**Proceedings of the Canadian Society for Mechanical Engineering International Congress 2022**
**CSME Congress 2022**
**June 5-8, 2022, Edmonton, AB, Canada**

# STATE-BASED PERIDYNAMICS MODELING APPROACH FOR COMPUTING CRACK-TIP ELASTIC STRESS FIELDS

Ali Khoshrou, Ayhan Ince*
Department of Mechanical, Industrial, and Aerospace Engineering, Concordia University, Montreal, Canada
*ayhan.ince@concordia.ca

***Abstract***— Crack-tip stress evaluation requires special treatments in the frame of classical elasticity theory for characterizing fatigue crack growth behavior. In previous studies, modified finite element methods have been used to remedy infinite stresses predicted by the classical elasticity theory at the tip of a crack. Peridynamics, as a recently developed non-local theory, has proved to be a powerful mathematical modeling approach to overcome the limitations of continuum mechanics e.g. analyzing discontinuous problems such as cracks without utilizing special treatments/functions or additional crack behavior criteria. In this study, a state-based peridynamics modeling approach is developed to model and simulate displacement and stress fields around a hole and crack shape geometry in a steel plate structure. The peridynamics model is first used to simulate displacement and stress fields for a central hole in a plate under the uniform tension. Furthermore, it is also shown that the peridynamics approach provides high accuracies in modeling displacement and stress fields for sharp elliptical-shaped crack geometry.

***Keywords-peridynamics; finite element method; stress field; crack growth***

## I. INTRODUCTION

The stress field near a crack tip is considered as a controlling driving force parameter in determining crack propagation behavior in fracture mechanics. Traditionally, finite element method (FEM)-based techniques have been implemented to numerically obtain stress fields in engineering components. However, FEM fails short in capturing the stress values at discontinuities such as a crack since the FEM solution includes spatial derivatives of displacement term and therefore its solution cannot be defined at the discontinuities. Therefore, special treatments need to be implemented, such as extended finite element method (xFEM), adding cohesive zone finite elements, elements deletions and other similar methods. Moreover, in mesh-based FEM, remeshing is required per incremental geometrical changes caused by the crack growth, leading to an increase in computation time.

Recently, a peridynamics theory is introduced based on a reformulation of continuum mechanics to overcome the limitation of continuum mechanics [1]. The peridynamics has shown to be successful to unify, and incorporate the mathematical modelling of continuous media, discontinuities, cracks, and particle mechanics within a single framework. Peridynamics uses a nonlocal model of force interactions, independent of deformation gradient, to determine displacement solution. By removing spatial derivatives of displacement and stress fields, peridynamics turns partial differentiation equations of classical continuum mechanics into integro-differential equations. The solution of peridynamics equations is valid to any material point inside a body regardless of discontinuities such as a crack geometry [1-5]. In peridynamic theory, all equations are derived based on the assumption that deformation at a single material point is caused by the summation of force interactions in a finite distance around the given material point. This distance is called the "horizon" and defines unique "neighbors" (i.e. force sources) for each points of a media. The concept of horizons and material points are visualized in Fig. 1. Fig.1 shows that the neighbors are defined at a finite distance $\delta$. The neighbors of a material point all contribute to the deformation at the central material point. Peridynamic studies have mainly focused on deformation and crack propagation of material bodies [6-12]. Madenci [7] illustrated the use of peridynamics in modelling the deformation of two rigid bodies in an impact event. Peridynamics works have been also carried out in modelling fatigue crack initiation and crack propagation [8-10]. Prakash [11] used a linear elastic constitutive model for bond-based peridynamics to model a plate with a center hole under a prescribed uniaxial displacement rate.

In literature, peridynamics approach is often not utilized to model stress fields, because the quantity of stress is a second derivate variable with respect to displacement if the stress formulation is derived based on the continuum mechanics. An original approach was to use continuum mechanics principles such as green strain matrix and tensor theories to derive the stress tensor [12]. The results while agreeable, are not representative of the peridynamics capabilities, as it defaults to continuum mechanics principles to derive the stresses. Fallah [13] has modeled a stress tensor of a plate with a central hole using bond-based peridynamics approach.

