Comparing the effects of two inquiry-based teaching strategies on secondary students' conceptual understanding and achievement in mathematics: A mixed-methods approach

by

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Abstract

In a recent study conducted in the Commonwealth of Dominica, Charles (2015) concluded that Dominican secondary mathematics teachers do not sufficiently expose students to inquiry-based approaches in teaching mathematics. One inquiry-based approach to teaching mathematics is through Investigations (Jaworski, 1986). A second inquiry-based approach is through learners' generating examples called Exemplification (Watson & Mason, 2005). Exemplification, however, has not yet taken root at the secondary level; thus, its effectiveness at that level is still questionable. This study investigated and compared the effects that Investigation and Exemplification had on secondary students' achievement and conceptual understanding of the three primary trigonometric ratios.

A pre-test–post-test, randomized, experimental design was used in this study. Thirty-five fourth form (grade 10) students from one secondary school in Dominica were randomly assigned to two groups. The researcher taught both groups for three weeks; one group using Investigation and the other using Exemplification. A mixed-methods approach was used to analyze students' responses on the pre-test and post-test to give measures of their achievement and conceptual understanding.

The study found that both the group taught by Investigation and the group taught by Exemplification had a statistically significant increase in achievement. It also found that the achievement for the group taught by Exemplification was statistically significantly higher than that of the group taught by Investigation. The study also found that the group taught by Exemplification had attained a higher level of conceptual understanding compared to the group taught by Investigation. The study recommended the use of both Investigation and Exemplification in Dominican secondary schools as a possible way of improving students' mathematics success in CSEC examinations.

Preface

This thesis is an original work by Christopher Charles. The research project, of which this thesis is a part, received research ethics approval from the University of Alberta Research Ethics Board, Project Name “Learners Generating Examples and Investigations”: Differences in Students’ Achievement and Conceptual Understanding of Trigonometry, I.D. Pro00072686, 08/29/2017.

Dedication

This thesis is dedicated to my mother, Rema Hilma Jacob. She was to me both mother and father and helped me at a tender age to understand the value of hard work and perseverance: they were both needed to complete this work. Her memory and teachings will always remain with me.

Acknowledgments

With God, all things are possible; I owe it all to God. He provided the opportunity, he put in place the many individuals who assisted me in one way or another, he protected me through this journey, and he blessed me with the intelligence I needed to take this thesis from start to finish. I thank you, Lord.

I thank the University of Alberta for providing the financial support needed to take me through this period of study. However, without the guidance of my graduate program advisor and the rest of my thesis committee, completing this thesis would not have been possible. Heartfelt thanks to Dr. Florence Glanfield, my program advisor, and Dr. Lynn McGarvey and Dr. Jacqueline P. Leighton, who completed my supervisory committee. Your guidance and contributions to my professional development will always be appreciated.

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Chapter 1: Introduction

Background of the Problem

Prolonged reports of students' dismal performances on the Caribbean Secondary Education Certificate (CSEC) mathematics examinations have teachers, ministry officials, policymakers, and other stakeholders in Dominica looking for answers. For instance, in the 2015 CSEC mathematics examination, Dominican youths who left secondary school obtained a meagre 42% pass rate (Caribbean Examination Council, 2015). This rate is consistent with the average 40% reported over the previous ten years by the Ministry of Education (MOE) (MOE, 2014). This consistent run of poor performances raises many questions which include but is not limited to how students are prepared to take this test, and the strategies they are exposed to in the teaching/learning process.

Several studies (Dahlberg & Housman, 1997; Dinkelman, 2013; Jaworski, 1986; Kidron & Tall, 2015; Meehan, 2007; Ponte & Matos, 1992; Rawson & Dunlosky, 2016; Sangster, 2012; Staples, 201; Watson & Mason, 2005; Watson & Shipman, 2008) have reported that inquiry-based approaches to teaching mathematics have led to improved mathematics performances in students at various educational levels. In a recent study, Charles (2015) found that, while most Dominican secondary mathematics teachers were aware of inquiry-based teaching approaches, such as the investigative approach, these teachers seldom used inquiry-based approaches in their teaching of mathematics at secondary schools. In his study, Charles uncovered two plausible reasons for this infrequent use of inquiry-based teaching approaches.

The first reason is that teachers claimed that the pressures of preparing students to take the CSEC mathematics examination forced them to mainly use strategies, such as direct instructions, which allow them to complete the CSEC syllabus on time (Charles, 2015). Direct instruction is portrayed in the literature as a teaching method that helps to facilitate the acquisition of knowledge, but its use for developing higher-order thinking skills in students is questionable (Ruthven et al., 2017). Further, Dominican teachers claimed that the use of inquiry-based teaching approaches consume too much time if they are to be adequately implemented (Charles, 2015).

The second reason is that, while most teacher participants knew about inquiry-based approaches for teaching mathematics, many of them did not fully understand how to use these approaches (Charles, 2015). For instance, many teachers reported in a survey that they used technology to teach mathematics (Charles, 2015). However, during a focus group discussion conducted among heads of mathematics departments from secondary schools across Dominica in that same study, the only use of technology that these senior teachers described is the use of powerpoint presentations. As a result, Charles recommended that further research is conducted into the teaching and learning of mathematics in Dominican schools. Hopefully, such research would lead to teaching practices that make more use of inquiry-based approaches to teaching mathematics (Charles, 2015). The greater use of inquiry-based teaching approaches could lead to improve mathematics performances, including on CSEC mathematics examinations.

CSEC mathematics is a high-stakes examination for Dominican youths. A passing grade can open many doors, such as job and scholarship opportunities, while a failing grade frequently leads to frustration and lost opportunities. This examination is taken by students leaving fifth forms of secondary schools in Dominica, and it is administered by the Caribbean Examination Council (CXC). CXC has been around since 1972 when it was instituted under an agreement by Caribbean governments, with a mandate to prepare and administer appropriate and relevant regional examinations. From its inception, the Council has been administering mathematics examinations to secondary school leavers. These examinations are driven by a syllabus developed by CXC and handed down to schools in participating countries; Dominica included.

Dominica, which was devastated by Hurricane Maria on September 18, 2018, is a small, developing country located in the Eastern Caribbean. It is an island of approximately 290 square miles (751 square kilometres), most of which are forested mountainous lands. Most settlements are alongside the coastal areas in small communities called villages with the central and hillier parts of the island used for agricultural purposes. The two largest cities are Roseau–the capital–in the South-Western part of the island and Portsmouth in the North. Roseau and Portsmouth are the main commercial towns, and they are home to a considerable portion of Dominica's 70 000 people. On the other hand, the remaining communities are home to many agricultural practices, such as farming and fishing.

Most secondary schools on the island are in the communities of Roseau and Portsmouth, but a few are in some agricultural-based, rural communities. Dominica has sixteen secondary schools, most of which are coeducational. Of the sixteen secondary schools, three are entirely funded and governed by private organizations. The government partly supports another three, but private organizations administrate them. The remaining ten secondary schools are fully funded and administrated by the government through the Ministry of Education. Three of the secondary schools are in the Portsmouth area, nine are in the Roseau area, and four can be found in rural communities. All four of these rural schools are fully funded and administrated by the government through the Ministry of Education. One of these rural schools was the site of this research study.

Secondary schools in Dominica have five grade levels, called forms, with the first-form being the lowest level and the fifth-form being the highest. First-form students typically come from the elementary schools (K – 6) after completing a grade six national assessment examination in mathematics, English language, science, and social studies. In many instances, this grade six national assessment is used to stream students into ranked classes at the first-form level. These ranked streams are usually maintained over the years as students move from one form level to the next. Hence, a school with more than one class at a form level will generally have these classes ranked. For example, the school where this study was conducted had four fourth-forms in the 2017 – 2018 school year. Form four-one comprised of the top performers, form four-two and form four-three had average performers, and form four-four had the lowest performers. Levels of performance are typically determined by students' subject grades, and students' performances at previous form levels are usually used to stream them at their current form level.

At the end of the fifth form, students take the CSEC examinations in the subject areas they study at the fourth-form and fifth-form levels. CSEC mathematics is usually taken by all students leaving the fifth-forms. The mathematics examination comprises of two papers: Paper 01 and Paper 02. In Paper 01, students are expected to answer 60 multiple-choice questions designed to assess their conceptual understanding and procedural fluency (CXC, 2016). In Paper 02, students are required to give written responses to ten questions designed to assess their conceptual understanding, procedural fluency, and problem-solving skills (CXC, 2016). Candidates are given 90 minutes for Paper 01 and two hours and 40 minutes for Paper 02. They can use calculators on Paper 02 but not on Paper 01. Approximately 1500 Dominican secondary school leavers take the CSEC mathematics examination every year.

Although students take CSEC mathematics examinations as a fifth-form exit test, the nature of mathematics dictates that sound and effective strategies cannot be left until the fifth-form level if students are to perform well in this examination. It is common knowledge within the mathematics education community that high levels of mathematical reasoning are best achieved when sound pedagogical practices are employed from foundational years through to the highest level of schooling (Tall, 2013). According to Tall (2013), this higher level of achievement can be attained because early mathematics concepts and strategies affect the way students understand mathematics in their later years. Hence, the way mathematics is taught in earlier years affects how students in fifth-form reason about and approach mathematics problems that are given on CSEC examinations. This research study focused on fourth-form students.

Statement of the Problem

Education officials, principals and teachers, parents, and the wider Dominican society want more students to obtain a passing grade (grades I, II, or III) on CSEC mathematics, given that it is a high-stakes examination. A greater number of CSEC passes in mathematics would help more Dominican youths to qualify for entry-level jobs in the public and private sectors, to meet the matriculation criteria for university programs, and to do further studies in STEM-related areas. Several studies (Dahlberg & Housman, 1997; Dinkelman, 2013; Jaworski, 1986; Kidron & Tall, 2015; Meehan, 2007; Ponte & Matos, 1992; Rawson & Dunlosky, 2016; Sangster, 2012; Staples, 201; Watson & Mason, 2005; Watson & Shipman, 2008) showed that students performances in mathematics improved when they were exposed to inquiry-based teaching approaches.

Instead, most Dominican secondary mathematics teachers use a direct instructional approach (Charles, 2015). While some studies (Ruthven et al., 2017; Rittle-Johnson, 2006; Sweller, 2003; Rittle-Johnson, Siegler, & Alibali, 2001) reported that direct instruction has a positive effect on students' learning, other studies (Hiebert et al., 1996); Kamii & Dominick, 1998) advocate for the use of inquiry-based approaches over direct instruction to help students develop a good understanding of mathematical concepts. Hence, students must be frequently exposed to inquiry-based teaching approaches if they are to develop a good understanding of mathematical concepts. This study supposes that the insufficient exposure to inquiry-based teaching approaches might be one reason why, on average, only approximately 40% of Dominican youths passed the CSEC mathematics examination each year between 2004 and 2013 (MOE, 2014).

In response to this problem, this study investigated the effects that two inquiry-based approaches to teaching mathematics had on students’ achievement and conceptual understanding of mathematics. These approaches are Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005), which are thoroughly discussed in chapter two. Studies such as Jaworski (1986), Ponte and Matos (1992), Sangster (2012), Staples (2011), and Kidron and Tall (2015) concluded that Investigation improves students’ understanding of and achievement in mathematics. Moreover, the CSEC mathematics syllabus, which drives the curriculum in Dominican secondary schools, advocates for the use of Investigation as a means of getting students to solve mathematics problems (CXC, 2003).

Other studies such as Rawson and Dunlosky (2016), Dinkelman (2013), Watson and Shipman (2008), Meehan (2007), Watson and Mason (2005), and Dahlberg and Housman (1997) concluded that the act of learners generating examples–Exemplification–have improved students’ understanding of and achievement in mathematics. However, both Charles (2015) and this current study found no research study investigating the effects of inquiry-based teaching approaches, such as Investigation and Exemplification, on students’ understanding and achievements in mathematics that were conducted in Dominica. This lack of such research studies in the Dominican context is a gap in the literature that this study addressed.

The claims that inquiry-based teaching approaches such as Investigation and Exemplification improve students' understanding and achievement in mathematics beg the question: how do students learn mathematics? Maybe a more appropriate question in the context of this study is: what classroom practices foster improvement in students' achievement and conceptual understanding of mathematical ideas? An immediate response to this question is the tasks through which teachers engage students for learning. Tasks are common features in most, if not all, mathematics classrooms. How useful these tasks are, however, is the critical issue. Stein et al. (2009) argue that, for tasks to be effective, their cognitive demands must match the intended outcome of the lesson. That is, if the intended outcome is to solve a routine mathematics question, then the task should be of low cognitive demand, involving memorization or procedure without connection (Stein et al., 2009). However, if the intended outcome is for students to gain a thorough understanding of a concept to solve novel mathematics problems, then the tasks should be of high cognitive demand (Stein et al., 2009).

According to Stein et al. (2009), high cognitively demanding tasks require teachers to build tasks on students' prior knowledge, provide scaffolding, encourage students to monitor their own progress, press students to give justifications for their reasoning, draw frequent connections among concepts, and provide optimal time to allow students to explore possibilities. These tasks go beyond the usual direct instruction approach, which appeared to be taking place in many Dominican mathematics classrooms (Charles, 2015). Drawing from Stein et al. (2009) and others, this researcher asserts that both Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) engage students in high cognitively demanding tasks. Hence, these approaches can foster the development of conceptual understanding in students, consequently leading to increased achievements.

Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) both engage students in tasks that require them to think independently and collaboratively with others, make connections among concepts, and justify their reasoning through discussions. These are all learning activities, which constitute a rich learning task (Griffin, 2009). According to Griffin (2009), a rich learning task is an activity in which students are called upon to behave like mathematicians. That is, students must use trial and error techniques while attempting to make sense of a mathematics concept, use their powers of observation and reasoning to figure out patterns and connections among concepts, and question the actions of their peers and themselves. Slavit et al. (2009) went further by asserting that rich mathematics tasks must allow different paths to solve problems, support collaborative activities where students can share ideas, develop understanding, and contain essential content in a relevant context. NCTM (2000) called for the use of such activities and claimed that these activities are essential for students to develop a conceptual understanding of mathematical ideas. This current study engaged students in rich learning tasks.

Purpose of the Study

The primary aim in undertaking this study was to explore how Dominican secondary students' achievement and conceptual understanding (Kilpatrick et al., 2001) of mathematics were affected by two inquiry-based teaching approaches–Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005). This researcher wanted to find out if they both helped students to increase their achievement in and conceptual understanding of mathematics and if they were equally effective in doing so. If shown to be both effective in this context, their use would be recommended to the relevant principals, teachers, and ministry officials.

That is, determining how Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) affected Dominican secondary students' achievement and conceptual understanding was an essential step in solving the problem of Dominican youth prolong poor performance on CSEC mathematics. Charles (2015) identified students' underexposure to inquiry-based approaches for teaching mathematics as one possible reason for their underachievement in CSEC mathematics. This study provided empirical evidence of the effects that Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) had on a group of Dominican fourth-form students' achievement and conceptual understanding of the three primary trigonometric ratios.

Objectives of the study.

While investigating the effects of Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) on Dominican secondary students' achievement and conceptual understanding (Kilpatrick et al., 2001) of the three primary trigonometric ratios, this study focused on four objectives.

1. To provide evidence on how Investigation affected the level of Dominican secondary students' achievement in mathematics.
2. To provide evidence on how Exemplification affected the level of Dominican secondary students' achievement in mathematics.
3. To provide evidence on how Exemplification affected the level of Dominican secondary students' achievement in mathematics compared to Investigation.
4. To provide evidence on how Exemplification affected Dominican secondary students' conceptual understanding of mathematics compared to Investigation.

The Significance of the Study

Dominican secondary school leavers are not the only ones who stand to benefit from the findings of this study. Business organizations, policymakers, Ministry of Education officials, school principals, teachers, current students, and parents might also benefit. Business organizations regularly employ young people, based on their qualifications from school, to carry out their functions. These business owners and managers frequently complain that many young employees are unable to solve job-related problems. With the problem-solving skills, students are likely to develop during Investigations and Exemplification; they should be better equipped to handle work-related challenges.

Policymakers and the Ministry of Education officials regularly deploy resources to combat the prolonged poor performance of students on CSEC mathematics. For instance, between 2007 and 2015, several mathematics teachers (this researcher included) received government scholarships to pursue Bachelor of Education (B.Ed.) degrees in Secondary Mathematics Education. The training of teachers is one step in solving this mathematics achievement problem. Knowing which practices are most likely to bring success in the Dominican context and knowing how to implement them is the next logical step. That is, these trained teachers with better pedagogy and pedagogical content knowledge, are now better positioned to utilize the findings of this study.

Principals, teachers, students, and parents may be direct beneficiaries of this study. When a student achieves good grades at CSEC mathematics, his parents or guardians are likely to feel a greater sense of satisfaction, knowing that more jobs and scholarship opportunities may be available to that child. With the knowledge of how these teaching strategies might affect students' achievement, teachers may be motivated to use them. With increased exposure to these inquiry-based approaches at schools, students might perform better in mathematics, and principals might see an improvement in their schools' overall mathematics performances.

This study may also be of benefit to researchers in the field of mathematics education, particularly Caribbean based researchers. In a personal conversation with a mathematics education professor at the University of the West Indies, the scarcity of scholarly research based in the Caribbean area came up. This professor noted that it is challenging to obtain research studies that focus on teaching and learning in the Caribbean in general and on teaching and learning mathematics specifically. Like Charles (2015), she said that more scholarly research in mathematics education should be done in the Caribbean and their findings published. This study once published, will provide scholarly work that may be used by others in the field of mathematics education.

Research Questions

Driving this study is the primary research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* Answering the following four sub-questions helped to formulate plausible answers to the main research question above. The sub-questions are:

1. How was the level of students' achievement affected after being taught by Investigation?
2. How was the level of students' achievement affected after being taught by Exemplification?
3. How do the levels of students' achievement differ after being taught using Investigation compared with Exemplification?
4. How does students' conceptual understanding differ after being taught using Investigation compared with Exemplification?

The answers to these research sub-questions provided a measure of how two inquiry-based approaches to teaching mathematics–Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005)–affected students' performances in mathematics in Dominican secondary schools.

Overview of Research Design

Comparing the effects of two inquiry-based teaching strategies on secondary students' conceptual understanding and achievement in mathematics: A mixed-methods approach used a two group, pre-test–post-test, independent measures (between group) experimental design. This design used a conceptual framework that drew from several theories. The teaching component of the study drew from the constructivist theory of learning (Piaget, 1977; Vygotsky, 1978; von Glasersfeld, 1995; Fosnot & Perry, 1996). The constructivist theory of learning is discussed in chapter two. The development of a pre-test and a post-test used in this study drew from the work of Danielson and Marquez (2016). The development of an assessment rubric drew from the work of Danielson and Marquez (2016), Kilpatrick et al. (2001), and Hiebert and Carpenter (1992). The framework used to assess students' conceptual understanding drew from the work of Kilpatrick et al. (2001) and Hiebert and Carpenter (1992). The assessment tools–pre-test, post-test, and assessment rubric–are described in chapter three, and the works of Danielson and Marquez, Kilpatrick et al., and Hiebert and Carpenter are discussed in chapter two. Students' responses on the pre-test and post-test were the only sources of data used in this study to assess their achievement and conceptual understanding.

For the teaching component, two classes of fourth-form students were mixed and randomly assigned to two groups. One group was taught using Investigation (Jaworski, 1986), and the other group was taught using Exemplification (Watson & Mason, 2005). This researcher taught the three primary trigonometric ratios to both groups of students; each group received instructions for the same length of time. Relevant aspects of the three primary trigonometric ratios are discussed in chapter two.

Two tests were administered to both groups to collect data for the study. A pre-test was given and completed before any teaching took place, and a post-test was administered after teaching took place. Both tests assessed the same mathematics content and constructs and were similarly constructed. They comprised both multiple-choice items, and question prompts that students responded to in writing. Details on the administration of the pre-test and post-test are provided in chapter three. All test items were marked, and raw scores (achievement), one for each test, were assigned to each student. These achievement scores were tabulated and analyzed using a mixed ANOVA to determine the differences in achievements within and across groups. The analysis from the ANOVA was presented using the appropriate tables and was interpreted to answer research sub-questions one, two, and three. These analyses are presented in chapter four.

Sub-question four was answered using a combination of qualitative and quantitative analyses. For the qualitative analysis, students' written responses were analyzed by themes with the help of the assessment rubric (describe in chapter three). Each theme represented a different aspect of students' use of representations and highlighted what they did correctly and incorrectly while using these different forms of representations. A separate analysis was done for each group, and the correct and incorrect use of representations were used to create group profiles. That is, a separate profile that highlighted correct and incorrect uses of representations was constructed for each group. These profiles were compared across groups to help determine how conceptual understanding differs across groups after teaching. This qualitative analysis is presented in chapter five. Differences in students' scores on their written responses and differences in the number of students who responded to written response items in each group were also used to determine how conceptual understanding differs across groups after teaching. These differences were analyzed quantitatively, and the results presented in chapter four. Full details on this design are provided in chapter three.

Background of the Researcher

As a secondary mathematics teacher for over twenty (20) years, I lived through the frustration, doubts, and anger of principals and supervisors who wanted better results but whom, in retrospect, I realized were at their wits end wondering how to approach Dominica's mathematics problem. As a young, untrained, and inexperienced teacher, I was part of the problem. My role within and my contribution to the problem changed as I graduated into a trained, matured, and more experienced teacher. As a Numeracy Specialist within the Ministry of Education, I grew more and more concerned and felt more and more inadequate in guiding mathematics teachers. It is these experiences that brought me to the University of Alberta and this doctoral thesis.

I started my mathematics teaching career at St. Andrews' High School (SAHS) in September of 1990 at the age of twenty. SAHS was (its doors were shut in 2006 giving way to the North East Comprehensive School), a coeducational institution located in a rural part of Dominica and served six rural communities. In my first year of teaching at SAHS, I taught mathematics to three grade levels: first form (grade 7), second form (grade 8), and third form (grade 9). The struggle had started. I was young, untrained, inexperienced, and above all, believed I knew how to teach CSEC mathematics because I had just completed an advance course (A – level) in mathematics. Three years of teaching, however, soon taught me that I needed more expertise in this area and when the opportunity came in 1993, I travelled to New Mexico to undertake undergraduate studies in mathematics education at New Mexico State University (NMSU).

The program of study I undertook at NMSU was a joint venture between NMSU and the Dominica Teachers' College (presently a department of the Dominica State College), which afforded me a certificate to teach mathematics at the secondary level. In the program, I undertook courses in mathematics, mathematics pedagogy, and some general education foundational courses. The program lasted two years: eight months (two semesters) at NMSU, followed by twelve months (three semesters) in Dominica. The Dominican leg involved course work regarding issues in the Dominican classroom and a supervised practicum period. I completed the program in 1995 and, after that, was certified in Dominica as a qualified teacher. With my new qualification, I continue to teach mathematics at the St. Andrews' High School.

Not only did I continue to teach mathematics, but soon after becoming qualified, I was given the position of head of the mathematics department at SAHS. As head of mathematics, I was partly responsible for supervising the work of the other mathematics teachers at the school. Included in my supervisory duties were: monitoring and providing feedback on teachers' lesson plans, unit plans, and exams; monitoring and providing feedback on teachers' classroom practices; identifying training and other immediate needs of the mathematics department; holding timely departmental meetings, and representing the department at school's management meetings. I continued to perform these duties and more at the North East Comprehensive School (NECS) after the doors of SAHS were shut in 2006. I also took with me to NECS experiences gained as deputy principal from SAHS.

I served as deputy principal at SAHS the year before its doors were closed and had the privilege to act as principal several times during that period. My positions were carried over to NECS, where I served as assistant principal, head of the mathematics department, and mathematics teacher. As part of my added responsibilities, I facilitated workshop sessions for younger teachers of mathematics and assisted principals on matters of discipline. I also served, as a member of a body of mathematics educators, with the mathematics learning support department in the Ministry of Education. This body gave rise to the National Association of Mathematics Educators (NAME), of which I was the first president.

NAME was registered in Dominica September of 2011 and had as its motto: *Empowering Mathematics Educators.* NAME, at its core, had the vision to increase the confidence, zeal, and effectiveness of mathematics educators in Dominica. I became the first president of NAME because my beliefs were inseparable from that of the association. In my stint as president, the constitution of NAME was developed and registered. By then, I had completed studies, which led to a B.Ed. in secondary mathematics education, with the University of the West Indies (UWI).

In my B.Ed. program, I was exposed to several areas of advanced mathematics content, mathematics pedagogy, foundational education courses, and technology in education. To cap this program, I conducted an action research study that focused on cooperative learning in my mathematics classroom. The study was part of a supervised practicum period. This qualification and the experiences mentioned above propelled me into the Ministry of Education.

In 2010, the government of Dominica, through the Ministry of Education enhancement unit, embarked upon an education enhancement project in which numeracy improvement was a significant component. The numeracy component was awarded to a consultancy firm from the United Kingdom (G2G Consultants) with two local persons attached as Numeracy Specialists. I served in one of these two positions for three years and gained some invaluable experiences in mathematics education during that period. As a numeracy specialist, I worked alongside the G2G principal consultant providing training to teachers of mathematics; visited, observed, and provided feedback to teachers; and prepared and demonstrated model mathematics lessons. I was also present in many meetings between the consultants and ministry officials when findings from the consultancy were presented and discussed. These encounters further opened my eyes to the frustrations and concerns of education policymakers regarding Dominica's mathematics problems. Consequently, I joined a mathematics task force to investigate factors affecting mathematics performance in Dominica. In September 2013, I came to the University of Alberta to read for a Master of Education (M.Ed.) in the area of mathematics education.

In my M.Ed. program, I took courses in research, curriculum and instruction, mathematics education, and quantitative analysis. I also worked as a research assistant, which helped to further my research skills. This program culminated with a thesis that focused on the factors affecting the instructional practices of Dominican secondary mathematics teachers. Data for the thesis were collected and analyzed through both qualitative and quantitative means (mixed methods) among Dominican secondary mathematics teachers. I completed the M.Ed. program in September of 2015 and started a Ph.D. program at the same time.

Besides taking further courses in research and mathematics education in my Ph.D. program, I took courses in program planning/instructional design, testing and measurement, and the use of multimedia in education. Furthermore, I taught a mathematics methods course to undergraduate students in the Department of Secondary Education at the University of Alberta for three years. For this course, I developed the course curriculum, identified suitable literature, and facilitated the development of mathematics content knowledge and mathematics pedagogical content knowledge of pre-service teachers. To accomplish these tasks, I drew from my knowledge and 20 plus years of experience teaching mathematics at the secondary level. Moreover, the experiences gained while completing these tasks, among others (discussed in detail in the relevant sections of this document) help shaped my thinking about teaching and learning. It is against this background that I acknowledge my assumptions and perspectives that could have introduced bias into the study (Bogdan & Biklen, 2007).

I believe that mathematics is best learned when students are actively engaged in activities that are guided by the teacher in an appropriate environment, which is the essence of constructivism. While both Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) draw from constructivism, I believe that the teacher has a more considerable influence on what students focus on in Exemplification; hence, Exemplification will have a more profound effect on students' conceptual understanding of mathematical ideas. This argument is in keeping with Vygotsky's (1978) notion of the zone of proximal development where teachers help students learn materials and solve problems that would usually be beyond their intellectual reach. I took steps to prevent this bias from being introduced into the study because I was conscious of the possible effects of this belief. That is, to the best of my abilities, I adhered to the Exemplification practices proposed by Watson and Mason (2005) and the Investigation practices proposed by Jaworski (1986) while using both teaching approaches.

My education and experiences provided the lens through which I analyzed and made meaning of the data (Patton, 2002). Therefore, my education and experiences were taken as positive features that added value to the study. I used my knowledge of education principles and my experiences in educational practices to influence the steps that I took to reduce, if not eliminate, the introduction of bias into the study. One step was to involve independent mathematics education experts in the development of the data collection instruments and the data analysis process. Another step taken was to ensure that both teaching approaches—Investigation and Exemplification—were implemented in an environment that allowed both groups of students to reap maximum benefit from their encounters with the materials. In taking these measures, I eliminated or significantly reduced, the influence of my assumptions on the way Exemplification and Investigation were implemented during the teaching sessions. The remainder of this document details the activities and results of the study. In these details, I am referred to as the researcher or researcher-teacher to satisfy Caribbean readers: academic writing is most accepted in the third person.

Definitions of Terms

The following terms are defined to help readers understand the context of the key terms in this study.

Achievement: Achievement is defined as the level of knowledge and understanding that a student has of the three primary trigonometric ratios as measured by his/her total scores on the pre-test and the post-test in this study.

Conceptual Understanding: Conceptual Understanding is defined by Kilpatrick et al. (2001) as "an integrated and functional grasp of mathematical ideas" (p. 118). In this study, Conceptual Understanding is defined as a student's ability to identify and discuss different representations of the primary trigonometry ratio, produce and discuss a diagram representation of a contextual problem based on the primary trigonometry ratios, and compare the same form of representation of these ratios as taken from students' written responses on the post-test given in this study.

Exemplification: According to Watson and Mason (2005), Exemplification is the act of learners generating examples when prompted by a teacher. In this study, Exemplification is defined as the act of a student generating examples of representations for the three primary trigonometric ratios, when prompted by the researcher-teacher, after they were introduced to these ratios using an inquiry-based activity.

Investigation: According to Jaworski (1986), “Investigation implies finding out” (p. 3). In this study, Investigation is defined as a teaching approach employed by the researcher-teacher, where students make use of rulers, protractors, squared paper, and prepared worksheets to produce and discuss different representations of the three primary trigonometric ratios.

Learning: According to Ambrose et al. (2010), learning is “A *process* that leads to *change*, which occurs as a result of *experience* and increases the potential of improved performance and future learning (p. 3).” Drawing from Ambrose (2010), this study defined learning as a change in students’ achievement and conceptual understanding of the three primary trigonometric ratios.

Representations: Representations are drawings (diagrams), tables, graphs, formulas, and worded statements used to depict a mathematical concept.

Student: A student is an average performer in fourth-form at a Dominican secondary school who participated in the study.

Assumptions, Limitations, and Delimitations

Methodological assumptions.

A mixed-methods approach was assumed to be the best to provide the desired outcomes due to the pragmatic nature of the research problem. According to Ling (2017), research in the pragmatic paradigm is driven by agenda and the practical needs of the researcher. Hence, a researcher may use methods and actions which might best satisfy their needs. Ling stated that mixed methods are well suited for the pragmatic paradigm. The researcher used and assumed that the mixed methods approach used in this study was the best methodology to investigate the research problem.

Theoretical assumptions.

Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) are both purported, by several studies, to raise learners' understanding of mathematics. This study assumed that Jaworski (1986), Watson and Mason (2005), and all articles used to support their theories are credible sources of information. This study also assumed that the articles used to provide and support the constructivist theory of learning, and the teaching model developed based on these constructivists' theories were credible sources of information. The study further assumed that the work of Kilpatrick et al. (2001), Danielson and Marquez (2016), and Heibert and Carpenter (1992) provided credible means of evaluating students' achievement and conceptual understanding.

This study also assumed that participants' learning of the three primary trigonometry ratios came about as a direct result of the strategies they encountered in the classroom. The study further assumed that the random assignment of participants into two groups accounted for all differences in their characteristics of the two groups. Based on these assumptions, the study associated the differences in students' achievement and conceptual understanding of the trigonometric ratios to their different group experiences during teaching.

Measurement assumptions.

A pre-test and a post-test were used to collect data for this study. Responses to a portion of the post-test were evaluated with the help of an assessment rubric. These instruments–pre-test, post-test, and assessment rubric–were validated by two mathematics education specialists, each having over ten years of experience teaching mathematics at the secondary level in Dominica. Polit and Beck (2016) asserted that the reliability and validity of research instruments could be assessed by experts in the relevant field. This study assumes that the two mathematics education specialists used to validate these instruments were experts in the field of mathematics education.

The study also assumes that they were indeed qualified to assess these instruments and that their assessments of these instruments were truthful.

Limitations.

The entire study was done in a single school where two groups of students were taught the same mathematics concepts, using different approaches, at different times. Students from the first group had the opportunity to share strategies with their friends in the second group. Such sharing could have confounded the results. Although participants were asked not to work across groups, it was not possible to monitor and avoid such sharing among students. Using groups from different schools would have minimized the risk of sure sharing. However, this study was conducted in the aftermath of Hurricane Maria, and other realistic sites were not available.

Carrying out this study in the aftermath of Hurricane Maria also posed a potential threat to the validity of the study results. It is well known throughout the educational community that a student's state of mind affects his learning. Hurricane Maria was a very traumatic event in most Dominicans' life, and students were no exceptions. Hence, the data obtained from the post-test might not reflect the full range of effects that the two inquiry-based approaches used in teaching might have had on students' achievement and conceptual understanding.

The random assignment of students into two groups did not result in balanced groups in terms of gender. There were six girls and ten boys in the group taught by Investigation and 11 girls and five boys in the group taught by Exemplification. In Dominica, girls have been performing better than boys in mathematics at all levels of schooling and in the CSEC examinations. Hence, the gender imbalance could have confounded the result of the study, which showed that the group taught by Exemplification performed better in the post-test than the group taught by Investigation. The study did not account for this gender imbalance. However, pre-tests at least showed that the two groups, prior to the intervention, showed similar levels of knowledge and skills. This does not preclude the possibility that girls learned the material more quickly in the Exemplification group.

Mathematics builds on itself, and students' knowledge of one area of mathematics can affect their understanding of other areas. Algebra permeates several areas of mathematics, including trigonometry, and the students' knowledge of algebra could have affected their understanding of the three primary trigonometric ratios. In this study, one group of students did algebra before the trigonometry unit for the study, while the other group did it after the trigonometric unit. The difference in the ordering of these contents could have confounded the results. Ideally, to maintain balance, both groups should have been taught algebra and trigonometry in the same order. However, the sequencing of the content was a school decision and was beyond the control of the researcher.

Sampling in an experimental study should be representative of the entire population if the findings are to be generalized to that population. A small sample is not likely to be representative of the entire population; hence, a small sample will limit the generalizability of the findings in the study. At the time of this study, the population comprised approximately 1500 students from fifteen secondary schools. The sample comprised 35 students from one secondary school. Hence, the sample was small (approximately 2%) and might not have been representative of the entire population. As a result, the findings of this study must be tentatively generalized to the target population. That is, the conclusions drawn from this study might not reflect the entire population.

Delimitation.

A delayed post-test could have been used to compare students' retention of the learned concepts after teaching. The use of this delayed post-test is particularly important for Exemplification because there appears to be no study reported in the literature that examined the effects that this approach has on students' retention. A delayed post-test was left out in this study because of the time constraints placed on the researcher in the aftermath of Hurricane Maria.

Format of the Thesis

Chapter one introduced this study that used a mixed-methods approach to compare the effects of Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005), two inquiry-based approaches to teaching mathematics, on secondary students conceptual understanding and achievement in mathematics. All previous works on the effects of Investigation and the effects of Exemplification on students' achievement and understanding of mathematics were situated outside of Dominican. The lack of such research in a Dominican context presented a gap in the literature that this research study addressed. Several groups of persons and organizations, business firms, policymakers, Ministry of Education officials, school principals, teachers, current students, parents, and mathematics education researchers might benefit from the results of this research study.

Chapter two is a comprehensive review of the literature relevant to this study. This literature review provides the foundation for choices made in this study. It deliberated on the relevant mathematics content–the three primary trigonometric ratios, the methodology used—mixed methods, the theoretical orientation—constructivism, teaching approaches used—Investigation and Exemplification, constructs assessed—conceptual understanding and achievement, assessment strategies, and the state of the literature as it relates to the effects of Investigation and Exemplification on students' understanding and achievement in mathematics.

Chapter three details the methodology, methods, and procedures that were used in conducting this study. In the details, attention is paid to the purpose of the study, the research questions and hypotheses, the research design, the participants—students and the researcher, teaching activities—Investigation and Exemplification, methods and procedures used in data collection and analyses—quantitative and qualitative, data collection instruments, validity and reliability, treatment of the results—mixing and prioritizing, strengths and limitations, and informed consent.

Chapter four presents the quantitative results of this study. It presents the results on the participants' demographic information; an analysis of the inter-rater consistency of the two mathematics education specialists who graded students' written responses; an ANOVA used to compare participants' achievements on the pre-test and post-test; and the comparisons made with participants' response rate on prompts soliciting written responses in the post-test, scores on these written responses, and correct responses on multiple-choice items from the pre-test and post-test. These results are presented with tables, graphs, and narratives.

Chapter five presents the qualitative results of this study. It does so by highlighting the levels of correctness with which participants addressed the written-response questions and the errors they made in addressing them. These results are presented by themes and are supported with extracts from students' written responses. Analyses of these themes were used to develop group profiles to show a measure of participants' conceptual understanding. Group profiles were compared to determine the differences in conceptual understanding between the group taught by Investigation and the group taught by Exemplification.

Chapter six presents the integration of the quantitative results from chapter four and the qualitative results from chapter five and the interpretation and discussion of these results. The full scope of the study is also presented to provide context and help readers understand the results and discussions. To that end, the chapter highlighted the general and research problems, the literature reviewed, the methodology used, and the results of chapter four and chapter five. The chapter also looks at how well the results answered the research questions and supported the hypotheses of the study. Finally, the chapter positioned the findings of the study in the literature and discussed the significance of these findings to Dominican stakeholders and the mathematics education community.

Chapter seven presents the conclusion, implications, and recommendations of the study. The study concluded that both Investigation and Exemplification increase students' conceptual understanding and achievement of mathematics, with Exemplification leading to a higher increase in both constructs. Considering this conclusion, it highlighted some possible implications for stakeholders in Dominican and the wider mathematics education community. Some recommendations for further research were also highlighted.

Chapter 2: Literature Review

A teacher's practice affects students' learning (Even, 1993; Baumert et al., 2010; Watson & Harel, 2013). One definition of learning is given by Ambrose et al. (2010) as: "A *process* that leads to *change*, which occurs as a result of *experience* and increases the potential of improved performance and future learning" (p. 3). Drawing from Ambrose et al. (2010), the changes in students' achievement and conceptual understanding of the three primary trigonometric ratios assessed in this study after teaching were measures of these students learning. Therefore, this study found it fitting to review articles that speak of the effects of teaching approaches on students' learning, along with articles that speak to the effects on their conceptual understand and on their achievement. The review of these articles, among others, helped lay the foundation for investigating the effects of Investigation and Exemplification on students' conceptual understanding and achievement in mathematics in a Dominican context.

So, what are the current teaching practices of Dominican secondary teachers that facilitate students' learning of mathematics? Charles (2015) found that most fourth-form and fifth-form mathematics teachers in Dominican used a direct instructional approach. Some studies (Rittle-Johnson, 2006; Rittle-Johnson, Siegler, & Alibali, 2001; Ruthven et al., 2017; Sweller, 2003) claimed that direct instruction, when properly implemented, can have positive effects on students' learning. Others advocate for the use of inquiry-based teaching approaches over direct instruction (Hiebert et al., 1996; Kamii & Dominick, 1998). These contradictory claims suggest that teaching should not be restricted to a single mode of instruction.

Can Dominican youth perform better on the CSEC mathematics examination with more exposure to inquiry-based approaches? More specifically, will the use of Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) in Dominica secondary classrooms improve students' levels of achievement and conceptual understanding of mathematical ideas? In this study, a student's level of achievement was taken as his aggregate score on a test, and his conceptual understanding was taken as his ability to manipulate representations of mathematical ideas and identify and use proper mathematical procedures. Both achievement and conceptual understanding were analyzed to answer the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* Both Investigation and Exemplification are inquiry-based approaches to teaching mathematics, and they draw from the constructivist's theory of learning. More details about these two teaching approaches and the constructivist theory of learning are given later in this chapter.

