Operator Amenability of Ultrapowers of the Fourier Algebra

by

Kyle Schlitt

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Abstract

It has been shown by Matthew Daws that the group algebra $\ell^1(G)$ of a discrete group G is never ultra-amenable. We explore the weak analogue to this statement and demonstrate that if any commutative $L^1(G)$ is ultra-weakly amenable, then G must necessarily be discrete. By showing that ultrapowers of complete maximal operator spaces are themselves maximal, we are able to demonstrate that the assumption of ultra-operator amenability of the Fourier algebra $A(G)$ forces G to be discrete. By considering a wide class of discrete groups, we find sufficient evidence to make reasonable the conjecture that such a property may well force G to be finite. We conclude with consideration of another weak analogue, showing that ultra-weak operator amenability of $A(G)$ already forces G to be discrete.

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Chapter 1

Introduction

In 1972, Barry Johnson established a groundbreaking result in [40], showing that amenability of a locally compact group can be entirely characterized by a cohomological criterion in terms of its group algebra. Since this framework exists for every Banach algebra, a suitable definition was provided for amenable Banach algebras which is compatible with the functor $L^1(\cdot)$. This sparked questions of which Banach algebras had this property; and a fresh field of research was born. In 1976, Alain Connes proved that for $\mathcal{C}^*$-algebras, amenability implies nuclearity [13]. The converse was established by Uffe Haagerup in 1983 [33]. At the end of his memoir, Johnson posed several questions which motivated further research. In particular, he asked whether $\mathcal{B}(E)$ is ever amenable for any infinite dimensional Banach space E.

For many years the answer was expected to be *no*. But even for the classical ℓ^p spaces, the question took some time to find an answer. Using the equivalence of amenability and nuclearity for $\mathcal{C}^*$-algebras, it was shown by Simon Wasserman in [66] that $\mathcal{B}(\mathcal{H})$ fails to be amenable for any infinite dimensional $\mathcal{H}$. In particular, this yielded non-amenability of $\mathcal{B}(\ell^2)$. In 2004, the late Charles Read made use of random hypergraphs to prove in [54] that amenability of $\mathcal{B}(\ell^1)$ would imply amenability of $\mathcal{F}_2$, the free group on two generators, a claim known to be false. Gilles Pisier took advantage of existing theory of expanding graphs in [51] to simplify the argument, which was subsequently adapted by Narutaka Ozawa in [48] to work simultaneously for all of $\mathcal{B}(\ell^1), \mathcal{B}(\ell^2)$, and $\mathcal{B}(\ell^\infty)$. Perhaps surprisingly, Johnson's question found an affirmative answer In 2009. In [2], Spiros Argyros and Richard Haydon constructed an infinite dimensional $\mathcal{L}^p$-space E with the property that $\mathcal{B}(E)$ is simply a unitarization of $\mathcal{K}(E)$. This was enough, as Niels Grønbæk proved in [31] that the algebra of compact operators on an $\mathcal{L}^p$-space is amenable.

In 2010, Volker Runde proved that $\ell^\infty(\mathcal{K}(E))$ fails to be amenable for all infinite dimensional $\mathcal{L}^p$-spaces E [61]. This resolved Johnson's question for the remaining ℓ^p spaces; for it was previously shown by the same author along with Matthew Daws in [60] that if any $\mathcal{B}(\ell^p)$ were amenable, this would force $\ell^\infty(\mathbb{I}, \mathcal{K}(\ell^p))$ to be as well, for every index set $\mathbb{I}$. Since ultrapowers are formed by taking quotients of such spaces, herein lies the connection with ultra-amenability, a definition first supplied by Daws in [15]. The ultraproduct construction, first introduced by Łoś in [46], keeps the category of Banach algebras closed. This makes reasonable the following question: *for a given Banach algebra, are all of its ultrapowers amenable?* Banach algebras for which the answer to this question is *yes* are said to be ultra-amenable. Since then, the issue has proved interesting in its own right. Two notable results are:

- The group algebra of an infinite discrete group is never ultra-amenable.
- A $\mathcal{C}^*$-algebra is ultra-amenable if and only if it is subhomogeneous.

An important consequence of these facts is that ultra-amenability is strictly stronger than amenability, yet weaker than contractability, the latter of which is a property yet to be witnessed by any infinite dimensional Banach algebra. Ultra-amenability was originally introduced as a property that the space $\mathcal{K}(\ell^p)$ ought *not* to have, which indeed turned out to be the case; but enough spaces have it to make it worthy of further study.

It was our original intent to generalize the non-amenability of group algebras to all infinite groups, a problem which naively seemed approachable. When such efforts turned out to be overly ambitious, it was suggested by the authors PhD supervisor, Volker Runde, to add operator space overtones to the definition, and instead consider the question of when ultrapowers of the Fourier algebra are operator-amenable. We pursue the theme of [15] by similarly introducing the concept of ultra-operator amenability. In Chapter 4, we show that the Fourier algebra of a non-discrete locally compact group can never have this property. The proof uses results in [15] to reduce the problem to a structure theory argument given in Chapter 3, which was an approach suggested by Brian Forrest.

In a version of the principle of local reflexivity given by Stefan Heinrich in [36], the second dual of every Banach space may be expressed as a quotient of one of its ultrapowers. In [17] it is shown that for a Banach algebra A, the Arens products on the second dual A'' drop to a well-defined product on $\mathrm{AP}(A')'$, the dual of the space of almost periodic functionals on A. Daws adapts Heinrich's technique in [16] and remarks that it can be used to identify $\mathrm{AP}(A')'$ with the Banach algebra quotient of a carefully chosen ultrapower of A. In Chapter 5, we work out in full the technical details behind this fact. In the case of a Fourier algebra, such an identification allows an important point derivation lift from the Measure algebra of an abelian subgroup, should one happen to be available. This is the key to an argument given in Chapter 6.

Two other variants of the notion of amenability are also of interest: weak amenability and operator weak amenability. Weak amenability was first introduced by George Bade, Phillip Curtis, and Garth Dales in [4]. It was shown by Johnson in [41] that every group algebra is weakly amenable. As with amenability, there is a nice analogue for the Fourier algebra which takes the structure of operator spaces into account. Nico Spronk proved in [63] that every Fourier algebra is operator weakly amenable. We introduce the terms ultra-weak amenability, and ultra-weak operator amenability, and consider in Chapter 6 the natural question of whether group algebras and Fourier algebras may have these respective properties. We show that for a non-discrete locally compact group G, the Fourier algebra can never be ultra-operator weakly amenable, adapting an argument in [28] given by Forrest which again reduces the problem to the structure theory argument. It is also shown that, at least for abelian G, for no infinite group may we ever find an ultra-weakly amenable group algebra, by showing that such a group must embed injectively into a discrete compactification, and is hence finite.

There are canonical functors $\min(\cdot)$ and $\max(\cdot)$ from the category of Banach algebras to that of completely contractive Banach algebras. As observed in [36], the class of commutative $\mathcal{C}^*$-algebras is closed under the ultrapower construction, it follows that $\min(\cdot)$ commutes with ultrapowers. In Chapter 2, we adapt the proof of a standard theorem in [22] to show that this also holds for $\max(\cdot)$.

1.1 Basics and Notation

Let E be a normed linear space. B_E shall denote the unit ball of E. For normed linear spaces E and F, an **operator** from E to F is a continuous linear map $T : E \to F$. Denote by $\mathcal{B}(E, F)$ the space of operators from E to F. We say $T \in \mathcal{B}(E, F)$ is **contractive** if $\|Tx\| \leq \|x\|$ for all $x \in E$, and an **isometry** if $\|Tx\| = \|x\|$ for all $x \in E$.

Let $E' = \mathcal{B}(E, \mathbb{C})$, which will be referred to as the **dual** of E. Any normed space may be endowed with the **weak topology** which is the locally convex topology induced by the family of seminorms $\{p_\varphi : \varphi \in E'\}$ defined by $p_\varphi(x) = |\varphi(x)|$, for all $x \in E$. Elements of E' will be referred to as **functionals** on E. There are two topologies on E' that are frequently relevant:

- the **norm topology**, where the norm on E' is defined by $\|\varphi\| = \sup\{|\varphi(x)| : x \in B_E\}$ for $\varphi \in E$, and
- the **weak-$*$ topology**, which is the locally convex topology induced by the family of seminorms $\{p_x : x \in E\}$, defined by $p_x(\varphi) = |\varphi(x)|$ for all $\varphi \in E'$.

We often abbreviate the weak and weak-$*$ topologies by w and w^*, respectively. An operator $T \in \mathcal{B}(E, F)$ is

- **compact** if $\overline{T(B_E)}$ is compact, and is
- **weakly compact** if $\overline{T(B_E)}^w$ is weakly compact.

If $\varphi \in E', x \in E$, then we often use $\langle \varphi, x \rangle$ to represent the value of φ at x, rather than simply writing $\varphi(x)$, which becomes impractical in many situations. For $T \in \mathcal{B}(E, F)$, $T' \in \mathcal{B}(F', E')$ is the operator defined by

$$\langle T'\varphi, x \rangle = \langle \varphi, Tx \rangle \text{ for all } \varphi \in F', x \in E.$$

Let $\kappa_E : E \to E''$ denote the canonical embedding defined by

$$\langle \kappa_E(x), \varphi \rangle = \langle \varphi, x \rangle \text{ for all } x \in E, \varphi \in E'.$$

We say that E is **reflexive** if κ_E is surjective.

For a Hilbert space $\mathcal{H}$, and an operator $T \in \mathcal{B}(\mathcal{H})$, the **adjoint** T^* of T is the unique $T^* \in \mathcal{B}(\mathcal{H})$ such that

$$\langle T\zeta, \eta \rangle = \langle \zeta, T^*\eta \rangle \text{ for all } \zeta, \eta \in \mathcal{H}.$$

We say that T is **unitary** if $T^* = T^{-1}$. Let $\mathcal{U}(\mathcal{H})$ denote the collection of unitary operators in $\mathcal{B}(\mathcal{H})$. By $\mathcal{H}^\infty$ we denote the Hilbert space of all $(\zeta_n)_{n=1}^\infty \in \prod_{n=1}^\infty \mathcal{H}$ such that $\sum_{n=1}^\infty \|\zeta_n\|^2 < \infty$, which is endowed with the inner product defined by

$$\langle (\zeta_n)_{n=1}^\infty, (\eta_n)_{n=1}^\infty \rangle = \lim_{n \to \mathcal{U}} \langle \zeta_n, \eta_n \rangle, \text{ for all } (\zeta_n)_{n=1}^\infty, (\eta_n)_{n=1}^\infty \in \mathcal{H}^\infty.$$

For $p \geq 1$, ℓ_n^p shall denote the space $\mathbb{C}^n$ normed by $\|x\| = \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}}$; and ℓ_n^∞ is the same space equipped with the norm $\|x\| = \max_{j=1}^n |x_j|$. Denote by ℓ^p the Banach space of all sequences $x = (x_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty |x_n|^p < \infty,$$

equipped with the norm $\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}}$. ℓ^∞ is the space of all bounded sequences, equipped with the supremum norm.

For Banach spaces E, F, and G, let $\mathfrak{B}(E \times F, G)$ denote the Banach space of all **bounded bilinear maps**, i.e. maps $T : E \times F \to G$ such that

- T is **bilinear,** i.e. both $u \mapsto T(u, v)$ and $v \mapsto T(u, v)$ are linear, and
- $\|T\| := \sup\{\|T(x, y)\| : x \in B_E, y \in B_F\} < \infty$.

A **Banach algebra** is a Banach space equipped with an associative submultiplicative product. For Banach algebras A and B, $T \in \mathcal{B}(A, B)$ is a **homomorphism** if $T(ab) = T(a)T(b)$ for all $a, b \in A$.

Example 1. Let Ω be a locally compact Hausdorff space. $\mathcal{C}(\Omega)$ shall denote the space of all continuous bounded complex valued functions on Ω, which is a Banach algebra under pointwise operations when equipped with the supremum norm:

$$\|f\|_\infty = \sup_{t \in \Omega} |f(t)|$$

$\mathcal{C}_0(\Omega)$ shall denote the subspace of all $f \in \mathcal{C}(\Omega)$ which **vanish at infinity**. That is, for all $\epsilon > 0$, there exists a compact $K \subset \Omega$ such that for all $t \notin K, |f(t)| < \epsilon$.

Example 2. Let E be a Banach space. $\mathcal{B}(E)$ is an Banach algebra with scalar product and addition defined pointwise, and product defined by function composition. The norm on $\mathcal{B}(E)$ is defined by

$$\|T\| = \sup_{x \in B_E} \|Tx\| \text{ for all } T \in \mathcal{B}(E).$$

It is immediate that point evaluation is a continous linear functional on $C(\Omega)$.

1.2 Tensor Products

We shall require properties of tensor products, which come equipped with bilinear operations that behave in a manner not unlike multiplication. First we recall the definition of a tensor product in the algebraic setting. The **algebraic tensor product** of linear spaces U and V, denoted $U \otimes V$, is the unique linear space which has the following universal property: there exists a bilinear map

$$U \times V \to U \otimes V, \text{ denoted by } (u, v) \mapsto u \otimes v,$$

with the property that for all bilinear $T : U \times V \to W$, there exists a unique linear $\widetilde{T} : U \otimes V \to W$ such that $\widetilde{T}(u \otimes v) = T(u, v)$ for all $u \in U, v \in V$. We call $\widetilde{T}$ the **linearization** of T.

A natural example of such a structure is given by the space of matrices over a linear space. For $m, n \geq 1$, denote by $\mathbb{M}_{m,n}$ the space of all m by n matrices over $\mathbb{C}$; and let $\mathbb{M}_n = \mathbb{M}_{n,n}$. Similarly, for a linear space V, $M_{m,n}[V]$ is the space of m by n matrices over V, with $M_n[V] = M_{n,n}[V]$. For $n \geq 1$, let $e^{[m,n]}_{i,j} \in \mathbb{M}_{m,n}$ denote the m by n matrix in which the (i, j)-th entry is 1 and all other entries are 0; and let $e^{[n]}_{i,j} = e^{[n,n]}_{i,j}$.

Example 3. Let V be a linear space. We may algebraically identify $M_n[V]$ with $\mathbb{M}_n \otimes V$ and $V \otimes \mathbb{M}_n$. Indeed if we pass the bilinear maps

$$\mathbb{M}_n \times V \to M_n[V], \text{ defined by } ([\lambda_{i,j}], x) \mapsto [\lambda_{i,j}x]$$

and

$$V \times \mathbb{M}_n \to M_n[V], \text{ defined by } (x, [\lambda_{i,j}]) \mapsto [\lambda_{i,j}x]$$

through the universal property of $\otimes$, we arrive at isomorphisms

$$V \otimes \mathbb{M}_n \to M_n[V], \text{ defined by } \sum_{i=1}^{n} \sum_{j=1}^{n} v_{i,j} \otimes e^{[n]}_{i,j} \mapsto [v_{i,j}]$$

and

$$\mathbb{M}_n \otimes V \to M_n[V], \text{ defined by } \sum_{i=1}^{n} \sum_{j=1}^{n} e^{[n]}_{i,j} \otimes v_{i,j} \mapsto [v_{i,j}].$$

Using these identifications, there is also a nice interaction between matrix spaces of tensor products and tensor products of matrix spaces.

Example 4. Let V, W be linear spaces. For $p, q \geq 1$, we have

$$\begin{aligned} M_{pq}[V \otimes W] &\cong \mathbb{M}_{pq} \otimes V \otimes W \\ &\cong \mathbb{M}_p \otimes \mathbb{M}_q \otimes V \otimes W \\ &\cong \mathbb{M}_p \otimes V \otimes \mathbb{M}_q \otimes W \\ &\cong M_p[V] \otimes M_q[W], \end{aligned}$$

thus providing an identification of $M_p[V] \otimes M_q[W]$ with $M_{pq}[V \otimes W]$.

When considering Banach spaces, we adjust the definition. Let E and F be Banach spaces. Define a norm on the algebraic tensor product $E \otimes F$ given by

$$\|u\|_\gamma = \inf \left\{ \sum_{j=1}^{n} \|u_j\| \cdot \|v_j\| : u_j \in U, v_j \in V, u = \sum_{j=1}^{n} u_j \otimes v_j \right\}, \text{ for all } u \in E \otimes F.$$

The **projective tensor product** $E \otimes^\gamma F$ is defined to be the completion of $E \otimes F$ with respect to $\|\cdot\|_\gamma$. As with the algebraic tensor product, $E \otimes^\gamma F$ may also be characterized by a universal property: $E \otimes^\gamma F$ is the unique Banach space such that for all Banach spaces G with $T \in \mathfrak{B}(E \times F, G)$, there exists a unique $\widetilde{T} \in \mathcal{B}(E \otimes^\gamma F, G)$ such that $\|\widetilde{T}\| = \|T\|$ and $\widetilde{T}(x \otimes y) = T(x, y)$ for all $x \in E, y \in F$. In particular, the Banach spaces $\mathfrak{B}(E \times F, G)$ and $\mathcal{B}(E \otimes^\gamma F, G)$ are isometrically isomorphic.

For Banach spaces E and F, the spaces $(E \otimes^\gamma F)'$ and $\mathcal{B}(E, F')$ are isometrically isomorphic. Indeed we may identify $T \in \mathcal{B}(E, F')$ with the functional on $E \otimes^\gamma F)$ defined by

$$\langle T, x \otimes y \rangle = \langle Tx, y \rangle \text{ for all } x \in E, y \in F, T \in \mathcal{B}(E, F').$$

There is also an isometric embedding $E \otimes^\gamma F' \hookrightarrow \mathcal{B}(E, F)'$, achieved by identifying $x \otimes \varphi \in E \otimes^\gamma F'$ with the functional on $\mathcal{B}(E, F)$ defined by

$$\langle x \otimes \varphi, T \rangle = \langle \varphi, Tx \rangle \text{ for all } T \in \mathcal{B}(E, F).$$

It is easy to see that if E is finite dimensional, this embedding is surjective. For a beautiful and thorough treatment of tensor products of Banach spaces, along with proofs of the facts collected in this section, see [57].

1.3 Ultrapowers

Ultraproducts were first formally introduced by Jerzy Łoś, who proved the Fundamental Theorem of Ultraproducts (also called Łoś's Theorem) in 1955 [46], which was the key ingredient for their construction. Originally stemming from Model theory, these ideas have since found their reaches in many branches of mathematics, with many categories of objects having a relevant version. Construction of the Hyperreals are a notable application of this, the first mention of which was made by Edwin Hewitt in [38]. In 1972, Didier Dacunha-Castelle and Jean-Louis Krivine provided a construction in [9] appropriate for Banach spaces. In 1980, Stefan Heinrich used ultrapowers to give a concise statement of the Principle of local reflexivity. This can be found in [36], which also happens to provide an excellent reference for the following.

Definition 5. Let $\mathbb{I}$ denote an arbitrary non-empty index set. An **ultrafilter** on $\mathbb{I}$ is a *maximal* collection $\mathcal{U}$ of subsets of $\mathbb{I}$ satisfying

- $\emptyset \notin \mathcal{U}$;
- If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$; and
- If $A \in \mathcal{U}$, and $A \subset B \subset \mathbb{I}$, then $B \in \mathcal{U}$.

Let X be a topological space, and suppose that for each $i \in \mathbb{I}$, we have $x_i \in X$. If $x \in X$ is such that for all neighborhoods U of x, $\{i \in \mathbb{I} : x_i \in U\} \in \mathcal{U}$. We say that x is **the limit of x_i along $\mathcal{U}$**, and write $\lim\limits_{i \to \mathcal{U}} x_i = x$. If X is Hausdorff, then x is unique. It is an important fact that if X is compact, then such an x always exists.

