Geometric and Graphical Study of 1-Dimensional N-Extended Supersymmetry Algebras

by

Minako Chinen

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Department of Mathematical and Statistical Sciences

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Abstract

In this thesis, we explore the relationship between graphical representations of the 1-dimensional N-extended supersymmetry algebra called *Adinkras*, compact Riemann surfaces, and quivers. An Adinkra is a graph which was originated in physics to study off-shell representations of the supersymmetry algebra. We focus on $N = 4$ Adinkras in this thesis. From a mathematical perspective, Adinkras are dessins d'enfants. Using this fact, we explain Adinkras as branched covering spaces for particular dessins. We also demonstrate how to generate a quiver Q_A from an Adinkra A and relate Q_A to the noncommutative generalization of Calabi-Yau varieties called Calabi-Yau algebras. More precisely, we construct a Jacobi algebra of Q_A with a superpotential. However, in general, a Jacobi algebra generated in this way need not be a Calabi-Yau algebra. We show that Jacobi algebras of quivers constructed from Adinkras are Calabi-Yau algebras of dimension 3. We also discuss Jacobians of the Riemann surfaces in which Adinkras are embedded using isogenous decompositions of Jacobians of the Adinkra Riemann surfaces.

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Introduction

An Adinka A is a graph introduced by physicists Faux and Gates to study off-shell representations of the 1-dimensional N-extended supersymmetry algebra. Namely, it is an N-regular bipartite graph whose edges are colored by N different colors. Adinkras are also equipped with odd-dashings and height assignment to capture all the information needed to encode the off-shell representations. However, for this thesis, our interest is Adinkras without odd-dashings and height assignment. We call such "pre-Adinkras" *Adinkra chromotopologies*. For the rest, we use Adinkras to mean Adinkra chromotopologies.

A simple example of an Adinkra is an N-dimensional hypercube A_N. Since vertices of N-hypercubes are expressed as elements of the vector space $\mathbb{F}_2^N$, one can construct a quotient graph of the N-hypercube by a subspace C of $\mathbb{F}_2^N$. According to [**DFGHILM**], the subspace C has to be a doubly even code to make A_N/C an Adinkra. Throughout the thesis, we denote such a quotient Adinkra by $A_{N,k}$ where k is a dimension of the doubly even code C. By definition, we see that $A_N = A_{N,0}$.

From a mathematical point of view, an adinkra is a *dessin d'enfant* (or *dessins* for short) which is a bipartite graphs A embedded in an oriented topological surface X [**DIKLM**]. We denote a dessin by (A, X). Note that any dessin can be embedded in a compact Riemann surface X [**JW**]. In this case, the Riemann suface X (when viewed as an algebraic curve) has to be defined over the field of algebraic numbers, or equivalently X admits a Belyi function $\beta : X \to \mathbb{P}^1$ that is a meromorphic function ramified at most over $\{0, 1, \infty\}$ (Belyi's theorem).

Considering Adinkras as dessins, we see them as either (A_N, X_N) or $(A_{N,k}, X_{N,k})$. We often call the Riemann surfaces X_N and $X_{N,k}$ *Adinkra Riemann surfaces*. In some situations, we view X_N and $X_{N,k}$ as Adinkras A_N and $A_{N,k}$ embedded in the surface X_N and $X_{N,k}$ respectively.

Doran et al. applied Belyi's theory to Adinkras and formulated covering space theory on Adinkras [**DIKLM**]. One of their results says that we have the following series of branched covering maps

$$X_N \to X_{N,k} \to B_N \to \mathbb{P}^1$$

where B_N is a dessin (Σ, S^2) consisting of a graph Σ with one black and one white vertex and N edges connecting them and the sphere S^2 as the underlying Riemann surface. We note that B_N is not an Adinkra because it is a quotient of A_N by a non doubly even code.

In this thesis, we use results from [**DIKLM**] and Bocklandt's work on quivers and dimer models [**B3**][**B4**][**B6**] to extend the study of $N = 4$ Adinkra chromotopologies. We are expecially interested in the $N = 4$ case because A_4 and $A_{4,k}$ are embedded in a torus as dessins, and Adinkras have an unexpected behavior when N is a multiple of 4. Our goals are to relate Adinkras to (noncommutative) Calabi-Yau variety and to explore the Riemann surfaces in which Adinkras are embedded. In addition to the fact that Adinkras are dessins, $N = 4$ Adinkras $A_4, A_{4,k}$ are especially *dimer models* which are bipartite graphs embedded in a torus. Since a torus is topologically equivalent to a square with opposite sides identified, the $N = 4$ Adinkras generate tilings on the torus. Taking the dual graph of A_4 or $A_{4,k}$, we get a directed graph called a quiver Q. We can define a relation W on Q called a *superpotential* which is a sum of counterclockwise cycles in Q minus a sum of clockwise cycles in Q. The quotient of the path algebra $\mathbb{C}Q$ by an ideal generated by taking cyclic derivatives of W

$$\mathbb{C}Q/\langle \partial_a W \mid a \in Q_1 \rangle$$

is called a *Jacobi algebra* of (Q, W). Remark that some Jacobi algebras appear as Calabi-Yau algebras of dimension 3. Roughly speaking, a Calabi-Yau algebra introduced by Ginzburg [**G**] is a noncommutative generalization of Calabi-Yau varieties. There are many methods for checking if a given Jacobi algebra of a pair (Q, W) is 3-Calabi-Yau. For our case of $N = 4$ Adinkras, we use the zigzag consistency condition given by Bocklandt [**B6**] to verify that Jacobi algebras for (A_4, W) and $(A_{4,k}, W)$ are 3-Calabi-Yau.

For the other goal, we explain the work done by Doran et al. [**DIKM**] on Jacobians of the Adinkra Riemann surfaces $X_N, X_{N,k}$ associated to $A_N, A_{N,k}$ respectively. They found that the hypercube Adinkras X_N are *generalized Humbert curves* which are a generalization of classical Humbert curves. Formally, a generalized Humbert curve is a compact Riemann surface X together with group $Aut(X) \cong (\mathbb{Z}/2\mathbb{Z})^k$ of conformal automorphisms such that the orbifold $X/Aut(X)$ is a sphere with $k+1$ singular points. Furthermore, by allowing $Aut(X)$ to be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^k$ with integer $p \geq 2$, we obtain a generalization of generalized Humbert curves called *generalized Fermat curves*. Carvacho, Hidalgo, and Quispe developed an isogenous decomposition theorem for Jacobians of generalized Fermat curves for prime integer $p \geq 2$ [**CHQ**], and Doran et al. were able to construct the Adinkra version of the isogenous decomposition theorem. Using this theorem specialized for the case of Adinkras, we give some examples for the isogenous decompositions of Jacobians for $N = 4, 5, 6$ Adinkra Riemann surfaces.

Chapter 1

Introduction to Adinkras

1.1 Physical Motivation to Adinkra

Elementary particles are the smallest particles which comprise physical matters, and they are divided into two groups: fermions and bosons. For example, photons belong to bosons, and electrons and quarks (the particles which compose neutrons and protons) belong to fermions.

Supersymmetry is a symmetry between bosons and fermions (i.e. invariance under exchanging bosons and fermions). Hence with supersymmetry, exchanging bosons and fermions does not produce any change of physical theory behind.

Roughly speaking, an Adinkra which was first studied in theoretical physics is a graph which explains such transformations between bosons and fermions. In this section, we want to give an elementary introduction to Adinkras from a physicist's perspective, and the content is based on [**Z**] and [**DFGHILM**].

1.1.1 Super Poincaré Lie Algebra

A *super vector space* is a vector space V with $\mathbb{Z}_2$-grading, that is,

$$V = V_0 \oplus V_1.$$

Elements in V_0 and V_1 are called *even* and *odd* respectively. We say the *parity* of $v \in V$ is 0 if $v \in V_0$, and 1 if $v \in V_1$. The dimension of a super vector space V is written by either (p, q) or $p|q$ where $\dim V_0 = p$ and $\dim V_1 = q$.

In the supersymmetry setting, we have an operator called *parity functor* Π which swaps even and odd components of a given super vector space $V = V_0 \oplus V_1$:

$$(\Pi V)_0 = V_1, \qquad (\Pi V)_1 = V_0$$

By considering a non-graded vector space W as $W = W_0$, we often use the parity functor Π to redefine W as W_1.

A *Lie superalgebra* (or *super Lie algebra*) is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

1. $[x, y] = (-1)^{p(x)p(y)}$[y, x]

2. $[x, [y, z]] = (-1)^{p(x)p(y)+p(x)p(z)}[y, [z, x]] + (-1)^{p(x)p(z)+p(y)p(z)}[z, [x, y]]$

Here $p(x)$ means the parity of $x \in \mathfrak{g}$. In particular, we are interested in the following Lie superalgebras which has a single even generator and several odd generators.

Definition 1.1.1 ([**DFGHIL**], [**Z**])**.** The N-extended *super Poincaré algebra* in 1-dimensional Minkowski space is the real Lie superalgebra

$$\mathfrak{po}^{1|N} = \mathbb{R}H \oplus \Pi Span\{Q_1, \ldots, Q_N\}$$

where $H = i\partial_t$ is a single even generator, and $Q_1, \ldots Q_N$ are odd generators (called *supersymmetry generators* which transform bosons to fermions and vice versa) such that

$$\{Q_I, Q_J\} = 2\delta_{IJ}H \tag{1.1}$$
$$[H, Q_I] = 0 \tag{1.2}$$

for $I, J = 1, \ldots, N$.

Note that δ_{IJ} is the Kronecker delta, $\{X, Y\} = XY + YX$ is the anticommutator, and $[X, Y] = XY - YX$ is the commutator. In 1-dimensional Minskowski space, there is a time-like direction t, and the symbol ∂_t denotes the translation in the direction t. We also use the parity functor Π to emphasize that the vector space $Span\{Q_1, \ldots, Q_N\}$ is an odd component of $\mathfrak{po}^{1|N}$. For more detail, we refer to [**DFGHIL**]. Note that we often refer to $\mathfrak{po}^{1|N}$ as the *supersymmetry algebra*.

1.1.2 Supermultiplets

We are interested in representations of $\mathfrak{po}^{1|N}$ called *supermultiplets*.

Definition 1.1.2 ([**DFGHILM**],[**Z**])**.** A real *supermultiplet* is a real linear representation of $\mathfrak{po}^{1|N}$, that means, a Lie superalgebra homomorpshim

$$\begin{aligned} \rho : \mathfrak{po}^{1|N} &\to End(\mathcal{M}) \\ H &\mapsto \rho(H) : \mathcal{M} \to \mathcal{M} \\ Q_I &\mapsto \rho(Q_I) : \mathcal{M} \to \mathcal{M} \end{aligned}$$

Furthermore, $\mathcal{M}$ is a $\mathbb{Z}_2$-graded vector space spanned by bosonic fields (or bosons) $\phi_1, \ldots, \phi_m$ and fermionic fields (or fermions) $\psi_1, \ldots, \psi_m$. Bosonic and fermionic fields all together are called *component fields*. We write $\rho(H)(s) := Hs$ and $\rho(Q_I)(s) := Q_I s$ where s is a component field. Most of the time, we call $\mathcal{M}$ a supermultiplet.

Especially we are interested in an *off-shell* supermultiplet which means the component fields ϕ_i, ψ_j do not satisfy any differential equations other than the equations (1.1) and (1.2). In off-shell representations, the number of bosons and the number of fermions are the same [**DFGHILM**].

An *Adinkra* is a graphical tool introduced by two physicists Faux and Gates to study off-shell representations (i.e. supermultiples) of $\mathfrak{po}^{1|N}$. It describes actions of N supersymetry generators $Q_1, \ldots, Q_N$ on bosons $\phi_1, \ldots, \phi_m$ and fermions $\psi_1, \ldots, \psi_m$. Roughly speaking, an Adinkra is a bipartite graph with a fixed collection of N edge colors such that

- edges are either dashed or solid
- edges are oriented
- each edge is colored by one of the N colors

An edge of color I corresponds to an action of the supersymmetry generator Q_I on component fields. Their vertex bipartition corresponds to bosons and fermions: black vertices for fermions and white vertices for bosons. We will give a mathematical definition of Adinkras in the next section.

Supermultiplets that can be represented by Adinkras have component fields and supersymmetry generators $Q_1, \ldots, Q_N$ which satisfy the following conditions [**DFGHILM**]:

- Given a bosonic field ϕ, a supersymmetry generators Q_I acting on ϕ is defined by either

$$Q_I \phi = \pm \psi, \ \ or \tag{1.3}$$
$$Q_I \phi = \pm \partial_\tau \psi \tag{1.4}$$

  for some fermonic field ψ.

- Given a fermionic field ν, a supersymmetry generator Q_I acting on ν is defined by either

$$Q_I \nu = \pm i B, \ \ or \tag{1.5}$$
$$Q_I \nu = \pm i \partial_\tau B \tag{1.6}$$

  for some bosonic field B.

Moreover, we call a supermultiple that has a corresponnding Adinkra expression *adinkraic*. The following definition summarizes the conditions for a supermultiplet to be adinkraic.

Definition 1.1.3. A supermultiplet $\mathcal{M}$ is called *adinkraic* if all of its supersymmetric transformation rules are of the following form: for fixed integers $1 \leq I \leq N$ and $1 \leq A \leq m$,

$$Q_I \phi_A = c \partial_t^{\lambda} \psi_B \tag{1.7}$$

$$Q_I \psi_B = \frac{i}{c} \partial_t^{1-\lambda} \phi_A \tag{1.8}$$

where $c \in \{-1, 1\}, \lambda \in \{0, 1\}$, and integer B with $1 \leq B \leq m$.

Note that there are non-adinkraic supermultiplets which do not satisfy the equations (1.7) and (1.8) [**DHIL**], but we do not consider them in this thesis. Also note that the supersymmetry algebra forces Q_I to map bosons ϕ and fermions ψ in the following way [**DFGHILM**]:

- $Q_I \phi = \pm \psi \iff Q_I \psi = \pm i \partial_\tau \phi$ (i.e. $\lambda = 0$ in both (1.7), (1.8))
- $Q_I \phi = \pm \partial_\tau \psi \iff Q_I \psi = \pm i \phi$ (i.e. $\lambda = 1$ in both (1.7), (1.8))

With these rules, we obtain the correspondence between Adinkras and adinkraic supermultiplets which we describe next.

1.1.3 Dictionary between Adikras and Adinkraic Supermultiplets

For any adinkraic supermultiplet $\mathcal{M}$, we can associate it with an Adinkra. Here is a dictionary between Adinkras and adinkraic supermultiplets:

- vertex bipartition corresponds to composition fields: black for fermions and white for bosons
- dashed edges corresponds to signs of Q_I; an edge is dashed if $c = -1$, and not dashed if $c = 1$
- orientation of edges corresponds to existence of ∂_τ; an edge goes from the vertex for ϕ_A (i.e. boson = white vertex) to the vertex for ψ_B (i.e. fermions = black vertex) if $\lambda = 0$, and an edge goes from the vertex for ψ_B to the vertex for ϕ_A if $\lambda = 1$.

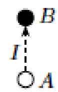
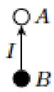
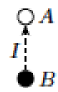
Adinkra	Q-action	Adinkra	Q-action
	$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$
	$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} \dot{\psi}_B \\ i\phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -\dot{\psi}_B \\ -i\phi_A \end{bmatrix}$

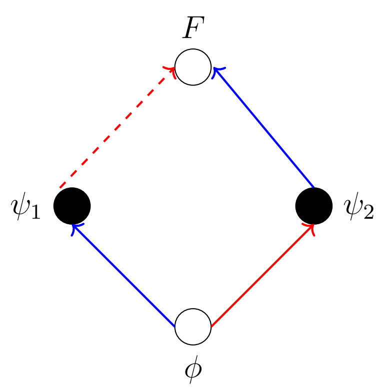
Figure 1.1: Visualization of the correspondence between Adinkras and action of supersymmetry generators (taken from [**DFGHILM**]); top left $c = 1, \lambda = 0$, top right $c = -1, \lambda = 0$, bottom left $c = 1, \lambda = 1$, bottom right $c = -1, \lambda = 1$

The example below demonstrates how an adikraic supermultiplets produces its corresponding Adinkra.

Example 1.1.4 (Adinkraic supermultiplet). A supermultiplet whose generators bosons ϕ and fermions ψ satisfy the set of rules given below:

$$\begin{aligned} Q_1\phi &= \psi_1 & Q_2\phi &= \psi_2 \\ Q_1\psi_1 &= i\partial_\tau\phi & Q_2\psi_1 &= -iF \\ Q_1\psi_2 &= iF & Q_2\psi_2 &= i\partial_\tau\phi \\ Q_1 F &= \partial_\tau\psi_2 & Q_2 F &= -\partial_\tau\psi_1 \end{aligned}$$

corresponds to the following Adinkra:



Each line of the Adinkra graph is represented by two of the eight equations as follows;

- The bottom left (blue, solid) is represented by $Q_1\phi = \psi_1$ and $Q_1\psi_1 = i\partial_\tau\phi$.
- The bottom right (red, solid) is represented by $Q_2\phi = \psi_2$ and $Q_2\psi_2 = i\partial_\tau\phi$.
- The top left (red, dashed) is represented by $Q_2\psi_1 = -iF$ and $Q_2F = -\partial_\tau\psi_1$.

- The top right (blue, solid) is represented by $Q_1\psi_2 = iF$ and $Q_1 F = \partial_\tau \psi_2$.

1.2 Mathematical View of Adinkras

A mathematical approach to Adinkras depends on graph theory. The rest of the thesis is based on the materials introduced in this and next section. Our discussion in sections 2 and 3 is also adapted from [**DFGHILM**] and [**Z**].

1.2.1 Mathematical Definition of Adinkras

As mathematicians, we define Adinkras as graphs which satisfy the following axioms.

Definition 1.2.1. An *Adinkra* is a finite connected graph satisfying the following conditions:

- N-regular: valency of each vertex of an Adinkra is N
- Bipartite: the set of vertices of an Adinkra can be divided into two disjoint subsets, a subset consisting of white vertices and a subset consisting of black vertices
- Odd-dashing: every 2-colored cycle of length 4 contains an odd number of dashed edges
- Height assignment [1]: a function $h : \{\text{vertices of Adinkra}\} \to \mathbb{Z}$ such that for any edge e going from a to b, $h(b) = h(a) + 1$
- Edge N-partite: each edge of an Adinkra belongs to N disjoint sets $E_1, \ldots, E_N$ of edges such that each vertex is incident to edges from each E_i

Example 1.2.2 (Hypercube Adinkras, [**Z**]). The simplest examples of Adinkras are N-dimensional hypercubes $[0,1]^N$. The Adinkra in Example 1.1.4 is an example of $N = 2$ hypercube Adinkra. For $N = 3$, we have a 3-dimensional cube whose each edge is colored by one of three colors red, blue, and green. We can define bipartition of vertices as follows; define $\{000, 011, 101, 110\}$ to be a set of bosons and $\{001, 010, 100, 111\}$ to be a set of fermions.

[1] The orientation of Adinkra edges (which is controlled by the value of λ) defines a height function on vertices of Adinkras [**DILM**].

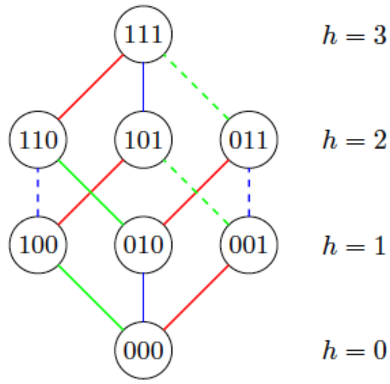


Figure 1.2: 3-cube Adinkra taken from [**Z**]

Example 1.2.3. (Valise form [**Z**]) Consider 3-cube Adinkra without odd-dashing (for simplicity). We can define our height function h to take 0 on all fermions and 1 on all bosons. Note that we could choose 0 for bosons and 1 for fermions. This produces a height-2 Adinkra called *valise*.

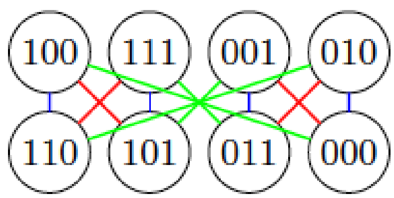
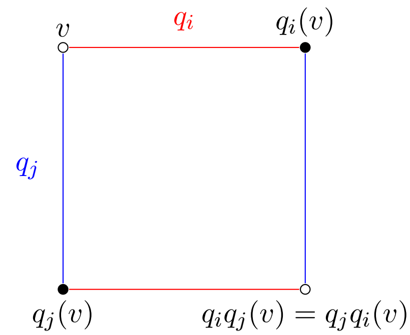


Figure 1.3: Valise Adinkra taken from [**Z**]

1.2.2 Chromotopologies

Definition 1.2.4. The edge set of an Adinkra consists of the N disjoint subsets of edges with colors $1, \cdots, N$. We call an ordering of these edge colors *rainbow*. In particular, an Adinkra without odd-dashing and height assignment is called a *chromotopology*, and a chromotopology without edge coloring (i.e. an Adinkra without odd-dashing, height assignment, and edge coloring) is called a *topology* of the Adinkra.

The condition of a graph A being a chromotopology is the same as existence of a map $q_i : \{\text{vertices of } A\} \to \{\text{vertices of } A\}$ defined as follows: for each color i, q_i sends a vertex v to a unique vertex $q_i(v)$ connected by an edge of color i such that $q_j q_i(v) = q_i q_j(v)$ for any $j \neq i$ [**Z**].



As the above diagram shows, the commutativity of the maps q_i and q_j corresponds to a 2-colored cycles of length 4.

1.2.3 Colored N-Cubes

From now on, we focus on chromotopologies of N-hypercube Adinkras, i.e. colored N-cubes $[0,1]^N$. It is an N-dimensional hypercube whose edges are partitioned into N disjoint subsets of edges of colors $1,\ldots,N$. Vertices of the N-cube correspond to elements of $(\mathbb{Z}/2\mathbb{Z})^N$. The *weight* of a vertex $v = (v_1,\ldots,v_N) \in (\mathbb{Z}/2\mathbb{Z})^N$ is a number of 1's in $(v_1,\ldots,v_N)$. Two vertices $v = (v_1,\ldots,v_N)$ and $w = (w_1,\ldots,w_N)$ are connected by an edge if and only if they are differ by exactly one component. If the vertices v and w are differ by i-th position, then the edge connecting these two vertices has a color i. The bipartite structure of a colored N-cube (as well as just N-cube) is given by defining vertices with even weights as white and vertices with odd weights as black . Since all the vertices of an N-cube $[0,1]^N$ are represented by the elements of $\mathbb{F}_2^N = \{0,1\}^N$, it is natural to use a binary code which is a vector subspace of $\mathbb{F}_2^N$ to study the nature of a colored N-cube.