This literature review provided the foundation for choices made in this study. It deliberated on the relevant mathematics content, methodology, theoretical orientation, teaching strategies, assessment, and state of the literature as it relates to the effects of Investigation and Exemplification on students' learning. Students' learning for this study refers to the changes in their conceptual understanding and achievement scores.

To obtain the articles that were reviewed, the researcher conducted manual searches of online journals and databases, resource books, and other printed materials. Some databases searched included but were not limited to Eric, Education Research Complete, SAGE online, and ProQuest. Most of the articles reviewed came from the University of Alberta Libraries and the World Wide Web. Google Scholar, on the World Wide Web, was particularly useful.

To focus database and web searches, keywords and phrases including 'conceptual understanding', 'conceptual knowledge', 'representations', 'achievement', 'achievement test', 'testing', 'investigation', 'mathematical investigations, 'investigative approach to teaching mathematics', 'learners generating examples', 'generating examples', 'exemplification', and several other combinations of these words and phrases were used. Some searches returned e-copies of relevant articles and others led to books that were borrowed or bought.

The Three Primary Trigonometric Ratios

The ratios of sine, cosine, and tangent are concepts in trigonometry, and in this study are referred to as the three primary trigonometric ratios. Trigonometry, the study of angles and their functions (Moritz, 1908), is a critical area in high-school mathematics. Different studies argue that it provides invaluable linkages among other fields in mathematics (Moritz, 1908; Weber, 2005), and it is a necessary pre-requisite for many practical areas such as surveying, engineering, and navigation (Bhattacharjee, 2012; Yiğit Koyunkaya, 2016; Weber, 2005). Weber (2005) claimed that trigonometry links algebraic, geometric, and graphical reasoning. Almost a hundred years before him, Moritz (1908) referred to plane trigonometry as the "lintel which bridges" (p. 392) the mathematics strands of algebra and geometry. The sine, cosine, and tangent ratios are aspects of plane trigonometry; hence, the importance of gaining a conceptual understanding of these concepts, in high-school mathematics, can be gleaned from the work of Weber and Moritz.

The concept of the right-angle triangle sometimes referred to as the right triangle, underpins these three primary trigonometric ratios. A right triangle is one which has an angle measure of exactly 90^0. The side opposite of the 90-degree angle is always the longest side of the triangle and is called the hypotenuse (see Figure 2.1).

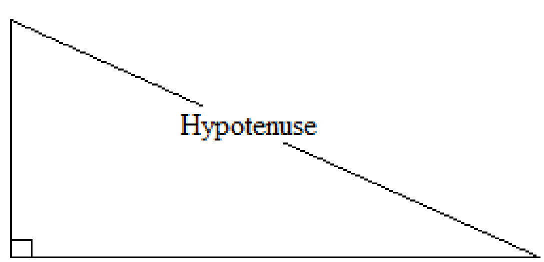


Figure 2.1: Drawing of a right-angled triangle to show the hypotenuse.

In the context of the trigonometric ratios, the two other sides, the legs of the right triangle, are named in reference to a marked angle called the reference angle. In reference to this angle, they are called the opposite side and the adjacent side. Unlike the hypotenuse, which is always located opposite the 90-degree angle, the positions of the adjacent and opposite sides are not fixed but change when the reference angle changes (see *Figure 2.2*).

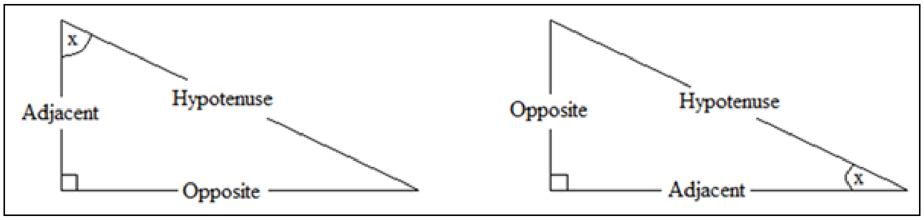


Figure 2.2: Drawings of right-angled triangles showing named sides and reference angles.

The sine, cosine, and tangent ratios are formed using the reference angle and two side lengths.

The sine of angle $x(sinx) = \frac{\text{Opposite side}}{Hypotenuse}$

The cosine of angle $x(cosx) = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$

The tangent of angle $x(tanx) = \frac{\text{Opposite side}}{\text{Adjacent side}}$

The ratios of sine, cosine, and tangent are constant for a given angle regardless of the size of the triangle. This principle is illustrated below using the concept of similar triangles (see Figure 2.3).

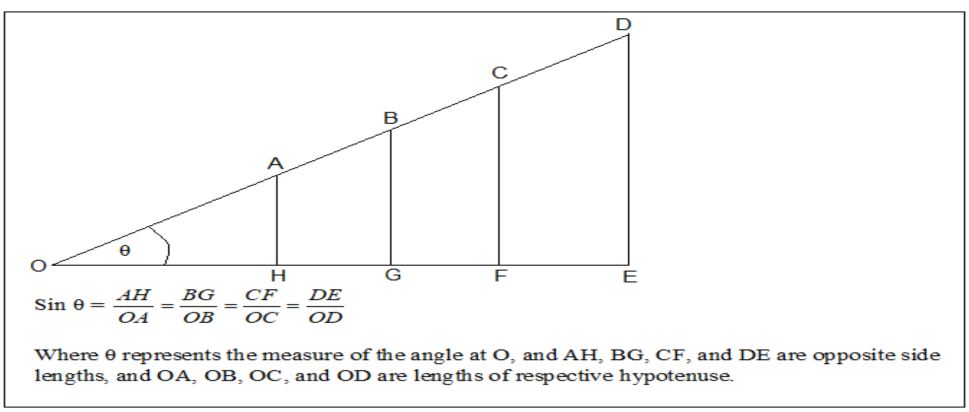


Figure 2.3: Illustration of the sine ratio through similar triangles.

However, the value of the ratio changes when the angle changes. How a change in the angle affects the value of the ratio is illustrated below using a unit circle.

In trigonometry, a unit circle is normally drawn on a pair of cartesian coordinate axes; it is centred at the origin (0, 0) with a radius of one unit. The radius is commonly represented by a straight line from the origin (0, 0) to a point (x, y) on the circumference of the circle (see *Figure 2.4*).

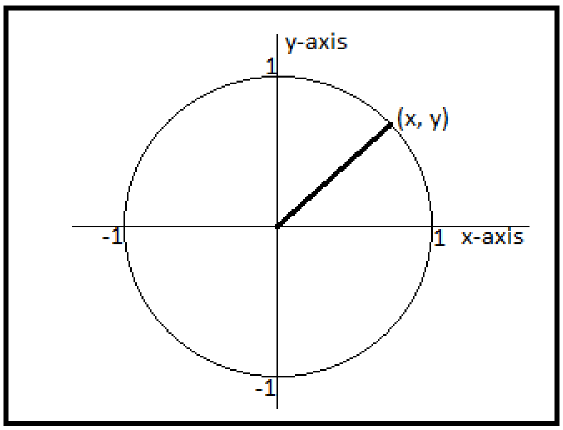


Figure 2.4: Diagram of the unit circle showing the cartesian coordinate axes and the radius.

In the unit circle, the reference angle (R) is always taken as the angle between the x-axis and the radius. However, the actual angle (θ) is between the positive x-axis and the radius taken in an anti-clockwise direction. Hence, the actual angle and the reference angle are the same in the first quadrant but are different in the second, third, and fourth quadrants (see *Figure 2.5*).

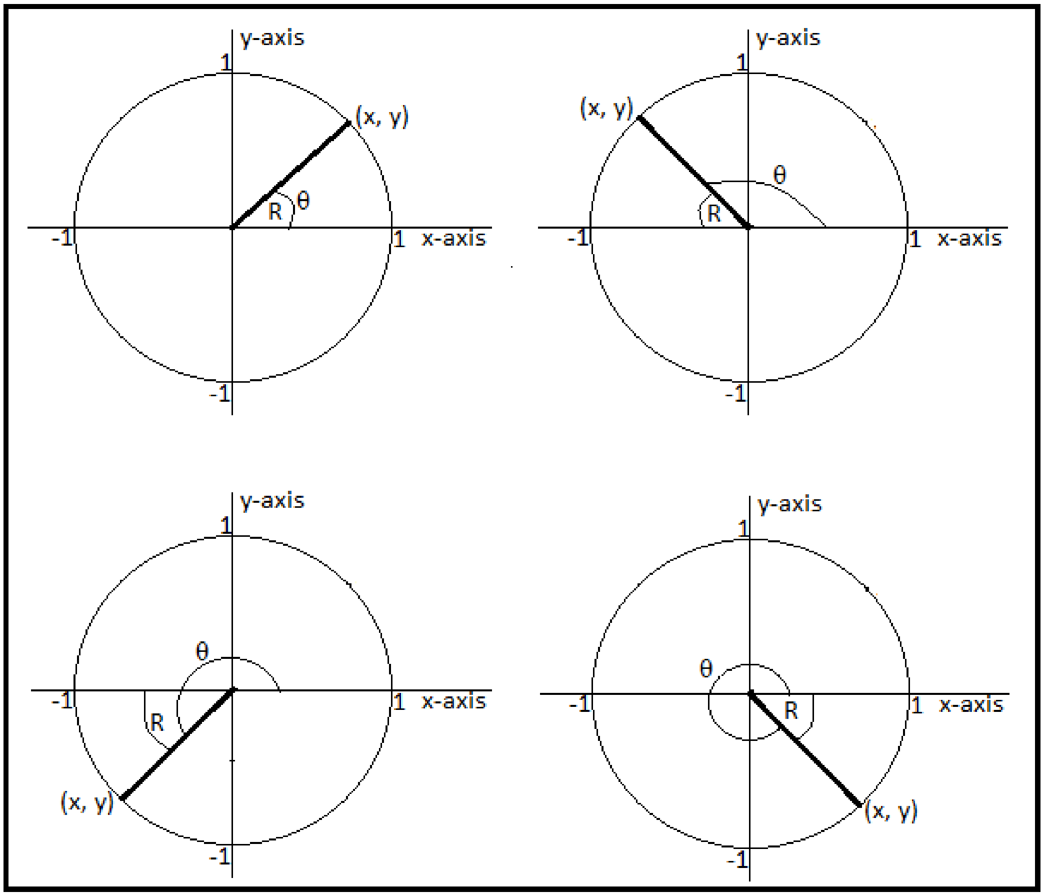


Figure 2.5: Diagram of the unit circle showing the difference between the reference angle (R) and the actual angle (θ) made by the radius and the positive x-axis.

A right-angle triangle is commonly drawn with a portion of the x-axis and a vertical line as its legs and the radius as its hypotenuse to illustrate the sine, cosine, and tangent ratios. The length of this vertical line is equivalent to the y-coordinate of the point on the circumference to which the radius is drawn; hence, the distance along this vertical line is commonly labelled y. The length of the portion of the x-axis used in the right triangle is equivalent to the x-coordinate of the point on the circumference to which the radius is drawn; hence, the distance along this portion of the x-axis is commonly labelled x. These features of the unit circle are depicted in *Figure 2.6* below.

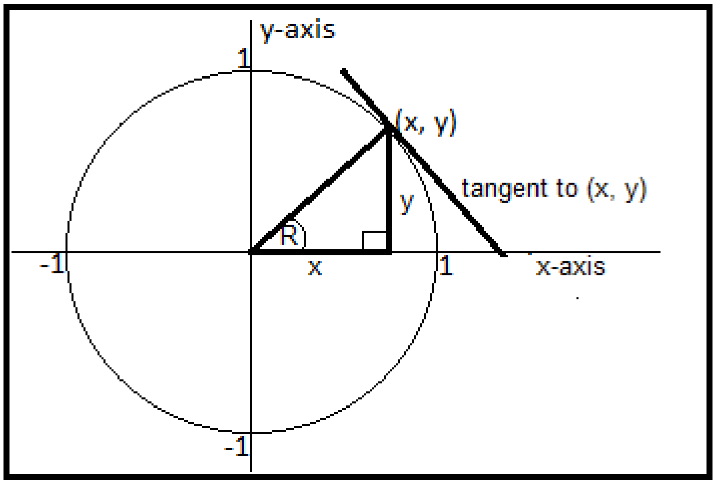


Figure 2.6: Diagram of the unit circle illustrating the features of a right-triangle from which the ratios can be formed.

The unit circle can be used to demonstrate how the sine, cosine, and tangent ratios behave as the reference angle changes (see *Figure 2.7* below). The focus of this illustration was restricted to angles between zero and 90 degrees because the study focused only on this range of angles, as is the CSEC mathematics syllabus (CXC, 2016).

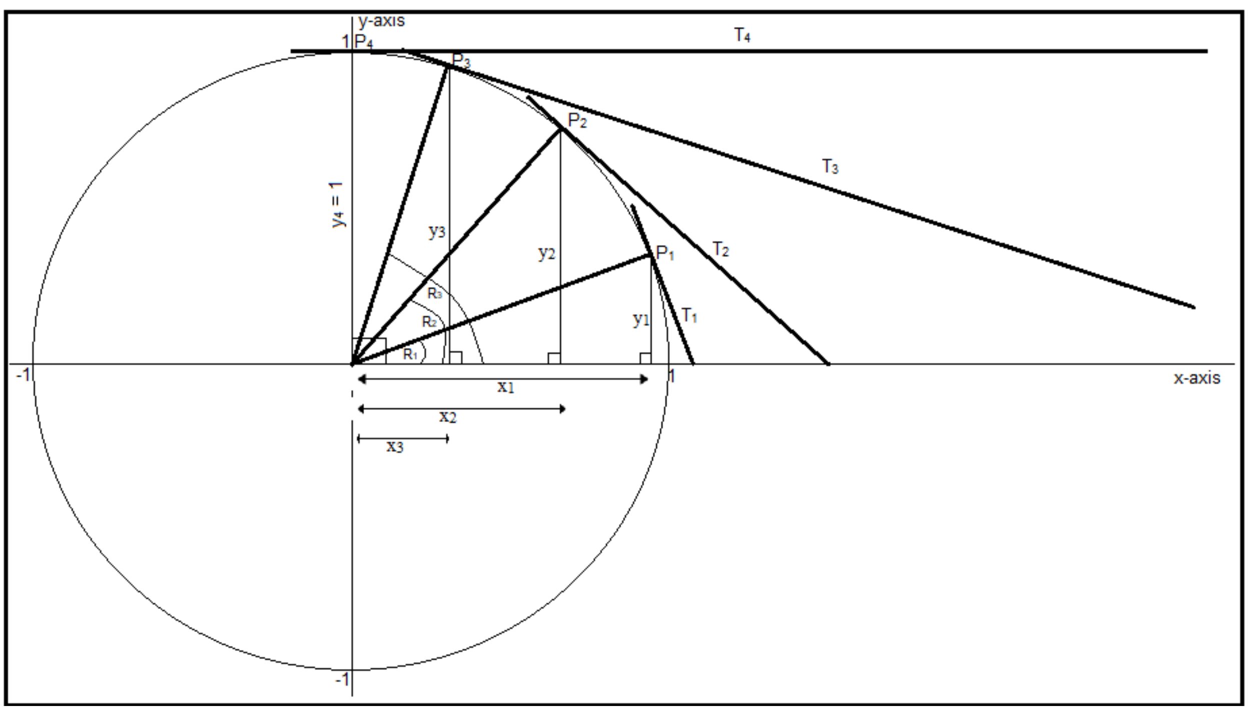


Figure 2.7: Diagram of the unit circle illustrating how the sine, cosine, and tangent ratios behave as the reference angle increases.

Figure 2.7 shows that as the value of the reference angle increases and approaches 90 degrees, from R_1 to R_2 to R_3 to 90^0, the value of y increases and approaches one, and the value of x decreases and approaches zero. *Figure 2.7* also shows that as the value of the reference angle increases and approaches 90 degrees, the lengths of the tangents (T_1, T_2, T_3, and T_4) to the respective points (P_1, P_2, P_3, and P_4) increase and approach infinity.

It is also possible to link the formulas for the sine, cosine, and tangent ratios to the unit circle. Using the fact that the radius, which is always one in a unit circle, forms the hypotenuse, for a unit circle the ratios are reduced to:

The sine of angle $x(sinx) = \frac{\text{Opposite side}}{Hypotenuse} = \frac{y}{1} = y$

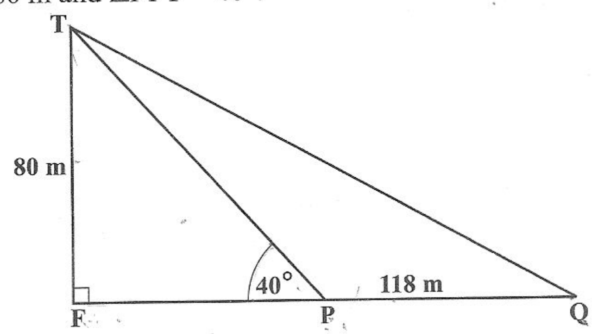
The cosine of angle $x(cosx) = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{x}{1} = x$

The tangent of angle $x(tanx) = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{y}{x}$

That is, for any point on the unit circle, the sine ratio of the angle between the radius to that point and the positive x-axis equals the value of the y-coordinate, and the cosine ratio equals the value of the x-coordinate. The tangent ratio is the quotient of the y-coordinate divided by the x-coordinate. Hence, as the reference angle increases and approaches 90 degrees, the value of x decreases and approaches zero; therefore, the value of tangent increases and approaches infinity. These facts can be used to determine the sine, cosine, and tangent ratios of any angle in a unit circle.

There are competing evidence for how best to teach the trigonometric ratios. Some studies (Blackett & Tall, 1991; Weber, 2005) argued that students develop a better understanding of the trigonometric ratios when taught using a geometric approach such as the unit circle. They claimed that in using the unit circle, students are better able to visualize and internalize how the values of the ratio changes with the changing measures of the angle. On the other hand, several studies (Carlson, & Silverman, 2007; Kendal & Stacey, 1998; Thompson, Yiğit Koyunkaya, 2016) argued that when taught using the ratio method with the use of the mnemonic, SOH–CAH–TOA, students performed better. Students' better performance was attributed to them being able to use the mnemonic and the diagram representations to select and apply the appropriate ratios to problems that require numeric solutions. These types of problems are prevalent in CSEC mathematics examinations; an example is provided in *Figure 2.8* below.

The diagram below, **not drawn to scale**, shows the positions of two ships, P and Q, at anchor. FT is the vertical face of a cliff jutting out of the water. P and Q are 118 m apart. $FT = 80$ m and $\angle FPT = 40°$.



Determine

(i) the angle of elevation of T from P **(1 mark)**

(ii) the length of FP **(2 marks)**

(iii) the angle of elevation of T from Q. **(3 marks)**

Figure 2.8: An example of a problem based on the three primary trigonometric ratios that requires a numeric solution (CSEC past paper, May/June 2014).

It is this very mnemonic and ratio approach that Dominican mathematics teachers employ when teaching the sine, cosine, and tangent ratios (personal experience), yet, students continue to perform poorly on problems based on the three primary trigonometric ratios at CSEC mathematics examination (CXC, 2015).

Therefore, does it matter if the trigonometric ratios are taught using the unit circle or as ratios using diagrams and the mnemonic SOH–CAH–TOA? Since there is conflicting evidence in that regard, it appears that both strategies could be useful. What then makes the difference? Could it be the instructional methods to which teachers expose students? After searching all available databases at the University of Alberta, google scholar, and the world wide web, all comparative studies found on teaching the trigonometric ratios were focused on the use of the unit circle versus the use of ratios in right triangles and the mnemonic. That is, no study was found comparing the teaching methods used (e.g., direct instruction and inquiry-based approaches).

Methodology

A methodology is mainly concern with how data is collected and analyzed. Mixed methods, as a methodological approach in educational research, has the potential to provide invaluable insights into educational practices (Johnson & Onwuegbuzie, 2004; Ercikan & Roth, 2006). This methodology makes use of both quantitative and qualitative means. Hence, it irons out the deficiencies of a single method and combines the strengths of the two (Johnson & Onwuegbuzie, 2004). This perspective is supported by Ercikan and Roth (2006), who argued that limiting research inquiries to only one type of research method can often lead to incomplete answers. Furthermore, Ling (2017) places the mixed methods approach in the pragmatic paradigm where the research agenda, social context, and practical needs drive methodological choices.

In recent times, several studies (Bal, 2016; Bosman & Schulze, 2018; Cabi, 2018; Callaghan et al., 2018; Tchoshanov, 2011) used the mixed-methods approach to investigate students' achievement in mathematics. Moreover, many of these studies investigated how teachers' classroom practices affected students' achievement. For instance, Bal (2016) used a mixed-methods approach to investigate the effects differentiated instruction had on sixth-grade students' achievement in algebra.

Learners' conceptual understanding of mathematical ideas was also investigated by several studies (Eichhorn, 2018; Gultepe, Celik, & Kilic, 2013; Prince & van Jaarsveld, 2017; Tatar & Zengin, 2016) using a mixed-methods approach. Of significance to this study is the approach used by Prince and van Jaarsveld (2017) to evaluate conceptual understanding. Prince and van Jaarsveld used students' written responses to open-ended questions to evaluate students' understanding of trigonometric ideas. From their study, Prince and van Jaarsveld concluded that "open-response questions reveal a learner's conceptual knowledge" (p. 174).

With the need to solve a practical and contextual research problem, it is reasonable to develop and use a pragmatic approach to investigate the research problem. Several studies (Creswell, 2009; Ling, 2017; Mertens, 2010) claimed that, in the pragmatic paradigm, the focus is on answering the research questions using the most suitable methods and procedures. It is also reasonable to use a mixed-methods approach to collect and analyze data to find plausible answers to the research question, *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* given its widespread use in similar research studies. Drawing from Johnson and Onwuegbuzie (2004) and Ercikan and Roth (2006), the goal of using a mixed-methods approach in this study is to uncover as complete a solution as possible.

Theoretical Orientation to the Study

The teaching component of this current studyis grounded in constructivism. The theory of constructivism was influenced by the work of several theorists, central among them are Jean Piaget and Lev Vygotsky with others such as von Glasersfeld, and Fosnot and Perry later adding to the idea. This section of chapter two discusses constructivism as a theory of learning with an elaboration on the work of Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996); as it relates to the roles of the students, teachers, and classroom environment in the teaching and learning of mathematics; and as a framework for conducting this study.

Constructivism: A theory of learning.

Constructivism is a theory of learning and not a method of teaching (Lesh, Doerr, Carmona, & Hjalmarson, 2003; Windschtil, 2000; Clements, 1997). The primary tenant of constructivism is that students construct knowledge when they actively engage in cognitive activities augmented by communication with their peers and with their teacher (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978). Jean Piaget, a major proponent of the constructivist theory of learning, focused on the cognitive aspects of learning, while Lev Vygotsky focused his work on the social aspect of learning. However, both theorists recognized that both social and cognitive factors are essential to learning (Piaget, 1977; Vygotsky, 1978).

Cognitive constructivism promotes the philosophy that students learn new materials by integrating them into existing mental structures (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995). According to Piaget (1977), learning takes place when a student consciously or unconsciously works towards maintaining a state of equilibrium when confronted with information or an experience that challenges his or her previous knowledge. That is, as students encounter new information or experiences, they move to a mental state of disequilibrium. This disequilibrium is resolved using Assimilation and Accommodation (Piaget, 1977).

Assimilation and Accommodation, according to Piaget (1977), are two mental processes through which adaptation in humans takes place. Piaget argued that intellectual growth is a result of adaptation. In his explanation of these processes, Piaget made use of the term schema, which refers to both an internal knowledge structure and the process of obtaining that structure. Both Assimilation and Accommodation have to do with how new knowledge is integrated into an individual's pre-existing knowledge structure. Assimilation takes place when a person absorbs new information about an idea without him or her having to change their previous understanding of that idea. That is, they use what they already know to make sense of new ideas. For example, an individual whose conception of a rectangle is that of a four-sided figure with two side lengths longer than the other two would have no problem identifying most computer screens as rectangles. However, if they encounter a computer screen with all four side lengths equal, they might not identify it as a rectangle. To identify it as a rectangle, Accommodation must take place.

Accommodation takes place when an individual is forced to make changes to his or her existing understanding of an idea to make sense of new information related to that idea (Piaget, 1977). Concerning the above example, that individual would have to change their conception or definition of a rectangle to accommodate a four-sided figure with all four side lengths equal—a square. This change in conception or definition will result in the person developing a new schema or knowledge structure about rectangles. Piaget refers to this as intellectual growth. Often, this happens when students' knowledge about a concept is incorrect or incomplete. Once accommodation has taken place, the learner returns to a state of equilibrium until he encounters new and unfamiliar information or experiences.

Social constructivism, on the other hand, espouses the philosophy that knowledge is socially constructed as a product of the environment and communications (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978). That is, social constructivism focuses on the role that language, social interaction, and culture play in the development of knowledge (Powell & Kalina, 2009). According to Vygotsky (1978), children learn two types of concepts: spontaneous concepts and scientific concepts. Spontaneous concepts are developed through everyday activities while learners develop scientific concepts in their classrooms through interactions with their peers and their teacher (Vygotsky, 1978). Moreover, Vygotsky claimed that students on their own cannot develop a scientific concept but can do so by collaborating with a more knowledgeable other.

Collaboration with a more knowledgeable other (a teacher) is of greatest worth when done within a student's Zone of Proximal Development (ZPD). "The zone of proximal development defines those functions that have not yet matured but are in the process of maturation" (Vygotsky, 1978, p. 86). That is, ZPD describes a stage of cognitive development where a learner cannot perform a task by himself but is capable of performing that task with the help of a teacher. Vygotsky (1978) described several ways a teacher might provide help to a child within his ZPD to help him perform a task that was initially beyond his intellectual reach. Vygotsky stated that "some might run through an entire demonstration and ask the children to repeat it, others might initiate the solution and ask the child to finish it, or [some may] offer leading questions" (Vygotsky, 1978, p. 86). In other words, a teacher must collaborate with learners in a way that best helps the learner to reach their intellectual goals.

Furthermore, collaboration requires participating individuals to understand each other in an environment (classroom culture) that is conducive to sharing. Part of this environment is language. According to Vygotsky (1978) and von Glasersfeld (1995), language serves a higher purpose than just communication; it is the mechanism through which students reason and come to know. Hence, a student develops knowledge and understanding as he or she engages in classroom activities with his or her peers and teacher, engages in classroom discussions, and reflects on classroom actions.

von Glasersfeld (1995) is the proponent of radical constructivism. His work builds on the theories of both Piaget's (1977) theory of cognitive constructivism and Vygotsky's (1978) theory of social constructivism. He accepted and built on Piaget's idea that learning is attained through mental activities. Like Piaget (1977), von Glasersfeld argued that learning takes place when students build on their prior mental structures–schemes–which are unique to each student. Like Vygotsky (1978), he claimed that social interactions with peers and a teacher help an individual to organize his cognitive processes.

Drawing from these assertions, Fosnot and Perry (1996) defined constructivism as a learning theory "that construes learning as an interpretive, recursive, non-linear building process by active learners interacting with their surround" (p. 34). This definition put forth by Fosnot and Perry, subsumes the theories of Piaget (1977), Vygotsky (1978), and von Glasersfeld (1995). That is, it calls for students to be actively involved in cognitive and social activities. Furthermore, this definition highlights three critical elements that should be present in classrooms where constructivism drives teaching and learning activities: the learner must be an active participant in his learning, the teacher must be a facilitator of learning, and the environment must be conducive to student-teacher and student-student interactions. The theoretical framework used in this study drew from the work of Fosnot and Perry (1996), Piaget (1977), Vygotsky (1978), and von Glasersfeld (1995).

A research model.

The model presented below (see *Figure 2.9*) helped to focus the teaching component of this study. It positions the students as active learners, the teacher as facilitator, and the classroom as an enabling environment where learning takes place. In this model, all three factors are given equal portions of the outer circle to signify that they are equally important in the learning process. That is, students are needed to carry out the mental and social activities through which learning takes place; the teacher is needed to prepare and focus learning activities to make sure that students stay on task, and to create and maintain a conducive environment; an enabling environment must be present for students to think and interact in meaningful ways. The middle circle represents students' conceptual understanding gained from the teaching and learning activities, and the innermost circle represents students' achievement, which is influenced by their conceptual understanding. This model was developed and constructed by this researcher and made use of his more than 20 years of experience teaching mathematics and the theories of Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996).

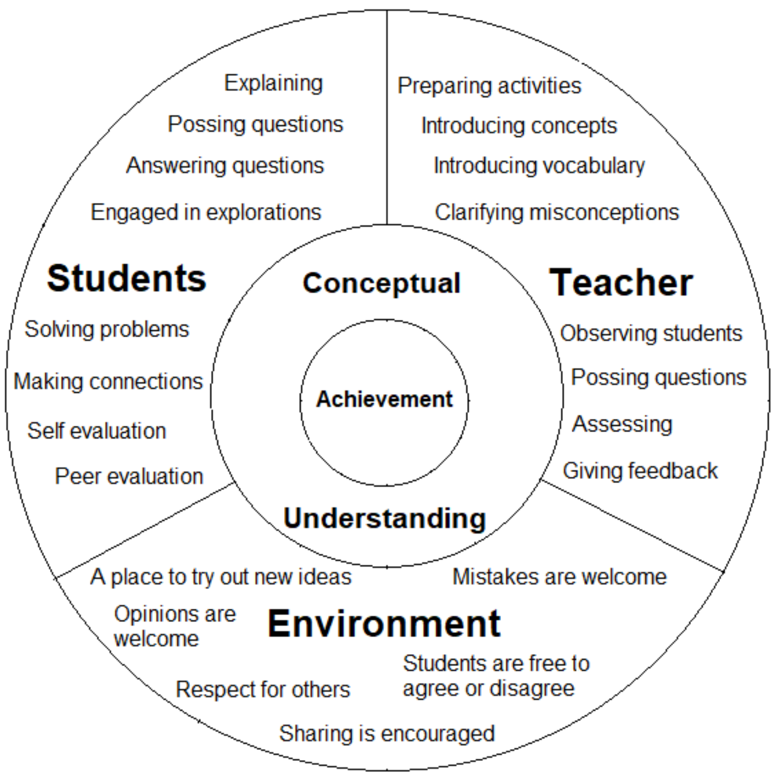


Figure 2.9: A model of a constructivist framework for teaching mathematics.

In this model, the roles of students are identified. As active learners, students must be allowed to engage in explorations, which will allow them to make connections among mathematical ideas (NCTM, 2000). Explorations involve but are not limited to, making conjectures and testing them (Jaworski, 1986), and generating examples of given concepts (Watson & Mason, 2005). They must also be encouraged to explain their reasoning, pose questions to their peers and teacher, and respond to questions from the same (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978). Problem-solving helps students to apply their learning in new situations (Hiebert & Carpenter, 1992; Kilpatrick et al., 200; NCTM, 2000), which gives them opportunities to evaluate themselves and their peers.

Teachers must act as facilitators of learning. As such, they must prepare meaningful activities for students, introduce new concepts and vocabularies to students, and help students to clarify misconceptions as they try to make meaning of these concepts. As students work on meaningful tasks, teachers must observe them and take preventative or corrective actions (e.g. disciplinary actions) when necessary, pose questions to help guide their action and thinking, assess their learning, and give meaningful and timely feedback. These actions are necessary for learning to be focused and meaningful.

Both the teacher and students have the responsibility to make and keep the environment conducive to learning. A conducive/enabling environment is one in which individual opinions are respected and valued. In such a space, students are more likely to share and try out new ideas without fear of making mistakes. Students are also more likely to express their agreement or disagreement with their peers' or teacher's views on mathematics and other matters. Such an environment provides the space for meaningful interactions to take place.

The result of a careful integration of these three factors: the student as an active learner, the teacher as facilitator, and an enabling environment is students' development of conceptual understanding (Fosnot & Perry, 1996). With conceptual understanding, it is expected that students will achieve higher on a test based on the relevant mathematics concepts. This model provided the framework in which teaching in this studywas designed and implemented.

Two Constructivist Approaches to Teaching

Constructivists, in their teaching, must be conscious of and cater to these three critical factors in the classroom: students as active learners, the teacher as facilitator, and an environment conducive to learning. The following are two approaches to the teaching of mathematics—Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005)—that make use of these factors.

Investigation as a constructivist approach to teaching.

An Investigation in this context refers to exploratory work students undertake to help them develop an understanding of a mathematical concept, a group of related mathematical concepts, or a mathematical domain. What exactly does such work entail? Several scholars working in the field of mathematics education have provided insights into this question by providing descriptions and explanations of an investigative approach to teaching and learning mathematics.

Ponte and Matos (1992) equate the roles that students play in the investigative approach to that of mathematicians. They argued that as mathematicians, students must be exposed to complex mathematical situations that require them to make meaning of the situations and find possible solution paths to the problem(s) embedded in the situation (Ponte & Matos, 1992). Important to this approach, students are expected to discover “patterns, relationships, similarities, and differences” (Ponte & Matos, 1992, p. 339) among concepts. They are also expected to use these findings along with group processes to develop generalizations, which in turn will help them solve other mathematical problems of a similar nature (Ponte & Matos, 1992). The idea of using what is discovered in one situation to inform other situations is supported by Sangster (2012).

Sangster (2012) described Investigation as an approach to teaching and learning where students develop knowledge, strategies, and skills that they can employ in other unfamiliar situations. Investigation referred to as an approach, speaks to the importance of using investigations continually in the mathematics classroom if students are to develop the strategies and skills that they can transfer to novel situations. In her explanation of the investigative approach, Sangster also emphasized that, while students must be led to discover new knowledge and construct understanding, the inquiry processes for acquiring these are of great importance (Sangster, 2012). According to Sangster, children must “have had the opportunity to choose how they progress, to seek patterns in the mathematics, to make decisions, to be required to justify their decision making, and to test their assumptions or hypotheses” (Sangster, 2012, p. 40).

These processes highlighted by Sangster are in line with the work of Kwang (2002) who earlier stated that: “Mathematical processes such as guessing, making conjectures and testing their conjectures, generalizing, and recording” (Kwang, 2002, p. 33) are critical to students’ developing conceptual understanding. The use of these processes supports the idea that attaining conceptual understanding is not a smooth and clean process, but rather students must be engaged in a messy motley of meaningful mathematical activities (Greenes, 1996).

Greenes (1996) argued that “investigations present curiosity provoking situations, problems, and questions that are intriguing and captivate students’ interest and attention” (p. 37). Intriguing problems allow students to create paths to a possible solution instead of following one that is already established. Rather, teachers should encourage the use of trials and errors and encourage students to struggle in their quest (Boaler, 2015). Such struggles can be achieved when students are given to use numerous methods to solve the problem, or the problem to be solved has multiple solutions (Greenes, 1996). In these instances, settling on an appropriate set of data and the best approach to finding the data are often messy and must be negotiated. Such negotiations can be confounded when students are working collaboratively, and team members have different interests when working on an open-ended task. For these reasons, an investigation may require one to several lessons (Nesbit, 2010) as students sift through the messiness of their activities in trying to find a meaningful consensus.

Jaworski (1986) asserted that "investigation in mathematics implies finding out" (p. 3) what strategies to use, whether such strategies already exist, what type of solution students should present, or if a solution exists. This description of Investigation provided by Jaworski subsumes the other descriptions discussed in the preceding paragraphs. That is, students exposure to complex mathematical situations referred to by Ponte and Matos (1992), students' exposure to inquiry processes written about by Sangster (2012) and Kwang (2002), and the creation of new paths to solutions referred to by others (Greenes, 1996; Kwang, 2002) are all captured in Jaworski's description of investigations as "finding out". For this reason, Jaworski (1986) approach to Investigation was used in this study.

Jaworski (1986) identified an investigative approach to teaching as one which encourages students to find out things about a mathematical concept for themselves. That is, the teacher should create situations that give students opportunities to explore mathematical ideas and principles and help them to arrive at generalizations (Jaworski,1986). To be meaningful and to cater to diversity in learners, investigative tasks must be open-ended, such that students can choose multiple paths (Jaworski, 1986). Selecting a path or several paths requires reasoning. Reasoning is a mental or cognitive process (Fosnot & Perry, 1996; Piaget, 1977: von Glasersfeld, 1995). To help develop conceptual understanding, Jaworski proposed that students explain their processes and reasoning. Hence, classroom discussion is a critical feature of this approach to teaching.

Moreover, Jaworski (1986) encouraged the use of collaborative groups. Not only does such grouping foster discussions among learners, but it also helps them generate knowledge and understanding as they bounce ideas off each other and their teacher (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995). Such activities must take place in an environment that is conducive to exploring, explaining, and sharing (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978) among others, in order to be meaningful. Hence, Jaworski's approach to classroom investigation caters to the integration of the three essential factors in constructivism: students as active learners, the teacher as facilitator, and a classroom environment that is conducive to learning. A closer look at Jaworski's approach to Investigation highlights these features.

Classroom features of Investigation.

According to Jaworski (1986), six processes are integrated into the investigative approach to teaching mathematics. Jaworski referred to them as Specializing, Conjecturing, Testing, Modifying, Generalizing, and Proving (Jaworski, 1986). These processes do not follow a linear sequence, and they may not all be included in a single Investigation (Jaworski, 1986; Pijls et al., 2003; Ponte & Matos, 1992; Staples, 2011; Yanik, Kurz, & Memis, 2014). That is, students must be allowed to move back and forth among these processes at all stages of an investigation.

Jaworski (1986) and others (Greenes, 1996; Kwang, 2002; Ponte & Matos, 1992) encourage the use of small collaborative groups. They claim that, in working in groups, students can "bounce" ideas of each other, thus, creating an environment of inquiry (Greenes, 1996; Jaworski, 1986; Kwang, 2002; Ponte & Matos, 1992). Furthermore, in collaborative groups, students might provide support for each other by debating ideas and sharing point-of-views (Vygotsky, 1978) that may not be accessible to a student working individually. In such an environment, the teacher must guide as the need becomes necessary (Sangster, 2012). The following are the six processes recommended by Jaworski.

Specializing is described as a systematic way of working with special cases so that students can identify a pattern (Jaworski, 1986). For instance, when presented with an unfamiliar mathematical problem, students are encouraged to try out special cases—often simpler problems—to discover a pattern that can be used to solve the original problem (Jaworski, 1986). These simpler problems, however, must be done systematically to make it easier to identify any pattern that may exist (Jaworski, 1986). A systematic approach in this context means that each case undertaken should contain only a slight variation from its predecessor, a variation that will allow students to assign any change in the results to the variation(s) in the problems. Not only does it matter that students do these simpler problems systematically, but how they record their results is also important (Jaworski, 1986; Ponte & Matos, 1992; Yanik et al., 2014). In many cases, the use of a table is recommended (Jaworski, 1986) because when a table is carefully constructed, it sequentially displays results. Such sequencing of results makes it easier to identify a pattern, which in turn allows students to make conjectures.

Conjecturing occurs when students predict what might happen in a case based on a pattern that is observed (Jaworski, 1986). This description is in line with Ponte and Matos (1992), who asserted that students involved in investigative work must reflect on their activities, identify patterns, and make conjectures based on the patterns identified. Both Jaworski (1986) and Ponte and Matos (1992) recognized and referred to the vital role that conjecturing plays in the investigative process. These researchers (Jaworski, 1986; Ponte & Matos, 1992) suggest that conjecturing is an expression of students' understanding because of the connections they make between the procedures and results during that process.