Let E be a Banach space and $\mathcal{U}$ be an ultrafilter on an index set $\mathbb{I}$. Let

$$\ell^\infty(\mathbb{I}, E) := \left\{ (x_i)_{i\in\mathbb{I}} \in \prod_{i\in\mathbb{I}} E : \sup_{i\in\mathbb{I}} \|x_i\| < \infty \right\} \text{ and } \mathcal{N} := \left\{ (x_i)_{i\in\mathbb{I}} \in \ell^\infty(\mathbb{I}, E) : \lim_{i\to\mathcal{U}} \|x_i\| = 0 \right\}.$$

We define the **ultrapower of E** as the quotient space

$$(E)_\mathcal{U} := \ell^\infty(\mathbb{I}, E)/\mathcal{N},$$

endowed with the quotient norm, where $\ell^\infty(\mathbb{I}, E)$ is normed by $\|(x_i)_{i\in\mathbb{I}}\|_\infty = \sup_{i\in\mathbb{I}} \|x_i\|$. For $x = (x_i)_{i\in\mathbb{I}} \in \ell^\infty(\mathbb{I}, E)$, we denote the element $x + \mathcal{N}$ by $(x_i)_\mathcal{U}$. The quotient norm on $(E)_\mathcal{U}$ can be calculated conveniently using the fact that $\|(x_i)_\mathcal{U}\| = \lim\limits_{i\to\mathcal{U}} \|x_i\|$.

An ultrafilter $\mathcal{U}$ on $\mathbb{I}$ is said to be **countably incomplete** if it contains a sequence

$$U_1 \supset U_2 \supset U_3 \supset \dots \text{ such that } \bigcap_{n=1}^{\infty} U_n = \emptyset.$$

Such ultrafilters are attractive, as they provide a technique which may be used to embed sequential convergence into convergence along $\mathcal{U}$, a technique which appears in [36, Proposition 6.3], for example. We extract the idea into the following proposition, which follows immediately from the definition of convergence along an ultrafilter.

Proposition 6. *Suppose $\mathcal{U}$ is countably incomplete on $\mathbb{I}$. Let X be a topological space, with $x_n \to x$ in X. For each $i \in \mathbb{I}$, suppose that $x_i = x_n$ whenever $i \in U_n \backslash U_{n+1}$ for some $n \geq 1$. Then*

$$\lim_{i \to \mathcal{U}} x_i = x.$$

There is a convenient consequence to this fact. If X is a topological space and $\mathcal{U}$ on $\mathbb{I}$ is countably incomplete, we may reduce to the case when $\mathbb{I} = \mathbb{N}$ when arguing existence of an $x \in X$ and $[x_i]_{\mathcal{U}} \in [X]_{\mathcal{U}}$ such that $\lim_{i \to \mathcal{U}} x_i = x$. This is precisely what is done in the proof of [15, Theorem 5.11], which we include as Theorem 46.

Assuming that an ultrafilter $\mathcal{U}$ on $\mathbb{I}$ is countably incomplete is hardly a loss of generality. Indeed existence of ultrafilters without this property would imply the existence of a so-called *measure cardinal*, an object whose existence does not follow from ZFC. See the remarks following Theorem 4.2.14 in [11] for more details.

There is a canonical isometric embedding $E \hookrightarrow [E]_{\mathcal{U}}$ which identifies each $x \in E$ with the element $[x_i]_{\mathcal{U}} \in [E]_{\mathcal{U}}$, where $x_i = x$ for every $i \in \mathbb{I}$. For Banach spaces E and F, with $T \in \mathcal{B}(E, F)$, the operator $[T]_{\mathcal{U}} \in \mathcal{B}\left([E]_{\mathcal{U}}, [F]_{\mathcal{U}}\right)$ is the operator $[x_i]_{\mathcal{U}} \mapsto [Tx_i]_{\mathcal{U}}$. This provides an isometry $\mathcal{B}(E, F) \hookrightarrow \mathcal{B}\left([E]_{\mathcal{U}}, [F]_{\mathcal{U}}\right)$ given by $T \mapsto [T]_{\mathcal{U}}$. This embedding preserves both quotient maps and isometries.

1.4 The Gelfand Transform and $\mathcal{C}^*$-algebras

Let A be a Banach algebra. A **character** of A is a non-zero multiplicative linear functional on A. Φ_A shall denote the set of all characters of A and is called the **character space of** A. As a subset of A', Φ_A inherits the w^*-topology, under which it becomes a locally compact Hausdorff space [27, Theorem 1.30]. One of the beautiful features of a Banach algebra is its Gelfand transform, which sends each of its elements to a confinuous function on its character space. We now outline some standard facts below, which can be found in many places; two notable sources are [27] and [42].

Recall that an **involutive Banach algebra** is a Banach algebra equipped with an **involution**: a map $A \to A$, denoted by $a \mapsto a^*$, such that the for every $a, b \in A$, $\lambda \in \mathbb{C}$,

- $a^{**} = a$,
- $(a + b)^* = a^* + b^*$,
- $(\lambda a)^* = \overline{\lambda} a^*$, and
- $(ab)^* = b^* a^*$.

If A and B are involutive Banach algebras, a $*$-homomorphism $T : A \to B$ is a homomorphism such that $T(a^*) = T(a)^*$ for all $a \in A$.

For a general involutive Banach algebra, there should be no expectation of a well behaved interaction between its norm and involution. A $\mathcal{C}^*$**-algebra** is a Banach algebra A, equipped with an involution, satisfying the $\mathcal{C}^*$*-identity*:

$$\|a\|^2 = \|a^*a\| \text{ for all } a \in A,$$

which is an analogue of the formula for complex modulus: $|\lambda|^2 = \lambda \cdot \overline{\lambda}$. By attaching this additional restriction on A, it immediately follows that an involution is an isometry. It is hence an easy exercise to check that $*$-homomorphism of $\mathcal{C}^*$-algebras is contractive, and is in fact isometric if it is injective.

Definition 7. The **Gelfand Transform** of a Banach algebra A is the map $\Gamma : A \to C(\Phi_A)$ defined by

$$\Gamma(a) : \varphi \mapsto \phi(a) \text{ for all } a \in A, \varphi \in \Phi_A.$$

For commutative Banach algebras, this map is a contractive algebra homomorphism. For commutative $\mathcal{C}^*$-algebras it is an isometric $*$-isomorphism. For any commutative $\mathcal{C}^*$-algebra A, we may therefore identify A with the space of bounded continuous functions on Φ_A, which is compact exactly when A is unital.

1.5 Operator Spaces and Completely Contractive Banach Algebras

It is a standard fact that every normed space E can be represented as a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space. For example, let $\mathcal{H}$ be the space of square-summable functions on $B_{E'}$. Treating each $x \in E$ as a bounded function on $B_{E'}$, represent x as the operator which acts on $\mathcal{H}$ via pointwise multiplication. This generates not only a norm on E, but also on $M_n[E]$ by identifying $M_n[\mathcal{B}(\mathcal{H})]$ with $\mathcal{B}(\mathcal{H}^n)$, for each $n \geq 1$. There is no reason to expect this representation to be unique. *Operator spaces* are normed spaces paired with such a *fixed* representation. That is, an **operator space** is a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Operator space are equipped with properties which entirely characterize them, thus providing an alternative abstract definition.

Definition 8. Let V be a linear space. A **matricial norm** $(\|\cdot\|)_{n=1}^{\infty}$ is a sequence of norms $\|\cdot\|_n$ on $M_n[V]$ satisfying the *Ruan axioms*:

(R1) $\|\alpha x \beta\|_n \leq |\alpha| \cdot \|x\|_n \cdot |\beta|$ for all $n \geq 1, x \in M_n[V], \alpha, \beta \in \mathbb{M}_n$ and

(R2) $\left\| \left[\begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array} \right] \right\|_{m+n} = \max\{\|x\|_n, \|y\|_m\}$ for all $m, n \geq 1, x \in M_n[V], y \in M_m[V]$.

An **abstract operator space** is a linear space paired with a matricial norm. By the foregoing, it is immediate that every operator space is an abstract operator space. It is also the case that every abstract operator space can be realized as a operator space (see [22, Theorem 2.3.5]). For this reason we drop the distinction between the two terms, and use whichever definition is more convenient at the time. A thorough treatment of operator spaces, along with proofs of the facts mentioned in this section can be found in [22].

Let V and W be operator spaces, and $T \in \mathcal{B}(V, W)$. The **n-th amplification**

$$T^{(n)} : M_n[V] \to M_n[W]$$

is defined by

$$T^{(n)} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} Tx_{11} & Tx_{12} & \dots & Tx_{1n} \\ Tx_{21} & Tx_{22} & \dots & Tx_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Tx_{n1} & Tx_{n2} & \cdots & Tx_{nn} \end{bmatrix}.$$

Definition 9. Let V and W be operator spaces and $T \in B(V, W)$. We say

- T is **completely bounded** if $\sup_{n \geq 1} \|T^{(n)}\| < \infty$,
- T is a **complete isometry (or contraction)** if $T^{(n)}$ is an isometry (or contraction) for all $n \geq 1$, and
- T is a **complete quotient map** if $T^{(n)}$ is a quotient map for every $n \geq 1$.

If E is an operator space which is not complete, it's completion $\widetilde{E}$ is also an operator space by virtue of the identification $M_n\left[\widetilde{E}\right] \cong \widetilde{M_n[E]}$.

Denote by $\mathcal{CB}(E, F)$ the space of all $T \in \mathcal{B}(E, F)$ which are completely bounded. The norm $\|T\|_{cb} := \sup_{n \geq 1} \|T^{(n)}\|$ turns $\mathcal{CB}(E, F)$ into a normed linear space which is complete if and only if F is. By [22, Corollary 2.2.3], every functional on an operator space is automatically completely bounded. It follows that $\mathcal{CB}(E, \mathbb{C}) = \mathcal{B}(E, \mathbb{C})$. In otherwords, the natural dual object of an operator coincides with its dual as a normed space, and we needn't worry about adding operator space overtones to functionals.

Examples.

- The **complex scalar operator space** is the space $\mathbb{C}$ equipped with the matricial norm determined by identifying elements of $M_n[\mathbb{C}] = \mathbb{M}_n$ with operators in $\mathcal{B}(\ell_n^2)$.
- The **product operator space structure** on the direct product $\prod_{i \in \mathbb{I}} V_i$ of operator spaces $(V_i)_{i \in \mathbb{I}}$ is the one whose matricial norm is generated by the identification

$$M_n\left[\prod_{i \in \mathbb{I}} V_i\right] \cong \prod_{i \in \mathbb{I}} M_n[V_i].$$

- Let A be a $\mathcal{C}^*$-algebra. A inherits **canonical operator structure** from its *universal representation* (see Section 1.11). The matricial norm appearing in this fashion is unique in the sense that it is the only matricial norm such that $M_n[A]$ is a $\mathcal{C}^*$-algebra for every $n \geq 1$.

- Let V be an operator space, and let N be a linear subspace of V. The inclusion $M_n[N] \subset M_n[V]$ for all $n \geq 1$ induces the **subspace operator space structure** on N.

- Let V be an operator space. Then V' is also an operator space. The **dual operator space structure** is determined by the matricial norm obtained by identifiying $M_n[V']$ with $\mathcal{CB}(V, \mathbb{M}_n)$. This definition of duality ensures that the canonical embedding $\kappa_V : V \hookrightarrow V''$ is a complete isometry (see [22, Section 3.2] for details).

- Let V be an operator space which is the dual of the normed space E. Then E inherits the **predual operator space structure** as a subspace of $E'' = V'$. In particular for any set $\mathcal{S}$, $\ell^1(\mathcal{S})$ is an operator space when consideed as the predual of the $\mathcal{C}^*$-algebra $\ell^\infty(\mathcal{S})$.

- Let V be an operator space. Then $[V]_\mathcal{U}$ receives the **ultrapower operator space structure** by considering it as a quotient of $\prod_{i \in \mathbb{I}} V$. It's matricial norm may also be obtained by algebraically identifying $M_n[[V]_\mathcal{U}]$ with $[M_n[V]]_\mathcal{U}$ for each $n \geq 1$.

In parallel with the Banach space setting, the map $[\mathcal{CB}(V,W)]_\mathcal{U} \to \mathcal{CB}([V]_\mathcal{U}, [W]_\mathcal{U})$ defined by $T \mapsto [T]_\mathcal{U}$ preserves complete isometries and complete quotient maps. In contrast, however, it need only be a complete contraction. Recall that for a normed space E, and a closed subspace $F \subset E$, the adjoint $E' \to F'$ of the isometric inclusion map $F \hookrightarrow E$ is a quotient map. In particular, we have

$$F' \cong E'/F^\perp \text{ and } (E/F)' \cong F^\perp,$$

where

$$F^\perp := \{\mu \in E' : \langle \mu, x \rangle = 0 \text{ for every } x \in F\}.$$

There is a convenient analogue for operator spaces.

Proposition 10. *Let V be an operator space and let $N \subset V$ be a closed subspace. Then*

$$N' \cong V'/N^\perp \text{ and } (V/N)' \cong N^\perp.$$

Proof. This is [22, Proposition 4.2.1]. □

Proposition 11. *Let A, B be $\mathcal{C}^*$-algebras equipped with their canonical operator space structures, and $T : A \to B$ an $*$-homorphism. T is a complete contraction. Moreover if T is injective, then T is a complete isometry.*

Proof. See the remarks following in [22, Proposition 2.2.6] for details. □

It is an immediate consequence of Proposition 11 that if A is a $\mathcal{C}^*$-algebra with a closed two-sided ideal I, the quotient operator space structure on the $\mathcal{C}^*$-algebra A/I coincides with its canonical one.

Given any normed space E, there exists matricial norms $(\|\cdot\|_{\min,n})_{n=1}^{\infty}$ and $(\|\cdot\|_{\max,n})_{n=1}^{\infty}$ with the property that if $(\|\cdot\|_n)_{n=1}^{\infty})$ is any matricial norm on E, then

$$(\|x\|_{\min,n})_{n=1}^{\infty} \leq (\|x\|_n)_{n=1}^{\infty} \leq (\|x\|_{\max,n})_{n=1}^{\infty} \text{ for all } x \in M_n[E].$$

Denote by $\min(E)$ and $\max(E)$ the operator space E equipped with these respective structures. They may be defined concretely as in [22, 3.3], but may also be characterized by the following universal properties.

Definition 12. Let E be a normed space space. $\min(E)$ and $\max(E)$ are the unique operator spaces such that

$$\mathcal{CB}(V, \min(E)) = \mathcal{B}(V, E) \text{ for every operator space } V,$$

and

$$\mathcal{CB}(\max(E), W) = \mathcal{B}(E, W) \text{ for every operator space } W.$$

E is **minimal** if it is completely isometric to $\min(E)$ and **maximal** if it is completely isometric to $\max(E)$. If $q : E \to F$ is a quotient map of normed spaces, it can be directly checked using the criterion above that $q : \max(E) \to \max(F)$ is a complete quotient map. These matricial norms enjoy attractive duality relationships. We have $\max(E)' = \min(E')$ and $\min(E)' = \max(E')$.

Proposition 13. *An operator space is minimal if and only if it is completely isometric to a subspace of a commutative $\mathcal{C}^*$-algebra.*

Proof. This is [22, Proposition 3.3.1]. □

Proposition 14. *A complete operator space is maximal if and only if it is completely isometric to a quotient of $\ell^1(\mathcal{S})$ for some set $\mathcal{S}$.*

Proof. This is [22, 3.3.2]. □

It follows from these results that for any measure space $(\mathcal{S}, \Sigma, \mu)$, $L^1(S, \Sigma, \mu)$ and $L^{\infty}(S, \Sigma, \mu)$ are maximal and minimal, respectively. Since ultrapowers of commutative $\mathcal{C}^*$-algebras are again commutative $\mathcal{C}^*$-algebras ([36, Proposition 3.1 (ii)] gives a proof, but it is easy to check), it is immediate that $[\min(V)]_{\mathcal{U}}$ is minimal for every operator space V. In Chapter 2 we show that $[\max(E)]_{\mathcal{U}}$ is maximal for every Banach space E. The following is part of Corollary 4.1.9 in [22].

Proposition 15. *Let E, F be complete operator spaces. A linear operator $T : E \to F$ is a complete quotient map if and only if T^* is a complete isometry.*

As with Banach spaces, the definition of tensor product may also be adapted for operator spaces. Let E and F be operator spaces, and define the following matricial norm on $E \otimes F$. For $n \geq 1$ and $u \in M_n [E \otimes F]$, define

$$\|u\|_{n,\wedge} := \inf \{|\alpha| \cdot \|v\| \cdot \|w\| \cdot |\beta|\}$$

where the infimum is taken over all $\alpha \in \mathbb{M}_{n,pq}, v \in M_p [E], w \in M_q [F], \beta \in \mathbb{M}_{pq,n}$ such that $u = \alpha(v \otimes w)\beta$. Note that this calculation uses the algebraic identification outlined in Example 4. The completion of $E \otimes F$ with respect to this norm is called the **operator space projective tensor product** of E and F, and is denoted $E \hat{\otimes} F$. While we do not make use of it here, the operator space projective tensor product may also be characterized by a universal property which linearizes so-called *jointly completely bounded bilinear maps* (see [6, Chapter 5] for details). Suppose A is Banach algebra which also has the structure of an operator space. We may pass the bounded bilinear product map

$$A \times A \to A$$

through the universal property of $\otimes$, obtaining an linear map

$$A \otimes A \to A, a \otimes b \mapsto ab.$$

If this map induces a completely contractive operator

$$\Delta_A : A \hat{\otimes} A \to A,$$

we say that A is **completely contractive**.

We now check that if A is a completely contractive Banach algebra, then so is $[A]_{\mathcal{U}}$, for any ultrafilter $\mathcal{U}$ on an index set $\mathbb{I}$. First, we require a particularly helpful operator. Consider the bounded bilinear map $[A]_{\mathcal{U}} \times [A]_{\mathcal{U}} \to [A \hat{\otimes} A]_{\mathcal{U}}$ given by

$$([a_i]_{\mathcal{U}}, [b_i]_{\mathcal{U}}) \mapsto [a_i \otimes b_i]_{\mathcal{U}} \text{ for all } [a_i]_{\mathcal{U}}, [b_i]_{\mathcal{U}} \in [A]_{\mathcal{U}}.$$

As before, we may use the universal property of $\otimes^\gamma$ to obtain a contractive operator

$$\psi_0 : [A]_{\mathcal{U}} \otimes [A]_{\mathcal{U}} \to [A \hat{\otimes} A]_{\mathcal{U}}$$

such that

$$\psi_0([a_i]_{\mathcal{U}} \otimes [b_i]_{\mathcal{U}}) = [a_i \otimes b_i]_{\mathcal{U}} \text{ for all } [a_i]_{\mathcal{U}}, [b_i]_{\mathcal{U}} \in [A]_{\mathcal{U}}.$$

It is immediate that ψ_0 is a contraction. If A comes with an operator space structure, we can say more.

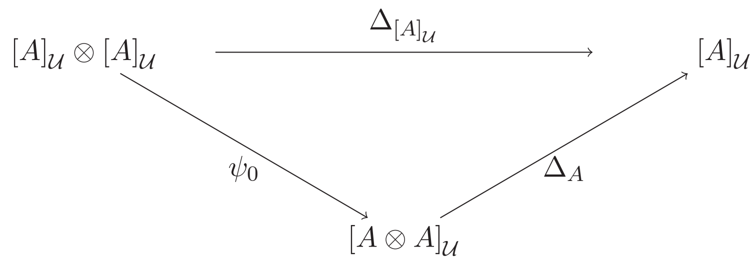
Proposition 16. *Let A be a completely contractive Banach algebra. $\psi_0 : [A]_{\mathcal{U}} \otimes [A]_{\mathcal{U}} \to [A \hat{\otimes} A]_{\mathcal{U}}$ is a complete contraction.*

Proof. For each $n \geq 1$, let $\alpha \in \mathbb{M}_{n,p}, u \in M_p\left[[A]_{\mathcal{U}}\right], v \in M_q\left[[A]_{\mathcal{U}}\right], \beta \in \mathbb{M}_{q,n}$ so that $\alpha(u \otimes v)\beta \in M_n\left[[A]_{\mathcal{U}} \otimes [A]_{\mathcal{U}}\right]$. Then identifying u and v respectively with $[u_i]_{\mathcal{U}} \in [M_p\,[A]]_{\mathcal{U}}$ and $[v_i]_{\mathcal{U}} \in [M_q\,[A]]_{\mathcal{U}}$ respectively, we may estimate

$$\begin{aligned}
\|\psi_0^{(n)}(\alpha(u \otimes v)\beta)\| &= \|\alpha \cdot \psi_0^{(pq)}(u \otimes v) \cdot \beta\| \\
&\leq |\alpha| \cdot |\beta| \cdot \|\psi_0^{(pq)}(u \otimes v)\| \\
&= |\alpha| \cdot |\beta| \cdot \|\,[u_i \otimes v_i]_{\mathcal{U}}\,\| \\
&= |\alpha| \cdot |\beta| \cdot \lim_{i \to \mathcal{U}} \|u_i \otimes v_i\| \\
&= |\alpha| \cdot |\beta| \cdot \lim_{i \to \mathcal{U}} \|u_i\|\|v_i\| \\
&= |\alpha| \cdot |\beta| \cdot \|u\| \cdot \|v\|
\end{aligned}$$

Taking infimums, the result follows. □

We may hence extend ψ_0 to a completely contractive operator $[A]_{\mathcal{U}} \hat{\otimes} [A]_{\mathcal{U}} \to [A \hat{\otimes} A]_{\mathcal{U}}$, which for ease of notation we shall also refer to as ψ_0. Finally, commutativity of the diagram



shows that $\Delta_{[A]_{\mathcal{U}}}$ is a composition of two completely contractive operators. Thus $[A]_{\mathcal{U}}$ is a completely contractive Banach algebra.

