University of Alberta

THE TROTTER-KATO APPROXIMATION IN UTILITY MAXIMIZATION

by

Lingyu Miao

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Abstract

This thesis deals with the Trotter-Kato approximation in utility maximization. The Trotter-Kato approximation is a method to split a differential equation into two parts, which are then solved iteratively over small time intervals. In the context of utility maximization, this procedure was introduced by Nadtochiy and Zariphopoulou [11] for partial differential equations (PDEs) in a Markovian setting, which we revisit in the first part of this thesis. We then study what the Trotter-Kato approximation can mean for backward stochastic differential equations (BSDEs), which do not need Markovian assumptions and allow for a probabilistic interpretation. We also discuss how the Trotter-Kato approximation can be implemented numerically in both the PDE and the BSDE case.

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Chapter 1

Introduction and setup

1.1 Introduction

In this thesis, we study a very classical problem of mathematical finance, but applying new methods. We consider an agent who maximizes expected utility from terminal wealth of an investment in stock and a contingent claim. Such a portfolio optimization in continuous time goes back at least to the seminal paper by Merton [8]. Here we consider an incomplete market, where the stock is driven by a Brownian motion while the contingent claim can depend on another correlated Brownian motion. In such a model, there exist essentially two approaches to dynamically characterize the optimal trading strategy and the value process related to the utility maximization problem. Under Markovian and regularity assumptions, the first approach is to derive and study a partial differential equation (PDE), called the Hamilton-Jacobi-Bellman equation, which is related to the utility maximization problem. The second approach uses backward stochastic differential equations (BSDEs) to characterize the value process, which do not require Markovian assumptions.

In general, neither the resulting PDE nor BSDE can be solved explicitly, and also existence and uniqueness questions of the solutions to the PDE and BSDE arise. In the PDE case, Nadtochiy and Zariphopoulou [11] recently introduced the so-called Trotter-Kato approximation. The theory behind this approximation goes back to Trotter [13] and Kato [3]. The main idea, which we will explain in Chapter 2 of this thesis, is to split the PDE into two parts and then solve the resulting two PDEs iteratively as an approximation to the original PDE. We will see that in our context, the splitting has the interpretation of separating parts related to complete and incomplete financial markets.

While the Trotter-Kato approximation has been recently introduced by Nadtochiy and Zariphopoulou [11] for PDEs related to utility maximization, a natural question is how such an approximation looks for the BSDE approach, which is the other dynamic approach to the utility maximization problem. To study this question, we first briefly review the theory of BSDEs and relate them to our utility maximization problem. We are then in position to give a meaning to the Trotter-Kato approximation in the BSDE context. While this study is far from being conclusive, it allows us to give a probabilistic interpretation to the Trotter-Kato approximation because the BSDE, as opposed to the PDE, is directly related to the underlying probabilistic framework. We will again see that the splitting leads to a natural financial interpretation in terms of complete and incomplete financial markets.

The remainder of this thesis is organized as follows. Section 1.2 introduces the mathematical model for the financial problem that we consider. Chapter 2 discusses how the Trotter-Kato approximation can be applied to PDEs related to this problem when we impose Markovian assumptions in the problem formulation. In Chapter 3, we first briefly introduce BSDEs, relate them to our financial problem and then discuss how the analogue of the Trotter-Kato approximation looks in this BSDE context. In Chapter 4, we give a numerical implementation of both PDE and BSDE approaches. The Appendices contain a brief overview of BMO-martingales, which are important for BSDEs, and the MATLAB code used in the numerical implementation.

1.2 Financial market model setup

The financial market we consider in this thesis consists of a risk-free asset, a risky asset and a contingent claim. The risk-free asset is a government bond with constant price at 1. It is straightforward to generalize to a situation with non-zero deterministic interest rate. The risky asset is available for trading for our agent.

The probabilistic framework consists of two Brownian motions W^1 and W, which have instantaneous correlation ρ and are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. For the formal definition, let W and $W^{\perp}$ be two independent Brownian motions, and denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the augmented filtration generated by them. To have a Brownian motion W^1 with instantaneous correlation ρ, we define

$$dW_s^1 = \rho_s \, dW_s + \sqrt{1 - \rho_s^2} \, dW_s^{\perp}, \quad 0 \le s \le T, \quad W_0^1 = 0 \tag{1.1}$$

for a predictable process ρ valued in (-1,1). Starting at time $t \in [0, T]$, the traded stock price, denoted by S, satisfies

$$dS_s = \mu_s S_s \, ds + \sigma_s S_s \, dW_s^1, \quad t \le s \le T, \quad S_t = \tilde{S}, \tag{1.2}$$

where the drift μ and the positive volatility σ are predictable stochastic processes. We assume that μ and σ are bounded and σ is bounded away from zero. This means that there exists a constant C such that $|\mu| \leq C$ and $\frac{1}{C} \leq \sigma \leq C$ $dt \otimes dP$-a.e.

The investor's risk preferences are modelled by an exponential utility function U such that

$$U(x) = -\mathrm{e}^{-\gamma x}, \quad \gamma > 0. \tag{1.3}$$

In order to maximize the expected utility of terminal wealth based on today's information, the investors can trade at any time $s \in [t, T]$ using a self-financing strategy. With the initial endowment $x > 0$ at time t, they will keep redistributing their money between the bond and the stock. We denote by π_s^0 and π_s the amount of money at time s invested in bond and stock, respectively. Therefore, since no exogenous infusion and consumption can occur, the total wealth $X_s = \pi_s^0 + \pi_s$ has the dynamics

$$dX_s = \mu_s \pi_s \, ds + \sigma_s \pi_s \, dW_s^1, \quad t \leq s \leq T, \tag{1.4}$$

with $X_t = x \in \mathbb{R}$. The process $(\pi_s)_{t \leq s \leq T}$ is called admissible if it is $(\mathcal{F}_s)_{t \leq s \leq T}$-predictable, satisfies $\int_t^T \pi_s^2 \, ds < \infty$ a.s. and is such that the corresponding $\exp(-\gamma X)$ is of class (D). The last condition means that

$$\{\exp(-\gamma X_\tau) : \tau \text{ is a stopping time}\} \quad \text{is uniformly integrable.}$$

We denote the set of admissible policies by $\mathcal{A}$, and write $\lambda_s = \frac{\mu_s}{\sigma_s}$ for the instantaneous Sharpe ratio, using that we assumed a zero interest rate.

Now, we are ready to define two value processes, which are related to our model set above. The first circumstance arises when the investor aims to maximize his expected utility without involving the derivative, namely

$$v_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} E^P \left[-\mathrm{e}^{-\gamma X_T} \middle| \mathcal{F}_t \right]. \tag{1.5}$$

Here we use the essential supremum because we take the supremum over a set of random variables. In contrast, the second circumstance happens when the investor takes the contingent claim G into consideration, namely

$$V_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} E^P \left[-\mathrm{e}^{-\gamma (X_T - G)} \middle| \mathcal{F}_t \right]. \tag{1.6}$$

Lemma 1.1. *If Sharpe ratio λ is deterministic, the value process without contingent claim* (1.5) *is given by*

$$v_t = -\exp\left(-\gamma x + \frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \quad a.s. \tag{1.7}$$

Proof. We have

$$\begin{aligned}
v_t &= \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} E^P \left[-\mathrm{e}^{-\gamma X_T} \middle| \mathcal{F}_t \right] \\
&= \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} E^P \left[-\exp\left(-\gamma x - \int_t^T \gamma (\mu_s \pi_s \, ds + \sigma_s \pi_s \, dW_s^1) \right) \middle| \mathcal{F}_t \right] \\
&= -\exp(-\gamma x) \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} E^P \left[\exp\left(-\int_t^T \gamma \pi_s \sigma_s \Big(\lambda_s \, ds + dW_s^1 \Big) \right) \middle| \mathcal{F}_t \right].
\end{aligned}$$

Under the probability measure Q defined by

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \lambda_s \, dW_s^1 - \frac{1}{2} \int_0^T \lambda_s^2 \, ds \right), \tag{1.8}$$

the process $\tilde{W}_s = W_s^1 + \int_0^s \lambda_u \, du$ is a Brownian motion. We first consider $\pi \in \mathcal{A}$ such that $\int \pi \, d\tilde{W}$ is a Q-martingale. For such π, we have

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\pi \in \mathcal{A}} E^P \left[\exp \left(- \int_t^T \gamma \pi_s \sigma_s \big(\lambda_s \, ds + dW_s^1 \big) \right) \middle| \mathcal{F}_t \right] \\
&= \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} E^Q \left[\exp \left(\int_t^T \lambda_s \, dW_s^1 + \frac{1}{2} \int_t^T \lambda_s^2 \, ds - \int_t^T \gamma \pi_s \sigma_s \, d\tilde{W}_s \right) \middle| \mathcal{F}_t \right] \\
&= \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} E^Q \left[\exp \left(\int_t^T (\lambda_s - \gamma \pi_s \sigma_s) \, d\tilde{W}_s - \frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \middle| \mathcal{F}_t \right] \\
&= \exp \left(-\frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} E^Q \left[\exp \left(\int_t^T (\lambda_s - \gamma \pi_s \sigma_s) \, d\tilde{W}_s \right) \middle| \mathcal{F}_t \right] \\
&\geq \exp \left(-\frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} \exp \left(E^Q \left[\int_t^T (\lambda_s - \gamma \pi_s \sigma_s) \, d\tilde{W}_s \middle| \mathcal{F}_t \right] \right) \\
&= \exp \left(-\frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \quad \text{a.s.,}
\end{aligned} \tag{1.9}$$

where we used Jensen's inequality for the second last line and that $\int \pi \, d\tilde{W}$ is a Q-martingale by assumption. For general $\pi \in \mathcal{A}$, we define the stopping times

$$\tau_n := \inf \left\{ s \geq t : \int_t^s \pi_u^2 \, du \geq n \right\} \wedge T$$

for $n \in \mathbb{N}$. For the stopped process, we obtain

$$E^P \left[\mathrm{e}^{-\gamma X_{\tau_n}} \middle| \mathcal{F}_t \right] \leq -\exp(-\gamma x) \exp \left(-\frac{1}{2} \int_t^T \lambda_s^2 \, ds \right) \quad \text{a.s.}$$

from the previous calculation because X_{τ_n} corresponds to the terminal wealth of the stopped strategy $\pi \mathbb{1}_{[\![0,\tau_n]\!]}$ for which $\int \pi \mathbb{1}_{[\![0,\tau_n]\!]} \, d\tilde{W}$ is a square-integrable Q-martingale. By the admissibility condition, $\left(\mathrm{e}^{-\gamma X_{\tau_n}}\right)_{n \in \mathbb{N}}$ is uniformly integrable and converges a.s. to $\mathrm{e}^{-\gamma X_T}$. Therefore, the conditional random variables $E^P \left[\mathrm{e}^{-\gamma X_{\tau_n}} \middle| \mathcal{F}_t \right]$ converge to $E^P \left[\mathrm{e}^{-\gamma X_T} \middle| \mathcal{F}_t \right]$ in L^1 and thus also a.s. along a subsequence. This implies

$$E^P\left[\mathrm{e}^{-\gamma X_T}\middle|\mathcal{F}_t\right] \leq -\exp(-\gamma x)\exp\left(-\frac{1}{2}\int_t^T \lambda_s^2\,ds\right) \quad \text{a.s.}$$

for all $\pi \in \mathcal{A}$. We conclude the proof by noting that equality in (1.9) holds if $\pi = \pi^* := \frac{\lambda}{\gamma\sigma}$ and observe that the corresponding $\exp(-\gamma X)$ can be written as the product of a uniformly integrable martingale and a deterministic process so that $\pi^* \in \mathcal{A}$. $\square$

Chapter 2

The Trotter-Kato approximation for PDEs

In this chapter, we explain how the Trotter-Kato approximation works for the PDE related to the value process, based on the example of exponential utility and the work by Nadtochiy and Zariphopoulou [11]. To obtain such a PDE representation, we need a Markovian setting, which we first introduce.

2.1 A Markovian setting

In this chapter, we assume that the instantaneous correlation ρ and the instantaneous Sharpe ratio λ of the traded stock are both constant. Moreover, we suppose that the contingent claim is of the form $G = g(Y_T)$ for a bounded and continuous function g, where Y_T is the terminal value of an observable asset Y, which is not tradable for our agent. The dynamics of Y is given by

$$dY_s = b(Y_s, s)\, ds + a(s)\, dW_s, \quad t \leq s \leq T, \tag{2.1}$$

with $Y_t = y \in \mathbb{R}$; the functions a and b and g are assumed to be bounded and continuous, and a is positive and bounded away from zero. Under these Markovian assumptions on G and S, the value process given in (1.6) is a function of the current value of Y and S, namely

$$V(t,x,y) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E^P\left[-\mathrm{e}^{-\gamma(X_T - g(Y_T))}\middle| X_t = x, Y_t = y\right].$$

Assuming sufficient smoothness of the value function V, it satisfies the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\begin{aligned} V_t + \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi(\mu V_x + \rho\sigma a(t) V_{xy})\right) \\ + \frac{1}{2}a^2(t)V_{yy} + b(y,t)V_y = 0 \end{aligned} \tag{2.2}$$

with terminal condition $V(x,y,T) = -\mathrm{e}^{-\gamma(x-g(y))}$; in this chapter, V_t and V_y denote the partial derivatives of V with respect to t and y, respectively. Thanks to the assumptions of exponential utility and Markovian dynamics, one can solve the PDE (2.2). Indeed, Musiela and Zariphopoulou [10] show that

$$V(x,y,t) = -\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2(T-t)}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)g(Y_T)}\middle| Y_t = y\right]\right)^{\frac{1}{1-\rho^2}},$$

where Q is defined in (1.8). Although we have here an explicit formula, we illustrate in the next section how the Trotter-Kato approximation can be applied to the PDE 2.2. The reasons why we do this despite the explicit result are that the Trotter-Kato approximation works for more general utility functions, where no explicit form is available, and the exposition of the next section serves as illustration and preparation for the application in the BSDE context.

