mp}{N+p} - \beta - \lambda_1 d_2 \end{bmatrix}. \end{aligned}$$

Then $\frac{2MK}{N} - \beta > 0$ since $0 < \eta < \frac{K-N}{2}$. If

$$\lambda_1 d_1 \geq r, \; \lambda_1 d_2 > \frac{2MK}{N} - \beta,$$

then

$$\begin{aligned} &\max\left\{ r\left(1 - \frac{2p}{K}\right) - \frac{MNq}{(N+p)^2} - \lambda_1 d_1 + \frac{MNq}{(N+p)^2}, \frac{Mp}{N+p} - \beta - \lambda_1 d_2 + \frac{Mp}{N+p} \right\} \\ &\leq \max\left\{ r\left(1 - \frac{2p}{K}\right) - \lambda_1 d_1, \frac{2MK}{N} - \beta - \lambda_1 d_2 \right\} \\ &< 0. \end{aligned}$$

Corollary 3.27 implies the following result.

Corollary 3.29. *Suppose that* $0 < \frac{N\beta}{M-\beta} < \frac{K-N}{2}$. *If* $\lambda_1 d_1 \geq r$, $\lambda_1 d_2 > \frac{2MK}{N} - \beta$, *then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system*

$$\begin{aligned} u_t &= d_1 u_{xx} + ru\left(1 - \frac{u}{K}\right) - \frac{Mu}{N+u}v, \\ v_t &= d_2 v_{xx} + \left(\frac{Mu}{N+u} - \beta\right)v, \quad \text{in } (0, 2\pi) \times (0, \infty), \end{aligned} \tag{3.64}$$

with Neumann boundary conditions (3.53) *where* $\lambda_1 = \frac{1}{4}$ *or periodic boundary conditions* (3.54) *where* $\lambda_1 = 1$.

The following example will show that neither of Corollary 3.26 and Corollary 3.27 is implied by the other. This example will also show how to find a concrete bound on $|d_1 - d_2|$ in Corollary 3.26.

Example 3.2. Consider a planar system

$$\begin{aligned} \frac{du}{dt} &= \beta u + (1 - u^2 - \alpha v^2)u := f(u, v), \\ \frac{dv}{dt} &= -\beta v + (1 - u^2 - \alpha v^2)v := g(u, v), \end{aligned} \tag{3.65}$$

where $1 \leq \alpha < 2$. Rewrite (3.65) in polar coordinators as

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2\cos^2(\theta) - \alpha r^2 \sin^2(\theta)), \\ \frac{d\theta}{dt} &= -\beta. \end{aligned} \tag{3.66}$$

Then $\frac{dr}{dt} > 0$ if $0 < r^2\cos^2(\theta) + \alpha r^2\sin^2(\theta) < 1$ and $\frac{dr}{dt} < 0$ if $r^2\cos^2(\theta) + \alpha r^2 > 1$. Thus, every solution of (3.65) except $(0,0)$ ultimately enters and remains in the annular region

$$G = \left\{(u, v) : \frac{1}{\alpha} \leq u^2 + v^2 \leq 1\right\}.$$

The Poincaré-Bendixson theorem implies that there exists a periodic solution $(p(t), q(t))$ in G. System (3.66) shows that the period of the periodic solution $(p(t), q(t))$ is $\frac{2\pi}{\beta}$. The linear variational equation of (3.65) at any solution $(p(t), q(t))$ is

$$\begin{aligned} \frac{du}{dt} &= (1 - 3p^2 - \alpha q^2)u + (\beta - 2\alpha pq)v, \\ \frac{dv}{dt} &= -(\beta + 2pq)u + (1 - p^2 - 3\alpha q^2)v. \end{aligned} \tag{3.67}$$

Since

$$1 - 3p^2 - \alpha q^2 + 1 - p^2 - 3\alpha q^2 = 2 - 4(p^2 + \alpha q^2) \le 2 - \frac{4}{\alpha} < 0, \qquad (3.68)$$

Poincaré's stability criterion implies that $(p(t), q(t))$ is asymptotically orbitally stable with asymptotic phase in G. A result in [60] implies that the periodic solution of (3.65) is unique. Let

$$\begin{aligned} M_1(t) &= \begin{bmatrix} f_u(p,q) - \lambda_1 d_1 & f_v(p,q) \\ g_u(p,q) & g_v(p,q) - \lambda_1 d_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 3p^2 - \alpha q^2 - \lambda_1 d_1 & \beta - 2\alpha pq \\ -(\beta + 2pq) & 1 - p^2 - 3\alpha q^2 - d_2 \end{bmatrix}. \end{aligned}$$

If

$$\lambda_1 d_1 > 1 + |\beta|, \ \lambda_1 d_2 > 2\alpha + |\beta|,$$

then it follows from $\frac{1}{\alpha} \le p^2 + q^2 \le 1$ that

$$\begin{aligned} & \max\{1 - 3p^2 - \alpha q^2 - \lambda_1 d_1 + |\beta + 2pq|, 1 - p^2 - 3\alpha q^2 - \lambda_1 d_2 + |\beta - 2\alpha pq|\} \\ < \ & \max\{1 + |\beta| - \lambda_1 d_1, |\beta| + 2\alpha - \lambda_1 d_2\} \\ < \ & 0. \end{aligned}$$

Corollary 3.27 implies the following result.

Corollary 3.30. *If* $d_1 > 1+|\beta|$ *and* $d_2 > 2\alpha+|\beta|$, *then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system*

$$\begin{aligned} u_t &= d_1 u_{xx} + \beta u + (1 - u^2 - \alpha v^2)u, \\ v_t &= d_2 v_{xx} - \beta v + (1 - u^2 - \alpha v^2)v, \qquad \text{in } (0, 2\pi) \times (0, \infty), \end{aligned} \qquad (3.69)$$

with Neumann boundary conditions (3.53) *where* $\lambda_1 = \frac{1}{4}$ *or periodic boundary conditions* (3.54) *where* $\lambda_1 = 1$.

From the argument developed in the proof of Proposition 4.15 in Section 4.2.1, if $1 \le \alpha < \frac{4}{3}$, then the solution of (3.67) satisfies

$$\sqrt{y^2(t) + z^2(t)} \le K\sqrt{y^2(s) + z^2(s)}, \quad 0 \le s \le t,$$

where $K = 4e^{\frac{2\pi}{\beta}+2T_2}\left(\frac{2\sqrt{\alpha(\beta^2+1)}}{\beta} + 1\right)$ and $T_2 = \frac{\frac{4\pi}{\beta}+4\ln(2)}{\frac{4}{\alpha}-3}$. Let $d = \frac{d_1+d_2}{2}$ and $\varepsilon = \frac{d_1-d_2}{2}$. Then

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} d+\varepsilon & 0 \\ 0 & d-\varepsilon \end{pmatrix}.$$

Corollary 3.20 implies the following corollary.

Corollary 3.31. *Suppose that* $d_1, d_2 \geq 0, d_1 + d_2 \neq 0$ *and* $1 \leq \alpha < \frac{4}{3}$. *Let* $K = 4e^{\frac{2\pi}{\beta}+2T_2}\left(\frac{2\sqrt{\alpha(\beta^2+1)}}{\beta}+1\right)$ *and* $T_2 = \frac{\frac{4\pi}{\beta}+4\ln(2)}{\frac{4}{\alpha}-3}$. *If* $|d_1 - d_2| \leq \frac{d_1+d_2}{K}$, *then* $(p(t), q(t))$ *is orbitally asymptotically stable with asymptotic phase for the diffusive system* (3.69) *with Neumann boundary conditions* (3.53) *or periodic boundary conditions* (3.54).

3.4 Scalar Reaction Diffusion Equations

Another type of reaction diffusion equation for which the existence of periodic solutions can be established is given in the form

$$u_t = u_{xx} + f(x, u, u_x), \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \ f \in C^2. \tag{3.70}$$

Fiedler and Mallet-Paret [30] proved that the classical theorem of Poincaré and Bendixson holds for the scalar reaction diffusion equation (3.70): the ω-limit set of any bounded solution satisfies exactly one of the following alternatives:
(i) it consists of precisely one periodic solution, or
(ii) for every orbit in the ω-limit set, its ω-limit set and α-limit set are contained in the set of steady state solutions.

When $f = f(u, u_x)$ is analytic, the Poincaré-Bendixson theorem was proved by Angenent and Fiedler [4]. It was shown in [4] that any ω-limit set contains either a rotating wave or a steady state. All periodic solutions of (3.70) in this case are rotating waves, that is solutions of the form $u = u(x - ct)$, which are always unstable. Independently, Massatt [68] proved that either the ω-limit set is a single rotating wave, or a set of equilibria which differ only by shifting x. The same result was also obtained by Matano [71], who further shows that the ω-limit set is a single equilibrium if $f = f(u, u_x)$ is even in the second argument.

The scalar reaction diffusion equation (3.70) with Dirichlet boundary conditions or Neumann boundary conditions is gradient-like with respect to a continuous Lyapunov functional F of the form

$$F(u) = \int_0^{2\pi} g(x, u, u_x)dx.$$

In particular, $\omega(u_0)$ consists entirely of equilibria if the orbit of $u(t, u_0)$ is bounded. In the general case of a scalar semilinear parabolic equation on the line segment (0, 1), Zelenyak [114] and, independently, Matano [70, 71] prove this result, where Zelenyak uses a special Lyapunov function to obtain this result and Matano uses a nontrivial application of the maximum principle. This result is also proved by Fiedler and Mallet-Paret [30], who show that the scalar reaction diffusion equation (3.70) with Dirichlet boundary conditions does not have periodic solutions (see page 339), and thus $\omega(u_0)$ consists entirely of equilibria. Hale and Raugel [39] consider an extension of a result of Hale and Massatt [38] on convergence of orbits to a single equilibrium point for gradient-like systems using the theory of integral manifolds in dynamical systems.

One of the tools in the analysis of ω-limit sets and the maximal compact attractor is the zero number, see Nickel [84], Matano [70], Henry [43], Brunovský and Fiedler [10]. For any continuous $u : S^1 \to \mathbb{R}$, the *zero number* $z(u)$ is the number of sign changes of u, not counting multiplicity, that is $z(u)$ is the maximal integer $n \leq \infty$ such that there exist $0 \leq x_{n+1} = x_0 < x_1 < \cdots < x_n < 2\pi$ with

$$u(x_i) \cdot u(x_{i+1}) < 0, \quad 0 \leq i \leq n.$$

Set $z(0) := 0$. The crucial property of $z(u)$ is the following. For any solution $u(t, u_0)$ of

$$u_t = u_{xx} + f(x, u, u_x), \quad x \in S^1,$$

with f sufficiently regular and $f(x, 0, 0) = 0$, the function $t \mapsto z(u(t, \cdot))$ is nonincreasing in t.

Now suppose that $u = \phi(t, x)$ is a nonconstant ω-periodic solution of

$$\begin{aligned} &u_t = u_{xx} + f(x, u, u_x), \quad 0 < x < 2\pi, \quad t > 0, \\ &u(t, 0) = u(t, 2\pi), u_x(t, 0) = u_x(t, 2\pi), \quad t \geq 0, \end{aligned} \tag{3.71}$$

where $f \in C^2((0, 2\pi) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$. The initial value problem associated to (3.71) generates a local semiflow on the Sobolev space X which contains all functions in $H^s(0, 2\pi)$ satisfying the periodic boundary conditions. Choose $s > \frac{5}{2}$ so that X embeds into $C^2(0, 2\pi)$. For $u_0 \in H^s(0, 2\pi)$, $u(t, x) =$

$u(t, u_0) \in C^2(0, 2\pi)$ is a solution which exists on $[0, T)$ for some $T > 0$. The linear variational equation of (3.71) at $\phi(t, x)$ is

$$\begin{aligned} &v_t = v_{xx} + a(t,x)v + b(t,x)v_x, \quad 0 < x < 2\pi, \quad t > 0, \\ &v(t,0) = v(t,2\pi), v_x(t,0) = v_x(t,2\pi), \quad t \geq 0, \end{aligned} \tag{3.72}$$

where

$$a(t,x) := \frac{\partial f}{\partial u}(x, \phi, \phi_x) = f_u(x, \phi, \phi_x), \quad b(t,x) := \frac{\partial f}{\partial u_x}(x, \phi, \phi_x) = f_{u_x}(x, \phi, \phi_x).$$

If $v_1, v_2, \cdots, v_k$ are solutions of (3.72), then $w(t) = (v_1 \wedge v_2 \wedge \cdots \wedge v_k)(t) \in \bigwedge^k X$ has pointwise representation

$$w(t, x_1, x_2, \cdots, x_4) = (v_1 \wedge v_2 \wedge \cdots \wedge v_k)(t, x_1, x_2) = \det(v_i(t, x_j)),$$

which satisfies the kth compound differential equation of (3.72) defined on $\bigwedge^k X$

$$w_t = \sum_{j=1}^{k} w_{x_j x_j} + \sum_{j=1}^{k} a(t, x_j) w_{x_j} + \sum_{j=1}^{k} b(t, x_j) w. \tag{3.73}$$

The periodic boundary conditions of v^i implies that

$$\begin{aligned} &w(t, x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_k) = w(t, x_1, \cdots, x_{j-1}, 2\pi, x_{j+1}, \cdots, x_k) \\ &w_{x_j}(t, x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_k) = w_{x_j}(t, x_1, \cdots, x_{j-1}, 2\pi, x_{j+1}, \cdots, x_k) \\ &\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \cdots, k. \end{aligned}$$

The eigenvalues and orthonormal eigenfunctions of the Laplace equation

$$\begin{aligned} &\Delta u = -\lambda_i u, \\ &u(0) = u(2\pi), u_x(0) = u_x(2\pi) \end{aligned} \tag{3.74}$$

are $\lambda_0 = 0, \lambda_{2n-1} = n^2, \lambda_{2n} = n^2, n = 1, 2, \cdots$, and $e^0(x) = 1, e^{2n-1}(x) = \frac{1}{\sqrt{\pi}} \cos nx, e^{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx, \cdots$ respectively.

A particular case of Theorem 3.8 is the following theorem.

Theorem 3.32. *Suppose that $f \in C^2((0, 2\pi) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and that $\phi(t, x)$ is an ω-periodic solution of (3.71). If the kth compound equation (3.73) is uniformly asymptotically stable in $\bigwedge^k X$, then $\phi(t, x)$ has a stable manifold with codimension at most $k - 2$.*

Consider a Lyapunov function

$$V(t) = \frac{1}{2}\int_{(0,2\pi)^k} w^2 = \frac{1}{2}\int_{(0,2\pi)^k} w^2(t, x_1, x_2, \cdots, x_k)dx_1 dx_2 \cdots dx_k.$$

It follows from Young's inequality that

$$\begin{aligned}
\frac{d^+V}{dt} &= \int_{(0,2\pi)^k} w w_t \\
&= \int_{(0,2\pi)^k} \left(\sum_{j=1}^k w w_{x_j x_j} + \sum_{j=1}^k a(t, x_j) w w_{x_j} + \sum_{j=1}^k b(t, x_j) w^2 \right) \\
&\le -\sum_{j=1}^k \int_{(0,2\pi)^k} (w_{x_j})^2 + \sum_{j=1}^k \int_{(0,2\pi)^k} \frac{1}{2}|a(t, x_j)| \left[\nu_j(x_j) w^2 + \frac{(w_{x_j})^2}{\nu_j(x_j)} \right] \\
&\quad + \int_{(0,2\pi)^k} \sum_{j=1}^k b(t, x_j) w^2 \\
&= \sum_{j=1}^k \int_{(0,2\pi)^k} \left(\frac{|a(t, x_j)|}{2\nu_j(x_j)} - 1 \right) (w_{x_j})^2 \\
&\quad + \int_{(0,2\pi)^k} \sum_{j=1}^k \left(b(t, x_j) + \frac{|a(t, x_j)|\nu_j(x_j)}{2} \right) w^2,
\end{aligned}$$

for any positive functions $\nu_1(x), \nu_2(x)$. If $\max_x |a(t, x)| > 0, 0 < x < 2\pi$, then a choice

$$\nu_j(x) = \max_{0<x<2\pi} |a(t, x)| := \nu(t), \quad j = 1, 2, \cdots, k,$$

implies that

$$\frac{d^+V}{dt} \le \mu(t)V,$$

where

$$\mu(t) = -\sum_{j=1}^{k-1} \lambda_j + 2k \max_x \left\{ b(t, x) + \frac{|a(t, x)|\nu(t)}{2} \right\}.$$

If $\max_x |a(t, x)| = 0$, then $a(t, x) \equiv 0$ for all $0 < x < 2\pi$ and

$$\mu(t) = -\sum_{j=1}^{k-1} \lambda_j + k \max_x b(t, x).$$

Theorem 3.32 implies the following corollaries.

Corollary 3.33. *Suppose that $f \in C^2((0,2\pi)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R})$ and $\phi(t,x)$ is an ω-periodic solution of* (3.71). *If $\max_x |a(t,x)| > 0,\ 0 < x < 2\pi$ and*

$$\int_0^\omega \max_x \left\{ b(t,x) + \frac{|a(t,x)|\nu(t)}{2} \right\} dt < \frac{\omega}{2k}\sum_{j=1}^{k-1}\lambda_j;$$

or if $a(t,x)\equiv 0$ for all $0 < x < 2\pi$ and

$$\int_0^\omega \max_x b(t,x) < \frac{\omega}{k}\sum_{j=1}^{k-1}\lambda_j$$

where $\lambda_{2j-1} = j^2, \lambda_{2j} = j^2, j = 1,2,\cdots$ and $\nu(t) = \max_x |a(t,x)|$, then $\phi(t,x)$ has a stable manifold with codimension at most $k-2$.

Example 3.3. Consider the van der Pol equation

$$\frac{d^2u}{dt^2} + \varepsilon(u^2-1)\frac{du}{dt} + u = 0. \tag{3.75}$$

Equation (3.75) has a unique stable limit cycle $\phi(t)$ with period ω for all values of $\varepsilon \geq 0$. The formula for ω as a function of ε can be found in [15, 25, 90, 106, 107]. Moreover, $\omega \geq 2\pi$. If $c > 0$, then $\phi(x+ct)$ is a $\frac{\omega}{c}$-periodic solution of

$$\begin{aligned} &u_t = u_{xx} + cu_x + \varepsilon(u^2-1)u_x + u, \quad 0 < x < \omega, \quad t > 0, \\ &u(t,0) = u(t,\omega), \quad u_x(t,0) = u_x(t,\omega), \quad t \geq 0. \end{aligned} \tag{3.76}$$

The initial value problem associated to (3.76) generates a local semiflow on the Sobolev space $X = H^s_{per}(0,\omega)$ which contains all functions in $H^s(0,2\pi)$ satisfying the periodic boundary conditions. Choose $s > \frac{5}{2}$ so that X embeds into $C^2(0,\omega)$. Now for $u_0 \in H^s_{per}(0,\omega)$, $u(t,x) \in C^2(0,\omega)$ is a classical solution which exists on $[0,T)$ for some $T > 0$. Since

$$\begin{aligned} \int_0^\omega u_t dx &= \int_0^\omega \left[u_{xx} + cu_x + \varepsilon(u^2-1)u_x + u\right] dx \\ &= u_x|_0^\omega + cu|_0^\omega + u^3|_0^\omega - u|_0^\omega + \int_0^\omega u dx \\ &= \int_0^\omega u dx, \end{aligned}$$

$G = \left\{ u \in H^s_{per}(0,\omega) : \int_0^\omega u(x)dx = 0 \right\}$ is an invariant set with respect to (3.76). In this invariant set G,

$$\int_0^\omega u u_{xx} dx \le -\frac{2\pi}{\omega} \int_0^\omega u^2 dx,$$

and thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^\omega u^2 dx &= \int_0^\omega \left[u u_{xx} + c u_x u + \varepsilon (u^2-1) u_x u + u^2 \right] dx \\ &= \int_0^\omega \left[u u_{xx} + u^2 \right] dx \\ &\le \left[-\left(\frac{2\pi}{\omega}\right)^2 + 1 \right] \int_0^\omega u^2 dx, \end{aligned}$$

which implies that the L^2 norm of u exists for all $t \ge 0$. For any $T > 0$,

$$\int_0^\omega u^2(t,x)dx \le \exp\left[\left(-4\pi^2/\omega^2 + 1\right) t \right] \int_0^\omega u^2(0,x)dx, \tag{3.77}$$

which implies that $\int_0^\omega u^2 dx$ is uniformly bounded for all $0 \le t \le T$.

For each $k \ge 2$, multiply u^{2^k-1} on both sides of (3.76) and integrate it to obtain

$$\int_0^\omega u^{2^k-1} u_t dx = \int_0^\omega \left[u^{2^k-1} u_{xx} + c u_x u^{2^k-1} + \varepsilon (u^2-1) u_x u^{2^k-1} + u^{2^k} \right] dx,$$

that is,

$$\frac{1}{2^k}\frac{d}{dt}\int_0^\omega u^{2^k} dx = -\frac{2^k-1}{2^{2k-2}} \int_0^\omega |\nabla u^{2^{k-1}}|^2 dx + \int_0^\omega u^{2^k} dx,$$

where $\nabla u = u_x$ or equivalently,

$$\frac{1}{2}\frac{d}{dt}\int_0^\omega (u^{2^{k-1}})^2 dx = -\frac{2^k-1}{2^{k-1}} \int_0^\omega |\nabla u^{2^{k-1}}|^2 dx + 2^{k-1} \int_0^\omega (u^{2^{k-1}})^2 dx.$$

Let

$$u^* = |u|^{2^{k-1}}, \quad \alpha_k = \frac{2^k-1}{2^{k-1}}, \quad \sigma_k = 2^{k-1}.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_0^\omega (u^*)^2 dx = -\alpha_k \int_0^\omega |\nabla u^*|^2 dx + \sigma_k \int_0^\omega (u^*)^2 dx. \tag{3.78}$$

Apply Young's inequality to a special case of the Nirenberg-Gagliardo inequality ([42], page 37)

$$\|\xi\|_{L^2} \le \sqrt{C}\|\xi\|_{W^{1,2}}^{1/2}\|\xi\|_{L^1}^{1/2},$$

to obtain

$$\|\xi\|_{L^2}^2 \le C\varepsilon^{-1}\|\xi\|_{L^1}^2 + C\varepsilon\|\xi\|_{W^{1,2}}^2 = C\varepsilon^{-1}\|\xi\|_{L^1}^2 + C\varepsilon\|\nabla\xi\|_{L^2}^2 + C\varepsilon\|\xi\|_{L^2}^2. \tag{3.79}$$

Substitute u^* in the place of ξ in (3.79) and ε_k in the place of ε to get

$$\int_0^\omega (u^*)^2 dx \le C\varepsilon_k^{-1}\left(\int_0^\omega u^* dx\right)^2 + C\varepsilon_k \int_0^\omega |\nabla u^*|^2 dx + C\varepsilon_k \int_0^\omega (u^*)^2 dx$$

and thus

$$\left[(C\varepsilon_k)^{-1} - 1\right]\int_0^\omega (u^*)^2 dx - \varepsilon_k^{-2}\left(\int_0^\omega u^* dx\right)^2 \le \int_0^\omega |\nabla u^*|^2 dx. \tag{3.80}$$

Choose $\varepsilon_k > 0$ small such that $-\frac{\alpha_k}{C} + (\alpha_k + \sigma_k)\varepsilon_k + (\varepsilon_k)^2 < 0$. It follows from (3.78) and (3.80) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\omega (u^*)^2 dx \\
\le\ & -\alpha_k\left[(C\varepsilon_k)^{-1} - 1\right]\int_0^\omega (u^*)^2 dx + \alpha_k\varepsilon_k^{-2}\left(\int_0^\omega u^* dx\right)^2 + \sigma_k \int_0^\omega (u^*)^2 dx \\
=\ & \left[\sigma_k - \alpha_k (C\varepsilon_k)^{-1} + \alpha_k\right]\int_0^\omega (u^*)^2 dx + \alpha_k\varepsilon_k^{-2}\left(\int_0^\omega u^* dx\right)^2 \\
\le\ & -\varepsilon_k \int_0^\omega (u^*)^2 dx + \alpha_k\varepsilon_k^{-2}\left(\int_0^\omega u^* dx\right)^2.
\end{aligned} \tag{3.81}$$

If $\int_0^\omega u^* dx$ is uniformly bounded for all $0 \le t \le T$, then

$$\int_0^\omega (u^*)^2 dx \le \max\left\{\frac{\alpha_k}{(\varepsilon_k)^3}\left[\sup_t \int_0^\omega u^* dx\right]^2, \int_0^\omega (u^*(0,x))^2 dx\right\}.$$

In particular, if $k = 2$, then $u^* = u^2, \sigma_2 = 2$ and $\alpha_2 = \frac{3}{2}$. Choose ε_2 such that $(2+\varepsilon_2)C\varepsilon_2 \le \frac{3}{2}$ and thus

$$\int_0^\omega u^4 dx \le \max\left\{\frac{3}{2(\varepsilon_2)^3}\left[\sup_t \int_0^\omega u^2 dx\right]^2, \int_0^\omega u^4(0,x) dx\right\}.$$

Therefore, $\sup_t \int_0^\omega u^4 dx$ is bounded since $\sup_t \int_0^\omega u^2 dx$ is bounded from (3.77). Induction will imply that for $k \geq 1$, $\sup_t \int_0^\omega u^{2^k} dx$ is bounded. By taking the $\frac{1}{2^k}$ power of both sides and passing to the limit, the L^∞ estimate is obtained. Therefore, for any $T > 0$, there exists a M, which depends on $\|u(0,x)\|_{L^\infty}$, such that

$$\|u(t)\|_\infty \leq M, \quad 0 \leq t \leq T.$$

Since

$$\begin{aligned}
\frac{d}{dt}\int_0^\omega u_x^2 dx &= -\int_0^\omega \left[u_{xx}^2 + cu_x u_{xx} + \varepsilon(u^2-1)u_x u_{xx} + uu_{xx}\right] dx \\
&= -\int_0^\omega [u_{xx}^2 dx + \varepsilon u^2 u_x u_{xx} dx - u_x^2] dx, \\
&\leq -\int_0^\omega u_{xx}^2 dx + M^2\varepsilon \int_0^\omega |u_x u_{xx}| dx + \int_0^\omega u_x^2 dx, \\
&\leq -\int_0^\omega u_{xx}^2 dx + \frac{M^4\varepsilon^2}{2}\int_0^\omega u_x^2 dx + \frac{1}{2}\int_0^\omega u_{xx}^2 dx + \int_0^\omega u_x^2 dx, \\
&\leq -\frac{1}{2}\int_0^\omega u_{xx}^2 dx + \left(1 + \frac{M^4\varepsilon^2}{2}\right)\int_0^\omega u_x^2 dx,
\end{aligned}$$

or

$$\int_0^\omega u_x^2(t,x)dx \leq \exp\left[\left(1 + M^4\varepsilon^2/2\right)t\right]\int_0^\omega u_x^2(0,x)dx,$$

the L^2 norm of u_x exists for all $0 \leq t \leq T$. Therefore $u(t)$ exists in $H^1_{per}(0,\omega)$ for all $t \geq 0$.