1.6 Abstract Harmonic Analysis

At the heart of abstract harmonic analysis lies the locally compact group, a source from which many of our structures draw their existence. By simultaneously possessing the virtues of an algebraic group alongside the right analytic framework, it was first established in 1933 by Alfred Haar in [35] that such an object comes equipped with a measure willing to interact beautifully with both its algebraic and topological structure. This remarkable fact is the key which unlocks the necessary integration theory required to generalize many tools from classical harmonic analysis, such as the Fourier and Fourier-Stieltjes transform. We now outline some important terms and facts, most of which can be found in [37] and [38]. [27] and [10] are two additional references, which belong in the toolbelt of anyone studying the landscape.

Definition 17. A **topological group** is a group G which is also a topological space on which the group operation

$$G \times G \to G,\ (s,t) \mapsto st$$

and inversion map

$$G \to G,\ s \mapsto s^{-1}$$

are continuous, where $G \times G$ is equipped with the product topology. A **locally compact group** G is a topological group whose underlying topology is both locally compact and Hausdorff.

Any group G which lacks such topological data may always be equipped with the discrete topology which turns it into a locally compact group. For any group G, locally compact or otherwise, G_d shall denote the same underlying group endowed with the discrete topology. In order to introduce a Haar measure on a locally compact group G, we first recall some basic facts from measure theory, which can be found in [10].

A collection Σ of subsets of a set $\mathcal{S}$ is a **σ-algebra on $\mathcal{S}$** if

- $\mathcal{S} \in \Sigma$,
- $A^c := \{x \in \mathcal{S} : x \notin A\} \in \Sigma$ whenever $A \in \Sigma$, and
- $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ whenever $A_n \in \Sigma$ for all $n \geq 1$.

A **measurable space** is a pair $(\mathcal{S}, \Sigma)$ where Σ is a σ-algebra on $\mathcal{S}$. A **measure** on a measurable space $(\mathcal{S}, \Sigma)$ is a function $\mu : \Sigma \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$ and
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in Σ.

A **measure space** is a triple $(\mathcal{S}, \Sigma, \mu)$ where $(\mathcal{S}, \Sigma)$ is a measurable space and μ is a measure on $(\mathcal{S}, \Sigma)$. If X is any locally compact Hausdorff space, denote by $\mathfrak{B}(X)$ the collection of **Borel sets**, which is the σ-algebra generated by the open subsets of X. A **Borel measure** on a locally compact Hausdorff space X is a measure on $(X, \mathfrak{B}(X))$. We shall call a Borel measure μ **regular** if

- $\mu(K) < \infty$ for every compact $K \subset G$;
- $\mu(U) = \sup_{K \subset U} \mu(K)$, where the supremum is taken over compact K, for all open subsets $U \subset G$; and
- $\mu(A) = \inf_{A \subset U} \mu(U)$, where the infimum is taken over open U, for all Borel subsets $A \subset G$.

Definition 18. Let G be a locally compact group. A **left Haar measure** on G is a regular Borel measure μ on G which is *left translation invariant*: i.e. $\mu(sA) = \mu(A)$ for all $s \in G, A \in \mathfrak{B}(G)$.

Theorem 19. *Every locally compact group G admits a left Haar measure, which is unique up to constant multiple.*

Proof. See [37, 4.15] or [10, Chapter 9] for details. □

For any locally compact group G, we will henceforth fix, and denote by m_G, a left Haar measure upon it. Note that for any $s \in G$, the measure m_G^s defined by

$$m_G^s(E) := m_G(Es), \text{ for all } E \in \mathfrak{B}(G),$$

is also a left Haar measure on G. By the uniqueness part of Theorem 19, there exists $K_s > 0$ such that $m_G(Es) = K_s m_G(E)$. This defines a continuous function $\Delta : G \to (0, \infty), s \mapsto K_s$ called the **modular function** of G. G is called **unimodular** if $\Delta(G) = \{1\}$. Note that all abelian groups are unimodular.

Let G be a locally compact group, $s \in G$, and f be any function on G. Denote by $L_s f$ and $R_s f$ the functions on G defined by

$$L_s f : t \mapsto f(st) \text{ and } R_s f : t \mapsto f(ts) \text{ for all } s, t \in G.$$

Two functions f and g, defined on G are said to be equivalent if

$$m_G\{s \in G : f(s) \neq g(s)\} = 0.$$

For any measure space $(\mathcal{S}, \Sigma, \mu)$, we may construct the classical *Lebesgue spaces* $L^p(\mathcal{S}, \Sigma, \mu)$, which are often abbreviated as $L^p(\mathcal{S})$ [20, III.3.4]. In the case of $(G, \mathfrak{B}(G), m_G)$, where G is a locally compact group, we arrive at the following special case. Recall that a function $f : G \to \mathbb{C}$ is **Borel measurable** if $f^{-1}(A) \in \mathfrak{B}(G)$ whenever $A \in \mathfrak{B}(\mathbb{C})$.

Definition 20. Let $p \in [1, \infty)$. $L^p(G)$ is the space of all equivalence classes of Borel measurable $f : G \to \mathbb{C}$ such that

$$\int_G |f(s)|^p m_G(ds) < \infty.$$

$L^\infty(G)$ is the space of all equivalences classes of Borel measurable $f : G \to \mathbb{C}$ such that

$$\operatorname{ess\,sup} |f| := \sup\{\alpha \geq 0 : m_G\{s \in G : |f(s)| > \alpha\} = 0\} < \infty.$$

$L^p(G)$ is a Banach space for each $p \in [1, \infty]$. $\|f\|_p = \left(\int_G |f(s)|^p m_G(ds) \right)^{\frac{1}{p}}$ for all $f \in L^p(G)$ with $p \in [1, \infty)$ and $\|f\|_\infty = \operatorname{ess\,sup} |f|$ for all $f \in L^\infty(G)$. $L^\infty(G)$ is isometrically isomorphic to the dual of $L^1(G)$ via the identification

$$\langle g, f \rangle = \int_G f(s) g(s) m_G(ds), \text{ for all } f \in L^1(G), g \in L^\infty(G).$$

If $p \in (1, \infty)$ and $q > 1$ is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(G)$ may be similarly identified with the dual of $L^p(G)$. In this case $L^p(G)$ is reflexive.

These spaces can be used to define an important class of locally compact groups: namely, *amenable* groups. Amenable groups were first considered by John von-Neumann in [47], where he defined *measurable* groups to be those which admits a finitely additive left invariant Borel measure. Mahlon Day showed in [19] that this is equivalent to the existence of a left invariant *mean* on G, and called such groups *amenable*. This was first used by the same author in [18]. It is the latter definition and terminology that has since become standard, which we shall now define.

Definition 21. A **left invariant mean** on G is a continuous linear functional $M \in L^\infty(G)'$ such that

- $\|M\| = \langle M, 1\rangle = 1$, and
- $M(L_s g) = M(g)$ for all $s \in G, g \in L^\infty(G)$.

G is **amenable** if it admits a left invariant mean.

Standard facts about amenable groups can be found in [59, Chapter 1]. In particular, the free group on 2 generators $\mathcal{F}_2$ is not amenable, while all compact and abelian groups are. A closed subgroup of an amenable group is again amenable.

There is no reason to expect, for a general measure space $(\mathcal{S}, \Sigma, \mu)$, any of the $L^p(\mathcal{S}, \Sigma, \mu)$ spaces for $p \in (1, \infty)$ to have an algebraic structure beyond that of a linear space. For a locally compact group G, however, $L^1(G)$ is a Banach algebra under the **convolution product**, which is defined for $f, g \in L^1(G)$ by

$$(f * g)(t) := \textstyle\int_G f(s)g(s^{-1}t)dm_G(s), \text{ for all } f, g \in L^1(G).$$

$L^1(G)$ is hence referred to as the **group algebra** of G. It may be checked directly that $(f, g) \mapsto f*g$ yields a contractive bilinear map $L^2(G) \times L^2(G) \to L^1(G)$ (see [37, (12.4)]).

Using the pointwise product, $L^\infty(G)$ is also a Banach algebra. $L^2(G)$ is a Hilbert space with inner product defined by

$$\langle f, g\rangle = \int_G f(s)g(s)m_G(ds) \text{ for all } f, g \in L^2(G).$$

For any function on G, let $\tilde{f}(s)$ denote the function defined by

$$\tilde{f}(s) := f(s^{-1}) \text{ for almost every } s \in G.$$

If G is unimodular, then for any $p \in [1, \infty)$, the map $L^p(G) \to L^p(G)$, $f \mapsto \tilde{f}$, is an isometric isomorphism [27, Section 2.4].

1.7 The Dual of an Abelian Locally Compact Group

Like many mathematical structures, there is a natural dual object associated with a locally compact abelian group. In a typical fashion, a class of appropriate morphisms into a fixed scalar object is formed, and is then endowed with the necessary structure to ensure it stays within the original category. In a category of hybrid objects such as locally compact groups, the appropriate morphisms are found to be continuous group homomorphisms which preserve both algebraic and topological structure.

The scalar group considered is the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, and hence we define a **character** of G to be a continuous group homomorphism $f : G \to \mathbb{T}$. The appropriate topology is as follows. The **compact-open topology** on $C(G, \mathbb{T})$, the continuous functions from G to $\mathbb{T}$, is the one generated by the subbase consisting of sets of the form

$$V(K, U) = \{f \in C(G, \mathbb{T}) : f(K) \subset U\} \text{ for some compact } K \subset G \text{ and open } U \subset \mathbb{T}.$$

For the duration of this section, we shall restrict our attention to a locally compact abelian group G. Let $\widehat{G}$ denote the set of all characters of G. $\widehat{G}$ is an abelian locally compact group under pointwise product, when equipped with compact open topology inherited as a subspace of $C(G, \mathbb{T})$. Note that $\widehat{G}$ is homeomorphically isomorphic to $\Phi_{L^1(G)}$ [27, Theorem 4.2]. Thus we may drop the distinction between characters of G and characters of $L^1(G)$.

Each $s \in G$ determines an element $F_s \in \widehat{\widehat{G}}$ given by $F_s(f) = f(s)$ for all $f \in \widehat{G}$. The map $s \mapsto F_s$ is an algebraic homeomorphism between G and $\widehat{\widehat{G}}$. This beautiful fact is referred to as the Pontrjagin Duality Theorem [27, Theorem 4.31]. It follows immediately that $\widehat{G}$ separates the points of G. If G is (discrete) compact, then $\widehat{G}$ is (compact) discrete [58, Theorem 1.2.5]. Hence, if G is non-compact, we may consider the compact group $\beta G := \widehat{\widehat{G}_d}$, known as the **Bohr compactification** of G. By Pontrjagin duality, there is a canonical injection $G \hookrightarrow \beta G$ which is continuous with dense range [27, Section 4.7].

The Bohr compactification possesses an attractive property.

Theorem 22. *Let G be a non-compact abelian group, and K be a compact group. Any continuous group homomorphism $G \to K$ extends to a continuous group homomorphism $\beta G \to K$.*

Proof. This is Proposition 4.78 in [27]. □

Note that this duality approach fails for non-abelian G. Indeed the only continuous homomorphism from a non-abelian locally compact group G to $\mathbb{T}$ is the constant function 1. See [38, 27.47 (b)] for details. For non-abelian G, however, a compactification still exists, and is given in Section 1.13.

1.8 Arens Products

Let A be a Banach algebra. There are two ways that the product of A can be extended to a product on A''. The results of these extensions are called the *Arens products* on A'', which are named after Richard Aren who introduced them in 1951 [1]. One attractive application is that they provide an algebraic criterion of an important class of functionals: those which possess a property called *weak almost periodicity*. These functionals play a key role in Chapter 5, and are defined in this section.

First define left and right actions $A \times A' \to A'$ and $A' \times A \to A'$ of A upon A' by

$$\langle a.\mu, b\rangle = \langle \mu, ba\rangle \text{ and } \langle \mu.a, b\rangle = \langle \mu, ab\rangle, \text{ for all } a, b \in A, \mu \in A'.$$

Next define bilinear maps $A' \times A'' \to A'$ and $A'' \times A' \to A'$ by setting

$$\langle \mu \cdot \Phi, a\rangle = \langle \Phi, a.\mu\rangle \text{ and } \langle \Phi \cdot \mu, a\rangle = \langle \Phi, \mu.a\rangle, \text{ for all } a \in A, \mu \in A', \Phi \in A''.$$

The **first** and **second Arens Products** on A'' are defined respectively as

$$\langle \Phi \Box \Psi, \mu\rangle = \langle \Phi, \Psi \cdot \mu\rangle \text{ and } \langle \Phi \Diamond \Psi, \mu\rangle = \langle \Psi, \mu \cdot \Phi\rangle, \text{ for all } \Phi, \Psi \in A'', \mu \in A'.$$

Both of these products turn A'' into a Banach algebra, and extend the natural product on $\kappa_A(A)$.

A functional $\mu \in A'$ is said to be **almost periodic** if the two operators $A \to A'$ defined by

$$a \mapsto a.\mu \text{ and } a \mapsto \mu.a$$

are compact. Let $\mathrm{AP}(A')$ denote the subspace of A' consisting of the almost periodic functionals. As established in [17, Proposition 2.4], the first Aren's product $\Box$ on A'' yields a well-defined product $\hat{\Box}$ on $\mathrm{AP}(A')'$ via the restriction map $\iota : A'' \to \mathrm{AP}(A')'$. That is, the bilinear operation $\hat{\Box}$ on $\mathrm{AP}(A')'$ given by $\iota(\Phi)\hat{\Box}\iota(\Psi) := \iota(\Phi\Box\Psi)$ is well-defined and turns $\mathrm{AP}(A')'$ into a Banach algebra. The key facts that allow this to work are extracted (from the proof) into the following observations made by Matthew Daws in [17], of which we shall make use of in Section 6.2.

Proposition 23. *The bilinear maps $A'' \times A' \to A'$ and $A' \times A'' \to A'$ have the following properties: If $\Phi, \Psi \in A''$ such that $\Phi - \Psi$ vanishes on $AP(A')$, then $\Phi \cdot \varphi = \Psi \cdot \varphi$ and $\varphi \cdot \Phi = \varphi \cdot \Psi$ for all $\varphi \in AP(A')$*

Proof. It is immediate that $\mathrm{AP}(A')$ is a submodule of A'. Therefore for all $a \in A$,

$$\begin{aligned}\langle \Phi \cdot \varphi, a\rangle &= \langle \Phi, \varphi.a\rangle \\ &= \langle \Phi, \varphi.a\rangle \\ &= \langle \Phi, \varphi.a\rangle + \langle \Psi - \Phi, \varphi.a\rangle \\ &= \langle \Psi, \varphi.a\rangle \\ &= \langle \Psi \cdot \varphi, a\rangle,\end{aligned}$$

as required. $\square$

Consequently, for $F \in \mathrm{AP}(A')'$ and $\varphi \in \mathrm{AP}(A')$, we may unambiguously write $F \cdot \varphi$ instead of $\Phi \cdot \varphi$, where $\Phi \in A''$ is chosen arbitrarily so that $\iota(\Phi) = F$.

Proposition 24. *If $F \in AP(A')', \varphi \in AP(A')$, then $F \cdot \varphi \in AP(A')$.*

Proof. It is sufficent to show that whenever $\Psi \in \mathrm{AP}(A')^{\perp}$, $\langle \Psi, F \cdot \varphi \rangle = 0$. Choose $\Phi \in A''$ such that $\iota(\Phi) = F$. Since $\varphi \in \mathrm{AP}(A')$, we have

$$\begin{aligned}
\langle \Psi, F.\varphi \rangle &= \langle \Psi, \Phi.\varphi \rangle \\
&= \langle \Psi \Box \Phi, \varphi \rangle \\
&= \langle \Psi \Diamond \Phi, \varphi \rangle \\
&= \langle \Psi \Diamond \Phi, \varphi \rangle \\
&= \langle \Phi, \varphi.\Psi \rangle \\
&= 0,
\end{aligned}$$

where we have used the fact that $\varphi.\Psi = 0$. Indeed for all $a \in A$ we have $\langle \varphi.\Psi, a \rangle = \langle \Psi, a.\varphi \rangle = 0$ since $a.\varphi \in \mathrm{AP}(A')$. □

These observations allow us to calculate the product of two elements in $\mathrm{AP}(A')$ directly, instead of first pulling them back to A''. That is, for $F, G \in \mathrm{AP}(A')'$, we may simply write $\langle F \hat{\Box} G, \varphi \rangle = \langle F, G \cdot \varphi \rangle$, whenever $\varphi \in \mathrm{AP}(A')$.

1.9 Amenability and Related Notions

The theory of amenable Banach algebras was birthed in 1972, when an appropriate definition for the term was cultivated. Given the aptitude of group algebras to determine characteristics of their underlying locally compact groups, an ideal definition of amenability for Banach algebras ought to be satisfied by group algebras $L^1(G)$ exactly when G is an amenable group. Thanks to Barry Johnson, who characterized group amenability in terms of a remarkable cohomological condition in [40], nobody was disappointed. In this section we state his groundbreaking result after first some outlining the necessary machinery, all of which can be found in [59, Chapter 2].

Let A be a Banach algebra. An A-**bimodule** is a Banach space E upon which A acts from both the left and the right, with the property that the actions

$$A \times E \to E, (a, x) \mapsto a.x$$

and

$$E \times A \to E, (x, a) \mapsto x.a$$

are bounded bilinear maps, and for all $a, b \in A, x \in E$, we have $(a.x).b = a.(x.b)$.

Example 25. Let A be a Banach algebra, and E be an A-bimodule. Then E' is also an A-bimodule via the *dual actions*:

$$\langle a.\varphi, x\rangle = \langle \varphi, x.a\rangle \text{ for all } a \in A, x \in E, \varphi \in E',$$

and

$$\langle \varphi.a, x\rangle = \langle \varphi, a.x\rangle \text{ for all } a \in A, x \in E, \varphi \in E'.$$

Example 26. Let A be a Banach algebra, and let $\varphi \in \Phi_A$. We may define the left and right actions

$$a.z := \varphi(a)z =: z.a \text{ for all } a \in A, z \in \mathbb{C}.$$

Let $\mathbb{C}_\varphi$ denote the complex numbers endowed with this A-bimodule structure.

