Bouncing Universes: Simple Cosmological Models with a Scalar Field

by

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Abstract

In this thesis, we consider a homogeneous and isotropic closed model of the Universe with a real massive scalar field. Hawking showed that if such a model of the Universe is fine-tuned, it can have an infinite number of bounces [1]. We study the case for the Universe that is microscopically time-symmetric about a homogeneous, isotropic bounce [2]. We begin by considering a classical periodic solution in which a Universe has a time-symmetric bounce and expands to a large maximum size with a fixed large number of zero crossings of the scalar field between each pair of consecutive bounces. After the inflation, the scalar field oscillates with a phase constant which we will call θ. A perturbation of the solution will be quantified by the change in this oscillation phase θ, and we are seeking to find the probability for two successive bounces.

Preface

This thesis is an original work by Elsad Cukali. No part of this thesis has been previously published. All results have been obtained using none other than the stated references. All references have been properly cited.

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my advisor Prof. Don Page for his guidance and support over the past two years. I am also grateful to Prof. Andrzej Czarnecki and my undergraduate project advisor Prof. Hakan Cebeci for his thoughtful remarks and suggestions he gave me during my undergraduate studies of differential geometry and gravity. Over the last years I sincerely appreciated the friendship and thoughtful discussions on physics with Nakul Aggarwal. Last but certainly not least, I would like to thank my wonderful family for their support and encouragement. I could not have done this without you.

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Homogeneous and Isotropic Universes

1.1 General Information

Cosmology is an ancient area of science and has reached many achievements recently due to the discoveries in quantum physics and the improvements in observational astronomy. There have been many attempts to explain the origin of the Universe [3] [4]. The good part of studying cosmology is that lots of theories are combined together in order to explain the cosmos. The toy models of cosmological models with homogeneous scalar fields seem to match the observations, enabling us to understand our cosmos better [5].

In this chapter, we will start with a toy model of the Universe with a massive homogeneous scalar field [6]. However, this toy model is not to be at all close to a realistic model for our Universe. We will first find the probability of successive bounces for a simple case and later we will try to find a general expression for when the Universe grows large and there there is a large amount of the oscillatory regime.

Usually, anisotropy tends to increase when the Universe gets smaller [2][7]. We will use this toy model to illustrate that even without the inflaton decaying to other matter, and even without anisotropy, the probability of successive bounces is still very low.

1.2 Established Facts

Based on the observations so far, the Universe seems to be homogeneous and isotropic on a large scale. Our Universe is expanding (described by Hubble's law) [8], and the expansion is accelerating due to the presence of dark energy [9], which represents about between 70 to 75 percent of the mass-energy of the Universe. The remaining 20 to 25 percent is composed of mainly dark matter and about an extra 5 percent accounts for baryons, radiation, and neutrinos [10][11][12].

Another exciting thing is that the whole Universe is filled with cosmic microwave background radiation also abbreviated as CMB [13]. It is a type of electromagnetic radiation that is believed to be a remnant from the early stage of the Universe. It provides us with a picture of the Universe at the time when neutral atoms were formed [10]. As we will see in the next chapter, the early Universe was approximately a de Sitter spacetime. In the next part, we will discuss the de Sitter spacetime.

1.3 The de Sitter Spacetime

To study the homogeneous Universe, it might be helpful to discuss the maximally-symmetric spaces, so in this part, we will focus on maximally-symmetric spaces. A maximally-symmetric space is an n-dimensional manifold and its construction is based on the embedding in $(n+1)$-dimensional space. Maximally-symmetric spaces are the same as spaces of constant curvature. In this part, we will give the Riemann tensor in any n-dimensional maximally-symmetric geometry of signature (p, q) and Ricci scalar curvature R. The Riemann curvature tensor for any maximally-symmetric n-manifold at any point and in any coordinate system is:

$$R_{abcd} = \frac{R}{n\,(n-1)}\,(g_{ac}g_{bd} - g_{ad}g_{bc})\,, \tag{1.1}$$

where n is the dimension of the space, R is the Ricci scalar, which is constant over M. In the case where the cosmological constant Λ is present, the Einstein equations are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{8\pi G_N}{c^4}T_{ab}. \tag{1.2}$$

Here, T_{ab} is the stress-energy tensor, R_{ab} is the Ricci curvature tensor, R is the Ricci scalar, g_{ab} is the metric tensor, G_N is Newton's gravitational constant which in 4-dimensional spacetime has the dimensions $[G] = M^{-1}L^3T^{-2}$, and c is the speed of light in vacuum. Restricting to n = 4, the Ricci tensor for a maximally symmetric space is equal to $R_{ab} = \frac{Rg_{ab}}{4}$. The de Sitter spacetime is a maximally symmetric vacuum solution of the Einstein field equations with constant curvature and cosmological constant $\Lambda > 0$. In this part, the discussions of de Sitter spacetime, which has topology $\mathbb{R}^1 \times S^3$, is for $n = 4$. It can be visualized as a hyperboloid

$$-v^2 + \omega^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{1.3}$$

in flat five-dimensional space $\mathbb{R}^5$ with metric

$$ds^2 = -dv^2 + d\omega^2 + dx^2 + dy^2 + dz^2. \tag{1.4}$$

Therefore, the maximally-symmetric solutions are regarded as solutions of Eq. (1.2) with $\Lambda = \frac{R}{4}$ and zero stress-energy tensor, $T_{ab} = 0$. Let us introduce coordinates (t, χ, θ, ϕ) on the hyperboloid by relations

$$v = \alpha \sinh\left(\frac{t}{\alpha}\right), \qquad \omega = \alpha \cosh\left(\frac{t}{\alpha}\right)\cos\chi, \qquad x = \alpha\cosh\frac{t}{\alpha}\sin\chi\cos\theta,$$

$$y = \alpha\cosh\left(\frac{t}{\alpha}\right)\sin\chi\sin\theta\cos\phi, \qquad z = \alpha\cosh\left(\frac{t}{\alpha}\right)\sin\chi\sin\theta\sin\phi, \tag{1.5}$$

where $\{t \in \mathbb{R}^5,\ \chi \in [0,\pi],\ \theta \in (0,\pi),\ \phi \in [0, 2\pi)\}$ and the coordinate χ is not to be confused with the notation we will use in the next chapter. In these standard coordinates, the de Sitter metric reads [14]

$$ds^2 = -dt^2 + \alpha^2\cosh^2\left(\frac{t}{\alpha}\right)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.6}$$

where $\alpha^2 = \frac{3}{\Lambda}$.

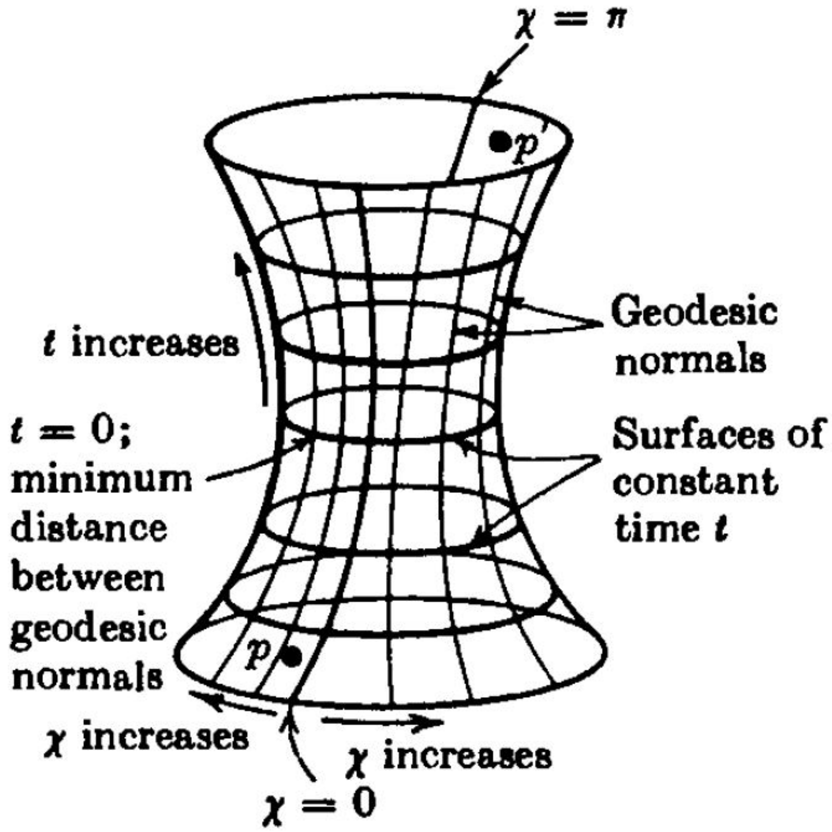


Figure 1.1: A maximally-symmetric Lorentzian manifold with constant positive scalar curvature, de Sitter space. The spatial sections of constant t are spheres S^3 of constant positive curvature and are Cauchy surfaces [14].

In the next part, we will discuss the Friedmann–Lemaître–Robertson–Walker spacetime, a model that is sometimes called the Standard Model of modern cosmology.

1.4 Robertson-Walker Spacetimes

As astronomical observations advanced, physicists have been trying to find a spacetime that gives a good representation of the large-scale properties of the observable Universe. On a spatial slice, our Universe is isotropic and homogeneous. Robertson-Walker spacetimes give a good approximate representation of our observable Universe. We can choose coordinates so that the metric of the Robertson-Walker spacetime has the form

$$ds^2 = -dt^2 + S^2(t)\, d\sigma^2, \tag{1.7}$$

where $d\sigma^2$ is the metric of three-space of constant curvature and is time-independent. The geometry of these three-spaces can be of constant positive, negative, or zero curvature. By rescaling the function S, one can normalize the curvature k of $d\sigma^2$ to be +1, -1, or 0. As a result, the metric $d\sigma^2$ can be written

$$d\sigma^2 = d\zeta^2 + f^2(\zeta)\left[d\theta^2 + \sin^2\theta d\phi^2\right], \tag{1.8}$$

where $f(\zeta)$ is a function of ζ only and for different values of k is has the form

$$f(\zeta) = \begin{cases} \sin\zeta & k = +1 \\ \zeta & k = 0 \\ \sinh\zeta & k = -1. \end{cases} \tag{1.9}$$

The coordinate $\zeta \in [0, \infty)$ for $k = 0$ or -1, and $\zeta \in [0, \pi]$ if $k = +1$. If $k = +1$, the three-spaces are diffeomorphic to a three-sphere S^3 and so are compact. For $k = 0$, the Friedmann–Lemaître–Robertson–Walker metric is conformal to the Minkowski spacetime, also called stretched Minkowski spacetime. We will discuss the Friedmann–Lemaître–Robertson–Walker metric in subsection 1.4.1.

Next, let us try to derive the physical velocity which we denoted by x_p. We are also going to denote the rescaled function $S(t)$ by $a(t)$. The distance over a hypersurface of constant t from the spatial origin $\zeta = 0$ to some other value of ζ is

$$x_p = a(t)\zeta. \tag{1.10}$$

The physical radial velocity is found to be

$$v_p = a(t)\frac{d\zeta}{dt} + \frac{da}{dt}\zeta = a\frac{d\zeta}{dt} + Hx_p, \tag{1.11}$$

where $a(t)$ is the scale factor depending on time only with t having the dimensions of time, and $H = \frac{1}{a}\frac{da}{dt}$ is the Hubble parameter. As it can be seen, when $\frac{d\zeta}{dt} = 0$, $v_p = Hx_p$ is equivalent to the Hubble law, the velocity due to the expansion of the Universe. In other cases, $a\frac{d\zeta}{dt}$ is called the peculiar velocity. One can also define the peculiar velocity as the velocity of an object relative to a rest frame.

1.4.1 The Friedmann–Lemaître–Robertson–Walker Metric

The Friedmann–Lemaître–Robertson–Walker metric, also known as the FLRW metric, describes the cosmic spacetime and assumes homogeneity and isotropy throughout the Universe and can be written as [15]

$$ds^2 = -dt^2 + \frac{a^2(t)}{\left(1+\frac{k}{4}r^2\right)^2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.12}$$

where we have used units with c=1, k is the curvature index and can take values -1, 0, or +1, and $a(t)$ is an arbitrary function of time only. The expression for the stress-energy tensor $T_{\mu\nu}$, corresponding to the perfect fluid, is

$$T_{\mu\nu} = P g_{\mu\nu} + (\rho + P)\, u_\mu u_\nu, \tag{1.13}$$

where u_ν is the 4-velocity of the fluid (comoving observer), ρ and P are the energy density and pressure measured in the rest frame of the fluid (by comoving observers), respectively.

Einstein's equations then lead to the following two independent equations

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2},$$

$$\frac{d\rho}{dt} = -3H(\rho + P), \tag{1.14}$$

where $H \equiv \frac{1}{a}\frac{da}{dt}$. In order to solve the Einstein field equations, we need to assume the equation of state as well, which we will do below. From the equation above, we note that one can define a critical density as

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G_N}. \tag{1.15}$$

As a result,

$$\frac{k}{a^2} = \frac{8\pi G_N}{3}(\rho - \rho_c). \tag{1.16}$$

By measuring the current density ρ and the Hubble parameter H, we can determine the sign of k, and therefore we would be able to tell whether the Universe is open, closed, or flat. One problem widely encountered by cosmologists is to explain why the Universe was created in such a way that at the beginning, the density of the Universe was close to the critical density. This is often called the flatness puzzle. This puzzle, together with many other ones, is successfully explained by inflation.

