University of Alberta

EXACT SOLUTION OF CYCLIC ELASTO-PLASTIC RESPONSE OF COMPOSITE SPHERES AND CYLINDERS.

By

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A thesis submitted to the Faculty of Graduate Studies And Research in partial fulfillment of the requirements for the degree of MASTERS OF SCIENCE

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Edmonton, Alberta

Spring, 1999

0-612-40023-9

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Title Of Thesis: Exact solution of cyclic elasto-plastic response of composite spheres and cylinders

Degree: MASTERS OF SCIENCE

Year This Degree Granted: 1999

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DEDICATION

This work is dedicated to my loving father (Mr. Yaw Yeboah Barimah) and mother (Madam Yaa Sefaah Mansah) by whose effort and care I have lived to complete this work.

ABSTRACT

This thesis deals with the determination of exact cyclic elastoplastic stress-strain relations for (a) an inclusion-matrix concentric sphere subjected to hydrostatic loading, and (b) a continuous fiber-matrix concentric cylinder subjected to plane strain, biaxial loading. Both phases are elastically isotropic and the inclusion is taken as elastically softer than the matrix. In addition, the matrix in the second problem is elastically incompressible. As well, the matrix is taken to be bilinear and isotropic hardening is assumed. Yielding is assumed to occur in the matrix by the vonMises' criterion. Using Hill's approach, the exact solution is first determined for a few alternate tensile and compressive loadings. Based on the developed equations, and using an inductive approach, the analytical relation for the overall stress and strain for the N^{th} loading sequence is *suggested*. In particular, the exact solution for the cylinder problem was validated by Finite Element Method. The Bauschinger effect for both composite geometries was then studied. It was seen that, in either case, the cyclic response is initially governed by isotropic hardening, whereas an asymptotic response is approached where both kinematic and isotropic mechanisms play equal roles.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. Abhijit Bhattacharyya, for the countless efforts and advice he offered me throughout the period of my study. I would like also to express my gratitude to my thesis committee members, Dr. C.Q. Ru, Dr. Z. Xia and Dr. D. Li, for their valuable suggestions. I thank my office mates, Sandra, Allen, Vesselin and Julian, for the various forms of assistance I received from them. Partial financial support from NSERC is acknowledged as also is the Mary Louise Imrie Graduate Student Award from the Faculty of Graduate Studies and Research , University of Alberta.

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$\boldsymbol{\sigma}^{(i)}$ stress tensors in the i^{th} phase

$\varepsilon_E^{(i)}$ elastic strain tensors in the i^{th} phase

$L_{jklm}^{(i)}$ fourth order elastic moduli tensor of the *i*th phase

κ_i bulk modulus of the i^{th} phase

μ_i shear modulus of the i^{th} phase

δ_{lm} kronecker delta

$tr(\boldsymbol{\sigma}^{(2)})$ trace of the stress tensor, $\boldsymbol{\sigma}^{(2)}$

$\varepsilon_{P,e}^{(2)}$ effective plastic strain in the matrix

σ_y yield stress

$\sigma_e^{(2)}$ effective stress

h tangent modulus

$\sigma_{rr}^{(i)}$ radial stress of the i^{th} phase

$u_r^{(i)}$ radial displacement of the i^{th} phase

$\sigma_{kk}^{(i)}$ local hydrostatic stress in the i^{th} phase

$\varepsilon_{kk}^{(i)}$ local hydrostatic strain in the i^{th} phase

a inclusion radius

b outer radius of composite sphere (and cylinder)

$\Delta\overline{\sigma}_{kk}^{(N)}$ additional stress during the N^{th} loading

$\Delta\overline{\varepsilon}_{kk}^{(N)}$ additional strain during the N^{th} loading

$R^{(N)}$ yield radius at N^{th} loading

E_i Young's modulus of the i^{th} phase

ν_i Poisson's ratio of the i^{th} phase

$\Delta u_r^{(i),(N)}$ additional displacement in the i^{th} during N^{th} loading

$\Delta\sigma_{\theta\theta}^{(i),(N)}$ additional hoop stress in the i^{th} phase at N^{th} loading

$\Delta\overline{\sigma}_{Y,kk}^{(N)}$ additional yeild stress at the N^{th} loading

E_P tangent modulus

α degree of isotropy

Chapter 1

INTRODUCTION

1.1 Literature review

Analytical approaches to the determination of the overall elastoplastic response of composites date back to Hill[1] when he addressed the issue of monotonic hydrostatic loading of a hollow sphere and cylinder, and determined the exact solution when a bilinear matrix undergoes plastic deformation. Since then, the aforementioned boundary value problems has served as an useful benchmark for either finite element computations or the determination of composite plasticity by effective medium theories. In particular, effective medium approaches seem to have yielded some success following the work of Berveiller and Zaoui[2]. Recognizing that the "constraint" power of a ductile material decreases with plastic deformation, they proposed the concept of secant moduli to characterize this weakening constraint under a monotonic proportional loading. Since then, Tandon and Weng[3] have used the concept of the secant moduli to propose an effective medium theory of particle-reinforced plasticity. Other related work which improve upon the originally suggested theory is due to Qiu and Weng[4] and Hu[5]. In the context of these developments, Hill's[1] solution has served as a benchmark against which the accuracy of the effective medium theories were assessed.

The current work stemmed from our interest in the application of effective medium approaches to model the effective(or overall) elastoplastic response of composites subjected to non-proportional and non-monotonic(or cyclic) loading(Lagoudas, Gavazzi and Nigam[6], Li and Chen[7]). The work of Lagoudas *et.al.*[6] deals with fiber-reinforced composites and compares the effective medium predictions with finite element calculations for the first 2~3 cycles of loading. Li and Chen[7] proposed an effective medium approach to the determination of non-proportional and cyclic loading of multiphase particulate composites(i.e.composites with spherical inclusions). In their work, they compare the predictions of their theory with experimental results for the special case of monotonic loading of a two-phase particulate composite. The question now is that, for the specific boundary value problem of a composite sphere subjected to cyclic hydrostatic loading, is it possible to compare the predictions of their model with an exact solution ? While one obvious approach is to implement finite element calculations for the aforementioned boundary value problem, another approach is to explore the possibility of *analytically* developing cyclic hydrostatic stress-strain relations for a composite sphere, that will be valid for not only the first 2~3 cycles of loading but for *any cycle* thereafter. An identical question may also be posed for the cyclic biaxial loading of a composite cylinder. A search of the open literature has failed to pinpoint any work along the aforementioned direction, probably due to the difficult nature of the problem.

In this work, we address two boundary value problems. In chapter 2, we address the analytical determination of cyclic stress-strain relations for a composite sphere subjected to hydrostatic loading at its boundary (see Fig.1). A spherical inclusion is taken to be concentrically embedded in a spherical matrix and perfectly bonded to it. The matrix is taken to be bilinear and assumed to undergo isotropic hardening. The yielding of the matrix is taken to follow vonMises' criterion. We now recall a well established fact(Qiu and Weng[4]) that the overall plastic deformation of a composite sphere subjected to monotonic hydrostatic loading becomes more pronounced as the inclusion becomes elastically softer than the matrix, and becomes significant when, in the limit, the inclusion becomes a void(a porous material). Therefore, we restrict the development of the cyclic hydrostatic stress-strain relations to a soft inclusion and a hard matrix. Based on the aforementioned assumptions, we first derive rigorously the exact stress-strain relations for the first five hydrostatic loading sequences(tensile and compressive loadings alternately) using Hill's approach[1]. In order to keep the problem tractable, we assume that during the load reversal of the composite sphere, the entire matrix is in a fully plastic state(the inclusion is always elastic). Examining the developed equations for the first five loading sequences, we use an inductive approach to suggest the equations corresponding to the *N*th loading sequence, where N is an integer such that $N \geq 1$.

In chapter 3, we address the analytical determination of cyclic biaxial stress-strain relations for a composite cylinder subjected to plane strain . The constitutive relations for the constituent phases are identical to the composite sphere problem, with additional restriction that the matrix is elastically incompressible. An inductive approach is again employed to determine the equations for the N^{th} loading. The predictions by the developed equations are then confirmed by comparing with finite element computations. In chapter 4, we give a summary of the work reported in the thesis, its advantages and limitations. This is followed by a discussion of the future extension of this work.

The developed equations for the N^{th} loading are used to study the influence of the inclusion volume fraction, the relative stiffness of the soft inclusion/hard matrix and the work-hardening of the matrix on the evolution of the cyclic stress-strain relations. Specifically, noting that the isotropic hardening character of the matrix does not necessarily imply isotropic hardening for the composite sphere or cylinder, we study the effect of the aforementioned factors on the evolution of the hardening (or the Bauschinger effect) during the cyclic stress-strain response of both composite geometries.

The scientific notation used in the thesis is briefly described. A second order tensor is denoted with a bold face, lower case Greek letter and a fourth-order tensor is denoted with a bold face, upper case Latin letter. The trace of a second order tensor, $\boldsymbol{\varepsilon}$, is denoted symbolically as $\mathrm{tr}(\boldsymbol{\varepsilon})$. In indicial notation, the trace of $\boldsymbol{\varepsilon}$ is ε_{kk}, where the Einstein summation convention has been used. The inner product of a fourth order tensor L with $\boldsymbol{\varepsilon}$ is a second order tensor and is symbolically denoted as $\mathbf{L}\boldsymbol{\varepsilon}$. On the other hand, the aforementioned second order tensor will have components, written in indicial notation as $L_{ijkl}\varepsilon_{kl}$.

1.2. Constitutive equations of the inclusion and the matrix

The inclusion (sphere or fiber) and the matrix are denoted as phase "1" and phase "2" respectively. The stress-strain relation is given by the Hooke's law as

$$\boldsymbol{\sigma}^{(i)} = \mathbf{L}^{(i)}\boldsymbol{\varepsilon}_E^{(i)} \qquad i = 1,2 \;,$$

1-1

where $\boldsymbol{\sigma}^{(i)}$ and $\boldsymbol{\varepsilon}_E^{(i)}$ are the stress and elastic strain tensors, respectively, in the i^{th} phase. The fourth order elastic moduli tensor of the ith phase is denoted as $\mathbf{L}^{(i)}$. Assuming both phases to be elastically isotropic, $\mathbf{L}^{(i)}$ is written explicitly as

$$L_{jklm}^{(i)} = \kappa_i \delta_{jk}\delta_{lm} + \mu_i(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl} - \frac{2}{3}\delta_{jk}\delta_{lm}) \;,$$

1-2

where κ_i and μ_i are the bulk and shear modulus, respectively, of the i^{th} phase.eq 1-1 and eq 1-2 may be written in a simpler form if the matrix is taken incompressible (necessary for the composite cylinder problem). The generalized Hooke's law for an elastically incompressible , isotropic material is given as

$$\boldsymbol{\sigma}^{(2)} = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^{(2)})\mathbf{I} + 2\mu_2\boldsymbol{\varepsilon}_E^{(2)} \;,$$

1-3

where $\mathrm{tr}(\boldsymbol{\sigma}^{(2)})$ is the trace of $\boldsymbol{\sigma}^{(2)}$, $\mathbf{I}$ is the second order identity tensor, μ_2 is the matrix shear modulus and $\boldsymbol{\varepsilon}_E^{(2)}$ is the deviatoric elastic strain tensor. We have invoked the condition of elastic incompressibility, $\mathrm{tr}(\boldsymbol{\varepsilon}^{(2)}) = 0$.

As we shall see later, the plastic deformation will only occur in the matrix. We shall assume that the effective stress controls the matrix plasticity, and is defined as

$$\sigma_e^{(2)} = \left(\frac{3}{2}\boldsymbol{\sigma}^{(2)\prime} : \boldsymbol{\sigma}^{(2)\prime}\right)^{1/2} \;,$$

in terms of the matrix deviatoric stress

$$\boldsymbol{\sigma}^{(2)'} = \boldsymbol{\sigma}^{(2)} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^{(2)})\,\mathbf{I} \ .$$

1-5

In 1-5, $\mathrm{tr}(\boldsymbol{\sigma}^{(2)})$ is defined as the trace of the stress tensor, $\boldsymbol{\sigma}^{(2)}$. When the matrix is loaded beyond the yield stress, the resulting plastic strain is denoted as $\boldsymbol{\varepsilon}_P^{(2)}$ and the corresponding effective plastic strain is defined as

$$\varepsilon_{P,e}^{(2)} = \left(\frac{2}{3}\boldsymbol{\varepsilon}_P^{(2)} : \boldsymbol{\varepsilon}_P^{(2)}\right)^{1/2} .$$

1-6

Assuming that the matrix is bilinear, the effective stress-effective plastic strain relation is taken as

$$\sigma_e^{(2)} = \sigma_y + h\,\varepsilon_{P,e}^{(2)} ,$$

1-7

where σ_y is the magnitude(or absolute value) of the yield stress and h is the strength coefficient respectively. The specific value of σ_y in a cyclic process will depend on the hardening character of the matrix. In this paper, the analysis is restricted to isotropic hardening. The corresponding expression for σ_y will be given in Sec.3. We emphasize that it is only the matrix that undergoes plastic deformation. In that context, the parameters σ_y and h are used without the superscript (2), with the understanding that these parameters pertain to the matrix only. The incremental plastic strains for a matrix with isotropic hardening is assumed to be given by the associated Prandtl-Reuss relation

$$d\,\boldsymbol{\varepsilon}_P^{(2)} = d\lambda\,\boldsymbol{\sigma}^{(2)'} \quad , \qquad\qquad d\lambda > 0 \quad , \tag{1-8}$$

where $d\lambda$ is a positive scalar parameter.

1.3 **References**

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Chapter 2

ANALYTICAL DETERMINATION OF CYCLIC HYDROSTATIC STRESS-STRAIN RELATIONS FOR A COMPOSITE SPHERE WITH A SOFT INCLUSION AND A HARD BILINEAR, ISOTROPICALLY HARDENING MATRIX

2.1 Boundary value problem for the Nth sequence of cyclic loading of the composite sphere and the solution procedure

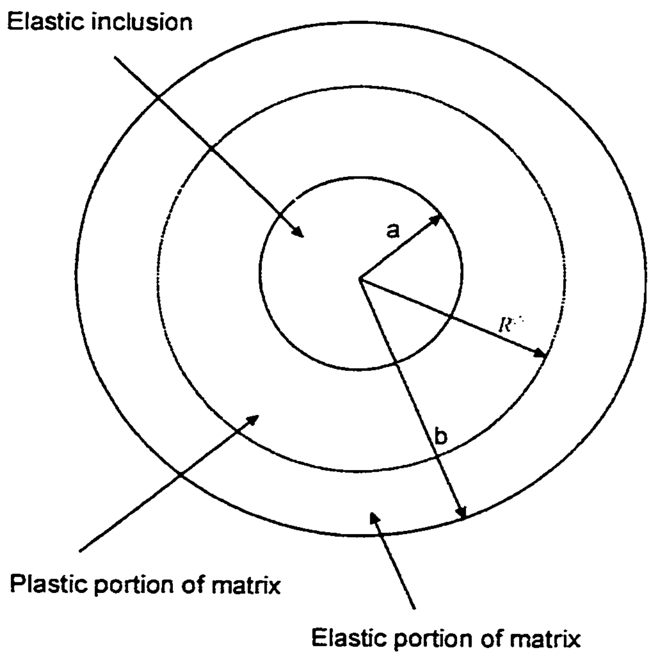

Figure 2.1: A schematic of the composite sphere with the elastic-plastic interface

We now outline the boundary value problem addressed in this chapter. A spherical inclusion of radius a is embedded concentrically in a spherical matrix of outer radius b, and perfectly bonded to it (Fig 2.1). The inclusion is elastic, whereas the matrix is taken to have a bilinear stress-strain relation and is assumed to undergo isotropic hardening. The composite sphere is subjected to either a uniform radial boundary traction, stated in a spherical co-ordinate system as

$$\sigma_{rr}^{(2)}(b) \neq 0 \quad , \quad \sigma_{r\theta}^{(2)}(b) = 0 \quad , \quad \sigma_{r\phi}^{(2)}(b) = 0 \ , \tag{2-1}$$

or a uniform radial boundary displacement

$$u_r^{(2)}(b) \neq 0 \quad , \quad u_\theta^{(2)}(b) = 0 \quad , \quad u_\phi^{(2)}(b) = 0 \ . \tag{2-2}$$

We define the composite volume average of the hydrostatic stress and strain as

$$\overline{\sigma}_{kk} = \frac{1}{\sum_{i=1}^{2} V^{(i)}} \left[\sum_{i=1}^{2} \int_{V^{(i)}} \sigma_{kk}^{(i)} \, dv \right] \qquad \text{and} \qquad \overline{\varepsilon}_{kk} = \frac{1}{\sum_{i=1}^{2} V^{(i)}} \left[\sum_{i=1}^{2} \int_{V^{(i)}} \varepsilon_{kk}^{(i)} \, dv \right] , \tag{2-3}$$

where $V^{(i)}$ is the volume, $\sigma_{kk}^{(i)}$ and $\varepsilon_{kk}^{(i)}$ are the local hydrostatic stress and strain in the *i*th phase. With these definitions, it may be shown that(Hill[1])

$$\sigma_{rr}^{(2)}(b) = \frac{1}{3} \overline{\sigma}_{kk} \qquad \text{and} \qquad u_r^{(2)}(b) = \frac{1}{3} b \, \overline{\varepsilon}_{kk} . \tag{2-4}$$

The issue is to determine the $\overline{\sigma}_{kk} - \overline{\varepsilon}_{kk}$ relation for the composite sphere when it is subjected to cyclic loading. The overall cyclic stress-strain relation will be viewed in the context of a sequence of tensile and compressive loadings. In Sec.2.2 to follow, we shall outline the boundary value problem corresponding to the N^{th} sequence of loading(the definition of the N^{th} sequence will be made precise in that section). The solution procedure to solve the boundary value problem is given in Sec.2.3.

2.2 Boundary value problem for the N^{th} sequence of cyclic loading

Hydrostatic loading from the virgin state of the composite sphere will be referred to as the 1^{st} sequence of cyclic loading. Reversal of loading from the 1^{st} sequence leads to the 2^{nd} sequence, and so on. The aim in this paper will be to determine the

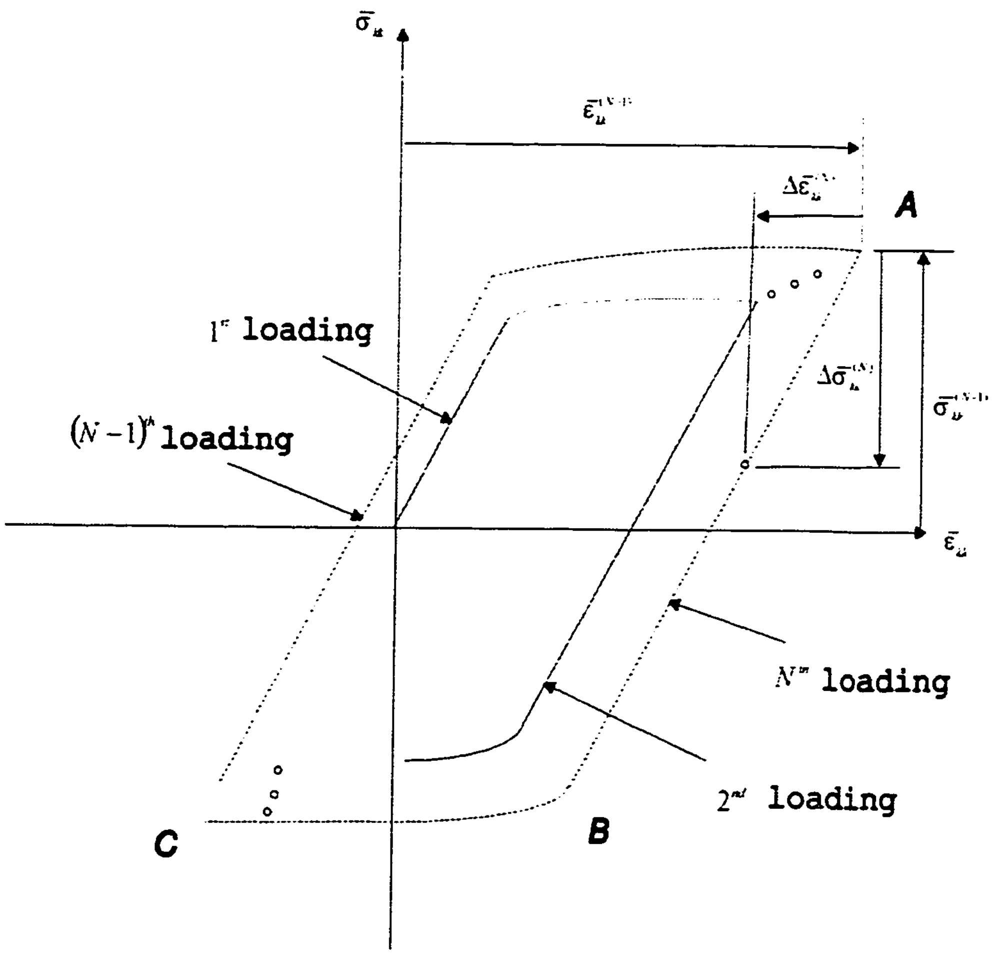

Figure 2.2: A schematic of the cyclic hydrostatic stress-strain curves for the composite sphere.