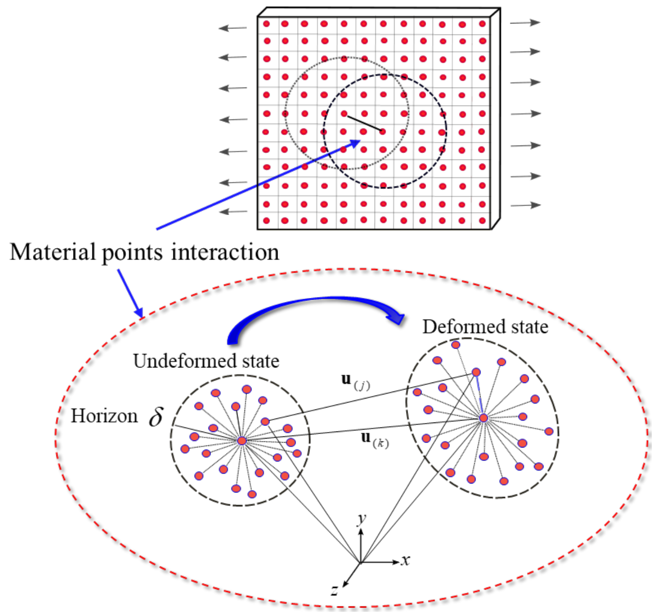



Figure 1. Material point interactions for undeformed and deformed states

Few studies focused on simulation of stress fields around a crack-shaped discontinues, since peridynamics stress modelling is not well understood. This paper focuses on using state-based peridynamics formulation to model displacement and stress fields for a steel plate with a central hole and then to simulate displacement and stress responses of the plate with crack-like geometries. The study is aimed to assess prediction accuracy of peridynamics modeling approach for obtaining both displacement and stress fields induced by different geometric discontinues including crack-like geometries. Model prediction results are verified with FEM data for uniaxial loading conditions. The developed peridynamic model can be applied to accurately assess stress states in different structures containing stress concentration, and crack-shape geometric discontinuities.

## II. Peridynamics Methodology

The peridynamics theory is based on the concept that the material consists of certain material points and deformation state is formulated based on the solution of the integro-differential equation of peridynamics governing equation in Eq. (1). In this equation, $\mathbf{u(x,t)}$ and $\ddot{\mathbf{u}}\mathbf{(x,t)}$ are displacement and acceleration for each material point at the time t, respectively. $\mathbf{b(x,t)}$ is the external body force, $\boldsymbol{\rho}\mathbf{(x)}$ is mass density. $\mathbf{t}$ denotes the force density that the material point at $\mathbf{x'}$ exerts on the material point at $\mathbf{x}$ and, $\mathbf{t'}$ represents the force density that the material point at $\mathbf{x}$ exerts on the material point at $\mathbf{x'}$. dH is an incremental volume of the material point.

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_H (\mathbf{t}(\mathbf{u}'-\mathbf{u},\mathbf{x}'-\mathbf{x},t) - \mathbf{t}'(\mathbf{u}-\mathbf{u}',\mathbf{x}-\mathbf{x}',t))dH + \mathbf{b}(\mathbf{x},t) \quad (1)$$

A plate specimen two different discontinuity configurations is used to determine displacement and stress responses computed by the peridynamics and FEM. TABLE I shows the material and geometric properties of the modeled plate specimen.

TABLE I. Plate and Central Hole Geometries and Attributes

| | | |
|---|---|---|
| **Plate Attributes** | E = 200GPa | ν = 0.3 |
| **Plate Size** | Width = 50cm | Length = 50cm |
| **Circular Hole** | Radius = 5cm | |
| **Crack-shape Discontinuity** | Major Radius = 5cm | Minor Radius = 300μm |
| **Applied Pressure** | P = 5GPa | Direction = y-axis |

The plate specimen is a 2D steel plate with the width and length of 50cm. The problem is assumed to be plane stress state and the material has elastic behavior to avoid complications related to 3D stresses and plastic deformation.