Next, we will discuss the equation of state which tells us how pressure is related to the density ($P \equiv P(\rho)$). One simple form is $P = w\rho$, where w is a real constant. This is not our Universe today but simple models help us to understand more complicated models. From the second equation above we get

$$\frac{d\rho}{dt} = -3H(\rho + w\rho) = -3\frac{\rho}{a}\frac{da}{dt}(1+w). \tag{1.17}$$

Integrating both sides we obtain

$$\rho \propto a^{-3(1+w)}, \tag{1.18}$$

where w takes different values for different components of the Universe. Below is a table which summarizes that possible values of w for different components of the Universe.

Table 1.1: Equation of state for different components of the Universe.

Typical component of Universe	**Equation of state**	**Density**
matter	$P = 0, w = 0$	$\rho \propto a^{-3}$
radiation	$P = \frac{1}{3}\rho, w = \frac{1}{3}$	$\rho \propto a^{-4}$
cosmological constant	$P = -\rho, w = -1$	$\rho =$const.

1.5 Geodesics Equation: Motion of Test Particles

In this section, we will study the motion of a test particle, either massive or massless, in the Friedmann–Lemaître–Robertson–Walker background. Consider the metric in Eq. (1.12) and the geodesic equation given by [15]

$$\frac{dp^\nu}{d\lambda} + \Gamma^\nu_{\beta\gamma} p^\beta p^\gamma = 0, \tag{1.19}$$

where $\nu = 0, 1, 2, 3$. The non-vanishing Christoffel symbols are [16]:

$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{1}{a}\frac{da}{dt},\ \Gamma^3_{13} = \Gamma^2_{12} = \frac{4 - kr^2}{(4 + kr^2)\, r},\ \Gamma^0_{11} = \frac{a\left(\frac{da}{dt}\right)}{(4 + kr^2)^2} = Hg_{11},$$

$$\Gamma^0_{22} = 16\frac{ar^2\left(\frac{da}{dt}\right)}{(4 + kr^2)^2} = Hg_{22},\ \Gamma^0_{33} = 16\frac{ar^2 \sin^2\theta\left(\frac{da}{dt}\right)}{(4 + kr^2)^2} = Hg_{22},\ \Gamma^1_{11} = -\frac{2kr}{4 + kr^2},$$

$$\Gamma^1_{22} = \frac{r\,(kr^2 - 4)}{4 + kr^2},\ \Gamma^1_{33} = \frac{r^2 \sin^2 (kr^2 - 4)}{4 + kr^2},\ \Gamma^3_{23} = \cot\theta,\ \Gamma^2_{33} = -\sin\theta\cos\theta. \tag{1.20}$$

For $\nu = 0$ we have:

$$\frac{dp^0}{d\lambda} + \Gamma^0_{11}\left(p^1\right)^2 + \Gamma^0_{22}\left(p^2\right)^2 + \Gamma^0_{33}\left(p^3\right)^2 = \frac{dp^0}{d\lambda} + Hg_{11}\left(p^1\right)^2 + Hg_{22}\left(p^2\right)^2$$

$$+Hg_{33}\left(p^3\right)^2 = \frac{dp^0}{d\lambda} + Hp^2 = 0. \tag{1.21}$$

Here, λ is an affine parameter such that $p^\nu = dx^\nu/d\lambda$ and $p^2 = g_{ij}p^i p^j$ is the square of the spatial magnitude of the momentum for both massive and massless particles. For a massive particle

$$-\left(p^0\right)^2 + p^2 = -m^2. \tag{1.22}$$

It is clear that for a massless particle like a photon, we have $p^\nu p_\nu = 0$. Differentiating both sides of the Eq. (1.22) we get $p^0 dp^0 = pdp$. Here, we obtain an expression for dp^0 and we substitute it in Eq. (1.21). Therefore,

$$p\frac{dp}{p^0 d\lambda} + Hp^2 = 0, \tag{1.23}$$

where p is the spatial momentum of the particle. This equation yields

$$p\left(\frac{dp}{dt}+\frac{1}{a}\frac{da}{dt}p\right)=0. \tag{1.24}$$

For all particles, either massive or massless, moving along geodesics we have momentum $p \propto \frac{1}{a}$. In the case of a photon, $m = 0$ and in units with $c = 1$ we have $E = p$, where E is the energy of the photon. This gives

$$p = E \propto \frac{1}{a}. \tag{1.25}$$

In other words, the energy of the photon is proportional to the inverse of the scale factor.

1.6 Radiation

For a massless particle like a photon, $p = p^0 = E \propto \omega$, where ω is the angular frequency of the photon. The cosmological redshift is found to be

$$\omega \propto \frac{1}{a}, \qquad \lambda \propto a. \tag{1.26}$$

We denoted the wavelength of the electromagnetic wave by λ. Now let us introduce the redshift parameter given by the equation

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}, \tag{1.27}$$

where $\lambda_{obs} = \lambda(t_0)$ is the wavelength of the photon measured on Earth now and $\lambda_{em} = \lambda(t_1)$ is the wavelength of the photon emitted by the galaxy at time t_1. Using the equation above we have

$$z = \frac{a(t_0)}{a(t_1)} - 1. \tag{1.28}$$

For nearby galaxies and for small z, we make the following expansion

$$a(t_1) = a(t_0)[1 + (t_1 - t_0)\, H\,(t_0) + ...]. \tag{1.29}$$

This gives us $z \approx \frac{Hd}{c} = \frac{v}{c}$, where d is the proper distance and v is the recessional velocity. Therefore, Hubble's law is obtained and given by

$$v \approx H_0 d, \tag{1.30}$$

with H_0 being the Hubble parameter at $t = t_0$.

1.7 Remarks on the Cosmological Constant Λ

The cosmological constant is usually understood as a form of matter with the following energy-momentum tensor

$$8\pi G_N T^{\Lambda}_{\mu\nu} = -\Lambda g_{\mu\nu}. \tag{1.31}$$

Since the energy density does not dilute, energy has to be created as the Universe expands. The energy density is thus found to be

$$\rho_\Lambda = -P_\Lambda = \frac{\Lambda}{8\pi G_N}. \tag{1.32}$$

There are many suggestions about the cosmological constant and the factors that determine its value. The value of Λ can be calculated from the observational parameters which is found to be approximately $\Lambda \approx 10^{-52}$ m^{-2}. It is also thought that the vacuum energy fluctuations contribute to the value of the cosmological constant. As predicted by quantum field theory, it has the following energy-momentum tensor:

$$T_{(vac)\mu\nu} = -\rho_{vac} g_{\mu\nu}. \tag{1.33}$$

A naive application of quantum field theory tells us that $\frac{\rho_{vac}}{\rho_{obs}} \sim 10^{120}$, a huge number which is also called the "old cosmological constant problem" [17].

Chapter 2

Solutions of a Simple Cosmological Model with a Massive Scalar Field

In this chapter, we will derive the solutions of equations of motion for a simple cosmological model with homogeneous massive scalar field $\varphi(t)$ with mass m, minimally coupled to a $k = +1$ FLRW Universe with zero cosmological constant and with metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)\, d\Omega_3^2, \tag{2.1}$$

where $d\Omega_3^2 = d\zeta^2 + \sin^2\zeta\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the standard line element on S^3 [18][19][20]. In the units with $\hbar = c = 1$ the Lorentzian action is [21][20]

$$S = \int 2\pi^2 a^3 N\left\{\frac{3}{8\pi G_N}\left[-\left(\frac{1}{Na}\frac{da}{dt}\right)^2 + \frac{1}{a^2}\right] + \frac{1}{2}\left(\frac{1}{N}\frac{d\varphi}{dt}\right)^2 - \frac{1}{2}m^2\varphi^2\right\}dt. \tag{2.2}$$

We are going to rescale the lapse function N as $n \equiv mN$, where n is dimensionless if t is considered to be dimensionless, and where N is dimensionless when the product mt is dimensionless. Notice that our scale factor a has the dimensions of length, so we will also introduce a dimensionless ma to be equal to $ma = e^\chi$. We can also obtain a dimensionless inflaton scalar field

$$\phi = \sqrt{\frac{4\pi G_N}{3}}\varphi. \tag{2.3}$$

Next, we will define the overdot as the dimensionless derivative with respect to mt and for any function of t, say $f(t)$, we have

$$\dot{f}(t) \equiv \frac{1}{m}\frac{df}{dt}. \tag{2.4}$$

Also, $G_N \equiv M_{Pl}^{-2}$ has the dimensions of the inverse mass squared. Using the definition for $\chi = \ln(ma)$, we can have another form of the action that we will be using in the derivation of the equations of motion and has the form

$$S = \frac{3\pi}{4}\frac{M_{Pl}^2}{m^2}\int ne^{3\chi}\left[-n^{-2}\left(\dot{\chi}^2 - \dot{\phi}^2\right) + e^{-2\chi} - \phi^2\right]dt = \int L\left(\chi, \dot{\chi}, \phi, \dot{\phi}, n\right)dt. \tag{2.5}$$

The variation of the action in Eq. (2.5) with respect to n and then setting $n = 1$ yields

$$\dot{\chi}^2 - \dot{\phi}^2 = \phi^2 - e^{-2\chi}. \tag{2.6}$$

Using the definition of the overdot, the Euler–Lagrange equation $\frac{1}{m}\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$ gives

$$\ddot{\phi} + 3\dot{\chi}\dot{\phi} + \phi = 0. \tag{2.7}$$

As we can see, m does not appear in the equations of motion, which can simplify our calculations where in the next chapter we will take m to be equal to unity.

Both Eq. (2.6) and Eq. (2.7) are very important and we will use them in the next chapter to construct a second-order differential equation for χ that will describe any trajectory in the (χ, ϕ) plane. For now let us use them by writing explicitly $\chi = \ln(ma)$ and the overdot as $\frac{1}{mN}\frac{d}{dt}$ with $N = 1$ and mt dimensionless, we can directly obtain from Eq. (2.6) the following:

$$H^2 \equiv \left(\frac{d\chi}{dt}\right)^2 = -\frac{1}{a^2} + \left(\frac{d\phi}{dt}\right)^2 + m^2\phi^2. \tag{2.8}$$

On the other hand, Eq. (2.7) will be

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + m^2\phi = 0, \tag{2.9}$$

where $H \equiv \frac{d\chi}{dt} \equiv m\dot{\chi} = \frac{1}{a}\frac{da}{dt} \equiv m\frac{\dot{a}}{a}$. It is worth reminding that ϕ is dimensionless and will be equal to $\sqrt{\frac{4\pi G_N}{3}}\varphi$ with φ measured in mass units as mentioned earlier.

Having obtained the equations of motion, we will give an approximate formula for the size of the Universe near the end of inflation which we will call a_e. Before we do this, we will discuss inflation, which is a theory of the exponential expansion of space in the early Universe. It occurs when the potential energy density dominates over the kinetic energy density. In this project, we will only focus on slow-roll inflation and before we move into the next step, let us discuss slow-roll inflation and the conditions for the slow-roll inflation.

2.1 Slow-Roll Inflation

Slow-roll models include inflation by scalar fields that are defined according to the model that is chosen. In our case, inflation by a single scalar field, also called an inflaton, with a suitable potential $V(\varphi)$ is a subset of slow-roll models. Assuming that the scalar field dominates the Universe let us define the energy density and pressure associated with the field. Eq. (2.6) can also be written in this form by using the rescaled scalar field $\phi = \sqrt{4\pi G_N/3}\varphi$:

$$H^2 \equiv \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = 2\left[\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}m^2\phi^2\right] - \frac{1}{a^2} = 2\rho - \frac{1}{a^2}, \tag{2.10}$$

where one sets the lapse $N = 1$ so that t has the dimensions of time or length, and ρ is $\frac{4\pi G_N}{3}$ times the energy density of a homogeneous scalar field that is the sum of kinetic and potential densities and given by

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}m^2\phi^2. \tag{2.11}$$

We know from Eq. (2.9) that $\frac{d^2\phi}{dt^2} + m^2\phi = -3H\frac{d\phi}{dt}$ so the time derivative of the rescaled energy density is

$$\frac{d\rho}{dt} = \left(\frac{d^2\phi}{dt^2} + m^2\phi\right)\frac{d\phi}{dt} = -3H\left(\frac{d\phi}{dt}\right)^2. \tag{2.12}$$

Comparing this with the continuity equation

$$\frac{d\rho}{dt} = -3H\left(\rho + P\right) \tag{2.13}$$

the similarly rescaled pressure induced by the field is

$$P = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{1}{2}m^2\phi^2. \tag{2.14}$$

An equation of state usually has the form $P = P(\rho)$, but here P does not depend only on ρ and temperature T if there is also entropy, so one would not normally say the scalar field has an equation of state. However, reminding the definition of the overdot, one can define an equation of state parameter w for the homogeneous scalar field as

$$w = \frac{P}{\rho} = \frac{\left(\frac{d\phi}{dt}\right)^2 - m^2\phi^2}{\left(\frac{d\phi}{dt}\right)^2 + m^2\phi^2} = \frac{\dot{\phi}^2 - \phi^2}{\dot{\phi}^2 + \phi^2}. \tag{2.15}$$

To make the equation of state parameter w a constant, we take two limits that are described below.

- Rapidly time-varying field: $m^2\phi^2 << \left(\frac{d\phi}{dt}\right)^2$, giving $w = 1$.

This would lead to $\rho \approx a^{-3(1+w)} \approx a^{-6}$.

- Slowly time-varying field: $m^2\phi^2 >> \left(\frac{d\phi}{dt}\right)^2$, giving $w = -1$.

This leads to $\rho \approx$ constant, and it is considered to mimic the cosmological constant.

A scalar field seems to be a good matter candidate for the inflaton, but it could have a potential $V(\varphi)$ different from that of the massive scalar field considered here, which has potential $V(\varphi) = (1/2)m^2\varphi^2 = \frac{3}{8\pi G_N}m^2\phi^2$.