$\bar{\sigma}_{kk} - \bar{\varepsilon}_{kk}$ relation during the N^{th} sequence of cyclic loading (the curve ABC in Fig.2.2). The point A represents the onset of the N^{th} sequence, the point B corresponds to the yield point and the point C represents the end of the N^{th} loading sequence. The current value of $\bar{\sigma}_{kk}$ and $\bar{\varepsilon}_{kk}$ during the N^{th} loading is referred as $\bar{\sigma}_{kk}^{(N)}$ and $\bar{\varepsilon}_{kk}^{(N)}$ respectively. Further, we define the parameters $\Delta\bar{\sigma}_{kk}^{(N)}$ and $\Delta\bar{\varepsilon}_{kk}^{(N)}$ (see Fig.2.2) which represent the additional stress and strain(i.e. after the (N-1)th loading) during the N^{th} loading. These are added to the total hydrostatic stress and strain, $\bar{\sigma}_{kk}^{(N-1)}$ and $\bar{\varepsilon}_{kk}^{(N-1)}$ accured at the end of the $(N\text{-}1)^{th}$ loading sequence(the point A in Fig. 2.2) to get the total stress and strain in the N^{th} loading sequence. Therefore

$$\bar{\sigma}_{kk} \equiv \bar{\sigma}_{kk}^{(N)} = \bar{\sigma}_{kk}^{(N-1)} + \Delta\bar{\sigma}_{kk}^{(N)} \qquad \text{and} \qquad \bar{\varepsilon}_{kk} \equiv \bar{\varepsilon}_{kk}^{(N)} = \bar{\varepsilon}_{kk}^{(N-1)} + \Delta\bar{\varepsilon}_{kk}^{(N)}$$

2-5

All other quantities may be similarly decomposed. Restricting the theory to small deformations and deformation gradients, the condition imposed at the composite boundary follows from 2-4 as

$$\Delta\sigma_{rr}^{(2),(N)}(b) = \frac{1}{3}\Delta\bar{\sigma}_{kk}^{(N)} \qquad \text{or} \qquad \Delta u_r^{(2),(N)}(b) = \frac{1}{3} b\, \Delta\bar{\varepsilon}_{kk}^{(N)} \ .$$

2-6

$\Delta u_r^{(i),(N)}$ additional displacement in the i^{th} during N^{th} loading

If $\Delta\sigma_{rr}^{(2),(N)}(b)$ is prescribed, then $\Delta u_r^{(2),(N)}(b)$ needs to be determined(or vice-versa). The boundary conditions are interchangeable and therefore will lead to identical stress-strain relations.

When the additional stress during the N^{th} loading sequence is positive(negative), we shall refer to the N^{th} loading as being tensile(compressive). The initial state of the composite sphere, represented by $\bar{\varepsilon}_{kk}^{(0)}$ and $\bar{\sigma}_{kk}^{(0)}$, needs to be given. We shall assume that the constituent phases are stress- and strain-free. Therefore, for the composite sphere, we have

$$\bar{\sigma}_{kk}^{(0)} = 0 \qquad , \qquad \bar{\varepsilon}_{kk}^{(0)} = 0 \ .$$

2-7

We begin by identifying the current value of the radial deformation field, $u_r^{(i)}(r)$, in the ith phase during the N^{th} loading sequence as $u_r^{(i),(N)}(r)$. An identical approach for $u_\theta^{(i)}$ and $u_\phi^{(i)}$ is taken. Invoking the radial symmetry inherent in the problem, the displacement components are

$$u_r^{(i)}(r) \equiv u_r^{(i),(N)}(r) \quad , \quad u_\theta^{(i)} \equiv u_\theta^{(i),(N)} = 0 \quad , \quad u_\phi^{(i)} \equiv u_\phi^{(i),(N)} = 0 \quad , \quad (i = 1,2) \ .$$

2-8

Based on 2-8, the additional displacements during the N[th] loading may be written as

$$\Delta u_r^{(i),(N)} \equiv \Delta u_r^{(i),(N)}(r), \qquad \Delta u_\theta^{(i),(N)} = 0, \qquad \Delta u_\phi^{(i),(N)} = 0 \ .$$

2-9

The nonvanishing additional strain components are(Malvern[2])

$$\Delta\varepsilon_{rr}^{(i),(N)} = \frac{\partial \Delta u_r^{(i),(N)}}{\partial r} \quad , \quad \Delta\varepsilon_{\theta\theta}^{(i),(N)} = \Delta\varepsilon_{\phi\phi}^{(i),(N)} = \frac{\Delta u_r^{(i),(N)}}{r} \ .$$

2-10

In general, the additional net strains[1](2-10) may be decomposed into an elastic and a plastic part, i.e.

$$\Delta\varepsilon^{(i),(N)} = \Delta\varepsilon_E^{(i),(N)} + \Delta\varepsilon_P^{(i),(N)} \ .$$

2-11

The nonvanishing components of the additional elastic strains are simply r-dependent, and the $(\theta\theta)$ component is equal to the $(\phi\phi)$ component. This is stated as

$$\Delta\varepsilon_{E.rr}^{(i),(N)} \equiv \Delta\varepsilon_{E.rr}^{(i),(N)}(r) \quad , \quad \Delta\varepsilon_{E.\theta\theta}^{(i),(N)} \equiv \Delta\varepsilon_{E.\theta\theta}^{(i),(N)}(r) = \varepsilon_{E.\phi\phi}^{(i),(N)} \ .$$

2-12

[1] The term "net strain" has been used here to indicate the sum of elastic and plastic strains at a continuum point. Contrast this with the term "total strain"(refer eq **2-5**) used to indicate the accumulated strain after N loading sequences.

The validity of eq 2-12 will be demonstrated in Sec. 2.3.2. Based on eq 1-1 and eq 2-12 the additional non-vanishing stress components are also r-dependent.These stress components are

$$\Delta\sigma_{rr}^{(i),(N)} = \left(\kappa_i + \frac{4}{3}\mu_i\right)\Delta\varepsilon_{E,rr}^{(i),(N)} + 2\left(\kappa_i - \frac{2}{3}\mu_i\right)\Delta\varepsilon_{E,\theta\theta}^{(i),(N)}$$

$$\Delta\sigma_{\theta\theta}^{(i),(N)} = \left(\kappa_i - \frac{2}{3}\mu_i\right)\Delta\varepsilon_{E,rr}^{(i),(N)} + 2\left(\kappa_i + \frac{1}{3}\mu_i\right)\Delta\varepsilon_{E,\theta\theta}^{(i),(N)} \quad .$$

$$\Delta\sigma_{\infty}^{(i),(N)} = \Delta\sigma_{\theta\theta}^{(i),(N)}$$

2-13

The sole non-trivial component of the equilibrium equation is written in terms of the additional stresses as(Malvern[2])

$$\frac{d\,\Delta\sigma_{rr}^{(i),(N)}}{dr} + \frac{2}{r}\left(\Delta\sigma_{rr}^{(i),(N)} - \Delta\sigma_{\theta\theta}^{(i),(N)}\right) = 0 \ .$$

2-14

The inclusion is assumed to be perfectly bonded to the matrix. Thus, the conditions at the inclusion-matrix interface(r = a) are written in terms of the additional displacements and stresses as

$$\Delta u_r^{(1),(N)}(a) = \Delta u_r^{(2),(N)}(a) \quad \text{and} \quad \Delta\sigma_{rr}^{(1),(N)}(a) = \Delta\sigma_{rr}^{(2),(N)}(a) \ .$$

2-15

2.3 Solution procedure for the Nth loading

Since the traction boundary condition and the displacement boundary condition are interchangeable, in this section, we give the solution procedure for the Nth loading with a traction-prescribed condition. During the Nth sequence, as $\Delta\bar{\sigma}_{kk}^{(N)}$ is increased(decreased) from zero during tensile loading(compressive loading), the overall composite response will be initially elastic. The commencement of yielding in the matrix will then occur and the matrix will be in an elastoplastic state. This will continue until the matrix has yielded in its entirety, and further loading will continue with the matrix in the fully plastic state. The solution procedure for $\Delta\overline{\sigma}_{kk}^{(N)}$ and $\Delta\overline{\varepsilon}_{kk}^{(N)}$ are now given for these three stages of deformation. The initial values of the displacement, strain and stress fields at the onset of the N^{th} loading will be given by $\mathbf{u}^{(i),(N-1)}$, $\boldsymbol{\varepsilon}^{(i),(N-1)}$ and $\boldsymbol{\sigma}^{(i),(N-1)}$; all these will be assumed to be known quantities.

2.3.1 *Elastic state of the composite sphere and the commencement of yielding*

The composite sphere is in an elastic state when both inclusion and matrix are in an elastic state. Solution of the elastic state in the inclusion and matrix may be obtained by combining eqs 2-9 to 2-11, eq 2-13 and eq 2-14 while setting the additional plastic strain in eq 2-11 to zero. This process results in a differential equation in $\Delta u_r^{(i),(N)}(r)$, the solution of which leads to

$$\Delta u_r^{(i),(N)}(r) = A_1^{(i),(N)} r + A_2^{(i),(N)} \frac{b^3}{r^2}, \qquad (i = 1,2).$$

2-16

The four constants $A_1^{(i),(N)}$ and $A_2^{(i),(N)}$ are determined using the interface conditions, 2-15., the boundary condition(the first of eq 2-6), and an additional condition where we require $\Delta u_r^{(1),(N)}(r)$ (i.e. the additional displacement in the inclusion during the N^{th} loading) to be finite as $r \to 0$. The constants turn out to be

$$A_1^{(1),(N)} = \frac{3\kappa_2 + 4\mu_2}{9[4\mu_2\kappa_2(1-c_1) + \kappa_1(3\kappa_2 + 4c_1\mu_2)]} \Delta\overline{\sigma}_{kk}^{(N)} \quad , \quad A_2^{(1),(N)} = 0$$

$$A_1^{(2),(N)} = \frac{3\kappa_1 + 4\mu_2}{3\kappa_2 + 4\mu_2} A_1^{(1),(N)} \quad , \quad A_2^{(2),(N)} = \frac{c_1(\kappa_2 - \kappa_1)}{3\kappa_2 + 4\mu_2} A_1^{(1),(N)}$$

2-17

With the constants (and thereby $\Delta u_r^{(i),(N)}(r)$) determined, the additional strains follow from eq 2-10 and the additional stresses from eq 2-13. Notice from the second of 2-17 and eq 2-16that the additional displacement in the inclusion is a linear function of *r*, i.e. $\Delta u_r^{(1),(N)}(r) = A_1^{(1),(N)} r$. This outcome, along with eq 2-10 and eq 2-13, means that the additional strain and stress field, in the inclusion are purely hydrostatic. Since the conclusion is valid for any $N \geq 1$, it means that the total stress field in the inclusion is always hydrostatic, and thus the inclusion can never yield(as long as plasticity is controlled by the effective stress). This conclusion was reached for N=1(or monotonic loading from the virgin state) by Hill[1].

The additional displacement in the matrix follows from 2-16, wherein due to the nonlinear term involving b^3/r^2, neither is the additional strain field (eq 2-10) nor is the additional stress field (eq 2-13) hydrostatic. The effective stress in the matrix is thus non-vanishing, and may be computed from eq 1-4 once the deviatoric stress components are determined. The total stress components in the matrix follow from the definition, $\boldsymbol{\sigma}^{(i),(N)} = \boldsymbol{\sigma}^{(i),(N-1)} + \Delta\boldsymbol{\sigma}^{(i),(N)}$. With the deviatoric stress defined by eq 1-5, the total deviatoric stress components during the N^th loading are

$$\sigma_{rr}^{(2)',(N)} = \frac{2}{3}(\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}) \quad , \quad \sigma_{\theta\theta}^{(2)',(N)} = \sigma_{\phi\phi}^{(2)',(N)} = -\frac{1}{2}\sigma_{rr}^{(2)',(N)}$$

$$\sigma_{r\theta}^{(2)',(N)} = 0 \quad , \quad \sigma_{\theta\phi}^{(2)',(N)} = 0 \quad , \quad \sigma_{r\phi}^{(2)',(N)} = 0$$

2-18

With 2-18, 1-4 results in

$$\sigma_e^{(2),(N)} = |\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| \quad .$$

2-19

Yielding commences when the magnitude, $\sigma_Y^{(N)}$, of the yield stress during the N^th loading is attained by the effective stress, $\sigma_e^{(2),(N)}$. This condition follows from eq 1-7 by identifying $\sigma_e^{(2)}$ with $\sigma_e^{(2),(N)}$, and σ_Y with $\sigma_Y^{(N)}$. The effective plastic strain, $\varepsilon_{P,e}^{(2)}$ is identified with the effective strain corresponding to the additional plasticity during the N^{th} loading; this is zero at commencement of yielding. Therefore, at the commencement of yielding, eq 1-7 becomes

$$|\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| = \sigma_Y^{(N)} \quad .$$

2-20

With reference to the N^{th} loading schematically depicted in Fig.2.2, $\sigma_Y^{(N)}$ corresponds to the magnitude of the stress at the point B on the curve ABC. Assuming that the matrix undergoes isotropic hardening, we set the magnitude of $\sigma_Y^{(N)}$ equal to the effective stress attained at the point of load reversal (point A). Therefore

$$\sigma_Y^{(N)} = \sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)} \quad ,$$

2-21

where $\sigma_Y^{(N-1)}$ is the magnitude of the yield stress during the $(N-1)^{th}$ loading and $\Delta\varepsilon_{P,e}^{(2),(N-1)}$ is the effective plastic strain computed from the additional plastic strain, $\Delta\varepsilon_P^{(2),(N-1)}$, in the matrix during the $(N-1)^{th}$ loading. eq 2-21 in eq 2-20 implies

$$|\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| = \sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)} \quad ,$$

2-22

or

$$\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} = \pm(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \qquad \text{if} \qquad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \gtrless 0 \quad .$$

2-23

We now use the decomposition

$$\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} = \sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)} + \Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)}$$

2-24

in eq 2-23 to get

$$\Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)} = \pm\left[(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \mp (\sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)})\right] \quad \text{if} \quad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \gtrless 0 \qquad \text{2-25}$$

On the other hand, using eq 2-10,eq 2-13 and eq 2-16, we have

$$\Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)} = -\frac{2\mu_2 c_1(\kappa_2-\kappa_1)}{3[4\mu_2\kappa_2(1-c_1)+\kappa_1(3\kappa_2+4c_1\mu_2)]}\left(\frac{b}{r}\right)^3 \Delta\bar{\sigma}_{kk}^{(N)} . \qquad \text{2-26}$$

Equating eq 2-25 and eq 2-26, we get the additional composite hydrostatic stress needed to initiate yielding during the N[th] loading. Denoting this critical stress as $\Delta\bar{\sigma}_{Y,kk}^{(N)}$, we have

$$\Delta\bar{\sigma}_{Y,kk}^{(N)} = \mp\left[\frac{2\mu_2 c_1(\kappa_2-\kappa_1)}{3[4\mu_2\kappa_2(1-c_1)+\kappa_1(3\kappa_2+4c_1\mu_2)]}\right]^{-1}\left(\frac{r}{b}\right)^3\left[(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \mp (\sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)})\right] \quad \text{if} \quad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \gtrless 0 , \qquad \text{2-27}$$

where, since $\kappa_2 > \kappa_1$ (matrix harder than the inclusion), it is easy to show that the entire term in the square parenthesis on the left is positive, and the sign of $\Delta\bar{\sigma}_{Y,kk}^{(N)}$ is controlled by the term on the right. In this work, we have not considered the case, $\kappa_2 < \kappa_1$ (inclusion harder than the matrix) as the plasticity of the composite sphere under monotonic hydrostatic loading is not very prominent(Qiu and Weng[3]). If we had done so, then the term in the left parenthesis would also have controlled the sign of $\Delta\overline{\sigma}_{Y.kk}^{(N)}$, and the rest of the theoretical development would also be influenced by such a consideration.

The total composite hydrostatic stress at the onset of yielding during the N^{th} loading follows from the first of eq 2-6. We denote this critical stress as $\overline{\sigma}_{Y}^{(N)}$ and define it as

$$\overline{\sigma}_{Y}^{(N)} = \overline{\sigma}_{kk}^{(N-1)} + \Delta\overline{\sigma}_{Y.kk}^{(N)} , \tag{2-28}$$

where $\overline{\sigma}_{kk}^{(N-1)}$ is a known quantity and $\Delta\overline{\sigma}_{Y.kk}^{(N)}$ follows from eq 2-27. The expression for $\Delta\overline{\sigma}_{Y.kk}^{(N)}$ in eq 2-27 is *r*-dependent, not only explicitly through the term, $(r/b)^3$, but also through the term in the square brackets on the right(this is apparent when the detailed expression for the term is derived for N=1,2,.... and so on). This *r*-dependence has to be examined to find the lowest magnitude of $\Delta\overline{\sigma}_{Y.kk}^{(N)}$; this is the magnitude of the additional hydrostatic stress at which the yielding commences during the N^{th} sequence,
and the corresponding value of *r*(where $a \le r \le b$) is the radius at which the onset of yielding occurs. The total hydrostatic stress at commencement of yielding will follow from eq 2-28 , whereas the corresponding additional composite hydrostatic strain will follow from eq 2-16 and the second of eq 2-6 , and the total strain will follow from the second of eq 2-5.

2.3.2 *Elastoplastic state of the composite sphere*

The composite sphere is in an elastoplastic state when the matrix undergoes elastoplastic deformation. During the elastoplastic deformation of the matrix, a portion of the matrix is in an elastic state and the remaining portion of the matrix is in a plastic state. A schematic of the elastoplastic state of the matrix has been shown in Fig.2.1 with the plastic state prevailing in the range, $a \le r \le R^{(N)}$, where $R^{(N)}$ defines the radius at which the plastic portion of the matrix ends, and the elastic portion of the matrix begins. With increasing loading, the radius $R^{(N)}$ gradually approaches the outer radius *b* of the composite sphere, and the matrix approaches a fully plastic state. Note that since the inclusion will always be in an elastic state, the solution for its displacement field will follow from eq 2-16 for i=1. The constant $A_1^{(1),(N)}$ (originally given in eq 2-17) will have to be determined afresh when the matrix is in an elastoplastic state, and also when the matrix enters the fully plastic state. The constant $A_2^{(1),(N)}$ (refer eq 2-17) will continue to be zero. It is not essential to give the expression for $A_1^{(1),(N)}$, though it may be easily found. We shall not give it explicitly. In the sequel, we derive the fields in the plastic portion and the elastic portion of the matrix, referred henceforth as phases 3 and 2 respectively.