Moreover, the importance of conjecturing can be gleaned from its use in investigative work undertaken by students in research studies. For example, Staples (2011) encouraged secondary students to formulate and test conjectures during a study in which they investigated the equation that best represented a hanging chain. The student who participated in Staples' (2011) study went through several rounds of conjecturing and testing before coming to a consensus on a plausible equation. These rounds of conjecturing involved two other processes– testing and modifying.

Testing and modifying take place recursively as students refine their understanding during the process of inquiry (Jaworski, 1986; Ponte & Matos, 1992). When a conjecture is made, it must be tested to verify whether it holds. If it holds, students must move on to conjecturing about another specific case that varies from the previous one (Jaworski, 1986; Ponte & Matos, 1992). However, if after testing a conjecture, it turns out to be false; students must modify their understanding of the pattern which they identified and proceed to test other conjectures (Jaworski, 1986; Ponte & Matos, 1992). Staples (2011) alluded to the high levels of demand placed on students when they are called upon to test conjectures and modify their understanding of patterns. In his deliberations, he pointed to the difficulties that students might face when measurements do not always yield accurate data, which makes it hard to recognize a pattern (Staples, 2011). However, Staples (2011) also argued that such experiences are useful because they help students understand the realities of mathematics in the real world.

The highlight of any mathematical investigation is the generalization arising from the various activities. Generalizing allows students to transfer and use the knowledge and understanding gained, to solve other similar mathematical problems (Jaworski, 1986). According to Jaworski (1986), generalizing means coming to a general statement about the concept(s) under investigation base on patterns observed and after testing and modifying of conjectures. Such a general statement is one form of representation. Hence, it is a demonstration of students' understanding of the concept(s) being investigated (Hiebert & Carpenter, 1992; Kilpatrick et al., 200; NCTM, 2000). Attaining such an understanding is exemplified by Staples (2011), where students were able to generalize in the form of an equation representing the shape of a catenary—hanging chain.

General statements may take other forms, such as an exposition, as demonstrated by Yanik et al. (2014). In their study, Yanik et al. (2014) worked with students to develop models for finding the height of ancient humans based on their stride lengths. These models were explained using words and arithmetic operations because these were the forms of representations available to the students at that time. However, while generalizing is the highpoint of a mathematical investigation, it is not the end.

Proving or convincing refers to the act of demonstrating that the generalization generated from investigative activities will hold for all cases within boundaries (Jaworski, 1986; Ponte & Matos, 1992). The way students convince their peers and teacher depends on their mathematical maturity and skills at the time of the investigation. In some instances, especially among mature students, elegant mathematical proofs are required (Jaworski, 1986; Ponte & Matos, 1992). In other cases, however, a simple justification may suffice (Jaworski, 1986). Such justification may take the form of an explanation in which students give reasons (verbally or in writing) why their generalization is a reasonable one.

However, there are some instances where neither proof nor justification is necessary. For example, Pijls et al. (2003) used an investigative approach to explore which of three conditions—using computer simulations before, during, or after instruction—improves 'mathematical level raising' in students. Level raising represents a change in students' understanding as they change their considerations of mathematical ideas from the perceptual level to the conceptual level (Pijls et al., 2003). Participants in that study were not asked to provide proof or justification. Therefore, it is useful to note that while these six processes may be included in the investigative approach to teaching mathematics, they do not all have to be present in a single Investigation.

Effects of Investigation.

In the literature, there is a multitude of studies that investigated the effects of the investigative approach to teaching mathematics on students' achievement and understanding of mathematical ideas, with most recent studies concluding that Investigation positively affects students' learning—increased their achievement in and conceptual understanding of mathematics. Investigation normally makes use of hands-on materials such as technology and other concrete materials, cooperative learning, and problem-solving in the classroom context. Hence, to capture the full scope of the impacts that Investigation has on students learning, four meta-analyses concerning the factors mentioned above are discussed. The discussions on these meta-analyses are followed by discussions on primary studies dealing with mathematical investigations.

One such meta-analysis was conducted by Li and Ma (2010), who examined and synthesized the effects that computer technology used in classroom investigations had on 36,793 K-12 students' learning of mathematics. The analysis considered 46 primary studies and concluded that, when used in classroom investigations, the use of computer technology had a significant positive impact on students' achievement in mathematics. Carbonneau, Marley, and Selig (2013) conducted a systematic review of the literature to examine the effects that the use of concrete materials used in classroom investigations had on students' learning. They looked at 55 studies that compared students' achievements when they were allowed to investigate mathematical ideas with the use of concrete materials versus teaching situations where students did not use concrete materials. Carbonneau et al. (2013) found a significant positive effect on the learning of students who investigated mathematical ideas with the use of concrete materials.

Cooperative learning and problem-posing are two essential tenants of a mathematical investigation. Johnson, Johnson, and Stanne (2000) examined 164 primary studies that investigated several cooperative learning strategies. They found that all cooperative learning strategies investigated had a significant positive impact on students' achievements. Rosli, Capraro, and Capraro (2014) synthesized research published between 1989 and 2011 to examine the effect of problem-posing on students' achievement in mathematics. The study concluded that problem-posing positively affected students' understanding and achievement in mathematics. Therefore, in as much as Investigations incorporate these factors, the widely accepted notion that it increases students' achievement and improves their conceptual understanding of mathematics is understandable. The following primary studies highlight some effects that Investigation had on students' learning.

Ponte and Matos (1992) used the logo software to investigate the impacts of recursion in geometry with eighth-grade students. Sessions with students that required them to manipulate the software, make conjectures, and pose and answer questions from their peers were videotaped and analyzed. The study concluded that the investigative activities that participants engaged in, helped them to develop and consolidate several geometric concepts. Ponte and Matos further stated that these activities help improve the students thinking skills.

Staples (2011) arrived at a similar conclusion when he stated that it is in having to deal with the imprecision of measurements that students learn to connect the mathematics they do to the real world. Staples (2011) reported on the investigative actions of upper secondary students who were attempting to find the equation of a hanging chain. In another study, Yanik et al.

(2014) had students measure and use the lengths of strides from footprints to estimate the height of primitive humans. Middle school students worked in small groups, and by comparing their stride lengths with that of ancient humans, they predicted the height of these ancient humans (Yanik et al., 2014).

Some studies into the investigative approach to teaching mathematics were comparative. Erbas and Yenmez (2011) investigated the effects of using Geometer's Sketchpad on sixth graders' learning of polygons. The study was a quasi-experimental, pre-test, post-test, and delayed post-test design. The control group was taught using traditional methods, while the experimental group investigated the characteristics of polygons using Geometer's Sketchpad. The work with students lasted two weeks. Participants who worked with the Geometer's Sketchpad performed better than their counterparts in both the post-test and the delayed post-test. Mainali and Heck (2017) conducted a similar study that utilized information technology and a pre-test–post-test, quasi-experimental design, and obtained similar results. Like Erbas and Yenmez, Mainali and Heck compared the effects of traditional teaching with an ICT-rich teaching environment on students' achievement and understanding. Mainali and Heck assessed students' achievement and understanding through both quantitative and qualitative data analyses. However, unlike Erbas and Yenmez, their study was conducted with secondary (ninth-grade) students.

Both Ekwueme Ekon, and Ezenwa-Nebife (2015) and Budak (2015) also used a pre-test–post-test, quasi-experimental design, but not using an ICT-rich teaching environment. In the Ekwueme et al. (2015) study, the experimental group investigated aspects of geometry, mensuration, and separation of mixtures with hands-on materials while the control group did not use hand-on materials. The results indicated that the experimental group performed better on the post-test than the control group. Similar results were obtained in Budak (2015), who investigated the impact of elementary schools' curriculums on students' achievement. The work of students who were taught using a conventional curriculum was compared with those of students taught using an investigative curriculum. The work of over 700 grade three students were analyzed. Budak found that students taught using an investigative curriculum performed better than students taught using a non-investigative curriculum.

Exemplification as a constructivist approach to teaching.

Exemplification in this study refers to the generation of examples in the mathematics classroom (Watson & Mason, 2005). However, what is prevalent is teachers producing and working through examples to demonstrate how to perform an algorithm or a mathematical process (Dinkelman & Cavey, 2015). Watson and Mason (2002) argued that many students could not generalize from worked examples; they have difficulties applying the processes demonstrated to them in a new situation. How then can examples be utilized in the classroom to maximize their benefits to students? Some scholars (Bills et al., 2006; Dinkelman, 2013; Dinkelman & Cavey, 2015; Meehan, 2007; Watson & Mason, 2005; Watson & Shipman, 2008) advocate for learners to generate examples.

Several studies (Bills et al., 2006; Dahlberg & Housman, 1997; Dinkelman, 2013; Meehan, 2007; Watson & Mason, 2005; Watson & Shipman 2008) have concluded that, when students generate examples, they develop a thorough understanding of mathematical ideas. Moreover, both Dinkelman (2013) and Watson and Mason (2005) claimed that by generating examples, students become exposed to a wide range of examples; hence, they may become aware of the many variations of a concept, the principle(s) that underpins it, and its many uses. Consequently, they may utilize such awareness in various problem situations, which may lead to them becoming better problem solvers (Bills et al., 2006; Dinkelman, 2013; Dinkelman & Cavey, 2015; Meehan, 2007; Watson & Mason, 2005; Watson & Shipman, 2008).

Bills et al. (2006) and Watson and Mason (2005) highlighted the pedagogical distinction between examples of a concept and examples of the use of a procedure. For instance, an image of a right-angle triangle is an example of a concept while the calculations used to determine a missing side or angle in a right angle triangle is an example of the application of a procedure. These represent two broad classes of examples, and both are essential to the learning of mathematics (Bills et al., 2006; Watson & Mason, 2005). Furthermore, Watson and Mason (2005) argued that the use of counterexamples and non-examples is necessary if learners are to develop a complete understanding of a concept. Watson and Mason are supported by Bills et al. (2006), who highlighted three categories of examples: generic examples, counter-examples, and non-examples.

A generic example is one which satisfies all the conditions detailed in the definition of a concept, a counter-example is one which appears to meet the necessary conditions but fails to fulfill an assertion about that concept, and a non-example is one that exists outside the boundaries detailed in the definition of a concept (Bills et al., 2006). For example, if an individual defines a prime number as an odd number with exactly two different factors, then numbers such as 3, 7, and 17 are generic examples; 2 is a counter-example because it not an odd number, and 1 is a non-example. Bills et al. (2006) called for learners to know and generate all three classes of examples. This call by Bills et al. supports Watson and Mason's (2005) notion of an example space.

Watson and Mason (2005) described an example space as a set of examples featuring several variances and nuances to which students have immediate access. They contended that "learning mathematics consists of exploring, rearranging, and extending [a learner's] example space and the relationships between and within them" (Watson & Mason, 2005, p. 6). Watson and Mason also argued that students who can extend their example space would experience flexibility in thinking and empowerment in the appreciation and adoption of new mathematics concepts. Their claim builds upon the findings of Dahlberg and Housman (1997), who found that undergraduate students who generated examples had a better understanding of functions over students who only read about the functions. Other studies (Dinkelman, 2013; Meehan, 2007; Watson & Shipman 2008) led to similar claims.

The act of expanding a learner's example space, as suggested by Watson and Mason (2005), falls within the ambit of the constructivist approach (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978) to teaching mathematics. A learner's example space is a mental or cognitive space that the learner may develop through exemplification activities (Watson & Mason, 2005). Watson and Mason (2005) used metaphors such as landscape, toolshed, kitchen cupboard, and ladder to help describe how one's example space works. All these metaphors suggest that a learner needs to search to find a suitable example of a concept to fulfill a set of conditions. In his or her search, a student may have to collaborate with peers to find suitable examples and understand them. Hence, asking students to generate examples forces them to engage in mental and, in some cases, collaborative activities which are based on the constructivist theory of learning (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995; Vygotsky, 1978), because students are actively constructing meaning through mental and social activities. These activities are most effective when they take place in a conducive classroom environment.

Classroom features of Exemplification.

Watson and Mason are the leading proponents of Learners' Generated Examples—Exemplification—as an approach to teaching mathematics. They argued that an example might stand for "illustrations of concepts and principles", "placeholders used instead of general definitions and theorems", "questions worked through", "questions to be worked on", "representatives of classes used as raw materials for inductive mathematical reasoning", and "specific contextual situations" (Watson & Mason, 2005, p. 3). Hence, a teacher's central role is to ensure that students are generating the most relevant examples, at the most suitable time, and in the most suitable environment, examples that lead to a thorough understanding of a concept. That is, a primary function of the teacher is to determine which classes of example that students should generate to achieve an intended goal. Thus, teachers' prompts become a critical factor in students' example generation.

When teaching through Exemplification, Watson and Mason (2005) advocate for teachers to start with a broad definition or explanation of a concept, then gradually apply different constraints to that prompt. They claimed that "often in mathematics, the action of adding constraints to problems open[s] up new possibilities for the learner and promotes creativity" (Watson & Mason, 2005, p. 11). According to Watson and Mason (2005), students' choices of examples within a task should start with a high degree of freedom that must be gradually constrained, thus, forcing them to "dig deeper" into the structure of a concept to come up with suitable examples. Starting with a high degree of freedom also allows for different entry points into the lesson for students.

Moreover, teachers must consider learners' experiences when developing and using prompts (Watson & Mason, 2005). According to Watson and Mason (2005), learners' "past experiences give them some dominant images to which they automatically refer, and which also limit other choices they can make" (p. 35). Essentially, Watson and Mason are arguing that, when learners are asked to generate examples of a mathematics concept, their first examples will reflect their previous experiences with the concept. With this argument, one can infer that when a teacher issues a prompt for students to generate examples, it triggers different images and several ways of thinking among students because of their various experiences. However, as constraints are added to an initial prompt, learners are forced to think deeper about the characteristics and underlying principles which govern the concept to generate other suitable examples.

For example, an initial prompt may ask students to draw a quadrilateral. Some students may produce a rectangle as a first example because that might be their only or most dominant conception of a quadrilateral. However, others may present a rhombus, a parallelogram, a trapezoid, or some other shape depending on their understanding of a quadrilateral at the time. If they are then asked to draw a rectangular quadrilateral which has all sides equal, students will have to produce a square. To produce a square, students will have to convince themselves that a square is rectangular. To accommodate this new information into their existing mental structure (Piaget, 1977), students need to restructure their current understanding of a rectangular quadrilateral. Thus, constraining the prompt in this manner provides the opportunity for students to learn that a square is a rectangle and why. That is, constraints that are carefully and strategically applied to prompts can lead students to discover features and principles that could lead them to a deeper understanding of the concept (Bills et al., 2006; Dinkelman, 2013; Dinkelman & Cavey, 2015; Meehan, 2007; Watson & Mason, 2005; Watson & Shipman, 2008). In other words, a student's understanding of a concept may deepen as he expands his example space (Watson & Mason, 2005).

The use of prompts to get students to generate examples, however, must be well structured if students are to increase their knowledge and understanding of mathematical concepts. Watson and Mason (2005) suggested some pedagogical practices that may facilitate changes in students' knowledge and understanding of mathematical concepts. They suggested that teachers should: (a) show learners new mathematical objects to increase their encounter with different types of examples; (b) allow learners to experience non-examples and counterexamples so that they are exposed to the full range within a class of examples; (c) get learners to restructure what they already know so that new knowledge can be attained; (d) get learners to use what they already know in novel situations, thus, building their problem-solving skills; and (e) get learners to create new objects from what they already know, thus, constructing more knowledge for themselves and others (Watson & Mason, 2005). The role of the teacher as a facilitator and the role of students as active learners are evident in these suggested activities. Furthermore, these activities must take place in a classroom environment where students are not impeded, in any way, from generating examples and sharing them.

Effects of Exemplification.

The practice of learners' generating examples has been around for years and has been shown to affect students' learning positively. For instance, Dahlberg and Housman (1997) used example generation activities with students years before Watson and Mason (2005). Dahlberg and Housman investigated strategies that affected third and fourth-year undergraduate students' understanding of a function. Of the different strategies that students employed, the study concluded that those who generated examples attained a better understanding of a function. Also, Watson and Mason (2002) reported on their observations and reflections of teachers working with students on example-generation tasks. Several classrooms with students working on different tasks were observed. Field notes were taken and analyzed to check for students' conceptual understanding of the materials. Watson and Mason (2002) concluded that example generation helped students to develop a conceptual understanding of mathematical ideas. Watson and Mason (2005) drew from the work of Dahlberg and Housman (1997) and Watson and Mason (2002). The rest of this section focused on the effects of Exemplification as proposed by Watson and Mason (2005).

According to the literature, students taught by Exemplification (Watson & Mason, 2005) gained conceptual understanding and increased their achievement in areas of mathematics. In one study, Abdul-Rahman (2005) used the examples that A-level mechanics and statistics students, in the UK, generated to investigate what they understood about mathematical objects. In this study, data were collected through semi-structured interviews where students were exposed to three different types of tasks regarding integration: answering questions that reveal their understanding, answering application questions, and generating examples. The study concluded that students who generated examples learned how to better manipulate the structure and the relationships among the elements in a given space.

Sandefur et al. (2012) conducted a study using Watson and Mason's (2005) notion of Exemplification. In this study, aspects of students' understanding were assessed while they generated examples in small groups. The purpose of the Sandefur et al. study was to investigate how example generation impacted undergraduate students' reasoning on proofs in problems solving tasks. The works of four groups were analyzed for differences in manipulation, use of syntactic, conceptual insight, and technical handle. The analysis of students' work revealed that generating examples helped students improved their understanding in these areas.

Other studies conducted at the post-secondary level produced similar findings. Furinghetti, Morselli, and Antonini (2011) investigated the use of Exemplification as a problem-solving strategy. They collected data through interviews, from students' notes as they worked through tasks, and from field notes taken by the researchers. Participants were students in their last two years at university. Furinghetti et al. (2011) found that generating examples help the students to develop the meaning of a concept and aided in their construction of proofs. Likewise, Scataglini-Belghitar and Mason (2012), working with first-year mathematics undergraduates, found that by generating examples of functions, students developed a better sense of their use.

However, the literature reports mixed effects of Exemplification (Watson & Mason, 2005) on students' achievement and conceptual understanding in comparative studies done at the undergraduate level. Iannone et al. (2011) investigated the differences in the achievements of two groups of undergraduate students who were asked to produce proofs of a function. One group was asked to generate examples of the function given prompts as proposed by Watson and Mason (2005), and the other group read about the function. After working for 20 minutes, both groups of participants were asked to work on proof production tasks. Their achievements in these tasks were analyzed using a statistical approach. The study found no significant differences in the learning of the two groups of students.

On the other hand, Rawson and Dunlosky (2016) conducted a study with 487 undergraduates to examine the effects that Exemplification (Watson & Mason, 2005) had on their learning of mathematical concepts. Two groups of students were used in the study. One group was asked to generate examples of the concepts after reading about them. The second group of students was asked only to read about the same concepts. Both groups were later tested in the same manner, and their test scores analyzed. The analysis showed that the example generation group yields better achievement scores than the non-generating group.

The literature also reports some studies conducted at the secondary school level that investigated the use of Exemplification (Watson & Mason, 2005). Watson and Shipman (2008) reported on their observations in secondary classrooms where students were generating examples and concluded that example generation helped students to develop a conceptual understanding of mathematical ideas. Through the analysis of field notes, Watson and Shipman concluded that students gained an understanding of the structure of a linear function and learned new materials. Dinkelman and Cavey (2015) also used Exemplification as proposed by Watson and Mason (2005) to assess high school students' understanding of functions. They exposed these students to tasks that required them to work with non-examples, determine a function from other relations, and generate a function given constraints. Dinkelman and Cavey concluded that the examples generated by students showed an improved understanding of functions.

However, studies that investigated the use of Exemplification (Watson & Mason, 2005) at the secondary school level appears to be sparse in the literature (Watson & Shipman, 2008). Of the two mentioned above (Watson & Shipman, 2008; Dinkelman & Cavey, 2015), one was referred to as being opportunistic (Watson & Shipman, 2008). Hence, there is a need for more studies to be done that investigate the use of Exemplification at the secondary school level.

A comparison of Investigation and Exemplification.

Investigation, as proposed by Jaworski (1986) and Exemplification, as proposed by Watson and Mason (2005), have many features in common. They integrate the three essential factors: students as active learners, the teacher as facilitator, and an environment that cater to various ways of thinking and of socializing in mathematics activities. Another similarity of note is that both Investigation and Exemplification cover a wide range of mathematical activities, for example, problem-solving, developing and applying procedures, and introducing students to new concepts. This study focuses on only one type of activity: activities geared towards introducing students to new concepts. The study would have taken a much longer period than the six weeks used in this study if it included other aspects, such as developing and applying procedures and problem-solving. The researcher did not have the luxury of this additional time in the aftermath of Hurricane Maria.

Therefore, this study is only interested in the activities a teacher would expose students to in the course of teaching them the ratios of sine, cosine, and tangent for the first time. With that in mind, this section focuses on the differences between the two teaching approaches—Investigation and Exemplification——concerning introducing students to a new mathematics concept. Based on the features of the two approaches detailed above, this researcher argues that Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) are set apart by the nature of their activities in which students are engaged.

In an investigation, students explore mathematics concepts, principles, and procedures through hands-on activities augmented by communication with their peers and their teacher (Jaworski, 1986). In a classroom Investigation, students are given an open-ended task geared towards helping them find out about a concept and materials to manipulate to investigate the mathematics embedded in that task. In doing so, they are called upon to gather data and make and test conjectures through physical activities such as measuring, recording, calculating, and discussing (Greenes, 1996; Jaworski, 1986; Kwang, 2002; Ponte & Matos, 1992; Sangster, 2012). However, to internalize the concept, mental activities such as identifying patterns, making and modifying conjectures, and generalizing from patterns identified are essential and encouraged (Jaworski, 1986; Ponte & Matos, 1992; Sangster, 2012).

However, in such a classroom, it is easy for students to focus on doing physical activities while neglecting the mental ones (Jaworski, 1986). That is, students might spend most of their time measuring, calculating, recording, and talking with their friends, and little attention is given to conjecturing, identifying patterns, and generalizing—cognitive activities. Therefore, it might appear that students are learning math while they might just be enjoying doing the physical activities that engage them. In essence, Investigation of that nature is highly influenced by physical and social activities, and care must be taken by the teacher to ensure that these activities by themselves do not become the focus of the lesson. That is, the teacher must provide the necessary scaffolding needed to help students engage in all activities, physical, social, and cognitive, in order for the desired learning to take place.

On the other hand, in Exemplification, students explore mathematics concepts, principles, and procedures by generating examples—a mental activity—augmented by communication with their peers and their teacher (Watson & Mason, 2005). In a typical exemplification class, students are given an initial introduction to the concept through a definition, explanation, or activity which may be hands-on or otherwise (Bills et al., 2006; Dahlberg & Housman, 1997; Dinkelman, 2013; Meehan, 2007; Watson & Mason, 2005; Watson & Shipman 2008). After this initial birth into the concept, where they are asked to generate an example, they are given further prompts, which are restricted versions of the initial prompt, to generate more examples. After each generation, the suitability of their examples is discussed with peers and the teacher to help clarify misconceptions. This sequence of activities may persist to include counter-examples and non-examples to ensure that students understand the full scope of the concept through a wide range of examples (Bills et al., 2006; Watson & Mason, 2005).

In Exemplification, the mental activity—generating examples—is the central activity that is augmented with communication and other classroom activities. That is, the principal material learners work with are the examples they generate through mental activities. It is through these mental activities that students assimilate and accommodate (Piaget, 1977) new ideas. By its nature, generating examples requires the direct involvement of the teacher who provides the prompts, restrictions, and who facilitates discussions about examples generated by students. That is, the teacher is directly involved in all stages of Exemplification.

However, while these activities appear to be well suited for developing understanding in students, one must be mindful of cognitive overload (Harrelson & Leaver-Dunn, 2003). According to Harrelson and Leaver-Dunn (2003), cognitive overload happens when a person working memory becomes full, and they can no longer efficiently process new information. They argue that, since the capacity of one's working memory depends on his prior-knowledge and past experiences, a novice learner—a student—will reach cognitive overload before an expert—the teacher. That is, with too much mental activity in a single teaching and learning session, students can suffer from cognitive overload, the result of with is frustration and mental shutdown (Harrelson & Leaver-Dunn, 2003).

Hence, Investigation and Exemplification are different in three major ways: by the nature of the principal materials that students work with, by the necessary involvement of the teacher in learning activities, and by the problems inherent to the activities in which students are engaged. However, having identified these as major differences does not mean that the principal materials identified, the direct involvement of the teacher identified, and the problems identified are exclusive to any one approach. For instance, in an investigation, students might suffer from cognitive overload and students might get distracted from the main focus of the lesson in Exemplification. Also, the teacher's direct involvement in learning activities in an investigation is as essential as in Exemplification if students are to gain a good conceptual understanding of mathematical ideas.

Conceptual Understanding

Several definitions and explanations for conceptual understanding can be found in the literature. Prevalent among them are definitions related to students' representations of a concepts (Heritage & Niemi, 2006; Hiebert and Carpenter, 1992; Hwang et al., 2007; Kilpatrick et al., 2001; Martinie & Bay-Williams, 2003; NCTM, 2000; Niemi, 2001; Panasuk, 2010; Pape & Tchoshanov, 2001; Thompson & Chappell, 2007). Hiebert and Carpenter (1992) described conceptual understanding as a network of mathematical representations wherein the degree of understanding can be assessed by the quantity and quality of connections among these representations. In this definition, Hiebert and Carpenter are drawing attention to both the number and strength of connections that an individual can make among representations as a measure of conceptual understanding.

Barmby et al. (2007) described conceptual understanding as a network of representations, but with a focus on mental activities. They defined conceptual understanding as the resulting network of connections among mental representations of mathematical concepts (Barmby et al., 2007). Barmby et al. definition suggested that conceptual understanding comes about as a result of mental activities that students engage in, therefore, suggesting that the learning of concepts is an active mental process. Dreyfus and Eisenberg (1996) also suggested the learning of concepts as an active mental process. Dreyfus and Eisenberg's idea can be gleaned from their assertion that conceptual understanding is associated with the confident and flexible manipulation of multiple representations of a mathematical concept (Dreyfus & Eisenberg, 1996).

Conceptual understanding is also defined in terms of relationships among mathematical concepts. Byrnes (1992) defined conceptual understanding as a set of linkages that can be formed as a result of the relationships among concepts in a domain. To form linkages leading to an accurate network of relationships, students must have a firm grasp of the underpinnings of each concept. To have a firm grasp, students must also have a good understanding of these relationships. This idea is emphasized in Panasuk (2011), who explains students' conceptual understanding as their ability to recognize beneficial relationships among related concepts. Panasuk exemplified this in terms of students' ability to identify functional relationships amongst various concepts in algebra and to distinguish and interpret amongst their different representations. Like Byrnes (1992) did before them, Rittle-Johnson et al. (2001) also stressed the essence of conceptual understanding within a mathematics domain. They defined conceptual understanding as the working knowledge of the principles by which a mathematical domain is governed (Rittle-Johnson et al., 2001). Unlike Byrnes (1992), whose definition focused on relationships among concepts, the focus of Rittle-Johnson et al. (2001) is on the principles—rules and codes—through which these relationships can be found.

Kilpatrick et al. (2001) have espoused an explanation of conceptual understanding that subsumes most of the definitions and explanations put forth in the preceding paragraphs. They argued that when students have attained conceptual understanding, they can integrate relevant mathematical ideas as a functional system wherein they have a sound and usable "comprehension of mathematical concepts, operations, and relations" (p. 116). Kilpatrick et al. (2001) further advocated that students with conceptual understanding can use multiple representations of mathematical concepts to communicate their knowledge of these concepts. The use of such representations includes, but is not limited to, identifying and connecting different representations depicting the same idea, selecting representations that are suitable for a situation, and explaining the similarities and differences among representations of connected concepts (Kilpatrick et al., 2001). To achieve these actions with representations, students must be actively and mentally engaged and must have a clear understanding of the relationships and principles which govern the relevant concepts.

Benefits of conceptual understanding.

There are several learning benefits to be derived by students as a result of attaining a conceptual understanding of mathematical ideas. Hiebert and Carpenter (1992) had much to say about such benefits. Hiebert and Carpenter (1992) was seminal in the study of conceptual understanding in mathematics. They drew from studies in cognitive science to investigate the understanding of mathematical ideas (Hiebert and Carpenter, 1992). Hiebert and Carpenter (1992) presented five assertions about understanding. (1) “Understanding Is Generative” (p. 74): Students who had attained understanding of a concept were more likely to develop procedures to solve problems in unfamiliar contexts. (2) “Understanding Promotes Remembering” (p. 74): Students who understand a concept structure the information about it and impose meaning on it to make the information easier to remember. (3) “Understanding Reduces the Amount that Must Be Remembered” (P. 75): Understanding helps students to connect information into a structured internal network that reduces the number of individual pieces of information to be remembered and retrieved. (4) “Understanding Enhances Transfer” (P. 75): With understanding, students are better able to use what they learn in one context to solve problems in another context. (5) “Understanding Influences Beliefs” (P. 77): With understanding, students are more likely to believe in the usefulness of mathematics. The work of Hiebert and Carpenter (1992) influenced many studies on conceptual understanding, and several of these studies are discussed in this section.

According to Hiebert and Carpenter (1992), students are more likely to develop procedures to solve novel mathematics problems when they attain a conceptual understanding of the relevant mathematics ideas. Hiebert and Carpenter (1992) reported on the work with fourth-grade students who were asked to connect decimal fraction numerals with physical representations of decimal quantities. Drawing on work within cognitive science, Hiebert and Carpenter further qualify this by arguing that mathematical ideas must be represented and connected internally, in the minds of students, so that students can act on them mentally, thus, producing and enacting meaningful ways to operate on these ideas.

This notion is supported by both Kilpatrick et al. (2001) and Schneider and Stern (2012). Schneider and Stern claimed that conceptual understanding helps students determine the most appropriate procedure to solve a problem in each situation. This claim is critical because problem-solving is an important feature of doing mathematics (NCTM, 2000). By definition, a mathematics problem represents a novel situation for which the intended group of students has no known procedure for solving it (NCTM, 2000). Kilpatrick et al. (2001) argued that "when students have acquired conceptual understanding in an area of mathematics, they see the connections among concepts and procedures" (p. 119), which allows them to solve unfamiliar problems.

Conceptual understanding also promotes long-term memory. Memory is essential if students are to develop fluency and flexibility in doing mathematics because it allows students to recall relevant facts when they are needed (Kilpatrick et al., 2001). Hiebert and Carpenter (1992) argue that teaching for conceptual understanding is a more efficient way of promoting memory in mathematics over rote memorization because of the way conceptual understanding allows students to integrate new information into a network of related information. That is, conceptual understanding enhances students' memory because it often structures new information "in such a way as to impose some meaning on it; by doing so, [it] often modif[ies] the information" (Hiebert & Carpenter, 1992**,** p. 74) to be remembered. According to Hiebert and Carpenter

(1992), connecting new information to a network also reduces the cognitive load or the amount of information to be remembered.

This idea is supported by Laswadi et al. (2016), who also argue that conceptual understanding allows students to connect the information to be learned to a network, thus, reducing the amount of information to be remembered. Laswadi et al., like Hiebert and Carpenter (1992), asserted that a well-structured network reduces the number of individual pieces of information that must be retrieved while students are engaged in doing mathematics. That is, connections forged through conceptual understanding help students to recall, use, manipulate, and transfer mathematics ideas and principles when needed.

Transfer is critical to problem-solving. Conceptual understanding helps students to transfer mathematical ideas and principles during problem-solving activities (Hiebert and Carpenter, 1992). Transfer refers to the ability of students to use mathematical facts, principles, and procedures learned in one context in other, sometimes unfamiliar contexts (Hiebert & Carpenter, 1992). According to Hiebert and Carpenter (1992), because of transfer, students can "improve their performance on some problems by learning to solve related ones" (p. 75). Miller and Hudson (2006) also support this notion of transfer. They argued that students who have attained conceptual understanding during their school years are usually more successful in using mathematics in postsecondary settings such as their workplace, colleges and universities, and even their daily lives (Miller & Hudson, 2006). It is also the case that individuals with a deep conceptual understanding also tend to hold positive beliefs about mathematics (Hiebert & Carpenter, 1992; Garofalo, 1989); thus, it can be argued that conceptual understanding has an impact on students' beliefs about mathematics (Hiebert & Carpenter, 1992).

NCTM (2000) claims that students who obtain a conceptual understanding of mathematical ideas are more likely to show flexibility in problem-solving. NCTM draws from the work of prominent scholars in the field of mathematics education such as Kilpatrick et al. (2001), Hiebert and Carpenter (1992), and Stein et al. (2009) to name a few. Further, a critical aspect of problem-solving is the ability to check the reasonableness of a solution. With conceptual understanding, students can achieve this goal. This assertion is in line with Garofalo and Lester (1985), who asserted that conceptual understanding could help students to check whether the solution to a problem is reasonable. Such a skill is important because it allows students to determine, without lengthy procedures, whether they should continue with a given strategy or change strategies during problem-solving activities. This idea was exemplified in a recent study by Gultepe et al. (2013), who concluded that conceptual understanding has a profound effect on the way students use algorithms to solve problems. Gultepe et al. investigated the "effects of students' conceptual understanding of chemical concepts and mathematical processing skills on algorithmic problem-solving skills" (p. 106).

Other studies (Hidayat & Iksan, 2015; Laswadi et al., 2016) have indicated that conceptual understanding has a positive influence on students' achievement. Achievement, in this case, refers to student performance on tests and other assessment tasks. Laswadi et al. (2016) attribute this influence on the ability of students who have attained conceptual understanding in a domain to "organize their knowledge and explain it as a coherent system" (p. 68). Laswadi et al. investigated, through a quasi-experimental design, the effects that model-facilitated learning versus traditional learning had on students' conceptual understanding. In their experiment, Laswadi et al. (2016) used a rubric to assign scores to students' representations. Laswadi et al. scored students' efforts in connecting several representations of the same mathematical concepts, explaining the similarities and differences among related concepts, and identifying the most appropriate representation for specific situations as proposed by Kilpatrick et al. (2001). Hidayat and Iksan (2015) also concluded that conceptual understanding has a strong connection with a student's mathematics achievement. Using a quasi-experimental design in which a Pearson correlational analysis was used to compare pre-test and post-test scores, Hidayat and Iksan concluded that there was a "significant relationship between conceptual understanding and achievement" (p. 2441).

In summary, conceptual understanding is a critical factor in students learning of mathematics. In that, it helps students develop procedural fluency (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; Schneider & Stern, 2012), promotes long-term memory (Hiebert & Carpenter, 1992; Laswadi et al., 2016), helps in problem solving (Garofalo & Lester, 1985; Gultepe et al., 2013; Miller & Hudson, 2006; NCTM, 2000), improves achievement in mathematics (Hidayat & Iksan, 2015; Laswadi et al., 2016), and positively impacts students' beliefs about mathematics (Garofalo, 1989; Hiebert & Carpenter, 1992).

Evaluating conceptual understanding.

Conceptual understanding is multifaceted and can be manifested in many ways. In the school setting, it can be seen in the way students work with representations of a concept (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; NTCM, 2000), in the way they identify and apply suitable procedures (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; Schneider & Stern, 2012), in their achievement scores (Hidayat & Iksan, 2015; Laswadi et al., 2016), and in their work on problem-solving task (Garofalo & Lester, 1985; Gultepe et al., 2013; Miller & Hudson, 2006; NCTM, 2000). However, problem-solving requires skills developed over time (NTCM, 2000; Stein et al., 2009). Therefore, a student's inability to solve a problem could be a result of their lack of problem-solving skills and not a lack of conceptual understanding. In this regard, this study did not evaluate students' conceptual understanding through problem-solving activities because the researcher did not know how proficient the participants were in problem-solving, nor did he had the time to develop students' problem-solving skills during the study.

Students' representations.

Hiebert and Carpenter (1992) recognized representation as a means by which students acquire and demonstrate mathematical understanding. In the words of Hiebert and Carpenter (1992), "The way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally" (p. 66). This acknowledgment highlighted the dual nature of representation: as a tool to help students organize their learning of a concept and as a means to gain an insight into a student's understanding of a concept. It is the assessment aspect of representation that is of most significance in the context of this study.

Besides Hiebert and Carpenter (1992), several other studies (Heritage & Niemi, 2006; Hiebert and Carpenter, 1992; Hwang et al., 2007; Kilpatrick et al., 2001; Martinie & Bay-Williams, 2003; NCTM, 2000; Niemi, 2001; Panasuk, 2010; Pape & Tchoshanov, 2001; Thompson & Chappell, 2007) advocated for the use of representations to develop and evaluate students' conceptual understanding. The studies listed all drew from the work of Hiebert & Carpenter (1992).

Heritage and Niemi (2006) advocated for the use of students' external representations such as "symbols, diagrams, maps, pictures, and language" (p. 266) to demonstrate mathematical understanding. They qualify this idea by stating that "representations are the indispensable medium through which students demonstrate whether and how they understand the ideas that have been introduced" (p. 267). Heritage and Niemi (2006) worked with 250 fifth-grade students whose teachers analyzed their representations of fractional concepts and made inferences about their conceptual understanding. These investigators found that advanced representational knowledge corresponded positively with better performance on problem-solving tasks. Similar inferences were made by Pape and Tchoshanov (2001), who stated: “Representations refer to the act of externalizing an internal mental abstraction” (p. 119). That is, a student with a superior conceptual understanding of a mathematical idea is likely to produce better representations of that idea over students with a weak conceptual understanding of the idea.

According to Panasuk (2010), students with a conceptual understanding of a mathematical domain can “discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations, and distinguish between and interpret different representations” (p. 237). This is in keeping with Kilpatrick et al. (2001) who asserted that students who have attained a conceptual understanding of a set of mathematics ideas should be able to distinguish among and interpret visual representations of these ideas, explain the similarities and differences among these representations, and produce representations: graphs, tables, diagrams, expositions, to name a few, of these concepts. Kilpatrick et al. point to the use of three indicators from which students’ conceptual understanding of mathematics can be evaluated.

On the other hand, Thompson and Chappell (2007) call for students to “explain their thinking, write their own problem, [and] compare and contrast mathematical concepts” (p. 186). Their ideas also call for multiple representations. However, unlike Kilpatrick et al. (2001), who advocate for both visual and language-based representation, Thompson & Chappell's work appears to focus on representations that are mainly language-based. This focus may be problematic because students who have problems in reading and writing may have difficulties in communicating their understanding. However, of critical importance in Thompson and

Chappell's work is the explicit call for the use of multiple representations, as is the case with Panasuk (2010) and Kilpatrick et al. (2001). In their call, Thompson and Chappell argued that "students who can graph an equation but who cannot develop a table of values for the equation or identify a real context in which such an equation would be used" have only a "limited understanding of equations" (Thompson & Chappell, 2007, pp. 193-194).

This argument is in line with Niemi (2001), Martinie and Bay-Williams (2003), and Hwang et al. (2007), who all concluded that superior representational knowledge is a viable measure of students' conceptual understanding. As an added measure, however, Niemi (2001) recommended that "assessment items should include incorrect representations and commonly misunderstood representations" (p. 360). This recommendation is important because these types of items can help assessors determine gaps in students' understanding, while their absence could leave assessors with a misguided notion that full understanding has been achieved (Niemi, 2001). The importance of this recommendation was highlighted by Martinie and Bay-Williams (2003), who reported that their assessment instrument, which used several representations, showed that students might appear to understand a concept using some representations while they still lacked a profound conceptual understanding of the concept.

