Optimal Benchmarks from a Trader's Perspective

by

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Abstract

The problem of benchmarking in financial markets is an important one. It could be a mutual fund looking to meet its cash inflows and outflows or a brokerage that has been contracted a benchmark price. There is also often incentive to manipulate benchmark. We introduce a discrete-time market model to analyze the trade-off between attainability of a benchmark and its resistance to manipulation. In our setting with a single asset and temporary price impact, an honest trader tries to minimize the costs and deviation to the benchmark while a manipulator pushes the benchmark price up. The resulting optimal benchmark is very similar to the VWAP (volume weighted average price) in that prices are weighted by traded volumes. We find another VWAP-like benchmark in a market that includes an auction with an imbalance announcement that has a permanent price impact.

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Chapter 1

Introduction

In financial markets, benchmarks are important. Such benchmarks include closing prices of stocks, the so-called 'fix' (an important reference point in foreign exchange markets), and benchmark interest rates. For instance, when clients purchase shares of a mutual fund, they are charged based on the trading day's closing price of the fund's constituent assets. This creates the need for mutual funds to buy and sell assets at a price as close to or better than the closing price. In other words, the closing price serves as a benchmark for the fund manager. As another example, suppose a client contracts a broker to purchase a large amount of shares. In order to minimize price impact the broker must split up the trade into multiple smaller trades. What price should the broker and client agree on? If a suitable benchmark is available, client and broker can contract on its price, which is observable for both client and broker, even if the market may not be fully transparent.

Benchmarks are often used as reference prices for valuing portfolios and underpinning other financial contracts. This leads to an incentive to manipulate benchmarks. For example, in the foreign exchange market, the above-mentioned 'fix' was based on prices observed during the 30 seconds before and after 16:00 GMT. The 'fix' was also used to value banks' holdings of foreign assets in their local currencies. In 2013, it was reported that traders were colluding to trade aggressively during this time period to manipulate the foreign exchange benchmark and distort values of their assets. This resulted in USD 10 billion in fines to the offending banks; see [22]. In 2015, the relevant window for the ‘fix’ was widened from one to five minutes to make this benchmark less prone to manipulation, but questions about its design remain; compare [9, 23].

While there is literature on benchmark design, it largely ignores the perspective of traders attempting to achieve prices close to the benchmark. Baldauf et al. [3] look to find a benchmark that is optimal for the principal-agent problem that occurs when a client contracts a brokerage to purchase a large amount of shares. The authors conclude that VWAP (volume weighted average price) is the optimal benchmark for such contracts in asset classes where such information is available; they also conclude that in more opaque markets, market administrators could help ease this principal-agent problem by setting their benchmark price to VWAP and reduce the likelihood of the agent trading inefficiently at the detriment of the principal. In response to benchmark manipulation scandals, Duffie and Dworczak [7] take the perspective of a benevolent benchmark administrator who wishes to accurately fix the value of an asset. When data from transactions or reports of agents whose profits depend on such benchmarks are available, they find a benchmark that puts small weight on small transactions and nearly equal weight on all large transactions. When such data is unavailable, VWAP emerges as the optimal benchmark.

Goetzmann et al. [14] examine benchmarks that are used as performance measures for money managers. Popular performance measures such as the Sharpe ratio and Alpha are shown to be prone to manipulation by managers, even in the presence of high transaction costs, allowing them to increase the perceived success of their funds and therefore compensation. They arrive at a manipulation proof performance measure that is the weighted average of a utility-like function. Duffie et al. [8] analyze over-the-counter markets, where traders cannot use a centralized exchange and must purchase assets from dealers, resulting in market opaqueness and search costs. The authors show that the introduction of a benchmark by dealers can increase market participation by investors, even to the point that it overcomes the dealers’ lost revenue from the increase of transparency. In addition, they show that, under certain conditions, the presence of a benchmark can increase social surplus.

We differ from the preceding literature in that we focus on the perspective of traders. Specifically, one honest trader who wishes to purchase an asset at a price as close to the benchmark as possible and one manipulating trader who desires to manipulate the benchmark upwards. We then attempt to design a benchmark that is resistant to the manipulating trader's deceptive orders, while still considering the honest trader's intent to attain the benchmark price. Although our problem formulation and model are very different from those in Baldauf et al. [3] and Duffie and Dworczak [7], we find optimal benchmarks to be a weighted average prices that resembles the VWAP, similarly to [3] and [7].

There is also literature that analyzes optimal trading strategies, where a broker is a targeting a certain trading benchmark. Common offerings by brokers include arrival price, TWAP (time weighted average price), VWAP, POV (percentage of volume), and MOC (market on close) benchmarks. Typically, the broker employs an algorithm for scheduling the trades to achieve an average price close to (or even better than) the benchmark. There is literature related to optimal execution problems for these different benchmarks. Particularly well studied is the optimal execution with an arrival price benchmark, going back to the seminal works by Bertsimas and Lo [4] and Almgren and Chriss [1]. Execution problems with a VWAP benchmark have been analyzed by Cartea and Jaimungal [5], Frei and Westray [10], Guéant and Royer [17], Humphery-Jenner [18], as well as Kato [19]. Strategies for POV have been addressed in Guéant [15], and Labadie and Lehalle [20] while problems with MOC benchmarks are the topics of Bacidore et al. [2] and Frei and Westray [11]. For a collection of relevant literature in this topic and overviews of algorithmic trading, we refer to the books by Cartea et al. [6], Guéant [16] as well as Lehalle and Laruelle [21]. While we do consider optimal trading strategies for the honest trader relative to a benchmark and the manipulating trader's strategy, our primary focus is on benchmark design that keeps these traders' goals in mind.

In Chapter 2, we start with a simple model that has two periods in which traders can make purchases. We examine how one would design a benchmark that minimizes the expected squared difference between the benchmark and the price paid by the honest trader, we show that a deterministic benchmark is preferable to the honest trader in terms of attainability. With this in mind, we examine how this honest trader would minimize their price paid. We can then introduce the manipulating trader and observe how the honest trader's strategy changes. We derive a sufficient condition for a benchmark to be optimal in resisting manipulation by solving the manipulating trader's optimization. With these conditions for our benchmark, we find the optimal convex combination of deterministic and stochastic benchmarks subject to a parameter that weights the preference of our two objectives.

Starting with a two-period model gives us more easily interpretable results that are generalized in Chapter 3, which follows the same structure as Chapter 2, but with N opportunities to purchase the asset. Additionally, in the final section of Chapter 3, we discuss our findings from numerical simulations. Chapters 2 and 3 employ a simple model for price paid by assuming that traders have temporary price impact when purchasing shares, that is price paid in the i^{th} period, $p_i = \tilde{p}_i + c\frac{\alpha_i}{u_i}$, where $\tilde{p}_i$ is some underlying random process that is independent of volumes, c is a constant that models the magnitude of price impact, α_i is the amount shares traded, and u_i is outside volume that is independent of previous prices.

Finally, in Chapter 4, we introduce a closing auction to our fictional market. Throughout the day orders may be submitted to the closing auction; these orders are executed at a single price after regular trading has ended. Such closing auctions are common in stock markets; see FTSE Russell [13] for an overview. At some point during the day an imbalance announcement occurs, where the difference in buy and sell orders to the auction is released; compare Bacidore et al. [2]. To model the imbalance announcement, we assume that it has a permanent price impact. We end with another numerical simulation that gives a comparison of our benchmark objectives in markets with and without an auction.

Chapter 2

Two-period model

We begin with a single asset with two trading periods, first taking the perspective of a trader who would like to purchase one unit of the asset at a price as close to the benchmark as possible. Then we introduce a price model where the trader has a temporary price impact that is inversely proportional to outside volume and find a strategy for minimizing the price paid. Switching our perspective to a trader who desires to manipulate the benchmark upwards, we investigate how such a trader would best accomplish this. This gives us a worst case scenario and optimality conditions for a benchmark that is robust against manipulation. Finally, we consider a benchmark that is a convex combination of two benchmarks, one being deterministic and focused on the attainability concern and the other being stochastic and focused on the manipulation concern.

2.1 Attainability

We start by considering a simple model with two trading periods and prices p_1 and p_2. A benchmark that is of the form

$$\beta = \beta_1 p_1 + \beta_2 p_2,$$

where we restrict the βs to be such that $\beta_1 + \beta_2 = 1$. A trader chooses to trade a fraction α in the first and $1 - \alpha$ in the second period. The trader wishes to minimize the squared difference between their price paid and the benchmark. Letting β be the weight in the first period, the optimization problem becomes

$$\min_{\alpha\in[0,1]} E\left[(\alpha p_1 + (1-\alpha)p_2 - \beta p_1 - (1-\beta)p_2)^2\right] = \min_{\alpha\in[0,1]} E\left[(\alpha-\beta)^2(p_1-p_2)^2\right].$$

If β is deterministic, then we have

$$E\left[(\alpha-\beta)^2(p_1-p_2)^2\right] = (\alpha-\beta)^2 E\left[(p_1-p_2)^2\right],$$

which is trivially minimized at $\alpha = \beta$. We also note that we only have optimality if the above objective function is zero, since it is non-negative. Additionally, for a stochastic β and deterministic α, there is non zero probability that $\alpha \neq \beta$, which means that

$$E\left[(\alpha p_1 + (1-\alpha)p_2 - \beta p_1 - (1-\beta)p_2)^2\right] > 0.$$

unless $P\left(p_1 = p_2 | \alpha \neq \beta\right) = 1$. Therefore, the trader cannot exactly attain the benchmark in the case of stochastic β. With deterministic β, it is trivial to choose β, for given α, to solve our attainability problem, or vice versa.

2.2 Minimizing trading cost without manipulator

We now consider a model with temporary price impact, where the price in period i is

$$p_i = \tilde{p}_i + c\frac{\alpha_i}{u_i},$$

where $\tilde{p}_i$ is the underlying price process at time i with constant expectation, c is a constant price impact coefficient, α_i is the amount of shares that the trader buys in the i^{th} period, and u_i is the outside volume, which is independent of prices and the other u_j for $j \neq i$.

We now examine stability from the perspective of our trader who wishes to obtain a long position at the lowest cost. This leads to minimizing

$$E\left[\alpha_1 p_1 + \alpha_2 p_2\right] = \alpha_1 E\left[\tilde{p}_1\right] + \alpha_2 E\left[\tilde{p}_2\right] + cE\left[\frac{\alpha_1^2}{u_1}\right] + cE\left[\frac{\alpha_2^2}{u_2}\right].$$

Using the fact that $\alpha_1 + \alpha_2 = 1$ and $E[\tilde{p}_1] = E[\tilde{p}_2]$ this becomes

$$E[\tilde{p}_1] + c\alpha_1^2 E\left[\frac{1}{u_1}\right] + c(1-\alpha_1)^2 E\left[\frac{1}{u_2}\right].$$

Differentiating with respect to α_1 and setting equal to 0, we have

$$\frac{\partial}{\partial \alpha_1} E\left[\alpha_1 p_1 + \alpha_2 p_2\right] = c\left(2\alpha_1 E\left[\frac{1}{u_1}\right] - 2(1-\alpha_1) E\left[\frac{1}{u_2}\right]\right) = 0,$$

which implies

$$\alpha_1^* = \frac{E\left[\frac{1}{u_2}\right]}{E\left[\frac{1}{u_1}\right] + E\left[\frac{1}{u_2}\right]}. \tag{2.1}$$

Note that if volume is deterministic, the honest trader's weight is allocated identically to outside order volume. If we wish to have optimality in the above attainability problem the benchmark administrator can simply set

$$\beta_1^* = \alpha_1^*.$$

2.3 Minimizing trading cost with manipulator

In order to address the stability problem, we examine how the trader from the previous section, now referred to as the honest trader, minimizes trading cost with the added presence of a manipulating trader. Assume we have a trader who wants to manipulate the benchmark and has capital restrictions such that they can only buy V units of our asset, which are broken into $v_1 + v_2 = V$. We can now model the price paid by a trader simply looking to obtain a position as

$$p_i = \tilde{p}_i + c\frac{\alpha_i + v_i}{u_i}. \tag{2.2}$$

We take v_1, v_2 as random variables so that the expected price paid by the honest trader can be expressed as

$$
\begin{aligned}
E\left[\alpha_1 p_1 + \alpha_2 p_2\right] &= \alpha_1 E\left[\tilde{p}_1\right] + \alpha_2 E\left[\tilde{p}_2\right] + c\alpha_1 E\left[\frac{\alpha_1 + v_1}{u_1}\right] + c\alpha_2 E\left[\frac{\alpha_2 + v_2}{u_2}\right] \\
&= E[\tilde{p}_1] + c\left(\alpha_1^2 E\left[\frac{1}{u_1}\right] + (1-\alpha)^2 E\left[\frac{1}{u_2}\right]\right) \\
&\quad + c\left(\alpha_1 E\left[\frac{v_1}{u_1}\right] + (1-\alpha_1) E\left[\frac{v_2}{u_2}\right]\right).
\end{aligned}
$$

Once again we differentiate with respect to α_1,

$$
\frac{\partial}{\partial \alpha_1} E\left[\alpha_1 p_1 + \alpha_2 p_2\right] = c\left(\alpha_1 E\left[\frac{2}{u_1}\right] - (1-\alpha_1) E\left[\frac{2}{u_2}\right] + E\left[\frac{v_1}{u_1}\right] - E\left[\frac{v_2}{u_2}\right]\right),
$$

which when set equal to zero gives us

$$
\begin{aligned}
\alpha_1 &= \frac{2E\left[\frac{1}{u_2}\right] - E\left[\frac{v_1}{u_1}\right] + E\left[\frac{v_2}{u_2}\right]}{2\left(E\left[\frac{1}{u_1}\right] + E\left[\frac{1}{u_2}\right]\right)} \\
&= \frac{E\left[\frac{1}{u_2}\right]}{E\left[\frac{1}{u_1}\right] + E\left[\frac{1}{u_2}\right]} - \frac{E\left[\frac{v_1}{u_1}\right] - E\left[\frac{v_2}{u_2}\right]}{2\left(E\left[\frac{1}{u_1}\right] + E\left[\frac{1}{u_2}\right]\right)}.
\end{aligned}
$$

This is the amount the honest trader would buy without a manipulator, shifted to account for the manipulators transactions. We see that in the presence of a manipulator, the honest trader will shift their purchases away from the period where it is expected that the manipulator's purchase will be a greater proportion of volume.

2.4 The manipulator's optimization

In order to determine how to create a benchmark that is resistant to manipulation, we first see how a manipulating trader would have the greatest effect. We consider a manipulator trying to make the benchmark as high as possible and constrain them to no short selling, i.e., $v_1 \geq 0$ and $v_2 \geq 0$, where the v_is are deterministic. We also assume that from the perspective of the manipulating trader, the honest trader's volume is negligible. The price the manipulating trader pays is

$$p_i = \tilde{p}_i + c\frac{v_i}{u_i}$$

and they carry out the following maximization

$$\begin{aligned}
\max_{v_1+v_2=V} E\left[\beta_1 p_1 + \beta_2 p_2\right] &= \max_{v_1+v_2=V} E\left[\beta_1\left(\tilde{p}_1 + \frac{v_1}{u_2}\right) + \beta_2\left(\tilde{p}_2 + \frac{v_2}{u_2}\right)\right] \\
&= \max_{0\leq v_1\leq V} E\left[\beta_1\tilde{p}_1\right] + v_1 E\left[\frac{\beta_1}{u_1}\right] + E\left[\beta_2\tilde{p}_2\right] \\
&\quad + (V - v_1)E\left[\frac{\beta_2}{u_2}\right] \\
&= E\left[\beta_1\tilde{p}_1\right] + E\left[\beta_2\tilde{p}_2\right] + VE\left[\frac{\beta_2}{u_2}\right] \\
&\quad + \max_{0\leq v_1\leq V} v_1 E\left[\frac{\beta_1}{u_1} - \frac{\beta_2}{u_2}\right] \\
&= \begin{cases} E\left[\beta_1\tilde{p}_1\right] + E\left[\beta_2\tilde{p}_2\right] + VE\left[\frac{\beta_1}{u_1}\right] & \text{if } E\left[\frac{\beta_1}{u_1}\right] > E\left[\frac{\beta_2}{u_2}\right], \\ E\left[\beta_1\tilde{p}_1\right] + E\left[\beta_2\tilde{p}_2\right] + VE\left[\frac{\beta_1}{u_1}\right] & \text{if } E\left[\frac{\beta_1}{u_1}\right] = E\left[\frac{\beta_2}{u_2}\right], \\ E\left[\beta_1\tilde{p}_1\right] + E\left[\beta_2\tilde{p}_2\right] + VE\left[\frac{\beta_2}{u_2}\right] & \text{if } E\left[\frac{\beta_1}{u_1}\right] < E\left[\frac{\beta_2}{u_2}\right]. \end{cases}
\end{aligned}$$

We see that the manipulator will trade all of V in the period with the largest $E\left[\frac{\beta_i}{u_i}\right]$. So the manipulator takes the expected benchmark weighting and volume into account, trying to impact the period with the larger β_i but also considering how the manipulating trade will be diluted by external volume. If we have $E\left[\frac{\beta_1}{u_1}\right] = E\left[\frac{\beta_2}{u_2}\right]$, then the manipulator is indifferent.