As long as the composite sphere was in an elastic state during the N^{th} loading, the additional elastic strains were given by eq 2-12 and the additional stresses by eq 2-13. It is easy to show that an identical situation results during the elastoplastic deformation also. We justify this assertion now. Specifically, the incremental plastic strains during the N^{th} loading will follow from eq 1-8, where $d\varepsilon_P^{(2)}$ is replaced with $d\Delta\varepsilon_P^{(3),(N)}$ and $\boldsymbol{\sigma}^{(2)'}$ is replaced with $\boldsymbol{\sigma}^{(3)',(N)}$. Therefore, the components of the incremental plastic strain are

$$d\,\Delta\varepsilon_{P,ij}^{(3),(N)} = d\lambda\,\sigma_{ij}^{(3)',(N)} \quad , \tag{2-29}$$

where the additional plastic strains accumulated during the N^{th} loading follow from the integration of the incremental plastic strains defined in 2-29, thus

$$\Delta\varepsilon_{P,ij}^{(3),(N)} = \int d\,\Delta\varepsilon_{P,ij}^{(3),(N)} \quad . \tag{2-30}$$

Note the use of the word "incremental" vis-a-vis the word "additional" when introducing eq 2-30. We emphasize the difference by stating that a "additional" quantity during the N^{th} loading follows from the integration of its "incremental" counterpart . With the components, $\sigma_{ij}^{(3)',(N)}$, given by eq 2-18 at the onset of plastic deformation, the nonvanishing incremental plastic strains(at the onset of plastic deformation) follow from eq 2-29 as

$$d\,\Delta\varepsilon_{P,rr}^{(3),(N)} \equiv d\,\Delta\varepsilon_{P,rr}^{(3),(N)}(r) \quad , \quad d\,\Delta\varepsilon_{P,\theta\theta}^{(3),(N)} = d\,\Delta\varepsilon_{P,\phi\phi}^{(3),(N)} = -\frac{1}{2}d\,\Delta\varepsilon_{P,rr}^{(3),(N)} \quad ,$$

2-31

where we underscore the fact that the incremental plastic strains at the onset of the plastic deformation are r-dependent since the deviatoric stresses(eq 2-18) are r-dependent. We recall that the relation between the components of the additional net strain $\Delta\varepsilon^{(3),(N)}$ in the matrix are given by eq 2-10. Therefore, the nonvanishing incremental net strains will also have an identical relation at the onset of plastic deformation

$$d\,\Delta\varepsilon_{rr}^{(3),(N)} \equiv d\,\Delta\varepsilon_{rr}^{(3),(N)}(r) \quad , \quad d\,\Delta\varepsilon_{\theta\theta}^{(3),(N)} = d\,\Delta\varepsilon_{\phi\phi}^{(3),(N)} \; .$$

2-32

Using eq 2-31, eq 2-32 and eq 2-11 (this equation is used in the incremental context), it is easy to show that the nonvanishing incremental elastic strain components are related as

$$d\,\Delta\varepsilon_{E,rr}^{(3),(N)} \equiv d\,\Delta\varepsilon_{E,rr}^{(3),(N)}(r) \quad , \quad d\,\Delta\varepsilon_{E,\theta\theta}^{(3),(N)} = d\,\Delta\varepsilon_{E,\phi\phi}^{(3),(N)} \; .$$

2-33

With eq 2-33, the incremental stresses will follow from eq 2-13 (except that the additional quantities in that equation will be replaced by incremental quantities). Note that this outcome follows after the "first" increment of plastic strain following the onset of yielding in the matrix. At the end of the "first" increment, the relation between the total stress components will continue to be given as with the additional stress components in eq 2-13, the deviatoric stress components will continue to be related as in 2-18 and thus the subsequent increment in plastic strains will continue to be given by eq 2-31. Therefore, the additional stress field during *the entire course of plastic deformation* will always be given by eq 2-13 and the total stress field will also have the same relation between its components. The nonvanishing deviatoric stress components follow from eq 2-18 and the incremental plastic strains follow from eq 2-29 as

$$d\,\Delta\varepsilon_{P,rr}^{(3),(N)} = \frac{2}{3} d\lambda \left(\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}\right) , \; d\,\Delta\varepsilon_{P,\theta\theta}^{(3),(N)} = d\,\Delta\varepsilon_{P,\infty}^{(3),(N)} = -\frac{1}{2} d\,\Delta\varepsilon_{P,rr}^{(3),(N)} ,$$

2-34

during the entire course of plastic deformation. Therefore, eq 2-32 and eq 2-33 corresponding to the incremental net strain and the incremental elastic strain components respectively are valid during the entire course of elastoplastic deformation. An identical relation is then implied between the *additional* net strain components and the *additional* elastic strain components, thus justifying the validity of the assumption represented by eq 2-12. Using eq 2-34 and eq 2-30, the nonvanishing additional plastic strain components are

$$\Delta\varepsilon_{P,\theta\theta}^{(3),(N)} = \Delta\varepsilon_{P,\infty}^{(3),(N)} = -\frac{1}{2} \Delta\varepsilon_{P,rr}^{(3),(N)} .$$

2-35

The effective plastic strain corresponding to the additional plasticity during the N^{th} loading follows from eq 1-6 as

$$\Delta\varepsilon_{P,e}^{(3),(N)} = \left(\frac{2}{3} \Delta\varepsilon_{P}^{(3),(N)} : \Delta\varepsilon_{P}^{(3),(N)}\right)^{1/2} ,$$

2-36

which, with eq 2-35, becomes

$$\Delta\varepsilon_{P,e}^{(3),(N)} = \left| \Delta\varepsilon_{P,rr}^{(3),(N)} \right| .$$

2-37

From the first of eq 2-34, we infer that since $d\lambda$ is positive, the sign of $d\,\Delta\varepsilon_{P,rr}^{(3),(N)}$ (and therefore the sign of $\Delta\varepsilon_{P,rr}^{(3),(N)}$) depends on the sign of $\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}$. Thus, eq 2-37 becomes

$$\Delta\varepsilon_{P,e}^{(3),(N)} = \pm\Delta\varepsilon_{P,rr}^{(3),(N)} \qquad \text{if} \qquad \sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)} \gtrless 0 \;.$$

2-38

We now define a parameter, $\Delta\varepsilon_{P}^{(N)}$, where

$$\Delta\varepsilon_{P}^{(N)} = \Delta\varepsilon_{P,rr}^{(3),(N)} \quad ,$$

2-39

with which, eq 2-38 is written as

$$\Delta\varepsilon_{P,e}^{(3),(N)} = \pm\Delta\varepsilon_{P}^{(N)} \qquad \text{if} \qquad \sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)} \gtrless 0 \;.$$

2-40

The plastic constitutive equation follows from eq 1-7 as

$$\sigma_{e}^{(3),(N)} = \sigma_{Y}^{(N)} + h\Delta\varepsilon_{P,e}^{(3),(N)} \quad ,$$

2-41

noting that, based on the assumption of isotropic hardening of the matrix, $\sigma_{Y}^{(N)}$, is defined in eq 2-21. With eq 2-19 (replacing the index 2 in that equation with 3) and 2-40, eq 2-41 becomes

$$\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)} = \pm\sigma_{Y}^{(N)} + h\Delta\varepsilon_{P}^{(N)} \qquad \text{if} \qquad \sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)} \gtrless 0 \;.$$

2-42

Using eq 2-24 (replacing index 2 with 3), eq 2-42 is re-written in terms of additional stresses during the N[th] loading as

$$\Delta\sigma_{rr}^{(3),(N)} - \Delta\sigma_{\theta\theta}^{(3),(N)} = \pm\sigma_Y^{(N)} + h\Delta\varepsilon_P^{(N)} - (\sigma_{rr}^{(3),(N-1)} - \sigma_{\theta\theta}^{(3),(N-1)}) \qquad \text{if} \qquad \sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)} \gtrless 0 \ .$$

2-43

The expression $\Delta\sigma_{rr}^{(3),(N)} - \Delta\sigma_{\theta\theta}^{(3),(N)}$ in eq 2-43 is fully determined once $\Delta\varepsilon_P^{(N)}$ is found. Note from eq 2-10 that we can write the following relation involving the additional strains in the matrix

$$\Delta\varepsilon_{rr}^{(3),(N)} = \frac{\partial}{\partial r}(r\Delta\varepsilon_{\theta\theta}^{(3),(N)}) \ .$$

2-44

But the additional net strain may be decomposed into an elastic and a plastic part,

$$\begin{aligned} \Delta\varepsilon_{rr}^{(3),(N)} &= \Delta\varepsilon_{E,rr}^{(3),(N)} + \Delta\varepsilon_{P,rr}^{(3),(N)} = \Delta\varepsilon_{E,rr}^{(3),(N)} + \Delta\varepsilon_P^{(N)} \\ \Delta\varepsilon_{\theta\theta}^{(3),(N)} &= \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} + \Delta\varepsilon_{P,\theta\theta}^{(3),(N)} = \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} - \frac{1}{2}\Delta\varepsilon_P^{(N)} \ , \end{aligned}$$

2-45

where the first of eq 2-35 and eq 2-39 have been used. eq 2-13 may now be used to show that

$$\Delta\varepsilon_{E,rr}^{(3),(N)} = \frac{\Delta\sigma_{rr}^{(3),(N)} - 2\nu_2\Delta\sigma_{\theta\theta}^{(3),(N)}}{E_2} \qquad , \qquad \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} = \frac{(1-\nu_2)\Delta\sigma_{\theta\theta}^{(3),(N)} - \nu_2\Delta\sigma_{rr}^{(3),(N)}}{E_2}$$

2-46

in terms of the Poisson's ratio, $\nu_2 = (3\kappa_2 - 2\mu_2)/(6\kappa_2 + 2\mu_2)$ and the Young's modulus of the matrix, $E_2 = 9\kappa_2\mu_2/(3\kappa_2 + \mu_2)$. With 2-46 in eq 2-45, and the result in eq 2-44, we get

$$\frac{1+\nu_2}{E_2}(\Delta\sigma_{\theta\theta}^{(3).(N)} - \Delta\sigma_{rr}^{(3).(N)}) + r\frac{1-\nu_2}{E_2}\frac{d\,\Delta\sigma_{\theta\theta}^{(3).(N)}}{dr} - \frac{\nu_2}{E_2}r\frac{d\,\Delta\sigma_{rr}^{(3).(N)}}{dr} = \left[\frac{3}{2}\Delta\varepsilon_P^{(N)} + \frac{1}{2}r\frac{d\,\Delta\varepsilon_P^{(N)}}{dr}\right]$$

2-47

Using the equilibrium equation eq 2-14 with eq 2-47, we can show that

$$r\frac{d(\Delta\sigma_{\theta\theta}^{(3).(N)} - \Delta\sigma_{rr}^{(3).(N)})}{dr} + 3(\Delta\sigma_{\theta\theta}^{(3).(N)} - \Delta\sigma_{rr}^{(3).(N)}) = \frac{E_2}{1-\nu_2}\left[\frac{r}{2}\frac{d\,\Delta\varepsilon_P^{(N)}}{dr} + \frac{3}{2}\Delta\varepsilon_P^{(N)}\right] .$$

2-48

When eq 2-43 is introduced in eq 2-48, we have a differential equation in terms of $\Delta\varepsilon_P^{(N)}$. This equation is solved subject to the condition that $\Delta\varepsilon_P^{(N)}$ vanishes at the elastic-plastic interface, $R^{(N)}$ (yet undetermined). Thus

$$\Delta\varepsilon_P^{(N)}(R^{(N)}) = 0 .$$

2-49

Once eq 2-48, is solved subject to eq 2-49, the resulting expression for $\Delta\varepsilon_P^{(N)}$ may be introduced in eq 2-43 to get an explicit expression for $\Delta\sigma_{\theta\theta}^{(3).(N)} - \Delta\sigma_{rr}^{(3).(N)}$. eq 2-14 may then be used to derive $\Delta\sigma_{rr}^{(3).(N)}$, thus

$$\Delta\sigma_{rr}^{(3).(N)} = -\int_a^r \frac{2}{r}(\Delta\sigma_{rr}^{(3).(N)} - \Delta\sigma_{\theta\theta}^{(3).(N)})dr + D$$

2-50

where D is a constant of integration to be determined.

We now have to determine a total of five parameters: $A_1^{(1).(N)}$, $A_1^{(2).(N)}$ and $A_2^{(2).(N)}$ (refer eq 2-16), the constant D(refer eq 2-50), and the radius $R^{(N)}$, of the elastic-plastic interface. These will follow from the two interface conditions(eq 2-15, in which the index 2 is replaced with 3), the traction boundary condition, given by the first of eq 2-6, and three additional conditions at the elastic-plastic interface

$$\Delta u_r^{(3),(N)}(R^{(N)}) = \Delta u_r^{(2),(N)}(R^{(N)})$$

$$\Delta \sigma_{rr}^{(3),(N)}(R^{(N)}) = \Delta \sigma_{rr}^{(2),(N)}(R^{(N)}) \qquad .$$

$$|\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}| = |\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| = \sigma_Y^{(N)} \qquad \text{at} \qquad r = R^{(N)}$$

2-51

The last of eq 2-51 represents the condition that the effective stress at the elastic-plastic interface is continuous, and is equal to the yield stress at that location. Therefore, while five parameters have to be determined, there are six conditions that need to be met. In fact, it may be shown that there are only five independent conditions, thus admitting a unique solution for the constants to be determined. This is now demonstrated. By the second of eq 2-10 and the first of eq 2-51, we see that the additional net circumferential strain at the elastic-plastic interface is continuous, and since yielding has not commenced yet at the interface, this additional net strain is all elastic. Therefore

$$\Delta \varepsilon_{E,\theta\theta}^{(3),(N)}(R^{(N)}) = \Delta \varepsilon_{E,\theta\theta}^{(2),(N)}(R^{(N)}) \ .$$

2-52

The continuity of the additional circumferential elastic strains imply the continuity of their total counterparts,i.e.

$$\varepsilon_{E,\theta\theta}^{(3),(N)}(R^{(N)}) = \varepsilon_{E,\theta\theta}^{(2),(N)}(R^{(N)}) \ .$$

2-53

eq 2-21 written in terms of total quantities) may be used to derive $\varepsilon_{E,\theta\theta}^{(i),(N)}$ in terms of $\sigma_{rr}^{(i),(N)}$ and $\sigma_{\theta\theta}^{(i),(N)}$, and then write eq 2-53 as

$$(1-2\nu_2)\sigma_{rr}^{(3),(N)} - (1-\nu_2)(\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}) = (1-2\nu_2)\sigma_{rr}^{(2),(N)} - (1-\nu_2)(\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}) \quad \text{at} \quad r = R^{(N)}$$

2-54

Notice that the second of eq 2-51 will imply continuity of the total radial stress at $r = R^{(N)}$

$$\sigma_{rr}^{(3),(N)}(R^{(N)}) = \sigma_{rr}^{(2),(N)}(R^{(N)})$$

using which, 2-54 may be written as

$$|\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}| = |\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| \qquad \text{at} \quad r = R^{(N)} .$$

2-55

Since the additional plastic strain at the elastic-plastic interface is zero (eq 2-49), 2-42 gives $|\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}| = \sigma_Y^{(N)}$. With this outcome, eq 2-55 coincides with the last of 2-51, thus proving that it is not an independent condition.

With the constants determined, $\Delta\sigma_{rr}^{(3),(N)}$ is given by eq 2-50, and $\Delta\sigma_{\theta\theta}^{(3),(N)}$ follows from eq 2-43. Now that the additional stresses have been found, the additional elastic strain in the circumferential direction, $\Delta\varepsilon_{E,\theta\theta}^{(3),(N)}$, follows from the second of eq 2-46, and the additional net circumferential strain, $\Delta\varepsilon_{\theta\theta}^{(3),(N)}$, follows from the second of eq 2-45. On the other hand, the additional displacement in the elastic portion of the matrix (phase 2) follows from the second of eq 2-16. The additional composite strain follows from eq 2-6 and the total composite strain follows from the second of eq 2-5.

The composite sphere enters into a fully plastic state when the matrix does. The equations corresponding to the fully plastic state are developed in a manner identical to the partially plastic state. In particular, the key equation in the fully plastic state is

2-43, which when substituted into eq 2-48, gives a differential equation in terms of $\Delta\varepsilon_P^{(N)}$. Solution of this differential equation in the context of a fully plastic state will be subject to the condition that

$$\Delta\varepsilon_P^{(N)}\big|_{r=b} = \Delta\varepsilon_P^{(N)}(b) \ , \qquad 2\text{-}56$$

where $\Delta\varepsilon_P^{(N)}(b)$ will have to be determined from boundary conditions. The subsequent procedure to find the deformation, strain and stress fields is identical to the procedure followed for the partially plastic state. The only difference between the fully plastic state and the partially plastic state is that in the former, we need to find only three parameters as opposed to the five parameters in case of the latter. These three parameters are: $A_I^{(I),(N)}$ (refer 2-16), $\Delta\varepsilon_P^{(N)}(b)$ (refer eq 2-55) and the constant D(refer eq 2-50). These three parameters will follow from the two interface conditions given by eq 2-15 and the traction boundary condition, given by the first of eq 2-6.

2.4 **The analytical solution**

The objective now is to develop an analytical solution for a Nth loading sequence based on the solution procedure given in Sec.2.3. The key issue during that process is the state of the composite sphere during load reversal(point A in Fig.2.2.). This is now explained. At the point of load reversal from the 1st sequence of loading to the 2nd sequence, it is possible that the matrix is in any one of the three states: elastic, partially plastic or fully plastic. In particular, if it is partially plastic, the term in the square brackets on the right of eq 2-27 (wherein we determine the composite hydrostatic yield stress in the 2nd sequence of loading) will represent two different expressions, the first expression being valid in the plastic portion of the matrix and the second expression being valid in the elastic portion of the matrix. Consideration of the two different expressions will arise also when the fields need to be found corresponding to the elastoplastic state and the fully plastic state of the matrix during the 2nd sequence of loading(see eq 2-43). If, now, the point of load reversal from the 2nd sequence to the 3rd sequence also occurs when the matrix is in a partially plastic state, it may be shown that the considered term in eq 2-27 will represent four different expressions. As the number of loadings increase, so will the number of expressions. Keeping track of so many expressions with continued loading is an impossible task, and in such a situation it is more efficient to resort to finite element computations. To keep the analytical solution tractable, we restrict the point of load reversal to a situation where the matrix is in a fully plastic state. With this restriction, the considered term in eq 2-27 will always be given by a single expression that represents the fully plastic state of the matrix during load reversal. As will be seen later, initial cyclic loading(e.g. under stress or strain control) of the composite sphere usually ensures that such a condition is met. However, with continued cycling accompanied by the isotropic hardening of the matrix, a loading sequence will be reached at the end of which the matrix fails to reach a fully plastic state. The solution will then not be valid beyond that critical loading sequence. This is also sensitive to input parameters such as inclusion volume fraction, relative stiffness of soft inclusion/hard matrix, and the strength coefficient of the matrix. The range of applicability of the analytical solution is discussed in Sec.2.5 on "Parametric studies". It is seen that even with the aforementioned restriction(that the matrix is in a fully plastic state at the point of load reversal), the solution is applicable over a large number of loading sequences for a given combination of input parameters.

We adopt an inductive approach to determine the analytical solution for the N^{th} loading. Specifically, assuming that the matrix is in a fully plastic state at the point of load reversal, we first rigorously determine the $\overline{\sigma}_{kk} - \overline{\varepsilon}_{kk}$ relation for $N = 1$(loading from the virgin state), following the procedure outlined in Sec.2.4. The solution for the first loading provides the initial condition for the 2^{nd} loading, with which we again rigorously determine the $\overline{\sigma}_{kk} - \overline{\varepsilon}_{kk}$ relation for $N = 2$. This is continued for $N = 5$. The solution for the first five loadings are then examined, and it is found that these yield a common pattern based on which, by an inductive process, the equations for the N^{th} loading are *suggested* (see sections 2.4.1 – 2.4.3) . In order to verify the suggested equations, we have checked that these coincide with the equations developed from first principles(i.e.based on Sec.2.4) for N=6 and 7.

The $\Delta\overline{\sigma}_{kk}^{(N)} - \Delta\overline{\varepsilon}_{kk}^{(N)}$ relation is now given for the N^{th} cycle of loading, whereas the $\overline{\sigma}_{kk} - \overline{\varepsilon}_{kk}$ relation will follow from eq 2-5. The loading process is taken to be alternately tensile and compressive with the first sequence of loading(from the virgin state) being tensile. If the first sequence of loading would have been compressive, one simply needs to reverse the sign of the entire loading process, use the equations below in Secs.2.5.1-2.5.3, and then reverse the sign of the calculated $\overline{\sigma}_{kk}$ and $\overline{\varepsilon}_{kk}$ to get the correct stress-strain curve. This approach will not work if the virgin state of the composite sphere is *not* strain- and stress-free. This is, however, outside the scope of this work.

2.4.1 *Elastic state of the composite sphere and the commencement of yielding*

The relation between the additional strain and stress is given by

$$\Delta\overline{\varepsilon}_{kk}^{(N)} = \frac{1}{3} P_1 \, \Delta\overline{\sigma}_{kk}^{(N)} ,$$

2-57

where P_1 is defined in eq.A.2-1 of the Appendix. With the incremental hydrostatic strain given by eq 2-56, the total hydrostatic stress and strain at the end of the N^{th} loading will follow from eq 2-5. Commencement of yielding during the N^{th} loading occurs when the additional hydrostatic stress attains the following value

$$\Delta\overline{\sigma}_{Y.kk}^{(N)} = \begin{cases} \dfrac{c_1}{2\mu_2 P_2}\sigma_Y^{(1)} & \text{for} \quad N=1 \\ \dfrac{1}{\mu_2 P_2}\left[-c_1 H\sigma_Y (1-2H)^{N-2} + (-1)^N h(P_4 + \Delta\varepsilon_p^{(1)}(b) - P_3)\right] & \text{for} \quad N>1 \end{cases}$$

2-58

where P_2, P_3 and P_4 are defined in eq.A.2-3 of the Appendix. The parameter $\Delta\varepsilon_p^{(1)}(b)$ follows from eq.A.2-9 and the parameter H is defined as

$$H = \left[1 + \frac{2h(1-\nu_2)}{E_2}\right]^{-1}$$

2.4.2 *Elastoplastic state of the composite sphere*

The elastoplastic state of the composite sphere is characterized by the radius of the elastoplastic interface(see Fig.2.1). The ratio, $R^{(N)}/b$, changes from $c_1^{1/3}$(corresponding to an elastic state of the matrix) to 1(a fully plastic state of the matrix). The additional strain and stress relation is given as

$$\Delta\bar{\varepsilon}_{kk}^{(N)} = \frac{1}{3\kappa_2}\Delta\bar{\sigma}_{kk}^{(N)} + P_6$$

2-59

During the elastoplastic deformation, it is convenient to treat $R^{(N)}/b$ as an input parameter, based on which $\Delta\bar{\sigma}_{kk}^{(N)}$ and P_6 are determined, leading to the determination of $\Delta\bar{\varepsilon}_{kk}^{(N)}$. The parameter $\Delta\bar{\sigma}_{kk}^{(N)}$ is given in eq.A. 2-11 and P_6 is given in eq.A. 2-4.