The studies reviewed in this section showed that students' use of multiple representations reflects their conceptual understanding of mathematics. One such study was Heritage and Niemi (2006). Heritage and Niemi investigated 250 fifth grade students' use of symbols, diagrams, maps, pictures, and language as a measure of their conceptual understanding of rational numbers. The study focused on students' conceptual understanding, learning gaps, and possible misconceptions. Heritage and Niemi (2006) concluded that students with high-quality representations had better conceptual understanding and performed better on problem-solving tasks than students with lower quality representations. As a caveat, however, Heritage and Niemi argued that proper assessment of students' conceptual understanding from their representations needs assessors who have the adequate content knowledge to be able to identify when learners have achieved what is intended, when they have not, and why. This assertion is in keeping with Pirie's (1998) declaration that the main function of mathematical representations is to communicate mathematical ideas. Through such communication, it becomes possible to gain insights into students' thinking; hence, their understanding of mathematical concepts and the principles that govern the manipulation of these concepts within a domain.

The studies mentioned in this section, drew from the work of Hiebert and Carpenter (1992). As stated earlier, Hiebert and Carpenter (1992) described conceptual understanding as a network of mathematical representations wherein the degree of understanding can be assessed by the quantity and quality of connections among these representations. Kilpatrick et al. (2001) highlighted three qualitative descriptors—students to distinguish among and interpret visual representations of mathematical concepts, students to explain the similarities and differences among representations of mathematical concepts, and students to produce representations of mathematical concepts—that this study used. This study also drew on Hiebert and Carpenter's assertions that the number of connections can assess conceptual understanding. These factors were integrated into an assessment rubric detailed in chapter three.

Assessment rubrics.

The use of rubrics as assessment tools is widespread in the mathematics education research community. The literature is full of examples where a rubric of some kind is used to assess different types of information collected from research participants. For example, Meier, Rich, and Cady (2006) used a rubric to assess students' work on problem-solving tasks; Wheland et al. (2009) used one to assess students' presentations on mathematical concepts, and Boote

(2014) used one to assess students' views collected through interviews. Therefore, the question is not whether rubrics can be used to assess students' mathematical representations, but rather, how are rubrics best employed, what type is most useful to evaluate students' representations, and what are the benefits and challenges of using a rubric.

Danielson and Marquez (2016) described a rubric as "a guide for evaluating performance" (p. 43). It is useful to note the use of the word 'guide' in this description. It reflects the importance of professional judgment by teachers and researchers while using a rubric and recognizes that it is difficult, if not impossible, for a rubric to cover every situation that may arise during the grading process (Thompson & Senk, 1998). A fuller description of a rubric is provided in the literature. Smit and Birri (2014) and Jonsson and Svingby (2007) both described a rubric as a tool used to assess students' work that is complex and authentic. This description indicates the importance of teachers' and researchers' knowledge and skills in using rubrics as with any other tool. That is, care must be taken in the development and subsequent use of a rubric.

When developing a rubric, one must be conscious that both technical and professional issues must be addressed (Danielson & Marquez, 2016). Issues such as validity and reliability are critical in research studies, and researchers must attend to them. The validity of a rubric is ascertained by whether it measures what it is intended to measure. Danielson and Marquez (2016) stated that "rubrics that are the product of many minds are generally superior to those created by an individual" (p. 50). The argument here is that rubrics that are developed by several experts tend to reflect more perspectives because they are embedded in the professional judgment of more than one person.

Danielson and Marquez (2016) also argued that developing a rubric and assessing students' work with a rubric takes time. Hence, researchers who are pressed for time may avoid developing and using a rubric and rely solely on their professional judgment or that of others. However, Thompson and Senk (1998) asserted that when raters become familiar with a rubric, then the scoring process becomes faster without sacrificing reliability. Furthermore, several studies (Jonsson & Svingby, 2007; Meier et al., 2006; Smit & Birri, 2014) suggest that the use of a rubric improves the reliability of raters scoring of complex tasks.

Reliability refers to the consistency in judging the merits of students' work (Thompson & Senk, 1998), and it is an essential aspect of any form of assessment. Attention to reliability is of particular importance; however, when more than one rater is used to judge students' work (Jonsson & Svingby, 2007). Jonsson and Svingby (2007) reviewed several studies that investigated reliability when more than one rater (inter-rater situations) used a rubric to grade the same work. They concluded that in inter-rater situations, reliability is improved by training raters to use the rubric. One form of training recommended by several researchers (Brown, Glasswell, & Harland, 2004; Danielson and Marquez, 2016; Jonsson and Svingby, 2007) is the grading and discussion of examples drawn from the work to be assessed. Jonsson and Svingby also asserted that if the same item on all papers is graded at the same time before moving on to another question, this tends to increase intra-rater consistency by helping a rater to apply the same standard to all papers (Jonsson & Svingby, 2007).

The type of rubric used in a research study is also important. Two types of rubrics are reported in the literature: holistic and analytic. Holistic rubrics are reportedly better suited for large scale assessment when the intended goal is a single grade (Boote, 2014; Jonsson & Svingby, 2007; Meier et al., 2006; Smit & Birri, 2014). On the other hand, analytic rubrics may help teachers and researchers recognize students' understanding and misconceptions (Jonsson & Svingby, 2007), and help them to provide feedback to these students (Panadero & Jonsson, 2013). The question is: what feature of an analytic rubric allows for such levels of assessment and feedback? The following are three features: defined criteria, categories, and task-specific.

Defined criteria refer to aspects of participants' performance that are simultaneously essential and independent of each other (Danielson & Marquez, 2016). That is, an analytic rubric should comprise the critical constructs that the assessment intends to measure. Moreover, these constructs, in so far as is possible, should not overlap (Danielson & Marquez, 2016; Jonsson & Svingby, 2007; Smit & Birri, 2014). On the other hand, a category refers to a quality indicator in which aspects of students' work can be placed (Boote, 2014; Danielson & Marquez, 2016; Meier et al., 2006). These levels may carry either quantitative or qualitative descriptor, or both (Boote, 2014; Danielson & Marquez, 2016; Jonsson & Svingby, 2007; Meier et al., 2006; Smit & Birri, 2014). When carefully developed and used in conjunction, these features allow for judgments that are clear and unambiguous (Boote, 2014; Danielson & Marquez, 2016; Meier et al., 2006).

Furthermore, the task to be assessed must be kept in focus. That is, the rubric should be task-specific instead of generic (Boote, 2014; Danielson & Marquez, 2016; Jonsson & Svingby, 2007; Meier et al., 2006; Smit & Birri, 2014). A task-specific rubric focuses on mathematical concepts and thinking that students are required to demonstrate (Boote, 2014; Meier et al., 2006; Smit & Birri, 2014). Hence, it allows for a more direct evaluation of students' understanding of these concepts. Furthermore, a task-specific rubric with categories carrying both quantitative and qualitative descriptors allows for the results to be included in students' achievement scores and thematic analysis. However, it is important for users of rubrics to keep in mind that even with the best-developed rubric, there are issues that will arise that might limit the use of the rubric (Boote, 2014; Danielson & Marquez, 2016; Meier et al., 2006; Smit & Birri, 2014).

For instance, some responses may be borderline and do not fall into a specific category (Boote, 2014; Danielson & Marquez, 2016; Meier et al., 2006; Smit & Birri, 2014). This occurrence is not uncommon because it is practically impossible for developers of a rubric to think of and cater to every solution or error that may surface in the course of grading students' work (Boote, 2014; Danielson & Marquez, 2016; Meier et al., 2006; Smit & Birri, 2014). That is, it is possible for unusual solution strategies and uncommon misconceptions to come up. In such instances, raters should draw on their knowledge and understanding of the subject area and trust their professional judgment (Danielson & Marquez, 2016). Hence, it is critical for users of a rubric to be knowledgeable about the content and constructs being evaluated by the rubric (Danielson & Marquez, 2016).

Achievement

Achievement scores derived from testing have pervaded western education systems. This researcher made this assertion because of the many studies (Akinsolaab & Awofalab, 2009; Galeshi, 2014; Harwell et al., 2007; Igbojinwaekwu, 2015; Kaeley, 1998; Mullis et al., 2012; Parke & Keener, 2011; PISA; Riordan & Noyce, 2001; TIMSS) that investigated factors affecting students' achievement scores or used students' achievement scores to investigate impeding and enabling factors (e.g. instructional practices, geographic location, economic condition, and curricula).

Achievement testing is still a driving force in Dominica's education system, and there are no signs that it will disappear anytime soon. Instead, testing bodies like the Caribbean Examination Council (CXC) have become more innovative with its move to have students take examinations over the internet instead of the long-standing paper and pencil approach. Testing to obtain achievement scores, whether using computers or a paper and pencil approach, is commonly referred to as traditional testing or traditional assessment (Ben-Hur, 2006; Danielson & Marquez, 2016; Romberg, 1992; Webb, 1992).

Instruments for these types of tests usually comprise “multiple-choice, true/false, matching–objective type–or short-answer” (Danielson & Marquez, 2016, p. 10) items. Scores obtained from these kinds of tests are commonly used by teachers, researchers, and the wider society to make judgments about students (e.g. TIMSS and PISA). Therefore, these tests must be well developed to enhance their usefulness (Danielson and Marquez, 2016). To enhance their usefulness, Danielson and Marquez (2016) argued that test items must be authentic and must contain a good measure of ambiguity. Danielson and Marquez argued that authentic items allow students “to produce work of good quality” when these items “seem plausible and worthwhile” (p. 13). Hence, items should be developed around practical situations and where possible to reflect students’ interests. Important is the measure of ambiguity used in multiple-choice items. According to Danielson and Marquez (2016), the distractors in multiple-choice items ought to be conceivable and yet unmistakably wrong. That is, the distractors should preferably reflect common and possible misconception students might have developed. Danielson and Marquez claimed that these features would better assist teachers and researchers in identifying errors in students’ knowledge and behaviour. This claim is in keeping with Cronbach’s (1984) description of testing as a systematic procedure for discerning behaviour and presenting it using a numerical scale.

Tests, when carefully designed, are well suited for assessing students’ understanding of content and procedures. It is argued that objective type questions might not be adequate to assess higher-order thinking, such as conceptual understanding (Ben-Hur, 2006; Danielson & Marquez, 2016). This claim is contradictory to that of CXC because the Council claims that their Paper 01, which comprises 60 multiple-choice items, evaluates candidates' knowledge and conceptual understanding of mathematics concepts. Furthermore, achievement scores obtained from such items are used worldwide to make important educational and economic decisions regarding students, educational programs, schools, and societies.

Traditional test scores are used to compare mathematics programs across countries. Well known within the mathematics education research community is the Trends in International Mathematics and Science Study (TIMSS) assessment, where thousands of fourth and eighth-grade students are tested. These test results are used to compare groups of students based on their country's economic development, geographical location, and population size (Mullis et al., 2012). Another assessment program that compares students across regions is the Programme for International Student Assessment (PISA). PISA assesses 15 and 16 years old students in mathematics, science, and reading and rank countries by subjects based on students' achievement scores (Galeshi, 2014). Multiple-choice items are included in both TIMSS and PISA assessments.

The results of these evaluations have become so significant in the mathematics education community that researchers analyze them further and use their findings to make judgments about countries' mathematics programs. One such study was conducted by Galeshi (2014), who investigated the differences in American and Taiwanese students' performance. In that study, the grade eight mathematics TIMSS results for 2007 were used. The analysis revealed a significant difference between American and Taiwanese students' mathematics performance and recommended that American educators review the way mathematics is taught and learned in American schools.

Other studies also used students' achievement scores to compare mathematics programs within the same jurisdiction. Riordan and Noyce (2001) conducted one such study to determine the impact of two mathematics programs in Massachusetts: Everyday Mathematics and Connected Mathematics. Everyday Mathematics refers to an elementary school program implemented in some schools in the Massachusetts district and Connected Mathematics relates to a similar program implemented at the middle school level. Both programs used curricula and pedagogy that were considered different from traditional programs. The study analyzed, statewide, fourth and eighth grades students' standardized mathematics test scores and concluded that fourth-grade students in Massachusetts using Everyday Mathematics and eighth-grade students in schools using Connected Mathematics significantly outperformed their peers from schools "using a mix of traditional programs and curricula" (p. 390).

In another but similar study, Harwell et al. (2007) investigated how standard-based curricula affected high school students' achievement patterns. According to Harwell et al., "standards-based curricula are curricula that were funded from a solicitation of proposals through the National Science Foundation" (p. 71). The investigation of curricula is necessary because research such as Kaeley (1998) showed that curricular items such as textbooks have a significant impact on students' learning. Kaeley argued that this is the case because teachers tend to base their instructional practice on strategies depicted in textbooks. Harwell et al.'s (2007) study analyzed data obtained from two groups of eleventh-grade students' scores obtained on a national achievement test. The study concluded that both urban and boundary suburban schools performed at the national average, but suburban schools that were more affluent performed better than their counterparts. While these studies in no way provide an exhaustive list of studies comparing mathematics programs, they exemplify a significant number of similar studies that are presently available in the literature.

Several studies (Akinsolaab & Awofalab, 2009; Igbojinwaekwu, 2015; Parke & Keener, 2011) have used achievement scores to compare and make judgments about curriculum and pedagogical practices. Parke and Keener (2011) investigated the impacts that mobility, course content, and sequencing have on students' mathematics performance. While mobility is a parental decision, contents to be covered and the sequencing of these contents are curricular and pedagogical decisions that must be taken by teachers and administrators. Parke and Keener used test scores and coursework scores of students, over four years, as the measure to determine if mobility and content decisions had any impact on students' performance. On the other hand, Akinsolaab and Awofalab (2009) investigated the effects of various types of groupings on students' mathematics achievement. Like Parke and Keener (2011), they made use of students' test scores and found significant differences in students' achievement among the various types of groupings. Like the selection and sequencing of content investigated by Parke and Keener, the use of different kinds of groupings in the classroom is a pedagogical decision that teachers are called upon to make regularly. Igbojinwaekwu (2015) also used a similar approach to investigate and compare instructional practices in the classroom. Therefore, it is reasonable to conclude that students' achievement scores on tests that are carefully designed to serve a specific purpose are appropriate measures for comparing the impacts of different instructional practices on students' learning.

Summary

This chapter discussed seven major ideas that underpin this current study. These ideas were the three primary trigonometric ratios—the mathematics content used in the study, the theoretical framework—constructivism—that grounded the teaching aspect of the study, the methodology used in the study to collect and analyze data—mixed-methods, Investigation and Exemplification—two approaches to teaching mathematics used in the study, and conceptual understanding and achievement—two constructs assessed in the study to compare students' learning of the three primary trigonometric ratios.

The mathematics concepts of sine, cosine, and tangent——the three primary trigonometric ratios—were discussed as concepts of trigonometry. The importance of trigonometry in school mathematics was also discussed. The chapter also discussed two approaches—the unit circle approach and the ratio and mnemonic (SOH–CAH–TOA) approach—widely used to teach the sine, cosine, and tangent ratios. Competing evidence for the use of these approaches were presented.

The chapter also discussed the methodology——mixed methods—used in this study. The literature on methodology reviewed and discussed, positioned mixed-methods in the pragmatic paradigm and provided evidence why mixed-methods was a suitable methodology for this study.

The theoretical framework which underpins teaching in this study is constructivism. Several constructivists' theories were presented and discussed, with attention paid to the theories of Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996). A research model that highlighted three critical factors—students as active participants, the teacher as facilitator, and an environment conducing to learning——in a lesson based on constructivism was presented and discussed.

Two sections, which dealt with mathematical investigations (Jaworski, 1986) and Exemplification (Watson and Mason, 2005), laid the foundation for the pedagogical practices that were used in this study. The features of each approach were discussed in the relevant section. Also highlighted were the relevant definitions or explanations of terms unique to the underlying principles of these instructional practices. Moreover, both approaches are portrayed as teaching approaches which reflect constructivism. The effects of both Investigation and Exemplification on students' achievement and conceptual understanding were also highlighted and discussed. Some inherent differences between these two approaches were also discussed.

One section also highlighted several definitions of conceptual understanding, its benefits to students, and assessment practices that can be employed to determine students' conceptual understanding. Attention was paid to the analysis of students' multiple representations as a measure of conceptual understanding. Also discussed was how a rubric could be used to enhance the analysis of students' representation. This section laid the foundation for assessing students' conceptual understanding in this study.

Achievement scores, some features of tests, and how scores obtained from tests were used in the education community were given. It highlighted the fact that achievement scores are used profusely to compare students, instructional programs, classroom pedagogy, and countries' curriculum practices. That is, achievement scores are presented as a viable option for comparing different groups of students. This section laid the foundation for the use of achievement scores to compare two groups of students in this study.

However, what is unclear from the literature is the extent to which Investigation and Exemplification are comparable as tools that can be used to facilitate students' learning of mathematics. This study shed light on the comparability of Exemplification and Investigation in improving students' achievement and developing their conceptual understanding of mathematics at the secondary school level.

Chapter 3: Methodology

The strength of an empirical research study rests on its design and the methods used to collect and analyze data. This study makes use of a mixed-methods design where data were collected and analyzed using both quantitative and qualitative approaches. These approaches and the study design are discussed in detail, paying attention to the following areas. The purpose of the study, research questions, hypotheses, research design, participants, data collection, data collection instruments, data analyses, validity and reliability, limitations, and informed consent.

Purpose of the Study

Most students exiting secondary schools in Dominica take the CSEC mathematics examination, which is a high stake test. Over the years, these students have not been achieving the types of grades that parents, teachers, and other stakeholders would like them to achieve; that is, they have been performing poorly. This researcher, a Dominican mathematics educator, is concerned and looking for ways to improve students' performances on this test. Drawing from a recent study (Charles, 2015) conducted among Dominican secondary teachers, this researcher thinks that the problem may lie in teachers' instructional approaches to teaching mathematics. Charles (2015) found that secondary students in Dominica were not sufficiently exposed to inquiry-based approaches to teaching mathematics.

Therefore, this doctoral research study presents data to show how students' achievement and conceptual understanding of the three primary trigonometric ratios were affected when they were taught using two inquiry-based approaches to teaching mathematics. These approaches are Exemplification, as proposed by Watson and Mason (2005) and Investigation, as suggested by Jaworski (1986). Both approaches are embedded in the constructivist's (Fosnot and Perry, 1996; Piaget, 1977; von Glasersfeld, 1995;Vygotsky, 1978) theory of learning and are expected to improve students' achievement scores and facilitate their development of a conceptual understanding of the three primary trigonometric ratios.

This study provided empirical evidence to show how Exemplification and Investigation affected students' achievement and conceptual understanding of the three primary trigonometric ratios. It also provided empirical evidence to show which of the two approaches to teaching mathematics had a greater effect on students' achievement and conceptual understanding of the three primary trigonometric ratios.

Research Questions

The central aim in collecting and analyzing data in this study is to answer the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* Data collection and analyses were focused on these sub-questions.

1. How was students' level of achievement affected after being taught by Investigation?
2. How was students' level of achievement affected after being taught by Exemplification?
3. How did students' levels of achievement differ after being taught by Investigation compared with Exemplification?
4. How did students' conceptual understanding differ after being taught by Investigation compared with Exemplification?

Data collected to answer sub-questions one, two, and three were analyzed using quantitative means, while data collected to answer sub-question four were analyzed using both quantitative and qualitative approaches.

Hypotheses

The following hypotheses are based on theories and findings found in the literature and this researcher's teaching experience. Several studies (Jaworski, 1986; NCTM, 2000; Pijls et al., 2003; Ponte & Matos, 1992; Sangster, 2012; Staples, 2011) concluded that Investigation facilitates the development of conceptual understanding of mathematical ideas, thus, allowing students to attain higher grades in mathematics achievement tests. Exemplification is also purported in the literature to improve students' achievement in and understanding of mathematical concepts (Dahlberg & Housman, 1997; Dinkelman, 2013; Meehan, 2007; Rawson & Dunlosky, 2016; Sandefur et al., 2012; Watson & Shipman, 2008). Thus, it is reasonable to expect that these two teaching approaches will improve students' achievement and conceptual understanding of the three primary trigonometric ratios.

However, this researcher supposes that Investigation and Exemplification have different effects on students' achievement and conceptual understanding of mathematics. This supposition was made because of the differences between these two teaching approaches: (a) the nature of the principal materials that students work with, (b) the necessary involvement of the teacher in learning activities, and (c) the problems inherent to the activities in which students are engaged. These differences were discussed in chapter two.

For instance, this researcher believes that the difference in "the necessary involvement of the teacher" can offset the balance in the three factors—students as active learners, the teacher as facilitator, and a conducive environment—that he believes are equally important in a constructivist approach to teaching. Several studies (Baumert et al., 2010; Even, 1993; Gales and Yan, 2001; Hill, Rowan, & Ball, 2005; Watson & Harel, 2013) showed that a teacher's knowledge and classroom actions are key factors in students' learning. Thus, the amount and type of teacher involvement are likely to impact the amount and type of learning that students experience, regardless of the teaching approach.

By its nature, Exemplification caters to the involvement of the teacher with all students at every stage of the lesson, while students remain meaningfully engaged. Meaningfully engaged in this context means that students are actively engaged in both cognitive activities and social interaction with their peers and the teacher. That is, the teacher leads every sequence of students' example generation and facilitate discussions during these generation sequences in Exemplification (Watson & Mason, 2005).

On the other hand, it is difficult for a teacher to facilitate the cognitive and social activities of all students during an Investigation, especially one involving small group activities. Dynamics within groups (e.g. synergy, students' interest and personality, and the amount of assistance needed) will dictate, to a large extent, the amount of time spent with each group or an individual. As the teacher works with one group, others are normally left unattended. That is, a teacher's input can be limited for some students during an Investigation. This imbalance in teacher facilitation could lead to significant differences in some students' learning.

Therefore, this researcher posited the following hypotheses:

1. In relation to sub-question 1: How was students' level of achievement affected after being taught by Investigation?

   Null hypothesis: There was no significant difference between the pre-test achievement and post-test achievement for students taught by Investigation.

   Alternate hypothesis: Post-test achievement is statistically significantly higher than pre-test achievement for students taught by Investigation.

2. In relation to sub-question 2: How was students' level of achievement affected after being taught by Exemplification?

Null hypothesis: There is no significant difference between the pre-test achievement and post-test achievement for students taught by Exemplification.

Alternate hypothesis: Post-test achievement is significantly higher than pre-test achievement for students taught by Exemplification.

3. In relation to sub-question 3: How did students' levels of achievement differ after being taught by Investigation compared with Exemplification?

   Null hypothesis: There is no significant difference in the level of achievement for students taught by Investigation compared to those taught by Exemplification.

   Alternate hypothesis: The level of achievement of students taught by Exemplification is significantly higher than the level of achievement of students taught by Investigation.

No hypothesis was posited for sub-question four: How did students' conceptual understanding differ after being taught using Investigation compared with Exemplification? The question asked how students' conceptual understanding differed across groups. Conceptual understanding was measured by students' appropriate production and use of multiple representations in this study. It was impossible to predict what representations students would produce and used, and the errors they would make in their production and use of representations.

Research Design

A two-group, pre-test–post-test, independent measures experimental design was used in this study. A convenient sample of two classes of fourth form students from one secondary school in Dominica were the participants. The sample was convenient because these students were the most accessible to the researcher in the aftermath of Hurricane Maria. The two classes accounted for 35 students. These two classes were mixed, and students were randomly assigned to two groups to account for differences that might have existed between the two classes (Gamst, Meyers, & Guarino, 2008). For the random assignment, numbers from 1 to 35 were written on pieces of paper, the papers were folded and placed in a cup, and students were asked to take one piece of paper. The 17 students who selected even numbers formed one group, and the 18 students with odd numbers formed another group.

The researcher taught the three primary trigonometric ratios to both groups of students in successive three-weeks periods. He taught the first group using Exemplification as proposed by Watson and Mason (2005). At the same time, another teacher taught algebra to the other group in a separate classroom. At the end of the trigonometry unit, groups were swapped to allow the researcher to teach the three primary trigonometric ratios to the second group using Investigation (Jaworski, 1986), while the other teacher taught algebra to the group the researcher taught first.

The researcher chose to teach both groups of students for several reasons. First, it controls the impact of the teacher as being a major cause of any difference between the groups' performances. Several studies (Baumert et al., 2010; Even, 1993; Gales and Yan, 2001; Hill, Rowan, & Ball, 2005; Watson & Harel, 2013) showed that a classroom teacher is a major factor in students' achievement. Hence, this study aims to reduce this teacher-factor. Second, it negated the need to spend time and resources training teachers to use Exemplification and Investigation. Finally, there was no need to have the teachers directly involved in the teaching and learning process because they were not the focus of the study, and their inclusion could bias the results given their prior history with the students.

A pre-test was given to each group of students immediately before any teaching of the three primary trigonometric ratios took place for that group. That is, the pre-test was given to the group of students taught by Exemplification during the first session that the researcher had with them. The completed pre-test papers for that group of students were enveloped, seal, and put away in a locked cupboard without being looked at or graded. During the last session with the

Exemplification group, immediately after all teaching had taken place, a post-test, which parallel the pre-test, was administered to the group. The completed post-test papers were also enveloped, seal, and put away in the locked cupboard without being looked at or graded. This same process—pre-test–teaching–post-test—was repeated three weeks later, after work was completed with the group taught by Exemplification, for the Investigation group before any test papers were graded.

A critical issue was considered in administering these pre-tests and post-tests. The pre-tests were given just before the teaching for both groups to ensure that it revealed students' achievement and conceptual understanding right before teaching. This action increased the chance that the level of achievement and conceptual understanding revealed by the post-test came from the teaching approach to which the researcher exposed students. For the same reason, the post-tests were given immediately after teaching. Testing immediately after teaching reduced the chance that others (teachers and other students) helped participants make sense of the content taught using other techniques. Such interference would distort (not a problem otherwise) the results.

A two-way (with one within-subjects factor and one between-groups factor), 2 x 2 mixed-factorial ANOVA was used to analyze students' achievement scores and answer the research questions related to differences in students' levels of achievement (sub-questions one, two, and three). Mixed methods were used to analyze students' conceptual understanding. The researcher conducted a quantitative analysis for the number of written responses students gave on the post-test and a quantitative analysis of the scores that students received for those written responses. No student, from either group, answered the prompts that solicited written responses on the pre-test. This lack of answers was likely due to the students' unfamiliarity with the concepts in these questions. Hence, no analysis was done that included that section of the pre-test. He also conducted a quantitative analysis of students' correct answers to multiple-choice items on both the pre-test and post-test. The researcher combined these quantitative analyses with the groups' profiles derived from a qualitative analysis of students' written responses on the post-test to answer the research sub-question four: How did students' conceptual understanding differ after being taught using Investigation compared with Exemplification?

All quantitative data analysis results are presented in Chapter four, and the result of the qualitative analysis is presented in Chapter five. Chapter six presents the mixing and interpretation of the quantitative and qualitative results related to conceptual understanding and discusses all results in relation to the present literature in the field of mathematics education. The researcher gave priority to the qualitative data when mixing results to answer the research question concerning differences in students' conceptual understanding (sub-question four).

Priority refers to the weight given to each, quantitative and qualitative, type of data collected and analyzed during a mixed-methods design (Creswell et al. 2003). According to Creswell (2013), priority is given to data types depending on the goals of the study, the scope of the research question(s), and the intention of each data type—qualitative and quantitative. In this study, the results of quantitative analyses were used to support the results of the qualitative analyses to answer sub-question four: How did students' conceptual understanding differ after being taught using Investigation compared with Exemplification? *Figure 3.1* presents a visual representation of the research design.

Visual representation.

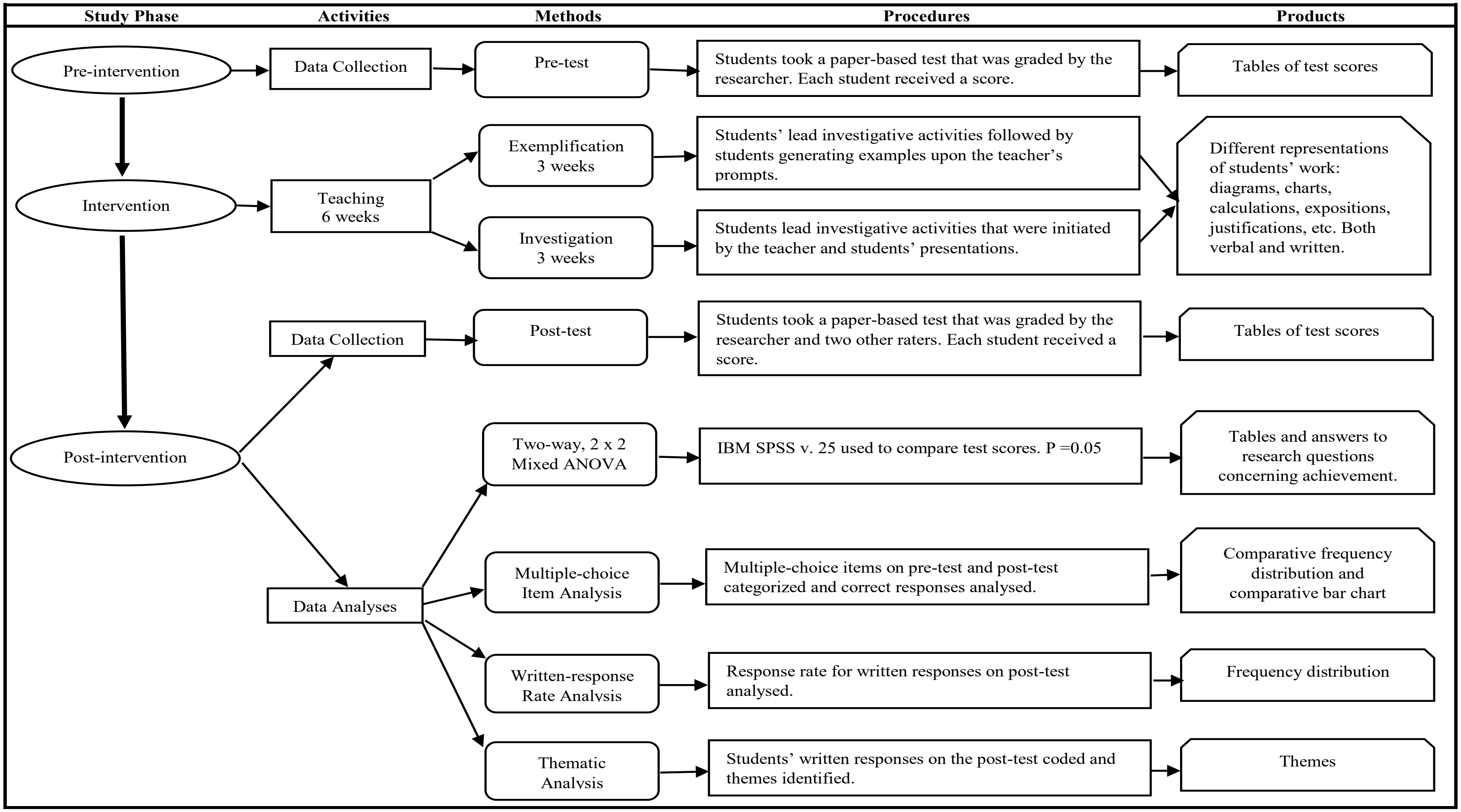


Figure 3.1: A diagram showing the research design used in this study. It is an experimental design. Depicted are the phases, data collection and analysis activities, methods employed, procedures used, and the resulting products.

Participants

The participants were fourth-form students from a secondary school in Dominica. In all schools in Dominica, fourth-form students are actively pursuing the CSEC mathematics syllabus. Hence, these students were preparing to take the CSEC mathematics examination in May/June of 2019. The sample consisted of 35 participants: 17 females and 18 males, with ages ranging from 15 years to 18 years. Participants were introduced to the study via an introductory letter (Appendix C) given to them one month before the study began. The letter outlined the procedures of the study, the consent and assent forms needed (Appendix D) and explained how confidentiality would be ensured. This sample of students represented a portion of the population of Dominican students who took the CSEC mathematics examination in May/June of 2019. At the time of this study, there were approximately 1200 students who fit this criterion in Dominica. The findings and conclusions of this study may be extrapolated to this population.

The researcher used convenience sampling (Mertens, 2010; Patton, 2002; Tashakkori & Teddlie, 1998) to identify and select participants. Given that several schools were not open for months after Hurricane Maria, the researcher had to carry out his research in an available school. The school chosen was the nearest to him that was operational, and its principal was willing to work with the researcher. He decided to conduct the entire research at that school. Based on the content to be taught (the three primary trigonometric ratios), the fourth form was identified as the relevant form level, and two teachers volunteered their classes. Students' assent and parental consent were obtained through assent/consent forms.

Gifted students and students identified as having learning difficulties were excluded from the study. That is, given the common practice of streaming in Dominican secondary schools, students of four-one, top performer, and students of four-four, low performers, were excluded. The selection of students with average performance abilities was essential to this research study because the researcher is aware that gifted students find ways to learn content regardless of how it is presented to them. On the other hand, students with learning challenges may need special attention. This study did not cater to either of these situations; thus, they were avoided.

A comparison of Exemplification and Investigation

This section discusses the similarities and differences between teaching by Exemplification and Investigation that took place in this study. The comparison focuses on the three essential factors in a lesson based on constructivism: the students as active learners, the teacher as facilitator, and the classroom as an environment conducive to learning. It also drew on the theoretical differences between Exemplification and Investigation identified by the researcher in Chapter two: the nature of the principal materials that students work with, the necessary involvement of the teacher in learning activities, and the problems inherent to the activities in which students are engaged. *Figure 3.2* summarises the involvment of the teacher-researcher and the students during the teaching sessions for both groups of students.

Factor of Constructivism	Teaching Methods	
	Investigation	Exemplification
Students as active learners	• Collected data through measuring and calculating • Made conjectures • Tested and modified conjectures • Participated in small-group and whole-class discussions • Identified patterns and arrive at generalizations.	• Collected data through measuring and calculating • Generated examples • Discussed examples • Participated in small-group and whole-class discussions • Identified patterns and arrive at generalizations.
The teacher as a facilitator	• Prepared materials for students' classroom activities • Helped students to make, test, and modify conjectures	• Prepared mathematical prompts

	• Facilitated discussions • Provided feedback to students • Demonstrated procedures • Provided scaffolding	• Prompted students to generate examples of mathematics concepts • Facilitated discussions • Provided feedback to students • Demonstrated procedures • Provided scaffolding
The classroom as an environment conducive to learning	Both groups of students cooperated with the teacher-researcher in making the classroom environment a safe space for learning.	

Figure 3.2: Table summarizing the involvement of the teacher and students during teaching.

The following are some similarities and differences between the two teaching approaches that are evident in *Figure 3.2*, and some justifications for their perceived differences.

Students as active learners: In Exemplification sessions, students were actively involved in generating examples and in discussing why they knew their examples met the necessary criteria set by the given prompts. These students were also engaged in small groups and teacher-led whole-class discussions. In Investigation sessions, students were actively involved in measuring, calculating, conjecturing, and testing and modifying their conjectures. They were also engaged in small groups and teacher-led whole-class discussions.

The teacher as a facilitator: The researcher-teacher prepared and delivered all prompts and facilitated discussions between prompts in Exemplification sessions. He also facilitated whole-class discussions to help students taught by Exemplification to arrive at general statements about the three primary trigonometric ratios. In Investigation sessions, the researcher-teacher prepared the materials students used to investigate the three primary trigonometric ratios, assisted students in their conjecturing and testing efforts, and facilitated whole-class discussions to help students arrive at general statements about the three primary trigonometric ratios. In both approaches, the researcher-teacher provided feedback to students in both small groups and whole-class settings.

The classroom as an environment conducive to learning: The researcher-teacher and students worked towards making the classroom environment a safe space for learning in both Exemplification and Investigation teaching sessions. From the onset, the researcher-teacher told students what was expected of them in terms of classroom behaviour. Students were asked to respect the views of others, never laugh at anyone's attempt to answer a question, or when they suggest an answer to a question. They were told that is was okay to make mistakes because that is how people learn. The students, as far as the researcher-teacher could tell, respected these guidelines. The researcher-teacher endeavoured to create a friendly and relaxed classroom atmosphere by entertaining students jokes while keeping them focus on the work, by attending to all students' suggestions, and by inviting students to approach him for clarification during and after class.

However, there were differences in two of the three factors, the students as active learners and the teacher as facilitator. This researcher attributes these differences (discussed below) to the differences in the nature of the principal materials that students work with and the necessary involvement of the teacher in learning activities in the two teaching approaches.

The principal material used in Exemplification sessions were the examples students generated. These examples were the results of students' cognitive processes. These students were also involved in teacher-led discussions that required them to use cognitive processes. Hence, students were cognitively engaged throughout Exemplification sessions. On the other hand, the principal materials used in Investigation sessions were measuring instruments and drawings prepared by the researcher-teacher. Using these instruments was of low cognitive demand (Stein et al., 2009) because students had already mastered the requisite skills. The conjecturing, the testing and modifying, and the discussions in which students were engaged used cognitive processes. However, the cognitive demand of these processes was reduced (Stein et al., 2009) by the necessary involvement of the teacher.

The necessary involvement of the teacher in the Exemplification sessions was to prompt students to generate examples and to facilitate discussions around those examples. Each subsequent prompt in a sequence increased the cognitive demand of the task (generating examples) in which students were engaged. That is, students were forced to dig deeper to come up with suitable examples (Watson & Mason, 2005). On the other hand, the necessary involvement of the teacher in Investigation sessions was to help students make conjectures and test and modify them. Assisting students to make, test, and modify conjectures was essential in this study because these students were not accustomed to making, testing, and modifying conjectures. However, providing students with this help reduce the cognitive demand of the tasks during an investigation (Stein et al., 2009). That is, helping students make conjectures reduced the amount of thinking that these students had to do to come up with suitable conjectures, which were tested and modified to identify patterns that led to general statements about the three primary trigonometric ratios. The possible effects of these differences on students' achievement and conceptual understanding of the three primary trigonometric ratios are discussed in Chapter six. The next section details the teacher’s and students’ involvment during each teaching and learning session.

Teaching

Teaching lasted approximately six weeks, 16 teaching sessions, with each group involved for approximately three weeks, eight teaching sessions. Each teaching session lasted 90 minutes, and there were three such sessions of mathematics every week. Out of the eight sessions per group, the first and last sessions were used for data collection. The other six sessions were used for teaching. The researcher used 85 minutes of teaching time for each session, allowing the first five minutes for the class to settle down. All 16 sessions were held in the same classroom.

For the remainder of this section, all sessions referred to are those in which teaching took place. One group of students were taught using Exemplification for three weeks before the other group of students were taught using Investigation. This move ensured that the researcher focused on only one teaching approach at a time. Focusing on only one teaching approach at a time was critical to ensure that there was no mixing of the two approaches (Exemplification and Investigation). Such mixing would have confounded the results of the post-test for each group. That is, the post-test scores could not be attributed to the effects of one teaching approach. *Figure 3.3* below illustrates the sequencing of the six teaching sessions for both Exemplification and Investigation.

