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Strongly Amenable Semigroups and Nonlinear Fixed Point Properties

by

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To my family and “ma petite puce”.

ABSTRACT

Left amenability, in it's modern form, was introduced by M. M. Day, in the 1940s. Amenability of groups and semigroups turned out to be quite common, and many interesting results are known, which motivated the introduction of extreme left amenability by Granirer in the 1960s. Extreme amenability turn out to be equivalent to a very strong nonlinear fixed point property, but examples of topological groups having this property are rather hard to construct. The purpose of this thesis is to study an intermediate property that we call strong left amenability.

If S is a semi-topological semigroup, and A denotes either $AP(S)$, $WAP(S)$ or $LUC(S)$ (the spaces of almost periodic, weakly almost periodic or left uniformly continuous functions on S respectively), then we say that A is strongly left amenable (SLA) if there is a compact left ideal group in the spectrum of A. We then say that S is SLA if $LUC(S)$ is SLA.

The first part of the thesis investigates the structure of such semigroups. We give some elementary properties, and characterize those semigroups for $AP(S)$, $WAP(S)$ and $LUC(S)$. We also characterize the strong left amenability of a semigroup when S is discrete, compact or connected. Finally, we show that homomorphic images of an SLA semigroup is SLA and so is the product of an extremely left amenable semigroup by a compact group. We conclude the first part of the thesis by giving some examples.

Amenability in general is closely related to non linear fixed point properties, and strong amenability is no exception. In the second part of this thesis, we characterize strong amenability in terms of a fixed compact set. We then obtain various fixed point properties related to jointly continuous actions and non-expansive mappings. We then extend some results on ultimately non-expansive mappings, a concept introduced by Kiang and Edelstein, to right reversible semigroups, and show that one of the conditions is always satisfied when the semigroup is indeed strongly amenable.

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Chapter 1

Introduction

Amenability traces its origins back to the work of Lebesgue on finitely additive measures in 1904 [48], and to the work of Banach and Tarski on their well-known paradox [3, 61]. The first study of this class of group was done by von Neumann in 1929 [64] as the class of non-paradoxical group. This class was later named *amenable* by Day, who started the modern period of amenability in the 1940s, when amenability shifted from the study of invariant measures to the study of invariant means [12]. For a more detailed account of the history of amenability, see [54].

Extremely amenable semigroups were introduced by Mitchell [52, 53] and Granirer [22] in the 60s. This is a very strong non-linear property, which is never possible for locally compact groups [23, 63] except in the case of the trivial group. More recent developments in the theory of extremely left amenable groups are presented in [19, 57]. A weaker version of extreme amenability, namely n-extreme amenability, was introduced in 1970 by Lau [37, 36]. The following year, Lau and Granirer [23] proved that all locally compact n-extremely left amenable (n-ELA) groups are finite groups, which was a major drawback since no non-trivial examples of n-ELA group were known. The first example

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of an extremely left amenable topological group was given by Herer and Christensen in 1975 [25]. It is now known that extremely left amenable topological groups are quite common and many examples are known [19]. The concept of strong left amenability, which is our main interest in this paper, sits somewhere between n-extreme left amenability and left amenability. The second chapter of this thesis, deals with all basic definitions, which are necessary later on to define strong left amenability. In particular we give the definition of extreme amenability and n-extreme amenability.

In the third chapter of this thesis, we study the structure of strongly left amenable (SLA) semigroups. We give the definition of such a semigroup, and characterize them. We also justify the use of the Banach algebra of left uniformly continuous functions for most of our work. In this chapter, we also provide the reader with some ways of constructing them, and give some examples. Some special classes of semigroup, such as discrete semigroup and compact semigroup are also studied.

Amenability is particularly interesting due to its closed relationship with fixed point properties. The fourth chapter of this thesis studies fixed point properties related to strong left amenability. In this chapter, we characterize strong left amenability in terms of a fixed compact set, and we get as corollary a fixed point theorem.

When K is a non-empty bounded closed convex subset of a Banach space E, we say that K has the fixed point property if for every non-expansive mapping $T : K \to K$, K contains a fixed point for T. We say that a Banach space E has the weak fixed point property if every weakly compact convex subset of E has the fixed point property. It is known [6] that if E has the weak fixed point property, then any weakly compact convex subset K has the common fixed point property for any commutative semigroup acting on K.

Every uniformly convex Banach space has the weak fixed point property. This was proved by Browder [5] in 1965, and improved on by Kirk [35], who proved that every weakly compact subset of a Banach space with normal structure has the fixed point property. Many examples of Banach space having the weak fixed point property are known [41]. These include c_0, l^1 and the Fourier algebra of a compact group. Notice that not all Banach spaces have the weak fixed point property. As was proved by Alspach [1], there exists a weakly compact convex subset K of $L^1[0,1]$ without any fixed point for some non-expansive map on K.

Representations of semigroups as non-expansive mappings are interesting to us since in many cases they characterize the existence of left invariant means on some C*-subalgebra of $l^\infty(S)$ [47]. In 2007, Kang [29] studied fixed point for representations of a semi-topological semigroup S as strongly continuous non-expansive mapping on a weakly compact convex subset of a Banach space, where $CB(S)$ is n-ELA. At the end of chapter 4, we generalize some of the work by Kang [29] to SLA semigroups.

Fixed point properties for a semigroup S acting on a Banach space X have been widely studied. Many results are known [28, 47, 39, 42, 30, 59, 58, 60], in particular, in the case where S is commutative, amenable or reversible. It is well known that every commutative semigroup is amenable, and every left amenable semigroup is left reversible. Papers investigating fixed point properties for left reversible semigroups include [49, 46, 34].

Similarly, fixed point properties for semigroup of non-expansive mappings have been studied extensively. In 1965, Kirk [35] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every non-expansive mapping T on C has a fixed point. As is well known, not every non-expansive action of a semigroup on a subset of a Banach space has a fixed point [1]. Many generalizations of this concept have also been investigated. For example, Holmes and Narayanaswami [28] introduced the concept of asymptotically non-expansive mapping and Kiang [32] studied eventually non-expansive mappings. In 1972, Goebel and Kirk [21] proved that if X is a uniformly convex Banach space, and C is a weakly compact convex subset of X, then every asymptotically non-expansive mapping on C has a fixed point.

In 1982, Edelstein and Kiang [16] introduced the concept of ultimately non-expansive mappings. They proved in particular that if S is a commutative semigroup of ultimately non-expansive mappings on a reflexive locally uniformly convex Banach space X such that $S(x)$ is precompact for some $x \in X^S$, then there is a common fixed point in X. Here X^S denotes the S-closure points of S.

In 1984, Edelstein [15] generalized this result to the case S is generated by a single map, replacing the precompactness condition by the existence of a S-closure point whose orbit is bounded. Finally, in 1985, Edelstein and Kiang [17] extended this result for general commutative semigroups. In the fifth chapter of this thesis, we extend this result to right-reversible semigroups, and show in corollary 5.4.2 that one of the condition of theorem 5.4.1 is always satisfied when the semigroup is SLA.

Chapter 2

Preliminaries

2.1 Introduction

In this section, we give some basic definitions and results related to the theory of analysis on semigroups, and we define amenability, extreme amenability and n-extreme amenability of a semigroup, and of a semi-topological semigroup. Since semi-topological semigroups is the basic structure that we will use to define strong amenability, those results are required for the rest of this thesis.

For a more complete treatment of the theory of semigroups and semigroup compactification, we refer the reader to the book of Berglund, Junghenn and Milness (see [4]).

2.2 Semigroup: Algebraic properties

Definition 2.2.1. A *semigroup* is a set S with a binary operation which is associative, *i.e.*, for any elements x, y and z in S, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

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Example 2.2.2. The sets $\mathbb{N}$ of natural numbers, $\mathbb{Z}$ of integers, $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ of real numbers, and $\mathbb{C}$ of complex numbers are all commutative semigroups with addition and multiplication.

Example 2.2.3. The set $M(n, \mathbb{C})$ of all $n \times n$ matrices over the complex numbers $\mathbb{C}$ under matrix multiplication is a noncommutative semigroup.

Example 2.2.4. If X is a set of cardinality greater than 1, then the set of all functions from X into X is a noncommutative semigroup under composition of functions.

An element $z \in S$ is a right zero, if $sz = z$ for any $s \in S$. A subset $I \subseteq S$ is a left ideal [resp. right ideal] if $SI \subseteq I$ [$IS \subseteq I$]. A left ideal [resp. right ideal] is minimal, if it does not contain any proper left ideal [resp. right ideal]. A semigroup does not need to contain any minimal left or right ideal, but if it does, every left ideal [resp. right ideal] contains a minimal left ideal [resp. minimal right ideal].

Proposition 2.2.5. *[4, Page 16, Proposition 2.4] Let S be a semigroup. Then a left ideal L is minimal if and only if $Ls = Ss = L$ for all $s \in L$. Also, if S has a minimal left ideal L, then $\{Ls : s \in S\}$ is the family of all minimal left ideals in S.*

Notice that the same result holds for right ideal if we change the order of the product. An element $e \in S$ is an idempotent if $e^2 = e$. An idempotent e is minimal, if it respects any one of the following equivalent properties:

Proposition 2.2.6. *[4, Page 17, Theorem 2.8] If $e \in S$ is an idempotent, then the following are equivalent:*

1. *Se is a minimal left ideal*

2. *eS is a minimal right ideal*

3. *eSe is a group*

Proposition 2.2.7. *[4, Page 19, Corollary 2.13] Let S be a semigroup with minimal idempotent. Then the minimal left ideals of S are groups if and only if S has a unique minimal right ideal.*

2.3 Semigroups with a topology

Now, let S be a semigroup with a Hausdorff topology. We say S is a right topological semigroup [resp. left topological semigroup] if for any net $s_\alpha \to s$ in S, and $t \in S$, we have that $s_\alpha t \to st$ [resp. $ts_\alpha \to ts$]. A semigroup with a Hausdorff topology is a semi-topological semigroup if it is both right and left topological (*i.e.* the product is separately continuous). Finally, a semigroup with a Hausdorff topology is a topological semigroup if for any net $s_\alpha \to s$ and $t_\beta \to t$ in S, we have that $s_\alpha t_\beta \to st$ (*i.e.* the product is jointly continuous). That is, if the product is continuous in the product topology.

Example 2.3.1. Any semigroup S with the discrete topology is a semi-topological semigroup.

Example 2.3.2. The set of real number $\mathbb{R}$ with multiplication and the usual topology is a semi-topological semigroup.

Proposition 2.3.3. *[4, page 31, theorem 3.11] If S is a compact, Hausdorff, right-topological semigroup, then S has a minimal idempotent. Moreover, all minimal left ideals of S are closed and pairwise homeomorphic.*

If S is a right-topological semigroup, then we define the topological center of S to be the set $\Lambda(S) = \{s \in S :$ the map $t \to st$ is continuous $\}$. It is known that every compact right topological group admits a left Haar measure, which is not necessarily unique unless some extra assumptions are satisfied [4, 51]. Notice that such a Haar measure is left invariant by members of the topological center, but not necessarily by all elements of the group. For some recent developments on compact right topological semigroups, see [40].

2.4 Function spaces

Let S be a semi-topological semigroup, and let $l^\infty(S)$ be the C*-algebra of all bounded complex-valued functions on S with the sup-norm topology, that is,

$$f \in l^\infty(S) \iff ||f|| = \sup_{s\in S} |f(s)| < \infty.$$

For any $f \in l^\infty(S)$ we define the left translation operator l_s by $l_s f(t) = f(st)$, and the right translation operator r_s by $r_s f(t) = f(ts)$, for any $s, t \in S$. We also define the following C*-subalgebras of $l^\infty(S)$:

1. $CB(S)$ is the C*-subalgebra of $l^\infty(S)$ consisting of all bounded norm continuous complex-valued functions.

2. $LUC(S)$ is the C*-subalgebra of $CB(S)$ consisting of all functions $f \in CB(S)$ for which the map $s \to l_s f$ is continuous. Similarly we can define $RUC(S)$ by replacing l_s by r_s.

3. $AP(S)$ is the C*-subalgebra of $CB(S)$ consisting of all functions $f \in CB(S)$ for which the set $\mathcal{L}O(f) = \{l_s f : s \in S\}$ is norm relatively compact in $CB(S)$.

4. $WAP(S)$ is the C*-subalgebra of $CB(S)$ consisting of all functions $f \in CB(S)$ for which the set $\mathcal{L}O(f)$ is weakly relatively compact in $CB(S)$.

For any semi-topological semigroup S, we have $AP(S) \subseteq WAP(S)$ and $AP(S) \subseteq LUC(S) \cap RUC(S)$. Also, we have $AP(S) = AP(S_d) \cap CB(S)$ and $WAP(S) = WAP(S_d) \cap CB(S)$, where S_d is the semigroup S with the discrete topology. If S is a compact topological semigroup, then $AP(S) = CB(S)$, and if S is a compact semi-topological semigroup, then $WAP(S) = CB(S)$ and $AP(S) = LUC(S) = RUC(S)$.

2.5 Amenability

Let A be a left translation invariant C*-subalgebra of $l^\infty(S)$ containing the constants. A mean on A is a linear functional $\mu : A \to \mathbb{C}$ such that $||\mu|| = 1$

and $\mu(1_S) = 1$, where 1_S is the constant one function on S. A mean μ is left invariant if $l_s^*\mu = \mu$ for all $s \in S$, where $l_s^*\mu(f) = \mu(l_s f)$. A mean μ is multiplicative if $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in A$. A is said to be [extremely] left amenable if there exists a [multiplicative] left invariant mean on A. S is said to be [extremely] left amenable, if $LUC(S)$ is [extremely] left amenable. Note that for a compact semi-topological semigroup S, $CB(S)$ is left amenable if and only if S has a unique minimal right ideal.

Example 2.5.1. If G is a compact group, then G is amenable [12].

Example 2.5.2. If G is an abelian group with the discrete topology, then G is amenable [12].

Now, let S^A denote the set of all multiplicative means on A with the weak*-topology, and $\Delta(S) = S^{LUC(S)}$. We define the left introversion operator determined by $\mu \in CB(S)^*$ by $T_\mu : A \to l^\infty(S)$, $(T_\mu f)(s) = \mu(l_s f)$ for all $f \in A, s \in S$, and we say A is m-left introverted if $T_\mu A \subseteq A$ for all $\mu \in S^A$. If A is m-left introverted, we define the Arens product $\odot$ on S^A by:

$$\mu \odot \nu(f) = \mu(\nu_l(f)), \quad \forall f \in A$$

$$\nu_l(f) = \nu(l_s f), \quad s \in S.$$

For any semi-topological semigroup, $AP(S)$, $WAP(S)$ and $LUC(S)$ are all translation invariant, left introverted C*-subalgebra of $CB(S)$ containing the constant functions. Also, with this product, $\Delta(S)$ is a compact right topological semigroup, and $S^{AP(S)}$ is a compact topological semigroup. For every $s \in S$, we define the point measure $\delta_s : A \to \mathbb{C}$ by $\delta_s(f) = f(s)$ for all $f \in A$. The point measures are dense in S^A, and define an embeding of S in S^A.