One issue that needs to be explained is how the was the matter created. There is still active research in an attempt to find how the Standard Model particles are produced from the energy of the scalar field and how hot the Universe was at the time of the big bang. One of the theories that explain the creation of elementary particles is the reheating process. In the reheating process, which happens at the end of inflation, the energy of the scalar field is transferred to the particles, and the hot big bang starts [4].

An important parameter to discuss in the inflation is the slow-roll parameter defined as

$$\epsilon \equiv -\frac{1}{H^2}\frac{dH}{dt}. \tag{2.16}$$

Inflation is defined as an epoch in which $\frac{d^2a}{dt^2} > 0$, which is

$$\epsilon = 1 - a\frac{\frac{d^2a}{dt^2}}{\left(\frac{da}{dt}\right)^2} \equiv 1 - \frac{a\ddot{a}}{\dot{a}^2} < 1. \tag{2.17}$$

Slow-roll inflation is called when $\epsilon << 1$. Let us find an expression for the slow-roll parameter. Differentiating Eq. (2.7) with respect to t and plugging it into Eq. (2.8), we get

$$H\frac{dH}{dt} = \frac{H}{a^2} - 3H\left(\frac{d\phi}{dt}\right)^2. \tag{2.18}$$

Therefore, the time derivative of H is

$$\frac{dH}{dt} = \frac{1}{a^2} - 3\left(\frac{d\phi}{dt}\right)^2. \tag{2.19}$$

Using the formula we got for $\frac{dH}{dt}$ and H^2 in the definition for the slow roll parameter we get

$$\epsilon = -\frac{1}{H^2}\frac{dH}{dt} = \frac{3\left(\frac{d\phi}{dt}\right)^2 - \frac{1}{a^2}}{\left(\frac{d\phi}{dt}\right)^2 + m^2\phi^2 - \frac{1}{a^2}}. \tag{2.20}$$

For the slow-roll inflation case, the kinetic energy density makes a small contribution to the total energy density. Therefore, the first condition to impose on the slow-roll parameter for the slow-roll inflation to occur is

$$\epsilon << 1. \tag{2.21}$$

However, just for the inflation to occur $\epsilon < 1$ will be enough.

Now that we have introduced slow-roll inflation, let us find an approximate formula for the size of the Universe after the inflation. Suppose that the Universe model starts with $\phi = \phi_i >> 1$ at $t = 0$. We also chose $\dot{\phi} = 0$, and $\dot{a} = 0$ and initially there is no kinetic energy $\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 = 0$, but $\rho = -P = \frac{1}{2}\phi_i^2$. Then the Friedmann equation sets the curvature term to $\frac{1}{a_i^2}$, and as such the initial value of the dimensionless rescaled scale factor ma_i is $\frac{1}{\phi_i}$. There is a step of increasing H from zero to some value $H \approx$ constant where the Universe is exponentially expanding. Then much later H will go back to zero at $a = a_m$, the maximum size of this epoch of expansion. After that, the Universe will shrink, and can either collapse to a singularity for most of the possible large ϕ_i, or for an infinite set of tiny ranges of ϕ_i, the collapse can reverse to give another bounce with $\dot{a} = 0$ at some small value of a, not necessarily at a_i. If it does bounce, there can be another epoch of inflation followed by dust dominance and then recollapse, which again usually will go to a singularity but for some small ranges of ϕ_i can lead to the third phase of expansion and contraction.

In the next chapter we have shown some graphs where for some different values of the maximum size of the Universe, say a_m, the Universe will have another bounce rather than going to a singularity. We will also use those graphs to demonstrate how one can find an approximate probability for another bounce or let us call it a strong bounce after deflation.

Consider a large initial scalar field $\phi \approx \phi_i >> 1$. If $m^2a^2\phi^2 >> a^2\left(\frac{d\phi}{dt}\right)^2$, then $\left(\frac{da}{dt}\right)^2 \approx (ma\phi)^2$-1. Taking the square root of both sides, with the sign giving t increasing from 0 as $(ma\phi)^2$ increases, we get

$$\frac{da}{dt} \approx \sqrt{m^2a^2\phi^2 - 1}. \tag{2.22}$$

Rearranging Eq. (2.22) and integrating with ϕ to be approximately a constant, say $\phi = \phi_i$, gives

$$t \approx \frac{1}{m\phi_i} \ln\left(\sqrt{m^2\phi_i^2a^2 - 1} + m\phi_i a\right)\Bigg|_{a_i}^{a} = \frac{1}{m\phi_i} \ln\left(\frac{\sqrt{m^2\phi_i^2a^2 - 1} + m\phi_i a}{\sqrt{m^2\phi_i^2a_i^2 - 1} + m\phi_i a_i}\right). \tag{2.23}$$

Again, the conditions in which this is a good approximation is that at $t = 0$, $\phi = \phi_i >> 1$, $\dot{\phi} = 0$, $\dot{a} = 0$, and $m^2a^2\phi^2 = 1$. For t not too large in the range $t << \frac{\phi_i}{m}$, one has $\phi \approx \phi_i$, so

$$a(t) \approx \frac{1}{m\phi_i} \cosh{(m\phi_i t)}. \tag{2.24}$$

As we discussed earlier, the early Universe was approximately a de Sitter spacetime. The scalar equation we have obtained in the equations of motion can be written as

$$\frac{d}{dt}\left[a^3(t)\frac{d\phi}{dt}\right] = -m^2 a^3(t). \tag{2.25}$$

Suppose $t << \frac{\phi_i}{m}$ and $\phi \approx \phi_i$. If we substitute the expression we found for $a(t)$ in the previous equation and if we integrate both sides we get

$$a^3\frac{d\phi}{dt} \approx -m^2\phi_i \frac{1}{(2m\phi_i)^3}\int_0^t \left(e^{m\phi_i t} + e^{-m\phi_i t}\right)^3 \, dt. \tag{2.26}$$

Let $x = e^{m\phi_i t}$, then one can easily get the following equation

$$\frac{d\phi}{dt} \approx -\frac{m}{3}\frac{x^6 + 9x^4 - 9x^2 - 1}{x^6 + 3x^4 + 3x^2 + 1}. \tag{2.27}$$

Assuming that the scalar field is sufficiently large, for $\frac{1}{m\phi_i} << t$ we have

$$\frac{d\phi}{dt} \approx -\frac{m}{3}. \tag{2.28}$$

Integrating both sides we get

$$\phi(t) \approx -\frac{m}{3}t + \phi_0, \tag{2.29}$$

where ϕ_0 and ϕ_i are slightly different. This is only an approximate equation for the time derivative of ϕ and is valid only for very large ϕ_i. We can use the approximate formula that we found for $\phi(t)$ in Eq. (2.9) so that one can find an approximate equation for the size of the Universe near the end of inflation.

Neglecting $\frac{1}{a^2}$ for $t >> \frac{1}{m\phi_i}$ we have

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 \approx \frac{m^2}{9} + m^2\left(\frac{m^2t^2}{9} - \frac{2m\phi_i t}{3} + \phi_i^2\right). \tag{2.30}$$

For most of the inflation, we assume $a^2\left(\frac{d\phi}{dt}\right)^2 << m^2a^2\phi^2$. As a result, the equation above reduces to

$$\frac{1}{a}\frac{da}{dt} \approx m\left(-\frac{m}{3}t + \phi_i\right), \tag{2.31}$$

and

$$\frac{d\ln a}{dt} \approx m\left(-\frac{m}{3}t + \phi_i\right). \tag{2.32}$$

Integrating both sides where time runs from zero to $t_e \approx \frac{3\phi_i}{m}$ we get

$$\ln\frac{a_e}{a_i} \approx \frac{3}{2}\phi_i^2. \tag{2.33}$$

Hence, at the end of inflation, the size of the Universe is

$$ma_e \sim e^{\frac{3}{2}\phi_i^2}. \tag{2.34}$$

It is worth mentioning that $\left(\frac{d\phi}{dt}\right)^2 << m^2\phi^2$ does not apply all the way down to $\phi = 0$. This result is only a crude approximation as we have assumed that the inflation lasts sufficiently long and have ignored $\ln\phi_i$ terms in $\ln\frac{a_e}{a_i}$. However, one can use another way of finding a_e by using Eq. (1.43). We assume that at the end of inflation, $\phi = 0$. Let $H = \frac{d}{dt}\ln a = \frac{d\phi}{dt}\frac{d\ln a}{d\phi}$, where $d\ln a = \frac{H}{\frac{d\phi}{dt}}d\phi$. Integrating both sides, one can easily get

$$\ln\frac{a_e}{a_i} = \int_{\phi_i}^{\phi_e}\frac{H}{\left(\frac{d\phi}{dt}\right)}\,d\phi. \tag{2.35}$$

Assuming a considerable amount of inflation and ignoring $\frac{d^2\phi}{dt^2}$ in Eq. (2.8) we get

$$3H\frac{d\phi}{dt} + m^2\phi \approx 0,$$

$$\ln\frac{a_e}{a_i} \approx -3\int_{\phi_i}^{\phi_e}\frac{H^2}{m^2\phi}\,d\phi. \tag{2.36}$$

Now we need to find an expression for H^2. If we use $\frac{d\phi}{dt} \approx -\frac{m^2\phi}{3H}$ in Eq. (2.7) we obtain

$$H^4 - \frac{m^4\phi^2}{9} - m^2\phi^2H^2 \approx 0. \tag{2.37}$$

After solving this equation for H^2, the only root that satisfies our conditions is

$$H^2 \approx \frac{m^2}{2}\left(\phi^2 + \sqrt{\phi^4 + \frac{4\phi^2}{9}}\right). \tag{2.38}$$

Substituting this result in the integral above we get

$$\ln\frac{a_e}{a_i} \approx -\frac{3}{2}\int_{\phi_i}^{0}\left(\phi^2 + \sqrt{\phi^4 + \frac{4\phi^2}{9}}\right)\frac{1}{\phi}\,d\phi =$$

$$= \frac{3}{4}\phi_i^2 + \frac{1}{4}\phi_i\sqrt{9\phi_i^2+4} + \frac{1}{3}\ln\left(\frac{3}{2}\phi_i + \sqrt{\frac{9}{4}\phi_i^2+1}\right). \tag{2.39}$$

However, there are other $\mathcal{O}(1)$ uncertainties in $\ln\frac{a_e}{a_i}$, so this is only an order of magnitude approximation. Note that for large ϕ_i we get $\ln\frac{a_e}{a_i} \approx \frac{3}{2}\phi_i^2 + \frac{1}{3}\ln(3\phi_i)$ or $ma_e \sim \frac{1}{\phi_i^{2/3}}e^{\frac{3}{2}\phi_i^2}$.

One can also analyze how ρ changes with time at the early Universe where $m\phi >> \frac{d\phi}{dt}$. The slow roll inflation requires $\frac{d\phi}{dt} \approx 0$, it can be seen that

$$P + \rho = \left(\frac{d\phi}{dt}\right)^2 \approx 0. \tag{2.40}$$

Let us approximate our case to the case of a perfect fluid to get

$$\frac{d\rho}{dt} = 3H\,(P+\rho) \approx 0. \tag{2.41}$$

That said, ρ changes slowly during the slow roll inflation.

At the end of inflation, all energy is concentrated in the inflaton field and then it produced many elementary particles which interacted with each other and came to a state of thermal equilibrium. However, as we mentioned earlier, we do not take in our model of the Universe a scalar field that decays into other matter. After the inflation, the scalar field oscillates and we will call it the oscillatory regime. In the next section, we will discuss this oscillatory regime, and further discussions will be done in the next chapter.

2.2 Oscillatory Regime

Let us do some calculations for the oscillatory regime and try to find an equation for ϕ that oscillates with an amplitude that is a function of the size of the Universe, a. Let $\phi \approx F(t)\cos(mt + \theta_0)$ where θ_0 is a constant phase and $F(t)$ a slowly varying function of time where the rescaled energy density $\rho << m^2$ (well into the oscillatory regime). Substituting this in the expression for ρ, we get roughly

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{m^2\phi^2}{2} \approx \frac{m^2F^2}{2}. \tag{2.42}$$

Therefore,

$$\phi \approx \frac{\sqrt{2\rho}}{m}\cos(mt + \theta_0). \tag{2.43}$$

Before we proceed we need to make an assumption that ρ at the end of inflation is close to the ρ at the beginning of oscillation $\rho_e \approx \rho_{osc}$. This way we can get a rough estimate on ρ. It is easy to confirm that $\rho_e \sim m^2$, which is an order of magnitude estimate. When the scalar field ϕ oscillates, the average over each oscillation of the pressure P is zero to a good approximation, so the stress-energy tensor of the scalar field is close to that of dust with zero-pressure matter, giving $a^3\rho$ approximately constant. Using this idea, if we denote $\phi \approx c_1 a^{-3/2}\cos(mt)$, at $t \approx t_e$, where t_e is the value of t at the end of inflation, we have

$$c_1^2 a_e^{-3} m^2 \sim m^2, \tag{2.44}$$

so $c_1 \sim a_e^{3/2}$. Substituting the expression we found for the constant c_1 back into the equation for ϕ we have

$$\phi \sim \left(\frac{a_e}{a}\right)^{3/2}\cos(mt + \theta_0). \tag{2.45}$$

By comparing this with the ϕ we have found, we can get a rough expression for ρ as a function of a

$$\rho \sim m^2\left(\frac{a_e}{a}\right)^3. \tag{2.46}$$

As we can see from the equation above, $\rho \propto a^{-3}$. One can easily find from Eq. (2.43) that the period of oscillation is equal to $\frac{2\pi}{m}$. Based on the definition of the rescaled pressure P that we made while discussing the slow-roll inflation and that the average value over one period of oscillation of either $\sin^2(mt + \theta_0)$ or $\cos^2(mt + \theta_0)$ is equal to $\frac{1}{2}$, the pressure will average to near zero over each oscillation. Hence, during the oscillatory regime the Universe will behave approximately as if it were filled with dust of total rest mass M that depends on ϕ_i and M is expressed in the conventional form as density times volume.