1.3 Coding Theory on Adinkras

Let us first introduce some terminologies from coding theory that we will use to study Adinkras.

Definition 1.3.1. A *(binary) code* C of length N is a linear subspace of $\mathbb{F}_2^N$. We write $x_1x_2\cdots x_n$ instead of $(x_1,x_2,\ldots,x_n) \in \mathbb{F}_2^N$, e.g. $1101 := (1,1,0,1) \in \mathbb{F}_2^4$. Each coordinate in the N-tuple of an element of $\mathbb{F}_2^N$ is called a *bit*. A *weight* of $x_1x_2\cdots x_n$ is the number of 1's in $x_1x_2\cdots x_n$, and the weight of x is denoted by $wt(x)$. In particular, elements of a code $C \subseteq \mathbb{F}_2^N$ are called *codewords*. Since a code C is a vector subspace, it comes with a basis which is called a *generating set*. Hence every codeword is a linear combination of basis elements in the generating set with coefficients in $\mathbb{F}_2 = \{0, 1\}$. The *dimension* of a code means the cardinality of the generating set of the code. We call a code C *even* if every codeword in C has weight divisible by 2. In particular, if every codeword of C has weight divisible by 4, C is called *doubly even*.

Let $v, w \in \mathbb{F}_2^N$. Define $v\&w$ to be the "bitwise and" which means i-th bit of $v\&w$ is 1 if and only if i-th bit of v and i-th bit of w are both 1. Note that $\mathbb{F}_2^N$ comes with the following property:

$$wt(v \boxplus w) = wt(v) + wt(w) - 2wt(v\&w)$$

where $\boxplus$ is the bitwise addition modulo 2 (i.e. the addition for the vector space $\mathbb{F}_2^N$) [**DFGHILM**].

We highlight that we can form a quotient space of a colored N-cube $[0,1]^N$ by a code $C \subseteq \mathbb{F}_2^N$, namely $[0,1]^N/C$. Its elements are cosets of C in $\mathbb{F}_2$. Thus two vertices of $[0,1]^N/C$ are identified when they differ by a codeword, i.e.

$$v \boxplus C = w \boxplus C \qquad \Longleftrightarrow \qquad v - w \in C$$

Unfortunately, it is not always the case that a quotient $[0,1]^N/C$ by a code $C \subseteq \mathbb{F}_2^N$ is an Adinkra chromotopology. The following theorem imposes a condition on the code C so that $[0,1]^N/C$ is an Adinkra chromotopology.

Theorem 1.3.2 ([**DFGHILM**]). *Every connected Adinkra chromotopology is isomorphic to a quotient of a colored N-dimensional cube by doubly even code.*

The following remark tells us an importance of using even codes, and it certainly applies to doubly even codes because they are also even codes.

Remark 1.3.3. *If C is an even code, the quotient of a colored N-cube $[0,1]^N$ by C gives a consistent coloring of the vertices. To see this, note that bipartition of the colored N-cube is given by the weight of a vertex modulo 2: white for even weights and black for odd weights. From the property of $\mathbb{F}_2^N$*

$$wt(v \boxplus w) = wt(v) + wt(w) - 2wt(v\&w),$$

we have that

$$wt(v \boxplus w) = wt(v) + wt(w) \qquad \text{mod } 2.$$

Then we see that $wt(v \boxplus w) = wt(v)$ *if and only if* $wt(w)$ *is even. If* $w \in C$*, then* $v \boxplus w$ *is an element of the coset* $v \boxplus C$*, and this coset corresponds to a vertex of* $[0,1]^N/C$*. Hence if* C *is a even code, then* $wt(w)$ *is even. This implies that the color of the vertex* $v \boxplus C$ *depends on the color of the vertex* v*. In other words, the assignment of vertex colors of* $[0,1]^N/C$ *is inherited from that of* $[0,1]^N$*.*

We list some doubly even codes for smaller N below. In general, it is not easy to find doubly even codes.

Example 1.3.4 ([**M**]). For N = 4, there is only doubly even code which is generated by a single element 1111. When $N = 6$, we have again only one doubly even code generated by $\{111100, 001111\}$ up to permutation equivalence. For $N = 8$, we have four doubly even codes C_1, C_2, C_3, C_4 whose generating sets are

$$\begin{aligned}
C_1 &= \langle 11111111 \rangle \\
C_2 &= \langle 11110000, 00001111 \rangle \\
C_3 &= \langle 11110000, 00111100, 00001111 \rangle \\
C_4 &= \langle 11110000, 00111100, 00001111, 10101010 \rangle.
\end{aligned}$$

As the Theorem 1.3.2 states, the quotients of $[0,1]^N$ by such double even codes $C \subseteq \mathbb{F}_2^N$ mentioned above give Adinkra chromotopologies.

Chapter 2

Belyi's Theory on Adinkras

2.1 Belyi's Theorem

A Riemann surface can be described as a zero set $\{F(x,y)=0\}$ where $F \in \mathbb{C}[x,y]$ is an irreducible polynomial. From an arithmetic point of view, it is natural to ask which Riemann surface is definable over $\bar{\mathbb{Q}}$, that is, which Riemann surface is given by a polynomial $f \in \bar{\mathbb{Q}}[x,y]$. Belyi's theorem imposes a condition in order for a Riemann surface to be definable over $\bar{\mathbb{Q}}$.

Theorem 2.1.1 (Belyi). *Let X be a compact Riemann surface. The following statements are equivalent:*

1. *X is defined over $\bar{\mathbb{Q}}$*
2. *X admits a morphism $f : X \to \mathbb{P}^1$ whose branch locus is a subset of $\{0,1,\infty\}$*

Here, a meromorphic function $f : X \to \mathbb{P}^1$ satisfying the condition 2 of the theorem is called a *Belyi function*, and a pair (X,f) is called a *Belyi pair*.

Our goal in this section is to prove Belyi's theorem, and our proof is based on the proof given in [**GO**]. Let us first consider the direction (1) $\implies$ (2). An idea of the proof for this direction is to reduce elements of branch locus of a given map $f : X \to \mathbb{P}^1$ one by one until we get $\{0,1,\infty\}$. The reduction of branch locus breaks up into two steps: "$\bar{\mathbb{Q}}$ to $\mathbb{Q}$ step" and "$\mathbb{Q}$ to $\{0,1,\infty\}$ step". Thus our proof of the direction (1) $\implies$ (2) consists of two parts.

If $\varphi : C_1 \to C_2$ is a map between smooth curves, let $B(\varphi)$ be the set of branch values of φ and $e_p(\varphi)$ be the ramification index of φ at a point $p \in C_1$. The following is a multiplicative property of ramification indices which we will use in the proof.

Lemma 2.1.2 ([**GO**])**.** *Suppose that*

$$C_1 \underset{\varphi}{\to} C_2 \underset{\psi}{\to} C_3$$

is a composition of maps of smooth curves. Then for all $p \in C_1$*,*

$$e_p(\psi \circ \varphi) = e_{\varphi(p)}(\psi) \cdot e_p(\varphi).$$

First we will consider the "$\bar{\mathbb{Q}}$ to $\mathbb{Q}$ step".

Proposition 2.1.3 (Reduction of Branch Locus from $\bar{\mathbb{Q}}P^1$ to $\mathbb{Q}P^1$)**.** *Let* $h_0 : X \to \mathbb{P}^1$ *be a map satisfying* $B(h_0) \subset \bar{\mathbb{Q}}P^1$*. Then there exists a map* $h_B : \mathbb{P}^1 \to \mathbb{P}^1$ *such that the composition*

$$h_B \circ h_0 : X \to \mathbb{P}^1$$

satisfies $B(h_B \circ h_0) \subset \mathbb{Q}P^1$*.*

Proof. Let $S \subset \bar{\mathbb{Q}}$ be a finite set and denote the minimal polynomial of S over $\mathbb{Q}$ by $m_{S,\mathbb{Q}}(x)$. In other words, $m_{S,\mathbb{Q}}(x)$ is a monic polynomial with rational coefficients such that all the elements of S are roots of $m_{S,\mathbb{Q}}(x)$. Then we set

$$B_0 := B(h_0) - \{\infty\}, \qquad h_1(x) := m_{B_0,\mathbb{Q}}(x).$$

We recursively define

$$B_i := B(h_i) - \{\infty\}, \qquad h_i(x) := m_{B_{i-1},\mathbb{Q}}(x).$$

For $i \geq 1$, the minimal polynomials h_i can be seen as maps $h_i : \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified at ∞ [**GO**].

<u>Claim</u>: $\forall i \geq 1$, if $deg(h_i) \geq 1$, then $deg(h_{i+1}) \leq deg(h_i) - 1$.

Note that the set of branch values B_i are *Galois stable* for $i \geq 1$ [**GO**] meaning that $\forall \sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ and $\forall \alpha \in B_i, \sigma\alpha \in B_i$. To demonstrate how this definition works, one can write the set of branch values as

$$B_i = \{h_i(a) \mid h_i'(a) = 0\} \qquad i \geq 1.$$

If $\alpha := h_i(a) \in B_i$, then

$$\sigma\alpha = \sigma h_i(a) = h_i(\sigma a)$$

because $h_i \in \mathbb{Q}[x]$. If we take derivative,

$$h_i'(\sigma a) = \sigma(h_i'(a)) = \sigma(0) = 0$$

since $h_i' \in \mathbb{Q}[x]$ and $\alpha = h_i(a) \in B_i$ (i.e. $h_i'(a) = 0$). Indeed $\sigma\alpha \in B_i$. By the definition of h_{i+1}, we have, for any i,

$$h_{i+1}(x) = \prod_{\alpha \in B_i} (x - \alpha).$$

In particular, $deg(h_{i+1}) = |B_i|$. Recall that $B_i = \{h_i(a) \mid h_i'(a) = 0\}$, so B_i knows about roots of h_i'. Then we see that

$$|B_i| \leq deg(h_i') < deg(h_i).$$

Here we have '$\leq$' instead of '$=$' on the left hand side of the above inequality because h_i' may have repeated roots which is the case when the number of distinct roots of h_i' is less than $deg(h_i')$. Since $deg(h_i') = deg(h) - 1$, we have

$$\begin{aligned}
&|B_i| \leq deg(h_i') = deg(h) - 1 \\
&\implies |B_i| \leq deg(h_i) - 1 \\
&\implies deg(h_{i+1}) \leq deg(h_i) - 1 \qquad \text{because } deg(h_{i+1}) = |B_i|
\end{aligned}$$

Thus Claim 1 is proved.

Claim 1 implies that there exists a positive integer ℓ such that h_ℓ is linear (i.e. $deg(h_\ell) = 1$). Now consider the composition

$$h_B := h_{\ell-1} \circ h_{\ell-2} \circ \cdots \circ h_1$$

and set

$$h := h_B \circ h_0.$$

Claim 2: $B(h) \subset \mathbb{Q}P^1$

Suppose $h : X \to \mathbb{P}^1$ is ramified at $p \in X$ over $q \in \mathbb{P}^1$ and $q \neq \infty$. This implies that $e_p(h) > 1$. But $h = h_B \circ h_0 = h_{\ell-1} \circ \cdots \circ h_1 \circ h_0$, so $e_p(h) = e_p(h_B \circ h_0) = e_{h_0(p)}(h_B) \cdot e_p(h_0) > 1$. Without loss of generality, assume $e_{h_0(p)}(h_B) > 1$. Since $h_B = h_{\ell-1} \circ \cdots \circ h_1$, Lemma 2.1.2 implies that

$$\begin{aligned}
e_{h_0(p)}(h_B) &= e_{h_0(p)}(h_{\ell-1} \circ \cdots \circ h_1) \\
&= e_{h_0(p)}(h_{\ell-1} \circ \cdots \circ h_k \circ k_{k-1} \circ \cdots \circ h_1) \\
&= e_{h_{k-1} \circ \cdots \circ h_1(h_0(p))}(h_{\ell-1} \circ \cdots h_k) \cdot e_{h_0(p)}(h_{k-1} \circ \cdots \circ h_1) > 1
\end{aligned}$$

for some integer k with $1 < k \leq \ell - 1$. Again without loss of generality, assume that

$e_{h_0(p)}(h_{k-1} \circ \cdots \circ h_1) > 1$. Apply Lemma 2.1.2 again to get

$$\begin{aligned} e_{h_0(p)}(h_{k-1} \circ \cdots \circ h_1) &= e_{h_0(p)}(h_{k-1} \circ \cdots \circ h_{i+1} \circ h_i \circ \cdots \circ h_1) \\ &= e_{h_i \circ \cdots \circ h_1(h_0(p))}(h_{k-1} \circ \cdots \circ h_{i+1}) \cdot e_{h_0(p)}(h_i \circ \cdots \circ h_1) > 1 \end{aligned}$$

where $1 \leq i < k-1$. Without loss of generality, assume $e_{h_0(p)}(h_i \circ \cdots \circ h_1) > 1$. Then Lemma 2.1.2 implies that

$$\begin{aligned} e_{h_0(p)}(h_i \circ \cdots \circ h_1) &= e_{h_0(p)}(h_i \circ h_{i-1} \circ \cdots \circ h_1) \\ &= e_{h_{i-1} \circ \cdots \circ h_1(h_0(p))}(h_i) \cdot e_{h_0(p)}(h_{i-1} \circ \cdots \circ h_1) > 1. \end{aligned}$$

Without loss of generality, take $e_{h_{i-1} \circ \cdots \circ h_1(h_0(p))}(h_i) > 1$. Hence we may say that there exists i such that $1 \leq i \leq \ell - 1$ with

$$e_{h_{i-1} \circ \cdots \circ h_1(h_0(p))}(h_i) = e_{h_{i-1} \circ \cdots \circ h_1 \circ h_0(p)}(h_i) > 1.$$

<u>Case I</u>: $i < \ell - 1$

$e_{h_{i-1} \circ \cdots \circ h_1 \circ h_0(p)}(h_i) > 1$ implies that h_i is ramified at the point $h_{i-1} \circ \cdots \circ h_1 \circ h_0(p)$. This means that $h_{i-1} \circ \cdots \circ h_1 \circ h_0(p)$ is a ramification point and $h_i \circ h_{i-1} \circ \cdots \circ h_1 \circ h_0(p) = h_i(h_{i-1} \circ \cdots \circ h_1 \circ h_0(p))$ is a branch value. Thus $h_i \circ h_{i-1} \circ \cdots h_1 \circ h_0(p) \in B_i$. Since $h_{i+1}(x) = m_{B_i,\mathbb{Q}}(x)$, we have $h_{i+1}(h_i \circ \cdots \circ h_1 \circ h_0(p)) = (h_{i+1} \circ h_i \circ \cdots \circ h_1 \circ h_0)(p) = 0$. Since $h_i \in \mathbb{Q}[x]$ for all $i \geq 1$, we have

$$\begin{aligned} & h_{\ell-1} \circ \cdots \circ h_{i+2} \circ h_{i+1} \circ h_i \circ \cdots \circ h_1 \circ h_0(p) \\ &= h_{\ell-1} \circ \cdots \circ h_{i+2}(h_{i+1} \circ h_i \circ \cdots \circ h_1 \circ h_0(p)) \\ &= h_{\ell-1} \circ \cdots \circ h_{i+2}(0) \in \mathbb{Q} \end{aligned}$$

where $h_{\ell-1} \circ \cdots h_{i+2}(0)$ is just a constant term. Hence $(h_{\ell-1} \circ \cdots \circ h_{i+1} \circ h_i \circ \cdots \circ h_1 \circ h_0)(p) = h(p) = q \in \mathbb{Q}$.

<u>Case II</u>: $i = \ell - 1$

Since $deg(h_\ell) = 1$ and $h_\ell(x) = \prod_{\alpha \in B_{\ell-1}}(x - \alpha) = (x - \alpha)$, we have $|B_{\ell-1}| = 1$. Because $h_\ell(x) = (x - \alpha) \in \mathbb{Q}[x]$, the only element of $B_{\ell-1}$, α, is in $\mathbb{Q}$, i.e. $B_{\ell-1} \subset \mathbb{Q}$. Since $e_{h_{i-1} \circ \cdots h_1 \circ h_0(p)}(h_i) > 1$ and $i = \ell - 1$, $h_{\ell-1}$ has a ramification point $h_{\ell-2} \circ \cdots \circ h_1 \circ h_0(p)$ and a branch value $h_{\ell-1}(h_{\ell-2} \circ \cdots \circ h_1 \circ h_0(p)) \in B_{\ell-1}$. But $h = h_{\ell-1} \circ \cdots \circ h_1 \circ h_0$ and by assumption, h is ramified at p over q,

$$\underbrace{h_{\ell-1} \circ \cdots \circ h_1 \circ h_0(p)}_{\in B_{\ell-1}} = h(p) = q.$$

Hence $q \in B_{\ell-1}$. Then $B_{\ell-1} \subset \mathbb{Q}$ together with $q \in B_{\ell-1}$ implies that $q \in \mathbb{Q}$. $\square$

Assume we have $g : \mathbb{P}^1 \to \mathbb{P}^1$. By Lemma 2.1.2, we see that $g \circ h : X \to \mathbb{P}^1$ is a Belyi map if and only if g satisfies the following two conditions [**GO**]:

1. $g(B(h)) \subseteq \{0, 1, \infty\}$
2. $B(g) \subseteq \{0, 1, \infty\}$

Now let us consider the "$\mathbb{Q} \to \{0, 1, \infty\}$ step".

Proposition 2.1.4 (Reduction of Branch Locus from $\mathbb{Q}P^1$ to $\{0, 1, \infty\}$)**.** *Assume $h : X \to \mathbb{P}^1$ is a meromorphic function on X with $B(h) \subset \mathbb{Q}P^1$. Then there exists a map $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that the composite $g \circ h$ is a Belyi map for X.*

Proof. Suppose $h : X \to \mathbb{P}^1$ with $B(h) \subset \mathbb{Q}P^1$(existence of such a map $h = h_B \circ h_0$ was proved in Proposition 2.1.3). For simplicity, we consider the case when $|B(h)| = 4$. In particular, we may assume that $B(h) = \{0, 1, \infty, s\} \subset \mathbb{Q}P^1$ where $0 < s < 1, s = m/(m + n)$ using Möbius transformation [**GO**]. We claim that there is a function $g : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying

1. $g(B(h)) \subseteq \{0, 1, \infty\}$
2. $B(g) \subseteq \{0, 1, \infty\}$

so that the composition $g \circ h : X \to \mathbb{P}^1$ is a Belyi function. Note that in order to prove the statement when $|B(h)| = 4$, it suffices to construct a function g which satisfies the conditions above [**GO**].

<u>Claim</u>: The function $g : \mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$g(x) = \frac{(m + n)^{m+n}}{m^m n^n} x^m (1 - x)^n$$

and it satisfies (i) $g(B(h)) \subseteq \{0, 1, \infty\}$ and (ii) $B(g) \subseteq \{0, 1, \infty\}$.

First consider a polynomial function given by

$$f_{c,k,\ell}(x) = cx^k (1 - x)^\ell$$

where $c \in (\bar{\mathbb{Q}})^*$ is a nonzero constant. For any $c \neq 0$ and any $k, \ell \geq 1$, we see that $f_{c,k,\ell}(0) = 0$, $f_{c,k,\ell}(1) = 0$, $f_{c,k,\ell}(\infty) = \infty$. For any k, ℓ, there must be a unique c such that $f_{c,k,\ell}(s) = 1$. Thus the condition (i) is satisfied (in this case $f_{c,k,\ell}(B(h)) \subseteq \{0, 1, \infty\}$). For the condition (ii) (in this case $B(f_{c,k,\ell}) \subseteq \{0, 1, \infty\}$), the branch values of $f_{c,k,\ell}$ are given by

$$B(f_{c,k,\ell}) - \{\infty\} = \{f_{c,k,\ell}(a) \mid f'_{c,k,\ell}(a) = 0\}$$

where

$$f'_{c,k,\ell}(x) = cx^{k-1}(1-x)^{\ell-1}[k(1-k) - \ell x].$$

The roots of $f'_{c,k,\ell}(x)$ are $0, 1$ and $x = k/(k+\ell)$. If one takes $k = m$ and $\ell = n$, the root of $f'_{c,k,\ell}$ other than 0 and 1 is $s = m/(m+n)$. Then the branch value corresponding to s is

$$f_{c,k,\ell}(s) = c \cdot (\frac{m}{m+n})^m (1 - \frac{m}{m+n})^n = c \cdot \frac{m^m n^n}{(m+n)^{m+n}}.$$

Hence taking $c = (m+n)^{m+n}/m^m n^n$ gives $f_{c,m,n}(s) = 1$, and the condition (ii) is satisfied. Via this construction, we obtained $g(x) = f_{c,m,n}(x) = cx^m(1-x)^n$ with $c = \frac{(m+n)^{m+n}}{m^m n^n}$. □

Remark 2.1.5. *The converse of this theorem, that is,* (2) $\Longrightarrow$ (1) *(so-called "obvious" part of Belyi's theorem) follows from Weil's work (Theorem 4 in [***W***]), and it is much more difficult to prove.*

Let us demonstrate how to construct a Belyi function on a particular Riemann surface.

Example 2.1.6 ([**GG**]). Let S be a Riemann surface

$$\{y^2 = x(x-1)(x-\sqrt{2})\} \cup \{\infty\}.$$

Since S is defined over $\overline{\mathbb{Q}}$, there is a Belyi function β on S (by Belyi's theorem). Our goal is to construct f.

Consider $h_0 : S \to \mathbb{P}^1$ given by $(x, y) \mapsto x$. Notice that h_0 has branch values $0, 1, \sqrt{2}$ other than ∞. This is clear from the following observation:

$$h_0^{-1}(0) = \{(0,0)\}, \quad h_0^{-1}(1) = \{(1,0)\}, \quad h_0^{-1}(\sqrt{2}) = \{(\sqrt{2}, 0)\}$$

while any point $\alpha \neq 0, 1, \sqrt{2}$ has two elements in its fiber:

$$h_0^{-1}(\alpha) = \{(\alpha, \sqrt{\alpha(\alpha-1)(\alpha-\sqrt{2})}), (\alpha, -\sqrt{\alpha(\alpha-1)(\alpha-\sqrt{2})})\}.$$

Hence $B(h_0) = \{0, 1, \sqrt{2}, \infty\}$, and $\sqrt{2}$ is the only irrational number.