If we now assume we want to minimize the expected value of our benchmark in order to counteract the manipulator's upward pressure, we have

$$\min_{\beta_1}\left(E\left[\beta_1(\tilde{p}_1 - \tilde{p}_2)\right] + E\left[\tilde{p}_2\right] + V\max\left\{E\left[\frac{\beta_1}{u_1}\right], E\left[\frac{1-\beta_1}{u_2}\right]\right\}\right).$$

Under the reasonable assumption that β_1 is independent of $\tilde{p}_i$ and with the fact that $E[\tilde{p}_1] = E[\tilde{p}_2]$, we see that

$$E[\beta_1(\tilde{p}_1 - \tilde{p}_2)] = 0.$$

All that remains is to minimize the last term,

$$\min_{\beta_1} \left(\max \left\{ E\left[\frac{\beta_1}{u_1}\right], E\left[\frac{1-\beta_1}{u_2}\right] \right\} \right).$$

Our optimal choice of β_1 is such that

$$E\left[\frac{\beta_1}{u_1}\right] = E\left[\frac{1-\beta_1}{u_2}\right].$$

Otherwise we would be able to make this maximum smaller by shifting weight to the smaller term. This gives us

$$E\left[\frac{\beta_1(u_1+u_2) - u_1}{u_1 u_2}\right] = 0. \tag{2.3}$$

While not unique, a solution to this is $\beta_1 = \frac{u_1}{u_1+u_2}$, so that our benchmark is the VWAP.

2.5 Combined problem

We assume that the honest trader's share requirement is small enough that there is no price impact or that their price impact is negligible to the benchmark administrator. The price observed is then $p_i = \tilde{p}_i + c\frac{v_i}{u_i}$. Let us now examine our choice of β_is when attainability *and* impact from manipulation are taken into account. The combined optimization is

$$\min_{\beta \in [0,1]} \min_{\alpha \in [0,1]} E\left[(\alpha p_1 + (1-\alpha)p_2 - \beta p_1 - (1-\beta)p_2)^2\right] + \lambda E\left[\beta p_1 + (1-\beta)p_2\right]. \tag{2.4}$$

2.5.1 Attainability subproblem

We first focus our attention on

$$f(\alpha) = E\left[(\alpha p_1 + (1-\alpha)p_2 - \beta p_1 - (1-\beta)p_2)^2\right],$$

which is the honest trader's optimization. We write it as

$$\begin{aligned}
f(\alpha) &= E\left[(\alpha-\beta)^2(p_1-p_2)^2\right] \\
&= \alpha^2 E\left[(p_1-p_2)^2\right] - 2\alpha E\left[\beta(p_1-p_2)^2\right] + E\left[\beta^2(p_1-p_2)^2\right] \\
&= E\left[(p_1-p_2)^2\right]\left(\alpha - \frac{E\left[\beta(p_1-p_2)^2\right]}{E\left[(p_1-p_2)^2\right]}\right)^2 \\
&\qquad + E\left[\beta^2(p_1-p_2)^2\right] - \frac{\left(E\left[\beta(p_1-p_1)^2\right]\right)^2}{E\left[(p_1-p_2)^2\right]}.
\end{aligned}$$

This is minimized at

$$\alpha^* = \frac{E\left[\beta(p_1-p_2)^2\right]}{E\left[(p_1-p_2)^2\right]},$$

and corresponding minimal value

$$f(\alpha^*) = E\left[\beta^2(p_1-p_2)^2\right] - \frac{\left(E\left[\beta(p_1-p_2)^2\right]\right)^2}{E\left[(p_1-p_2)^2\right]}.$$

Combined minimization after attainability problem is solved

This choice of α reduces the minimization in (2.4) to

$$\min_\beta \left(E\left[\beta^2(p_1-p_2)^2\right] - \frac{\left(E\left[\beta(p_1-p_2)^2\right]\right)^2}{E\left[(p_1-p_2)^2\right]} + \lambda E\left[\beta p_1 + (1-\beta)p_2\right] \right).$$

From Section 2.4, we see that we can rewrite this as

$$\min_\beta \left(E\left[\beta^2(p_1-p_2)^2\right] - \frac{\left(E\left[\beta(p_1-p_2)^2\right]\right)^2}{E\left[(p_1-p_2)^2\right]} + \lambda \max\left\{ E\left[\frac{\beta}{u_1}\right], E\left[\frac{1-\beta}{u_2}\right] \right\} \right).$$

Using our assumptions that $E[\tilde{p}_1] = E[\tilde{p}_2]$ and that β and u_is are independent of $\tilde{p}_i$s, we calculate the first term,

$$\begin{aligned} E\left[\beta^2(p_1-p_2)^2\right] &= E\left[\beta^2\left(\tilde{p}_1-\tilde{p}_2+c\frac{v_1}{u_1}-c\frac{v_2}{u_2}\right)^2\right] \\ &= E\left[\beta^2\left(\tilde{p}_1-\tilde{p}_2\right)^2+2\beta^2\left(\tilde{p}_1-\tilde{p}_2\right)\left(c\frac{v_1}{u_1}-c\frac{v_2}{u_2}\right)\right. \\ &\qquad \left.+\beta^2\left(c\frac{v_1}{u_1}-c\frac{v_2}{u_2}\right)^2\right] \\ &= E\left[\beta^2\right]E\left[(\tilde{p}_1-\tilde{p}_2)^2\right]+E\left[\beta^2\left(c\frac{v_1}{u_1}-c\frac{v_2}{u_2}\right)^2\right]. \end{aligned}$$

Letting $a = \tilde{p}_1 - \tilde{p}_2$ and $b = c\frac{v_1}{u_1} - c\frac{V-v_1}{u_2}$, we can write the first two terms in the minimization as

$$E\left[\beta^2(p_1-p_2)^2\right] - \frac{\left(E\left[\beta(p_1-p_1)^2\right]\right)^2}{E\left[(p_1-p_2)^2\right]},$$

as

$$\frac{\left(E\left[\beta^2\right]E\left[a^2\right]+E\left[\beta^2b^2\right]\right)\left(E\left[a^2\right]+E\left[b^2\right]\right)-\left(E\left[\beta\right]E\left[a^2\right]+E\left[\beta b^2\right]\right)^2}{E\left[a^2\right]+E\left[b^2\right]}.$$

2.5.2 Minimizing manipulator impact with deterministic β

The combined minimization is now

$$\begin{aligned} \min_{\beta}\Bigg\{ &\frac{\left(E\left[\beta^2\right]E\left[a^2\right]+E\left[\beta^2b^2\right]\right)\left(E\left[a^2\right]+E\left[b^2\right]\right)-\left(E\left[\beta\right]E\left[a^2\right]+E\left[\beta b^2\right]\right)^2}{E\left[a^2\right]+E\left[b^2\right]} \\ &+\lambda\max\left\{E\left[\frac{\beta}{u_1}\right], E\left[\frac{1-\beta}{u_2}\right]\right\}\Bigg\}. \end{aligned} \tag{2.5}$$

From Section 2.4 we know that the second term is minimized when $\beta = \frac{u_1}{u_1+u_2}$, and that first term is minimized and equal to zero if β is deterministic, as is discussed in Section 2.1. With deterministic β our second term becomes

$$\min\left\{\beta E\left[\frac{1}{u_1}\right],(1-\beta)E\left[\frac{1}{u_2}\right]\right\},$$

which is again minimized by finding β such that

$$\beta E\left[\frac{1}{u_1}\right]=(1-\beta)E\left[\frac{1}{u_2}\right],$$

which results in

$$\beta=\frac{E\left[1/u_2\right]}{E\left[1/u_1\right]+E\left[1/u_2\right]}.$$

2.5.3 Convex combination of deterministic and stochastic β

We consider a β that is a convex combination of the deterministic and stochastic solutions. That is

$$\beta=\mu\frac{u_1}{u_1+u_2}+(1-\mu)\frac{E\left[1/u_2\right]}{E\left[1/u_1\right]+E\left[1/u_2\right]}, \tag{2.6}$$

for $\mu\in[0,1]$.

We motivate this choice as follows. For a constant c, we write β

$$\beta=\gamma\frac{u_1}{u_1+u_2}+c \tag{2.7}$$

for a random variable γ. This is a definition of γ as

$$\gamma=\frac{u_1+u_2}{u_1}(\beta-c).$$

We note that in the minimization problem (2.5), the first term

$$f_1(\beta)=\frac{\left(E\left[\beta^2\right]E\left[a^2\right]+E\left[\beta^2b^2\right]\right)\left(E\left[a^2\right]+E\left[b^2\right]\right)-\left(E\left[\beta\right]E\left[a^2\right]+E\left[\beta b^2\right]\right)^2}{E\left[a^2\right]+E\left[b^2\right]}$$

is translation invariant in the sense that $f_1(\beta+c)=f_1(\beta)$ for any constant c

because

$$
\begin{aligned}
f_1(\beta + c) &= \frac{\left(E\left[(\beta+c)^2\right]E\left[a^2\right] + E\left[(\beta+c)^2 b^2\right]\right)\left(E\left[a^2\right] + E\left[b^2\right]\right)}{E\left[a^2\right] + E\left[b^2\right]} \\
&- \frac{\left(E\left[\beta+c\right]E\left[a^2\right] + E\left[(\beta+c)b^2\right]\right)^2}{E\left[a^2\right] + E\left[b^2\right]} \\
&= \frac{\left(E\left[\beta^2\right]E\left[a^2\right] + E\left[\beta^2 b^2\right]\right)\left(E\left[a^2\right] + E\left[b^2\right]\right)}{E\left[a^2\right] + E\left[b^2\right]} \\
&- \frac{\left(E\left[\beta\right]E\left[a^2\right] + E\left[\beta b^2\right]\right)^2}{E\left[a^2\right] + E\left[b^2\right]} \\
&+ \frac{c^2\left(E\left[a^2\right] + E\left[b^2\right]\right)\left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(cE\left[a^2\right] + cE\left[b^2\right]\right)^2}{E\left[a^2\right] + E\left[b^2\right]} \\
&+ \frac{2c\left(E\left[\beta\right]E\left[a^2\right] + E\left[\beta b^2\right]\right)\left(E\left[a^2\right] + E\left[b^2\right]\right)}{E\left[a^2\right] + E\left[b^2\right]} \\
&- \frac{2cE\left[a^2\right]\left(E\left[\beta b^2\right] + E\left[\beta\right]E\left[b^2\right]\right)}{E\left[a^2\right] + E\left[b^2\right]} \\
&- \frac{2cE\left[\beta\right]\left(E\left[a^2\right]\right)^2 + 2cE\left[\beta b^2\right]E\left[b^2\right]}{E\left[a^2\right] + E\left[b^2\right]} \\
&= \frac{\left(E\left[\beta^2\right]E\left[a^2\right] + E\left[\beta^2 b^2\right]\right)\left(E\left[a^2\right] + E\left[b^2\right]\right)}{E\left[a^2\right] + E\left[b^2\right]} \\
&- \frac{\left(E\left[\beta\right]E\left[a^2\right] + E\left[\beta b^2\right]\right)^2}{E\left[a^2\right] + E\left[b^2\right]} \\
&= f_1(\beta).
\end{aligned}
$$

Therefore, the optimal c in (2.7) is determined by the second term in (2.5) so that we have the equality

$$
E\left[\frac{\beta}{u_1}\right] = E\left[\frac{1-\beta}{u_2}\right].
$$

Plugging in (2.7) yields

$$
E\left[\frac{\gamma}{u_1+u_2}\right] + E\left[\frac{c}{u_1}\right] = E\left[\frac{1}{u_2}\left(1 - \gamma\frac{u_1}{u_1+u_2}\right)\right] - E\left[\frac{c}{u_2}\right],
$$

hence

$$c = \frac{E\left[\frac{1}{u_2}\left(1-\gamma\frac{u_1}{u_1+u_2}\right)\right] - E\left[\frac{\gamma}{u_1+u_2}\right]}{E\left[\frac{1}{u_1}+\frac{1}{u_2}\right]}.$$

Assuming that γ is independent from u_1 and u_2. We see that $f_1(\beta)$ depends on γ only through $E[\gamma]$ because

$$f_2(\beta) = E\left[\frac{\beta}{u_1}\right] = E\left[\gamma\right] E\left[\frac{1}{u_1+u_2}\right] + E\left[\frac{c}{u_1}\right] = f_2\left(E[\gamma]\frac{u_1}{u_1+u_2}+c\right),$$

where $f_2(\beta)$ denotes the the second term in (2.5). Moreover, the first term can be written as

$$f_1(\beta) = (E[\gamma])^2\left(\frac{E[\gamma^2]}{(E[\gamma])^2}c_1 - c_2\right)$$

for some constants $c_1 > c_2$. Therefore, we minimize $E[\gamma^2]$ for given $E[\gamma]$. Jensen's inequality implies $E[\gamma^2] \geq (E[\gamma])^2$, with equality for deterministic γ. Therefore γ is deterministic in the optimum. Setting $\mu = E[\gamma] = \gamma$, (2.7) becomes (2.6). Finally, we have $\mu \in [0,1]$ to guarantee that $\beta \geq 0$ almost surely.

Using (2.6), we obtain

$$\begin{aligned}
&\max\left\{E\left[\frac{\beta}{u_1}\right], E\left[\frac{1-\beta}{u_2}\right]\right\}\\
&= \max\left\{\mu E\left[\frac{1}{u_1+u_2}\right] + (1-\mu)\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]},\right.\\
&\qquad\qquad \left. E\left[\frac{\mu\frac{u_2}{u_1+u_2} + (1-\mu)\frac{E[1/u_1]}{E[1/u_1]+E[1/u_2]}}{u_2}\right]\right\}\\
&= \max\left\{\mu E\left[\frac{1}{u_1+u_2}\right] + (1-\mu)\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]},\right.\\
&\qquad\qquad \left.\mu E\left[\frac{1}{u_1+u_2}\right] + (1-\mu)\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}\right\}\\
&= \mu E\left[\frac{1}{u_1+u_2}\right] + (1-\mu)\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}.
\end{aligned}$$

Note that this also shows that we are in a situation where $E\left[\frac{\beta}{u_1}\right] = E\left[\frac{1-\beta}{u_2}\right]$, so the manipulating trader is indifferent to when they place their trades.