	Content Focus	
Sessions	**Exemplification (weeks 1, 2 & 3)**	**Investigation (weeks 4, 5 & 6)**
1. Same for both groups but taught at different time.	Review of right-angle triangles What makes a triangle right-angled? Hypotenuse, opposite, adjacent Does orientation maters?	
2	Measuring sides and angles - Measure and record sides and angles - Complete table - Draw graphs	Sine ratio - Measure and record sides and angles - Complete table - Draw graphs - Presentations
3	The Sine Ratio Students produced examples of different representations and angles.	Cosine ratio - Measure and record sides and angles - Complete table - Draw graphs - Presentations
4	The Cosine Ratio Students produced examples of different representations and angles.	Tangent ratio - Measure and record sides and angles - Complete table - Draw graphs - Presentations

5	The Tangent Ratio Students produced examples of different representations and angles.	Identifying similarities and differences - Formulas - Tables - Graphs - Presentations
6. Same for both groups but taught at different time.	Solving right angle triangle, angle of elevation, angle of depression Identifying appropriate ratios, finding missing sides, finding missing angles.	

Figure 3.3: A table showing the sequences of teaching sessions for both groups of students.

For exemplification, the concept of the right-angle triangle and other related concepts (e.g. hypotenuse, opposite side, adjacent side, reference angle) were reviewed in session one. Students were introduced to the sine, cosine, and tangent ratios through an inquiry-based activity in session two. Students generated and discussed examples of different representations (e.g. diagram, angle, table, graph) of the sine ratio in session three, the cosine ratio in session four, and the tangent ratio in session five. In session six, students solved problems that made use of the sine, cosine, and tangent ratios (finding missing sides or angles in right triangles and problems dealing with angles of elevation and depression). Details of each session are given under the relevant section below.

For Investigation, the concept of the right-angle triangle and other related concepts (e.g. hypotenuse, opposite side, adjacent side, reference angle) were reviewed in session one. Students investigated the sine ratio in session two, the cosine ratio in session three, and the tangent ratio in session four. Students compared representations of the sine, cosine, and tangent ratios in session five. In session six, students solved problems that made use of the sine, cosine, and tangent ratios (finding missing sides or angles in right triangles and problems dealing with angles of elevation and depression). Details of each session are given under the relevant section below.

Teaching activities in session one and session six were the same for both groups of students; therefore, activities in session one and session six are discussed as separate sections under teaching, instead of repeating them under respective sections for Exemplification and

Investigation. Session one is presented before discussions for sessions related to the two teaching approaches, and session six is presented after all discussions related to the two teaching approaches (Exemplification and Investigation) are completed.

Session one: Common to both groups.

Students were assigned to groups of four; they formed their groups (these groupings were maintained throughout the three weeks of teaching). Then, the researcher-teacher reviewed the prerequisite knowledge, the right-angle triangle, students needed before studying the three primary trigonometric ratios. Although this was part of students' previous knowledge, their understanding of it needed to be refreshed because of its importance to the teaching and learning of the trigonometric ratios (Fosnot & Perry, 1996; Piaget, 1977; von Glasersfeld, 1995). In this review lesson, students, in groups of four, were presented with a sheet containing several triangles, some right triangles and some non-right triangles, and were asked to identify the right triangles from the set by putting an **"R"** in right-angle triangles. *Figure 3.4* shows a sample of the triangles that students worked with; the full range of triangles are presented in Appendix H.

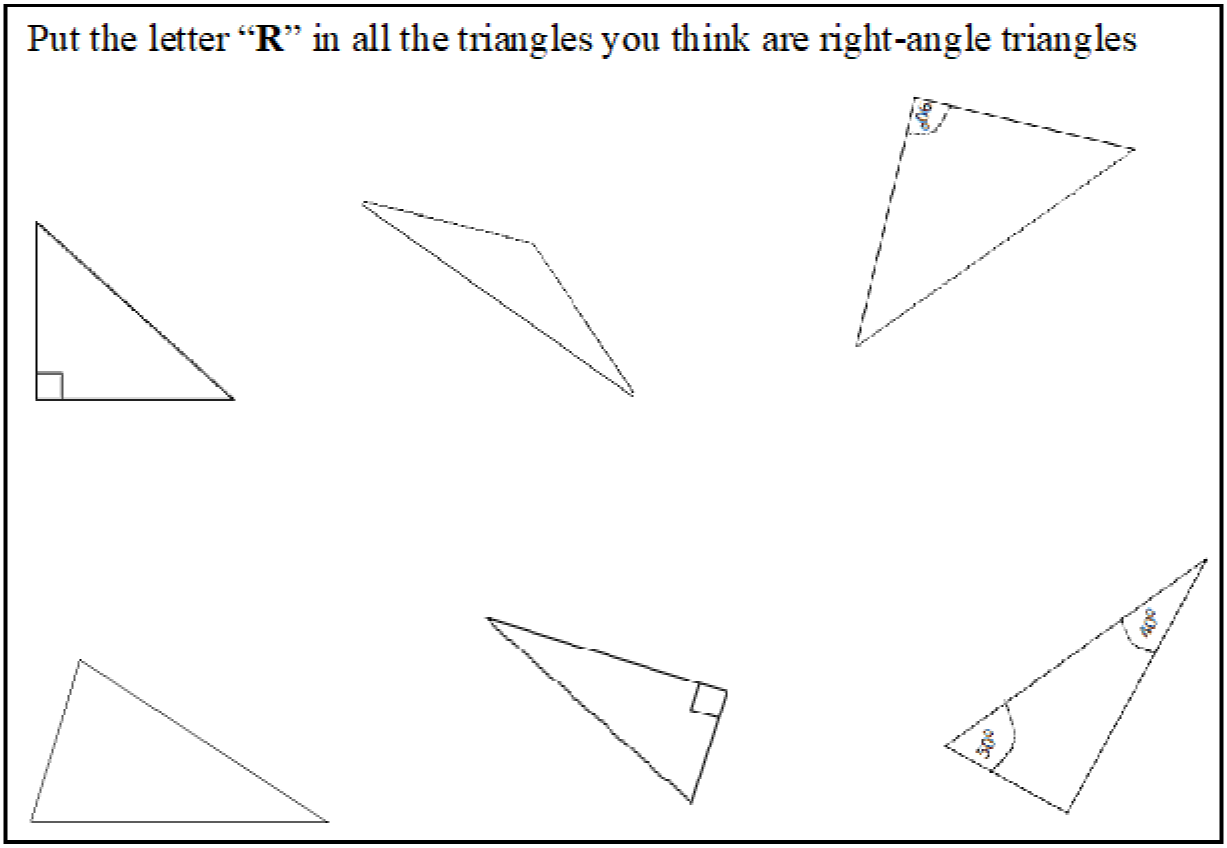


Figure 3.4: A sample of the triangles that students worked with during session one.

Figure 3.4 shows right-angle triangles drawn at different orientations and with different ways of identifying them: the 'box' at one angle, one angle labelled as 90^0, and two angles that add to 90^0. Students, as groups, were asked to show the triangles that they marked with **R** and give reasons for identifying them as right-angle triangles. During the discussion, students were probed with these questions.

- What makes a triangle right-angled?
- Does the orientation of the triangle matter?

These questions were geared towards helping students understand that the only feature of a triangle that makes it a right-angle triangle is the measure of 90^0 as one of its internal angles.

Then, students were introduced to the concept of the hypotenuse. The researcher-teacher drew a diagram of a right triangle on the board and asked students to identify the longest side. A discussion followed, prompted by the questions:

- Where is the longest side of any triangle located?
- For a right-angle triangle, where would the longest side be located?

These questions were to help students understand that the longest side of a right-angle triangle, called the hypotenuse, is always located opposite the 90^0 angles in the triangle (see *Figure 2.1*, p. 24).

Then, the researcher-teacher introduced students to the concepts of the reference angle, opposite side, and adjacent side. The researcher-teacher labelled the right-angle triangle on the chalkboard with a reference angle and identified the opposite side and adjacent side. Following the teacher's demonstration, students (groups) were asked to mark a given reference angle on each of the right triangle they identified on their sheets. Then they were asked to identify the hypotenuse, opposite side, and adjacent side on each of these triangles. The researcher-teacher moved among the student-groups, checking for understanding and helping students to clear up misconceptions. *Figure 3.5* shows an extract from one group's work that exemplifies the common errors among students' work.

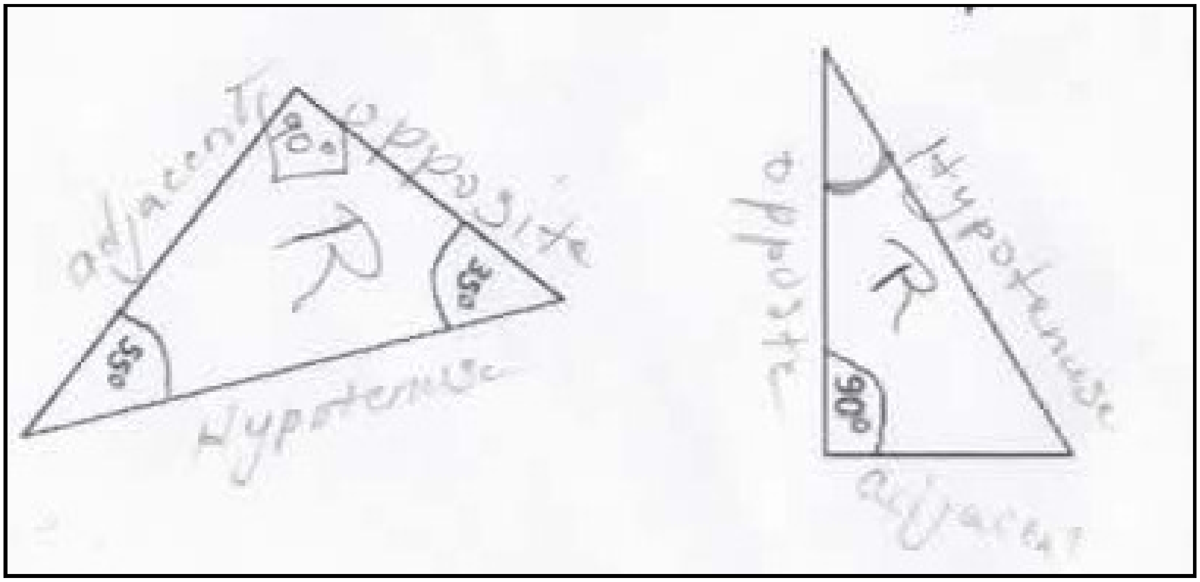


Figure 3.5: Triangles extracted from students' work showing common errors made in identifying the opposite and adjacent sides.

Two errors are evident in *Figure 3.5.* One, the reference angle is not identified in the first triangle. It is impossible to say which is the opposite side or adjacent side without having identified the reference angle. The researcher-teacher pointed out to students the importance of pinpointing the reference angle when they are called upon to identify the opposite and adjacent sides. It is important to pinpoint the reference angle because theses sides are always in reference to an angle, not the 90^0 angle, in a right-angle triangle. Two, the adjacent and opposite sides were incorrectly identified in the second triangle. Students were reminded of the meaning of the words opposite and adjacent, which are relative terms, and that these words are always in relation to the reference angles—opposite the reference angle and adjacent (beside) the reference angle. Session one, for both groups of students, ended with the researcher-teacher helping student to correct these errors.

Teaching through Exemplification: Sessions 2–5.

Teaching by Exemplification comprised four sessions; they were session two to session five. In session two, students were introduced to the three primary trigonometric ratios through a guided discovery activity. This activity presented students with their first set of examples of representations of the sine, cosine, and tangent ratios. Moreover, this introductory activity was used to give students a solid foundation of these ratios; thus, ensuring that students were not severely harmed during the study. Taking this step was crucial in the aftermath of Hurricane Maria because there was little to no time for remedial work if students' performances were severely negatively affected using Exemplification because there is a limited number of studies showing the effects of Exemplification on students' learning at the secondary school level. In sessions three, four, and five, students were taught using mostly Exemplification. That is, students generated examples of different representations of the three primary trigonometric ratios.

In session two, students (working in groups of four) were given worksheets that contained six right-angle triangles. A reference angle, the hypotenuse, the opposite side, and the adjacent side were marked on each triangle. Reference angles were selected to cover a wide range of angles between zero and 90^0. They were approximately 10^0, 20^0, 40^0, 50^0, 70^0, and 85^0. *Figure 3.6* shows images of the six triangles; the actual triangles are presented in Appendix I.

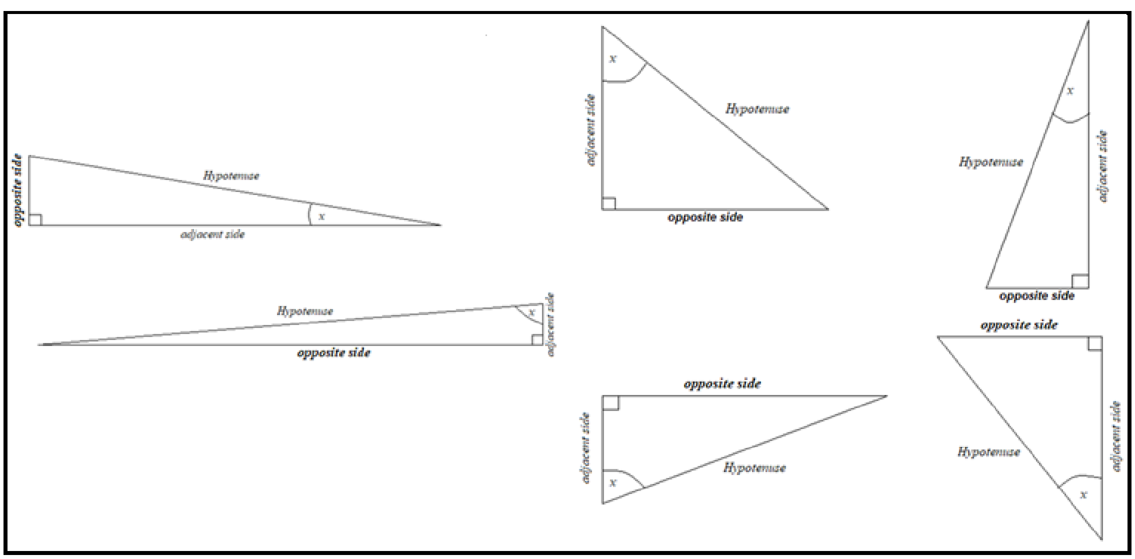


Figure 3.6: Images of the triangles used in all measurement-based activities for both Exemplification and Investigation.

Students were asked to use their rulers, protractors, and calculators to find values to fill a given table. *Figure 3.7* shows one group of students table that they filled during session two.

Measurements				Calculations			From Calculator		
Marked angle	Hypotenuse	Opposite side	Adjacent side	$\frac{Opp.}{Hyp.}$	$\frac{Adj.}{Hyp.}$	$\frac{Opp.}{Adj.}$	Sine	Cosine	Tangent
50°	8.3cm	6.5cm	5.3cm	0.783	0.639	1.23	0.766	0.642	1.191
20°	8.3cm	2.9cm	7.8cm	0.350	0.876	0.372	0.342	0.940	0.363
70°	8.9cm	8.3cm	3.1cm	0.932	0.350	2.678	0.940	0.342	2.747
39°	7.6cm	4.9cm	5.9cm	0.644	0.776	0.830	0.630	0.777	0.810
10°	10.8cm	1.8cm	10.7cm	0.167	0.991	0.168	0.173	0.985	0.176
88°	13.2	13.1cm	1.1cm	0.992	0.083	11.91	0.999	0.349	28.63

Figure 3.7: A table filled during the Exemplification guided discovery activity.

The first four columns—angle, hypotenuse, opposite side, adjacent side—were filled with measurement values, the next three columns were filled with values found by dividing the relevant measurements (sides), and the last three columns were filled with values found using the trigonometric functions on calculators. Calculator values were rounded and taken to three decimal places.

Once the table was completed, students (table groups) were asked to examine the last six columns and determine formulas for the sine, cosine, and tangent ratios. All groups identified the formulas with some assistance from the researcher-teacher. Initially, they identified the formulas as:

$$sine = \frac{opposite\ side}{hypotenuse}$$

$$cosine = \frac{adjacent\ side}{hypotenuse}$$

$$tangent = \frac{opposite\ side}{adjacent\ side}$$

The researcher-teacher pointed out that each of these ratios included an angle. Hence, it was not,

$$sine = \frac{opposite\ side}{hypotenuse},$$

but, sine of the reference angle, or:

$$\sin(\theta) = \frac{opposite\ side}{hypotenuse}.$$

Similar corrections were made to the formulas for the cosine and tangent ratios. At this point, the session ended. For homework, students were asked to draw graphs for the sine, cosine, and tangent ratios using the values from their tables and the given axes that were already scaled.

The first fifteen minutes of session three was spent checking students' graphs, identifying flaws and ways they could be improved. *Figure 3.8* shows a student's graphs of the sine ratio that typifies the initial sine curves drawn by most students.

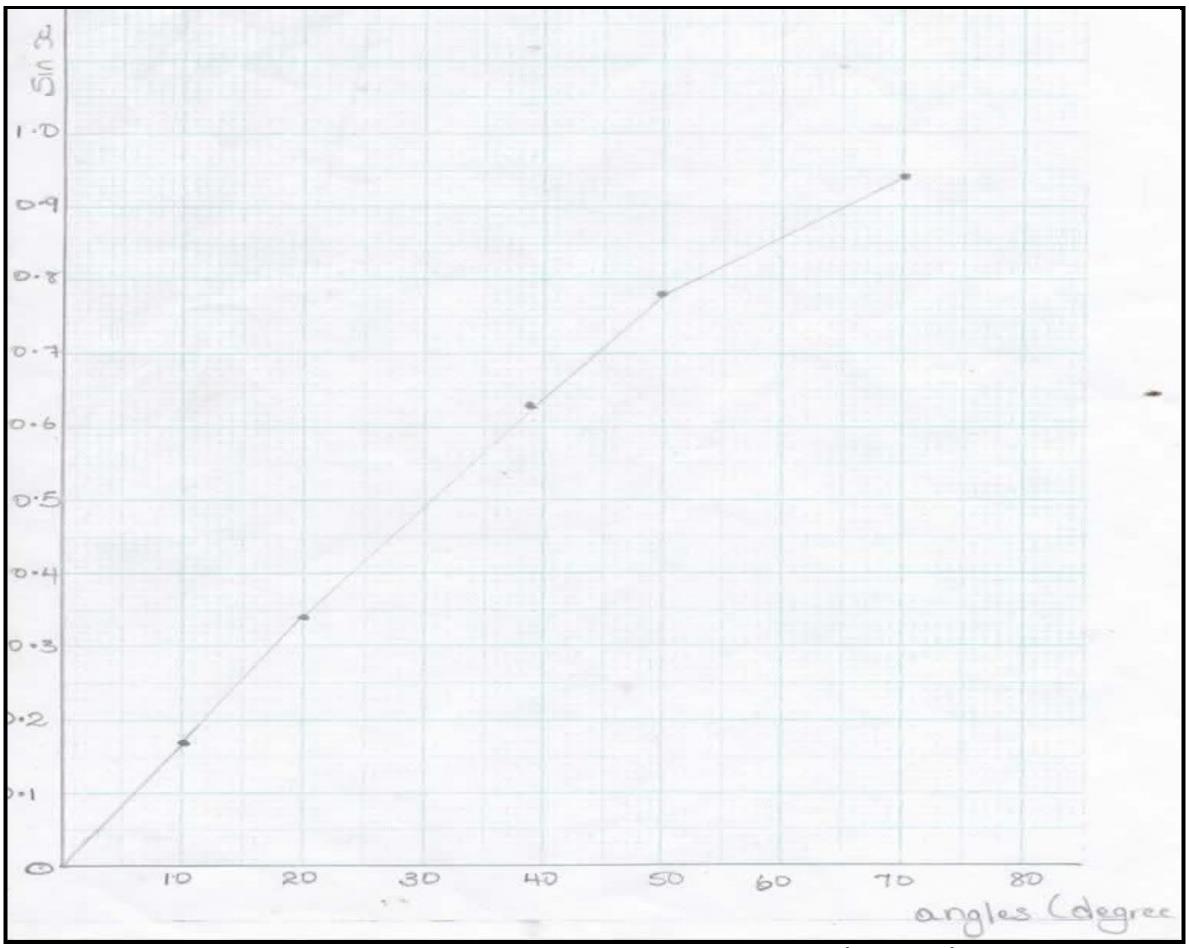


Figure 3.8: An initial graph of the sine ratio for angles between 0^0 and 90^0 drawn by an Exemplification student.

Figure 3.8 shows the value of sine increasing from zero and going towards one as the values of the angles increase from zero to 90^0. The next point was for an angle value greater than 85^0; the sine value of this angle could not be represented on the graph. This student stopped the graph at the last point that she/he could plot. The graph depicted in *Figure 3.8* also shows the student using straight lines to connect the plotted points. Using straight lines to join the points was the most common error made by students. The researcher-teacher helped students to correct that error by getting them to replace the straight-line sections of their graphs with curves. *Figure 3.9* shows the same student corrected vision of the graph of the sine ratio for angles between zero and 90^0. Instead of straight lines, the student is using ‘curved’ lines to join the plotted points. The student also extended the graph to show that it is approaching a value of one at 90^0.

Figure 3.9: A corrected graph of the sine ratio for angles between 0^0 and 90^0 drawn by an Exemplification student.

The remaining 70 minutes were spent generating and discussing examples of representations of the sine ratio. There were two rounds of this activity, with each round focusing on a different attribute of the sine ratio. The first round focused on diagrams, and the second focused on angles. The first round lasted approximately 30 minutes and contained four prompts. The second round lasted approximately 40 minutes and contained five prompts. Watson and Mason (2005) advocated for a minimum of three prompts. They argued that with a greater number of prompts, learners are forced to dig deeper to find a suitable example. Between prompts, the researcher-teacher checked students (groups) work, asked for clarification, provided guidance where and when needed, and in some cases, provided an example to help students generate other relevant examples (Watson & Mason, 2005). All prompts were given verbally and followed the sequences below.

These prompts were given about students generating examples of diagrams for the sine ratio. Students wrote their answers to these prompts on paper and displayed them when asked.

1. The sine ratio uses the opposite side and hypotenuse of a right-angle triangle. Draw a diagram showing the sides and the angle involved in the sine ratio.
2. Draw another diagram with the angle in a different position.
3. Draw a diagram in which the sine ratio cannot be used. Why can't the sine ratio be used in this case?
4. Draw another diagram in which the sine ratio cannot be used but for a different reason. Explain your reason.

The following are the prompts given to students for generating examples of angles for the sine ratio. Answers to these prompts were given verbally.

1. Give an angle whose sine is less than the sine of 45^0.

2. Give one whose sine is greater than the sine of 45^0.
3. Give one whose sine is equal to 1.
4. Give one whose sine is greater than the sine of 60^0 but less than the sine of 30^0.
5. Give one whose sine is greater than 1.

Students, in their groups, were asked to make a statement about the sine ratio for angles between zero and 90^0 to end the session. After some discussions, most groups stated that the value of the sine of an angle increases from zero to one as the angles increase from zero to 90^0.

Teaching in session four focused on the cosine ratio with reference made to the sine ratio. Such references allowed students the opportunity to compare the sine and cosine ratios and help them see the ratios as connected concepts. In this teaching session, students generated examples in two rounds, with each round focusing on a different attribute of the sine and cosine ratios. Like in session three, the first round focused on diagrams, and the second focused on angles. The first round lasted approximately 30 minutes and contained five prompts. The second round lasted approximately 50 minutes and contained five prompts. As in the previous session, between prompts, the researcher-teacher checked students (groups) work, asked for clarification, provided guidance where and when needed, and provided examples to help students figure out other relevant examples. The following prompts were given verbally.

These prompts related to students generating examples of diagrams for the cosine ratio. Students wrote their answers to these prompts on paper and displayed them when asked.

1. The cosine ratio deals with the adjacent side and hypotenuse of a right-angle triangle. Draw a diagram showing the sides and the angle involved in the cosine ratio.
2. Draw another diagram with the angle in a different position.

3. Draw a diagram in which the cosine ratio cannot be used. Why can't the cosine ratio be used in this case?
4. Draw another diagram in which the cosine ratio cannot be used but for a different reason. Explain your reason.
5. Draw a diagram in which the cosine ratio cannot be used, but the sine ratio can be used. Say why.

The following are the prompts given to students for generating examples of angles for the cosine ratio. Answers to these prompts were all given verbally.

1. Look at the sine and cosine curves or tables you drew. Observe that the sine of 40^0 is equal to the cosine of 50^0 and the sine of 30^0 is equal to the cosine of 60^0. Give an example of two angles for which the sine of one angle is equal to the cosine of the other.
2. Give another example of two such angles.
3. Give an example of two angles between 25^0 and 75^0 that you have not given before for which the sine of one angle is equal to the cosine of the other.
4. Give an example of two angles that are less than 45^0 for which the sine of one angle is equal to the cosine of the other.
5. Give one angle that has the same value for both sine and cosine.

Students (groups) were asked to make a statement explaining the link between the sine and cosine of angles between zero degrees and 90^0 to culminate this activity. After some deliberation, groups were asked to share their statements with the class. All groups came up with a statement claiming that if two angles add up to 90^0, the sine (cosine) of one is equal to the cosine (sine) of the other.

Session five was geared towards helping students develop an understanding of the tangent ratio, and to position it in relation to the sine and cosine ratios. In this session, students generated examples in two rounds. Like in sessions three and four, the first round focused on diagrams, and the second round focused on angles. The first round lasted approximately 20 minutes and contained six prompts. The second round lasted approximately 60 minutes and contained six prompts. As in the previous sessions, between prompts, the researcher-teacher checked students (groups) work, asked for clarification, provided guidance where and when needed, and provided examples to help students figure out other relevant examples. The following prompts were given verbally.

These prompts related to students generating examples of diagrams for the tangent ratio. Students wrote their answers to these prompts on paper and displayed them when asked.

1. The tangent ratio uses the opposite and adjacent sides of a right-angle triangle. Draw a diagram for which the tangent ratio can be used.
2. Draw another diagram with the angle in a different position.
3. Draw a diagram for which the tangent ratio cannot be used. Why can't the tangent ratio be used?
4. Draw a diagram for which the tangent ratio cannot be used but for which the cosine ratio can be used. Give reason(s) why this is so.
5. Draw a diagram in which any of the ratios; sine, cosine, or tangent; can be used.
6. Draw a diagram in which none of the ratios can be used. Explain why.

The following are the prompts that were given to students for generating examples of angles for the tangent ratio. Answers to these prompts were all given verbally.

1. Look at the sine, cosine, and tangent cures and tables you drew. Observe how they increase or decrease with their changing angles. Give an example of an angle whose tangent is less than the tangent of 45^0.
2. Give an example of an angle whose tangent is greater than one.
3. Give an example of an angle whose tangent is too large to call.
4. Give an example of an angle whose tangent is less than that of the sine of 30^0.
5. Give an example of an angle whose tangent is less than that of the cosine of 30^0.
6. Give an example of an angle whose tangent lies between the sine of 30^0 and the cosine of 45^0.

Students (groups) were asked to come up with a statement that speaks to the major similarities and differences among the three ratios for angles between zero degrees and 90^0 to culminate this activity. After some deliberation, groups were asked to share their statements with the class. Several similarities and differences were identified and shared, some trivial (e.g., all the ratios use division) and others significant. Among the significant similarities were the values of both sine and tangent increases as the values of the angles increase from 0^0 to 90^0, and the values of sine and cosine are between zero and one. Among the significant differences were: for increasing values of angles, values of sine increase while values of cosine decrease; tangent has values greater than one while for both sine and cosine, the highest value is one; for increasing values of angles, the values of sine and tangent increases but at different rates.

Teaching through Investigation: Sessions 2–5.

Teaching by Investigation also comprised four teaching sessions: session two to session five. Session two focused on the sine ratio, session three on the cosine ratio, session four on the tangent ratio, and session five on comparing the three ratios.

In investigating the sine ratio, students investigated the formula, developed a table of values for angles between 0^0 and 90^0, and drew the graph of the sine ratio for angles between 0^0 and 90^0. Students, working in groups of four, were given a worksheet (see Appendix I) that contained six right-angled triangles (see *Figure 3.6* above) and a table, and they were also given a graph sheet containing a pair of scaled axes. For each triangle, a reference angle, the hypotenuse, the adjacent side, and the opposite sides were marked. Reference angles were selected to cover a wide range of angles between 0^0 and 90^0: They were approximately 10^0, 20^0, 40^0, 50^0, 70^0, and 85^0. Students were asked to use their rulers, protractors, and calculators to find values to fill the table on the given worksheet. *Figure 3.10* shows one group of students table that they filled while investigating the sine ratio.

Angle *x*	Adjacent side	Opposite side	Hypotenuse	Sine (using a calculator)
50^0	6.6 cm	6.5 cm	8.8 cm	0.766
20^0	8 cm	3cm	8.4 cm	0.342
70^0	3.4cm.	8.4cm	9cm.	0.939
40^0	6cm	4.8cm	7.4cm	0.642
10^0	10.4cm	1.6cm	8.6 cm	0.173
88^0	1cm	13.2	13.1	0.999

Figure 3.10: A table filled by a group of Investigation students while investigating the sine ratio.

The first four columns in *Figure 3.10*—angle, hypotenuse, opposite side, and adjacent side—were filled with measurement values, and the fifth column was filled with values found using the sine function on their calculators. Calculator values were rounded and taken to three decimal places. Once the table was completed, students (table groups) were asked to carry out calculations with the measured values to determine the formula for the sine ratio. They were looking for the calculations that would produce the same or approximate values obtained from the sine function from the calculator. After much calculations, all groups identified the formula of the sine ratio as:

$$sine = \frac{opposite\ side}{hypotenuse}.$$

The teacher/researcher gave them, with a reason (the ratios always include an angle), the correct version as:

$$sin\ (\theta) = \frac{opposite\ side}{hypotenuse}.$$

Students were assisted by the teacher-researcher, in making conjectures about the sine ratio and in testing them. It was essential for the teacher-researcher to help students initially to make conjecture, test and modify them, and refine their claims in this session because students were not familiar with these learning processes. They were assisted in making the following conjectures, and they tested them with the aid of calculators.

1. The sine of 90^0 is one. Students tested this conjecture and found it to be true.
2. The sine of zero is zero. Students tested this conjecture and found it to be true.
3. The sine of 45^0 is 0.5. Students tested this conjecture and found that it was false. These students were asked to explain why they thought that this conjecture was false. After small groups and whole-class discussions, the students concluded that it was because the sine ratio does not produce a straight line graph.
4. The sine of 20^0 is less than the sine of angle 60^0. Students tested this conjecture and found it to be true.

Students were asked to make a statement about the sine ratio based on the results of their conjecturing and testing. Most students claimed that the values of sine increase from zero to one as the angles increase from zero to 90^0. The session ended at this juncture. For homework, students were asked to use their table of values to draw the graph of the sine ratio between 0^0 and 90^0.

The first ten minutes of session three was spent checking students' graphs that they drew for homework. *Figure 3.11* shows one student's graph for the sine ratio.

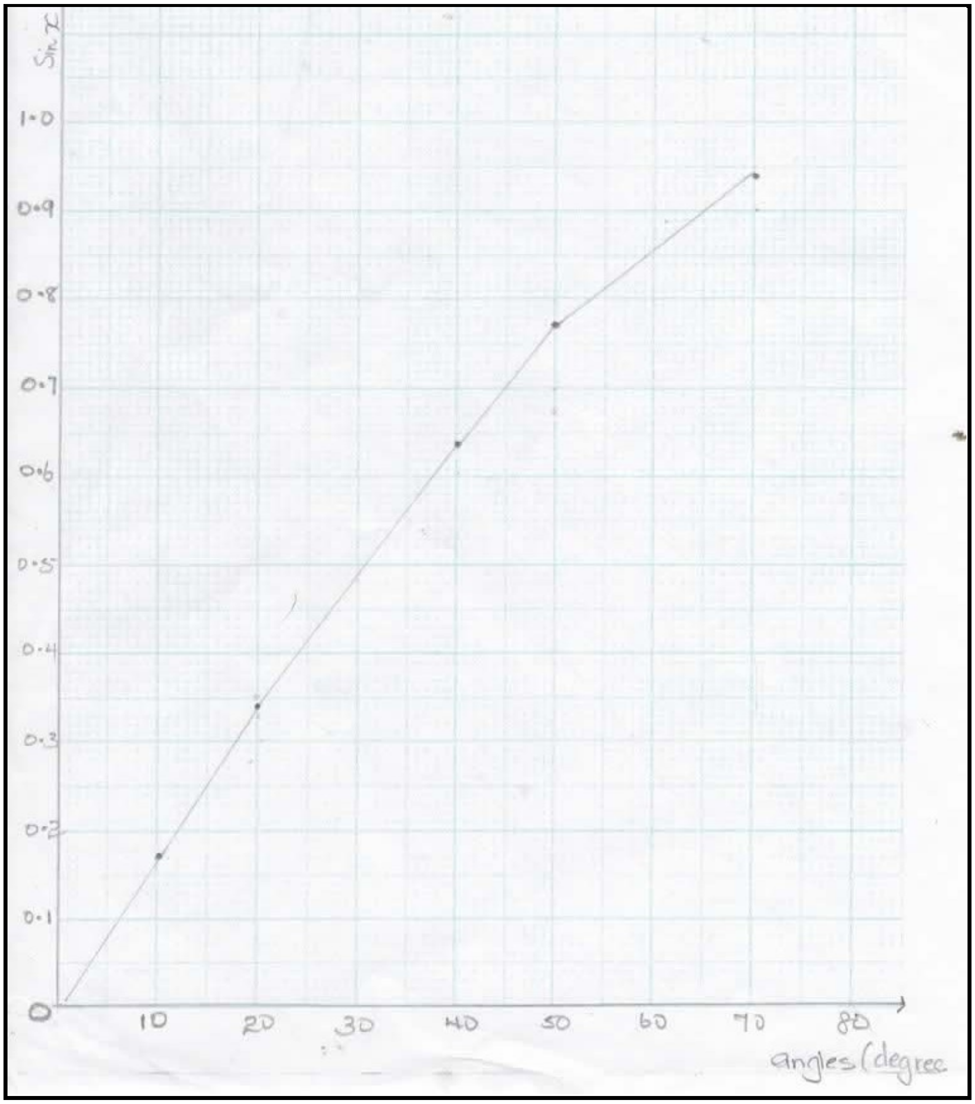


Figure 3.11: An Investigation student's graph of the sine ratio for angles between 0^0 and 90^0.

Figure 3.11 typifies the sine curve produced by most students; it shows the values of sine increasing from zero going towards one as the values of angles increase from zero to 90^0. The next point was for an angle value greater than 85^0; the sine value of this angle could not be represented on the graph. This student stopped the graph at the last point that she could plot.

However, as was evident with the group taught by Exemplification, students used straight lines to connect the plotted points. The researcher-teacher help them to make the necessary correction by getting them to change the straight line to curves.

Then, a similar approach to session two was taken to investigate the cosine ratio. In investigating the cosine ratio, students verified the formula, developed a table of values for angles between 0^0 and 90^0, and drew the graph of the cosine ratio. Students, working in groups of four, were given a worksheet (Appendix I) that contained the same six right triangles as in the previous session, a table like the previous one, and a copy of a graph sheet containing a pair of scaled axes. For each triangle, a reference angle, the hypotenuse, opposite side, and the adjacent sides were marked. Students were asked to use their rulers, protractors, and calculators to find values to fill in the table on the given worksheet. *Figure 3.12* shows one group's table that they filled while investigating the cosine ratio.

Angle *x*	Adjacent side (mm)	Opposite side (mm)	Hypotenuse (mm)	Cosine (using a calculator)
51°	53	62	85	0.63
21°	78	30	84	0.93
70°	32	85	90	0.34
40°	60	48	77	0.77
10°	108	19	109	0.98
86°	11	132	133	0.070

Figure 3.12: A table of values filled a group of Investigation students while investigating the cosine ratio.

The first four columns in *Figure 3.12*—angle, hypotenuse, opposite side, and adjacent side—were filled with measurement values, and the fifth column was filled with values found using the cosine function on their calculators. Once the table was completed, students (table groups) carried out several calculations to find the values that match those in the table column that was filled with values from the cosine function on the calculator. After much calculations, all groups identified the formula of the cosine ratio as:

$$cos\ (\theta) = \frac{adjacent\ side}{hypotenuse}.$$

Students were then encouraged to make conjectures about the cosine ratio and test them. Students were encouraged to make conjectures similar to those made in session two—e.g. the cosine of 0^0 is …, the cosine of 90^0 is …, the cosine of 45^0 is …, the cosine of 30^0 is …, the cosine of angle ‘a’ is higher or lower than the cosine of angle ‘y’. The researcher-teacher suggested to the students the above conjectures, but allowed them, in their groups, to choose what conjectures to make and test; while moving around to check the activities of each group. He also provided some help where and when necessary (e.g. assure students that they were on the right track, give an example of a relevant conjecture, answer direct questions).

Students, in their small groups, were also asked to come up with a statement about the cosine ratio. At the end of their conjecturing, testing, and modifying, the class was brought together, and groups asked for their statements about the cosine ratio. The following statement summarizes students’ utterances about the cosine ratio: As the values of the angles increased from zero to 90^0, the values of cosine decreased from one to zero.

Students then used their table of values to draw a graph of the cosine ratio between 0^0 and 90^0. *Figure 3.13* shows a student’s graph of the cosine ratio that exemplifies the graph of the cosine ratio drawn by most students. *Figure 3.13* shows the values of cosine decreasing from one and going towards zero (the point it reaches zero not shown), in a non-linear manner, as the angles increased from zero to 90^0 (the 90^0 not shown). The session ended with a discussion about the shape of the graph of the cosine ratio. During that discussion, students stated that the shape of the cosine graph was the "opposite" to that of the sine graph.