Next consider the minimal polynomial m of $\sqrt{2}$, that is

$$m(x) = x^2 - 2.$$

Since $m'(x) = 2x$ and its root is $x = 0$, the branch value of m other than ∞ is $m(0) = -2$. Note that $m : x \mapsto x^2 - 2$ sends $0 \mapsto -2, 1 \mapsto -1, \sqrt{2} \mapsto 0$, and $\infty \mapsto \infty$, so $m(B(h_0)) = \{-2, -1, 0, \infty\}$. Also note that $B(\psi \circ \varphi) = B(\psi) \cup \psi(B(\varphi))$ for any composition of branch covers $C_1 \xrightarrow{\varphi} C_2 \xrightarrow{\psi} C_3$ [**GG**]. Using this property, we have

$$B(m \circ h_0) = B(m) \cup m(B(h_0)) = \{-2, -1, 0, \infty\} \subset \mathbb{Q}P^1.$$

Applying a Möbius transformation $T : x \mapsto -1/x$, we have

$$T(B(m \circ h_0)) = \{1/2, 1, \infty, 0\}$$

so all the branch values except for ∞ now lie in the interval $[0, 1]$.

As a final step, we compose $T \circ m \circ h_0$ with the following Belyi polynomial

$$g(x) = 4x(1 - x).$$

In fact, g maps $1/2 \mapsto 1, 1 \mapsto 0, 0 \mapsto 0$, and $\infty \mapsto \infty$. Hence using this Belyi polynomial g, we reduced branch locus from $\{0, 1, 1/2, \infty\}$ to $\{0, 1, \infty\}$.

Considering this long sequence of compositions,

$$\begin{tikzcd}
\{y^2 = x(x-1)(x-\sqrt{2})\} \cup \{\infty\} \arrow[d, "h_0:(x,y)\mapsto x"] \arrow[dddd, bend right, "\beta"'] \\
\mathbb{P}^1 \arrow[d, "m:x\mapsto x^2-2"] \\
\mathbb{P}^1 \arrow[d, "T:u\mapsto -1/u"] \\
\mathbb{P}^1 \arrow[d, "g:z\mapsto 4z(1-z)"] \\
\mathbb{P}^1
\end{tikzcd}$$

we see that a Belyi function on S is $\beta := g \circ T \circ m \circ h_0$ given by

$$\beta(x) = \frac{-4(x^2 - 1)}{(x^2 - 2)^2}.$$

2.2 Dessins d'Enfants

Belyi's work inspired Grothendieck to explore interactions between Riemann surfaces defined over number fields and bipartite graphs drawn on topological surfaces. The following result from Grothendieck gives the combinatorial description of Belyi functions:

Theorem 2.2.1 (Grothendieck's Correspondence [**GG**]). *There is a 1-1 correspondence between the following sets:*

1. *Bipartite connected graphs embedded in a oriented topological surface*
2. *Belyi pairs* (X, β)

The graphs in the theorem which correspond to Belyi pairs are known as *dessins d'enfants* ("children's drawings" in French). We have already seen Belyi pairs in the previous section, so in this section we will introduce dessins d'enfants. The contents of the sections 2 and 3 are adapted from [**GG**] and [**HS1**].

2.2.1 Definition and an Example

Definition 2.2.2. A *dessin d'enfant* (or simply a *dessin*) is a pair $(X, \mathcal{D})$ where X is an oriented topological surface and $\mathcal{D} \subset X$ is a finite graph satisfying the following conditions:

1. $\mathcal{D}$ is connected
2. $\mathcal{D}$ is bipartite
3. $X \setminus \mathcal{D}$ is the union of finitely many topological discs, called *faces* of $\mathcal{D}$

The genus of a dessin $(X, \mathcal{D})$ is the genus of the underlying topological space X. The heart of dessins is that they can be obtained from a Belyi pair (X, β). Elements in the fibers $\beta^{-1}(0)$ and $\beta^{-1}(1)$ are white and black vertices of $\mathcal{D}$ respectively (we can choose $\beta^{-1}(0)$ to be black vertices and $\beta^{-1}(1)$ to be white vertices as well). Elements of $\beta^{-1}(\infty)$ corresponds to centers of faces in $\mathcal{D}$, and $\beta^{-1}([0,1])$ corresponds the embedded graph $\mathcal{D}$. Moreover, ramification index at a point in the fiber $\beta^{-1}(0)$ or $\beta^{-1}(1)$ is the valency of the vertex representing that point. If $\beta : X \to \mathbb{P}^1$ is a branched cover of degree d, then the dessin $\mathcal{D}$ has d edges in total. Note that $\mathcal{D}$ decomposes the surface X into open cells containing exactly one preimage of ∞.

Remark 2.2.3. *([**JW**]) A dessin $(X, \mathcal{D})$ gives the underlying topological surface X a Riemann surface structure. The Riemann surfaces (seen as algebraic curves) arising in this way are the ones that can be defined over $\overline{\mathbb{Q}}$.*

Example 2.2.4. In this example, we will construct a dessin that corresponds to the Belyi function $\beta : S \to \mathbb{P}^1$ given by $\beta(x) = \frac{x^2}{2x-1}$ where S is an elliptic curve defined by $y^2 = x(x-1)(x-\frac{1}{2})$. We first compute preimages of $0, 1$ and ∞.

$$\begin{aligned} \beta^{-1}(0) &= \{(0,0)\} && \text{when the numerator is 0} \\ \beta^{-1}(1) &= \{(1,0)\} && \text{when } x^2 = 2x - 1 \\ \beta^{-1}(\infty) &= \{(1/2, 0), (\infty, \infty)\} && \text{when the denominator approaches to 0} \end{aligned}$$

Next we need to find out the degree of β. Since $deg(\beta) = \max\{|\ \beta^{-1}(p)| : p \in \mathbb{P}^1\}$, we will compute the cardinality of the fiber over an unramified value. Without loss of generality, we compute the cardinality of the fiber $\beta^{-1}(2)$.

$$\begin{aligned}\beta^{-1}(2) &= \{(x,y) \mid x^2/(2x-1) = 2\} \\ &= \{(x,y) \mid x^2 - 4x + 2 = 0\} \\ &= \{(x,y) \mid x = 2 \pm \sqrt{2}\}\end{aligned}$$

Next we need to find the y-values when $x = 2 \pm \sqrt{2}$. Since $(2+\sqrt{2}, y_1)$ and $(2-\sqrt{2}, y_2)$ are both points in the elliptic curve, they should satisfy the equation $y^2 = x(x-1)(x-\frac{1}{2})$. Substituting $x = 2+\sqrt{2}$ into $y^2 = x(x-1)(x-\frac{1}{2})$, we get $y = \pm\sqrt{(2+\sqrt{2})(1+\sqrt{2})(\frac{3}{2}+\sqrt{2})}$. Similarly, substituting $x = 2-\sqrt{2}$ into the equation gives $y = \pm\sqrt{(2-\sqrt{2})(1-\sqrt{2})(\frac{3}{2}-\sqrt{2})}$. So we have

$$\beta^{-1}(2) = \{(2+\sqrt{2}, y_1), (2+\sqrt{2}, -y_1), (2-\sqrt{2}, y_2), (2-\sqrt{2}, -y_2)\}$$

where $y_1 = \sqrt{(2+\sqrt{2})(1+\sqrt{2})(\frac{3}{2}+\sqrt{2})}$ and $y_2 = \sqrt{(2-\sqrt{2})(1-\sqrt{2})(\frac{3}{2}-\sqrt{2})}$. Thus $|\beta^{-1}(2)| = 4$ which folds for any unramified value and so $deg(\beta) = 4$. Since $deg(\beta) = 4$, for any $q \in \mathbb{P}^1$,

$$\sum_{p \in \beta^{-1}(q)} e_p = deg(\beta) = 4 \tag{2.1}$$

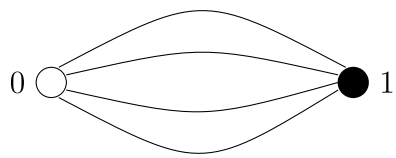
where e_p is a ramification index at p. Consider the equation (2.1) when $q = 0, 1, \infty$:

$$\begin{aligned} q = 0 : &\sum_{p \in \beta^{-1}(0)} e_p = e_{(0,0)} = 4 \\ q = 1 : &\sum_{p \in \beta^{-1}(1)} e_p = e_{(1,0)} = 4 \\ q = \infty : &\sum_{p \in \beta^{-1}(\infty)} e_p = e_{(1/2,0)} + e_{(\infty,\infty)} = 4 \implies e_{(1/2,0)} = 2 = e_{(\infty,\infty)} \end{aligned}$$

The following table summarizes the results of all these computations.

critical points x	critical values $\beta(x)$	ramification index
$x = 0$	0	4
$x = 1$	1	4
$x = 1/2$	∞	2
$x = \infty$	∞	2

The dessin for this Belyi pair $(X, \frac{x^2}{2x-1})$ where X is an elliptic curve given by $y^2 = x(x-1)(x-\frac{1}{2})$ consists of single black vertex of valency 4, single white vertex of valency 4, two faces, and 4 edges.



The Euler characteristic for this dessin is $\chi = 2-4+2 = 0$, so its genus is 1 which implies that the graph is embedded in a torus.

In subsequent sections, we will look at various ways to represent dessins.

2.2.2 Dessins as Bipartite Ribbon Graphs

All dessins have (bipartite) ribbon graph structures, and in fact these two are equivalent notions. This property enables us to represent dessins using permutations.

Definition 2.2.5. A *ribbon graph* $\Gamma = (H, \nu, \epsilon)$ consists of a finite set of half edges H, and the permutations ν, ϵ on H such that $\epsilon^2 = id$ and it has no fixed points. Cycles in ν represent vertices of the ribbon graph, and similarly, cycles in ϵ represent edges of the graph. Furthermore, cycles in the composition $\phi := \nu \circ \epsilon$ correspond to *faces* of the graph.

Example 2.2.6. Consider a ribbon graph (H, ϵ, ν) given by

$$\begin{aligned} H &= \{1, 2, \ldots, 8\} \\ \epsilon &= (1,5)(2,6)(3,7)(4,8) \\ \nu &= (1,2,3,4)(8,7,6,5). \end{aligned}$$

Composing ϵ and ν, we get

$$\phi = \nu \circ \epsilon = (1,8)(2,5)(3,6)(4,7).$$

Hence this ribbon graph Γ has 4 edges, 2 vertices and 4 faces. Its Euler characteristic is $\chi = 2-4+4 = 2$. This means $2 = \chi = 2-2g$, so $g = 0$. Hence this ribbon graph Γ is embedded in a sphere S^2. As a 1-skeleton, this graph is the same as the graph we constructed in Example 2.2.4. Because they are embedded in different surfaces, they are not the same as a ribbon graph.

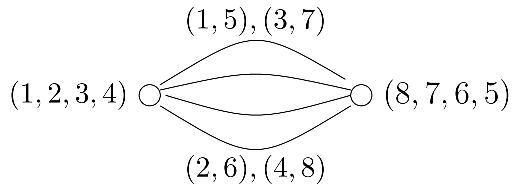


Figure 2.1: A ribbon graph

To make Γ bipartite[1], we simply set the cycle $(1, 2, 3, 4)$ to be black (or white) and the other cycle to be the opposite vertex color. We are going to revisit this example in chapter 3.

From Dessins to (Bipartite) Ribbon Graphs

Let us first explain how to obtain a ribbon graph from a dessin $(X, \mathcal{D})$. All we need is to assign each vertex a cyclic permutation of half edges which are incident to that vertex. Since ribbon graphs are defined in terms of half edges, we divide each edge of $\mathcal{D}$ into half so that we have $2d$ half edges where d is the number of edges of $\mathcal{D}$. For each vertex v of $\mathcal{D}$, take a coordinate chart (U_v, ψ) where U_v is an open neighborhood of v in the surface X and $\psi : U_v \to \mathbb{R}^2$ such that the image $\psi(\mathcal{D} \cap U_v) \subset \mathbb{R}^2$ is a "star" graph with a single vertex $\psi(v)$ in the center and all edges dispersing from the vertex $\psi(v)$. Let ν_v be a permutation on the half edges of the star graph, and in particular it traverses edges incident to $\psi(v)$ in counterclockwise direction. Hence the product $\nu = \prod_v \nu_v$ is a permutation for vertices, and $\nu \in S_{2d}$. The permutation ϵ for edges is a product of transpositions which permute one half edge with the other so that these two half edges form an edge. At the end, the pair (ν, ϵ) defines a ribbon graph.

From (Bipartite) Ribbon Graphs to Dessins

On the other hand, we can obtain a dessin from a ribbon graph $\Gamma = (H, \nu, \epsilon)$ as follows. Recall that the composition $\phi = \nu \circ \epsilon$ represents faces of Γ. Take an edge e of Γ and construct a cyclic permutation $(e, \phi(e), \phi^2(e), \ldots, \phi^k(e))$. This defines a closed path which is a union of half edges. Then we glue a 2-cell into this closed path to get a face. Repeat this process for different edges of Γ. At the end, we get a closed surface X with embedded graph $\mathcal{D}$ on it.

2.2.3 Permutation Representations of Dessins

If we use permutations on a set of edges instead of a set of half edges, we get the permutation representation of a dessin. Let $(X, \mathcal{D})$ be a dessin with N edges labelled

[1] In general, a graph G is a bipartite graph if and only if each cycle of G has even length [**ADH**]. We see the ribbon graph Γ in this example satisfies this property; each of the four faces are of length 2.

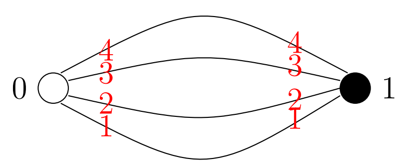
by integers from 1 through N. Define $\sigma_0 \in S_N$ by $\sigma_0(i) = j$ where both i and j are edges incident to the same white vertex. The same construction defines the permutation $\sigma_1 \in S_N$ for black vertices. Note that both σ_0 and σ_1 permute edges along the same orientation. Similar to the definition of the ribbon graph structure on the dessin $(X, \mathcal{D})$, each cycle in the permutation σ_0 (or σ_1) correspond to a white (or black) vertex of $\mathcal{D}$. The length of a cycle in σ_0 (or σ_1) implies the valency of the corresponding vertex. Moreover, cycles in the composition $\sigma_1\sigma_0$ (or equivalently $\sigma_0\sigma_1$) corresponds to the faces of $\mathcal{D}$. To be more precise, repeated application of $\sigma_1\sigma_0$ on a single vertex i,

$$i, \quad \sigma_1\sigma_0(i), \quad (\sigma_1\sigma_0)^2(i), \quad \ldots, \quad (\sigma_1\sigma_0)^{k-1}(i), \quad (\sigma_1\sigma_0)^k(i) = i$$

forms a k-cycle in $\sigma_1\sigma_0$ and this cycle lists in a counterclockwise direction a half of $2k$ edges of a face containing the edge i. If we use the composition $\sigma_0\sigma_1$ instead of $\sigma_1\sigma_0$, we enumerate edges in a clockwise direction.

Definition 2.2.7. The pair (σ_0, σ_1) is called the *permutation representation* of the dessin.

Example 2.2.8. We want to find the permutation representation of the dessin $(X, \mathcal{D})$ in Example 2.2.4. We choose counterclockwise orientation for the underlying surface X. The embedded graph $\mathcal{D}$ has one black and one white vertices and each has valency of 4. Since $\mathcal{D}$ has 4 edges in total, the permutations σ_0 and σ_1 are elements of S_4 and each of them consists of one cycle of length 4. Therefore $\sigma_0 = (1, 2, 3, 4), \sigma_1 = (1, 2, 3, 4)$ and the composition $\sigma_1\sigma_0 = (1, 3)(2, 4)$ for the faces of $\mathcal{D}$.



2.3 Monodromy Representation of Belyi Functions

In the previous subsection, we have seen the combinatorial representation of a Belyi function, which is a dessin. In this subsection, let us introduce another way to look at Belyi functions, which is monodromy representations of Belyi functions.

Let $\beta : S \to \mathbb{P}^1$ be a Belyi function of degree d. Then it has the following monodromy morphism

$$Mon_\beta : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, y) \to S_d$$

where $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, y)$ is the fundamental group generated by two loops γ_0 and γ_1 based at $y = 1/2$, winding around 0 and 1 counterclockwise respectively. Hence Mon_β is determined by two permutations[2] $Mon_\beta(\gamma_0) =: \sigma_{\gamma_0}^{-1}$ and $Mon_\beta(\gamma_1) =: \sigma_{\gamma_1}^{-1}$. Observe that if the pair (σ_0, σ_1) is a permutation representation of the dessin $\mathcal{D}_\beta$ corresponding to the Belyi function β, then $\sigma_0 = \sigma_{\gamma_0}$ and $\sigma_1 = \sigma_{\gamma_1}$. To see this, first consider a point $x_e \in \beta^{-1}(y)$ lying on the edge e of $\mathcal{D}_\beta \subset S$. Then the lift $\tilde{\gamma_0}$ of the loop γ_0 with initial point x_e has a terminal point at $x_{\sigma_0(e)} \in \beta^{-1}(y)$. This is because β is of the form $z \mapsto z^n$ with $n > 1$ in a neighborhood of an element in $\beta^{-1}(0)$. We can show $\sigma_1 = \sigma_{\gamma_1}$ in a similar fashion. This discussion is summarized in the following remark.

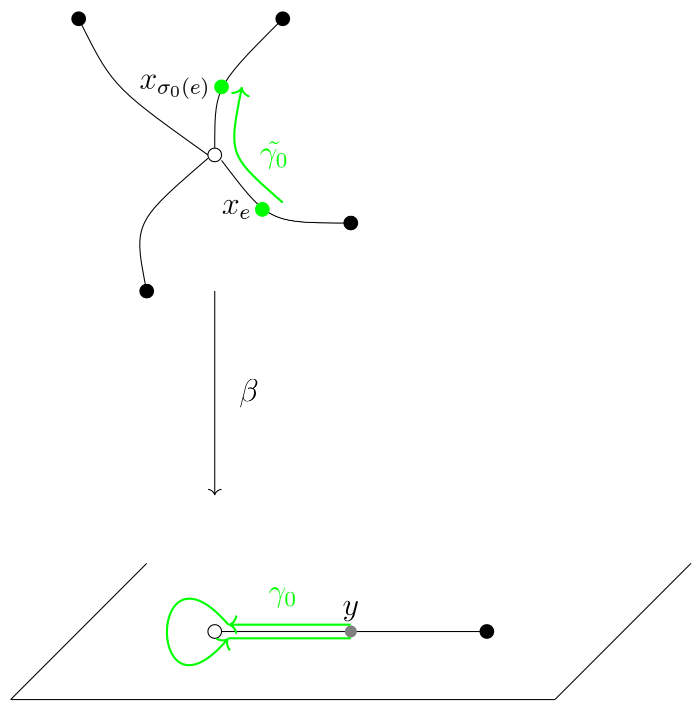


Figure 2.2: Monodromy action

Remark 2.3.1 ([**GG**])**.** *The permutation representation pair of a dessin and the monodromy of the corresponding Belyi pair are equivalent.*

If (S, β) is a Belyi pair and $(X, \mathcal{D})$ is the corresponding dessin whose permutation representation pair is (σ_0, σ_1), the permutation group generated by σ_0, σ_1 is called the *monodromy group* $Mon(\mathcal{D})$ of the dessin (or *monodromy group* $Mon(\beta)$ of the Belyi

[2]If we defined the monodromy homomorphism Mon_β by $Mon_\beta(\gamma_0) = \sigma_{\gamma_0}$ and $Mon_\beta(\gamma_1) = \sigma_{\gamma_1}$, Mon_β would be an anti-homomorphism [**GG**].

pair).

2.4 Applications to Adinkra Chromotopologies

In this section, we describe Adinkra chromotopologies using tools from the previous discussion including permutation representations of dessins. Our discussion in the rest of the chapter is adapted from [**DIKLM**].

2.4.1 Adinkra Chromotopologies as Bipartite Ribbon Graphs

First of all, let us consider the ribbon graph structure on a Adinkra chromotopology which is a connected N-regular bipartite graph with colored edges (no height function and no odd dashing). Note that each vertex has edges of all colors of the rainbow. For any vertex, the rainbow gives a cyclic permutation of half edges incident to that vertex. For a white vertex, the corresponding cycle in the permutation ν enumerate half edges incident to the white vertex in the order of rainbow. For a black vertex, the corresponding cycle in the permutation ν permutes half edges incident to the black vertex in the opposite order of the rainbow. Remark that the cycles representing black vertices should have listed along the order of rainbow, but in this way we lose the bipartite structure of Adinkra [**DIKLM**].

2.4.2 Underlying Surfaces of Adinkra Chromotopologies

Since ribbon graphs are equivalent to dessins, we can view an Adinkra chromotpology as a dessin as well. In general, underlying Riemann surface X of a dessinis obtained by taking a ribbon graph as a 1-skeleton of X and then filling in loops of the 1-skeleton by attaching 2-cells to them. The ribbon graph structure of an Adinkra determines which loops to attach 2-cells.

Suppose our Adinkra chromotopology A has a rainbow $(C_1, \ldots, C_N)$. Let w be a white vertex of A and pick a color C_i from the rainbow. If we leave w with the edge of color C_i, we reach to a black vertex $b_1 = w + e_i$, where e_i is a standard basis vector of $\mathbb{R}^N$. Because the order of the permutation for black vertices is opposite to that of the permutation for white vertices, we leave b_1 with the edge of color C_{i-1}, and we get to the white vertex $w' = b_1 + e_{i-1} = w + e_i + e_{i-1}$. Again because the order of the permutation for white vertices is opposite to that of the permutation for black vertices, we leave w' with the edge of color C_i. This edge takes us to the black vertex $b_2 = w' + e_i = w + e_i + e_{i-1} + e_i = w + e_{i-1} \neq b_1$. Similarly, we leave b_2 with the edge of color C_{i-1} and end up at the white vertex $w'' = b_2 + e_{i-1} = w + e_{i-1} + e_{i-1} = w$. Hence we completed a (C_i, C_{i-1})-colored loop.

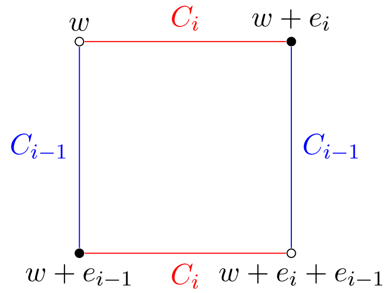


Figure 2.3: The structure of a (C_{i-1}, C_i)-colored loop

We apply the same procedure for the other white vertices so that we can attach 2-cells to all the (C_i, C_{i+1})-loops. This is how we construct a Riemann surface associated to an Adinkra chromotopology A.

Remark 2.4.1 ([**DIKLM**]). *All the faces generated in this way are* 4*-gons. If we choose the order of colors in the rainbow to be the same for both white and black vertices, we will get* $2N$*-gonal faces.*

2.4.3 Monodromy Permutations for Adinkra Chromotopologies

Let us fix some notations:

- $C_{N,max} :=$ maximal even code in $\mathbb{F}_2^N$
- $A_N := N$-dimensional hypercube Adinkra chromotopology
- $A_{N,k} := A_N / C_k$ where C_k is a k-dimensional doubly even code.
- $\mathcal{A}_{N,k} :=$ set of (N, k)-Adinkra chromotopologies
- $\mathcal{X}_{N,k} :=$ set of Riemann surfaces obtained from Adinkras $A_{N,k}$

Adinkra chromotopologies are dessin, and dessins are associated to Belyi functions. We have seen that those Belyi functions (or dessins) have monodromy representations. We want to describe monodromy action on Adinkra chromotopologies.