Chapter 3

Bouncing Universes

3.1 Introduction

In our previous discussion, we considered a model of the Universe with a homogeneous massive scalar field φ with mass m, minimally coupled to a $k = +1$ FLRW metric with zero cosmological constant. The same model of the Universe will be considered in this chapter as well. Suppose that the Universe starts to expand and we have the inflationary regime, where $P \approx -\rho$ and later on the oscillatory regime with the pressure averaging to approximately zero. Eventually, the Universe will stop expanding at some larger a which we called a_{max}, or local maximum. We are going to fix the scalar field to zero at $a = a_{max}$ and then we will have a deflationary regime where the Universe gets smaller and we will have two possibilities. One possibility is that the Universe goes to the stiff regime and thus goes to singularity, and the other possibility is that there is a bounce at some smaller a.

In this chapter, we will be starting with a random initial condition near the local maximum, say near $a = a_{max}$ and a will be measured in units of $\frac{1}{m}$. We will focus on the simplest case where the cosmological constant is set to zero and we will briefly discuss some kind of non-singular trajectories which we will be calling the periodic solutions, first noticed by Hawking in 1983 [1].

Before we do any numerical work, we will also discuss the strong bounces, traverses, and some simple calculations which demonstrate what a periodic solution looks like for some small values of the initial conditions of $a = a_m$.

The same idea will be implemented when we have a Universe that goes through a large amount of inflation and oscillation of the scalar field. In order to discuss further the oscillatory regime, we will also introduce the phase of oscillation of the scalar field which we will call the phase constant θ, asymptotically approaching a constant that we will calculate numerically and then compare it to the value already published. If this phase constant changes by a small amount then as the Universe reaches a maximum size and then contracts, it will either have another bounce at some other value of either positive or negative value of the scalar field or there will be another singularity.

In order to explain the main goal of the project, we will be approximately calculating the probability of a strong bounce at small $a = e^{\chi}$ to return back to much larger $a = e^{\chi}$.

Before we further discuss the results, let us first introduce the equations of motion. In order to get them into a simpler form, let us define $f' \equiv \frac{df}{d\phi} = \frac{\dot{f}}{\dot{\phi}}$, where f is a function of time t. Using Eq. (2.7) and Eq. (2.8), we can find a second order differential equation of the form [21]:

$$\chi'' = \frac{(1 - \chi'^2)\left(\phi\chi' + 3\phi^2 - 2e^{-2\chi}\right)}{e^{-2\chi} - \phi^2}, \tag{3.1}$$

for any trajectory representing the evolution of this model in the (χ, ϕ) plane. If we look at Eq. (3.1) carefully, the denominator becomes zero when $\phi = \pm e^{-\chi}$. We are going to call these two lines as the boundaries in the (χ, ϕ) plane. Therefore, we can say that there are three separate regions [21]:

- Region I: $e^{-\chi} < \phi$
- Region II: $-e^{-\chi} < \phi < e^{-\chi}$
- Region III: $-e^{-\chi} > \phi$.

The boundary I separates region I and II whereas boundary II separates the region II and III.

If the trajectory crosses one of the two boundaries, then at the boundary we have either $\chi' = \pm 1$ with $\dot{\phi} \neq 0$ or $\chi' = -\phi$ if $\frac{d\phi}{dt} = 0$. The latter one can be interpreted as the trajectory momentarily halting at the boundary and then turning directly around going back out the same path it came in. This is a similar case to what we refer to as the time-symmetric bounce. The idea of bounces and traverses is described below [21]:

- Bounce: The trajectory goes from either region I or III into region II (or just to its boundary), and then returning back to the region it came from, the region I or III. We are interested in strong bounces where there is another inflation after a bounce with many traverses but we will also consider weak bounces where the Universe goes to singularity after the bounce. The special case is the time-symmetric bounce where there will be a special case of periodic solutions that we will further discuss below.

- Traverse: The trajectory goes from region I to region II and the crosses $\phi = 0$ which is the χ-axis and then goes to region III or the inverse.

Numerical results show that no traverse segment within region II can have any points with $\chi < 0$, so we expect the traverse segment to have all points with $\chi > 0$.

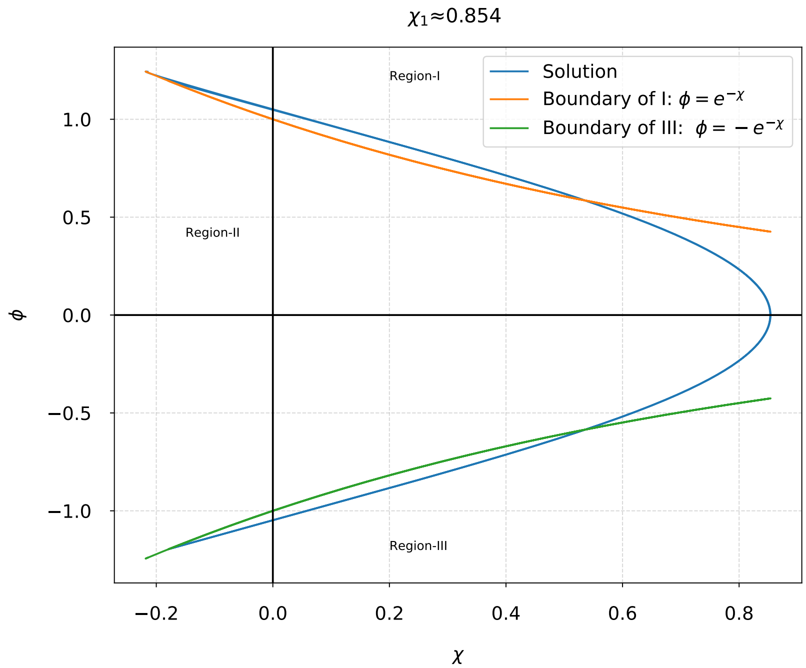
Eq. (3.1) is very important in understanding the evolution of the Universe model and we will also use it later to derive an approximate series expression for what we shall call the repeller solution. Any solution for which ϕ changes significantly during the bounce, will either follow the repeller solution in region II up to some value of positive or negative ϕ until it turns to the right, towards larger χ and hence gives another bounce, or will cross the repeller solution and go to the singularity, so the repeller solution seems to be in between.

3.1.1 Time Symmetric Bounce and Periodic Solutions

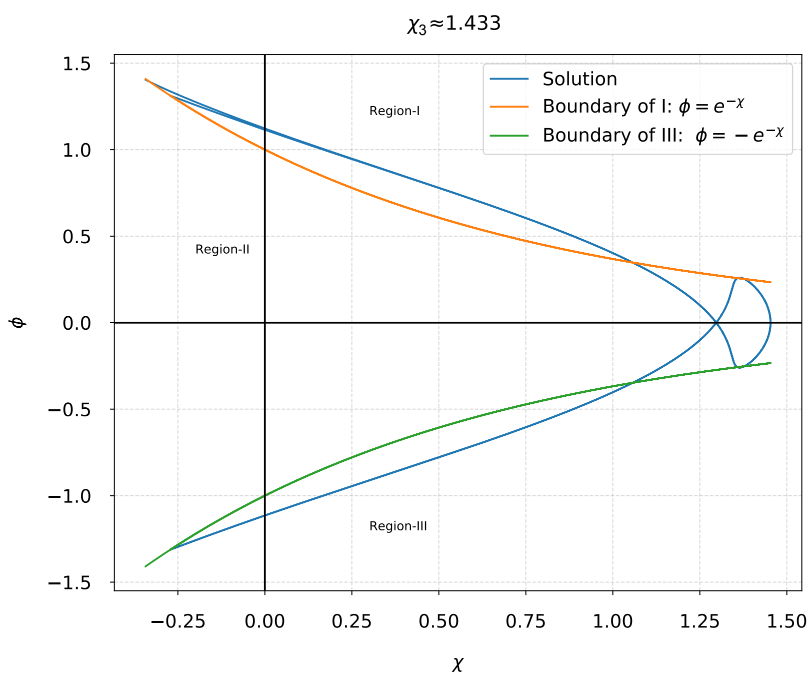
Hawking noticed that there exist non-singular trajectories that are time-symmetric about the bounce [1]. That said, $a(-t) = a(t)$, $\phi(-t) = \phi(t)$ and the time derivative of both a and ϕ reverses sign. In other words $\frac{da(-t)}{d(-t)} = -\frac{da(t)}{d(t)}$ and $\frac{d\phi(-t)}{d(-t)} = -\frac{d\phi(t)}{d(t)}$. The trajectory halts at the boundary and returns to the same path it came from. Each successive bounce is separated by an odd number of traverses for the periodic solutions that are also symmetric under reflections about the χ-axis.

We did some research to find the initial value of χ at the boundary between regions I and II, which gives one traverse between each successive pair of time-symmetric bounces for this perhaps simplest periodic solution. The numerical results show that $\chi \approx -0.271723$ at the boundary between regions I and II or regions II and III. The trajectory moved to higher χ until it reaches a maximum χ at $\phi = 0$ which we will denote by χ_1. If the trajectory intersected the χ-axis at the exact value of χ_1, the curve would go to the boundary between regions II and III, halt, return back the same path it came from, and reach χ_1 again. However, we found an approximate value of χ_1 so the trajectory crossed the χ-axis at $\chi = 0.85399$ and after coming from boundary II it intersected the χ-axis at $\chi = 0.855237$. Since the deviation from the true periodic solution is growing with time, the approximate value of χ_1 is $\chi_1 \approx 0.854$ numerically. In the case of three traverses between two successive bounces, we will denote the maximum size of the Universe after a bounce by χ_3 where the solution crosses the χ-axis. We want to find as precise values of both χ_1 and χ_3 as possible in order to get an approximated probability for another strong bounce which we will denote by p_i, where i denotes the number of traverses between successive bounces. Before we calculate this probability, we need to show a few more graphs later on and use the numerical results to illustrate the probability for the case of one traverse but in the end, we will try to find a general formula of this probability for a large number of traverses. If we were able to find the exact value of χ_1, the shape of the solution curve would look like a reverse letter C where one traverse after each bounce repeats forever and hence forming a sequence of traverses, say $\{...1, 1, ...\}$. A similar case would happen if we knew exactly what χ_3 was. After three traverses we would have one bounce and then three traverses and so on, forming a sequence of traverses $\{...3, 3, ...\}$. The numerical value of χ_3 is approximately $\chi_3 \approx 1.43328$. We showed what this curve would approximately look in Figure 3.1. As Hawking noticed, these kinds of periodic solutions will have an odd number of traverses between two bounces and we illustrated them with one and three traverses as in the graphs below. In other words, i will be a positive odd number and suppose $i = 1$. For an exact value of $\chi_1 = \ln a$ there would be only one $\phi = 0$ crossing at χ_{max} but since we cannot calculate it precisely there will be two such ϕ crossings close to each other and we got an average of both them giving the desired approximate value for $a_i = e^{\chi_i}$. In other words, the curve does not come back exactly the same path it came from. A similar case would happen when $i = 3$ and so on.

The two graphs below start from the time symmetric bounce and then the curve crosses the χ-axis at $\phi = 0$ then goes to boundary II without entering region II. The curve come back on the same path it came from and intersected the χ-axis again and reached boundary I again. For the exact value of χ_i, this pattern repeats forever.



(a)



(b)

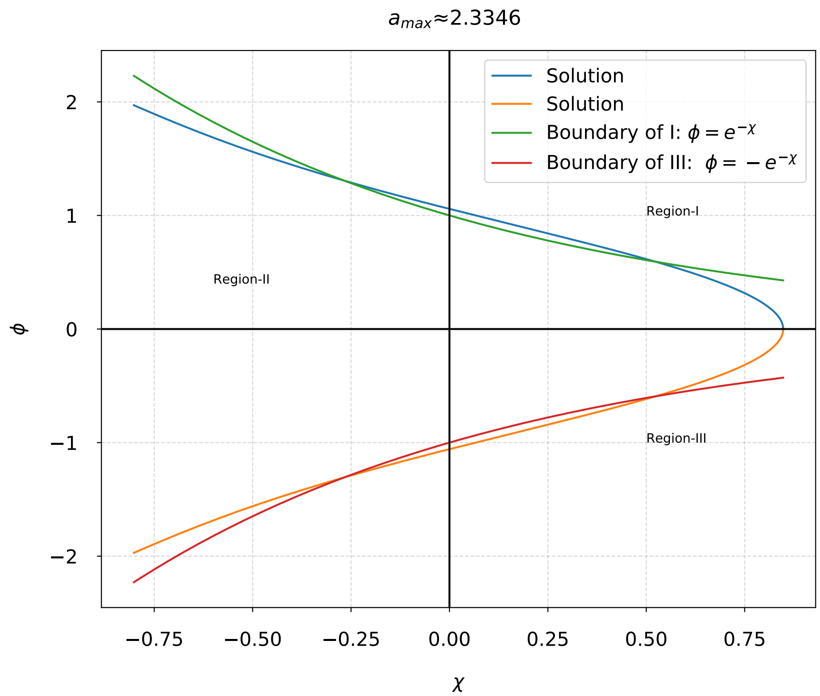
Figure 3.1: A plot of time-symmetric bounces with $i = 1, 3$. (a) The value of ϕ at the time-symmetric bounce is $\phi \approx \pm 1.1956$ or $\varphi \approx \pm 0.5843 M_{Pl}$. (b) The value of ϕ at the time-symmetric bounce is $\phi \approx \pm 1.312$ or $\varphi \approx \pm 0.6412 M_{Pl}$.