The linear variational equation of (3.75) at $\phi(x+ct)$ is

$$\begin{aligned}
&v_t = v_{xx} + \left(\varepsilon(\phi^2 - 1) + c\right)v_x + (1 + 2\varepsilon\phi\phi_x)v, \quad 0 < x < \omega, \quad t > 0, \\
&v(t,0) = v(t,\omega), \quad v_x(t,0) = v_x(t,\omega), \quad t \geq 0, \\
&v(0,x) = v_0(x), \quad 0 \leq x \leq \omega.
\end{aligned} \tag{3.82}$$

It can be shown that $u(t)$ exists in $H^1_{per}(0,\omega)$ for all $t \geq 0$ since $\phi(x+ct) \in C^2(0,\omega)$. The pointwise representation of the kth compound equation of (3.82) defined on $\bigwedge^k G$ is

$$w_t = \sum_{j=1}^k w_{x_j x_j} + \sum_{j=1}^k a(t,x_j)w_{x_j} + \sum_{j=1}^k b(t,x_j)w, \tag{3.83}$$

where

$$a(t,x) = \varepsilon(\phi^2(x+ct)-1)+c,$$

$$b(t,x) = 1+2\varepsilon\phi(x+ct)\phi_x(x+ct).$$

Consider a Lyapunov function

$$V(t) = \frac{1}{2}\int_{(0,2\pi)^k} w^2 = \frac{1}{2}\int_{(0,2\pi)^k} w^2(t,x_1,x_2,\cdots,x_k)dx_1dx_2\cdots dx_n.$$

Then

$$\begin{aligned}
\frac{d^+V}{dt} &= \int_{(0,2\pi)^k} ww_t \\
&= \int_{(0,2\pi)^k}\left(w\sum_{j=1}^k w_{x_jx_j} + \sum_{j=1}^k a(t,x_j)ww_{x_j} + \sum_{j=1}^k b(t,x_j)w^2\right) \\
&= \int_{(0,2\pi)^k}\left(w\sum_{j=1}^k w_{x_jx_j} + \sum_{j=1}^k \left(\varepsilon(\phi^2(x_j+ct)-1)+c\right)ww_{x_j}\right) \\
&\quad + \int_{(0,2\pi)^k}\left(kw^2 + 2\varepsilon\sum_{j=1}^k \phi(x_j+ct)\phi_x(x_j+ct)w^2\right) \\
&= \int_{(0,2\pi)^k}\left(w\sum_{j=1}^k w_{x_jx_j} + kw^2\right) + \varepsilon\int_\Omega \sum_{j=1}^k \phi(x_j+ct)\phi_x(x_j+ct)w^2 \\
&\le 2\left(\sum_{j=1}^k \lambda_j + k + k\varepsilon\|\phi(x+ct)\phi_x(x+ct)\|_\infty\right)\cdot 2V \\
&= 2\left(\sum_{j=1}^k \lambda_j + k + k\varepsilon\|\phi\phi_x\|_\infty\right)V,
\end{aligned}$$

where

$$\lambda_{2j-1} = \lambda_{2j} = -j^2\left(\frac{2\pi}{\omega}\right)^2, \quad j=1,2,\cdots.$$

Thus, for any k characteristic multipliers $\mu_1,\mu_2,\cdots,\mu_k$,

$$|\mu_1\mu_2\cdots\mu_k| \le \exp\left(\sum_{j=1}^k \lambda_j + k + k\varepsilon\|\phi\phi_x\|_\infty\right).$$

Since 1 is a characteristic multiplier of the periodic solution $\phi(x+ct)$, say $\mu_1 = 1$, the other $k-1$ characteristic multipliers

$$|\mu_2\mu_2\cdots\mu_k| \le \exp\left(\sum_{j=1}^{k}\lambda_j + k + k\varepsilon\|\phi\phi_x\|_\infty\right). \tag{3.84}$$

Case 1: $\varepsilon = 0.2$

An approximation of the periodic solution of the ordinary differential equation (3.75) is shown in Figure 3.1, which implies $2\|\phi\phi_x\|_\infty \le (2.2)^2$. The Urabe formula of the period implies that $\omega \approx 6.3$. Then

$$\sum_{j=1}^{3}\lambda_j + 3 + 3\varepsilon\|\phi\phi_x\|_\infty \le -6\left(\frac{2\pi}{\omega}\right)^2 + 3 + \frac{3}{2}\cdot 0.2\cdot(2.2)^2 \approx -1.516.$$

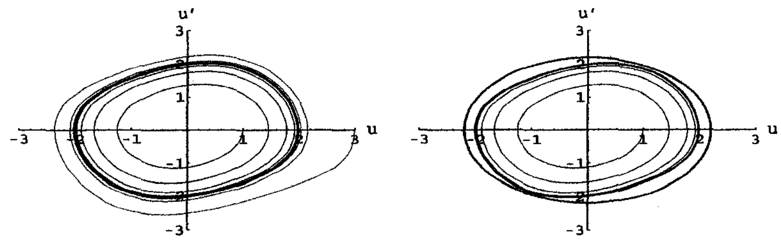

Figure 3.1: $\varepsilon = 0.2$. The black trajectory is $u^2 + (u')^2 = 2.2^2$.

Theorem 3.32 implies the following corollary.

Corollary 3.34. *If* $\varepsilon = 0.2$, *then the periodic solution* $\phi(x+ct)$ *of the reaction diffusion equation* (3.75) *has a stable manifold with codimension at most* 1 *in* G.

Case 2: $\varepsilon = 1$

An approximation of the periodic solution of the ordinary differential equation (3.75) is shown in Figure 3.2, which implies $\|\phi\|_\infty \le A \approx 2.009$, $2\|\phi\phi_x\|_\infty \le (2.9)^2$. The Urabe formula of the period implies that $\omega \approx 6.687$. Then

$$\sum_{j=1}^{9}\lambda_j + 9 + 9\varepsilon\|\phi\phi_x\|_\infty \le -85\left(\frac{2\pi}{\omega}\right)^2 + 9 + 9\cdot 1\cdot 2.009\cdot 2.9 \approx -13.609.$$

Theorem 3.32 implies the following corollary.

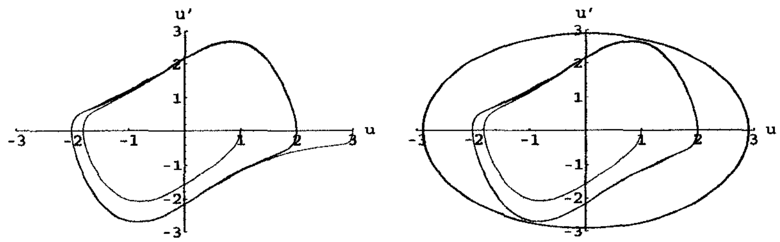

Figure 3.2: $\varepsilon = 1$. The black trajectory is $u^2 + (u')^2 = 2.9^2$.

Corollary 3.35. *If* $\varepsilon = 1$, *then the periodic solution* $\phi(x + ct)$ *of the reaction diffusion equation* (3.75) *has a stable manifold with codimension at most* 7 *in* G.

Case 3: $\varepsilon = 10$

An approximation of the periodic solution of the ordinary differential equation (3.75) is shown in Figure 3.3, which implies $\|\phi\|_\infty \le A \approx 2.0145$, $\|\phi\phi_x\|_\infty \le 16$. The Urabe formula of the period implies that $\omega \approx 19.1550$. Then

$$\sum_{j=1}^{1} 28\lambda_j + 128 + 128\varepsilon\|\phi\phi_x\|_\infty \le -178880\left(\frac{2\pi}{\omega}\right)^2 + 9 + 9\cdot 1\cdot 2.009\cdot 2.9 \approx -488.181.$$

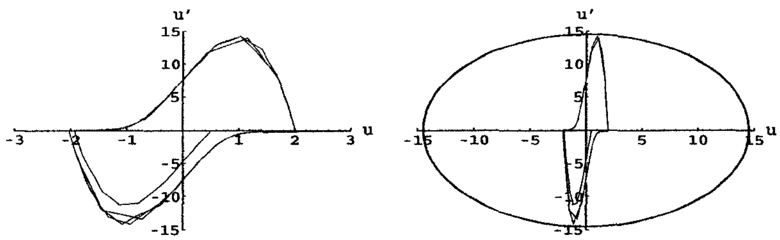

Figure 3.3: $\varepsilon = 10$. The black trajectory is $u^2 + (u')^2 = 14.5^2$.

Theorem 3.32 implies the following corollary.

Corollary 3.36. *If* $\varepsilon = 10$, *then the periodic solution* $\phi(x + ct)$ *of the reaction diffusion equation* (3.75) *has a stable manifold with at most codimension* 126 *in* G.

For the case $c = \varepsilon > 0$, it can be shown that $u = \phi(x + \varepsilon t)$ is the only nonconstant periodic solution of the reaction diffusion equation (3.76). The full discussion is as follows. If $c = \varepsilon > 0$, then the reaction diffusion equation (3.76) is

$$\begin{aligned} &u_t = u_{xx} + \varepsilon u^2 u_x + u, \quad 0 < x < \omega, \quad t > 0, \\ &u(t,0) = u(t,\omega), \quad u_x(t,0) = u_x(t,\omega), \quad t \geq 0, \\ &u(0,x) = u_0(x), \quad 0 \leq x \leq \omega. \end{aligned} \tag{3.85}$$

The steady state solutions of (3.85) satisfies

$$\begin{aligned} &0 = u_{xx} + \varepsilon u^2 u_x + u, \quad 0 < x < \omega, \quad t > 0, \\ &u(0) = u(\omega), \quad u_x(0) = u_x(\omega). \end{aligned} \tag{3.86}$$

Let $v = u_x$. Then (3.86) is equivalent to

$$\begin{aligned} &u_x = v := f(u,v), \\ &v_x = -\varepsilon u^2 v + u := g(u,v), \quad 0 < x < \omega, \quad t > 0, \\ &u(0) = u(\omega), \quad v(0) = v(\omega). \end{aligned}$$

Since

$$\operatorname{div}(f,g) = f_u(u,v) + g_v(u,v) = -\varepsilon u^2,$$

Bendixson's Condition implies that the only solution of (3.86) is $u = 0$. The linear variational equation of (3.86) at $u = 0$ is

$$\begin{aligned} &v_t = v_{xx} + v, \quad 0 < x < \omega, \\ &v(0) = v(\omega), \quad v_x(0) = v_x(\omega), \end{aligned}$$

whose eigenvalue problem

$$\begin{aligned} &\lambda\phi = \phi'' + \phi, \quad 0 < x < \omega, \\ &\phi(0) = \phi(\omega), \quad \phi(0) = \phi(\omega), \end{aligned}$$

has solutions

$$\begin{aligned} &\lambda_0 = 1, && \phi_0(x) = 1, \\ &\lambda_{2j-1} = -j^2 + 1, && \phi_{2j-1}(x) = \cos\left(\tfrac{2\pi j}{\omega}x\right), \\ &\lambda_{2j-1} = -j^2 + 1, && \phi_{2j}(x) = \sin\left(\tfrac{2\pi j}{\omega}x\right), \end{aligned}$$

where $j = 1, 2, 3, \cdots$.

Corollary 3.37. *The solution* $u = 0$ *of the reaction diffusion equation* (3.85) *has a* 2*-dimensional center manifold,* 1*-dimensional unstable manifold and a stable manifold with codimension* 3.

The result of Angenent and Fiedler [4] implies that the periodic solutions of (3.85) are in form of $u = u(x - c_1 t)$. Let $z = x - c_1 t$. Then

$$-c_1 \frac{du}{dz} = \frac{d^2u}{dz^2} + \varepsilon u^2 \frac{du}{dz} + u,$$
$$u(0) = u(\omega), \quad u_x(0) = u_x(\omega).$$

or

$$\begin{aligned} &\frac{d^2u}{dz^2} + (\varepsilon u^2 + c_1)\frac{du}{dz} + u = 0, \\ &u(0) = u(\omega), \quad u_x(0) = u_x(\omega), \end{aligned} \tag{3.87}$$

which is equivalent to

$$\begin{aligned} &\frac{du}{dz} = v := f^1(u, v), \\ &\frac{dv}{dz} = -(\varepsilon u^2 + c_1)v - u := f^2(u, v) \\ &u(0) = u(\omega), \quad v(0) = v(\omega). \end{aligned}$$

Since

$$\operatorname{div}(f, g) = f_u^1(u, v) + f_v^2(u, v) = -(\varepsilon u^2 + c_1),$$

Bendixson's condition implies that the only solution of (3.87) is $u = 0$ if $c_1 \geq 0$. Thus the periodic solutions of (3.85) are in form of $u = u(x + c_1 t)$ where $c_1 > 0$. Let $p = \sqrt{\frac{c_1}{\varepsilon}} u$. Then

$$\sqrt{\frac{c_1}{\varepsilon}} \frac{d^2p}{dz^2} + \left(\varepsilon \frac{c_1}{\varepsilon} p^2 - c\right) \sqrt{\frac{c_1}{\varepsilon}} \frac{dp}{dz} + \sqrt{\frac{c_1}{\varepsilon}} p = 0,$$
$$p(0) = p(\omega), \quad p_x(0) = p_x(\omega),$$

or

$$\begin{aligned} &\frac{d^2p}{dz^2} + c_1 \left(p^2 - 1\right) \frac{dp}{dz} + p = 0, \\ &p(0) = p(\omega), \quad p_x(0) = p_x(\omega). \end{aligned} \tag{3.88}$$

The result of van de Pol equation implies that equation

$$\frac{d^2p}{dz^2} + c_1 \left(p^2 - 1\right) \frac{dp}{dz} + p = 0$$

has an ω-periodic solution if and only if $c_1 = \varepsilon$. Therefore, the only periodic solution of the reaction diffusion equation (3.85) is $\phi(x + \varepsilon t)$ where ϕ is a solution of the van der Pol equation

$$\frac{d^2p}{dz^2} + \varepsilon\left(p^2 - 1\right)\frac{dp}{dz} + p = 0.$$

The result in Angenent and Fiedler [4] implies that the ω-limit set of any bounded solution of the reaction diffusion equation (3.85) is either $u = 0$ or $u = \phi(x + \varepsilon t)$ whose stability is given by Corollary 3.34- Corollary 3.37.

Example 3.4. Consider a scalar reaction diffusion equation

$$\begin{cases} u_t = u_{xx} + u + \varepsilon u(1 - u^2 - u_x^2) + cu_x, & 0 < x < 2\pi, \quad t > 0, \\ u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi), \end{cases} \tag{3.89}$$

where $\varepsilon, c > 0$. Similar arguments used in Chapter 2 show that the initial value problem associated to (3.89) generates a global semiflow on the Sobolev space $X = H^1_{per}(0,2\pi) = \{\phi \in L^2(0,2\pi) : \phi' \in L^2(0,2\pi), \phi(0) = \phi(2\pi),\ \phi'(0) = \phi'(2\pi)\}$. Since $u = \cos(x)$ is a solution of

$$\begin{cases} u_t = u_{xx} + u + \varepsilon u(1 - u^2 - u_x^2), & 0 < x < 2\pi, \quad t > 0, \\ u(t,0) = u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi), \end{cases}$$

equation (3.89) has a $\frac{2\pi}{c}$-periodic solution $u = \cos(x+ct)$. In fact, the circle of equilibria $\cos(x+a), a \in \mathbb{R}$ of (3.4) becomes the periodic orbit $u = \cos(x+ct)$ of (3.89).

The linear variational equation of (3.89) at $u = \cos(x + ct)$ is

$$\begin{aligned} &v_t = v_{xx} + (c + \varepsilon\sin(2x + 2ct))v_x + (1 - \varepsilon - \varepsilon\cos(2x + 2ct))v, \\ &v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \end{aligned} \tag{3.90}$$

which has solutions

$$e_1 = \sin(x + ct) \quad \text{and} \quad e_2 = e^{-2\varepsilon t}\cos(x + ct),$$

with the characteristic multipliers

$$\mu_1 = 1 \quad \text{and} \quad \mu_2 = e^{-2\varepsilon\cdot\frac{2\pi}{c}} < 1.$$

The linear variational equation of (3.4) at $u = \cos(x)$ is

$$\begin{aligned} &v_t = v_{xx} + \varepsilon\sin(2x)v_x + (1-\varepsilon-\varepsilon\cos(2x))v, \ 0 < x < 2\pi, \ t > 0, \\ &v(t,0) = v(t,2\pi), \quad v_x(t,0) = v_x(t,2\pi), \quad t \geq 0, \end{aligned}$$

whose eigenvalue problem is given by

$$\begin{aligned} &\lambda\phi = \phi'' + \varepsilon\sin(2x)\phi' + (1-\varepsilon-\varepsilon\cos(2x))\phi, \\ &\phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi). \end{aligned} \tag{3.91}$$

If (3.91) has an eigenvalue λ with the corresponding eigenfunction $\phi(x)$, then (3.90) has a characteristic multiplier $e^{\lambda\cdot\frac{2\pi}{c}}$ with a corresponding nonzero solution $e^{\lambda t}\phi(x+ct)$. In fact, substitute $v = e^{\lambda t}\phi(x+ct)$ into (3.90) to get

$$\begin{aligned} &\lambda e^{\lambda t}\phi(x+ct) + e^{\lambda t}\phi_x(x+ct)c \\ = \ & e^{\lambda t}\phi_{xx}(x+ct) + (c+\varepsilon\sin(2x+2ct))e^{\lambda t}\phi_x(x+ct) \\ &+(1-\varepsilon-\varepsilon\cos(2x+2ct))e^{\lambda t}\phi(x+ct), \end{aligned}$$

or

$$\lambda\phi(x+ct) = \phi_{xx}(x+ct)+\varepsilon\sin(2x+2ct)\phi_x(x+ct)+(1-\varepsilon-\varepsilon\cos(2x+2ct))\phi(x+ct),$$

which always holds from (3.91). The stability of the periodic solution $u = \cos(x+ct)$ of (3.89) can be discussed by studying the stability of the steady state solution $u = \cos(x)$ of (3.4), which has been studied in Example 2.1 of Section 2.2. In particular, Corollary 2.13 and Corollary 2.14 imply the following results on the estimate of the stability of $u = \cos(x+ct)$.

Corollary 3.38. *If* $0 < \varepsilon < 2(\sqrt{202} - 14)$, *then* $u = \cos(x+ct)$ *has a 1-dimensional center manifold, a 1-dimensional unstable manifold and a stable manifold with codimension* 2.

Corollary 3.39. *If* $2(\sqrt{202} - 14) \leq \varepsilon < 2(13\sqrt{2} - 18)$, *then* $u = \cos(x+ct)$ *has a stable manifold with codimension at most* 3.

Chapter 4

Convergence Theorems

In this chapter the implications of various types of stability of a solution of a differential equation for the structure of its omega limit set are considered.

A fundamental result of this type is the Poincaré-Bendixson theorem for a planar autonomous system of ordinary differential equations. The ω-limit set of a bounded solution of such a system is a periodic orbit if it does not contain an equilibrium. Massera's theorem [69] infers the existence of a periodic solution to a nonautonomous time-periodic scalar differential equation from the existence of a bounded solution. Other expositions and extensions of the Poincaré-Bendixson theorem were obtained by Hirsch [45], R. A. Smith [98, 99, 101], Mallet-Paret and H. L. Smith [66]. An application of the Poincaré-Bendixson theorem is that it can be used to deduce the existence of a periodic solution in the case when no equilibrium occurs in the ω-limit set of a bounded solution.

In dynamics, it is natural to enquire how much of the nontransient behaviour can be detected from an analysis of an individual orbit and its relationship with its neighbours. Sell [96] proved for a general semiflow on a metric space that a Lagrange stable orbit has a phase asymptotically stable periodic orbit as its omega limit set if the orbit itself is phase asymptotically stable. A Lagrange stable orbit means that the closure of the orbit is compact. Good expositions of Sell's results may be found in Cronin [22] Chapter 6 and Saperstone [93] Chapter III. Yoshizawa [112] also discussed these results and extended the application to functional differential equations. Pliss [89] established a closely related result for autonomous differential equations in $\mathbb{R}^n$ where the stability requirements are somewhat different from those of Sell: Lyapunov stability is not required but a certain uniformity is imposed on the manner in which the orbit attracts its neighbours. Li and Muldowney [54] and Muldowney [79] extended these results in the context of dynamical systems on a general metric space with greatly simplified proofs. A good summary of these results may also be found in Muldowney [80].

Another result of Muldowney in [80] is to deduce the existence of an equilibrium, even when the algebraic equations yielding the equilibria can not be solved explicitly. For example, Muldowney proved that, for an autonomous ordinary differential equation

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^n,$$

the ω-limit set of a bounded solution $u(t)$ is a stable hyperbolic equilibrium if and only if the linear variational equation at $u(t)$,

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(u(t))v,$$

is uniformly asymptotically stable. In particular, if the bounded solution $u(t) = u^*$ is an equilibrium, it reduces to the familiar observation that u^* is stable hyperbolic if and only if the linear variational equation at u^* is uniformly asymptotically stable. Li and Muldowney [55] and Muldowney [79] also proved that, if the ω-limit set of a bounded solution $u(t)$ contains no equilibrium, it is a stable hyperbolic periodic orbit if and only if the second compound differential equation

$$\frac{dw}{dt} = \frac{\partial f}{\partial u}^{[2]}(u(t))w$$

is uniformly asymptotically stable. This is in fact the necessary and sufficient condition cited in Theorem 3.15, Section 3.3.3 for stable hyperbolicity when $u(t)$ is itself a periodic solution.

An infinite dimensional version of the Poincaré-Bendixson theorem for a scalar reaction diffusion equation was studied by Henry [43], Brunovský and Fiedler [10, 11], Massatt [68], Matano [71], Angenent and Fielder [4], Fiedler and Mallet-Paret [30]. It was proved by Fiedler and Mallet-Paret that, for a general scalar reaction diffusion equation

$$u_t = u_{xx} + f(x, u, u_x), \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \; f \in C^2,$$

the ω-limit set of any bounded solution satisfies exactly one of the following alternatives:
(i) it consists in precisely one periodic solution, or
(ii) for every orbit in the ω-limit set, its ω-limit set and α-limit set are contained in the set of steady state solutions.

In this chapter, an extension is given of the results in [55, 79, 80] to an infinite dimensional differential equation

$$\frac{du}{dt} + Au = f(u), \quad u(\cdot) \in X, \tag{4.1}$$

where A is a sectorial operator in a Banach space X. The characterization of the stability of steady state solutions in terms of stability of linearizations and orbital stability of periodic orbits in terms of stability of second compound differential equations, discussed in Chapter 2 and Chapter 3 respectively, are special cases of this chapter.

4.1 Existence of Stable Steady State Solutions

The following notation, definitions, and lemmas are taken from Henry [42], page 53-54, page 91-92 and page 98-104 and Smoller [103], page 114-122.

Let A be a sectorial operator in a Banach space X and f be continuously differentiable from X^α into X where $0 \leq \alpha < 1$. Let $\mathcal{D}(A)$ denote the domain of A. Consider an initial value problem

$$\begin{aligned} &\frac{du}{dt} + Au = f(u), \; u \in X, \\ &u(0) = u_0. \end{aligned} \tag{4.2}$$

In the following, let $u(t) = u(t, u_0)$ denote the solution of (4.2). A solution is a *steady state solution* or (an *equilibrium*) if $u = u^* \in \mathcal{D}(A)$ and $Au^* = f(u^*)$.

Definition 4.1. A solution $u(t, u_0)$ of (4.2) on $[0, \infty)$ is said to be *stable* if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $u(t, u_1)$ with $\|u_1 - u_0\|_\alpha < \delta$ exists on $[0, \infty)$ and satisfies $\|u(t, u_1) - u(t, u_0)\|_\alpha < \varepsilon$ for all $t \geq 0$. A solution $u(t, u_0)$ of (4.2) on $[0, \infty)$ is said to be *uniformly stable* if there exists a $\delta > 0$ such that for each $\varepsilon > 0$, any solution $u(t, u_1)$ with

$\|u_1 - u_0\|_\alpha < \delta$ exists and satisfies $\|u(t, u_1) - u(t, u_0)\|_\alpha < \varepsilon$ for all $t \geq 0$. A solution $u(t, u_0)$ of (4.2) on $[0, \infty)$ is said to be *uniformly asymptotically stable* if it is uniformly stable and there exists a $\delta_0 > 0$ and for each $\varepsilon > 0$, there is a corresponding $T = T(\varepsilon)$ such that if $\|u_1 - u_0\|_\alpha \leq \delta_0$, then $\|u(t, u_1) - u(t, u_0)\|_\alpha < \varepsilon$ for all $t \geq T$.

The linear variational equation of (4.2) at the steady state solution $u(t) = u^*$ is

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u^*)v. \tag{4.3}$$

Lemma 4.1. *Let f be continuously differentiable from X^α into X in a neighbourhood of a steady state solution u^* of (4.2). If the spectrum of $A - \frac{\partial f}{\partial u}(u^*)$ lies in $\{Re\ \lambda > \beta\}$ for some $\beta > 0$, then the steady state solution u^* of (4.2) is uniformly asymptotically stable in X^α. More precisely, there exist $\rho > 0, M \geq 1$ such that if $\|u_0 - u^*\|_\alpha \leq \frac{\rho}{2M}$, then there is a unique solution of*

$$\frac{du}{dt} + Au = f(u), \ u \in X,$$
$$u(0) = u_0,$$

satisfying for $0 \leq t < \infty$,

$$\|u(t, u_0) - u^*\|_\alpha \leq 2Me^{-\beta t}\|u_0 - u^*\|_\alpha.$$

Lemma 4.2. *Let f be continuously differentiable from X^α into X in a neighbourhood of a steady state solution u^* of (4.2). Let $L = A - \frac{\partial f}{\partial u}(u^*)$. If $\sigma(L) \cap \{Re\ \lambda < 0\}$ is a nonempty spectral set, then the equilibrium u^* of (4.2) is unstable. Specifically, there exist $\varepsilon_0 > 0$ and $\{u_n, n \geq 1\}$ with $\|u_n - u^*\|_\alpha \to 0$ as $n \to \infty$, but for all n*

$$\sup_{t \geq 0} \|u(t, u_n) - u^*\|_\alpha \geq \varepsilon_0 > 0.$$

Here the supremum is taken over the maximal interval of existence of $u(t, u_n)$.