Letting $\beta^{(1)} = \frac{u_1}{u_1+u_2}$, and $\beta^{(2)} = \frac{E[1/u_2]}{E[1/u_1]+E[1/u_2]}$, and breaking down the first term in the overall minimization (2.5),

$$\frac{\left(E\left[\beta^2\right]E\left[a^2\right]+E\left[\beta^2 b^2\right]\right)\left(E\left[a^2\right]+E\left[b^2\right]\right)-\left(E\left[\beta\right]E\left[a^2\right]+E\left[\beta b^2\right]\right)^2}{E\left[a^2\right]+E\left[b^2\right]},$$

we calculate the following

$$\begin{aligned}
E[\beta^2] &= E\left[\left(\mu\beta^{(1)}+(1-\mu)\beta^{(2)}\right)^2\right] \\
&= \mu^2 E\left[\left(\beta^{(1)}\right)^2\right] + 2\mu(1-\mu)E\left[\beta^{(1)}\right]\beta^{(2)} \\
&\quad + (1-\mu)^2\left(\beta^{(2)}\right)^2,
\end{aligned}$$

$$\begin{aligned}
E[\beta^2 b^2] &= \mu^2 E\left[\left(\beta^{(1)}b\right)^2\right] + 2\mu(1-\mu)E\left[\beta^{(1)}b^2\right]\beta^{(2)} \\
&\quad + (1-\mu)^2\left(\beta^{(2)}\right)^2 E\left[b^2\right],
\end{aligned}$$

$$\begin{aligned}
E\left[\beta^2\right]E\left[a^2\right]+E\left[\beta^2 b^2\right] &= \mu^2 E\left[\left(\beta^{(1)}\right)^2\left(a^2+b^2\right)\right] \\
&\quad + 2\mu(1-\mu)\left\{\beta^{(2)}E\left[\beta^{(1)}\left(a^2+b^2\right)\right]\right\} \\
&\quad + (1-\mu)^2\left(\beta^{(2)}\right)^2\left(E\left[a^2+b^2\right]\right) \\
&= \mu^2 A + 2\mu(1-\mu)B + (1-\mu)^2 C,
\end{aligned}$$

$$\begin{aligned}
\left(E\left[\beta\right]E\left[a^2\right]+E\left[\beta b^2\right]\right)^2 &= \left\{\left(\mu E\left[\beta^{(1)}\right]+(1-\mu)E\left[\beta^{(2)}\right]\right)E\left[a^2\right]\right. \\
&\qquad \left. +\mu E\left[\beta^{(1)}b^2\right]+(1-\mu)E\left[\beta^{(2)}b^2\right]\right\}^2 \\
&= \left\{\mu\left(E\left[\beta^{(1)}\right]E\left[a^2\right]+E\left[\beta^{(1)}b^2\right]\right)\right. \\
&\qquad \left. +(1-\mu)\beta^{(2)}\left(E\left[a^2+b^2\right]\right)\right\}^2 \\
&= \mu^2\left(E\left[\beta^{(1)}\right]E\left[a^2\right]+E\left[\beta^{(1)}b^2\right]\right)^2 \\
&\quad + 2\mu(1-\mu)\beta^{(2)}\left(E\left[\beta^{(1)}\right]E\left[a^2\right]\right. \\
&\qquad\qquad \left. +E\left[\beta^{(1)}b^2\right]\right)\left(E\left[a^2+b^2\right]\right) \\
&\quad + (1-\mu)^2\left(\beta^{(2)}\right)^2\left(E\left[a^2+b^2\right]\right)^2.
\end{aligned}$$

The numerator is now,

$$\mu^2 \left\{ E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[\beta^{(1)}\right] E\left[a^2\right] + E\left[\beta^{(1)} b^2\right] \right)^2 \right\}$$
$$+ 2\mu(1-\mu)\beta^{(2)} \left\{ E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \left(E\left[a^2\right] + E\left[b^2\right]\right) \right.$$
$$\left. - \left(E\left[\beta^{(1)}\right] E\left[a^2\right] + E\left[\beta^{(1)} b^2\right] \right) \left(E\left[a^2+b^2\right] \right) \right\}$$
$$+ (1-\mu)^2 \left(\beta^{(2)}\right)^2 \left\{ E\left[a^2+b^2\right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[a^2+b^2\right] \right)^2 \right\}$$

$$= \mu^2 \left\{ E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \right)^2 \right\}$$
$$+ 2\mu(1-\mu)\beta^{(2)} \left\{ E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \left(E\left[a^2+b^2\right]\right) \right.$$
$$\left. - E\left[\beta^{(1)}\left(a^2+b^2\right)\right] E\left[a^2+b^2\right] \right\}$$
$$+ (1-\mu)^2 \left(\beta^{(2)}\right)^2 \left\{ \left(E\left[a^2+b^2\right] \right)^2 - \left(E\left[a^2+b^2\right] \right)^2 \right\}.$$

The overall combined problem is

$$\min_{\mu\in[0,1]} \frac{\mu^2 \left\{ E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \right)^2 \right\}}{E\left[a^2\right] + E\left[b^2\right]}$$
$$+ \lambda\mu E\left[\frac{1}{u_1+u_2}\right] + \lambda(1-\mu)\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}$$

$$= \min_{\mu\in[0,1]} \frac{\mu^2 \left\{ E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \right)^2 \right\}}{E\left[a^2\right] + E\left[b^2\right]}$$
$$+ \lambda\mu \left(E\left[\frac{1}{u_1+u_2}\right] - \frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]} \right) + \lambda\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}$$

$$= \min_{\mu\in[0,1]} \frac{E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - \left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \right)^2}{E\left[a^2\right] + E\left[b^2\right]}$$
$$\cdot \left(\mu + \frac{\lambda \left(E\left[\frac{1}{u_1+u_2}\right] - \frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]} \right) \left(E\left[a^2\right] + E\left[b^2\right]\right)}{2E\left[\left(\beta^{(1)}\right)^2 (a^2+b^2) \right] \left(E\left[a^2\right] + E\left[b^2\right]\right) - 2\left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right] \right)^2} \right)^2$$

$$+ \lambda \frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}$$

$$- \frac{\lambda^2 \left(E\left[\frac{1}{u_1+u_2}\right] - \frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]} \right)^2 \left(E\left[a^2\right] + E\left[b^2\right]\right)}{4E\left[\left(\beta^{(1)}\right)^2 \left(a^2+b^2\right)\right] \left(E\left[a^2\right]+E\left[b^2\right]\right) - 4\left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right]\right)^2},$$

which is minimized at

$$\mu = \min \left\{ \frac{\lambda \left(\frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]} - E\left[\frac{1}{u_1+u_2}\right] \right) \left(E\left[a^2\right]+E\left[b^2\right]\right)}{2E\left[\left(\beta^{(1)}\right)^2\left(a^2+b^2\right)\right] E\left[a^2+b^2\right] - 2\left(E\left[\beta^{(1)}\left(a^2+b^2\right)\right]\right)^2}, 1 \right\}. \tag{2.8}$$

Where we require $\mu < 1$. We will now note a few properties of μ.

By applying Jensen's inequality under the probability measure Q defined as

$$Q[A] = E\left[\mathbb{1}_A \frac{a^2+b^2}{E[a^2+b^2]}\right], \tag{2.9}$$

we show the denominator is positive

$$E\left[\left(\beta^{(1)}\right)^2 \frac{a^2+b^2}{E[a^2+b^2]}\right] = E^Q\left[\left(\beta^{(1)}\right)^2\right]$$
$$> \left(E^Q\left[\beta^{(1)}\right]\right)^2 = \left(E\left[\beta^{(1)} \frac{a^2+b^2}{E[a^2+b^2]}\right]\right)^2.$$

This implies that

$$E\left[(\beta^{(1)})^2(a^2+b^2)\right] E[a^2+b^2] > \left(E\left[\beta^{(1)}(a^2+b^2)\right]\right)^2.$$

For $\lambda \geq 0$, the numerator is nonnegative. First note that the function

$$f(x) = \frac{1}{1/x+c}, \quad x > 0, c > 0$$

is concave because

$$f'(x) = -(1/x+c)^{-2}(-1/x^2) = \frac{1}{(1+cx)^2}, \quad f''(x) = \frac{-2c}{(1+cx)^3} < 0.$$

For positive, independent random variables X and Y, applying Jensen's inequality twice yields

$$\begin{aligned}
E\left[\frac{1}{1/X+1/Y}\right] &= E\left[E\left[\frac{1}{1/X+1/Y}\middle| X\right]\right] \\
&\leq E\left[\frac{1}{1/X+1/E[Y|X]}\right] \\
&= E\left[\frac{1}{1/X+1/E[Y]}\right] \\
&\leq \frac{1}{1/E[X]+1/E[Y]}.
\end{aligned}$$

Setting $X = 1/u_1$ and $Y = 1/u_2$ implies

$$E\left[\frac{1}{u_1+u_2}\right] \leq \frac{1}{1/E[1/u_1]+1/E[1/u_2]} = \frac{E[1/u_1]E[1/u_2]}{E[1/u_1]+E[1/u_2]}.$$

Using the probability measure Q from (2.9), we can write

$$\mu = \min\left\{\frac{\lambda\left(\beta^{(2)}E[1/u_1] - E\left[\beta^{(1)}/u_1\right]\right)}{2\left(E\left[a^2+b^2\right]\right)\operatorname{Var}^Q\left(\beta^{(1)}\right)}, 1\right\}.$$

This has a similarity to the slope coefficient of a simple linear regression, where we have a covariance like term in the numerator and a variance term that is scaled by $E[a^2+b^2]$ in the denominator.

We also look at the interpretation of $\frac{\partial\beta}{\partial\lambda}$. Where $\frac{\partial\beta}{\partial\lambda}$ is the marginal change in the optimal first weight of the benchmark for increased manipulation concern. Note that

$$\frac{\partial\beta}{\partial\lambda} = \left(\beta^{(1)}-\beta^{(2)}\right)\frac{\partial\mu}{\partial\lambda} = \left(\beta^{(1)}-\beta^{(2)}\right)\frac{\left(\beta^{(2)}E[1/u_1] - E\left[\beta^{(1)}/u_1\right]\right)}{2\left(E\left[a^2\right]+E\left[b^2\right]\right)\operatorname{Var}^Q\left(\beta^{(1)}\right)}$$

for λ such that the minimum in (2.8) is not binding. Moreover, a probability measure R defined by

$$R[A] = E\left[\mathbb{1}_A\frac{(p_1-p_2)^2}{E[(p_1-p_2)^2]}\right]$$

satisfies $R[A] = Q[A]$ for all events A that are measurable with respect to u_1

and u_2, i.e., events that do not depend on price fluctuations. In particular, we obtain $\mathrm{Var}^R\left(\beta^{(1)}\right) = \mathrm{Var}^Q\left(\beta^{(1)}\right)$. Rewrite $\frac{\partial\beta}{\partial\lambda}$ as

$$\frac{\partial\beta}{\partial\lambda} = \left(\beta^{(1)} - \beta^{(2)}\right) \frac{\left(\beta^{(2)}E[1/u_1] - E\left[\beta^{(1)}/u_1\right]\right)}{2E\left[(p_1-p_2)^2\right]\mathrm{Var}^R\left(\beta^{(1)}\right)}$$
$$= \left(\beta^{(1)} - \beta^{(2)}\right) \frac{\left(E^R\left[\frac{\beta^{(2)}-\beta^{(1)}}{u_1(p_1-p_2)^2}\right]\right)}{2\mathrm{Var}^R\left(\beta^{(1)}\right)}$$

Because $\beta^{(2)}$ is deterministic, we have $\mathrm{Var}^R\left(\beta^{(1)}\right) = \mathrm{Var}^R\left(\beta^{(2)} - \beta^{(1)}\right)$. Therefore, we can write

$$\frac{\partial\beta}{\partial\lambda} = -\Delta\beta\frac{\mathrm{objective2}(\Delta\beta)}{2\,\mathrm{objective1}(\Delta\beta)},$$

where $\Delta\beta = \beta^{(2)} - \beta^{(1)}$ and

$$\mathrm{objective1}(\Delta\beta) = \mathrm{Var}^R(\Delta\beta)\mathrm{Var}(p_1 - p_2), \quad \mathrm{objective2}(\Delta\beta) = E[\Delta\beta/u_1].$$

That is, objective1$(\Delta\beta)$ relates to our attainability goal and objective2$(\Delta\beta)$ relates to the goal of keeping the benchmark resistant to upward manipulation. So, for an increase in λ (an increase in weight to resistance against manipulation) our μ (the weight given to the minimizer of our first objective) changes by the ratio $\frac{\mathrm{objective2}(\Delta\beta)}{2\,\mathrm{objective1}(\Delta\beta)}$.

Chapter 3

Multiperiod model without auction

In this chapter, we repeat the structure of Chapter 2 but with N trading periods, starting with attainability, then minimizing trading cost, moving to the introduction of a manipulating trader, and finishing with a benchmark that is the convex combination of deterministic and stochastic benchmarks.

3.1 Attainability

We again take the perspective of a trader who wishes to obtain a long position at as close a price to the benchmark as possible. The trader has N opportunities to purchase α_i, $i = 1, 2, \ldots, N$, units in each period with $\sum_{i=1}^{N} \alpha_i = 1$ and predictable α_is. Predictability is here with respect to the joint filtration of randomness in prices and outside volumes. Similarly, our benchmark is again of the form $\sum_{i=1}^{N} \beta_i p_i$, a linear combination of prices observed with weights that sum to one. That is, $\sum_{i=1}^{N} \beta_i = 1$, and our β_is are a function of volume only. The trader faces the minimization problem

$$\min_{\alpha} E\left[\left(\sum_{i=1}^{N} \alpha_i p_i - \sum_{i=1}^{N} \beta_i p_i\right)^2\right] = \min_{\alpha} E\left[\left(\sum_{i=1}^{N} (\alpha_i - \beta_i) p_i\right)^2\right] \qquad (3.1)$$

If β_is are deterministic or even predictable the trader can set $\alpha_i = \beta_i$ for all i to minimize this. We also note that $\min_\alpha E\left[\left(\sum_{i=1}^N \alpha_i p_i - \sum_{i=1}^N \beta_i p_i\right)^2\right] > 0$ for not predictable β_is. Using that $\sum_{i=1}^N \alpha_i = 1$ and $\sum_{i=1}^N \beta_i = 1$, the constrained problem (3.1) is equivalent to the unconstrained problem

$$\min_\alpha E\left[\left(\sum_{i=1}^{N-1}(\alpha_i - \beta_i)(p_i - p_N)\right)^2\right].$$

The minimum is equal to zero if and only if

$$\sum_{i=1}^{N-1}(\alpha_i - \beta_i)(p_i - p_N) = 0 \quad \text{a.s.}$$

If $p_i - p_N$ are not almost surely linearly dependent for coefficients that depend only on volume dynamics (for example, if prices contain noise), then the only way to achieve this equality is by setting $\alpha_i = \beta_i$. Which is possible only if all β_i are predictable because we require α_i to be predictable. Hence, if some β_i are not predictable, then we must have $\min_\alpha E\left[\left(\sum_{i=1}^N \alpha_i p_i - \sum_{i=1}^N \beta_i p_i\right)^2\right] > 0$ and the trader cannot exactly match the benchmark.