Two different discontinuity geometries are generated, one with a circular hole ($K_t = 3$) to and another with a crack-shape central crack geometry ($K_t = 150$). The study is aimed to analyze the accuracy of the peridynamic model for the whole material body and specifically in high stress areas. The dimensions of the crack-shape geometry are chosen in such a way so that the crack tip induces a high stress concentration factor of $K_t = 150$, for a representative crack geometry. Different number of material points are tested for each plate configurations to determine the optimum material grid points that yields accurate displacement and stress fields while maintaining computational efficiency. The predicted results are then compared with FEM data.

The peridynamic model of the plate is implemented using in-house built MATLAB code to construct material points representing the plate as a whole body. If the plate that goes under deformation is denoted as Ψ, then for all points **x** Є Ψ, horizon is defined as

$$H(\mathbf{x}) = \{x' \in \Psi \mid 0 \le \|\mathbf{x}'-\mathbf{x}\| < \delta_x\} \subset \Psi \quad (2)$$

With $\delta_x$ as the horizon size for material point **x**. The force density **t** applied by **x'** on material point **x** is calculated as

$$\mathbf{t} = 2\delta\{ad\frac{\Lambda_{xx'}}{|\mathbf{x}-\mathbf{x}'|}\theta_x + bs_{xx'}\} \times \frac{\mathbf{y}-\mathbf{y}'}{|\mathbf{y}-\mathbf{y}'|} \quad (3)$$

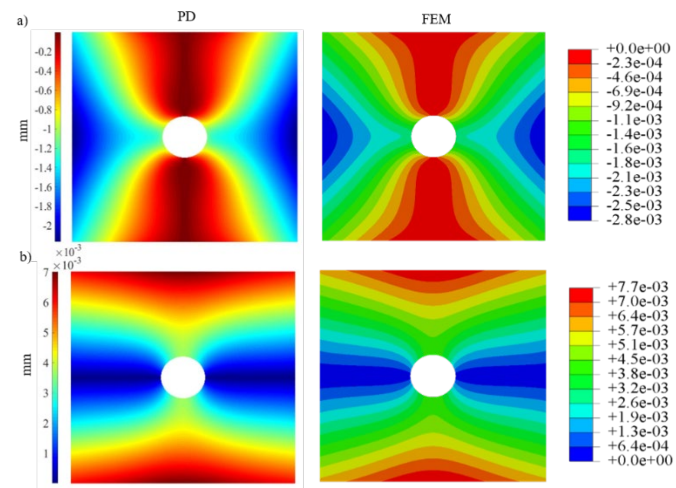
In which *a*, *d* and *b* are material constants, Λ is a geometrical parameter, $\theta_x$ is a dilatation parameter dependent on the strain energy density, and $s_{xx'}$ is the "stretch" of the bond between **x** and **x'** when deformed. Equations (2) and (3) are then substituted in Eq. (1) to find the displacement solution.

Numerical solution of peridynamic governing equations can be determined by using programing code e.g. MATLAB. By using explicit forwards and backwards difference techniques, a time integration of peridynamic equations can be derived and programmed [7]. Steady state solutions have been derived by using adaptive dynamic relaxation (ADR) technique since the given deformation problem is regarded as quasi-static [7].

Finally, the force densities are used to determine the stress field. By attributing the normal elements of the stress tensor to their respective cross sections of the material points, stress elements can be calculated by summing all the force densities applied by the neighbors that pass through the relative cross section. For details see [14].

## III. RESULTS AND DISCUSSION

The stress fields predicted by the peridynamic model are compared with FEM solutions for two plate configurations: a plate with a hole and a plate with a crack-shape discontinuity. Fig. 2 and Fig. 3 show the normal displacement and stress fields for the plate with the central hole. It can be seen that the values of stress and displacement components match very closely to results of FEM. The extremum values in both displacement and stress fields are identical in both models. As seen from Fig. 2 and Fig. 3, both displacement and stress responses predicted by both models are in good agreement for the complete material body including material points in the proximity of the hole.