3.1.2 Calculation of an Approximate Probability p_i for Strong Bounces with $i = 1$

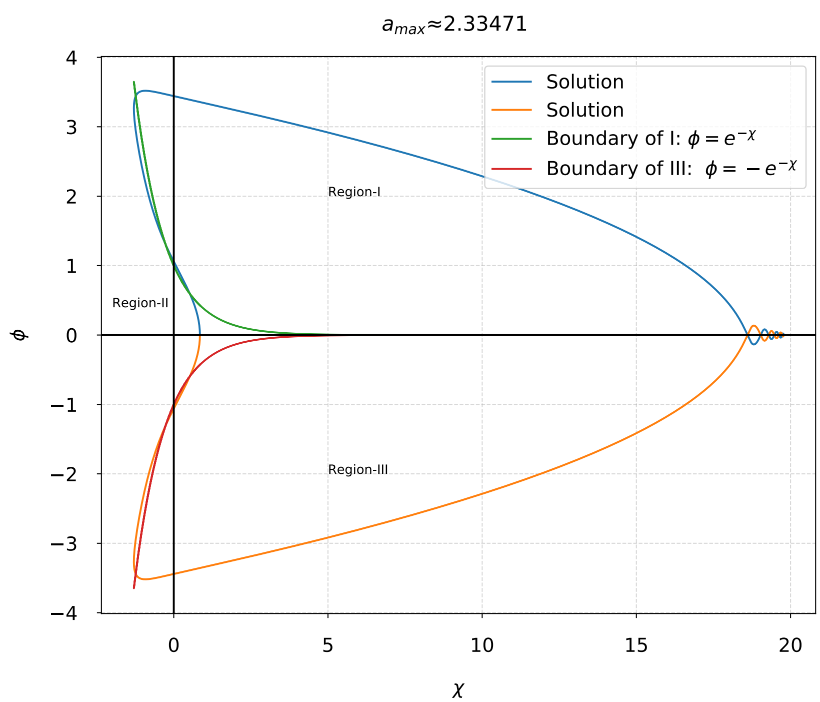
In this section, we will calculate an approximate probability of another strong bounce for the case $i = 1$, where i is an odd number and is equal to the number of traverses between successive bounces. After we set $m = 1$, we did some numerical research on finding how much a_{max} can be different from a_i so that the solution curve will approach the repeller solution without crossing it and hence have another bounce at some other value of either positive or negative ϕ.

In this part, we will consider $t = 0$ at the maximum size of a which we denoted it by a_{max}. We are going to denote by $a_{i-0.5}$ and $a_{i+0.5}$ the value of a at the farthest distance from a_i for which we have a bounce. It is worth mentioning that a and χ can be used interchangeably and given by the relation $\chi = \ln a$. Let us consider the case with one traverse between two successive bounces, $i = 1$. Ideally, if the solution curve starts from $a_{max} = a_{0.5}$ then it will approach the repeller solution to $\phi = \pm\infty$. Assuming that the round-off and the step size are not much larger than a few times 10^{-6} the approximate value of $a_{0.5}$ is $a_{0.5} \approx 2.3347$ numerically. If the trajectory starts with a value slightly different than this value, then the trajectory will either follow the repeller solution up to some positive value of ϕ and then turn right to have another inflationary period followed by many traverses or turn left, cross the repeller solution and go singular.

Below are two graphs that represent this case. We started with the initial conditions $\dot{a}(t = 0) = 0$, $\phi(t = 0) = 0$, $a(t = 0) = a_{max}$, $\dot{\phi}(t = 0) = 1/a_{max}$. For values of a_{max} less than $a_{1.5}$ but above a_1 the solution approaches the repeller solution in negative ϕ and then turns right to have another period of inflation followed by many traverses.



(a)

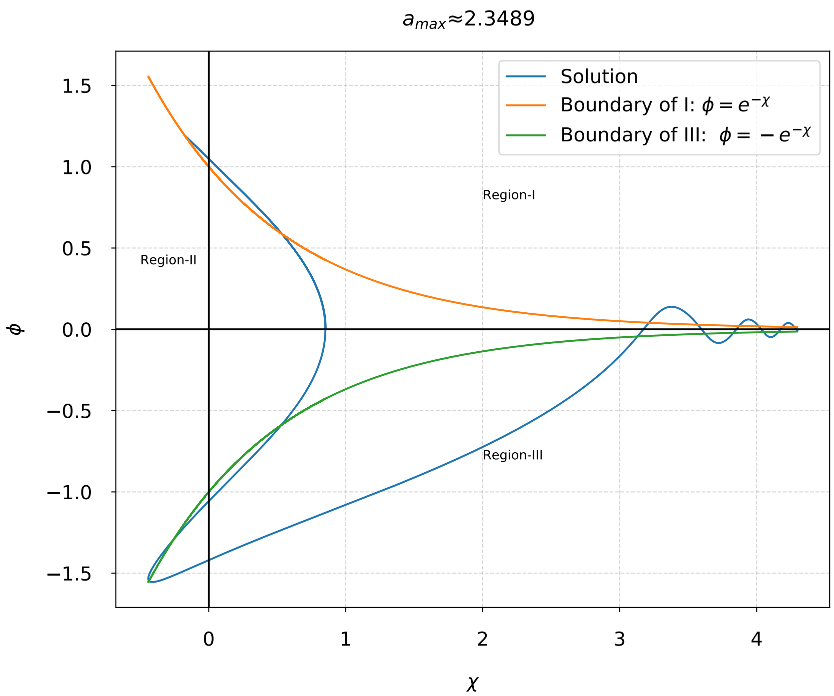


(b)

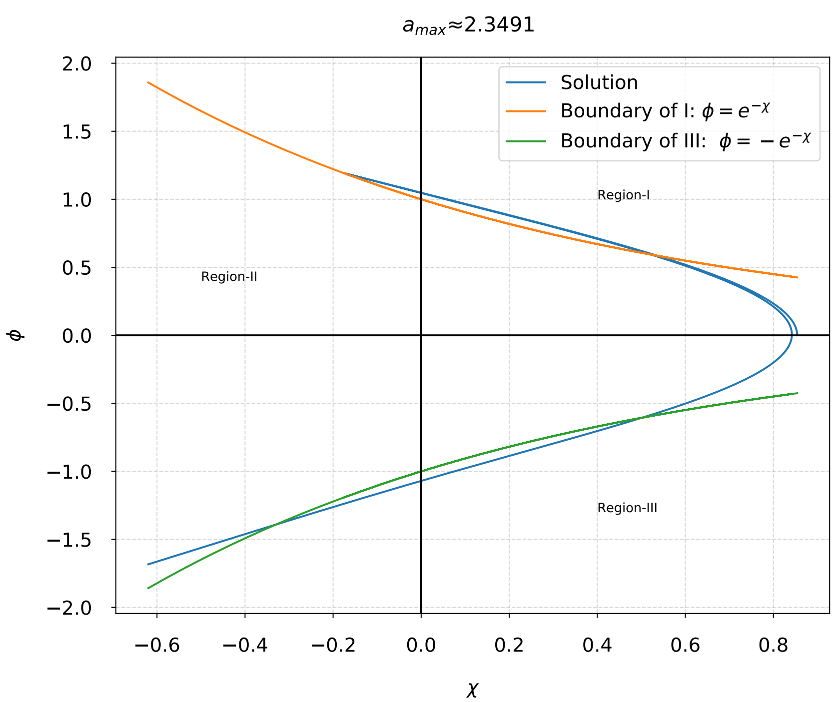
Figure 3.2: (a) $a_{max} = 2.3346 < a_{0.5}$, $m = 1$, (b) $a_{max} = 2.33471 > a_{0.5}$, $m = 1$

If a_{max} is slightly bigger then a_1 up to the value which we will call $a_{max} = a_{1.5}$ the solution will approach the repeller solution in negative ϕ. If $a_{max} = m_{1.5}$ then the solution will approach the repeller solution to $\phi = -\infty$. This is similar to the graphs above but this time the solution curve is following the repeller solution in the negative ϕ. We started with the initial conditions $\dot{a}(t=0) = 0$, $\phi(t=0) = 0$, $a(t=0) = a_{max}$, and $\dot{\phi}(t=0) = a/a_{max}$. For values of a_{max} less than $a_{1.5}$ but above a_1 the solution approaches the repeller solution in negative ϕ and then turns right to have another period of inflation followed by traverses. On the other hand, for values of a_{max} more than $a_{1.5}$ the solution approaches the repeller solution in negative ϕ and then crosses it and goes to the left until reaches the singularity. Numerical calculations show that $a_{1.5} \approx 2.349$. As it can be seen in Figure 3.3 (a), we started at $a = a_{max}$ and $\phi = 0$ with $\frac{da}{dt} = 0$ and $\frac{d\phi}{dt} = +\frac{1}{a_{max}}$ initially, evolved up to an approximate time-symmetric bounce at a minimum for a, evolved back down to so close to the initial point that the difference is not visible in the graph, and then we evolved further down to negative phi where the bounce is visibly not time-symmetric but still led to inflation for $\phi < 0$.

So far we have found numerically the approximate values of $a_{0.5}$, a_1, and $a_{1.5}$. In order to find the approximate probability of a strong bounce with a_{max} near a_1 we also need to find the approximate value of $a_{2.5}$. If the curve starts from $a_{max} = a_{2.5}$ then the solution will approach the repeller solution at $\phi = -\infty$ after three traverses. The value of $a_{2.5}$ will be slightly smaller than the value of a_3. The higher the value of the positive odd number i gets, the closer $a_{i-0.5}$ to a_i gets. When i is large, there will be many traverses before the strong bounce and hence the phase of the oscillation θ will be well defined and we will see later that it approaches a constant. We will calculate its numerical value later. Numerical calculations suggest that $a_{2.5} \approx 4.2736$. When the curve starts from a_{max} slightly smaller than $a_{2.5}$ then the solution will follow the repeller solution up to some negative ϕ and cross the repeller solution and go singular. When the curve starts from a_{max} slightly greater than $a_{2.5}$ then the solution will follow the repeller solution up to some negative ϕ and turn right to have another period of inflation. We chose $a_{max} = 4.271$ and $a_{max} = 4.274$ to show this in Figure 3.4.

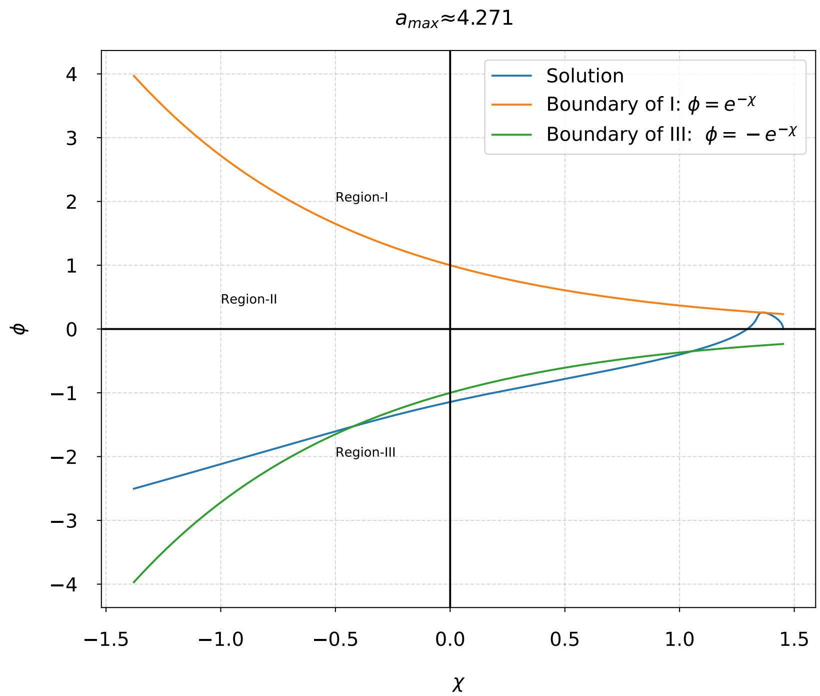


(a)

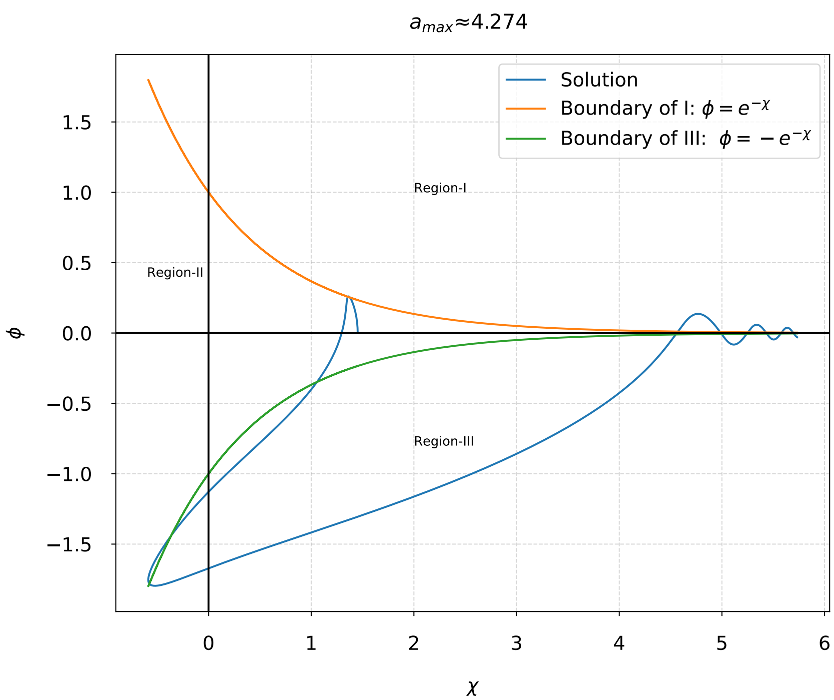


(b)