For any Banach algebra A and any A-bimodule E, there is an associated *Hochschild cochain complex* which may formed, giving rise to a sequence of *Hochschild cohomology groups*, which we shall now introduce. First we assemble the basics of Banach space cohomology. A **cochain complex** of Banach spaces

$$\dots \xrightarrow{d^{-3}} E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^{0} \xrightarrow{d^{0}} E^{1} \xrightarrow{d^{1}} E^{2} \xrightarrow{d^{2}} \dots$$

is a pair of families $(E^n)_{n\in\mathbb{Z}}$ and $(d^n)_{n\in\mathbb{Z}}$, abbreviated $(E^\bullet, d^\bullet)$, such that

- E^n is a Banach space for all $n \in \mathbb{Z}$,
- $d^n \in \mathcal{B}(E^n, E^{n+1})$ for all $n \in \mathbb{Z}$, and
- $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

We refer to

- d_n as the n**-coboundary operator**,
- elements of E_n as n**-cochains**,
- elements of $\mathcal{B}^n(E) := \operatorname{im}(d_{n-1})$ as n**-coboundaries**, and
- elements of $\mathcal{Z}^n(E) := \ker(d_n)$ as n**-cocycles**.

Example 27. Let A be a Banach algebra, E be an A-bimodule. Let $\mathcal{L}^n(A, E)$ denote the Banach space of all bounded n-linear maps $T : A^n \to E$ equipped with the supremum norm. That is, $\mathcal{L}^n(A, E)$ is the collection of maps $T : A^n \to E$ which are linear in each variable with

$$\|T\| := \sup \|T(x_1, \dots, x_n)\| < \infty,$$

where the infimum is taken over all $x_1, \dots, x_n \in B_A$. Define $\delta_0 : E \to \mathcal{B}(A, E)$ by

$$[\delta_0(x)]\,(a) = a.x - x.a, \text{ for all } x \in E, a \in A;$$

and for $n \geq 1$, define $\delta^n : \mathcal{L}^n(A,E) \to \mathcal{L}^{n+1}(A,E)$ by

$$\begin{aligned}[\delta^n(T)]\,(a_1,\ldots,a_{n+1}) := a_1.T(a_2,\ldots,a_{n+1})& \\ + \sum_{k=1}^{n}(-1)^k T(a_1,\ldots,a_k a_{k+1},\ldots,a_{n+1})& \\ + (-1)^{n+1}T(a_1,\ldots,a_n).a_{n+1},&\end{aligned}$$

for all $T \in \mathcal{L}^n(A,E), a_1,\ldots,a_{n+1} \in A$. To check that the **Hochschild cochain complex**

$$0 \hookrightarrow E \xrightarrow{\delta^0} \mathcal{B}(A,E) \xrightarrow{\delta^1} \mathcal{B}^2(A,E) \xrightarrow{\delta^2} \mathcal{B}^3(A,E) \xrightarrow{\delta^3} \mathcal{B}^4(A,E) \xrightarrow{\delta^4} \ldots$$

is indeed a cochain complex is both vapid and tiresome, yet straightforward.

Starting with any cochain complex $(E^\bullet, d^\bullet)$, we obtain for each $n \in \mathbb{Z}$ the ***n*-th cohomology group** by forming the quotient

$$\mathcal{H}^n(E) := \mathcal{Z}^n(E)/\mathcal{B}^n(E).$$

Let $\mathcal{Z}^n(A,E)$ and $\mathcal{B}^n(A,E)$ denote respectively the spaces n-cocycles and n-coboundaries of the Hochschild cochain complex. Then for $n \geq 1$, the ***n*-th Hochschild cohomology group of A with coefficients in E** is the quotient space

$$\mathcal{H}^n(A,E) := \mathcal{Z}^n(A,E)/\mathcal{B}^n(A,E).$$

We may now state the remarkable fact owed to Barry Johnson.

Theorem 28. *A locally compact group G is amenable if and only if $\mathcal{H}^1(L^1(G),E') = 0$ for every $L^1(G)$-bimodule E.*

Proof. This is [40, Theorem 2.5]. A fully detailed and very readable exposition is also delivered in [59, 2.1]. □

It is this fact which motivates defining a Banach algebra A to be **amenable** if $\mathcal{H}^1(A,E') = 0$ for all A-bimodules E. If one is not ardent to enjoy the preceding construction, this machinery may be bypassed using the following terminology, which is common practice: a **derivation** from a Banach algebra A to an A-bimodule E is an operator $D : A \to E$ satisfying the *product rule*

$$D(ab) = a.D(b) + D(a).b \text{ for all } a,b \in A.$$

We say D is **inner** if there exists $x \in E$ such that $D(a) = a.x - x.a$ for all $a \in A$.

It is immediate that A is amenable if and only if for all A-bimodules E, every derivation $D : A \to E'$ is inner.

There are many interesting variations of amenability which permit corresponding structures associated to a locally compact group to have their own analogues for Theorem 28. One example, which we now introduce, is *operator amenability*, which incorporates appropriate overtones which account for the operator space structure of a completely contractive Banach algebra.

Let A be a completely contractive Banach algebra. An **operator A-bimodule** is both an operator space and an A-bimodule such that the maps:

$$A\hat{\otimes}V \to V, a \otimes v \mapsto a.v$$

and

$$V\hat{\otimes}A \to V, v \otimes a \mapsto v.a$$

are completely bounded. As with bimodules, if V is an operator A-bimodule, then V' is an operator A-bimodule when equipped with the dual actions. We say that A is **operator amenable** whenever V is an operator A-bimodule V, every completely bounded derivation $D : A \to V'$ is inner. This definition has a nice analogue of Johnson's result for the Fourier algebra. $A(G)$ is operator amenable if and only if G is amenable [23, Theorem 3.6]. Conveniently, it is shown in the same reference that if A is operator amenable and $q : A \to B$ is a completely quotient map, then B is operator amenable.

We shall also consider two weakened notions and apply them to ultrapowers of the objects to which they are relevant:

- A Banach algebra A is **weakly amenable** if every bounded derivation $D : A \to A'$ is inner.
- A completely contractive Banach algebra A is **operator weakly amenable** if every completely bounded derivation $D : A \to A'$ is inner.

Barry Johnson proved in [41] that $L^1(G)$ is always weakly amenable. and Nico Spronk established the dual claim in [63]: $A(G)$ is always operator weakly amenable. In Chapter 6 we obtain some negative results about the much stronger ultra-theoretic variants of these properties. A particular brand of scalar valued derivations, called *point derivations*, are a useful tool in accomplishing this. It is shown in [67] that a weakly amenable Banach algebra may not admit any point derivations. For the convenience of the reader we supply the argument below. First, recall the following standard fact about normed linear spaces.

Lemma 29. *Let E be a normed linear space, and $\varphi, \varphi_1, \ldots, \varphi_n \in E'$. If $\bigcap_{j=1}^{n} ker(\varphi_j) \subset ker(\varphi)$, then φ is a linear combination of $\varphi_1, \ldots, \varphi_n$.*

Proof. This is Lemma 3.9 from [26]. □

Eshaghi Madjid and Taher Yazdanpanah proved in [67, Theorem 1.4] that weakly amenable Banach algebras may never admit point derivations. Their argument, which requires no machinery but the preceding lemma, is supplied below for convenience.

Proposition 30. *A weakly amenable Banach algebra may admit no point derivations.*

Proof. Suppose first that A is a weakly amenable Banach algebra. Let $\phi \in \Phi_A$ and $d : A \to \mathbb{C}_\varphi$ be a non-zero point derivation. Define $D : A \to A'$ by

$$D(a) = d(a)\varphi \text{ for all } a \in A, \varphi \in A'.$$

Then D is a derivation, since for all $c \in A$,

$$\begin{aligned}
\langle D(ab), c\rangle &= \langle d(ab)\varphi, c\rangle \\
&= \langle [\varphi(a)d(b) + \varphi(b)d(a)]\,\varphi, c\rangle \\
&= \langle \varphi(a)d(b)\varphi + \varphi(b)d(a)\varphi, c\rangle \\
&= \langle \varphi(a)d(b)\varphi, c\rangle + \langle \varphi(b)d(a)\varphi, c\rangle \\
&= \varphi(a)d(b)\varphi(c) + \varphi(b)d(a)\varphi(c) \\
&= d(b)\varphi(ca) + d(a)\varphi(bc) \\
&= \langle D(b), ca\rangle + \langle D(a), bc\rangle \\
&= \langle a.D(b), c\rangle + \langle D(a).b, c\rangle \\
&= \langle a.D(b) + D(a).b, c\rangle
\end{aligned}$$

Since A is weakly amenable, there exists $\psi \in A'$ such that $D(a) = a.\psi - \psi.a$ for all $a \in A$. Choose $z \in A$ such that $\varphi(z) = 1$. If $\ker(\varphi) \subset \ker(d)$, then by the foregoing with $n = 1$, there exists $\alpha \in \mathbb{C}$ such that $d = \alpha\varphi$. Then

$$\begin{aligned}
2\alpha &= 2\alpha\varphi(z) \\
&= 2d(z) \\
&= 2d(z)\varphi(z) \\
&= d(z^2), \text{ since } d \text{ is a derivation} \\
&= \alpha\varphi(z^2) \\
&= \alpha
\end{aligned}$$

forcing $\alpha = 0$, contradicting the fact that $d \neq 0$.

Otherwise, choose $a \in A$ such that $\varphi(a) = 0$ and $d(a) = 1$. Set $z' = z + (1 - d(z))za$. Then both $\varphi(z')$ and $d(z')$ are equal to 1, and therefore

$$\begin{aligned}1 &= \langle D(z'), z' \rangle \\ &= \langle a.\psi - \psi.a, z' \rangle \\ &= \langle \psi, z'^2 \rangle - \langle \psi, z'^2 \rangle \\ &= 0,\end{aligned}$$

a contradiction. Therefore A can admit no point derivations. □

Nothing is harmed when considering the weaker property of operator weak amenability, and hence we may deduce the following slightly stronger statement.

Corollary 31. *A operator weakly amenable completely contractive Banach algebra A may admit no point derivations.*

Proof. Let φ, d, and D be as in Proposition 30. It has already been seen that D is a derivation and cannot be inner. For all $A \in M_n[A]$, we have $D^{(n)}(A) = d^{(n)}(A) \otimes \varphi$. Thus $\|D^{(n)}\| = \|d^{(n)}\|$ for all $n \geq 1$ and since d is completely bounded, so is D. □

1.10 The Measure Algebra

For discrete groups G, there is a canonical identification between the dual of $C_0(G)$ and $L^1(G)$ given by

$$\langle f, g \rangle = \sum_{t \in G} f(t)g(t) \text{ for all } f \in L^1(G), g \in C_0(G).$$

For non-discrete G, the elements of $L^1(G)$ only supply *some* of the functionals on $C_0(G)$; a larger space, which we now introduce, is needed to play the role of its dual. We first recall some necessary facts about complex measures, which can be found in [10].

For the following, let a measurable space $(\mathcal{S}, \Sigma)$ be fixed. A **complex measure** on $(\mathcal{S}, \Sigma)$ is a function $\mu : \Sigma \to \mathbb{C}$ such that

- $\mu(\emptyset) = 0$ and
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in Σ.

A collection $\{A_j\}_{j=1}^{n} \subset \Sigma$ of disjoint sets is a **finite partition** of a set $A \in \Sigma$ if $A = \bigcup_{j=1}^{n} A_j$. For any complex measure μ on $(\mathcal{S}, \Sigma)$ there exists a finite real-valued measure $|\mu|$ on $(\mathcal{S}, \Sigma)$, called the **variation** of μ, defined by

$$|\mu|(A) = \sup\left\{\sum_{j=1}^{n} |\mu(A_j)| : \{A_j\}_{j=1}^{n} \text{ is a finite partition of } A\right\}, \text{ for all } A \in \Sigma.$$

A complex Borel measure μ on a locally compact Hausdorff space X is **regular** if its variation $|\mu|$ is regular.

The collection $M(X)$ of all complex regular Borel measures on the locally compact Hausdorff space X is a Banach space, where each measure is normed by its variation over X. That is,

$$\|\mu\| = |\mu|(X) \text{ for all } \mu \in M(X).$$

For each $x \in X$, the measure $\delta_x \in M(X)$ defined for all $E \in \mathfrak{B}(X)$ by

$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases}$$

is called the **point mass at** x. By the Riesz representation theorem, $M(X)$ may be identified with $\mathcal{C}_0(X)'$ via

$$\langle \mu, f\rangle = \int_X f(x)\mu(dx), \text{ for all } \mu \in M(X), f \in \mathcal{C}_0(X).$$

When G is a locally compact group and $\mu, \nu \in M(G)$, we may thus identify the functional on $C_0(G)$ defined by

$$f \mapsto \int_G f(st)\mu(ds)\nu(dt).$$

with a unique element $\mu * \nu \in M(G)$, which we call **the convolution product of** μ **and** ν. It is straightforward to check that this turns $M(G)$ into a Banach algebra. The convolution product $\mu * \nu$ may also be calculated directly by the formulas

$$(\mu * \nu)(E) = \int_G \nu(s^{-1}E)\mu(ds) = \int_G \mu(Es^{-1})\nu(ds), \text{ for all } E \in \mathfrak{B}(G).$$

For any locally compact group G, there is an isometric embedding of $L^1(G)$ into $M(G)$

$$L^1(G) \hookrightarrow M(G), f \mapsto \mu_f$$

defined by

$$\mu_f(E) = \textstyle\int_G f(s)m_G(ds) \text{ for all } E \in \mathfrak{B}(G).$$

This embedding preserves convolutions, and we may describe its image concretely: $\mu \in M(G)$ is said to be **absolutely continuous** if whenever $E \in \mathfrak{B}(G)$ with $m_G(E) = 0$, we necessarily have $\mu(E) = 0$. By the Radon-Nikodym Theorem [10, Theorem 4.2.2], the self-adjoint subalgebra of absolutely continuous $\mu \in M(G)$, denoted by $M_a(G)$, is the image of $L^1(G)$ in $M(G)$ (see [27, Section 2.3] for an explanation of why the hypothesis of σ-finiteness in the Radon-Nikodym Theorem can be ignored). It is not hard to see that this embedding is surjective exactly when $M_a(G)$ contains the point masses, i.e. when G is discrete. When this is the case, for each $t \in G$ we may identify δ_t with the function in $L^1(G)$ defined by

$$\delta_t(s) = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}, \text{ for every } s \in G.$$

Fereidoun Ghahramani and Garth Dales proved in [14] that the class of Banach algebras which take the form $M(G)$ for some locally compact group G provides no new examples of amenable Banach algebras. That is, $M(G)$ is amenable if and only of G is discrete and amenable. In fact it is also shown in the same reference that the $M(G)$ admits point derivations exactly when G is non-discrete, a result which we make use of in both of the main arguments in Chapter 6.

1.11 Representation Theory

Given a locally compact group G, it is often useful to realize its elements concretely as operators on a Hilbert space. For example, we may always identify each $s \in G$ with the left translation operator $L_{s^{-1}} \in \mathcal{B}(L^2(G))$, which is defined by

$$[L_{s^{-1}}\zeta](t) = \zeta(s^{-1}t) \text{ for almost every } t \in G.$$

This results in a group homomorphism $G \to \mathcal{U}(L^2(G))$ which is continuous when $\mathcal{B}(L^2(G))$ is given either the *weak* or *strong operator topology*, which we now recall (see [27, Appendix 1] for a detailed treatment). For any Hilbert space $\mathcal{H}$, denote by $\omega_{\zeta,\eta}$ the functional on $\mathcal{B}(\mathcal{H})$ defined by

$$\omega_{\zeta,\eta}(T) = \langle T\zeta, \eta\rangle \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

The **weak operator topology** on $\mathcal{B}(\mathcal{H})$ is the locally convex topology induced by the family of seminorms $\{p_{\zeta,\eta} : \zeta, \eta \in \mathcal{H}\}$ defined by

$$p_{\zeta,\eta}(T) = |\omega_{\zeta,\eta}(T)| \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

That is, in the weak operator topology,

$$T_\alpha \to 0 \text{ if and only if } \langle T_\alpha\zeta, \eta\rangle \to 0 \text{ for all } \zeta, \eta \in \mathcal{H}.$$

The **strong operator topology** is the locally convex topology induced by the family of seminorms $\{p_\zeta : \zeta \in \mathcal{H}\}$, defined by

$$p_\zeta(T) = \|T\zeta\| : \zeta \in \mathcal{H}, \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

This yields a convergence criterion of

$$T_\alpha \to 0 \text{ if and only if } \|T_\alpha\zeta\| \to 0 \text{ for all } \zeta \in \mathcal{H}.$$

Note that the strong and weak operator topologies, abbreviated SOT and WOT respectively, coincide on $\mathcal{U}(\mathcal{H})$ [27, Section 3.1]. Representations are generalizations of the foregoing, with a general Hilbert space playing the role of $L^2(G)$. We now collect some facts about them, which can be found in [27, Chapter 3].

Let G be a locally compact group. A **continuous unitary representation** of G is an ordered pair $(\pi, \mathcal{H}_\pi)$ such that $\mathcal{H}_\pi$ is a Hilbert Space and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous homomorphism with respect to either the strong or weak operator topology. Two representations $\pi_1 : G \to \mathcal{U}(\mathcal{H}_{\pi_1}), \pi_2 : G \to \mathcal{U}(\mathcal{H}_{\pi_2})$ are **unitarily equivalent** if there exists an unitary operator $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $T\pi_1(s) = \pi_2(s)T$ for all $s \in G$. Let Σ_G denote the collection of equivalence classes of continuous unitary representations of G. We will assume that a fixed representative from each equivalence class $[\pi] \in \Sigma_G$ has been chosen, and abuse notation by writing $\pi \in \Sigma_G$.

Let A be an involutive Banach algebra. A **$*$-representation of A** is an ordered pair $(\pi, \mathcal{H}_\pi)$ such that $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. $(\pi, \mathcal{H}_\pi)$ is **degenerate** if there exists a $\zeta \in \mathcal{H}$ such that $\pi(A)\zeta = \{0\}$. As shown in [27, Section 3.2], the continuous unitary representations of G and non-degenerate $*$-representations of $L^1(G)$ are in bijective correspondance. In particular, let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation. Define the $*$-representation $(\pi_*, \mathcal{H}_{\pi_*})$ on $L^1(G)$ by setting $\mathcal{H}_{\pi_*} = \mathcal{H}_\pi$ and defining

$$\pi_*(f) := \int_G f(s)\pi(s)\mu(ds), \text{ for all } f \in L^1(G),$$

where the integral is to be understood in the Bochner sense. See [57, Section 2.3] for a thorough treatment of the Bochner integral. Alternatively, the virtues of Hilbert spaces allow avoidance of such details by using the fact that

$$\langle \pi_*(f)\zeta, \eta\rangle = \int_G f(s)\langle \pi(s)\zeta, \eta\rangle \mu(ds) \text{ for all } f \in L^1(G) \text{ and } \zeta, \eta \in \mathcal{H}_\pi.$$

To simplify notation we shall not distinguish between a continuous unitary representation $\pi \in \Sigma_G$ and its lift π_* to $L^1(G)$. There are two particular representations that arise with any locally compact group G.

- $\lambda : G \to \mathcal{B}(L^2(G))$ defined by $\lambda(s) : \zeta \mapsto \zeta(s^{-1}\cdot)$ is the **left regular representation** of G, and
- $\pi_G := \bigoplus_{\pi \in \Sigma_G} \pi$ is the **universal representation** of G.

If $\pi \in \Sigma_G$, and $\zeta, \eta \in \mathcal{H}_\pi$, let $\pi_{\zeta,\eta}$ denote the **coefficient function** $\omega_{\zeta,\eta} \circ \pi$. In Section 6.1, when the symbol δ is used to denote an element of Σ_{G_d}, $\delta_{\zeta,\eta}$ should not be confused with the Delta Dirac functional, which shall not be required at present.

1.12 $\mathcal{C}^*$-algebras Associated With a Locally Compact Group

We have so far already seen several Banach algebras which are associated with any locally compact group. Starting with a general locally compact group G, its continuous unitary representations will now give rise to a family of $\mathcal{C}^*$-algebras. We now supply the details behind this, right after assembling the required background, which can be found in [49] and [50]. A **concrete von-Neumann algebra** $\mathcal{M}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, which is closed in the weak operator topology. An **abstract von-Neumann algebra** $\mathcal{M}$ is a unital $\mathcal{C}^*$-algebra which is the dual of some Banach space $\mathcal{M}_*$.