Let $A \in \mathcal{A}_{(N,k)}$ and $X := X_A$ be the Riemann surface associated to A. The Belyi pair (X, β) can be represented by a monodromy map $F_2 \cong \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to S_d$ where d is the degree of β [**HS1**]. In other words, the Belyi pair is determined by the monodromy permutations $\sigma_0, \sigma_1 \in S_d$. In the earlier section, we looked at the action of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ on the fiber $\beta^{-1}(y)$ of unbranched point $y \in \mathbb{P}^1$: for $\tilde{x} \in \beta^{-1}(y)$, such action is defined by $\tilde{x} \cdot \tilde{\gamma} = \tilde{\gamma}(1)$. Here γ is a loop winding around 0 or 1 based at y, and $\tilde{\gamma}$ is the lift of γ such that $\tilde{\gamma}(0) = \tilde{x}$. In this case, σ_0 describes action of the loop winding around 0, and σ_1 describes action of the loop winding around 1.

Now we are going to describe those permutations σ_0 and σ_1 for the case of Adinkra chromotopologies. We see them as permutations of edges of Adinkra chromotopologies because the Belyi function is unramified there. At each white vertex w, we define an N-cycle $\sigma_0^{(w)}$ by listing the N edges incident to w in the order of the rainbow. Similarly, an N-cycle $\sigma_1^{(b)}$ is given by listing N edges incident to a black vertex b in the opposite order of the rainbow. We define σ_0 and σ_1 as products of $\sigma_0^{(w)}$ and $\sigma_1^{(b)}$ taken over 2^{N-k-1} white and 2^{N-k-1} black vertices respectively:

$$\sigma_0 = \prod_w \sigma_0^{(w)} \qquad \sigma_1 = \prod_b \sigma_1^{(b)}.$$

The pair (σ_0, σ_1) of permutations defined above is the permutation representation pair of the Belyi pair (X_A, β) where X_A is the Riemann surface associated to $A \in \mathcal{A}_{N,k}$.

Monodromy actions at 0 and 1 given by σ_0 and σ_1

Let us describe how σ_0 and σ_1 define monodromy action. By definition of β, we know that $\beta(A) = [0, 1]$. Now take a loop γ_0 winding around 0 in counterclockwise direction. Take the base point of γ_0 to be the intersection of γ_0 and $[0, 1]$. This base point can be lifted to $2^{N-k-1}N$ different points in X, and each of these lifted points lie in a unique edge in A. Then the lift of γ_0 with initial point p_i on the edge of color C_i incident to the white vertex w terminates at point p_{i+1} on the edge of color C_{i+1} incident to the same white vertex w (see Figure 2.2). Thus the monodromy action at 0 sends the edge of color C_i to the edge of color C_{i+1} both incident to the white vertex w. The monodromy action at 1 can be described by applying the same argument for a loop winding around 1.

Monodromy action at ∞ given by σ_∞

The monodromy over ∞ is given by the composition $\sigma_\infty = \sigma_1\sigma_0$. This permutation σ_∞ is the product of $2^{N-k-2}N$ disjoint 2-cycles representing edges of the same color composing a 2-colored face of length 4. The monodromy action at ∞ is given as follows: take a loop γ_∞ winding clockwise around ∞. Then the lift of γ_∞ to the Riemann surface X has an initial point lying on the edge of color C_i and terminal point lying on the edge of color C_i. The edges on which the initial and terminal point of the lift lie compose a (C_i, C_{i+1})-colored face of length 4.

However, it is much more convenient if each cycle in σ_∞ lists all four edges which make up a 2-colored face of length 4. So we introduce another permutation π_∞ which consists of $2^{N-k-2}N$ disjoint 4-cycles. Each 4-cycle in π_∞ lists the edges that make up each face as we traverse the face clockwise with respect to the orientation. The original monodromy σ_∞ can be recovered from π_∞ by forgetting two edges that start from white vertices and head to black vertices.

2.5 Covering Space Theory for Adinkras

In this section, we study a Adinkra chromotopology (seen as a dessin) as covering space of a certain dessin, called *beach balls* B_N. This section is also adapted from [**DIKLM**].

2.5.1 Beach Balls B_N

We consider a Belyi pair $(B_N, \tilde{\beta})$ consisting of the Riemann surface B_N whose corresponding dessin has one black and one white vertex and N edges connecting them, and the Belyi function $\tilde{\beta} : \mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$\tilde{\beta}(x) = \frac{x^N}{x^N + 1}.$$

In this case $\tilde{\beta}$ is a covering map of degree N branched over 0 and 1 only, and thus the fibers over 0 and 1 are $\beta^{-1}(0) = \{0\}$ and $\beta^{-1}(1) = \{\infty\}$. Aside to that, we have $\beta^{-1}(\infty) = \{N\text{th roots of } -1\}$, and indeed β^{-1} is unramified over ∞. Since the Euler characteristic for B_N is $\chi = 2 = 2 - 2g$, its genus is $g = 0$. Thus the graph is embedded in a sphere. We call the dessin B_N the *beach ball*. Notice that this graph is the same graph (seen as a 1-skeleton) as the one we saw in Example 2.2.4, but they are considered to be different dessins since they are embedded in different surfaces; the one is embedded in a sphere, and the other is embedded in a torus.

Remark 2.5.1. *From [*__J__*], we have $B_N = X_N/C_{N,max}$ where X_N is the Adinkra Riemann surface corresponding to the hypercube Adinkra A_N. Since $C_{N,max}$ is not doubly even (it is just a maximal even code), so the Riemann surface B_N is not an Adinkra Riemann surface.*

2.5.2 Monodromy Permutations Revisited

Our goal in this section is to show that the Belyi pair (X, β_X) for $X \in \mathcal{X}_{N,K}$ factors through $(B_N, \tilde{\beta})$. The first step is to give more details on the monodromy elements σ_0, σ_1 for Adinkra chromotopologies.

Let $C_k \subseteq C_{N,max}$ be a doubly even code of dimension k, and $C_{N,max}$ is a maximal even code in $\mathbb{F}_2^N$. Let A be an Adinkra chromotopology with rainbow $(1, 2, \ldots, N)$ and $X \in \mathcal{X}_{N,k}$ be the Riemann surface associated to A. The white vertices of A are elements of the space $C_{N,\max}/C_k$, and the black vertices are elements of D/C_k where $D \subset \mathbb{F}_2^N$ is a set of elements with odd weights. We denote an edge of color i incident to a white vertex c by i_c. Let $I := \{1_c, 2_c, \ldots, N_c \mid c \in C_{N,max}/C_k\}$. We consider monodromy elements σ_0, σ_1 as elements of symmetric group S_I on the set I. Recall that we defined an N-cycle $\sigma_0^{(w)}$ by listing N edges incident to the white vertex w following the order of the rainbow, and similarly the N-cycle $\sigma_1^{(b)}$ is given by listing N edges incident to the black vertex b in the opposite order of the rainbow. For a white vertex $c \in C_{N,max}/C_k$, we have

$$\sigma_0^{(c)} = (1_c, \ldots, N_c)$$

$$\sigma_0 = \prod_{c \in C_{N,max}/C_k} \sigma_0^{(c)} = \prod_{c \in C_{N,max}/C_k} (1_c, \ldots, N_c).$$

Next we describe the monodromy element σ_1 for black vertices. Note that the edge of color i incident to a black vertex $d \in D/C_k$ is incident to the white vertex $d + e_i \in C_{N,max}/C_k$ where e_i is the standard basis vector for $\mathbb{F}_2^N$. Thus the N-cycle $\sigma_1^{(d)}$ can be defined by listing colored edges incident to white vertices written in terms of the black vertex d:

$$\sigma_1^{(d)} = (N_{d+e_N}, \ldots, 1_{d+e_1})$$

$$\sigma_1 = \prod_{d \in D/C_k} \sigma_1^{(d)} = \prod_{d \in D/C_k} (N_{d+e_N}, \ldots, 1_{d+e_1}).$$

Next, we describe the composition of $\sigma_\infty = \sigma_1\sigma_0$ in more detail. For any $i = 1, \ldots, N-1$, let c_i denote an element of $C_{N,max}$ whose only i-th and $(i+1)$-th components are 1 and the rest is 0, that is,

$$c_i := (0, \ldots, 0, 1, 1, 0, \ldots, 0) \in C_{N,\max}.$$

In other words, $c_i = e_i + e_{i+1}$. In fact, the set $\{c_i \mid i = 1, \ldots, N-1\}$ is a generating set for $C_{N,max}$ [**DIKLM**]. When $i = N$, we have

$$c_N := \sum_{i=1}^{N-1} c_i = (1, 0, \ldots, 0, 1).$$

Assume we are given an edge i_c incident a white vertex c. Then i_c is mapped to $(i+1)_c$ via σ_0. Note that if we start with an edge N_c, σ_0 maps it to 1_c. Also note that $(i+1)_c = (i+1)_{c+e_{i+1}}$ where $c + e_{i+1}$ is a black vertex. Hence σ_1 maps $(i+1)_{c+e_{i+1}}$ to $i_{c+e_{i+1}+e_i} = i_{c+c_i}$. Thus the composition $\sigma_1\sigma_0$ takes i_c to i_{c+c_i}.

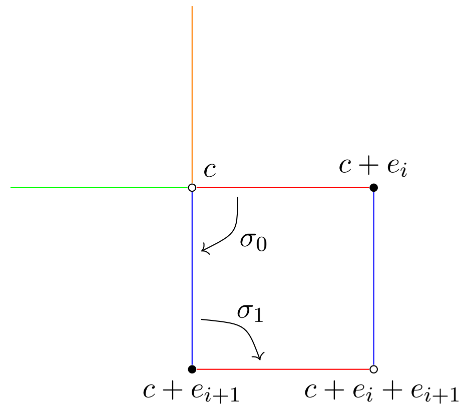


Figure 2.4: Pictorial description of $\sigma_1\sigma_0$

Let $H_i = \{(c, c+c_i) \mid c, c+c_i$ belong to the same face and $c \in C_{N,\max}/C_k\}$. Then we write

$$\sigma_\infty = \prod_{i=1}^{N} \prod_{c \in H_i} (i_c, i_{c+c_i}).$$

Since we follow edges of color i and then $i+1$ to go from the white vertex c to another white vertex $c+c_i$, we can see that

$$\pi_\infty = \prod_{i=1}^{N} \prod_{c \in H_i} (i_c, (i+1)_c, i_{c+c_i}(i+1)_{c+c_i}).$$

Now we are ready to prove the following theorem.

Theorem 2.5.2 ([**DIKLM**])**.** *The Belyi pair (X, β) where $X \in \mathcal{X}_{N,k}$ with rainbow $(1, \dots, N)$ factors through the Belyi pair $(B_N, \tilde{\beta})$ with rainbow $(1, \dots, N)$; that is there exists a map $f_X : X \to B_N$ such that $\beta = \tilde{\beta} \circ f_X$.*

$$\begin{tikzcd} X \arrow[r, dotted, "\exists f_X"] \arrow[dr, "\beta"'] & B_N \arrow[d, "\tilde{\beta}"] \\ & \mathbb{P}^1 \end{tikzcd}$$

Proof. We first consider the Riemann surface X_N associated to the N-hypercube Adinkra A_N. From Remark 2.5.1, we see that the map

$$f_{X_N} : X_N \to B_N$$

is given by taking a quotient of X_N by the maximal even code $C_{N,max} \subseteq \mathbb{F}_2^N$. On the other hand, taking quotient of A_N by $C_{N,max}$ gives the embedded graph $A_N/C_{N,max}$ of B_N, and we denote $A_N/C_{N,max}$ by Σ_N. Since white vertices are the vectors with even weights, $C_{N,max}$ contains all the white vertices of A_N while all the black vertices of A_N lie outside of $C_{N,max}$. Quotienting A_N by $C_{N,max}$ implies that all white vertices of A_N are identified with one another, and thus all the black vertices of A_N can be considered to be identical to each other as well. From [**DFGHILM**], edges of the same color incident to the equivalent vertices are equal, so f_{X_N} maps the hypercube Adinkra A_N onto the quotient $\Sigma_N = A_N/C_{N,max}$:

$$f_{X_N}(A_N) = \Sigma_N.$$

Note that the commutativity of the diagram below implies the desired factorization[3]:

$$\begin{array}{ccc}
\pi_1(\mathbb{C} \setminus \{0,1,\infty\}) \times \beta^{-1}(e) & \xrightarrow{\sigma_0,\sigma_1} & \beta^{-1}(e) \\
\Big\downarrow{\scriptstyle id \times f_{X_N}} & & \Big\downarrow{\scriptstyle f_{X_N}} \\
\pi_1(\mathbb{C} \setminus \{0,1,\infty\}) \times \tilde{\beta}^{-1}(e) & \xrightarrow{\tilde{\sigma}_0,\tilde{\sigma}_1} & \tilde{\beta}^{-1}(e)
\end{array}$$

where e is the single edge in $\mathbb{P}^1$ and the horizontal arrows are monodromy actions. The monodromy elements for the Belyi pair $\tilde{\beta} : B_N \to \mathbb{P}^1$ are $\tilde{\sigma}_0 = (1, 2, \ldots, N)$ and $\tilde{\sigma}_1 = (N, N-1, \ldots, 1)$ [**DIKLM**], and note that the edges of color i are denoted by i. Assume we are given an edge i_c incident to a vertex c on the top-left corner and follow the diagram in the clockwise direction. Then we have

$$f_{X_N}(\sigma_0(i_c)) = f_{X_N}((i+1)_c) = i+1$$

by the construction of f_{X_N}. If we follow the diagram counterclockwise, we obtain

$$\tilde{\sigma}_0(f_{X_N}(i_c)) = \tilde{\sigma}_0(i) = i+1.$$

Hence $f_{X_N}(\sigma_0(i_c)) = \tilde{\sigma}_0(f_{X_N}(i_c))$. We apply the similar argument to σ_1 and obtain

$$f_{X_N}(\sigma_1(i_c)) = \tilde{\sigma}_1(f_{X_N}(i_c)).$$

By the proof given in subsection 2.5.4, we have $\beta = \tilde{\beta} \circ f_{X_N}$.

Now consider the general case which is when $X_{N,k} \in \mathcal{X}_{N,k}$ for a doubly even code C_k. Let $A_{N,k}$ be the corresponding Adinkra chromotopology. Since $C_k \subseteq C_{N,max}$ and $A_{N,k} = A_N/C_k$, we have the following isomorphism [**DIKLM**]:

$$A_{N,k}/C_{N,max} \cong A_N/C_{N,max} = \Sigma_N.$$

[3]Proof for this statement is at the end of section 5.

Thus $A_{N,k} \to A_{N,k}/C_{N,max}$ induces a well-defined map

$$f_{X_{N,k}} : X_{N,k} \to B_N = X_N/C_{N,max}.$$

The map $f_{X_{N,k}}$ takes all the white vertices in $X_{N,k}$ to the single white vertex in Σ_N, and maps all the black vertices in $X_{N,k}$ to the single black vertex in Σ_N. Then it identifies all the edges of given color i with the edge of color i in Σ_N. We can give the similar argument that we gave for the case of X_N to show that the monodromy actions are compatible. □

Let us summarize the branching structures of two Belyi functions on an Adinkra Riemann surface X and B_N.

- $\beta : X \to \mathbb{P}^1$ is ramified over $0, 1$ and ∞. The ramification indices over 0 and 1 are N, and the ramification index over ∞ is 2.
- $\tilde{\beta} : B_N \to \mathbb{P}^1$ is ramified over 0 and 1 only, and the ramification indices at the points over 1 and 0 are N. We highlight that $\tilde{\beta}$ is unramified over ∞.

From the diagram below

$$\begin{tikzcd} X \arrow[d, "f_X"] \arrow[dd, bend left=60, "\beta"] \\ B_N \arrow[d, "\tilde{\beta}"] \\ \mathbb{P}^1 \end{tikzcd}$$

we see that the branching structure of $\beta : X \to \mathbb{P}^1$ splits into two parts. All the branch values with ramification index of N, the points 0 and 1, occur in $\tilde{\beta} : B_N \to \mathbb{P}^1$. The branch value with ramification index of 2, ∞, occurs in $f_X : X \to B_N$.

In Theorem 2.5.2, we showed the existence of maps $f_{X_N} : X_N \to B_N$ and $f_{X_{N,k}} : X_{N,k} \to B_N$ with $X_N \in \mathcal{X}_{N,0}$ and $X_{N,k} \in \mathcal{X}_{N,k}$, so we have the following commutative diagram:

$$\begin{tikzcd} X_N \arrow[r, dotted, "\exists p"] \arrow[d, "f_{X_N}"] \arrow[dd, bend right=60, "\beta_{X_N}"'] & X_{N,k} \arrow[dl, bend left, "f_{X_{N,k}}"] \arrow[ddl, bend left=40, "\beta_{X_{N,k}}"] \\ B_N \arrow[d, "\tilde{\beta}"] & \\ \mathbb{P}^1 & \end{tikzcd}$$

We want to show that the map $f_{X_N} : X_N \to B_N$ factors through $f_{X_{N,k}} : X_{N,k} \to B_N$, that means there is another map $p : X_N \to X_{N,k}$ so that $f_{X_N} = f_{X_{N,k}} \circ p$. In order to prove this statement, we need to use the following remark.

Remark 2.5.3 ([**DIKLM**]). *The monodromy group of* $(X_{N,k}, f_{X_{N,k}} : X_{N,k} \to B_N)$ *is given by the elements*

$$\rho_i = \prod_{c \in H_i} (c, c + c_i)$$

of the symmetric group $S_{c_{N,max}/C_k}$ *on the set* $C_{N,max}/C_k$*,* $1 \leq i \leq N$ *and* $H_i = \{(c, c + c_i) \mid c, c + c_i$ *belong to the same face and* $c \in C_{N,max}/C_k\}$*.*

Note that the monodromy of $f_{X_{N,k}}$ is permutations on a set of white vertices because $f_{X_{N,k}}$ is unramified over white vertices.

Theorem 2.5.4 ([**DIKLM**]). *The Belyi pair* $(X_N, f_{X_N} : X_N \to B_N)$ *factors through* $(X_{N,k}, f_{X_{N,k}} : X_{N,k} \to B_N)$ *for any* $X_{N,k} \in \mathcal{X}_{N,k}$*.*

Proof. Let $p : X_N \to X_{N,k}$ be the projection map obtained from $A_N \to A_N/C_k = A_{N,k}$ where C_k is a doubly even code. We want to show that the monodromy actions are compatible. Then the desired factorization follows.

We need to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B_N^*) \times f_{X_N}^{-1}(w) & \xrightarrow{\rho_i} & f_{X_N}^{-1}(w) \\ \Big\downarrow{\scriptstyle id \times p} & & \Big\downarrow{\scriptstyle p} \\ \pi_1(B_N^*) \times f_{X_{N,k}}^{-1}(w) & \xrightarrow{\bar{\rho}_i} & f_{X_{N,k}}^{-1}(w) \end{array}$$

where w is a white vertex and $B_N^* := B_N - \{\text{centers of faces}\}$. Let c be a white vertex and ρ_i the monodromy generator. Following the diagram in the clockwise direction gives

$$p(\rho_i(c)) = p(c + c_i) = [c + c_i]$$

where $[\cdot]$ is an equivalence class in $C_{N,\max}/C_k$. Similarly, tracing the diagram in counterclockwise direction gives

$$\bar{\rho}_i(p(c)) = \bar{\rho}_i([c]) = [c + c_i].$$

Hence the factorization of (X_N, f_{X_N}) through $(X_{N,k}, f_{X_{N,k}})$ follows. ☐

Corollary 2.5.5 ([**DIKLM**]). *The Belyi pair* (X_N, β_{X_N}) *factors through the Belyi pair* $(X_{N,k}, \beta_{X_{N,k}})$ *for* $X_{N,k} \in \mathcal{X}_{N,k}$ *(i.e.* $\beta_{X_N} = \beta_{X_{N,k}} \circ p$*).*

2.5.3 Tower of $N = 4$ Covering Spaces

From Corollary 2.5.5, we have a tower of covering spaces:

$$\begin{tikzcd} X_N \arrow[d, "p"] \\ X_{N,k} \arrow[d, "f_{X_{N,k}}"] \\ B_N \arrow[d, "\tilde{\beta}"] \\ \mathbb{P}^1 \end{tikzcd}$$

From Theorem 2.5.2 and 2.5.4, we should view Adinkra Riemann surfaces in $\mathcal{X}_{N,k}$ as branched covers of B_N rather than $\mathbb{P}^1$ because we have less branch values if the base is B_N. In particular, we are interested in the tower of covering spaces when N = 4, that is

$$\begin{tikzcd} X_4 \arrow[d, "p"] \\ X_{4,k} \arrow[d, "f_{X_{4,k}}"] \\ B_4 \arrow[d, "\tilde{\beta}"] \\ \mathbb{P}^1 \end{tikzcd}$$

Note that we actually have an extra intermediate cover which is an elliptic curve E between $X_{4,k}$ and B_4. For any subset $C \subseteq C_{N,max}$, consider the map

$$X_N \to X_N/C \to B_N.$$

If C is doubly even, the quotient X_N/C is an Adinkra Riemann surface. However if C is not doubly even, X_N/C is not an Adinkra Riemann surface. When $N = 4$, the vector space $(\mathbb{F}_2)^4$ is 4-dimensional, and the maximal even code $C_{4,\max} \cong (\mathbb{Z}/2\mathbb{Z})^3$ is 3-dimensional. Note that when $N = 4$, the only doubly even code is $C_{DE} = \langle 1111 \rangle = \{1111, 0000\}$ so $C_{DE} \subseteq C_{4,\max}$ is 1-dimensional. Thus there should be a 2-dimensional even code C' lying between $C_{4,\max}$ and C_{DB}:

$$C_{DE} \subseteq C' \subseteq C_{4,\max}.$$

It should be $C' = \langle 1010, 0101 \rangle$, and thus

$$\langle 1111 \rangle \subseteq \langle 1010, 0101 \rangle \subseteq \langle 1100, 0110, 0011 \rangle.$$

From the previous discussion, we know that $X_4/C_{DE} = X_4/\langle 1111 \rangle = X_{4,1} \in \mathcal{X}_{4,1}$ and $X_4/C_{4,\max} = X_4/\langle 1100, 0110, 0011 \rangle = B_4$. Since C' is not doubly even, $E := X_4/C' = X_4/\langle 1010, 0101 \rangle$ is not an Adinkra Riemann surface. However it is still a Riemann surface, more specifically it is an elliptic curve (which is a double cover of B_4). Hence a complete picture for the tower of covering spaces for $N = 4$ is

$$\begin{tikzcd} X_4 \arrow[d, "p"] \\ X_{4,1} \arrow[d, "f_{X_{4,1}}"] \\ E \arrow[d] \\ B_4 \arrow[d, "\tilde{\beta}"] \\ \mathbb{P}^1 \end{tikzcd}$$

We will investigate this covering space tower from theories of quivers and dimer models in the next chapter.