Figure 3.3: (a) $a_{max} = 2.3489 < a_{1.5}$, m=1 (b) $a_{max} = 2.3491 > a_{1.5}$, $m = 1$



(a)



(b)

Figure 3.4: (a) $a_{max} = 4.271 < a_{2.5}$ $m = 1$, (b) $a_{max} = 4.274 > a_{2.5}$, $m = 1$

Having found numerically what $a_{0.5}$, a_1, $a_{1.5}$, and $a_{2.5}$ are approximately, we can find a rough estimate what the probability p_1 is. We estimate p_1 as:

$$p_1 \approx \frac{\chi_{1.5} - \chi_{0.5}}{\chi_{2.5} - \chi_{0.5}}. \tag{3.2}$$

Since $\chi_{0.5} \approx 0.8479$, $\chi_1 \approx 0.8499$, $\chi_{1.5} \approx 0.8539$, and $\chi_{2.5} \approx 1.4526$ one can approximately calculate the probability p_1 as:

$$p_1 \approx \frac{0.8539 - 0.8479}{1.4526 - 0.8479} = 0.01. \tag{3.3}$$

Now, our final goal is to find an asymptotic formula for p_i for very large i. To determine this, we need the behavior of the solutions in the dust-like oscillatory regime, inflation, bounce, repeller solution, and the stiff behavior going to the singularity at $a = 0$, including the perturbations from the behaviour for the periodic solutions with $a_m = a_i$. Therefore, in the next part, we will discuss the approximate solutions and the various perturbations from the periodic solutions.

3.2 An Approximate Series Expression for the Repeller Solution

In this part, we will discuss the repeller solution and we will find an approximate series solution for it. Every trajectory during a bounce with a large change in ϕ, or after crossing into Region II for $\chi < 0$ has a large change in ϕ before entering the stiff regime with $\dot{\phi}^2 \gg \phi^2$ to go singular, will follow close to the repeller solution for a large range of ϕ.

For large $|\phi|$, let the repeller solution be approximated by the series

$$\chi = -\ln\phi + \ln\sqrt{\frac{2}{3}} + \frac{c_2}{\phi^2} + \frac{c_3}{\phi^4} + \frac{c_4}{\phi^6} + \mathcal{O}(\phi^{-8}). \tag{3.4}$$

First, we need to determine the coefficients c_2, c_3, c_4 so that the curve satisfies our conditions mentioned above. It can be immediately determined that $\chi' = -\frac{1}{\phi} - \frac{2c_2}{\phi^3} - \frac{4c_3}{\phi^5} - \frac{5c_4}{\phi^7} + \mathcal{O}(\phi^{-9})$ and $\chi'' = \frac{1}{\phi^2} + \frac{6c_2}{\phi^4} + \frac{20c_3}{\phi^6} + \mathcal{O}(\phi^{-8})$. Before we proceed, we make an expansion for $e^{-2\chi}$ as follows

$$e^{-2\chi} = \frac{3}{2}\phi^2 \left(1 - \frac{2c_2}{\phi^2} - \frac{12c_3 - 2c_2^2}{\phi^4} - \frac{4c_2^3 + 4c_2 c_3}{3\phi^6} + \mathcal{O}(\phi^{-8})\right). \tag{3.5}$$

We can get the coefficients by plugging these in Eq. (3.1):

$$\left(e^{-2\chi} - \phi^2\right)\chi'' = \left(1 - \chi'^2\right)\left(\phi\chi' + 3\phi^2 - 2e^{-2\chi}\right), \tag{3.6}$$

obtaining

$$\left(\frac{\phi^2}{2} - 3c_2 + \frac{3(c_2^2 - c_3)}{\phi^2} + \frac{18c_2c_3 - 6c_2^2}{\phi^4} + \mathcal{O}(\phi^{-6})\right)\left(\frac{1}{\phi^2} + \frac{6c_2}{\phi^4} + \frac{20c_3}{\phi^6} + \mathcal{O}\left(\phi^{-8}\right)\right) =$$

$$= \left(1 - \frac{1}{\phi^2} + \frac{4c_2}{\phi^4} + \frac{4c_2^2}{\phi^6} + \frac{8c_3}{\phi^6} + \mathcal{O}(\phi^{-8})\right)\left(-1 - \frac{2c_2}{\phi^2} - \frac{4c_3}{\phi^4} + 6c_2 - 6\frac{6c_2c_3 - 2c_2^2}{\phi^4}\right.$$

$$\left. + \mathcal{O}(\phi^{-8})\right). \tag{3.7}$$

Following the same procedure for c_4 and c_5, we got the following expression for the unstable repeller solution, $\chi(\phi)$, up to order $\mathcal{O}(\phi^{-10})$

$$\chi = -\ln\phi + \ln\sqrt{\frac{2}{3}} + \frac{1}{4\phi^2} + \frac{11}{48\phi^4} + \frac{77}{144\phi^6} + \frac{297}{128\phi^8} + \mathcal{O}(\phi^{-10}). \tag{3.8}$$

3.3 Linearized Equation for the Perturbation of the Repeller Solution

In this part, we will analyze the perturbation of the repeller solution, β. Suppose $\chi = \chi_r + \beta$. Then $\chi' = \chi_r' + \beta'$ and $\chi'' = \chi_r'' + \beta''$. To linear order in β,

$$e^{-2\chi} \approx (1 - 2\beta)\, e^{-2\chi_r}. \tag{3.9}$$

Using these in the Eq. (3.1) and using the fact that χ_r obeys that equation, we get the following linearized equation

$$\left(e^{-2\chi_r} - \phi^2\right)\beta'' + \left(3\phi\chi_r'^2 + 6\phi^2\chi_r' - \phi - 4\chi_r' e^{-2\chi_r}\right)\beta'$$

$$+ \left(-2e^{-2\chi_2}\chi_r'' + 4\chi_r'^2 e^{-2\chi_r} - 4e^{-2\chi_r}\right)\beta = 0. \tag{3.10}$$

If we make the approximation $\chi_r \approx -\ln\phi + \ln\sqrt{\frac{2}{3}}$, then $\chi_r' \approx -\frac{1}{\phi}$ and $\chi_r'' \approx \frac{1}{\phi^2}$. We plug these relations in the equation above and we obtain

$$\frac{\phi^2}{2}\beta'' + \left(\frac{3}{\phi} - \phi\right)\beta' + \left(3 - 6\phi^2\right)\beta \approx 0. \tag{3.11}$$

For $\phi >> 1$, we can approximate the differential equation above as

$$\beta'' - \frac{2}{\phi}\beta' - 12\beta \approx 0. \tag{3.12}$$

This has a general approximate solution for $\phi >> 1$ and which does not lead to collapse of the Universe to $a = 0$ of the form

$$\beta \approx C\phi\cosh\left[2\sqrt{3}(\phi - \phi_*)\right]. \tag{3.13}$$

Here, the minimum distance between the curve and the repeller solution happens at ϕ_*. For the particular solution given in Figure 3.2 (b), the minimum value of $\beta = \chi - \chi_r$ was $\beta \approx 0.1352$, occurring at $\phi_* \approx 1.4234$ numerically. This corresponds to $C \approx 9.5 \times 10^{-2}$. As a result, the perturbation of our particular solution has the form found numerically as

$$\beta = 9.5 \times 10^{-2}\phi\cosh\left[\sqrt{12}(\phi - 1.4234)\right]. \tag{3.14}$$

3.4 Approximate Solutions for the Inflationary Regime

In this part, we will find an approximate equation for the attractor solution. As the Universe expands, $m^2a^2 >> 1$ so the solution approaches the attractor solution which we will find an approximate series expression for it. With $\chi = \ln(ma)$, let the function U be defined as $U = -\frac{d\chi}{d\phi}$. It easy to check that $\dot{\chi} = \frac{\dot{a}}{a}$ and $\dot{\phi} = -\frac{\dot{\chi}}{U}$. If we use these in Eq. (2.6), we get $\dot{\chi}^2 = \frac{\left(m^2\phi^2a^2-1\right)U^2}{a^2(U^2-1)}$ and $\ddot{\chi} = -3\dot{\phi}^2 + \frac{1}{a^2}$. These will be very useful to find

$$\ddot{\phi} = -\frac{\ddot{\chi}U - \dot{U}\dot{\chi}}{U^2} = -\frac{\dot{\chi}^2U'}{U^3} - \frac{1}{Ua^2} + \frac{3\dot{\chi}^2}{U^3}, \tag{3.15}$$

where $' = \frac{df}{d\phi}$. Using the expression for $\ddot{\phi}$ in Eq. (2.7) yields

$$\frac{dU}{d\phi} = \frac{1-U^2}{m^2a^2\phi^2-1} + 3 - 3U^2 + \frac{m^2\phi a^2 U\left(U^2-1\right)}{m^2\phi^2a^2-1}$$

$$= \left(U^2-1\right)\left(\frac{U}{\phi} - 3 + \frac{\frac{U}{\phi}-1}{m^2a^2\phi^2-1}\right). \tag{3.16}$$

As Universe expands, the Universe grows very large and $m^2a^2\phi^2 >> 1$, so

$$\frac{dU}{d\phi} \approx 3 - 3U^2 + \frac{U^3-U}{\phi} = \left(1-U^2\right)\left(3-\frac{U}{\phi}\right). \tag{3.17}$$

The equation for $\frac{dU}{d\phi}$ is exact with the assumption that $1/a^2 = 0$ except near $a = a_m$ where $1/a^2$ becomes important again. It is worth mentioning that it does not apply near each point where ϕ crosses zero and when a gets close to its maximum value. During slow-roll inflation with $\phi >> 1$, the solution will rapidly approach the attractor solution

$$U = 3\phi + \frac{a}{\phi} + \frac{b}{\phi^3} + \frac{c}{\phi^5} + \frac{f}{\phi^7} + \mathcal{O}(\phi^{-9}). \tag{3.18}$$

In order to find the coefficients a, b, c, and f, we plug this expression into the differential equation for U, obtaining the following expression:

$$3 - \frac{a}{\phi^2} - \frac{3b}{\phi^4} - \frac{5c}{\phi^6} - \frac{7f}{\phi^8} + \mathcal{O}(\phi^{-10}) = 9a + \left(6a^2 - a + 9b\right)\frac{1}{\phi^2} + \left(a^3 - b + 12ab + 9c\right)\frac{1}{\phi^4}$$

$$+ \left(3a^2b + 6b^2 - c + 12ac + 9f\right)\frac{1}{\phi^6} + \left(3ab^2 + 3a^2c + 12bc - f + 12af\right)\frac{1}{\phi^8}. \tag{3.19}$$

By comparing both sides of the expression, we found the following coefficients $a = \frac{1}{3}$, $b = \frac{-2}{27}$, $c = \frac{11}{243}$, $f = \frac{-10}{243}$. Therefore, the attractor solution has the form

$$U = 3\phi + \frac{1}{3\phi} - \frac{2}{27\phi^3} + \frac{11}{243\phi^5} - \frac{10}{243\phi^7} + \mathcal{O}(\phi^{-9}). \tag{3.20}$$

We also want to remark that since Eq. (3.20) is a good approximation only for $m^2a^2\phi^2 - 1 >> \frac{\frac{U}{\phi}-1}{\frac{U}{\phi}-3} \approx 6\phi^2$, the attractor solution will only become a good approximation for $m^2a^2 >> 1$ and not just $m^2a^2\phi^2 >> 1$.

3.5 Approximate Solutions in Both the Inflationary and Oscillatory Regimes

In this part, we will use a different representation that applies to both the inflationary and oscillatory regime, so long as the dimensionless $1/a^2$ can be neglected. Let us define T as

$$T = \frac{2}{3h}, \tag{3.21}$$

where $h \equiv \dot{\chi}$, a dimensionless Hubble expansion rate and we have defined the overdot so that it means the dimensionless derivative $\frac{1}{m}\frac{d}{dt}$ when the lapse is $N = 1$ so that t is the comoving proper time.