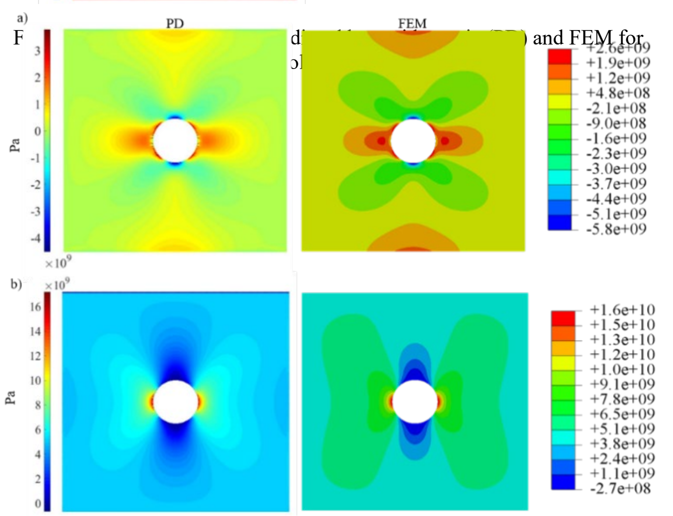




Figure 3. Stress field predicted by peridynamic (PD) and FEM for plate with a hole a) $\sigma_{xx}$ and b) $\sigma_{yy}$

Fig. 4 and Fig. 5 present displacement and stress fields computed by the peridynamics and FEM for the crack-shape central discontinuity, respectively. The total number of material points of 800 is used for the peridynamics model. The values of displacement closely match for all material points of the plate including at the vicinity of the crack tip in Fig. 4. As for the stress fields, the stress responses predicted by the peridynamic model match well with the FEM results far and near the crack tip as shown in Fig. 5. Fig. 5 shows also the deviation of the stress field predicted by the peridynamic model at the crack tip. In Fig. 6, the stress distribution ahead of the crack tip are graphed on the basis of the normalized distance from the crack tip to better assess the comparison of the peridynamic and FEM results. The stress values deviate at the crack tip but converge immediately near the crack tip. This is attributed to the fact that the nonlocality capability of the peridynamic solution unlike the FEM solution. Furthermore, solutions of both modeling techniques depend on mesh size and a number of material points used in areas with high stress concentrations.

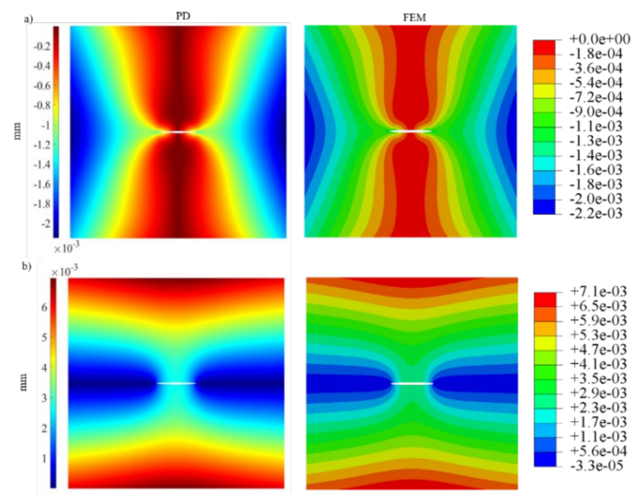



Figure 4. Displacement field predicted by peridynamic (PD) and FEM for plate with a crack-like geometry a) $u_{xx}$ and b) $u_{yy}$