As is well known, a semigroup S is extremely left amenable (ELA) if and only if there exists a right zero in $\Delta(S)$ (See [52, 53]). Also, when S is a discrete semigroup, then S is ELA if and only if every two elements of S have a common right zero (*i.e.* for all $s, t \in S$, there exists $z \in S$ such that $sz = tz = z$, see [22]).

For any $n \in \mathbb{N}$, we say A is n-ELA if there exists a set $F \subseteq \Delta(S)$ such that $|F| = n$ and which is minimal with respect to the property $l_s^* F = F$ for all $s \in S$. A semigroup S is n-ELA if and only if there exists a left ideal group of order n in $\Delta(S)$ (See [37, 36]). When S is a discrete semigroup with finite intersection property for right ideals, we define the equivalence relation (r) by: $a(r)b \Leftrightarrow ac = bc$ for some $c \in S$. We denote by $S/(r)$ the semigroup S with this equivalence relation, and we have the following characterization due to Lau:

Theorem 2.5.3. *(See [37])*

1. *If S is n-ELA and F_0 is a coset representative of $S/(r)$, then for each finite subset σ of S, there exists $t_\sigma \in S$, depending on σ, such that $aF_0 t_\sigma = F_0 t_\sigma$ for all $a \in \sigma$.*

2. *If for any finite subset σ of S, there exists $F_\sigma \subseteq S$, $|F_\sigma| = n$, such that $aF_\sigma = F_\sigma$ for all $a \in S$, then S is m-ELA for some $m \leq n$, m dividing n.*

Chapter 3

Strong left amenability

3.1 Introduction

In this chapter, we want to study the general structure of strongly left amenable semigroups. Strong amenability is a property that sits somewhere between amenability and extreme amenability. In section 3.2 we give the definition of such a semigroup and explore some of its elementary properties, most of which are necessary later on in this chapter.

In section 3.3, we give a characterization of strong amenability of $LUC(S)$, $WAP(S)$ and $AP(S)$. This section also explains why most of our study has centered around the $LUC(S)$ case, since we prove that the two other cases are nothing else than amenability.

In section 3.4, we prove that every homomorphic image of a strongly amenable semigroup is also strongly amenable, and the product of an extremely amenable semigroup by a compact group is strongly amenable. This will allow us in section 3.6 to give some examples of non-trivial strongly amenable groups. But before we give examples, we study in section 3.5 the strong amenability of some special classes of semigroups, which includes the

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compact and discrete cases.

3.2 Strong left amenability

Definition 3.2.1. A semi-topological semigroup S is strongly left amenable (SLA) if there exists a left ideal group in $\Delta(S)$.

Notice that all n-ELA semigroups are SLA. Also, all compact groups are SLA, this is because if G is a compact group, then $G = \Delta(G)$.

Theorem 3.2.2. *If S is a semi-topological semigroup with a left ideal group K in $\Delta(S)$, then the following holds:*

1. *K is a minimal left ideal*
2. *K is compact*
3. *K is a right topological group*
4. *$l_s^* K = K$ for all $s \in S$.*
5. *For any $\mu, \nu \in K$ and $s \in S$, $l_s^* \mu = l_s^* \nu \Rightarrow \mu = \nu$*
6. *$K \odot \mu$ is a compact left ideal group for all $\mu \in \Delta(S)$.*

Proof.

1. Let $M \subseteq K$ be another left ideal in $\Delta(S)$ and let $m \in M$, then: $K = K \odot m \subseteq M$. Therefore $K = M$, and K is a minimal left ideal.
2. It suffices to show that K is closed since $\Delta(S)$ is compact. Take a $k \in K$. Then $\Delta(S) \odot k$ is a left ideal of $\Delta(S)$ that sits in K. It is closed since $\Delta(S)$ is compact and the multiplication is right continuous. But K is a minimal left ideal from (1). Therefore $K = \Delta(S) \odot k$ and thus is closed.
3. By assumption, K is a group. Since $\Delta(S)$ is right topological then so is K.

4. Let $s \in S$, $\mu \in K$ and $\delta_s \odot \mu = \nu$. Then since K is a left ideal, we have that $\delta_s \odot K \subseteq K$, also, since K is a group, we have $\nu \odot K = \mu \odot K = K$. Therefore, we have that:

$$l_s^* K = \delta_s \odot K = \delta_s \odot (\mu \odot K) = (\delta_s \odot \mu) \odot K = \nu \odot K = K.$$

5. Let ψ be the identity in K, then we have:

$$\begin{array}{rcl} l_s^*\mu & = & l_s^*\nu \\ & \Rightarrow & \delta_s \odot \mu \\ & = & \delta_s \odot \nu \\ & \Rightarrow & \delta_s \odot (\psi \odot \mu) \\ & = & \delta_s \odot (\psi \odot \nu) \\ & \Rightarrow & (\delta_s \odot \psi) \odot \mu \\ & = & (\delta_s \odot \psi) \odot \nu \Rightarrow \mu \\ & = & \nu, \end{array}$$

where the last implication follows from the fact that K is a left ideal, and therefore $\delta_s \odot \psi$ is in the group K.

6. Consider the map $f_a : K \to \Delta(S)$ defined by $f_a(\mu) = \mu \odot a$. This map is continuous by continuity of the Arens product in the first variable, and therefore $K \odot a$ is a continuous image of a compact set, and is therefore compact. Since K is a minimal left ideal, then by proposition 2.2.5, $K \odot a$ is also a minimal left ideal. $K \odot a$ is associative since $\Delta(S)$ is a semigroup. We only need to prove the existence of an identity and the existence of an inverse for any element in $K \odot a$ to complete the proof. The identity is given by $e_a = x \odot (a \odot x)^{-1} \odot a$ for any $x \in K$, where the inverse is taken in the group K. This e_a is independent of the choice of

x since if $y \in K$, we have

$$\begin{aligned}
y \odot (a \odot y)^{-1} \odot a &= (x \odot x^{-1}) \odot y \odot (a \odot y)^{-1} \odot a \\
&= x \odot (y^{-1} \odot x)^{-1} \odot (a \odot y)^{-1} \odot a \\
&= x \odot (a \odot y \odot y^{-1} \odot x)^{-1} \odot a \\
&= x \odot (a \odot x)^{-1} \odot a \\
&= e_a.
\end{aligned}$$

To see that e_a is an identity for $K \odot a$. Let $(y \odot a) \in K \odot a$, then we have:

$$\begin{aligned}
(y \odot a) \odot e_a &= (y \odot a) \odot x \odot (a \odot x)^{-1} \odot a \\
&= y \odot (a \odot x) \odot (a \odot x)^{-1} \odot a \\
&= y \odot a \\
e_a \odot (y \odot a) &= x \odot (a \odot x)^{-1} \odot a \odot (y \odot a) \\
&= x \odot (a \odot x)^{-1} \odot (a \odot x) \odot (x^{-1} \odot y) \odot a \\
&= (x \odot x^{-1}) \odot (y \odot a) \\
&= y \odot a
\end{aligned}$$

Also, for any $y \odot a \in K \odot a$, its inverse in $K \odot a$ is given by

$$y \odot (a \odot y)^{-1} \odot (a \odot y)^{-1} \odot a.$$

This is true since if $y \odot a \in K \odot a$, then:

$$\begin{aligned}
&(y \odot a) \odot [y \odot (a \odot y)^{-1} \odot (a \odot y)^{-1} \odot a] \\
&= y \odot (a \odot y) \odot (a \odot y)^{-1} \odot (a \odot y)^{-1} \odot a \\
&= y \odot (a \odot y)^{-1} \odot a \\
&= e_a
\end{aligned}$$

Similarly. we have: $[y \odot (a \odot y)^{-1} \odot (a \odot y)^{-1} \odot a] \odot (y \odot a) = e_a$, and therefore, it follows that $\Delta(S) \odot a = K \odot a$ is a group.

$\square$

Theorem 3.2.3. *Let S be a strongly left amenable semigroup and let K_1 and K_2 be two left ideal groups in $\Delta(S)$. Then K_1 and K_2 are isomorphic (as right-topological group).*

Proof. We define the map $f : K_1 \to K_2$ by $f(x) = x \odot e_2$, and the map $g : K_2 \to K_1$ by $g(y) = y \odot e_1$, where e_1 is the identity in K_1, and e_2 is the identity in K_2. This is well defined since K_1 and K_2 are left ideal. We first show that those maps are onto. To see that, first notice that $f(K_1)$ is a left ideal contained in K_2. Let y be any element in $\Delta(S)$, and x be any element of K_1. Then we have $y \odot f(x) = y \odot (x \odot e_2) = (y \odot x) \odot e_2 \in K_1 \odot e_2 = f(K_1)$, and therefore $f(K_1)$ is a left ideal in K_2, so by minimality of K_2 this implies that $f(K_1) = K_2$. It follows that f is onto. We now want to show that those maps are homomorphisms:

$$\begin{aligned} f(x \odot y) &= (x \odot y) \odot e_2 = x \odot (y \odot e_2) = x \odot [e_2 \odot (y \odot e_2)] \\ &= (x \odot e_2) \odot (y \odot e_2) = f(x) \odot f(y). \end{aligned}$$

Therefore, f is a (semigroup) homomorphism. The same proof, replacing e_2 by e_1, shows that g is also a homomorphism. To see that they are actually group homomorphisms, we need to show: $e_i \odot e_j = e_j$. To prove that, let $y \in K_2$. Since f is onto, there exists $x \in K_1$ such that $f(x) = y$. It follows that:

$$y \odot (e_1 \odot e_2) = f(x) \odot (e_1 \odot e_2) = f(x) \odot f(e_1) = f(x \odot e_1) = f(x) = y$$

$$(e_1 \odot e_2) \odot y = (e_1 \odot e_2) \odot f(x) = f(e_1) \odot f(x) = f(e_1 \odot x) = f(x) = y.$$

Therefore, $e_1 \odot e_2$ is the identity of K_2, and therefore $e_1 \odot e_2 = e_2$. The same proof can be used to show $e_i \odot e_j = e_j$, where $i, j \in \{1, 2\}$. And to finish the proof that f is a (group) homomorphism, we have: $[f(x^{-1})] \odot [f(x)] = f(x^{-1} \odot x) = f(e_1) = e_2$. Since this is true for any $x \in K_1$, it follows: $[f(x)]^{-1} = f(x^{-1})$. Therefore, it is a group homomorphism. To finish the proof of this theorem, we only have left to prove that f is continuous. Let x_α

be a net in K_1 which converges to $x \in K_1$. Then it follows: $f(x_\alpha) = x_\alpha \odot e_2 \to x \odot e_2 = f(x)$. To finish the proof we need to show $f \circ g = id$ and $g \circ f = id$.

$$(f \circ g)(y) = f(y \odot e_1) = y \odot e_1 \odot e_2 = y \odot e_2 = y, \quad \forall y \in K_2,$$

$$(g \circ f)(x) = g(x \odot e_2) = x \odot e_2 \odot e_1 = x \odot e_1 = x, \quad \forall x \in K_1,$$

and therefore, K_1 and K_2 are isomorphic as right topological group. $\square$

Theorem 3.2.4. *Every strongly left amenable semigroups is left amenable.*

Proof. This proof is inspired by a similar proof in [2]. Since K is a compact right topological group, it admits a left Haar measure ν [4]. Then we can define the following function:

$$\lambda(f) = \int_K \langle f, \mu \rangle d\nu(\mu).$$

We want to show this is a left invariant mean on $LUC(S)$. First, if 1 is the constant one function on S, then we have

$$\lambda(1) = \int_K \langle 1, \mu \rangle d\nu(\mu) = \int_K 1 d\nu(\mu) = 1.$$

Also, we have

$$||\lambda|| = \sup_{||f|| \leq 1} |\lambda(f)| \leq \sup_{||f|| \leq 1} \sup_{\mu \in K} |\mu(f)| \leq \sup_{||f|| \leq 1} \sup_{\mu \in K} ||\mu|| \quad ||f|| = \sup_{||f|| \leq 1} ||f|| \leq 1,$$

and therefore, λ is a mean on $LUC(S)$. Now, we want to show that this mean is left invariant. Let $s \in S$, and ψ be the identity of K. Then we have:

$$\begin{aligned} l_s^* \lambda(f) &= \lambda(l_s f) = \int_K \langle l_s f, \mu \rangle d\nu(\mu) = \int_K \langle f, l_s^* \mu \rangle d\nu(\mu) \\ &= \int_K \langle f, l_s^*(\psi \odot \mu) \rangle d\nu(\mu) = \int_K \langle f, (l_s^* \psi) \odot \mu \rangle d\nu(\mu) \\ &= \int_K \langle f, \mu \rangle d\nu(\mu) = \lambda(f). \end{aligned}$$

Notice that the 4th and 6th equalities follow from the fact that the Haar measure is left invariant by elements of the topological center $\Lambda(K)$, and both ψ and $l_s^* \psi$ are in $\Lambda(K)$. It follows that λ is left invariant, and therefore if S is strongly left amenable, then S is left amenable. $\square$

3.3 Characterization

We are now ready for our characterization theorem for SLA semigroup. Recall first that since $\Delta(S)$ is a compact right-topological semigroup, it follows from proposition 2.3.3 that $\Delta(S)$ has a minimal idempotent. Therefore by proposition 2.2.6, there exists a minimal left ideal and a minimal right ideal in $\Delta(S)$.

Theorem 3.3.1. *Let S be a semi-topological semigroup. Then the following are equivalent:*

1. *There exists a left ideal group in $\Delta(S)$.*
2. *There exists a compact left ideal group in $\Delta(S)$.*
3. *All minimal left ideals of $\Delta(S)$ are compact groups.*
4. *There exists $a \in \Delta(S)$ for which $\Delta(S) \odot a$ is a group.*
5. *There exists a compact group $K \subseteq \Delta(S)$ such that $l_s^* K = K$ for all $s \in S$.*
6. *There exists a unique minimal right ideal in $\Delta(S)$.*

Proof.
(1) $\Rightarrow$ **(2)**: This follows from theorem 3.2.2.
(2) $\Rightarrow$ **(1)**: Trivial.
(3) $\Leftrightarrow$ **(6)**: Since $\Delta(S)$ is a compact right topological semigroup, it follows by Proposition 2.3.3 that $\Delta(S)$ has a minimal idempotent, so we can apply Proposition 2.2.7, which gives us that all minimal left ideals of $\Delta(S)$ are groups if and only if there is a unique minimal right ideal in $\Delta(S)$.
(3) $\Rightarrow$ **(2)**: By proposition 2.3.3, we know that $\Delta(S)$ has (at least one) a minimal left ideal. Since all minimal left ideal of $\Delta(S)$ are groups, it follows there exists a compact left ideal group in $\Delta(S)$.
(2) $\Rightarrow$ **(3)**: Let K be a compact left ideal group in $\Delta(S)$. Then $\{K \odot \mu : \mu \in \Delta(S)\}$ is the collection of all minimal left ideals in $\Delta(S)$ by Proposition 2.2.5.