3.2 Minimizing trading cost without manipulation

We retain our model of temporary price impact from Chapter 2, where the price paid in period i is,

$$p_i = \tilde{p}_i + c\frac{\alpha_i}{u_i},$$

with α_is being predictable and $\tilde{p}_i$ and u_i not predictable. Moreover, we assume that $\tilde{p}_i$ are such that $E\left[\tilde{p}_{i+1} - \tilde{p}_i | \mathcal{F}_{i-1}\right] = 0$ for all $i = 1, \ldots, N-1$, where $\mathcal{F}_i$ is the σ-algebra generated by $u_1, \ldots, u_i, \tilde{p}_1, \ldots, \tilde{p}_i$. Additionally, we assume u_i are independent of all other prices and volumes. If the trader wishes to minimize trading cost, they face the below minimization,

$$\min_{\alpha} E\left[\sum_{i=1}^{N} \alpha_i p_i\right] = \min_{\alpha} \sum_{i=1}^{N} E\left[\alpha_i \left(\tilde{p}_i + c\frac{\alpha_i}{u_i}\right)\right] = \min_{\alpha} \sum_{i=1}^{N} E\left[c\frac{\alpha_i^2}{u_i} + \alpha_i \tilde{p}_i\right].$$

Using the fact that $\sum_{i=1}^{N} \alpha_i = 1$

$$\begin{aligned}
\sum_{i=1}^{N} E\left[\alpha_i \tilde{p}_i\right] &= \sum_{i=1}^{N-1} E\left[\alpha_i (\tilde{p}_i - \tilde{p}_N)\right] + E\left[\tilde{p}_N\right] \\
&= \sum_{i=1}^{N-1} E\left[\alpha_i \underbrace{E\left[(\tilde{p}_i - \tilde{p}_N)|\mathcal{F}_{i-1}\right]}_{=0}\right] + E\left[\tilde{p}_N\right] \\
&= E\left[\tilde{p}_N\right].
\end{aligned}$$

By Lemma 7 in Baldauf et al. [3], our assumption

$$E[\tilde{p}_{i+1} - \tilde{p}_i|\mathcal{F}_{i-1}] = 0 \quad \text{for all } i = 1, \ldots, N-1,$$

gives us

$$E[\tilde{p}_i - \tilde{p}_N|\mathcal{F}_{i-1}] = 0 \quad \text{for all } i = 1, \ldots, N-1.$$

We can now write

$$\begin{aligned}
\min_{\alpha} \sum_{i=1}^{N} E\left[c\frac{\alpha_i^2}{u_i} + \alpha_i \tilde{p}_i\right] &= \min_{\alpha} \sum_{i=1}^{N} E\left[c\frac{\alpha_i^2}{u_i}\right] + E[\tilde{p}_N], \\
&= E[\tilde{p}_N] + \min_{\alpha} \sum_{i=1}^{N} E\left[c\alpha_i^2 E\left[\frac{1}{u_i}\middle|\mathcal{F}_{i-1}\right]\right] \\
&= E[\tilde{p}_N] + \min_{\alpha} \sum_{i=1}^{N} E\left[c\alpha_i^2 E\left[\frac{1}{u_i}\right]\right].
\end{aligned}$$

There is no longer any randomness in this minimization, so our optimizer, α, is deterministic. Starting with

$$\min_{\alpha} \sum_{i=1}^{N} c\alpha_i^2 E\left[\frac{1}{u_i}\right] \leq \sum_{i=1}^{N} c\tilde{\alpha}_i{}^2 E\left[\frac{1}{u_i}\right]$$

We can take expectation of both sides,

$$\min_{\alpha} \sum_{i=1}^{N} c\alpha_i^2 E\left[\frac{1}{u_i}\right] \leq E\left[\sum_{i=1}^{N} c\tilde{\alpha_i}^2 E\left[\frac{1}{u_i}\right]\right]$$

thus,

$$\min_{\alpha} \sum_{i=1}^{N} c\alpha_i^2 E\left[\frac{1}{u_i}\right] \leq \min_{\tilde{\alpha}} E\left[\sum_{i=1}^{N} c\tilde{\alpha_i}^2 E\left[\frac{1}{u_i}\right]\right].$$

This shows that we need only consider $\min_\alpha \sum_{i=1}^{N} c\alpha_i^2 E\left[\frac{1}{u_i}\right]$. Using the constraint $\sum_{i=1}^{N} \alpha_i = 1$, this becomes

$$\min_{\alpha} \left\{ \sum_{i=1}^{N-1} c\alpha_i^2 E\left[\frac{1}{u_i}\right] + c\left(1 - \sum_{i=1}^{N-1} \alpha_i\right)^2 E\left[\frac{1}{u_N}\right] \right\}.$$

Differentiating with respect to α_j and setting equal to zero we obtain

$$2c\alpha_j E\left[\frac{1}{u_j}\right] - 2c\left(1 - \sum_{i=1}^{N-1} \alpha_i\right) E\left[\frac{1}{u_N}\right] = 0 \quad \text{for all} \quad j = 1, \ldots, N-1$$

which gives us

$$\alpha_j = \left(1 - \sum_{i=1}^{N-1} \alpha_i\right) \frac{E\left[1/u_N\right]}{E\left[1/u_j\right]} \quad \text{for all} \quad j = 1, \ldots, N-1. \tag{3.2}$$

Summing over j, we have,

$$\begin{aligned} \sum_{j=1}^{N-1} \alpha_j &= \left(1 - \sum_{i=1}^{N-1} \alpha_i\right) \sum_{j=1}^{N-1} \frac{E\left[1/u_N\right]}{E\left[1/u_j\right]} \\ &= \frac{\sum_{j=1}^{N-1} \frac{E[1/u_N]}{E[1/u_j]}}{1 + \sum_{j=1}^{N-1} \frac{E[1/u_N]}{E[1/u_j]}}. \end{aligned}$$

Substituting this into (3.2), we obtain

$$\begin{aligned}
\alpha_j &= \left(1 - \sum_{i=1}^{N-1} \alpha_i\right) \frac{E\left[1/u_N\right]}{E\left[1/u_j\right]} \\
&= \frac{1}{1 + \sum_{j=1}^{N-1} \frac{E[1/u_N]}{E[1/u_j]}} \frac{E\left[1/u_N\right]}{E\left[1/u_j\right]} \\
&= \frac{1}{\sum_{j=1}^{N} \frac{E[1/u_N]}{E[1/u_j]}} \frac{E\left[1/u_N\right]}{E\left[1/u_j\right]} \\
&= \frac{\frac{1}{E[1/u_j]}}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}} \quad \text{for all} \quad j = 1 \dots N-1.
\end{aligned}$$

Finally,

$$\alpha_N = 1 - \sum_{i=1}^{N-1} \alpha_i = 1 - \sum_{i=1}^{N-1} \frac{\frac{1}{E[1/u_j]}}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}} = \frac{\frac{1}{E[1/u_N]}}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}}.$$

and we see that our α_i are consistent with the case of $N = 2$.

3.3 Minimizing trading cost with manipulation

We now examine how the honest trader's strategy in the previous section is affected by including another trader who is placing upward pressure on prices to manipulate the benchmark. That is we now have

$$p_i = \tilde{p}_i + c\frac{\alpha_i + v_i}{u_i},$$

where v_is are manipulating trades with $\sum_{i=1}^{N} v_i = V$. The problem is now

$$\begin{aligned}
\min_{\alpha} E\left[\sum_{i=1}^{N} \alpha_i p_i\right] &= \min_{\alpha} \sum_{i=1}^{N} E\left[\alpha_i \left(\tilde{p}_i + c\frac{\alpha_i + v_i}{u_i}\right)\right] \\
&= \min_{\alpha} \left\{ \sum_{i=1}^{N} E\left[\alpha_i \tilde{p}_i\right] + \sum_{i=1}^{N} E\left[\alpha_i c\frac{\alpha_i + v_i}{u_i}\right] \right\},
\end{aligned}$$

which can be written as

$$
\begin{aligned}
&E\left[\tilde{p}_N\right] + \min_{\alpha} \sum_{i=1}^{N} E\left[c\alpha_i^2 \frac{1}{u_i} + c\alpha_i \frac{v_i}{u_i}\right] \\
&= E\left[\tilde{p}_N\right] + \min_{\alpha} \sum_{i=1}^{N} E\left[c\alpha_i^2 E\left[\frac{1}{u_i}\right]\right] + E\left[c\alpha_i \frac{v_i}{u_i}\right].
\end{aligned}
$$

If we assume that v_i are deterministic and using the independence of u_is, we have

$$
\min_{\alpha} \sum_{i=1}^{N} E\left[c\alpha_i^2 E\left[\frac{1}{u_i}\right]\right] + v_i E\left[c\alpha_i E\left[\frac{1}{u_i}\right]\right].
$$

By the same argument as in the previous section we can show that it is optimal to have deterministic α. We can now focus on

$$
\begin{aligned}
\min_{\alpha} \sum_{i=1}^{N} c\alpha_i^2 E\left[\frac{1}{u_i}\right] + c\alpha_i E\left[\frac{v_i}{u_i}\right] = \min_{\alpha} \sum_{i=1}^{N-1} &\left(c\alpha_i^2 E\left[\frac{1}{u_i}\right] + c\alpha_i E\left[\frac{v_i}{u_i}\right]\right) \\
&+ c\left(1 - \sum_{i=1}^{N-1} \alpha_i\right)^2 E\left[\frac{1}{u_N}\right] \\
&+ c\left(1 - \sum_{i=1}^{N-1} \alpha_i\right) E\left[\frac{v_N}{u_N}\right],
\end{aligned}
$$

differentiating the interior with respect to α_j and setting equal to zero, we obtain

$$
2c\alpha_j E\left[\frac{1}{u_j}\right] - 2c\left(1 - \sum_{i=1}^{N-1} \alpha_i\right) E\left[\frac{1}{u_N}\right] + cE\left[\frac{v_j}{u_j}\right] - cE\left[\frac{v_N}{u_N}\right] = 0,
$$

for $j = 1, \ldots, N-1$, and

$$
\alpha_j = \frac{\left(1 - \sum_{i=1}^{N-1} \alpha_i\right) E\left[\frac{1}{u_N}\right] - \frac{1}{2}\left(E\left[\frac{v_j}{u_j}\right] - E\left[\frac{v_N}{u_N}\right]\right)}{E\left[\frac{1}{u_j}\right]}, \quad j = 1, \ldots, N-1. \tag{3.3}
$$

Summing over j

$$\sum_{j=1}^{N-1} \alpha_j = \left(1 - \sum_{i=1}^{N-1} \alpha_i\right) \sum_{j=1}^{N-1} \frac{E\left[\frac{1}{u_N}\right]}{E\left[\frac{1}{u_j}\right]} - \frac{1}{2} \sum_{j=1}^{N-1} \frac{\left(E\left[\frac{v_j}{u_j}\right] - E\left[\frac{v_N}{u_N}\right]\right)}{E\left[\frac{1}{u_j}\right]}$$

$$= \frac{\sum_{j=1}^{N-1} \frac{E[1/u_N] - \frac{1}{2}(E[v_j/u_j] - E[v_N/u_N])}{E[1/u_j]}}{1 + \sum_{j=1}^{N-1} \frac{E[1/u_N]}{E[1/u_j]}}$$

and

$$1 - \sum_{j=1}^{N-1} \alpha_j = \frac{1 + \sum_{j=1}^{N-1} \frac{\frac{1}{2}(E[v_j/u_j] - E[v_N/u_N])}{E[1/u_j]}}{\sum_{j=1}^{N} \frac{E[1/u_N]}{E[1/u_j]}} = \alpha_N,$$

and finally substituting into (3.3),

$$\begin{aligned}
\alpha_j &= \frac{\frac{1 + \sum_{i=1}^{N-1} \frac{\frac{1}{2}(E[v_i/u_i] - E[v_N/u_N])}{E[1/u_i]}}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} - \frac{1}{2}\left(E\left[v_j/u_j\right] - E\left[v_N/u_N\right]\right)}{E\left[1/u_j\right]} \\
&= \frac{\frac{1}{E[1/u_j]}\left(1 + \sum_{i=1}^{N-1} \frac{(E[v_i/u_i] - E[v_N/u_N])}{2E[1/u_i]}\right)}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} - \frac{E\left[v_j/u_j\right] - E\left[v_N/u_N\right]}{2E\left[1/u_j\right]} \\
&= \frac{\frac{1}{E[1/u_j]}}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} + \frac{\frac{1}{E[1/u_j]} \sum_{i=1}^{N-1} \frac{E[v_i/u_i] - E[v_N/u_N]}{2E[1/u_i]}}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} - \frac{E\left[v_j/u_j\right] - E\left[v_N/u_N\right]}{2E\left[1/u_j\right]} \\
&= \frac{\frac{1}{E[1/u_j]}}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} \qquad (3.4) \\
&\quad + \frac{2\sum_{i=1}^{N-1} \frac{E[v_i/u_i] - E[v_N/u_N]}{2E[1/u_i]} - \sum_{i=1}^{N} \frac{1}{E[1/u_i]}\left(E\left[v_j/u_j\right] - E\left[v_N/u_N\right]\right)}{\sum_{i=1}^{N} \frac{2E[1/u_j]}{E[1/u_i]}} \\
&= \frac{\frac{1}{E[1/u_j]}}{\sum_{i=1}^{N} \frac{1}{E[1/u_i]}} + \frac{\sum_{i=1}^{N} \frac{E[v_i/u_i] - E[v_j/u_j]}{2E[1/u_i]}}{\sum_{i=1}^{N} \frac{E[1/u_j]}{E[1/u_i]}}, \quad j = 1, \ldots, N-1. \qquad (3.5)
\end{aligned}$$

Additionally,

$$\alpha_N = \frac{1+\sum_{i=1}^{N-1}\frac{\frac{1}{2}(E[v_i/u_i]-E[v_N/u_N])}{E[1/u_i]}}{\sum_{i=1}^{N}\frac{E[1/u_N]}{E[1/u_i]}}$$
$$= \frac{\frac{1}{E[1/u_N]}}{\sum_{i=1}^{N}\frac{1}{E[1/u_i]}} + \frac{\sum_{i=1}^{N}\frac{(E[v_i/u_i]-E[v_N/u_N])}{2E[1/u_i]}}{\sum_{i=1}^{N}\frac{E[1/u_N]}{E[1/u_i]}},$$

so the formula in (3.5) is true for $j = 1, \ldots, N$ and this is consistent with the formula in the two-period case.

3.4 The manipulator's optimization

Once again, we consider a manipulator who would like to exert as much upward pressure on the benchmark as possible, with no short selling. The manipulating trader can purchase V units and clearly it is optimal for them to purchase all of them. We also require that our v_j are predictable and assume that the benchmark weights are of the form

$$\beta_j = Xu_j + y_j(u_j), \tag{3.6}$$

where X is a random variable and $y_j(u_j)$ depend only on u_j. The benchmarks that we will consider later are of this form.

The manipulator faces the following maximization,

$$\max_{\sum_{j=1}^N v_j = V} E\left[\sum_{j=1}^N \beta_j p_j\right] = \max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N E\left[\beta_j\left(\tilde{p}_j + c\frac{v_j}{u_j}\right)\right]$$
$$= \sum_{j=1}^N E\left[\beta_j\tilde{p}_j\right] + \max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N E\left[v_j X + v_j E\left[\frac{y_j(u_j)}{u_j}\middle|\mathcal{F}_{j-1}\right]\right]$$
$$= \sum_{j=1}^N E\left[\beta_j\tilde{p}_j\right] + VE[X] + \max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N E\left[v_j E\left[\frac{y_j(u_j)}{u_j}\right]\right].$$

Focusing on $\max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N E\left[v_j E\left[\frac{y_j(u_j)}{u_j}\right]\right]$, we start with

$$\max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N v_j E\left[\frac{y_j(u_j)}{u_j}\right] \geq \sum_{j=1}^N \tilde{v}_j E\left[\frac{y_j(u_j)}{u_j}\right],$$

taking expectation of both sides,

$$\max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N v_j E\left[\frac{y_j(u_j)}{u_j}\right] \geq E\left[\sum_{j=1}^N \tilde{v}_j E\left[\frac{y_j(u_j)}{u_j}\right]\right],$$

$$\max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N v_j E\left[\frac{y_j(u_j)}{u_j}\right] \geq \max_{\sum_{j=1}^N \tilde{v}_j = V} E\left[\sum_{j=1}^N \tilde{v}_j E\left[\frac{y_j(u_j)}{u_j}\right]\right].$$

So with deterministic v_i, we find the optimal value

$$\begin{aligned}\max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N v_j E\left[\frac{y_j(u_j)}{u_j}\right] &= \max_{\sum_{j=1}^N v_j = V} \sum_{j=1}^N v_j E\left[\frac{y_j(u_j)}{u_j}\right] \\ &= V \max_{j=1,\ldots,N} E\left[\frac{y_j(u_j)}{u_j}\right].\end{aligned}$$

Therefore, the manipulator will purchase all of their shares in the period with the largest $E\left[\frac{y_j(u_j)}{u_j}\right]$, and

$$\begin{aligned}\max_{j=1,\ldots,N} E\left[\frac{y_j(u_j)}{u_j}\right] + E[X] &= \max_{j=1,\ldots,N} E\left[\frac{y_j(u_j) + X u_j}{u_j}\right] \\ &= \max_{j=1,\ldots,N} E\left[\frac{\beta_j}{u_j}\right]\end{aligned}$$

So as before, the manipulator with will put all of their capital in the period with the largest $E\left[\frac{\beta_j}{u_j}\right]$, that is the period where their purchase has the most impact on the benchmark. We also see that our assumption of v_j being deterministic in the preceding Section 3.3, is justified.