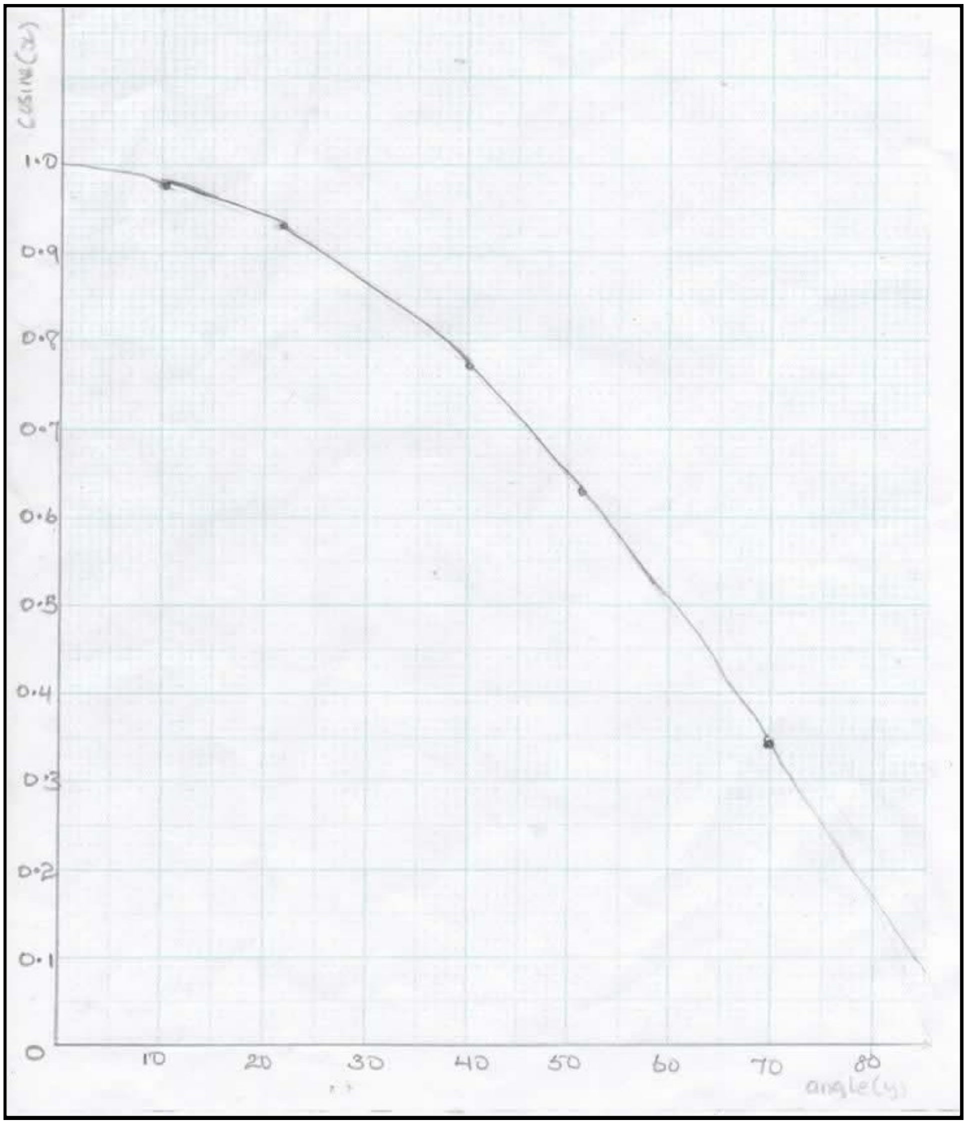


Figure 3.13: An Investigation student's graph of the cosine ratio for angles between 0^0 and 90^0.

A similar approach to sessions two and three was used to investigate the tangent ratio in session four. In investigating the tangent ratio, students verified the formula, developed a table of values for angles between 0^0 and 90^0, and drew the graph of the tangent ratio. Students, working in groups of four, were given a worksheet (Appendix I) that contained the same six right triangles as in the previous sessions, a table like those given previously, and a copy of a graph sheet containing a pair of scaled axes. For each triangle, a reference angle, the hypotenuse, the opposite side, and the adjacent sides were marked. Students were asked to use their rulers, protractors, and calculators to find values to fill the table on the given worksheet. *Figure 3.14* shows one group's table that they filled while investigating the tangent ratio.

Angle *x*	Adjacent side	Opposite side	Hypotenuse	Tangent (using a calculator)
51°	5·3cm	6·5 cm	8·3 cm	1·234
21°	7·8 cm	2·9 cm	8·3 cm	0·383
70°	3·1 cm	8·3 cm	8·9 cm	2·747
40°	5·9 cm	4·9 cm	7 6 cm	0·839
10°	10·7cm	1·8 cm	10·8 cm	0·176
86°	1·1 cm	13·1 cm	13·2cm	14·300

Figure 3.14: A table filled by a group of Investigation students while investigating the tangent ratio.

The first four columns in *Figure 3.14*—angle, hypotenuse, opposite side, and adjacent side—were filled with measurement values, and the fifth column was filled with values found using the tangent function on their calculators. Calculator values were rounded and taken to three decimal places.

Once the table was completed, the researcher-teacher suggested the following conjecture about the formula for the tangent ratio: Now that you know the formula for the sine and cosine ratios, what do you think is the formula for the tangent ratio? Most students suggested that the formula for the tangent ratio contained the opposite and adjacent sides because these were the only two sides that were not already combined to make a formula for a ratio. Students were asked to make further conjectures and test and modify them if necessary. Some groups further conjectured that the formula for the tangent ratio was the opposite side divided by the adjacent side, others conjectured that it was the adjacent side divided by the opposite side. Students tested their conjectures by dividing the relevant sides and by comparing their answers with the values in the last column of their table. After some work, all groups identified the formula of the tangent ratio as:

$$tan\ (\theta) = \frac{opposite\ side}{adjacent\ side}.$$

Students were then encouraged to make other conjectures about the tangent ratio and test them. Students were encouraged to make conjectures similar to those made in session three–e.g. the tangent of 0^0 is …, the tangent of 90^0 is …, the tangent of 45^0 is …, the tangent of 30^0 is …, the tangent of angle ‘a’ is higher or lower than the tangent of angle ‘y’. The researcher-teacher suggested to the students the above conjectures, but allowed them, in their groups, to choose what conjectures to make and test, while moving around to check the activities of each group. He also provided some help where and when necessary (e.g. assure students that they were on the right track, give an example of a relevant conjecture, answer direct questions).

Students, in their small groups, were also asked to come up with a statement about the tangent ratio. At the end of their conjecturing, testing, and modifying, the class was brought together, and groups asked for their statements about the tangent ratio. The following statement summarizes students’ utterances about the tangent ratio: As the values of the angles increased from 0^0 to 90^0, the values of tangent increases, but to very large values.

Then, students used their table of values to draw the graph of the tangent ratio for values between 0^0 and 90^0. *Figure 3.15* shows a student’s graph of the tangent ratio that exemplifies the graphs of the tangent ratio drawn by most students.

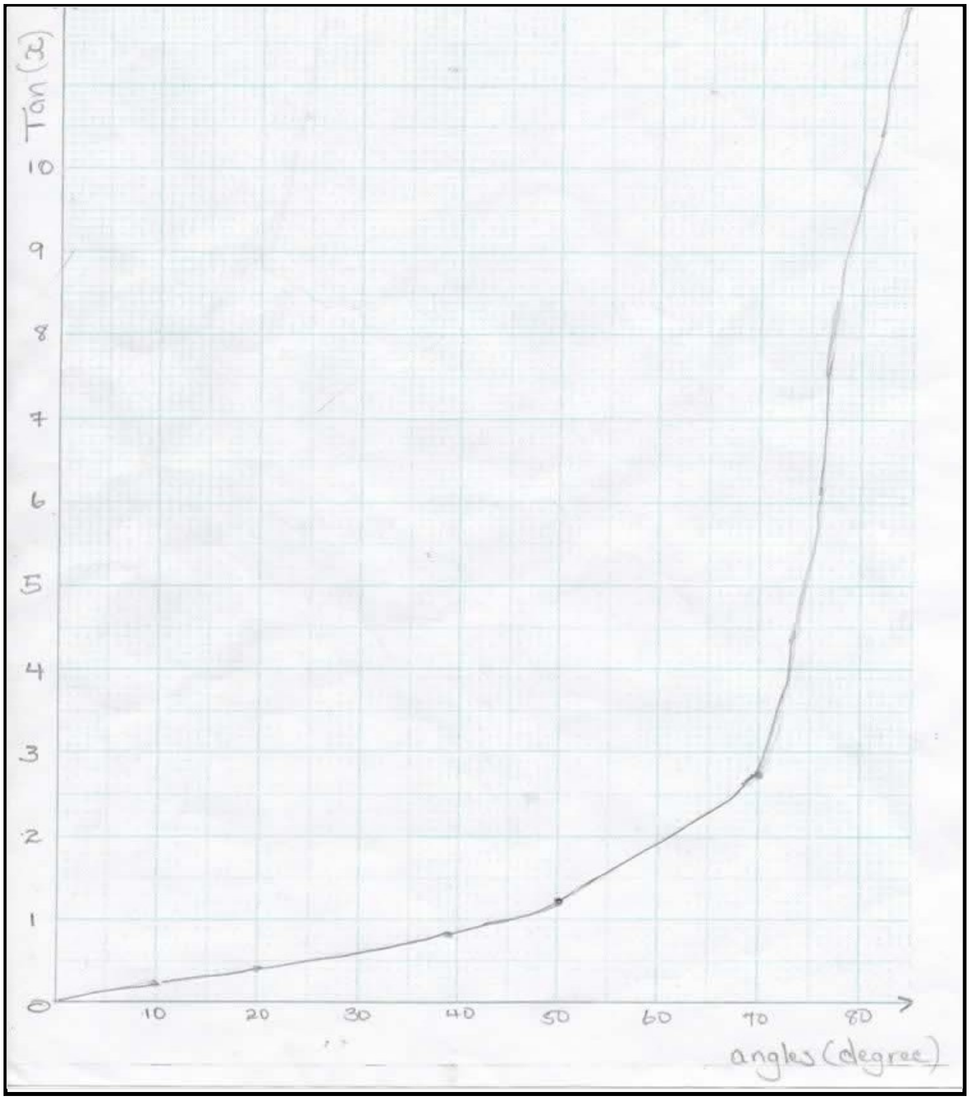


Figure 3.15: An Investigation student's graph of the tangent ratio for angles between 0^0 and 90^0.

Figure 3.15 shows the values of tangent increasing exponentially between 0^0 and 90^0 (90^0 not shown); that is, going towards an asymptote at 90^0. The session ended with a discussion about the shape of the graph of the tangent ratio. During that discussion, students stated that the shape of the graph of the tangent ratio was different from both the sine curve and the cosine curve. The students also stated that they could not find a value for tangent at 90^0.

In session five, students identified the similarities and differences among representations of the sine, cosine, and tangent ratios. Working in small groups of four (same table groups) and using the formulas they identified, the tables of values created, and the graphs drawn from previous classes, they identified and listed similarities and differences among the different representations of the three primary trigonometric ratios. Each group wrote its list of similarities (e.g.: all ratios uses two sides of a right-angle triangle, none of the ratios produced straight-line graphs, the sine ratio and the tangent ratio increase for angles between 0^0 and 90^0) and differences (e.g.: the sine and tangent ratios increase while the cosine ratio decrease for values between 0^0 and 90^0; the sine ratio has a value of one at 90^0, but the cosine ratio has a value of zero at 90^0) on a sheet of manila/presentation paper. Each group presented their findings (list of similarities and differences) to the rest of the class. Each group presented for approximately five minutes. During each presentation, both the teacher-researcher and the other students sought clarifications through questions. The session ended with the teacher-researcher reviewing the formulas for sine, cosine, and tangent ratios and informing students that they will be using them to solve problems involving right triangles in their next session.

Session six: Common to both groups.

This session, which was the same for both groups of students, focused on students' application of the three primary trigonometric ratios to find missing sides and missing angles in right-angle triangles and on solving contextual problems. This session catered to students developing the skills needed to solve CSEC type questions, like the one depicted in *Figure 2.8* (p. 30), using their knowledge and understanding of the three primary trigonometric ratios.

The session started with a review of the formulas for the ratios of sine, cosine, and tangent. In the review, students were reminded of the sides that form each ratio and how they can be identified in a right-angle triangle (hypotenuse is always the longest side and is opposite the right angle, and the opposite and adjacent sides are in reference to a marked angle). Students were reminded that the formulas for the ratios are:

$$sin(\theta) = \frac{opposite\ side}{hypotenuse}$$

$$cos(\theta) = \frac{adjacent\ side}{hypotenuse}$$

$$tan(\theta) = \frac{opposite\ side}{adjacent\ side}$$

Then students were introduced to the concepts of the angle of elevation and the angle of depression. Students were told, with the aid of diagrams, that an angle of elevation is the angle formed between the horizontal and the line of sight when looking upwards (see *Figure 3.16*), and an angle of depression is formed between the horizontal and the line of sight when looking downwards (see *Figure 3.17*).

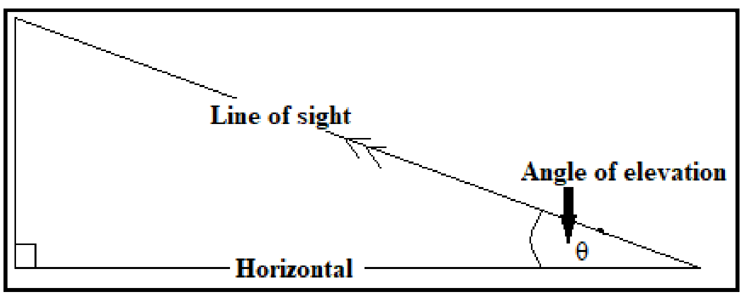


Figure 3.16: Diagram showing an angle of elevation.

Figure 3.16 illustrates the diagram used to demonstrate the angle of elevation to students. The angle of elevation is shown as θ and is between the line of sight and the horizontal. The horizontal is shown as one leg of the right-angle triangle, and the line of sight is the hypotenuse. The upward arrows placed on the line of sight depicted an upward look from the horizontal.

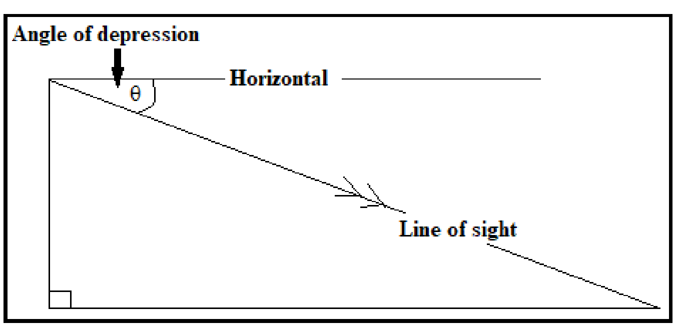


Figure 3.17: Diagram showing an angle of depression.

Figure 3.17 illustrates the diagram used to demonstrate the angle of depression to students. The angle of depression is shown as θ and is between the line of sight and the horizontal. The horizontal is not a leg of the right-angle triangle in this case, but the line of sight is the hypotenuse. The downward arrows placed on the line of sight depicted a downward look from the horizontal.

The researcher-teacher then demonstrated the procedure for calculating a missing side (see *Figure 3.18*) and a missing angle (see *Figure 3.19*) using formulas for trigonometric ratios. This action (demonstrating to students at this point) was in keeping with Vygotsky' (1978) notion of Zone of Proximal Development (see chapter two, pp. 35–36), where a more knowledgeable other (in this case the researcher-teacher) helps students to solve problems that are beyond their intellectual reach. The researcher-teacher knew, from conversations with their mathematics teachers, that these students had not solved linear equations before; hence, they would have difficulties manipulating the sine, cosine, and tangent formulas to find a missing side or a missing angle without the teacher's direct assistance. That is, this researcher-teacher was working with these students within their ZPD when he demonstrated the manipulation of the trigonometric formulas to them (see *Figure 3.18* and *Figure 3.19*).

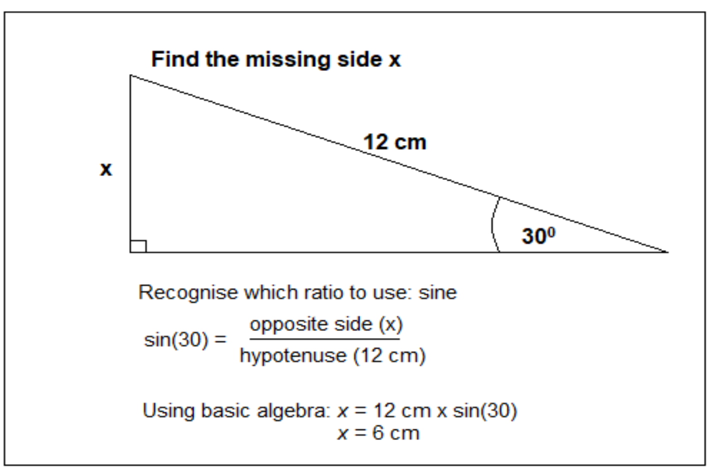


Figure 3.18: A diagram and calculations showing how to calculate a missing side of a right triangle using the sine ratio.

Figure 3.18 shows a right-angle triangle with a reference angle of 30^0, a hypotenuse of 12 centimetres, and the opposite side (x) as unknown. The figure also shows sine as the appropriate ratio to find the unknown side and the calculations done, using basic algebra, to find the side x.

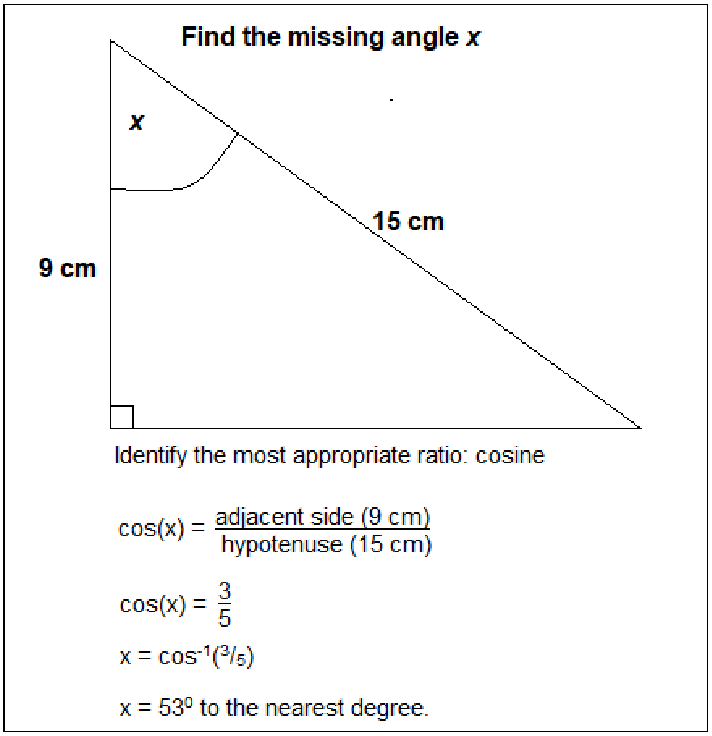


Figure 3.19: A diagram and calculations showing how to calculate a missing angle of a right triangle using the sine ratio.

Figure 3.19 shows a right-angle triangle with an unknown angle (x), a hypotenuse of 15 centimetres, and the adjacent side of nine centimetres. The figure also shows cosine as the appropriate ratio to find the unknown angle and the calculations done (using the inverse ratio) to find the angle x.

Students then applied their knowledge and understanding of the sine, cosine, and tangent ratios to solve right-angle triangles and angle of elevation and depression problems. Each person was given a worksheet (Appendix J), but they were encouraged to collaborate as table groups. The worksheet comprised of several right-angle triangles with either a side or the reference angle missing, one question about angle of elevation, and one question about angle of depression. The students had to select the appropriate ratio and apply an appropriate procedure to find the missing side or the missing angle in the given triangles. Students had to draw diagrams to represent the angle of elevation and angle of depression problems and solve them. Students completed the worksheet while the researcher-teacher moved among the students giving assistance where needed and helping students to clear up misconceptions.

Methods and Procedures

A mixed-methods approach was used in the collection and analysis of data for this research study. These methods and their procedures are discussed in this section to include: the methods and procedures for data collection; the instruments used for data collection; the methods and procedures used for data analysis; the methods and procedures used for presentation of data, findings, and results; and the methods and procedures adopted to maintain data security.

Data collection.

Data was collected using a pre-test and a post-test—these data collection instruments are detailed in the subsequent section. The pre-test was administered first to the group of students taught through Exemplification. This test was done in the classroom during a session just before the teaching started. They were given 90 minutes to complete the paper-based test. To complete this test, they were required to circle the letters corresponding to the correct answers in section one, and to write their responses in the space provided on the test papers for given prompts in section two. They were not allowed to use calculators. At the end of the 90 minutes, all test papers with students' names were collected, enveloped, sealed, and locked away. This pre-test was supervised by the researcher-teacher. In the session immediately after teaching, these students were given a post-test under the same conditions, using the same procedures. These completed test papers were also collected, enveloped, sealed, and locked away.

The same procedure was followed for students taught using Investigation. That is, they were given the same pre-test as the group taught by Exemplification, under the same conditions, in the session just before their teaching started. They were also given the same post-test as the group taught by Exemplification, under the same conditions, in the session immediately after their teaching ended. The completed pre-test and post-test papers with students' names were also collected, enveloped, sealed, and locked away until grading. Test papers for both the pre-test and the post-test were marked at the same time as this was likely to increase consistency in grading the written responses (Danielson & Marquez, 2016). *Figure 3.20* shows the sequence of activities described above during the data collection period.

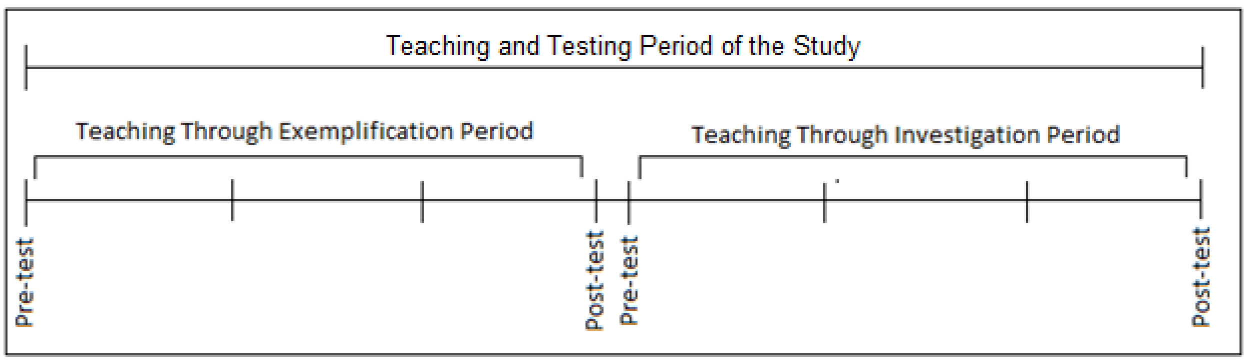


Figure 3.20: A diagram showing the sequence of activities during the data collection and teaching periods.

The grading of the multiple-choice items was done by the researcher. Each multiple-choice item was awarded one mark for a correct answer, and no mark was deducted for a wrong answer. Grading of the written responses was done by two experienced secondary mathematics teachers. Both teachers have a B.Ed. in Secondary Mathematics Education (one has an M.Ed. in Instructional Design), have been teaching mathematics at the secondary level for more than ten years, and have been grading CSEC mathematics examination papers for the past five or more years. The decision to stay away from grading the written responses was taken by the researcher to prevent his bias, or the appearance of it, from tainting the data and using two highly experienced raters allowed for a fairer assessment of students' work because it incorporates more than one expert's perspectives (Danielson & Marquez, 2016).

Although grading of the written responses was done separately by two raters, the same assessment rubric (Appendix M) was used by both. The assessment rubric was developed by the researcher, who drew from Kilpatrick et al. (2001) and Hiebert and Carpenter (1992). This rubric was developed after all teaching but before any grading had taken place. The rubric was discussed and used with the raters before any grading took place. In a two-hour workshop-like meeting, the two raters and the researcher used the rubric to assess an anonymized sample of three post-test scripts. This workshop-like meeting followed this sequence of activities:

1. A copy of the assessment rubric was given to each rater, and its major tenants discussed (the categories and the standards within each category). The group looked at the reasons for assigning marks, and the number of marks to be assigned or deducted.
2. The raters and researcher collaboratively graded one test paper using the rubric.
3. Each rater and the researcher separately graded another test paper. The team (the two raters and the researcher) then came together to discuss the assignment of marks and some changes were made to marks assigned by individuals in some areas.
4. Each rater and the researcher, again, separately graded another test paper. The team, again, came together to discuss the grading. The researcher and one rater had assigned the same mark, and the other rater's assigned mark was half a point lower. The researcher was satisfied with that level of consistency and brought the workshop to a close.
5. Anonymized copies of all participants' post-test papers were given to the raters for grading. Only participants' written responses were given.

At the end of grading, each student was given a raw score, which came from adding marks obtained from both sections of the tests–multiple-choice in section one and written responses in section two. A student's mark for the written response section was taken as the mean of the two raters' marks. Two raters marked the written response section of the post-test (no one had attempted this section on the pre-test), resulting in two marks for each participant for that section. To decide how to arrive at a single mark for each participant, the researcher needed to know whether one rater scores were consistently higher than the other. A Spearman-rho correlation analysis was conducted to determine this level of consistency. This analysis is presented in chapter four. The analysis showed that one rater consistently graded higher (r_s = 0.853) than the other. Therefore, a mean of these two raters' grades presented a fair assessment of students' work on the written response questions. Moreover, the main purpose was to evaluate the comparaibility of the two teaching approaches and less on the individual scores students produced.

Instruments.

Three instruments were used in the data collection process: the pre-test, post-test, and an assessment rubric used in the grading of the written response section of the tests. The following are descriptions of these instruments and the procedures used in their development.

Pre-test and post-test.

Both the pre-test (Appendix K) and the post-test (Appendix L) were developed by the researcher with inputs from two subject matter specialists–experienced teachers who also served as raters. Both tests comprised of 12 multiple-choice items and three prompts for students' written responses that assessed students' understanding of representations of the sine ratio, cosine ratio, and tangent ratio. The tests were parallel but not the same, in that, the positions of some multiple-choice items in the pre-test were changed in the post-test; the positions of some options in the pre-test were changed in the post-test; and some numbers in items on the pre-test that required calculation, were changed in the post-test. Each multiple-choice item had four options with only one correct answer. Students were given instructions to select the correct option.

In the multiple-choice section, items one, two, and three assessed students' ability to identify the structures of the three primary trigonometric ratios. Items four, five, and six assessed students' knowledge of the relationship between sine and cosine. Items seven, eight, and nine assessed students' ability to identify different representations of the ratios–diagram, graph, and table. And items 10, 11, and 12 assessed students' ability to perform calculations involving trigonometric ratios.

Question one of the written response section asked students to draw a diagram to represent a contextual situation, and discuss how that diagram can be used to solve the problem embedded in the situation. No calculations were required, but students had to discuss/show the calculation procedure. Question two required students to show and discuss multiple representations of the cosine ratio. Question three required students to show the three trigonometric ratios using the same form of representation—formula, graph, or table—and compare these representations. These prompts were identical for both the pre-test and post-test.

The development of these instruments (pre-test and post-test) drew from the researcher's and the two subject matter specialists' knowledge and experiences working with CSEC mathematics. Ali (2014), a book of CSEC sample multiple-choice questions were used to help develop the 12 items for the multiple-choice section of the pre-test and post-test. The researcher developed the written response section drawing from CSEC past papers. The first prompt in the written response section of the post-test (last on the pre-test) modelled a typical CSEC trigonometric ratios question, and the other prompts were developed by the researcher to reflect the work of Kilpatrick et al. (2001). Kilpatrick et al. (2001) claimed that learners show conceptual understanding when they can represent a concept in multiple ways, see and discuss the connections among these representations, and see and discuss the similarities and differences among different representations of a concept.

Once developed, the test was passed on to the two subject-matter specialists for review. The test papers and an information letter (Appendix E) explaining the purpose and structure of the test were emailed to the two teachers. They were asked to speak to the suitability of the test to assess students' understanding of the sine, cosine, and tangent ratios, and to make suggestions for changes that might enhance the test. Both teachers indicated that the test was assessing students' understanding of the trigonometric ratios (Appendix G). However, one teacher suggested that a question with a diagram (right-angle triangle) could have been included among the first three multiple-choice items to assess students understanding that the trigonometric ratios as they relate to right-angle triangles. Both the researcher and the other teacher/subject-matter specialist felt that it was not necessary to change any of these items because all three made explicit reference to the right-angle triangle. Hence, the test was kept as circulated.

Ideally, this test should have been field-tested and its psychometrics reported in this study. Field testing was not done given the chaos which existed in the education sector in Dominica after Hurricane Maria. Instead, two experts in the area of Mathematics Education were used to verify the suitability of the data collection instrument—the tests. This approach is in line with Polit and Beck (2016), who are leaders in the field of research in nursing education. Also, a search of the literature produced no established instruments that could have been used for this study.

Assessment rubric.

The researcher developed an assessment rubric after all teaching had taken place, but before the completed test-papers were accessed for grading. That is, the researcher did not know students' written responses on the pre-test or post-test when he developed the assessment rubric. Such timing was necessary to eliminate the researcher's bias towards one group of students–the group taught by Exemplification or the group taught by Investigation. For the same reason (elimination of researcher bias), the researcher developed the assessment rubric after all teaching had taken place. In doing so, he could not have tailored his teaching to benefit one group over the other.

Participants' written responses were graded with the help of the assessment rubric (Appendix M). This rubric was developed by the researcher, who drew from the work of Kilpatrick et al. (2001) and Hiebert and Carpenter (1992). It comprised of three domains based on Kilpatrick et al. (2001) work on representations. The domains were: representing a contextual problem, multiple representations of a single concept, and comparing representations of related concepts. These representation domains matched the prompts on the pre-test and post-test that solicited students' written responses.

The rubric also had five quality categories comprising of both qualitative and quantitative labels (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001).The categories and labels are 4–Full representational knowledge, 3–Good representational knowledge, 2– Fair representational knowledge, 1–Very limited representational knowledge, and 0– No representational knowledge. The numbers represent the marks that could be awarded for work falling within a labelled category. Furthermore, standards for the labelled categories and themes were included.

These standards were developed by the researcher and discussed with the two raters in a workshop designed to help them understand how to best use this rubric. The standards have both quantity and quality indicators (Hiebert & Carpenter, 1992). Quantity refers to the number and variety of meaningful representations students can produce for the domain: multiple representations of a single concept, and the number and variety of significant similarities and differences they can identify among representations for the domain: comparing representations for related concepts. Quality refers to levels of accuracy in students' representations in the domain: multiple representations of a single concept, in their comparisons in the domain: comparing representations of related concepts, and in their representation of a contextual problem in the domain: representing a contextual problem. Raters were asked to use their professional judgment together with these indicators to assign the most appropriate marks to students' written responses.

Quantitative data analysis.

Five types of data analyses were conducted on the data collected through the pre-test and post-test. They were a Spearman-rho correlation, a two-way, 2 x 2 mixed ANOVA, an analysis of students' correct responses on the multiple-choice items, a response rate analysis of students' written response on the post-test (student did not respond to questions on the pre-test that required a written answer), and a categorical analysis of students' scores for their written responses on the post-test. This categorical analysis included a Chi-square test of homogeneity used to determine whether there was a difference in the conceptual understanding of both groups of students. These analyses are detailed below.

Inter-rater consistency.

A Spearman-rho correlation analysis was conducted to determine the level of consistency between the scores of the two mathematics education specialists who graded the written response section of the post-test, and to determine a method for obtaining a single score for each participant. The scores derived from the written section of the post-test provided the information for this analysis. Both specialists had graded the written response section of the post-test for both groups of students, the group taught by Investigation and the group taught by Exemplification. Thus, there were two sets of scores for each student for that section of the post-test. The researcher wanted to determine the degree of consistency between the raters, allowing for the mean of their scores to suffice as the single score for a participant. Hence, a Spearman-rho correlation coefficient was determined using IBM SPSS software version 25 at a significance level of 0.05. A significance level of 0.05 is commonly used in research in the social sciences.

Analysis of achievement.

For the two-way, 2 x 2 mixed ANCOVA, IBM SPSS software version 25 was also used to analyze test scores. Scores were extracted from the relevant spreadsheets containing students' pre-test scores and post-test scores. The researcher performed the specified analysis. In this analysis, there were two categorical independent variables, one within-subject variable, and one between-subject variable. The within subject-variable was Test-type with two levels: pre-test and post-test. The between-subject variable was Methods (Teaching methods) with two levels: Investigations and Exemplification. The one continuous dependent variable was test scores, which was taken as students' achievement of the three primary trigonometric ratios.

For this analysis, both the sphericity and homogeneity assumption was tested. The sphericity assumption was tested using Mauchly's test, and the homogeneity assumption was tested using Levene's test. The analysis generated a descriptive statistics table showing the means and standard deviations of the pre-test and post-test scores for both groups. It also generated a table depicting the test of within-subject effects showing the main effect and the interaction effect. Another table showed the test of between-subject main effect and a profile plot gave a pictorial representation of the differences within and between the two groups. The results of this analysis were provided in chapter four.

Analysis of the number of correct responses on multiple-choice items.

All students completed the multiple-choice section on both the pre-test and the post-test. Both tests were parallel and designed to assess students' understanding of the three primary trigonometric ratios on four constructs. Three out of the 12 questions were designed to evaluate students' understanding of the structures of the ratios. Another three questions were designed to evaluate students' understanding of the relationship between the sine and cosine ratios. Three more questions were designed to evaluate students' understanding of representations of the sine, cosine, and tangent ratios, and three questions were designed to evaluate students' understanding of calculations with the formulas of the three primary trigonometric ratios. *Figure 3.21* shows the table used to record the number of correct answers for each construct.

Constructs	**Exemplification**		**Investigation**	
	Pre	**Post**	**Pre**	**Post**
Structure of the ratios.				
The relationship between the sine ratio and the cosine ratio.				
Representations of the ratios.				
Calculations with formulas of the ratios.				

Figure 3.21: Table used in analyzing correct responses on multiple-choice items.

The first column of the table shows the four constructs assessed by the multiple-choice items in the pre-test and the post-test. The number of correct answers for each construct for the Exemplification group on the pre-test and post-test were recorded in the second column, which was sub-divided, and the number of correct answers for each construct for the Investigation group on the pre-test and post-test were recorded in the third column, which was also sub-divided. That is, each correct response on a test was manually counted, and the total number (frequency) placed in the appropriate row and column to show the total number of correct responses obtained by the group for a construct. Further analyses were performed on these frequency distributions to show the construct in which the groups performed the worst and best, the construct in which the most and least gains were achieved, and the two groups of students were compared based on their performance in the different constructs. The result of this analysis was integrated with the results of other analyses to give a measure of students' conceptual understanding.

Analysis of the number of written responses.

Students' use of representations of a mathematical concept is closely linked to their understanding of that concept (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001; NCTM, 2000). Students were asked to respond in writing to three prompts that required them to use representations of the three primary trigonometric ratios. These prompts were developed around the work of Kilpatrick et al., (2001) and Hiebert and Carpenter (1992) to help gauge students' understanding of these trigonometric ratios.

The first prompt required students to demonstrate their understanding of the trigonometric ratios by representing and discussing a contextual problem. The second prompt required students to represent and discuss the cosine ratio using multiple forms of representations. The third prompt required students to represent the three primary trig-ratios using the same form of representation and discuss the similarities and differences among these representations. The number of students within a group (the group taught by exemplification and the group taught by Investigation) who responded to these prompts was taken as a measure of the group's understanding of the three primary trigonometric ratios.

No student responded to any of these prompts in the pre-test, which indicated that they were unfamiliar with concepts of the three primary trigonometric ratios before the teaching of these concepts. However, students responded to these prompts in the post-test, which indicated that they had gained a measure of understanding of the three primary trigonometric ratios during the teaching of these concepts. An analysis of the number of written responses (response rate) per prompt per group was shown using a frequency distribution. *Figure 3.22* shows the frequency distribution (table) used to record the number of written responses on the post-test.

Representation Domains (Prompts)	**Parts of the Prompts**	**Number of Responses**	
		Investigation	**Exemplification**
Representing a contextual problem.	1. Drawing a diagram to represent an angle of elevation problem.		
	2. Discussing a procedure to find the height of a building.		
Multiple representations of a single concept.	1. Representing the cosine ratio using three different forms of representations.		
	2. Justifying your choices of representations.		
Comparing related concepts.	1. Representing the three primary trig-ratios using the same form of representation.		
	2. Comparing the representations used to show the different ratios.		
Total			

Figure 3.22: Table used in analyzing the number of written responses on the post-test.

The first column in *Figure 3.22* shows the prompts reduced to what the study calls "Representation Domains"—Representing a contextual problem, Multiple representations of a single concept, and Comparing related concepts. The study used these representational domains to identify what students were able to do with representations of the three primary trigonometric ratios. Students' work within these representational domains provided a measure of their conceptual understanding of the three primary trigonometric ratios.

The second column shows the different parts of the three prompts: each prompt had two parts. The third column, Number of Responses, was further split into two columns: the first of the two was used for recording the number of responses from the group taught by Investigation in the pre-test and post-test, and the second of the two was used for recording the number of responses from the group taught by Exemplification in the pre-test and post-test. Each written response for a group was manually counted, and the total number (frequency) placed in the appropriate row and column. The total number of responses for each group was also calculated by adding all values within the respective columns.

Further analyses were performed on this frequency distributions to show the percentage rate of responses for each part of a prompt, for each representation domain, and for the total number of responses for each group of students. These analyses were done only for the post-test because no students had responded to these prompts in the pre-test. The percentage response rate for a single part was calculated by dividing the number of written responses by the number of students (16) who completed the post-test. The percentage response rate for a representation domain (prompt) was calculated by adding the number of responses for both parts of the prompt and dividing the answer by 32: the total number of possible responses for a prompt. The overall percentage response rate for a group was calculated by adding the number of responses within a group column and dividing the answer by 96: the total number of possible written responses by a group. The result of this analysis was integrated with the results of other analyses to give a measure of students' conceptual understanding.

Categorical analysis of scores for written responses.

The score obtained by students on each prompt soliciting their written responses on the post-test was also an indicator of their understanding of the three primary trigonometric ratios. Hence, an analysis was done on these scores to get a numerical measure of each group's understanding of the three primary trigonometric ratios. scores were analyzed based on the three representation domains identified previously: representing a contextual problem, multiple representations of a single concept, and comparing related concepts. Students' scores in each domain were taken as the mean of the scores given by the two people who marked their post-test papers. Four was the maximum score that could be obtained in each domain, and students who did not provide an answer to a prompt were given zero for the relevant section of that prompt.

For this analysis, students' scores for the written responses on the post-test were placed in three categories. Scores of three or higher represented a good understanding, scores from two to less than three represented an average understanding, and scores of less than two represented a low understanding. These ranges were used to reflect what obtains at most schools in Dominica. From personal experience, this researcher knows that the pass mark at most schools in Dominica is 50% (55% at some schools). That is, students who obtain 50% or 55% in a test, in an exam, or on an assignment would have obtained a passing score. A score of 75% is considered a good score by most Dominican schools, and a student who obtains a score of 75% or higher would be a good performer. Hence, in this context using a four-point scale, it was reasonable to use a range of three to four to represent a good understanding, a range of two to less than three to represent an average understanding, and a range of zero to less than two to represent a low understanding. *Figure 3.23* shows the table used to analyze students' scores of their written responses on the post-test.

Representation Domains (Prompts)	**Investigation**				**Exemplification**		
	$M \geq 3$	$2 \leq M < 3$	$M < 2$		$M \geq 3$	$2 \leq M < 3$	$M < 2$
Representing a contextual problem.							
Multiple representations of a single concept.							
Comparing related concepts.							
Totals							

Figure 3.23: Table used in analyzing scores on written-response in post-test.

The first column in *Figure 3.23* shows the representation domains. The second column shows the data associated with the group taught by Investigation, with further sub-division to show score ranges. The third column shows the data associated with the group taught by Exemplification, also with further sub-division to show score ranges. $S \geq 3$ denotes the range of scores that are three or higher and signifies a good conceptual understanding of the trigonometric ratios, $2 \leq S < 3$ denotes the range of scores from two to less than three and signifies an average conceptual understanding of the trigonometric ratios, and $S < 2$ denotes the range of scores that are less than two and signifies a low conceptual understanding of the trigonometric ratios. The totals represent the total number scores (frequency) within a range across all prompts. Scores that fell in each range for each representation domain per group were manually counted and the total placed in the appropriate row and column of the distribution table.