2.4.3 *Fully plastic state of the composite sphere*

The elastoplastic state of the matrix gives away to a full plastic state when $R^{(N)}/b = 1$. In this situation, the additional stress-strain relation is given by

$$\Delta\bar{\varepsilon}_{kk}^{(N)} = \frac{1}{3\kappa_2}\Delta\bar{\sigma}_{kk}^{(N)} + P_{10} ,$$

2-60

where P_{10} is given by Eq.A. 2-10.

2.5 Parametric studies

The cyclic hydrostatic stress-strain relations for a composite sphere are studied in this section. We assume that the matrix has the elastoplastic properties of a 6061 aluminium(Arsenault[4], Nieh and Chellman[5])

$$E_2 = 68.3 \text{ GPa} \quad , \quad v_2 = 1/3 \quad , \quad \sigma_Y^{(1)} = 250 \text{ MPa} \quad , \quad h = 173 \text{ MPa} \quad . \tag{2-61}$$

We note that the material parameters are used only for the purpose of the parametric study. However, if the results of the study are used to predict experimental results, other factors such as the ultimate strength of matrix needs to be considered. This is beyond the scope of the current work.

We shall focus on studying the influence of the following parameters on the composite elastoplastic response: (1) the inclusion volume fraction, c_1, (2) the relative stiffness of the Young's modulus of inclusion to matrix, E_1/E_2, (3) the strength coefficient, h, of the matrix. An alternative parameter is the tangent modulus of the matrix, E_P, defined as (Qiu and Weng[3])

$$E_P = E_2\left(1+\frac{E_2}{h}\right)^{-1} \quad . \tag{2-62}$$

Note that for a perfectly plastic material($h \to 0$), $E_P/E_2 \to 0$, and for a perfectly elastic material($h \to \infty$), $E_P/E_2 \to 1$.

We now study the evolution in the $\bar{\sigma}_{kk} - \bar{\varepsilon}_{kk}$ relation for a stress-controlled process and a strain-controlled process. In either case, we take $c_1 = 80\%$, $E_P/E_2 = 0.01$ and $E_1/E_2 = 0.1$ (inclusions softer than the matrix). The stress-controlled process is shown in Fig.2.3, wherein the stress has been limited to +1100 MPa and −1100 MPa. The magnitude of the stress-control value has been denoted as

$\overline{\sigma}_{kk}^{CR}$ =1100 MPa. The first five sequences of loading are shown with a solid line. Continued cycling results in a linear stress-strain response, shown for N=119th and 120th sequence of loading. Recall that we assumed the strength coefficient of the matrix at a constant value of h = 173 MPa. The strain-controlled process is shown in Fig.2.4 wherein the magnitude of strain control is fixed at $\overline{\varepsilon}_{kk}^{CR} = 5\%$. The hardening of the composite response is clearly visible. Due to the interchangeability of the traction and displacement prescribed boundary conditions, the rest of the paper gives a parametric study of a stress-controlled process.

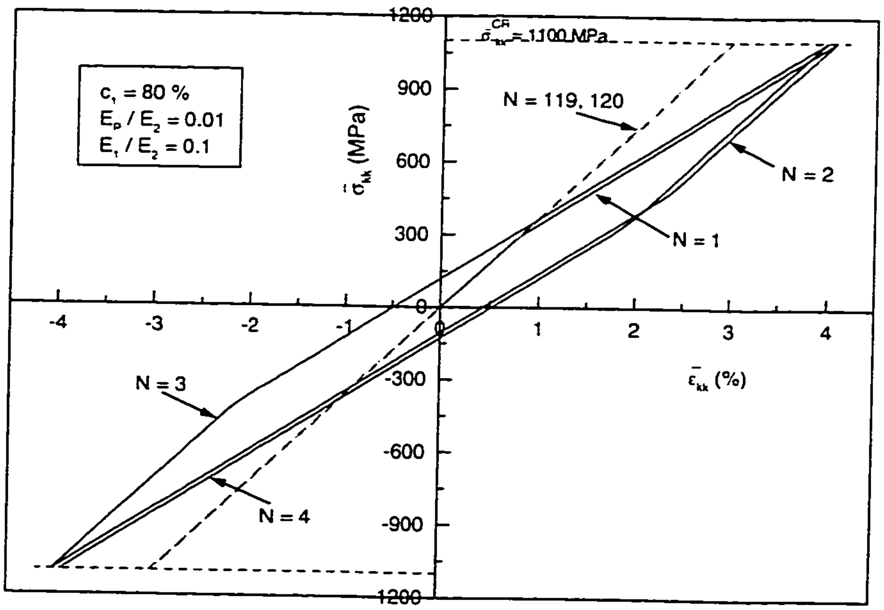

Figure 2.3: The cyclic hydrostatic stress-strain curve under stress control.

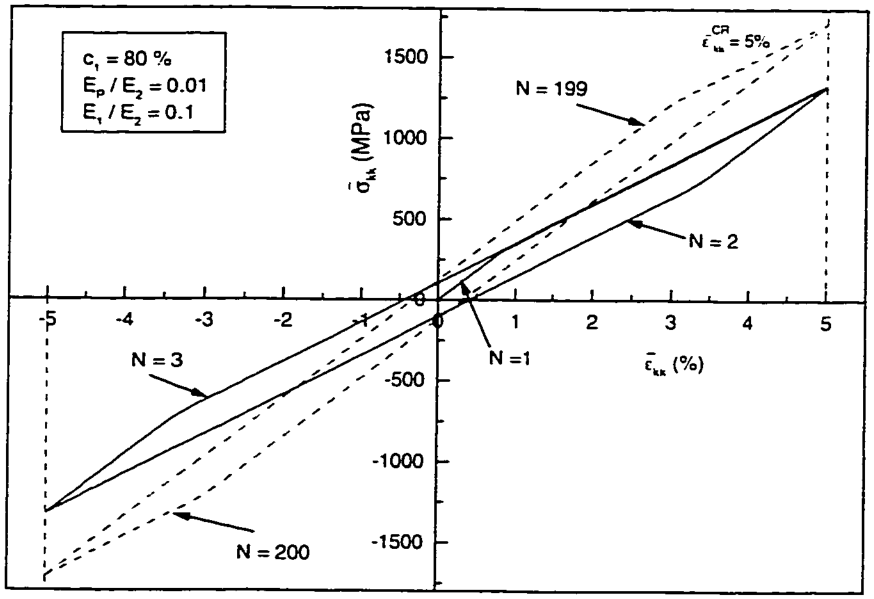

Figure 2.4: The cyclic hydrostatic stress-strain curve under strain control.

2.5.1 *Range of applicability of the model*

Recall that the analytical solution is valid for the N^{th} loading as long as at the end of the$(N\text{-}1)^{th}$ loading, the composite is in a fully plastic state. Alternatively, it is desirable to know, for example, that during a stress-controlled process, how many loadings are possible before the matrix does not fully yield when the stress control value is reached. We present the results of such a study in Figs.2.5-2.6. In Fig.2.5, we give the critical number of loadings, N, for the entire range of inclusion volume fraction, c_1. The presented data corresponds to an inclusion 10 times softer than the matrix($E_1 / E_2 = 0.1$) whereas Fig.2.6 corresponds to the case of a void in the matrix($E_1 / E_2 = 0.0$). Let us now carefully examine Fig.2.5. We see that when the strength coefficient of the matrix is $E_p / E_2 = 0.1$ and the stress control value is set at $\bar{\sigma}^{CR}_{kk} = 1100$ MPa, the value of N remains quite low for a wide range of volume fraction(the lowermost curve). Beyond $c_1 = 80\%$, N starts increasing from around 25 and then shoots up to about 450. The general trend of an increasing N with increasing volume fraction can be anticipated. This is because with increasing inclusion volume fraction or more of the elastic phase, higher internal stresses result in the matrix for a given composite stress and leads to more plasticity in the matrix. This, in turn, implies that the matrix will be partially plastic at a given stress control value only at a higher value of N. An almost identical response is shown over a wide range of inclusion volume fraction, $0 \le c_1 \le 80\%$, when we set $\overline{\sigma}_{kk}^{CR} = 1500$ MPa. This demonstrates that the stress-control value has almost no effect over almost the entire range of c_1 for the combination of properties studied. Now compare the lowermost curve with the top curve. The difference between the two is that the top curve corresponds to a matrix with a lower strength coefficient, where $E_P / E_2 = 0.01$. It is seen that the parameter N at a given volume fraction is much higher as compared to when we had a matrix with a higher strength coefficient(the lowermost curve). This is because with a lower work hardening of the matrix, the plasticity in the matrix is enhanced at a given stress, and hence the partially plastic state at a given stress control value is delayed, leading to a larger value for N. Fig. 2.6 gives results corresponding to a void in the

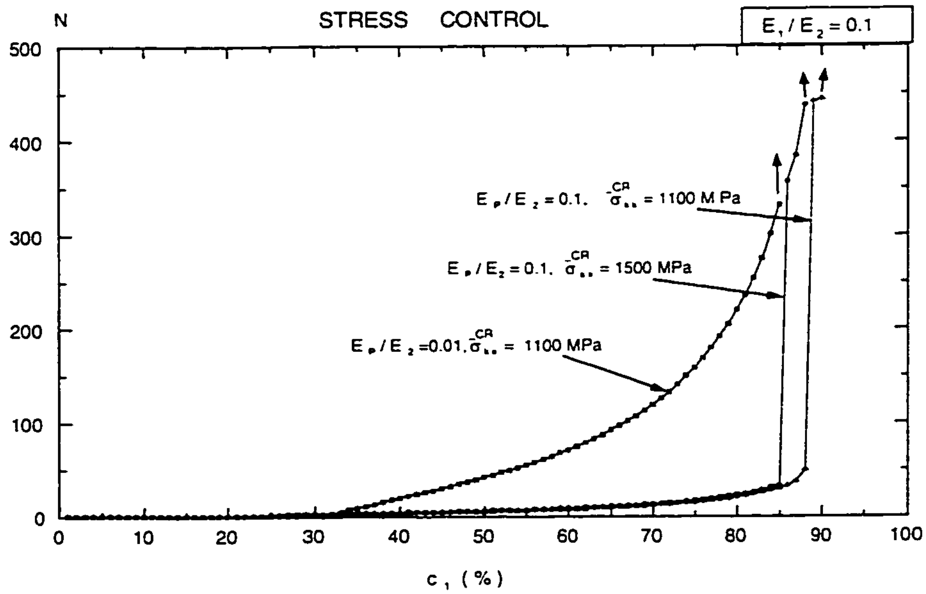

Figure 2.5: The inclusion volume fraction dependence of the critical number of loading sequences upto which the matrix is fully plastic under stress control and when the inclusion is 10 times softer than the matrix.

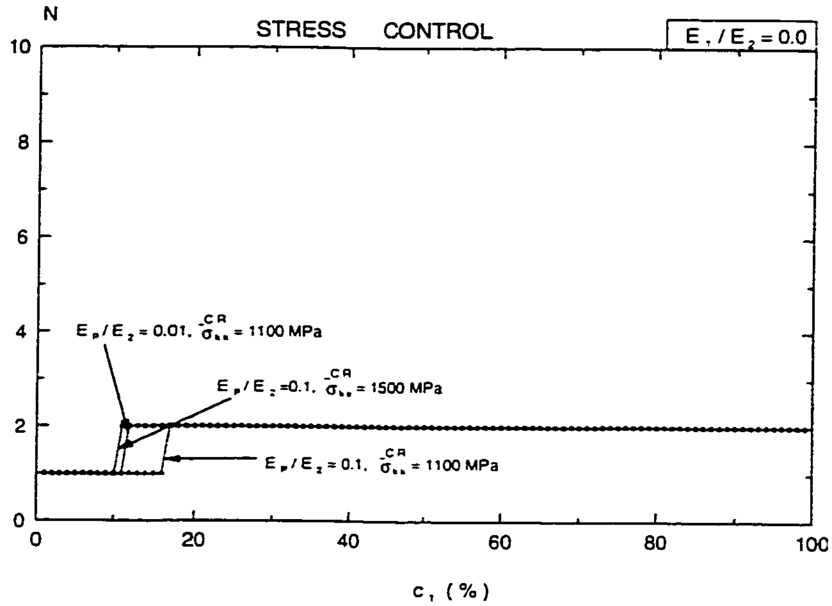

Figure 2.6: The inclusion volume fraction dependence of the critical number of loading sequences upto which the matrix is fully plastic under stress control and when the inclusion is a void.

matrix(i.e. $E_1 / E_2 = 0.0$); the remaining parameters are identical to what had been used for Fig.2.5. It is seen from Fig. 2.6 that during stress control for a composite sphere with a void, the number of critical loadings is just 2. Summarizing these results, we see that the total number of sequences in the cyclic loading process over which the analytical solution is applicable increases when (1) the inclusion volume fraction increases, (2) the matrix is plastically softer(or has a lower strength coefficient), and (3) when a soft inclusion approaches the stiffness of a hard matrix.

2.5.2 *The Bauschinger effect of the composite sphere*

We now study the bauschinger effect of the composite sphere within the range of parameters discussed above. The bauschinger effect refers to the phenomenon in which a material displays different magnitudes of yield stress when loaded in compression after tensile loading (or vice-versa). This has important consequences in applications where sign reversal of stresses occurs (Dieter [6]). Before we study the composite bauschinger effect, we parametrize it in terms of $\alpha^{(N)}$, which is defined by the equation

$$\alpha^{(N)} = \frac{1}{2}\left(1 + \frac{\left|\overline{\sigma}_Y^{(N)}\right| - \left|\overline{\sigma}_Y^{(N-1)}\right|}{\left|\overline{\sigma}_{kk}^{(N-1)} - \overline{\sigma}_Y^{(N-1)}\right|}\right),$$

2-63

where $\overline{\sigma}_Y^{(N)}$ is the magnitude of the composite hydrostatic yield stress during the N^{th} loading(defined in eq 2-28) and $\overline{\sigma}_{kk}^{(N-1)}$ in eq 2-62 is the total hydrostatic stress at the end of the (N-1)th loading. Note that the limiting values of $\alpha = 0$ and 1 corresponds to the case of pure kinematic and isotropic hardening respectively in the overall composite response. A value of $\alpha = 0.5$ is then attributed to a situation where both kinematic and isotropic hardening mechanisms play an equally important role.

The parameters on the right of eq 2-63 follow from the analytical solution, using which α is calculated. Note that as these parameters change during the course of loading, α also will change. The evolution of α over a sequence of loadings are given in Figs.2.7 to 2.10.

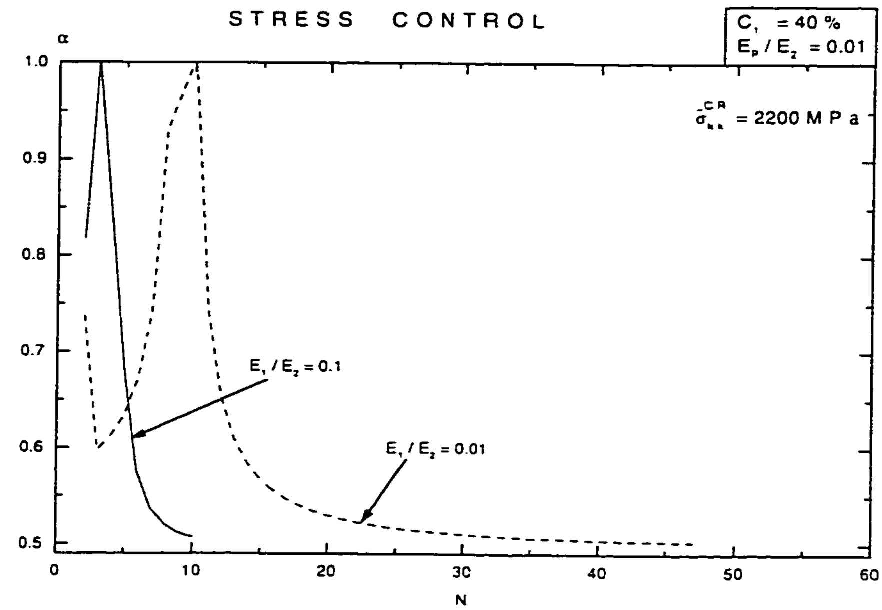

Figure 2.7: The evolution of the Bauschinger effect for the composite sphere under stress control and the influence of the relative stiffness of inclusion/matrix.

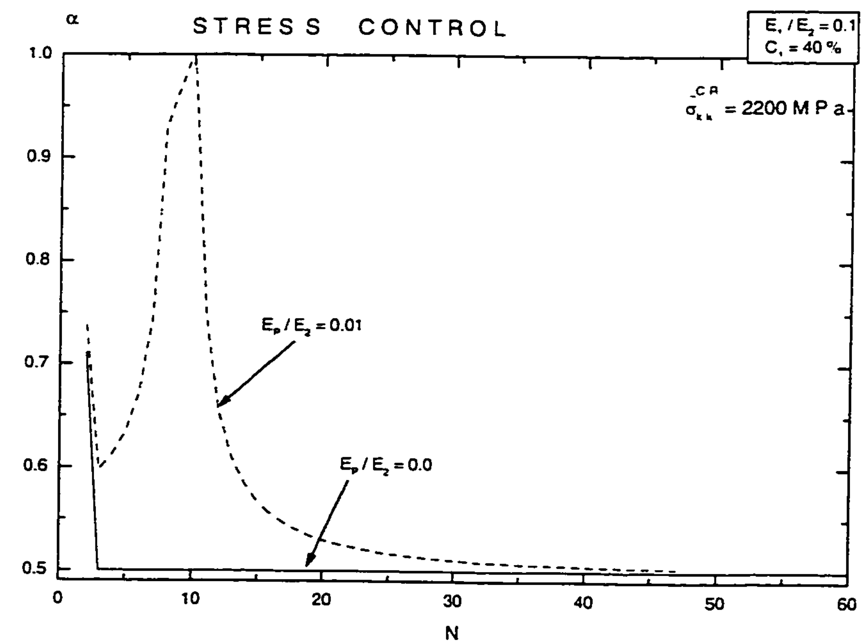

Figure 2.8: The evolution of the Bauschinger effect for the composite sphere under stress control and the influence of the matrix tangent modulus.

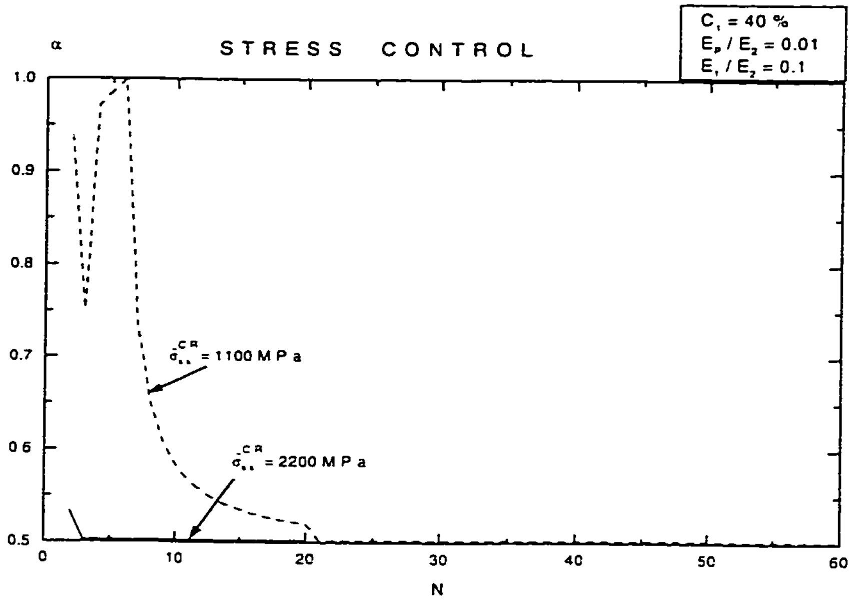

Figure 2.9: The evolution of the Bauschinger effect for the composite sphere under stress control and the influence of the stress control value, $\overline{\sigma}_{kk}^{CR}$.