Remark. We showed that the manipulator's strategy is deterministic if the benchmark is of the form (3.6). To see why such a condition is needed, we consider a simple example with three trading periods. Assume that the external volumes u_1, u_2, and u_3 are independent, $1/u_2$, and $1/u_3$ have the same mean M, and u_1 has the distribution

$$u_1 = \begin{cases} 3/2 & \text{with probability } 3/4, \\ 1/2 & \text{with probability } 1/4. \end{cases}$$

Suppose further that the benchmark weights are given by $\beta_1 = 0$, $\beta_2 = \mathbb{1}_{u_1=3/2}$, and $\beta_3 = \mathbb{1}_{u_1=1/2}$. We can compute

$$E\left[\frac{\beta_1}{u_1}\right] = 0, \;\; E\left[\frac{\beta_2}{u_2}\right] = E[\beta_2]E\left[\frac{1}{u_2}\right] = 3M/4, \;\; E\left[\frac{\beta_3}{u_3}\right] = E[\beta_3]E\left[\frac{1}{u_3}\right] = M/4.$$

Therefore, the optimal deterministic strategy for the manipulator is to purchase all shares in the second period, resulting in an expected benchmark manipulation of

$$E\left[\beta_2 c \frac{V}{u_2}\right] = E[\beta_2]cVE\left[\frac{1}{u_2}\right] = \frac{3}{4}cVM.$$

However, if the manipulator chooses the predictable strategy

$$v_2 = V\mathbb{1}_{u_1=3/2}, \quad v_3 = V\mathbb{1}_{u_1=1/2},$$

the expected benchmark manipulation will be larger, namely,

$$\begin{aligned} E\left[\beta_2 c \frac{v_2}{u_2}\right] + E\left[\beta_3 c \frac{v_3}{u_3}\right] &= cVE[\mathbb{1}_{u_1=3/2}]E\left[\frac{1}{u_2}\right] + cVE[\mathbb{1}_{u_1=1/2}]E\left[\frac{1}{u_3}\right] \\ &= \frac{3}{4}cVM + \frac{1}{4}cVM = cVM. \end{aligned}$$

This example shows that the manipulator can learn from observing the volume and choose a strategy that has a bigger impact on the benchmark than any deterministic strategy. Such learning cannot be used in a profitable way if $\frac{\beta_j}{u_j}$ does not depend on j or if it depends only on u_j. In the former case, the manipulator's marginal impact will be the same for each period, namely $c\frac{\beta_j}{u_j}$, so that it does not matter in which period they are buying. In the latter case, the manipulator's marginal impact in period j will be a function of u_j. Because the manipulator needs to choose a predictable strategy and u_j is independent, they cannot find a stochastic strategy that is better than the optimal deterministic strategy. These two cases give rise to the benchmark form (3.6).

Let us now try and minimize the expected value of our benchmark with the presence of a manipulating trader.

$$\min_{\beta} E\left[\sum_{i=1}^{N} \beta_i p_i\right] = \min_{\beta}\left\{\sum_{i=1}^{N} E\left[\beta_i \tilde{p}_i\right] + \max_i VE\left[\frac{\beta_i}{u_i}\right]\right\},$$

using the facts, $E[\tilde{p}_i] = E[\tilde{p}_j]$, $\sum_{i=1}^{N} \beta_i = 1$, and β_i are independent of u_i,

$$\min_{\beta} E\left[\sum_{i=1}^{N} \beta_i p_i\right] = E\left[\tilde{p}_1\right] + \min_{\beta}\left\{\max_i VE\left[\frac{\beta_i}{u_i}\right]\right\}.$$

This implies that we desire our benchmark to satisfy $E\left[\frac{\beta_i}{u_i}\right] = E\left[\frac{\beta_j}{u_j}\right]$ for all $i, j = 1, \ldots, N$. We can justify this by assuming that there is an optimal solution where $E\left[\frac{\beta_i}{u_i}\right] \neq E\left[\frac{\beta_j}{u_j}\right]$ for some $i \neq j$. This solution can then be improved upon by moving weight away from the β_i larger $E\left[\frac{\beta_j}{u_j}\right]$ and it is therefore not optimal. One solution for this is $\beta_i = \frac{u_i}{\sum_{j=1}^{N} u_j}$. We have

$$\min_{\beta} E\left[\sum_{i=1}^{N} \beta_i p_i\right] = E\left[\tilde{p}_1\right] + VE\left[\frac{1}{\sum_{j=1}^{N} u_j}\right].$$

3.5 Combined problem

Once again, assume that the honest trader's share requirement is small enough that their trades do not affect prices from the benchmark administrator's position. Following the assumptions and price model below

1. $\sum_{i=1}^{N} \alpha_i = 1$ and α_i are predictable

2. $\sum_{i=1}^{N} \beta_i = 1$ and β_i are functions of volume

3. $p_i = \tilde{p}_i + c\frac{v_i}{u_i}$

our combined problem is

$$\min_{\beta}\min_{\alpha} E\left[\left(\sum_{i=1}^{N}(\alpha_i-\beta_i)p_i\right)^2\right]+\lambda E\left[\sum_{i=1}^{N}\beta_i p_i\right].$$

3.5.1 Attainability subproblem

We assume that the α_i are deterministic, and examine the honest trader's optimization

$$\min_{\alpha} E\left[\left(\sum_{i=1}^{N}(\alpha_i-\beta_i)p_i\right)^2\right]=\min_{\alpha} E\left[\left(\sum_{i=1}^{N-1}(\alpha_i-\beta_i)(p_i-p_N)\right)^2\right].$$

Differentiating the interior with respect to α_j and setting equal to 0, we have

$$E\left[2\left(p_j-p_N\right)\sum_{i=1}^{N-1}\left(\alpha_i-\beta_i\right)\left(p_i-p_N\right)\right]=0,\quad \text{for all}\quad j=1,\ldots,N-1,$$

which yields

$$E\left[\left(\alpha_j-\beta_j\right)\left(p_j-p_N\right)^2\right]=-E\left[\left(p_j-p_N\right)\sum_{i=1,i\neq j}^{N-1}\left(\alpha_i-\beta_i\right)\left(p_i-p_N\right)\right]$$

for $j=1,\ldots,N-1$. Since we have deterministic α_j,

$$\alpha_j=\frac{E\left[\beta_j(p_j-p_N)^2\right]-E\left[\left(p_j-p_N\right)\sum_{i=1,i\neq j}^{N-1}\left(\alpha_i-\beta_i\right)\left(p_i-p_N\right)\right]}{E\left[(p_j-p_N)^2\right]},$$

Again for $j=1,\ldots,N$. This is the same as before for the case $N=2$. This can be restated as requiring α_js to satisfy

$$\sum_{i=1}^{N-1}\alpha_i E[(p_i-p_N)(p_j-p_N)]=\sum_{i=1}^{N-1}E[\beta_i(p_i-p_N)(p_j-p_N)],\quad\forall\quad j=1,\ldots,N-1.$$

We write this in a matrix multiplication form $A\alpha = b$, where

$$A_{ij} = E\left[(p_i - p_N)(p_j - p_N)\right],$$

and

$$b_j = \sum_{i=1}^{N-1} E[\beta_i (p_i - p_N)(p_j - p_N)].$$

This allows us to write

$$\alpha^* = A^{-1} b,$$

where α^* is a vector of the first $N-1$ optimal α_js and $\alpha_N = 1 - \sum_{j=1}^{N-1} \alpha_j$. The the optimal value is

$$f(\alpha^*) = E\left[\left((p - p_N)^\top \left(A^{-1}b - \beta\right)\right)^2\right]. \tag{3.7}$$

Note that in Subsection 2.5.1, we have that

$$\alpha^* = \frac{E\left[\beta(p_1 - p_2)^2\right]}{E\left[(p_1 - p_2)^2\right]},$$

which is consistent with this more generalized solution.

3.5.2 Convex combination in N periods

We recall from Section 3.1 that the first term in our optimization is zero only if we have predictable or deterministic β_is. We now find deterministic β_is that satisfy the manipulation part of our optimization. That is the second term with deterministic β_i,

$$\min_{\beta} E\left[\sum_{i=1}^{N} \beta_i p_i\right],$$

we know from Section 3.4 that we need our β_is to satisfy

1. $E\left[\frac{\beta_i}{u_i}\right] = E\left[\frac{\beta_j}{u_j}\right] \quad \text{for all} \quad i, j = 1, \ldots, N$

2. $\sum_{i=1}^{N} \beta_i = 1.$

This is achieved for

$$\beta_i = \frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}}.$$

Recall also that $E\left[\sum_{i=1}^{N} \beta_i p_i\right]$ is minimized over stochastic β_i summing up to 1 by $\beta_i = \frac{u_i}{\sum_{j=1}^{N} u_j}$. So let us now proceed looking at a convex combination of deterministic and stochastic solutions,

$$\begin{aligned}\beta_i &= \mu \frac{u_i}{\sum_{j=1}^{N} u_j} + (1-\mu) \frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}} \\ &= \mu \beta^{(1)} + (1-\mu)\beta^{(2)}.\end{aligned}$$

We calculate the first term, noting that the vector $b = \mu b^{(1)} + (1-\mu) b^{(2)}$, where $b_j^{(i)} = \sum_{i=1}^{N-1} E[\beta_i^{(i)}(p_i - p_N)(p_j - p_N)]$ for $j = 1, \ldots, N-1$, and plugging in the optimal value from (3.7),

$$\begin{aligned}E\left[\left(\sum_{i=1}^{N} (\alpha_i - \beta_i) p_i\right)^2\right] &= E\left[\left((p-p_N)^\top (A^{-1}b - \beta)\right)^2\right] \\ &= E\Big[\Big\{(p-p_N)^\top \left(A^{-1}\left(\mu b^{(1)} + (1-\mu) b^{(2)}\right) - \mu\beta^{(1)}\right. \\ &\qquad\qquad \left. -(1-\mu)\beta^{(2)}\right)\Big\}^2\Big] \\ &= E\Big[\Big\{(p-p_N)^\top \left(\mu A^{-1}\left(b^{(1)} - b^{(2)}\right) + A^{-1}b^{(2)}\right. \\ &\qquad\qquad \left. -\mu(\beta^{(1)} - \beta^{(2)}) - \beta^{(2)}\right)\Big\}^2\Big] \\ &= E\Big[\Big\{\mu(p-p_N)^\top \left(A^{-1}\left(b^{(1)} - b^{(2)}\right) - (\beta^{(1)} - \beta^{(2)})\right) \\ &\qquad + (p-p_N)^\top \left(A^{-1}b^{(2)} - \beta^{(2)}\right)\Big\}^2\Big].\end{aligned}$$

We note that $\beta^{(2)}$ is deterministic, so $b^{(2)}$ can be written as $A\beta^{(2)}$, meaning that $A^{-1}b^{(2)} = \beta^{(2)}$. Therefore $A^{-1}b^{(2)} - \beta^{(2)} = 0$, and

$$E\left[\left(\sum_{i=1}^{N} (\alpha_i - \beta_i) p_i\right)^2\right] = \mu^2 E\left[\left((p-p_N)^\top \left(A^{-1}b^{(1)} - \beta^{(1)}\right)\right)^2\right].$$

The second term in the overall minimization becomes

$$
\begin{aligned}
E\left[\sum_{i=1}^{N}\beta_i p_i\right] &= \sum_{i=1}^{N} E\left[\left(\mu\frac{u_i}{\sum_{j=1}^{N}u_j} + (1-\mu)\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}\right)p_i\right] \\
&= \sum_{i=1}^{N}\mu E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] + (1-\mu)\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right] \\
&= \mu\sum_{i=1}^{N} E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] + (1-\mu)\sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right].
\end{aligned}
$$

We differentiate our simplified objective,

$$
\begin{aligned}
&\mu^2 E\left[\left((p-p_N)^\top\left(A^{-1}b^{(1)}-\beta^{(1)}\right)\right)^2\right] \\
&\quad + \lambda\left(\mu\sum_{i=1}^{N}E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] + (1-\mu)\sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right]\right),
\end{aligned}
$$

with respect to μ and set it equal to zero,

$$
\begin{aligned}
&2\mu E\left[\left((p-p_N)^\top\left(A^{-1}b^{(1)}-\beta^{(1)}\right)\right)^2\right] \\
&\quad + \lambda\left(\sum_{i=1}^{N}E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] - \sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right]\right) = 0,
\end{aligned}
$$

which gives us

$$
\mu = \frac{\lambda}{2}\frac{\sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E\left[1/u_j\right]}}E\left[p_i\right] - \sum_{i=1}^{N}E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right]}{E\left[\left((p-p_N)^\top\left(A^{-1}b^{(1)}-\beta^{(1)}\right)\right)^2\right]}.
$$

We examine the numerator and check if it is positive. To do so, we compute

$$
\begin{aligned}
&\sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right] - \sum_{i=1}^{N}E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] \\
&= \sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[\tilde{p}_i + c\frac{v_i}{u_i}\right] - \sum_{i=1}^{N}E\left[\left(\tilde{p}_i + c\frac{v_i}{u_i}\right)\frac{u_i}{\sum_{j=1}^{N}u_j}\right].
\end{aligned}
$$

Under deterministic v_i, this numerator becomes

$$\sum_{i=1}^{N} E\left[\tilde{p}_i\right]\left(\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{u_i}{\sum_{j=1}^{N}u_j}\right]\right) + \sum_{i=1}^{N} cv_i\left(\frac{1}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{1}{\sum_{j=1}^{N}u_j}\right]\right).$$

Proposition 1. *If u_i are independent, a is a non negative constant, and b is a positive constant, then*

$$\frac{1}{a+b\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{1}{a+b\sum_{j=1}^{N}u_j}\right] \geq 0.$$

Proof. Proceeding in a method similar to Chapter 2, note that $f(x) = \frac{1}{b/x+c}$ is a concave function with $f''(x) = \frac{-2bc}{(b+cx)^3}$ which is less than zero for $c > 0$ and $x > 0$. Next, consider the function $g_N(z) = \frac{1}{a+b\left(\sum_{j=1}^{N-1}1/X_j+1/z\right)}$, where X_j are independent and non negative random variables. $g_N(z)$ is concave and $g_N(X_N) = \frac{1}{a+b\sum_{j=1}^{N}1/X_j}$. Then,

$$\begin{aligned}
E\left[g_N(X_N)\right] &= E\left[\left[g_N(X_N)|X_1,\ldots,X_{N-1}\right]\right] \\
&\leq E\left[g_N(E\left[X_N|X_1,\ldots,X_N\right])\right] \quad \text{(Jensen's inequality)} \\
&= E\left[\frac{1}{a+b\sum_{j=1}^{N-1}1/X_j+b/E[X_N|X_1,\ldots,X_N]}\right] \\
&= E\left[\frac{1}{a+b\sum_{j=1}^{N-1}1/X_j+b/E[X_N]}\right].
\end{aligned}$$

Then take $g_{N-1}(z) = \frac{1}{a+b\sum_{j=1}^{N-2}X_j+b/E[X_N]+b/z}$, which is again concave, and apply Jensen's inequality to obtain

$$\begin{aligned} E\left[g_{N-1}(X_{N-1})\right] &= E\left[E\left[g_{N-1}(X_{N-1})|X_1,\ldots,X_{N-2}\right]\right] \\ &\leq E\left[g_{N-1}(E[X_{N-1}|X_1,\ldots,X_{N-1}])\right] \\ &= E\left[g_{N-1}(E[X_{N-1}])\right] \\ &= E\left[\frac{1}{a+b\sum_{j=1}^{N-2}1/X_j+b/E[X_{N-1}]+b/E[X_N]}\right]. \end{aligned}$$

Doing this N times we see

$$E\left[\frac{1}{a+b\sum_{j=1}^{N}1/X_j}\right] \leq E\left[\frac{1}{a+b\sum_{j=1}^{N}1/E[X_j]}\right] = \frac{1}{a+b\sum_{j=1}^{N}1/E[X_j]}.$$

Letting $X_i = 1/u_i$ we have, $E\left[\frac{1}{a+b\sum_{j=1}^{N}u_j}\right] \leq \frac{1}{a+b\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}$, which shows us $\frac{1}{a+b\sum_{j=1}^{N}\frac{1}{E\left[1/u_j\right]}} - E\left[\frac{1}{a+b\sum_{j=1}^{N}u_j}\right] \geq 0.$ □