Every concrete von-Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is an abstract von-Neumann algebra. Indeed by [42, Section 7.4] we may take $\mathcal{M}_*$ to be the subspace of $\mathcal{M}'$ consisting of all functionals of the form $\sum_{n=1}^{\infty} \omega_{\zeta_n,\eta_n}$ for some $\zeta_n, \eta_n \in \mathcal{H}$ with

- $\sum_{n=1}^{\infty} \|\zeta_n\| \cdot \|\eta_n\| < \infty$,
- $\sum_{n=1}^{\infty} \|\zeta_n\|^2 < \infty$, and
- $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$.

On the other hand, every abstract von-Neumann algebra is canonically isometrically $*$-isomorphic to a concrete von-Neumann algebra [62, Theorem 1.16.7]. The distinction between the two objects is thus dropped.

By definition, a von-Neumann algebra immediately receives an operator structure from its realization $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$, which coincides with its canonical operator space structure as a $\mathcal{C}^*$-algebra. From this structure, the predual $\mathcal{M}_*$ of $\mathcal{M}$ thus becomes an operator space.

Theorem 32. *Let $\mathcal{M}$ be a von-Neumann algebra. $(\mathcal{M}_*)'$ is completely isometric to $\mathcal{M}$.*

Proof. This is [7, Theorem 2.9]. □

Let A be a $\mathcal{C}^*$ algebra. It follows from the Sherman-Takeda theorem [70, Theorem I], that the second dual A'' is a von Neumann algebra. Consequently, there are *prima facie* two ways to obtain an operator space structure on A''. We may use the canonical one that A'' inherits as a $\mathcal{C}^*$-algebra, or the one obtained by taking twice the operator space dual of A. A result by David Blecher establishes that these two structures coincide [7, Corollary 2.6].

In contrast with completely contractive Banach algebras, the ultrapower $[\mathcal{M}]_{\mathcal{U}}$ of a von-Neumann algebra $\mathcal{M}$ need not again be a von-Neumann algebra. To rectify this, the following approach is adopted. If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von-Neumann algebra. Let $\mathfrak{H} = [\mathcal{H}]_{\mathcal{U}}$, which is a Hilbert space when equipped with the inner product

$$\langle [x_i]_{\mathcal{U}}, [y_i]_{\mathcal{U}} \rangle := \lim_{i \to \mathcal{U}} \langle x_i, y_i \rangle \text{ for all } [x_i]_{\mathcal{U}}, [y_i]_{\mathcal{U}} \in \mathfrak{H}.$$

Identify $[\mathcal{M}]_{\mathcal{U}}$ with a subspace of $\mathcal{B}(\mathfrak{H})$ via

$$[T_i]_{\mathcal{U}} : [x_i]_{\mathcal{U}} \mapsto [T_i x_i]_{\mathcal{U}}$$

and define the **von-Neumann ultrapower of** $\mathcal{M}$ by

$$[\mathcal{M}]_{\mathcal{U}^*} := \overline{[\mathcal{M}]_{\mathcal{U}}}^{\text{WOT}}.$$

By [53, Theorem 1.1], there is a w^*-continuous isometric isomorphism between $[\mathcal{M}]_{\mathcal{U}^*}$ and $[\mathcal{M}_*]'_{\mathcal{U}}$, providing an alternative realization for the underlying Banach space structure of $[\mathcal{M}]_{\mathcal{U}^*}$.

The **group von-Neumann algebra** of a locally compact group G is defined to be

$$VN(G) := \overline{\lambda(L^1(G))}^{\text{WOT}},$$

which inherits a natural operator space structure as a subspace of $\mathcal{B}(L^2(G))$.

The **Fourier algebra** of G is the predual of its group von-Neumann algebra, which may be be concretely realized as the commutative closed subalgebra of $C(G)$ given by

$$A(G) := \{\lambda_{\zeta,\eta} : \zeta, \eta \in L^2(G)\},$$

equipped with pointwise operations and normed by

$$\|f\|_{A(G)} := \inf\{\|\zeta\| \cdot \|\eta\| : f = \lambda_{\zeta,\eta}, \zeta, \eta \in L^2(G)\} \text{ for all } f \in A(G).$$

This along with all subsequent claims in this section about $A(G)$ may be found in [25]. The duality pairing of $VN(G)$ with $A(G)$ is given by

$$\langle T, f \rangle = \langle T\zeta, \eta \rangle \text{ for all } T \in VN(G),$$

whenever $f \in A(G)$ is written $f = \lambda_{\zeta,\eta}$. As the predual of the operator space $VN(G)$, $A(G)$ inherits an operator space structure. For abelian G, $VN(G)$ is a commutative C^*-algebra and thus has minimal operator space structure. In this case it follows that $A(G)$ maximal. Brian Forrest and Peter Wood proved in [29] that the Fourier algebra of a closed subgroup H of G is completely isometrically isomorphic to a quotient of $A(G)$. In particular, we have the following.

Proposition 33. *If $H \leq G$ is a closed subgroup, then the restriction map $A(G) \to A(H)$ is a complete quotient map.*

Proof. This is [29, Proposition 4.2]. □

For abelian G, and $f \in A(G), \zeta, \eta \in L^2(G)$ such that $f(s) = \langle \lambda(s)\zeta, \eta \rangle$ for all $s \in G$, it may be directly checked that $f(s) = (\zeta * \tilde{\eta})(s)$. It follows from [58, Theorem 1.2.4] and [58, Theorem 1.6.3] that $L^1(G)$ is isometrically isomorphic to $A(\widehat{G})$ via the **Fourier transform** $\mathcal{F} : L^1(G) \to A(\widehat{G})$ defined by

$$\mathcal{F}f : \gamma \mapsto \int_G f(s)\gamma(s^{-1})m_G(ds) \text{ for all } f \in L^1(G), \gamma \in \widehat{G}.$$

See [58, Section 1.2] for more details.

Aside from the spaces $C(G), C_0(G)$ and $VN(G)$, each element of Σ_G also supplies a $\mathcal{C}^*$-algebra associated with G. For each $\pi \in \Sigma_G$, define $\mathcal{C}^*_\pi(G)$ to be the (norm) closure of $\pi(L^1(G))$ in $\mathcal{B}(\mathcal{H}_\pi)$; and define $B_\pi(G) := [\mathcal{C}^*_\pi(G)]^*$. We take special interest in the spaces obtained by applying this to the left regular and universal representation of G:

- $\mathcal{C}^*(G) := \mathcal{C}^*_{\pi_G}(G)$ is the **group $\mathcal{C}^*$-algebra of** G;
- $\mathcal{C}^*_\lambda(G)$ is the **reduced group $\mathcal{C}^*$-algebra of** G;
- $B(G) := B_{\pi_G}(G)$ is the **Fourier-Stieltjes algebra of** G; and
- $B_\lambda(G)$ is the **reduced Fourier-Stieltjes algebra of** G**.**

$B(G)$ may be concretely realized as the Banach algebra $\{\pi_{\zeta,\eta} : \pi \in \Sigma_G, \zeta, \eta \in \mathcal{H}_\pi\}$. Under this identification, $A(G)$ is a closed two-sided ideal of $B(G)$. For each $\pi \in \Sigma_G$, $B_\pi(G)$ may be identified with the w^*-closure of

$$A_\pi(G) := \text{span}\{\pi_{\zeta,\eta} : \zeta, \eta \in \mathcal{H}_\pi\}$$

in $B(G)$. It is shown in [39] that $B_\lambda(G) = B(G)$ if and only if G is amenable. Thus for amenable G, a w^*-closed subspace of $B(G)$ containing $A(G)$ is already equal to all of $B(G)$.

It may be checked directly that for discrete G, the w^*-topology on any bounded subset of $B(G)$ is that of pointwise convergence. Moreover if G is discrete, then the restriction $B(G) \to B(H)$ is a quotient map. In fact this is true whenever G is a [SIN]-group [55, Theorem 2]. If G is abelian, there exists an algebra homomorphism $M(\widehat{G}) \to B(G)$, $\mu \mapsto \phi_\mu$, defined by

$$\phi_\mu(s) := \textstyle\int_{\widehat{G}} \gamma(s) d\mu(\gamma) \text{ for all } s \in G, \mu \in M(\widehat{G}),$$

which is called the **Fourier-Stieltjes** transform. It can be directly seen that this is nothing but the adjoint of the Gelfand transform

$$\Gamma_{\mathcal{C}^*(G)} : \mathcal{C}^*(G) \to C_0(\widehat{G}) \text{ which,}$$

since $\mathcal{C}^*(G)$ is a commutative $\mathcal{C}^*$-algebra, is a isometric isomorphism. Thus so too is the Fourier-Stieltjes transform.

1.13 Almost Periodic Functions

In Section 1.8, almost periodic functionals on a general Banach algebra A were introduced. We may also speak of almost periodic *functions* defined directly upon a locally compact group. A function $f \in C(G)$ is **almost periodic** if the set $\overline{\{L_s f : s \in G\}}$ is compact in $C(G)$. Let $\mathrm{AP}(G)$ denote the space of almost periodic functions on G, which is a Banach algebra when equipped with pointwise product. The previous definition of almost periodic is in fact an extension of this one. Indeed it may be checked that each element of $\mathrm{AP}(G)$ is an element of $L^\infty(G)$ which acts upon $L^1(G)$ as an almost periodic functional in the sense of Section 1.8. On the other hand, by [65, Theorem 2], each $f \in \mathrm{AP}(L^\infty(G))$ is equal almost everywhere to a unique element of $\mathrm{AP}(G)$. Thus we may identify $\mathrm{AP}(L^\infty(G))$ with $\mathrm{AP}(G)$, the latter space consisting of the unique continuous representative from each equivalence class in the former.

Note that $\mathrm{AP}(G)$ is a unital $\mathcal{C}^*$-subalgebra of $L^\infty(G)$, and thus its character space $G^{\mathrm{AP}} := \Phi_{\mathrm{AP}(G)}$ is compact. G^{AP} is called the **almost periodic compactification** of G. Since point evalulation is a character of $\mathrm{AP}(G)$, there is a natural map $G \to G^{\mathrm{AP}}$, $s \mapsto \widehat{s}$, which has dense range and induces compact group structure on G^{AP} [49, Section 3.2.16]. The map $G \to G^{\mathrm{AP}}$ is an injection exactly when $\mathrm{AP}(G)$ separates the points of G, in which case we say G is **maximally almost periodic** [49, Section 3.2.17]. If G is abelian, then $\widehat{G} \subset \mathrm{AP}(G)$ [27, Theorem 4.79], and it immediately follows that G is maximally almost periodic, which is relied upon in Section 6.2. In this case, the continuous extension of the canonical injection $G \hookrightarrow G^{\mathrm{AP}}$ is a homeomorphic isomorphism from $\beta G \to G^{\mathrm{AP}}$. Thus the Bohr compactification and the almost periodic compactification of G indeed coincide.

Some observations are now made about the product in G^{AP}. Note first that for any $\varphi \in G^{\mathrm{AP}}$, and $f \in \mathrm{AP}(G)$ we may compute

$$\begin{aligned}\langle \varphi, f \rangle &= \lim_\alpha \langle \widehat{s_\alpha}, f \rangle \\ &= \lim_\alpha f(s_\alpha),\end{aligned}$$

where (s_α) is any net in G such that $\widehat{s_\alpha} \to \varphi$ in G^{AP}. Note that we have the actions of G upon $\mathrm{AP}(G)$ via $s.f : t \mapsto f(ts)$, and $f.s : t \mapsto f(st)$. For any $s \in G, \varphi \in G^{\mathrm{AP}}$ we may thus evaluate

$$\begin{aligned}
\langle \varphi \cdot \widehat{s}, f \rangle &= \lim_\alpha \langle \widehat{t_\alpha} \cdot \widehat{s}, f \rangle \\
&= \lim_\alpha \langle \widehat{t_\alpha s}, f \rangle \\
&= \lim_\alpha f(t_\alpha s) \\
&= \lim_\alpha s.f(t_\alpha) \\
&= \lim_\alpha \langle \widehat{t_\alpha}, s.f \rangle \\
&= \langle \varphi, s.f \rangle
\end{aligned}$$

Finally, for $\varphi, \psi \in G^{\mathrm{AP}}, f \in \mathrm{AP}(G)$, choose a net s_α in G such that $\widehat{s_\alpha} \to \psi$ in G^{AP}. Then

$$\begin{aligned}
\langle \varphi \cdot \psi, f \rangle &= \lim_\alpha \langle \varphi \cdot \widehat{s_\alpha}, f \rangle \\
&= \lim_\alpha \langle \varphi, s_\alpha.f \rangle \\
&= \langle \varphi, \psi.f \rangle
\end{aligned}$$

since for all $g \in L^1(G), \langle \psi.f, g \rangle = \langle \psi, f * g \rangle = \lim_\alpha \langle s_\alpha, f * g \rangle = \langle s_\alpha.f, g \rangle$. Note that we have used the fact that pointwise convergence is enough to ensure that $s_\alpha.f \to \psi.f$ in norm, as f is almost periodic. One consequence to the foregoing is that we need not pull back elements of G^{AP} to nets in G in order to compute products. We may instead obtain the result of multiplying elements of G^{AP} in an algebraic fashion, as with the Arens products. An important application of this observation appears in the proof of Proposition 49, upon which the main result of Chapter 4 depends.

Chapter 2

Ultrapowers of Maximal Spaces

As mentioned in Section 1.5, an ultrapower of a minimal operator space is again minimal. We show in this section that the analagous statement for maximality holds for complete operator spaces. In 1984, Ulrich Groh proved in [30, Proposition 2.2] that for any von-Neumann algebra $\mathcal{M}$ and any non-principle ultrafilter $\mathcal{U}$, the ultrapower $[\mathcal{M}_*]_{\mathcal{U}}$ of its predual is isometrically isomorphic to the predual of a von-Neumann algebra.

In [22, Proposition 10.3.6] the authors present the special case where $\mathcal{M}_*$ is the *infinite-dimensional trace class matrix space*, and adapt the argument to show that the foregoing isomorphism is in fact a complete isometry. We verify in this section that the adaptation remains adequate in the general case. In light of Proposition 14 we depend on it for the case $\mathcal{M}_* = \ell^1(\mathcal{S})$ for a set $\mathcal{S}$ which, since all Banach spaces appear as quotients of such spaces (see Theorem 5.9 in [26] and the remark thereafter), will allow us to show that $[\max(E)]_{\mathcal{U}} = \max([E]_{\mathcal{U}})$ for all Banach spaces E and ultrafilters $\mathcal{U}$. We begin with some background facts.

Let $\mathcal{M}$ be a von-Neumann algebra. Note that under the canonical left and right actions of $\mathcal{M}$ upon $\mathcal{M}'$ given by

$$\langle a.\varphi, b\rangle = \langle \varphi, ba\rangle \text{ and } \langle \varphi.a, b\rangle = \langle \varphi, ab\rangle, \text{ for all } a, b \in \mathcal{M},\ \varphi \in \mathcal{M}',$$

the separate w^*-continuity of multiplication in $\mathcal{M}$ implies that $\kappa_{\mathcal{M}_*}(\mathcal{M}_*)$ is invariant. This supplies $\mathcal{M}_*$ with an $\mathcal{M}$-bimodule structure.

A projection $e \in \mathcal{M}$ is **central** if it commutes with all elements in $\mathcal{M}$. It was shown by Edward Effros in [24] that the map $e \mapsto \mathcal{M}_*.e$ is a bijective correspondence between the set of central projections in $\mathcal{M}$ and that of closed $\mathcal{M}$-submodules of $\mathcal{M}_*$.

For any von-Neumann algebra $\mathcal{M}$, the map

$$\prod_{i\in\mathbb{I}} \mathcal{M} \to [\mathcal{M}]_{\mathcal{U}}, (g_i)_{i\in\mathbb{I}} \mapsto [g_i]_{\mathcal{U}}$$

is a complete quotient map. Hence its adjoint $[\mathcal{M}]'_{\mathcal{U}} \hookrightarrow \left[\prod_{i\in\mathbb{I}} \mathcal{M}\right]'$ is a complete isometry.

Theorem 34. *Let $\mathcal{M}$ be a von-Neumann algebra. $[\mathcal{M}_*]'_{\mathcal{U}}$ is completely isometric to the predual of a von-Neumann algebra.*

Proof. Let $\mathcal{R} = \left[\prod_{i\in\mathbb{I}} \mathcal{M}\right]$. Note that $[\mathcal{M}_*]_{\mathcal{U}}$ is completely isometric to its image $\mathcal{T}$ under the chain of embeddings

$$[\mathcal{M}_*]_{\mathcal{U}} \hookrightarrow [\mathcal{M}']_{\mathcal{U}} \hookrightarrow [\mathcal{M}]'_{\mathcal{U}} \hookrightarrow \left[\prod_{i\in\mathbb{I}} \mathcal{M}\right]',$$

which is a closed $\mathcal{R}$-submodule of $\mathcal{R}'$. There is hence a central projection $e \in \mathcal{R}''$ such that $\mathcal{T} = \mathcal{R}'.e$, which is completely isometric the predual of the von-Neumann algebra $\mathcal{R}''e \subset \mathcal{R}''$. Indeed as a subspace of $\mathcal{R}'$, $\mathcal{R}'.e$ is completely isometric to $\mathcal{R}''/[\mathcal{R}'.e]^{\perp}$, which possesses the operator space structure of a $\mathcal{C}^*$-algebra. Finally, note that $\mathcal{R}''/[\mathcal{R}'.e]^{\perp}$ is $*$-isomorphic to $\mathcal{R}''e$. Since both are $\mathcal{C}^*$-algebras with canonical operator spaces structure, this completes the proof. □

By Theorem 34, the operator space dual $[\mathcal{M}_*]_{\mathcal{U}}$ is completely isometric to a von-Neumann algebra with canonical operator space structure. This yields the following convenient consequence.

Corollary 35. *Let $\mathcal{M}$ be a von-Neumann algebra. $[\mathcal{M}]_{\mathcal{U}^*}$ is completely isometric to the dual operator space structure on $[\mathcal{M}_*]'_{\mathcal{U}}$.*

For any Banach space E, Proposition 14 yields a quotient map $\ell^1(\mathcal{S}) \to \max(E)$ for some set $\mathcal{S}$. The associated complete quotient map

$$[\ell^1(\mathcal{S})]_{\mathcal{U}} \to [\max(E)]_{\mathcal{U}},$$

ensures that if $[\ell^1(\mathcal{S})]_{\mathcal{U}}$ is maximal, then $[\max(E)]_{\mathcal{U}}$ is too.

Corollary 36. *Let E be a Banach space $[\max(E)]_{\mathcal{U}}$ is maximal.*

Proof. Fix a set $\mathcal{S}$ and let $\mathcal{M} = \ell^\infty(\mathcal{S}), \mathcal{M}_* = \ell^1(\mathcal{S})$. The proof of Theorem 34 shows that $[\ell^1(\mathcal{S})]_{\mathcal{U}}$ is completely isometric to the predual of a commutative von-Neumann algebra, and is hence minimal. By the foregoing, this is sufficient. □

An important application Corollary 36 appears in the proof of Proposition 49, upon which the main result of Chapter 4 depends. Corollary 35 is used without comment.

Chapter 3

Structure of Locally Compact Groups

Many of our results take advantage of structure theory of locally compact groups. It is the goal of this section to present such material in isolation from the harmonic analysis structures to which it is applied. The main result of this section is that if every abelian subgroup of G is finite, then G is discrete. Recall that a topological space X is **connected** if it cannot be written as the disjoint union of non-empty open sets. Locally compact groups whose underlying topological space is connected are well understood.

Theorem 37. *Let G be a connected locally compact group. There exists a compact subgroup $K \subset G$, and subgroups $H_1, \ldots, H_n \subset G$, each isomorphic to the additive group $\mathbb{R}$, such that the map*

$$H_1 \times \cdots \times H_n \times K \mapsto G,\ (h_1, \ldots, h_n, k) \mapsto h_1 \cdots h_n k$$

is a homeomorphism.