Definition 4.2. For any $u_0 \in X^\alpha$, the *positive orbit* of $u(t, u_0)$ is $C_+(u_0) = \{u(t, u_0) : t \geq 0\}$ and the *ω-limit set* of u_0 is

$$\omega(u_0) = \{u \in X^\alpha : \text{there exist } t_n \to \infty \text{ such that } u(t_n, u_0) \to u\}.$$

Remark 4.1. If $u_1 \in C_+(u_0)$, then $\omega(u_1) = \omega(u_0)$. Also

$$\omega(u_0) = \bigcap_{t \geq 0} \text{cl } C_+(u(t, u_0)),$$

where cl denotes the topological closure.

Lemma 4.3. *If $C_+(u_0)$ lies in a compact set in X^α, then $\omega(u_0)$ is nonempty, closed, connected, invariant and dist$(u(t, u_0), \omega(u_0)) \to 0$ as $t \to \infty$.*

The following Lemma 4.4, Lemma 4.5 and Lemma 4.6 are taken from Smoller [103], page 114-122.

Lemma 4.4. *Let $u(t, u_0)$ be a solution of* (4.2) *which exists on $[0, T)$. Then there exists a neighborhood N of u_0 such that if $u_1 \in N$, there is a solution $u(t, u_1)$ of* (4.2) *which exists on $[0, T)$. Moreover, there is a constant $c > 0$ such that for all such $u_1 \in N$,*

$$\|u(t, u_1) - u(t, u_0)\|_\alpha \leq c\|u_1 - u_0\|_\alpha. \tag{4.4}$$

Let $u(t, u_0)$ be a solution of (4.2) which exists on $[0, T)$. Let N be as in Lemma 4.4. Define the solution operator $(t, u_0) \mapsto S(t)u_0$ on the set $[0, T) \times X^\alpha$ such that the solution $u(t, u_0) = S(t)u_0$ satisfies $u(0, u_0) = u_0$.

Lemma 4.5. *Let $u(t, u_0)$ be a solution of* (4.2) *and N be as in Lemma* 4.4. *If $u_1 \in N$, then $S(t), 0 \leq t \leq T$, is (Fréchet) differentiable, and if $u_0 \in N$, then $v(t, v_0) = dS(t)_{u_0} v_0$ solves the linear variational equation of* (4.2) *at $u(t, u_0)$,*

$$\frac{dv}{dt} = Av + \frac{\partial f}{\partial u}(u(t, u_0))v,$$
$$v(0) = v_0.$$

Lemma 4.6. *$S(t)$ is continuous (Fréchet) differentiable on N.*

For an autonomous finite dimensional ordinary differential equation, a necessary and sufficient condition for the ω-limit set to be a stable hyperbolic equilibrium was proved by Muldowney [80]. Consider an autonomous differential equation

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^n, \tag{4.5}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. This defines a semiflow $(t, u_0) \mapsto \phi(t, u_0)$ on the set $\mathbb{R} \times \mathbb{R}^n$ such that the solution $u(t, u_0) = \phi(t, u_0)$ satisfies $u(0, u_0) = u_0$. The linear variational equation of (4.5) at a solution $u(t, u_0)$ is

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(u(t, u_0))v. \tag{4.6}$$

Because of linearity, every solution has the same stability properties as the zero solution. Therefore, it is permissible to say that a linear system is stable, uniformly stable, and so on.

Proposition 4.7. *The equation* (4.6) *is uniformly asymptotically stable if and only if there exist constants* $K, \sigma > 0$ *such that if* $u \in C_+(u_0)$,

$$\left\| \frac{\partial}{\partial u}\phi(t, u) \right\| \le Ke^{-\sigma t}, \quad \text{for } t \ge 0.$$

Definition 4.3. An equilibrium u^*, $f(u^*) = 0$, is *stable hyperbolic* if $\operatorname{Re}\lambda < 0$ for every eigenvalue of $\frac{\partial f}{\partial u}(u^*)$. This is equivalent to the uniform asymptotic stability of (4.6) with $u(t, u_0) = u^*$.

Theorem 4.8. *Suppose that* $u(t, u_0)$ *is a bounded solution of* (4.5). *Then* $\lim_{t \to +\infty} u(t, u_0) = u^*$, *where* u^* *is a stable hyperbolic equilibrium if and only if the linear variational equation* (4.6) *of* (4.5) *with respect to* $u(t, u_0)$ *is uniformly asymptotically stable.*

The generalizations of Definition 4.3 and Theorem 4.8 for a differential equation (4.2) in a Banach space X are as follows.

Definition 4.4. A steady state solution u^* is said to be stable hyperbolic if the spectrum of $A - \frac{\partial f}{\partial u}(u^*)$ lies in $\{Re\lambda > \beta\}$ for some $\beta > 0$.

Remark 4.2. Definition 4.4 is equivalent to Definition 11.19 in Smoller [103], page 120. Theorem 11.20 in Smoller [103], page 120, shows that the hyperbolic stability of u^* is equivalent to the uniform asymptotic stability of (4.3).

Definition 4.5. The equation

$$\begin{aligned} \frac{dv}{dt} &= Av + \frac{\partial f}{\partial u}(u(t,u_0))v, \\ v(0) &= v_0 \end{aligned} \tag{4.7}$$

is said to be *uniformly asymptotically stable* if there exist constants $K, \sigma > 0$ such that $\|dS(t)_u\|_\alpha \le Ke^{-\sigma t}$ for $t \ge 0$ if $u \in C_+(u_0)$.

Theorem 4.9. *Suppose that $u(t,u_0)$ is a solution of (4.2) and that $C_+(u_0) = \{u(t,u_0), t \ge 0\}$ is relatively compact in X^α with $0 < \alpha < 1$. Then $\lim_{t\to+\infty} u(t,u_0) = u^*$, where u^* is a stable hyperbolic steady state solution if and only if the linear variational equation (4.3) of (4.2) with respect to $u(t,u_0)$ is uniformly asymptotically stable.*

Theorem 4.9 will be proved in Section 4.1.1.

Remark 4.3. For a steady state solution $u = u^*$, its omega limit set is itself and thus Theorem 2.8 in Section 2.2 is a special case of Theorem 4.9.

Remark 4.4. An example of a diffusive epidemiology disease model will be discussed in Chapter 6 as an application of Theorem 4.9.

4.1.1 Proof of Theorem 4.9

To prove Theorem 4.9, a particular upper bound of the estimate in the inequality in Lemma 7.1.1 in Henry [42] is given in Lemma 4.10 .

Lemma 4.10. *Suppose that $a, b \ge 0, \beta > 0$ and $u(t)$ is nonnegative and locally integrable on $0 \le t < \infty$ with*

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1}u(s)ds.$$

Then

$$u(t) \le \left(\left[\frac{1}{\beta}\right] + 1\right) ae^{2\rho t}, \tag{4.8}$$

where $\rho = (b\Gamma(\beta))^{\frac{1}{\beta}}$ and $[\cdot]$ is the greatest integer function.

Proof. Lemma 7.1.1 in Henry [42] shows that

$$u(t) \le a\sum_{n=0}^{\infty} \frac{(b\Gamma(\beta)t^\beta)^n}{\Gamma(n\beta+1)}. \tag{4.9}$$

An alternative proof of the above inequality can be obtained by first showing that $a\sum_{n=0}^{\infty}\frac{(b\Gamma(\beta)t^{\beta})^n}{\Gamma(n\beta+1)}$ is a solution of $v(t)=a+b\int_0^t(t-s)^{\beta-1}v(s)ds$, and then showing that $u(t)\le v(t)$ for all $0\le t<\infty$. Let $\rho=(b\Gamma(\beta))^{\frac{1}{\beta}}$. Notice that

$$([\beta n])!\le\Gamma(\beta n),\quad t^{\beta n}\le\begin{cases}t^{[\beta n]}, & 0\le t<1,\\ t^{[\beta n]+1}, & t\ge 1.\end{cases}$$

If $\rho t\ge 1$, then

$$a\sum_{n=0}^{\infty}\frac{(\rho t)^{\beta n}}{\Gamma(\beta n)}\le a\sum_{n=0}^{\infty}\frac{(\rho t)^{[\beta n]+1}}{([\beta n])!}\le\left(\left[\frac{1}{\beta}\right]+1\right)a\rho t\sum_{k=0}^{\infty}\frac{(\rho t)^k}{k!}\le\left(\left[\frac{1}{\beta}\right]+1\right)a\rho te^{\rho t},$$

and thus,

$$u(t)\le\left(\left[\frac{1}{\beta}\right]+1\right)ae^{2\rho t}.$$

If $0\le\rho t<1$, then

$$a\sum_{n=0}^{\infty}\frac{(\rho t)^{\beta n}}{\Gamma(\beta n)}\le a\sum_{n=0}^{\infty}\frac{(\rho t)^{[\beta n]}}{([\beta n])!}\le\left(\left[\frac{1}{\beta}\right]+1\right)a\sum_{k=0}^{\infty}\frac{(\rho t)^k}{k!}\le\left(\left[\frac{1}{\beta}\right]+1\right)ae^{\rho t},$$

and thus,

$$u(t)\le\left(\left[\frac{1}{\beta}\right]+1\right)ae^{2\rho t}.$$

Therefore for $0\le t<\infty$,

$$u(t)\le\left(\left[\frac{1}{\beta}\right]+1\right)ae^{2\rho t}.$$

■

The proof of Theorem 4.9 adapts the proof of Theorem 11.12 in Smoller [103], page 121-122 and Theorem 5.1 in Muldowney [80].

Proof of Theorem 4.9:

Proof. Suppose that $\lim_{t\to+\infty}u(t,u_0)=u^*$ and u^* is a stable hyperbolic steady state solution. Then the spectrum of $L=A-\frac{\partial f}{\partial u}(u^*)$ lies in $\{\text{Re }\lambda>\beta\}$ for some $\beta>0$. If $0<\beta<\beta_1<\text{Re }\sigma(L)$, there exists $M_1\ge 1$ such that for $t>0$, $v\in X^{\alpha}$,

$$\begin{aligned}\|e^{-Lt}v\|_{\alpha}&\le M_1e^{-\beta_1 t}\|v\|_{\alpha},\\ \|e^{-Lt}v\|_{\alpha}&\le M_1t^{-\alpha}e^{-\beta_1 t}\|v\|.\end{aligned}\tag{4.10}$$

In fact, $dS(t)_{u^*} = e^{-Lt}$. Since f is continuous differentiable from X^α in to X, choose $T > 0$ and an open set $N = \{u \in X^\alpha : \|u - u^*\|_\alpha < r\}$ with the property that if $u_1, u_2 \in N$ and $u(t, u_1), u(t, u_2)$ are solutions of (4.2), then there exists $M_2 \geq 1$ such that for $0 \leq s \leq T$,

$$\begin{aligned} & \left\| f(u(s,u_1)) - f(u(s,u_2)) - \frac{\partial f}{\partial u}(u^*)u(s,u_1) + \frac{\partial f}{\partial u}(u^*)u(s,u_2) \right\| \\ \leq \; & M_2 \|u(s,u_1) - u(s,u_2)\|_\alpha. \end{aligned}$$

Rewrite (4.2) as

$$\frac{du}{dt} + \left(A - \frac{\partial f}{\partial u}(u^*)\right) u = f(u) - \frac{\partial f}{\partial u}(u^*)u.$$

Then any solution $u(t, u_0)$ of (4.2) satisfies

$$u(t,u_0) = e^{-Lt}u_0 + \int_0^t e^{-L(t-s)} \left(f(u(s,u_0)) - \frac{\partial f}{\partial u}(u^*)u(s,u_0) \right) ds.$$

Thus,

$$\begin{aligned} & \|u(t,u_1) - u(t,u_2)\|_\alpha \\ \leq \; & \|e^{-Lt}(u_1 - u_2)\|_\alpha \\ & + \int_0^t \left\| e^{-L(t-s)} \left(f(u(s,u_1)) - f(u(s,u_2)) - \tfrac{\partial f}{\partial u}(u^*)u(s,u_1) + \tfrac{\partial f}{\partial u}(u^*)u(s,u_2) \right) \right\|_\alpha ds \\ \leq \; & M_1 e^{-\beta_1 t} \|u_1 - u_2\|_\alpha \\ & + \int_0^t M_1 (t-s)^{-\alpha} e^{-\beta_1(t-s)} \cdot \\ & \quad \cdot \left\| f(u(s,u_1)) - f(u(s,u_2) - \tfrac{\partial f}{\partial u}(u^*)u(s,u_1) + \tfrac{\partial f}{\partial u}(u^*)u(s,u_2) \right\| ds \\ \leq \; & M_1 e^{-\beta_1 t} \|u_1 - u_2\|_\alpha + \int_0^t M_1 (t-s)^{-\alpha} e^{-\beta_1(t-s)} M_2 \|u(s,u_1) - u(s,u_2)\|_\alpha, \end{aligned}$$

Lemma 4.10 implies that there exist constants $r, M, \beta_2 > 0$ such that if $\|u_i - u^*\|_\alpha < r,\ i = 1, 2$, then

$$\|u(t,u_1) - u(t,u_2)\|_\alpha \leq M e^{-\beta_2 t} \|u_1 - u_2\|_\alpha, \quad \text{for } 0 \leq t \leq T. \tag{4.11}$$

If $\|u_1 - u^*\|_\alpha < \frac{r}{2}$ and $\|u_2 - u_1\|_\alpha < \frac{r}{2}$, then $\|u_2 - u^*\|_\alpha < r$. Choose $0 < \gamma < \beta_2 (< \beta_1)$, so that $Me^{-\beta_2 T} < e^{-\gamma T}$. For any $t > 0$, there exists a nonnegative integer n_1 such that $n_1T \le t < (n_1+1)T$. Let $t_1 = t - n_1T$. Then $0 \le t_1 < T$. Notice that

$$\begin{aligned}
\|u(nT, u_1) - u^*\|_\alpha &\le Me^{-\beta_2 T}\|u((n-1)T, u_1) - u^*\|_\alpha \\
&\le \|u((n-1)T, u_1 - u^*\|_\alpha \\
&\le \cdots \le \|u_1 - u^*\|_\alpha < \tfrac{r}{2}, \\
\|u(nT, u_2) - u^*\|_\alpha &\le Me^{-\beta_2 T}\|u((n-1)T, u_2) - u^*\|_\alpha \\
&\le \|u((n-1)T, u_2) - u^*\|_\alpha \\
&\le \cdots \le \|u_2 - u^*\|_\alpha < r.
\end{aligned}$$

Thus from (4.11),

$$\begin{aligned}
&\|u(t, u_1) - u(t, u_2)\|_\alpha \\
= {} & \|u(t_1 + n_1T, u_1) - u(t_1 + n_1T_1, u_2)\|_\alpha \\
\le {} & Me^{-\beta_2 t_1}\|u(n_1T, u_1) - u(n_1T, u_2)\|_\alpha \\
\le {} & Me^{-\beta_2 t_1}M^{n_1}e^{-n_1\beta_2 T}\|u_1 - u_2\|_\alpha \\
\le {} & Me^{-\gamma(t-n_1T)}e^{-n_1\gamma T}\|u_1 - u_2\|_\alpha = Me^{-\gamma t}\|u_1 - u_2\|_\alpha.
\end{aligned}$$

Hence if $\|u_1 - u^*\|_\alpha < \frac{r}{2}$ and $\|u_2 - u_1\|_\alpha < \frac{r}{2}$, then

$$\|u(t, u_1) - u(t, u_2)\|_\alpha \le Me^{-\gamma t}\|u_1 - u_2\|_\alpha, \quad \text{for } t \ge 0. \tag{4.12}$$

This implies that there exists $r_1 > 0$ such that if $u_1 \in X^\alpha$, $\|u_1 - u^*\|_\alpha < r_1$, then $v(t, v_0) = dS(t)_{u_1}v_0$ solves the linear variational equation of (4.2) at $u(t, u_1)$,

$$\begin{aligned}
&\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u(t, u_1))v, \\
&v(0) = v_0,
\end{aligned}$$

where $v_0 \in X^\alpha$ and

$$\|dS(t)_{u_1}\|_\alpha \le K_1 e^{-\gamma_1 t}, \quad \text{for } t \ge 0, \tag{4.13}$$

for some constants K_1 and $\gamma_1 > 0$. Notice that if $\lim_{t\to+\infty} u(t, u_0) = u^*$, then there exists a constant $T_1 > 0$ such that for $t \ge T_1$, $\|u(t, u_0) - u^*\|_\alpha < r_1$. Inequality (4.13) implies that if $u \in \{u(t, u_0) : t \ge T_1\}$,

$$\|dS(t)_u\|_\alpha \le K_1 e^{-\gamma_1 t}, \quad \text{for } t \ge 0.$$

Since the orbit $C_+(u_0)$ is relatively compact, Lemma 4.4 and Lemma 4.6 imply that there exist constants $K_2, \gamma_3 > 0$ such that if $u \in \{u(t,u_0) : 0 \le t \le T_1\}$,

$$\|dS(t)_u\|_\alpha \le K_2 e^{-\gamma_2 t}, \quad \text{for } t \in [0, T_1].$$

Let $K = \max\{K_1, K_2\}$ and $\sigma = \max\{\gamma_1, \gamma_2\}$. Then if $u \in C_+(u_0) = \{u(t,u_0) : t \ge 0\}$,

$$\|dS(t)_u\|_\alpha \le K e^{-\sigma t}, \quad \text{for } t \ge 0.$$

Therefore, the linear variational equation (4.7) of (4.2) at $u(t,u_0)$ is uniformly asymptotically stable.

Conversely, suppose that the linear variational equation (4.7) of (4.2) at $u(t,u_0)$ is uniformly asymptotically stable: there exist constants $K, \sigma > 0$ such that if $u \in C_+(u_0) = \{u(t,u_0) : t \ge 0\}$,

$$\|dS(t)_u\|_\alpha \le K e^{-\sigma t}, \quad \text{for } t \ge 0. \tag{4.14}$$

If $u(t,u_0)$ is a solution of (4.2), then

$$\frac{d^2u}{dt^2} = \frac{d}{dt}(-Au + f(u)) = -A\frac{du}{dt} + \frac{\partial f}{\partial u}\frac{du}{dt}.$$

Thus $\frac{du}{dt}(t,u_0)$ is a solution of the linear variational equation (4.7) and $\frac{du}{dt}(t,u_0) = dS(t)_{u_0}\frac{du}{dt}(0,u_0)$. Inequality (4.14) implies that $\lim_{t\to+\infty} \frac{du}{dt}(t,u_0) = 0$ and thus

$$\lim_{t\to+\infty} -Au(t,u_0) + f(u(t,u_0)) = 0. \tag{4.15}$$

Since $C_+(u_0)$ is relatively compact in X^α, $\omega(u_0)$ is nonempty and connected. If $u^* \in \omega(u_0)$, then there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to+\infty} u(t_n,u_0) = u^*$. From (4.15),

$$\lim_{n\to+\infty} -Au(t_n,u_0) + f(u(t_n,u_0)) = 0.$$

Since f is continuous and A is closed on X^α when $0 < \alpha < 1$,

$$\lim_{n\to+\infty} -Au(t_n,u_0) + f(u(t_n,u_0)) = -Au^* + f(u^*).$$

Thus u^* is a steady state solution of (4.2). From Lemma 4.4 and Lemma 4.6, inequality (4.14) is satisfied with $u = u^*$ which means that u^* is a stable hyperbolic steady state solution. Hence there exists an open neighbourhood

N of u^* such that any solution $u(t, u_1)$ of (4.2) with initial condition $u_1 \in N$ stays in N when t is large enough. Consequently, u^* is the unique steady state solution in N since the omega limit set $\omega(u_0)$ is connected.

■

4.2 Existence of Stable Periodic Solutions

In this section, conditions for an omega limit set to be a stable hyperbolic periodic solution will be studied. A result of Li and Muldowney [55] for n-dimensional ordinary differential equation is the following. Consider an autonomous differential equation

$$\frac{du}{dt} = f(u), \quad u(\cdot) \in \mathbb{R}^n, \tag{4.16}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. This defines a semiflow $(t, u_0) \mapsto \phi(t, u_0)$ on the set $\mathbb{R} \times \mathbb{R}^n$ such that the solution $u(t, u_0) = \phi(t, u_0)$ satisfies $u(0, u_0) = u_0$.

The linear variational equation of (4.16) at a solution $u(t, u_0)$ is

$$\frac{dv}{dt} = \frac{\partial f}{\partial u}(u(t, u_0))v. \tag{4.17}$$

The second compound differential equation of (4.17) is

$$\frac{dw}{dt} = \frac{\partial f}{\partial u}^{[2]}(u(t, u_0))w. \tag{4.18}$$

Theorem 4.11. *Suppose that $u(t, u_0)$ is a bounded solution of (4.16) whose omega limit set $\omega(u_0)$ contains no equilibrium. Then $\omega(u_0)$ is a stable hyperbolic periodic orbit if and only if the second compound differential equation (4.18) of (4.17) is uniformly asymptotically stable.*

A generalization of the sufficiency statement in Theorem 4.11 for a differential equation in a Banach space X is Theorem 4.12.

Theorem 4.12. *Let A be a sectorial operator in a Banach space X and f be continuously differentiable from X^α into X and $0 \leq \alpha < 1$. Suppose that $u(t, u_0)$ is a solution of*

$$\begin{aligned} &\frac{du}{dt} + Au = f(u), \\ &u(0) = u_0. \end{aligned} \tag{4.19}$$

such that
(i) $C_+(u_0) = \{u(t, u_0) : t \geq 0\}$ *is relatively compact in* X^α*;*
(ii) $v^1(t) = \frac{du}{dt}(t, u_0)$ *is uniformly bounded in* X^α *with respect to all* $t \geq 0$;.
(iii) the omega limit set $\omega(u_0)$ *of* $u(t, u_0)$ *contains no steady state solution;*
(iv) the second compound differential equation

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(u(t, u_0))w, \tag{4.20}$$

is uniformly asymptotically stable in $\bigwedge^2 X^\alpha$.
Then $\omega(u_0)$ *is a stable hyperbolic periodic solution in* X^α.

The proof of Theorem 4.12 will be discussed in Section 4.2.1. The key consequence of the asymptotic stability assumption (iv) is that $v^1(t)$, which is tangent to the periodic orbit and a solution of the linear variational equation

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u(t, u_0))v \tag{4.21}$$

satisfies

$$\|(v^1 \wedge v)(t)\| \leq K\|(v^1 \wedge v)(s)\|e^{-\gamma(t-s)}, \quad 0 \leq s \leq t, \tag{4.22}$$

for some constants $K, \gamma > 0$. The exponential decay of $v^1 \wedge v$ is deduced from the uniform asymptotic stability of the second compound differential equation (4.20). This means that, infinitesimally, the area of a parallelogram with one side tangent to the periodic orbit decays exponentially in the dynamics.

The remainder of this section will be devoted to discussion of an application of Theorem 4.12. Consider a reaction diffusion system

$$\frac{\partial u}{\partial t} = D\Delta u + f(x, u), \qquad x \in \Omega, \tag{4.23}$$

with Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \tag{4.24}$$

or Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.25}$$

or a more general boundary condition of the form

$$Bu|_{\partial\Omega} = 0, \tag{4.26}$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous in u, uniformly in x and $D = \text{diag}(d_1, \cdots, d_n)$, $d_i \geq 0$. Let $X = L^2(\Omega, \mathbb{R}^n)$ and $A : \mathcal{D}(A) \subset X \to X$ be the linear unbounded operator $Au = -\Delta u$, where

$$\mathcal{D}(A) = \left\{ u \in H^2(\Omega, \mathbb{R}^n),\ Bu|_{\partial\Omega} = 0 \right\}.$$

The initial value problem of (4.23) is well-posed in X^α. Two representations of the second compound differential equation of (4.23) are (3.15) and (3.17) in Section 3.2 where the coefficient $a_{jk}(t,x) = \frac{\partial f_j}{\partial u_k}(x, u(t,x))$. Theorem 4.12 gives a sufficient condition for the ω-limit set of a bounded solution $u(t, u_0)$ to be a periodic orbit in X^α.

Example 4.1. Consider a special case of (4.23) with $n = 2$ and $\Omega = (0, L)$:

$$\begin{aligned} &u_t = d_1 u_{xx} + f(x,u,v), \quad v_t = d_2 v_{xx} + g(x,u,v), \qquad 0 < x < L, \quad t > 0, \\ &u(t,0) = u(t,L) = 0, \quad v(t,0) = v(t,L) = 0, \quad t \geq 0, \end{aligned} \tag{4.27}$$

where $f, g \in C^2((0,L) \times \mathbb{R}^2 \to \mathbb{R}^2)$. Suppose that $(u(t,x), v(t,x))$ is a bounded solution of (4.27) in $X = H_0^1(0,L) \times H_0^1(0,L)$. Then $\left(\frac{\partial u}{\partial t}(t,x), \frac{\partial v}{\partial t}(t,x)\right) \in X$ (see Henry [42], Theorem 3.5.2, page 71). A pointwise representation of the second compound differential equation of the linear variation equation at any solution $(u(t,x), v(t,x))$ is

$$\begin{aligned} \frac{\partial w_{11}}{\partial t} &= d_1(\Delta_1 w_{11} + \Delta_2 w_{11}) + (a_{11}(t,x_1) + a_{11}(t,x_2)) w_{11} \\ &\quad + a_{12}(t,x_1) w_{21} + a_{12}(t,x_2) w_{12}, \\ \frac{\partial w_{22}}{\partial t} &= d_2(\Delta_1 w_{22} + \Delta_2 w_{22}) + (a_{22}(t,x_1) + a_{22}(t,x_2)) w_{22} \\ &\quad + a_{21}(t,x_1) w_{12} + a_{21}(t,x_2) w_{21}, \\ \frac{\partial w_{12}}{\partial t} &= d_1 \Delta_1 w_{12} + d_2 \Delta_2 w_{12} + (a_{11}(t,x_1) + a_{22}(t,x_2)) w_{12} \\ &\quad + a_{12}(t,x_1) w_{22} + a_{21}(t,x_2) w_{11}, \end{aligned} \tag{4.28}$$

where

$$\begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{21}(t,x) & a_{22}(t,x) \end{bmatrix} := \begin{bmatrix} \dfrac{\partial f}{\partial u}(x, u(t,x), v(t,x)) & \dfrac{\partial f}{\partial v}(x, u(t,x), v(t,x)) \\ \dfrac{\partial g}{\partial u}(x, u(t,x), v(t,x)) & \dfrac{\partial g}{\partial v}(x, u(t,x), v(t,x)) \end{bmatrix}.$$

Let $\lambda_1 \le \lambda_2 \le \cdots$ denote the eigenvalues of the Laplace equation

$$\begin{array}{ll} \Delta u = -\lambda_i u, & \Delta v = -\lambda_i v \\ u(t,0) = u(t,L) = 0, & v(t,0) = v(t,L) = 0. \end{array}$$

Consider a Lyapunov function

$$V(t) = \frac{1}{2!}\int_{(0,L)^2} \left[(w_{11})^2 + (w_{22})^2 + 2(w_{12})^2\right].$$

Then the derivative of V calculated for (3.28) of Section 3.2 implies that for any positive functions $\nu_1(x), \nu_2(x)$,

$$\begin{aligned} \frac{d^+V}{dt} &\le \int_{(0,L)^2} \left[\mu_1(t)(w_{11})^2 + \mu_2(t)(w_{22})^2 + 2\mu_3(t)(w_{12})^2\right] \\ &\le 2\mu(t)V, \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} \mu_1(t) &= -(\lambda_1+\lambda_2)\,d_1 + \max_x\left\{2a_{11}(t,x) + \frac{1}{\nu_1(x)}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\}, \\ \mu_2(t) &= -(\lambda_1+\lambda_2)\,d_2 + \max_x\left\{2a_{22}(t,x) + \frac{1}{\nu_2(x)}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\}, \\ \mu_3(t) &= -\lambda_1(d_1+d_2) + \max_x\left\{a_{11}(t,x) + \frac{\nu_2(x)}{2}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\} \\ &\quad + \max_x\left\{a_{22}(t,x) + \frac{\nu_1(x)}{2}\left|a_{12}(t,x)+a_{21}(t,x)\right|\right\}, \\ \mu(t) &= \max\{\mu_1(t),\mu_2(t),\mu_3(t)\}, \end{aligned} \tag{4.30}$$

and the maximum is taken over $0 < x < L$.