Letting $a = 0$ and $b = 1$, we see that the μ is non negative. Looking at $\sum_{i=1}^{N}E\left[\tilde{p}_i\right]\left(\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E\left[1/u_j\right]}} - E\left[\frac{u_i}{\sum_{j=1}^{N}u_j}\right]\right)$, with $E[\tilde{p}_i] = E[\tilde{p}_j]$, for $i,j = 1,\ldots,N$, this is

$$E\left[\tilde{p}_1\right]\left(\frac{\sum_{i=1}^{N}\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{\sum_{i=1}^{N}u_i}{\sum_{j=1}^{N}u_j}\right]\right) = 0.$$

So the numerator is nonnegative. That is

$$\begin{aligned} &\sum_{i=1}^{N}\frac{\frac{1}{E[1/u_i]}}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}}E\left[p_i\right] - \sum_{i=1}^{N}E\left[p_i\frac{u_i}{\sum_{j=1}^{N}u_j}\right] \\ &= \left(\frac{1}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{1}{\sum_{j=1}^{N}u_j}\right]\right)c\sum_{i=1}^{N}v_i \\ &= Vc\left(\frac{1}{\sum_{j=1}^{N}\frac{1}{E[1/u_j]}} - E\left[\frac{1}{\sum_{j=1}^{N}u_j}\right]\right) \geq 0. \end{aligned}$$

Finally we also require that $\mu \leq 1$, so simply put

$$\mu = \min \left(\frac{\lambda}{2} \frac{Vc \left(\frac{1}{\sum_{j=1}^{N} \frac{1}{E[1/u_j]}} - E\left[\frac{1}{\sum_{j=1}^{N} u_j} \right] \right)}{E\left[\left((p - p_N)^\top \left(A^{-1} b^{(1)} - \beta^{(1)} \right) \right)^2 \right]}, 1 \right). \tag{3.8}$$

If we let $B = (p - p_T)(p - p_T)^\top$, we can write

$$A^{-1}b^{(1)} = (E[B])^{-1}E[B\beta^{(1)}],$$

Then the denominator in (3.8) can be interpreted as a form of variance of $\beta^{(1)}$ under an adjusted measure, taking the price structure into account. Similarly, the numerator in (3.8) can be seen as measuring a weighted distance between the denominators of $\beta^{(1)}$ and $\beta^{(2)}$. Therefore, the numerator in (3.8) describes how much we should go in the direction of $\beta^{(2)}$ from $\beta^{(1)}$, with the denominator normalizing the expression.

3.6 Numerical example

A series of numerical experiments were run to observe how μ and the resulting benchmark changes with various model parameters. 10,000 days of 100 periods were simulated with volume at each period following a gamma distribution. We model underlying prices $\tilde{p}_i$ as

$$\tilde{p}_i = \tilde{\mu} + \sum_{l=1}^{i} z_l,$$

where z_l are independent normal random variables with mean 0 and variance σ^2. Looking at $E\left[(\tilde{p}_i - \tilde{p}_N)(\tilde{p}_j - \tilde{p}_N)\right]$, we note that,

$$\tilde{p}_i - \tilde{p}_N = \sum_{l=1}^{i} z_l + \tilde{\mu} - \left(\sum_{l=1}^{N} z_l + \tilde{\mu} \right) = - \sum_{l=i+1}^{N} z_l.$$

So

$$
\begin{aligned}
E\left[(\tilde{p}_i - \tilde{p}_N)(\tilde{p}_j - \tilde{p}_N)\right] &= E\left[\left(-\sum_{l=i+1}^{N} z_l\right)\left(-\sum_{k=j+1}^{N} z_k\right)\right] \\
&= \sum_{k=\max(i,j)+1}^{N} E[z_k^2] \\
&= (N - \max(i,j))\sigma^2,
\end{aligned}
$$

using that z_i are independent of each other and identically distributed. We now look to simplify the matrix A,

$$
\begin{aligned}
A_{ij} &= E\left[(p_i - p_N)\left((p_j - p_N)\right)\right] \\
&= E\left[\left(\tilde{p}_i - \tilde{p}_N + c\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\right)\left(\tilde{p}_j - \tilde{p}_N + c\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right)\right] \\
&= E\left[\left(-\sum_{l=i+1}^{N} z_l + c\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\right)\left(-\sum_{l=j+1}^{N} z_l + c\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right)\right] \\
&= E\left[\left(-\sum_{l=i+1}^{N} z_l\right)\left(-\sum_{l=j+1}^{N} z_l\right)\right] - cE\left[\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\sum_{l=i+1}^{N} z_l\right] \\
&\quad - cE\left[\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\sum_{l=j+1}^{N} z_l\right] + c^2 E\left[\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right].
\end{aligned}
$$

Noting that the z_ls are independent of volumes and have zero mean, as well as the above facts, we have

$$
A_{ij} = (N - \max(i,j))\sigma^2 + c^2 E\left[\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right].
$$

Let's also simplify $b^{(1)}$ with this more concrete price model:

$$
\begin{aligned}
b_j^{(1)} &= \sum_{i=1}^{N-1} E[\beta_i^{(1)}(p_i - p_N)(p_j - p_N)] \\
&= \sum_{i=1}^{N-1} E\left[\beta_i^{(1)}\left(-\sum_{l=i+1}^{N} z_l + c\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\right)\left(-\sum_{l=j+1}^{N} z_l + c\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right)\right] \\
&= \sum_{i=1}^{N-1} E\left[\beta_i^{(1)}\right](N - \max(i,j))\sigma^2 + c^2 E\left[\beta_i^{(1)}\left(\frac{v_i}{u_i} - \frac{v_N}{u_N}\right)\left(\frac{v_j}{u_j} - \frac{v_N}{u_N}\right)\right]
\end{aligned}
$$

Letting $\tilde{z}$ be an $N-1$ vector with $\tilde{z}_i = \sum_{l=i+1}^{N} z_l$, we can re write the denominator as

$$
\begin{aligned}
&E\left[\left((p - p_N)^\top \left(A^{-1}b^{(1)} - \beta^{(1)}\right)\right)^2\right] \\
&\quad = E\left[\left(\left(c\left(\frac{u}{v} - \frac{v_N}{u_N}\right) - \tilde{z}\right)^\top \left(A^{-1}b^{(1)} - \beta^{(1)}\right)\right)^2\right]
\end{aligned}
$$

These expansions are used when calculating these objects in the simulations and we also see that μ depends on price volatility and volumes.

Choosing c

To carry out our simulation, we need to find a reasonable value for the price impact coefficient, c, in the expression

$$p_i = \tilde{p}_i + c\frac{v_i}{u_i}.$$

We use the trading rule of thumb referenced in [10], that trading one day's volume costs about one day's volatility in basis points. This gives us

$$c \approx 100\frac{\sqrt{N\sigma^2}}{U},$$

where N is the number of periods, σ is the volatility of underlying prices described above, and U is the total volume in a day.

Simulating volumes

Outside volume is modelled by independent gamma random variable with varying parameters. Two scenarios of volume distribution over time were considered: volume that is on average largest at the beginning and end of the day, eg. $u_i \sim \text{Gamma}(100, 100(\frac{i}{100} - 0.5)^2)$, where the Gamma distribution is defined by shape and scale parameters and volume that is constant throughout the day on average, eg. $u_i \sim \text{Gamma}(100, 35)$ for all i.

Distribution of benchmarks

10,000 days were simulated with with given manipulator capital, volume distributions, starting asset price, and σ^2. Kernel densities were then estimated for the deterministic, VWAP, and optimal benchmark using a Gaussian kernel. U shaped volume and constant expectation volume appeared to have no effect on the distribution of the benchmarks. The distributions can be seen below in Figure 3.1.

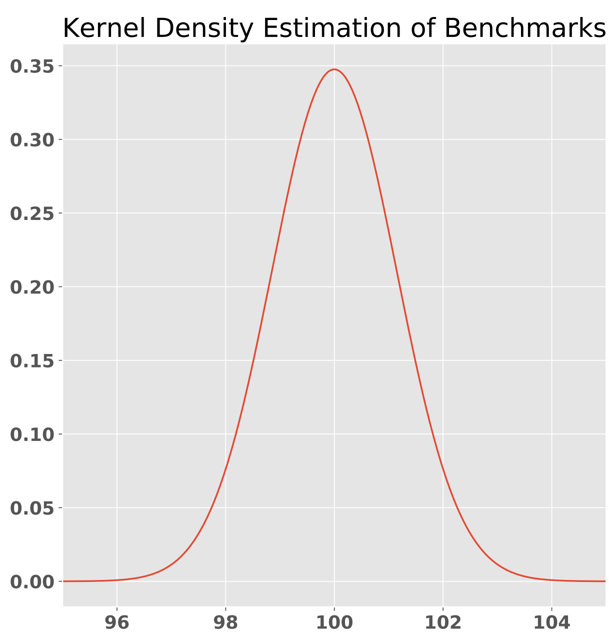

Figure 3.1: Distribution of benchmarks over 10,000 simulated days, with "U" shaped volume, $\sigma^2 = 0.01$, manipulator capital of \$100, and a starting asset price of \$100

Objective breakdown as a function of λ

We will now examine how our two concerns, attainability and manipulation, comprise our total objective function. The below Figure 3.2 the averages of these values over 10,000 samples. Since the manipulation objective is just the benchmark value we subtract the "true" value, 100, from this component and also scale it by λ, `shifted_total_objective` is then the sum of this and `attainability_objective`. As a reminder, the attainability objective is $E\left[\left(\sum_{i=1}^{N}\alpha_i p_i - \sum_{i=1}^{N}\beta_i p_i\right)^2\right]$.

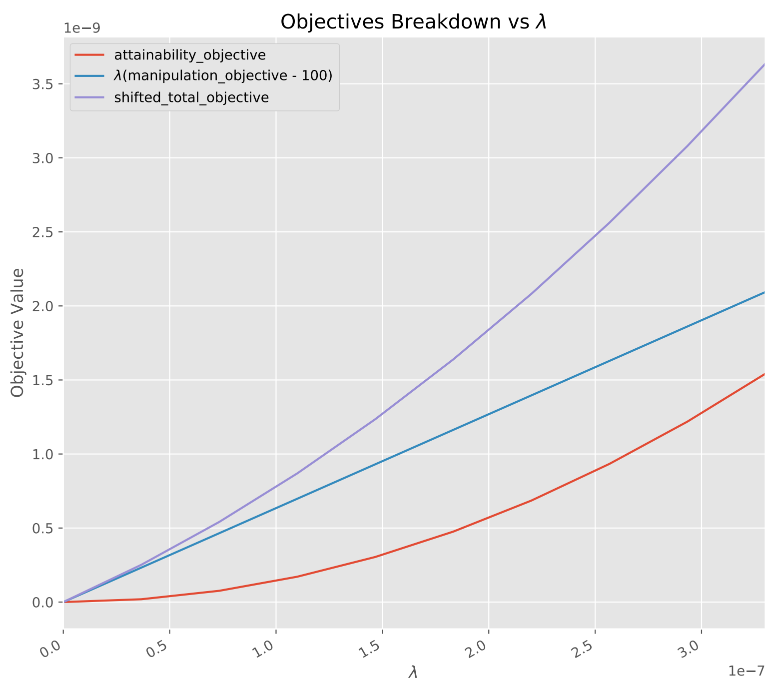

Figure 3.2: Average objective components vs. λ with constant expectation volumes throughout the day.

μ and volume shape

How does our value of μ, the weighting of our two potential benchmarks, change with different models of volume. We look at the value of μ as the volume distribution throughout the day goes from constant to "U" shaped while keeping total volume constant. Figure 3.3 shows that as volume becomes more "U" shaped, μ, the amount of weight place on the stochastic benchmark, decreases.

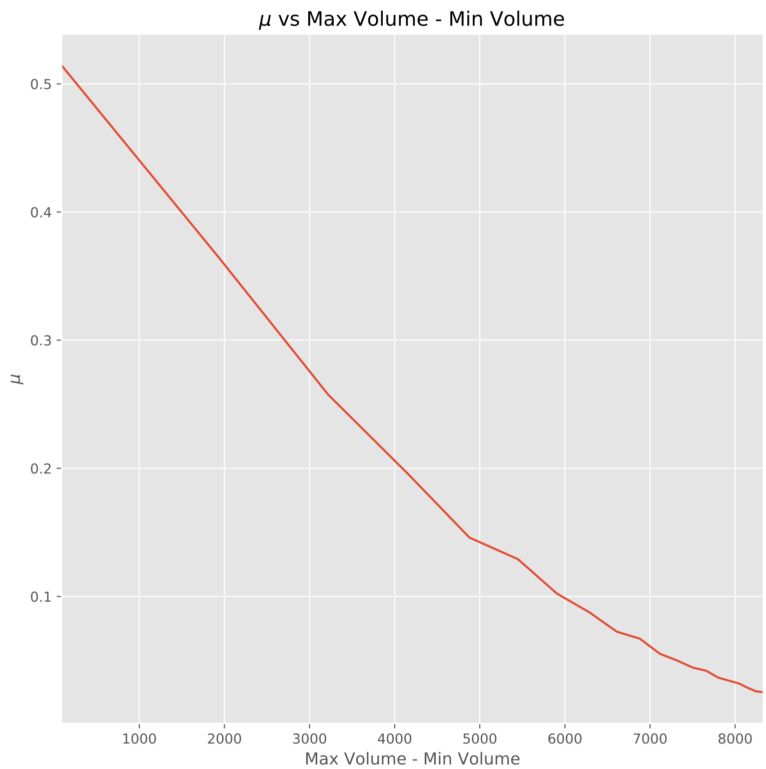

Figure 3.3: μ vs. the difference in expected maximum volume and minimum expected volume in the day.

Chapter 4

Multiperiod model with auction

We now introduce a closing auction to our model, following a similar structure to the previous two chapters. However, we do not find the optimal strategy for minimizing trading costs as this is covered in Frei and Yan [12]. We start with a trader who wishes to attain the asset at the benchmark price, then introduce a manipulating trader and see how they would best artificially inflate the benchmark. We then find an optimal convex combination of deterministic and stochastic benchmarks and end with a numerical simulation.

4.1 Attainability

We retain the parties from the previous chapters, a benchmark administrator, a trader wishes to buy shares at the benchmark, and trader who wishes to manipulate the benchmark upwards. The benchmark is again of the form $\beta = \sum_{i=1}^{T} \beta_i p_i$ with $\sum_{i=1}^{T} \beta_i = 1$.