Also, we proved in proposition 3.2.2 that $K \odot \mu$ is a group for all $\mu \in \Delta(S)$, which completes the proof.

(4) $\Rightarrow$ (1): $\Delta(S) \odot a$ is a left ideal since if $\mu \in \Delta(S)$ we have $\mu \odot (\Delta(S) \odot a) = (\mu \odot \Delta(S)) \odot a \subseteq \Delta(S) \odot a$. Therefore $\Delta(S) \odot a$ is a left ideal group in $\Delta(S)$.

(2) $\Rightarrow$ (4): Let K be a compact left ideal group in $\Delta(S)$, Then $K \odot \mu = \Delta(S) \odot \mu = K$ for all $\mu \in K$. Therefore $\Delta(S) \odot \mu$ is a group.

(1) $\Rightarrow$ (5): If K is a left ideal group in $\Delta(S)$, then $\mu \odot K \subseteq K$ for any $\mu \in \Delta(S)$, but since K is also a group, then for any $x \in K$ we have that $K = K \odot x = x \odot K$, which implies for any $\mu \in \Delta(S)$,

$$\mu \odot K = \mu \odot (x \odot K) = (\mu \odot x) \odot K = \nu \odot K = K,$$

for some $\nu \in K$. Therefore, for any $s \in S$, we have that $l_s^* K = \delta_s \odot K = K$.

(5) $\Rightarrow$ (1): By continuity of the Arens product in the first variable, $l_s^* K = \delta_s \odot K$ is a left ideal. By assumption this is also a group. $\square$

If S is a semi-topological semigroup, and A is a left translation invariant C^*-subalgebra of $CB(S)$ containing the constants, then we say that A is m-admissible if the function $s \to (T_\mu f)(s) = \mu(l_s f)$ is in A for all $f \in A$ and $\mu \in S^A$ [4]. In this case, S^A is a semigroup with the Arens product. A closed invariant (left and right) subalgebra of $WAP(S)$ containing the constants is always m-admissible [10]. If A is an m-admissible subalgebra of $LUC(S)$, then we say A is strongly left amenable, if S^A is strongly left amenable.

Theorem 3.3.2. *Let A be an m-admissible subalgebra of $WAP(S)$. If A is left amenable, then A is strongly left amenable.*

Proof. We define the following maps:

$$\epsilon : S \to S^A, \quad \epsilon(s) = \delta_s, \quad \forall s \in S$$

$$\epsilon^* : CB(S^A) \to A, \quad \epsilon^*(\tilde{f})(s) = \tilde{f}(\epsilon(s)), \quad \forall \tilde{f} \in CB(S^A), \forall s \in S$$

$$\epsilon^{**} : A^* \to CB(S^A)^*, \quad \epsilon^{**}(\mu)(\tilde{f}) = \mu(\epsilon^*(\tilde{f})), \quad \forall \mu \in A^*, \forall \tilde{f} \in CB(S^A)$$

Notice that if λ is a mean on A, then $\mu = \epsilon^{**}(\lambda)$ is a mean on $CB(S^A)$. We want to prove that if λ is also left invariant, then so is μ. Let $s, t \in S$ and

$\tilde{f} \in CB(S^A)$. We define $f \in CB(S)$ to be the restriction of $\tilde{f}$ to S. Then we have $\epsilon^*(l_{\epsilon(s)}\tilde{f})(t) = l_{\epsilon(s)}(\tilde{f})(\epsilon(t)) = \tilde{f}(\epsilon(st)) = f(st) = l_s f(t)$, and also we have that $\epsilon^*(\tilde{f})(t) = \tilde{f}(\epsilon(t)) = f(t)$. Therefore we have

$$\begin{aligned} l_{\epsilon(s)}\mu(\tilde{f}) &= \mu(l_{\epsilon(s)}\tilde{f}) = \epsilon^{**}(\lambda)(l_{\epsilon(s)}\tilde{f}) = \lambda(\epsilon^*(l_{\epsilon(s)}\tilde{f})) = \lambda(l_s f) \\ &= \lambda(f) = \lambda(\epsilon^*(\tilde{f})) = \epsilon^{**}(\lambda)(\tilde{f}) = \mu(\tilde{f}). \end{aligned}$$

Now, using the continuity of the map $s \to l_s^*\mu$, we get that $l_x^*\mu = \mu$ for all $x \in S^A$, which shows that μ is a left invariant mean on $CB(S^A)$.

Now, we want to show that that S^A has a unique minimal right ideal. Suppose that R_1 and R_2 are two distincts minimal right ideals in S^A. Since A is a subalgebra of $WAP(S)$, S^A is a semi-topological semigroup, and therefore minimal right ideals of S^A are closed. Therefore, we can apply Urysohn's lemma [65] to construct a function g which is 0 on R_1 and 1 on R_2. If we take $x \in R_1$ and $y \in R_2$, then we get that: $\mu(l_x g) = 0$ and $\mu(l_y g) = 1$, which contradict the left invariance of μ. Therefore, there is unique minimal right ideal. Now, pick any minimal left ideal L in S^A, then RL is a left ideal contained in L, therefore by minimality of L, we have that $RL = L$. But we also know that the product of a right ideal by a left ideal is a group. It follows that L is a left ideal group in S^A, and therefore S^A is strongly left amenable. □

Notice that the only place in the proof where we use the fact that A is an m-admissible subalgebra of $WAP(S)$ is to prove that minimal right ideals are closed. Also, the previous theorem justifies our definition of strong left amenability of a semi-topological semigroup S using $LUC(S)$, and the choice of focusing almost all our efforts on this algebra.

Corollary 3.3.3. *Let S be a semi-topological semigroup. If $LUC(S)$ is left amenable and if the minimal right ideals of S^{LUC} are closed, then $LUC(S)$ is strongly left amenable.*

Proof. If the minimal right ideals of S^{LUC} are closed, then we can apply the same proof as in the theorem. □

Corollary 3.3.4. *Let S be a left amenable locally compact, non compact group. Then the minimal right ideals of S^{LUC} are not closed.*

Proof. If the minimal right ideals of S^{LUC} are closed, then S is strongly left amenable. But, we know that the only locally compact strongly left amenable groups are the compact groups. Therefore it is impossible. $\square$

Theorem 3.3.5. *If S is a semi-topological semigroup, then the following are equivalent:*

1. *$AP(S)$ is left amenable.*

2. *$AP(S)$ is strongly left amenable.*

3. *There exists a compact set $K \subseteq S^{AP(S)}$ which is minimal with respect to the property $l_s^* K = K$ for all $s \in S$.*

Proof.
(1) $\Rightarrow$ (2): $AP(S)$ is an m-admissible subalgebra of $WAP(S)$, therefore we can apply theorem 3.3.2.
(2) $\Rightarrow$ (1): This is the same proof as in the $LUC(S)$ case.
(2) $\Rightarrow$ (3): See theorem 3.2.2.
(3) $\Rightarrow$ (2): Since $l_s^* K = K$ for all $s \in S$, it follows that $\mu \odot K \subseteq K$ for all $\mu \in S^{AP(S)}$, and therefore K is a left ideal. Let $\mu \in K$, then $K \odot \mu$ is also a left ideal in $S^{AP(S)}$, and $K \odot \mu \subseteq K$. Therefore it follows from the minimality of K that $K = K \odot \mu$. Now, we want to show $\mu \odot K = K$. Let $\psi \in K$, and s_α be a net in S such that $\delta_{s_\alpha} \to \mu$, then since $l_{s_\alpha}^* K = K$ for all α, there is a net φ_α in K such that $\delta_{s_\alpha} \odot \varphi_\alpha = \psi$ for all α. Now since K is compact, there exists a subnet φ_β of φ_α such that φ_β converges to some $\varphi \in K$. It follows that $\delta_{s_\beta} \odot \varphi_\beta = \psi$ for all β, and $\delta_{s_\beta} \odot \varphi_\beta \to \mu \odot \varphi$. Therefore $\mu \odot \varphi = \psi$, and it follows that $\mu \odot K = K$. Now since K is both left and right simple, it must be a group, and $AP(S)$ is strongly left amenable. $\square$

3.4 Construction

In the next theorem, we want to establish that every homomorphic image of a SLA semigroup is also SLA. Let S, T be semi-topological semigroups, and let $f : S \to T$ be a continuous surjective homomorphism. We define $\tilde{f} : LUC(T) \to LUC(S)$ by $[\tilde{f}(g)](s) = g[f(s)]$ for all $s \in S$ and we define $\tilde{\tilde{f}} : \Delta(S) \to \Delta(T)$ by $[\tilde{\tilde{f}}(\mu)](g) = \mu[\tilde{f}(g)]$ for all $g \in LUC(T)$. We first want to show that those maps are properly defined. Let $g \in LUC(T)$, we want to show $\tilde{f}(g) \in LUC(S)$. Let s_α be a net in S which converges to $s \in S$. Then we have $f(s_\alpha) \to f(s)$ since f is continuous, therefore since g is also continuous, $g(f(s_\alpha)) \to g(f(s))$. It follows that:

$$[\tilde{f}(g)](s_\alpha) = g(f(s_\alpha)) \to g(f(s)) = [\tilde{f}(g)](s).$$

Therefore $\tilde{f}(g)$ is continuous. To see it is actually left uniformly continuous, let $t \in S$. Then we have

$$\begin{aligned}
&\sup_{t\in S} |l_{s_\alpha}(\tilde{f}(g))(t) - l_s(\tilde{f}(g))(t)| = \sup_{t\in S} |\tilde{f}(g)(s_\alpha t) - \tilde{f}(g)(st)| \\
= \ &\sup_{t\in S} |g(f(s_\alpha t)) - g(f(st))| = \sup_{t\in S} |g(f(s_\alpha)f(t)) - g(f(s)f(t))| \\
= \ &\sup_{t\in S} |l_{f(s_\alpha)}g(f(t)) - l_{f(s)}g(f(t))| \to 0,
\end{aligned}$$

since $g \in LUC(T)$ and $f(s_\alpha) \to f(s)$. Therefore $\tilde{f}(g) \in LUC(S)$.

Now, we want to show that $\tilde{\tilde{f}}(\mu) \in \Delta(T)$. First, notice that if $e : T \to \mathbb{C}$, defined by $e(t) = 1$ for all $t \in T$, then $[\tilde{f}(e)](s) = e(f(s)) = 1$. Therefore, $[\tilde{\tilde{f}}(\mu)](e) = \mu(\tilde{f}(e)) = 1$ for all $\mu \in \Delta(S)$. Now, let $g \in LUC(T)$ be such that $||g|| \leq 1$. Then

$$||\tilde{f}(g)|| = \sup_{s\in S} |[\tilde{f}(g)](s)| = \sup_{s\in S} |g(f(s))| \leq \sup_{t\in T} |g(t)| \leq 1.$$

It follows that

$$||\tilde{\tilde{f}}(\mu)|| = \sup_{||g||\leq 1} |\tilde{\tilde{f}}(\mu)(g)| = \sup_{||g||\leq 1} |\mu(\tilde{f}(g))| \leq 1,$$

and therefore $\tilde{\tilde{f}}(\mu) \in \Delta(T)$ for all $\mu \in \Delta(S)$.

Finally, we define the map $\pi : S \to \Delta(S)$ by $\pi(s) = \delta_s$ and we define the map $\rho : T \to \Delta(T)$ by $\rho(t) = \delta_t$. Now, let K be a compact left ideal group in $\Delta(S)$, and let $K' = \tilde{\tilde{f}}(K)$. We will show that K' is a compact left ideal group in $\Delta(T)$, but first we need a lemma.

Lemma 3.4.1. *The following is true for the maps we just defined:*

1. $l_s\tilde{f}(g) = \tilde{f}(l_{f(s)}g)$ *for all* $s \in S$ *and* $g \in LUC(T)$.
2. $l^*_{f(s)}\tilde{\tilde{f}}(\mu) = \tilde{\tilde{f}}(l^*_s\mu)$ *for all* $s \in S$ *and* $\mu \in \Delta(S)$.
3. $\tilde{\tilde{f}}(\delta_s) = \delta_{f(s)}$ *for all* $s \in S$.
4. $\rho \circ f = \tilde{\tilde{f}} \circ \pi$.
5. $\nu_l(\tilde{f}(g)) = \tilde{f}(\nu'_l(g))$ *for all* $g \in LUC(T)$ *and* $\nu \in \Delta(S)$.
6. $\tilde{f}$ *and* $\tilde{\tilde{f}}$ *are homomorphism.*
7. $\tilde{f}$ *and* $\tilde{\tilde{f}}$ *are continuous.*
8. $l^*_t K' = K'$ *for all* $t \in T$.
9. K' *is a group.*
10. K' *is compact.*
11. $\tilde{\tilde{f}}$ *is onto.*

Proof.

1. For all $x \in S$ we have that:

$$\begin{aligned} [l_s\tilde{f}(g)](x) &= \tilde{f}(g)(sx) = g(f(sx)) = g(f(s)f(x)) \\ &= (l_{f(s)}g)(f(x)) = [\tilde{f}(l_{f(s)}g)](x) \end{aligned}$$

and therefore $l_s\tilde{f}(g) = \tilde{f}(l_{f(s)}g)$.

2. For all $g \in LUC(T)$ we have that:

$$\begin{aligned}[l^*_{f(s)}\tilde{\tilde{f}}(\mu)](g) &= \tilde{\tilde{f}}(\mu)(l_{f(s)}g) = \mu(\tilde{f}(l_{f(s)}g)) = \mu(l_s\tilde{f}(g)) \\ &= (l^*_s\mu)(\tilde{f}(g)) = [\tilde{\tilde{f}}(l^*_s\mu)](g)\end{aligned}$$

   And therefore $l^*_{f(s)}\tilde{\tilde{f}}(\mu) = \tilde{\tilde{f}}(l^*_s\mu)$.

3. Let $g \in LUC(T)$, then we have:

$$\tilde{\tilde{f}}(\delta_s)(g) = \delta_s(\tilde{f}(g)) = (\tilde{f}(g))(s) = g[f(s)] = \delta_{f(s)}(g)$$

   And therefore, it follows that $\tilde{\tilde{f}}(\delta_s) = \delta_{f(s)}$.

4. Let $s \in S$, then we have that:

$$(\rho \circ f)(s) = \rho(f(s)) = \delta_{f(s)} = \tilde{\tilde{f}}(\delta_s) = (\tilde{\tilde{f}} \circ \pi)(s)$$

   and therefore $\rho \circ f = \tilde{\tilde{f}} \circ \pi$.