Proof. This theorem and its proof can be found at the beginning of [69, 4.13]. □

For each $s \in G$, there exists a maximally connected subset of G containing s, called the **connected component of** s, denoted G_s. G_e is equal to the intersection of all open subgroups of G, and is itself a closed normal subgroup of G. For each $s \in G$, $G_s = sG_e$. Every group can be partitioned into connected components. G is **almost connected** if G/G_e is compact. Recall that a topological space X is a T_0**-space** if for each $x, y \in X$, there is either a neighborhood of x not containing y, or a neighborhood of y not containing x.

Proposition 38. *Let G be a locally compact group, and let H be a T_0 group. $f : G \to H$ be a continuous open homorphic map. Then $\overline{f(G_e)} = H_e$.*

Proof. This is Theorem 7.12 in [37]. □

We take special interest in a particular class of groups G for which G_e is open: namely that of Lie groups. For this we require some geometric notions. Recall that for subsets $U \subset \mathbb{R}^m$, and $V \subset \mathbb{R}^n$, a function $f : U \to V$ is **smooth** if all partial derivatives of all orders exist and are defined on all of U. A **differential manifold of dimension** n is a second countable Hausdorff space X with coordinate chart $\{(h_\alpha, U_\alpha)\}_{\alpha \in \mathbb{A}}$, where

- $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ is an open covering of X,
- each $h_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image, and
- the transition function $h_{\alpha,\beta} = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$ is smooth for each $\alpha, \beta \in \mathbb{A}$

If X and Y are differential manifolds with coordinate charts $\{(h_\alpha, U_\alpha)\}_{\alpha\in\mathbb{A}}$ and $\{(t_\beta, V_\beta)\}_{\beta\in\mathbb{B}}$, then $f : X \to Y$ is **smooth** if $t_\beta \circ f \circ h_\alpha^{-1}$ is smooth for all $\alpha \in \mathbb{A}, \beta \in \mathbb{B}$. A **Lie group** is a locally compact group G which has the structure of a differential manifold such that both group multiplication and inversion are smooth.

Definition 39. Let G be a locally compact group. G is a **pro-Lie group** if for all neighborhoods H of the identity $e \in G$, there exists a compact subgroup $K \subset H$ such that G/K is a Lie group.

The following is a theorem of Yamabe.

Theorem 40. *Let G be a locally compact group. There exists an open, almost connected subgroup $H \leq G$ which is a pro-Lie group.*

Proof. This is Theorem 12.2.15 in [50]. □

Definition 41. Let X and Y be topological spaces. Let $p : X \to Y$ be a continuous surjection. p is called a **covering map** if for all $y \in Y$ there exists an open $U \subset Y$ containing y and a collection $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ of disjoint open subsets of X, with

- $p^{-1}(U) = \bigcup_{\alpha\in\mathbb{A}} U_\alpha$, and
- for each $\alpha \in \mathbb{A}$, $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism.

It is immediate that such maps are local homeomorphisms. If a set U and a collection $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ satisfy Definition 41, we call the sets U_α **slices** and say that they **evenly cover** U.

Lemma 42. *Let G be a locally compact group, and H a Lie group. If $p : G \to H$ is a homomorphic covering map, then G is a Lie group.*

Proof. Let $\{(h_\beta, V_\beta)\}_{\alpha\in\mathbb{B}}$ be a coordinate chart for H. To construct a coordinate chart for G, let $s \in G$ be arbitrary, and let (h_β, V_β) be a patch with $p(s) \in V_\beta$. Let U be an open subset of H containing $p(s)$ which is covered evenly by slices $\{U_\alpha\}_{\alpha\in\mathbb{A}}$. Without loss of generality, assume $U \subset V_\beta$; and let $U_\alpha \subset G$ be the slice which contains s. Then $(h_\beta|_U \circ p|_{U_\alpha}, U_\alpha)$ is a coordinate patch with $s \in U_\alpha$. Repeating this process for all $s \in G$ yields an open cover $\{U_\alpha\}$ of G. It can be checked directly that the transition functions for these patches are nothing but restrictions of the transitions functions for the coordinate chart $\{(h_\beta, V_\beta)\}_{\alpha\in\mathbb{B}}$ and are hence smooth. By construction of this coordinate chart, p is smooth.

To see that multiplication in G is smooth, let $s, t \in G$, and choose neighborhoods A of s and and B of t sufficiently small so that $p|_{AB} : AB \to p(AB)$ is a homeomorphism. By construction of the coordinate chart on G, this restriction along with its inverse is smooth. Let $\cdot_G : G \times G \to G$ and $\cdot_H : H \times H \to H$ denote group multiplication in G and H respectively. Since p is a homomorphism, $p \circ \cdot_G = \cdot_H \circ (p \times p)$. Therefore, when restricted to $A \times B$, $\cdot_G$ is the composition of smooth functions $\left(p|_{AB}\right)^{-1} \circ \cdot_H \circ (p \times p)$. Therefore $\cdot_G$ is smooth on $A \times B$. Since $s, t \in G$ are arbitrary, $\cdot_G$ is smooth.

Similarly, if i_G and i_H are the inversion maps defined on G and H, we have $p \circ i_G = i_H \circ p$. Thus for arbitrary $s \in G$, choose a neighborhood A of s so that $p|_{A^{-1}} : A^{-1} \to p(A^{-1})$ is a homeomorphism. Then when restricted to A, i_G is the composition of smooth functions $\left(p|_{A^{-1}}\right)^{-1} \circ i_H \circ p$.

Hence G is in fact a Lie group. □

The main result of this section involves an application of Theorem 40, at which point it will be of use to observe that the quotient map $G \to G/K$ of a locally compact group by a finite subgroup cannot create the structure of a Lie group where before there was none. That is, if K is finite and G/K is a Lie group, then G must have already been one.

Lemma 43. *Let G be a locally compact group, and $K \leq G$ a finite normal subgroup. The quotient map $q : G \to K$ is a covering map.*

Proof. Write $K = \{k_1, \ldots, k_n\}$. Let $s \in G$, and choose open $V \subset G/K$ containing sK. By continuity of q, since G is Hausdorff we may choose an open $U \subset G$ containing e such that

- $sk_jU \cap sk_lU = \phi$ whenever $j \neq l$
- $q[sk_jU] \subset V$ for each $j = 1, \ldots, n$

Set $W = q(U)$. For each $j = 1, ..., n$, the restricted quotient map $q|_{sk_jU} : sk_jU \to sK \cdot W$ is a homeomorphism. Thus $sK \cdot W$ is evenly covered by slices $\{sk_1U, \ldots, sk_nU\}$. □

Given the structure of connected groups described by Theorem 37, we may always find an infinite subgroup of any connected group which is either compact or abelian. The following result due to Efim Zelmanov says we may always find one which is abelian.

Theorem 44. *Every infinite compact group has an infinite abelian subgroup.*

Proof. This is Theorem 2 in [68]. □

We may now state this Chapter's main result, which is the key which unlocks the main results of this paper.

Proposition 45. *Let G be a locally compact group with the property that every abelian subgroup is finite. Then G is discrete.*

Proof. By Theorem 40, we may choose an open almost connected subgroup H of G which is a pro-Lie group. Choose a compact normal subgroup K of H such that $e \in K$ and H/K is a Lie group. Let $\pi : H \to H/K$ denote the quotient map, which is a covering map by Lemma 43. By Lemma 41, H is also a Lie group and thus H_e is open. By Theorem 37, there exists abelian subgroups $H_1, \ldots, H_n \leq H_e$ and compact $C \leq H_e$ such that H_e is homeomorphic to $H_1 \times \cdots \times H_n \times C$. By assumption, and Theorem 44, H_e is finite. Since H_e is open in G and finite, G is discrete. □

A recurring strategy in chapters 4 and 6 will be to show that a property may not be possessed by any locally compact group which admits an infinite abelian subgroup. Proposition 45 will then imply that any group which *does* have it must be discrete.

Chapter 4

Ultra-operator Amenability

In [15], Matthew Daws introduced and investigated the idea of *ultra-amenability*: a Banach algebra A is **ultra-amenable** if $[A]_{\mathcal{U}}$ is amenable for every ultrafilter $\mathcal{U}$ on an index set $\mathbb{I}$. The concept originally drew its motivation from his joint work with Volker Runde in [60], to be used as a tool for establishing non-amenability for $\mathcal{B}(\ell^p)$ for $p \in (1, \infty), p \neq 2$. The property is very restrictive, as ultrapowers of Banach algebras are quite large. A property which asks every ultrapower of a Banach algebra to be amenable seems to be one we ought to expect to *fail*. Indeed finding non-trivial examples of Banach algebras which posess it has so far proven elusive. For instance, Daws shows in [15] that for any discrete group G, not only is $\ell^1(G)$ not ultra-amenable, but in fact the following much stronger claim.

Theorem 46. *Let G be an infinite discrete group and $\mathcal{U}$ be a countably incomplete ultrafilter on $\mathbb{I}$. $[\ell^1(G)]_{\mathcal{U}}$ is not amenable.*

Proof. This is Theorem 5.11 in [15]. □

For non-discrete groups, the following is known.

Theorem 47. *Let G be an infinite locally compact group which is either abelian or compact. $L^1(G)$ is not ultra-amenable.*

Proof. This is Theorem 5.9 in [15]. □

In this section we modify the definition of ultra-amenability to incorporate operator space structure: a completely contractive Banach algebra A is **ultra-operator amenable** if $[A]_{\mathcal{U}}$ is operator amenable for every ultrapower $\mathcal{U}$ on any index set $\mathbb{I}$. Following in the spirit of the foregoing work done by Daws, we ask when the Fourier algebra $A(G)$ of a locally compact group G may possess this property. In contrast to the question of ultra-amenability of $L^1(G)$, which finds a negative answer for all discrete G, we show that $A(G)$ fails to be ultra-operator amenable for all *non-discrete* G. The following observations allow us to make use of Theorem 47 to reduce the problem to the argument given in Chapter 3.

Proposition 48. *Let G be a locally compact group such that $[A(G)]_{\mathcal{U}}$ is operator amenable. If $H \leq G$ is a closed subgroup, then $[A(H)]_{\mathcal{U}}$ is operator amenable. In particular, if $A(G)$ is ultra-operator amenable, so is $A(H)$.*

Proof. Let $\mathcal{U}$ be a ultrafilter on an index set $\mathbb{I}$. Then the complete quotient map $\alpha : A(G) \to A(H)$ induces a complete quotient map $[\alpha]_{\mathcal{U}} : [A(G)]_{\mathcal{U}} \to [A(H)]_{\mathcal{U}}$. Since such maps preserve operator amenability, the result follows. □

Proposition 49. *Let $H \leq G$ be an infinite subgroup which is abelian. $A(G)$ is not ultra-operator amenable.*

Proof. Suppose first that $H \leq G$ is abelian. By replacing H with its closure if necessary, we may assume it is closed. By Theorem 47, there exists an ultrapower $\mathcal{U}$ on an index set $\mathbb{I}$ such that $\left[L^1(\widehat{H})\right]_{\mathcal{U}}$ is not amenable. Therefore since $[A(H)]_{\mathcal{U}}$ is maximal, it cannot be operator amenable. Thus neither can $[A(G)]_{\mathcal{U}}$ by Proposition 48. □

Combining Proposition 49 and Proposition 45, we arrive at the following.

Theorem 50. *Let G be a locally compact group such that $A(G)$ is ultra-operator amenable. Then G is discrete.*

4.1 Discrete Groups

When seeking to answer the question of whether an infinite group G exists with the property that that $A(G)$ is ultra-operator amenable, in light of Theorem 50, we naturally restrict our attention to discrete groups. It will be seen in this section that for a wide selection of infinite discrete groups, $A(G)$ yet still fails to be ultra-operator amenable. For the remainder of this section, we shall assume that G is a discrete group. We begin by assembling some algebraic facts and terminology.

Let Q and N be groups. G is an **extension of Q by N** if there exists a short exact sequence

$$1 \hookrightarrow N \hookrightarrow G \longrightarrow Q \longrightarrow 1$$

where 1 denotes a group with a single element.

Let G be a locally compact group, and let $H \subset G$. Recall that the **subgroup of G generated by H** is the smallest subgroup of G which contains H. We say that G is

- **periodic** if $\{s^n : n \geq 1\}$ is finite for all $s \in G$, and
- **locally finite** if all of its finite subsets generate finite subgroups.

By considering the finite set $\{s\}$ for each $s \in G$, it is immediate that all locally finite groups are periodic. However the converse fails to hold in general.

We now turn out attention to the category of **elementary amenable groups**, denoted by $\mathcal{E}$, which is the smallest subcategory (of the category of groups) satisfying the following properties:

- if G is abelian or finite, then $G \in \mathcal{E}$;
- If G is isomorphic to H, and $G \in \mathcal{E}$, then $H \in \mathcal{E}$;
- If G is an extension of H by N, and $N, H \in \mathcal{E}$, then $G \in \mathcal{E}$;
- If H is a subgroup of G or a quotient of G, and $G \in \mathcal{E}$, then $H \in \mathcal{E}$; and
- $\mathcal{E}$ is closed under **directed unions**: if (H_α) is a net of subgroups of G such that each $H_\alpha \in \mathcal{E}$ and $H_\alpha \leq H_\beta$ whenever $\alpha \leq \beta$, then $G \in \mathcal{E}$.

Ching Chou proved that periodicity and local finiteness are equivalent for all groups in $\mathcal{E}$.

Theorem 51. *If $G \in \mathcal{E}$ is periodic, then G is locally finite.*

Proof. This is [12, Proposition 2.3]. □

Note that G is any infinite locally finite group, then $A(G)$ is not ultra-operator amenable. This is a consequence of Corollary 49 as well as the following theorem by Philip Hall and C. R. Kulatilaka.

Theorem 52. *Every infinite locally finite group admits an infinite abelian subgroup.*

Proof. This is [34, Theorem I]. □

Corollary 53. *Let $G \in \mathcal{E}$ be infinite. Then $A(G)$ is not ultra-operator amenable.*

Proof. Suppose for a contradiction that G is finite with $A(G)$ ultra-operator amenable. Then G is periodic by Corollary 49 since $\{s^n : n \geq 1\}$ is an abelian subgroup of G whenever $s \in G$. Since $G \in \mathcal{E}$, G is locally finite by Theorem 51 and thus admits an infinite abelian subgroup by Theorem 52. This contradicts Corollary 49. □

We may also rule out an additional type of group from supplying infinite examples which have ultra-operator amenable Fourier algebras. Recall that, for a linear space V, the **general linear group** $\mathrm{GL}(V, \mathbb{F})$ is the group of all isomorphisms $V \to V$. A **linear group** G **over** $\mathbb{F}$ is a subgroup of $\mathrm{GL}(V, \mathbb{F})$ for some linear space V.

Recall that a group G is solvable if there exists $n \geq 1$ and a descending chain

$$G = G_1 \supset G_2 \supset \cdots \supset G_n = 1 \text{ such that}$$

- G_{k+1} is a normal subgroup of G_k for each $k = 1, \ldots, n-1$, and
- G_k/G_{k+1} is abelian.

Lemma 54. *Let G be amenable. If G is solvable or locally finite, then $G \in \mathcal{E}$.*

Proof. If G is solvable, it follows inductively that $G \in \mathcal{E}$. If G is locally finite, each of its finitely generated subgroups, of which it is a directed union, are in $\mathcal{E}$. □

The following theorem, along with its corollary, is due to Jacques Tits.

Theorem 55. *Let G be a finitely generated linear group. G either contains $\mathcal{F}_2$ or a solvable subgroup of finite index.*

Proof. This is [64, Corollary 1]. □

Note that if G contains a solvable group H of finite index, it also contains a normal solvable subgroup N which is contained in H and also has finite index. The following consequence is also due to J. Tits.

Corollary 56. *If G is a linear amenable group, then $G \in \mathcal{E}$.*

Proof. Since each group G is the directed union of its finitely generated subgroups, it is sufficient to show that the result holds whenever G is finitely generated. By Theorem 55 and the foregoing, we may choose a normal solvable subgroup N of G with finite index. Since $N \in \mathcal{E}$ and $G/N \in \mathcal{E}$, the exactness of

$$1 \hookrightarrow N \hookrightarrow G \longrightarrow G/N \longrightarrow 1$$

implies that $G \in \mathcal{E}$. □

Recall that the **center** of a group G, denoted $\mathcal{Z}(G)$, is the set $\{s \in G : st = ts \text{ for all } t \in G\}$. A **central series** for a group G is an ascending chain

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

with the property that

- G_k is a normal subgroup of G for each $k = 1, \ldots, n$,
- $G_k/G_{k+1} \subset \mathcal{Z}(G/G_{k+1})$ for all $k = 0, \ldots, n-1$.

A group G is **nilpotent** if it admits a central series. It may be checked directly that every nilpotent group is solvable.

Recall that for $p \geq 1$, a function $f : \mathbb{N} \to \mathbb{N}$ is in $\mathcal{O}(n^p)$ if there exists $M, m \geq 0$ such that

$$f(n) \leq M \cdot n^p$$

for all $n \geq m$. A group G is of **polynomial growth** if for every finite subset $F \subset G$, the function $f_F : \mathbb{N} \to \mathbb{N}$ is in $\mathcal{O}(n^p)$, where f_F is defined by $f_F(n) = |F^n|$ and

$$F^n = \{x_1 \cdots x_n : x_j \in F \text{ for each } j = 1, \dots, n\}.$$

The following is due to Mikhael Gromov.

Theorem 57. *Let G be a finitely generated group with polynomial growth. G admits a nilpotent subgroup of finite index.*

Proof. This is the main theorem of [32]. □

Corollary 58. *Let G be a group with polynomial growth such that $A(G)$ is ultra-operator amenable. Then G is finite.*

Proof. By Theorem 52, it is sufficient to show that G is locally finite. Let H be a finitely generated subgroup of G. It is immediate that H is also of polynomial growth. By Theorem 57, H admits a nilpotent subgroup of finite index. It follows that H admits a solvable normal subgroup of finite index. Thus $H \in \mathcal{E}$. By Corollary 53, H must be finite. □

We may summarize the work of this chapter to conclude the following

Theorem 59. *Let G be an infinite group such that $A(G)$ is ultra-operator amenable, then*

- *G is discrete and amenable,*
- *G is not of polynomial growth,*
- *$G \notin \mathcal{E}$, and*
- *G is not linear.*

It seems to be reasonable to put forth the conjecture that the only groups G for which $A(G)$ are ultra-operator amenable are the finite ones. It would be interesting to know if this is true; yet in this regard we are unable to draw any further concrete conclusions.

Chapter 5

Functionals upon the Space of Almost Periodic Functionals

As for the non-discrete case of the problem of whether a locally compact group may have an ultra-amenable group algebra, Daws shows that the group algebra of any infinite abelian or compact group is never ultra-amenable. A key ingredient in his argument is an observation in [16] (see the second remark after Proposition 5.5) by the same author, that there exists an ultrapower $\mathcal{U}$ on an index set $\mathbb{I}$ such that $\mathrm{AP}(G)'$ is isometrically isomorphic to a quotient of $\left[L^1(G)\right]_{\mathcal{U}}$. To prove this, the following technique by Stefan Heinrich is adapted. Recall the principle of local reflexivity, due to Haskell Rosenthal and Joram Lindenstrauss, which is worked out in detail in [57, Theorem 5.54].