2.5.4 Proof: Compatible Monodromy Action Implies The Factorization

In the last section of chapter 2, we want to give a proof for the statement in the proof of Theorem 2.5.2: the commutativity of the diagram

$$\begin{tikzcd} \pi_1(\mathbb{C} \setminus \{0,1,\infty\}) \times \beta^{-1}(e) \arrow[r, "\sigma_0{,}\sigma_1"] \arrow[d, "id \times f_{X_N}"] & \beta^{-1}(e) \arrow[d, "f_{X_N}"] \\ \pi_1(\mathbb{C} \setminus \{0,1,\infty\}) \times \tilde{\beta}^{-1}(e) \arrow[r, "\tilde{\sigma}_0{,}\tilde{\sigma}_1"] & \tilde{\beta}^{-1}(e) \end{tikzcd}$$

implies the factorization $\beta = \tilde{\beta} \circ f_X$, i.e.

$$\begin{tikzcd} X \arrow[r, dotted, "\exists f_X"] \arrow[dr, "\beta"'] & B_N \arrow[d, "\tilde{\beta}"] \\ & \mathbb{P}^1 \end{tikzcd}$$

To prove this statement, we will use some facts about Fuchsian uniformizations of Adinkra Riemann surfaces [**DIKLM**], [**GG**]:

- $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ is uniformized by $\Gamma_{\infty,\infty,\infty} = \langle x_1, x_2, x_3 \mid x_1x_2x_3 = 1\rangle$
- $\mathbb{P}^1$ is uniformized by $\Gamma_{N,N,2} = \langle x_1, x_2, x_3 \mid x_1^N = x_2^N = x_3^N = x_1x_2x_3 = 1\rangle$
- B_N is uniformized by $\Gamma_N = \langle y_1, \ldots, y_N \mid y_1^2 = \cdots = y_N^2 = y_1 \cdots y_N = 1\rangle$
- X_N is uniformized by $\Gamma_{(N,0)} = [\Gamma_N, \Gamma_N]$

Proof. We consider the case when $N \geq 5$ (i.e. genus of $X_N \geq 2$).

Consider $X_N \to \mathbb{P}^1$. In particular,

$$\beta : X_N \setminus \beta^{-1}(\{0, 1, \infty\}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

being a covering map implies that

$$\pi_1(\beta) : \pi_1(X_N \setminus \Sigma_\beta) \hookrightarrow \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$$

where $\Sigma_\beta = \beta^{-1}(\{0, 1, \infty\})$. Also note that under Fuchsian uniformization, β can be written as

$$\mathbb{H}/\Gamma \to \mathbb{H}/\Gamma_{\infty,\infty,\infty}$$

where Γ is a Fuchsian group uniformizing $X_N \setminus \Sigma_\beta$. The monodromy homomorphism for the cover $\beta : X_N \setminus \Sigma_\beta \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is

$$\begin{aligned} \rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) &\to S_I \\ \gamma_0 &\mapsto \sigma_{\gamma_0}^{-1} \\ \gamma_1 &\mapsto \sigma_{\gamma_1}^{-1} \end{aligned}$$

where $I = \{1_c, \ldots, N_c \mid c \in C_{N,\max}/C_k\}$ and $C_k \subseteq C_{N,\max}$ is a k-dimensional doubly even code. $ker(\rho) \subseteq \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ corresponds to

$$\underbrace{\mathbb{H}/ker(\rho)}_{X_N \setminus \Sigma_\beta} \to \underbrace{\mathbb{H}/\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}$$

by Theorem 1.69 (v) [**GG**] taking $\tilde{X} = \mathbb{H}$ and $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Also note that $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \Gamma_{\infty,\infty,\infty}$ and $ker(\rho) = \Gamma$.

Next we consider the beach ball $B_N, N \geq 5$. The covering map

$$\tilde{\beta} : B_N \setminus \Sigma_{\tilde{\beta}} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

where $\Sigma_{\tilde{\beta}} = \tilde{\beta}^{-1}(\{0, 1, \infty\})$ has an expression in terms of Fuchsian uniformizations:

$$\mathbb{H}/\tilde{\Gamma} \to \mathbb{H}/\Gamma_{\infty,\infty,\infty}$$

where $\tilde{\Gamma}$ is a Fuchsian group uniformizing $B_N \setminus \Sigma_{\tilde{\beta}}$. Since $\tilde{\beta}$ is a covering map, we have

$$\pi_1(\tilde{\beta}) : \pi_1(B_N \setminus \Sigma_{\tilde{\beta}}) \hookrightarrow \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}).$$

The monodromy homomorphism for the cover $\tilde{\beta} : B_N \setminus \Sigma_{\tilde{\beta}} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is

$$\begin{aligned} \tilde{\rho} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) &\to S_N \\ \gamma_0 &\mapsto \tilde{\sigma}_{\gamma_0}^{-1} \\ \gamma_1 &\mapsto \tilde{\sigma}_{\gamma_1}^{-1}. \end{aligned}$$

$ker(\tilde{\rho}) \subseteq \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ corresponds to

$$\underbrace{\mathbb{H}/ker(\tilde{\rho})}_{B_N \setminus \Sigma_{\tilde{\beta}}} \to \underbrace{\mathbb{H}/\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}$$

by Theorem 1.69 (v) [**GG**]. Also note that $ker(\tilde{\rho}) = \tilde{\Gamma}$.

We want to show that $ker(\rho) \subseteq \ker(\tilde{\rho})$ in $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. Consider the following diagram:

$$\begin{tikzcd}
S_I \arrow[r, "i_c \mapsto i"] & S_N \\
Im(\rho) \arrow[u, hook] \arrow[r] & Im(\tilde{\rho}) \arrow[u, hook] \\
\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \arrow[u, "\rho"] \arrow[r, equal] & \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \arrow[u, "\tilde{\rho}"'] \\
ker(\rho) \arrow[u, hook] & ker(\tilde{\rho}) \arrow[u, hook]
\end{tikzcd}$$

Let $x \in ker(\rho)$ which means $\rho(x) = id_\rho \in Im(\rho)$. Using the map $Im(\rho) \rightarrow Im(\tilde{\rho})$, $id_\rho \in Im(\rho)$ gets mapped to $id_{\tilde{\rho}} \in Im(\tilde{\rho})$. Thus the composition map $F_2 \rightarrow Im(\rho) \rightarrow Im(\tilde{\rho})$ takes the kernel element $x \in F_2$ to the identity element $id_{\tilde{\rho}} \in Im(\tilde{\rho})$. This means $\tilde{\rho}(x) = id_{\tilde{\rho}}$ and thus $x \in ker(\tilde{\rho})$. Hence $ker(\rho) \subseteq \ker(\tilde{\rho})$ in $F_2 \cong \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. By the Corollary 2.10 from [**LOO**], $ker(\rho) \subseteq ker(\tilde{\rho})$ implies $\beta = \tilde{\beta} \circ f_{X_N}$. □

Chapter 3

Quiver Theory on Adinkras

3.1 Ribbon Graphs and Quivers

This chapter is based on Bocklandt's work [**B3**][**B4**][**B6**]. We first set orientation which we use for the entire chapter.

Orientation

- counterclockwise cycles are positive cycles
- clockwise cycles are negative cycles
- counterclockwise about white vertices and clockwise about black vertices (when taking the dual of a bipartite ribbon graph and defining its orientation)

3.1.1 Ribbon Graphs

Recall that Adinkras also have a ribbon graph structure, and a ribbon graph is equivalent to a graph embedded in a surface. Let us restate the definition of ribbon graphs.

Definition 3.1.1. A *ribbon graph* is a graph $\Gamma = (H, \nu, \epsilon)$ consisting of

- a set of half edges H
- two permutations $\nu, \epsilon : H \to H$

such that ϵ is an involution (i.e. $\epsilon^2 = id$). In the cycle decompositions of ν and ϵ, each cycle of ν is called a *vertex* of Γ, and that of ϵ is called an *edge*. Moreover, cycles in the cycle decomposition of $\psi := \nu \circ \epsilon$ are called *faces* of Γ.

In general, swapping the roles of vertices and faces of a plane graph gives a dual graph. This construction of dual graphs also applies to ribbon graphs.

Definition 3.1.2. The *dual* of a ribbon graph $\Gamma = (H, \nu, \epsilon)$ is the graph $\Gamma^\vee = (H, \psi, \epsilon)$.

Hence $\Gamma^\vee$ have the same set H of half edges as Γ, but the roles of vertices ν and faces ψ are swapped. It is not guaranteed that $\Gamma^\vee$ is isomorphic to Γ.

Example 3.1.3. In Example 2.2.6, we showed that a ribbon graph $\Gamma = (H, \nu, \epsilon)$ where

$$\begin{aligned} \epsilon &= (1,5)(2,6)(3,7)(4,8) \\ \nu &= (1,2,3,4)(8,7,6,5) \\ \psi &= \nu \circ \epsilon = (1,8)(2,5)(3,6)(4,7). \end{aligned}$$

gives the following ribbon graph

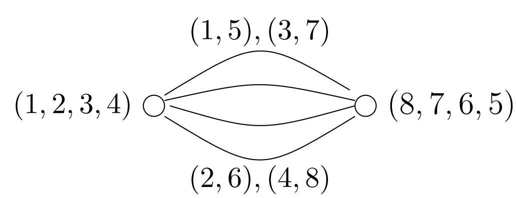


Figure 3.1: The ribbon graph Γ from Example 2.2.6

The dual $\Gamma^\vee$ of the ribbon graph Γ is obtained by switching the role of ν and ψ. As a result, ν represents faces and ψ represents vertices of $\Gamma^\vee$.

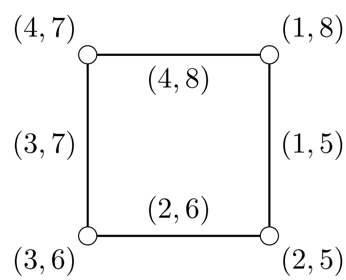


Figure 3.2: Dual of the ribbon graph Γ

It is possible to put a bipartite graph structure on a ribbon graph by breaking up the vertex set of the ribbon graph into two disjoint subsets. Note that we can do this only when the ribbon graph has no cycles of odd length (by Köning's theorem [**ADH**]). In the case of Example 3.1.3, the ribbon graph Γ only has faces whose boundary cycles are of length 2, so it can be turned into a bipartite ribbon graph. We set $(1,2,3,4)$ to be black and $(8,7,6,5)$ to be white vertex.

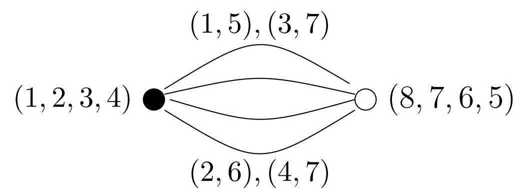


Figure 3.3: The ribbon graph Γ with a bipartite structure

In particular a bipartite ribbon graph embedded in a torus is often called a *dimer model*. Moreover, we have another operation on bipartite ribbon graphs called the *mirror* of a bipartite ribbon graph. It is defined by reversing the cycles representing white vertices [**B3**].

Definition 3.1.4. The *mirror* (a.k.a. *twist, specular dual*) of a bipartite ribbon graph $\Gamma = (H, \nu, \epsilon)$ is a ribbon graph $\Gamma^{\bowtie} = (H, \nu', \epsilon)$ where

$$\nu'(x) = \begin{cases} \nu(x) & \text{if } \nu\text{-orbit of } x \text{ is black} \\ \nu^{-1}(x) & \text{if } \nu\text{-orbit of } x \text{ is white} \end{cases}$$

This mirror operation produces the same graph as what we started with, but the surface in which it is embedded might be different. Note that the mirror operation is an involution, i.e. $(\Gamma^{\bowtie})^{\bowtie} = \Gamma$ [**B1**]. The following example demonstrates the definition of mirror ribbon graph.

Example 3.1.5. Consider the ribbon graph Γ in Example 3.1.3 which is embedded in a sphere. If we identify the cycle $(1, 2, 3, 4)$ in the cycle decompotision of ν with a black vertex and $(8, 7, 6, 5)$ with a white vertex, then its mirror ribbon graph $\Gamma^{\bowtie} = (H, \epsilon, \nu')$ is given by

$$\begin{aligned} H &= \{1, 2, \dots, 8\}, \\ \epsilon &= (1, 5)(2, 6)(3, 7)(4, 8), \\ \nu' &= (1, 2, 3, 4)(5, 6, 7, 8). \end{aligned}$$

This gives

$$\psi' = \nu' \circ \epsilon = (1, 6, 3, 8)(2, 7, 4, 5).$$

Its Euler characteristic is $\chi = 2 - 4 + 2 = 0$ which means $0 = \chi = 2 - 2g$, so $g = 1$. Therefore the mirror ribbon graph $\Gamma^{\bowtie}$ is embedded in a torus, and it is a dimer model.

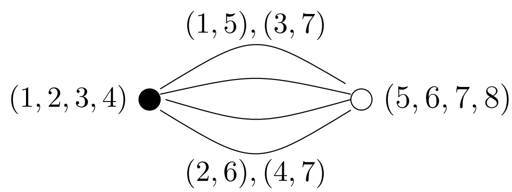


Figure 3.4: Mirror of the ribbon graph Γ in Example 3.1.3

3.1.2 Quivers

Let us introduce a type of directed graphs called *quivers*. Unlike ordinary directed graphs (in graph theoretical sense), quivers are allowed to have loops and multiple arrows between vertices. We will see later how dimer models (bipartite ribbon graphs) give rise to quivers.

Definition 3.1.6. A *quiver* Q is a directed graph given by a set Q_0 of vertices, a set Q_1 of arrows, and maps $h, t : Q_1 \to Q_0$ which specify head and tail of a given arrow.

$$\underset{t(a)}{\bullet} \xrightarrow{\quad a \quad} \underset{h(a)}{\bullet}$$

A *path* p of Q is a sequence of arrows $p := a_1 \cdots a_k$ in Q such that $h(a_i) = t(a_{i+1})$ for $a_1, \ldots, a_k \in Q_1, 1 \leq i < k$ read from left to right (use Craw's convention [**C1**]):

$$\bullet \xrightarrow[a_1]{} \overset{h(a_1)=t(a_2)}{\bullet} \xrightarrow[a_2]{} \overset{h(a_2)=t(a_3)}{\bullet} \xrightarrow[a_3]{} \bullet$$

Figure 3.5: path $p := a_1 a_2 a_3$

We set $t(p) = t(a_1)$ and $h(p) = h(a_k)$. In addition, for each vertex $v_i \in Q_0$, we define a *trivial path* e_i which is a path of length 0, i.e. $h(e_i) = v_i = t(e_i)$. A nontrivial path $p = a_1 \cdots a_k$ is called a *cycle* if $h(p) = t(p)$. If Q contains no cycles, it is called *acyclic*.

Next we define algebras which encode structures of quivers.

Definition 3.1.7. Let Q be a quiver. The *path algebra* of Q is a $\mathbb{C}$-algebra $\mathbb{C}Q$ whose underlying $\mathbb{C}$-vector space has a basis consisting of all paths of Q. Its multiplication is a concatenation of paths in Q if possible, and zero otherwise.

Note that a path algebra $\mathbb{C}Q$ is graded by path length [**C1**]:

$$\mathbb{C}Q = \bigoplus_{k=0} (\mathbb{C}Q)_k$$

where $(\mathbb{C}Q)_k$ is the vector subspace spanned by paths of length k, and $(\mathbb{C}Q)_k \cdot (\mathbb{C}Q)_\ell \subseteq (\mathbb{C}Q)_{k+\ell}$. Furthermore, $(\mathbb{C}Q)_0$ is a semisimple ring whose basis elements e_i are orthogonal idempotents, i.e. $e_i^2 = e_i$, $e_i e_j = e_i$ if $i = j$ and 0 otherwise.

Remark 3.1.8. *The identity element of $\mathbb{C}Q$ is $\sum_{i \in Q_0} e_i$ where e_i is the trivial path (so its length is 0). This is because for any $a \in Q_1$, $ae_i \neq 0$ when $i = h(a)$ and $ae_i = 0$ otherwise. Thus we have*

$$a(\sum e_i) = 0 + \cdots + 0 + ae_{h(a)} + 0 + \cdots + 0 = ae_{h(a)} = a$$

Similarly we can show $(\sum e_i)a = a$ by using the fact that $e_i a \neq 0$ when $i = t(a)$ and zero otherwise.

3.1.3 Quivers Generated From Ribbon Graphs

Given a bipartite ribbon graph Γ, one can take its dual $\Gamma^\vee$, and add the orientation (we defined at the beginning) to each edge of $\Gamma^\vee$. The resulting directed graph is a quiver Q (more precisely, it is an embedded quiver).

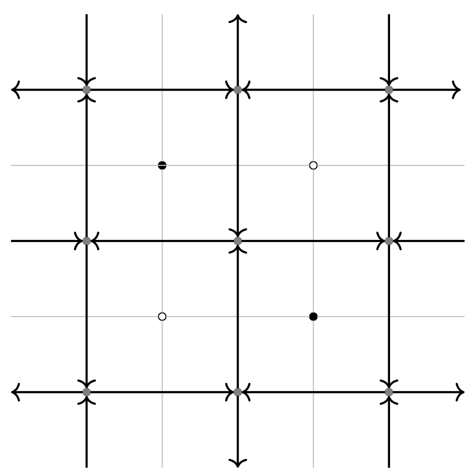


Figure 3.6: A quiver generated from a ribbon graph

Note that one can also start with the mirror $\Gamma^{\bowtie}$ of Γ and follow the same procedure to get another embedded quiver corresponding to the dual $\Gamma^{\bowtie\vee}$ of $\Gamma^{\bowtie}$. The following correlation diagram describes how dual and mirror ribbon graphs are related in terms of permutations.

dual

mirror

Γ	$\Gamma^{\vee}$
vertices: ν	vertices: ψ
edges: ϵ	edges: ϵ
faces: $\psi = \nu \circ \epsilon$	edges: ν
$\Gamma^{\bowtie}$	$\Gamma^{\bowtie\vee} = \Gamma^{\vee\bowtie}$
vertices: ν'	vertices: ψ'
edges: ϵ	edges: ϵ
faces: $\psi' = \nu' \circ \epsilon$	faces: ν'

Figure 3.7: Dual and Mirror Operations on a Ribbon Graph

The top left corner represents a (bipartite) ribbon graph Γ before we take the dual or mirror. Taking orientation into account, the column on the right represents embedded quivers generated from the bipartite ribbon graphs Γ and $\Gamma^{\bowtie}$. By abuse of notation, we write quivers obtained from $\Gamma, \Gamma^{\bowtie}$ as $\Gamma^{\vee}, \Gamma^{\bowtie\vee}$ respectively.

By imposing relations on a quiver, we can construct a quotient algebra of the path algebra. In particular, we are interested in a relation given by a sum of counterclockwise cycles minus a sum of clockwise cycles.

Definition 3.1.9. Let Q be a quiver and $\mathbb{C}Q$ be its path algebra. A *superpotential* $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ is a linear combination of cyclic paths of Q. The *cyclic derivative* of a boundary cycle $p = a_1 \cdots a_k$ with respect to an arrow a_i is

$$\partial_{a_i} p = a_{i+1} a_{i+2} \cdots a_k a_1 \cdots a_{i-1}.$$

Note that p is equivalent up to permutation. The definition of cyclic derivatives linearly extends to cyclic derivatives of the superpotential W.

If Q is a quiver obtained from a dimer model, a superpotential W can be defined as a sum of all couterclockwise cycles of Q minus a sum of all clockwise cycles of Q:

$$W = \sum_{c \in Q_2^+} c - \sum_{c \in Q_2^-} c$$

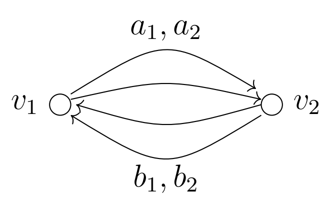
where Q_2^+ is a set of couterclockwise cycles of Q and Q_2^- is a set of clockwisecycles of Q.

Definition 3.1.10. The *Jacobi algebra* of a quiver Q with superpotential W is

$$J(Q) = \frac{\mathbb{C}Q}{\langle \partial_a W \mid a \in Q_1 \rangle}.$$

Let us give an example of a Jacobi algebra of a quiver with superpotential.

Example 3.1.11. Consider a quiver Q with a superpotential $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$.



In fact this quiver is obtained from the dimer model $\Gamma^{\bowtie}$ in Example 3.1.5. Since dual operation has no effect on the underlying surface, Q is still embedded in a torus. That means Q can be viewed as a periodic quiver drawn on a plane, and it has the following fundamental domain:

$$\begin{tikzcd}
v_2 & v_1 \arrow[l, "a_2"'] \\
v_1 \arrow[u, leftarrow, "b_2"] \arrow[r, "a_1"] & v_2 \arrow[u, "b_1"'] \arrow[d, "b_2"] \\
v_2 \arrow[u, "b_1"] & v_1 \arrow[l, "a_2"]
\end{tikzcd}$$

From this diagram, we see that there is one couterclockwise cycle (i.e. positive cycle) $a_1b_1a_2b_2$ and one clockwise cycle (i.e. negative cycle) $a_1b_2a_2b_1$. Hence a superpotential for this quiver is $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$. Then the corresponding Jacobi algebra is

$$J(Q) = \frac{\mathbb{C}Q}{\langle \partial_{a_1}W, \partial_{a_2}W, \partial_{b_1}W, \partial_{b_2}W \rangle}$$

where

$$\begin{aligned}
\partial_{a_1}W &= b_1a_2b_2 - b_2a_2b_1, \\
\partial_{a_2}W &= b_2a_1b_1 - b_1a_1b_2, \\
\partial_{b_1}W &= a_2b_2a_1 - a_1b_2a_2, \\
\partial_{b_2}W &= a_1b_2a_2 - a_2b_1a_1.
\end{aligned}$$

3.2 Applications to Adinkras

3.2.1 Smash Product Expression

Our goal for this chapter is to apply all these technologies to $N = 4$ covering space tower of Adinkras. Especially we are interested in quivers obtained from mirrors of $N = 4$ Adinkras (i.e. the bottom right corner of the chart):

$$\begin{tikzcd}
X_4^{\bowtie\vee} \arrow[d, "p^{\bowtie\vee}"] & X_4^{\vee} \arrow[d, "p^{\vee}"] \\
X_{4,1}^{\bowtie\vee} \arrow[d, "f_x^{\bowtie\vee}"] & X_{4,1}^{\vee} \arrow[d, "f_x^{\vee}"] \\
E^{\bowtie\vee} \arrow[d] & E^{\vee} \arrow[d] \\
B_4^{\bowtie\vee} \arrow[d, "\tilde{\beta}^{\bowtie\vee}"] & B_4^{\vee} \arrow[d, "\tilde{\beta}^{\vee}"] \\
(\mathbb{P}^1)^{\bowtie\vee} & (\mathbb{P}^1)^{\vee}
\end{tikzcd}$$

Figure 3.8: Left: the tower of the mirror duals, Right: the tower of the duals.