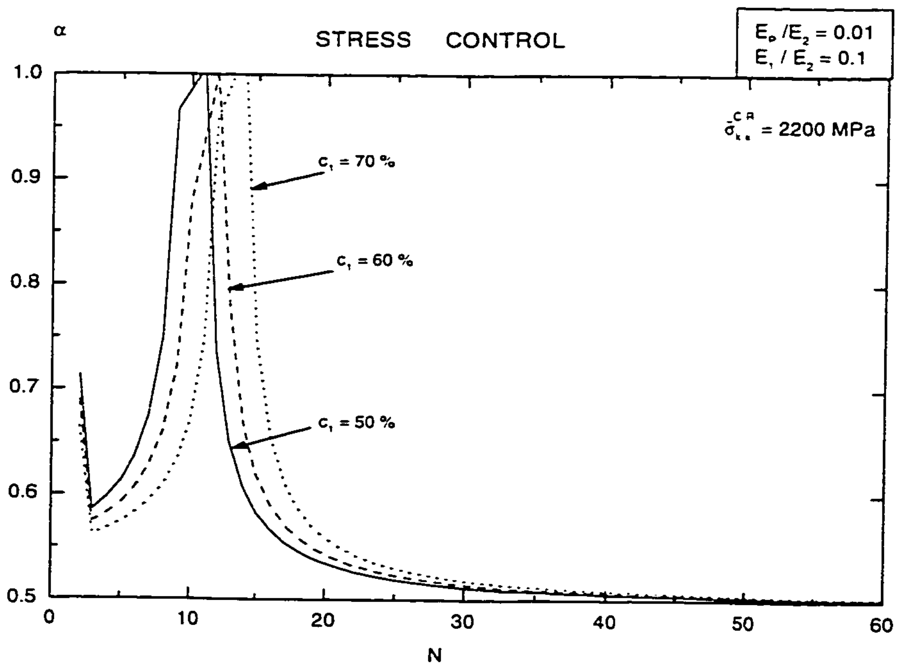

Figure 2.10: The evolution of the Bauschinger effect for the composite sphere under stress control and the influence of the inclusion volume fraction.

In Fig.2.7, we examine the effect of the relative stiffness parameter, E_1 / E_2 during a stress-controlled process. We see that when $E_1 / E_2 = 0.1$(the solid line in Fig.2.7), the Bauschinger effect parameter initially increases to 1(indicates a trend towards isotropic hardening), and then decreases monotonically towards $\alpha = 0.5$. The curve does not extend beyond N=10 as the fully plastic state of the matrix is not achieved beyond the aforementioned value of N. As the stiffness parameter decreases to 0.01(the dashed line), the evolution in α shows an oscillation during the initial cycling but beyond N=10, the value of the parameter monotonically decreases towards a value of 0.5. Fig.2.8 shows the evolution in α during stress control as the relative tangent modulus, E_P / E_2 is changed. Specifically, it is seen that if the tangent modulus parameter is decreased until the matrix is perfectly plastic(the lower curve in both figures), the value of α approaches 0.5 much "faster". The effect of the stress control value, $\overline{\sigma}_{kk}^{CR}$, on α is shown in Fig.2.9. It is apparent that while the initial response of the composite sphere is primarily governed by isotropic hardening, the value of α subsequently approaches 0.5. The effect of the inclusion volume fraction on the evolution of α is depicted in Fig.2.10, where a similar trend is noticed.

The analysis of the composite Bauschinger effect given in Figs.2.7-2.10 indicates that irrespective of the input parameters studied, the hydrostatic stress-strain response of the composite sphere is initially primarily isotropic to begin with(influenced no doubt by the isotropic hardening character of the matrix). Gradually , however, with continued cyclic loading, the kinematic hardening mechanism asserts itself and the composite tends towards a response where both kinematic and isotropic mechanisms play an equally important role(i.e. $\alpha = 0.5$). This outcome is attributed principally to the evolution in the internal stresses of the composite sphere.

2.6 References

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Chapter 3

ON THE EXACT SOLUTION OF CYCLIC ELASTOPLASTIC RESPONSE OF A INFINITELY LONG COMPOSITE CYLINDER SUBJECTED TO UNIFORM IN-PLANE RADIAL LOADING

3.1 Boundary value problem for the Nth loading sequence

The boundary value problem addressed in this paper is now outlined. A circular fiber is concentrically embedded in a cylinder with a circular cross-section and perfectly bonded to it. The radius of the fiber (phase 1) - is "a" whereas the radius of the outer cylinder (phase 2) is "b" (see fig. 3.1). The composite cylinder is subjected to a biaxial traction, stated in a cylindrical co-ordinate system (r, θ, z) as

$$\sigma_{rr}^{(2)}(b) = \overline{\sigma} \quad , \quad \sigma_{r\theta}^{(2)}(b) = 0 \quad , \quad \sigma_{rz}^{(2)}(b) = 0 \, ,$$

3-1

where the z-axis is taken to coincide with the line of axisymmetry of the composite cylinder. The composite is subjected to plane strain by setting

$$u_z^{(i)} = 0 \qquad (i = 1,2) \ .$$

3-2

The two phases are assumed to be perfectly bonded to each other, resulting in the continuity of tractions and displacements at the interface. Thus at r = a, we have

$$\sigma_{rr}^{(1)}(a) = \sigma_{rr}^{(2)}(a) \quad , \quad \sigma_{r\theta}^{(1)}(a) = \sigma_{r\theta}^{(2)}(a) \quad , \quad \sigma_{rz}^{(1)}(a) = \sigma_{rz}^{(2)}(a) \, ,$$

and

$$u_r^{(1)}(a) = u_r^{(2)}(a) \ , \ u_\theta^{(1)}(a) = u_\theta^{(2)}(a) \ , \ u_z^{(1)}(a) = u_z^{(2)}(a) \ .$$

3-3

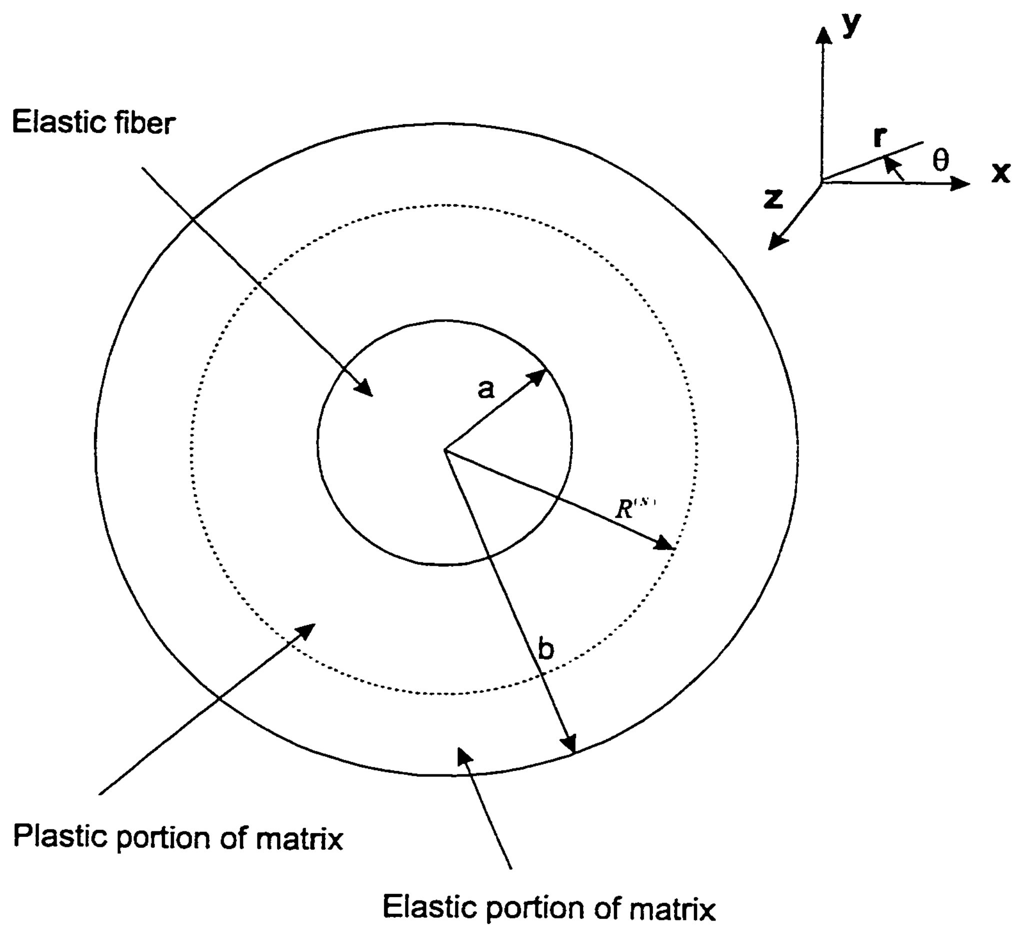

Figure 3.1: A schematic of the composite cylinder cross-section with the elastic-plastic interface.

We define the biaxial strain, $\bar{\varepsilon}$, as

$$\bar{\varepsilon} = \frac{1}{2\sum_{i=1}^{2} V^{(i)}}\left[\sum_{i=1}^{2} \int_{V^{(i)}} \varepsilon_{kk}^{(i)}\, dv\right] , \qquad \text{3-4}$$

where $V^{(i)}$ is the volume of the ith phase and $\varepsilon_{kk}^{(i)} = tr(\varepsilon^{(i)})$. Using the plane strain condition and the Gauss divergence theorem, it is possible to show that

$$\bar{\varepsilon} = \frac{1}{b} u_r^{(2)}(b) \quad . \tag{3-5}$$

The objective is to determine the $\bar{\sigma} - \bar{\varepsilon}$ relation during cyclic loading of the composite cylinder.

3.2 **Boundary value problem for the Nth sequence of cyclic loading**

Loading of the composite cylinder from its virgin state will be referred to as the 1st sequence in the cyclic loading process. Reversal of loading from the first sequence leads to the 2nd sequence, and so on. The objective is to determine the $\bar{\sigma} - \bar{\varepsilon}$ relation during the Nth sequence of cyclic loading (the curve ABC in fig 3.2). The point A represents the onset of the Nth sequence, the point B is the yield point and point C represents the end of the Nth loading sequence. In the sequel, we shall use the following notation to denote different field quantities during the Nth loading sequence. The current value of $\bar{\sigma}$ and $\bar{\varepsilon}$ during the Nth loading is referred to as $\bar{\sigma}^{(N)}$ and $\bar{\varepsilon}^{(N)}$ respectively. The total stress, $\bar{\sigma}^{(N)}$, may be written as the sum of the stress, $\bar{\sigma}^{(N-1)}$, accrued at the end of the (N-1)th loading and the additional stress, $\Delta\bar{\sigma}^{(N)}$, imposed during the Nth loading sequence (see fig 3.2). A similar decomposition may also be admitted for $\bar{\varepsilon}^{(N)}$; thus

$$\bar{\sigma} \equiv \bar{\sigma}^{(N)} = \bar{\sigma}^{(N-1)} + \Delta\bar{\sigma}^{(N)} \qquad \text{and} \qquad \bar{\varepsilon} \equiv \bar{\varepsilon}^{(N)} = \bar{\varepsilon}^{(N-1)} + \Delta\bar{\varepsilon}^{(N)} \quad . \tag{3-6}$$

Note that when N =1, the initial condition of the composite cylinder is represented by $\bar{\sigma}^{(0)}$ and $\bar{\varepsilon}^{(0)}$. We shall assume that the composite is stress- and strain-free in its virgin state, thus

$$\bar{\sigma}^{(0)} = 0 \qquad \text{and} \qquad \bar{\varepsilon}^{(0)} = 0 \ .$$

3-7

All other quantities may be similarly decomposed. Based on such a decomposition, the first of eq 3-1 and eq 3-5 become,

$$\Delta\bar{\sigma}^{(N)} = \Delta\sigma_{rr}^{(2),(N)}(b) \qquad \text{and} \qquad \Delta\bar{\varepsilon}^{(N)} = \frac{1}{b}\,\Delta u_r^{(2),(N)}(b) \ .$$

3-8

The additional imposed traction, $\Delta\bar{\sigma}^{(N)}$, is the specified parameter and the objective is to determine $\Delta\bar{\varepsilon}^{(N)}$. Once the $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation is established, the total quantities

may be found using eq 3-6 and thereby the overall cyclic stress-strain response of the composite cylinder is determined.

Due to the axisymmetry inherent in the boundary value problem, the radial displacement field is simply r-dependent whereas the circumferential displacement vanishes. Identifying the radial displacement during the N^{th} loading as $u_r^{(i),(N)}(r)$, the displacement components are

$$u_r^{(i)}(r) \equiv u_r^{(i),(N)}(r), \qquad u_\theta^{(i)} = 0, \qquad u_z^{(i)} = 0 \ ,$$

3-9

where the latter is simply the plane strain condition, stated originally in eq 3-2. The non-zero component of the additional displacement during the N^{th} loading is

$$\Delta u_r^{(i)}(r) = \Delta u_r^{(i),(N)}(r) \ .$$

3-10

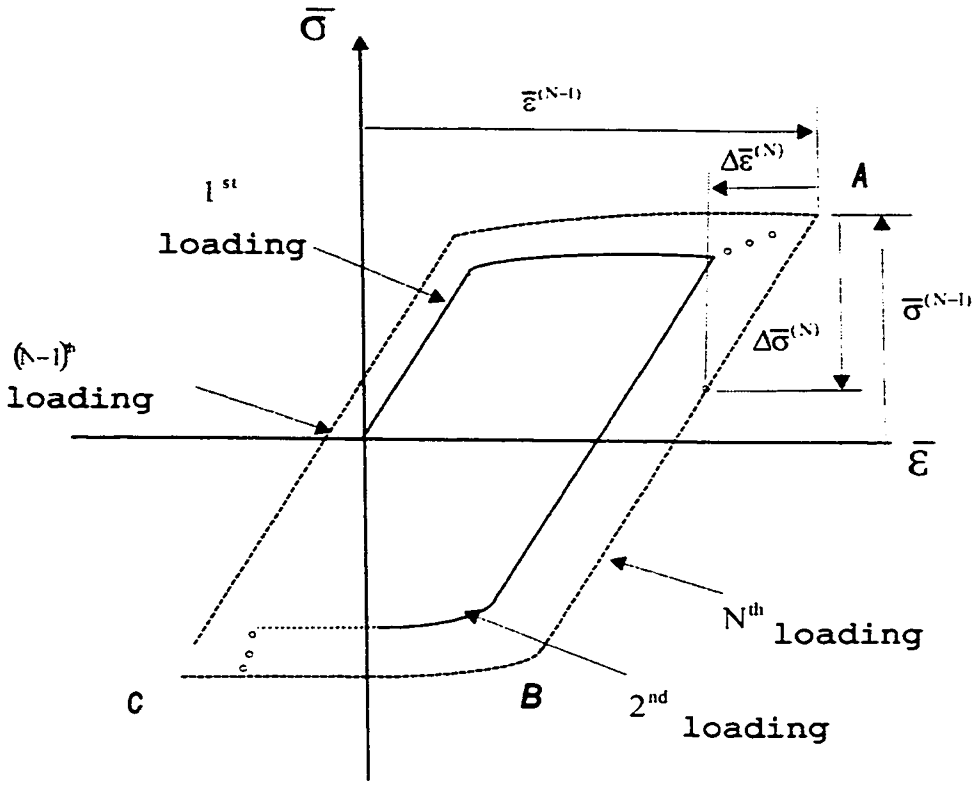

Figure 3.2: A schematic of the cyclic biaxial stress-strain response of the composite cylinder

The non-vanishing additional strain components are (Malvern, [1])

$$\Delta\varepsilon_{rr}^{(i),(N)} = \frac{\partial \Delta u_r^{(i),(N)}}{\partial r} \quad , \quad \Delta\varepsilon_{\theta\theta}^{(i),(N)} = \frac{\Delta u_r^{(i),(N)}}{r} \quad .$$

3-11

The additional non-vanishing stress components follow from 1-1for the fiber and eq 1-3 for the matrix. During the elastic response of the matrix (and thereby the composite cylinder), the plane strain condition gives $\Delta\varepsilon_{zz}^{(2),(N)} = \Delta\varepsilon_{E.zz}^{(2),(N)} = 0$. An identical condition always holds for the elastic fiber. Using these conditions, the additional stress components for both phases may be summarized as

$$\Delta\sigma_{rr}^{(i),(N)} = 2\mu_i\,\Delta\varepsilon_{E,rr}^{(i),(N)} + \frac{1}{3}\left(1-\frac{2\mu_i}{3\kappa_i}\right)\Delta\sigma_{kk}^{(i),(N)}$$

$$\Delta\sigma_{\theta\theta}^{(i),(N)} = 2\mu_i\,\Delta\varepsilon_{E,\theta\theta}^{(i),(N)} + \frac{1}{3}\left(1-\frac{2\mu_i}{3\kappa_i}\right)\Delta\sigma_{kk}^{(i),(N)}$$

$$\Delta\sigma_{zz}^{(i),(N)} = \frac{1}{3}\left(1-\frac{2\mu_i}{3\kappa_i}\right)\Delta\sigma_{kk}^{(i),(N)} .$$

3-12

Simpler equations for the matrix (i=2) follow from eq 3-12 by using the incompressibility condition, $\kappa_2 \rightarrow \infty$ (which also implies $\Delta\varepsilon_{E,rr}^{(2),(N)} = -\Delta\varepsilon_{E,\theta\theta}^{(2),(N)}$). During the plastic deformation in the matrix, eq 3-12 continue to be valid (as we shall see later). The sole non-trivial component of the equilibrium equation is written in terms of the additional stress components as (Malvern, [1])

$$\frac{d\,\Delta\sigma_{rr}^{(i),(N)}}{dr} + \frac{1}{r}\left(\Delta\sigma_{rr}^{(i),(N)} - \Delta\sigma_{\theta\theta}^{(i),(N)}\right) = 0 \quad .$$

3-13

For the current displacement and stress fields, eq 3-3 are used to write only the non-trivial interface conditions in terms of the addition displacement and stress fields during the N^{th} loading as

$$\Delta u_r^{(1),(N)}(a) = \Delta u_r^{(2),(N)}(a) \quad \text{and} \quad \Delta\sigma_{rr}^{(1),(N)}(a) = \Delta\sigma_{rr}^{(2),(N)}(a) \quad .$$

3-14

3.3 Solution procedure for the N^{th} loading

The fiber is assumed to always remain elastic whereas the matrix may undergo elasto-plastic deformation. During the N^{th} sequence of loading , as $\Delta\bar{\sigma}^{(N)}$ is increased (decreased) from zero during tensile loading (compressive loading), the overall composite response will be initially elastic. The matrix will then yield and be in an elastic- plastic state. This will continue until the matrix has completely yielded. The solution procedure for $\Delta\bar{\sigma}^{(N)}$ and $\Delta\bar{\varepsilon}^{(N)}$ will be now given for these three stages of deformation. The initial values of the strain and stress fields at the onset of the N^{th} loading will be given by $\varepsilon^{(i),(N-1)}$ and $\sigma^{(i),(N-1)}$; all these are assumed to be known.