Since $H^1(0,L) \subset L^2(0,L)$ is compact, Theorem 4.12 implies the following corollary.

Corollary 4.13. *Suppose that $n = 2$ and $u(t,u_0)$ is a bounded solution of (4.23) in X and that there exist constants $a, b > 0$ such that*

$$\int_s^t \mu < -a(t-s) + b, \quad 0 \le s \le t < \infty, \tag{4.31}$$

where $\mu(\cdot)$ is defined by (4.30). If the omega limit set $\omega(u_0)$ of $u(t,u_0)$ contains no steady state solution, then it is a stable hyperbolic periodic solution in $L^2(0,L)$.

Remark 4.5. The above procedure also applies to other boundary conditions where λ_i are the eigenvalues of the Laplace equation with correspond boundary conditions.

Remark 4.6. To apply Theorem 4.12, it is not necessary to find an explicit bounded solution $u(t, u_0)$. For example, if S is an invariant bounded set and no steady state solutions exist in S, then there exists a stable hyperbolic periodic solution in S as long as inequality (4.31) is true for an upper bound of $\mu(t)$ in S.

Fiedler and Mallet-Paret [30] proved the classical Poincaré-Bendixson theorem for a scalar reaction diffusion equation

$$\begin{aligned} &u_t = u_{xx} + f(x, u, u_x), \quad 0 < x < 2\pi, \quad t > 0, \\ &u(t,0) = u(t, 2\pi), \quad u_x(t,0) = u_x(t, 2\pi), \quad t \geq 0, \end{aligned} \tag{4.32}$$

where $f \in C^2((0, 2\pi) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$. However, Theorem 4.12 establishes both the existence and stability of periodic solutions. Let $X = \{u \in H^2(0, 2\pi) \; : \; u(0) = u(2\pi), \;\; u_x(0) = u_x(2\pi)\}$. The initial value problem associated to (4.32) generates a local semiflow on X. The linear variational equation of (4.32) at a solution $u = u(t, x)$ is

$$\begin{aligned} &v_t = v_{xx} + a(t,x)v_x + b(t,x)v_x, \quad 0 < x < 2\pi, \quad t > 0, \\ &v(t,0) = v(t, 2\pi), \quad v_x(t,0) = v_x(t, 2\pi), \quad t \geq 0, \end{aligned} \tag{4.33}$$

where

$$a(t,x) = \frac{\partial f}{\partial u_x}(x, u, u_x) = f_{u_x}(x, u, u_x), \quad b(t,x) = \frac{\partial f}{\partial u}(x, u, u_x) = f_u(x, u, u_x).$$

A similar argument to that in Corollary 3.33 of Section 3.4 implies Corollary 4.14.

Corollary 4.14. *Suppose that $u(t, x)$ is a bounded solution in X and that there exist $a, b > 0$ such that $\max_x |a(t, x)| > 0, \; 0 \leq x \leq 2\pi$ and*

$$\int_s^t \max_{0 \leq x \leq 2\pi} \{4b(t,x) + 2|a(t,x)|\nu(t)\} \, dt < (1-a)(t-s) + b;$$

or $a(t,x) \equiv 0$ for $0 \leq x \leq 2\pi$ and

$$2 \int_s^t \max_x b(t,x) < (1-a)(t-s) + b,$$

where $\nu(t) = \max_x |a(t,x)|$. If the omega limit set $\omega(u_0)$ of $u(t,x)$ contains no steady state solution, then it is a stable hyperbolic periodic solution in $L^2(0, L)$.

4.2.1 Proof of Theorem 4.12:

The proof of Theorem 4.12 is adapted from the argument used in the proof of Theorem 4.1 in Li and Muldowney [55]. To prove Theorem 4.12, the following Proposition 4.15 is established to show that, under the assumptions of the theorem, the solution space of the linear variational equation (4.21) is the direct sum of two subspaces X_1 and X_2; the 1-dimensional subspace X_1 is strongly stable and spanned by $v^1(t) = \frac{du}{dt}(t, u_0)$ and the codimension 1 subspace X_2 is uniformly asymptotically stable.

Let X be a normed space and $t \in [0, \infty)$ and X^* be its (continuous) dual space. Let V be a linear space of maps $t \mapsto v(t)$ from $[0, \infty)$ to X. If $x \in X$ and $\psi \in X^*$, define

$$\psi(x) = \langle \psi, x \rangle .$$

For $x^1, x^2 \in X$, define a norm of $x^1 \wedge x^2$ as follows (see Section 1.1):

$$\|x^1 \wedge x^2\|_{\bigwedge^2 X} = \sup_{\psi_i} \left| \det \begin{bmatrix} \langle \psi_1, x^1 \rangle & \langle \psi_1, x^2 \rangle \\ \langle \psi_2, x^1 \rangle & \langle \psi_2, x^2 \rangle \end{bmatrix} \right| . \tag{4.34}$$

Here the supreme is taken over $\psi_i \in X'$, $\|\psi_i\|_{X'} \leq 1$, $i = 1, 2$. For simplicity, the symbol $\|\cdot\|$ will be used instead of $\|\cdot\|_{\bigwedge^k X}$ except when the relationship with the norm in X is to be emphasized.

Proposition 4.15. *Suppose that assumptions (i), (ii), (iii) are satisfied:*

(i) there exists a constant $L > 1$ and a nonzero $v^1 \in V$ such that

$$\|v^1(t)\| \leq L\|v^1(s)\|, \quad 0 \leq s, t;$$

(ii) there exist constants $K, \gamma > 0$ such that for every $v \in V$,

$$\|(v^1 \wedge v)(t)\| \leq K\|(v^1 \wedge v)(s)\| e^{-\gamma(t-s)}, \quad 0 \leq s \leq t;$$

(iii) there exist constants $H > 0$ and $\beta \geq 0$ such that for every $v \in V$,

$$\|v(t)\| \leq H\|v(s)\| e^{\beta(t-s)}, \quad 0 \leq s \leq t.$$

Then there exists a constant $C > 0$ such that

$$V = V_1 \oplus V_2$$

where

$$V_1 = span\{v^1\}, \quad V_2 = \left\{y \in V : \; \|v(t)\| \le C\|y(s)\|e^{-\gamma(t-s)}, \; 0 \le s \le t\right\}.$$

Remark 4.7. When $X = \mathbb{R}^n$, Proposition 4.15 is proved by Li and Muldowney [55].

Proposition 4.15 will be proved by the following sequence of lemmas.

Lemma 4.16. *For $x^1, x^2 \in X$,*

$$\eta\|x^1\|\|x^2\| \le \|x^1 \wedge x^2\| \le 2\|x^1\|\|x^2\|,$$

where

$$\eta = \min_{-1 \le \nu \le 1} \left\{ \left\| \frac{x^1}{\|x^1\|} - \nu \frac{x^2}{\|x^2\|} \right\| \right\}.$$

Proof. For $x^i \in X$ and $\psi_i \in X'$, $\|\psi_i\| \le 1$, $i = 1, 2$,

$$\begin{aligned} & \left| \det \begin{bmatrix} \langle \psi_1, x^1 \rangle & \langle \psi_1, x^2 \rangle \\ \langle \psi_2, x^1 \rangle & \langle \psi_2, x^2 \rangle \end{bmatrix} \right| \\ \le \; & | \langle \psi_1, x^1 \rangle \cdot \langle \psi_2, x^2 \rangle | + | \langle \psi_1, x^2 \rangle \cdot \langle \psi_2, x^1 \rangle | \\ \le \; & 2\|x^1\|\|x^2\|. \end{aligned}$$

Thus

$$\|x^1 \wedge x^2\| \le 2\|x^1\|\|x^2\|.$$

Let $(x^1)^* \in X'$ such that $\|x\| = \langle x^*, x \rangle$, $\|x^*\|_{X'} = 1$. Then

$$\begin{aligned} \|x^1 \wedge x^2\| & \ge \sup_{\psi} | \; \|x^1\| \cdot \langle \psi, x^2 \rangle - \langle (x^1)^*, x^2 \rangle \cdot \langle \psi, x^1 \rangle \; | \\ & \ge \|x^1\|\|x^2\| \sup_{\psi} \left| \frac{\langle \psi, x^2 \rangle}{\|x^2\|} - \frac{\langle \psi, x^1 \rangle}{\|x^1\|} \cdot \frac{\langle (x^1)^*, x^2 \rangle}{\|x^2\|} \right| \\ & = \|x^1\|\|x^2\| \sup_{\psi} \left| \left\langle \psi, \frac{x^2}{\|x^2\|} - \nu \frac{x^1}{\|x^1\|} \right\rangle \right| \\ & = \|x^1\|\|x^2\| \left\| \frac{x^2}{\|x^2\|} - \nu \frac{x^1}{\|x^1\|} \right\|, \end{aligned}$$

where $\nu = \frac{\langle (x^1)^*, x^2 \rangle}{\|x^2\|}$ and $-1 \le \nu \le 1$. Therefore,

$$\|x^1 \wedge x^2\| \ge \eta \|x^1\| \|x^2\|,$$

where

$$\eta = \min_{-1 \le \nu \le 1} \left\{ \left\| \frac{x^1}{\|x^1\|} - \nu \frac{x^2}{\|x^2\|} \right\| \right\}.$$

■

Remark 4.8. The number η in Lemma 4.16 is a measure of the angle between x^1 and x^2, see Figure 4.1. When X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$,

$$\|x^1 \wedge x^2\|^2 = \det \begin{vmatrix} \langle x^1, x^1 \rangle & \langle x^1, x^2 \rangle \\ \langle x^2, x^1 \rangle & \langle x^2, x^2 \rangle \end{vmatrix} = \sin^2 \theta \|x^1\|^2 \|x^2\|^2,$$

where θ is the angle of the vector x^1, x^2 and

$$\cos \theta = \frac{\langle x^1, x^2 \rangle}{\|x^1\| \|x^2\|}.$$

Thus

$$\|x^1 \wedge x^2\|^2 = \eta^2 \|x^1\|^2 \|x^2\|^2,$$

where $\eta = \left\| \frac{x^1}{\|x^1\|} - \nu \frac{x^2}{\|x^2\|} \right\|$ with $\nu = \frac{\langle x^1, x^2 \rangle}{\|x^1\|}$.

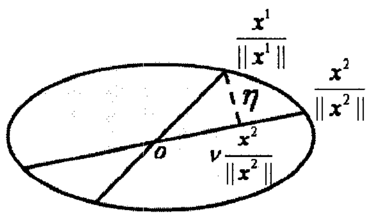

Figure 4.1: A measure of the angle between x^1 and x^2

Lemma 4.17 is from Daleckiĭ and Kreĭn [23], page 156.

Lemma 4.17. *Suppose that the normed space X decomposes into a direct sum $X = X_1 \oplus X_2$ of closed subspaces and $P_1, P_2 = I - P_1$ are the corresponding projections. Then*

$$\inf_{x_i \in X_i} \left\{ \left\| \frac{x^1}{\|x^1\|} - \frac{x^2}{\|x^2\|} \right\| \right\} \le \frac{2}{\|P_i\|}, \quad i = 1, 2.$$

Proof. Suppose that $x^i \in X_i, i = 1,2$. Let $x = x^1 + x^2$ where $x^i = P_i x, i = 1,2$. Then

$$\begin{aligned}
\left\| \frac{x^1}{\|x^1\|} - \frac{x^2}{\|x^2\|} \right\| &= \left\| \frac{P_1 x}{\|P_1 x\|} - \frac{P_2 x}{\|P_2 x\|} \right\| \\
&= \frac{1}{\|P_1 x\|} \left\| P_1 x - \frac{\|P_1 x\|}{\|P_2 x\|} P_2 x \right\| \\
&= \frac{1}{\|P_1 x\|} \left\| x - \frac{\|P_1 x\| - \|P_2 x\|}{\|P_2 x\|} P_2 x \right\| \\
&\leq \frac{1}{\|P_1 x\|} \left(\|x\| + \frac{\|P_1 x + P_2 x\|}{\|P_2 x\|} \|P_2 x\| \right) \\
&= \frac{2\|x\|}{\|P_1 x\|},
\end{aligned}$$

and thus

$$\inf_{x_i \in X_i} \left\{ \left\| \frac{x^1}{\|x^1\|} - \frac{x^2}{\|x^2\|} \right\| \right\} \leq 2 \inf_{x \in X} \frac{\|x\|}{\|P_1 x\|} = \frac{2}{\|P_1\|}.$$

Similarly,

$$\inf_{x_i \in X_i} \left\{ \left\| \frac{x^1}{\|x^1\|} - \frac{x^2}{\|x^2\|} \right\| \right\} \leq \frac{2}{\|P_2\|}.$$

■

Lemma 4.18 is Corollary 2.3 proved in Section 2.1.2.

Lemma 4.18. *Suppose that for every $v \in V$, the following two conditions are satisfied:*

(i) $\limsup_{t\to\infty} \|v(t)\| < \infty$;

(ii) $\liminf_{t\to\infty} \|v(t)\| = 0$ *implies* $\lim_{t\to\infty} \|v(t)\| = 0$.

Then

$$\operatorname{codim}\left\{ v \in V : \lim_{t\to\infty} \|v(t)\| = 0 \right\} < 2$$

if and only if, for all $v^1, v^2 \in V$,

$$\lim_{t\to\infty} \left\|(v^1 \wedge v^2)(t)\right\| = 0.$$

The proof of Proposition 4.15 is adapted from Li and Muldowney [55].

Proof of Proposition 4.15:

Proof. The proof of Proposition 4.15 will be divided into two parts according to the value of β in assumption (iii).

Case 1: $\beta = 0$ in assumption (iii)

Let $V_0 = \{v \in V : \lim_{t\to\infty} \|v(t)\| = 0\}$. For any $v \in V_0$ and $t \geq 0$, there exists a T such that $\|v(t+T)\| \leq \frac{1}{2L}\|v(t)\|$. It follows from assumption (i) that for any $-1 \leq \nu \leq 1$,

$$\frac{1}{2L} = \frac{1}{L} - \frac{1}{2L} \leq \left\| \frac{v^1(t+T)}{\|v^1(t)\|} - \nu \frac{v(t+T)}{\|v(t)\|} \right\| \leq H \left\| \frac{v^1(t)}{\|v^1(t)\|} - \nu \frac{v(t)}{\|v(t)\|} \right\|,$$

or

$$\frac{1}{2HL} \leq \left\| \frac{v^1(t)}{\|v^1(t)\|} - \nu \frac{v(t)}{\|v(t)\|} \right\|.$$

Thus, Lemma 4.16 implies that

$$\begin{aligned} \frac{1}{2HL}\|v^1(t)\|\|v(t)\| &\leq \|(v^1 \wedge v)(t)\| \leq K\|(v^1 \wedge v)(s)\|e^{-\gamma(t-s)} \\ &\leq 2K\|v^1(s)\|\|v(s)\|e^{-\gamma(t-s)}. \end{aligned}$$

It follows from assumption (i) again that

$$\frac{1}{2HL}\|v(t)\| \leq 2KL\|v(s)\|e^{-\gamma(t-s)},$$

or

$$\|v(t)\| \leq 4HKL^2\|v(s)\|e^{-\gamma(t-s)}.$$

Therefore,

$$V_0 = \{v \in V : \|v(t)\| \leq 4HKL^2\|v(s)\|e^{-\gamma(t-s)}, \text{ for all } 0 \leq s \leq t\}.$$

Now under assumptions (ii) and (iii), Lemma 4.18 implies that for every $v \in V$, there exists a nontrivial $v_0 = c_1 v^1 + c_2 v$ such that

$$\lim_{t\to\infty} \|v_0(t)\| = 0.$$

From assumption (i), $c_2 \neq 0$ and thus $v = -\frac{c_1}{c_2}v^1 + \frac{1}{c_2}v_0$. Therefore $V = \text{span}\{v^1\} \bigoplus V_0$.

Case 2: $\beta > 0$ in assumption (iii)

In the following, it will be shown that assumptions (i)-(iii) imply that there exists a constant $H' > 0$ such that

$$\|v(t)\| \leq H'\|v(s)\|, \ 0 \leq s \leq t,$$

and thus Proposition 4.15 follows from **Case 1**.

For each $t \geq 0$, let $(v^1)^*(t) \in X'$ be such that $\|v^1(t)\| = \langle (v^1)^*(t), v^1(t)\rangle$ and $\|(v^1)^*(t)\|_{X'} = 1$. It follows from assumption (i) that $\|v^1(t)\| \neq 0$. For each $v \in V$, let

$$\mathcal{V} = \text{span}\{v^1, v\}.$$

Given $0 < \sigma < \tau$, define

$$v^2(t) = -\frac{\langle (v^1)^*(\tau), v(\tau)\rangle}{\|v^1(\tau)\|} v^1(t) + v(t). \tag{4.35}$$

Then $v_2 \in \mathcal{V}$ and $\langle (v^1)^*(\tau), v^2(\tau)\rangle = 0$. For each $t \geq 0$, let $(v^2)^*(t) \in X'$ be such that $\|v^2(t)\| = \langle (v^2)^*(t), v^2(t)\rangle$ and $\|(v^2)^*(t)\|_{X'} = 1$. Then

$$\begin{aligned} \|v^1(\tau)\|\|v^2(\tau)\| &= \left|\det\begin{bmatrix} \langle (v^1)^*, v^1\rangle & \langle (v^1)^*, v^2\rangle \\ \langle (v^2)^*, v^1\rangle & \langle (v^2)^*, v^2\rangle \end{bmatrix}(\tau)\right| \\ &\leq \|(v^1 \wedge v^2)(\tau)\| \\ &\leq K\|(v^1 \wedge v^2)(\sigma)\|e^{-\gamma(\tau-\sigma)} \\ &\leq 2K\|v^1(\sigma)\|\|v^2(\sigma)\|e^{-\gamma(\tau-\sigma)}. \end{aligned}$$

Thus assumption (i) implies that

$$\begin{aligned} \|v^2(\tau)\| &\leq 2K\frac{\|v^1(\sigma)\|}{\|v^1(\tau)\|}\|v^2(\sigma)\|e^{-\gamma(\tau-\sigma)} \\ &\leq 2KL\|v^2(\sigma)\|e^{-\gamma(\tau-\sigma)}. \end{aligned}$$

Therefore, every $v \in V$ can be written as

$$v(t) = \frac{\langle (v^1)^*(\tau), v(\tau)\rangle}{\|v^1(\tau)\|} v^1(t) + v^2(t), \tag{4.36}$$

where $v^2 \in \mathcal{V}$ and $\|v^2(\tau)\| \leq 2KL\|v^2(\sigma)\|e^{-\gamma(\tau-\sigma)}$. Choose $T > 0$ such that

$$\delta := 2KLe^{-\gamma T} < \frac{1}{L} < 1. \tag{4.37}$$

Let $s_0 \geq 0$ and define $s_k = s_0 + kT,\ k = 0, 1, 2, \cdots$. Then every $v \in V$ can be written as

$$v(t) = c_k v^1(t) + v^{2,k}(t), \qquad k = 0, 1, 2, \cdots, \tag{4.38}$$

where $v^{2,k} \in \mathcal{V}$ and

$$c_k = \frac{\langle (v^1)^*(s_{k+1}), v(s_{k+1})\rangle}{\|v^1(s_{k+1})\|}, \quad \langle (v^1)^*(s_{k+1}), v^{2,k}(s_{k+1})\rangle = 0,$$

$$\|v^{2,k}(s_{k+1})\| \leq \delta \|v^{2,k}(s_k)\|.$$

Let

$$\mathcal{V}_1 = \text{span}\{v^1\}, \quad \mathcal{V}_{2,k} = \{v \in \mathcal{V} : \|v(s_{k+1})\| \leq \delta \|v(s_k)\|\}.$$

Then (4.38) implies that on the interval $[s_k, s_{k+1}]$, $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_{2,k}$ and $\mathcal{V}_1, \mathcal{V}_{2,k}$ are closed subspaces of $\mathcal{V}$. Notice that if $v^2 \in \mathcal{V}_{2,k}$, then

$$\begin{aligned} He^{\beta T}\left\|\frac{v^1(s_k)}{\|v^1(s_k)\|} - \frac{v^2(s_k)}{\|v^2(s_k)\|}\right\| &\geq \left\|\frac{v^1(s_{k+1})}{\|v^1(s_k)\|} - \frac{v^2(s_{k+1})}{\|v^2(s_k)\|}\right\| \\ &\geq \frac{1}{L} - \delta > 0. \end{aligned}$$

Thus, on the interval $[s_k, s_{k+1}]$, Lemma 4.17 implies that the projections $P_i(\cdot), i = 1, 2$ onto $\mathcal{V}_1$ and $\mathcal{V}_{2,k}$ respectively satisfy

$$P_i(s_k) \leq \mu H e^{\beta T} := \rho,$$

where $\mu = \frac{2L}{1 - L\delta}$. For every $v \in V$, it follows from (4.38) that

$$v(s_k) = c_k v^1(s_k) + v^{2,k}(s_k) = c_{k-1} v^1(s_k) + v^{2,k-1}(s_k).$$

Then

$$\begin{aligned} v^{2,k}(s_k) &= P_2(s_k) v^{2,k-1}(s_k), \\ c_k v^1(s_k) &= c_{k-1} v^1(s_k) + P_1(s_k) v^{2,k-1}(s_k). \end{aligned}$$

Thus, for $i = 1, 2$,

$$\begin{aligned} \|P_i(s_k) v^{2,k-1}(s_k)\| &\leq \rho \|v^{2,k-1}(s_k)\| \\ &\leq \rho\delta \|v^{2,k-1}(s_{k-1})\| := R\|v^{2,k-1}(s_{k-1})\| \\ &\leq \cdots \leq R^k \|v^{2,0}(s_0)\| \\ &\leq \rho R^k \|v(s_0)\|, \end{aligned}$$

since $\|v^{2,0}(s_0)\| = \|P_2(s_0)v(s_0)\| \leq \rho\|v(s_0)\|$, where

$$R = \rho\delta = 2KL\mu He^{(\beta-\gamma)T}. \tag{4.39}$$

Hence

$$\begin{aligned} |c_k|\|v^1(s_k)\| &\leq |c_{k-1}|\|v^1(s_k)\| + \|P_1(s_k)v^{2,k-1}(s_k)\| \\ &\leq |c_{k-1}|\|v^1(s_k)\| + \rho R^k\|v(s_0)\|, \end{aligned}$$

or

$$\begin{aligned} |c_k| &\leq |c_{k-1}| + \rho\|v(s_0)\|\frac{R^k}{\|v^1(s_k)\|}, \\ &\leq \cdots \leq |c_0| + \rho\|v(s_0)\|\sum_{i=1}^{k}\frac{R^i}{\|v^1(s_i)\|}, \\ &\leq \rho\frac{\|v(s_0)\|}{\|v^1(s_0)\|} + \rho\|v(s_0)\|\sum_{i=1}^{k}\frac{R^i}{\|v^1(s_i)\|}, \\ &\leq \rho\|v(s_0)\|\sum_{i=0}^{k}\frac{R^i}{m}. \end{aligned} \tag{4.40}$$

since $|c_0| = \frac{\|P_1(s_0)v(s_0)\|}{\|v^1(s_0)\|} \leq \rho\frac{\|v(s_0)\|}{\|v^1(s_0)\|}$, where $m := \frac{\|v^1(s_0)\|}{L} \leq \|v^1(t)\|$ for all $t \geq s_0$.

Case 2.1: $\beta < \gamma$

It can be assumed that $R < 1$ in addition to (4.37) by choosing T sufficiently large if necessary. Now inequality (4.40) implies

$$|c_k| \leq \frac{\rho}{m(1-R)}\|v(s_0)\|,$$

and thus

$$\begin{aligned} \|v(s_k)\| &\leq |c_k|\|v^1(s_k)\| + \|v^{2,k}(s_k)\| \\ &\leq \frac{\rho}{m(1-R)}\|y(s_0)\|\|v^1(s_k)\| + \rho R^k\|v(s_0)\| \\ &\leq \rho\left(\frac{M}{m(1-R)} + 1\right)\|v(s_0)\|, \end{aligned}$$

where $M := L\|v^1(s_0\|) \geq \|v^1(t)\|$ for all $t \geq s_0$. It follows from assumption (iii) that for every $v \in V$,

$$\|v(t)\| \leq He^{\beta T}\|v(s_k)\| \quad \text{for } s_k \leq t < s_{k+1},$$

and thus

$$\|v(t)\| \leq H'\|v(s_0)\|, \quad 0 \leq s_0 \leq t,$$

where $H' = He^{\beta T}\rho\left(\frac{M}{m(1-R)}+1\right)$ which is independent of s_0.