2.2 Splitting the PDE

As presented in Nadtochiy and Zariphopoulou [11], the idea of the Trotter-Kato approximation method is first to rewrite the form (2.2) as

$$V_t + \mathcal{H}^\rho(V) + \mathcal{L}^{\sqrt{1-\rho^2}}(V) = 0, \tag{2.3}$$

where the corresponding complete market part equals

$$\mathcal{H}^\rho(V) = \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi(\mu V_x + \rho\sigma a(t) V_{xy}) \right) + \frac{1}{2}\rho^2 a^2(t) V_{yy} \tag{2.4}$$

and the incomplete market part is

$$\mathcal{L}^{\sqrt{1-\rho^2}}(V) = \frac{1}{2}(1-\rho^2)a^2(t)V_{yy} + b(y,t)V_y.$$

The appropriate way of splitting means to consider two auxiliary PDE problems,

$$\begin{cases} V_t^{(1)} + \mathcal{H}^\rho(V^{(1)}) = 0 \\ V^{(1)}(x,y,T) = -\mathrm{e}^{-\gamma\left(x - g^{(1)}(y)\right)} \end{cases} \tag{2.5}$$

and

$$\begin{cases} V_t^{(2)} + \mathcal{L}^{\sqrt{1-\rho^2}}(V^{(2)}) = 0 \\ V^{(2)}(x,y,T) = -\mathrm{e}^{-\gamma\left(x - g^{(2)}(y)\right)}, \end{cases} \tag{2.6}$$

where we assumed terminal conditions of the same form as for the value function V. We compare graphically the direct method in Figure 2.1 with the split method in Figure 2.2.

In the case of a utility function defined on the positive half axis, Nadtochiy and Zariphopoulou [11] show that solving (2.5) and (2.6) iteratively on the

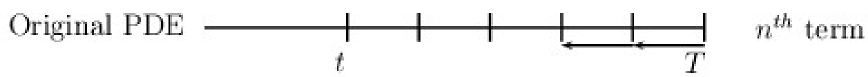

Figure 2.1: Illustration of direct method

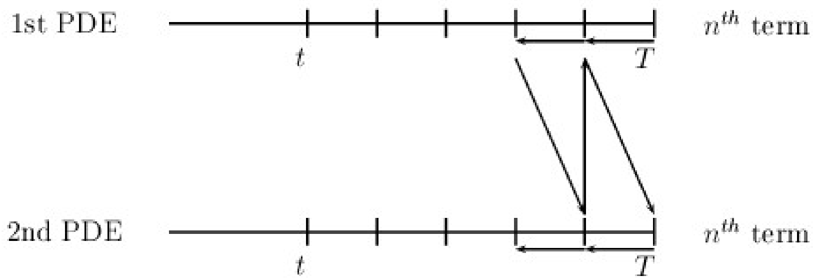

Figure 2.2: Illustration of split method

intervals $[t+\frac{n-1}{n}(T-t),T]$, $[t+\frac{n-2}{n}(T-t),t+\frac{n-1}{n}(T-t)]$, ..., $[t,t+\frac{1}{n}(T-t)]$ converges to the same result as (2.3) when the number n of steps goes to infinity.

Starting by solving the linear case (2.6), we define a new function $F^{(2)}$ by

$$V^{(2)}(x,y,t) = -\mathrm{e}^{-\gamma x}F^{(2)}(y,t).$$

Substituting this in (2.6), we derive that

$$\begin{cases} F_t^{(2)} + \dfrac{1}{2}(1-\rho^2)a^2(t)F_{yy}^{(2)} + b(y,t)F_y^{(2)} = 0 \\ F^{(2)}(y,T) = \mathrm{e}^{\gamma g^{(2)}(y)}, \end{cases} \tag{2.7}$$

which is a linear partial differential equation. We set

$$\tilde{Y}_t = Y_t\sqrt{1-\rho^2} \tag{2.8}$$

and define a new probability measure $Q^{(2)}$ by

$$\frac{dQ^{(2)}}{dP} = \exp\Bigg(-\int_0^T \frac{1}{a(s)}\Big(b(Y_s,s) - \frac{b(\tilde{Y}_s,s)}{\sqrt{1-\rho^2}}\Big)\, dW_s \\ - \frac{1}{2}\int_0^T \frac{1}{a^2(s)}\Big(b(Y_s,s) - \frac{b(\tilde{Y}_s,s)}{\sqrt{1-\rho^2}}\Big)^2\, ds\Bigg). \tag{2.9}$$

The process $W^{(2)}$ given by $dW_s^{(2)} = dW_s + \frac{1}{a(s)}\Big(b(Y_s,s) - \frac{b(\tilde{Y}_s,s)}{\sqrt{1-\rho^2}}\Big)\, ds$ is a Brownian motion under $Q^{(2)}$ and the dynamic of Y is given by

$$dY_s = \frac{b(\tilde{Y}_s,s)}{\sqrt{1-\rho^2}}\, ds + a(s)\, dW_s^{(2)}, \quad t \le s \le T,$$

with $Y_t = y \in \mathbb{R}$. Thus, we have

$$d\tilde{Y}_s = b(\tilde{Y}_s,s)\, ds + a(s)\sqrt{1-\rho^2}\, dW_s^{(2)}, \quad t \le s \le T,$$

with $\tilde{Y}_t = \tilde{y} := y\sqrt{1-\rho^2}$. Using the Feynman-Kac representation (see Theorem 8.3.1 of Øksendal [6]) of the solution to (2.7), we deduce that

$$F^{(2)}(\tilde{y},t) = E^{Q^{(2)}}\left[\mathrm{e}^{\gamma g^{(2)}(\tilde{Y}_T)} \Big| \tilde{Y}_t = \tilde{y} \right],$$

which gives us

$$\begin{aligned} V^{(2)}(x,y,t) &= -\mathrm{e}^{-\gamma x} F^{(2)}(y,t) \\ &= -\mathrm{e}^{-\gamma x} E^{Q^{(2)}}\left[\mathrm{e}^{\gamma g^{(2)}\left(Y_T\sqrt{1-\rho^2}\right)} \Bigg| Y_t = \frac{y}{\sqrt{1-\rho^2}} \right]. \end{aligned} \tag{2.10}$$

Later, we will use this solution as terminal condition for the first PDE. We next solve (2.5) for the generic terminal condition $g^{(1)}$.

In order to derive the optimal π^* in (2.4), we can simply take the π-partial derivative and equate it to zero, which is $\sigma^2 V_{xx}\pi + \mu V_x + \rho\sigma a(t)V_{xy} = 0$. Assuming $V_{xx} < 0$, this yields the maximizing π as

$$\pi^* = -\frac{\lambda V_x}{\sigma V_{xx}} - \frac{\rho a(t) V_{xy}}{\sigma V_{xx}}, \tag{2.11}$$

using $\lambda = \frac{\mu}{\sigma}$. Moreover, we still need to construct a new function $F^{(1)}$ by

$$V^{(1)}(x,y,t) = -\mathrm{e}^{-\gamma x} F^{(1)}(y,t).$$

In terms of this specific value function $V^{(1)}$, we have $V_x^{(1)} = -\gamma V^{(1)}$, $V_{xx}^{(1)} = \gamma^2 V^{(1)} < 0$ and $V_{xy}^{(1)} = -\gamma V_y^{(1)}$; therefore, the general form of π^* in (2.11) can be simplified to

$$\pi^* = -\frac{\lambda V_x^{(1)}}{\sigma V_{xx}^{(1)}} - \frac{\rho a(t) V_{xy}^{(1)}}{\sigma V_{xx}^{(1)}} = \frac{\lambda}{\sigma\gamma} + \frac{\rho a(t) V_y^{(1)}}{\sigma\gamma V^{(1)}}.$$

Substituting both π^* and $V^{(1)}$ in (2.5), we derive that

$$\begin{cases} F_t^{(1)} + \dfrac{1}{2}\rho^2 a^2(t) F_{yy}^{(1)} - \dfrac{1}{2}\lambda^2 F^{(1)} - \dfrac{1}{2}\dfrac{\rho^2 a^2(t)\left(F_y^{(1)}\right)^2}{F^{(1)}} - \rho a(t)\lambda F_y^{(1)} = 0 \\ F^{(1)}(y,T) = \mathrm{e}^{\gamma g^{(1)}(y)}, \end{cases}$$

which is a non-linear partial differential equation. However, it can be linearized via a logarithmic transformation. In this sense, setting $F^{(1)}(y,t) = \mathrm{e}^{v(y,t)}$, we need to solve

$$v_t + \frac{1}{2}\rho^2 a^2(t) v_{yy} - \rho a(t)\lambda v_y - \frac{1}{2}\lambda^2 = 0 \tag{2.12}$$

with terminal condition $v(y,T) = \gamma g^{(1)}(y)$, which is a linear partial differential equation. Similarly to (2.8), we set

$$\hat{Y}_t = \rho Y_t$$

and define a new probability measure $Q^{(1)}$ by

$$\frac{dQ^{(1)}}{dP} = \exp\left(-\int_0^T \left(\frac{b(Y_s,s)}{a(s)} + \lambda\right) dW_s - \frac{1}{2}\int_0^T \left(\frac{b(Y_s,s)}{a(s)} + \lambda\right)^2 ds\right).$$

The process $W^{(1)}$ given by $dW_s^{(1)} = dW_s + \left(\frac{b(Y_s,s)}{a(s)} + \lambda\right) ds$ is a Brownian motion under $Q^{(1)}$ and the dynamic of Y is given by

$$dY_s = -a(s)\lambda\, ds + a(s)\, dW_s^{(1)}, \quad t \le s \le T,$$

with $Y_t = y \in \mathbb{R}$. Thus, we have

$$d\hat{Y}_s = -\rho a(s)\lambda\, ds + \rho a(s)\, dW_s^{(1)}, \quad t \le s \le T,$$

with $\hat{Y}_t = \hat{y} := \rho y$. Using the Feynman-Kac representation of the solution to (2.12), we deduce that

$$v(\hat{y},t) = -\frac{1}{2}\lambda^2(T-t) + E^{Q^{(1)}}\left[\gamma g^{(1)}\big(\hat{Y}_T\big)\Big|\hat{Y}_t = \hat{y}\right],$$

which yields

$$\begin{aligned} V^{(1)}(x,y,t) &= -\mathrm{e}^{-\gamma x}F^{(1)}(y,t) \\ &= -\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2(T-t) + E^{Q^{(1)}}\left[\gamma g^{(1)}(\rho Y_T)\big|Y_t = \frac{y}{\rho}\right]}. \end{aligned} \tag{2.13}$$

2.3 Three examples

Now we are ready to consider some specific examples to test the convergence. In order to solve by both direct and split methods, for simplicity, we start to assume that the payoff function $G = g(Y_T)$ satisfies $g(y) = y$ and the functions a and b only depend on time. According to Musiela and Zariphopoulou [10], the direct solution is given by

$$
\begin{aligned}
&V\Big(x, y, \frac{n-1}{n}T\Big) \\
&= -\mathrm{e}^{-\gamma x}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)g(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-\frac{n-1}{n}T)}\Big|Y_{\frac{n-1}{n}T} = y\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2\frac{T}{n}}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)\left(y+\int_{\frac{n-1}{n}T}^{T}\left((b(s)-\rho\lambda a(s))\,ds+a(s)\,dW_s^Q\right)\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{-\gamma x+\gamma y-\frac{1}{2}\lambda^2\frac{T}{n}+\gamma\int_{\frac{n-1}{n}T}^{T}\left(b(s)-\rho\lambda a(s)+\frac{1}{2}\gamma a^2(s)(1-\rho^2)\right)ds}.
\end{aligned}
$$

Based on (2.10), when $t = \frac{n-1}{n}T$, the solution of the second PDE is given by

$$
\begin{aligned}
&V^{(2)}\Big(x, y, \frac{n-1}{n}T\Big) \\
&= -\mathrm{e}^{-\gamma x}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma g^{(2)}\left(Y_T\sqrt{1-\rho^2}\right)}\Big|Y_{\frac{n-1}{n}T} = \frac{y}{\sqrt{1-\rho^2}}\right] \\
&= -\mathrm{e}^{-\gamma x}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\sqrt{1-\rho^2}\left(\frac{y}{\sqrt{1-\rho^2}}+\int_{\frac{n-1}{n}T}^{T}\left(\frac{b(s)}{\sqrt{1-\rho^2}}\,ds+a(s)\,dW_s^{(2)}\right)\right)}\right] \\
&= -\mathrm{e}^{-\gamma\left(x-g^{(1)}(y)\right)},
\end{aligned}
$$

where $g^{(1)}(y) = y + \int_{\frac{n-1}{n}T}^{T}\left(b(s) + \frac{1}{2}\gamma a^2(s)(1-\rho^2)\right)ds$. As a result, based on (2.13), the solution of the first PDE is given by

$$
V^{(1)}\Big(x, y, \frac{n-1}{n}T\Big) = -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2(T-\frac{n-1}{n}T)+E^{Q^{(1)}}\left[\gamma g^{(1)}(\rho Y_T)\Big|Y_{\frac{n-1}{n}T}=\frac{y}{\rho}\right]},
$$

which can be simplified to

$$
\begin{aligned}
&V^{(1)}\Big(x, y, \frac{n-1}{n}T\Big) \\
&= -\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2 \frac{T}{n} + E^{Q^{(1)}}\left[\gamma\left(y + \rho\int_{\frac{n-1}{n}T}^{T}\left(-a(s)\lambda\, ds + a(s)\, dW_s^{(1)}\right) + \int_{\frac{n-1}{n}T}^{T}\left(b(s) + \frac{1}{2}\gamma a^2(s)(1-\rho^2)\right) ds\right)\right]} \\
&= -\mathrm{e}^{-\gamma x + \gamma y - \frac{1}{2}\lambda^2 \frac{T}{n} + \gamma\int_{\frac{n-1}{n}T}^{T}\left(b(s) - \rho\lambda a(s) + \frac{1}{2}\gamma a^2(s)(1-\rho^2)\right) ds},
\end{aligned}
$$

which clearly coincides with the direct method after one step, hence it coincides after any step.