Once again, the honest trader's objective is

$$\min_{\alpha_i} E\left[\left(\sum_{i=1}^{T}(\alpha_i - \beta_i)p_i\right)^2\right],$$

which as in the previous section is minimized for deterministic β_i. In the case of stochastic β_i, we assume deterministic α_js and differentiate the interior with respect to α_j, first noting that,

$$f(\alpha) = E\left[\left(\sum_{i=1}^{T}(\alpha_i - \beta_i)p_i\right)^2\right] = E\left[\left(\sum_{i=1}^{T-1}(\alpha_i - \beta_i)(p_i - p_T)\right)^2\right].$$

Then differentiating with respect to α_j

$$\frac{\partial f(\alpha)}{\partial \alpha_j} = E\left[2(p_j - p_T)\left(\sum_{i=1}^{T-1}(\alpha_i - \beta_i)(p_i - p_T)\right)\right] = 0 \text{ for } j = 1, \ldots, T-1,$$

which is equivalent to

$$\sum_{i=1}^{T-1}\alpha_i E\left[(p_i - p_T)(p_j - p_T)\right] = \sum_{i=1}^{T-1} E\left[\beta_i(p_i - p_T)(p_j - p_T)\right] \text{ for } j = 1, \ldots, T-1.$$

We can represent this system as

$$A\alpha = b$$

where

$$b_j = \sum_{i=1}^{T-1} E[\beta_i(p_i - p_T)(p_j - p_T)]$$

and

$$A_{ij} = E[(p_i - p_T)(p_j - p_T)].$$

For $i, j = 1, \ldots, T-1$ and $\alpha_T = 1 - \sum_{i=1}^{T-1}\alpha_i$. Assuming that A is invertible then the α_i that satisfy these equations are

$$\alpha* = A^{-1}b.$$

Finally, this gives the value

$$f(\alpha*) = E\big[\big((p - p_T)^\top(A^{-1}b - \beta)\big)^2\big].$$

4.2 The manipulator's optimization

We now introduce our model for price impact, which consists of temporary price impact from trades made during open trading and permanent price impact from trades submitted to the auction at the time of the imbalance announcement. We use a similar set up to the one found in [12], including the "stylized feature that the order imbalance is cleared immediately and no further orders are submitted to the auction". That is, our market has continuous trading followed by an auction at the end of the trading day. Additionally, at some point in time before the auction, the imbalance in the auction is announced. Our price process remains relatively unchanged to the previous section with changes made to include the imbalance announcement and auction. Letting τ be the time of the imbalance announcement and T be the time of the auction. Our model consists of

- the underlying price process, $\tilde{p}_i = \tilde{p}_{i-1} + z_i$ for $i = 1, \ldots, \tau - 1, \tau + 1, \ldots, T - 1$, where z_i are independent random variables with mean 0 and constant variance σ^2

- the underlying price at the time of the imbalance announcement $\tilde{p}_\tau = \tilde{p}_{\tau-1} + z_\tau + c_\tau N$ where $N = u_T + v_T$, the imbalance from outside orders submitted to the auction, u_T and the manipulator's orders submitted to the auction, v_T, and c_τ the price impact of the imbalance announcement

- the final underlying price is $\tilde{p}_T = \tilde{p}_{T-1} + y$ where y is the price fluctuation in the auction and is independent of $z_i s$

- the price paid, $p_i = \tilde{p}_i + c\frac{v_i}{u_i}$ for $i \neq T$, where v_i is the order submitted, u_i is outside volume and c is a coefficient for price impact

- the final price paid $p_T = \tilde{p}_T$ since the trader's price impact to the auction is already included in p_τ and subsequent p_is.

With our model for price impact solidified, we will analyze how a benchmark can be impacted. We assume the trader can purchase V shares to exert upwards pressure on the benchmark. The trader purchases v_i shares in each period and each v_i is predictable except for v_T which is assumed to be observable at time $\tau - 1$. We also assume that the trader will not short sell. Naturally, it is optimal for the trader to spend all of V. We assume our benchmark is of the form $\beta_j = Xu_j + y_j(u_j)$ for $j = 1, \dots, T-1$ and $\beta_T = \frac{c}{c_\tau}X - X\sum_{j=\tau}^{T-1} u_j + y_T(u_T)$. The benchmark that we consider below will satisfy these conditions, which imply

$$\sum_{j=\tau}^{T} \beta_j = \frac{c}{c_\tau}X + \sum_{j=\tau}^{T} y_j(u_j). \tag{4.1}$$

The manipulating trader faces the optimization

$$\begin{aligned}
&\max_{\sum_{i=1}^T v_i = V} E\left[\sum_{i=1}^{T} \beta_i p_i\right] = \max_{\sum_{i=1}^T v_i = V} \sum_{i=1}^{T-1} E\left[\beta_i\left(\tilde{p}_i + c\frac{v_i}{u_i}\right)\right] + E\left[\beta_T(\tilde{p}_{T-1} + y)\right] \\
&= \max_{\sum_{i=1}^T v_i = V} \left\{\sum_{i=\tau}^{T-1} E\left[\beta_i\left(\tilde{p}_{\tau-1} + \sum_{l=\tau}^{i} z_l + c_\tau(u_T + v_T)\right)\right] + \sum_{i=1}^{T-1} E\left[\beta_i c\frac{v_i}{u_i}\right]\right. \\
&\qquad \left. + E\left[\beta_T\left(\tilde{p}_{\tau-1} + \sum_{l=\tau}^{T-1} z_l + c_\tau(u_T + v_T)\right)\right]\right\} + \sum_{i=1}^{\tau-1} E\left[\beta_i \tilde{p}_i\right] + E\left[\beta_T y\right] \\
&= \sum_{i=\tau}^{T-1} E\left[\beta_i\left(\tilde{p}_{\tau-1} + \sum_{l=\tau}^{i} z_l + c_\tau u_T\right)\right] + E\left[\beta_T\left(\tilde{p}_{\tau-1} + \sum_{l=\tau}^{T-1} z_l + c_\tau u_T\right)\right] \\
&\quad + \max_{\sum_{i=1}^T v_i = V} \left\{\sum_{i=1}^{T-1} E\left[\beta_i c\frac{v_i}{u_i}\right] + \sum_{i=\tau}^{T} E\left[\beta_i c_\tau v_T\right]\right\} + \sum_{i=1}^{\tau-1} E\left[\beta_i \tilde{p}_i\right] + E\left[\beta_T y\right].
\end{aligned}$$

Using (4.1), we focus on

$$\begin{aligned}
&\max_{\sum_{i=1}^T v_i = V} \left\{\sum_{i=1}^{T-1} E\left[\beta_i c\frac{v_i}{u_i}\right] + \sum_{i=\tau}^{T} E\left[c_\tau v_T \beta_i\right]\right\} \\
&= \max_{\sum_{i=1}^T v_i = V} \left\{\sum_{i=1}^{T-1} E\left[(Xu_i + y_i(u_i))c\frac{v_i}{u_i}\right] + E\left[cv_T X + c_\tau v_T \sum_{j=\tau}^{T} y_j(u_j)\right]\right\}
\end{aligned}$$

Applying the facts that v_T is $\mathcal{F}_{\tau-1}$-measurable, v_i are $\mathcal{F}_{i-1}$-measurable and u_i

are all independent, this becomes

$$\max_{\sum_{i=1}^T v_i = V} \left\{ cVE[X] + c\sum_{i=1}^{T-1} E\left[E\left[\frac{v_i y_i(u_i)}{u_i}\middle|\mathcal{F}_{i-1}\right]\right]\right.$$
$$\left. + E\left[E\left[c_\tau v_T \sum_{j=\tau}^{T} y_j(u_j)\middle|\mathcal{F}_{\tau-1}\right]\right]\right\}$$
$$= \max_{\sum_{i=1}^T v_i = V} \left\{ cVE[X] + c\sum_{i=1}^{T-1} E\left[v_i E\left[\frac{y_i(u_i)}{u_i}\right]\right] + E\left[c_\tau v_T E\left[\sum_{j=\tau}^{T} y_j(u_j)\right]\right]\right\}$$
$$= cVE[X] + \max_{\sum_{i=1}^T v_i = V} \left\{ c\sum_{i=1}^{T-1} E[v_i]\, E\left[\frac{y_i(u_i)}{u_i}\right] + c_\tau E[v_T]\, E\left[\sum_{j=\tau}^{T} y_j(u_j)\right]\right\}$$
$$= \max_{\sum_{i=1}^T v_i = V} \left\{ c\sum_{i=1}^{T-1} E[v_i]\, E\left[\frac{\beta_i}{u_i}\right] + c_\tau E[v_T]\, E\left[\sum_{j=\tau}^{T} \beta_j\right]\right\},$$

using again (4.1) in the last equality. Similarly to Section 3.4, this implies that the manipulator's optimal strategy is deterministic. If $c_\tau E\big[\sum_{j=\tau}^T \beta_j\big] > cE\big[\frac{\beta_i}{u_i}\big]$ for all $i = 1, \ldots, T-1$, it is optimal for the manipulator to purchase all shares in the auction, i.e., $v_T = V$. Otherwise, it is optimal to purchase all shares in the period $i = 1, \ldots, T-1$ with the largest $E\big[\frac{\beta_i}{u_i}\big]$.

This results from the permanent price impact of orders submitted to the auction. If there is a period where the trader can have a larger price impact than all of the β_is after and including τ, then it is worth it for the trader to buy V shares in that period.

We minimize the manipulator's impact to our benchmark. Recalling that our β_is are functions of volumes only and $\sum_{i=1}^T \beta_i = 1$, we proceed with the minimization,

$$\min_\beta E\left[\sum_{i=1}^T \beta_i p_i\right].$$

From the above we see that we require our β_is to satisfy

$$cE\left[\frac{\beta_i}{u_i}\right] = c_\tau \sum_{j=\tau}^{T} E[\beta_j] \quad \forall \quad i = 1, \ldots, T-1.$$

This can be rewritten as

$$cE\left[\frac{\beta_i}{u_i}\right] = c_\tau\left(\sum_{j=\tau}^{T-1} E[\beta_j] + 1 - \sum_{j=1}^{T-1} E[\beta_j]\right) \quad \forall \quad i = 1, \ldots, T-1,$$
$$cE\left[\frac{\beta_i}{u_i}\right] = c_\tau\left(1 - \sum_{j=1}^{\tau-1} E[\beta_j]\right) \quad \forall \quad i = 1, \ldots, T-1.$$

Since $E\left[\frac{\beta_i}{u_i}\right]$ does not depend on i, we consider an approach of the form $\beta_i = u_i X$ for some random variable X, which needs to satisfy

$$cE[X] = c_\tau\left(1 - \sum_{j=1}^{\tau-1} E[X u_j]\right),$$
$$c_\tau = E\left[X\left(c + c_\tau \sum_{j=1}^{\tau-1} u_j\right)\right].$$

One solution is $X = \frac{c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j}$ so that

$$\beta_i = \frac{u_i c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \quad \forall \quad i = 1, \ldots, T-1.$$

Note that in the limit $c_\tau \to \infty$ (infinite price impact of auction imbalance), β_i converges to $\frac{u_i}{\sum_{j=1}^{\tau-1} u_j}$, which corresponds to the formula in the non-auction case for periods $i = 1, \ldots, \tau - 1$. We further have

$$\beta_T = 1 - \sum_{i=1}^{T-1} \beta_i = 1 - \frac{\sum_{i=1}^{T-1} u_i c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} = \frac{c - c_\tau \sum_{i=\tau}^{T-1} u_i}{c + c_\tau \sum_{j=1}^{\tau-1} u_j}.$$

Note that this choice of the benchmark satisfies (4.1) because

$$\sum_{j=\tau}^{T} \beta_j = \frac{\sum_{i=\tau}^{T-1} u_i c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} + \frac{c - c_\tau \sum_{i=\tau}^{T-1} u_i}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} = \frac{c}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} = \frac{c}{c_\tau}\frac{\beta_i}{u_i}$$

for all $i = 1, \ldots, T-1$.

4.3 Combined problem

Now that we know to be optimal in terms of attainability our β_i should be deterministic and to be optimal in response to manipulation, the β_i need to satisfy

$$cE\left[\frac{\beta_i}{u_i}\right] = c_\tau \sum_{j=\tau}^{T} E[\beta_j] \quad \forall \quad i = 1, \ldots, T-1,$$

let us now find a deterministic solution that satisfies the above. We will then look at a convex combination of the two, as in the previous section. One possibility is

$$\beta_i = \frac{c_\tau}{E\left[1/u_i\right]\left(c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]\right)} \quad \forall \quad i = 1, \ldots, T-1$$

and

$$\begin{aligned}
\beta_T &= 1 - \sum_{i=1}^{T-1} \frac{c_\tau}{E\left[1/u_i\right]\left(c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]\right)} \\
&= 1 - \frac{c_\tau \sum_{i=1}^{T-1} 1/E[1/u_i]}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]} \\
&= \frac{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right] - c_\tau \sum_{i=1}^{T-1} 1/E[1/u_i]}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]} \\
&= \frac{c - c_\tau \sum_{i=\tau}^{T-1} 1/E[1/u_i]}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]}.
\end{aligned}$$

We verify the deterministic solution

$$\begin{aligned}
&cE\left[\frac{c_\tau}{u_i E\left[1/u_i\right]\left(c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]\right)}\right] \\
&= c_\tau \left(\sum_{j=\tau}^{T-1} \frac{c_\tau}{E\left[1/u_j\right]\left(c + c_\tau \sum_{l=1}^{\tau-1} 1/E\left[1/u_l\right]\right)} + \frac{c - c_\tau \sum_{i=\tau}^{T-1} 1/E[1/u_i]}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]}\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{cc_\tau}{E\left[1/u_i\right]\left(c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]\right)}E[1/u_i] \\
&\quad = \frac{c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]}\left(c_\tau\sum_{j=\tau}^{T-1}1/E\left[1/u_j\right]+c-c_\tau\sum_{i=\tau}^{T-1}1/E[1/u_i]\right),
\end{aligned}$$

and also equivalent to

$$c = c.$$

So this deterministic solution does indeed satisfy the desired condition.

Optimizing over a convex combination

We will now proceed as in the previous section and optimize over a convex combination. The proposed benchmark weights are now

$$\begin{aligned}
\beta_i &= \mu\beta_i^{(1)} + (1-\mu)\beta_i^{(2)} \\
&= \mu\frac{u_i c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}u_j} + (1-\mu)\frac{c_\tau}{E\left[1/u_i\right]\left(c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]\right)}
\end{aligned}$$

for all $i = 1, \ldots, T-1$ and

$$\beta_T = \mu\frac{c-c_\tau\sum_{i=\tau}^{T-1}u_i}{c+c_\tau\sum_{j=1}^{\tau-1}u_j} + (1-\mu)\frac{c-c_\tau\sum_{i=\tau}^{T-1}1/E[1/u_i]}{c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]}.$$

Note that both the stochastic and deterministic benchmark have the property that as the price impact from regular trading goes to infinity, the benchmarks are solely based on the auction price.

The optimization is

$$\min_{\mu\in[0,1]}\left\{\min_{\alpha_i} E\left[\left(\sum_{i=1}^{T}(\alpha_i-\beta_i)p_i\right)^2\right] + \lambda E\left[\sum_{i=1}^{T}\beta_i p_i\right]\right\}.$$

Plugging in the value from Section 4.1 and noting that $b = \mu b^{(1)} + (1-\mu)b^{(2)}$, where $b_j^{(i)} = \sum_{i=1}^{T-1} E[\beta_i^{(i)}(p_i - p_T)(p_j - p_T)]$, we find

$$
\begin{aligned}
&E\left[\left(\sum_{i=1}^{T}(\alpha_i - \beta_i)p_i\right)^2\right] = E\left[\left((p - p_T)^\top (A^{-1}b - \beta)\right)^2\right] \\
&= E\left[\left((p - p_T)^\top (A^{-1}\left(\mu b^{(1)} + (1-\mu)b^{(2)}\right) - \mu\beta^{(1)} - (1-\mu)\beta^{(2)})\right)^2\right] \\
&= E\left[\left((p - p_T)^\top \left(\mu A^{-1}\left(b^{(1)} - b^{(2)}\right) + A^{-1}b^{(2)} - \mu(\beta^{(1)} - \beta^{(2)}) - \beta^{(2)}\right)\right)^2\right] \\
&= E\Big[\Big\{\mu(p - p_T)^\top \left(A^{-1}\left(b^{(1)} - b^{(2)}\right) - (\beta^{(1)} - \beta^{(2)})\right) \\
&\qquad + (p - p_T)^\top \left(A^{-1}b^{(2)} - \beta^{(2)}\right)\Big\}^2\Big].
\end{aligned}
$$

We also see that since $\beta^{(2)}$ is deterministic, $b^{(2)}$ can be written as $A\beta^{(2)}$, meaning that $A^{-1}b^{(2)} = \beta^{(2)}$. This reduces the above to

$$
\begin{aligned}
&E\left[\left(\sum_{i=1}^{T}(\alpha_i - \beta_i)p_i\right)^2\right] \\
&= \mu^2 E\left[\left((p - p_T)^\top \left(A^{-1}\left(b^{(1)} - b^{(2)}\right) - (\beta^{(1)} - \beta^{(2)})\right)\right)^2\right] \\
&= \mu^2 E\left[\left((p - p_T)^\top \left(A^{-1}b^{(1)} - \beta^{(1)}\right)\right)^2\right].
\end{aligned}
$$