Case 2.2: $\beta \geq \gamma$

The case $\beta = \gamma$ can be included in the case $\beta > \gamma$ by replacing γ by a slightly smaller constant in assumption (*ii*) or β by a slightly larger constant in assumption (*iii*). Thus, without loss of generality, only the case $\beta > \gamma$ will be discussed.

Step 1: Inequality (4.40) implies

$$|c_k| \leq \rho\frac{R^k - 1}{m(R-1)}\|v(s_0)\|,$$

and thus

$$\begin{aligned}\|v(s_k)\| &\leq |c_k|\|v^1(s_k)\| + \|v^{2,k}(s_k)\| \\ &\leq \rho\frac{R^k-1}{m(R-1)}\|v(s_0)\|\|v^1(s_k)\| + \rho R^k\|v(s_0)\| \\ &\leq \rho\left(\frac{M}{m(R-1)}+1\right)R^k\|v(s_0)\|.\end{aligned}$$

Let $0 < \gamma_1 < \gamma$. Choose $T > \frac{1}{\gamma_1}\ln(4KL^2H + 2KL^2)$ so that

$$2KL\mu He^{-\gamma_1 T} < 1.$$

Notice that if $\beta_1 = \beta + \gamma_1 - \gamma < \beta$, then

$$R = 2KL\mu He^{-\gamma_1 T}e^{\beta_1 T} \leq e^{\beta_1 T},$$

and thus

$$\begin{aligned}\|v(s_k)\| &\leq \rho\left(\frac{M}{m(R-1)}+1\right)e^{k\beta_1 T}\|v(s_0)\| \\ &\leq \rho\left(\frac{M}{m(R-1)}+1\right)e^{\beta_1(s_k-s_0)}\|v(s_0)\|.\end{aligned}$$

Hence, for $s_k \leq t < s_{k+1}$,

$$\|v(t)\| \leq He^{\beta T}\|v(s_k)\| \leq H_1\|v(s_0)\|e^{\beta_1(t-s_0)}, \quad 0 \leq s_0 \leq t,$$

where $H_1 = He^{\beta T}\rho\left(\frac{M}{m(R-1)} + 1\right)$ which is independent of s_0.
Step 2: Repeat *Step* 1 until $\beta_n < \gamma$ for some n and

$$\|v(t)\| \leq H_n \|v(s_0)\| e^{\beta_n (t-s_0)}, \quad 0 \leq s_0 \leq t,$$

where H_n is independent of s_0. In fact, this can be done by choosing T sufficiently large in each step. Since $\beta_n < \gamma$, **Case 2.1** implies that there exists a constant $H' > 0$ such that

$$\|v(t)\| \leq H' \|v(s)\|, \; 0 \leq s \leq t.$$

Therefore, Proposition 4.15 is proved.

■

Proof of Theorem 4.12:

Proof. Since $u(t, u_0)$ is bounded and $\frac{\partial f}{\partial u}(u(t, u_0)) \in \mathscr{L}(X^\alpha, X)$, there exists a constant C_1 such that $\|\frac{\partial f}{\partial u}(u(t, u_0))\|_{\mathscr{L}(X^\alpha, X)} \leq C_1$. For a sectorial operator A, there exists a constant M_1 such that for $0 \leq \beta_1 \leq \alpha < 1$,

$$\|e^{-A(t-s)} y(s)\|_\alpha \leq M_1 (t-s)^{\beta_1 - \alpha} \|y(s)\|_{\beta_1}.$$

The solution of the linear variational equation of (4.19) at $u(t, u_0)$,

$$\begin{aligned} &\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u(t, u_0))v, \\ &v(0) = v_0, \end{aligned} \tag{4.41}$$

satisfies

$$\begin{aligned} \|v(t, v_0)\|_\alpha &\leq \|e^{-A(t-t_0)} v(0)\|_\alpha + \int_0^t \left\| e^{-A(t-s)} \frac{df}{du}(u(s, u_0)) v(s, v_0) \right\|_\alpha ds \\ &\leq M_1 \|v(0)\|_\alpha + \int_0^t \left\| e^{-A(t-s)} \frac{\partial f}{\partial u}(u(s, u_0)) v(s, v_0) \right\|_\alpha ds \\ &\leq M_1 \|v(0)\|_\alpha + M_1 \int_0^t (t-s)^{-\alpha} \left\| \frac{\partial f}{\partial u}(u(s, u_0)) v(s, v_0) \right\| ds \\ &\leq M_1 \|v(0)\|_\alpha + M_1 C_1 \int_0^t (t-s)^{-\alpha} \|v(s, v_0)\|_\alpha ds. \end{aligned}$$

Thus, Lemma 4.10 implies that there exist constants $M, \beta > 0$ such that

$$\|v(t, v_0)\|_\alpha \leq M e^{\beta t} \|v_0\|_\alpha. \tag{4.42}$$

Since $v^1(t) = \frac{du}{dt}(u(t, u_0)) \in X^\alpha$ (see Henry [42], Theorem 3.5.2, page 71) is uniformly bounded, there exists a constant $a > 0$ such that

$$\|v^1(t)\|_\alpha < a, \qquad t \geq 0. \tag{4.43}$$

Moreover, since its omega limit set $\omega(u_0)$ contains no steady state solution, there exists a constant $b > 0$ such that

$$\|v^1(s)\|_\alpha \geq b > 0, \qquad s \geq 0. \tag{4.44}$$

If not, then there exists a sequence $\{s_n\}$, $s_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that

$$\lim_{n \to \infty} -Au(s_n, u_0) + f(u(s_n, u_0)) = \lim_{n \to \infty} v^1(s_n) = 0.$$

Since f is continuous and A is closed on X^α when $0 < \alpha < 1$, this implies that $\omega(u_0)$ contains a steady state solution contradicting the assumption that $\omega(u_0)$ contains no such solution. Thus, (4.43) and (4.44) imply

$$\|v^1(t)\|_\alpha \leq L\|v^1(s)\|_\alpha, \quad \text{for } t, s \geq 0, \tag{4.45}$$

where $L = \frac{a}{b} > 1$. Let $\{T(t, s), 0 \leq s \leq t\}$ be the evolution operator generated by the solution of (4.41). The space X decomposes into a direct sum $X = X_1(t) \oplus X_2(t)$ where $X_1(t) = \text{span}\{\frac{du}{dt}(t, u_0)\} \subset X^\alpha$ is the tangent vector space at time t of the bounded solution $u(t, u_0)$. If $v(t) = v(t, v_0)$ is a

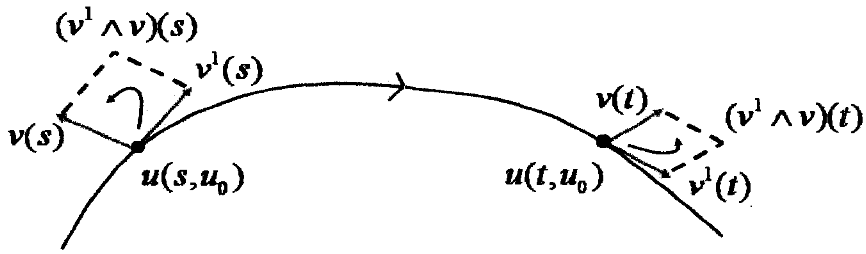

Figure 4.2: Evolution of the oriented infinitesimal parallelogram $v^1 \wedge v$

solution of the linear variational equation (4.41), then $(v^1 \wedge v)(t)$ is a solution of the second compound differential equation (4.20); see Figure 4.2. Thus the uniform asymptotic stability of (4.20) implies that there exist constant $K, \gamma > 0$ such that

$$\|(v^1 \wedge v)(t)\| \leq K\|(v^1 \wedge v)(s)\|e^{-\gamma(t-s)}, \quad 0 \leq s \leq t. \tag{4.46}$$

From (4.42), (4.45) and (4.46), Proposition 4.15 implies that there exists a constant $C > 0$ such that

$$\begin{aligned} \|T(t,s)x_1\|_\alpha &\leq C\|x_1\|_\alpha \quad \text{for } x_1 \in X_1(s), \quad 0 \leq s,t, \\ \|T(t,s)x_2\|_\alpha &\leq Ce^{-\gamma(t-s)}\|x_2\|_\alpha \quad \text{for } x_2 \in X_2^\alpha(s), \quad 0 \leq s \leq t. \end{aligned} \tag{4.47}$$

Since the stability of the second compound differential equation (4.20) is sufficiently robust, the above argument can be applied to any solution in the omega limit set $\omega(u_0)$.

In the following, consider $u = u(t,u_1)$ where $u_1 \in \omega(u_0)$ and so $u(t,u_1) \in \omega(u_0)$. Let $X = X_1(t) \oplus X_2(t)$ where $X_1(t) = \text{span}\{\frac{du}{dt}(t,u_1)\} \subset X^\alpha$ and $P_i(s)$ be the projections onto $X_i(s), i = 1,2$. Next apply an argument used in Coppel [19], page 82-85, Henry [42], page 251-253 and Li and Muldowney [55]. For any solution $u(t)$ of (4.19), let $z = u - u(t,u_1)$. Then (4.19) is equivalent to

$$\frac{dz}{dt} + \left(A - \frac{\partial f}{\partial u}(u(t,u_1))\right) z = g(t,z), \tag{4.48}$$

where

$$g(t,z) = f(u(t,u_1) + z) - f(u(t,u_1)) - \frac{\partial f}{\partial u}(u(t,u_1))z$$

satisfies $g(t,0) = 0$ and

$$\|g(t,z_1) - g(t,z_2)\| \leq k(\delta)\|z_1 - z_2\|_\alpha, \quad \text{if } \|z_1\|_\alpha, \|z_2\|_\alpha \leq \delta,$$

with $k(\delta) \to 0$ as $\delta \to 0^+$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|z_1\|_\alpha, \|z_2\|_\alpha \leq \delta$ implies

$$\|g(t,z_1) - g(t,z_2)\| \leq \varepsilon\|z_1 - z_2\|_\alpha. \tag{4.49}$$

If $0 < \beta < \gamma$, consider a Banach space

$$\mathcal{B}_{\alpha,\beta} = \{z \in C([0,\infty), X^\alpha) : \|z\|_{\alpha,\beta} < \infty\},$$

where $\|z\|_{\alpha,\beta} = \sup_{t\geq 0} \|z(t)\|_\alpha e^{\beta t}$. If $z \in \mathcal{B}_{\alpha,\beta}$, $\|z\|_{\alpha,\beta} \leq \delta$ and $a \in X_2^\alpha(0)$, define $F_a(z)$ as

$$\begin{aligned} F_a(z)(t) &= T(t,0)a + \int_0^t T(t,s)P_2(s)g(s,z(s))ds \\ &\quad - \int_t^\infty T(t,s)P_1(s)g(s,z(s))ds, \quad t \geq 0. \end{aligned} \tag{4.50}$$

From (4.47), (4.49) and (4.50),

$$\begin{aligned}
\|F_a(z)(t)\|_\alpha &\le C\left[e^{-\gamma t}\|a\|_\alpha + \varepsilon\int_0^t e^{-\gamma(t-s)}\|z(s)\|_\alpha ds + \varepsilon\int_t^\infty \|z(s)\|_\alpha ds\right] \\
&\le C\left[e^{-\gamma t}\|a\|_\alpha + \varepsilon\|z\|_{\alpha,\beta}\left(\int_0^t e^{-\gamma(t-s)}e^{-\beta s}ds + \int_t^\infty e^{-\beta s}ds\right)\right] \\
&\le C\left[e^{-\gamma t}\|a\|_\alpha + \varepsilon\|z\|_{\alpha,\beta}\left(\frac{e^{-\beta t}}{\gamma-\beta} + \frac{e^{-\beta t}}{\beta}\right)\right] \\
&= Ce^{-\gamma t}\|a\|_\alpha + \theta\|z\|_{\alpha,\beta}e^{-\beta t}, \qquad \theta = C\varepsilon\frac{\gamma}{\beta(\gamma-\beta)}.
\end{aligned} \tag{4.51}$$

Choose ε and δ so that $0 < \theta < 1$ and $C\|a\|_\alpha \le (1-\theta)\delta$ and thus

$$\begin{aligned}
\|F_a(z)(t)\|_{\alpha,\beta} &\le C\|a\|_\alpha + \theta\|z\|_{\alpha,\beta} \\
&\le (1-\theta)\delta + \theta\delta = \delta.
\end{aligned} \tag{4.52}$$

Let $z, z_i \in \mathcal{B}_{\alpha,\beta}, \|z\|_{\alpha,\beta} \le \delta, \|z_i\|_{\alpha,\beta} \le \delta, i = 1,2$ and $a \in X_2^\alpha(0), \|a\|_\alpha \le \frac{(1-\theta)}{C}\delta$. Then $F_a(z) \in \mathcal{B}_{\alpha,\beta}$ and $\|F_a(z)\|_{\alpha,\beta} \le \delta$. A similar estimate shows that

$$\|F_a(z_1) - F_a(z_2)\|_{\alpha,\beta} \le \theta\|z_1 - z_2\|_{\alpha,\beta}.$$

Hence the equation $z = F_\xi(z)$ has a unique solution $z^* = z^*(t,a)$. Notice that

$$(t,a) \mapsto u(t,\xi) := u(t,u_1) + z^*(t,a)$$

is a solution of (4.19) with initial condition

$$\xi = u(0,u_1) + z^*(0,a) = u_1 + a - \int_0^\infty T(0,s)P_1(s)g(s,z(s))ds. \tag{4.53}$$

From the first inequality of (4.52),

$$\|z^*(t,a)\|_{\alpha,\beta} \le \frac{C}{1-\theta}\|a\|_\alpha.$$

Since $\|g(t,z)\|_\alpha = o(\|z\|_\alpha)$ uniformly in t for $\|z\|_\alpha \to 0$, it follows that

$$z^*(0,a) = a + o(\|a\|_\alpha)\dot{u}(0,u_1). \tag{4.54}$$

If ρ is sufficiently small, the set of all ξ satisfying (4.53) with $\|\xi\|_\alpha < \rho$ and $P_1(0)\xi = 0$ is a manifold S_ρ of codimension 1 and if $\xi \in S_\rho$,

$$\lim_{t\to\infty}\|u(t,\xi) - u(t,u_1)\|_\alpha = 0. \tag{4.55}$$

Let $u(t, \xi)$ be a solution of (4.19) with $u(0, \xi) = \xi$. For $\xi = u_1$, equation

$$u(t, \xi) - u_1 - z^*(0, a) = 0 \tag{4.56}$$

has a solution when $t = 0, a = 0$. Observe that the linear map $(t, a) \mapsto t\frac{du}{dt}(0, u_1) - a$ is invertible. From (4.54), the implicit function theorem implies that equation (4.56) is solvable when $\|\xi - u_1\|_\alpha < \sigma$ for some constant $\sigma > 0$, that is, there exist $t' = t'(\xi)$ and $a' \in X_2^\alpha(0)$ such that

$$u(t', \xi) = u_1 + z^*(0, a'),$$

where

$$\|a'\|_\alpha \le \frac{1 - \theta}{C}\delta.$$

Since $u_1 \in \omega(u_0)$, there exists $\xi_1, \xi_2 \in S_\rho$ such that $u(t_1, \xi_1) = \xi_2, t_1 > 0$ and a sequence $t_k \to \infty, u(t_k, \xi_1) \to u_1, k \to \infty$. Now (4.55) implies

$$\lim_{t\to\infty} \|u(t, \xi_1) - u(t, \xi_2)\|_\alpha = 0$$

so that

$$\lim_{t\to\infty} \|u(t, \xi_1) - u(t_1, u(t, \xi_1))\|_\alpha = 0 \tag{4.57}$$

since $u(t, \xi_2) = u(t, u(t_1, \xi_1)) = u(t + t_1, \xi_1) = u(t_1, u(t, \xi_1))$. Let $t = t_k$ in (4.57). Then

$$\lim_{k\to\infty} \|u(t_k, \xi_1) - u(t_1, u(t_k, \xi_1))\|_\alpha = 0$$

and thus $u_1 = u(t_1, u_1)$, which shows that $u(t, u_1)$ is periodic with period t_1. From Theorem 3.7 in Section 3.1, this orbit is stable hyperbolic, and thus attracts all nearby orbits. Therefore this orbit is the whole set $\omega(u_0)$.

■

Remark 4.9. If the bounded solution $u = u(t, u_0)$ is a periodic solution, then its omega limit set is itself. Thus Theorem 3.7 in Section 3.1 is a special case of Theorem 4.12.

Chapter 5

Bendixson Criterion

In this chapter a generalization of the Bendixson criterion for the nonexistence of periodic orbits to differential equations in Banach spaces is established.

In the early 1900s, I. Bendixson and H. Dulac gave conditions for a 2-dimensional autonomous system of ordinary differential equations to rule out the existence of nontrivial periodic solutions, which are called the Bendixson and Dulac criteria. In particular, Bendixson [6] showed that

$$\frac{du}{dt} = f(u), \ u \in \mathbb{R}^n \tag{5.1}$$

has no nonconstant periodic solution if $n = 2$ and

$$\text{div } f \neq 0 \quad \text{on } \mathbb{R}^2.$$

Dulac [26] generalized this to the statement that if div $(\alpha f) \neq 0$ on a simply connected open subset D of $\mathbb{R}^2$, where α is a real-valued function on D, then there is no closed path of (5.1) which lies entirely in D.

Higher dimensional Bendixson criteria have been developed by Busenberg and van den Driessche [12] and by R. A. Smith [100, 102]. Busenberg and van den Driessche obtained conditions which preclude the possibility of certain oriented loops occurring in the dynamics of (5.1) and are not confined to finite dimensional spaces. In particular, an extension is obtained for functional differential equations. R. A. Smith [102] shows that, if the system (5.1) is dissipative and $\lambda_1(x) + \lambda_2(x) < 0$ or $\lambda_{n-1}(x) + \lambda_n(x) > 0$, then each bounded solution converges to an equilibrium. Here $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$

are the eigenvalues of $\frac{1}{2}\left((\frac{\partial f}{\partial x})^* + \frac{\partial f}{\partial x}\right)$, $\frac{\partial f}{\partial x}$ is the Jacobian matrix of f and the asterisk denotes transposition. In particular, there are no simple closed curves that are invariant with respect to (5.1). Smith's proof contains an error; see Li and Muldowney [53], page 465. However the result is correct.

Based on compound differential equations, Muldowney [78] observed that Smith's conditions imply that, in the dynamics of (5.1) the usual Euclidean measure of 2-dimensional surface area decreases with increasing or decreasing time respectively. Also it was noted that, for any 2-dimensional surface which has a given simple closed curve as its boundary, there is a positive lower bound on the area of the surface that depends only on the boundary curve. These observations were used in [78] to develop two new approaches to higher dimensional Bendixson conditions based on various measures of surface area. Both approaches give the Bendixson-Dulac results when $n = 2$.

The first approach in [78] uses the fact that, if some measure of 2-dimensional surface area decreases in the dynamics of (5.1), then no simply connected open region $D \subset \mathbb{R}^n$ can contain a periodic orbit or any invariant simple closed curve if such a curve is the boundary of a 2-dimensional surface of minimum area. Since the boundary is invariant and the surface area decreases strictly, the minimality of the area is contradicted.

The preceding approach has the advantage that it requires only local existence of solutions. However, it places a restriction on the region D to which it is applied: it must have a shape that permits the existence of a minimal surface for any simple closed curve in D. As explained in [78], the "minimal surface" can exist in a fairly abstract sense, such as the existence of a minimizing sequence of surfaces.

The second approach requires that solutions originating in D exist globally, but replaces the requirement that 2-dimensional surface area decreases with a condition that the area of any surface originating in D tends to zero as t tends to infinity under (5.1). If the boundary is an invariant simple closed curve in D, this would contradict the existence of a positive lower bound for the surface area.

Both of these approaches are facilitated through consideration of the second compound differential equation

$$\frac{dw}{dt} = \frac{\partial f^{[2]}}{\partial u}(u(t,u_0))w \tag{5.2}$$

with respect to a solution $u(t,u_0), u_0 \in D$, which governs the evolution of 2-dimensional surface areas near $u(t,u_0)$. In particular, the second approach can be implemented in terms of Bendixson conditions related to stability requirement on (5.2); see Li and Muldowney [58], Muldowney [78].

This chapter begins with the establishment of a positive lower bound for a 2-dimensional surface area whose boundary is a given simple closed curve in a Banach space X; the bound depends only on the curve. Bendixson conditions for differential equations in Banach spaces are developed in terms of stability of associated compound linear differential equations.

5.1 Surfaces and Boundaries in Banach Spaces

In this section, a measure of 2-dimensional surface is introduced and the existence of a positive lower bound for 2-dimensional surface area whose boundary is a given simple closed curve in a Banach space X is obtained.

Throughout this section, the real Banach space X is assumed to satisfy the *Radon-Nikodym property* (see [5, 27, 88]) and its norm is *Gâteaux differentiable* (see [29, 46, 72]).

Definition 5.1. Suppose that G is a nonempty open subset of a real Banach space X and that $f : G \to Y$ is a map from G into a real Banach space Y. The map f is *Gâteaux differentiable* at $x \in G$ if for each $h \in X$, the limit

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} \tag{5.3}$$

exists in the norm topology of Y and defines a linear (in h) map which is continuous from X to Y. The map f is *Frechét differentiable* at $x \in G$ if the limit (5.3) is uniform for $h \in S_X = \{x \in X \ : \ \|x\| = 1\}$. The *Gâteaux* derivative of f at x is denoted by $df(x)$.

Definition 5.2. A real Banach space X is said to satisfy the *Radon-Nikodym property* if every function of bounded variation from $[0,1]$ into X is Gâteaux differentiable almost everywhere.

Earlier results of Dunford and Pettis (see [27]) showed that such spaces include separable dual spaces and reflexive spaces. The Aronszajn Theorem (see [88]) shows that every Lipschitz continuous map $f : G \to Y$ is Gâteaux differentiable almost everywhere provided that X is separable and Y satisfies the Radon-Nikodym property.

Definition 5.3. A norm $\|\cdot\|$ on a Banach space X is *Gâteaux differentiable* if it is Gâteaux differentiable at every point of $X\backslash\{0\}$.

Lemma 5.1. *Let X be a Banach space whose norm is Gâteaux differentiable at the point $x \neq 0$ and X^* be its (continuous) dual space. Then* $\lim\limits_{t\to 0} \dfrac{\|x+ty\|-\|x\|}{t} = G_x(y)$ *defines an element $G_x \in X^*$ such that $\|G_x\| \leq 1$. Furthermore, if $x \in X$ and $|x| \leq 1$, then $\|G_x\| = 1$.*

Let $B = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$ be the Euclidean unit ball in $\mathbb{R}^2$ centered at the origin, whose closure and boundary are denoted by $\overline{B}$ and ∂B. The boundary ∂B can also be associated with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The following definitions are similar to those in Li [51] and Li and Muldowney [53, 58].

Definition 5.4.
(*i*) A map $\psi \in \mathrm{Lip}(S^1 \to X)$ is a *simple rectifiable closed curve* if it is one to one.
(*ii*) A map $h \in \mathrm{Lip}(\overline{B} \to X)$ is a *rectifiable 2-dimensional surface* in X, and the restriction $h|_{\partial B}$ is the *boundary* of h. The map h is a *simple rectifiable 2-dimensional surface* if $h|_{\partial B} : \partial B \to X$ is one to one. Henceforth, for brevity, the term "rectifiable" will be omitted.

Define

$$\sum(\psi, X) := \left\{h \in \mathrm{Lip}(\overline{B} \to X),\ h|_{\partial B} \text{ is one to one and } h(\partial B) = \psi(S^1)\right\}, \tag{5.4}$$

where $\psi \in \mathrm{Lip}(S^1 \to X)$ is a simple closed curve. If D is a simply connected open subset of X, then

$$\sum(\psi, D) := \left\{h \in \mathrm{Lip}(\overline{B} \to D),\ h|_{\partial B} \text{ is one to one and } h(\partial B) = \psi(S^1)\right\}, \tag{5.5}$$

is nonempty (see Li and Muldowney [58]).

Definition 5.5. If X satisfies the Radon-Nikodym property and its norm is Gâteaux differentiable, define a surface area measure σ_2 on $\mathrm{Lip}(\overline{B} \to X)$ by

$$\sigma_2(h) := \int_B \|d^{(2)}h\| = \int_B \left\| \frac{\partial h}{\partial r_1}(r) \wedge \frac{\partial h}{\partial r_2}(r) \right\| dr, \qquad h \in \mathrm{Lip}(\overline{B} \to X), \tag{5.6}$$

where $d\ h$ denotes the Gâteaux derivative of h. Here $\|\cdot\|$ is any norm on $\bigwedge^2 X$ and the associated operator norm on $\mathscr{L}(\bigwedge^2\mathbb{R}^2, \bigwedge^2 X)$.

The integral in (5.6) exists and is finite since h is Gâteaux differentiable almost everywhere in B and the integrand is bounded.

Example 5.1. Suppose that $X = \{(x_1, x_2, \cdots), x_i \in \mathbb{R}, i = 1, 2, \cdots\}$ is a sequence space. Then $h(r) = (h_1(r), h_1(r), \cdots)$ and for each $h \in \mathrm{Lip}(\overline{B} \to X)$,

$$\int_B \left(\sum_{1 \le i < j}^{\infty} \left| \frac{\partial(h_i, h_j)}{\partial(r_1, r_2)} \right|^p \right)^{\frac{1}{p}} dr_1 dr_2 \quad \text{and} \quad \int_B \sup_{1 \le i < j} \left| \frac{\partial(h_i, h_j)}{\partial(r_1, r_2)} \right| dr_1 dr_2$$

are examples of measures $\sigma_2(h)$ of 2-dimensional surface area if the integrals exist. When $X = \mathbb{R}^n$ and $p = 2$, this is the usual Euclidean measure of the surface area.

Example 5.2. Let X be a space of functions $f : s \mapsto f_s$, where $s \in \Omega$, a measurable subset in $\mathbb{R}^n$, and $f_s \in \mathbb{R}$. A surface h in X is a Lipschitz function $r \in \overline{B} \mapsto h(r) \in X$ and may be expressed in the form $(r, s) \mapsto h_s(r), r \in \overline{B}, s \in \Omega$ and $h_s(r) \in \mathbb{R}$. Then

$$\int_B \left(\int_{\Omega^2} \left| \frac{\partial(h_{s_1}, h_{s_2})}{\partial(r_1, r_2)} \right|^p ds_1 ds_2 \right)^{\frac{1}{p}} dr_1 dr_2 \quad \text{and} \quad \int_B \sup_{s_1, s_2 \in \Omega} \left| \frac{\partial(h_{s_1}, h_{s_2})}{\partial(r_1, r_2)} \right| dr_1 dr_2$$

are measures $\sigma_2(h)$ of 2-dimensional surface area if the integrals exist.