We now consider a second example. A stochastic process $Y = (Y_t)_{t\geq 0}$ is said to be an Ornstein-Uhlenbeck process if it satisfies the linear stochastic differential equation

$$dY_s = \theta(\nu - Y_s)\, ds + \phi\, dW_s, \quad t \leq s \leq T$$

with parameters θ, $\phi \in \mathbb{R}_+$ and $\nu \in \mathbb{R}$. In this case, ν is the long-run equilibrium level or long-run mean price of the asset Y, θ is the speed of reversion. Apply Itō's lemma to the function $f(Y_s, s) = Y_s \mathrm{e}^{\theta s}$ to derive

$$df(Y_s, s) = \theta Y_s \mathrm{e}^{\theta s}\, ds + \mathrm{e}^{\theta s}\, dY_s = \theta\nu \mathrm{e}^{\theta s}\, ds + \phi \mathrm{e}^{\theta s}\, dW_s.$$

Integrating from t to T we get

$$Y_T \mathrm{e}^{\theta T} = Y_t \mathrm{e}^{\theta t} + \int_t^T \theta\nu \mathrm{e}^{\theta s}\, ds + \int_t^T \phi \mathrm{e}^{\theta s}\, dW_s$$

whereupon we obtain the solution as

$$Y_T = Y_t \mathrm{e}^{\theta(t-T)} + \nu \left(1 - \mathrm{e}^{\theta(t-T)}\right) + \int_t^T \phi \mathrm{e}^{\theta(s-T)}\, dW_s. \tag{2.14}$$

Hence the direct solution is given by

$$\begin{aligned}
&V\Big(x, y, \frac{n-1}{n}T\Big) \\
&= -\mathrm{e}^{-\gamma x} \left(E^Q \left[\mathrm{e}^{\gamma(1-\rho^2)g(Y_T) - \frac{1}{2}(1-\rho^2)\lambda^2(T - \frac{n-1}{n}T)} \Big| Y_{\frac{n-1}{n}T} = y\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2 \frac{T}{n} + \gamma y \mathrm{e}^{-\theta \frac{T}{n}} + \gamma\nu\left(1 - \mathrm{e}^{-\theta\frac{T}{n}}\right)} \\
&\quad \times \left(E^Q \left[\mathrm{e}^{(1-\rho^2)\left(-\int_{\frac{n-1}{n}T}^{T} \left(\gamma\rho\lambda\phi \mathrm{e}^{\theta(s-T)}\, ds - \gamma\phi \mathrm{e}^{\theta(s-T)}\, dW_s^Q\right)\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2 \frac{T}{n} + \gamma y \mathrm{e}^{-\theta \frac{T}{n}} + \gamma\nu\left(1 - \mathrm{e}^{-\theta\frac{T}{n}}\right) - \gamma\rho\lambda\frac{\phi}{\theta}\left(1 - \mathrm{e}^{-\theta\frac{T}{n}}\right) + \frac{1}{4\theta}\gamma^2(1-\rho^2)\phi^2\left(1 - \mathrm{e}^{-2\theta\frac{T}{n}}\right)}.
\end{aligned}$$

Under the probability measure $Q^{(2)}$ defined in (2.9), the dynamic of $\tilde{Y}$ is given by

$$d\tilde{Y}_s = \theta(\nu - \tilde{Y}_s)\, ds + \sqrt{1-\rho^2}\phi\, dW_s^{(2)}, \quad t \le s \le T.$$

Hence, (2.8) and (2.14) imply that

$$Y_T = \frac{\tilde{Y}_T}{\sqrt{1-\rho^2}} = Y_t \mathrm{e}^{\theta(t-T)} + \frac{\nu}{\sqrt{1-\rho^2}}(1 - \mathrm{e}^{\theta(t-T)}) + \int_t^T \phi \mathrm{e}^{\theta(s-T)}\, dW_s^{(2)}.$$

Based on (2.10), when $t = \frac{n-1}{n}T$, the solution of the second PDE is given by

$$V^{(2)}\Big(x, y, \frac{n-1}{n}T\Big) = -\mathrm{e}^{-\gamma x} E^{Q^{(2)}} \left[\mathrm{e}^{\gamma g^{(2)}\left(Y_T\sqrt{1-\rho^2}\right)} \Big| Y_{\frac{n-1}{n}T} = \frac{y}{\sqrt{1-\rho^2}}\right],$$

which can be rewritten as

$$
\begin{aligned}
&V^{(2)}\Big(x, y, \frac{n-1}{n}T\Big)\\
&= -\mathrm{e}^{-\gamma x} E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\sqrt{1-\rho^2}\left(\frac{y}{\sqrt{1-\rho^2}}\mathrm{e}^{-\theta\frac{T}{n}}+\frac{\nu}{\sqrt{1-\rho^2}}\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right)+\int_{\frac{n-1}{n}T}^{T}\phi\mathrm{e}^{\theta(s-T)}\,dW_s^{(2)}\right)}\right]\\
&= -\mathrm{e}^{-\gamma x+\gamma y\mathrm{e}^{-\theta\frac{T}{n}}+\gamma\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right)+\frac{1}{4\theta}\gamma^2(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right)}\\
&= -\mathrm{e}^{-\gamma\left(x-g^{(1)}(y)\right)},
\end{aligned}
$$

where $g^{(1)}(y) = y\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right)$. As a result, based on (2.13), the solution of the first PDE is given by

$$
\begin{aligned}
&V^{(1)}\Big(x, y, \frac{n-1}{n}T\Big)\\
&= -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2(T-\frac{n-1}{n}T)+E^{Q^{(1)}}\left[\gamma g^{(1)}(\rho Y_T)\middle| Y_{\frac{n-1}{n}T}=\frac{y}{\rho}\right]}\\
&= -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2\frac{T}{n}}\\
&\quad\times \mathrm{e}^{E^{Q^{(1)}}\left[\gamma\left(\left(y+\rho\int_{\frac{n-1}{n}T}^{T}\left(-\phi\lambda\,ds+\phi\,dW_s^{(1)}\right)\right)\mathrm{e}^{-\theta\frac{T}{n}}+\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right)+\frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right)\right)\right]}\\
&= -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2\frac{T}{n}+\gamma y\mathrm{e}^{-\theta\frac{T}{n}}+\gamma\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right)-\gamma\rho\lambda\phi\frac{T}{n}\mathrm{e}^{-\theta\frac{T}{n}}+\frac{1}{4\theta}\gamma^2(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right)},
\end{aligned}
$$

which does not coincide with the direct method after one step. Indeed, it is clear to see that both methods do not coincide at any step this time. However, similarly to Nadtochiy and Zariphopoulou [11], the solution of the split method should converge to the solution of the direct method when n, the number of steps, goes to infinity. At time $t = 0$ with n steps, the solution using the split method can be expressed as

$$
\begin{aligned}
f^{(n)}(x,y) :=& -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2 T+\gamma y\mathrm{e}^{-\theta T}+\gamma\nu\left(1-\mathrm{e}^{-\theta T}\right)+\frac{1}{4\theta}\gamma^2(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta T}\right)} \\
&\times \mathrm{e}^{-\frac{\gamma\rho\lambda\phi\frac{T}{n}\mathrm{e}^{-\theta\frac{T}{n}}\left(1-\mathrm{e}^{-\theta T}\right)}{1-\mathrm{e}^{-\theta\frac{T}{n}}}}. \qquad (2.15)
\end{aligned}
$$

Meanwhile, the solution of the direct method is given by

$$
\begin{aligned}
V(x,y,0) =& -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2 T+\gamma y\mathrm{e}^{-\theta T}+\gamma\nu\left(1-\mathrm{e}^{-\theta T}\right)+\frac{1}{4\theta}\gamma^2(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta T}\right)} \\
&\times \mathrm{e}^{-\gamma\rho\lambda\frac{\phi}{\theta}\left(1-\mathrm{e}^{-\theta T}\right)}. \qquad (2.16)
\end{aligned}
$$

We can see that (2.15) and (2.16) are almost the same, except for the last term. Hence, in order to show that the value $f^{(n)}(x,y)$ of the split method converges to $V(x,y,0)$, it is enough to prove that $\frac{\frac{T}{n}\mathrm{e}^{-\theta\frac{T}{n}}}{1-\mathrm{e}^{-\theta\frac{T}{n}}}$ converges to $\frac{1}{\theta}$. Indeed, we have

$$
\lim_{n\to\infty}\frac{\frac{T}{n}\mathrm{e}^{-\theta\frac{T}{n}}}{1-\mathrm{e}^{-\theta\frac{T}{n}}} = \lim_{n\to\infty}\frac{\frac{T}{n}}{\mathrm{e}^{\theta\frac{T}{n}}-1} = \lim_{n\to\infty}\frac{-\frac{T}{n^2}}{-\frac{\theta T}{n^2}\mathrm{e}^{\theta\frac{T}{n}}} = \lim_{n\to\infty}\frac{1}{\theta\mathrm{e}^{\theta\frac{T}{n}}} = \frac{1}{\theta},
$$

where we used L'Hôspital's rule for the second equality.

Remark 2.1. *This example of an Ornstein-Uhlenbeck process can be generalized to dynamics of the form*

$$
dY_s = \theta_s(\nu_s - Y_s)\,ds + \phi_s\,dW_s, \quad t \le s \le T, \qquad (2.17)
$$

where θ, ν and ϕ are time-dependent integrable functions with θ and ϕ non-negative. Indeed, similarly to the derivation of (2.15) *and* (2.16)*, we obtain* *in this case*

$$
\begin{aligned}
f^{(n)}(x,y) :=& -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2 T+\gamma y \mathrm{e}^{\int_T^0 \theta_s\,ds}+\gamma\int_0^T \theta_s\nu_s \mathrm{e}^{\int_T^s \theta_u\,du}\,ds+\frac{\gamma^2}{2}(1-\rho^2)\int_0^T \phi_s^2 \mathrm{e}^{2\int_T^s \theta_u\,du}\,ds} \\
&\times \exp\left(-\gamma\rho\lambda \sum_{j=0}^{n-1} \mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du} \int_{\frac{j}{n}T}^{\frac{j+1}{n}T} \phi_s\,ds\right), \\
V(x,y,0) :=& -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2 T+\gamma y \mathrm{e}^{\int_T^0 \theta_s\,ds}+\gamma\int_0^T \theta_s\nu_s \mathrm{e}^{\int_T^s \theta_u\,du}\,ds+\frac{\gamma^2}{2}(1-\rho^2)\int_0^T \phi_s^2 \mathrm{e}^{2\int_T^s \theta_u\,du}\,ds} \\
&\times \exp\left(-\gamma\rho\lambda \int_0^T \phi_s \mathrm{e}^{\int_T^s \theta_u\,du}\,ds\right).
\end{aligned}
$$

We can rewrite

$$
\sum_{j=0}^{n-1} \mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du} \int_{\frac{j}{n}T}^{\frac{j+1}{n}T} \phi_s\,ds = \int_0^T \phi_s \sum_{j=0}^{n-1} \mathbb{1}_{(\frac{j}{n}T,\frac{j+1}{n}T]}(s)\mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du}\,ds.
$$

Because θ is nonnegative, we have

$$
0 < \sum_{j=0}^{n-1} \mathbb{1}_{(\frac{j}{n}T,\frac{j+1}{n}T]}(s)\mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du} \leq 1,
$$

so that we can apply dominated convergence, which yields

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T \phi_s \sum_{j=0}^{n-1} \mathbb{1}_{(\frac{j}{n}T,\frac{j+1}{n}T]}(s)\mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du}\,ds \\
&= \int_0^T \phi_s \lim_{n\to\infty}\sum_{j=0}^{n-1} \mathbb{1}_{(\frac{j}{n}T,\frac{j+1}{n}T]}(s)\mathrm{e}^{\int_T^{\frac{j}{n}T}\theta_u\,du}\,ds \\
&= \int_0^T \phi_s \mathrm{e}^{\int_T^s \theta_u\,du}\,ds.
\end{aligned}
$$

This implies that $f^{(n)}(x,y)$ converges to $V(x,y,0)$ when Y is of the form (2.17).