5. Let $s \in S$, then we have:

$$\begin{aligned}\nu_l(\tilde{f}(g))(s) &= (l^*_s\nu)(\tilde{f}(g)) = \nu(l_s\tilde{f}(g)) = \nu(\tilde{f}(l_{f(s)}g)) \\ &= [\tilde{\tilde{f}}(\nu)](l_{f(s)}g) = \nu'(l_{f(s)}g) = (l^*_{f(s)}\nu')(g) \\ &= (\nu'_l(g))(f(s)) = \tilde{f}(\nu'_l(g))(s)\end{aligned}$$

   And therefore: $\nu_l(\tilde{f}(g)) = \tilde{f}(\nu'_l(g))$.

6. Let $g, h \in LUC(T)$ and $s \in S$. Then we have:

$$\begin{aligned}\tilde{f}(gh)(s) &= (gh)(f(s)) = g(f(s))h(f(s)) \\ &= \tilde{f}(g)(s)\ \tilde{f}(h)(s) = [\tilde{f}(g)\tilde{f}(h)](s)\end{aligned}$$

   And therefore $\tilde{f}$ is a (semigroup) homomorphism. Now, to show that $\tilde{\tilde{f}}$ is also a (semigroup) homomorphism, let $\mu, \nu \in \Delta(S)$ and $g \in LUC(T)$. Let $\tilde{\tilde{f}}(\mu) = \mu'$ and $\tilde{\tilde{f}}(\nu) = \nu'$. Then we have:

$$\begin{aligned}\tilde{\tilde{f}}(\mu \odot \nu)(g) &= (\mu \odot \nu)(\tilde{f}(g)) = \mu(\nu_l(\tilde{f}(g))) \\ &= \mu(\tilde{f}(\nu'_l(g))) = \tilde{\tilde{f}}(\mu)[\nu'_l(g)] \\ &= \mu'(\nu'_l(g)) = (\mu' \odot \nu')(g) \\ &= [\tilde{\tilde{f}}(\mu) \odot \tilde{\tilde{f}}(\nu)](g)\end{aligned}$$

And therefore, $\tilde{\tilde{f}}$ is also a (semigroup) homomorphism.

7. Let g_α be a net in $LUC(T)$ which converges to g. That is

$$\sup_{t \in T} |g_\alpha(t) - g(t)| \to 0.$$

We want to prove that $\tilde{f}(g_\alpha) \to \tilde{f}(g)$.

$$\begin{aligned} \sup_{s \in S} |\tilde{f}(g_\alpha)(s) - \tilde{f}(g)(s)| &= \sup_{s \in S} |g_\alpha(f(s)) - g(f(s))| \\ &= \sup_{t \in T} |g_\alpha(t) - g(t)| \\ &\to 0 \end{aligned}$$

And therefore $\tilde{f}$ is continuous. Now, to show that $\tilde{\tilde{f}}$ is continuous, let $\mu_\alpha \to \mu$, that is $\mu_\alpha(h) \to \mu(h)$ for all $h \in LUC(S)$. It follows that:

$$\tilde{\tilde{f}}(\mu_\alpha)(h) = \mu_\alpha(\tilde{f}(h)) \to \mu(\tilde{f}(h)) = \tilde{\tilde{f}}(\mu)(h)$$

And therefore $\tilde{\tilde{f}}$ is continuous.

8. We have that: $l_t^* K' = l_{f(s)}^* \tilde{\tilde{f}}(K) = \tilde{\tilde{f}}(l_s^* K) = \tilde{\tilde{f}}(K) = K'$.

9. This is a direct consequence of the fact that $\tilde{\tilde{f}}$ is a homomorphism onto K', and K is a group.

10. Since $\tilde{\tilde{f}}$ is continuous and K is compact, then $K' = \tilde{\tilde{f}}(K)$ is compact.

11. Let ν be any element in $\Delta(T)$. Then there exists a net of point measure δ_{t_α} in $\Delta(T)$ which converges to ν. Now, since f is onto, there exists a net s_α in S such that $f(s_\alpha) = t_\alpha$, and therefore $\tilde{\tilde{f}}(\delta_{s_\alpha}) = \delta_{t_\alpha}$. Now, since $\Delta(S)$ is compact, there exists a subnet δ_{s_β} of δ_{s_α} which converges to some μ in $\Delta(S)$. Now to prove that $\tilde{\tilde{f}}(\mu) = \nu$, we use the continuity of $\tilde{\tilde{f}}$. Since $\delta_{s_\beta} \to \mu$ we have that $\delta_{t_\beta} = \tilde{\tilde{f}}(s_\beta) \to \tilde{\tilde{f}}(\mu)$. But since δ_{t_β} is a subnet of δ_{t_α}, it also converges to ν. Therefore, by unicity of the limit in a Hausdorff space, it follows that $\tilde{\tilde{f}}(\mu) = \nu$, and $\tilde{\tilde{f}}$ is therefore onto.

□

Theorem 3.4.2. *Every homomorphic image of a strongly left amenable semigroup is strongly left amenable.*

Proof. Let S be a SLA semigroup, and $f : S \to T$ be a homomorphism onto a semi-topological semigroup T. If S is SLA, with K a compact left ideal group in $\Delta(S)$, then $\tilde{\tilde{f}}(K) = K'$ is a compact left ideal group in $\Delta(T)$. That K' is a compact group follows from Lemma 3.4.1. To see that is is also a left ideal, let $\mu \in \Delta(T)$. Then there is a net t_β in T such that $\delta_{t_\beta} \to \mu$. Since $l^*_{t_\beta} K' = K'$ for all β, it follows from the compactness of K' that for any $\nu \in K'$, $l^*_{t_\beta}\nu \to \mu \odot \nu \in K'$. Therefore $\mu \odot K' = K'$ for all $\mu \in \Delta(T)$. So K' is a left ideal. □

We already know that every compact group, ELA semigroup and n-ELA semigroup is strongly left amenable. Now, we want to prove that the product of an extremely left amenable semigroup by a compact group is also strongly left amenable, which will prove, using Pestov [19], that the unitary group $U(M)$ of an injective von Neumann algebra M with the $s(M, M_*)$-topology is strongly left amenable.

Let S be an extremely left amenable and G be a compact group. We define $S \times G$ using the Cartesian product, with the product topology. If $\mu \in \Delta(S)$ and $\nu \in \Delta(G) = \{\delta_g : g \in G\}$, then we define $\mu \times \nu \in \Delta(S \times G)$ by

$$(\mu \times \nu)(f) = \nu(h), \qquad h(t) = \mu(\pi_t f), \qquad \pi_t f(s) = f(s,t),$$

for any $f \in LUC(S \times G)$. Notice that whenever $||f|| \leq 1$, then $||\pi_t f|| \leq 1$. Therefore, we have $||\mu \times \nu|| = \sup_{||f|| \leq 1} |(\mu \times \nu)(f)| \leq 1$. Also, since $(\mu \times \nu)(1) = 1$, then $||\mu \times \nu|| = 1$. Therefore, $\mu \times \nu \in \Delta(S \times G)$.

Lemma 3.4.3. *Let S be an extremely left amenable semigroup with $\mu \in \Delta(S)$ be a right zero, and let $\varphi \in \Delta(S)$. Let G be a compact group, and let $\psi, \nu \in \Delta(G)$, with $\psi = \delta_g$ and $\nu = \delta_{g'}$. Then the following holds:*

1. $\delta_s \times \delta_g = \delta_{(s,g)}$.

2. $(\mu \times \delta_g)(f) = \mu(\pi_g f)$.

3. $\pi_g(l_{(s,g')}f) = l_s(\pi_{g'g}f)$.

4. $\pi_g(\mu \times \delta_{g'})_l(f) = \mu_l(\pi_{gg'}f)$.

5. $(\varphi \times \delta_g) \odot (\mu \times \delta_{g'}) = \mu \times \delta_{gg'}$.

6. $\delta_{(s,g)} \odot (\mu \times \delta_{g'}) = \mu \times \delta_{gg'}$.

Proof.

1. $(\delta_s \times \delta_g)(f) = \delta_g(h) = h(g) = \delta_s(\pi_g f) = \pi_g f(s) = f(s,g) = \delta_{(s,g)}(f)$

2. $(\mu \times \delta_g)(f) = \delta_g(h) = h(g) = \mu(\pi_g f)$

3. $\pi_g(l_{(s,g')}(f))(t) = l_{(s,g')}(f)(t,g) = f(st, g'g) = (\pi_{g'g}f)(st) = (l_s \pi_{g'g} f)(t)$

4.

$$\begin{aligned}
\pi_g(\mu \times \delta_{g'})_l(f)(s) &= (\mu \times \delta_{g'})_l(f)(s,g) = l^*_{(s,g)}(\mu \times \delta_{g'})(f) \\
&= (\mu \times \delta_{g'})(l_{(s,g)}f) = \delta_{g'}(h) = h(g') \\
&= \mu(\pi_{g'} l_{(s,g)} f) = \mu(l_s \pi_{gg'} f) \\
&= l^*_s \mu(\pi_{gg'} f) = \mu_l(\pi_{gg'} f)(s)
\end{aligned}$$

5.

$$\begin{aligned}
(\varphi \times \delta_g) \odot (\mu \times \delta_{g'})(f) &= (\varphi \times \delta_g)((\mu \times \delta_{g'})_l(f)) = \delta_g(h) \\
&= h(g) = \varphi[\pi_g(\mu \times \delta_{g'})(f)] \\
&= \varphi[\mu_l(\pi_{gg'} f)] = (\varphi \odot \mu)(\pi_{gg'} f) \\
&= \mu(\pi_{gg'} f) = (\mu \times \delta_{gg'})(f)
\end{aligned}$$

6.

$$\begin{aligned}
\delta_{(s,g)} \odot (\mu \times \delta_{g'}))(f) &= (\delta_s \times \delta_g) \odot (\mu \times \delta_{g'}))(f) \\
&= (\mu \times \delta_{gg'})(f)
\end{aligned}$$

□

Theorem 3.4.4. *The product of an extremely left amenable semigroup by a compact group is strongly left amenable.*

Proof. This follows from lemma 3.4.3. Let $\mu \in \Delta(S)$ be a right zero, then we want to show $K = \mu \times \{\delta_g : g \in G\}$ is a left ideal group in $\Delta(S \times G)$. Since the point measures are dense in $\Delta(S \times G)$, it is enough to show that K is a group, and $l^*_{(s,g)}K \subseteq K$. We have $l^*_{(s,g)}K = \delta_{(s,g)} \odot K \subseteq K$ by property (6) of lemma 3.4.3. Also, by property (6) of lemma 3.4.3, it is clear that K is a group. Therefore $S \times G$ is strongly left amenable. □

3.5 Special classes of semigroups

In this section, we want to characterize strong left amenability for some particular classes of semigroups that include compact semigroups, locally compact groups, discrete semigroups and connected semigroups.

Theorem 3.5.1. *If S is a compact semi-topological semigroup or a totally bounded topological group, then $LUC(S)$ is strongly left amenable if and only if $LUC(S)$ is left amenable.*

Proof. It is known [4, Proposition 4.4.8] that if S is either a compact semi-topological semigroup or a totally bounded topological group, then $AP(S) = LUC(S) = RUC(S)$. Therefore we can apply corollary 3.3.5, which gives the result. □

In the particular case where the semigroup is indeed a group, it is known that the only locally compact groups, which are strongly left amenable, are compact. (See [43]).

Let S be a SLA semigroup and K be a compact left ideal group in $\Delta(S)$. We define an equivalence relation $\sim$ by $s \sim t$ if and only if $l^*_s\mu = l^*_t\mu$ for all $\mu \in K$. We denote this semigroup by S/K. On this semigroup we use the topology induced by S. Let $\psi \in K$ be the identity in K. We define the map

$f : S \to K$ by $f(s) = l_s^* \psi$, the map $\pi : S \to S/K$ by $\pi(s) = \overline{s}$ and the map $\hat{f} : S/K \to K$ by $\hat{f}(\overline{s}) = f(s)$. Then f is a homomorphism since if $s, t \in S$ then

$$\begin{aligned} f(st) &= l_{st}^* \psi = l_s^* l_t^* \psi = \delta_s \odot \delta_t \odot \psi = \delta_s \odot [\psi \odot (\delta_t \odot \psi)] \\ &= (\delta_s \odot \psi) \odot (\delta_t \odot \psi) = f(s) \odot f(t). \end{aligned}$$

Also, f is continuous since if s_α is a net in S which converges to some $s \in S$, then $f(s_\alpha) = l_{s_\alpha}^* \psi = \delta_{s_\alpha} \odot \psi \to \delta_s \odot \psi = l_s^* \psi = f(s)$. Also, the map $\hat{f}$ is well defined, since if $s, t \in S$ such that $\overline{s} = \overline{t}$ then $\hat{f}(\overline{s}) = l_s^* \psi = l_t^* \psi = \hat{f}(\overline{t})$.