Further analyses were performed on this frequency distributions to show the number and percentage rate for each category for each representation domain and an overall percentage rate for all domains. These percentages were compared across groups to provide a measure of students' conceptual understanding. The results of this analysis was integrated with the results of other analyses to give a measure of students' conceptual understanding and the differences in conceptual understanding between the two groups of students. As an added measure, a Chi-square test of homogeneity was conducted to determine if the conceptual understanding of students taught by Investigation differed from that of the students taught by Exemplification. The test was conducted using version 25 of the IBM SPSS statistics software with a confidence level of 0.05. More details about these analyses are provided in Chapter four.

Qualitative data analysis.

This study drew on the definitions and explanations of conceptual understanding as espoused by Kilpatrick et al. (2001) and Hiebert and Carpenter (1992) in its analysis and interpretation of students' conceptual understanding. Kilpatrick et al.'s work was used for its qualitative indicators (identifying and connecting different representations depicting the same idea, selecting representations that are suitable for a situation, and explaining the similarities and differences among representations). Hiebert and Carpenter (1992) was used primarily for its quantitative indicator (quantity of connections among representations) in this mixed methods research study. Below are the details of the qualitative analysis.

A form of document analysis (Bowen, 2009) was conducted on students' written responses for the post-test, with each group's work being analyzed separately. According to

Bowen (2009), a "[d]ocument analysis is a procedure for reviewing or evaluating documents" (p. 27). Documents are printed or e-copies of texts and images (Bowen, 2009). Students' test papers with their written responses were considered documents in this study. Document analysis combines content analysis and thematic analysis to generate themes (Bowen, 2009). Bowen (2009) described content analysis as "the process of organizing information into categories" (p. 32) based on the research question. He described thematic analysis as "a form of pattern recognition within the data, with emerging themes becoming the categories for analysis" (Bowen, 2009, p. 32). According to Bowen (2009), document analysis involves the iterative processes of (a) superficially examining the documents to get a sense of what they obtain, (b) thoroughly reading and examining the contents of the documents to develop categories, and (c) interpreting the contents to develop themes. These three processes, superficially examining the documents, thoroughly reading and examining the contents of the documents, and interpreting the contents to develop themes, were used to analyzed students' written responses on the post-test.

First, the researcher read superficially (skimmed) the written responses of all 32 students to get an overall impression of their answers. Skimming highlighted the prompts to which students responded the most, some errors that students made in their responses, and the form of representation students gravitated towards. There were three prompts soliciting students' written responses: (1) Draw a diagram to represent a contextual situation, and discuss how that diagram can be used to solve the problem embedded in the situation. (2) Show and discuss multiple representations of the cosine ratio. (3) Show and compare the three trigonometric ratios using the same form of representation. All 32 students' responses to a single prompt were skimmed before going to responses for another prompt. This helped the researcher to keep in view the full range of responses for a single prompt. Also, focusing on a single prompt at a time helped the researcher to apply the same perspective to all answers for anyone prompt (Jonsson & Svingby, 2007). This process helped the researcher devise his strategy for further analysis.

Second, the researcher carefully read through students' answers, obtained relevant content information, and listed these contents information for each student's work in a table from which themes were identified. One table was used for each prompt, and a different table was used for each group. That is, the analysis for each group made use of three tables; one for analysing the prompt that asked students to represent and discuss a contextual problem, one for the prompt that asked students to represent and discuss the cosine ratio, and one for the prompt that asked students to represent and compare the three ratios. Thus, a total of six tables were used in this analysis. *Figure 3.24* shows an example of the tables used for the group taught by Investigation.

I.D	**Correct answer**	**Error**	**Themes**	
			Title	**content**
I-1				
I-2				
I-3				
Etc.				

Figure 3.24: Table used to develop themes for written responses.

The first three columns (I.D, Correct answer, and Error) of the table were used during the second stage of the analysis. The column labelled I.D showed the identifier on students' papers from which information was obtained. I-1 is a student in the group taught by Investigation with and arbitrary number 1. E-4 (not shown) is a student from the group taught by Exemplification with an arbitrary number 4. These identifiers were written on each paper before any marking or analysis was done, and the originals papers locked away. These original papers were never referred to at any point during this analysis. However, identifying the papers in the analysis help the researcher to continually review and refer to a specific piece of work. The parts of students' answers that were appropriate, or information about them, were obtained and placed in the second column–Correct answer–and the errors, or information about them, were obtained and placed in the third column–Error. This information (contents) was obtained from all 32 post-test papers for the first prompt and placed in the appropriate table–a table for the group taught by Exemplification and one for the group taught by Investigation.

Students' correct answers in column two, the Correct answer column, were colour coded based on the part of the prompt they answered correctly. For instance, the first prompt asked students to represent a contextual problem (angle of elevation) and discuss an appropriate procedure for finding a dimension (the height of a building). The analysis of students' answers to this prompt focused on three considerations: students drew a diagram to represent the problem, students placed information on the diagram, and students discussed a procedure for calculating the height of the building. The correct portion, or a note describing what was correct, for each consideration was placed in the second column of the table for each student. The correct portions, or notes, for the same consideration, was coloured the same. That is, there was a maximum of three colour codes in the second column of the tables depicted in *Figure 3.24* above. These were further analyzed to identify themes among students' correct responses. On the other hand, the incorrect portion, or a note describing what was incorrect, for each consideration was placed in the Error column (the third column) of the table for each student. These errors were coloured differently based on the type of error. That is, errors of a similar nature were coloured the same for all students. Different colours to those used in the second column were used. These were also further analyzed to identify themes among students' errors.

The third stage of the analysis used the colour coded contents of the second column–Correct answer–and the third column–Error–to find themes to fill the fourth column–Title–under the heading; Themes (see *Figure 3.24* above). The fifth column, which is under the heading, Themes, was used to make the identification of themes easier. The contents from the second column and the third column that were coded with the same colours were brought together, without repetition, in column five. That is, several students had the same correct answers and the same errors; however, each correct answer and each error, regardless of the number of students who produced them, was presented once in column five, with contents with the same colour grouped. Themes were identified by looking at the contents within each colour coded group. These themes were written in the fourth column labelled as Title in *Figure 3.24* above.

The processes described above were repeated for the second and third prompts that asked students to show and discuss multiple representations of the cosine ratio and show and compare the three trigonometric ratios using the same form of representation. However, the considerations for each prompt were different. For the second prompt, which asked students to show and discuss multiple representations of the cosine ratio, the considerations focused on students' representations of the cosine ratio; the cosine formula, the cosine curve for angles between zero and 90^0, and a table showing the behavior of values for angles between zero and 90^0; and students reasons for how they knew that each representation shows the cosine ratio. For the third prompt, which asked students to to show the three trigonometric ratios using the same form of representation and to compare these representations, the considerations focused on the use of the same representations: the sine, cosine, and tangent formulas; or the sine, cosine, and tangent curves for angles between zero and 90^0; or a table showing the behavior of values for angles between zero and 90^0 for the sine, cosine, and tangent ratios; and similarities and differences identified among the sine, cosine, and tangent ratios. Themes were identified using students' correct answers and errors, as was done for the first prompt. All theses were presented in chapter five and are support with extracts from students' work.

The analysis also presented two profiles based on the themes identified; a profile for the group taught by Exemplification and one for the group taught by Investigation. These group profiles reflected the portions of each prompt that students within a group (group taught by Exemplification and group taught by Investigation) answered correctly and the types of errors these students made. These profiles, which are presented in chapter five, are mixed with other quantitative analyses in chapter six to give a measure of students' conceptual understanding of the three primary trigonometric ratios.

Treatment of Results

The results of the analyses are presented in two chapters. The result of the Spearman-rho correlation, the ANOVA, the correct multiple-choice items analyses, the analysis of scores on written responses, and the analysis of the number of written responses (response rate) are presented in chapter four. The correlation showed a high level of consistency between the two raters. This high level of consistency indicated that participants' true scores were likely to lie between the two raters' scores, thus, allowing the mean of these two scores to be taken as a participant's true score. The results of the ANOVA were used to answer the research questions related to differences in students' levels of achievement of the three primary trigonometric ratios. The results of the other three quantitative analyses; the analysis of correct multiple-choice items, the analysis of scores on written responses, and the analysis of the number of written responses; were integrated with the results of the qualitative analysis, presented in chapter five, to answer the research question related to differences in students' conceptual understanding of the three primary trigonometric ratios. This integration was presented in chapter six.

Integration.

Integration refers to the stages of the research process where the quantitative and qualitative data are combined, with greater priority given to either the qualitative or quantitative results or equal priority given to both (Creswell et al., 2003). According to Creswell et al. (2003), priority refers to the weight given to each, quantitative and qualitative, type of data collected during a mixed-methods design. Creswell et al. further argued that priority is given to data types depending on the goals of the study, the scope of the research question(s), and the intention of each data type. In this study, priority was given to the qualitative results to answer the sub-question: How did students' conceptual understanding differ after being taught using Investigation compared with Exemplification? Conceptual understanding was determined by students' use of representations (Hiebert & Carpenter, 1992; Kilpatrick et al., 2001). Students' representations, which they produced and discussed in the post-test, were qualitative data.

However, the results of the three quantitative analyses that were integrated with the qualitative analysis helped to determine the differences between the conceptual understanding of the two groups of students–those taught by Exemplification and those taught by Investigation. The analysis of the number of written responses (response rate) showed the number of students who contributed to the group profile presented in chapter five. The analysis of the scores for the written-responses showed the quality of scores obtained by students in each group. The analysis of correct multiple-choice items showed the differences in the groups' performances on the following constructs: structure of the three primary trigonometric ratios, the relationship between the sine ratio and the cosine ratio, representations of the three primary trigonometric ratios, and calculations with formulas of the three primary trigonometric ratios. Steps were also taken to ensure the validity and reliability of all data collected and analyzed.

Validity

Several steps were taken to ensure the internal validity of this study. One was using two experts for validating the data collection instruments; pre-test, post-test, and assessment rubric used to grade students' written responses. Using this measure to validate a data collection instrument was recommended by Polit and Beck (2016) in the absence of field testing the instruments. These same experts were also used to grade students' written responses. Using more than one expert to grade students' written responses was in line with Danielson and Marquez (2016), who advocated for several experts to be involved in judging students' written work. The qualifications and actions of these experts were discussed in the data collection section of this chapter. Danielson and Marquez argued that instruments developed by and performances judged by several experts usually provide more objective assessments because they embrace the professional judgment of more than one person.

A second validity measure taken was in the timing of the pre-test and the post-test. The pre-test was given immediately before the teaching to ensure that it measured the achievement and conceptual understanding of the three primary trigonometric ratios that students took into the teaching. The post-test was given immediately after the teaching to ensure, as much as was possible, it measured only the impact of the teaching on students' achievement and conceptual understanding of the three primary trigonometric ratios.

A third measure taken was in the timing of the development of the assessment rubric. The assessment rubric was developed by the researcher after all teaching was finished, but before any of the test papers were retrieved from the locked cupboard for grading. This timing helped the researcher reduced, if not eliminate, the introduction of bias into the study. Developing the assessment rubric before teaching would have given the researcher-teacher a keen sense of what was to be assessed by the rubric; thus, he might have taught to favour one group over the other.

Developing the rubric after retrieving the test papers and reading students' answers would have given him a keen sense of both the correct answers and errors of both groups. Thus, he might have developed the rubric to favour the answers of one group over the other. This researcher avoided both unwanted situations, teaching that favours one group over the other and a rubric that favours the answers of one group over the other by developing the rubric between teaching and retrieving the papers for grading. This measure helped to enhance the validity of the study.

A fourth measure taken to ensure the internal validity of the study was in the experimental design. The study used an experimental design in which students were randomly assigned to two groups. According to Gamst et al. (2008), a randomized design relies on randomization to control for the effects of confounding variables. Confounding variables are potential causal variables that were not explicitly included in the study. By randomly assigning participants to treatment groups, the researcher assumes that, on average, confounding variables will affect each group equally, so any significant differences between conditions can fairly be attributed to the independent variables. Students from two existing fourth forms were randomly assigned to two new groups in this study.

Strengths and Limitations

This study has both strengths and limitations. Among its strengths is the mixed-methods approach that allowed the study to account for both students' achievement and their conceptual understanding. Achievement scores are the most widely used measure of mathematics performance in most western society, and it is used to make decisions about students' future. This study kept students' achievement scores in focus. Conceptual understanding of mathematics concepts is critical to students' future educational and other mathematics-related endeavours. This study also kept conceptual understanding, a construct that is difficult to measure through quantitative means, in focus. That is, through a mixed-methods approach, this study ironed out the deficiencies of a purely quantitative design or a purely qualitative design by combining the two (Ercikan & Roth, 2006; Johnson & Onwuegbuzie, 2004).

The experimental design was another strength of this study. In this design, participants were randomly assigned to groups; thus, the researcher was able to control for many, if not all, of the differences that existed between the established classes of students before the study.

As a further strength, the data collection instruments were developed and validated by trained and experienced subject matter specialists, who worked within the same education system as the students for over ten years. This action is significant if inferences made from this study are to be taken seriously. These experts also graded the participants' work with a high degree of consistency.

The study, however, was not without limitations. The researcher could not account for or completely prevent collaboration between students from both groups because they were all from the same school and lived in the same communities. However, he administered the pre-test just before teaching and the post-test immediately after teaching to minimize the confounding effects of collaboration. Also, he ensured that while one group was engaged in studying the three primary trigonometric ratios, the other group was engaged in studying a seemingly unrelated topic. Other factors such as gender imbalance between the groups, the effects of hurricane maria on students, and the reverse sequencing of the two content strands—algebra and the primary trigonometric ratios—for the groups might have also affected the internal validity of the study. Furthermore, the small sample size may have affected the external validity (generalizability) of the study. These limitations were discussed in Chapter one.

Ethical Considerations

Ethics review is an important aspect of research projects to be undertaken by students attending the University of Alberta. It is of particular importance when the research involves human subjects. The University has strict guidelines to govern such research, and a board exists that reviews and gives permission to conduct research. This researcher got permission from the Research Ethics Board at the University of Alberta to conduct his study into the effects that Exemplification and Investigation have on students' achievement and conceptual understanding of the three primary trigonometric ratios.

The University of Alberta's research ethics board indicates that ethical reviews be focused on the protection of the participants. That is, research studies involving humans should maximize benefits and minimize the chance that harm may come to them (University of Alberta, 2013). This consideration is in keeping with provincial and federal legislation and regulations. In adherence to these policies, legislation, and regulations, the researcher got informed consent from all necessary parties. The researcher is also aware that consent must be voluntary, based on enough information, and an adequate understanding of both the proposed research and its implications (University of Alberta, 2013). This researcher adhered to these demands by obtaining permission from relevant authorities and persons. He informed all interested groups and individuals of the details of the study and its possible benefits and risks to students, the school, and the Ministry of Education.

The researcher also ensured and continue to ensure that the collection and management of data adhere to the guidelines of the research ethics board of the University of Alberta. All participants' information was held in strict confidence and will be properly disposed of at the appropriate time. As such, students' responses to question prompts were anonymized before exposing them to anyone. Further, these data sets were kept on a password-protected computer, and the hard copies were kept in a locked cupboard at my home. They will be kept for a minimum of five years after a successful defence, after which they will be shredded and burnt.

The researcher has done all in his power to ensure that no individual or institution was negatively affected by this study.

This chapter presented the data collection methods and procedures used in this current study. The next chapter, Chapter four, presents the quantitative results of this study. These quantitative results were used to answer the study three hypotheses and provide evidence of measures of students' understanding. The quantitative results presented in Chapter four were based on the data provided by the pre-test and post-test described in chapter three.

Chapter 4: Quantitative Results

A mixed-methods approach, quantitative and qualitative, was used in analyzing the data in this research study. This chapter presents the quantitative data analyses aimed at answering the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* The chapter focuses on quantitative data analyses used to determine differences in levels of achievement within and between the two groups; the group taught by Investigation and the group taught by Exemplification. It also presents quantitative data analyses that relate to students' understanding of representations of the three primary trigonometric ratios. Students' understanding of representations of the three primary trigonometrical ratios was used to determine their conceptual understanding of these ratios.

The contents of this chapter are presented in sections. The first section presents participants' demographic information, followed by a description of an analysis to determine the inter-rater consistency of the two mathematics education specialists who graded students' written responses. This analysis helped to determine a method for computing a single score for each participant from the scores assigned by the two raters. The next section focuses on students' achievement. It comprised of an ANOVA and answers to the three research questions related to effects on students' achievements. The penultimate section provided information about students' understanding of representations, which is critical in answering the research question related to the effects on students' conceptual understanding. The chapter ends with a summary of the quantitative analyses performed.

Participants' Demographics

Thirty-five fourth-form students originally participated in this Research Study. Three participants did not do the post-test; hence, the results of the study will compare how the teaching affected 32 participants' achievement and conceptual understanding of the three primary trigonometric ratios. Of these 32 fourth-form students, approximately 53% ($N = 17$) were female and approximately 47% ($N = 15$) were male. The ages of the students ranged from 15 to 18, with the mean age of 16 and a standard deviation of 1.23 (M = 16, SD = 1.23). All 32 participants were working towards taking the CSEC mathematics examination in May of 2019 and followed the same mathematics curriculum. Analyses of participants' final scores for mathematics at the end of fourth-form are shown in *Figure 4.1* and *Table 4.1*.

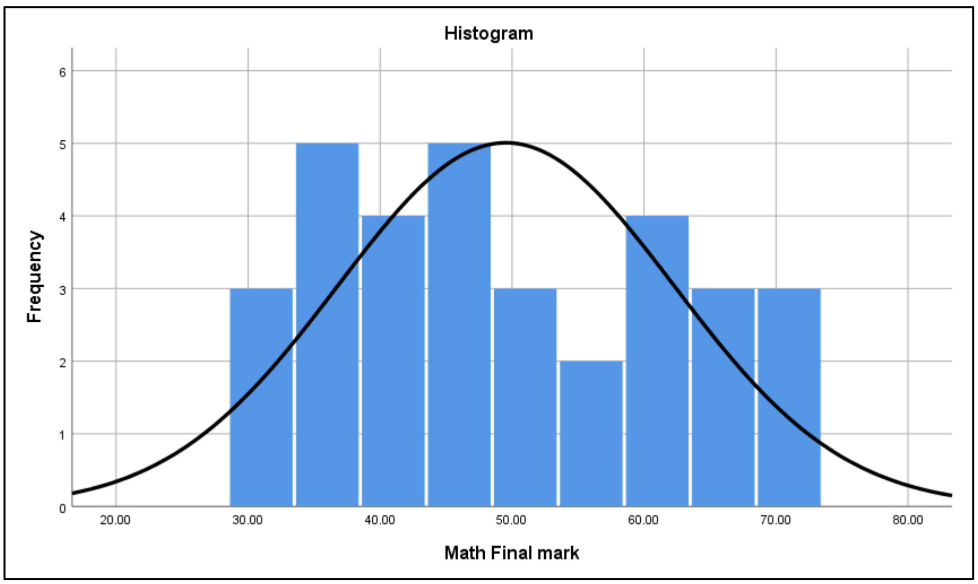


Figure 4.1: Histogram and curve showing participants' scores for mathematics at the end of fourth-form.

Figure 4.1 shows a histogram accompanied by a curve. The curve indicates that participants' mean score for mathematics at the end of fourth form was close to 50. The histogram shows that there were no outliers. *Table 4.1* provides more details about the mean, median, maximum and minimum values, and standard deviation for the participants.

Table 4.1

Statistics of participants' final scores for mathematics at the end of fourth-form.

N	Valid	32
	Missing	0
Mean		49.53
Median		48.00
Std. Deviation		12.75
Minimum		31.00
Maximum		70.00

Table 4.1 shows that the 32 participants final scores for mathematics at the end of fourth-form had a mean of 49.53 and a standard deviation of 12.75. The median score was 48.00, with a minimum score of 31 and a maximum score of 70. These scores show that, in general, these 32 students were average performers in mathematics.

Spearman Rho Correlation

Two raters marked the written response section of the post-test resulting in two scores for each participant. The scores for each rater were ranked and a Spearman rho correlation coefficient was used to show how consistently higher or lower one rater scored these written responses in relation to the other rater. The Spearman correlation was chosen instead of the Pearson correlation because the data used to obtain the coefficient was the ranking of the scores (ordinal data) given by the two raters. Pearson correlation is most suited when the data are continuous. Table 4.2 shows the result of this analysis conducted in version 25 of the IBM SPSS software.

Table 4.2

Spearman ranked correlation coefficient for inter-rater consistency.

			Rater A	Rater B
Spearman's rho	Rater A	Correlation Coefficient	1.000	.853**
		Sig. (2-tailed)	.	.000
		N	32	32
	Rater B	Correlation Coefficient	.853**	1.000
		Sig. (2-tailed)	.000	.
		N	32	32

**. Correlation is significant at the 0.01 level (2-tailed).

The analysis presented in Table 4.2 shows a Spearman rho correlation coefficient of 0.853 (r_s = 0.853). This correlation coefficient shows a very strong positive linear relationship between the two raters scores (Garmst et al., 2008). That is, one rater consistently graded participants higher than the other. The analysis also shows that there is a more than 99% chance that this level of consistency did not happen by chance. Hence, it is reasonable to assume that a participant's true score falls between the scores given by the two raters. Therefore, participants' scores for the written response section on the post-test were taken as the means of the two raters' scores.

Achievement

A participant's level of achievement, hereafter called achievement (A), was taken as the raw score (RS) obtained from adding his score on the written response (WR) section to his score on the multiple-choice (MC) section. There were 12 multiple-choice items, each contributing one mark for a correct answer, and three written response prompts, each contributing a maximum of four marks. Thus, a participants' raw score ranged from zero to 24, with 24 being the maximum attainable raw score. Table 4.3 shows the raw scores both groups of participants obtained in the pre-test and post-test.

Table 4.3

Participants raw scores obtained in the pre-test and post-test

Investigation						Exemplification					
Pre-test			Post-test			Pre-test			Post-test		
MC	WR	**TRS(A)**	MC	WR	**TRS(A)**	MC	WR	**TRS(A)**	MC	WR	**TRS(A)**
2	0	**2**	10	8.75	**18.75**	4	0	**4**	9	10.00	**19.00**
3	0	**3**	8	10.25	**18.25**	3	0	**3**	7	11.50	**18.50**
1	0	**1**	8	8.50	**16.50**	2	0	**2**	9	9.25	**18.25**
2	0	**2**	7	7.75	**14.75**	2	0	**2**	9	9.00	**18.00**
4	0	**4**	7	5.75	**12.75**	1	0	**1**	8	8.75	**16.75**
1	0	**1**	6	6.00	**12.00**	4	0	**4**	8	8.75	**16.75**
3	0	**3**	6	5.75	**11.75**	2	0	**2**	9	7.75	**16.75**
2	0	**2**	5	6.25	**11.25**	5	0	**5**	8	7.75	**15.75**
1	0	**1**	7	3.50	**10.50**	2	0	**2**	8	7.13	**15.13**
9	0	**9**	6	4.00	**10.00**	1	0	**1**	5	9.50	**14.50**
2	0	**2**	5	4.50	**9.50**	6	0	**6**	6	7.63	**13.63**
1	0	**1**	5	3.75	**8.75**	1	0	**1**	5	7.38	**12.38**
1	0	**1**	3	4.00	**7.00**	5	0	**5**	8	3.25	**11.25**
2	0	**2**	3	2.50	**5.50**	2	0	**2**	5	4.25	**9.25**
7	0	**7**	2	1.25	**3.25**	1	0	**1**	4	4.00	**8.00**
2	0	**2**	1	1.25	**2.25**	4	0	**4**	4	3.75	**7.75**

In Table 4.3, MC denotes Multiple-Choice, WR denotes Written Response, TRS denotes Total Raw Score, and A denotes Achievement, which was taken as the total raw score. The numbers under each item (MC, WR, TRS) are the marks that students received for that section of the test; there are 16 marks under each item signifying that there were 16 participants (students) in each group. The marks for the written responses (WR) on the pre-test in both groups are all zeros. No student attempted that section in the pre-test and was given a zero for that section of the pre-test. The total raw score (Achievement) for a participant was determined by adding his MC mark to his WR mark. WR marks were taken as the average of the two raters marks (see Appendix N) for that section. An ANOVA (described later in this chapter) was used to determine the differences in achievement (TRS) between the group taught by Investigation and the group taught by Exemplification. Following is a summary of the findings obtained from the ANOVA, and details of the ANOVA follows this summary of findings.

Summary of findings.

The ANOVA that follows this summary was used to answer the research sub-questions one, two, and three and their related hypotheses. These are the findings of the ANOVA:

Research question one: How was students' level of achievement affected after being taught by Investigation?

Result: *Students taught by Investigation had a statistically significant increase in their level of achievement.*

Research question two: How was students' level of achievement affected after being taught by Exemplification?

Result: *Students taught by Exemplification had a statistically significant increase in their level of achievement.*

Research question three: How did students' levels of achievements differ after being taught by Investigation compared with Exemplification?

Result: *Students taught by Exemplification level of achievement was statistically significantly higher than students taught by Investigation.*

Analysis of variance in achievement.

This ANOVA was conducted to answer research sub-questions one, two, and three and their related hypotheses. There were two independent variables. One was the teaching approach (Method) with two levels: Investigation and Exemplification. The other was Test-type, also with two levels: Pre-test and Post-test. The dependent variable was Achievement (Total Raw Score) that was given as students' test scores. It was a two-way, 2 x 2 mixed ANOVA with repeated measures on the Test-type variable. The analysis was performed using version 25 of the IBM SPSS software. The Omnibus test was performed using a significance level of 0.05. The significance level of 0.05 is the most commonly used in research in the social sciences; it provides a 95% level of confidence that the null hypothesis is not falsely rejected.

Descriptive statistics.

Descriptive statistics are coefficients used to summarize a data set. These coefficients may include measures of central tendencies such as mean, mode, and median and measures of variability such as range (the difference between maximum and minimum values) and standard deviation. Table 4.3 shows the descriptive statistics for students' pre-test and post-test scores that were analyzed in the ANOVA. The table presents the mean and standard deviation for students taught by Exemplification and those taught by Investigation.

Table 4.4

Descriptive statistics for both groups of students.

Test Type	Method	Item Type	Mean	Std. Deviation	Frequency
Pre-test	Investigation	MC	2.69	2.27	16
		RR	0	0	16
		RS	**2.69**	**2.27**	16
	Exemplification	MC	2.81	1.64	16
		RR	0	0	16
		RS	**2.81**	**1.64**	16
Post-test	Investigation	MC	5.56	2.39	16
		RR	5.23	2.64	16
		RS	**10.80**	**4.85**	16
	Exemplification	MC	7.00	1.86	16
		RR	7.48	2.45	16
		RS	**14.48**	**3.75**	16

In Table 4.4, a breakdown of the pre-test scores showed that students taught by Investigation had a mean score of 2.69 (11.21%) with a standard deviation of 2.27. Those taught by Exemplification had a mean score of 2.81 (11.71%) with a standard deviation of 1.64. These statistics indicate that the difference in achievement between the two groups in the pre-test was minimal (0.5%) in favour of the group taught by Exemplification with students taught by Investigation obtaining a wider range of scores.

Also in Table 4.4, a breakdown of the post-test scores showed that students taught by Investigation had a mean score of 10.80 (45%) with a standard deviation of 4.85. Those taught by Exemplification had a mean score of 14.48 (60%) with a standard deviation of 3.75. These statistics indicate that the difference in achievement between the two groups in the post-test was 3.68 (15%) in favour of the group taught by Exemplification with students taught by Investigation obtaining a wider range of scores.

Assumptions.

Violation of the sphericity assumption was not an issue for the ANOVA described above. Sphericity is the equal variance between combinations of two levels of a repeated measure in an ANOVA (Gamst et al., 2008). If the variance between different combinations of two levels of the repeated measure are not equal, the sphericity assumption is violated, and there is an increased chance that the result of the ANOVA gives a Type 1 error: show a significant statistical difference between levels of the repeated measure when there is none (Gamst et al., 2008). However, the violation of the sphericity assumption is not an issue when there are only two levels of the repeated measure (Gamst et al., 2008). There were only two levels of the repeated measure in the ANOVA for this study (pre-test and post-test); hence, violation of the sphericity assumption was not an issue.

Violation of the homogeneity assumption was also not an issue because the design was a balanced one with an equal number (16) of participants in both groups (Gamst et al., 2008). Homogeneity is the equal variance between groups in an ANOVA (Gamst et al., 2008). If the variance between groups is not equal, the homogeneity assumption is violated, and there is an increased chance that the result of the ANOVA gives a Type 1 error: show a significant statistical difference between groups when there is none (Gamst et al., 2008). However, the violation of the homogeneity assumption is hardly an issue when there is the same number of participants in both groups (Gamst et al., 2008). Table 4.5 shows the result for Levene's test of homogeneity.

Table 4.5

Results for Levene's test of homogeneity

		Levene Statistic	df1	df2	Sig.
Pre-test	Based on Mean	.126	1	30	.725
	Based on Median	.000	1	30	1.000
	Based on Median and with adjusted df	.000	1	25.494	1.000
	Based on trimmed mean	.022	1	30	.883
Post-test	Based on Mean	.490	1	30	.489
	Based on Median	.563	1	30	.459
	Based on Median and with adjusted df	.563	1	28.430	.459
	Based on trimmed mean	.514	1	30	.479

Table 4.5 shows that the significance values were higher than .05 for both the pre-test ($p = .725$) and the post-test ($p = .489$) based on the mean. These p-values indicate that the homogenous assumption was not violated for the pre-test or the post-test.

Within-subjects' effects.

A within-subject effect is the amount of variance between two levels of a repeated measure for an individual or a group of participants (Gamst et al., 2008). The amount of variance between the mean of students' scores on the post-test and the mean of students' scores on the pre-test is the within-subject effect determined in this study. This within-subject effect was determined for the group taught by Exemplification and for the group taught by Investigation. Table 4.6 shows the ANOVA results for the within-subject effect for both groups of students.

Table 4.6

Results for within-subject effects

Source		Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Tests	Sphericity Assumed	1563.955	1	1563.955	126.404	.000	.808
	Greenhouse-Geisser	1563.955	1.000	1563.955	126.404	.000	.808
	Huynh-Feldt	1563.955	1.000	1563.955	126.404	.000	.808
	Lower-bound	1563.955	1.000	1563.955	126.404	.000	.808
Tests * Methods	Sphericity Assumed	50.543	1	50.543	4.085	.052	.120
	Greenhouse-Geisser	50.543	1.000	50.543	4.085	.052	.120
	Huynh-Feldt	50.543	1.000	50.543	4.085	.052	.120
	Lower-bound	50.543	1.000	50.543	4.085	.052	.120
Error (Tests)	Sphericity Assumed	371.181	30	12.373			
	Greenhouse-Geisser	371.181	30.000	12.373			
	Huynh-Feldt	371.181	30.000	12.373			
	Lower-bound	371.181	30.000	12.373			

Table 4.6 is the summary table for the within-subject effect of the ANOVA. It shows a significant main effect on Test-type, $F(1, 30) = 126.404$, $p < .001$, $\eta_p^2 = 0.808$. This main effect indicates that if the method by which the groups were taught is ignored, the post-test scores for at least one group is significantly higher than the pre-test scores for that group. Also, approximately 81% of the variance between the post-test scores and pre-tests scores is accounted for by the teaching. Also, Table 4.6 shows that Test-type did not interact significantly with Method, Test-type * Methods, $F(1, 30) = 4.085$, $p = .052$, $\eta_p^2 = 0.120$. This non-significant interaction indicates that the post-test scores were significantly higher than the pre-test scores for both levels of Methods (Exemplification and Investigation). That is, the post-test scores for both the group of students taught by Investigation and the group of students taught by Exemplification were significantly higher than their respective pre-test scores.

Between-subjects' effects.

A between-subject effect is the amount of variance between two groups of participants (Gamst et al., 2008). The amount of variance between the mean score for students taught by

Exemplification and the mean score for students taught by Investigation is the between-subject effect determined in this study. This between-subject effect was determined for both pre-test and the post-test. Table 4.7 shows the ANOVA result for this between-subject effect.

Table 4.7

Results for the between-subject effects.

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Intercept	3788.018	1	3788.018	366.571	.000	.924
Methods	57.903	1	57.903	5.603	.025	.157
Error	310.009	30	10.334			

Table 4.7 is the summary table for the between-subject effect in the ANOVA. It shows that there was a significant main effect of Methods, $F(1, 30) = 5.603$, $p = .025$, $\eta_p^2 = 0.157$. This effect indicates that there is a significant difference between the two levels of Method, Investigation and Exemplification, for at least one level of the Test-type. The difference in achievement between the two groups in the pre-test was minimal (0.5%), as indicated in Table 4.4; thus, the significant difference between the groups came in the post-test. Table 4.7 also indicates that the teaching accounts for approximately 16% of the variance that existed between the groups.

Effects on achievement.

The effects of the teaching on students' achievement are shown in the answers to the following sub-questions and their related hypotheses:

1. How was students' level of achievement affected after being taught by Investigation?

H_{01}: There was no significant difference between the pre-test achievement and post-test achievement for students taught by Investigation (H_{01}: $\mu_{pre} = \mu_{post}$).

H_{A1}: Post-test achievement is statistically significantly higher than pre-test achievement for students taught by Investigation (H_{A1}: $\mu_{post} > \mu_{pre}$).

2. How was students' level of achievement affected after being taught by Exemplification?

   H_{02}: There is no significant difference between the pre-test achievement and post-test achievement for students taught by Exemplification (H_{02}: $\mu_{pre} = \mu_{post}$).

   H_{A2}: Post-test achievement is significantly higher than pre-test achievement for students taught by Exemplification (H_{A2}: $\mu_{post} > \mu_{pre}$).

3. How did students' levels of achievement differ after being taught by Investigation compared with Exemplification?

   H_{03}: There is no significant difference in the level achievement for students taught by Investigation compared to those taught by Exemplification (H_{03}: $\mu_I = \mu_E$).

   H_{A3}: The level of achievement of students taught by Exemplification is significantly higher than the level of achievement of students taught by Investigation. (H_{A3}: $\mu_E > \mu_I$).

Sub-question one and its related hypothesis.

Table 4.6 and Table 4.4 show that there is a significant difference ($P < 0.001$) between the pre-test scores (M = 2.69, SD = 2.27) and the post-test scores (M = 10.80, SD = 4.85) for students taught by Investigation. These statistics indicate that the method Investigation had significantly increased students' achievement scores. Therefore, the study rejects the null hypothesis H_{01} and concludes that students taught by Investigation had a significant increase in their level of achievement of the three primary trigonometric ratios.

Sub-question two and its related hypothesis.

Table 4.6 and Table 4.4 show that there is a significant difference ($P < 0.001$) between the pre-test scores (M = 2.81, SD = 1.64) and the post-test scores (M = 14.48, SD = 3.75) for students taught by Exemplification. These statistics indicate that the method Exemplification significantly increased students' achievement scores. Therefore, the study rejects the null hypothesis H_{02} and concludes that students taught by Exemplification had a significant increase in their level of achievement of the three primary trigonometric ratios.

Sub-question three and its related hypothesis.

Table 4.7 and Table 4.4 show that there is a significant difference ($P = 0.025$) between the post-test scores for the method Investigation (M = 10.80, SD = 4.85) and the post-test score for the method Exemplification (M = 14.48, SD = 3.75). These statistics indicate that the achievement of students taught by Exemplification was significantly higher than the achievement of students taught by Investigation. Therefore, the study rejects the null hypothesis H_{03} and concludes that the level of achievement of students who were taught the three primary trigonometric ratios by Exemplification was significantly higher than the level of achievement of students who were taught by Investigation.

The results for sub-questions one, two, and three can be seen in *Figure 4.2*, which is a profile plot showing the differences in marginal means (see Table 4.4) for achievement scores.

Figure 4.2: Profile plot showing marginal means for achievement scores.

Figure 4.2 shows a minimal difference between marginal means of both groups of students on the pre-test (Test-type 1), and a larger difference between their marginal means on the post-test (Test-type 2). The post-test marginal mean for the group taught by Exemplification was higher than that of the group taught by Investigation. Also, it shows the marginal means of both groups on the post-test being higher than their marginal means on the pre-test.

Conclusion.

A two-way, 2 x 2 mixed ANOVA design was used to assess the effects of the teaching on students' achievement of the three primary trigonometric ratios. The group of fourth-form students who took the pre-test was exposed to teaching for 16 teaching sessions of 90 minutes each session. One group of students was taught using Investigation and the other using Exemplification. They were given a post-test that parallel the pre-test to obtain achievement scores (achievement). The independent within-subjects variable was Test-type with two levels: pre-test and post-test. The independent between-subject variable was Methods with two levels: Investigation and Exemplification. The dependent variable was students' achievement scores on both the pre-test and the post-test.

The two-way interaction (Test-type*Methods) was not significant, $F(1, 30) = 4.085$, $p =$ 0. 052, $\eta_p^2 = 0.120$. However, all main effects were significant at $p < .05$. For the effects of the teaching on students' achievement, students taught by Investigation had a significant increase in their achievement, students taught by Exemplification also had a significant increase in their achievement, and students taught using Exemplification achievement was significantly higher than students taught using Investigation.

Conceptual Understanding

Conceptual understanding is a construct that cannot be measured directly; hence, it must be measured indirectly. The analysis of students' use of representations (Kilpatrick et al., 2001; Hiebert & Carpenter, 1992) was the indirect approach used to measure their conceptual understanding in this study. That is, a student's conceptual understanding of the three primary trigonometric ratios was measured across these three representational domains: his ability to identify and discuss different representations of the ratios, to produce and discuss a diagram representation of a contextual problem based on the ratios, and to compare the same form of representation of these ratios. Three prompts that solicited written responses from students were used to obtain data that were in line with these representational domains. Students' responses to these prompts were graded using marks from zero to four.

A mark of zero for a prompt indicated that the student either did not respond to the prompt or his representation was not related to the prompt. In both cases, a zero was taken to mean that the student had no conceptual understanding of the three primary trigonometric ratios. Marks that were less than two suggested a low conceptual understanding, marks that were greater than or equal to two by less than three suggested an average conceptual understanding, and marks that were greater than or equal to three suggested a good conceptual understanding. The reason for this categorization was given in Chapter two. This categorization was applied to both the pre-test and post-test. However, no student responded to the written response prompts in the pre-test, thus, showing their lack of understanding of the three primary trigonometric ratios before the teaching.

Therefore, the following two analyses were conducted to compare the conceptual understanding of the two groups of students, those taught by Investigation and those taught by Exemplification, based on their written responses on the post-test. One analysis compared the numbers of written responses completed by both groups, and the other analysis compared the number of written responses from both groups that fell into the three categories of conceptual understanding—Good, Average, and Low. Below are the details of these analyses.

Summary of findings.

Three comparisons of the conceptual understanding of the three primary trigonometric ratios of students' taught by Investigation or exemplification were made in this section. Two comparisons dealt with percentages and one was a Chi-square test of homogeneity. One comparison of percentages found that students taught by Exemplification submitted more written responses on the post-test than the group taught by Investigation, which was interpreted as higher attainment of conceptual understanding. The second comparison of percentages found that more students taught by Exemplification obtained grades that depict an average–good level of conceptual understanding on the post-test than the group taught by Investigation, which had more students obtaining grades that depict a low level of conceptual understanding. The Chi-square test of homogeneity showed no statistically significant difference between the groups' levels of conceptual understanding. The analyses that led to these comparisons are described below.

Analysis of the number of written responses.