Finally, what happens if all conditions remain the same except that b is a function of both time and the nontraded asset Y? Similarly as above, the direct solution is given by

$$
\begin{aligned}
&V(x,y,0)\\
&= -\mathrm{e}^{-\gamma x}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)g(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2 T}\middle| Y_0=y\right]\right)^{\frac{1}{1-\rho^2}}\\
&= -\mathrm{e}^{-\gamma x-\frac{1}{2}\lambda^2 T}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)\left(y+\int_0^T\left((b(Y_s,s)-\rho\lambda a(s))\,ds+a(s)\,dW_s^Q\right)\right)}\middle| Y_0=y\right]\right)^{\frac{1}{1-\rho^2}}\\
&= -\mathrm{e}^{-\gamma x+\gamma y-\frac{1}{2}\lambda^2 T}\left(E^Q\left[\mathrm{e}^{\gamma(1-\rho^2)\left(\int_0^T\left((b(Y_s,s)-\rho\lambda a(s))\,ds+a(s)\,dW_s^Q\right)\right)}\middle| Y_0=y\right]\right)^{\frac{1}{1-\rho^2}}.
\end{aligned}
$$

When $t=\frac{n-1}{n}T$, the solution of the second PDE is given by

$$
\begin{aligned}
&V^{(2)}\left(x,y,\frac{n-1}{n}T\right)\\
&= -\mathrm{e}^{-\gamma x}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma g^{(2)}\left(Y_T\sqrt{1-\rho^2}\right)}\middle| Y_{\frac{n-1}{n}T}=\frac{y}{\sqrt{1-\rho^2}}\right]\\
&= -\mathrm{e}^{-\gamma x}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\sqrt{1-\rho^2}\left(\frac{y}{\sqrt{1-\rho^2}}+\int_{\frac{n-1}{n}T}^T\left(\frac{b(\tilde{Y}_s,s)}{\sqrt{1-\rho^2}}\,ds+a(s)\,dW_s^{(2)}\right)\right)}\middle| Y_{\frac{n-1}{n}T}=\frac{y}{\sqrt{1-\rho^2}}\right]\\
&= -\mathrm{e}^{-\gamma x+\gamma y}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\int_{\frac{n-1}{n}T}^T\left(b(\sqrt{1-\rho^2}Y_s,s)\,ds+\sqrt{1-\rho^2}a(s)\,dW_s^{(2)}\right)}\middle| Y_{\frac{n-1}{n}T}=\frac{y}{\sqrt{1-\rho^2}}\right]\\
&= -\mathrm{e}^{-\gamma x+\gamma y}E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\Xi_1}\middle| Y_0=\frac{y}{\sqrt{1-\rho^2}}\right]\\
&= -\mathrm{e}^{-\gamma\left(x-g^{(1)}(y)\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi_j &:= \int_0^{\frac{T}{n}}\left(b\left(\sqrt{1-\rho^2}Y_s,s+\frac{n-j}{n}T\right)ds+\sqrt{1-\rho^2}a\left(s+\frac{n-j}{n}T\right)dW_s^{(2)}\right),\\
g^{(1)}(y) &:= y+\frac{1}{\gamma}\ln\left(E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\Xi_1}\middle| Y_0=\frac{y}{\sqrt{1-\rho^2}}\right]\right).
\end{aligned}
$$

Therefore, the solution of the first PDE is given by

$$
\begin{aligned}
V^{(1)}\Big(x, y, \frac{n-1}{n}T\Big) &= -\,\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2 (T - \frac{n-1}{n}T) + E^{Q^{(1)}}\left[\gamma g^{(1)}(\rho Y_T) \middle| Y_{\frac{n-1}{n}T} = \frac{y}{\rho}\right]} \\
&= -\,\mathrm{e}^{-\gamma x - \frac{1}{2}\lambda^2 \frac{T}{n} + E^{Q^{(1)}}\left[\gamma \rho Y_T \middle| Y_{\frac{n-1}{n}T} = \frac{y}{\rho}\right]} \\
&\quad \times \mathrm{e}^{E^{Q^{(1)}}\left[\ln\left(E^{Q^{(2)}}\left[\mathrm{e}^{\gamma \Xi_1} \middle| Y_0 = \frac{y_1}{\sqrt{1-\rho^2}}\right]\Big|_{y_1 = \rho Y_T}\right) \middle| Y_{\frac{n-1}{n}T} = \frac{y}{\rho}\right]} \\
&= -\,\mathrm{e}^{-\gamma x + \gamma y - \frac{1}{2}\lambda^2 \frac{T}{n} - \gamma\rho \int_{\frac{n-1}{n}T}^{T} a(s)\lambda\, ds} \\
&\quad \times \mathrm{e}^{E^{Q^{(1)}}\left[\ln\left(E^{Q^{(2)}}\left[\mathrm{e}^{\gamma \Xi_1} \middle| Y_0 = \frac{y - \rho \int_{\frac{n-1}{n}T}^{T} a(s)\lambda\, ds + z}{\sqrt{1-\rho^2}}\right]\Big|_{z=N}\right)\right]}
\end{aligned}
$$

for N normally distributed with mean 0 and variance $\int_{\frac{n-1}{n}T}^{T} \rho^2 a^2(s)\, ds$ under the probability measure $Q^{(1)}$. We can see that after one step, unlike the first example, the split method does not perfectly coincide with the direct method this time. In order to test the convergence theorem in general, we need to keep solving functions $g^{(1)}$ and $g^{(2)}$ and replace them into the corresponding PDEs for n steps. After repeating n times, we derive the following pattern

$$
\begin{aligned}
&V(x, y, 0) \\
&= -\mathrm{e}^{-\gamma x + \gamma y - \frac{1}{2}\lambda^2 T - \gamma\rho \int_0^T a(s)\lambda\, ds} \\
&\quad \times \mathrm{e}^{\cdots E^{Q^{(1)}}\left[\ln\left(E^{Q^{(2)}}\left[\mathrm{e}^{\gamma \Xi_2 + E^{Q^{(1)}}\left[\ln\left(E^{Q^{(2)}}\left[\mathrm{e}^{\gamma\Xi_1} \middle| Y_0 = \frac{y_1}{\sqrt{1-\rho^2}}\right]\Big|_{y_1=\rho Y_T}\right)\middle| Y_{\frac{n-1}{n}T} = \frac{y}{\rho}\right]} \cdots \right.\right.\right.} \\
&\qquad\qquad \cdots \left.\left.\left.\middle| Y_0 = \frac{y_2}{\sqrt{1-\rho^2}}\right]\Big|_{y_2 = \rho Y_{\frac{n-1}{n}T}}\right)\middle| Y_{\frac{n-2}{n}T} = \frac{y}{\rho}\right] \cdots .
\end{aligned}
$$

We will study in Chapter 4 a numerical implementation to test the convergence. We already note the iterative pattern involving expectations, logarithmic and exponential functions, which will also appear in the next chapter when we consider the Trotter-Kato approximation for BSDEs.

Chapter 3

The Trotter-Kato approximation for BSDEs

In this chapter, we briefly introduce backward stochastic differential equations (BSDEs) and relate their application to our financial optimal control problem. We then discuss what an application of the Trotter-Kato approximation to these BSDEs means.

3.1 Quadratic BSDEs

We consider a d-dimensional Brownian motion W on a complete probability space $(\Omega, \mathcal{F}_T, P)$ with $(\mathcal{F}_t)_{0\le t\le T}$ the augmented filtration generated by W up to time T. A BSDE is a stochastic differential equation with given terminal value. It is of the form

$$d\Gamma_t = -f(t, \Gamma_t, Z_t)\, dt + Z_t\, dW_t, \quad \Gamma_T = \xi, \tag{3.1}$$

or, equivalently,

$$\Gamma_t = \xi + \int_t^T f(s, \Gamma_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \le T,$$

where the terminal value ξ is an $\mathcal{F}_T$-measurable random variable, and the mapping $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, called *generator*, is $\mathcal{P}\otimes\mathcal{B}\otimes\mathcal{B}^d$-measurable. Here $\mathcal{P}$ denotes the σ-algebra generated by the predictable processes and $\mathcal{B}^d$ is the Borel-σ-algebra on $\mathbb{R}^d$. We denote the BSDE (3.1) by BSDE(f,ξ). As opposed to SDEs with initial conditions, the solutions to BSDEs consist of two components, namely a semimartingale Γ and a predictable $\mathbb{R}^d$-valued process Z. Moreover, if the BSDE (3.1) satisfies the following conditions: there exists constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
&\|\xi\|_{L^\infty} < \infty, \quad |f(t,y,z)| \le C_1(1 + |y| + |z|^2),\\
&(y_1 - y_2)|f(t,y_1,z) - f(t,y_2,z)| \le C_2|y_1 - y_2|^2\\
&|f(t,y,z_1) - f(t,y,z_2)| \le C_3(1 + |z_1| + |z_2|)|z_1 - z_2|
\end{aligned}$$

for all $t \in [0,T], y, y_1, y_2 \in \mathbb{R}, z, z_1, z_2 \in \mathbb{R}^d$, we call it *quadratic* BSDE because the conditions imply that f is of quadratic growth in the most important variable z.

Before applying, I will first list two main results about quadratic BSDEs without proof. These results go back to Kobylanski [5] and have been generalized to martingales in continuous filtrations by Morlais [9] under slightly weaker conditions. The next two theorems follow from Theorems 2.5–2.7 of Morlais [9]. We denote by $\mathcal{S}^\infty$ the space of bounded semimartingales and by $\mathbb{H}^2$ the space of predictable $\mathbb{R}^d$-valued processes Z, satisfying $E\left[\int_0^T |Z_t|^2\, dt\right] < \infty$.

The first result is the existence and uniqueness.

Theorem 3.1. *Every quadratic BSDE has a unique solution* $(\Gamma, Z) \in \mathcal{S}^\infty \times \mathbb{H}^2$.

The other result is the comparison principle.

Theorem 3.2. *Let* (Γ^1, Z^1) *and* (Γ^2, Z^2) *be the associated solutions of the quadratic BSDE* (f^1, ξ^1) *and BSDE* (f^2, ξ^2)*, respectively. Assume that*

$$\xi^1 \geq \xi^2 \quad \text{and} \quad f^1(t, \Gamma_t^2, Z_t^2) \geq f^2(t, \Gamma_t^2, Z_t^2), \quad dt \otimes dP - a.e.$$

Then we have that almost surely for any time t, $\Gamma_t^1 \geq \Gamma_t^2$.

3.2 BSDE characterization of the value process

We now relate our optimization problem (1.6) to BSDEs. To do so, we consider the financial market model introduced in Section 1.2. Moreover, because we are using exponential utility, we can assume without loss of generality that the initial capital at time t is zero: $X_t = 0$. Indeed, for general $X_t = x$, the corresponding value process equals $\mathrm{e}^{-\gamma x} V_s$, $t \leq s \leq T$, where V is the value process with $X_t = 0$. To give a BSDE characterization in a situation similar to Chapter 2, we impose the following assumptions. G is $\mathcal{W}_T$-measurable and ρ, σ and μ are $\mathbb{W}$-predictable, where $\mathbb{W} = (\mathcal{W}_t)_{0 \leq t \leq T}$ is the augmented filtration generated by W. This generalizes the assumption $G = g(Y_T)$ in Chapter 2. The above setting allows us to give a BSDE characterization for the value process V, using the results of Section 3.1 on quadratic BSDEs.

Proposition 3.3. *The BSDE*

$$\Gamma_t = G + \int_t^T \left(\frac{1}{2}\gamma(1-\rho_s^2)Z_s^2 - Z_s\rho_s\lambda_s - \frac{\lambda_s^2}{2\gamma} \right) ds - \int_t^T Z_s\, dW_s \qquad (3.2)$$

for $0 \leq t \leq T$ *has a unique solution* (Γ, Z) *where* Γ *is a real-valued bounded continuous* $\mathbb{W}$*-semimartingale and* Z *is a* $\mathbb{W}$*-predictable process which satisfies* $E^P\left[\int_0^T Z_s^2\, ds\right] < \infty$. *Moreover, there exists a continuous version* V *such that* $V = -\mathrm{e}^{\gamma\Gamma}$, *and its optimal strategy* π^* *is given by* $\pi^* = \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma}$.

Because of the importance of Proposition 3.3, we give two different ways of proof, which give additional insight into the relation between BSDEs and utility maximization. The first proof uses arguments similar to Lemma 4.1 of Frei [1] while the second method is based on the martingale optimality principle similar to Hu et al. [2] and Touzi [12].

Proof of Proposition 3.3 (first method). Existence and uniqueness of a solution (Γ, Z) of (3.3) follow from Theorem 3.1 with $\xi := G$ and

$$f(t,z) = \frac{1}{2}\gamma(1-\rho_t^2)z^2 - z\rho_t\lambda_t - \frac{\lambda_t^2}{2\gamma} \text{ for } t \in [0,T] \text{ and } z \in \mathbb{R}.$$

It remains to prove $V = -\mathrm{e}^{\gamma\Gamma}$ and $\pi^* = \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma}$. Proposition 7 of Mania and Schweizer [7] implies that $\int Z\, dW$ is a BMO-martingale with respect to both filtrations $\mathbb{F}$ and $\mathbb{W}$. Using

$$\exp(-\gamma X_s) = \exp(-\gamma X_s)\exp(\gamma\Gamma_t - \gamma\Gamma_s)\exp(\gamma\Gamma_s - \gamma\Gamma_t),$$

as well as (1.4) and (3.2), we can rewrite it as

$$
\begin{aligned}
&\exp(-\gamma X_s) \\
&= \exp\left(-\gamma\int_t^s \mu_r\pi_r\,dr - \gamma\int_t^s \pi_r\sigma_r\,dW_r^1\right)\exp(\gamma\Gamma_t - \gamma\Gamma_s) \\
&\quad\times\exp\left(-\gamma\int_t^s\left(\frac{1}{2}\gamma(1-\rho_r^2)Z_r^2 - Z_r\rho_r\lambda_r - \frac{\lambda_r^2}{2\gamma}\right)dr + \gamma\int_t^s Z_r\,dW_r\right) \\
&= \exp(\gamma\Gamma_t - \gamma\Gamma_s)\frac{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_s}{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_t} \\
&\quad\times\exp\left(\frac{1}{2}\int_t^s(\gamma\rho_r Z_r + \lambda_r - \gamma\pi_r\sigma_r)^2\,dr\right) \\
&\geq \exp(\gamma\Gamma_t - \gamma\Gamma_s)\frac{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_s}{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_t}, \quad t \leq s \leq T, \qquad (3.3)
\end{aligned}
$$

where we are using the notation $\mathcal{E}(M) = \exp\left(M - \frac{1}{2}\langle M\rangle\right)$ for the stochastic exponential of a continuous martingale M. Therefore, when $s = T$, we obtain

$$
\exp(-\gamma X_T + \gamma G) \geq \exp(\gamma\Gamma_t)\frac{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_T}{\mathcal{E}\left(\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1\right)_t}
$$

since $\Gamma_T = G$. After taking the expectation conditional on $\mathcal{F}_t$ on both sides, we have

$$
E^P\left[\exp(-\gamma X_T + \gamma G)|\mathcal{F}_t\right] \geq \exp(\gamma\Gamma_t)
$$

because Theorem A.2 shows that the stochastic exponential of the continuous BMO-martingale $\int\gamma Z\,dW - \int\gamma\pi\sigma\,dW^1$ is a true martingale provided that $\int\pi\,dW^1$ is a BMO-martingale. For general $\pi\in\mathcal{A}$, we argue analogously to the proof of Lemma 2.1 so that

$$
\operatorname*{ess\,inf}_{\pi\in\mathcal{A}} E^P\left[\mathrm{e}^{-\gamma X_T + \gamma G}\middle|\mathcal{F}_t\right] \geq \exp(\gamma\Gamma_t),
$$

which means

$$V_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E^P \left[-\mathrm{e}^{-\gamma X_T + \gamma G} \middle| \mathcal{F}_t \right] \leq -\exp\left(\gamma \Gamma_t\right).$$

As a result, $V = -\mathrm{e}^{\gamma\Gamma}$, and the equality holds for $\pi = \pi^* := \frac{\rho}{\sigma} Z + \frac{\lambda}{\gamma\sigma}$. We have $\pi^* \in \mathcal{A}$ because (3.3) implies that the corresponding $\exp(-\gamma X)$ is the product of a uniformly integrable martingale and a bounded process. $\square$

Furthermore, we can use another method to prove Proposition 3.3 by applying the martingale optimality principle.