We refer to the appendix for the images of duals and mirror duals of B_4, E, $X_{4,1}$, and X_4. Note that all the covers except for $B_4^{\bowtie\vee} \to (\mathbb{P}^1)^{\bowtie\vee}$ and $B_4^{\vee} \to (\mathbb{P}^1)^{\vee}$ are covers of degree 2, and we will focus on those degree-2 covers for the rest of the chapter. Also we are particularly interested in the mirror dual side since we have an unexpected observation which we will explain soon.

Let us introduce smash products first. Later, we will write Jacobi algebras of quivers associated to $B_4^{\bowtie}, E^{\bowtie}$, and $X_{4,1}^{\bowtie}$ as smash products of Jacobi algebras of the base quivers.

Definition 3.2.1. Let A be an algebra and G a finite group acting on A. A *smash product* $A * G$ is defined by

$$A * G := A \otimes \mathbb{C}G.$$

The multiplication in $A * G$ is given by

$$(a_1 \otimes g_1) \cdot (a_2 \otimes g_2) = a_1 g_1(a_2) \otimes g_1 g_2$$

for $a_1, a_2 \in A$ and $g_1, g_2 \in G$. There is an injective homomorphism

$$\begin{aligned} A &\to A * G \\ a &\mapsto a \otimes id_G \end{aligned}$$

Hence A can be viewed as a subalgebra of $A * G$.

Next, we start investigation on the mirror dual side. We count numbers of cycles in the cycle decompositions of ϵ, ψ', and ν' for each mirror dual ribbon graphs in the tower to compute their genera:

$$g(X_4^{\bowtie\vee}) = 5, \quad g(X_{4,1}^{\bowtie\vee}) = 1, \quad g(E^{\bowtie\vee}) = 1, \quad g(B_4^{\bowtie\vee}) = 1.$$

By applying Riemann-Hurwitz formula for each covering map, we see that only the covering map $X_4^{\bowtie\vee} \to X_{4,1}^{\bowtie\vee}$ is ramified over 8 vertices, and everything else is unramified. In fact, we don't observe this phenomenon in the dual side (the tower on the right in Figure 3.8); all the covering maps in the dual side are unramified. According to Bocklandt's definition of Galois covers [**B3**], the covering maps $X_{4,1}^{\bowtie\vee} \to E^{\bowtie\vee}$ and $E^{\bowtie\vee} \to B_4^{\bowtie\vee}$ are Galois and $X_4^{\bowtie\vee} \to X_{4,1}^{\bowtie\vee}$ is not (because it is branched). Hence, we can express Jacobi algebras of covers as smash products of Jacobi algebras of their base mirror dual graphs and $\mathbb{Z}/2\mathbb{Z}$ [**B3**]:

$$J(X_{4,1}^{\bowtie\vee}) = J(E^{\bowtie\vee}) * \mathbb{Z}/2\mathbb{Z},$$

$$J(E^{\bowtie\vee}) = J(B_4^{\bowtie\vee}) * \mathbb{Z}/2\mathbb{Z}.$$

However, we cannot express $J(X_4^{\bowtie\vee})$ in terms of a smash product of $J(X_{4,1}^{\bowtie\vee})$ because $X_4^{\bowtie\vee} \to X_{4,1}^{\bowtie\vee}$ is not a Galois cover. We would like to point out that $J(B_4^{\bowtie\vee})$ which is the Jacobi algebra $J(Q)$ from Example 3.1.11 can be written as a smash product of the down-up algebra $A(0,1,0) = \langle x, y \mid x^2y - yx^2, y^2x - xy^2\rangle$ with $\mathbb{Z}/2\mathbb{Z}$ [**RRZ**]:

$$J(B_4^{\bowtie\vee}) = A(0,1,0) * \mathbb{Z}/2\mathbb{Z}.$$

Unlike the mirror dual side, all the covering maps in the dual quiver side are Galois, so a Jacobi algebra of a cover is a smash product of a Jacobi algebra of a base quiver with $\mathbb{Z}/2\mathbb{Z}$.

3.2.2 Calabi-Yau Algebras

Jacobi algebras of embedded quivers with superpotentials has a strong connection to a noncommutative generalization of Calabi-Yau varieties, which are called *Calabi-Yau algebras*.

Definition 3.2.2 ([**G**])**.** Let A be an ordinary algebra over $\mathbb{C}$. Then A is called a *Calabi-Yau algebra of dimension* d (in short CY-d) if it satisfies these two conditions as a bimodule over itsetlf:

- A is *homologically smooth* which means A has a bounded resolution by finitely generated projective $A-A$-bimodules
- $Ext^k_{\text{A-Bimod}}(A, A\otimes A) \cong \begin{cases} A & \text{if } k = d \\ 0 & \text{if } k \neq d \end{cases}$

A Calabi-Yau variety X has a property that for any $\mathcal{F}, \mathcal{G} \in D(coh(X))$,

$$Hom(\mathcal{F}, \mathcal{G}) \cong Hom(\mathcal{G}, \mathcal{F}[d])^\vee.$$

A Calabi-Yau algebra A was defined so that it also satisfies a similar condition in a category $modA$ of finitely generated projective modules over A [**B1**][**BR**]: if A is a Calabi-Yau algebra of dimension d, it satisfies that for any $M, N \in D(modA)$,

$$Hom(M, N) \cong Hom(N, M[d])^\vee.$$

Famous examples of Calabi-Yau algebras are algebras $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n variables and a coordinate ring $\mathbb{C}[X]$ of a Calabi-Yau variety X [**LM**]. The first one is of dimension n and the last one is of dimension $dimX$.

Remark 3.2.3 ([**B4**], [**RRZ**])**.** *The Jacobi algebra $J(B_4^{\bowtie\vee})$ is a Calabi-Yau algebra of dimension* 3.

We note that any CY-3 algebra is a Jacobi algebra of some embedded quiver with a superpotential, but not all Jacobi algebras are CY-3 [**BR**]. However there is a way to check if a given Jacobi algebra of an embedded quiver is CY-3.

Theorem 3.2.4 ([**B6**]). *If Q is a quiver embedded in a surface of genus $g \geq 1$ and zigzag consistent, then $J(Q)$ is CY-3.*

To describe what zigzag consistent means, we first want to introduce two kinds of zigzag path called zig ray and zag ray. Let Q be an embedded quiver and consider its universal cover $\tilde{Q}$.

Definition 3.2.5. For any arrow $a \in \tilde{Q}$, an infinite path $a_0a_1a_2a_3\dots$ such that $a_0 = a$ and

- $a_{\text{odd}}a_{\text{even}}$ sits in a positive cycle (i.e. counterclockwise cycle)
- $a_{\text{even}}a_{\text{odd}}$ sits in a negative cycle (i.e. clockwise cycle)

is called a *zig ray* of a. Similarly, an infinite path $a_0a_1a_2a_3\dots$ such that $a_0 = a$ and

- $a_{\text{even}}a_{\text{odd}}$ sits in a positive cycle (i.e. counterclockwise cycle)
- $a_{\text{odd}}a_{\text{even}}$ sits in a negative cycle (i.e. clockwise cycle)

is called *zag ray* of a.

Figure 3.9 is an example of a zig ray (blue) $a'_0a'_1a'_2a'_3\dots$ and a zag ray (red) $a_0a_1a_2a_3\dots$ of an edge $a = a_0 = a'_0$ in universal cover $\tilde{Q}$ of a quiver Q obtained from $B_4^{\bowtie\vee}$.

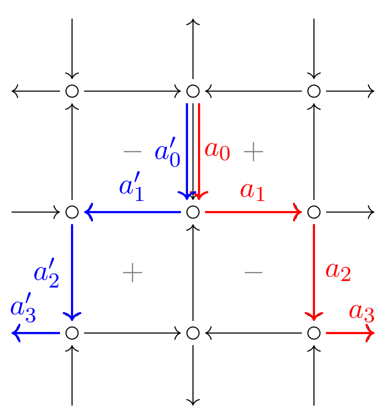


Figure 3.9: zig ray (blue) and zag ray(red) of an arrow $a \in \tilde{Q}$

Definition 3.2.6. An embedded quiver Q is called *zigzag consistent* if for every arrow $\tilde{a} \in \tilde{Q}$, the zig ray and the zag ray of $\tilde{a}$ intersect only at $\tilde{a}$.

Let $\tilde{Q}$ denote the universal cover of the quiver Q obtained from $B_4^{\bowtie\vee}$. We see that $\tilde{Q}$ is exactly the periodic quiver in Figure 3.9, and thus Q is zigzag consistent. This is because zig ray and zag ray of each arrow in $\tilde{Q}$ are isomorphic to the ones in Figure 3.9. In fact, this universal cover $\tilde{Q}$ in Figure 3.9 is also the universal covers of $E^{\vee} = E^{\bowtie\vee}, X_{4,1}^{\vee} = X_{4,1}^{\bowtie\vee}$, and $X_4^{\vee}$ because doubling the fundamental domain of $B_4^{\bowtie\vee}$ gives that of $E^{\bowtie\vee}$, and doubling the fundamental domain of $E^{\bowtie\vee}$ gives that of $X_{4,1}^{\bowtie\vee}$ and so on. Hence they are all zigzag consistent which implies that their Jacobi algebras are CY-3.

We also claim that $X_4^{\bowtie\vee}$ is also zigzag consistent. Note that only $X_4^{\bowtie\vee}$ is a genus-5 curve in the compositions $X_4^{\bowtie\vee} \to X_{4,1}^{\bowtie\vee} \to E^{\bowtie\vee} \to B_4^{\bowtie\vee}$. To prove the claim, we consider the quiver generated from $X_4^{\bowtie\vee}$ lifted to the universal cover $\tilde{X}_4^{\bowtie\vee}$. We also use the following property of hyperbolic tessellation: Consider a geodesic in a Poincaré disk and two non-geodesic arcs equidistant from the geodesic (such a pair of non-geodesic curves always exist [**DU**]). Then those two equidistant arcs bound a zigzag path consisting of edges of boundary cycles of the tesselation, and the geodesic passes through the midpoints of the edges of the zigzag path (see Figure 3.10).

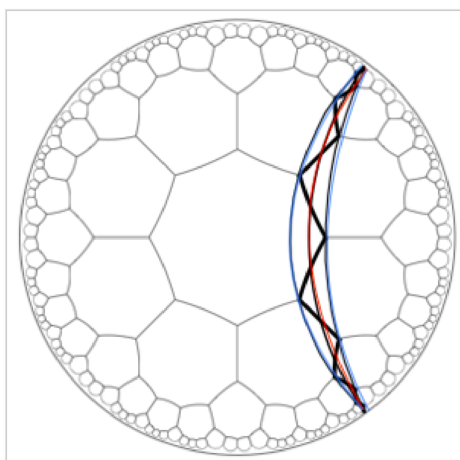


Figure 3.10: A geodesic (red) and two equidistant curves (blue) from the geodesic (image taken from [**ESC**])

Claim 3.2.7. $X_4^{\bowtie\vee}$ *is zigzag consistent.*

Proof. Assume we are given a geodesic and two equidistant curves which bound a zigzag path γ with a starting arrow α in the universal cover $\tilde{X}_4^{\bowtie\vee}$. There is another zigzag path γ' with the same starting arrow α. This is because if γ is constructed in a way that it turns right (at $h(\alpha)$) first and turns left next and so on, then the other zigzag path γ' is constructed by making a left tern (at $h(\alpha)$) first then a right turn next and so on. Hence we have one geodesic and two bounding "parallel" arcs for each construction of a zigzag path with the starting edge α. Such geodesics intersect only at the midpoint of α [**BE**]. This means that the zig ray and the zag ray of α intersect only at α. This argument works for any starting arrow $\alpha \in \tilde{X}_4^{\bowtie\vee}$. Thus $X_4^{\bowtie\vee}$ is zigzag consistent. $\square$

The images below explain the idea of the proof of the claim. The geodesic and bounding curves in Figure 3.11 capture zigzag paths of the blue edge connecting the vertices 0000 and 0100. Although the universal cover of the quiver T generated from $X_4^{\bowtie}$ is not included in the images, using these images is sufficient[1]for us to understand the idea.

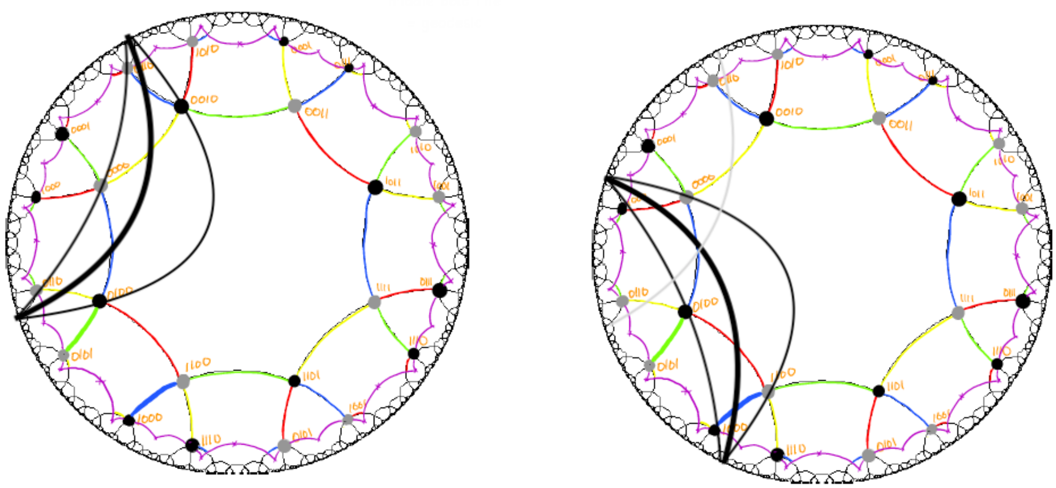


Figure 3.11: Two different geodesics passing through the midpoints of two different zigzag paths (images taken from [**D**])

[1]Note that the quiver T generated from $X_4^{\bowtie}$ is $X_4^{\bowtie\vee}$ with orientation. As a graph, X_4 is isomorphic to $X_4^{\vee}$ [**DIKLM**]. The mirror operation $\bowtie$ changes underlying surface of a bipartite ribbon graph without altering the graph, so we see that

- X_4 and $X_4^{\bowtie}$ are the same as a graph (but X_4 is embedded in a torus and $X_4^{\bowtie}$ is embedded in a genus-5 curve)
- $X_4^{\vee}$ and $X_4^{\vee\bowtie}$ are the same as a graph (but $X_4^{\vee}$ is embedded in a torus and $X_4^{\vee\bowtie}$ is embedded in a genus-5 curve)

Thus as a graph (i.e. ignoring the underlying surface), X_4, $X_4^{\bowtie}$, $X_4^{\vee}$, and $X_4^{\vee\bowtie}$ are all isomorphic to each other. Then, in particular, the midpoint of an edge of $X_4^{\bowtie}$ is equivalent to the midpoint of the corresponding edge in $X_4^{\bowtie\vee}$. Therefore in their universal cover $\tilde{X}_4^{\bowtie} = \tilde{X}_4^{\bowtie\vee} = \mathbb{H}$, a geodesic in $\tilde{X}_4^{\bowtie}$ is equivalent to a geodesic in $\tilde{X}_4^{\bowtie\vee}$ (by seeing a geodesic as path passing through midpoints

The main result of this chapter is the following:

Theorem 3.2.8. *The embedded quivers $X_{4,1}^{\bowtie\vee} = X_{4,1}^{\vee}, X_4^{\vee}$, and $X_4^{\bowtie\vee}$ obtained from the Adinkra Riemann surfaces $X_4, X_{4,1}$ and their mirrors $X_4^{\bowtie}, X_{4,1}^{\bowtie}$ are zigzag consistent. Hence their Jacobi algebras are CY-3.*

---

of a zigzag path in $\mathbb{H}$).

To sum up this argument, Figure 3.2.2 implies that we have a geodesic which passes through midpoints of arrows forming a zigzag path in the universal cover of the quiver T generated from $X_4^{\bowtie}$. Moreover, if any pair of such geodesics intersect, they intersect at most at a single point.

Chapter 4

Jacobians of Adinkra Riemann Surfaces

4.1 Preliminaries

The main purpose of this section is to introduce Jacobians of compact Riemann surfaces which we will use in section 3. We begin with the notion of abelian varieties and present Jacobians as abelian varieties with an additional structure.

4.1.1 Abelian Varieties

An *algebraic group* G is an algebraic variety equipped with group structure. More precisely, it has a multiplication

$$\begin{aligned} G \cdot G &\to G \\ (g_1, g_2) &\mapsto gh \end{aligned}$$

and inverse operation

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

both given by regular maps on the variety G.

Definition 4.1.1. An *abelian variety* is a projective variety that is also an algebraic group. In particular, an abelian variety defined over $\mathbb{C}$ is a complex torus with an embedding into a complex projective space.

A standard example of abelian varieties is elliptic curves which are abelian varieties of dimension 1.

Definition 4.1.2. A morphism $f : A_1 \to A_2$ between abelian varieties is called an *isogeny* if it is surjective and $ker(f)$ is finite. Here, A_1 and A_2 are said to be *isogenous.* The cardinality of the kernel is called the *degree* of f.

An equivalent statement of the definition of an isogey is that f is isogeny if and only if it is surjective and $dim(A_1) = dim(A_2)$. The following theorem is known as Poincaré complete reducibility theorem which allows one to decompose an abelian variety A into simple abelian varieties.

Theorem 4.1.3 (Poincaré)**.** *If A is an abelian variety, then there exist simple abelian varieties A_i and positive integers n_i such that A is isogenous to the product*

$$A \cong_{isog} A_1^{n_1} \times \cdots \times A_r^{n_r}. \tag{4.1}$$

The factors A_i and n_i are unique up to isogeny and permutation of the factors.

In particular, we call such decomposition the *isogenous decomposition of A*. We call an abelian variety A *decomposable* if it has the isogenous decomposition. Otherwise A is called *simple*. In particular, A is said to be *completely reducible* if it has the isogenous decomposition, and the factors in the decomposition are elliptic curves. We are especially interested in abelian varieties which come equipped with principal polarization.

Definition 4.1.4. A pair (X, H) is called a *polarized abelian variety* if $X = \mathbb{C}^n/\Lambda$ is a complex torus, and the *polarization* $H = c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of an ample line bundle $L \to X$. The polarization H is called *principal* if $\dim H^0(X, L) = 1$, i.e. the line bundle L has one holomorphic section (up to constant).

An interesting case of principally polarized abelian varieties are Jacobians of compact Riemann surfaces. What is nice about working with Jacobians is that a compact Riemann surface X is uniquely determined by the corresponding Jacobian $Jac(X)$ (Torelli's theorem), and $Jac(X)$ has a group structure which the curve X doesn't have.

4.1.2 Jacobians of Compact Riemann Surfaces

We are going to to construct a Jacobian of a compact Riemann surface X of genus g in two ways.

Definition 4.1.5. Let ω be a closed 1-form on X. The integral of ω along any homology class $[c] \in H_1(X,\mathbb{Z})$, $\int_c \omega$, is called a *period* of ω. In general an integral of a closed k-form over a k-chain is called *period* of that closed k-form.

Since $H_1(X,\mathbb{Z}) \cong \pi_1(X)/[\pi_1(X),\pi_1(X)]$, a 1-cycle c can be considered as a loop. So the definition of a period given above is equivalent to an integral of 1-form ω over a loop in X. Next, consider the following group homomorphism

$$\phi : H_1(X,\mathbb{Z}) \to \Gamma(X,K_X)^* = Hom(\Gamma(X,K_X),\mathbb{C})$$
$$[c] \mapsto (\omega \mapsto \int_c \omega)$$

where K_X is the canonical bundle over X. We claim that ϕ is injective.

Lemma 4.1.6. *$\phi : H_1(X,\mathbb{Z}) \to \Gamma(X,K_X)^*$ is injective.*

Proof. By the universal coefficient theorem, we have an exact sequence

$$0 \to H_1(X,\mathbb{Z}) \otimes \mathbb{C} \to H_1(X,\mathbb{C}) \to Tor(H_0(X,\mathbb{Z}),\mathbb{C}) \to 0.$$

In particular, we have an injection

$$H_1(X,\mathbb{Z}) \otimes \mathbb{C} \hookrightarrow H_1(X,\mathbb{C}).$$

Since a map $H_1(X,\mathbb{Z}) \hookrightarrow H_1(X,\mathbb{Z}) \otimes \mathbb{C}$ is also injective, the composition

$$H_1(X,\mathbb{Z}) \hookrightarrow H_1(X,\mathbb{Z}) \otimes \mathbb{C} \hookrightarrow H_1(X,\mathbb{C})$$

is again injective. Hence we obtain $H_1(X,\mathbb{Z}) \hookrightarrow H_1(X,\mathbb{C})$. By de Rham's Theorem, $H^1_{dR}(X) \cong H_1(X,\mathbb{C})^*$. By taking dual on both sides, $H^1_{dR}(X)^* \cong H_1(X,\mathbb{C})^{**} = H_1(X,\mathbb{C})$. Thus we have the following injection

$$H_1(X,\mathbb{Z}) \hookrightarrow H_1(X,\mathbb{C}) \cong H^1_{dR}(X)^*$$
$$c \mapsto (\omega \mapsto \int_c \omega)$$

The Hodge decomposition for compact Riemann surfaces says $H^1_{dR}(X) = \Gamma(X,K_X) \oplus \overline{\Gamma(X,K_X)}$ [**BL**]. Thus we now have a composition

$$H_1(X,\mathbb{Z}) \hookrightarrow H^1_{dR}(X)^* = \Gamma(X,K_X)^* \oplus \overline{\Gamma(X,K_X)^*} \twoheadrightarrow \Gamma(X,K_X)^*$$

which is given by

$$c \mapsto c = l + \bar{l} \mapsto l.$$

The kernel of this composition $\phi : H_1(X, \mathbb{Z}) \to \Gamma(X, K_X)^*$ is

$$\begin{aligned} ker(\phi) &= \{c \in H_1(X, \mathbb{Z}) \mid c = l + \bar{l} \text{ with } l = 0\} \\ &= \{c \in H_1(X, \mathbb{Z}) \mid c = 0 + 0 = 0\} \\ &= \{c \in H_1(X, \mathbb{Z}) \mid c = 0\} = \{0\} \end{aligned}$$

Hence the composition is injective. □

Moreover, since $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, the image $\phi(H_1(X, \mathbb{Z}))$ is a discrete subgroup of rank $2g$ (i.e. a lattice) in $\Gamma(X, K_X)^*$.