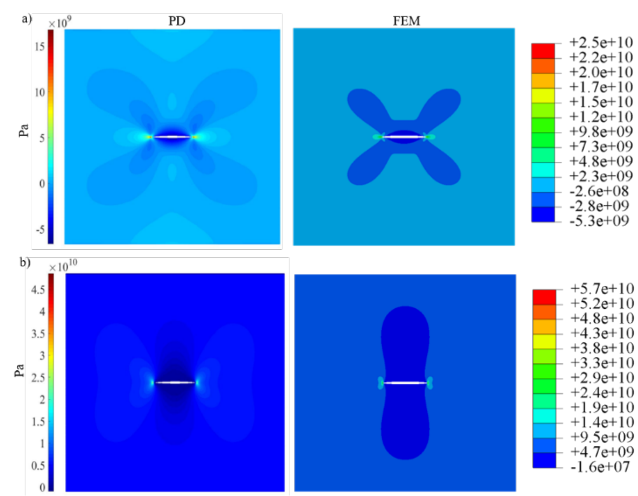



Figure 5. Stress field predicted by peridynamic (PD) and FEM for plate with a crack-like geometry a) $\sigma_{xx}$ and b) $\sigma_{yy}$

To capture accurate stress distribution in the close proximity of the crack tip, extremely fine meshes are required in FEM. Whereas for any peridynamic model developed, a greater number of material points per area need to be employed to capture accurate stress gradient in the close proximity of the crack tip. As the stress concentration at the crack tip increases, values predicted by the peridynamic model tend to deviate from the FEM solution at the crack tip. However, the peridynamic and the FEM solutions converge ahead of the crack tip. Since the FEM solution is based on the classical continuum theory in the view of the singularity of stress and strain fields at the crack tip, the peridynamic model predictions are considered to be more accurate at the close vicinity of the crack. Therefore, the peridynamic displacement and stress results can be used to for evaluation of crack propagation behavior e.g. derivation of the J-integral. To improve the computation time for the peridynamic model, a dual-horizon concept can be implemented to allow more material grid points near local discontinuities and lesser material points far away from discontinuities.

## IV. Conclusion

In this paper, a state-based peridynamic model is developed to simulate displacement and stress fields for two different plate configurations representing hole, and crack-shape geometric discontinuities i.e. low and high stress concentrations. The predicted results are compared with numerical FEM data. Peridynamic predicted results show that the peridynamic model is accurate in predicting displacement and stress responses at all material points of the plate except at the geometrical boundaries of the domain, including the plate edges and hole surface. The predicted results also show that in high stress concentration area(s), the peridynamic model deviates from FEM results. Even though the nonlocality nature of the peridynamics along with mesh size and a number of material points sensitivity of both models prevents an accurate comparison in these areas, the peridynamic model is considered to be more accurate in the close vicinity and the crack tip due to the inherent singularity problem of the FEM solution on the basis of the classical continuum theory. Therefore, the peridynamics modeling method can be used to derive components of stress tensor at and near the crack tip. Considering that crack growth behavior is dependent on the stress field near the crack tip. Such modeling capability offers a significant potential to predict accurate displacement and stress fields in the vicinity of crack tips for the fatigue crack growth assessment. Therefore, the developed peridynamic model can be applied to study crack growth behavior based on the J-integral values, without the need for any special treatments otherwise required by FEM techniques.


## Acknowledgment

The authors would like to acknowledge the financial support of Natural Sciences and Engineering Research Council of Canada (NSERC) (DGECR-2018-00232).