Proof of Proposition 3.3 (second method). For every $\pi \in \mathcal{A}$, we define the process

$$V_t^\pi = -\mathrm{e}^{-\gamma(X_t - \Gamma_t)}, \quad t \in [0, T],$$

where Γ is the first component of the solution to a quadratic BSDE with terminal condition G and whose generator $f(t, z)$ we will specify later.

We compute first by Itô's formula that

$$\begin{aligned} dV_t^\pi =& -\gamma V_t^\pi (dX_t - d\Gamma_t) + \frac{\gamma^2}{2} V_t^\pi d\langle X - \Gamma\rangle_t \\ =& -\gamma V_t^\pi \left(\mu_t \pi_t \, dt + \sigma_t \pi_t \, dW_t^1 + f(t, Z_t)\, dt - Z_t \, dW_t \right) \\ &+ \frac{\gamma^2}{2} V_t^\pi \left(\sigma_t^2 \pi_t^2 \, dt + Z_t^2 \, dt - 2\rho_t \sigma_t Z_t \pi_t \, dt \right) \\ =& -\gamma V_t^\pi \left[\left(f(t, Z_t) - \varphi(t, Z_t, \pi_t) \right) \, dt + \sigma_t \pi_t \, dW_t^1 - Z_t \, dW_t \right] \end{aligned}$$

where we used the notation

$$\varphi(t, z, \pi) := \frac{\gamma}{2} \sigma_t^2 \pi^2 - \mu_t \pi - \gamma \rho_t \sigma_t z \pi + \frac{\gamma}{2} z^2.$$

We choose $f(t,z) = \inf_{r\in\mathbb{R}} \varphi(t,z,r) = \frac{1}{2}\gamma(1-\rho_t^2)z^2 - z\rho_t\lambda_t - \frac{\lambda_t^2}{2\gamma}$ so that the process V^π is a local supermartingale due to its nonpositive drift. Then, it follows from the admissibility condition that the process V^π is a supermartingale, which implies that $E^P[V_T^\pi|\mathcal{F}_t] \leq -\mathrm{e}^{-\gamma(X_t-\Gamma_t)}$ a.s. It follows from the arbitrariness of $\pi \in \mathcal{A}$ that

$$V_t \leq -\mathrm{e}^{\gamma\Gamma_t}. \tag{3.4}$$

To prove the equality, we notice that when $\pi^* := \mathrm{argmin}_{r\in\mathbb{R}}\varphi(.,z,r) = \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma}$, the dynamics of the process V^{π^*} is

$$dV_t^{\pi^*} = -\gamma V_t^{\pi^*}(\sigma_t\pi_t^*\,dW_t^1 - Z_t\,dW_t).$$

By Proposition 7 of Mania and Schweizer [7], $\int Z\,dW$ is a BMO-martingale and so is $\int \sigma\pi^*\,dW^1$ because π^* is a linear combination of Z with bounded processes. We again deduce from Theorem A.2 that V^{π^*} is a true martingale. Consequently,

$$\pi^* := \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma} \quad \text{and} \quad E^P[V_T^{\pi^*}|\mathcal{F}_t] = -\mathrm{e}^{\gamma\Gamma_t}$$

Together with (3.4), this shows that $V = -\mathrm{e}^{\gamma\Gamma}$ and π^* is an optimal portfolio strategy. Indeed, $\mathrm{e}^{-\gamma X}$ is uniformly integrable since $\mathrm{e}^{-\gamma X} = -V^{\pi^*}\mathrm{e}^{-\gamma\Gamma}$ is a product of a uniformly integrable martingale and a bounded process, hence $\pi^* \in \mathcal{A}$. □

Proposition 3.3 gives us a general characterization of the value process V in terms of the solution of the BSDE (3.2), but there is no explicit formula available so far. The next proposition shows a specific form of V by assuming a constant correlation ρ.

Proposition 3.4. *Assume that ρ is constant. Then the value process V satisfies*

$$V_t = -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma G - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)}\middle|\mathcal{F}_t\right]\right)^{\frac{1}{1-\rho^2}},$$

where the probability $\hat{P}$ is defined as $\frac{d\hat{P}}{dP} := \exp\left(-\int_0^T \lambda_s\, dW_s^1 - \frac{1}{2}\int_0^T \lambda_s^2\, ds\right)$.

Proof. We know that the process $\hat{W} := W + \int \rho\lambda\, ds$ is a Brownian motion under the probability measure $\hat{P}$ defined above. Because ρ is constant, we can rewrite the BSDE (3.2) as

$$\begin{aligned}
\Gamma_t &= G + \int_t^T \left(\frac{1}{2}\gamma(1-\rho^2)Z_s^2 - \frac{\lambda_s^2}{2\gamma}\right) ds - \int_t^T Z_s\,(dW_s + \rho\lambda_s\, ds) \\
&= G - \frac{1}{2\gamma}\int_t^T \lambda_s^2\, ds + \frac{1}{2}\gamma(1-\rho^2)\int_t^T Z_s^2\, ds - \int_t^T Z_s\, d\hat{W}_s \\
&= G - \frac{1}{2\gamma}\int_t^T \lambda_s^2\, ds - \frac{1}{\gamma(1-\rho^2)}\ln\left(\frac{\mathcal{E}\left(\int \gamma(1-\rho^2)Z\, d\hat{W}\right)_T}{\mathcal{E}\left(\int \gamma(1-\rho^2)Z\, d\hat{W}\right)_t}\right).
\end{aligned}$$

Therefore, we derive that

$$\mathrm{e}^{\gamma(1-\rho^2)\Gamma_t}\frac{\mathcal{E}\left(\int \gamma(1-\rho^2)Z\, d\hat{W}\right)_T}{\mathcal{E}\left(\int \gamma(1-\rho^2)Z\, d\hat{W}\right)_t} = \mathrm{e}^{(1-\rho^2)\left(\gamma G - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)}.$$

Similarly to the first proof of Proposition 3.3, we set $M = \int Z\, dW$, which is a BMO-martingale under the probability measure P, and set $N = \int \lambda\, dW$, which is also a BMO-martingale under P due to the boundedness of λ. Hence $M - \langle M, N\rangle = \int Z\, d\hat{W}$ is a BMO-martingale under $\hat{P}$ by Theorem A.3. According to Theorem A.2, the stochastic exponential of $\int \gamma(1-\rho^2)Z\, d\hat{W}$ is a true martingale under $\hat{P}$. After taking the expectation conditional on $\mathcal{F}_t$ on both sides under the probability measure $\hat{P}$, we get

$$\mathrm{e}^{\gamma(1-\rho^2)\Gamma_t} = E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma G - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)}\middle| \mathcal{F}_t\right],$$

then Proposition 3.3 implies that

$$V_t = -\mathrm{e}^{\gamma\Gamma_t} = -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma G - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)}\middle| \mathcal{F}_t\right]\right)^{\frac{1}{1-\rho^2}}. \qquad \square$$

3.3 Splitting the BSDE

In this section, we discuss the idea of the Trotter-Kato approximation applied to BSDEs. We start with an auxiliary result for Brownian motions.

Lemma 3.5. *Assume $\int_0^T \frac{1}{1-\rho_s^2}\, ds < \infty$ a.s. Then $W^{1,\perp} = (W_t^{1,\perp})_{0\le t\le T}$ given by*

$$W_t^{1,\perp} = \int_0^t \frac{1}{\sqrt{1-\rho_s^2}}\, dW_s - \int_0^t \frac{\rho_s}{\sqrt{1-\rho_s^2}}\, dW_s^1 \tag{3.5}$$

is a Brownian motion which is independent of W^1.

Proof. 1) It is clear that $W^{1,\perp}$ is a continuous local martingale with $W_0^{1,\perp} = 0$ and its quadratic variation satisfies

$$\begin{aligned}
&d\langle W^{1,\perp}, W^{1,\perp}\rangle_t = \\
&d\left\langle \int \frac{1}{\sqrt{1-\rho^2}}\, dW - \int \frac{\rho}{\sqrt{1-\rho^2}}\, dW^1, \int \frac{1}{\sqrt{1-\rho^2}}\, dW - \int \frac{\rho}{\sqrt{1-\rho^2}}\, dW^1 \right\rangle_t \\
&= \frac{1}{1-\rho_t^2}\, dt - \frac{\rho_t}{1-\rho_t^2}\, d\langle W, W^1\rangle_t - \frac{\rho_t}{1-\rho_t^2}\, d\langle W^1, W\rangle_t + \frac{\rho_t^2}{1-\rho_t^2}\, dt \\
&= \frac{1}{1-\rho_t^2}\, dt - \frac{\rho_t^2}{1-\rho_t^2}\, dt - \frac{\rho_t^2}{1-\rho_t^2}\, dt + \frac{\rho_t^2}{1-\rho_t^2}\, dt \\
&= dt.
\end{aligned}$$

Therefore, *Lévy's Characterization of Brownian Motion* implies that $W^{1,\perp}$ is a Brownian motion.

2) We can derive that

$$\begin{aligned} d\langle W^{1,\perp}, W^1\rangle_t &= d\left\langle \int \frac{1}{\sqrt{1-\rho^2}}\, dW - \int \frac{\rho}{\sqrt{1-\rho^2}}\, dW^1, W^1 \right\rangle_t \\ &= \frac{1}{\sqrt{1-\rho_t^2}}\, d\langle W, W^1\rangle_t - \frac{\rho_t}{\sqrt{1-\rho_t^2}}\, dt \\ &= \frac{\rho_t}{\sqrt{1-\rho_t^2}}\, dt - \frac{\rho_t}{\sqrt{1-\rho_t^2}}\, dt \\ &= 0. \end{aligned}$$

Therefore, $W^{1,\perp}$ and W^1 are uncorrelated Brownian motions and hence independent. □

Lemma 3.5 shows that $W^{1,\perp}$ and W^1 are two independent Brownian motions. Similarly to (1.1), we derive from (3.5) that

$$dW_s = \rho_s\, dW_s^1 + \sqrt{1-\rho_s^2}\, dW_s^{1,\perp}, \quad 0 \le s \le T, \quad W_0 = 0. \tag{3.6}$$

As a result, we can rewrite the BSDE (3.2) as

$$\begin{aligned} \Gamma_t = G &+ \int_t^T \left(\frac{1}{2}\gamma(1-\rho_s^2) Z_s^2 - Z_s \rho_s \lambda_s - \frac{\lambda_s^2}{2\gamma} \right) ds \\ &- \int_t^T Z_s \rho_s\, dW_s^1 - \int_t^T Z_s \sqrt{1-\rho_s^2}\, dW_s^{1,\perp}. \end{aligned}$$

After setting $Z_t^{(1)} = Z_t \rho_t$ and $Z_t^{(2)} = Z_t \sqrt{1-\rho_t^2}$, we have

$$\Gamma_t = G + \int_t^T \left(\frac{1}{2}\gamma \left|Z_s^{(2)}\right|^2 - \frac{\lambda_s^2}{2\gamma} \right) ds - \int_t^T Z_s^{(1)} (\, dW_s^1 + \lambda_s\, ds) - \int_t^T Z_s^{(2)}\, dW_s^{1,\perp}.$$

This can be seen as a stochastic analogue to the HJB equation (2.3). Recalling the dynamics (1.2) of the traded stock price S, we can interpret

$$\int_t^T Z_s^{(1)}(\,dW_s^1 + \lambda_s\,ds) = \int_t^T Z_s^{(1)} \frac{1}{\sigma_s S_s}\,dS_s$$

as the complete part. On the other hand, we have $d\langle S, W^{1,\perp}\rangle = 0$ so that we interpret

$$\int_t^T \frac{1}{2}\gamma\left|Z_s^{(2)}\right|^2\,ds - \int_t^T Z_s^{(2)}\,dW_s^{1,\perp}$$

as the incomplete part. Therefore, we have a very similar decomposition as for the PDE (2.3) in Chapter 2. While the Trotter-Kato approximation for PDEs is well established, we next introduce and study a similar scheme for the BSDEs based on the above-mentioned observation of complete and incomplete parts. We consider the two BSDEs

$$\begin{cases} \Gamma_t^{(1)} = G^{(1)} - \displaystyle\int_t^T Z_s^{(1)}(\,dW_s^1 + \lambda_s\,ds) & (3.7) \\ \Gamma_t^{(2)} = G^{(2)} - \displaystyle\int_t^T \frac{\lambda_s^2}{2\gamma}\,ds + \int_t^T \frac{1}{2}\gamma\left|Z_s^{(2)}\right|^2\,ds - \int_t^T Z_s^{(2)}\,dW_s^{1,\perp}, & (3.8) \end{cases}$$

where $\Gamma^{(1)}$ corresponds to a complete market problem and $\Gamma^{(2)}$ represents the orthogonal, incomplete market component, respectively.