Theorem 60. *Let E be a Banach space. For any finite dimensional subpsaces $M \subset E'', N \subset E'$ and $\epsilon > 0$, there exists $T \in \mathcal{B}(M, E)$ such that*

- $\|\Phi\| - \epsilon < \|T\Phi\| \leq \|\Phi\| + \epsilon$ *for all $x \in E$,*
- $\langle \varphi, T\Phi \rangle = \langle \Phi, \varphi \rangle$ *for all $\Phi \in M, \varphi \in N$, and*
- $T(\kappa_E(x)) = x$ *for all $x \in E$ with $\kappa_E(x) \in M$.*

Heinrich proved in [36] that this implies that the second dual of any Banach space E embeds isometrically into a carefully chosen ultrapower $[E]_{\mathcal{U}}$ as a complemented subspace. In the case of a Banach algebra A, the natural product on $[A]_{\mathcal{U}}$ thus provides a product on A'' which need not agree with either Arens product on A. In [16, Theorem 4.1 and 4.2], Daws adapts Heinrich's argument to show that if A is Arens regular, $\mathcal{U}$ can be chosen so that the resulting product on A'' agrees with both Arens products. This results in an ultrapower $\mathcal{U}$ and a quotient map $\sigma_{\mathcal{U}} : [A]_{\mathcal{U}} \to A''$ defined by

$$\langle \sigma_{\mathcal{U}} \left[a_i\right]_{\mathcal{U}}, \varphi \rangle = \lim_{i \to \mathcal{U}} \langle \varphi, a_i \rangle$$

which is an algebra homomorphism (note that if $\mathcal{U}$ is chosen arbitrarily, the above map is always well-defined and remains a contraction). He then remarks that if the Arens regularity assumption is dropped, the argument behind this fact may be adapted further to show that $\mathcal{U}$ may still be chosen to allow the map $\sigma_{\mathcal{U}}^{\mathrm{AP}} : [A]_{\mathcal{U}} \to \mathrm{AP}(A')'$ defined by

$$\sigma_{\mathcal{U}}^{\mathrm{AP}} = \sigma_{\mathcal{U}}|_{\mathrm{AP}(A')}$$

to be an algebra homomorphism. We now work out in full the details behind this.

5.1 AP$(A')'$ as a Quotient of $[A]_{\mathcal{U}}$

For Banach spaces E and F, $T \in B(E,F)$ is a $(1+\epsilon)$**-isomorphism onto its range** if

$$\|x\| - \epsilon \le \|Tx\| \le \|x\| + \epsilon$$

for every $x \in E$. For any Banach space E, $\mathrm{FIN}\,(E)$ shall denote the collection of finite-dimensional subspaces of E. If $M \in \mathrm{FIN}\,(E''), N \in \mathrm{FIN}\,(E')$, then an $(1+\epsilon)$-isomorphism $T \in \mathcal{B}(E,F)$ onto its range is an ϵ**-isomorphism along** N if $\langle \Phi, \mu \rangle = \langle \mu, T\Phi \rangle$ for all $\Phi \in M, \mu \in N$. For $S \subset E$ and $\epsilon > 0$, let $(S)_\epsilon := \{x \in E : \|x - y\| \le \epsilon$ for some $y \in S\}$.

Note that the Principle of local reflexivity may be stated using ultrapowers as follows.

Theorem 61. *Let E be a Banach space. There exists an ultrapower $\mathcal{U}$ on an index set $\mathbb{I}$ and an isometric embedding $K : E'' \hookrightarrow [E]_{\mathcal{U}}$ such that the $K(\kappa_E(x)) = [x]_{\mathcal{U}}$.*

Proof. This is Proposition 6.7 in [36]. □

The technique used to prove this was adapted by Matthew Daws to to prove Corollary 65, the details of which we shall now supply. First, a technical fact will be required, which can be found in [5]. For the remainder of this section, fix the following setting. Let E be a Banach space, $M \in \mathrm{FIN}\,(E'')$, and let $m, n \ge 1$. For each $1 \le i \le n$ and $1 \le j \le m$, let

- F_i and G_j be Banach spaces,
- $A_i : B(M,E) \to F_i$ and $\psi_j : B(M,E) \to G_j$ be bounded linear operators,
- $y_i \in F_i$, and
- $C_j \subset G_j$ be convex.

We shall say that

- M **satisfies the exact conditions** $(A_i, y_i)_{i=1}^n$ if for each $N \in \mathrm{FIN}\,(E'), \epsilon > 0$ there exists an ϵ-isomorphism T along N such that $A_i(T) = y_i$ for each $i = 1, \dots, n$;
- M **satisfies the approximate conditions** $(\psi_j, C_j)_{j=1}^m$ if for each $N \in \mathrm{FIN}\,(E'), \epsilon > 0$ there exists an ϵ-isomorphism T along N such that $\psi_j(T) \in (C_j)_\epsilon$ for each $j = 1, \dots, m$;

- M satisfies the exact conditions $(A_i, y_i)_{i=1}^n$ and the approximate conditions $(\psi_j, C_j)_{j=1}^m$ **simultaneously** if both of the above conditions are satisfied by virtue of *the same* operator $T \in \mathcal{B}(M, E)$.

By virtue of the tensor product identifications outlined in Section 1.2, we may treat A_i' and A_i'' as operators

$$F_i' \to M \otimes^\gamma E' \text{ and } \mathcal{B}(M, E'') \to F_i''$$

respectively. Let ι_M denote the inclusion $M \subset E''$. The following is Theorem 2.3 in [5], with advantage taken of the remark that succeeds it.

Theorem 62. *Suppose that the map* $\mathcal{B}(M, E) \to \bigoplus_{i=1}^n F_i$, $S \mapsto (A_i(S))_{i=1}^n$ *has closed range. The following are equivalent:*

- *M satisfies the exact conditions $(F_i, y_i)_{i=1}^n$ and the exact conditions $(G_j, C_j)_{j=1}^m$* simultaneously*.*

- *There exists $T \in \mathcal{B}(M, E)$ such that $A_i(T) = y_i, A_i''(\iota_M) = \kappa_{F_i}(y_i)$ for each $1 \leq i \leq n$; and $\psi_j''(\iota_M)$ is in the w^*-closure of $\kappa_{G_j}(C_j)$ for each $1 \leq j \leq m$.*

The advantage to Theorem 62 is that, at least for purposes at hand, the first statement is desirable as an assumption, whereas the second is straightforward to demonstrate.

5.2 Banach Algebras and Almost Periodic Functionals

By the end of this section we shall have constructed an ultrapower $\mathcal{U}$ on an index set $\mathbb{I}$ such that the dual of $\mathrm{AP}(A')$ is isometrically isomorphic (as a Banach algebra) to a quotient of $[A]_{\mathcal{U}}$. In order to accomplish this, we begin by constructing a collection of approximate conditions along with a single exact condition to which Theorem 62 will be applied. Let A be a Banach algebra with $M \in \mathrm{FIN}\,(A'')\,, N \in \mathrm{FIN}\,(\mathrm{AP}(A'))$. Note that $M_0 = M + M \Box M \in \mathrm{FIN}\,(A'')$ and define

$$A_{M_0} : B(M_0, A) \to B(\kappa_A(A) \cap M_0, A)$$

to be the restriction map (which is surjective and hence has closed range).

Let $B_{M_0} \in B(\kappa_A(A) \cap M_0, A)$ be the map given by

$$B_{M_0}(\kappa_A(x)) = x.$$

Since N is finite dimensional, its unit sphere is compact so for any $\delta > 0$ we may choose $(\mu_i)_{i=1}^n$ such that

$$\min_{i=1}^n \|\mu_i - \mu\| < \delta \text{ for each } \mu \in N \text{ with } \|\mu\| = 1.$$

For each $1 \leq i \leq n$, define $\psi_i : B(M_0, A) \to B(M_0, A')$ by

$$\psi_i(T)(\Phi) = T(\Phi) \cdot \mu_i \text{ for all } T \in B(M_0, A),\ \Phi \in M_0,$$

and $T_i \in B(M_0, A')$ by

$$T_i(\Phi) = \Phi \cdot \mu_i \text{ for all } \Phi \in M_0.$$

Let us agree to call such a choice of $(\mu_i, \psi_i, T_i)_{i=1}^n$ **subordinate** to δ.

Proposition 63. *M_0 satisfies the exact condition (A_{M_0}, B_{M_0}) and the approximate conditions $(\psi_i, \{T_i\})_{i=1}^n$ simultaneously.*

Proof. We proceed by applying Theorem 62 to the spaces $F = B(M_0 \cap \kappa_A(A), A)$, $G_i = B(M_0, A')$ for $i = 1, \ldots, n$. We now check that $\psi_i''(\iota_{M_0}) = \kappa_{G_i}(T_i)$. Observe that treating ψ_i' as a map $M_0 \otimes^\gamma A'' \to B(M_0, A)'$, we have

$$\begin{aligned}
\langle \psi_i'(\Phi \otimes \Psi), T\rangle &= \langle \Phi \otimes \Psi, \psi_i(T)\rangle \\
&= \langle \Psi, \psi_i(T)(\Phi)\rangle \\
&= \langle \Psi, T(\Phi).\mu_i\rangle \\
&= \langle \mu_i.\Psi, T(\Phi)\rangle \\
&= \langle \Phi \otimes \mu_i.\Psi, T\rangle
\end{aligned}$$

and thus $\psi_i'(\Phi \otimes \Psi) = \Phi \otimes \mu_i.\Psi$ for all $\Phi \in M_0, \Psi \in A''$. Therefore identifying $\psi_i''(\iota_{M_0})$ and $\kappa_{G_i}(T_i)$ with functionals on $M_0 \otimes^\gamma A''$, we see that

$$\begin{aligned}
\langle \psi_i''(\iota_{M_0}), \Phi \otimes \Psi\rangle &= \langle \iota_{M_0}, \psi_i(\Phi \otimes \Psi)\rangle \\
&= \langle \iota_{M_0}, \Phi \otimes \mu_i.\Psi\rangle \\
&= \langle \Phi, \mu_i.\Psi\rangle \\
&= \langle \Psi \Diamond \Phi, \mu_i\rangle \\
&= \langle \Psi \Box \Phi, \mu_i\rangle \\
&= \langle \Psi, \Phi.\mu_i\rangle \\
&= \langle \Psi, T_i(\Phi)\rangle \\
&= \langle \Phi \otimes \Psi, T_i\rangle \\
&= \langle \kappa_{G_i}(T_i), \Phi \otimes \Psi\rangle,
\end{aligned}$$

as required.

Lastly, it is instantaneous that $A''_{M_0}(\iota_{M_0})$ and $\kappa_F(B_{M_0})$ are equal to the inclusion

$$M_0 \cap \kappa_A(A) \hookrightarrow A''.$$

By Theorem 62, it follows that M_0 satisfies the exact condition (A_{M_0}, B_{M_0}) and the approximate conditions $(\psi_i, \{T_i\})_{i=1}^n$. $\square$

We now use this fact to create an analogue of Theorem 60 which allows us to ask the witnessing operator T to "approximately" behave as an algebra homomorphism when restricted to AP(A').

Theorem 64. *Let A be a Banach algebra. Let $M \in FIN(A'')$, $N \in FIN(AP(A'))$, $\hat{N} \in FIN(A')$, and $\epsilon > 0$. There exists an ϵ-isomorphism $T \in \mathcal{B}(M, A)$ along $\hat{N}$ such that*

- *$T(\kappa_A(a)) = a$ for all $\kappa_A(a) \in M \cap \kappa_A(A)$, and*
- *$|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi)\rangle| \leq \epsilon \|\mu\| \|\Phi\| \|\Psi\|$ for all $\mu \in N$, $\Psi, \Phi \in M$.*

Proof. Choose δ such that $\delta(1+\delta)(3+\delta) < \epsilon$, and $(\mu_i, \psi_i, T_i)_{i=1}^n$ to be subordinate to δ. Let $M_0 = \text{span}\{M, M\Box M\}$ and $N_0 = \text{span}\{\hat{N}, M \cdot \hat{N}, N\}$. By Theorem 63, there exists δ-isomorphism $T \in B(M_0, A)$ along N_0 such that

- $T(\kappa_A(a)) = a$ for all $a \in M_0 \cap \kappa_A(A)$, and
- $\|\psi_i(T) - T_i\| < \delta$ for each $1 \leq i \leq n$.

It remains only to show that

$$|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi)\rangle| \leq \epsilon \|\mu\| \|\Phi\| \|\Psi\| \text{ for all } \mu \in N,\ \Psi, \Phi \in M.$$

Let $\mu \in N$, $\Psi, \Phi \in M$ all be of norm 1. By the way μ_i's were chosen, we may choose j such that $\|\mu_j - \mu\| < \delta$. By the foregoing, since $\Phi \Box \Psi \in M_0$ and $\Psi.\mu \in \hat{N}_0$, we have

$$\begin{aligned}\langle \mu, T(\Phi \Box \Psi)\rangle &= \langle \Phi \Box \Psi, \mu\rangle \\ &= \langle \Phi, \Psi.\mu\rangle \\ &= \langle \Psi.\mu, T(\Phi)\rangle\end{aligned}$$

For each i, we also have

$$\|\psi_i(T)(\Psi) - T_i(\Psi)\| \leq \|\psi_i(T) - T_i\| \cdot \|\Psi\| \leq \|\psi_i(T) - T_i\| < \delta.$$

Therefore,

$$\begin{aligned}\|T(\Psi).\mu - \Psi.\mu\| &\leq \|T(\Psi).\mu - T(\Psi).\mu_i\| + \|T(\Psi).\mu_i - \Psi.\mu_i\| + \|\Psi.\mu_i - \Psi.\mu\| \\ &\leq \|T(\Psi)\| \cdot \|\mu - \mu_i\| + \|\psi_i(T)(\Psi) - T_i(\Psi)\| + \|\Psi\| \cdot \|\mu_i - \mu\| \\ &\leq (1+\delta) \cdot \delta + \delta + \delta \\ &= \delta(1+\delta) + 2\delta\end{aligned}$$

Finally, since $\mu \in N_0$ and $\Phi\Box\Psi \in M_0$,

$$\begin{aligned}
|\langle \mu, T(\Phi\Box\Psi) - T(\Phi)T(\Psi)\rangle| &= |\langle \mu, T(\Phi\Box\Psi)\rangle - \langle \mu, T(\Phi)T(\Psi)\rangle| \\
&= |\langle \Phi\Box\Psi, \mu\rangle - \langle T(\Psi).\mu, T(\Phi)\rangle| \\
&= |\langle \Phi, \Psi.\mu\rangle - \langle T(\Psi).\mu, T(\Phi)\rangle| \\
&= |\langle \Psi.\mu, T(\Phi)\rangle - \langle T(\Psi).\mu, T(\Phi)\rangle| \\
&< \|T(\Phi)\| \cdot \|\Psi.\mu - T(\Psi).\mu\| \\
&\leq (1+\delta)(\delta(1+\delta) + 2\delta) \\
&< \epsilon
\end{aligned}$$

The restriction $T|_M$ gives the desired operator, and completes the proof. □

In the spirit of Theorem 64 by Heinrich, Proposition 63 is now used to construct an index set for the desired ultrapower.

Corollary 65. *[16, Theorem 5.4] Let A be a Banach algebra. There exists an ultrafilter $\mathcal{U}$ and an isometric embedding $K : A'' \to (A)_{\mathcal{U}}$ such that*

- *$\sigma_{\mathcal{U}} \circ K$ is the identity on A'';*
- *$K \circ \kappa_A$ is the canonical map $A \to (A)_{\mathcal{U}}$;*
- *Let $\iota : A'' \to AP(A')'$ be the quotient (or restriction) map. For $\Phi, \Psi \in A''$, we have*

$$\iota(\Phi) \cdot \iota(\Psi) = \sigma_{\mathcal{U}}^{AP}(K(\Phi)K(\Psi)).$$

Proof. Define an index set by

$$I = \{(M, N, \hat{N}, \epsilon) : M \in \mathrm{FIN}\,(A''), N \in \mathrm{FIN}\,(\mathrm{AP}(A')), \hat{N} \in \mathrm{FIN}\,(A'), \epsilon > 0\}.$$

Define a partial order on $\mathbb{I}$ by letting $(M_1, N_1, \hat{N}_1, \epsilon_1) \leq (M_2, N_2, \hat{N}_2, \epsilon_2)$ if:

- $M_1 \subset M_2$,
- $N_1 \subset N_2$,
- $\hat{N}_1 \subset \hat{N}_2$, and
- $\epsilon_1 \geq \epsilon_2$.

Let $\mathcal{U}$ be an ultrafilter dominating the order filter on $\mathbb{I}$.

For $i \in \mathbb{I}$, write $i = (M_i, N_i, \hat{N}_i, \epsilon_i)$. By Theorem 64, choose an ϵ_i-isomorphism $T_i \in \mathcal{B}(M_i, A)$ along $\hat{N}_i$ such that

- $T(\kappa_A(a)) = a$ for all $\kappa_A(a) \in M \cap \kappa_A(A)$, and

- $|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi)\rangle| \leq \epsilon \|\mu\| \|\Phi\| \|\Psi\|$ for all $\mu \in N$, $\Psi, \Phi \in M$.

Define $K : A'' \to (A)_{\mathcal{U}}$ by setting $K(\Phi) = [x_i]_{\mathcal{U}}$ where

$$x_i = \begin{cases} T_i(\Phi) & \text{if } \Phi \in M_i \\ 0 & \text{otherwise} \end{cases}$$

By the choice of $\mathcal{U}$, we have $\{j \in \mathbb{I} : x_j = T_j(\Phi)\} \in \mathcal{U}$. Therefore the first two desired conditions are satisfied immediately.

For $\Phi, \Psi \in A''$, let $K(\Phi) = [x_i]_{\mathcal{U}}$, and $K(\Psi) = [y_i]_{\mathcal{U}}$, and $K(\Psi \Box \Psi) = [z_i]_{\mathcal{U}}$. By the second condition of Theorem 64, and by choice of $\mathcal{U}$, we have

$$\lim_{i \to \mathcal{U}} \langle \mu, x_i y_i - z_i \rangle = 0 \text{ for all } \mu \in \mathrm{AP}(A').$$

Thus for all $\mu \in \mathrm{AP}(A')$, we may compute

$$\begin{aligned} \left\langle \sigma_{\mathcal{U}}^{\mathrm{AP}}(K(\Phi)K(\Psi)), \mu \right\rangle &= \left\langle \sigma_{\mathcal{U}}^{\mathrm{AP}}\left([x_i y_i]_{\mathcal{U}}\right), \mu \right\rangle \\ &= \lim_{i \to \mathcal{U}} \langle \mu, x_i y_i \rangle \\ &= \lim_{i \to \mathcal{U}} \langle \mu, z_i \rangle \\ &= \left\langle \sigma_{\mathcal{U}}^{\mathrm{AP}}\left([z_i]_{\mathcal{U}}\right), \mu \right\rangle \\ &= \left\langle \sigma_{\mathcal{U}}^{\mathrm{AP}}(K(\Phi \Box \Psi)), \mu \right\rangle \\ &= \langle \iota(\Phi \Box \Psi), \mu \rangle \\ &= \langle \iota(\Phi) \cdot \iota(\Psi), \mu \rangle \end{aligned}$$

□

Using the ultrapower chosen in Corollary 65, this is now sufficient to express $\mathrm{AP}(A')'$ as a quotient of $[A]_{\mathcal{U}}$. To see this, we need one last observation, which nicely illustrates an attractive feature of almost periodic functionals.

Lemma 66. *Let $[x_i]_{\mathcal{U}}, [y_i]_{\mathcal{U}}, [w_i]_{\mathcal{U}}, [z_i]_{\mathcal{U}} \in [A]_{\mathcal{U}}$ be such that*

$$\lim_{i \to \mathcal{U}} \langle \mu, x_i \rangle = \lim_{i \to \mathcal{U}} \langle \mu, w_i \rangle \text{ and } \lim_{i \to \mathcal{U}} \langle \mu, y_i \rangle = \lim_{i \to \mathcal{U}} \langle \mu, z_i \rangle \text{ for all } \mu \in A'.$$

Then $\lim_{i \to \mathcal{U}} \langle \mu, x_i y_i \rangle = \lim_{i \to \mathcal{U}} \langle \mu, w_i z_i \rangle$ for all $\mu \in AP(A')$.