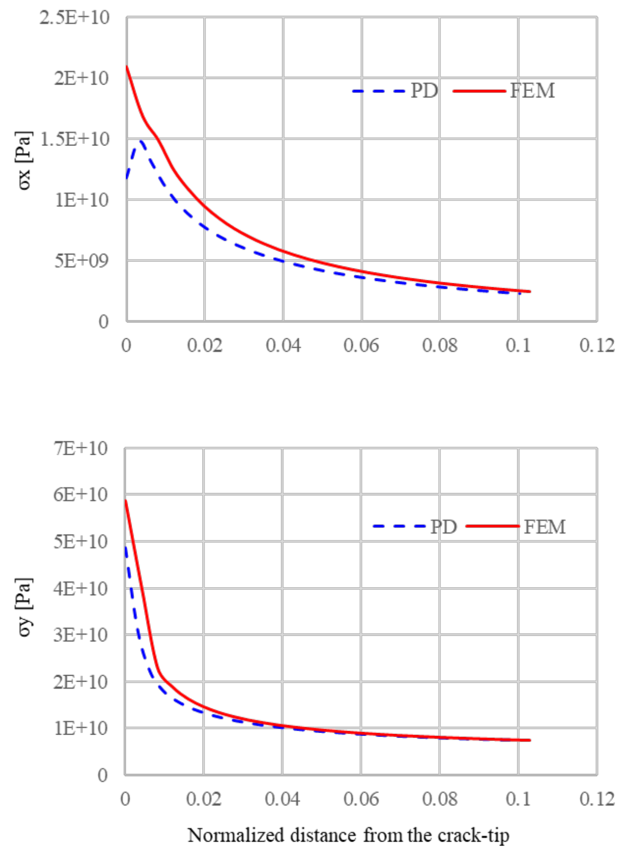




Figure 6. Near crack-tip stress field comparison for plate with a crack-like geometry

## V. References


[1] S. A. Silling, "Reformulation of elasticity theory for discontinuities and long-range forces," *Journal of the Mechanics and Physics of Solids,* vol. 48, no. 1, pp. 175-209, 2000.

[2] S. A. Silling and R. B. Lehoucq, "Peridynamic Theory of Solid Mechanics," *Advances in Applied Mechanics,* vol. 44, pp. 73-168, 2010.

[3] Y. D. Ha and F. Bobaru, "Studies of dynamic crack propagation and crack branching with peridynamics," *International Journal of Fracture,* vol. 162, pp. 229-244, 2010.

[4] R. W. Macek and S. A. Silling, "Peridynamics via finite element analysis," *Finite Elements in Analysis and Design,* vol. 43, no. 15, pp. 1169-1178, 2007.

[5] B. Kilic and E. Madenci, "Peridynamic Theory for Thermomechanical Analysis," *IEEE Transactions on Advanced Packaging,* vol. 33, no. 1, pp. 97-105, 2010.

[6] S. A. Silling and E. Askari, "A meshfree method based on the peridynamic model of solid mechanics," Computers & Structures, vol. 83, no. 17, pp. 1526-1535, 2005..

[7] E. Madenci and E. Oterkus, Peridynamic Theory and Its Applications, New York: Springer, 2014.

[8] S.A. Silling and A. Askari, “Peridynamic model for fatigue cracking”, *Tech. rep. no. SAND2014-18590, Sandia National Laboratories (SNL-NM)*, 2014..

[9] D.J. Bang, A. Ince, E. Oterkus, S. Oterkus, “Crack growth modeling and simulation of a peridynamic fatigue model based on numerical and analytical solution approaches”, *Theor. Appl. Fract. Mech*, vol. 114, pp.103026, 2021..

[10] D.J. Bang, A. Ince, “Integration of a peridynamic fatigue model with two-parameter crack driving force”, *Engineering with* Computers, 2022. https://doi.org/10.1007/s00366-022-01619-4.

[11] N. Prakash and G. D. Seidel, "A novel two-parameter linear elastic constitutive model for bond based peridynamics," 2015.

[12] X. W. Jiang and H. Wang, "Crack-tip stress evaluation of multi-scale Griffith crack subjected to tensile loading by using peridynamics," 2017.

[13] A. S. Fallah, I. N. Giannakeas, R. Mella, M. R. Wenman, Y. Safa, H, Bahai, "On the Computational Derivation of Bond-Based Peridynamic Stress Tensor," *Journal of Peridynamics and Nonlocal Modeling,* vol. 2, pp. 352-378, 2020.

[14] W. Liu and J. W. Hong, "Discretized peridynamics for brittle and ductile solids," *International Journal for Numerical Methods in Engineering,* vol. 89, no. 8, pp. 1028-1046, 2012.