As in the Trotter-Kato approximation, we start by studying the incomplete component (3.8). The idea follows directly from the proof of Proposition 3.4 by constructing a stochastic exponential term, then $\Gamma^{(2)}$ becomes

$$\Gamma_t^{(2)} = G^{(2)} - \int_t^T \frac{\lambda_s^2}{2\gamma}\,ds - \frac{1}{\gamma}\ln\left(\frac{\mathcal{E}\left(\int \gamma Z^{(2)}\,dW^{1,\perp}\right)_T}{\mathcal{E}\left(\int \gamma Z^{(2)}\,dW^{1,\perp}\right)_t}\right).$$

This implies that

$$\mathrm{e}^{\gamma\Gamma_t^{(2)}}\frac{\mathcal{E}\left(\int\gamma Z^{(2)}\,dW^{1,\perp}\right)_T}{\mathcal{E}\left(\int\gamma Z^{(2)}\,dW^{1,\perp}\right)_t}=\mathrm{e}^{\gamma G^{(2)}-\int_t^T\frac{\lambda_s^2}{2}\,ds}$$

We know that $\int\gamma Z^{(2)}\,dW^{1,\perp}$ is a BMO-martingale, and hence its stochastic exponential is a true martingale by Theorem A.2. Because $W^{1,\perp}$ is orthogonal to W^1 by Lemma 3.5, this holds in both filtrations $(\mathcal{F}_t)_{0\le t\le T}$ and $(\mathcal{F}_t\vee\mathcal{W}_T^1)_{0\le t\le T}$ where $\mathcal{W}_T^1$ is the σ-algebra generated by W^1. We would like to get rid of only the incomplete component and hence we condition on $\mathcal{F}_t\vee\mathcal{W}_T^1$. This yields

$$\mathrm{e}^{\gamma\Gamma_t^{(2)}}=E^P\left[\mathrm{e}^{\gamma G^{(2)}-\int_t^T\frac{\lambda_s^2}{2}\,ds}\middle|\mathcal{F}_t\vee\mathcal{W}_T^1\right],$$

which can be simplified as the form, namely,

$$\Gamma_t^{(2)}=\frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\gamma G^{(2)}-\int_t^T\frac{\lambda_s^2}{2}\,ds}\middle|\mathcal{F}_t\vee\mathcal{W}_T^1\right]\right). \tag{3.9}$$

For the first auxiliary BSDE, the solution is easy to calculate via taking the expectation conditional on $\mathcal{F}_t$ on both sides under the probability measure $\hat{P}$ defined in Proposition 3.4. The explicit solution of (3.7) is given by

$$\Gamma_t^{(1)}=E^{\hat{P}}\left[G^{(1)}\middle|\mathcal{F}_t\right]. \tag{3.10}$$

Using a method similar to the Trotter-Kato approximation, we obtain after n

steps that

$$\tilde{\Gamma}_0^{(n)} = \cdots E^{\hat{P}}\left[\ln\left(E^{P}\left[\mathrm{e}^{E^{\hat{P}}\left[\ln\left(E^{P}\left[\mathrm{e}^{\gamma G-\int_{\frac{n-1}{n}T}^{T}\frac{\lambda^2}{2}\,ds}\middle|\mathcal{F}_{\frac{n-1}{n}T}\vee\mathcal{W}_T^1\right]\right)\middle|\mathcal{F}_{\frac{n-1}{n}T}\right]\cdots\cdots-\int_{\frac{n-2}{n}T}^{\frac{n-1}{n}T}\frac{\lambda^2}{2}\,ds}\middle|\mathcal{F}_{\frac{n-2}{n}T}\vee\mathcal{W}^1_{\frac{n-1}{n}T}\right]\right)\middle|\mathcal{F}_{\frac{n-2}{n}T}\right]\cdots$$

3.4 Two examples

While proving the convergence of $\tilde{\Gamma}_0^{(n)}$ to Γ_0 looks very involved, we check the claim first in the first two examples of Section 2.3 and then look in Chapter 4 at a numerical implementation. We assume that ρ is constant and the payoff function G satisfies $G = Y_T$, where the nontraded asset Y has the dynamics

$$dY_s = b_s\,ds + a_s\,dW_s, \quad t \le s \le T, \quad Y_t = y.$$

We suppose that the coefficients a, b, μ and σ are deterministic functions. Using Proposition 3.4, the direct solution after one step is given by

$$\begin{aligned}
&V_{\frac{n-1}{n}T} \\
&= -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma Y_T-\frac{1}{2}\int_{\frac{n-1}{n}T}^{T}\lambda_s^2\,ds\right)}\middle|\mathcal{F}_{\frac{n-1}{n}T}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma Y_{\frac{n-1}{n}T}+\gamma\int_{\frac{n-1}{n}T}^{T}((b_s-\rho\lambda_s a_s)\,ds+a_s\,d\hat{W}_s)-\frac{1}{2}\int_{\frac{n-1}{n}T}^{T}\lambda_s^2\,ds\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T}}\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\int_{\frac{n-1}{n}T}^{T}\left(\left(\gamma b_s-\gamma\rho\lambda_s a_s-\frac{1}{2}\lambda_s^2\right)ds+\gamma a_s\,d\hat{W}_s\right)\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T}+\int_{\frac{n-1}{n}T}^{T}\left(\gamma b_s-\gamma\rho\lambda_s a_s-\frac{1}{2}\lambda_s^2+\frac{1}{2}\gamma^2(1-\rho^2)a_s^2\right)ds}.
\end{aligned} \tag{3.11}$$

Based on (3.6) and (3.9), when $t = \frac{n-1}{n}T$, the solution of the step for the incomplete component is given by

$$
\begin{aligned}
&\Gamma^{(2)}_{\frac{n-1}{n}T} \\
&= \frac{1}{\gamma} \ln \left(E^P \left[\mathrm{e}^{\gamma Y_T - \int_{\frac{n-1}{n}T}^T \frac{\lambda_s^2}{2}\, ds} \middle| \mathcal{F}_{\frac{n-1}{n}T} \vee \mathcal{W}_T^1 \right] \right) \\
&= \frac{1}{\gamma} \ln \left(E^P \left[\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T} + \gamma \int_{\frac{n-1}{n}T}^T (b_s\, ds + a_s\, dW_s) - \int_{\frac{n-1}{n}T}^T \frac{\lambda_s^2}{2}\, ds} \middle| \mathcal{F}_{\frac{n-1}{n}T} \vee \mathcal{W}_T^1 \right] \right) \\
&= \frac{1}{\gamma} \ln \left(E^P \left[\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T} + \gamma \int_{\frac{n-1}{n}T}^T \left(\left(b_s - \frac{1}{2\gamma}\lambda_s^2 \right) ds + a_s (\rho\, dW_s^1 + \sqrt{1-\rho^2}\, dW_s^{1,\perp}) \right)} \middle| \mathcal{F}_{\frac{n-1}{n}T} \vee \mathcal{W}_T^1 \right] \right) \\
&= \frac{1}{\gamma} \ln \left(\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \gamma a_s \rho\, dW_s^1} E^P \left[\mathrm{e}^{\int_{\frac{n-1}{n}T}^T \left(\gamma b_s - \frac{1}{2}\lambda_s^2 \right) ds + \gamma a_s \sqrt{1-\rho^2}\, dW_s^{1,\perp}} \right] \right) \\
&= \frac{1}{\gamma} \ln \left(\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \gamma a_s \rho\, dW_s^1 + \int_{\frac{n-1}{n}T}^T \left(\gamma b_s - \frac{1}{2}\lambda_s^2 + \frac{1}{2}\gamma^2 (1-\rho^2) a_s^2 \right) ds} \right) \\
&= Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \left(\left(b_s - \frac{1}{2\gamma}\lambda_s^2 + \frac{1}{2}\gamma(1-\rho^2)a_s^2 \right) ds + a_s \rho\, dW_s^1 \right),
\end{aligned}
$$

using that $\int_{\frac{n-1}{n}T}^T \gamma a_s \rho\, dW_s^1$ is measurable with respect to $\mathcal{F}_{\frac{n-1}{n}T} \vee \mathcal{W}_T^1$ and $\int_{\frac{n-1}{n}T}^T \gamma a_s \sqrt{1-\rho^2}\, dW_s^{1,\perp}$ is independent of $\mathcal{F}_{\frac{n-1}{n}T} \vee \mathcal{W}_T^1$. Therefore, after substituting the above result as the terminal value into the equation (3.10), we derive the solution of the step for the complete component as

$$
\begin{aligned}
&\Gamma^{(1)}_{\frac{n-1}{n}T} \\
&= E^{\hat{P}} \left[Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \left(\left(b_s - \frac{1}{2\gamma}\lambda_s^2 + \frac{1}{2}\gamma(1-\rho^2)a_s^2 \right) ds + a_s \rho\, dW_s^1 \right) \middle| \mathcal{F}_{\frac{n-1}{n}T} \right] \\
&= Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \left(b_s - \rho \lambda_s a_s - \frac{1}{2\gamma}\lambda_s^2 + \frac{1}{2}\gamma(1-\rho^2)a_s^2 \right) ds \\
&\quad + E^{\hat{P}} \left[\int_{\frac{n-1}{n}T}^T a_s \rho (\lambda_s\, ds + dW_s^1) \right] \\
&= Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^T \left(b_s - \rho \lambda_s a_s - \frac{1}{2\gamma}\lambda_s^2 + \frac{1}{2}\gamma(1-\rho^2)a_s^2 \right) ds,
\end{aligned}
$$

because $W^1 + \int \lambda\, ds$ is a Brownian motion under the probability measure $\hat{P}$ defined in Proposition 3.4. Then Proposition 3.3 implies that

$$V_{\frac{n-1}{n}T} = -\mathrm{e}^{\gamma \Gamma^{(1)}_{\frac{n-1}{n}T}} = -\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T} + \int_{\frac{n-1}{n}T}^{T} \left(\gamma b_s - \gamma\rho\lambda_s a_s - \frac{1}{2}\lambda_s^2 + \frac{1}{2}\gamma^2(1-\rho^2)a_s^2\right) ds},$$

which is exactly the same as the direct solution (3.11) after one step. Consequently, it converges after any step as well.

Moreover, let us reconsider the Ornstein-Uhlenbeck process defined in Section 2.3 and see how the BSDE methods work this time. In this case, b is stochastic but the other conditions remain the same. From (2.14), setting $t = \frac{n-1}{n}T$, we obtain

$$Y_T = Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1 - \mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}}^{T} \phi \mathrm{e}^{\theta(s-T)}\, dW_s.$$

Similarly as the previous example, Proposition 3.4 yields that the direct solution after one step is given by

$$\begin{aligned}
&V_{\frac{n-1}{n}T} \\
&= -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma Y_T - \frac{1}{2}\int_{\frac{n-1}{n}T}^{T}\lambda_s^2\, ds\right)}\middle|\mathcal{F}_{\frac{n-1}{n}T}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\gamma Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \gamma\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}}^{T}\gamma\phi\mathrm{e}^{\theta(s-T)}\, dW_s - \frac{1}{2}\int_{\frac{n-1}{n}T}^{T}\lambda_s^2\, ds\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \gamma\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right)} \\
&\quad \times \left(E^{\hat{P}}\left[\mathrm{e}^{(1-\rho^2)\left(\int_{\frac{n-1}{n}T}^{T}\left(-\gamma\rho\lambda_s\phi\mathrm{e}^{\theta(s-T)} - \frac{1}{2}\lambda_s^2\right) ds + \int_{\frac{n-1}{n}T}^{T}\gamma\phi\mathrm{e}^{\theta(s-T)}\, d\hat{W}_s\right)}\right]\right)^{\frac{1}{1-\rho^2}} \\
&= -\mathrm{e}^{\gamma\Gamma_{\frac{n-1}{n}T}},
\end{aligned}$$

where

$$\Gamma_{\frac{n-1}{n}T} = Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) - \int_{\frac{n-1}{n}T}^{T}\left(\rho\lambda_s\phi\mathrm{e}^{\theta(s-T)} + \frac{1}{2\gamma}\lambda_s^2\right)ds$$
$$+ \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right). \tag{3.12}$$

Based on (3.6) and (3.9), when $t = \frac{n-1}{n}T$, the solution of the step for the incomplete component is given by

$$\begin{aligned}
&\Gamma^{(2)}_{\frac{n-1}{n}T} \\
&= \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\gamma Y_T - \int_{\frac{n-1}{n}T}^{T}\frac{\lambda_s^2}{2}\,ds}\middle|\mathcal{F}_{\frac{n-1}{n}T}\vee\mathcal{W}_T^1\right]\right) \\
&= \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\gamma Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \gamma\nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}}^{T}\left(\gamma\phi\mathrm{e}^{\theta(s-T)}\,dW_s - \frac{1}{2}\lambda_s^2\,ds\right)}\middle|\mathcal{F}_{\frac{n-1}{n}T}\vee\mathcal{W}_T^1\right]\right) \\
&= Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) \\
&\quad + \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\int_{\frac{n-1}{n}T}^{T}\left(-\frac{1}{2}\lambda_s^2\,ds + \gamma\phi\mathrm{e}^{\theta(s-T)}(\rho\,dW_s^1 + \sqrt{1-\rho^2}\,dW_s^{1,\perp})\right)}\middle|\mathcal{F}_{\frac{n-1}{n}T}\vee\mathcal{W}_T^1\right]\right) \\
&= Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}T}^{T}\phi\mathrm{e}^{\theta(s-T)}\rho\,dW_s^1 \\
&\quad + \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\int_{\frac{n-1}{n}T}^{T}\left(-\frac{1}{2}\lambda_s^2\,ds + \gamma\phi\mathrm{e}^{\theta(s-T)}\sqrt{1-\rho^2}\,dW_s^{1,\perp}\right)}\right]\right) \\
&= Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}T}^{T}\phi\mathrm{e}^{\theta(s-T)}\rho\,dW_s^1 - \int_{\frac{n-1}{n}T}^{T}\frac{1}{2\gamma}\lambda_s^2\,ds \\
&\quad + \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right),
\end{aligned}$$

which is based on the measurability and independence properties with respect to the filtration $\mathcal{F}_{\frac{n-1}{n}T}\vee\mathcal{W}_T^1$ similarly to the previous example. Therefore, after substituting the above outcome as the terminal value into the solution of the first BSDE (3.10), we derive the solution of the step for the complete component as