3.3.1 *Elastic state of the composite cylinder and the commencement of yielding*

The solution for the additional displacement field follows from eq 3-10 to eq 3-13 as

$$\Delta u_r^{(i),(N)}(r) = A_1^{(i),(N)} r + A_2^{(i),(N)} \frac{b^2}{r}, \qquad (i = 1,2) \; .$$

3-15

The four constants $A_{(j)}^{(i),(N)}$ (i,j=1,2) are found using the two interface conditions, the traction boundary condition (eq 3-1) and an additional condition where we require $\Delta u_r^{(1),(N)}(r)$ (i.e. additional displacement in the fiber during the N^{th} loading) to be finite as $r \rightarrow 0$. The constants are

$$A_1^{(1),(N)} = \frac{3}{2[3\kappa_1 + \mu_1 + 3\mu_2(1-c_1)]} \; \Delta\bar{\sigma}^{(N)}, \quad A_2^{(1),(N)} = 0, \quad A_1^{(2),(N)} = 0, \quad A_2^{(2),(N)} = c_1 A_1^{(1),(N)} \; ,$$

3-16

where $c_1 = (a/b)^2$ is the fiber volume fraction. The onset of plastic deformation in the matrix during the N^{th} loading is assumed to occur when the effective stress in the matrix $\sigma_e^{(2),(N)}$ equals the yield stress $\sigma_Y^{(N)}$. The effective stress follow from eq 1-5 and the total form of eq 3-12 as

$$\sigma_e^{(2),(N)} = \frac{\sqrt{3}}{2}\left|\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}\right| \quad .$$

3-17

The yield condition is then stated as

$$\left|\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}\right| = \frac{2}{\sqrt{3}}\sigma_Y^{(N)} \quad .$$

3-18

We assume that the matrix undergoes isotropic hardening ; therefore , the magnitude of $\sigma_Y^{(N)}$ is set equal to the effective stress attained at the point of load reversal (at the end of the (N-1) th loading) . Therefore

$$\sigma_Y^{(N)} = \sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)} \quad ,$$

3-19

where $\sigma_Y^{(N-1)}$ is the magnitude of the yield stress during the $(N\text{-}1)^{th}$ loading and $\varepsilon_{p,e}^{(2),(N-1)}$ is the effective plastic strain computed from the additional plastic strain, $\varepsilon_p^{(2),(N-1)}$, in the matrix during the N^{th} loading. eq 3-18 and eq 3-19 may be combined to write

$$\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} = \pm\frac{2}{\sqrt{3}}(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \qquad \text{if} \qquad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \begin{smallmatrix} > \\ < \end{smallmatrix} 0$$

3-20

We now use the decomposition

$$\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} = \sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)} + \Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)} \quad ,$$

3-21

in eq 3-20 to write

$$\Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)} = \pm\left[\frac{2}{\sqrt{3}}(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \mp (\sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)})\right]$$

$$\text{if} \quad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \gtrless 0$$

3-22

At the onset of yielding during the Nth loading, eq 3-15, eq3-16, eq 3-11 and eq 3-12 give

$$\Delta\sigma_{rr}^{(2),(N)} - \Delta\sigma_{\theta\theta}^{(2),(N)} = -\frac{6\mu_2 c_1}{[3k_1 + \mu_1 + 3\mu_2(1-c_1)]}\left(\frac{b}{r}\right)^2 \Delta\bar{\sigma}^{(N)} \quad ;$$

3-23

We equate eq 3-22 and eq 3-23 and write $\Delta\bar{\sigma}^{(N)}$ as

$$\Delta\bar{\sigma}^{(N)} = \mp\left[\frac{3k_1 + \mu_1 + 3\mu_2(1-c_1)}{6c_1\mu_2}\right]^{-1}\left(\frac{r}{b}\right)^2\left[\frac{2}{\sqrt{3}}(\sigma_Y^{(N-1)} + h\Delta\varepsilon_{P,e}^{(2),(N-1)}) \mp (\sigma_{rr}^{(2),(N-1)} - \sigma_{\theta\theta}^{(2),(N-1)})\right]$$

$$\text{if} \quad \sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)} \gtrless 0$$

3-24

Note that the expression represented by eq 3-24 gives the value of $\Delta\bar{\sigma}^{(N)}$ corresponding to commencement of yielding in the matrix during the Nth loading. However, since $\Delta\bar{\sigma}^{(N)}$ is r-dependent explicitly through the term $(r/b)^2$ and implicitly through the term in the square brackets, we need to identify the value of r in the range , $a \le r \le b$, at which $\Delta\overline{\sigma}^{(N)}$, has the minimum magnitude. This value of $\Delta\overline{\sigma}^{(N)}$ is identified as the additional composite yield stress, $\Delta\overline{\sigma}_Y^{(N)}$ during the N[th] loading and the corresponding value of r identifies the location where the yielding commences. It may be shown that the lowest value of $\Delta\overline{\sigma}^{(N)}$ corresponds to r = a , i.e. yielding during the N[th] loading commences at the interface. The additional composite biaxial stress at the onset of yielding during the N[th] is then

$$\Delta\overline{\sigma}_Y^{(N)} = \Delta\overline{\sigma}^{(N)}\Big|_{r=a} \quad ,$$

3-25

and the corresponding total composite biaxial yield stress during the N[th] loading follows from the first of eq 3-6 as

$$\overline{\sigma}_Y^{(N)} = \overline{\sigma}^{(N-1)} + \Delta\overline{\sigma}^{(N-1)}\Big|_{r=a} \quad .$$

3-26

3.3.2 *Elastoplastic state of the composite cylinder*

During the elastoplastic deformation of the matrix , the interface between the plastic and elastic zones in the matrix originates at r = a. With an increase in the biaxial stress , the radius of the plastic zone gradually increases until it reaches the periphery of the matrix , r = b. At this stage , the matrix is fully yielded. We denote $R^{(N)}$ $(a \le R^{(N)} \le b)$ as the radius of the plastic-elastic interface during the elastoplastic deformation of the matrix (see fig 3.1). Since the fiber is assumed to be elastic, the solution for the strain and stress fields in the fiber will continue to be the same as before. Here, we derive the fields in the plastic portion and elastic portion of the matrix, referred henceforth as phases 3 and 2 respectively.

The plastic portion of the matrix

The incremental plastic strains follow from eq 1-8 as

$$d\Delta\varepsilon_{p.ij}^{(3).(N)} = d\lambda \ \ \sigma_{ij}^{(3)'.(N)} ,$$

3-27

whereas the additional plastic strains accumulated during the Nth loading of the composite cylinder follow as

$$\Delta\varepsilon_{p.ij}^{(3).(N)} = \int d\Delta\varepsilon_{p.ij}^{(3).(N)} .$$

3-28

Using eq 3-12 and eq 3-27, the non-vanishing incremental plastic strains during the Nth loading are

$$d\,\Delta\varepsilon_{p.rr}^{(3).(N)} = \frac{1}{2} d\lambda\left(\sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)}\right) , \quad d\,\Delta\varepsilon_{p.\theta\theta}^{(3).(N)} = -d\,\Delta\varepsilon_{p.rr}^{(3).(N)} .$$

3-29

With the latter of eq 3-29, the additional plastic strain components are related as

$$\Delta\varepsilon_{p.\theta\theta}^{(3).(N)} = -\,\Delta\varepsilon_{p.rr}^{(3).(N)} .$$

3-30

Using eq 1-6, the effective plastic strain corresponding to the additional plasticity during the Nth loading becomes

$$\Delta\varepsilon_{P.e}^{(3).(N)} = \left(\frac{2}{3}\Delta\varepsilon_{P}^{(3).(N)} : \Delta\varepsilon_{P}^{(3).(N)}\right)^{1/2} ,$$

3-31

which, using eq 3-30, reduces to

$$\Delta\varepsilon_{P.e}^{(3).(N)} = \frac{2}{\sqrt{3}} \left| \Delta\varepsilon_{P.rr}^{(3).(N)} \right| .$$

3-32

From eq 3-29 we may infer that since $d\lambda$ in positive , the sign of the incremental plastic strain, $d\Delta\varepsilon_{p.rr}^{(3).(N)}$ (and thereby, the sign of the additional plastic strain $\Delta\varepsilon_{p.rr}^{(3).(N)}$) depends on the sign of $\sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)}$. Defining $\Delta\varepsilon_{p.rr}^{(3).(N)} = \Delta\varepsilon_{p}^{(N)}$, eq 3-32 may be written as

$$\Delta\varepsilon_{P.e}^{(3).(N)} = \pm\frac{2}{\sqrt{3}}\Delta\varepsilon_{P}^{(N)} \qquad \text{if} \qquad \sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)} \gtrless 0 \ .$$

3-33

The plastic constitutive equation follows from eq 1-7 as

$$\sigma_{e}^{(3).(N)} = \sigma_{Y}^{(N)} + h\Delta\varepsilon_{P.e}^{(3).(N)} \ ,$$

3-34

where $\sigma_{Y}^{(N)}$ has been defined in eq 3-26. We may now combine 3-33 and 3-34 and use eq 3-12 to write

$$\sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)} = \pm\sigma_{Y}^{(N)} + \frac{2}{\sqrt{3}}h\Delta\varepsilon_{P}^{(N)} \qquad \text{if} \qquad \sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)} \gtrless 0 \ .$$

3-35

eq 3-35 is re-written in terms of additional stresses during the N^{th} loading as

$$\Delta\sigma_{rr}^{(3).(N)} - \Delta\sigma_{\theta\theta}^{(3).(N)} = \pm\sigma_{Y}^{(N)} + \frac{2}{\sqrt{3}}h\Delta\varepsilon_{P}^{(N)} - (\sigma_{rr}^{(3).(N-1)} - \sigma_{\theta\theta}^{(3).(N-1)}) \qquad \text{if} \qquad \sigma_{rr}^{(3).(N)} - \sigma_{\theta\theta}^{(3).(N)} \gtrless 0 \ .$$

3-36

The sole unknown quantity is eq 3-36 is $\Delta\varepsilon_{p}^{(N)}$; this is what we set out to determine next. eq 3-11 allows us to write

$$\Delta\varepsilon_{rr}^{(3).(N)} = \frac{\partial}{\partial r}(r\Delta\varepsilon_{\theta\theta}^{(3).(N)}) \ .$$

3-37

The additional net strains are decomposed into an elastic and plastic part

$$\Delta\varepsilon_{rr}^{(3),(N)} = \Delta\varepsilon_{E,rr}^{(3),(N)} + \Delta\varepsilon_{P,rr}^{(3),(N)} = \Delta\varepsilon_{E,rr}^{(3),(N)} + \Delta\varepsilon_{P}^{(N)}$$
$$\Delta\varepsilon_{\theta\theta}^{(3),(N)} = \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} + \Delta\varepsilon_{P,\theta\theta}^{(3),(N)} = \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} - \Delta\varepsilon_{P}^{(N)}$$

3-38

eq 3-12 allows us to write

$$\Delta\varepsilon_{E,rr}^{(3),(N)} = \frac{3}{4E_2}\left(\Delta\sigma_{rr}^{(3),(N)} - \Delta\sigma_{\theta\theta}^{(3),(N)}\right) , \qquad \Delta\varepsilon_{E,\theta\theta}^{(3),(N)} = -\Delta\varepsilon_{E,rr}^{(3),(N)} ,$$

3-39

where $E_2 = 3\mu_2$. With eq 3-39 in eq 3-38 and eq 3-37, we have

$$\frac{3}{2E_2}(\Delta\sigma_{\theta\theta}^{(3),(N)} - \Delta\sigma_{rr}^{(3),(N)}) + \frac{3r}{4E_2}\frac{d\,\Delta\sigma_{\theta\theta}^{(3),(N)}}{dr} - \frac{3}{4E_2}\frac{d\,\Delta\sigma_{rr}^{(3),(N)}}{dr} = \left[2\Delta\varepsilon_{P}^{(N)} + r\frac{d\,\Delta\varepsilon_{P}^{(N)}}{dr}\right] .$$

3-40

The equilibrium equation, eq 3-13 may now be used in combination with eq 3-40 to get

$$r\frac{d(\Delta\sigma_{\theta\theta}^{(3),(N)} - \Delta\sigma_{rr}^{(3),(N)})}{dr} + 2(\Delta\sigma_{\theta\theta}^{(3),(N)} - \Delta\sigma_{rr}^{(3),(N)}) = \frac{4E_2}{3}\left[r\frac{d\,\Delta\varepsilon_{P}^{(N)}}{dr} + 2\Delta\varepsilon_{P}^{(N)}\right] .$$

3-41

When eq 3-36 is introduced in eq3-41, the resulting differential equation may be solved for $\Delta\varepsilon_{p}^{(N)}$, subject to the condition that $\Delta\varepsilon_{p}^{(N)}$ vanishes at the elastic-plastic interface, $R^{(N)}$ (yet undetermined). Thus

$$\Delta\varepsilon_{P}^{(N)}(R^{(N)}) = 0 .$$

3-42

The resulting expression for $\Delta\varepsilon_p^{(N)}$ will now be introduced to find the $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation during the composite elasto-plastic response. As a first step, eq 3-13 is re-written in the following integral form

$$\Delta\sigma_{rr}^{(3),(N)} = -\int_a^r \frac{1}{r}(\Delta\sigma_{rr}^{(3),(N)} - \Delta\sigma_{\theta\theta}^{(3),(N)})dr + D \ ,$$

3-43

in terms of a parameter D where the integrand may be written in terms of $\Delta\varepsilon_p^{(N)}$ (using 1-5).At this stage, a total of five parameters have to be determined: $A_1^{(1),(N)}, A_2^{(1),(N)}, A_2^{(2),(N)}$ (refer eq 3-15) the constant D (refer eq 3-43)and the radius $R^{(N)}$ of the elastic-plastic interface. On the other hand, there are six conditions that need to be met: the two interface conditions (eq 3-14 where index 2 in replaced by 3) , the traction boundary condition (first of eq 3-8) and the following three additional conditions at the elastic-plastic interface

$$\Delta u_r^{(3),(N)}(R^{(N)}) = \Delta u_r^{(2),(N)}(R^{(N)}) \quad ,$$

$$\Delta\sigma_{rr}^{(3),(N)}(R^{(N)}) = \Delta\sigma_{rr}^{(2),(N)}(R^{(N)}) \quad ,$$

$$|\sigma_{rr}^{(3),(N)} - \sigma_{\theta\theta}^{(3),(N)}| = |\sigma_{rr}^{(2),(N)} - \sigma_{\theta\theta}^{(2),(N)}| = \sigma_Y^{(N)} \quad \text{at} \quad r = R^{(N)}$$

3-44

The first two conditions are the continuity of displacements and the traction at the elastoplastic interface. The third condition states that the effective stress is continuous and is equal to the yield stress at $r = R^{(N)}$. Thus, while five parameters have to be determined there are six conditions to be met. In fact, it is possible to prove that the last condition is not an independent one, and does, in fact, follow from the first two of eq 3-44. We shall not go into the details here.

The five unknown parameters may now be determined. The additional radial stress, $\Delta\sigma_{rr}^{(3),(N)}$, will follow from eq 3-43, and the additional circumferential stress, $\Delta\sigma_{\theta\theta}^{(3),(N)}$, will follow from eq 3-36. The additional elastic strains follow from eq 3-39 and the additional net strain in the plastic portion of the matrix will follow from eq 3-38. In particular, the additional displacement , $\Delta u_r^{(2),(N)}$ in the elastic portion of the matrix will follow from eq 3-15. The additional composite strain will follow from the second of eq 3-8 and the total composite strain will follow from the second of eq 3-6.

3.3.3 ***Fully plastic state of the composite cylinder***

As the matrix enters into a fully plastic state, so does the composite cylinder. The equations corresponding to fully plastic state may be developed in a manner identical to the procedure for the partially plastic state (see section 3.2.2) The key equations are eq 3-36 and eq3-41 which, when combined, give a differential equation $\Delta\varepsilon_p^{(N)}$. The differential equation then solved subject to the condition that

$$\Delta\varepsilon_p^{(N)}\,|_{r=b} = \Delta\varepsilon_p^{(N)}(b) \quad .$$

3-45

where $\Delta\varepsilon_p^{(N)}(b)$ will nave to be determined. Three unknown parameters have to be determined: $A_I^{(1),(N)}$ (refer eq 3-15), $\Delta\varepsilon_p^{(N)}(b)$ (refer eq 3-45) and the constant D (refer eq 3-43) . These three parameters will follow from the two interface conditions eq 3-14 and the traction boundary condition (first of eq 3-8) .

The inductive approach and the analytical solution follow next. In order to get a tractable analytical solution, we have restricted it to the situation where the matrix is in a fully plastic state at the onset of the N^{th} loading . This assumption is identical to the one made for the spherical problem, as outlined in sec. 2.5.

3.4. **The inductive approach**

The crux of the inductive approach is explained by the following example. Suppose we are required to predict the Nth term in a sequence of numbers

$$1, \quad x, \quad x^2, \quad x^3 \ldots$$

It is obvious by induction that the Nth term is x^{N-1}. A similar approach is taken in the current problem where we search for a function that will represent the $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation for any N. The steps that have been followed in developing the equations are now given.

1 Develop the $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation rigorously (i.e. based on Sec. 3.3) for the first q loadings ($1 \le N \le q$).

2. Attempt to find a function in terms of the parameter N (and material properties) such that it gives the $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation for all of the first q loadings ($1 \le N \le q$).
3. If step 2 is successfully completed, then use the developed function to predict the $\Delta\bar{\sigma}^{(q+1)} - \Delta\bar{\varepsilon}^{(q+1)}$ relation.
4. Rigorously develop the $\Delta\bar{\sigma}^{(q+1)} - \Delta\bar{\varepsilon}^{(q+1)}$ relation based on Sec. 3.3.
5. Compare the $\Delta\bar{\sigma}^{(q+1)} - \Delta\bar{\varepsilon}^{(q+1)}$ relation developed in Steps 3 and 4. If the outcome is identical in both cases, then the generalized $\Delta\bar{\sigma}^{(N)} - \Delta\bar{\varepsilon}^{(N)}$ relation is considered to have been found. Otherwise, the entire process is repeated by going back to Step 1 and increasing q by 1.

The above iterative process becomes increasingly complicated as q increases. This is because with increasing number of cyclic loadings, it becomes considerably harder to find the necessary function (see step 2). We were successful in identifying a function that represented the first 10 loadings. The developed function was also able to successfully predict the 11th, 12th and 13th loadings developed rigorously based on sec. 3.3 . At this stage, we assume that the function for any N has been determined . This assumption is confirmed with finite element calculations in sec 3.6 . We now turn to the issue of explaining how we identified the generalized function.

Without any loss in generality, we focus on the fully plastic state of the composite cylinder during the 7th loading sequence. It is possible to represent the $\Delta\overline{\sigma}^{(7)} - \Delta\overline{\varepsilon}^{(7)}$ relation as

$$\Delta\overline{\varepsilon}^{(7)} = P_{10} c_1 \Delta\overline{\sigma}^{(7)} + P_{11}$$

3-46

where the parameter P_{10} and P_{11} are given in the Appendix. While both parameters are given in terms of material properties and the fiber volume fraction, c_1, in particular, P_{11} is also given in terms of P_4 and P_{14} . These latter parameters represent two different series (see Appendix). The number of terms in each series increase with increasing N. The key (and the most difficult) issue was to find a general expression that could each describe the terms P_4 and P_{14} . We shall attempt to explain this issue with regard to P_4. If P_4 was to be written out explicitly for N = 7, it will have several terms, all involving different exponents of 'hP_3'. For N = 7, we have

$$P_4 = \ldots - 2(2)^5 (hP_3)^5 - 10(2)^4 (hP_3)^5 + (2)^6 (hP_3)^6 + \ldots$$

3-47

where only some of the terms in P_4 have been given. All the terms in P_4 is arranged in Table 1. In the first column, all terms involving identical exponents of 'hP_3' have been grouped in blocks (i.e. block A has all terms involving $(hP_3)^5$). The numbers in the top row refer to the loading sequence. Thus in order to generate the terms in P_4 for N = 7 we look at the column for N =7 (9th column from left). As we go down this column, we see that some of the cells are empty whereas some have non-zero values.