Lemma 3.5.2. *The map $\hat{f}$ is a continuous, one-to-one, homomorphism. Also, $\hat{f}(S/K)$ is dense in K. Therefore, $|S/K| \leq |K|$ and if $|K| = n$ we have that S/K is a group of order n.*

Proof. The continuity and the fact that $\hat{f}$ is a homomorphism, is a direct consequence of the fact that f is a continuous homomorphism. Now to see it is injective, let $\overline{s}, \overline{t} \in S/K$ such that $\hat{f}(\overline{s}) = \hat{f}(\overline{t})$. For any $\mu \in K$ we have:

$$\begin{aligned} \hat{f}(\overline{s}) = \hat{f}(\overline{t}) &\Rightarrow l_s^* \psi = l_t^* \psi \\ &\Rightarrow \delta_s \odot \psi = \delta_t \odot \psi \\ &\Rightarrow \delta_s \odot \psi \odot \mu = \delta_t \odot \psi \odot \mu \\ &\Rightarrow \delta_s \odot \mu = \delta_t \odot \mu. \end{aligned}$$

Since this is true for all $\mu \in K$, it follows that $\overline{s} = \overline{t}$ and the map $\hat{f}$ is injective. Now, to show that $\tilde{f}(S/K)$ is dense in K, let $\mu \in K$, then there is a net $\{s_\alpha\}$ in S which converges to μ, then

$$\hat{f}(\overline{s_\alpha}) = \delta_{s_\alpha} \odot \psi \to \mu \odot \psi = \mu.$$

By injectivity of $\hat{f}$, we have $|S/K| \leq |K|$. Finally if $|K| = n$, then K is discrete, and since $\hat{f}(S/K)$ is dense in a discrete group, $|S/K| = |K|$. $\square$

Lemma 3.5.3. *If S is a SLA semigroup with a compact left ideal group K in $\Delta(S)$, then S/K is a cancellative semigroup.*

Proof. Let $\overline{r}, \overline{s}, \overline{t} \in S/K$ and suppose that $\overline{s}\overline{t} = \overline{r}\overline{t}$. Then we have

$$\begin{aligned}
\overline{s}\overline{t} = \overline{r}\overline{t} &\Rightarrow \tilde{f}(\overline{s}\overline{t}) = \tilde{f}(\overline{r}\overline{t}) \\
&\Rightarrow \tilde{f}(\overline{s}) \odot \tilde{f}(\overline{t}) = \tilde{f}(\overline{r}) \odot \tilde{f}(\overline{t}) \text{ since } \tilde{f} \text{ is a homomorphism} \\
&\Rightarrow \tilde{f}(\overline{s}) = \tilde{f}(\overline{r}) \text{ since } K \text{ is a group} \\
&\Rightarrow \overline{s} = \overline{r} \text{ since } \tilde{f} \text{ is one-to-one.}
\end{aligned}$$

A similar proof shows that $\overline{s}\overline{t} = \overline{s}\overline{r} \Rightarrow \overline{t} = \overline{r}$, and therefore S/K is a cancellative semigroup. □

Corollary 3.5.4. *If K is finite or if S is compact, then S/K is isomorphic to K.*

Proof. In either case, using Lemma 3.5.2, it follows the map f is onto. Therefore $\hat{f}$ is a continuous bijective homomorphism; hence it is an isomorphism between S/K and K. □

Theorem 3.5.5. *If S is a discrete semigroup, then S is strongly left amenable if and only if S is n-extremely left amenable for some $n \in \mathbb{N}$, that is, if S is a discrete semigroup, then the Stone-Čech compactification βS never contains a compact left ideal group of infinite order.*

Proof. We know that if S is SLA, then S/K is a cancellative SLA semigroup. Therefore $\beta(S/K)$ has a unique minimal right ideal. In [11, Proposition 6.23], it is proved that if T is an infinite cancellative semigroup, then βT has at least 2^c minimal right ideal. Therefore it follows that S/K is finite. Now, since $\hat{f}(S/K)$ is dense in K, it follows K is also finite, and therefore S is n-ELA for some $n \in \mathbb{N}$. □

There is a characterization of n-ELA discrete semigroups and of SLA discrete semigroups in [26]. In view of our last theorem, those two characterizations actually describe the same objects. Let S be strongly left amenable. If $|K| = \infty$ we will say S is ∞-ELA. Otherwise it is n-ELA for some $n \in \mathbb{N}$.

Theorem 3.5.6. *If S is a connected SLA semigroup, then S is either ELA or ∞-ELA.*

Proof. We know that every connected n-ELA semigroup is ELA (See [37]). Also every SLA semigroups are either n-ELA or ∞-ELA. So we only need to see that a connected SLA semigroup can be ∞-ELA. For example, if we take the circle group $\mathbb{T}$ with its usual topology induced by $\mathbb{R}^2$. By compactness of $\mathbb{T}$, we can take $K = \mathbb{T}$, which show that S is ∞-ELA. $\square$

3.6 Examples

We conclude this chapter by giving some examples of strongly amenable semigroups and groups. The first set of examples were actually mentioned at the beginning of this chapter, but we thought it would be worth rewriting them here.

Example 3.6.1.

1. All compact groups are strongly amenable.
2. All extremely amenable semigroups are strongly amenable.
3. All n-ELA semigroups are strongly amenable.

Many examples of extremely amenable groups are now known. This allow us to give the following non-trivial examples of strongly amenable groups:

Example 3.6.2.

1. The unitary group $U(l^2)$ with the strong operator topology is strongly amenable [24].
2. The group $L^0(\mathbb{I}, U(1))$ of measurable maps from the standard Lebesgue space to the circle rotation group $U(1)$, equipped with the topology of convergence in measure, is strongly amenable (Glasner [20], also unpublished Furstenberg and Weiss).
3. The group $L^0(\mathbb{I}, G)$ of measurable maps from the standard Lebesgue space to an amenable locally compact group G, equipped with the topology of convergence in measure, is strongly amenable [56]

4. The group $Aut(\mathbb{Q}, \leq)$ of all order-preserving self-bijections of the set of rational numbers, equipped with the topology of simple convergence on $\mathbb{Q}$, when $\mathbb{Q}$ is discrete. [55]

If G is a locally compact group, then for any function $f \in L^2(G)$, we define $\tilde{f}(x) = \overline{f(x^{-1})}$. We then define the Fourier algebra $A(G)$ to be the Banach algebra

$$A(G) = \{f * \tilde{g} : f, g \in L^2(G)\}$$

with the norm defined as $||f * \tilde{g}||_{A(G)} = ||f||_2||g||_2$ (See [18]). We define the group von Neumann algebra to be the dual of $A(G)$. Alternatively, one can define $VN(G)$ to be the von Neumann algebra generated in $\mathcal{B}(L^2(G))$ by the left regular representation. In the case where G is a locally compact abelian group, then $A(G)$ can be defined as the Banach algebra of all Fourier transform of functions in $L^1(\hat{G})$, where $\hat{G}$ is the dual group of G, and $VN(G) = L^\infty(\hat{G})$.

Finally, using theorem 3.4.4, we get the following examples of strongly left amenable semigroups as given in the following theorem.

Theorem 3.6.3.

1. *The unitary group $U(M)$ of an injective von Neumann algebra M with the $s(M, M_*)$ topology is strongly left amenable.*

2. *The unitary group of the group von Neumann algebra $VN(G)$ of an amenable locally compact group G is strongly left amenable.*

Proof. **(1):** By Pestov [19], we know that $U(M)$ is the product of an extremely left amenable semigroup by a compact group. Therefore the result follows from theorem 3.4.4.

(2): The group von Neumann algebra $VN(G)$ of an amenable locally compact group G is an injective von Neuman algebra [45], and therefore its unitary group is strongly left amenable by (1). □

Chapter 4

Fixed point properties

4.1 Introduction

In this chapter, we want to investigate some fixed point properties related to strongly amenable semigroups.

In section 4.2 we characterize strong amenability by the existence of a fixed compact set whenever S acts on a compact Hausdorff space X. We also obtain in this section as a corollary a result for strongly amenable semigroup similar in flavor to the Banach fixed point theorem.

In section 4.3, we look at strongly continuous representations of semigroup as non-expansive mappings, and extend some result of Kang [29].

4.2 Fixed point properties

We define an anti-action of S on $l^\infty(S)$ by:

$$S \times l^\infty(S) \to l^\infty(S),$$

$$(s, f) = s \cdot f = l_s f,$$

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$$l_s f(t) = f(st), \quad \forall s, t \in S, \forall f \in l^\infty(S).$$

This map restricted to $CB(S)$ define an anti-action on $CB(S)$, and similarly for $LUC(S)$.

We define the following action of S on $\Delta(S)$:

$$S \times \Delta(S) \to \Delta(S),$$

$$(s, \mu) = s \cdot \mu = l_s^* \mu,$$

$$l_s^* \mu(f) = \mu(l_s f).$$

When we consider S as a subset of $\Delta(S)$, this map coincides with the Arens product already defined: $s \cdot \mu = \delta_s \odot \mu$. If X is a Hausdorff topological space, we define $CB(X)$ to be the set of all continuous, bounded, complex valued functions on X, and $LUC(S, X)$ to be the subalgebra of $CB(X)$ for which the map $S \to CB(X)$ defined by $s \to l_s f$ is continuous (See [38]). As for $LUC(S)$, we define an anti-action of S on $LUC(S, X)$ by:

$$S \times LUC(S, X) \to LUC(S, X),$$

$$(s, f) = s \cdot f = l_s f.$$

Fix $x \in X$. We define the following map:

$$\Phi : LUC(S, X) \to LUC(S),$$

$$\Phi(f) = T_x(f),$$

$$T_x f(s) = f(s \cdot x).$$

Notice that this map commutes with the action of S, i.e. $s \cdot \Phi(f) = \Phi(s \cdot f)$ for all $s \in S$, and for all $f \in LUC(S, X)$. To see that, let $t \in S$ then we have:

$$\begin{aligned} s \cdot (\Phi(f))(t) &= (\Phi(f))(st) = (T_x f)(st) = f(st \cdot x) = (s \cdot f)(t \cdot x) \\ &= (T_x(s \cdot f))(t) = (\Phi(s \cdot f))(t). \end{aligned}$$

Now, let $\Delta(LUC(S, X))$ denote the set of all multiplicative means on $LUC(S, X)$. We define the action of S on $\Delta(LUC(S, X))$ by:

$$S \times \Delta(LUC(S, X)) \to \Delta(LUC(S, X)),$$

$$(s, \mu) = s \cdot \mu = l_s^* \mu,$$

and we define the following map:

$$\Psi : \Delta(S) \to \Delta(LUC(S, X)),$$

$$\Psi(\mu) = \mu',$$

$$\mu'(f) = \mu(\Phi(f)).$$

Once again, this map commutes with the action of S. If $f \in LUC(S, X)$, we have

$$\begin{aligned} (s \cdot (\Psi(\mu)))(f) &= (\Psi(\mu))(s \cdot f) = \mu(\Phi(s \cdot f)) = \mu(s \cdot \Phi(f)) = (s \cdot \mu)(\Phi(f)) \\ &= (\Psi(s \cdot \mu))(f), \end{aligned}$$

which proves that $s \cdot (\Psi(\mu)) = \Psi(s \cdot \mu)$ for all $s \in S$ and for all $\mu \in \Delta(S)$. Now, if we associate X with the point measures on $\Delta(LUC(S, X))$, then notice that the actions of S on X correspond to the action of S on δ_X. That is, for any $f \in LUC(S, X)$, we have

$$\delta_{s \cdot x}(f) = f(s \cdot x) = (s \cdot f)(x) = \delta_x(s \cdot f) = (s \cdot \delta_x)(f),$$

and therefore we have $\delta_{s \cdot x} = s \cdot \delta_x$ for all $s \in S$ and for all $x \in X$. We now want to prove that Ψ is continuous. Let $\mu_\alpha \to \mu$ be a net in $\Delta(S)$. Then we have.

$$(\Psi(\mu_\alpha))(f) = \mu_\alpha(T_x f) \to \mu(T_x f) = (\Psi(\mu))(f),$$

which proves the continuity of Ψ.

Theorem 4.2.1. *If S is a semi-topological semigroup, then the following are equivalent:*

1. *There exists a compact set $K \subseteq \Delta(S)$ such that $l_s^* K = K$.*

2. *Whenever S acts on a compact Hausdorff space X, where the action is jointly continuous, there exists a compact set $K' \subseteq X$ such that $s \cdot K' = K'$ for all $s \in S$.*

Also, $|K'| \leq |K|$*. Moreover, if* S *is a SLA semigroup, then the two conditions holds.*

Proof.
(1) $\Rightarrow$ (2): If X is compact, then $LUC(S, X) = C(X)$ since the action of S on X is jointly continuous. So $\Delta(LUC(S, X)) = X$. Let $K' = \Psi(K)$. Then K' is compact and $s \cdot K' = s \cdot \Psi(K) = \Psi(s \cdot K) = \Psi(K) = K'$.
(2) $\Rightarrow$ (1): Since $\Delta(S)$ is a compact Hausdorff space, and the action of S on $\Delta(S)$ defined by $(s, \mu) = l_s^* \mu$ is jointly continuous, there exists $K \subseteq \Delta(S)$ such that $s \cdot K = l_s^* K = K$ for all $s \in S$.

(SLA) $\Rightarrow$ (1): This is trivial using theorem 3.3.1.

□

Corollary 4.2.2. *Let* S *be a strongly left amenable semi-topological semigroup acting on a compact subset* X *of a complete metric space. Suppose that the action is jointly continuous and for at least one* $s \in S$*, the map* $s : X \to X$ *is contractive, then there is in* X *a common fixed point for* S*.*

Proof. Since $s : X \to X$ is a contractive map, there exists a point $x \in X$ such that $s^n y \to x$ for all $y \in X$. Therefore, $diam(s^n X) \to 0$ as $n \to \infty$. Now, by the theorem, we know there exists $K \subseteq X$ such that $t \cdot K = K$ for all $t \in S$. Now, since $K \subseteq s^n X$ for all n, we have that $K = \{x\}$, and $x \in X$ is a common fixed point. □

Now, if S is a semi-topological semigroup which acts on a compact Hausdorff space X, where the action is jointly continuous, then we can extend the action $S \times X \to X$, to an action $\Delta(S) \times X \to X$, which is continuous in the first variable in the following way:

$$\mu \cdot x = \Psi_x(\mu), \quad \mu \in \Delta(S), \quad x \in X$$

Lemma 4.2.3. *Let* S *be a semi-topological semigroup, and let* $K \subseteq \Delta(S)$ *such that* $\mu \odot K = K$ *for all* $\mu \in \Delta(S)$*. Then there is a compact left ideal group in* $\Delta(S)$*, i.e.,* S *is strongly left amenable.*

Proof. First, notice that K is a left ideal, but we don't know if K is minimal. Let $\varphi \in K$ and define $J = K \odot \varphi$. Therefore for any $\mu \in \Delta(S)$,

$$\mu \odot J = \mu \odot (K \odot \varphi) = (\mu \odot K) \odot \varphi = K \odot \varphi = J$$

But also, since $\Delta(S)$ is a compact right topological semigroup, $\Delta(S)$ contains a minimal left ideal, and therefore any left ideal of $\Delta(S)$ contains a minimal left ideal. Let $J = K \odot \varphi \subseteq \Delta(S) \odot \varphi$ which is a minimal left ideal (See [4]). Therefore J is left and right simple, which implies that J is a left ideal group in $\Delta(S)$. It follows that S is strongly left amenable. □

Theorem 4.2.4. *If S is a semi-topological semigroup, then the following are equivalent:*

1. *S is strongly left amenable.*
2. *Whenever S acts on a compact Hausdorff space X, where the action is jointly continuous, there exists a compact set $K' \subseteq X$ such that $\mu \cdot K' = K'$ for all $\mu \in \Delta(S)$, where $\mu \cdot K'$ denotes the extension of the action to an (right continuous) action of $\Delta(S)$ on X.*

Proof.
(1) $\Rightarrow$ **(2)** If K is a compact left ideal group in $\Delta(S)$, let $K' = \Psi(K)$. For all $\mu \in \Delta(S)$, we have

$$\mu \cdot K' = \mu \cdot \Psi(K) = \mu \cdot (K \cdot x) = (\mu \odot K) \cdot x = K \cdot x = \Psi(K) = K'.$$

(2) $\Rightarrow$ **(1)** Let $X = \Delta(S)$, then we have $K \subseteq \Delta(S)$ such that $\mu \odot K = K$ for all $\mu \in \Delta(S)$. Now, we can apply lemma 4.2.3 to prove that S is strongly left amenable. □

4.3 Non-expansive mappings

For this section, E will denote a Banach space, and C a weakly compact, bounded convex subset of E. Also, S will denote a semi-topological semigroup, and $\mathcal{S}$ will be a strongly continuous representation of S as non-expensive mapping on C, that is, $\mathcal{S} = \{T_s : s \in S\}$ such that:

1. $T_s : C \to C, \forall s \in S$,

2. $T_{st} = T_s T_t, \forall s, t \in S$,

3. $||T_s x - T_s y|| \leq ||x - y||, \forall s \in S, \forall x, y \in C$,

4. $s \to T_s x$ is continuous on $S, \forall x \in C$.