$$\begin{aligned}\Gamma^{(1)}_{\frac{n-1}{n}T} &= E^{\hat{P}}\Bigg[Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) + \int_{\frac{n-1}{n}T}^{T}\phi\mathrm{e}^{\theta(s-T)}\rho\, dW^1_s \\ &\quad - \int_{\frac{n-1}{n}T}^{T}\frac{1}{2\gamma}\lambda_s^2\, ds + \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right)\Bigg|\mathcal{F}_{\frac{n-1}{n}T}\Bigg] \\ &= Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) - \int_{\frac{n-1}{n}T}^{T}\left(\rho\lambda_s\phi\mathrm{e}^{\theta(s-T)} + \frac{1}{2\gamma}\lambda_s^2\right) ds \\ &\quad + \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right) + E^{\hat{P}}\left[\int_{\frac{n-1}{n}T}^{T}\phi\mathrm{e}^{\theta(s-T)}\rho(\lambda_s\, ds + dW^1_s)\right] \\ &= Y_{\frac{n-1}{n}T}\mathrm{e}^{-\theta\frac{T}{n}} + \nu\left(1-\mathrm{e}^{-\theta\frac{T}{n}}\right) - \int_{\frac{n-1}{n}T}^{T}\left(\rho\lambda_s\phi\mathrm{e}^{\theta(s-T)} + \frac{1}{2\gamma}\lambda_s^2\right) ds \\ &\quad + \frac{1}{4\theta}\gamma(1-\rho^2)\phi^2\left(1-\mathrm{e}^{-2\theta\frac{T}{n}}\right),\end{aligned}$$

which clearly coincides with $\Gamma_{\frac{n-1}{n}T}$ calculated in (3.12). Hence, for Ornstein-Uhlenbeck processes, the split BSDE method is the same as the direct BSDE method for every step. Note that this differs from the spitting method for PDEs, where we get that the two values differ after any step, but converge in the limit as the number of steps tends to infinity. The reason for this difference are different techniques: in the BSDE case, we evaluate the complete part directly by taking expectations under $\hat{P}$ while in the PDE case, we essentially replace the drift of Y by the drift of S through the probability measure $Q^{(1)}$. This gives the same in the limit, but at every step we get different values. In general, since $G^{(2)} = G$ in (3.9), we can rewrite (3.9) as

$$\begin{aligned}\Gamma^{(2)}_t &= \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\gamma G - \int_t^T \frac{\lambda_s^2}{2}\, ds}\Big|\mathcal{F}_t \vee \mathcal{W}^1_T\right]\right) \\ &= \frac{1}{\gamma}\ln\left(E^P\left[\mathrm{e}^{\gamma\Gamma_t + \gamma\int_t^T Z^{(1)}_s(dW^1_s + \lambda_s\, ds)}\mathcal{E}\left(\int \gamma Z^{(2)}\, dW^{1,\perp}\right)_{t,T}\Bigg|\mathcal{F}_t \vee \mathcal{W}^1_T\right]\right).\end{aligned}$$

In the Ornstein-Uhlenbeck case, $Z^{(1)}$ is $(\mathcal{W}_s^1)_{0\le s\le T}$-predictable because $Z_t^{(1)} = \phi e^{\theta(t-T)}\rho$, which is deterministic. This implies $\Gamma_t^{(2)} = \Gamma_t + \int_t^T Z_s^{(1)}(\,dW_s^1 + \lambda_s\,ds)$. Consequently, we rewrite (3.10) as

$$\Gamma_t^{(1)} = E^{\hat{P}}\left[G^{(1)}\middle|\mathcal{F}_t\right] = E^{\hat{P}}\left[\Gamma_t + \int_t^T Z_s^{(1)}(\,dW_s^1 + \lambda_s\,ds)\middle|\mathcal{F}_t\right] = \Gamma_t,$$

which gives us the same value as the direct method for just one step.

Chapter 4

Numerical implementation

In this chapter, we numerically test and verify the convergence of the Trotter-Kato approximation for both PDE and BSDE cases. For illustration, we consider again the Ornstein-Uhlenbeck process because this example can be explicitly solved in both direct and split methods. When we simulate the price process using Monte Carlo method, there are two main approximation errors: one is due to the finite number of steps, the other is due to the finite number of paths. Additionally to the Ornstein-Uhlenbeck process, I put two MATLAB codes for the general case at the very end of Appendix B. However, these codes only work for low-dimensional numbers of steps and paths because of their high complexity and the limited virtual memory of the computer.

4.1 PDE case

Figure 4.1 shows the different utility values under the PDE methods when we fix a high number of paths. We can see that the result of the direct method is always close to the true value. However, the utility values of the split method are below the true value at the beginning and starts converging after 10 steps. The values of the split method using Monte Carlo simulation fluctuate around the values from the explicit formula (2.15).

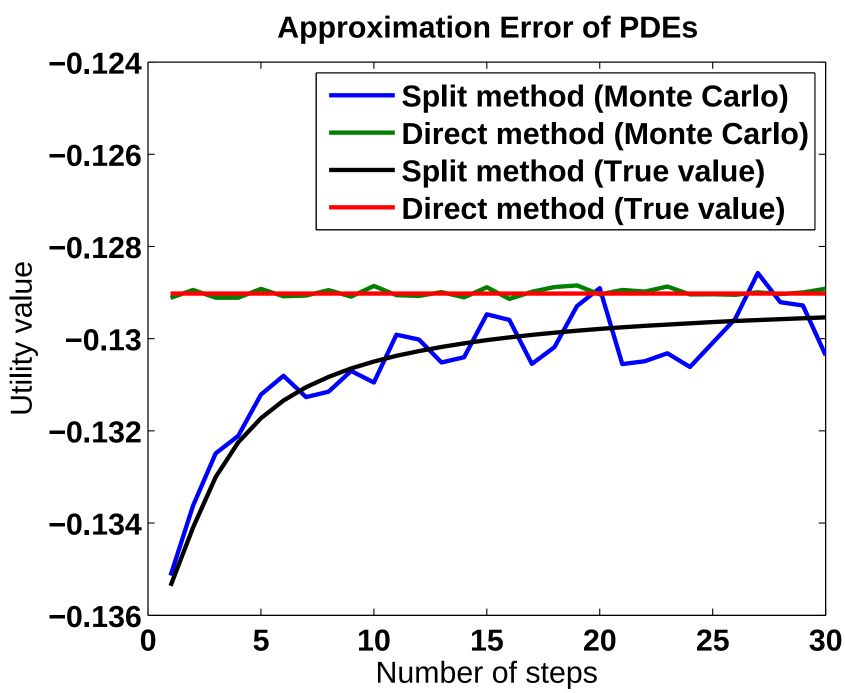

Figure 4.1: Approximation error of PDEs with different number of steps

Figure 4.2 shows the different utility values under the PDE methods when we fix a low number of steps. We chose a low number of steps because both direct and split methods converge really fast under a high number of steps. Even though we set the number of steps equal to 8, we still find that both methods converge fast and give values close to the true value already after 20 paths.

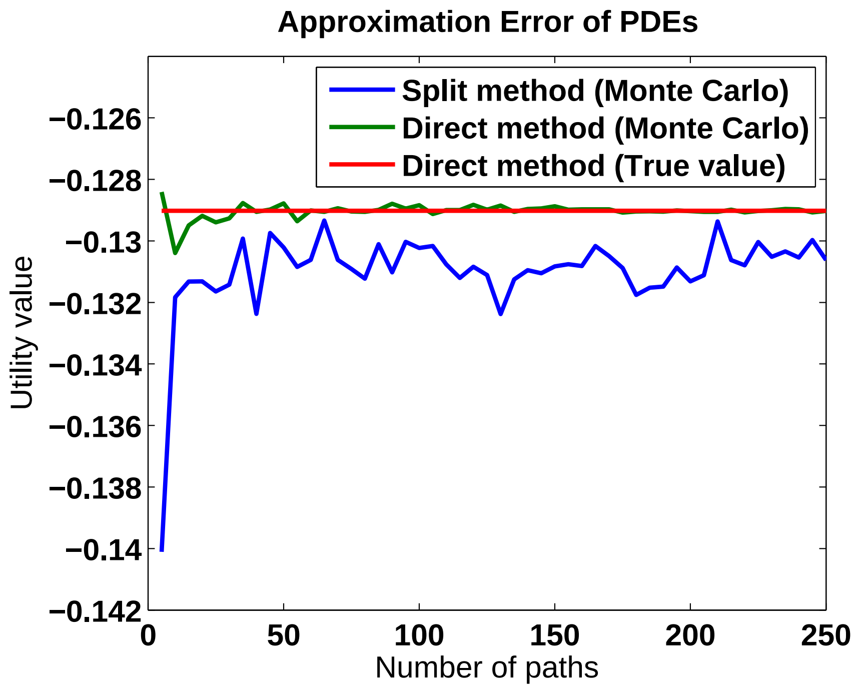

Figure 4.2: Approximation error of PDEs with different number of paths

4.2 BSDE case

Figure 4.3 shows the different utility values under the BSDE methods when we fix a high number of paths. The direct method is still close to the true value, no matter how many steps we have. Unlike Figure 4.1 there is no convergence trend for the split method, but the outcome is around the true value for any number of steps since we have already proven at the end of Chapter 3 that, in Ornstein-Uhlenbeck case, the split BSDE method is always the same as the direct BSDE method for every step.

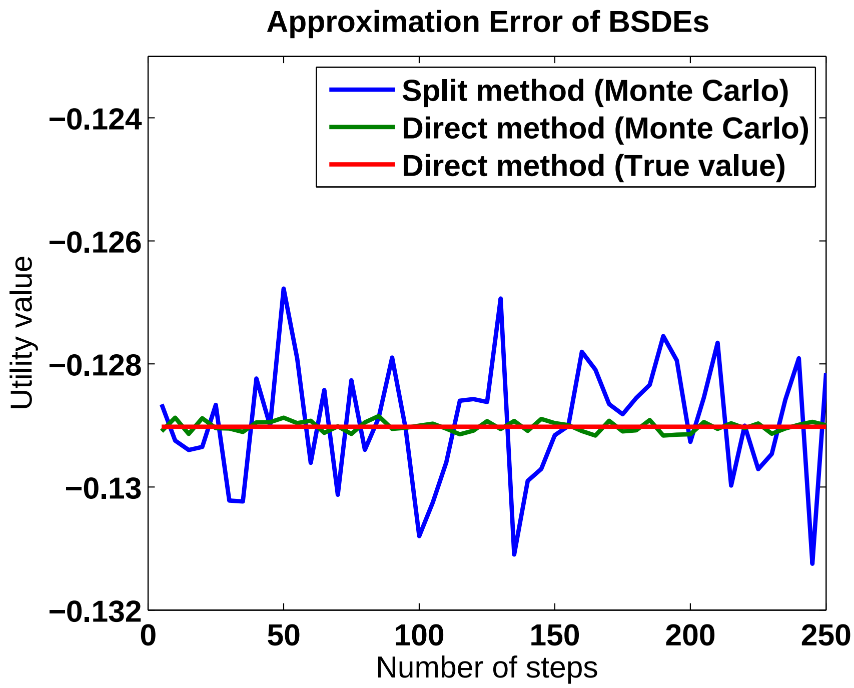

Figure 4.3: Approximation error of BSDEs with different number of steps

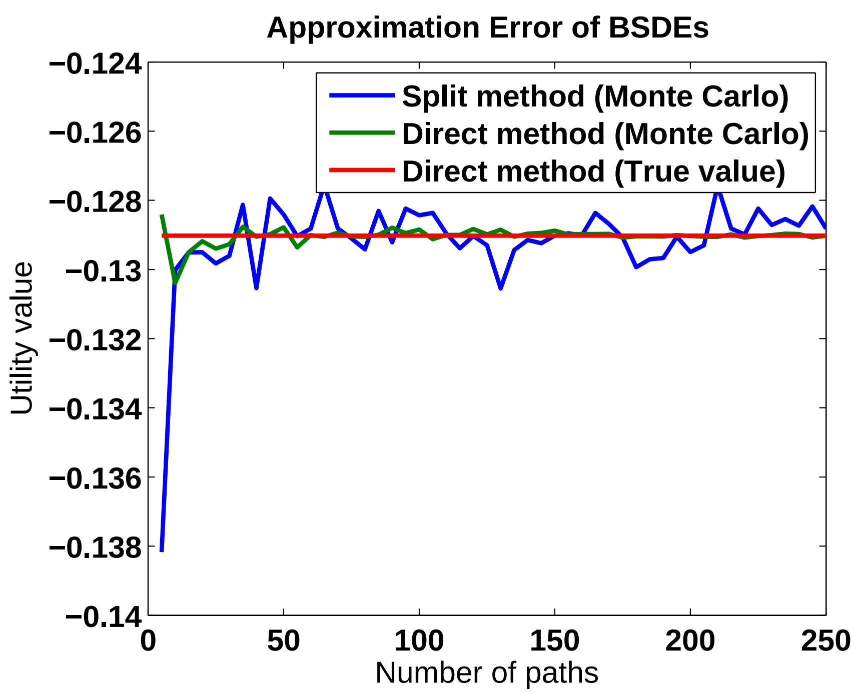

Figure 4.4: Approximation error of BSDEs with different number of paths

Figure 4.4 shows the different utility values under the BSDE methods when we fix a low number of steps. It is not surprising that this figure looks almost the same as Figure 4.2 since we set all parameters equal. The only difference to Figure 4.2 is that there is no narrowing gap between true value and the values of the split method. The reason is again that, for Ornstein-Uhlenbeck process, there is no approximation error due to the finite number of steps in the split BSDE method.

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Appendix A

BMO-martingales

In this appendix, we give a very brief summary of some concepts of continuous BMO-martingales which is based on Kazamaki [4]. First, we define the BMO-martingale as follows

Definition A.1. *A continuous local martingale $M = (M_t)_{0 \le t \le T}$ is a BMO-martingale if there is a constant C such that*

$$E\left[|M_T - M_\tau|^2 \middle| \mathcal{F}_\tau\right] \le C \quad a.s.$$

for every stopping time τ.

Next, I introduce two important theorems which are frequently used in the main part of this thesis.

In general, the stochastic exponential of a martingale does not need to be a true martingale. However, the next theorem ensures that the stochastic exponential of a BMO-martingale is a martingale. This is particularly helpful because this allows us to use BMO-martingales to define new probability measures via their stochastic exponentials.