N =Number of Loadings

N			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Block A	$2^5h^5P_3^5$	1						1	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2
	$2^4h^5P_3^5$	10							-1	3	-5	7	-9	11	-13	15	-17	19	-21	23	-25	27
	$2^3h^5P_3^5$	40								1	-4	9	-16	25	-36	49	-64	81	-100	121	-144	169
	$2^2h^5P_3^5$	80									-1	5	-14	30	-55	91	-140	204	-285	385	-506	650
	$2h^5P_3^5$	80										1	-6	20	-50	105	-196	336	-540	825	-1210	1716
	$h^5P_3^5$	32											-1	7	-27	77	-182	378	-714	1254	-2079	3795
Block B	$2^6h^6P_3^6$	1							1	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2
	$2^5h^6P_3^6$	12								-1	3	-5	7	-9	11	-13	15	-17	19	-21	23	-25
	$2^4h^6P_3^6$	60									1	-4	9	-16	25	-36	49	-64	81	-100	121	-144
	$2^3h^6P_3^6$	160										-1	5	-14	30	-55	91	-140	204	-285	385	-506
	$2^2h^6P_3^6$	240											1	-6	20	-50	105	-196	336	-540	825	-1210
	$2h^6P_3^6$	192												-1	7	-27	77	-182	378	-714	1254	-2079
	$h^6P_3^6$	64													1	-8	35	-112	294	-672	1386	-2640
Block C	$2^7h^7P_3^7$	1								1	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2
	$2^6h^7P_3^7$	14									-1	3	-5	7	-9	11	-13	15	-17	19	-21	23
	$2^5h^7P_3^7$	84										1	-4	9	-16	25	-36	49	-64	81	-100	121
	$2^4h^7P_3^7$	280											-1	5	-14	30	-55	91	-140	204	-285	385
	$2^3h^7P_3^7$	560												1	-6	20	-50	105	-196	336	-540	825
	$2^2h^7P_3^7$	672													-1	7	-27	77	-182	378	-714	1254
	$2h^7P_3^7$	...														1	-8	35	-112	294	-672	1386
	$h^7P_3^7$	...															-1	9	-44	156	-450	1122

Table 1: The sample matrix used to explain the inductive approach

The empty cells are assigned a value of zero. With this in mind, let us generate one term in P_4 for N= 7. The second cell for N = 7 has an entry, -1. We multiply this entry with the term $(2)^4(hP_3)^5$ (3^{rd} row , 1^{st} column) and the entry 10 (3^{rd} row , 2^{nd} column) to get $-10(2)^4(hP_3)^5$ Other terms in P_4 are generated in a similar manner by going down the column for N =7 . Note that we have shown only a portion of the complete table. The terms in P_4 are arranged in the tabular form for $1 \le N \le 10$. Interestingly enough the entire table may be generated by two expressions, written in terms of the parameter N. We shall explain the development of one of these expressions in detail. Let us focus on the entries for $1 \le N \le 10$ in block A (the first six rows) . A repetitive pattern may be identified . Note the three cell entries within the L-shaped box. The entry ,3, is the negative of the sum of -1 and -2 . If this L-shaped box is translated to include any other group of 3 entries in block A and within the columns, $1 \le N \le 10$, an identical relation is observed . Based on this feature, the remaining cells in block A beyond N> 10 have been filled up. Also note that the entire set of cell entries in block A are repeated in block B (and in block C), except with a one-column shift (and a two-column shift) to the right. After careful study the entry in a generic cell is found to be represented by the term ${}^{(L_2-1)}c_{(r-1)} + 2\,{}^{(L_2-1)}c_r$ in P_4 (see Appendix). The entries in the first column and second columns also may be represented by generalized expressions (not discussed in detail here). The overall outcome is the series for P_4 given in the Appendix. An identical approach is taken to develop P_{14}. These two terms appear in the equations for the N^{th} loading sequence, during the elastic state, elasto-plastic and fully plastic state of the composite cylinder

3.5. The analytical solution

The $\Delta\overline{\sigma}^{(N)} - \Delta\overline{\varepsilon}^{(N)}$ relation is now given for the three stages of deformation - elastic state, elastoplastic state and fully plastic state of the composite cylinder - during the Nth loading. Once $\Delta\overline{\sigma}^{(N)} - \Delta\overline{\varepsilon}^{(N)}$ is known for any N, the $\Delta\overline{\sigma}^{(N)} - \Delta\overline{\varepsilon}^{(N)}$ relation follows from eq 3-39.

3.5.1 ***Elastic state of the composite cylinder and the commencement of yielding***

$$\Delta\bar{\varepsilon}^{(N)} = P_1 \, \Delta\bar{\sigma}^{(N)} \ , \tag{3-48}$$

where P_1 is given in the Appendix. The additional biaxial yield stress of the composite cylinder during the N^{th} loading (defined in eq 3-25) is given as

$$\Delta\bar{\sigma}_Y^{(N)} = \frac{\sqrt{3}\sigma_y P_2}{6}\left[(2hP_3)^{(N-1)} + P_4 + P_5\right] - \frac{hP_6P_2P_7}{3} \, . \tag{3-49}$$

where P_2, P_3, P_4, P_5, P_6, and P_7 are given in the Appendix.

3.5.2 ***Elastoplastic state of the composite cylinder***

$$\Delta\bar{\varepsilon}^{(N)} = \frac{1}{P_8}\Delta\bar{\sigma}^{(N)} + \left(\frac{P_9}{P_8} - \frac{hP_6P_7c_1}{2\mu_2(1-\nu_2)}\right) \ , \tag{3-50}$$

where P_6, P_7, P_8, and P_9 are given in the Appendix.

3.5.3 ***Fully plastic state of the composite cylinder***

$$\Delta\bar{\varepsilon}^{(N)} = P_{10}c_1\Delta\bar{\sigma}^{(N)} + P_{11} \quad , \tag{3-51}$$

where P_{10} and P_{11} are given in the Appendix.

3.6. Parametric studies

The cyclic biaxial response of the composite cylinder is studied in this section. We shall assume that the matrix has the elastoplastic properties of a 6061 aluminum (Arsenault [2], Nieh and Chellman [3])

$$E_2 = 68.3 \text{ GPa}, \qquad \nu_2 = \frac{1}{3}, \qquad \sigma_y = 250 \text{ MPa}, \qquad h = 173 \text{ MPa} . \tag{3-52}$$

3.6.1 ***Finite element validation***

Since the derivation of the exact solution is based on an inductive approach (or an "educated guess" approach), it is desirable to validate the results; this is done by the finite element method. The lateral cross-section of the composite cylinder is shown in fig 3.3.

Due to the axisymmetry of the problem, it is sufficient to consider the rectangular domain (of width ΔL) inscribed in that figure. This rectangular domain is considered as the representative volume element for the numerical implementation. The RVE is depicted in fig 3.4; assuming u and v are the x- and y-displacements of a continuum point , the boundary conditions applied to the RVE $(0 \le x \le \Delta L, 0 \le y \le b)$ are given 3-53 to eq 3-55 next.

$$u(0, y) = 0, \qquad u(\Delta L, y) = 0 , \tag{3-53}$$

$$v(x,0) = 0 , \tag{3-54}$$

$$\sigma_{yy}(x, b) = \bar{\sigma}, \qquad \sigma_{yx}(x, b) = 0 . \tag{3-55}$$

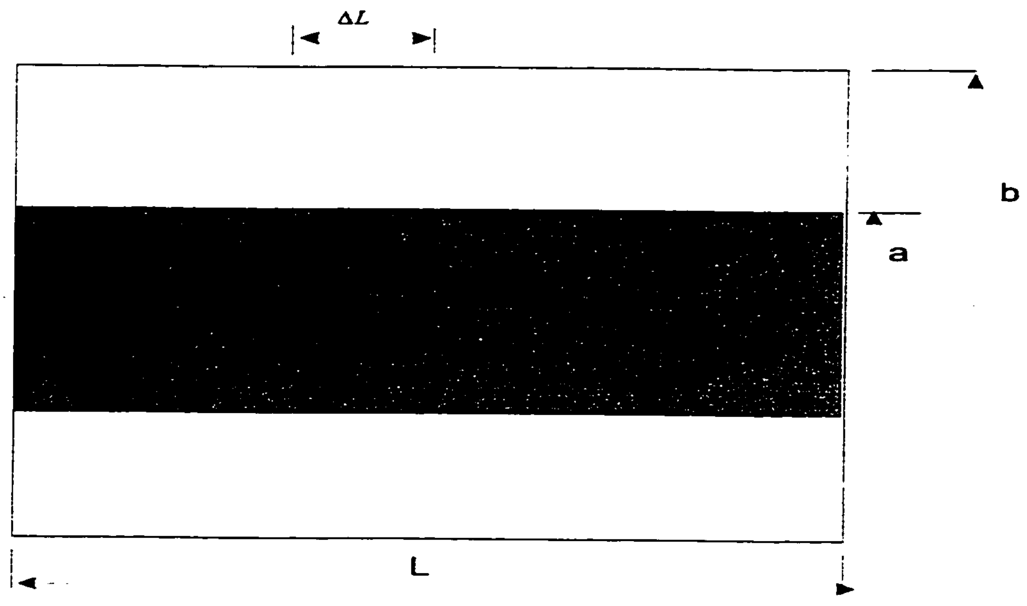

Figure 3.3 Cross section of a cylindrical composite

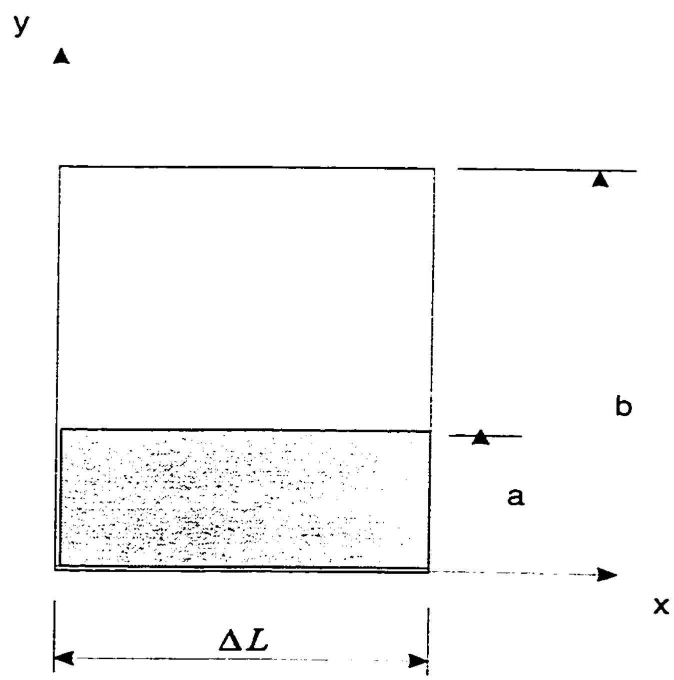

Figure 3.4 The representative volume element (RVE).

The condition of axisymmetry about the axis of the composite cylinder (x-axis in fig 3.4) is enforced in the finite element software, (ANSYS 52), by invoking the " axisymmetry" option appropriately. After some numerical experimentation, it was determined that the optimum number of elements was 25 x 25 for the fiber ($y \leq a$) and 25 x 25 for the matrix ($a \leq y \leq b$). The comparison between the exact solution and the finite element results are shown in fig 3.6. The excellent agreement between the two approaches is taken as a validation of the exact solution.

The parametric study will attempt to demonstrate the influence of the following parameters on the biaxial response of the composite cylinder: 1). The inclusion volume fraction, c_1 2). The relative stiffness of the Young's modulus of inclusion to matrix, E_1 / E_2, and 3). The tangent modulus, E_p, of the matrix.

The evolution in the $\overline{\sigma} - \overline{\varepsilon}$ response is demonstrated for a stress controlled fig 3.5 and a strain controlled process fig 3.6. The parameters, $\overline{\sigma}^{CR}$ and $\overline{\varepsilon}^{CR}$ (see fig 3.5 and fig 3.6) respectively denote the stress control and strain-controlled values. In either case, the effect of the fiber is to induce hardening with continued cyclic loading, leading to an overall linear stress-strain relation. Due to the interchangeability of the traction and displacement boundary condition, we shall focus on stress controlled process in the rest of the parametric study.

3.6.2 *The range of applicability of the exact solution*

Recall that the exact solution for the N^{th} loading is contingent upon the assumption that at its onset (or at the end of the (N-1) loading) the matrix is in a fully plastic state. fig 3.7 and fig 3.8 gives the value of N beyond which the matrix fails to yield completely when the stress-control value is reached . It is below this critical value of N that the solution will then be applicable. fig 3.7 demonstrates that when the fiber is elastically softer than the matrix by an order of magnitude ($E_1 / E_2 = 0.1$), the critical value of N increases when the fiber volume fraction increases.

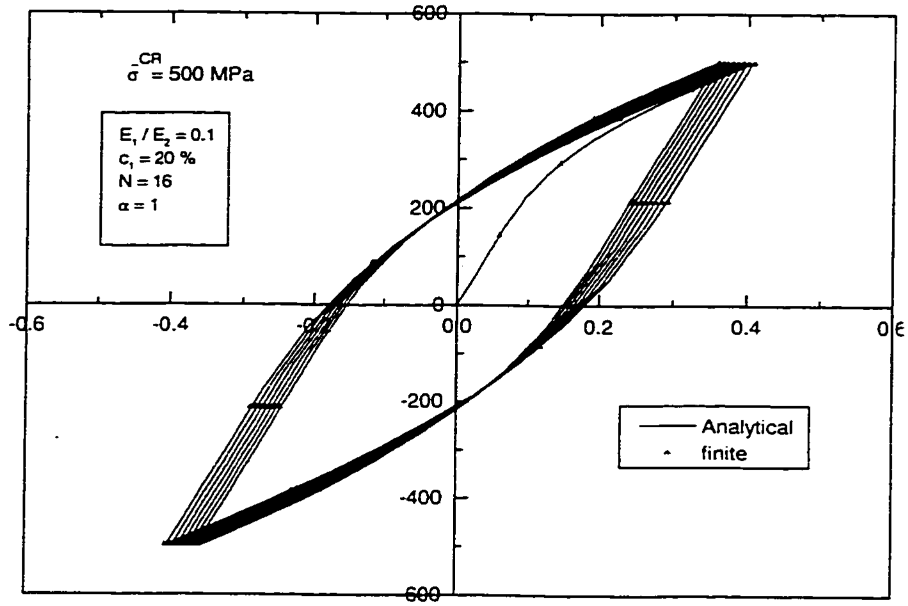

Figure 3.5: The cyclic stress-strain curve under stress control (fiber composite)

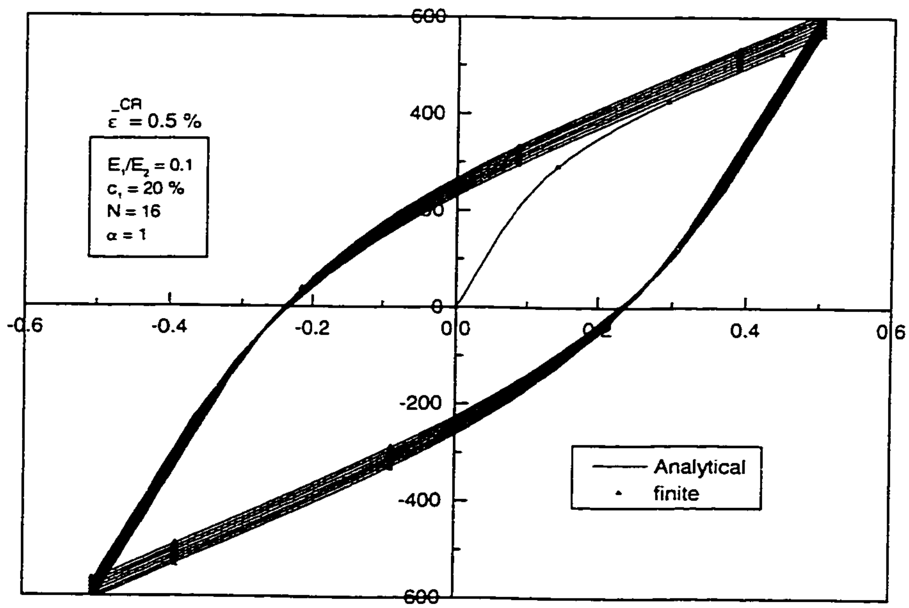

Figure 3.6:The cyclic stress-strain curve under strain control (fiber composite)

While this value is relatively insensitive to the stress control value , $\overline{\sigma}^{CR}$ (see bottom two curves), it is highly sensitive (and increases) as the bilinear matrix becomes plastically more compliant (the bottom and top curve in fig 3.7. However, if the cylinder has a co-axial cylindrical void ($E_1 / E_2 = 0.0$), the critical value of N reduces drastically over almost the entire range of c_1 . Summarizing these results, it is seen that the range of N over which the exact solution is applicable increases with (1) a higher fiber volume fraction, (2) the matrix is plastically more compliant, and (3) when a softer fiber approaches the stiffness of a harder matrix.

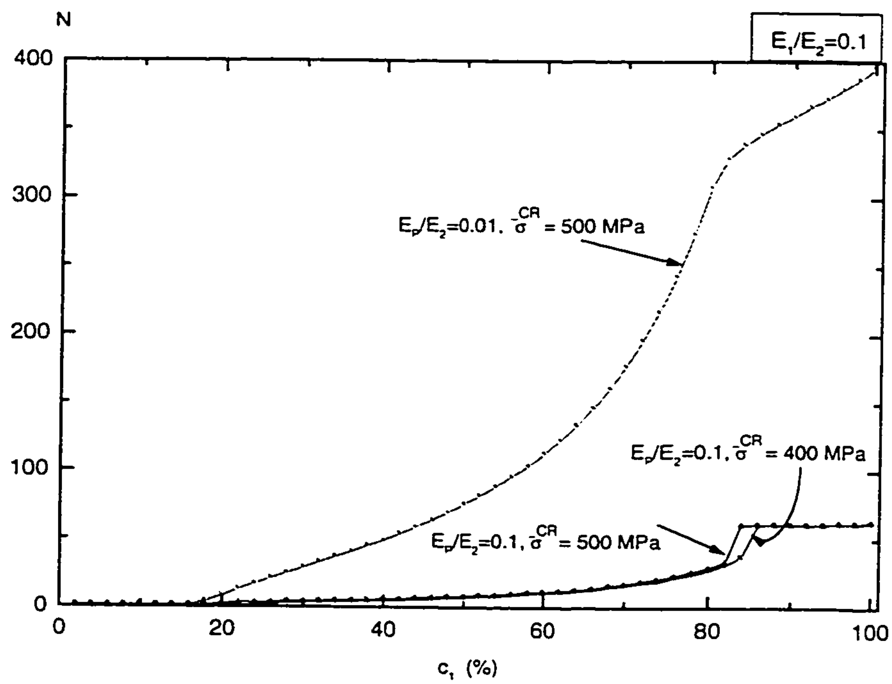

Figure 3.7: Inclusion volume fraction dependence on the critical number of loading

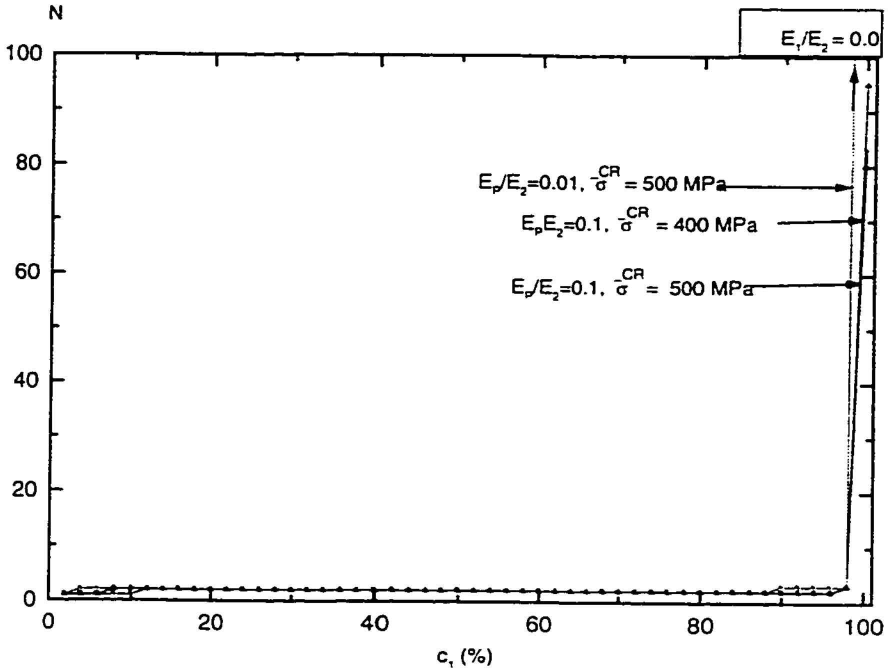

Figure 3.8: Inclusion volume fraction dependence of the critical number of loading

3.6.3 *The Bauschinger effect of the composite cylinder*

The Bauschinger effect of the composite cylinder is now studied. This is parameterized by α defined as

$$\alpha^{(N)} = \frac{1}{2}\left(1 + \frac{\left|\overline{\sigma}_Y^{(N)}\right| - \left|\overline{\sigma}_Y^{(N-1)}\right|}{\left|\overline{\sigma}^{(N-1)} - \overline{\sigma}_Y^{(N-1)}\right|}\right),$$

3-56

where $\left|\overline{\sigma}_Y^{(N)}\right|$ is the magnitude of the composite biaxial yield stress during the N^{th} loading and $\overline{\sigma}^{(N-1)}$ is the total biaxial stress at the end of the $(N-1)^{th}$ loading . The limiting values of $\alpha = 0$ and 1 correspond to pure kinematic and isotropic hardening respectively. The parameters on the right of eq 3-56 follow from the exact solution. As these parameters change during loading , so does α. The evolution of α is discussed in fig 3.9 to fig 3.11. In (fig 3.9), we set the stress-control value, $\overline{\sigma}^{CR} = 500\,\text{MPa}$. It is seen that if the ratio $E_1 / E_2 = 0.1$, the initial response of the composite cylinder is primarily isotropic (see solid line) , following which the α decreases to 0.5 , indicating that the long term response of the cylinder is governed equally by kinematic and isotropic mechanisms. With a softer fiber, $E_1 / E_2 = 0.01$, the transition of α to a value of 0.5 occurs faster. The same trend is seen irrespective of the strength coefficient of the matrix (fig 3.10) or the fiber volume fraction (fig 3.11). These lead us to conclude that the effect of the cyclic loading is to induce kinematic hardening in the overall biaxial stress-strain response of the composite cylinder .