For most of this section, we will be using $S^{CB(S)}$ instead of $S^{LUC(S)}$, but notice that $S^{CB(S)}$ is not in general a semigroup, since it is not necessarily closed under the Arens product. Let μ be any multiplicative mean on $CB(S)$. Define the equivalence relation $\sim_\mu$ on S by $s \sim_\mu t$ if and only if $l_s^*\mu = l_t^*\mu$, and we denote by S/μ the quotient of S with this equivalence relation. Also, we define the set $S_\mu = \{s \in S : l_s^*\mu = \mu\}$. Notice that this section is mostly based on [44, 29].

Now, for any $\mu \in S^{CB(S)}$ and $z \in C$, we define $T_\mu z$ weakly by: $\langle T_\mu z, x^* \rangle = \mu_s \langle T_s z, x^* \rangle$, for all $x^* \in E^*$. This map is well defined, since if $\{s_\alpha\}$ is a net in S such that $\delta_{s_\alpha} \to^{w^*} \mu$, then $\langle T_{s_\alpha} z, x^* \rangle = \delta_{s_\alpha} \langle T_s z, x^* \rangle \to \mu_s \langle T_s z, x^* \rangle = \langle T_\mu z, x^* \rangle$. It follows that $T_{s_\alpha} z \to^w T_\mu z$. By weak compactness, we then have that $T_\mu z \in C$. Finaly, if $z \in C$ and $x^* \in E^*$ are fixed, we define $f_{x^*}(s) = \langle T_\mu z, x^* \rangle$ for all $s \in S$. The map f_{x^*} is in $CB(S)$, but not necessarily in $LUC(S)$, which is why we need to work with $CB(S)$ instead of $LUC(S)$.

Lemma 4.3.1. *Let S be a semi-topological semigroup. Let $\{s_\alpha\}$ be a net in S such that $\delta_{s_\alpha} \to^{w^*} \mu \in S^{CB(S)}$, let $s \in S_\mu$, and let $r, t \in S$ such that $r \sim_\mu t$. Also, let $z \in C$. Then:*

1. $T_{ss_\alpha} z - T_{s_\alpha} z \to^w 0$,

2. $T_{rs_\alpha} z - T_{ts_\alpha} z \to^w 0$.

Proof.

1. If $x^* \in E^*$, then we have:

$$\begin{aligned}
\langle T_{ss_\alpha} z - T_{s_\alpha} z, x^* \rangle &= f_{x^*}(ss_\alpha) - f_{x^*}(s_\alpha) \\
&= (l_s f_{x^*} - f_{x^*})(s_\alpha) \\
&= \delta_{s_\alpha}(l_s f_{x^*} - f_{x^*}) \\
&\to \mu(l_s f_{x^*} - f_{x^*}) \\
&= (l_s^* \mu - \mu)(f_{x^*}) \\
&= (\mu - \mu)(f_{x^*}) \\
&= 0.
\end{aligned}$$

   and therefore, it follows that $T_{ss_\alpha} z - T_{s_\alpha} z \to^w 0$.

2. If $x^* \in E^*$, then we have:

$$\begin{aligned}
\langle T_{rs_\alpha} z - T_{ts_\alpha} z, x^* \rangle &= f_{x^*}(rs_\alpha) - f_{x^*}(ts_\alpha) \\
&= (l_r f_{x^*} - l_t f_{x^*})(s_\alpha) \\
&= \delta_{s_\alpha}(l_r f_{x^*} - l_t f_{x^*}) \\
&\to \mu(l_r f_{x^*} - l_t f_{x^*}) \\
&= (l_r^* \mu - l_t^* \mu)(f_{x^*}) \\
&= 0.
\end{aligned}$$

   Therefore, it follows that $T_{rs_\alpha} z - T_{ts_\alpha} z \to^w 0$.

□

Theorem 4.3.2. *Let E be a Banach space, and C be a compact convex subset of E. Let S be a semi-topological semigroup and let $\mu \in S^{CB(S)}$. Assume S_μ is non-empty, $\mathcal{S} = \{T_s : s \in S\}$ is a strongly continuous representation of S as non-expansive mappins, and $T_\mu z = z$ for some $z \in C$.*

1. *Then $T_s z = z$ for all $s \in S_\mu$.*

2. *If $s \in S$ is such that $T_s z = z$. Then $T_t z = z$ for all $t \sim_\mu s$ in S/μ.*

3. *If there exists an element s in each equivalence class of S/μ such that $T_s z = z$, then $T_t z = z$ for all $t \in S$.*

Proof.

1. Since C is compact, the weak and norm topologies coincides. Therefore, if s_α is a net in S such that $\delta_{s_\alpha} \rightarrow^{w^*} \mu$, then we have $||T_{s_\alpha} z - T_\mu z|| \rightarrow 0$. Also,

$$\begin{aligned} ||T_s z - T_{s_\alpha} z|| &= ||T_s z - T_{ss_\alpha} z + T_{ss_\alpha} z - T_{s_\alpha} z|| \\ &\leq ||T_s z - T_{ss_\alpha} z|| + ||T_{ss_\alpha} z - T_{s_\alpha} z|| \\ &\leq ||z - T_{s_\alpha} z|| + ||T_{ss_\alpha} z - T_{s_\alpha} z|| \\ &\rightarrow 0 + 0 = 0. \end{aligned}$$

   Notice that for the last inequality, we used the fact that $T_\mu z = z$ and lemma 4.3.1 (1). Therefore, $T_{s_\alpha} z \rightarrow T_\mu z = z$ and also $T_{s_\alpha} z \rightarrow T_s z$. By unicity of the limit in a Hausdorff space, it follows that: $T_s z = z$.

2. Let $\{s_\alpha\}$ be a net in S such that $\delta_{s_\alpha} \rightarrow \mu$ .We know that $T_{s_\alpha} z \rightarrow T_\mu z = z$ and therefore, by the continuity of T_s and T_t, we have $T_s(T_{s_\alpha} z) \rightarrow T_s z$ and $T_t(T_{s_\alpha} z) \rightarrow T_t z$. Therefore, using lemma 4.3.1 (2), we have:

$$0 \leftarrow ||T_{ss_\alpha} z - T_{ts_\alpha} z|| = ||T_s(T_{s_\alpha} z) - T_t(T_{s_\alpha} z)|| \rightarrow ||T_s z - T_t z||.$$

   It follows, by the unicity of the limit, that $T_s z = T_t z = z$.

3. We apply (2) for each equivalence class.

□

Theorem 4.3.3. *Let E be a Banach space, and C be a weakly compact convex subset of E. Let S be a semi-topological semigroup and let $\mu \in S^{CB(S)}$. Assume S_μ is non-empty, $\mathcal{S} = \{T_s : s \in S\}$ is a strongly continuous representation of S as non-expansive mappings, T_s is weak-to-weak continuous for all $s \in S$, and $T_\mu z = z$ for some $z \in C$.*

1. *Then $T_s z = z$ for all $s \in S_\mu$.*

2. *If $s \in S$ is such that $T_s z = z$. Then $T_t z = z$ for all $t \sim_\mu s$.*

3. *If every equivalence class of S/μ contains an element $s \in S$ such that $T_s z = z$, then $T_t z = z$ for all $t \in S$*

Proof.

1. Let s_α be a net in S such that $\delta_{s_\alpha} \to \mu$, then by lemma 4.3.1, $T_{s_\alpha} z \to T_\mu z$. By the weak-to-weak continuity of T_s, $T_{ss_\alpha} z = T_s(T_{s_\alpha} z) \to^w T_s(T_\mu z) = T_s z$. Also, by lemma 4.3.1, $T_{s_\alpha} z - T_{ss_\alpha} z \to^w 0$. Hence $T_{ss_\alpha} z - T_s z \to^w T_s z - T_s z = 0$. Therefore $T_{s_\alpha} z - T_s z = (T_{s_\alpha} z - T_{ss_\alpha} z) + (T_{ss_\alpha} z - T_s z) \to 0$. Thus $T_\mu z - T_s z = 0$ and $T_s z - T_\mu z = z$.

2. Let $\{s_\alpha\}$ be a net in S such that $\delta_{s_\alpha} \to \mu$. We have that $T_s z - T_t z \ {}^w\!\leftarrow T_{ss_\alpha} z - T_{ts_\alpha} z \to^w 0$. Therefore, it follows that $T_s z = T_t z = z$.

3. We apply (2) for each equivalence class.

□

Notice that most of the results at the beginning of this section are based on the requirement that the map $f_{x^*}(s) = \langle T_s z, x^* \rangle$ is in $CB(S)$. If we add the extra assumption that the semigroup S is such that the map f_{x^*} is in $LUC(S)$, the last two theorems are still valid if $CB(S)$ is replaced by $LUC(S)$. This is the case in particular, if we assume S to be discrete. Also, some of the theorems require the assumption that for some multiplicative mean $\mu \in \Delta(S)$, the set S_μ is non-empty. It would be interesting to know when this set is non-empty.

Theorem 4.3.4. *Let S be a strongly left amenable semi-topological semigroup with compact left ideal group K in $\Delta(S)$. Let C be a nonempty compact convex subset of a Banach space E, and let $\mathcal{S} = \{T_s : s \in S\}$ be a strongly continuous representation of S as non-expansive mappings from C into C.*

1. *If S/K is a group, then S_μ is non-empty for all $\mu \in K$.*

2. If there exists a $z \in C$ such that $T_\mu z = z$ for some $\mu \in K$. Then $T_e z = z$ where e is the identity of K.

Proof.

1. Let (s_α) be a net in S such that $\delta_{s_\alpha} \to e$. Then we have

$$T_e z = \lim_\alpha T_{s_\alpha} z = \lim_\alpha T_{s_\alpha}(T_\mu z) = \lim_\alpha T_{l^*_{s_\alpha}\mu} z = T_{e\odot\mu} z = T_\mu z = z.$$

2. Recall that for any $s \in S$, we have $l_s^* K = K$. Let $\bar{e}$ be the identity in S/K, and let e be any element of S in the equivalence class of $\bar{e}$. Now for any element $\mu \in K$, there exists $\nu \in K$ such that $l_s^*\nu = \mu$. Therefore, we have $l_e^* l_s^* \nu = l_s^* \nu$ and $l_e^*\mu = \mu$. Now, since this is true for any $\mu \in K$, it follows that

$$e \in S_0 = \{s \in S : l_s^*\mu = \mu, \forall \mu \in K\}.$$

Finally, we have
$$S_0 = \bigcap_{\mu \in K} S_\mu,$$
and therefore $e \in S_\mu$ for any $\mu \in K$.

□

Theorem 4.3.5. *Let S be a semi-topological semigroup for which there exists a compact set $K \subseteq S^{CB(S)}$ which is left invariant, that is $l_s^* K = K$ for all $s \in S$. Let C be a nonempty compact convex subset of a Banach space E, and let $\mathcal{S} = \{T_s : s \in S\}$ be a strongly continuous representation of S as non-expansive mappings from C into C. If $z \in C$, then the following are equivalent:*

1. $T_s z = z$ for all $s \in S$,

2. $T_\mu z = z$ for all $\mu \in K$.

Proof.
(1) ⇒ (2) Suppose $T_s z = z$ for all $s \in S$, and let μ be any element in K. Then we have

$$\langle T_\mu z, x^* \rangle = \mu_s \langle T_s z, x^* \rangle = \mu_s \langle z, x^* \rangle = \langle z, x^* \rangle, \quad \forall x^* \in E^*.$$

Therefore, it follows that $T_\mu z = z$ for all $\mu \in K$.
(2) ⇒ (1) Suppose $T_\mu z = z$ for all $\mu \in K$. Then we have

$$T_s z = T_s T_\mu z = T_{l_s^* \mu} = T_\nu z = z, \quad \forall s \in S,$$

where $l_s^* \mu = \nu \in K$. □

Theorem 4.3.6. *Let S be a SLA semi-topological semigroup, E be a Banach space, C be a compact subset of E, and let $\mathcal{S} = \{T_s : s \in S\}$ be a jointly continuous representation of S as non-expansive mappings on C. Then at least one of the following is true:*

1. *There exists (a unique) $z \in C$ such that $T_s z = z$ for all $s \in S$.*

2. *For each $s \in S$, there exists $x, y \in C$ such that $||T_s x - T_s y|| = ||x - y||$.*

Proof. Since the action of S on C is jointly continuous, by theorem 4.2.1 there exists a compact set $K \subseteq C$ such that $T_s K = K$ for all $s \in S$. If $|K| = 1$, then (1) is true. Otherwise, fix $s \in S$. Since K is compact, there exists $x', y' \in K$ such that $||x' - y'|| = diam(K)$. Now, since $T_s K = K$, there is $x, y \in K$ such that $T_s x = x'$ and $T_s y = y'$. By non-expensiveness, we then have:

$$||x' - y'|| = ||T_s x - T_s y|| \leq ||x - y|| \leq ||x' - y'|| = diam(C).$$

Therefore, $||T_s x - T_s y|| = ||x - y||$ and (2) is satisfied.

□

Chapter 5

Ultimately non-expansive mapping

5.1 Introduction

In this chapter, we want to extend some results of Edelstein and Kiang to right reversible semigroups. The original work was done for abelian group. This chapter is not as much about strong left amenability as the previous two chapters were, but we notice that one of the conditions necessary for our theorem to work is always satisfied for strongly amenable semigroups.

In section 5.2 we define the concept of ultimately non-expansive mappings, and provide the reader with some examples. Our main theorem is given in section 5.4, but the proof being lengthy, we separate the proof in many lemmas which are proven in section 5.3.

A similar version of this chapter has been accepted for publication in N. Bouffard, Strongly amenable semigroups and nonlinear fixed point properties, Journal of Nonlinear Analysis

5.2 Ultimately non-expansive mappings

Let S be a (discrete) semigroup, we say that S is right reversible if for any $a, b \in S$ we have that $Sa \cap Sb \neq \emptyset$, that is for each $a, b \in S$ there exists $c, d \in S$ such that $ca = db$. Similarly, S is left reversible if $aS \cap bS \neq \emptyset$, for all $a, b \in S$. In this paper, any semigroup could be considered semi-topological, as long as the semigroup is considered reversible in the discrete sense. Let S be a semigroup which acts on a Banach space X where the action is continuous in the second variable. We say that the action is ultimately non-expansive if for all $u, v \in X$ and for all $\alpha > 0$, there is a left ideal $I \subseteq S$ such that:

$$||s \cdot u - s \cdot v|| \leq (1 + \alpha)||u - v||, \quad \forall s \in I.$$

We can show that the action of S on X is ultimately non-expansive if and only if for every $u, v \in X$ and every $\alpha > 0$ there is a $s \in S$ such that for all $t \in S$:

$$||ts \cdot u - ts \cdot v|| \leq (1 + \alpha)||u - v||.$$

Notice that this definition is slightly different from the one found in [17], but in the case S is abelian, the two definitions are equivalent. If the inequality still holds for $\alpha = 0$, then we call the action asymptotically non-expansive.