Next, by assuming $a^2\phi^2 >> 1$ let us define $\phi = h\cos\psi$, $\dot{\phi} = -h\sin\psi$, and $\dot{h} = \ddot{\chi} \approx -3\dot{\phi}^2$. Henceforth in this section, we shall use $=$ instead of $\approx$ for the equations we get when we ignore terms going as inverse powers of $m^2a^2\phi^2$. We will later see that here we can get the phase constant θ, basically the phase of the oscillation, and we will calculate its asymptotic value numerically. The dimensionless time derivative of the dimensionless T is

$$\dot{T} = -\frac{2}{3h^2}\dot{h} = \frac{2}{h^2}\dot{\phi}^2 = 2\sin^2\psi. \tag{3.22}$$

Similarly, we can find the time derivative of ψ by calculating the time derivative for ϕ:

$$\dot{\phi} = \dot{h}\cos\psi - (h\sin\psi)\,\dot{\psi} = -h\sin\psi. \tag{3.23}$$

Using the fact that $\dot{h} = -3\dot{\phi}^2$, we have

$$\dot{\psi} = 1 - \frac{1}{T}\sin(2\psi). \tag{3.24}$$

In order to find $\frac{dT}{d\psi}$ we need to divide $\frac{dT}{dt}$ by $\frac{d\psi}{dt}$ and we obtain

$$\frac{\dot{T}}{\dot{\psi}} = \frac{dT}{d\psi} = \frac{2\sin^2\psi}{1 - \frac{1}{T}\sin(2\psi)}. \tag{3.25}$$

3.5.1 Approximate Expression of T for Very Small ψ

In this part, we will find a series form for T so that we can find the value of T for some small ψ. To do so, we need to go back to the attractor solution, where ψ and T are both small and we consider a great amount of inflation so that we can neglect the spatial curvature. Consider a series with increasing positive orders of ψ of the form

$$T = c_1\psi + c_2\psi^2 + c_3\psi^3 + c_4\psi^4 + c_5\psi^5 + c_6\psi^6 + c_7\psi^7 + c_8\psi^8 + \mathcal{O}(\psi^9). \tag{3.26}$$

We can use this expression in the differential equation for $\frac{dT}{d\psi}$ but in order to get rid of T in the denominator, we make the following rearrangement for the differential equation

$$\Big(T - \sin(2\psi)\Big)\frac{dT}{d\psi} = T\Big(1 - \cos(2\psi)\Big). \tag{3.27}$$

A more detailed calculation of the coefficients is given in Appendix B. The coefficients we got are

$$c_1 = 2,\ c_2 = 0,\ c_3 = \frac{2}{3},\ c_4 = 0,\ c_5 = -\frac{26}{15},\ c_6 = 0,\ c_7 = \frac{2764}{315}, ... \tag{3.28}$$

As a result, in this regime where $\psi << 1$ the series for T in terms of increasing positive powers of ψ is

$$T = 2\psi + \frac{2}{3}\psi^3 - \frac{26}{15}\psi^5 + \frac{2764}{315}\psi^7 + \mathcal{O}(\psi^9). \tag{3.29}$$

The reason why we searched for such a series is that we are trying to find an initial value for T when we do the numerical integration for the first order differential equation for $\frac{dT}{d\psi}$. Even though the initial value will be accurate up to some order, in fact, it will help us to find the integration constant θ up to some order and satisfactory enough to give some meaning to the model. However, starting at too high a value for ψ, the error would depend on the starting value mainly from the error from truncating the infinite series in ψ of $T(\psi)$, but on the other hand, if we start at too low a value for ψ, we would expect that the error would depend on the starting value mainly from round-off error in doing the integration.

3.5.2 The Linearized Perturbation for T

In this part, we will find a linear perturbation equation for T, which we can use it to determine the power of a_m when finding an asymptotic expression for $\delta\theta$, the range of the phase constant that gives another bounce. Let us introduce a small perturbation of T denoted by t, here not to be confused with the time. Therefore, $T' = T + t$. If we substitute this into the Eq. (3.79), we get

$$\left(T - \sin(2\psi) + t\right)\left(\frac{dt}{d\psi} + \frac{dT}{d\psi}\right) = (T + t)\left(1 - \cos(2\psi)\right). \tag{3.30}$$

As a result, the general differential equation for the linearized perturbation t in terms of T and ψ is

$$\left(T - \sin(2\psi)\right)\frac{dt}{d\psi} = t\left(1 - \frac{dT}{d\psi} - \cos(2\psi)\right). \tag{3.31}$$

In the case of small ψ we can use the series expression we found for T above and substitute this instead of T and we obtain

$$\left(2\psi^3 - 2\psi^5 + \frac{44}{5}\psi^7\right)\frac{dt}{d\psi} = t\left(-2 + 8\psi^4 - \frac{184}{3}\psi^6 - \frac{2}{315}\psi^8\right). \tag{3.32}$$

This is a first-order ODE and integrating both sides we have a truncated series for t whose leading term for small ψ gives

$$t \approx \frac{c}{\psi}\exp\left(\frac{1}{2\psi^2}\right). \tag{3.33}$$

For very small ψ, the leading term in the exponent is $\frac{1}{2\psi^2}$. As can be seen in the equation above, the linearized perturbation diverges at $\psi = 0$. Numerical integration of Eq. (3.31) to large ψ will then give the coefficient of the expression for $t(\psi)$ for small ψ.

3.5.3 Approximate Expression of T for Very Large ψ, Oscillatory Regime

In this section, we will find an approximate series expression for T when $\psi >> 1$, or in other words, well into the oscillatory regime but before the Universe reaches the maximum size. Eq. (3.25) will be very important in order to find a Fourier series for T in the oscillatory regime in terms of ψ, and there will also be a phase constant for each solution which we call θ. This constant will come from the integration to be done in the coming steps and we want to find out how much of a deviation from this θ will give another bounce after the Universe starts to re-collapse. For a large number of zero crossings of ϕ, this phase constant is well defined. In the research paper for the symmetric bounce quantum state of the Universe, the phase constant is found by the formula [2]

$$\theta = \frac{2}{3\dot{\chi}} - \psi + \sin\psi\cos\psi. \tag{3.34}$$

In this part, we will find a better approximation for the phase constant θ, and to do so we need to rearrange the differential equation above into this form

$$\left(1 - \frac{1}{T}\sin(2\psi)\right)\frac{dT}{d\psi} - \left(1 - \cos(2\psi)\right) = 0. \tag{3.35}$$

Let T be expressed as a series sum,

$$T = \psi + f_0 + \frac{f_1}{\psi} + \frac{f_2}{\psi^2} + \frac{f_3}{\psi^3} + ... = \psi + \sum_{m=0}^{\infty}\frac{f_m}{\psi^m}, \tag{3.36}$$

where $f_0, f_1, f_2, ...$ will be linear combinations of only constants, $\sin(2m\psi)$, and $\cos(2m\psi)$ with integer values of m.

In Appendix B, we have found the following four differential equations for the first of these f_m functions:

$$\frac{df_0}{d\psi} = -\cos(2\psi), \tag{3.37}$$

$$\frac{df_1}{d\psi} = 2\sin(2\psi)\sin^2(\psi) = \sin(2\psi) - \frac{1}{2}\sin(4\psi), \tag{3.38}$$

$$\frac{df_2}{d\psi} = f_1 - f_0\sin(2\psi) + f_0\frac{\sin(4\psi)}{2} + 2\sin^2(2\psi)\sin^2(\psi), \tag{3.39}$$

$$\frac{df_3}{d\psi} = 2f_2 + f_0^2\sin(2\psi) - 2f_1\sin(2\psi) + f_0^2\frac{df_0}{d\psi}\sin(2\psi) - f_1\frac{df_0}{d\psi}\sin(2\psi) - f_0\frac{df_1}{d\psi}\sin(2\psi)$$

$$+\sin(2\psi)\frac{df_2}{d\psi}. \tag{3.40}$$

We can find the integral on both sides for each equation obtained but the problem is that we will end up with the integration constants which will have to be determined. For example,

$$f_0 = \int \frac{df_0}{d\psi}\, d\psi = -\int \cos(2\psi)\, d\psi = -\frac{1}{2}\sin(2\psi) + \theta. \tag{3.41}$$

Similarly, we can also find f_1, f_2 and so on

$$\int \frac{df_1}{d\psi}\, d\psi = \int \left[\sin(2\psi) - \frac{1}{2}\sin(4\psi)\right] d\psi. \tag{3.42}$$

This will give us

$$f_1 = \frac{1}{8}\cos(4\psi) - \frac{1}{2}\cos(2\psi) + \theta_1. \tag{3.43}$$

To find the value of θ_1 and the other integration constants θ_n, we make sure that the average value of $\frac{df_{n+1}}{d\psi}$ vanishes over one period of 2π and therefore no increase in f_{n+1} over one period will happen. We can get recursion relations for f_n''s in terms of the f_n's and their derivatives with smaller n. This will determine the integration constant for f_{n-1}. The functions f_n will be oscillatory functions of ψ that do not have monotonically growing or shrinking terms. Imposing all of these we have,

$$\int_0^{2\pi} \frac{df_2}{d\psi}\, d\psi = \int_0^{2\pi} \left(f_1 - f_0 \sin(2\psi) + f_0 \frac{\sin(4\psi)}{2} + 2\sin^2(2\psi)\sin^2(\psi) \right) d\psi = 0. \tag{3.44}$$

After we did this integration, we found out that

$$\theta_1 = -\frac{3}{4}. \tag{3.45}$$

As a result,

$$f_1 = \frac{1}{8}\cos(4\psi) - \frac{1}{2}\cos(2\psi) - \frac{3}{4}. \tag{3.46}$$

In order to get a reasonably accurate value for the integration constant θ, we need to truncate the series for T up to order ψ^{-3} so that our results would be satisfactory enough and assuming that the series converges fast enough. However, we did not prove and it seems quite cumbersome to prove that the actual Fourier series for T will converge when we consider all the terms. Since this is not the aim of our project, we tend to truncate the series up to this order. Integrating $\frac{df_2}{d\psi}$ we get

$$\int \frac{df_2}{d\psi} d\psi = \int \left[f_1 - f_0 \sin(2\psi) + f_0 \frac{\sin(4\psi)}{2} + 2\sin^2(2\psi)\sin^2(\psi) \right] d\psi$$

$$= \frac{1}{16}\sin(6\psi) - \frac{5}{32}\sin(4\psi) - \frac{7}{16}\sin(2\psi) + \theta \left[\frac{1}{2}\cos(2\psi) - \frac{1}{8}\cos(4\psi) \right] + \theta_2. \quad (3.47)$$

As we mentioned earlier, we set the average of $\frac{df_3}{d\psi}$ over one period of 2π to be equal to zero so that will give the value of θ_2. Therefore

$$\int_0^{2\pi} \left[2f_2 + f_0^2 \sin(2\psi) - 2f_1 \sin(2\psi) + f_0^2 \frac{df_0}{d\psi} \sin(2\psi) - f_1 \frac{df_0}{d\psi} \sin(2\psi) - f_0 \frac{df_1}{d\psi} \sin(2\psi) \right.$$

$$\left. + \frac{df_2}{d\psi} \sin(2\psi) \right] d\psi = 0. \quad (3.48)$$

After doing this integration, one can obtain $\theta_2 = \frac{3\theta}{4}$. As a result, f_2 is equal to

$$f_2 = -\frac{7}{16}\sin(2\psi) - \frac{5}{32}\sin(4\psi) + \frac{1}{16}\sin(6\psi) + \theta \left(\frac{3}{4} + \frac{1}{2}\cos(2\psi) - \frac{1}{8}\cos(4\psi) \right). \quad (3.49)$$

The more terms we get, the better the approximations will be but we think that truncating the series up to order ψ^{-3} would be sufficient enough to get satisfactory results. Substituting these expressions in the series for T we get

$$T = \psi + f_0 + \frac{f_1}{\psi} + \frac{f_2}{\psi^2} + \mathcal{O}(\psi^{-3}) = \psi - \frac{1}{2}\sin(2\psi) + \theta + \left[-\frac{3}{4} - \frac{1}{2}\cos(2\psi) + \frac{1}{8}\cos(4\psi) \right] \frac{1}{\psi}$$

$$+ \left\{ -\frac{7}{16}\sin(2\psi) - \frac{5}{32}\sin(4\psi) + \frac{1}{16}\sin(6\psi) + \theta \left[\frac{3}{4} + \frac{1}{2}\cos(2\psi) - \frac{1}{8}\cos(4\psi) \right] \right\} \frac{1}{\psi^2}$$

$$+ \mathcal{O}(\psi^{-3}). \quad (3.50)$$

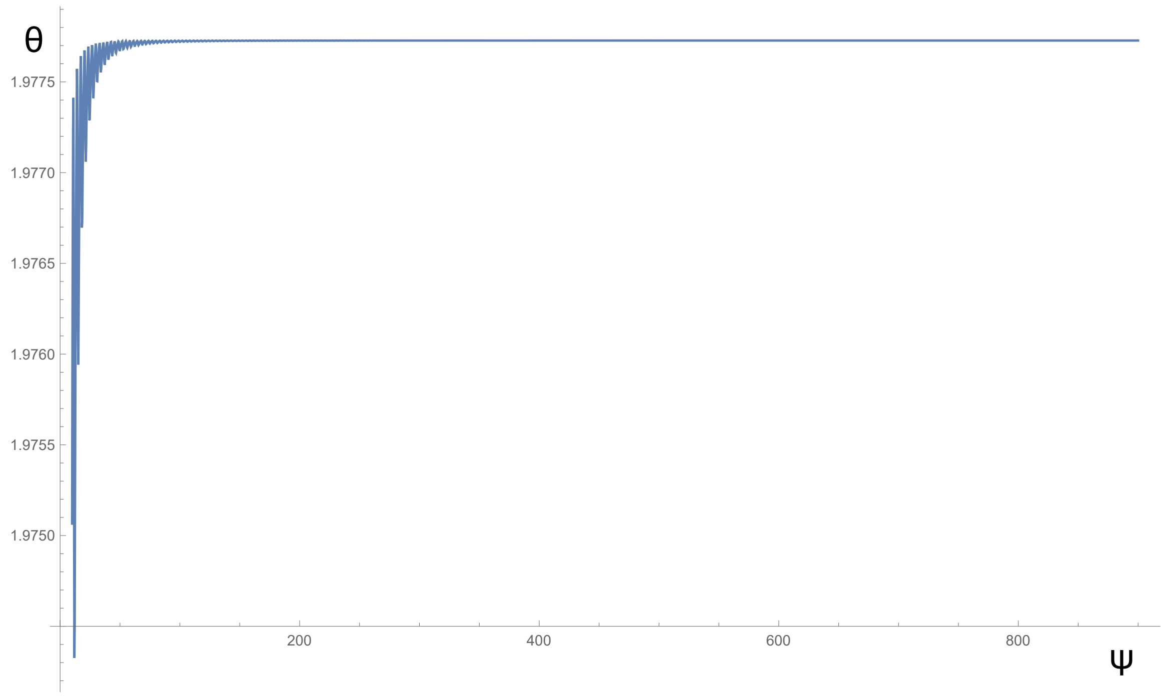
3.6 Numerical Calculations for the Phase Constant

In this part, we will find numerically the value of the phase constant θ for the attractor solution that represents a large amount of inflation.