Definition 4.1.7. The Jacobian of X is the quotient group

$$Jac(X) := \Gamma(X, K_X)^*/\phi(H_1(X, \mathbb{Z})).$$

Note that $\Gamma(X, K_X)^* \cong \mathbb{C}^g$ where g is a genus of X. To see this, we need to show $\dim \Gamma(X, K_X) = g$ using the Riemann-Roch theorem for line bundles: Take the line bundle L to be the trivial bundle, then $K_X \otimes L^{-1} = K_X$, $h^0(X, L) = 1$ and $\deg(L) = 0$ [**CA**]. Then the relation

$$h^0(X, L) - h^0(X, L^{-1} \otimes K_X) = \deg(L) - g + 1$$

becomes

$$1 - h^0(X, K_X) = 0 - g + 1$$

which gives $h^0(X, K_X) = g$. Thus $\dim H^0(X, K_X) = \dim H^0(X, K_X)^* = \dim \Gamma(X, K_X)^* = g$. Hence $\Gamma(X, K_X)^* \cong \mathbb{C}^g$.

Next, let $\omega_1, \ldots, \omega_g$ be a basis of $\Gamma(X, K_X)$ and $a_1, b_1, \ldots, a_g, b_g$ the canonical basis of $H_1(X, \mathbb{Z})$. Consider the following column vectors whose components are periods of the 1-forms $\omega_1, \ldots, \omega_g$:

$$A_i := (\int_{a_i} \omega_1, \ldots, \int_{a_i} \omega_g)^T \qquad B_i := (\int_{b_i} \omega_1, \ldots, \int_{b_i} \omega_g)^T \qquad i = 1, \ldots, g.$$

These 2g column vectors are linearly independent over $\mathbb{R}$ and thus generate a lattice

$$\Lambda = \{a_1 A_1 + \cdots + a_g A_g + b_1 B_1 + \cdots + b_g B_g \mid a_i, b_i \in \mathbb{Z}\}$$

in $\mathbb{C}^g$. Hence the Jacobian $Jac(X)$ can be rewritten as

$$Jac(X) = \mathbb{C}^g/\Lambda.$$

Example 4.1.8. The Jacobian of the Riemann sphere is trivial, i.e. $Jac(\mathbb{P}^1) = 0$.

Example 4.1.9. The Jacobian of a 1-dimensional complex torus X is again X. To see this, take $X = \mathbb{C}/L$ where $L = \tau_1\mathbb{Z} \oplus \tau_2\mathbb{Z}$, $\tau_1, \tau_2 \in \mathbb{C}, \tau_1/\tau_2 \notin \mathbb{R}$ is a lattice. By definition, the Jacobian of X is $Jac(X) = \mathbb{C}^1/\Lambda$ with the lattice $\Lambda = (\int_a \omega)\mathbb{Z} \oplus (\int_b \omega)\mathbb{Z}$. Clearly $Jac(X)$ is a 1-dimensional complex torus. In order to show $Jac(X) \cong X$, we only need to prove that $L = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^*$.

Since the genus of X is 1, we have $\dim \Gamma(X, K_X) = 1$ from the previous discussion. It follows that the basis of $\Gamma(X, K_X)$ is dz, and so the lattice is

$$\Lambda = (\int_a dz)\mathbb{Z} \oplus (\int_b dz)\mathbb{Z}.$$

Let $\pi : \mathbb{C} \to \mathbb{C}/L = X$ be the canonical projection. Consider the two curves in $\mathbb{C}$:

$$\begin{aligned} \gamma_1 &: [0,1] \to \mathbb{C}, \quad t \mapsto t\tau_1 \\ \gamma_2 &: [0,1] \to \mathbb{C}, \quad t \mapsto t\tau_2. \end{aligned}$$

Composing these curves γ_1, γ_2 with the projection π, we obtain curves in $X = \mathbb{C}/(\tau_1\mathbb{Z} \oplus \tau_2\mathbb{Z})$:

$$\begin{aligned} a = \pi \circ \gamma_1 &: [0,1] \to X, \quad t \mapsto t\tau_1 \mapsto 0 \\ b = \pi \circ \gamma_2 &: [0,1] \to X, \quad t \mapsto t\tau_2 \mapsto 0. \end{aligned}$$

Notice that the curves a and b are closed, and so they generate $H_1(X, \mathbb{Z})$. Since pullback of the 1-form dz on $X = \mathbb{C}/L$ is a 1-form $\pi^*(dz)$ on $\mathbb{C}$, $\pi^*(dz) = dz$. Thus we have

$$\int_a dz = \int_{\gamma_1} \pi^*(dz) = \int_{\gamma_1} dz.$$

This is a contour integral of a constant function 1 along a path γ_1 given by $\gamma_1(t) = t\tau_1, t \in [0,1]$, so we have

$$\int_a dz = \int_{\gamma_1} dz = \int_0^1 \gamma_1'(t)dt = \int_0^1 \tau_1 dt = \tau_1.$$

For the same reason, we also have

$$\int_b dz = \int_{\gamma_2} dz = \int_0^1 \gamma_2'(t)dt = \int_0^1 \tau_2 dt = \tau_2.$$

Hence

$$\Lambda = (\int_a dz)\mathbb{Z} \oplus (\int_b dz)\mathbb{Z} = \tau_1\mathbb{Z} \oplus \tau_2\mathbb{Z} = L$$

which implies that $Jac(X) \cong X$.

4.2 Generalized Humbert Curves

Let us set aside the story of Jacobians and introduce a notion of generalized Humbert curves. It is a closed Riemann surface X with a property that the quotient space $X/Aut(X)$ is a sphere with some singular points (i.e. an orbifold). In the first subsection, we briefly discuss orbifolds and then give a formal definition of generalized Humbert curve. At the end of the section, we describe a generalized Hubert curve structure of hypercube Adinkra Riemann surfaces X_N.

4.2.1 Orbifolds

Roughly speaking, an orbifold is a topological space which is locally homeomorphic to an open subset of a quotient space $\mathbb{R}^n/G$ where G is a finite group acting faithfully on $\mathbb{R}^n$. The following definition of orbifolds is taken from [**TH**].

Definition 4.2.1. An *orbifold* O is a Hausdorff topological space X_O (called the *underlying space*) with an open cover $\{U_i\}$ such that for each U_i, there is

- a finite group G_i associated to U_i
- an open subset $V_i \subseteq \mathbb{R}^n$ on which G_i acts and a homeomorphism $\varphi_i : U_i \to V_i/G_i$

If $U_i \subset U_j$, then there is an injective group homomorphism $f_{ij} : G_i \hookrightarrow G_j$ and an embedding $h_{ij} : V_i \hookrightarrow V_j$ such that

- for $g \in G_i, h_{ij}(gv) = f_{ij}(g)h_{ij}(v)$
- the following diagram commutes:

$$\begin{tikzcd}
V_i \arrow[r, hook, "h_{ij}"] \arrow[d] & V_j \arrow[d] \\
V_i/G_i \arrow[r, "h_{ij}/G_i"] & V_j/G_i \arrow[d] \\
 & V_j/G_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, hook] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

Example 4.2.2. A manifold M can be seen as an orbifold by taking $G_i = 1$ for each i. This makes each open subset $U_i \subseteq M$ homeomorphic to $V_i \subseteq \mathbb{R}^n$.

Example 4.2.3. The group $\mathbb{Z}_2$ acts on $\mathbb{R}^1$ by $x \mapsto -x$, so the orbifold $\mathbb{R}^1/\mathbb{Z}_2$ can be identified with $[0, \infty)$.

Unlike manifolds, orbifolds might have singular points. For each point x in an orbifold O, there is an open subset $V \subset \mathbb{R}^n$ together with a group G acting on V. Let $G_x \subset G$ denote the stabilizer subgroup of a point in V corresponding to x. Then

- If $G_x = 1$, x is a non-singular point
- If $G_x \neq 1$, x is a singular point

In particular, when $G_x \neq 1$, the order of the group G_x is called the *order* of the singular point x. A collection of singular points of O is called a *singular locus* of O. Hence when an orbifold has an empty singular locus, that orbifold is in fact a manifold [**TH**].

Example 4.2.4. Consider the orbifold $\mathbb{R}^1/\mathbb{Z}_2$ in Example 4.2.3 again. Identifying points x with $-x$ produces the fixed point 0, so 0 is a singular point of $\mathbb{R}^1/\mathbb{Z}_2$.

Example 4.2.5. Let M be a manifold and G a group acting on M properly discontinuously[1]. Then the quotient space M/G is an orbifold [**TH**]. However, not every orbifold arises in this way. The teardrop is an example of an orbifold which is not a quotient of a manifold by a group [**RO**]. The underlying topological space of the teardop is S^2, and it has just one singular point whose neighborhood is homeomorphic to $\mathbb{R}^2/\mathbb{Z}_n$.

Definition 4.2.6. A *signature* of an orbifold O is defined to be a tuple $(g, k; n_1, \ldots, n_k)$ where g is a genus of the underlying topological space X_O, and k is the number of singular points of order $n_1, \ldots, n_k$.

4.2.2 Generalized Humbert Curves and Their Algebraic Models

We are now going to give a formal definition of *generalized Humbert curve* which is a generalization of a *classical Humbert curve*. We note that a classical Humbert curve

[1]Let X be a topological space and G be a group of homeomorphisms $X \to X$. We say G acts on X *properly discontinuously* if for each $x \in X$, there are at most finitely many elements $id = g_1, \ldots, g_r \in G$, and there exists a neighborhood U_x of x such that $g(U_x) \cap U_x = \emptyset$ for all $g \in G - \{g_1, \ldots, g_r\}$.

is a generalized Humbert curve of type 4 [**HR-C**], and the contents of this subsection is based on [**CG-AHR**].

Definition 4.2.7. A closed Riemann surface S of genus g is called a *generalized Humbert curve of type* $k \geq 1$ if it has a group of conformal automorphisms $H \cong (\mathbb{Z}/2\mathbb{Z})^k$ such that S/H has signature $(0, k+1; 2, \cdots, 2)$ (meaning that $S/H = \mathbb{P}^1$ with $k+1$ cone points of order 2). The group H is called a *generalized Humbert group of type* k, and the pair (S, H) is called a *generalized Humbert pair of type* k.

If S/H has signature $(0, k+1; n, \cdots, n)$, then $H \cong (\mathbb{Z}/n\mathbb{Z})^k$ and S becomes a *generalized Fermat curve pair*, which is a generalization of generalized Humbert curves [**CHQ**]. Next we look at certain type of algebraic curves which turn out to be an algebraic model of generalized Humbert curves.

Definition 4.2.8. Let $x_1, \ldots, x_{k+1}$ be coordinates for $\mathbb{P}^1$. Define $C(\lambda_1, \ldots, \lambda_{k-2}) \subset \mathbb{P}^1$ to be an algebraic curve which is defined by the following $k-1$ homogeneous equations of degree 2:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 0 \\ \lambda_1 x_1^2 + x_2^2 + x_4^2 &= 0 \\ \lambda_2 x_1^2 + x_2^2 + x_5^2 &= 0 \\ &\vdots \\ \lambda_{k-2} x_1^2 + x_2^2 + x_{k+1}^2 &= 0 \end{aligned}$$

where $\lambda_j \in \mathbb{C} - \{0, 1\}, \lambda_i \neq \lambda_j$ for $i \neq j$.

Note that the condition on λ_j guarantees that the curve $C(\lambda_1, \ldots, \lambda_{k-2})$ defined above is a smooth algebraic curve, hence it is a compact Riemann surface [**CG-AHR**]. The group of conformal automorphisms of $C(\lambda_1, \ldots, \lambda_{k-2})$ is $H_0 \cong (\mathbb{Z}/2\mathbb{Z})^k$, and this group is generated by transformations a_j given by

$$a_j([x_1, \ldots, x_{k+1}]) = [x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{k+1}], \quad j = 1, \ldots, k.$$

Consider a holomorphic map of degree 2^k:

$$\begin{aligned} \pi : C(\lambda_1, \ldots, \lambda_{k-2}) &\to \mathbb{P}^1 \\ [x_1, \ldots, x_{k+1}] &\mapsto (x_2/x_1)^2. \end{aligned}$$

Then for any $j \in \{1, \ldots, k\}$, we have $\pi a_j = \pi$. Note that the set of branch values of the holomorphic map π is $\{\infty, 0, -1, -\lambda_1, \ldots, -\lambda_{k-2}\}$, and it follows that

$(C(\lambda_1, \ldots, \lambda_{k-2}), H_0)$ is in fact a generalized Humbert pair of type k [**CG-AHR**]. Let us state a remarkable observation about this curve $C(\lambda_1, \ldots, \lambda_{k-2})$.

Theorem 4.2.9 ([**CG-AHR**]). *Let (S, H) be a generalized Humbert pair of type k. Let $T : S/H \to \mathbb{P}^1$ to be a conformal homeomorphism which maps the set of cone points of S/H to the set $\{\infty, 0, -1, -\lambda_1, -\lambda_2, \ldots, -\lambda_{k-2}\}$ of branch values of $\pi : C(\lambda_1, \ldots, \lambda_{k-2}) \to \mathbb{P}^1$. Then (S, H) is conformally equivalent to $(C(\lambda_1, \ldots, \lambda_{k-2}), H_0)$, meaning that there is a conformal homeomorphism $f : S \to C(\lambda_1, \ldots \lambda_{k-2})$ such that $f^{-1}H_0 f = H$.*

4.2.3 Adinkra Riemann Surfaces X_N as Generalized Humbert Curves

Note that the rest of the chapter is adapted from [**DIKM**] and [**CHQ**]. Recall that we can write B_N as an orbifold $B_N \cong X_N/C_{N,max}$ where $C_{N,max} \cong (\mathbb{Z}/2\mathbb{Z})^{N-1}$. Furthermore, B_N has orbifold signature $(0, N; 2, \ldots, 2)$, i.e. there are N cone points whose order are all 2 (these points occur at the branch values of $f_{x_N} : X_N \to B_N$ which are N-th roots of -1) [**DIKLM**]. We first prove that X_N has $C_{N,max} \cong (\mathbb{Z}/2\mathbb{Z})^{N-1}$ as a group of conformal automorphisms, and then conclude that X_N is a generalized Humbert curve. To prove this statement, we use some materials from Fuchsian uniformizations of Adinkra Riemann surfaces [**DIKLM**]. Even though we briefly mentioned Fuchsian groups which uniformize Adinkra Riemann surfaces and B_N in chapter 2, we will describe them again.

- B_N is uniformized by a Fuchsian group $\Gamma_N = \langle y_1, \ldots, y_N \mid y_1^2 = \cdots = y_N^2 = y_1 \cdots y_N = 1\rangle$, that means B_N can be written as $\mathbb{H}/\Gamma_N$ for $N \geq 5$ and $\mathbb{C}/\Gamma_N$ for $N = 4$
- X_N is uniformized by $\Gamma_{(N,0)} := [\Gamma_N, \Gamma_N] \subseteq \Gamma_N$, meaning that X_N can be written as $\mathbb{H}/\Gamma_{(N,0)}$ for $N \geq 5$ and $\mathbb{C}/\Gamma_{(N,0)}$ for $N = 4$

The proof consists of two parts, and here is the first part.

Lemma 4.2.10. $\Gamma_{(N,0)} = ker(\varphi)$ *where $\varphi : \Gamma_N \to \mathbb{F}_2^N$ is a group homomorphism such that $\varphi(\Gamma_N) = C_{N,max}$.*

Proof. Consider a group homomorphism

$$\begin{aligned} \varphi : (\Gamma_N, \cdot) &\to (\mathbb{F}_2^N, +) \\ y_i &\mapsto c_i \end{aligned}$$

where y_i and c_i are generators of Γ_N and $C_{N,max}$ respectively. Let $x \in \Gamma_{(N,0)} = [\Gamma_N, \Gamma_N]$ so we write $x = y_i y_j y_i^{-1} y_j^{-1}$. Then

$$\varphi(x) = \varphi(y_i y_j y_i^{-1} y_j^{-1}) = c_i + c_j - c_i - c_j = (0, \cdots, 0).$$

This means $x \in ker\varphi$, and thus $\Gamma_{(N,0)} \subseteq ker\varphi$.

For the opposite direction, let $x \in ker\varphi$ so we have $\varphi(x) = (0, \cdots, 0)$. There are three possibilities for x:

- $x = y_i y_i$: Then $\varphi(x) = c_i + c_i = (0, \cdots, 0, 2, 2, 0, \cdots, 0) = (0, \cdots, 0)$ in $\mathbb{F}_2^N$. Thus $x \in ker\varphi$. Since $y_i^2 = 1$ by definition of Γ_N, we have $y_i = y_i^{-1}$. Hence $x = y_i y_i = y_i 1 y_i^{-1} 1^{-1} \in [\Gamma_N, \Gamma_N]$.
- $x = y_i y_i^{-1} = id$: Then $\phi(x) = c_i - c_i = (0, \cdots, 0)$ and indeed $x \in ker\varphi$. We see that $x = y_i y_i^{-1} = y_i 1 y_i^{-1} 1^{-1} \in [\Gamma_N, \Gamma_N]$.
- $x = y_i y_j y_i^{-1} y_j^{-1}$: Then $\varphi(x) = c_i + c_j - c_i - c_j = (0, \cdots, 0)$ so $x \in ker\varphi$ and clearly $x \in [\Gamma_N, \Gamma_N]$.

In all the cases, we have $x \in [\Gamma_N, \Gamma_N] = \Gamma_{(N,0)}$. Therefore $\ker\varphi \subseteq \Gamma_{(N,0)}$ and we conclude that $ker\varphi = \Gamma_{(N,0)}$. □

Proposition 4.2.11. *$Aut(X_N) \cong C_{N,max}$.*

Proof. Again consider the group homomorphism

$$\begin{aligned} \varphi : \Gamma_N &\to \mathbb{F}_2^N \\ y_i &\mapsto c_i. \end{aligned}$$

Since $\varphi(\Gamma_N) = C_{N,max}$ by definition of φ, we have

$$\Gamma_N / \ker\varphi \cong C_{N,max}.$$

On the other hand, Proposition 2.35 from [**GG**] implies that

$$Aut(X_N) = Aut(\mathbb{H}/\Gamma_{(N,0)}) = N(\Gamma_{(N,0)})/\Gamma_{(N,0)}$$

where $N(\Gamma_{(N,0)}) = \{y \in \Gamma_N \mid y\Gamma_{(N,0)} = \Gamma_{(N,0)}y\}$ is the normalizer of $\Gamma_{(N,0)}$ in Γ_N. Note that commutator groups and normalizers have the following properties:

- $[\Gamma_N, \Gamma_N] = \Gamma_{(N,0)} \lhd \Gamma_N$
- $N(\Gamma_{(N,0)})$ is the largest subgroup of Γ_N such that $\Gamma_{(N,0)} \lhd N(\Gamma_{(N,0)})$

By maximality of $N(\Gamma_{(N,0)})$, we have $N(\Gamma_{(N,0))}) = \Gamma_N$. This implies that

$$Aut(X_N) \cong N(\Gamma_{(N,0))})/\Gamma_{(N,0)} = \Gamma_N/\Gamma_{(N,0)}.$$

By Lemma 4.2.10, we have

$$Aut(X_N) = \Gamma_N/\Gamma_{(N,0)} = \Gamma_N/ker\varphi \cong C_{N,max}.$$

Therefore $Aut(X_N) \cong C_{N,max} \cong (\mathbb{Z}/2\mathbb{Z})^{N-1}$. □

Hence by definition, a hypercube Adinkra Riemann surface X_N is a generalized Humbert curve of type $N-1$. According to Theorem 4.2.9, X_N is conformally equivalent to $C(\lambda_1, \ldots, \lambda_{N-3})$ in this case. Let's look at this statement more carefully. Define $T : B_N = X_N/C_{N,max} \to \mathbb{P}^1$ be a Möbius transformation which maps the N-th roots of -1 to the set $\{\infty, 0, -1, -\lambda_1, \ldots, -\lambda_{N-3}\}$ of branch values of $C(\lambda_1, \ldots, \lambda_{N-3}) \to \mathbb{P}^1$. Such T can be given by

$$T(z) = \frac{z-\zeta}{z-\zeta^{-1}} \cdot \frac{\zeta^3 - \zeta^{-1}}{\zeta - \zeta^3}$$

where $\zeta = e^{i\frac{\pi}{N}}$ and ζ^j with $j = 1, 3, 5, \ldots, 2N-1$ are N-th roots of -1. Then we see that T maps $\zeta, \zeta^3, \zeta^{2N-1}$ to $0, -1, \infty$ respectively; otherwise $T(z) \in (-\infty, 0)$ for $z = \zeta^j$ with $j \in \{1, 3, 5, \ldots, 2N-1\} - \{1, 3, 2N-1\}$. Now reorder the N-th roots of -1 by setting

$$\zeta_k := \zeta^{2k-1}.$$

For $k = 1, \ldots, N$, denote

$$\mu_k := T(\zeta_k).$$

Using this new notation, we have

$$\begin{aligned}
\mu_1 &= T(\zeta_1) = T(\zeta^{2(1)-1}) = T(\zeta^1) = 0 \\
\mu_2 &= T(\zeta_2) = T(\zeta^{2(2)-1}) = T(\zeta^3) = -1 \\
\mu_N &= T(\zeta_N) = T(\zeta^{2(N)-1}) = T(\zeta^{2N-1}) = \infty
\end{aligned}$$

Then X_N is conformally equivalent to the curve $C(\mu_3, \ldots, \mu_{N-1})$ which is a zero set of the following system of equations

$$\begin{aligned}
x_1^2 + x_2^2 + x_3^2 &= 0 \\
\mu_3 x_1^2 + x_2^2 + x_4^2 &= 0 \\
&\vdots \\
\mu_{N-1} x_1^2 + x_2^2 + x_N^2 &= 0
\end{aligned}$$

where μ_k are the images of the N-th roots of -1 under the Möbius transformation T.

4.3 Isogenous Decomposition of Jacobians of Adinkra Riemann Surfaces

Every generalized Humbert curves are generalized Fermat curves. More precisely, a generalized Humbert curve is a generalized Fermat curve of type $(2, n)$. Carvacho, Hidalgo, and Quispe formulated an isogenous decomposition theorem for Jacobians of generalized Fermat curves of type (p, n) with $p \geq 2$ a prime integer. Since the pair $(X_N, C_{N,max})$ of Adinkra Riemann surfaces and the maximal even code is also a generalized Fermat curve of type $(N-1, 2)$, Doran, Iga, Kostiuk, and Méndez-Diez used the isogenous decomposition theorem given by Carvacho, Hidalgo, and Quispe to develop an Adinkra Riemann surface version of it.