Example 5.2.1. Every non-expansive or asymptotically non-expansive mapping [27] is ultimately non-expansive.

Example 5.2.2. Let K be the closed unit disc in $\mathbb{R}^2$ with Euclidiean norm and polar coordinates. We define the following map on K:

$$f(r, \theta) = (r/2, \theta),$$

$$g(r, \theta) = (r, 2\theta \bmod 2\pi),$$

then the semigroup of continuous mappings generated by f, g is asymptotically non-expansive, and therefore ultimately non-expansive [27].

Example 5.2.3. Let X be a normed linear space. For $x \in X$, define $f : X \to X$ by:

$$f(x) = \begin{cases} \frac{x}{4} & \text{if } ||x|| \leq 1 \\ \left(\frac{(3/2)||x||-5/4}{||x||}\right)x & \text{if } 1 \leq ||x|| \leq 3/2 \\ \left(\frac{||x||-1/2}{||x||}\right)x & \text{If } ||x|| \geq 3/2 \end{cases}$$

Then f is an asymptotically non-expansive map on X, and hence the semigroup generated by f is also asymptotically non-expansive [31].

A point $x \in X$ is said to be S-left-recurrent if for any $\varepsilon > 0$ and any left ideal $I \subseteq S$ there exists $s \in I$ such that $||s \cdot x - x|| < \varepsilon$. This is the case if and only if for any $\epsilon > 0$ and any $s \in S$ there is a $t \in S$ such that $||ts(x) - x|| < \epsilon$. We define right-recurrent by replacing left ideal by right ideal in the definition. Once again, this definition is slightly different from the one given in [17].

Example 5.2.4. If G is a group acting on a Banach space X, then $e \cdot x$ is left-recurrent for all $x \in X$, where e is the identity of G. If the action is a group action (and not just a semigroup action), then all points of X are left-recurrent.

Example 5.2.5. If S is a strongly left amenable semigroup acting on a compact subset X of a Banach space where the action is jointly continuous, then there is in X a left recurrent point. To see that, we first need to extend the action of S on X to an action of $\Delta(S)$ on X. For a fixed $x \in X$, let $f_x : S \to X, f_x(s) = s \cdot x$. Next we define $\tilde{f}_x : LUC(S, X) \to LUC(S)$ by $\tilde{f}_x(g)(s) = g(f_x(s))$, and finally we define the map: $\tilde{\tilde{f}}_x : \Delta(S) \to \Delta(LUC(S, X)) = X$ by $\tilde{\tilde{f}}_x(\mu)(g) = \mu(\tilde{f}_x(g))$. We define the action of a point $\mu \in \Delta(S)$ on $x \in X$, by $\mu \cdot x = \tilde{\tilde{f}}_x(\mu)$. Direct calculation shows that this map is continuous in the first variable, and extends the action from S. Now, let $x \in X$ and $e \in K$ be the identity of the left ideal group K in $\Delta(S)$. Define $z = e \cdot x$. Then z is a left-recurrent point in X since if $s \in S$ and $\mu = (se)^{-1}$ then $||\mu s(z) - z|| = 0$. Now since $\mu \in \Delta(S)$, there exists a net $\{s_\alpha\}$ in S such that $s_\alpha \to^{w^*} \mu$ and therefore $s_\alpha s(z) \to \mu s(z) = z$. This means that for every $n \in \mathbb{N}$ there is a $s_n \in S$ such that $||s_n s(z) - z|| < \frac{1}{n}$, which implies that z is left-recurrent.

We say that $x \in X$ has the approximate identity property for the orbit if for every net s_α in S such that $s_\alpha x \to x$, then $s_\alpha y \to y$ for all $y \in Orb(x)$, where $Orb(x) = \{s \cdot x : s \in S\}$. That is, if for every net s_α in S such that $s_\alpha x \to x$, then $s_\alpha tx \to tx$ for all $t \in S$. This last definition is similar to property (B) found in [27].

If X is a normed linear space, then X is locally uniformly convex [50] if and only if given $\epsilon > 0$ and an element $x \in X$ with $||x|| = 1$, there exists $\delta(\epsilon, x) > 0$ such that

$$\frac{||x+y||}{2} \leq 1 - \delta(\epsilon, x) \text{ whenever } ||x-y|| \geq \epsilon \text{ and } ||y|| = 1.$$

If δ depends only of ϵ, then X is uniformly convex [8]. Also, X is called strictly convex, if the unit sphere of X does not contain any line segment (i.e. each point of the unit sphere of X is an extreme point of the unit ball of X). It is known that every uniformly convex space is locally uniformly convex, and every locally uniformly convex space is strictly convex [50].

If X is a locally uniformly convex space and (x_n) is a sequence in X such that $||x_n|| \leq 1$ for all $n \in \mathbb{N}$, which converges weakly to some $x \in X$ such that $||x|| = 1$ then $||x_n - x|| \to 0$ (The proof is similar to the one found in [14, page 32 theorem 4(iii)]). Also, if X, Y are normed spaces, and Y is strictly convex, then every isometry $\pi : X \to Y$ is affine. The idea of the proof is given in [62].

For more information on strict convexity, local uniform convexity and uniform convexity, we refer the reader to the books of Diestel [14], Carothers [7] and Day [13].

5.3 Some lemmas

We will write the proof of the main result using 5 lemmas. These lemmas are inspired by those found in [17]

Lemma 5.3.1. *Let S be an ultimately non-expansive right reversible semigroup of continuous mappings of a Banach space X into itself. Let $u_1, u_2, ..., u_n \in X$*

and $s \in S$. Then to any positive integer k there is a s_k in S with the property that, for any $t \in S$ and each $i, j \in \{1, 2, ..., n\}, i \neq j$,

$$||ts_k s(u_i) - ts_k s(u_j)|| \leq (1 + 1/k)||u_i - u_j||, \tag{5.1}$$

$$||ts_k s(u_i) - ts_k s(u_j)|| \leq (1 + 1/k)||s(u_i) - s(u_j)||. \tag{5.2}$$

Proof. For every $i, j \in \{1, 2, ...n\}$, there exists $s_k^{i,j} \in S$ such that

$$||ts_k^{i,j}(u_i) - ts_k^{i,j}(u_j)|| \leq \left(1 + \frac{1}{k}\right)||u_i - u_j||.$$

Now let $\{t_1, t_2, ..., t_m\}$ be all permutation of the product $\prod_{i,j} s_k^{i,j}$. Now, by right reversibility, there exists $c_1, c_2, ...c_m \in S$ such that $c_1 t_1 = c_2 t_2 = ... = c_m t_m$. Then we define $s_k = c_1 t_1$. Therefore we have

$$||ts_k(u_i) - ts_k(u_j)|| \leq \left(1 + \frac{1}{k}\right)||u_i - u_j||, \quad \forall t \in S \text{ and } i, j \in \{1, 2, ..., n\}.$$

Now, let $s \in S$. Then by right reversibility, there exists sequences c_k and c'_k such that $c_k s_k s = c'_k s s_k$ for all k. Then we have

$$||tc_k s_k s(u_i) - tc_k s_k s(u_j)|| = ||tc'_k s s_k(u_i) - tc'_k s s_k(u_j)|| \leq \left(1 + \frac{1}{k}\right)||u_i - u_j||.$$

Now, if we let $s'_k = c_k s_k$, then we have

$$||ts'_k s(u_i) - ts'_k s(u_j)|| \leq \left(1 + \frac{1}{k}\right)||u_i - u_j||.$$

Now, by replacing all u_i by $s(u_i)$, we can construct a sequence s''_k such that

$$||ts''_k s(u_i) - ts''_k s(u_j)|| \leq \left(1 + \frac{1}{k}\right)||s(u_i) - s(u_j)||.$$

Now, using right reversibility once again, we can define d_k, d'_k such that $d_k s'_k s''_k = d'_k s''_k s'_k$, and we define $s_k = d_k s'_k s''_k$. Therefore

$$||ts_k s(u_i) - ts_k s(u_j)|| \leq \left(1 + \frac{1}{k}\right)||u_i - u_j||,$$

and

$$||ts_k s(u_i) - ts_k s(u_j)|| \leq \left(1 + \frac{1}{k}\right)||s(u_i) - s(u_j)||,$$

for all $t \in S$. $\square$

Lemma 5.3.2. *Let S be an ultimately non-expansive right reversible semigroup of continuous mappings of a Banach space X into itself. Let $z_1, z_2 \in X, s \in S$. Suppose there are sequences t_k, t'_k in S such that*

$$\lim_{k\to\infty} t'_k s_k s(z_i) = s(z_i) \text{ and } \lim_{k\to\infty} t_k s_k s(z_i) = z_i, \quad i \in \{1,2\},$$

where s_k is the sequence constructed in lemma 5.3.1 with $n = 2$ and $u_1 = z_1, u_2 = z_2$. then $||s(z_1) - s(z_2)|| = ||z_1 - z_2||$.

Proof. By lemma 5.3.1, there exists a sequence s_k such that:

$$||ss_k t(z_1) - ss_k t(z_2)|| \leq (1 + 1/k)||z_1 - z_2||, \tag{5.3}$$

$$||ss_k t(z_1) - ss_k t(z_2)|| \leq (1 + 1/k)||t(z_1) - t(z_2)||, \tag{5.4}$$

for all $s \in S$. By substituting t'_k and t_k for s in equation (5.3) and (5.4) respectively, we obtain

$$||s(z_1) - s(z_2)|| = \lim_{k\to\infty} ||t'_k s_k t(z_1) - t'_k s_k t(z_2)|| \leq (1 + \frac{1}{k})||z_1 - z_2||,$$

and

$$||z_1 - z_2|| = \lim_{k\to\infty} ||t_k s_k t(z_1) - t_k s_k t(z_2)|| \leq (1 + \frac{1}{k})||s(z_1) - s(z_2)||.$$

Therefore it follows that $||s(z_1) - s(z_2)|| = ||z_1 - z_2||$. □

Lemma 5.3.3. *Let X be a reflexive locally uniformly convex Banach space, and let S be an ultimately non-expansive right reversible semigroup of continuous mappings of X into itself. Suppose that $p, q, z \in X$ and $\{s_k\}$ in S are such that $z = \lambda p + (1-\lambda)q$ for some $\lambda, 0 < \lambda < 1$ and*

$$||ss_k(p) - ss_k(z)|| \leq (1 + 1/k)||p - z||,$$

$$||ss_k(q) - ss_k(z)|| \leq (1 + 1/k)||q - z||,$$

for all $s \in S$. Suppose further that a sequence $\{t_k\}$ in S exists such that $\{t_k s_k(p)\}$ and $\{t_k s_k(q)\}$ converge and $\lim_{k\to\infty} t_k s_k(p) = p$, $\lim_{k\to\infty} t_k s_k(q) = q$. Then $\{t_k s_k(z)\}$ converges and

$$\lim_{k\to\infty} t_k s_k(z) = z.$$

Proof. Replacing s by t_k in both inequalities shows that $t_k s_k(z)$ is bounded since both $t_k s_k(p)$ and $t_k s_k(q)$ are bounded. Therefore, by reflexivity, we can apply the Eberlein-Smulian theorem [9], which gives us that there exists a subsequence $t_{k_j} s_{k_j}(z)$ which is weakly convergent to some $w \in X$. Now, since norm closure is equivalent to weak closure for convex sets, it follows that $||p-w|| \leq ||p-z||$ and $||q-w|| \leq ||q-z||$, and therefore

$$||p-q|| = ||p-z|| + ||q-z|| \geq ||p-w|| + ||q-w|| \geq ||p-q||.$$

It follows that $||p-w|| = ||p-z||$ and $||q-w|| = ||q-z||$, and therefore using the strict convexity of X, $z = w$. Now, since this is true for any convergent subsequence, it follows that $t_k s_k(z)$ converges weakly to z. Now consider the following sequence

$$\frac{t_k s_k(p) - t_k s_k(z)}{(1+1/k)||p-z||}.$$

All those points are in the unit ball of X, and this sequence converges weakly to.

$$\frac{p-z}{||p-z||}.$$

Since X is locally uniformly convex, it follows that this sequence also converges in norm. Therefore:

$$\lim_{k\to\infty} t_k s_k(p) - t_k s_k(z)) = p - z \text{ and } \lim_{k\to\infty} t_k s_k(z) = z.$$

□

Lemma 5.3.4. *Let X be a reflexive locally uniformly convex Banach space and S an ultimately non-expansive right reversible semigroup of continuous mappings of X into itself. Suppose $x \in X$ is left-recurrent under S and x has the approximate identity property for the orbit. Let $u_1, u_2 \in Orb(x)$. Then the restriction of each member $s \in S$ to the line segment $[u_1, u_2]$ is an affine isometry.*

Proof. Let z_1, z_2 be points on the line segment $[u_1, u_2]$. Since all isometries in a strictly convex Banach space are affine, we only need to prove that $||s \cdot z_1 -$

$s \cdot z_2|| = ||z_1 - z_2||$. By lemma 5.3.1 there exists a sequence s_k in S such that

$$||ts_k s(u) - ts_k s(v)|| \leq \Big(1 + \frac{1}{k}\Big)||u - v||,$$

$$||ts_k s(u) - ts_k s(v)|| \leq \Big(1 + \frac{1}{k}\Big)||s(u) - s(v)||,$$

for all u, v in the set $\{u_1, u_2, z_1, z_2, s(u_1), s(u_2), s(z_1), s(z_2)\}$. Now, since x is left-recurrent, we can construct a sequence t_k such that $\lim_{k\to\infty} t_k s_k s(x) = x$. Now, since $u_i \in Orb(x)$, we have, using the approximate identity property for the orbit, $\lim_{k\to\infty} t_k s_k s(u_i) = u_i$. Using lemma 5.3.3, it follows $\lim_{k\to\infty} t_k s_k s(z_i) = z_i$. Now, let $t'_k = st_k$, then we have

$$\lim_{k\to\infty} t'_k s_k s(z_i) = \lim_{k\to\infty} st_k s_k s(z_i) = s \lim_{k\to\infty} t_k s_k s(z_i) = s(z_i),$$

and therefore by lemma 5.3.2, it follows that $||sz_1 - sz_2|| = ||z_1 - z_2||$. Therefore s is affine on the line segment $[u_1, u_2]$. □

Lemma 5.3.5. *Let X be a reflexive locally uniformly convex Banach space and S an ultimately non-expansive right reversible semigroup of continuous mappings of X into itself. If $x \in X$ is left-recurrent under S, and x has the approximate identity property for the orbit, then the restriction of each $s \in S$ to $\overline{co}(Orb(x))$ is an affine isometry.*

Proof. We do the proof by induction. Consider the set $C = co(u_1, u_2, ..., u_n)$, where each $u_i \in Orb(x)$. We want to show that the action of S on C is affine. If $n = 2$, then this is true by lemma 5.3.4. Suppose it is true for any $m < n$. Let z_1, z_2 be any two elements of C, and let p_1, p_2 be the extreme point of the line segment defined by the intersection of the line passing by z_1, z_2, and the set C. Notice that p_1 and p_2 are on a facet of the polytope $co(u_1, u_2, ..., u_n)$, therefore p_1, p_2 are affine combinations of at most $n - 1$ elements of $\{u_1, u_2, ..., u_n\}$.