While Example 5.1 is a special case of Example 5.2, it is useful to state the examples separately. For instance, the first example might be used to define surface area for a function space in terms of the Fourier coefficients. The second example uses the pointwise representation of the functions.

When $X = \mathbb{R}^n$ and ψ is a simple closed curve in $\mathbb{R}^n$, the existence of a positive lower bound for $\sigma_2(h)$ where $h \in \sum(\psi, \mathbb{R}^n)$ has been established by Li and Muldowney [51, 53, 78]. In the following, the existence of a positive lower bound for $\sigma_2(h)$ for general 2-dimensional surfaces in X with a given boundary will be proved.

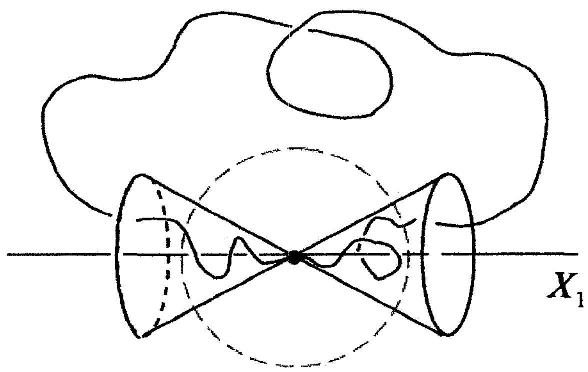

Figure 5.1: The boundary of the 2-dimensional surface h in X

In general outline, the procedure is as follows. At any point on the curve where there is a tangent, a whole segment of the curve lies inside a cone whose axis is the tangent line. Because ψ is one to one, there is a ball centered on the point of tangency which contains no points of the curve other than those inside the cone. A rotation about the tangent is a map from X to $\mathbb{R}^2$ which does not increase the area. The region inside the ball and outside the cone "traps" a section of the surface which is mapped onto a sector of a disc in the upper half-plane of $\mathbb{R}^2$; see Figure 5.1. The area of the sector is a lower bound for the area of any surface in X that has this boundary curve.

Suppose that $\psi \in \text{Lip}(S^1 \to X)$ is a simple closed curve. Then ψ is Gâteaux differentiable almost everywhere since X satisfies the Radon-Nikodym property. Moreover, ψ is Frechét differentiable almost everywhere since Frechét differentiability is equivalent to Gâteaux differentiability in $\mathbb{R}$. Without loss of generality, assume that ψ is Frechét differentiable at $0 \in S^1$ and $\psi(0) = 0 \in X$. Let $X_1 = d\psi(0)\mathbb{R}$ and e be a unit vector in X_1. If $x = ae \in X_1$ and $a \in \mathbb{R}$, let $q(x) = a$ be the coordinate functional of $x_1 \in X_1$ referred to e, whose norm $\|q\|_{X_1} = 1$. By the Hahn-Banach theorem, the linear functional q admits an extension on the whole space X denoted also by q with the same norm. Define $P_1 x = q(x)e$ and $P_2 = I - P_1$. Then P_1 and P_2 are projections and

$$\|P_1\| = 1, \qquad \|P_2\| \leq 2 \tag{5.7}$$

(see Fabian et al [29], page 139 or Li and Muldowney [53], page 459-460). Now $X = X_1 \oplus X_2$ where $X_2 = P_2 X$. Define a map $\mathcal{R} : X \to \mathbb{R}^2$ by

$$\mathcal{R}(x) = (q(x), \|P_2 x\|), \tag{5.8}$$

where $P_1 x = q(x)e \in X_1$, $q(x) \in \mathbb{R}$.

Remark 5.1. The map $\mathcal{R}$ is Lipschitz continuous. Let $\prod_0, \prod_+$ and $\prod_-$ denote the sets of points $(y_1, y_2) \in \mathbb{R}^2$ such that $y_2 = 0, y_2 > 0$ and $y_2 < 0$, respectively. Then $\mathcal{R}$ can be viewed as a rigid rotation of X about $X_1 \cong \prod_0$ into $\overline{\prod}_+$ with $\mathcal{R}X_1 = \prod_0$.

Remark 5.2. If X is a separable Hilbert space with an orthonormal basis $\{e^i\}_1^\infty$, then the rotation map $\mathcal{R} : X \to \mathbb{R}^2$ about $X_1 = \text{span}\{e^1\}$ is

$$x \mapsto \mathcal{R}(x) = \left(x_1, \left(\sum_{i=2}^{\infty} (x_i)^2 \right)^{\frac{1}{2}} \right),$$

where $x = \sum_{i=1}^{\infty} x_i e^i$.

Remark 5.3. Since it is assumed that the norm $\|\cdot\|$ of the Banach space X is Gâteaux differentiable, it follows from Lemma 5.1 that Gâteaux derivative of $\mathcal{R}$ exists and is uniformly bounded for all $x \in X$. Thus, there exists a constant $M > 0$ such that

$$d^{(2)}\mathcal{R}(x) : \bigwedge^2 X \to \bigwedge^2 \mathbb{R}^2 \simeq \mathbb{R}$$

satisfies

$$\left\| d^{(2)}\mathcal{R}(x) \right\| \leq M.$$

Let $v = \mathcal{R} \circ \psi : S^1 \to \overline{\prod}_+$. Since ψ is Frechét differentiable at $0 \in S^1$ and $\psi(0) = 0 \in X$, it can be assumed that

$$\psi(r) = d\psi(0)s + g(s)$$

where $g(s) = o(s)$ near $s = 0$. Thus,

$$P_1\psi(s) = d\psi(0)s + P_1g(s) \in X_1,$$
$$P_2\psi(s) = P_2g(s) \in X_2,$$

and $v(s) = (v_1(s), v_2(s))$ where $v_1(s) = q(P_1\psi(s)), v_2(s) = \|P_2\psi(s)\|$ implies that

$$v_1(s) = ms + o(s), \quad 0 \le v_2(s) = o(s), \tag{5.9}$$

near $s = 0$, where $0 \neq m = q(P_1d\psi(0))(1)$. Without loss of generality, assume that $m > 0$. If $a > m$, there exists $0 < \varepsilon < \pi$ such that

$$0 \le v_2(s) \le a|v_1(s)|, \quad -\varepsilon < s < \varepsilon$$

and $sv(s) > 0, s \neq 0$. Thus $v((-\varepsilon, \varepsilon))$ is a curve from $v(-\varepsilon)$ to $v(\varepsilon)$ through 0 and lying in the sectorial region

$$T = \left\{(v_1, v_2) \in R^2 \ : \ 0 \le v_2 \le a|v_1|\right\},$$

see Figure 5.2.

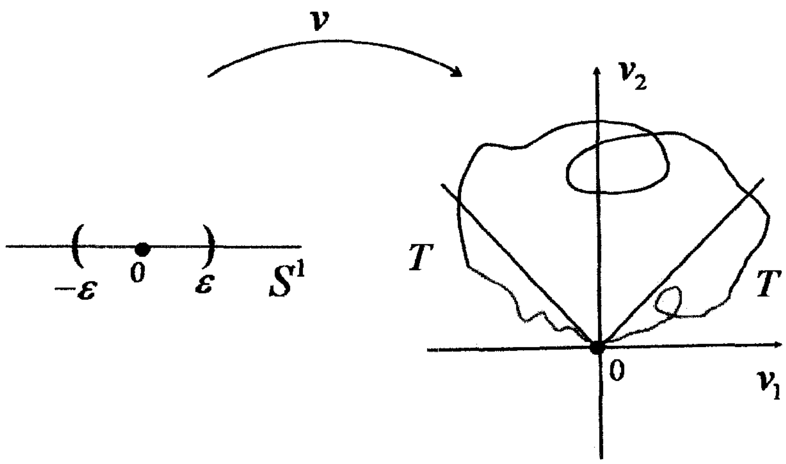

Figure 5.2: A sector T at 0 in $\mathbb{R}^2$.

Next define $\tilde{\psi} : S^1 \to X$ by $\tilde{\psi}(s) = \psi(s)$ on $s \in S^1 \setminus (-\varepsilon, \varepsilon)$ and $\tilde{\psi} : [-\varepsilon, \varepsilon] \to X$ is one-to-one defined by linear interpolation from $\psi(-\varepsilon)$ to $P_1\psi(-\varepsilon)$, from $P_1\psi(-\varepsilon)$ to $P_1\psi(\varepsilon)$ with $\tilde{\psi}(0) = 0$ and from $P_1\psi(\varepsilon)$ to $\psi(\varepsilon)$. If $\tilde{v} = \mathcal{R} \circ \tilde{\psi} : S^1 \to \mathbb{R}^2$, then $v, \tilde{v}$ differ only possibly on $(-\varepsilon, \varepsilon)$. Both

v and $\tilde{v}$ map $[-\varepsilon, \varepsilon]$ to curves from $v(-\varepsilon)$ to $v(\varepsilon)$ through $0 = v(0) = \tilde{v}(0)$ and passing from one half of T to the other as s passes through 0.

The main result in this section, Proposition 5.2, shows the existence of a positive lower bound for $\sigma_2(h)$ independent of $h \in \sum(\psi, X)$. This was deduced in the original paper of Muldowney [78] on systems in $\mathbb{R}^n$ from a classical result on the existence of a surface of minimum area. The proof given here for a general Banach space is adapted from a Brouwer degree argument used in Li [51] and Li and Muldowney [53]. A good reference for degree theory is the book by Lloyd [59].

Proposition 5.2. *Let X be a Banach space satisfying the Radon-Nikodym property whose norm is Gâteaux differentiable. Suppose that $\psi \in Lip(\partial B \to X)$ is a simple closed curve in X. Then there exists $\delta > 0$ such that*

$$\sigma_2(h) \geq \delta$$

for all $h \in \sum(\psi, X)$.

Proof. As in the preceding discussion, it is assumed that $\psi(0) = 0 \in X$ and ψ is Frechét differentiable at 0. Let $h \in \sum(\psi, X)$ and $\mathfrak{R} = \mathcal{R} \circ h : \overline{B} \to \overline{\prod}_+$ and $v = \mathcal{R} \circ \psi : S^1 \to \overline{\prod}_+$ so that $\mathfrak{R}(\partial B) = v(S^1)$, where $\mathcal{R}$ is defined by (5.8). Remark 5.3 implies that there exists a constant M such that

$$\left\| d^{(2)} \mathcal{R}(x) \right\| \leq M$$

for all $x \in X$. Therefore,

$$\begin{aligned} \int_B \left| d^{(2)} \mathfrak{R} \right| &= \int_B \left| d^{(2)} \mathcal{R}(h) \circ d^{(2)} h(r) \right| dr \\ &\leq \int_B \left\| d^{(2)} \mathcal{R}(h) \right\| \left\| d^{(2)} h(r) \right\| dr \\ &\leq M \int_B \left\| d^{(2)} h(r) \right\| \\ &\leq M \int_B \left\| d^{(2)} h \right\| = M \sigma_2(h). \end{aligned} \tag{5.10}$$

Since $h|_{\partial B}$ is one-to-one and $h(\partial B) = \psi(S^1)$, the connected components of $\mathbb{R}^2 \setminus \mathfrak{R}(\partial B) = \mathbb{R}^2 \setminus v(\partial B)$ are the same for every $h \in \sum(\psi, X)$ and thus the

Brouwer degree of $\mathfrak{R}$ relative to B, $\deg(\mathfrak{R}, B, \cdot)$, is a constant and independent of h on each component. It will be shown that there is a component Δ of $\mathbb{R}^2 \setminus \mathfrak{R}(\partial B)$ such that $\deg(\mathfrak{R}, B, p) = \pm 1$ if $p \in \Delta$. It follows from (5.10) that

$$\delta = \frac{1}{M}\text{area}\Delta \leq \sigma_2(h)$$

for all $h \in \sum(\psi, X)$.

Let $\tilde{\psi}$ be as in the preliminary discussion. It follows that there exists $\tilde{h} \in \sum(\tilde{\psi}, X)$ and an open arc $\gamma \subset \partial B$ such that $h(\gamma) = \psi((-\varepsilon, \varepsilon)), \tilde{h}(\gamma) = \tilde{\psi}((-\varepsilon, \varepsilon))$ and $r_0 \in \gamma, h(r_0) = \tilde{h}(r_0) = 0$. Recall $\mathfrak{R} = \mathcal{R} \circ h$ and let $\tilde{\mathfrak{R}} = \mathcal{R} \circ \tilde{h}$; $r \in \partial B \backslash \gamma$; $\mathfrak{R}(\gamma) \subset T, \tilde{\mathfrak{R}}(\gamma) \subset T$ and both curves pass from one convex half of T to the other half as r passed through r_0. Further, $\tilde{\mathfrak{R}}(\gamma)$ consists of three line

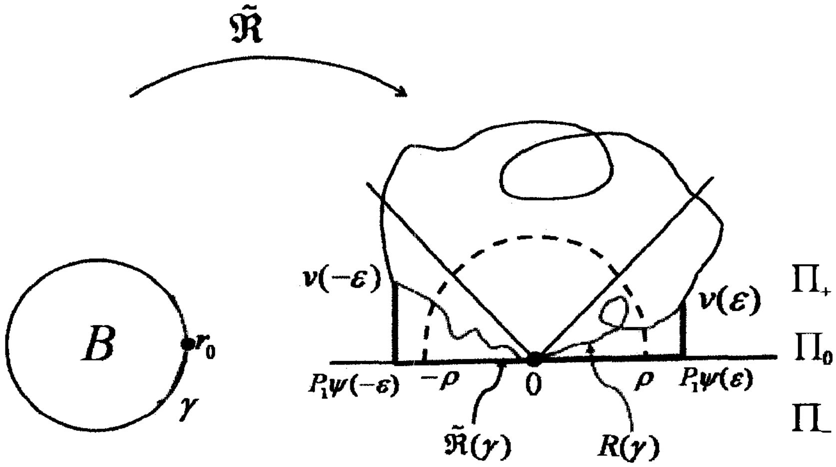

Figure 5.3: The map $\tilde{\mathfrak{R}}$. Note $v(\pm\varepsilon) = (P_1\psi\,(\pm\varepsilon)\,, \|P_2\psi\,(\pm\varepsilon)\,\|)$.

segments $[\psi(-\varepsilon), P_1\psi(-\varepsilon)], [P_1\psi(-\varepsilon), P_1\psi(\varepsilon)] \in \Pi_0$ and $[P_1\psi(\varepsilon), \psi(\varepsilon)]$. Since $\mathfrak{R}(x) = 0$ if and only if $x = 0$, it follows that, if $r \in \partial B$, $0 = \tilde{\mathfrak{R}}(s) = \mathcal{R} \circ \tilde{h}(s)$ if and only if $r = r_0$. Therefore, the compact set $\tilde{\mathfrak{R}}(\partial B \setminus \gamma) = \mathfrak{R}(\partial B \setminus \gamma)$ is a positive distance from $0 \in \mathbb{R}^2$ and so does not intersect $B(0, \rho) \subset \mathbb{R}^2$ for some $\rho > 0$; see Figure 5.3. In particular,

$$\tilde{\mathfrak{R}}(\partial B) \cap B(0, \rho) = (-\rho, \rho) \subset \Pi_0. \tag{5.11}$$

Since $\mathfrak{R}(s) = \tilde{\mathfrak{R}}(s)$ if $r \in \partial B \setminus \gamma$ and the line segment $[\mathfrak{R}(s), \tilde{\mathfrak{R}}(s)] \subset T$ if $r \in \gamma$, the Poincaré-Bohl theorem, [59], page 25, implies that $\deg(\mathfrak{R}, B, \cdot) = \deg(\tilde{\mathfrak{R}}, B, \cdot)$ on $\mathbb{R}^2 \setminus T$. From (5.11), the sets $\Pi_-, B(0,\rho) \cap \Pi_+$ are subsets of Δ_1, Δ_2, connected components of $\mathbb{R}^2 \setminus \tilde{\mathfrak{R}}(\partial B)$. The interval $[-\rho, \rho] \subset \prod_0$ is a subset of $\tilde{\mathfrak{R}}(\partial B)$ and is in the boundary of each of these sets. Further, $\tilde{\mathfrak{R}}$ is one-to-one on $\tilde{\mathfrak{R}}^{-1}([-\rho, \rho])$ and $\deg(\tilde{\mathfrak{R}}, B, \cdot) = 0$ on Δ_1 since $\mathfrak{R}(\partial B) \subset \overline{\prod_+}$. Therefore, $\deg(\tilde{\mathfrak{R}}, B, \cdot) = \pm 1$ on $B(0, \rho) \cap \prod_+ \subset \Delta_2$. The Poincaré-Bohl theorem then implies $\deg(\mathfrak{R}, B, \cdot) = \pm 1$ on $B(0,\rho) \setminus T$ and hence

$$\delta = \frac{1}{M}\text{area}(B(0,\rho) \setminus T) \leq \sigma_2(h).$$

■

5.2 Bendixson Criterion

In this section, a generalization of the Bendixson criterion to autonomous differential equations in a Banach spaces will be developed. Let A be a sectorial operator in a Banach space X and $f(u)$ be continuously differentiable from X^α into X and $0 \leq \alpha < 1$. Consider the initial value problem

$$\begin{aligned} \frac{du}{dt} + Au &= f(u), \\ u(0) &= u_0 \in X^\alpha. \end{aligned} \tag{5.12}$$

In the following, let $u(t, u_0) \in X^\alpha$ denote the solution of (5.12). The linear variational differential equation of (5.12) at a solution $u(t, u_0)$ is

$$\frac{dv}{dt} + Av = \frac{\partial f}{\partial u}(u(t, u_0))v, \tag{5.13}$$

and its second compound differential equation defined on $\bigwedge^2 X^\alpha$ is

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(u(t, u_0))w. \tag{5.14}$$

Suppose that X^α is a Banach space satisfying the Radon-Nikodym property whose norm is Gâteaux differentiable. Let $D \subset X^\alpha$ be a simply connected open subset and $B = \{x \in \mathbb{R}^2 : |x| < 1\}$. Consider a 2-dimensional surface $h_0 \in \text{Lip}(\overline{B} \to D)$. Suppose that $h_t(r) = u(t, h_0(r))$ is defined for all

t and $r \in \overline{B}$. Then h_t is a 2-dimensional surface and

$$\sigma_2(h_t) = \int\limits_B \left\|d^{(2)}h_t\right\|_\alpha = \int\limits_B \left\|\frac{\partial h_t}{\partial r_1} \wedge \frac{\partial h_t}{\partial r_2}\right\|_\alpha \tag{5.15}$$

where $\|\cdot\|_\alpha$ is any norm on $\bigwedge^2 X^\alpha$. Proposition 5.2 implies the following Bendixson criterion.

Bendixson Criterion. *Suppose that*

(*i*) *$D \subset X^\alpha$ is a simply connected open set;*

(*ii*) *the solutions of* (5.12) *exist for all t if $u_0 \in D$;*

(*iii*) *ψ is a simple closed curve in D;*

(*iv*) *there is a simple 2-dimensional surface $h_0 \in \sum(\psi, D)$ such that $\sigma_2(h_t)$ tends to zero as t tends to infinity under* (5.12).

Then ψ cannot be invariant with respect to (5.12).

Remark 5.4. Condition (*iv*) is irrespective of the norm on $\bigwedge^2 X^\alpha$ that is used to define $\sigma_2(h_t)$.

Since

$$dh_t(r) = \frac{\partial u(t, h_0(r))}{\partial u_0} \cdot dh_0(r)$$

satisfies the linear variational differential equation (5.13) with $u_0 = h_0(r)$, it follows from the Binet-Cauchy identity that

$$d^{(2)}h_t(r) = \frac{\partial u(t, h_0(r))}{\partial u_0}^{(2)} \cdot d^{(2)}h_0(r)$$

which satisfies the second compound differential equation (5.14). This observation can be used to derive a concrete sufficient condition for Assumption (*iv*) of **Bendixson Criterion.**

Theorem 5.3. *Suppose that*

(*i*) *$D \subset X^\alpha$ is a simply connected open set;*

(*ii*) *the solutions of* (5.12) *exist for all t if $u_0 \in D$;*

(*iii*) *the family of linear systems*

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(u(t, u_0))w, \qquad u_0 \in S,$$

is equi-asymptotically stable in $\bigwedge^2 X^\alpha$ if S is a compact subset of D.

Then D contains no simple closed curve that is invariant with respect to (5.12).

The equi-asymptotic stability in Assumption (*iii*) of Theorem 5.3 means that if S is a compact subset of D, then, for all $u_0 \in S$, the linear systems (5.14) are asymptotically stable and the limit

$$\lim_{t \to \infty} \|w(t)\|_\alpha = 0$$

is uniform with respect to $u_0 \in S$, where $w(t) \in \bigwedge^2 X^\alpha$ is a solution of (5.14) and $\|\cdot\|_\alpha$ is any norm on $\bigwedge^2 X^\alpha$.

Proof. Suppose that $\psi \in \mathrm{Lip}(S^1 \to D)$ is a simple closed curve which is invariant with respect to (5.12). Let $h_0 \in \sum(\psi, D)$ be a simple 2-dimensional surface which is one-to-one on ∂B and $h_0(\partial B) = \psi(S^1)$, where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$. Then $r \mapsto h_t(r) = u(t, h_0(r))$ is also a simple 2-dimensional surface which is one-to-one on ∂B and $h_t(\partial B) = \psi(S^1)$. Since $dh_t(r) = \dfrac{\partial u(t, h_0(r))}{\partial u_0} \cdot dh_0(r)$ satisfies the linear variational differential equation (5.13) with $u_0 = h_0(r)$, $d^{(2)}h_t(r) = \dfrac{\partial u(t, h_0(r))}{\partial u_0}^{(2)} \cdot d^{(2)}h_0(r)$ is a solution of the second compound differential equation

$$\frac{dw}{dt} + A^{[2]}w = \frac{\partial f}{\partial u}^{[2]}(u(t, u_0))w, \qquad u_0 = h_0(r), \quad r \in \overline{B}.$$

These linear systems are equi-asymptotically stable and

$$\lim_{t \to \infty} \left\| \frac{\partial u(t, h_0(r))}{\partial u_0}^{(2)} \right\|_\alpha = 0$$

uniformly with respect to $r \in \overline{B}$, which implies that

$$\lim_{t \to \infty} \|d^{(2)}h_t(r)\|_\alpha = 0$$

uniformly with respect to $r \in \overline{B}$. Thus $\lim_{t \to \infty} \sigma_2(h_t) = 0$. But Proposition 5.2 shows that $\delta \leq \sigma_2(h_t)$ for some $\delta > 0$. Therefore, ψ can not be invariant with respect to (5.12).

∎

Remark 5.5. Theorem 5.3 rules out the existence of nontrivial periodic orbits, homoclinic loops and heteroclinic loops. For some systems, the equi-asymptotic stability of the family of linear systems (5.14) can be established by constructing a suitable Lyapunov function in $\bigwedge^2 X^\alpha$. For instance, (5.14) is equi-asymptotically stable if there exists a positive definite function $V(w)$, such that $\frac{d^+V}{dt}\Big|_{(5.14)}$ is negative definite, and V and $\frac{d^+V}{dt}\Big|_{(5.14)}$ are both independent of u_0.

Remark 5.6. The results in this section are motivated by the high-dimensional Bendixson criterion for an autonomous ordinary differential equation in $\mathbb{R}^n$ established by Muldowney [78]. The proof of Theorem 5.3 is adapted from the idea used in Theorem 4.1 of [78].

Remark 5.7. An application of Theorem 5.3 is to combine with Theorem 4.12, Section 4.2 to show the existence of steady state solutions.

Consider a scalar reaction diffusion equation

$$\begin{aligned} &u_t = u_{xx} + f(x, u, u_x), \quad 0 < x < L, \quad t > 0, \\ &u(t, 0) = u(t, L), \quad u_x(t, 0) = u_x(t, L). \end{aligned} \tag{5.16}$$

Here $f \in C^2((0, L) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$. Let $X = H^1_{per}(0, L) = \{\phi \in L^2(0, L) : \phi' \in L^2(0, L), \phi(0) = \phi(L), \ \phi'(0) = \phi'(L)\}$. Suppose that if $u_0 \in X$, $u(t, x) = u(t, u_0) \in X$ is a solution which exists for all $t \geq 0$. The linear variational equation of (5.16) at a solution $u(t, x)$ is

$$\begin{aligned} &v_t = v_{xx} + a(t, x)v_x + b(t, x)v, \quad 0 < x < L, \quad t > 0, \\ &v(t, 0) = v(t, L), \quad v_x(t, 0) = v_x(t, L), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} a(t, x) &= \frac{\partial f}{\partial u_x}(x, u(t, x, u_0(x)), u_x(t, x, u_0(x)), \\ b(t, x) &= \frac{\partial f}{\partial u}(x, u(t, x, u_0(x)), u_x(t, x, u_0(x))). \end{aligned} \tag{5.18}$$

The pointwise representation of $w(t) \in \bigwedge^2 X$ is

$$w(t, x_1, x_2) = \det\left[\langle x_i, w(t)\rangle\right]$$

which satisfies the second compound differential equation of (5.17) defined on $\bigwedge^2 X$

$$w_t = \sum_{j=1}^{2} w_{x_j x_j} + \sum_{j=1}^{2} a(t, x_j) w_{x_j} + \sum_{j=1}^{2} b(t, x_j) w. \tag{5.19}$$

The eigenvalues of the Laplace equation

$$\begin{aligned} \Delta u &= -\lambda_i u, \\ u(0) &= u(L), \quad u_x(0) = u_x(L) \end{aligned} \tag{5.20}$$

are $\lambda_1 = 0, \lambda_{2n} = \left(\frac{n\pi}{L}\right)^2, \lambda_{2n+1} = \left(\frac{n\pi}{L}\right)^2$.