Proposition 4.3.1 (for hypercube Adinkra Riemann surfaces X_N [**DIKM**])**.** *The Jacobian of X_N is isogenous to the product of the Jacobians of X_N/C where C runs over all subgroups of $C_{N,max}$ of index 2 such that the genus of X_N/C is at least 1. Moreover, $X_N/C \to B_N$ is a cover branched over a subset S of the Nth roots of -1 and $|S| = 2k$ with $k \geq 2$. Conversely, every such hyperelliptic curve arises from a subgroup of index 2 in $C_{N,max}$.*

Corollary 4.3.2 (for quotient Adinkra Riemann surfaces $X_{N,k} = X_N/C$ [**DIKM**])**.** *Let $C \subseteq C_{N,max}$ be a doubly even code of dimension k. If $X_{N,k} = X_N/C$ is a quotient Adinkra Riemann surface, then the isogenous decomposition of $Jac(X_{N,k})$ is given by the product of $Jac(X_N/K)$ where K runs over all subgroups of index 2 in $C_{N,max}$ such that $K \supseteq C$.*

4.3.1 Examples

In this section, we apply Proposition 4.3.1 and Corollary 4.3.2 to Adinkra Riemann surfaces when $N = 4, 5, 6$. We use the following properties from [**DIKM**] in order to make computations easier:

- Consider index 2 subgroup of $C_{N,\max}$ as $ker(\phi : C_{N,max} \to \mathbb{Z}/2\mathbb{Z})$ (in fact this statement holds as long as ϕ is nontrivial)

- The map $X_N/ker(\phi) \to B_N$ is a double cover branched over μ_i for which $c_i \notin ker(\phi)$ (Recall: μ_i are images of N-th roots of -1 under the Möbius transformation T, and c_i are the generators of $C_{N,max}$)

- The curve $X_N/ker(\phi)$ has positive genus if and only if the set $C_{N,\max} \setminus ker(\phi) = \{c_i \in C_{N,\max} \mid c_i \notin ker(\phi)\}$ has at least 4 elements

- A subgroup $K \subseteq C_{N,\max}$ contains a doubly even code C if and only if $\phi(C) = 0$

Example 4.3.3 ($N = 3, 4$ [**DIKM**]). When $N \leq 3$, all the Adinkra Riemann surfaces are genus 0 curves, thus their Jacobians are $\{0\}$. When $N = 4$, we have two Adinkra Riemann surfaces X_4 and $X_{4,1}$ whose genera are both 1. Hence $J(X_4) = X_4$ and $J(X_{4,1}) = X_{4,1}$.

First we consider the isogenous decomposition of $J(X_4)$. Notice that there is only one subset S of 4th roots of -1 of even cardinality containing at least 4 elements, namely the entire set $S = \{z \in \mathbb{C} \mid z^4 = -1\} = \{z = e^{\frac{\pi(1+2k)}{4}} \mid k = 0, 1, 2, 3\}$. Therefore, by Proposition 4.3.1, there is only one curve E of genus $g \geq 1$ such that the map $E \to B_4$ is branched over the subset S, and the isogenous decomposition of $Jac(X_4)$ is

$$Jac(X_4) \cong_{isog} Jac(E)$$

where $E = X_4/K$ and $K = ker(\phi : C_{4,\max} \to \mathbb{Z}/2\mathbb{Z})$. In fact E is an elliptic curve which is a double cover of B_4 over the 4th roots of -1, and it is given by the equation $y^2 = x^4 + 1$ [**DIKM**]. Since Jacobian of elliptic curve is isomorphic to the elliptic curve itself, the isogenous decomposition of $Jac(X_4)$ can be rewritten as

$$X_4 \cong_{isog} E = X_4/K.$$

Next let's consider the isogenous decomposition of $Jac(X_{4,1})$. Note that the maximal even code $C_{4,max}$ is generated by three elements $c_1 = (1100), c_2 = (0110), c_3 = (0011)$ with the relation $c_1+c_2+c_3 = (1001) =: c_4$. Also note that the homomorphism $\phi : C_{4,max} \to \mathbb{Z}/2\mathbb{Z}$ corresponding to the curve $X_4/ker(\phi)$ is determined by sending all of c_i to 1. Hence define the homomorphism $\phi : C_{4,max} \to \mathbb{Z}/2\mathbb{Z}$ as follow:

$$\begin{aligned} \phi : C_{4,max} &\to \mathbb{Z}/2\mathbb{Z} \\ c_1, c_2, c_3 &\mapsto 1 \\ c_4 &\mapsto 1 \end{aligned}$$

The corresponding index 2 subgroup is

$$ker_{4,1} := ker(\phi) = \{(0000), (1010), (0101), (1111)\} = C_{4,\max} \setminus \{c_1, c_2, c_3, c_4\},$$

and this subgroup $ker_{4,1}$ contains the doubly even code $C = \{(0000), (1111)\}$. Then by Corollary 3.2, the isogenous decomposition of $J(X_{4,1})$ is

$$\begin{aligned} Jac(X_{4,1}) &\cong_{isog} Jac(X_4/ker_{4,1}) \\ &\implies X_{4,1} \cong_{isog} X_4/ker_{4,1}. \end{aligned}$$

because both $X_{4,1}$ and $X_4/ker_{4,1}$ are elliptic curves [**DIKM**]. In fact, the elliptic curves $X_4, X_{4,1}, X_4/ker_{4,1}$ are all isomorphic to each other [**DIKM**]. However as a dessin (i.e. embedded graph plus underlying surface), they are different.

Example 4.3.4 (N = 5 [**DIKM**]). First we consider the Adinkra Riemann surface X_5 associated to the hypercube Adinkra A_5. There are $\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \frac{5!}{4!(5-4)!} = 5$ ways to choose a subset consisting of four of 5th roots of -1. This implies that there are five possible branch loci S_i of $X_5/K \to B_5$ which means there are five different branch covers $X_5/K \to B_5$ for each S_i. By Proposition 4.3.1, this implies that there are five factors X_5/K_i in the isogenous decomposition of $Jac(X_5)$, so we have five index 2 subgroups $K_i \subseteq C_{N,max}$. Thus

$$Jac(X_5) \cong_{isog} Jac(X_5/K_1) \times Jac(X_5/K_2) \times Jac(X_5/K_3) \times Jac(X_5/K_4) \times Jac(X_5/K_5).$$

Each X_5/K_i is in fact an elliptic curve [**DIKM**], so we have $Jac(X_5/K_i) \cong X_5/K_i$. These five elliptic curves X_5/K_i are isomorphic to each other because every branch locus S_i is related to one another by Möbius transformations corresponding to a rotation by multiples of $\frac{2\pi}{5}$.

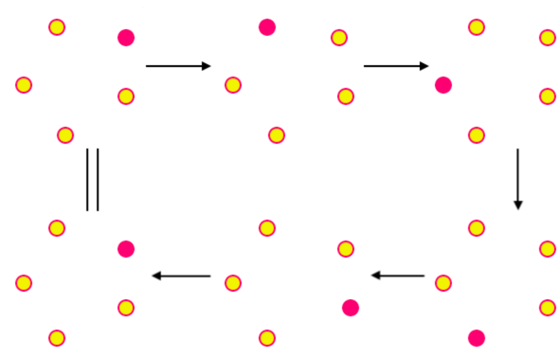


Figure 4.1: Configurations of 5th roots of -1

Hence we have

$$Jac(X_5) \cong_{isog} E \times E \times E \times E \times E = E^5$$

where the elliptic curve E is given by $y^2 = x^4 - x^3 + x^2 - x + 1$ [**DIKM**].

Next we look at the isogenous decomposition of $Jac(X_{5,1})$ where $X_{5,1} = X_5/C$ and $C \subseteq C_{5,max}$ is a 1-dimensional doubly even code. Note that the maximal even code $C_{5,max}$ is generated by $c_1 = (11000), c_2 = (01100), c_3 = (00110), c_4 = (00011)$, and those vectors satisfy the relation $c_1 + c_2 + c_3 + c_4 = (10001) =: c_5$. There are five homomorphisms $C_{5,max} \to \mathbb{Z}/2\mathbb{Z}$ defined by the following way: for $i = 1, \ldots, 5$,

$$\begin{aligned} \phi_i : C_{5,max} &\to \mathbb{Z}/2\mathbb{Z} \\ c_i &\mapsto 0 \\ c_{j\neq i} &\mapsto 1 \end{aligned}$$

Since a 1-dimensional doubly even code [2] C for $N = 5$ is generated by a single element of the form $c_{\alpha_1} + c_{\alpha_2}$ where $c_{\alpha_1}, c_{\alpha_2}$ are two of the generators of $C_{5,max}$. By the definition of ϕ_i, there are three homomorphisms which kill $C = \langle c_{\alpha_1} + c_{\alpha_2} \rangle$, namely

$$\begin{aligned} \phi_i : C_{5,\text{max}} &\to \mathbb{Z}/2\mathbb{Z} \\ \phi_j : C_{5,\text{max}} &\to \mathbb{Z}/2\mathbb{Z} \\ \phi_k : C_{5,\text{max}} &\to \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

where $i, j, k \neq \alpha_1, \alpha_2$. This is because

$$\phi_i(c_{\alpha_1} + c_{\alpha_2}) = \phi_i(c_{\alpha_1}) + \phi_i(c_{\alpha_2}) = 1 + 1 = 0$$

which implies that $\phi_i(C) = 0$. Thus $ker(\phi_i)$ contains C. The same argument applies to ϕ_j and ϕ_k. On the other hand, the homomorphisms ϕ_{α_1} and ϕ_{α_2} do not kill C:

$$\begin{aligned} \phi_{\alpha_1}(c_{\alpha_1} + c_{\alpha_2}) &= \phi_{\alpha_1}(c_{\alpha_1}) + \phi_{\alpha_1}(c_{\alpha_2}) = 0 + 1 = 1 \\ \phi_{\alpha_2}(c_{\alpha_1} + c_{\alpha_2}) &= \phi_{\alpha_2}(c_{\alpha_1}) + \phi_{\alpha_2}(c_{\alpha_2}) = 1 + 0 = 1 \end{aligned}$$

Thus $\phi_{\alpha_1}(C) \neq 0 \neq \phi_{\alpha_2}(C)$, and so the factors $X_5/ker(\phi_{\alpha_1})$ and $X_5/ker(\phi_{\alpha_2})$ do not appear in the isogenous factorization of $Jac(X_{5,1})$.

For example, consider the following 1-dimensional doubly even code $C = \langle(11110)\rangle = \langle c_1 + c_3 \rangle$. For any $i = 2, 4, 5$, the image $\phi_i(C)$ of C under the homomorphism $\phi_i : C_{5,max} \to B_5$ is determined by $\phi_i(c_1 + c_3)$. For $i = 2, 4, 5$,

$$\phi_i(c_1 + c_3) = \phi_i(c_1) + \phi_i(c_3) = 1 + 1 = 0.$$

Hence we have $\phi_i(C) = 0$ for $i = 2, 4, 5$. This implies that $ker(\phi_i) \supseteq C$. If $i = 1$, we have

$$\phi_1(c_1 + c_3) = \phi_1(c_1) + \phi_1(c_3) = 0 + 1 = 1.$$

[2] In N=5, there are only 1-dimensional doubly even codes and no higher dimensional doubly even code. [**DFGHILM**]

Hence $\phi_1(C) \neq 0$ which implies that $ker(\phi_1) \not\supseteq C$. We can show that $ker(\phi_3) \not\supseteq C$ using the same argument. Therefore, by Corollary 4.3.2,

$$Jac(X_{5,1}) \cong_{isog} Jac(X_5/ker(\phi_2)) \times Jac(X_5/ker(\phi_4)) \times Jac(X_5/ker(\phi_5))$$

where $X_{5,1} = X_5/C$ and $C = \langle(11110)\rangle = \langle c_1 + c_3\rangle$.

Example 4.3.5 ($N = 6$ [**DIKM**]). We consider the isogenous decomposition of $Jac(X_6)$. Let S be a subset of the set of the roots of $z^6 = -1$. There are two choices for the cardinality of the subset S. The resulting isogenous decomposition of $Jac(X_6)$ is a product of factors obtained from these two choices.

Choice 1: $|S| = 6$, i.e. $S = \{6\text{th roots of } -1\}$
There is a unique map $X_6/ker(\phi : C_{6,\max} \to \mathbb{Z}/2\mathbb{Z}) \to B_6$ whose branch locus is S. This implies that there is a unique hyperelliptic curve $H := X_6/ker(\phi)$ given by the equation $y^2 = x^6 + 1$ [**DIKM**]. Since the degree of the polynomial $x^6 + 1$ is $6 = 2(2)+2 = 2g+2$, the genus of this hyperelliptic curve H is $g = 2$. Thus the Jacobian $Jac(H)$ is a 2-dimensional complex torus. Notice that H possesses an involution $(x, y) \mapsto (-x, y)$ which is different from the hyperelliptic involution $(x, y) \mapsto (x, -y)$. There is a fact that for a closed Riemann surface of genus 2 admitting an involution different from the hyperbolic involution, its Jacobian is isogenous to a product of two elliptic curves [**CHQ**]. This fact definitely applies to our case, so $Jac(H)$ is isogenous to the product of two elliptic curves E_1 and E_2,

$$Jac(H) \cong_{isog} E_1 \times E_2.$$

In fact, E_1 and E_2 are isomorphic to each other, and they are given by the equation $y^2 = x^3 + 1$ [**DIKM**].

Choice 2: $|S| = 4$, i.e. $S \subset \{6\text{th roots of} - 1\}$
There are $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{6!}{4!(6-4)!} = 15$ ways to choose subsets S_i consisting of four 6th roots of -1. This implies that there are 15 branched covers $X_6/ker(\phi : C_{6,\max} \to \mathbb{Z}/2\mathbb{Z}) \to B_6$ with branch locus S_i with $i = 1, \ldots, 15$. By Proposition 4.3.1, this means that there are 15 additional factors in the isogenous decomposition of $Jac(X_6)$, and each factor is an elliptic curve [**DIKM**]. Next, we want to consider configurations of four of 6th roots of -1. There are three variations of the configurations.

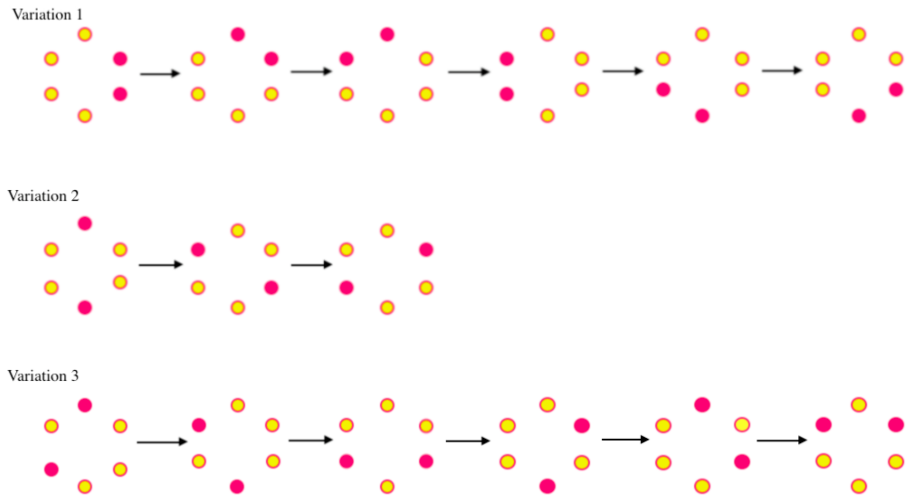


Figure 4.2: Configurations of four 6th roots of -1

In variation 1, the six configurations are considered to be equivalent, and for variation 2, the three configurations are equivalent to each other. Similarly for variation 3, the six configurations are equivalent to each other. If we let D_i be an elliptic curve corresponding to the variation i, the following product of elliptic curves

$$(D_1)^{\oplus 6} \times (D_2)^{\oplus 3} \times (D_3)^{\oplus 6}$$

constitute the other part of the isogenous decomposition of $Jac(X_6)$.

Combining the results from Choice 1 and Choice 2 gives the isogenous decomposition of $J(X_6)$:

$$\begin{aligned} Jac(X_6) &\cong_{isog} Jac(H) \times (D_1)^{\oplus 6} \times (D_2)^{\oplus 3} \times (D_3)^{\oplus 6} \\ &\cong_{isog} \underbrace{E_1 \times E_1}_{\text{Choice 1}} \times \underbrace{(D_1)^{\oplus 6} \times (D_2)^{\oplus 3} \times (D_3)^{\oplus 6}}_{\text{Choice 2}}. \end{aligned}$$

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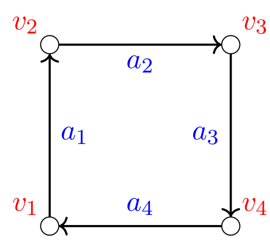
Appendix A

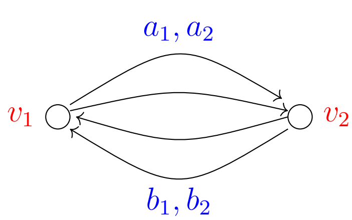
Dual and Mirror Dual Quivers of The $N = 4$ Covering Space Tower

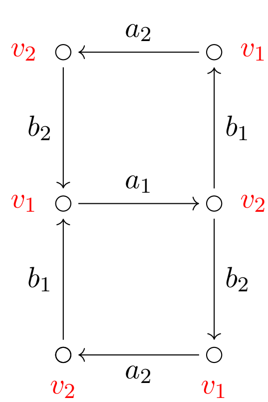
- $B_4^{\vee}$ $\quad \#v = 4, \#e = 4, \#f = 2, g = 0$



- $B_4^{\bowtie\vee}$ $\quad \#v = 2, \#e = 4, \#f = 2, g = 1$

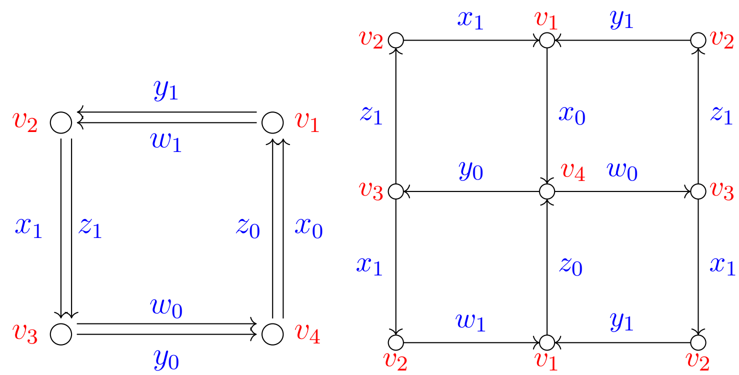
The quiver on the right represents when the left quiver is embedded in a torus (fundamental domain).





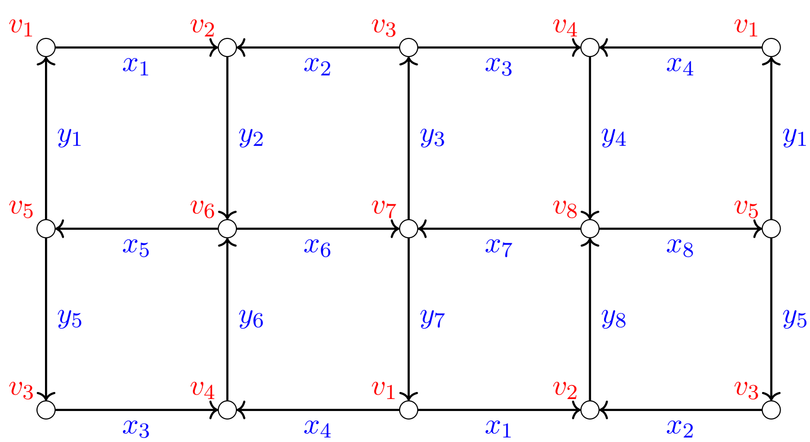
• $E_4^{\bowtie\vee} = E^\vee$ $\quad \#v = 4, \#e = 8, \#f = 4, g = 1$

The quiver on the right shows when the left quiver is embedded in a torus (i.e. the fundamental domain of $E^{\bowtie\vee}$.



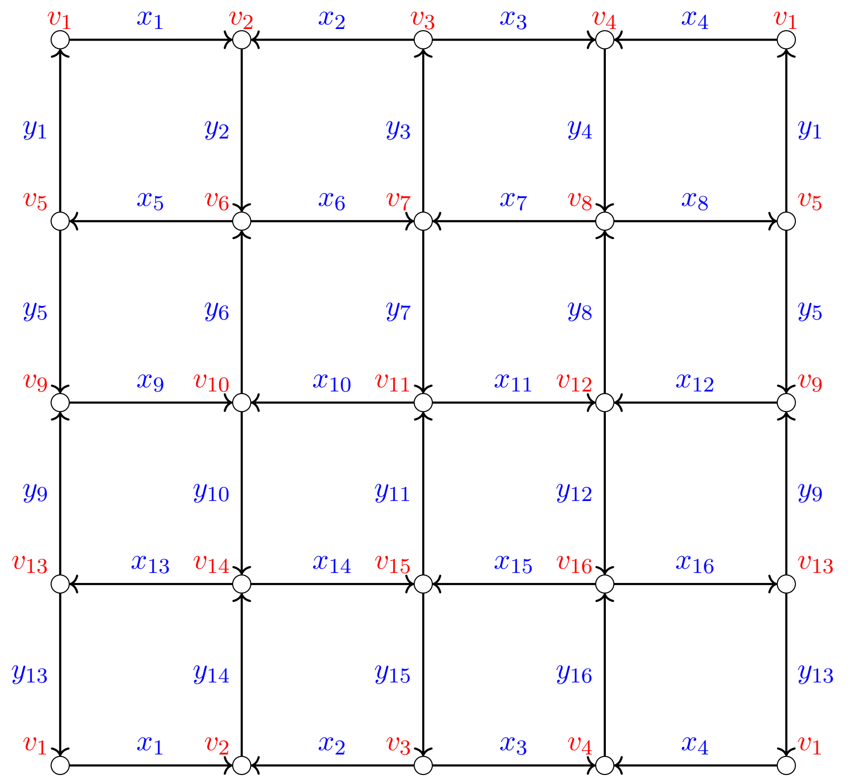
• $X_{4,1}^{\bowtie,\vee} = X_{4,1}^\vee$ $\quad \#v = 8, \#e = 16, \#f = 8, g = 1$

The fundamental domain of $X_{4,1}^{\bowtie\vee}$



• $X_4^\vee$ $\quad \#v = 16, \#e = 32, \#f = 16, g = 1$

The fundamental domain of $X_4^\vee$



- $X_4^{\bowtie,\vee}$ $\#v = 8, \#e = 32, \#f = 16, g = 5$