Now, let s_k be a sequence such that

$$||ts_k s(u) - ts_k s(v)|| \leq (1 + 1/k)||u - v||,$$

$$||ts_k s(u) - ts_k s(v)|| \leq (1 + 1/k)||s(u) - s(v)||,$$

for all $u, v \in \{p_1, p_2, z_1, z_2\}$ and all $t \in S$. Now since x is left-recurrent, for each k there exists t_k such that

$$d(t_k s_k s(x), x) < \frac{1}{k}.$$

Therefore $\lim_{k\to\infty} t_k s_k s(x) = x$. Now since x has the approximate identity property for the orbit, and each u_i are in $Orb(x)$ it follows that

$$\lim_{k\to\infty} t_k s_k s(u_i) = u_i.$$

Since p_i is the convex hull of $n-1$ of the u_i, say, $p_i = \sum_{i=1}^{n-1} \lambda_i u_i$, with $\sum_{i=1}^{n-1} \lambda_i = 1$, then we have

$$\lim_{k\to\infty} t_k s_k s(p_i) = \lim_{k\to\infty} t_k s_k s\Big(\sum_{i=1}^{n-1} \lambda_i u_i\Big) = \sum_{i=1}^{n-1} \lambda_i \lim_{k\to\infty} (t_k s_k s u_i) = \sum_{i=1}^{n-1} \lambda_i u_i = p_i.$$

Using similar construction, we can find a sequence t'_k such that

$$\lim_{k\to\infty} t'_k s_k s(p_i) = s(p_i)$$

and therefore applying lemma 5.3.3 we get that $\lim_{k\to\infty} t'_k s_k s(z_i) = s(z_i)$, which finally gives us, by lemma 5.3.2, that $||s(z_1) - s(z_2)|| = ||z_1 - z_2||$. Since the same holds for any $z_1, z_2 \in co(u_1, u_2, ..., u_n)$, it follows that S acts on $co(Orb(x))$ isometrically, and since all isometry in a strictly convex Banach space are affine, the action of S on $co(Orb(x))$ is affine. Finally, by continuity we get the result. $\square$

5.4 Main theorem

In order to complete the proof of our main theorem, we need to apply the Ryll-Nardzewski fixed point theorem which states that

If X is a locally convex space, K is a weakly compact convex subset of K, and S is a noncontracting semigroup of weakly continuous affine maps of K into K, then there is a fixed point x_0 in K such that $s(x_0) = x_0$ for every $s \in S$. Here noncontracting means that for any pair of distinct point $x, y \in K$:

$$0 \notin \overline{\{s(x) - s(y) : s \in S\}}.$$

Theorem 5.4.1. *Let X be a reflexive locally uniformly convex Banach space and S an ultimately nonexpansive right reversible semigroup of continuous self-maps of X. If an $x \in X$ exists such that $S(x)$ is bounded, x is a left-recurrent point under S and x has the approximate identity property for the orbit, then $\overline{co}(Orb(x))$ contains a point ξ such that $S(\xi) = \{\xi\}$.*

Proof. We know that the action of each $s \in S$ on $\overline{co}(Orb(x))$ is an affine isometry by lemma 5.3.5. So first we need to show that the action is non-contracting. Let s_α be a net in S, then for any $x, y \in X$, we have that $||s_\alpha(x) - s_\alpha(y)|| = ||x - y||$, and therefore $s_\alpha(x) - s_\alpha(y)$ can not converge to 0, which means it is noncontracting. Next, we need to show that $Orb(x)$ is weakly compact. Since $Orb(x)$ is bounded, $\overline{co}(Orb(x))$ is also bounded; therefore it is contained in a closed ball, which is w*-compact by Alaoglu's theorem. It follows that $\overline{co}(Orb(x))$ is w^*-compact. Now since X is reflexive, the weak and weak* topologies coincides, which means that $\overline{co}(Orb(x))$ is weakly compact. We can therefore apply the Ryll-Nardzewski fixed point theorem to give us a $\xi \in \overline{co}(Orb(x))$ such that $s(\xi) = \xi$ for all $s \in S$. □

Corollary 5.4.2. *Let X be a compact subset of a reflexive locally uniformly convex Banach space and S an ultimately non-expansive right reversible strongly left amenable semigroup of continuous self-maps of X. If whenever $s_\alpha x \to x$ implies that $s_\alpha tx \to tx$ for all $t \in S$, there is a point $\xi \in X$ such that $S(\xi) = \{\xi\}$.*

Proof. This is a direct consequence of theorem 5.4.1 and example 5.2.5. □

For the last result of this section, we want to make use of the fact that the action of S on $\overline{co}(Orb(x))$, as defined in the previous theorem, is an isometry to prove that it is actually a bijective map on $\overline{Orb(x)}$. This is based on a similar result by Kiang (See [33]). But in order to prove our result, we will need a few more assumptions on S.

Theorem 5.4.3. *Let X be a reflexive locally uniformly convex Banach space and S an ultimately non-expansive left and right reversible semigroup of continuous self-maps of X. If an $x \in X$ exists such that $Orb(x)$ is bounded, x*

is a left and right-recurrent point under S and x has the approximate identity property for the orbit, then the action of S on $\overline{Orb(x)}$ is a bijective isometry.

Proof. By lemma 5.3.5, we know the action of S is isometric. To see that it is injective, let $y, z \in \overline{Orb(x)}$ and $s \in S$ be such that $s(y) = s(z)$. Then $||y - z|| = ||s(y) - s(z)|| = 0$, and therefore $y = z$. Finally, to show that it is surjective, let $s \in S$ and $y \in \overline{Orb(x)}$. We want to find $z \in \overline{Orb(x)}$ such that $s(z) = y$. Since $y \in \overline{Orb(x)}$, there exists a sequence s_n in S such that

$$||s_n x - y|| < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Now, by left reversibility there are sequences $\{a_n\}$ and $\{b_n\}$ in S such that $ss_n a_n = s_n s b_n$ for all $n \in \mathbb{N}$. Also, by right-recurrence, we have a sequence $\{t_n\}$ in S such that

$$||sb_n t_n x - x|| \leq \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

It follows that

$$\begin{aligned} ||ss_n a_n t_n x - y|| &\leq ||ss_n a_n t_n x - s_n x|| + ||s_n x - y|| \\ &= ||s_n s b_n t_n x - s_n x|| + ||s_n x - y|| \\ &= ||sb_n t_n x - x|| + ||s_n x - y|| \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}. \end{aligned}$$

It follows that $\{ss_n a_n t_n x\}$ converges to y. We want to show that $\{s_n a_n t_n x\}$ is also convergent.

$$\begin{aligned} ||s_n a_n t_n x - s_m a_m t_m x|| &= ||ss_n a_n t_n x - ss_m a_m t_m x|| \\ &\leq ||ss_n a_n t_n x - y|| + ||y - ss_m a_m t_m x|| \to_{m,n\to\infty} 0. \end{aligned}$$

Therefore, $\{s_n a_n t_n(x)\}$ is a Cauchy sequence. Since X is complete, $\{s_n a_n t_n x\}$ converges to some $z \in \overline{Orb(x)}$. Now, since $\{s_n a_n t_n x\} \to z$ and $\{ss_n a_n t_n x\} \to y$, we conclude that $s(z) = y$, which proves the surjectivity.

□

Chapter 6

Conclusion

6.1 Summary

In this section, we give a short overview of the work that has been accomplished in this thesis, and we provide the reader with a diagram showing some of the main implications and equivalences that were either already known, or proven in the text.

In the third chapter, we showed that all extremely amenable or n-extremely amenable semigroups are strongly amenable, and all strongly amenable semigroups are amenable. We also showed that strong amenability in the case of compact semigroups is the same as amenability, and in the case of discrete semigroups is the same as n-extreme amenability. We also characterize strong amenability for other algebras than $LUC(S)$. In particular, for $WAP(S)$ and $AP(S)$, we showed that strong amenability is the same as amenability. In this chapter, we also showed how to construct strong amenable semigroups using products and homomorphisms, and provided the reader with many examples of topological groups which are strongly amenable.

In the fourth chapter, we investigated fixed point properties related to strong amenability. In particular, we characterized strongly amenable semigroups in terms of the existence of a fixed compact set when the semigroup acts on a compact set. We also obtained some fixed point properties related to non-expansive mappings when $CB(S)$ has a property similar to strong amenability.

Finally, in chapter 5, we extended a result of Kiang and Edelstein related to ultimately non-expansive mappings, and showed that one of the conditions is always satisfied when the semigroup is indeed strongly amenable.

To conclude this section, we summarize many of the important properties related to strong amenability in the diagram found on the following page. In the diagram, S denotes a semi-topological semigroup, and the arrows indicate an implication. Also, this is a list the abbreviations used in the diagram:

Abreviation	**Full name**
AP	almost periodic
ELA	extremely left amenable
LA	left amenable
LR	left reversible
LUC	left uniformly continuous
n-ELA	n-extremely left amenable
SLA	strongly left amenable
WAP	weakly almost periodic

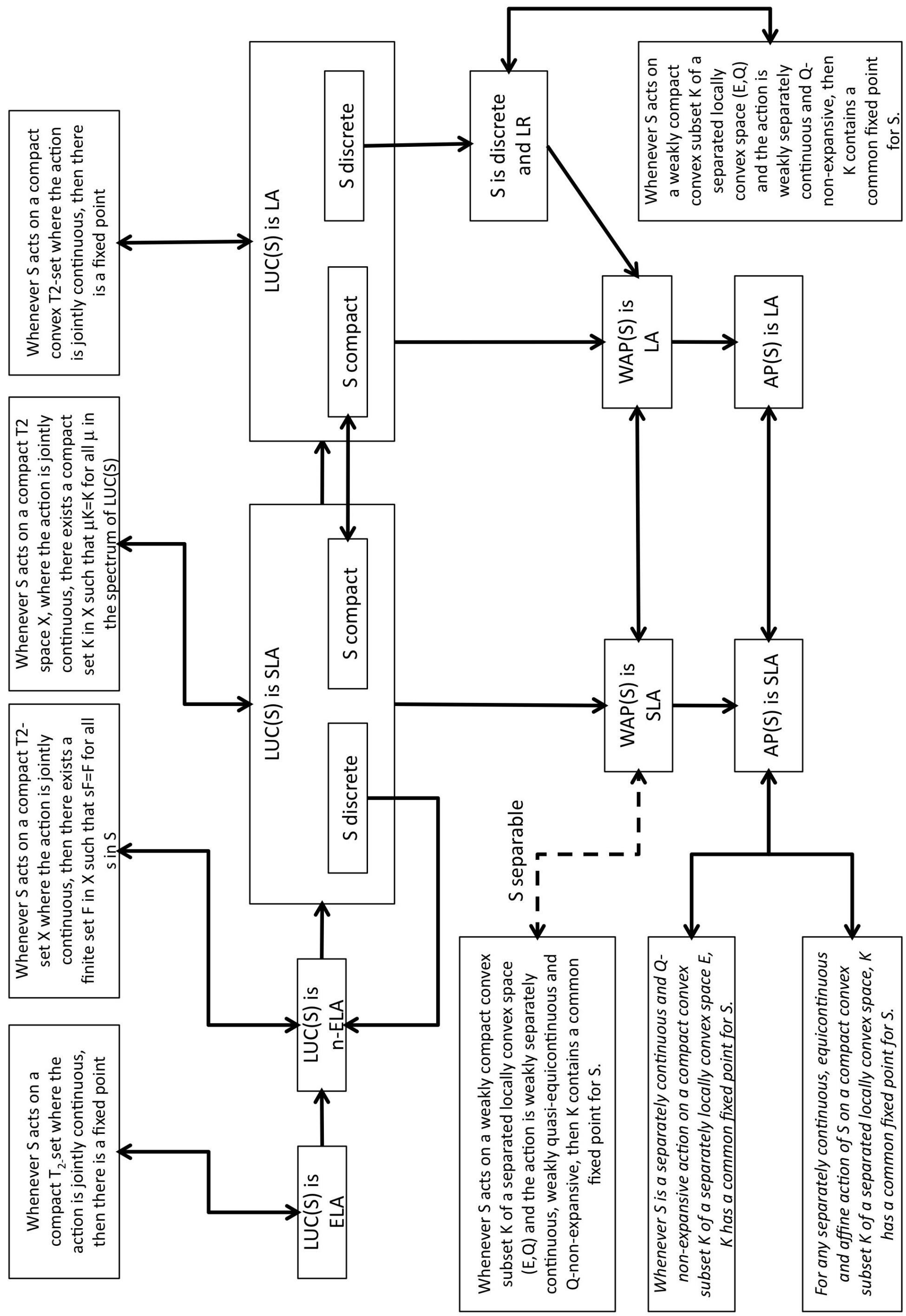

Figure 6.1: Diagram showing some of the main implications related to strongly amenable semigroups

6.2 Future work

To complete this thesis, we provide the reader with a few problems that remain open at this point. Those are problems that we either could not solve or did not have time to investigate enough. The two main questions that remain to be solved related to the work in this thesis are the following:

Question 1. Many examples of strongly left amenable groups are known, but all of them are of the form of the product of an extremely amenable group by a compact group. Would it be possible to find other examples of such groups?

Question 2. We proved that a semi-topological semigroup is strongly amenable if and only if whenever S acts on a compact Hausdorff space X, where the action is jointly continuous, there exists a compact set $K \subseteq X$ such that $\mu \cdot K' = K'$ for all $\mu \in \Delta(S)$, where $\mu \cdot K'$ denotes the extension of the action to an (right continuous) action of $\Delta(S)$ on X. Would it possible to find a characterization of strong amenability in terms of a fixed compact set without relying on the extension of the action ? Or even better, in terms of an actual fixed point, and not just a fixed compact set ?

There are also many other questions that remain open, and all of them would be interesting to investigate. Here are a few examples:

Question 3. Are the theorems in section 4.3 still valid if $CB(S)$ is replaced by $LUC(S)$?

Question 4. Can we define the Fourier or the Fourier-Stiejes algebra for a strongly left amenable semigroup ? This was done by Lau and Loy (See [40]) in the case S is a compact right-topological group.

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