Proof. Let $\mu \in \mathrm{AP}(A')$, so that $\{\mu.x_i : i \in \mathbb{I}\}$ and $\{z_i.\mu : i \in \mathbb{I}\}$ are relatively compact. Set

$\varphi := \lim_{i\to\mathcal{U}} \mu.x_i$ and $\psi := \lim_{i\to\mathcal{U}} z_i.\mu$ which are well-defined by the fact that $\mu \in \mathrm{AP}(A')$. Then

$$\begin{aligned}
\lim_{i\to\mathcal{U}}\langle \mu, x_i y_i\rangle &= \lim_{i\to\mathcal{U}}\langle \mu.x_i, y_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \mu.x_i, y_i\rangle - \lim_{i\to\mathcal{U}}\langle \varphi, y_i\rangle + \lim_{i\to\mathcal{U}}\langle \varphi, y_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \varphi, z_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \varphi, z_i\rangle - \lim_{i\to\mathcal{U}}\langle \mu.x_i, z_i\rangle + \lim_{i\to\mathcal{U}}\langle \mu.x_i, z_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle z_i.\mu, x_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle z_i.\mu, x_i\rangle - \lim_{i\to\mathcal{U}}\langle \psi, x_i\rangle + \lim_{i\to\mathcal{U}}\langle \psi, x_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \psi, w_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \psi, w_i\rangle - \lim_{i\to\mathcal{U}}\langle z_i.\mu, w_i\rangle + \lim_{i\to\mathcal{U}}\langle z_i.\mu, w_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \mu, w_i z_i\rangle
\end{aligned}$$

as required. □

Proposition 67. $\sigma_{\mathcal{U}}^{AP}$ *is an algebra homomorphism.*

Proof. Let $[x_i]_{\mathcal{U}}, [y_i]_{\mathcal{U}} \in [A]_{\mathcal{U}}$ and let $[w_i]_{\mathcal{U}} = K(\sigma_{\mathcal{U}}[x_i]_{\mathcal{U}}), [z_i]_{\mathcal{U}} = K(\sigma_{\mathcal{U}}([y_i]_{\mathcal{U}})$. Since $\sigma_{\mathcal{U}} \circ K$ is the identity map, the hypotheses of Lemma 66 are satisfied and therefore for all $\mu \in \mathrm{AP}(A')$, we have

$$\begin{aligned}
\langle \sigma_{\mathcal{U}}^{\mathrm{AP}}([x_i]_{\mathcal{U}}) \cdot \sigma_{\mathcal{U}}^{\mathrm{AP}}([y_i]_{\mathcal{U}}), \mu\rangle &= \langle \iota\left[\sigma_{\mathcal{U}}([x_i]_{\mathcal{U}})\right] \cdot \iota\left[\sigma_{\mathcal{U}}([y_i]_{\mathcal{U}})\right], \mu\rangle \\
&= \langle \sigma_{\mathcal{U}}^{\mathrm{AP}}\left(K\left[\sigma_{\mathcal{U}}([x_i]_{\mathcal{U}})\right] K\left[\sigma_{\mathcal{U}}([y_i]_{\mathcal{U}})\right]\right), \mu\rangle \\
&= \langle \sigma_{\mathcal{U}}^{\mathrm{AP}}\left([w_i z_i]_{\mathcal{U}}\right), \mu\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \mu, w_i z_i\rangle \\
&= \lim_{i\to\mathcal{U}}\langle \mu, x_i y_i\rangle \\
&= \langle \sigma_{\mathcal{U}}^{\mathrm{AP}}\left[[x_i]_{\mathcal{U}}[y_i]_{\mathcal{U}}\right], \mu\rangle
\end{aligned}$$

□

Corollary 68. *Let A be a Banach algebra. There exists an ultrapower $\mathcal{U}$ on an index set $\mathbb{I}$ such that $AP(A')'$ is isometrically isomorphic to a quotient of $[A]_{\mathcal{U}}$.*

In the case of a completely contractive Banach algebra A, the space $\mathrm{AP}(A')'$ also has operator space structure. It is not obvious that $\sigma_{\mathcal{U}}^{\mathrm{AP}}$ should be a complete quotient map. However in the case where $A = M_*$ is the predual of a von-Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ we may draw a positive conclusion. In this case we may identify $\sigma_{\mathcal{U}}$ with a map $[M_*]_{\mathcal{U}} \to M'$ given by

$$\langle \sigma_{\mathcal{U}}[f_i]_{\mathcal{U}}, T\rangle = \lim_{i\to\mathcal{U}}\langle Tx_i, y_i\rangle$$

where each $x_i, y_i \in \mathcal{H}$ are chosen so that $\langle T, f_i\rangle = \langle Tx_i, y_i\rangle$ for all $T \in M$. Moreover, its adjoint $\sigma_{\mathcal{U}}'$ may be identified with a map $M'' \to [M]_{\mathcal{U}^*}$, which by weak-$*$ continuity may be computed as follows.

Let $\Phi \in M''$ and choose a net T_α in M such that $\kappa_M(T_\alpha)$ converges to Φ in $\sigma(M'', M')$. Then for all $x = [x_i]_{\mathcal{U}}, y = [y_i]_{\mathcal{U}} \in [\mathcal{H}]_{\mathcal{U}}$ we have

$$\begin{aligned}\langle \sigma'_{\mathcal{U}}(\Phi)x, y\rangle &= \lim_\alpha \langle \sigma'_{\mathcal{U}}(\kappa_M(T_\alpha))x, y\rangle \\ &= \lim_\alpha \lim_{i\to\mathcal{U}} \langle T_\alpha x_i, y_i\rangle.\end{aligned}$$

Proposition 69. *If A is the predual of a von-Neumann algebra, $\sigma_{\mathcal{U}}^{AP}$ is a complete quotient map.*

Proof. Since the restriction map $M' \to \mathrm{AP}(M)'$ is the adjoint of the completely isometric inclusion map $\mathrm{AP}(M) \subset M$, it is sufficient to check that $\sigma_{\mathcal{U}} : [M_*]_{\mathcal{U}} \to M'$ is a complete quotient map. By [22, Corollary 4.1.9], this is equivalent to checking that $\sigma'_{\mathcal{U}}$ is a complete isometry. Since $\sigma'_{\mathcal{U}}$ is clearly injective, it only remains to check that $\sigma'_{\mathcal{U}}$ preseves involution.

Let $\Phi \in M''$. Choose a net T_α in M so that $\kappa_M(T_\alpha)$ and $\kappa_M(T_\alpha^*)$ converge respectively to Φ and Φ^* in $\sigma(M'', M')$. Then for all $x = [x_i]_{\mathcal{U}}, y = [y_i]_{\mathcal{U}} \in [\mathcal{H}]_{\mathcal{U}}$ we have

$$\begin{aligned}\langle \sigma'_{\mathcal{U}}(\Phi^*)x, y\rangle &= \lim_\alpha \langle \sigma'_{\mathcal{U}}(\kappa_M(T_\alpha^*))x, y\rangle \\ &= \lim_\alpha \lim_{i\to\mathcal{U}} \langle T_\alpha^* x_i, y_i\rangle \\ &= \lim_\alpha \lim_{i\to\mathcal{U}} \langle x_i, T_\alpha y_i\rangle \\ &= \lim_\alpha \lim_{i\to\mathcal{U}} \overline{\langle T_\alpha y_i, x_i\rangle} \\ &= \overline{\lim_\alpha \lim_{i\to\mathcal{U}} \langle T_\alpha y_i, x_i\rangle} \\ &= \overline{\lim_\alpha \langle \sigma'_{\mathcal{U}}(\kappa_M(T_\alpha))y, x\rangle} \\ &= \overline{\langle \sigma'_{\mathcal{U}}(\Phi)y, x\rangle} \\ &= \langle x, \sigma'_{\mathcal{U}}(\Phi)y\rangle,\end{aligned}$$

as required. □

Corollary 70. *If A is the predual of a von-Neumann algebra, there exists an ultrafilter $\mathcal{U}$ on an index set $\mathbb{I}$ such that $AP(A')$ is completely isometric to the quotient of $[A]_{\mathcal{U}}$.*

As mentioned at the beginning of this chapter, the ultrapower constructed above and Corollary 68 were key components to one of Daws' results, namely that $L^1(G)$ fails to be ultra-amenable whenever G is infinite and abelian. It will also be of use in Chapter 6, where the weak analogues of both this statement and Theorem 50 are explored. The strength of Corollary 70 allows us not only to obtain a weak analogue for Theorem 50, but also to show in Section 6.1 that at least one ultrapower of $A(G)$ admits point derivations whenever G is non-discrete.

Chapter 6

Weak Analogues

It it worthy of note that by this point all of the results about ultra-amenability and ultra-operator amenability have been *negative*, which may not be surprising given the origins of the terms. It may seem prudent to ask whether we ought to expect any reasonble spaces to ever possess these properties. This motivates consideration of their weak analogues, which takes place in this chapter. We shall say that a (completely contractive) Banach algebra A is **ultra-weakly (operator) amenable** if $[A]_{\mathcal{U}}$ is weakly (operator) amenable for all ultrafilters $\mathcal{U}$ on $\mathbb{I}$. To justify why this is weak enough to yield hope for at least *some* positive results, observe that since $\mathcal{C}^*$-algebras are always weakly amenable [8], and every ultrapower of a $\mathcal{C}^*$-algebra is again a $\mathcal{C}^*$-algebra, we immediately arrive at the conclusion that every $\mathcal{C}^*$-algebra is ultra-weakly amenable. This provides a large enough class of examples which make the following questions interesting when G is an infinite locally compact group:

- Is $L^1(G)$ ever ultra-weakly amenable?
- Is $A(G)$ every ultra-weakly operator amenable?

In light of the fact that $L^1(G)$ is always weakly amenable [41] and $A(G)$ is always weakly operator amenable [63], these questions naturally present themselves. We answer the first question in the negative whenever G is abelian. The second question finds the same conclusion whenever G has an infinite abelian subgroup, which is sufficient to unlock the results of Chapter 3 to force a negative answer for all non-discrete groups. A nice little Corollary follows from this: if the Fourier algebra is ultra-weakly operator amenable, then the reduced Fourier stieltjes algebra is weakly operator amenable.

6.1 Ultra-weak Operator Amenability of $A(G)$

The goal of this section is to establish that if $A(G)$ is ultra-weakly operator amenable, then G is discrete. As in Chapter 4, we shall reduce the problem to the results of Chapter 3. That is, we shall show that this cannot occur if G has an infinite abelian subgroup. The technique, starting with such a subgroup H along with the observation that $\widehat{H_d}$ is non-discrete, will be to lift a point derivation on $M(\widehat{H_d})$ back through a chain of quotient maps back to $\mathrm{AP}(\widehat{G})$. This is an application of a technique illustrated by Brian Forrest in [28]. To proceed, we require some observations about representations of G_d for an arbitrary locally compact group G.

Taking the continuous composition of the left regular representation λ with the identity map $G_d \to G$, we obtain a representation of $\delta : G_d \to \mathcal{B}(L^2(G))$. Note the distinction between δ and the left regular representation of G_d, the latter of which has image in $\mathcal{B}(\ell^2(G))$. As in Section 1.12, $B_\delta(G_d)$ is a w^*-closed subspace of $B(G_d)$.

As the second dual of $A(G)$, the first and second Arens products turn $VN(G)'$ into a Banach algebra. As explained in [28, Section 2], the technique used to define the first Arens product also works to provide a product on the dual of a certain type of subspace of $VN(G)$, which we now outline. Recall from Section 1.8 the action of $A(G)$ upon $VN(G)$:

$$\langle f \cdot T, g \rangle = \langle T, fg \rangle$$

A subset $\mathcal{X} \subset VN(G)$ is **topologically invariant** if $f \cdot T \in \mathcal{X}$ whenever $f \in A(G)$ and $T \in \mathcal{X}$. If $\mathcal{X}$ is topologically invariant, then for each $\Phi \in \mathcal{X}', T \in \mathcal{X}$, we may define $\Phi \odot T \in VN(G)$ by

$$\langle \Phi \odot T, f \rangle = \langle \Phi, f.T \rangle \text{ for all } f \in A(G).$$

In this case, we say that $\mathcal{X}$ is **topologically introverted** if $\Phi \odot T \in \mathcal{X}$ whenever $\Phi \in \mathcal{X}', T \in \mathcal{X}$.

Let $\mathcal{X}$ be a topologically introverted subspace of $VN(G)$. Then $\mathcal{X}'$ is a Banach algebra under the product

$$\langle \Phi \hat{\odot} \Psi, T \rangle := \langle \Phi, \Psi \odot T \rangle \text{ for all } T \in \mathcal{X}.$$

Moreover, $K_\delta(\mathcal{X}) := \{\Gamma \in \mathcal{X} : \Gamma \circ \lambda = 0\}$ is a w^*-closed (two sided) ideal of $\mathcal{X}'$; and if $\mathcal{C}^*_\delta(G_d) \subset \mathcal{X}$, then $B_\delta(G_d)$ may be isometrically isomorphically identified with the algebra quotient $\mathcal{X}'/K_\delta(\mathcal{X})$ [28, Lemma 3.1]. By [21, Theorem 7.3] and [21, Chapter 8 (ii)], $\mathcal{C}^*_\delta(G_d) \subset \mathrm{AP}(\widehat{G})$. Since $\mathrm{AP}(\widehat{G})$ is topologically introverted [45, Lemma 7.1], $B_\delta(G_d)$ is thus an algebra quotient of $\mathrm{AP}(\widehat{G})'$. Note that for any subgroup H of G, the representation $\delta : G_d \to \mathcal{U}(L^2(G))$ can be restricted to $\delta : H_d \to \mathcal{U}(L^2(G))$. Note the distinction between this and the representation of H_d on $L^2(H)$, which will not be considered here.

Lemma 71. *Let G be a locally compact group. Let H be a subgroup of G. $B_\lambda(H_d) \subset B_\delta(H_d)$.*

Proof. Since $B_\delta(H_d)$ is w^*-closed (see Section 1.12), it is enough to show that $A(H_d) \subset B_\delta(H_d)$. To see this, we first check that $B_\delta(H_d)$ contains the point masses. Let $\{E_\alpha\}_{\alpha \in \mathbb{A}}$ be a compact neighborhood basis for G. For each $\alpha \in \mathbb{A}, t \in H$, define

$$F^t_\alpha = \frac{1}{\sqrt{m_G(E_\alpha)}} \chi_{tE_\alpha}.$$

For each $\alpha \in \mathbb{A}$, $F_\alpha \in L^2(G)$ by compactness of E_α. Thus $\delta_{F_\alpha^e, F_\alpha^t} \in B_\delta(H_d)$. Since H is Hausdorff, it may be checked directly that $\delta_{F_\alpha^e, F_\alpha^t}$ is bounded and converges in the w^*-topology of $B(H_d)$ to the point mass at t. It immediately follows that $B_\delta(H_d)$ contains all $f \in A(H_d)$ with finite support. Since such functions are dense in $A(H_d)$, this completes the proof. □

Lemma 72. *Let G be a locally compact group. Let H be an abelian subgroup of G. Then there is a quotient map (of Banach algebras) $B_\delta(G_d) \to B_\delta(H_d)$.*

Proof. The adjoint of the inclusion map $\mathcal{C}_\delta^*(H_d) \hookrightarrow \mathcal{C}_\delta^*(G_d)$ is the restriction map $B_\delta(G_d) \to B_\delta(H_d)$, which contains $B_\lambda(H_d)$ by Lemma 71. Since H_d is abelian and thus amenable, the result follows. □

We may now state the main result of this section.

Theorem 73. *Let G be a locally compact group which admits an infinite abelian subgroup. $AP(\widehat{G})'$ admits point derivations. In particular, $AP(\widehat{G})$ is not weakly operator amenable.*

Proof. Take an infinite abelian subgroup $H \subset G$ by Proposition 45. Since H_d is not compact, $\widehat{H_d}$ is not discrete. We may thus choose a point derivation $M(\widehat{H_d}) \to \mathbb{C}$. Using the Fourier-Stieltjes transform, this yields a point derivation $B(H_d) \to \mathbb{C}$. By Lemmas 72 and the foregoing realization of $B_\delta(G_d)$ as an algebra quotient of $\mathrm{AP}(\widehat{G})'$, we arrive at the required point derivation on $\mathrm{AP}(\widehat{G})'$. □

The following corollaries are immediate when we apply Proposition 69 and Proposition 45, respectively.

Corollary 74. *Let G be a locally compact group which admits an infinite abelian subgroup. $A(G)$ is not ultra-weakly operator amenable.*

Corollary 75. *Let G be a non-discrete locally compact group. $A(G)$ is not ultra-weakly operator amenable.*

We also have the following consequence for the reduced Fourier-Stieltjes algebra.

Corollary 76. *If $A(G)$ is ultra-weakly operator amenable, then $B_\lambda(G)$ is weakly operator amenable.*

Proof. If $A(G)$ is ultra-weakly operator amenable, then $\mathrm{AP}(\widehat{G})$ is weakly operator amenable, and G is discrete by Corollary 75. By the proof of Theorem 73, $B_\delta(G_d)$ is a quotient of $\mathrm{AP}(\widehat{G})$. Since G is discrete, we have $B_\lambda(G) = B_\delta(G_d)$. □

The two main results of this paper, namely Theorem 50 and Corollary 75 have proofs which use a similar strategy. Beginning with the assumption that $A(G)$ is ultra-operator amenable (or ultra-weakly operator amenable) it is shown that G may not have any infinite abelian subgroup. From here Proposition 45 forces G to be discrete. It would be interesting to know if $A(G)$ is ever ultra-weakly operator amenable for an infinite G.

6.2 Ultra-weak Amenability of $L^1(G)$

We now demonstrate that the group algebra of an infinite abelian G, is never ultra-weakly amenable. In order to do this, we make use of the Bohr compactification of G^{AP} of G, which will be shown to be discrete whenever $L^1(G)$ is ultra-weakly amenable. Since an abelian groups embeds into their Bohr compactifications (see section 1.7) this forces G to be finite.

To see the details behind this argument, it is advantageous to make use of the fact that G^{AP} may be realized as the character space of $\mathrm{AP}(G)$. Recall that the Gelfand transform

$$\Gamma : \mathrm{AP}(G) \to C(G^{\mathrm{AP}})$$

of the Banach algebra $\mathrm{AP}(G)$ is a algebraic $*$-isomorphism. It's adjoint

$$\Gamma' : M(G^{\mathrm{AP}}) \to \mathrm{AP}(G)'$$

is thus a surjective isometry. Observe that it is also an algebra homomorphism. Indeed we may compute

$$\begin{aligned}
\langle \Gamma'(\mu * \nu), f\rangle &= \int_{G^{\mathrm{AP}}} \int_{G^{\mathrm{AP}}} \langle \varphi \cdot \psi, f\rangle d\mu(\varphi) d\nu(\psi) \\
&= \int_{G^{\mathrm{AP}}} \int_{G^{\mathrm{AP}}} \langle \varphi, \psi \cdot f\rangle d\mu(\varphi) d\nu(\psi) \\
&= \int_{G^{\mathrm{AP}}} \langle \Gamma'(\mu), \psi \cdot f\rangle d\nu(\psi) \\
&= \int_{G^{\mathrm{AP}}} \langle \Gamma'(\mu) \hat{\Box} \psi, f\rangle d\nu(\psi) \\
&= \int_{G^{\mathrm{AP}}} \langle \psi, f \cdot \Gamma'(\mu)\rangle d\nu(\psi) \\
&= \langle \Gamma'(\nu), f \cdot \Gamma'(\mu)\rangle \\
&= \langle \Gamma'(\mu) \hat{\Box} \Gamma'(\mu)\rangle
\end{aligned}$$

If G is infinite, then so is G^{AP} which is thus non-discrete, forcing $M(G^{\mathrm{AP}})$, and hence $\mathrm{AP}(G)'$, to admit point-derivations. By Corollary 68, since $\left[L^1(G)\right]_{\mathcal{U}}$ is commutative, we have the following.

Proposition 77. *If G is an infinite abelian locally compact group, $L^1(G)$ is not ultra-weakly amenable.*

It is tempting to expect this result to imply that a general locally compact group G must be discrete in order for $L^1(G)$ to be ultra-weakly amenable. But in fact this is not the same as establishing an analogue of Proposition 49 for weak amenability of $L^1(G)$, which rules out infinite abelian *subgroups*; and doing so is problematic, since for a non-open subgroup H of G, m_H and m_G need not have any relation. This prevents us from reducing $L^1(G)$ to $L^1(H)$ (when H is not open) as is done with the Fourier algebra in Proposition 48. The question phrased by Matthew Daws which motivated this entire study, namely whether there exists an infinite group with an ultra-amenable group algebra, yet remains open.

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