Theorem A.2 (Kazamaki [4] Theorem 2.3)**.** *If M is a BMO-martingale, then $\mathcal{E}(M)$ is a martingale.*

The next result says that the BMO-property is invariant under suitable changes of measure.

Theorem A.3 (Kazamaki [4] Theorem 3.4)**.** *If both M and N are BMO-martingales under a probability measure P, then $M - \langle M, N\rangle$ is a BMO-martingale under the probability measure Q given by $\frac{dQ}{dP} = \mathcal{E}(N)_T$.*

Appendix B

MATLAB Code

Code to generate Figure 4.1:

```
n = 30;
rng('default')
x = zeros(n,1);
y = zeros(n,1);
z = zeros(n,1);
c = zeros(n,1);
for i=1:n
x(i) = i;
y(i) = OUsplitPDE(5,5,10,x(i),100,100,0.5,0.1,1,5,1,1);
z(i) = Direct(5,5,10,x(i),10000,0.5,0.1);
y(i) = OUsplitPDE(5,5,10,x(i),100,100,0.5,0.1,1,5,1,1);
c(i) = - exp(1^2*(5-10)/2 + 0.1*5*exp(1*(5-10)) + ...
    0.1*5*(1-exp(1*(5-10))) + 0.1*0.5*1*1*exp(1*(5-10)/x(i)) ...
    * (1-exp(1*(5-10)))*((5-10)/x(i))/(1-exp(1*(5-10)/x(i))) ...
    + 0.1^2*(1-0.5^2)*1^2*(1-exp(2*1*(5-10)))/(4*1));
end
R = Truevalue(5,5,10,0.5,0.1,1,5,1,1);
r(1:n) = R;
figure, plot(x,y,x,z,x,c,'k',x,r,'r','LineWidth',2),
xlabel('Number of steps','fontsize',14), ylabel('Utility ...
    value','fontsize',14), title('Approximation Error of ...
    PDEs','fontsize',14,'FontWeight','bold'), ...
    set(gca,'fontsize',14,'FontWeight','bold'),
legend('Split method (Monte Carlo)','Direct method (Monte ...
    Carlo)','Split method (True value)','Direct method (True ...
    value)')
```

Code to generate Figure 4.2:

```
n = 50;
rng('default')
x = zeros(n,1);
y = zeros(n,1);
```

```
z = zeros(n,1);
for i=1:n
x(i) = 5*i;
y(i) = OUsplitPDE(5,5,10,8,x(i),x(i),0.5,0.1,1,5,1,1);
z(i) = Direct(5,5,10,8,x(i)^2,0.5,0.1);
end
R = Truevalue(5,5,10,0.5,0.1,1,5,1,1);
r(1:n) = R;
figure, plot(x,y,x,z,x,r,'LineWidth',2),
xlabel('Number of paths','fontsize',14), ylabel('Utility ...
    value','fontsize',14), title('Approximation Error of ...
    PDEs','fontsize',14,'FontWeight','bold'), ...
    set(gca,'fontsize',14,'FontWeight','bold'),
legend('Split method (Monte Carlo)','Direct method (Monte ...
    Carlo)','Direct method (True value)')
```

Code to generate Figure 4.3:

```
n = 50;
rng('default')
x = zeros(n,1);
y = zeros(n,1);
z = zeros(n,1);
for i=1:n
x(i) = 5*i;
y(i) = OUsplitBSDE(5,5,10,x(i),100,100,0.5,0.1,1,5,1,1);
z(i) = Direct(5,5,10,x(i),10000,0.5,0.1);
end
R = Truevalue(5,5,10,0.5,0.1,1,5,1,1);
r(1:n) = R;
figure, plot(x,y,x,z,x,r,'LineWidth',2),
xlabel('Number of steps','fontsize',14), ylabel('Utility ...
    value','fontsize',14), title('Approximation Error of ...
    BSDEs','fontsize',14,'FontWeight','bold'), ...
    set(gca,'fontsize',14,'FontWeight','bold'),
legend('Split method (Monte Carlo)','Direct method (Monte ...
    Carlo)','Direct method (True value)')
```

Code to generate Figure 4.4:

```
n = 50;
rng('default')
x = zeros(n,1);
y = zeros(n,1);
z = zeros(n,1);
for i=1:n
x(i) = 5*i;
y(i) = OUsplitBSDE(5,5,10,8,x(i),x(i),0.5,0.1,1,5,1,1);
z(i) = Direct(5,5,10,8,x(i)^2,0.5,0.1);
```

```
end
R = Truevalue(5,5,10,0.5,0.1,1,5,1,1);
r(1:n) = R;
figure, plot(x,y,x,z,x,r,'LineWidth',2),
xlabel('Number of paths','fontsize',14), ylabel('Utility ...
    value','fontsize',14), title('Approximation Error of ...
    BSDEs','fontsize',14,'FontWeight','bold'), ...
    set(gca,'fontsize',14,'FontWeight','bold'),
legend('Split method (Monte Carlo)','Direct method (Monte ...
    Carlo)','Direct method (True value)')
```

Code to generate the true utility value of O-U process based on (2.16):

```
function value = Truevalue(Yt, t, T, rho, gamma, theta, nu, ...
    phi, lambda)
value = - exp(lambda^2*(t-T)/2 + gamma*Yt*exp(theta*(t-T)) ...
    + gamma*nu*(1-exp(theta*(t-T))) - ...
    gamma*rho*lambda*phi*(1-exp(theta*(t-T)))/theta + ...
    gamma^2*(1-rho^2)*phi^2*(1-exp(2*theta*(t-T)))/(4*theta));
end
```

Code to generate the utility value of O-U process with direct method:

```
function value = Direct(Yt,t,T,Nsteps,Npaths,rho,gamma)
    function B = b(Y,s)
        B = 1*(5 - Y); % theta = 1, nu = 5
    end
    function A = a(Y,s)
        A = 1;
    end
    function Lambda = lambda(Y,s)
        Lambda = 1;
    end
    function G = g(Y)
        G = Y;
    end
% insert a, b, lambda and g as nested functions
l = (T-t)/(100*Nsteps);
lsq = l^.5;
Ssample = zeros(100*Nsteps+1,Npaths);
Ssample(1,:) = Yt;
for j=1:(100*Nsteps)
    Ssample(j+1,:) = Ssample(j,:) + ...
        b(Ssample(j,:),(j-1)*l+t)*l - ...
        rho*lambda(Ssample(j,:), (j-1)*l+t)*a(Ssample(j,:), ...
        (j-1)*l+t)*l + a(Ssample(j,:), ...
        (j-1)*l+t)*lsq.*randn(1,Npaths);
end
Lambda = zeros(Nsteps,Npaths);
```

```
for j=1:Nsteps
    Lambda(j,:) = ...
        lambda(Ssample(j*100+1,:),(j-1)*(T-t)/Nsteps+t);
end
YT = Ssample(end,:);
value = - (mean(exp((1-rho^2)*(gamma*g(YT) - ...
    sum(Lambda.^2,1)*(T-t)/Nsteps/2))))^(1/(1-rho^2));
end
```

Code to generate the utility value of O-U process with split PDE method:

```
function value = OUsplitPDE(Yt, t, T, Nsteps, Npaths1, ...
    Npaths2, rho, gamma, theta, nu, phi, lambda)
l = (T-t)/Nsteps;
lsq = l^.5;
V = 1;
for j=1:Nsteps
    R1 = randn(Npaths1,1);
    % R1 corresponds to the probability measure Q(1)
    R2 = randn(Npaths2,1);
    % R2 corresponds to the probability measure Q(2)
    Ssample1 = gamma*phi*exp(-theta*l)*...
        sqrt(1-rho^2)*lsq.*R2* exp(-theta*l*(j-1));
    V1 = mean(exp(Ssample1));
    Ssample2 = (-gamma*rho*lambda*phi*l + ...
        gamma*rho*phi*lsq.*R1) * exp(-theta*l*j);
    V2 = exp(mean(Ssample2))*V1;
    V = V * V2;
end
value = - exp((t-T)*lambda^2/2 + gamma*(Yt*exp(theta*(t-T)) ...
    + nu*(1-exp(theta*(t-T)))))*V;
end
```

Code to generate the utility value of O-U process with split BSDE method:

```
function value = OUsplitBSDE(Yt, t, T, Nsteps, Npaths1, ...
    Npaths2, rho, gamma, theta, nu, phi, lambda)
l = (T-t)/Nsteps;
lsq = l^.5;
Gamma = 0;
for j=1:Nsteps
    R1 = repmat(randn(Npaths1,1),[1,Npaths2]);
    % R1 corresponds to the Brownian motion W(1)
    R2 = repmat(randn(1,Npaths2),[Npaths1,1]);
    % R2 corresponds to the Brownian motion W(1,orthogonal)
    Ssample = ...
        (gamma*(-rho*lambda*phi/theta)*(1-exp(-theta*l)) + ...
        gamma*phi*exp(-theta*l)*rho*lsq.*R1 + ...
        gamma*phi*exp(-theta*l)*sqrt(1-rho^2)*lsq.*R2) * ...
```

```
        exp(-theta*l*(j-1));
    Gamma2 = (1/gamma)*log(mean(exp(Ssample),2));
    Gamma1 = mean(Gamma2);
    Gamma = Gamma + Gamma1;
end
value = - exp((t-T)*lambda^2/2 + ...
    gamma*(Gamma+Yt*exp(theta*(t-T)) + ...
    nu*(1-exp(theta*(t-T)))));
end
```

Code to generate the utility value of general process with split PDE method:

```
function value = SplitPDE(Yt,t,T,Nsteps,Npaths,rho,gamma)
    function Lambda = lambda(Y,s)
        Lambda = 1;
    end
    function B = b(Y,s)
        B = 1*(5 - Y);
    end
    function A = a(s)
        A = 1;
    end
    function G = g(Y)
        G = Y;
    end
% insert lambda, b, a and g as nested functions
l = (T-t)/Nsteps;
lsq = l^.5;
Ssample = Yt;
Lambda = 0;
for j=1:Nsteps
    if j < 2
        Dim = 0;
    else
        Dim = ndims(Ssample);
    end
    Lambda = repmat(Lambda + ...
        lambda(Ssample,(j-1)*(T-t)/Nsteps+t).^2*l/2, ...
        [ones(1,Dim),Npaths,Npaths]);
    Ssample = repmat(Ssample,[ones(1,Dim),Npaths,Npaths]);
    S1 = size(Ssample);
    S1(2*j-1) = 1;
    R1 = repmat(randn([ones(1,2*j-2), Npaths, ...
        ones(1,ndims(Ssample)-2*j+1)]), [S1]);
    % R1 corresponds to the probability measure Q(1)
    S2 = size(Ssample);
    S2(2*j) = 1;
    R2 = repmat(randn([ones(1,2*j-1), Npaths, ...
        ones(1,ndims(Ssample)-2*j)]), [S2]);
    % R2 corresponds to the probability measure Q(2)
```

```
    Ssample = Ssample - ...
        a((j-1)*(T-t)/Nsteps+t)*lambda(Ssample, ...
        (j-1)*(T-t)/Nsteps+t)*rho*l + ...
        a((j-1)*(T-t)/Nsteps+t)*rho*lsq.*R1;
    Ssample = Ssample + b(Ssample,(j-1)*(T-t)/Nsteps+t)*l + ...
        a((j-1)*(T-t)/Nsteps+t)*sqrt(1-rho^2)*lsq.*R2;
end
G = g(Ssample) - Lambda/gamma;
for j=1:Nsteps
    G = (1/gamma)*log(mean(exp(gamma*G),2*Nsteps-2*j+2));
    G = mean(G,2*Nsteps-2*j+1);
end
value = - exp(gamma*G);
end
```

Code to generate the utility value of general process with split BSDE method:

```
function value = ...
    SplitBSDE(Yt,t,T,Nsteps,Npaths1,Npaths2,rho,gamma)
    function Lambda = lambda(Y,s)
        Lambda = 1;
    end
    function B = b(Y,s)
        B = 1*(5 - Y);
    end
    function A = a(Y,s)
        A = 1;
    end
    function G = g(Y)
        G = Y;
    end
% insert lambda, b, a and g as nested functions
l = (T-t)/Nsteps;
lsq = l^.5;
Ssample = Yt;
Lambda = 0;
for j=1:Nsteps
    if j < 2
        Dim = 0;
    else
        Dim = ndims(Ssample);
    end
    Lambda = repmat(Lambda + ...
        lambda(Ssample,(j-1)*(T-t)/Nsteps+t).^2*l/2, ...
        [ones(1,Dim),Npaths1,Npaths2]);
    Ssample = repmat(Ssample,[ones(1,Dim),Npaths1,Npaths2]);
    S1 = size(Ssample);
    S1(2*j-1) = 1;
    R1 = repmat(randn([ones(1,2*j-2), Npaths1, ...
        ones(1,ndims(Ssample)-2*j+1)]), [S1]);
```

```
    % R1 corresponds to the Brownian motion W(1)
    S2 = size(Ssample);
    S2(2*j) = 1;
    R2 = repmat(randn([ones(1,2*j-1), Npaths2, ...
        ones(1,ndims(Ssample)-2*j)]), [S2]);
    % R2 corresponds to the Brownian motion W(1,orthogonal)
    Ssample = Ssample + b(Ssample,(j-1)*(T-t)/Nsteps+t)*l - ...
        a(Ssample, (j-1)*(T-t)/Nsteps+t)*lambda(Ssample, ...
        (j-1)*(T-t)/Nsteps+t)*rho*l + a(Ssample, ...
        (j-1)*(T-t)/Nsteps+t)*rho*lsq.*R1 + a(Ssample, ...
        (j-1)*(T-t)/Nsteps+t)*sqrt(1-rho^2)*lsq.*R2;
end
Gamma = g(Ssample) - Lambda/gamma;
for j=1:Nsteps
    Gamma = ...
        (1/gamma)*log(mean(exp(gamma*Gamma),2*Nsteps-2*j+2));
    Gamma = mean(Gamma,2*Nsteps-2*j+1);
end
value = - exp(gamma*Gamma);
end
```