Looking at the second term in the optimization over μ, we have

$$
\begin{aligned}
\lambda E\left[\sum_{i=1}^{T}\beta_i p_i\right] &= \lambda E\left[\sum_{i=1}^{T}\left(\mu\beta_i^{(1)} + (1-\mu)\beta_i^{(2)}\right)p_i\right] \\
&= \lambda \sum_{i=1}^{T}\left(\mu E\left[\left(\beta_i^{(1)} - \beta_i^{(2)}\right)p_i\right] + \beta_i^{(2)}E[p_i]\right).
\end{aligned}
$$

We differentiate the objective with respect to μ,

$$
\begin{aligned}
&\frac{\partial}{\partial\mu}\left(\min_{\alpha_i} E\left[\left(\sum_{i=1}^{T}(\alpha_i - \beta_i)p_i\right)^2\right] + \lambda E\left[\sum_{i=1}^{T}\beta_i p_i\right]\right) \\
&\quad = 2\mu E\left[\left((p - p_T)^\top \left(A^{-1}b^{(1)} - \beta^{(1)}\right)\right)^2\right] + \lambda\sum_{i=1}^{T}\left(E\left[\left(\beta_i^{(1)} - \beta_i^{(2)}\right)p_i\right]\right).
\end{aligned}
$$

When we set this equal to zero we obtain

$$\mu = \frac{\lambda}{2} \frac{\sum_{i=1}^{T} \left(E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) p_i \right] \right)}{E\left[\left((p - p_T)^\top \left(A^{-1} b^{(1)} - \beta^{(1)} \right) \right)^2 \right]}. \tag{4.2}$$

We now simplify the numerator. For $i < \tau$

$$\begin{aligned} E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) p_i \right] &= E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) \left(\tilde{p}_i + c\frac{v_i}{u_i} \right) \right] \\ &= p_0 E\left[\beta_i^{(2)} - \beta_i^{(1)} \right] + cv_i \left(\beta_i^{(2)} E\left[1/u_i\right] - E\left[\beta_i^{(1)}/u_i \right] \right) \\ &= p_0 E\left[\beta_i^{(2)} - \beta_i^{(1)} \right] \\ &\quad + cv_i \left(\frac{c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E[1/u_j]} - E\left[\frac{c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right), \end{aligned}$$

for $\tau \leq i \leq T-1$,

$$E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) p_i \right] = E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) \left(p_0 + \sum_{l=1}^{i} z_l + c_\tau (u_T + v_T) + c\frac{v_i}{u_i} \right) \right]$$

using the facts the $E[z_l] = 0, z_l$ are independent of all other things, $E[u_T] = 0$, and u_T is independent of all β_is,

$$\begin{aligned} &= (p_0 + c_\tau v_T) E\left[\beta_i^{(2)} - \beta_i^{(1)} \right] \\ &\quad + cv_i \left(\frac{c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E[1/u_j]} - E\left[\frac{c_\tau}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right), \end{aligned}$$

and for $i = T$

$$\begin{aligned} E\left[\left(\beta_T^{(2)} - \beta_T^{(1)} \right) p_T \right] &= E\left[\left(\beta_T^{(2)} - \beta_T^{(1)} \right) \left(p_0 + \sum_{l=1}^{T-1} z_l + c_\tau (u_T + v_T) + y \right) \right] \\ &= (p_0 + c_\tau v_T) E\left[\beta_T^{(2)} - \beta_T^{(1)} \right], \end{aligned}$$

assuming that y is similarly distributed to z_l with zero mean and independent of everything. We can now write

$$\begin{aligned}
&\sum_{i=1}^{T}\left(E\left[\left(\beta_i^{(2)}-\beta_i^{(1)}\right)p_i\right]\right)\\
&\quad= p_0\sum_{i=1}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]+c_\tau v_T\sum_{i=\tau}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]\\
&\qquad+\sum_{i=1}^{T-1}cv_i\left(\frac{c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}1/E[1/u_j]}-E\left[\frac{c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}u_j}\right]\right)\\
&\quad= p_0\sum_{i=1}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]+c_\tau v_T\sum_{i=\tau}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]\\
&\qquad+c(V-v_T)\left(\frac{c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}1/E[1/u_j]}-E\left[\frac{c_\tau}{c+c_\tau\sum_{j=1}^{\tau-1}u_j}\right]\right).
\end{aligned}$$

Let us further analyze

$$\begin{aligned}
\sum_{i=\tau}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]&=c_\tau\left(\frac{\sum_{i=\tau}^{T-1}1/E[1/u_i]}{c+c_\tau\sum_{j=\tau}^{\tau-1}1/E[1/u_j]}-E\left[\frac{\sum_{i=1}^{T-1}u_i}{c+c_\tau\sum_{j=1}^{\tau-1}u_j}\right]\right)\\
&\quad+\frac{c-c_\tau\sum_{i=\tau}^{T-1}1/E[1/u_i]}{c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]}-E\left[\frac{c-c_\tau\sum_{i=\tau}^{T-1}u_i}{c+c_\tau\sum_{j=1}^{\tau-1}u_j}\right]\\
&=c\left(\frac{1}{c+c_\tau\sum_{j=1}^{\tau-1}1/E\left[1/u_j\right]}-E\left[\frac{1}{c+c_\tau\sum_{j=1}^{\tau-1}u_j}\right]\right).
\end{aligned}$$

We also note that

$$\sum_{i=1}^{T}E\left[\beta_i^{(2)}-\beta_i^{(1)}\right]=\sum_{i=1}^{T}\beta_i^{(2)}-E\left[\sum_{i=1}^{T}\beta_i^{(1)}\right]=1-1=0.$$

We now write our numerator as

$$
\begin{aligned}
\sum_{i=1}^{T} & \left(E\left[\left(\beta_i^{(2)} - \beta_i^{(1)} \right) p_i \right] \right) \\
&= cc_\tau v_T \left(\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E\left[1/u_j\right]} - E\left[\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right) \\
&\quad + cc_\tau (V - v_T) \left(\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E[1/u_j]} - E\left[\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right) \\
&= V cc_\tau \left(\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E[1/u_j]} - E\left[\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right).
\end{aligned}
$$

A simple application of Proposition 1 shows us that this numerator is nonnegative. Finally, including the requirement that $\mu \leq 1$, we have

$$
\mu = \min \left(\frac{\lambda}{2} \frac{V cc_\tau \left(\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} 1/E[1/u_j]} - E\left[\frac{1}{c + c_\tau \sum_{j=1}^{\tau-1} u_j} \right] \right)}{E\left[\left((p - p_T)^\top \left(A^{-1} b^{(1)} - \beta^{(1)} \right) \right)^2 \right]}, 1 \right).
$$

Remembering that

$$
A_{ij} = E[(p_i - p_T)(p_j - p_T)],
$$

If we let $B = (p - p_T)(p - p_T)^\top)$, we can write

$$
A^{-1} b^{(1)} = (E[B])^{-1} E[B \beta^{(1)}],
$$

Then the denominator in (4.2) can be interpreted as a form of variance of $\beta^{(1)}$ under an adjusted measure, taking the price structure into account. Similarly, the numerator in (4.2) can be seen as measuring a weighted distance between $\beta^{(1)}$ and $\beta^{(2)}$. Therefore, the numerator in (4.2) describes how much we should go in the direction of $\beta^{(2)}$ from $\beta^{(1)}$, with the denominator normalizing the expression.

4.4 Numerical example

We proceed similarly to Section 3.6 and run some numerical experiments to observe how the new μ and benchmarks change with model parameters. Again, 10,000 trading days of 100 periods were simulated. The following parameters and distributions were used:

- $\tilde{p}_i = \tilde{\mu} + \sum_{l=1}^{i} z_l$, the underlying price process where z_l are independent normal random variables with mean 0 and variance σ^2.

- y, the random difference from the last regular trading period to the auction price have the same distribution as z_ls.

- $c = 100\frac{\sqrt{(T-1)\sigma^2}}{U}$, where U is the day's total volume.

- $c_\tau = 6 \times 10^{-6}$ as taken from Yan [24]

- u_i follow a Gamma distribution for $i = 1, \ldots, T-1$.

- u_T follows a normal distribution with mean zero and a variance of 10% of the day's total volume.

Kernel density estimation of benchmarks

We first note that the kernel density estimation of the benchmarks looks identical to the no-auction case in Chapter 3, as is show in Figure 4.1

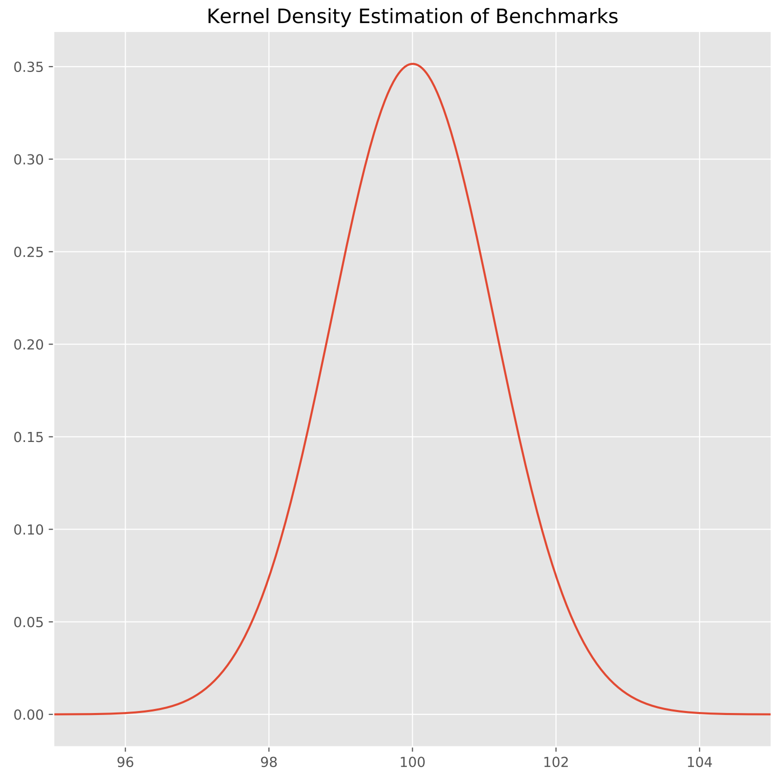

Figure 4.1: Kernel density estimation of the deterministic, stochastic, and optimal benchmarks (identical curves) in a market with an auction.

Honest trader's strategy with deterministic and stochastic benchmarks

Here we show the honest trader's strategy for a day with "U" shaped volume. In Figure 4.2 we see that the honest traders share purchases mirror the benchmarks and therefore volume except for in the auction period where the benchmarks are also negative. Note that the honest trader will purchase more shares than necessary in regular trading and sell them in the auction to better match the benchmark.

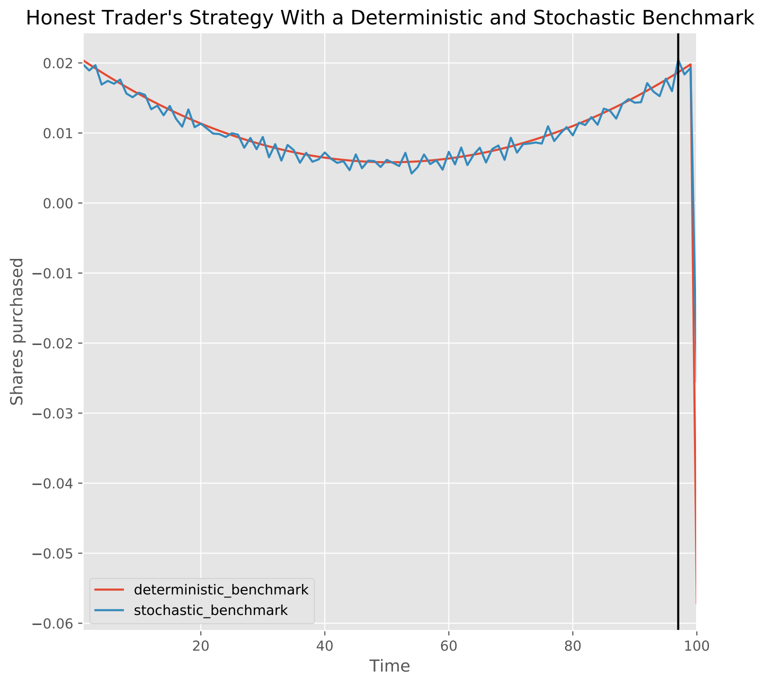

Figure 4.2: Proportion of shares that the honest trader buys throughout the day with a deterministic and stochastic benchmark. The black vertical line highlights the time of the imbalance announcement.

The attainability objective in auction vs no auction case

We find that μ grows much slower in relation to λ in the auction case. This leads to the attainability objective growing much slower as a function of λ in this case. In Figure 4.3 the attainability objective in the no auction case grows almost instantaneously compared to the auction case. We also see they grow to nearly identical values.

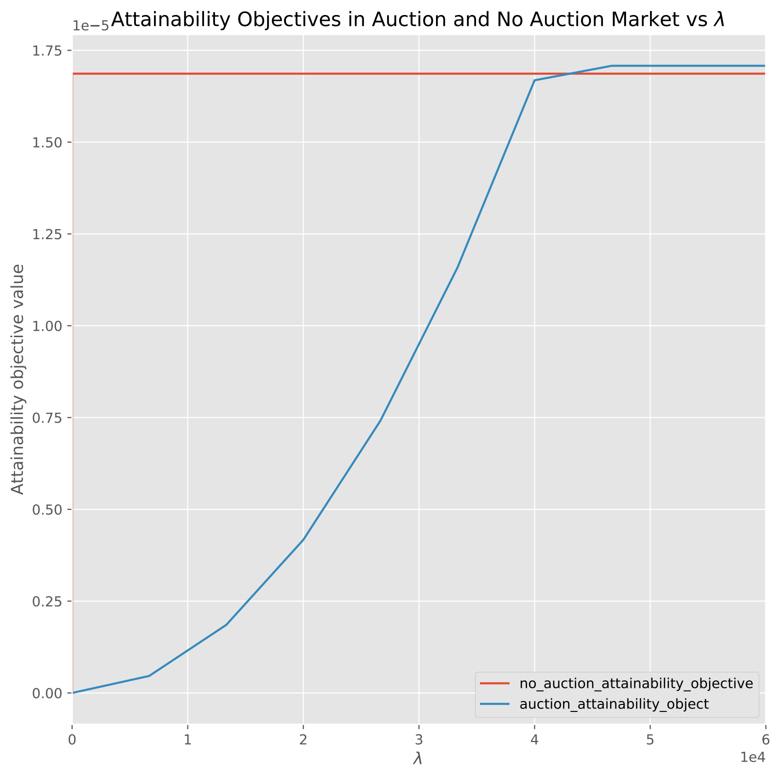

Figure 4.3: The attainability portion of our objective functions vs λ

Chapter 5

Conclusion

We have found that in a world where market administrators care only about attainability, a deterministic benchmark is preferable, and in a market with no auction, VWAP is optimal in preventing against manipulation. Taking both objectives into account, we determined the optimal weight of a convex combination of VWAP and a deterministic form of VWAP. This weight bears similarity to a simple linear regression coefficient and is a linear function in the weight of the manipulation concern, with the slope of the function depending on the amount of randomness in market prices and volumes.

In a market with an auction that has an imbalance announcement with permanent price impact, the optimal benchmark in terms of manipulation concern is no longer VWAP, but of a similar form that takes into account the permanent price impact of the imbalance announcement. It also appears that in a simulation with the same parameters of markets with and without an auction, the optimal benchmark weight grows much more slowly in the weight of the manipulation concern, implying that manipulation is more difficult with the presence of an auction.

This work contributes to a growing body of literature on the benefits of VWAP-like benchmarks. In markets with no auctions under the model employed, a VWAP benchmark is optimal for combating manipulation, a well-documented problem across markets. Further work into comparing a market with and without an auction would be useful in decision making around the administration of derivatives markets like the Montreal Exchange, which currently does not have an auction.

It may seem that our model of a permanent price impact from the imbalance announcement produces unnatural results, specifically, a negative weight being placed on the auction price. Further investigation into the validity of this model may be necessary to determine if this is an idiosyncrasy of the model employed or indeed present in reality. If this model is indeed reflective of reality, then the case may be that in order to combat manipulation in a market with an auction we are required to place a negative weight on the auction price.

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