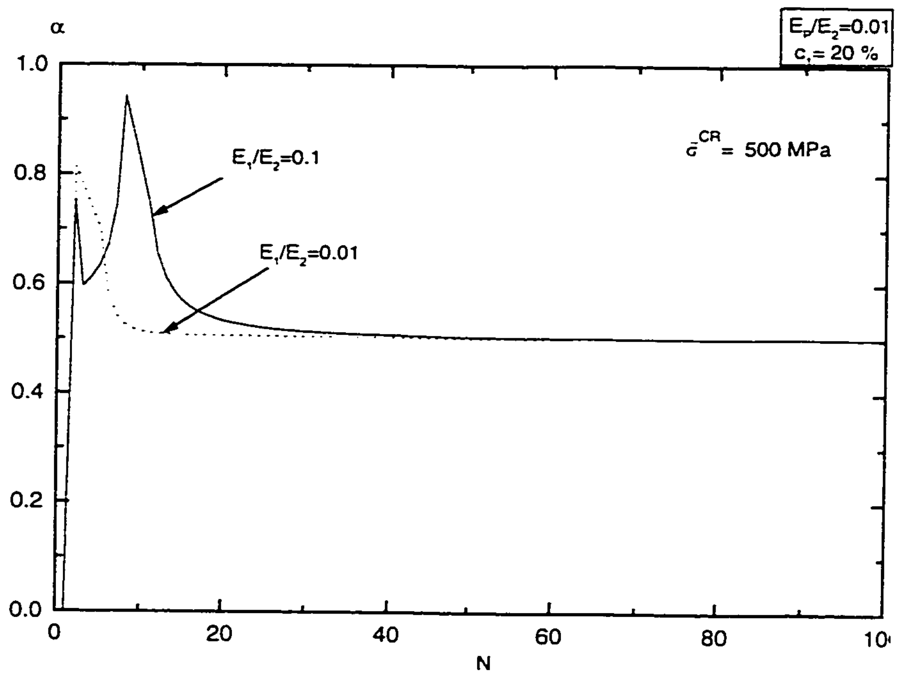

Figure 3.9: Bauschinger effect and the influence of the relative stiffness of inclusion/matrix

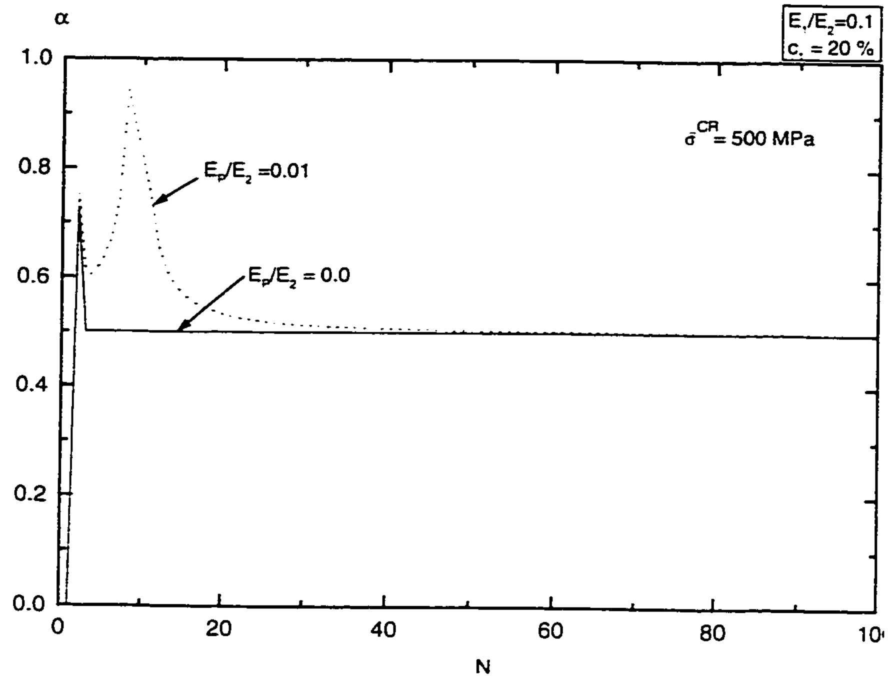

Figure 3.10: Bauschinger effect and the influence of the matrix tangent modulus

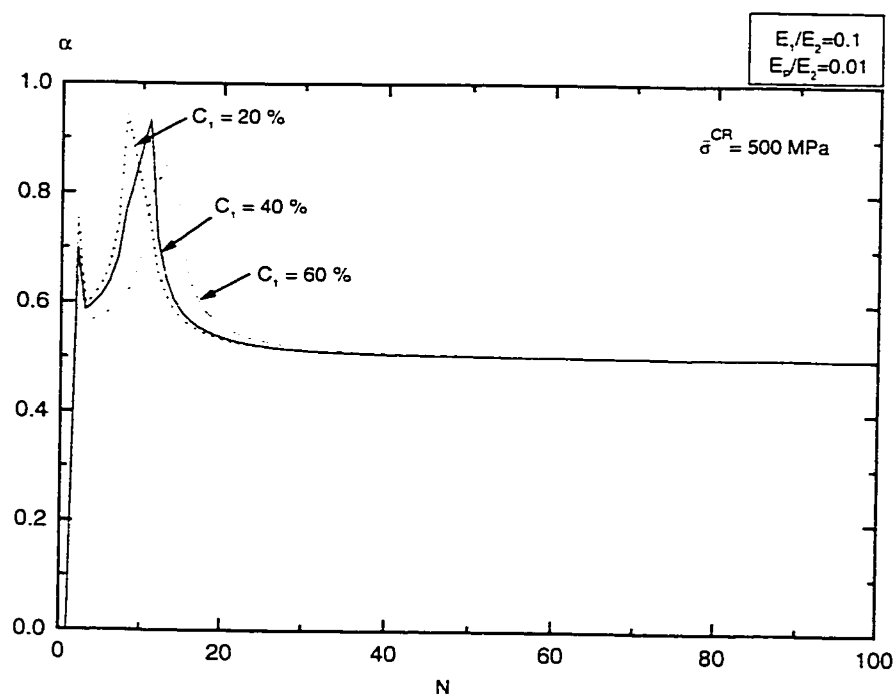

Figure 3.11: Bauschinger effect and the influence of the inclusion volume fraction

3.7 **Reference**

[1] Malvern, L.E.: Introduction to the Mechanics of a Continuous Medium. Prentice-Hall, Inc., New Jersey, pg.670 (1969).

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Chapter 4

CONCLUSIONS AND FUTURE WORK

4.1 Conclusions

We have determined exact solutions to the following two boundary value problems: (1) Cyclic elastoplastic hydrostatic loading of a composite sphere, (b) Cyclic elastoplastic biaxial loading of an infinitely long composite cylinder. The inclusion and matrix in both problems have been taken to be elastically isotropic. As well, the matrix in the latter problem is taken as elastically incompressible. The matrix is assumed to have a bilinear stress-strain relation and isotropic hardening is considered. Yielding in the matrix is taken to occur by the vonMises's criterion. Based on Hill's approach, the stress-strain relation is rigorously worked out for the first five(alternately tensile and compressive) sequences of cyclic loading in case of the first problem, and for the first thirteen sequences in case of the second problem. An inductive approach is then employed in either case to determine the stress-strain relation for the Nth sequence of cyclic loading(where N is an integer, $N \geq 1$). The developed relations for the cylinder problem are confirmed by comparing their predictions to finite element computations. Finally, the Bauschinger effect of the composite is studied in both cases and it is seen that the composite response initially shows almost pure isotropic hardening, but then moves towards a stable response where isotropic and kinematic hardening play equally important roles.

4.2 Advantages of the current work

The boundary value problems addressed in this thesis are probably the first of its kind involving cyclic elastoplastic loading of composites. While the problems are quite idealized, their utility lies in that the exact solutions may be used as benchmark problems against which the accuracy of finite element codes or approximate theories of composite plasticity may be tested. Moreover, implementation of the exact solutions on a computer are atleast two orders of magnitude faster than the corresponding finite element computations. Further, these solutions may be used to predict cyclic stress-strain relations for ductile materials containing spherical voids subjected to hydrostatic loading or containing aligned cylindrical voids and subjected to biaxial loading in the transverse plane.

4.3 Limitations of the current work

Several assumptions have been made in developing the exact solutions. The issue as to whether these limitations may be removed in a future extension of the current work is discussed in the next section. These are

(i) The case of hard inclusion/soft matrix has not been considered. This is relevant in case of several technologically important composites, e.g. ceramic particle reinforced and carbon fiber metal matrix composites.

(ii) The assumption of perfect bonding between the inclusion and the matrix may be appropriate during the initial stages of cyclic loading, but it may be more realistic to include the effects of interface degradation with continued loading. This has, however, not been included in the current work.

(iii) Another restriction in the model is that the matrix has been assumed to undergo isotropic hardening. In fact, a ductile matrix usually shows a response that is neither isotropic or kinematic(the two extremes of hardening behavior), but is usually a combination of the two.

(iv) We require that the matrix has to be in a fully yielded state during the point of composite load reversal.

4.4 Future Work

Any future extension of the results reported in this dissertation should ideally try to remove the restrictions listed in the previous section. In that context, we mention that the first restriction is the easiest to address; the derivation of the exact solution for the case of hard inclusion/soft matrix mimics the process outlined in this thesis.

The second limitation may be removed by assuming that the interface between the inclusion and matrix can be modeled as springs, such that the traction at thee interface is still continuous but the displacement has a jump. This jump in the displacement is then taken linearly proportional to the traction at the interface, where the constant of proportionality is a measure of the interfacial strength. This constant is infinite if the interface is perfect, and is zero if the interface is totally debonded. Usually, an interface degrades from being perfect(or near perfect) to being totally debonded, an effect which can be included in the model by using a constitutive law for the spring constant.

The third restriction may also be addressed in the context of the current approach. The hardening character of the matrix will be characterized by the use of a Bauschinger effect parameter, and the overall cyclic stress-strain relations will also be a function of this parameter.

The fourth restriction had been originally introduced in order to keep the problem tractable. In other words, it appears at this time that any future extension of the current work will have to retain this restriction in order to get tractable analytical solutions of cyclic elastoplastic loading of composite materials.

APPENDIX

The parameters in 2-57 to 2-60 are

$$P_1 = \frac{4\mu_2 + 3\kappa_1 + 3c_1(\kappa_2 - \kappa_1)}{\kappa_2(4\mu_2 + 3\kappa_1) - 4\mu_2 c_1(\kappa_2 - \kappa_1)}, \qquad P_2 = \frac{c_1(\kappa_2 - \kappa_1)}{\kappa_2(4\mu_2 + 3\kappa_1) - 4\mu_2 c_1(\kappa_2 - \kappa_1)}$$

$$P_3 = \frac{2(1 - \nu_2)(3\kappa_2 + \mu_2)\sigma_y}{9\kappa_2\mu_2 + 2h(1 - \nu_2)(3\kappa_2 + \mu_2)} \qquad \text{A 2-1}$$

$$P_4 = \begin{cases} 0 & N = 2 \\ \sum_{i=2}^{N-1} \left[\Delta\varepsilon_p^{(i)}(b) + 2HP_3(-1)^{i-1}(1-2H)^{i-2}\right] & N > 2 \end{cases}, \qquad \text{A 2-2}$$

where $\Delta\varepsilon_p^{(N)}(b)$ follows from eq. A 2-7.

$$P_5 = \frac{3\kappa_1\kappa_2}{2(\kappa_2 - \kappa_1)Hc_1} \qquad \text{A 2-3}$$

$$P_6 = \begin{cases} \frac{1}{2}\sigma_Y^{(1)}\left[\frac{4\mu_2 + 3k_2}{3\mu_2 k_2}\right]\left\{\frac{R^{(N)}}{b}\right\}^3 & \text{for} \quad N = 1 \\ \frac{1}{2}\sigma_Y^{(1)}\left[-2H(1-2H)^{N-2}\right]\left[\frac{4\mu_2 + 3k_2}{3k_2\mu_2}\right]\left\{\frac{R^{(N)}}{b}\right\}^3 + (-1)^N h\left[\frac{4\mu_2 + 3k_2}{3k_2\mu_2}\right]\left[P_4 + \Delta\varepsilon_p^{(1)}(b) - P_3\right] & \text{for} \quad N > 1 \end{cases}$$

A 2-4

$$P_7 = (-1)^N \frac{2h}{3}\left[P_4 + \Delta\varepsilon_p^{(1)}(b) - P_3\right]\left[\frac{2}{c_1} - 2\left(\frac{b}{R^{(N)}}\right)^3 + \frac{3P_3P_5}{\sigma_Y^{(1)}}\right] ,$$ A 2-5

$$P_8 = (-1)^N 4H^2(1-2H)^{N-2}\sigma_Y^{(1)} \ln c_1 + \left[P_4 + \Delta\varepsilon_p^{(1)}(b) - P_3\right]\left[4h\left(\frac{1}{c_1} - 1\right) + \frac{6P_3P_5h}{\sigma_Y^{(1)}}\right] , + (-1)^{N-1}\Delta\bar{\sigma}_{kk}^{(N)}$$

A 2-6

$$\Delta\varepsilon_p^{(N)}(b) = \begin{cases} \left(2H\sigma_Y^{(1)} \ln c_1 + \Delta\bar{\sigma}_{kk}^{(N)}\right)\left[2h\left(1 - \frac{1}{c_1}\right) - 3P_5\right]^{-1} + P_3 & \text{for} \quad N = 1 \\ P_8\left[2h\left(1 - \frac{1}{c_1}\right) - 3P_5\right]^{-1} + (-1)^N 2HP_3(1-2H)^{N-2} & \text{for} \quad N > 1 \end{cases}$$

A 2-7

$$\Delta\bar{\sigma}_{kk}^{(N)} = \begin{cases} -\left[2H\sigma_y \ln\left[\left(\frac{R^{(N)}}{b}\right)^3 c_1\right] + 3P_3P_5\left(\frac{R^{(N)}}{b}\right)^3 - 2hP_3\left(1 - \frac{1}{c_1}\left(\frac{R^{(N)}}{b}\right)^3\right) + 2\sigma_Y^{(1)}\left(1 - \left(\frac{R^{(N)}}{b}\right)^3\right)\right] & \text{for} \quad N = 1 \\ 3\Delta\sigma_{rr}^{(2),(N)}(b) + (-1)^{N-1} 4\sigma_y\left(1 - \left(\frac{R^{(N)}}{b}\right)^3\right) & \text{for} \quad N > 1 \end{cases}$$

A 2-8

where $\Delta\sigma_{rr}^{(2),(N)}(b)$ is given by eq.A. 2-9 below.

$$\Delta\sigma_{rr}^{(2),(N)}(b)=\frac{2}{3}H(1-2H)^{N-2}\left[2H\sigma_Y^{(1)}\ln\left[\left(\frac{R^{(N)}}{b}\right)^3 c_1\right]+2hP_3\left(1-\left(\frac{R^{(N)}}{b}\right)^3\frac{1}{c_1}\right)-3P_3P_5\left(\frac{R^{(N)}}{b}\right)^3\right]+P_7$$

for $N>1$.

A 2-9

$$P_{10}=\begin{cases}-\frac{3}{2H}\left[\Delta\varepsilon_p^{(1)}(b)-P_3\right] & N=1\\ (-1)^N\frac{3}{2H}\left[\Delta\varepsilon_p^{(N)}(b)+(-1)^{N-1}2HP_3(1-2H)^{N-2}\right]+3(-1)^N\left(1-\frac{1}{H}\right)\left[P_4+\Delta\varepsilon_p^{(1)}(b)-P_3\right] & N>1\end{cases}$$

A 2-10

The parameters used in eq 3-48 to eq 3-51 are given below.

$$P_1=\frac{c_1}{F+2\mu_2(1-c_1)}\ \text{where}\quad F=2\left(\kappa_1+\frac{1}{3}\mu_1\right)$$ A. 3-1

$$P_2=\frac{1}{\mu_2P_1}\quad,\quad P_3=\frac{1}{E_2+h}.$$ A. 3-2

$$P_4 = \begin{cases} 0 & \text{if} \quad N \le 2 \\ \sum_{c=1}^{N-2} \left[\sum_{r=1}^{j_1} \left\{ (-1)^{L_2} \, {}^{c}C_r \left({}^{(L_2-1)}C_{(r-1)} + 2\, {}^{(L_2-1)}C_r \right) (2hP_3)^c \right\} \right] & \text{if} \quad N > 2 \end{cases} \qquad \text{A. 3-3}$$

where $L_2 = N - c - 1$ and j_1 is defined by equation A. 3-9 by replacing k with N.

Note that, in equation A. 3-3 and A. 3-8 we define

$$ {}^{A}C_{B} = \begin{cases} 0 & \text{if} \quad A < B \\ \dfrac{A!}{B!(A-B)!} & \text{if} \quad A > B \end{cases} \qquad \text{A. 3-4}$$

$$P_5 = \begin{cases} 0 & \text{if} \quad N = 1 \\ 2\sum_{i=1}^{N-1} \left[(-1)^{(i+N)} (2hP_3)^{(i-1)} \right] & \text{if} \quad N > 1 \end{cases} \qquad \text{A. 3-5}$$

$$P_6 = E_2 P_3 \qquad \text{A. 3-6}$$

$$P_7 = \begin{cases} 0 & \text{if} \quad N = 1 \\ \sum_{i=1}^{N-1} \left[\dfrac{P_{12}^{(i)}}{a^2} P_{13}^{(N-i)} \right] & \text{if} \quad N > 1 \end{cases} \qquad \text{A. 3-7}$$

where

$$P_{12}^{(i)} = \frac{\Delta\bar{\varepsilon}^{(i)}}{c_1} a^2 \quad \text{and} \quad P_{13}^{(N-i)} = 2(2hP_3)^{(K-2)} + P_{14} \ .$$

$$P_{14} = \begin{cases} 0 & \text{if} \quad N \le 2 \\ \sum_{c=1}^{k-2} [\sum_{r=1}^{j_2} 2^c \{(-1)^{L_1} \ {}^{c}C_r \ {}^{(L_1-1)}C_{(r-1)} \ (hP_3)^{(c-1)}\}] & \text{if} \quad N > 2 \end{cases} \quad . \qquad \text{A. 3-8}$$

In the above equation, we define

$$k = N - i + 1 \quad , \quad L_1 = k - c - 1 \quad \text{and}$$

$$j2 = \begin{cases} c & \text{if} \quad c = j3 \\ cH(j3 - c) + (k - c - 1)H(c - j3) & \text{if} \quad c \ne j3 \end{cases} \quad , \qquad \text{A. 3-9}$$

where

$$j_3 = \begin{cases} \frac{1}{2}(k - 1) & \text{for odd } k \\ \frac{1}{2}(k - 2) & \text{for even } k \end{cases} \quad . \qquad \text{A. 3-10}$$

and H(x) is the Heaviside function define as

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \end{cases} \qquad \text{A. 3-11}$$

$$P_8 = 2\left[\frac{P_6}{3E_2}\ln\left(\frac{R^{(N)}}{a}\right)^2 + \frac{F}{2}\left(\frac{R^{(N)}}{a}\right)^2 + \mu_2\left(1-\left(\frac{R^{(N)}}{b}\right)^2\right) - \frac{P_6 h}{3}\left(1-\left(\frac{R^{(N)}}{a}\right)^2\right)\right]\left(\frac{b}{R^{(N)}}\right)^2 .$$

A. 3-12

$$P_9 = \left[P_3 F - 2\mu_2 P_3\left(c_1 - \left(\frac{a}{R^{(N)}}\right)^2\right) + \frac{2P_6}{3}\left(1-\left(\frac{a}{R^{(N)}}\right)^2\right)\right] h P_7 \quad . \qquad \text{A. 3-13}$$

$$P_{10} = \left[\frac{2hP_6(1-c_1)}{3} + F\right]^{-1} \quad . \qquad \text{A. 3-14}$$

$$P_{11} = P_{10} c_1 \left[\frac{\sigma_y P_6}{\sqrt{3}} \ln(c_1)\left\{(2hP_3)^{(N-1)} + P_4 + P_5\right\} + 2hP_6^2 P_7 (1-c_1)\right] \quad , \qquad \textbf{A. 3-1}$$

where σ_y is the yield stress of the virgin matrix.