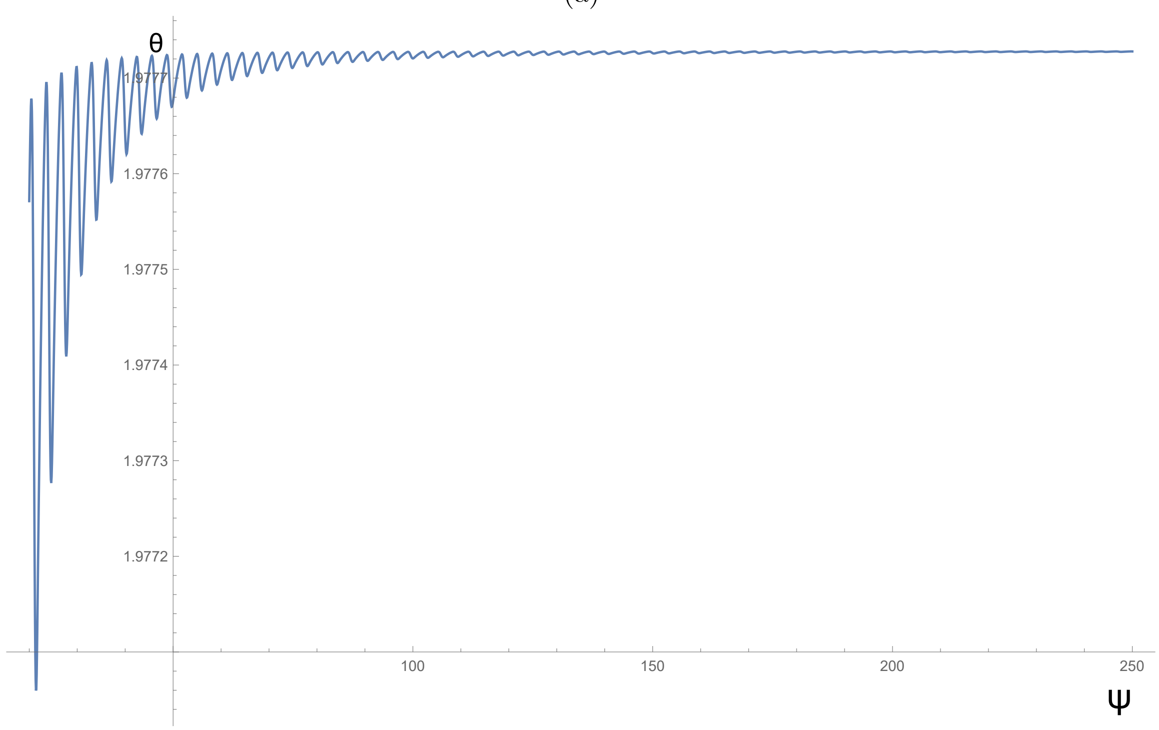
$$\theta \approx \left[1 + \frac{1}{\psi^2}\left(\frac{3}{4} + \frac{\cos(2\psi)}{2} - \frac{\cos(4\psi)}{8}\right)\right]^{-1} \left\{T - \psi + \frac{\sin(2\psi)}{2} - \frac{1}{\psi}\left[\frac{\cos(4\psi)}{8} - \frac{\cos(2\psi)}{2} - \frac{3}{4}\right] + \frac{1}{\psi^2}\left[\frac{7}{16}\sin(2\psi) + \frac{5\sin(4\psi)}{32} - \frac{1}{16}\sin(6\psi)\right]\right\}. \tag{3.51}$$

The phase constant θ is an asymptotic constant late in the 'dust regime' but before the spatial curvature term becomes important. As shown in Figure 3.5, for large ψ, the value of θ approaches a constant. For the attractor solution, the initial value for T can be roughly calculated from the series expression of T at ψ small enough. The oscillations of the truncated formula for theta should fairly rapidly damp to almost zero oscillations, especially in the range of $\psi \in [0, 100]$. We did different integrations starting at different initial values of ψ and we found a range where the asymptotic value of theta stays the same to several decimal places when you change the starting value of ψ within this range. Numerical calculations showed that θ approached 1.9777 asymptotically as ψ goes to infinity.

Below are two graphs that show our numerical results for the asymptotic value of θ. The previously published value of θ was 1.978 [2]. In the next step, we will find the linearized perturbation of T. We plan to integrate the linearized perturbation numerically across the intermediate region between large ψ and small ψ. We will calculate the perturbation in θ from its value in the attractor solution.



(a)



(b)

Figure 3.5: Numerical plot of the asymptotic expression for the phase constant θ. It seems that the expression goes to 1.9777 asymptotically.

Chapter 4

Conclusions, Recommendations, & Future Work

4.1 Conclusions

In this thesis, we considered a Universe that has a bounce and expands to a large maximum size with a large number of zero crossings of the scalar field between two consecutive bounces. We tried to find the perturbation of the solution as a change in a certain phase θ defined in Section 3.5. We are still seeking to find for what fraction of this phase change would the Universe have another bounce as a function of the maximum size of the Universe, a_{max}.

We have discussed the inflationary regime and the oscillatory regime for a symmetric bounce solution as well as the perturbations around them. We have found an approximate numerical value of the phase constant θ for the attractor solution representing a large amount of inflation as shown in Figure 3.5. As it can be seen in the figures mentioned, the asymptotic expression for this phase constant in the limit that the initial value of the scalar field, ϕ_i, is taken to be arbitrarily large and rapidly damps and approaches to a constant as the Universe gets very large.

4.2 Future Work

Although we have found numerically what the value of the phase constant is, up to about five significant figures, when the solution in the inflationary regime approaches very near to the attractor solution, and we found the equation for linearized perturbations around T, we still need to use this perturbation to determine what fraction of this phase gives another bounce, or in other words stays close enough to the attractor solution so that it does not go to a singularity.

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Appendix A: Derivation of an Approximate Expression of T for Very Small ψ

Let us consider a series with increasing positive powers of ψ of the form

$$T \approx c_1\psi + c_2\psi^2 + c_3\psi^3 + c_4\psi^4 + c_5\psi^5 + c_6\psi^6 + c_7\psi^7. \tag{A.1}$$

Before we plug this series for T in this equation

$$\Big(T - \sin(2\psi)\Big)\frac{dT}{d\psi} = T\Big(1 - \cos(2\psi)\Big), \tag{A.2}$$

we will use the expansion of both $\sin(2\psi)$ and $\cos(2\psi)$ given by the Taylor expansion

$$\sin(2\psi) \approx 2\psi - \frac{4}{3}\psi^3 + \frac{4}{15}\psi^5 - \frac{8}{315}\psi^7, \tag{A.3}$$

and

$$\cos(2\psi) \approx 1 - 2\psi^2 + \frac{2}{3}\psi^4 - \frac{4}{45}\psi^6. \tag{A.4}$$

Also the approximate expression for $\frac{dT}{d\psi}$ is given by

$$\frac{dT}{d\psi} \approx c_1 + 2c_2\psi + 3c_3\psi^2 + 4c_4\psi^3 + 5c_5\psi^4 + 6c_6\psi^5 + 7c_7\psi^6. \tag{A.5}$$

With all this information, we plug them in the Eq. (A.2) and we get

$$\left[c_1\psi + c_2\psi^2 + c_3\psi^3 + c_4\psi^4 + c_5\psi^5 + c_6\psi^6 + c_7\psi^7 - \left(2\psi - \frac{4}{3}\psi^3 + \frac{4}{15}\psi^5 - \frac{8}{315}\psi^7\right)\right]\Big[c_1$$

$$+2c_2\psi + 3c_3\psi^2 + 4c_4\psi^3 + 5c_5\psi^4 + 6c_6\psi^5 + 7c_7\psi^6\Big] \approx \Big(c_1\psi + c_2\psi^2 + c_3\psi^3 + c_4\psi^4 + c_5\psi^5$$

$$+c_6\psi^6 + c_7\psi^7\Big) \left(2\psi^2 - \frac{2}{3}\psi^4 + \frac{4}{45}\psi^6\right). \qquad \text{(A.6)}$$

Comparing both sides with respect to each power of ψ we conclude that $c_1 = 2$ and $c_2 = 0$. For the constant c_3 we get the expression

$$c_1\left(c_3 + \frac{4}{3}\right) + 3c_3(c_1 - 2) - 4 = 0. \qquad \text{(A.7)}$$

Therefore $c_3 = \frac{2}{3}$. Also $c_4 = c_6 = 0$. Next, $c_5 = -\frac{26}{15}$ and $c_7 = \frac{2764}{315}$.

Appendix B: Derivation of Differential Equations for the Oscillatory Terms in the Series of T

For a large quantity like T with leading term $T_0 = \psi$, we expand in $\frac{(T-\psi)}{\psi} = x$. As a result,

$$x = \frac{f_0}{\psi} + \frac{f_1}{\psi^2} + \frac{f_2}{\psi^3} + \frac{f_3}{\psi^4} + ..., \tag{B.1}$$

also the derivative of T with respect to ψ can be calculated easily as

$$\frac{dT}{d\psi} = 1 + \frac{df_0}{d\psi} + \frac{df_1}{d\psi}\frac{1}{\psi} + \left(\frac{df_2}{d\psi} - f_1\right)\frac{1}{\psi^2} + \left(\frac{df_3}{d\psi} - 2f_2\right)\frac{1}{\psi^3} + ...$$
$$+ \left(\frac{df_n}{d\psi} - (n-1)f_{(n-1)}\right)\frac{1}{\psi^n} + ... \tag{B.2}$$

The expansion of $\frac{1}{T}$ is

$$\frac{1}{T} = \frac{1}{\psi}(1 - x + x^2 - x^3 + x^4 + ...) = \frac{1}{\psi} - \frac{f_0}{\psi^2} + \left(f_0^2 - f_1\right)\frac{1}{\psi^3} + \left(2f_0f_1 - f_2 - f_0^3\right)\frac{1}{\psi^4}$$
$$+\mathcal{O}(\psi^{-5}). \tag{B.3}$$

Inserting these in the differential equation

$$\left(1 - \frac{\sin(2\psi)}{T}\right)\frac{dT}{d\psi} = \left(1 - \cos(2\psi)\right) \tag{B.4}$$

we get the following long expression

$$\frac{df_0}{d\psi} + \cos(2\psi) + \left[\frac{df_1}{d\psi} - \left(\frac{df_0}{d\psi} + 1\right)\sin(2\psi)\right]\frac{1}{\psi}$$

$$+\left[\frac{df_2}{d\psi} - f_1 + \left(f_0 + f_0\frac{df_0}{d\psi} - \frac{df_1}{d\psi}\right)\sin(2\psi)\right]\frac{1}{\psi^2} + \left[\frac{df_3}{d\psi} - 2f_2 + \left(2f_1 - f_0^2 - f_0^2\frac{df_0}{d\psi}\right.\right.$$

$$\left.\left. + f_1\frac{df_0}{d\psi} + f_0\frac{df_1}{d\psi} - \frac{df_2}{d\psi}\right)\sin(2\psi)\right]\frac{1}{\psi^3} + O(\psi^{-4}) = 0. \tag{B.5}$$

Equating the coefficients in front of ψ^{-n} to zero one gets the following first three differential equations

$$\frac{df_0}{d\psi} = -\cos(2\psi), \tag{B.6}$$

$$\frac{df_1}{d\psi} = 2\sin(2\psi)\sin^2(\psi) = \sin(2\psi) - \frac{1}{2}\sin(4\psi), \tag{B.7}$$

$$\frac{df_2}{d\psi} = f_1 - f_0\sin(2\psi) + \frac{1}{2}f_0\sin(4\psi) - \frac{1}{4}\cos(2\psi) - \frac{1}{2}\cos(4\psi) + \frac{1}{4}\cos(6\psi) + \frac{1}{2}. \tag{B.8}$$

Hence,

$$\frac{df_2}{d\psi} = -\frac{7}{8}\cos(2\psi) - \frac{5}{8}\cos(4\psi) + \frac{3}{8}\cos(6\psi) + \theta\left[-\sin(2\psi) + \frac{1}{2}\sin(4\psi)\right]. \tag{B.9}$$

Next, we can find $\frac{df_3}{d\psi}$ as

$$\frac{df_3}{d\psi} = 2f_2 + \left(f_0^2 - 2f_1 + f_0^2\frac{df_0}{d\psi} - f_1\frac{df_0}{d\psi} - f_0\frac{df_1}{d\psi} + \frac{df_2}{d\psi}\right)\sin(2\psi). \tag{B.10}$$

In order to calculate the coefficients f_1, f_2, and f_3 we integrate on both sides the equations that we got above and it looks as if one gets a new constant of integration at each stage when one integrates f_i' to get f_i, where $f_i' = \frac{df_i}{d\psi}$. However, we have to set each integration constant after the θ in $f_0 = \theta - \frac{1}{2}\sin(2\psi)$ so that the next f_i' does not have a nonzero average over one period of ψ.