Suppose that, for $u, v \in \mathbb{R}$,

$$\begin{aligned} \left|\frac{\partial f}{\partial v}(x,u,v)\right| &\le A(x), \\ \frac{\partial f}{\partial u}(x,u,v) &\le B(x). \end{aligned} \tag{5.21}$$

Consider a Lyapunov function

$$V := \frac{1}{2}\int_{(0,L)^2} w^2 = \frac{1}{2}\int_{(0,L)^2} w^2(t,x_1,x_2)dx_1dx_2.$$

The discussion in the proof of Corollary 3.33, Section 3.4 implies that

$$\begin{aligned} \frac{d^+V}{dt} \le & \sum_{j=1}^{2}\int_{(0,L)^2}\left(\frac{A(x_j)}{2\nu_j(x_j)} - 1\right)(w_{x_j})^2 \\ & + \int_{(0,L)^2}\sum_{j=1}^{2}\left(B(x_j) + \frac{A(x_j)\nu_j(x_j)}{2}\right)w^2, \end{aligned}$$

for any positive functions $\nu_1(x), \nu_2(x)$. If $\max_x A(x) > 0, 0 \le x \le L$, then a choice

$$\nu_j(x) = \max_x A(x) := \nu, \quad j = 1, 2,$$

implies that

$$\frac{d^+V}{dt} \le \mu V,$$

where

$$\mu = -\lambda_1 - \lambda_2 + 4\max_x\left\{B(x) + \frac{A(x)\nu}{2}\right\}.$$

If $\max_x A(x) = 0$, then the function $f(x,u,u_x) = f(x,u)$ which is independent of u_x, and thus

$$\mu = -\lambda_1 - \lambda_2 + 2\max_x B(x).$$

Theorem 5.3 implies the following corollary.

Corollary 5.4. *Suppose that the solutions of* (5.16) *exist for all* $t \geq 0$ *and* $u_0 \in X$. *If* $\max_x |A(x)| > 0,\ 0 \leq x \leq L$, *and*

$$\max_x \{2B(x) + A(x)\nu\} < \frac{\pi^2}{2L^2},$$

where $\nu = \max_x A(x)$ *or if* $f(x,u,u_x) = f(x,u)$ *and*

$$\max_x B(x) < \frac{\pi^2}{2L^2},$$

where $A(x), B(x)$ *is defined by* (5.21), *then there is no simple closed curve that is invariant with respect to* (5.16) *in* $L^2(0,L)$.

Example 5.3. Consider a scalar reaction diffusion equation

$$\begin{aligned} &u_t = u_{xx} + pu - u^3 + q(x), \quad 0 < x < L, t > 0, \\ &u(t,0) = u(t,L), \quad u_x(t,0) = u_x(t,L) \end{aligned} \tag{5.22}$$

Suppose that p is a positive constant and $q(x)$ is nonnegative continuous in x and $\max_x q(x) \leq Q,\ 0 \leq x \leq L$. Let $X = H^1_{per}(0,L)$. Since

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u^2 dx &= \int_0^L u_t u dx \\ &= \int_0^L [-u_x^2 + pu^2 - u^4 + p(x)u] dx \\ &\leq \left(p + \frac{Q}{2}\right)\int_0^L u^2 dx + \frac{QL}{2}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u_x^2 &= -\int_0^L u_t u_{xx} dx \\ &= \int_0^L -u_{xx}^2 + pu_x^2 - q(x)u_{xx}] dx \\ &\leq \left(p + \frac{Q}{2}\right)\int_0^L u_x^2 dx + \frac{QL}{2}, \end{aligned}$$

the Gronwall inequality implies that the solution $u(t,x)$ of (5.22) exists in X for all $t \geq 0$. The linear variational equation of (5.22) at a solution $u(t,x)$ is

$$v_t = v_{xx} + (p - 2u^2)v,$$

and the pointwise representation of the second compound differential equation defined on $\bigwedge^2 X$ is

$$w_t = w_{x_1x_1} + w_{x_2x_2} + (2p - u^2(t,x_1) - u^2(t,x_2))w. \tag{5.23}$$

If

$$-\lambda_1 - \lambda_2 + 2p < 0,$$

Corollary 5.4 implies that there is no simple closed curve that is invariant with respect to (5.16) in $L^2(0,L)$. Since the ω limit set of a bounded solution is nonempty, Theorem 4.12 of Section 4.2 implies that (5.22) has at least one steady state solution, which is nonconstant if $p(x)$ is nonconstant.

Example 5.4. Consider a reaction diffusion system

$$\begin{aligned} u_t &= d_1 u_{xx} + au - u^3 - uv^2, \\ v_t &= d_2 v_{xx} - bv - v^3 + u^2 v, \quad 0 < x < L, \quad t > 0, \end{aligned} \tag{5.24}$$

with Dirichlet boundary conditions or Neumann boundary conditions or periodic boundary conditions. Suppose that $a, b, d_1, d_2 > 0$. Let X be a subspace $H^1(0,L) \times H^1(0,L)$ with corresponding boundary conditions. Let X be a subspace $H^1_{per}(0,L)$. Since

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L (u^2+v^2)dx &= \int_0^L (u_t u + v_t v)dx \\ &= \int_0^L [-d_1 u_x^2 + au^2 - u^4 - u^2v^2]dx \\ &\quad + \int_0^L [-d_2 v_x^2 - bv^2 - v^4 + u^2 v^2]dx \\ &\le a\int_0^L u^2 dx, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L (u_x^2+v_x^2)dx &= -\int_0^{2\pi} (u_t u_{xx} + v_t v_{xx})dx \\ &= \int_0^L [-d_1 u_{xx}^2 + au_x^2 - 3u^2u_x^2 - 2uu_x v v_x]dx \\ &\quad + \int_0^L [-d_2 v_{xx}^2 - bv_x^2 - 3v^2 v_x^2 + 2uu_x v v_x]dx \\ &\le a\int_0^{2\pi} u_x^2 dx, \end{aligned}$$

the solutions of (5.24) exist in $X \times X$ for all $t \geq 0$. The linear variational equation of (5.24) at a solution $(u(t,x), v(t,x))$ is

$$\begin{aligned} p_t &= d_1 p_{xx} + (a - 2u^2 - v^2)p - 2uvq, \\ q_t &= d_2 q_{xx} + (-b - 2v^2 - u^2)q + 2uvp, \quad 0 < x < L, \quad t > 0, \end{aligned}$$

and the pointwise representation of the second compound differential equation defined on $\bigwedge^2 X$ is

$$\begin{aligned} \frac{\partial w_{11}}{\partial t} &= d_1(\Delta_1 w_{11} + \Delta_2 w_{11}) + (a_{11}(t,x_1) + a_{11}(t,x_2))w_{11} \\ &\quad + a_{12}(t,x_1)w_{21} + a_{12}(t,x_2)w_{12}, \\ \frac{\partial w_{22}}{\partial t} &= d_2(\Delta_1 w_{22} + \Delta_2 w_{22}) + (a_{22}(t,x_1) + a_{22}(t,x_2))w_{22} \\ &\quad + a_{21}(t,x_1)w_{12} + a_{21}(t,x_2)w_{21}, \\ \frac{\partial w_{12}}{\partial t} &= d_1\Delta_1 w_{12} + d_2\Delta_2 w_{12} + (a_{11}(t,x_1) + a_{22}(t,x_2))w_{12} \\ &\quad + a_{12}(t,x_1)w_{22} + a_{21}(t,x_2)w_{11}, \end{aligned}$$

where $\Delta_i w = \frac{\partial^2}{\partial x_i \partial x_i}$ and

$$a_{11} = a - 2u^2 - v^2, \quad a_{12} = -2uv, \quad a_{21} = 2uv, \quad a_{22} = b - 2v^2 - u^2$$

(see (3.26) of Section 3.2 for details). Consider a Lyapunov function

$$V(t) := \frac{1}{2!}\int_{(0,L)^2} \left((w_{11})^2 + (w_{22})^2 + 2(w_{12})^2\right).$$

Then the derivative of V calculated for (3.28) of Section 3.2 implies that

$$\frac{d^+V}{dt} \leq \int_{(0,L)^2} \left(\mu_1(t)(w_{11})^2 + \mu_2(t)(w_{22})^2 + 2\mu_3(t)(w_{12})^2\right) \qquad (5.25)$$

where

$$\begin{aligned} \mu_1(t) &= -(\lambda_1 + \lambda_2)\, d_1 + \max_x \{2a - 4u^2(t,x) - 2v^2(t,x)\} \\ &\leq -(\lambda_1 + \lambda_2)\, d_1 + 2a, \\ \mu_2(t) &= -(\lambda_1 + \lambda_2)\, d_2 + \max_x \{-2b - 4v^2(t,x) - 2u^2(t,x)\} \\ &\leq -(\lambda_1 + \lambda_2)\, d_2 - 2b, \\ \mu_3(t) &= -\lambda_1(d_1 + d_2) + \max_x \{a - u^2(t,x) - v^2(t,x)\} \\ &\quad + \max_x \{-b - 2v^2(t,x) - u^2(t,x)\} \\ &\leq -\lambda_1(d_1 + d_2) + a - b, \end{aligned}$$

and $\lambda_1 \leq \lambda_2 \leq \cdots$ are eigenvalues of the Laplace equation $u_{xx} = -\lambda_i u$ with corresponding boundary conditions.

Corollary 5.5. *If*

$$a < \min\left\{\frac{\lambda_1 + \lambda_2}{2} d_1, b + \lambda_1(d_1 + d_2)\right\},$$

then the system (5.24) *contains no simple closed curve that is invariant in* $L^2(0, L)$.

Chapter 6

An Example from Epidemiology

A general account of spatial dispersion of biological populations is contained in the work of Skellam [97], and in the recent books by Okubo [86] and Murray [81]. Early results on models for the spatial dependence of the spread of epidemics are presented by Fisher [32], Kolmogorov, Petrovsky and Piscounov [49] and Turing [105]. In particular, studies on infectious diseases with diffusion have been discussed by Capasso [13], Capasso and Fortunato [14], deMottoni, Orlandi and Tesei [24], Gurtin and MacCamy [36], MacCamy [64], Kallen, Arcuri and Murrary [48], and Webb [108].

In this chapter, as an example, the Bendixson criterion of Chapter 6 is used to preclude the existence of nontrivial periodic solutions to a diffusive epidemiological SIR model

$$\begin{cases} S_t = d_1 S_{xx} + \Lambda - \beta(x) IS - b_1 S, & \\ I_t = d_2 I_{xx} + \beta(x) IS - (b_2 + \gamma) I, & 0 < x < 1, \quad t > 0 \\ R_t = d_3 R_{xx} + \gamma I - b_3 R, & \end{cases} \tag{6.1}$$

with homogeneous Neumann boundary conditions:

$$\begin{cases} S_x(t,0) = S_x(t,1) = 0, & \\ I_x(t,0) = I_x(t,1) = 0, & t \geq 0 \\ R_x(t,0) = R_x(t,1) = 0, & \end{cases} \tag{6.2}$$

and initial conditions:

$$\begin{cases} S(0,x) = S_0(x) \geq 0, \\ I(0,x) = I_0(x) \geq 0, & 0 \leq x \leq 1, \\ R(0,x) = R_0(x) \geq 0. \end{cases} \tag{6.3}$$

Here S, I, R denote the populations that are susceptible, infectious, and recovered, see Figure 6.1. The parameter $b_i > 0, i = 1,2,3$ is the death rate, $\gamma > 0$ is the recovery rate and $d_i \geq 0, i = 1,2,3$ is the diffusion coefficients. The parameter $\beta(x)$ is the effective per capita contact rate among individuals which is assumed to be spatially dependent and $\beta(x) \in C([0,1])$ are nonnegative, $\|\beta\|_\infty = \max\limits_{0\leq x\leq 1} \beta(x) > 0$. It is also assumed that there is a constant recruitment $\Lambda > 0$ for this model.

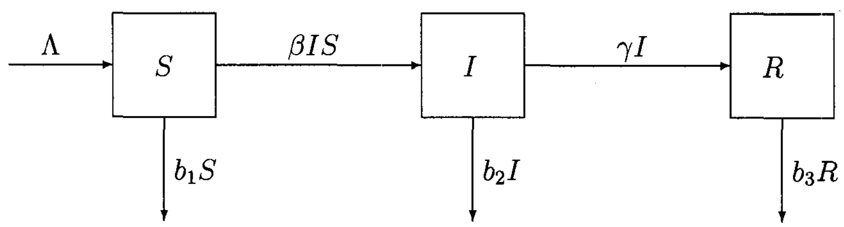

Figure 6.1: A diffusive SIR model.

The model without diffusion, a system of ordinary differential equations, is well understood; see Hethcote [44] and Brauer, van den Driessche and Wu [9]. This chapter presents some results on the existence and the structure of the global attractor of (6.3), and the nonexistence of periodic solutions, when diffusion is added and the contact rate β is spatial dependent. Consequently, the existence of at least one positive, x-dependent endemic steady state solution is obtained.

The approach to the existence and boundedness is motivated by the work of Dung and Smith [28] in their study of a parabolic system modeling microbial populations with competition.

6.1 Existence and Boundedness

Let $C_+([0,1])$ be the cone of nonnegative functions in the Banach space $C([0,1])$ with the usual supremum norm $\|\cdot\|_\infty$. For $u = (u_1, \cdots, u_k) \in$

$X_+ \times \cdots \times X_+$, define $\|u\|_\infty = \max_{1\le i\le k} |u_i|$.

Theorem 6.1. *The system* (6.1)-(6.3) *generates a nonlinear local semidynamical system* $T(t)_{t\ge0}$ *on the space* $C_+([0,1])\times C_+([0,1])\times C_+([0,1])$. *Moreover, the solutions to* (6.1)-(6.3) *are in fact classical solutions.*

Remark 6.1. The system (6.1)-(6.3) is a system of semilinear parabolic partial differential equations. The standard analysis on the existence by fixed point theory, see Rothe [92], Pazy [87], Henry [42] and Wu [111].

Let $\tau(S_0, I_0, R_0)$ be the maximal interval of existence of the system (6.1)-(6.3). The first two equations in (6.1) do not contain R, which allows the reduction of (6.1) to

$$\begin{cases} S_t = d_1 S_{xx} + \Lambda - \beta(x)IS - b_1 S, \\ I_t = d_2 I_{xx} + \beta(x)IS - (b_2+\gamma)I. \end{cases} \tag{6.4}$$

In the following, the system (6.4) is studied instead of (6.1). For simplicity, let $X_+ = C_+([0,1]) \times C_+([0,1])$.

Consider the following ordinary differential equation

$$\begin{cases} u_t = \Lambda - \beta(x)Iu - b_1 u, \quad 0 < x < 1, \quad t > 0 \\ u(0) = \|S_0\|_\infty. \end{cases} \tag{6.5}$$

Then

$$\begin{aligned} u(t) &\le \left(\|S_0\|_\infty - \frac{\Lambda}{b_1}\right) e^{-b_1 t} + \frac{\Lambda}{b_1} \\ &\le \max\left\{\|S_0\|_\infty, \frac{\Lambda}{b_1}\right\} := S_{max}, \end{aligned}$$

and thus

$$\limsup_{t\to\infty} u(t) \le \frac{\Lambda}{b_1}\ .$$

Since

$$\begin{aligned} -S_t + d_1 S_{xx} + \Lambda - \beta(x)IS - b_1 S &= -u_t + \Lambda - \beta(x)Iu - b_1 u, \\ S(0,x) &\le u(0), \end{aligned}$$

the comparison theorem (see Smoller [103], page 94) implies that

$$\begin{aligned} &S(t,x) \le u(t) \le S_{max}, && \text{for all } 0 \le x \le 1,\ t \ge 0, \\ &\limsup_{t\to\infty} S(t,x) \le \tfrac{\Lambda}{b_1}, && \text{for all } 0 \le x \le 1. \end{aligned} \tag{6.6}$$

Now for any $\varepsilon \geq 0$, define a feasible region

$$\Gamma_\varepsilon := \left\{ (S(x), I(x)) \in X_+ : \|S\|_\infty \leq \frac{\Lambda}{b_1} + \varepsilon \right\}. \tag{6.7}$$

Proposition 6.2. *Γ_ε is positively invariant for* (6.2)-(6.4) *and for $\varepsilon > 0$, Γ_ε is an absorbing set in X_+.*

Proof. For any $\varepsilon \geq 0$ and any solution $(S(t), I(t))$ of (6.2)-(6.4) with $\|S_0\|_\infty \leq \frac{\Lambda}{b_1} + \varepsilon$, since

$$\begin{aligned}
-S_t + d_1 S_{xx} + \Lambda - \beta(x) IS - b_1 S &\geq -\beta(x) I \left(\frac{\Lambda}{b_1} + \varepsilon \right) - b_1 \varepsilon, \\
S_x(t,0) = S_x(t,1) &= 0, \\
S(0,x) &\leq \frac{\Lambda}{b_1} + \varepsilon,
\end{aligned}$$

a comparison theorem implies that

$$S(t,x) \leq \frac{\Lambda}{b_1} + \varepsilon, \qquad \text{for all } 0 \leq x \leq 1,\ t \geq 0.$$

Thus Γ_ε is positively invariant.

For any bounded set $\mathcal{B}$ in X_+, there exists a M such that $\|S\|_\infty \leq M$ for every $(S(x), I(x)) \in \mathcal{B}$. Suppose that $\varepsilon > 0$. If $M \leq \frac{\Lambda}{b_1} + \varepsilon$, choose $T = 0$, otherwise choose $T = \frac{1}{b_1} \ln \frac{M - \frac{\Lambda}{b_1}}{\varepsilon}$. Then, for $t \geq T$,

$$\begin{aligned}
S(t,x) &\leq \left(\|S_0\|_\infty - \frac{\Lambda}{b_1} \right) e^{-b_1 t} + \frac{\Lambda}{b_1} \\
&\leq \left(M - \frac{\Lambda}{b_1} \right) e^{-b_1 T} + \frac{\Lambda}{b_1} \\
&\leq \frac{\Lambda}{b_1} + \varepsilon,
\end{aligned}$$

which implies that Γ_ε is an absorbing set in X_+ when $\varepsilon > 0$. ∎

Since the solution of (6.2)-(6.4) is attracted to the set

$$\Gamma_0 = \left\{ (S(x), I(x)) \in X_+ : \|S\|_\infty \leq \frac{\Lambda}{b_1} \right\}$$

for larger time, in the following the dynamics of (6.2)-(6.4) in Γ_0 is studied. It follows from

$$(S+I)_t = d_1 S_{xx} + d_2 I_{xx} + \Lambda - b_1 S - (b_2+\gamma) I$$

that

$$\begin{aligned}
\int_0^1 (S+I)_t dx &= \int_0^1 (d_1 S_{xx} + d_2 I_{xx} + \Lambda - b_1 S - (b_2+\gamma)I)\, dx \\
&= d_1 S_x \big|_{x=0}^{x=1} + d_2 I_x \big|_{x=0}^{x=1} + \int_0^1 (\Lambda - b_1 S - (b_2+\gamma)I) dx \\
&\le \Lambda - b_m \int_0^1 (S+I) dx,
\end{aligned}$$

where $b_m = \min\{b_1, b_2+\gamma\}$, which implies that

$$\begin{aligned}
\int_0^1 (S(t,x)+I(t,x))dx &\le \left(\int_0^1 (S_0(x)+I_0(x))dx - \frac{\Lambda}{b_m}\right) e^{-b_m t} + \frac{\Lambda}{b_m} \\
&\le \max\left\{\int_0^1 (S_0(x)+I_0(x))dx, \frac{\Lambda}{b_m}\right\} \\
&\le \frac{\Lambda}{b_m} + \|I_0\|_\infty := C_1(\|I_0\|_\infty),
\end{aligned} \tag{6.8}$$

where $C_1(\|I_0\|_\infty)$ is a positive continuous function, and thus

$$\limsup_{t\to\infty} \int_0^1 (S(t,x)+I(t,x))dx \le \frac{\Lambda}{b_m}.$$

In particular,

$$\int_0^1 I(t,x)dx \le C_1(\|I_0\|_\infty), \tag{6.9}$$

and

$$\limsup_{t\to\infty} \int_0^1 I(t,x)dx \le \frac{\Lambda}{b_m} := c_1. \tag{6.10}$$

The next proposition uses mathematical induction to estimate the L^p norm of I. The argument is standard and has been used by Dung and Smith [28].

Proposition 6.3. *For each $0 < p < \infty$, there exists a positive continuous function $C_p(\|I_0\|_\infty)$ and a positive constant c_p such that*

$$\|I(t)\|_p \le C_p(\|I_0\|_\infty), \qquad t \ge 0 \tag{6.11}$$

and

$$\limsup_{t\to\infty} \|I(t)\|_p \le c_p. \tag{6.12}$$

Proof. Assume that (6.11) holds for some $p \ge 1$, in particular, it holds for $p = 1$ by (6.9) and (6.10). It is sufficient to prove that it holds for exponent $2p$. Here only the bound on L_2 norm is proved. The general case can be obtained similarly and will be omitted. Since

$$\begin{aligned}\frac{d}{dt}\int_0^1 I^2 dx &= 2d_2\int_0^1 II_{xx}dx + 2\int_0^1 \beta(x)SI^2dx - 2(b_2+\gamma)\int_0^1 I^2dx \\ &\le -2d_2\int_0^1 I_x^2dx + 2\|\beta\|_\infty\frac{\Lambda}{b_1}\int_0^1 I^2dx - 2(b_2+\gamma)\int_0^1 I^2dx,\end{aligned}$$

the Nirenberg-Gagliardo inequality (see Henry [42], page 37) and Young's inequality imply that

$$\begin{aligned}\int_0^1 I^2dx &\le C\left(\int_0^1 I_x^2dx + \int_0^1 I^2dx\right)^{\frac{1}{2}}\left(\int_0^1 Idx\right) \\ &\le C\varepsilon_1\left(\int_0^1 I_x^2dx + \int_0^1 I^2dx\right) + \frac{C}{\varepsilon_1}\left(\int_0^1 Idx\right)^2,\end{aligned} \tag{6.13}$$

where C and ε_1 are positive constants. Thus,

$$\begin{aligned}&\frac{d}{dt}\int_0^1 I^2dx \\ \le\ & -2d_2\int_0^1 I_x^2dx + 2\|\beta\|_\infty\frac{\Lambda}{b_1}C\varepsilon_1\left(\int_0^1 I_x^2dx + \int_0^1 I^2dx\right) \\ & +2\|\beta\|_\infty\tfrac{\Lambda}{b_1}\tfrac{C}{\varepsilon_1}\left(\int_0^1 Idx\right)^2 - 2(b_2+\gamma)\int_0^1 I^2dx \\ \le\ & 2\left(-d_2 + \|\beta\|_\infty\frac{\Lambda}{b_1}C\varepsilon_1\right)\int_0^1 I_x^2dx + 2\left(-(b_2+\gamma) + \|\beta\|_\infty\frac{\Lambda}{b_1}C\varepsilon_1\right)\int_0^1 I^2dx \\ & +2\|\beta\|_\infty\frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}\left(\int_0^1 Idx\right)^2 \\ \le\ & -2\left(d_2 - \|\beta\|_\infty\frac{\Lambda}{b_1}C\varepsilon_1\right)\int_0^1 I_x^2dx - 2\left((b_2+\gamma) - \|\beta\|_\infty\frac{\Lambda}{b_1}C\varepsilon_1\right)\int_0^1 I^2dx \\ & +2M(\|I(t)\|_1) \\ :=\ & -2\alpha_1\int_0^1 I_x^2dx - 2\alpha_2\int_0^1 I^2dx + 2M(\|I(t)\|_1),\end{aligned}$$

where $M(\|I(t)\|_1) = \|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}\left(\int_0^1 I(t)dx\right)^2$. Moreover, from (6.9) and (6.10),

$$M(\|I(t)\|_1) \le \|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}(C_1(\|I_0\|_\infty))^2,$$

and

$$\limsup_{t\to\infty} M(\|I(t)\|_1) \le \|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}\left(\frac{\Lambda}{b_m}\right)^2 = \|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}(c_1)^2.$$

Choose $\varepsilon_1 < \frac{1}{C\|\beta\|_\infty \frac{\Lambda}{b_1}} \min\{d_2, b_2+\gamma\}$. Then $\alpha_1, \alpha_2 > 0$ and

$$\begin{aligned}\int_0^1 I^2(t,x)dx &\le \left(\int_0^1 I_0^2(x)dx - \frac{M(\|I(t)\|_1)}{\alpha_2}\right)e^{-2\alpha_2 t} + \frac{M(\|I(t)\|_1)}{\alpha_2} \\ &\le \max\left\{\|I_0\|_\infty^2, \frac{\|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}(C_1(\|I_0\|_\infty))^2}{\alpha_2}\right\} \\ &\le \max\left\{\|I_0\|_\infty^2, \frac{\|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}(\frac{\Lambda}{b_m}+\|I_0\|_\infty)^2}{\alpha_2}\right\} := (C_2(\|I_0\|_\infty))^2.\end{aligned} \tag{6.14}$$

Therefore, $\|I(t)\|_2$ is defined for all $t \ge 0$ and there exists a positive constant $c_2 = \sqrt{\frac{\|\beta\|_\infty \frac{\Lambda}{b_1}\frac{C}{\varepsilon_1}(c_1)^2}{\alpha_2}}$

$$\limsup_{t\to\infty} \|I(t)\|_2 \le c_2.$$

■

Theorem 6.4 shows an upper bound of $\|I\|_\infty$, whose proof is based on the integral equation of $I(t)$ and the Sobolev compact embedding property.

Theorem 6.4. *The solution of (6.2)-(6.4) exists for all $t \ge 0$ in Γ_0. Furthermore, there exists a positive continuous function $C(\|I_0\|_\infty)$ and a positive constant c, independent of the initial data I_0, such that*

$$\|I(t)\|_\infty \le C(\|I_0\|_\infty), \qquad t \ge 0, \tag{6.15}$$

and

$$\limsup_{t\to\infty} \|I(t)\|_\infty \le c. \tag{6.16}$$

Proof. First, consider problem (6.2)-(6.4) in the larger space $Y = L^p((0,1))$. The solution $I(t)$ satisfies

$$I(t) = e^{-Bt}I_0 + \int_0^t e^{-B(t-s)}g_2(S(s), I(s))ds, \tag{6.17}$$

where $B = -d_2\triangle + (b_2 + \gamma)I$ with homogeneous Neumann boundary conditions,

$$g_2(S(t), I(t))(x) = g_2(x, S(t,x), I(t,x)) = \beta(x)S(t,x)I(t,x).$$

The operator B generates an analytic semigroup $\{T(t) = e^{-Bt}\}_{t>0}$ on Y with $0 \in \rho(B)$. The semigroup of operators $\{T(t) = e^{-Bt}\}_{t>0}$ map Y into the space $Y^\alpha = D(B^\alpha)$ with the norm $\|u\|_{Y^\alpha} = \|B^\alpha u\|_p$, where $0 < \alpha < 1$ and B^α is the fractional power of B, see Pazy [87], page 74 and page 242. Choose p such that $\frac{1}{2p} < \alpha < 1$ and the embedding

$$Y^\alpha \xrightarrow[\text{cont.}]{} C^\nu([0,1]), \qquad 0 \leq \nu < 2\alpha - \frac{1}{p} \tag{6.18}$$

is continuous, see Henry [42], page 39 and Pazy [87], page 243. For each $I_0 \in X_+$,

$$B^\alpha I(t) = B^\alpha e^{-Bt} I_0 + \int_0^t B^\alpha e^{-B(t-s)} g_2(S(s), I(s))ds.$$

From the L^p estimates of $I(t)$ in Proposition 6.3 and $\|S(t)\|_\infty \leq \frac{\Lambda}{b_1}$, for $t \geq 0$, we have that there is a positive function $G_p = G_p(\|I_0\|_\infty) = \|\beta\|_\infty \frac{\Lambda}{b_1} C_p(\|I_0\|_\infty)$ such that

$$\begin{aligned} \|g_2(S(t), I(t))\|_p &= \left(\int_0^1 (\beta(x)S(t,x)I(t,x))^p dx\right)^{\frac{1}{p}} \\ &\leq \|\beta\|_\infty \frac{\Lambda}{b_1} \left(\int_0^1 I^p dx\right)^{\frac{1}{p}} \\ &\leq G_p(\|I_0\|_\infty), \end{aligned}$$

and there is a positive constant $g_p = \|\beta\|_\infty \frac{\Lambda}{b_1} c_p$, independent of $\|I_0\|_\infty$, such that

$$\limsup_{t\to\infty} \|g_2(S(t), I(t))\|_p \leq g_p.$$

Therefore, there is a $\eta = \eta(I_0) > 0$ such that $\|g_2(S(t), I(t))\|_p \leq 2g_p$ for

$t \geq \eta$. Consequently, for $t \geq \eta$,

$$
\begin{aligned}
&\|I(t)\|_{Y^\alpha} = \|B^\alpha I(t)\|_p \\
\leq\ & \|B^\alpha e^{-Bt} I_0\|_p + \int_0^t \|B^\alpha e^{-B(t-s)}\|_{\mathscr{L}(Y,Y)} \|g_2(S(s), I(s))\|_p ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} \|g_2(S(s), I(s))\|_p ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \int_0^\eta G_p(\|I_0\|_\infty) C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} ds \\
& + \int_\eta^t 2g_p C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \eta G_p(\|I_0\|_\infty) C_\alpha (t-\eta)^{-\alpha} e^{-\delta(t-\eta)} \\
& + 2g_p C_\alpha \int_0^{t-\eta} r^{-\alpha} e^{-\delta r} dr \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \eta G_p(\|I_0\|_\infty) C_\alpha (t-\eta)^{-\alpha} e^{-\delta(t-\eta)} \\
& + 2g_p C_\alpha \int_0^\infty r^{-\alpha} e^{-\delta r} dr,
\end{aligned}
$$

and thus $\|I(t)\|_{Y^\alpha}$ is defined for all $t \geq 0$ and

$$
\limsup_{t \to \infty} \|I(t)\|_{Y^\alpha} \leq 2g_p C_\alpha \int_0^\infty r^{-\alpha} e^{-\delta r} dr = 2g_p C_\alpha \delta^{-1+\alpha} \Gamma(1-\alpha). \tag{6.19}
$$

Furthermore, the estimate (6.19) and the embedding (6.18) implies that (6.16) holds. For $t \geq 1$,

$$
\begin{aligned}
&\|I(t)\|_{Y^\alpha} = \|B^\alpha I(t)\|_p \\
\leq\ & \|B^\alpha e^{-Bt} I_0\|_p + \int_0^t \|B^\alpha e^{-B(t-s)}\|_{\mathscr{L}(Y,Y)} \|g_2(S(s), I(s))\|_p ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} \|g_2(S(s), I(s))\|_p ds \\
=\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + G_p(\|I_0\|_\infty) C_\alpha \int_0^t r^{-\alpha} e^{-\delta r} dr \\
\leq\ & C_\alpha e^{-\delta} \|I_0\|_p + G_p(\|I_0\|_\infty) C_\alpha \int_0^\infty r^{-\alpha} e^{-\delta r} dr \\
\leq\ & C_\alpha e^{-\delta} \|I_0\|_\infty + G_p(\|I_0\|_\infty) C_\alpha \delta^{-1+\alpha} \Gamma(1-\alpha),
\end{aligned} \tag{6.20}
$$

and thus the embedding (6.18) implies that (6.15) holds for $t \geq 1$. For $0 \leq t \leq 1$, it follows from (6.17) and $\|e^{-Bt}\|_{\mathscr{L}(X_+,X_+)} \leq e^{-(d_2+\gamma)t}$ that

$$\begin{aligned}
&\|I(t)\|_\infty \\
\leq\ & \|e^{-Bt}I_0\|_\infty + \int_0^t \|e^{-B(t-s)}g_2(S(s),I(s))\|_\infty ds \\
\leq\ & \|e^{-Bt}\|_{\mathscr{L}(X,X)}\|I_0\|_\infty + C\int_0^t \|e^{-B(t-s)}g_2(S(s),I(s))\|_{Y^\alpha} ds \\
\leq\ & e^{-(d_2+\gamma)t}\|I_0\|_\infty + C\int_0^t \|B^\alpha e^{-B(t-s)}g_2(S(s),I(s))\|_p ds \\
\leq\ & \|I_0\|_\infty + C\int_0^t \|B^\alpha e^{-B(t-s)}\|_{\mathscr{L}(Y,Y)}\|g_2(S(s),I(s))\|_p ds \\
\leq\ & \|I_0\|_\infty + C\int_0^t C_\alpha (t-s)^{-\alpha}e^{-\delta(t-s)}\|g_2(S(s),I(s))\|_p ds \\
\leq\ & \|I_0\|_\infty + CG_p(\|I_0\|_\infty)C_\alpha \int_0^t r^{-\alpha}e^{-\delta r}dr \\
\leq\ & \|I_0\|_\infty + CG_p(\|I_0\|_\infty)C_\alpha \int_0^\infty r^{-\alpha}e^{-\delta r}dr
\end{aligned}$$

and thus

$$\|I(t)\|_\infty \leq \|I_0\|_\infty + CG_p(\|I_0\|_\infty)C_\alpha\delta^{-1+\alpha}\Gamma(1-\alpha). \tag{6.21}$$

Therefore, (6.15) holds for all $t \geq 0$.

■

Now define a new feasible region

$$\Theta := \{(S(x),I(x))^T \in X_+ : \|S\|_\infty \leq \frac{\Lambda}{b_1},\ \|I\|_\infty \leq 2c\}, \tag{6.22}$$

where c is from Theorem 6.4.

Corollary 6.5. *Θ is a bounded absorbing set for* (6.2)-(6.4) *in* Γ_0.

Proof. For any even number $p \geq 2$ such that $\frac{1}{2p} < \alpha < 1$,

$$\begin{aligned}
&\|g_2(S(t),I(t))\|_p \\
\leq\ & \|\beta\|_\infty \frac{\Lambda}{b_1}\left(\int_0^1 I^p dx\right)^{\frac{1}{p}} \\
\leq\ & \|\beta\|_\infty \frac{\Lambda}{b_1}\left(\left(\int_0^1 I_0^p dx - \frac{M(\|I(t)\|_{\frac{p}{2}})}{\alpha_2}\right)e^{-2p\alpha_2 t} + \frac{M(\|I(t)\|_{\frac{p}{2}})}{\alpha_2}\right)^{\frac{1}{p}}.
\end{aligned}$$

It can be shown that for any bounded set $\mathcal{B}$ in Γ_0, there is a $\eta_1 = \eta_1(\mathcal{B}) > 0$ such that for $t \geq \eta_1$,

$$\|g_2(S(t), I(t))\|_p \leq \|\beta\|_\infty \frac{\Lambda}{b_1} 2 \left(\frac{\|\beta\|_\infty \frac{\Lambda}{b_1} \frac{C}{\varepsilon_1} (c_{\frac{p}{2}})^p}{\alpha_2} \right)^{\frac{1}{p}} = 2g_p$$

holds for all $(S(t), I(t))^T \in \mathcal{B}$. Consequently, for $t \geq \eta_1$,

$$\begin{aligned}
&\|I(t)\|_{Y^\alpha} = \|B^\alpha I(t)\|_p \\
\leq\ & \|B^\alpha e^{-Bt} I_0\|_p + \int_0^t \|B^\alpha e^{-B(t-s)}\|_{\mathscr{L}(Y,Y)} \|g_2(S(s), I(s))\|_p ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_p + \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} \|g_2(S(s), I(s))\|_p ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_\infty + \int_0^{\eta_1} G_p(\|I_0\|_\infty) C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} ds \\
& + \int_{\eta_1}^t 2g_p C_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} ds \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_\infty + \eta_1 G_p(\|I_0\|_\infty) C_\alpha (t-\eta_1)^{-\alpha} e^{-\delta(t-\eta_1)} \\
& + 2g_p C_\alpha \int_0^{t-\eta_1} r^{-\alpha} e^{-\delta r} dr \\
\leq\ & C_\alpha t^{-\alpha} e^{-\delta t} \|I_0\|_\infty + \eta_1 G_p(\|I_0\|_\infty) C_\alpha (t-\eta_1)^{-\alpha} e^{-\delta(t-\eta_1)} \\
& + 2g_p C_\alpha \int_0^\infty r^{-\alpha} e^{-\delta r} dr,
\end{aligned}$$

and thus there exists a $\eta_2 = \eta_2(\mathcal{B}) > \eta_1$ such that for $t > \eta_2$,

$$\|I(t)\|_{Y^\alpha} \leq 4g_p C_\alpha \int_0^\infty r^{-\alpha} e^{-\delta r} dr = 4g_p C_\alpha \delta^{-1+\alpha} \Gamma(1-\alpha).$$

Furthermore, for $t > \eta_2$,

$$\|I(t)\|_\infty \leq C \|I(t)\|_{Y^\alpha} \leq 4C g_p C_\alpha \delta^{-1+\alpha} \Gamma(1-\alpha) = 2c,$$

where C is the embedding constant in (6.18). Therefore, Θ is a bounded absorbing set in Γ_0.

■

Theorem 6.6. *There exists a compact, connected attractor $\mathcal{A} = \omega(\Theta)$ of (6.2)-(6.4) which attracts the bounded sets in X_+ and is the maximal bounded attractor in X_+.*

Proof. The above result shows that system (6.2)-(6.4) generates a nonlinear semi-dynamical system $T(t)_{t\geq 0}:\Gamma_0\to\Gamma_0$. Let $T(t)=(T_1(t),T_2(t))$, where $S(t)=T_1(t)S_0$ and $I(t)=T_2(t)I_0$. From (6.20), for any $t_1\geq 1$,

$$\|I(t_1)\|_{Y^\alpha}\leq C_\alpha e^{-\delta}\|I_0\|_\infty + M_p(\|I_0\|_\infty)C_\alpha\delta^{-1+\alpha}\Gamma(1-\alpha):=M(\|I_0\|_\infty)$$

and thus

$$\|I(t_1)\|_{C^\nu}\leq C\|I(t_1)\|_{Y^\alpha}\leq CM(\|I_0\|_\infty).$$

Since

$$C^\nu([0,1])\xrightarrow[\text{comp.}]{} C([0,1]), \tag{6.23}$$

$T_2(t_1)$ is compact. A similar argument shows that $T_1(t_1)$ is also compact. Thus, for every bounded set $\mathcal{B}\subseteq\Gamma_0$ and $t_1\geq 1$,

$$T(t_1)\mathcal{B}\subseteq\{(S(x),I(x))\in X_+:\|S(t_1)\|_{C^\nu}\leq N,\|I(t_1)\|_{C^\nu}\leq M\},$$

where N and M depend on $\mathcal{B}$. Theorem 1.1 of Temam [104], page 23 implies that the ω-limit set of Θ, $\mathcal{A}=\omega(\Theta)$, is a compact, connected attractor of (6.2)-(6.4) which attracts the bounded sets in Γ_0 and is the maximal bounded attractor in Γ_0. Furthermore, Proposition 6.2 implies that Γ_ε is an absorbing set in X_+ when $\varepsilon>0$. It can be shown that $\omega(\Gamma_\varepsilon)\subseteq\Gamma_0$. Therefore, $\mathcal{A}=\omega(\Theta)$ is a global attractor for (6.2)-(6.4) in X_+. ∎

6.2 Nonexistence of Periodic Solutions

In this section, Theorem 5.3 of Section 5.2 will be used to rule out the existence of nontrivial periodic orbits. A particular conclusion that can be drawn from this, as a consequence of Theorem 4.12 of Section 4.2, is the existence of at least one positive steady state solution is obtained.

The linear variational equation of (6.4) at any solution $(S(t,x),I(t,x))\in\Theta$ of (6.2)-(6.4) is

$$\begin{cases} u_t &= d_1u_{xx}-(\beta(x)I+b_1)u-\beta(x)Sv,\\ v_t &= d_2v_{xx}+\beta(x)Iu+(\beta(x)S-(b_2+\gamma))v,\end{cases} \tag{6.24}$$

with homogeneous Neumann boundary conditions. The pointwise representation of the second compound differential equation of (6.24) is

$$\begin{aligned}
\frac{\partial w_{11}}{\partial t} &= d_1(\Delta_1 w_{11} + \Delta_2 w_{11}) + (a_{11}(t,x_1) + a_{11}(t,x_2))w_{11} \\
&\quad + a_{12}(t,x_1)w_{21} + a_{12}(t,x_2)w_{12}, \\
\frac{\partial w_{22}}{\partial t} &= d_2(\Delta_1 w_{22} + \Delta_2 w_{22}) + (a_{22}(t,x_1) + a_{22}(t,x_2))w_{22} \\
&\quad + a_{21}(t,x_1)w_{12} + a_{21}(t,x_2)w_{21}, \\
\frac{\partial w_{12}}{\partial t} &= d_1\Delta_1 w_{12} + d_2\Delta_2 w_{12} + (a_{11}(t,x_1) + a_{22}(t,x_2))w_{12} \\
&\quad + a_{12}(t,x_1)w_{22} + a_{21}(t,x_2)w_{11}.
\end{aligned} \tag{6.25}$$

where $\Delta_i w = \frac{\partial^2}{\partial x_i \partial x_i}$ and

$a_{11} = -(\beta(x)I + b_1)$, $a_{12} = -\beta(x)S$, $a_{21} = \beta(x)I$, $a_{22} = (\beta(x)S - (b_2+\gamma))$

(see (3.26) of Section 3.2 for details). Consider a Lyapunov function

$$V(t) := \frac{1}{2!}\int_{(0,L)^2} \left((w_{11})^2 + (w_{22})^2 + 2(w_{12})^2\right).$$

Then the derivative of V calculated for (3.28) of Section 3.2 implies that

$$\frac{d^+V}{dt} \le \int_{(0,L)^2} \left(\mu_1(t)(w_{11})^2 + \mu_2(t)(w_{22})^2 + 2\mu_3(t)(w_{12})^2\right) \tag{6.26}$$

where

$$\begin{aligned}
\mu_1(t) &= -d_1\pi^2 - 2b_1 + \frac{1}{\varepsilon}\|\beta\|_\infty\frac{\Lambda}{b_1} + \frac{2c}{\varepsilon}\|\beta\|_\infty \\
\mu_2(t) &= -d_2\pi^2 - 2(b_2+\gamma) + 2\|\beta\|_\infty\frac{\Lambda}{b_1} + \frac{2c}{\varepsilon}\|\beta\|_\infty + \frac{1}{\varepsilon}\|\beta\|_\infty\frac{\Lambda}{b_1} \\
\mu_3(t) &= -b_1 - (b_2+\gamma) + \|\beta\|_\infty\frac{\Lambda}{b_1} + \varepsilon\|\beta\|_\infty\frac{\Lambda}{b_1} + 2c\varepsilon\|\beta\|_\infty
\end{aligned}$$

where $\|\beta\|_\infty = \max_{0\le x\le 1}\beta(x)$, $\varepsilon >$ is an arbitrary constant and the constant c is defined by (6.22).

Theorem 6.7. *If the diffusion terms d_1 and d_2 are sufficiently large and*

$$b_1 + (b_2+\gamma) > \|\beta\|_\infty\frac{\Lambda}{b_1},$$

then system (6.2)-(6.4) has no periodic solutions in Θ and thus there exists a positive, x-dependent steady state solution.

Appendix A

Lozinskiĭ Measures

This appendix is a summary on Lozinskiĭ measures. Detailed information on Lozinskiĭ measures of matrices will be referred to Coppel [19], Muldowney [76–78] and of a bounded linear operators on a Banach space will be referred to Daleckiĭ and Kreĭn [23], page 61.

Let X be a Banach space. For any $x, y \in X$, the limit

$$\partial_y \|x\| := \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h}, \tag{A.1}$$

always exists, see Coppel [19], page 3 and Daleckiĭ and Kreĭn [23], page 61. Let $\mathscr{B}(X)$ denote the space of all bounded linear operators from X to X.

Definition A.1. For any $A \in \mathscr{B}(X)$, the *Lozinskiĭ measure* of A is defined by

$$\mu(A) := \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}. \tag{A.2}$$

Remark A.1. It follows from $(A.1)$ that $\mu(A)$ exists. The "Lozinskiĭ measure" μ had previously been called the "Lozinskii logarithmic norm"; however, it is not a norm since it can take negative values. The term "Lozinskiĭ measure" has been used by Coppel [19].

The following propositions are stated in Daleckiĭ and Kreĭn [23], page 61-62.

Proposition A.1. *For any* $A, B \in \mathscr{B}(X)$,

(*i*) $\mu(\alpha A) = \alpha \mu(A)$, *if* $\alpha \geq 0$;

(*ii*) $|\mu(A)| \leq \|A\|$;

(*iii*) $\mu(A+B) \leq \mu(A) + \mu(B)$;

(*iv*) $|\mu(A) - \mu(B)| \leq \|A - B\|$.

Proposition A.2. *If X is a Banach space, then*

$$-\mu(-A) \leq Re\lambda(A) \leq \mu(A), \quad \text{for all } \lambda \in \sigma(A)$$

where $\sigma(A)$ denotes the spectrum of A.

Proposition A.3. *If X is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, then*

$$\mu(A) = \lambda_M(A_{\mathcal{R}}), \qquad A_{\mathcal{R}} = Re\ A = \frac{A + A^*}{2}$$

where A^* is the adjoint operator of A and $\lambda_M(A_{\mathcal{R}}) = \sup_{\|x\|=1} \langle A_{\mathcal{R}} x, x \rangle$. Moreover,

$$-\mu(-A) = \lambda_m(A_{\mathcal{R}})$$

where $\lambda_m(A_{\mathcal{R}}) = \inf_{\|x\|=1} \langle A_{\mathcal{R}} x, x \rangle$.

Example A.1. Let X be a Banach space of sequences of complex or real numbers, $x = \{\xi_j\}$. Then each operator $A \in \mathscr{B}(X)$ corresponds to a matrix $\left[a_j^k(t)\right]$. .

X	$\|A\|$	$\mu(A)$
$l_\infty = \left\{ x : \|x\|_\infty = \sup_j \|\xi_j\| < \infty \right\}$	$\sup_i \sum_k \|a_i^k\|$	$\sup_i \left(\text{Re } a_i^i + \sum_{k \neq i} \|a_i^k\| \right)$
$l_1 = \left\{ x : \|x\|_{l_1} = \sum_{j=1}^{\infty} \|\xi_j\| < \infty \right\}$	$\sup_k \sum_i \|a_i^k\|$	$\sup_k \left(\text{Re } a_k^k + \sum_{i \neq k} \|a_i^k\| \right)$
$l_2 = \left\{ x : \|x\|_{l_2} = \left(\sum_{j=1}^{\infty} \|\xi_i\|^2 \right)^{\frac{1}{2}} < \infty \right\}$	$\sqrt{\lambda_M(A^*A)}$	$\lambda_M \left(\frac{A + A^*}{2} \right)$

Table A.1: Norms and Lozinskiĭ measures of A

Remark A.2. The above results are given by Daleckiĭ and Kreĭn [23] as exercises, see page 61-62. When $X = \mathbb{C}^n$ and $A = [a_i^j]$ is a $n \times n$ matrix, a similar table is given by Coppel [19], page 41.

Proposition A.4. *Suppose that $x(t)$ is a solution of*

$$\frac{dx}{dt} = A(t)x, \quad x \in X, \tag{A.3}$$

where $A(t) \in \mathscr{B}(X)$. Then, for $t \geq t_0$,

$$\|x(t_0)\| \exp\left(-\int_{t_0}^{t} \mu(-A(s))ds \right) \leq \|x(t)\| \leq \|x(t_0)\| \exp\left(\int_{t_0}^{t} \mu(A(s))ds \right). \tag{A.4}$$

Corollary A.5. *The equation (A.3) is*
unstable if

$$\liminf_{t\to\infty} \int_{t_0}^{t} \mu(-A(s))ds = -\infty;$$

stable if

$$\limsup_{t\to\infty} \int_{t_0}^{t} \mu(A(s))ds < \infty;$$

asymptotically stable if

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu(A(s))ds = -\infty;$$

uniformly stable if

$$\mu(A(t)) \leq 0, \quad \textit{for } t \geq t_0;$$

uniformly asymptotically stable if

$$\mu(A(t)) \leq -\delta < 0, \quad \textit{for } t \geq t_0.$$

Appendix B

Sectorial Operators

This appendix is a summary on sectorial operators and detailed information is referred to Henry [42], page 16-29.

B.1 Sectorial Operators and Analytic Semigroups

Definition B.1. A linear operator A in a Banach space X is called a *sectorial operator* if it is a closed densely defined operator such that, for some $\phi \in (0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,\phi} = \{\lambda \mid \phi \leq |arg(\lambda - a)| \leq \pi, \ \lambda \neq a\}$$

is in the resolvent set of A and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \quad \text{for all } \lambda \in S_{a,\phi}.$$

Example B.1.

1. If A is a bounded linear operator on a Banach space, then A is sectorial.

2. If A is a self adjoint densely defined operator in a Hilbert space, and if A is bounded below, then A is sectorial.

3. If $Au(x) = -\triangle u(x)$, $x \in \Omega$, when $u \in C_0^2(\Omega)$ where Ω is an open set in $\mathbb{R}^n$, and A is the closure in $L_p(\Omega)$ of $-\triangle|_{C_0^2(\Omega)}, 1 \leq p < \infty$, then A is sectorial if its resolvent set meets the left half-plane.

Definition B.2. An *analytic semigroup* on a Banach space is a family of continuous linear operators on X, $\{T(t)\}_{t\geq 0}$, satisfying

(i) $T(0) = I,\ T(t)T(s) = T(t+S)$ for $t, s \geq 0$,

(ii) $T(t)x \mapsto x$ as $t \to 0^+$, for each $x \in X$,

(iii) $t \mapsto T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.

The infinitesimal generator L of this semigroup is defined by

$$Lx = \lim_{t \to 0^+} \frac{T(t)x - x}{t},$$

its domain $\mathcal{D}(L)$ consisting of all $x \in X$ for which this limit (in X) exists and $T(t)$ is usually written as $T(t) = e^{Lt}$.

Theorem B.1. *If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$,*

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda,$$

where Γ is a contour in $\rho(-A)$ with $arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$.

Further e^{-tA} can be continued analytically into a sector $\{t \neq 0 \ : \ |arg\ t| < \varepsilon\}$ containing the positive real axis, and if $Re\ \sigma(A) > a$, that is, if $Re\lambda > a$ whenever $\lambda \in \sigma(A)$, then for $t > 0$

$$\|e^{-tA}\| \leq Ce^{-\alpha t}, \quad \|Ae^{-tA}\| \leq \frac{C}{t}e^{-\alpha t}$$

for some constant C.

Finally $\frac{d}{dt}e^{-tA} = -Ae^{-tA}$ for $t > 0$.

B.2 Fractional Powers of Operators

Definition B.3. Suppose A is a sectorial operator and $Re\ \sigma(A) > 0$, define for any $\alpha > 0$,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-tA}dt,$$

where $\Gamma(\cdot)$ is the Gamma function.

Theorem B.2. *If A is a sectorial operator and Re $\sigma(A) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-one and satisfies*

$$A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$$

whenever $\alpha, \beta > 0$. Also for $0 < \alpha < 1$,

$$A^{-\alpha} = \frac{sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}d\lambda.$$

Definition B.4. Define A^α to be the inverse of $A^{-\alpha}$ $(\alpha > 0)$, $\mathcal{D}(A^\alpha) = \mathcal{R}(A^{-\alpha})$. Set $A^0 = I$ on X.

Remark B.1.

1. $A^\alpha = AA^{\alpha-1} = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\infty t^{-\alpha}Ae^{-At}dt.$
2. If $\alpha > 0, A^\alpha$ is a closed and densely defined.
3. If $\alpha \geq \beta$, then $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^\beta)$.
4. $A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ on $\mathcal{D}(A^\gamma)$ where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$.
5. $A^\alpha e^{-At} = e^{-At}A^\alpha$ on $\mathcal{D}(A^\alpha), t > 0$.

Theorem B.3. *If A is a sectorial operator and Re $\sigma(A) > \delta > 0$, then for any $\alpha \geq 0$, there exists $C_\alpha < \infty$ such that*

$$\|A^{-\alpha}e^{-At}\| \leq C_\alpha t^{-\alpha}e^{-\delta t} \quad for\, t > 0,$$

and if $0 < \alpha \leq 1, x \in D(A^\alpha)$,

$$\|(e^{-At} - 1)x\| \leq \frac{1}{\alpha}C_{1-\alpha}t^\alpha\|A^\alpha x\|.$$

In addition, C_α is bounded for all α in any compact interval of $(0, \infty)$, *and bounded as $\alpha \to 0^+$.*

Definition B.5. Suppose A is a sectorial operator, define for any $\alpha \geq 0$,

$$X^\alpha = D(A_1^\alpha)$$

with the graph norm

$$\|x\|_\alpha = \|A_1^\alpha x\|,$$

where $A_1 = A + aI$ with a chosen so *Re* $\sigma(A_1) > 0$.

Remark B.2. It can be shown that different choices of a give equivalent norms on X^α.

Theorem B.4. *If A is a sectorial operator in a Banach space X, then X^α is a Banach space in the norm $\|\cdot\|_\alpha$ for $\alpha \geq 0, X^0 = X$, and for $\alpha \geq \beta \geq 0, X^\alpha$ is dense subspace of X^β with continuous inclusion. If A has compact resolvent, the inclusion $X^\alpha \subset X^\beta$ is compact when $\alpha > \beta \geq 0$.*

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