No student responded to a single prompt on the pre-test that solicited a written response, which was taken to mean that students had no conceptual understanding of the three primary trigonometric ratios before the teaching. However, students from both groups responded to prompts on the post-test that solicited a written response, which was taken to mean that students had gained a measure of conceptual understanding of the three primary trigonometric ratios from the teaching. The section of the post-test that solicited students' written responses contained three prompts, and each prompt had two parts:

1. A dog is lying on the ground 25 metres away from the foot of a building. It observes a bird on top of the building at an angle of elevation of 20^0.
    i. Draw a diagram (triangle) to represent the situation described above. Mark clearly on your diagram: the distance of 25 metres, its line of sight, the 20^0 angle, and the height of the building.

   ii. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).
2. This question requires you to use and discuss multiple representations of the cosine ratio.
   i. Including the formula, show **<u>at least</u>** three (3) different representations of the cosine ratio.
   ii. Discuss how you know that each of these representation shows the cosine ratio.
3. This question requires you to show the three trigonometric ratios using the **<u>same form</u>** of representation, then to compare and contrast these representations.
   i. Use the **<u>same form</u>** of representation to show the sine, cosine, and tangent ratios.
   ii. Discuss the similarities and differences among these representations.

Part one of each prompt required students to produce at least one representation—formula, graph, table, or diagram—related to the three primary trigonometric ratios. Part two of each prompt required students to discuss, explain, justify, or compare the representation(s) they produced in part one of the same prompt. Table 4.7 shows the prompts (reduced to representation domains), the two parts of each prompt that students responded to, and the number of students who responded to the different parts of each prompt, which indicated the number of student who gained a measure of conceptual understanding of the different aspects of the ratios. Table 4.7 shows the number of written responses for both the group taught by Exemplification and the group taught by Investigation.

Table 4.8

Number of written responses on the post-test for both groups of students.

Representation Domains (Prompts)	**Parts**	**Number of Responses**	
		Investigation	**Exemplification**
Representing a contextual problem.	1. Drawing a diagram to represent an angle of elevation problem.	15	16
	2. Discussing a procedure to find the height of a building.	12	16
Multiple representations of a single concept.	1. Representing the cosine ratio using three different forms of representations.	13	16
	2. Justifying choices of representations.	10	15
Comparing related concepts.	1. Representing the three primary trig-ratios using the same form of representation.	12	15
	2. Comparing the representations used to show the different ratios.	11	16
Totals		**73**	**94**

Group taught by Investigation.

There were 16 students in the group taught by Investigation, who completed the post-test. Fifteen (94%) responded to part one, and 12 (75%) responded to part two of the prompt categorized as representing a contextual problem (the first prompt on the post-test). Thirteen (81%) responded to part one, and 10 (63%) responded to part two of the prompt categorized as multiple representations of a single concept (the second prompt on the post-test). Twelve (75%) responded to part one, and 11 (69%) responded to part two of the prompt categorized as comparing related concepts (the third prompt on the post-test).

No prompt received a 100% response rate for this group of students, which indicated that for every aspect of the trigonometric ratios that were tested, some students did not gain a conceptual understanding. The prompt categorized as representing a contextual problem received the highest response rate of 84% (27 out of a possible 32 responses). The other two prompts, multiple representations of a single concept and comparing related concepts, received a response rate of 72% (23 out of a possible 32 responses). The overall response rate for this group of students was 76% (73 out of a possible 96 responses), which indicated that overall, 76% of this group of students gained a measure of conceptual understanding of the three primary trigonometric ratios.

Group taught by Exemplification.

There were also 16 students in the group taught by Exemplification who completed the post-test. Sixteen (100%) responded to part one, and 16 (100%) responded to part two of the prompt categorized as representing a contextual problem (the first prompt on the post-test). Sixteen (100%) responded to part one, and 15 (94%) responded to part two of the prompt categorized as multiple representations of a single concept (the second prompt on the post-test). Fifteen (94%) responded to part one, and 16 (100%) responded to part two of the prompt categorized as comparing related concepts (the third prompt on the post-test). The prompt categorized as representing a contextual problem received the highest response rate of 100% (32 out of a possible 32 responses). The other two prompts, multiple representations of a single concept and comparing related concepts, received a response rate of 97% (31 out of a possible 32 responses). The overall response rate for this group of students was 98% (94 out of a possible 96 responses), which indicated that overall, 98% of this group of students gained a measure of conceptual understanding of the three primary trigonometric ratios.

A comparison of the two groups number of written responses.

Students in both groups responded to all three prompts, but with different response rates. Students in the Exemplification group submitted more responses than students in the Investigation group for every prompt, which indicated that students in the group taught by Exemplification had gained a greater measure of conceptual understanding of the three primary trigonometric ratios than the group taught by Investigation. The group taught by Exemplification had a 100% response rate (submitted 32 out of a possible 32 responses) for the prompt categorized as representing a contextual problem compared to an 84% response rate (submitted 27 out of a possible 32 responses) for the group taught by Investigation. The group taught by Exemplification had a 97% response rate (submitted 31 out of a possible 32 responses) for the prompt categorized as multiple representations of a single concept compared to a 72% response rate (submitted 23 out of a possible 32 responses) for the group taught by Investigation. The group taught by Exemplification had a 97% response rate (submitted 31 out of a possible 32 responses) for the prompt categorized as comparing related concepts compared to a 72% response rate (submitted 23 out of a possible 32 responses) for the group taught by Investigation. The group taught by Exemplification had an overall 98% response rate (submitted 94 out of a possible 96 responses) compared to an overall 76% response rate (submitted 73 out of a possible 96 responses) for the group taught by Investigation.

Categorical analysis of scores for written responses.

This analysis was done on scores obtained from the written-response section of the post-test; no student responded to any prompt on the pre-test soliciting their written response. A student’s score for each prompt was taken as the mean of the scores given by the two people who marked students' written responses on the post-test. Four was the maximum attainable score for each prompt, and students who did not respond to a prompt were given zero for that prompt. Scores were tabulated based on the three representational domains (prompts) identified previously: representing a contextual problem, multiple representations of a single concept, and comparing related concepts. Table 4.9 shows the results of the analysis of scores obtained by both groups of students in the written response section of the post-test.

Table 4.9

Analysis of scores on the written-responses for both groups of students

Representation Domains (Prompts)	Investigation			Exemplification		
	$S \geq 3$	$2 \leq S < 3$	$S < 2$	$S \geq 3$	$2 \leq S < 3$	$S < 2$
Representing a contextual problem.	1	10	5	2	8	6
Multiple representations of a single concept.	2	3	11	8	2	6
Comparing related concepts.	7	1	8	10	0	6
Totals	**10**	**14**	**24**	**20**	**10**	**18**

In Table 4.9, $S \geq 3$ denotes the range of scores that are three or higher and signifies a good conceptual understanding of the trigonometric ratios, $2 \leq S < 3$ denotes the range of scores from two to less than three and signifies an average conceptual understanding of the trigonometric ratios, and $S < 2$ denotes the range of scores that are less than two and signifies a low conceptual understanding of the trigonometric ratios. The numbers (e.g. 1, 2, and 7 under $S \geq 3$**)** represent the frequencies of scores. The totals represent the total number scores within a range across all prompts. The same ranges are presented for both Investigation and Exemplification.

Group taught by Investigation.

For the prompt categorized as representing a contextual problem, one student (6%) obtained a score within the good conceptual understanding range, ten students (63%) obtained a score within the average conceptual understanding range, and five students (31%) obtained a score within the low conceptual understanding range. For the prompt categorized as multiple representations of a single concept, two students (12%) obtained a score within the good conceptual understanding range, three students (19%) obtained a score within the average conceptual understanding range, and 11 students (69%) obtained a score within the low conceptual understanding range. For the prompt categorized as comparing related concepts, seven students (44%) obtained a score within the good conceptual understanding range, one student (6%) obtained a score within the average conceptual understanding range, and eight students (50%) obtained a score within the low conceptual understanding range. Out of a total of 48 possible responses for the section, ten responses (21%) were scored within the good conceptual understanding range, 14 responses (29%) were scored within the average conceptual understanding range, and 24 responses (50%) were scored within the low conceptual understanding range.

Group taught by Exemplification.

For the prompt categorized as representing a contextual problem, two students (12%) obtained a score within the good conceptual understanding range, eight students (50%) obtained a score within the average conceptual understanding range, and six students (38%) obtained a score within the low conceptual understanding range. For the prompt categorized as multiple representations of a single concept, eight students (50%) obtained within the good conceptual understanding range, two students (12%) obtained a score within the average conceptual understanding range, and six students (38%) obtained a score within the low conceptual understanding range. For the prompt categorized as comparing related concepts, ten students (62%) obtained within the good conceptual understanding range, zero students obtained a score within the average conceptual understanding range, and six students (38%) obtained a score within the low conceptual understanding range. Out of a total of 48 possible responses, 20 responses (41.5%) were scored within the good conceptual understanding range, ten responses (21%) were a score within the average conceptual understanding range, and 18 responses (37.5%) were scored within the low conceptual understanding range.

A comparison of the groups' scores within categories.

Students in both groups obtained scores in all three scoring categories on all three prompts, which indicates a wide range of conceptual understanding within each group.

Approximately 42% of the responses submitted by the group taught by Exemplification received a score in the good conceptual understanding category compared to approximately 21% of the responses submitted by the group taught by Investigation. Approximately 21% of the responses submitted by the group taught by Exemplification received a score in the average conceptual understanding category compared to approximately 29% of the responses submitted by the group taught by Investigation. Approximately 38% of the responses submitted by the group taught by Exemplification received a score in the low conceptual understanding category compared to the 50% of the responses submitted by the group taught by Investigation. That is, approximately 63% of the responses submitted by the group taught by Exemplification received a score in at least the average conceptual understanding category compared with 50% of the responses submitted by the group taught by Investigation. These comparisons indicate that the group taught by Exemplification gained a greater measure of conceptual understanding of the three primary trigonometric ratios than the group taght by Investigation.

Chi-square test of homogeniety.

As a further comparison, a Chi-square test of homogeneity was performed to determine whether the conceptual understanding of the group taught by Investigation differed significantly from that of the group taught by Exemplification in any of the three categories (Good, Average, and Low). There were two categorical variables, Method with two levels—Investigation and Exemplification—and Conceptual understanding with three levels—Good, Average, and Low. The analysis was performed using version 25 of the IBM SPSS software using a significance level of 0.05. The significance level of 0.05 is the most used in research in the social sciences; it provides a 95% level of confidence that the null hypothesis is not falsely rejected. Table 4.10 shows the contingency table used in this Chi-square test.

Table 4.10

Contingency table showing the number of scores within categories for both groups

Method	**Conceptual Understanding**			
	Good	Average	Low	Total
Investigation	10	14	24	48
Exemplification	20	10	18	48
Total	30	24	42	96

In Table 4.10, a good conceptual understanding was associated with the range of scores that were three or higher ($S \geq 3$), an average conceptual understanding was associated with the range of scores from two to less than three ($2 \leq S < 3$), and a low conceptual understanding was associated the range of scores that were less than two ($S < 2$). The numbers (e.g. 10 and 20 under good) represent the frequencies of scores in the conceptual understanding categories.

There were three *Null hypotheses* for this Chi-square test of homogeneity:

- The proportion of responses within the good conceptual understanding category given by students taught by Investigation is identical to the proportion of responses given by students in that category taught by Exemplification. (H_o:$P_{\text{Investigation in Good}} = P_{\text{Exemplification in Good}}$)
- The proportion of responses within the average conceptual understanding category given by students taught by Investigation is identical to the proportion of responses given by students in that category taught by Exemplification. (H_o:$P_{\text{Investigation in Average}} = P_{\text{Exemplification in Average}}$)
- The proportion of responses within the average conceptual understanding category given by students taught by Investigation is identical to the proportion of responses given by students in that category taught by Exemplification. (H_o:$P_{\text{Investigation in Low}} = P_{\text{Exemplification in Low}}$)

The *Alternate hypothesis* for this Chi-square test was:

The proportion of responses within at least one of the conceptual understanding categories is not identical for both groups of students.

The following are the results of the Chi-square test of homogeneity performed on the contingency table above (Table 4.10) using IBM SPSS version 25 with a confidence level of 0.05.

Table 4.11

*Method*Conceptual Understanding crosstabulation table*

			Conceptual Understanding			
			Good	Average	Low	Total
Methods	Exemplification	Count	20	10	18	48
		Expected Count	15	12	21	48
		% within Methods	41.7%	20.8%	37.5%	100.0%
	Investigation	Count	10	14	24	48
		Expected Count	15	12	21	48
		% within Methods	20.8%	29.2%	50.0%	100.0%
Total		Count	30	24	42	96
		Expected Count	30	24	42	96
		% within Methods	31.3%	25.0%	43.8%	100.0%

Table 4.11 shows the count, expected count, and the percentage (proportion) for each method in each conceptual understanding category. Of great significance is the expected count because it speaks to the suitability in using the Chi-square test for the given data set. A suitable condition for the use of the Chi-square test is for the expected count in each cell to be five or more. The expected count for Good was 15, for Average was 12, and for Low was 21 for both Investigation and Exemplification. Hence, the Chi-square test was appropriate for the above data set. Table 4.12 shows the result of the Chi-square test of Homogeneity conducted on the data set with a confidence level of 0.05.

Table 4.12

Result for the Chi-square test of homogeneity

	Value	df	Asymptotic Significance (2-sided)
Pearson Chi-Square	4.857[a]	2	.088
Likelihood Ratio	4.928	2	.085
N of Valid Cases	96		

Table 4.12 shows a test statistic of 4.857, a degree of freedom of two, and a *p*-value of .088 for the Pearson Chi-Square. The *p*-value of .088 indicates that the test statistic was **not** significant. Therefore, for the Chi-square test of homogeneity that was performed to determine whether the conceptual understanding of the group taught by Investigation differed significantly from that of the group taught by Exemplification in any of the three categories (Good, Average, and Low), there was no significant difference between the two groups in any of the conceptual understanding categories, X^2 $(2, N = 96) = 4.86$, $p = .088$. That is, both groups of students had the same conceptual understanding in all categories.

While this Chi-square test shows that there was no significant difference in the conceptual understanding of both groups of students, the comparison of the groups based on percentages suggested that the group taught by Exemplification may have attained a higher level of conceptual understanding than the group taught by Investigation. This apparent contradictory evidence warrants that the power of the Chi-square test be determined. Statistical power is the probability of rejecting a false null hypothesis and it is equal to one minus Beta $(1 - \beta)$ where Beta is the probability of accepting a false null hypothesis (making a type II error). An acceptable level of Beta for studies in the social sciences is 20% or lower (Gamst et al., 2008), thus, making the acceptable levels of power being 80% or higher (0.80 or higher).

According to Cohen (1992), power is affected by the alpha level, effect size, and sample size. For this Chi-square test, the alpha level was set at 0.05, which is common for research studies in the social sciences, and the effect size was set at 0.5 which represents a large effect size for Chi-square tests (Cohen, 1992). A large effect size was chosen to determine the smallest sample needed to obtain an acceptable power of 0.80; thus, increasing the likelihood of rejecting the null hypothesis if there is a positive effect—a significant difference in the conceptual understanding of the group taught by Exemplification and the group taught be Investigation. Table 4. 13 shows the results of the power calculation done using an online power calculator (https://www.masc.org.au/stats/PowerCalculator/PowerChiSquare).

Table 4.13

Power calculation results for the Chi-square test

Statistical Test	Sample Size	Degrees of Freedom	Effect size	Significance Level	Power
Chi-Square	32	2	0.5	0.05	0.7176

Table 4.13 shows a calculated power of 0.7176 (approximately 0.7 or 70%) using the sample size of 32, two degrees of freedom, a large effect size of 0.5, and an alpha level of 0.05. A power of 0.7 (70%) is below the level accepted for studies in the social sciences with 0.8 being the lowest acceptable power. This low power is most likely as a result of the small sample size which was identified as a limitation of the study. With a power of approximately 0.7, there is an approximate Beta level of 0.3 or 30%. Thus, there was a near 30% chance that a type II error was made in this Chi-square test. A 30% chance of a type II error is unacceptable for research in the social sciences; hence, greater emphasis was placed on the analysis of percentages.

Conclusion.

Three analyses—percentages of the number of written responses, percentages of scores for the written responses within categories, and a Chi-square test of homogeneity—were done to compare the conceptual understanding of the group of students taught by Investigation to that of the group taught by Exemplification. The two analyses that compared percentages indicated that the group taught by Exemplification might have attained a higher level of conceptual understanding of the three primary trigonometric ratios than the group taught by Investigation. A Chi-square test of homogeneity indicated that the apparent higher level of conceptual understanding attained by the group taught by Exemplification was not statistically significant. However, this Chi-square test had low statistical power, thus, greater emphasis was placed on the analyses of percentages. These findings will be integrated with the findings from Chapter five, where the conceptual understanding of the two groups were analyzed and compared using qualitative methods, to provide more details about the differences in conceptual understanding attained by both groups of students. The following analysis helps provide some of these details.

Analysis of Correct Responses on Multiple-choice Items

All students answered the multiple-choice items on both the pre-test and post-test. These tests were developed to assess students' understanding of three constructs related to the three primary trigonometric ratios: the structures of the three primary trigonometric ratios, relationships between the sine and cosine ratios, the different forms of representations (e.g. formulas, graphs, tables) of the three primary trigonometric ratios, and calculations with the formulas of the three primary trigonometric ratios. The first three multiple-choice items—items one, two, and three—on both the pre-test and post-test assessed students' understanding of the structures of the three primary trigonometric ratios. The next three multiple-choice items—items four, five, and six—on both the pre-test and post-test assessed students' understanding of relationships between the sine and cosine ratios, The next three multiple-choice items—items seven, eight, and nine—on both the pre-test and post-test assessed students' understanding of the different forms of representations of the three primary trigonometric ratios. The last three multiple-choice items—items ten, 11, and 12—on both the pre-test and post-test assessed students' understanding of calculations with the formulas of three primary trigonometric ratios. Table 4.14 shows the number of correct responses obtained by both groups for each construct in both the pre-test and the post-test.

Table 4.14

Analysis of correct responses to MC items for both groups

Constructs	Investigation		Exemplification	
	Pre	**Post**	**Pre**	**Post**
Structure of the ratios.	11	31	8	39
The relationship between the sine ratio and the cosine ratio.	9	13	12	15
Representations of the ratios.	7	29	14	32
Calculations with formulas of the ratios.	12	20	11	26
Totals	**39**	**93**	**45**	**112**

In Table 4.14, Pre denotes pre-test and Post denotes post-test. The numbers (e.g. 11, 9, 7, and 12 under Pre**)** represent the number of correct responses (frequencies) in a category. The totals represent the total number of correct responses within a category across all constructs. The maximum possible correct responses for a construct was 48 (three questions per construct times 16 students per group) in both the pre-test and post-test for a group of students.The same categories are presented for both Investigation and Exemplification.

Group taught by Investigation.

This group of students obtained 11(23%) correct responses on the pre-test and 31(65%) correct responses on the post-test in the construct identified as the structure of the three primary trigonometric ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 42% (20 additional correct responses). Students obtained nine (19%) correct responses on the pre-test and 13(27%) correct responses on the post-test in the construct identified as relationships between the sine and cosine ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 8% (four additional correct responses). Students obtained seven (15%) correct responses on the pre-test and 29(60%) correct responses on the post-test in the construct identified as representations of the three primary trigonometric ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 45% (22 additional correct responses). Students obtained 12 (25%) correct responses on the pre-test and 20(42%) correct responses on the post-test in the construct identified as calculations with formulas of the ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 17% (eight additional correct responses). Overall, there were a possible 192 responses across constructs. This group of students obtained 39 (20%) correct responses on the pre-test and 93(48%) correct responses on the post-test. This difference in the number of correct responses (54) between the post-test and the pre-test represented a gain of approximately 28%.

These students registered the highest gain (45%) in the construct identified as representations of the three primary trigonometric ratios and their lowest gain (8%) in the construct identified as relationships between the sine and cosine ratios. They obtained the highest rate (65%) of correct responses in the construct identified as the structure of the three primary trigonometric ratios and the lowest rate (27%) of correct responses in the construct identified as relationships between the sine and cosine ratio on the post-test.

Group taught by Exemplification.

This group of students obtained 8(17%) correct responses on the pre-test and 39(81%) correct responses on the post-test in the construct identified as the structure of the three primary trigonometric ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 65% (31 additional correct responses). Students obtained 12(25%) correct responses on the pre-test and 15(31%) correct responses on the post-test in the construct identified as relationships between the sine ratio and the cosine ratio. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 6% (three additional correct responses). Students obtained 14(29%) correct responses on the pre-test and 32(67%) correct responses on the post-test in the construct identified as representations of the three primary trigonometric ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 38% (18 additional correct responses). Students obtained 11(23%) correct responses on the pre-test and 26(54%) correct responses on the post-test in the construct identified as calculations with formulas of the ratios. The difference between the correct responses on the post-test and on the pre-test for this construct represented a gain of approximately 31% (15 additional correct responses). Overall, there were a possible 192 responses across constructs. This group of students obtained 45 (23%) correct responses on the pre-test and 112(58%) correct responses on the post-test. This difference in the number of correct responses (67) between the post-test and the pre-test represented a gain of approximately 35%.

These students registered the highest gain (65%) in the construct identified as the structure of the three primary trigonometric ratios and their lowest gain (6%) in the construct identified as relationships between the sine and cosine ratios. They obtained the highest rate (81%) of correct responses in the construct identified as the structure of the three primary trigonometric ratios and the lowest rate (31%) of correct responses in the construct identified as relationships between the sine and cosine ratios on the post-test.

A comparison of gains across constructs.

Both groups of students, those taught by Investigation and those taught by Exemplification, increased the number of correct responses in all constructs—structures of the three primary trigonometric ratios, relationships between the sine and cosine ratios, representations of the three primary trigonometric ratios, and calculations with formulas of the ratios—from the pre-test to the post-test. The group taught by Exemplification had a higher percentage gain than the group taught by Investigation in the construct identified as structures of the three primary trigonometric ratios: 65% more correct responses compared to 42% more correct responses. The group taught by Exemplification also had a higher percentage gain than the group taught by Investigation in the construct identified as calculations with the formulas of three primary trigonometric ratios: 31% more correct responses compared to 17% more correct responses. On the other hand, the group taught by Investigation had a higher percentage gain than the group taught by Exemplification in the construct identified as relationships between sine and cosine ratios: 8% more correct responses compared to 6% more correct responses. Also, the group taught by Investigation had a higher percentage gain than the group taught by Exemplification in the construct identified as representations of the three primary trigonometric ratios: 45% more correct responses compared to 38% more correct responses. Overall, the group taught by Exemplification had a higher percentage gain than the group taught by Investigation: 35% more correct responses compared to 28% more correct responses.

Summary

Chapter four presented the quantitative results for this research study: *Comparing the effects of two inquiry-based teaching strategies on secondary students' conceptual understanding and achievement in mathematics: A mixed-methods approach.* Seven quantitative analyses were done and presented in this chapter. An analysis of students end of year fourth-form mathematics grades was presented to show that they were average learners. A Spearman-rho correlation was done to get a measure of consistency between the scores for the two people who graded students' written responses. The result of this Spearman-rho correlation allowed the mean of the raters' scores to be used as a single score for each student. A two-way, 2 x 2, mixed ANOVA was conducted to determine the differences in achievement between the groups. Four other quantitative analyses were presented to provide details about students' understanding of the three primary trigonometric ratios. One that analyzed the number of written responses on the post-test for each group of students, one that analyzed categories of scores students obtained on the written responses, a Chi-square test that analyzed the differences in conceptual understanding of the groups, and one that analyzed the number of correct responses for the multiple-choice items of both the pre- and post-test.

The ANOVA was used to answer the research questions related to the effects of Exemplification and Investigation on students' levels of achievement. It showed that both Investigation and Exemplification significantly increased students' achievement of the three primary trigonometric ratios. It also showed that Exemplification had a significantly higher effect than Investigation on students' achievement of those trigonometric ratios. Three of the other analyses—analysis of the number of written responses, analysis of the scores obtained for written responses, and the analysis of the number of correct responses on the multiple-choice items—were combined with the qualitative analysis of Chapter five, to answer the research question related to the effects that Exemplification and Investigation had on students' conceptual understanding.

Chapter 5: Qualitative Results

This chapter presents the qualitative data analysis aimed at answering the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* It presents the themes generated by a document-type analysis of participants' (*N* = 32) written responses. These themes provide insights into the differences in the two groups of students' (the group taught by Exemplification and the group taught by Investigation) conceptual understanding of the three primary trigonometric ratios after being taught by different teaching approaches—Exemplification or Investigation. That is, while chapter four presented the significant numerical differences between the group taught by Investigation compared to the group taught by Exemplification, chapter five presents the aspects of the trigonometric ratios that both groups of students [mis]understood conceptually. It does so by highlighting the levels of correctness with which participants addressed the prompts soliciting written responses and the errors they made in addressing them.

The chapter presents the themes based on the three representation domains to which the prompts soliciting students' written responses were reduced: representing a contextual problem, multiple representations of a single concept, and comparing related concepts. Each theme is supported with extracts from students' written responses and the researchers' analysis and evaluation of these responses. These analyses were used to develop group profiles: a separate profile for the group taught by Exemplification and the group taught by Investigation. These group profiles provide a measure of each group's conceptual understanding. These profiles are compared across groups. The chapter ends with a summary that recapitulates the main points of the chapter.

Representation Domains

There were three prompts soliciting participants' written responses. The first prompt, which asked students to draw a diagram to represent an angle of elevation problem and discuss how to find the height of a building, solicited information for the representation domain: representing a contextual problem. The second prompt, which asked students to show and discuss different representations of the cosine ratio, solicited information for the representation domain: multiple representations of a single concept. The third prompt, which asked students to represent the three primary trig-ratios using the same form of representation and discuss their similarities and differences, solicited information for the representation domain: comparing related concepts. Each representation domain focused on a qualitative indicator based on the work of Kilpatrick et al. (2001). According to Kilpatrick et al. (2001), students who have attained conceptual understanding can: represent and discuss a contextual problem, present a concept using multiple forms of representations, and discuss similarities and differences among representations of related concepts. How well participants were able to perform these after they were taught the three primary trigonometric ratios is the focus of this chapter.

Group Taught by Investigation

Sixteen fourth-form students participated and completed the post-test for this group. Hence, this analysis shows how teaching through Investigation affected these 16 participants' conceptual understanding of the three primary trigonometric ratios. Of these 16 fourth-form students, approximately 62.5% ($N = 10$) were male and approximately 37.5% ($N = 6$) were female. The ages of the students in this group ranged from 15 to 18, with the mean age of 16.5 years and a standard deviation of 1.79 (M = 16.5, SD = 1.79). The following are the themes generated based on the analysis of their written responses on the post-test.

Representing a contextual problem.

The result of the analysis of students' responses to the following prompt is presented in this section.

> A dog is lying on the ground 25 metres away from the foot of a building. It observes a bird on top of the building at an angle of elevation of 20^0.
>
> iii. Draw a diagram (triangle) to represent the situation described above. Mark clearly on your diagram: the distance of 25 metres, its line of sight, the 20^0 angle, and the height of the building.
>
> iv. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

The analysis of students' responses to this prompt yields four themes: diagram used to represent the problem, selection of the correct ratio, selection of an incorrect ratio, and calculation procedure. The contents of these themes comprised of both correct and incorrect responses and are supported by extracts from students' responses. Extracts are cropped sections of students' written responses that were selected to support the relevant theme.

Theme 1: Diagram used to represent the problem.

Students who responded to section one of this prompt correctly represented the problem with a right-angle triangle. In some cases, the most relevant information was accurately placed on that diagram. *Figure 5.1* presents one such response.

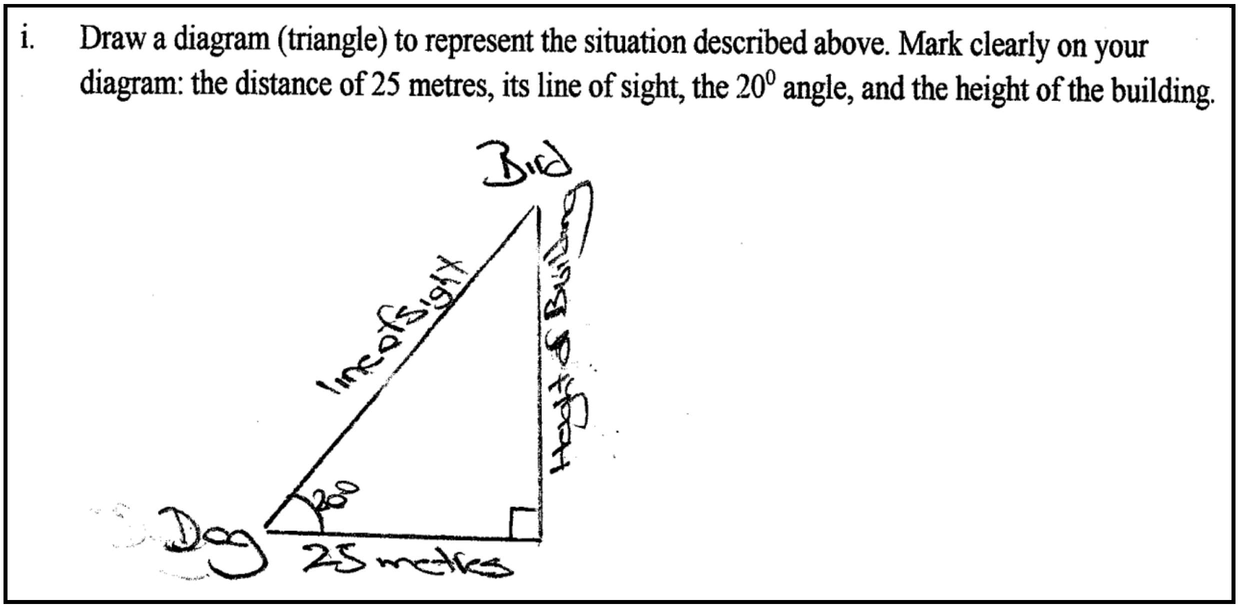


Figure 5.1: Participant I-1 representation of the problem.

Figure 5.1 shows the position of the dog on the ground (horizontal line) 25 meters away from the foot of the building, and the bird positioned at the top of the building. It shows a vertical line representing the building with a right angle between the building and the ground. A diagonal line connects the positions of the dog and the bird; this line represents the dog's line of sight (the direction in which the dog is looking). The 20^0 angle between the diagonal line and the ground represents the angle of elevation. An upward arrow should have been placed on the diagonal line to show that it is the dog looking upwards and not the bird looking downwards; thus, giving significance to the concept of angle of elevation.

In some instances, some relevant information was not placed on the diagram, but the information placed on the diagram were accurate and essential. *Figure 5.2* shows one such response.

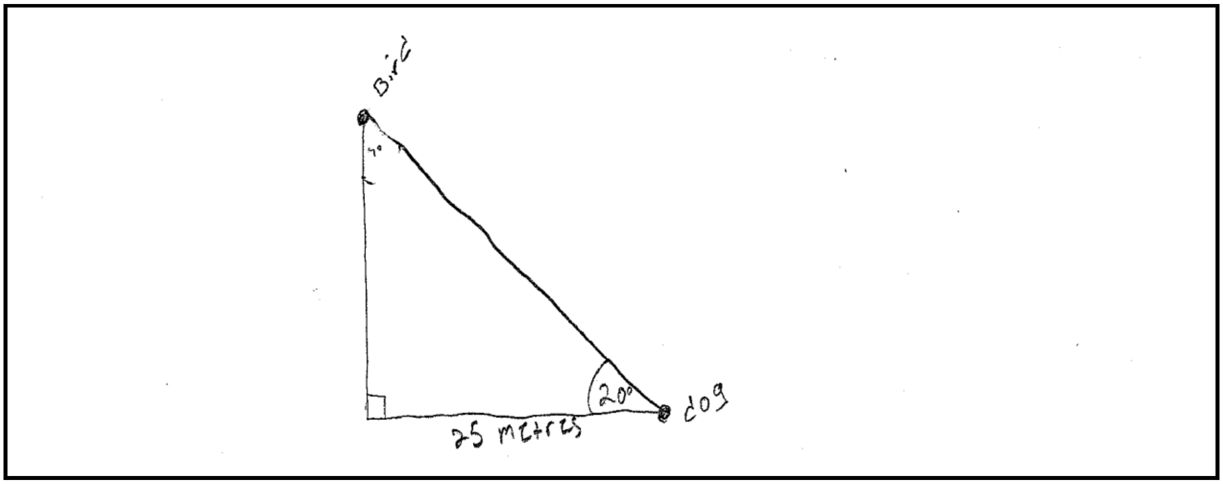


Figure 5.2: Participant I-5 representation of the problem.

Figure 5.2 shows several essential and accurately placed pieces of information. A right-angle triangle is used to represent the problem. The dog is on the ground (horizontal line) 25 meters away from the lower end of the vertical line, and the bird is at the upper end of the vertical line. The 20^0 angle, which is the angle of elevation, is between the diagonal line and the horizontal. Several important pieces of information are missing from this representation: the vertical line should have been labelled to show the building, the diagonal line should have been labelled to show the dog's line of sight, and arrows should have been placed on the diagonal line to show that it is the dog that is looking upwards; thus, an angle of elevation.

Theme 2: Selection of the correct ratio.

Some students correctly selected tangent as the most appropriate ratio to solve the problem but provided a limited justification for that selection. *Figure 5.3* shows one such justification.

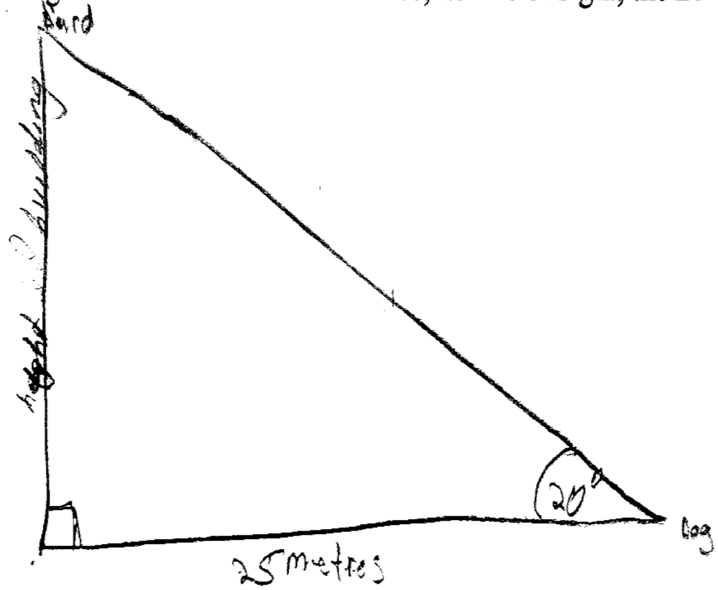


diagram: the distance of 25 metres, its line of sight, the 20° angle, and the height of the building

ii. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

I would find the height of the building by using the tangent ratio to find the height because I have it Adjacent - and the opposite side I would use the formula to find the height

Figure 5.3: Participant I-10 justification for choosing the tangent ratio.

In the justification presented in *Figure 5.3*, the student relates the tangent ratio to the appropriate side lengths—the adjacent and opposite sides. However, this student did not specify the aspect of the diagram that represents the opposite side (building) or the adjacent side (25 meters). Also, this student refers to a formula but did not provide the formula, which would have shown the relationship between the adjacent and opposite sides in the tangent ratio.

Some students gave a flawed justification for selecting the tangent ratio. In these instances, they did not correctly identify at least one side on the diagram representing the problem or they stated at least one incorrect side that formed the tangent ratio. *Figure 5.4* shows one such justification.

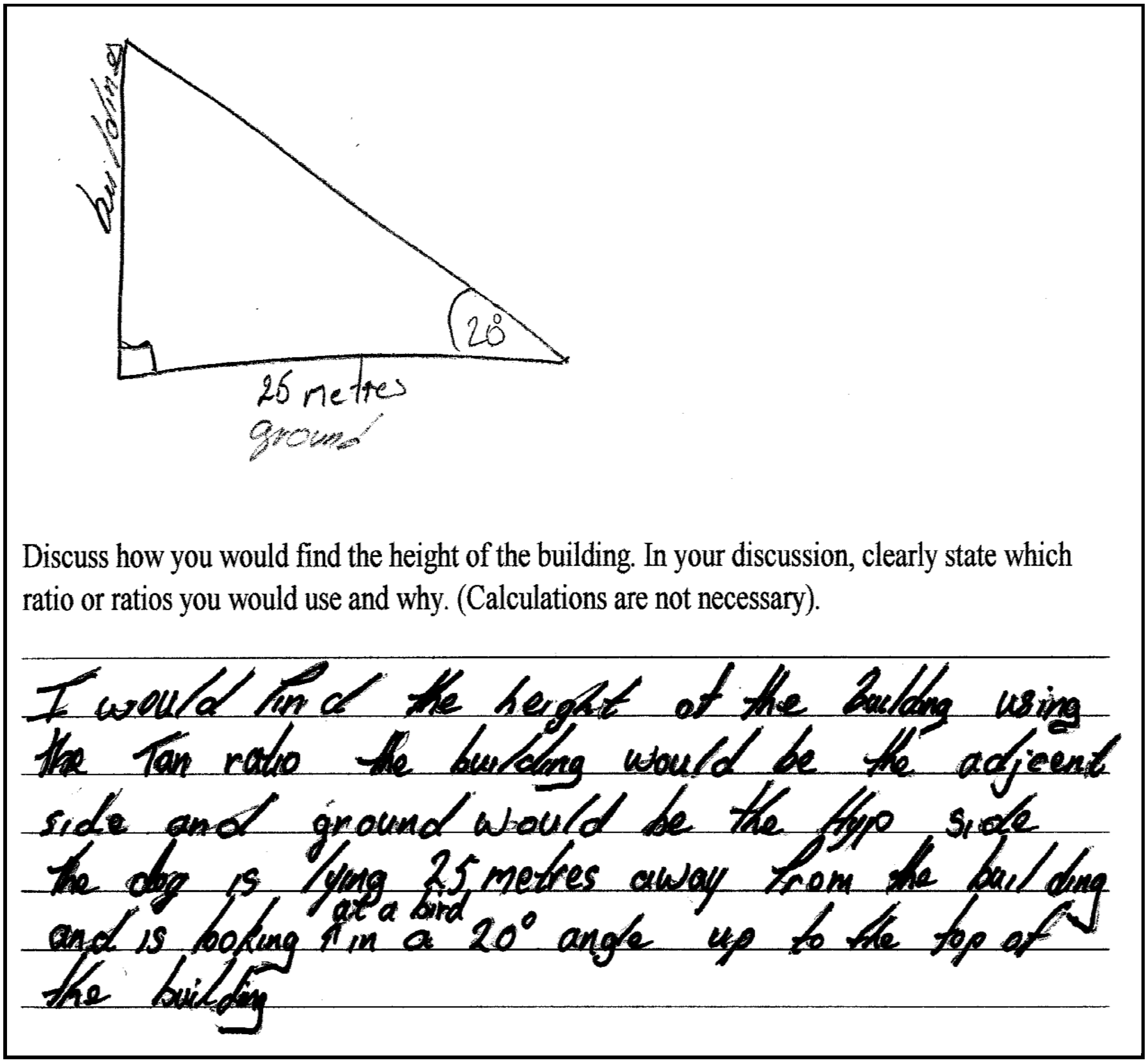


Figure 5.4: Participant I-11 justification for choosing the tangent ratio. The student's writing is written over for clarity.

In the justification presented in *Figure 5.4*, the student incorrectly identified the distance along the ground as the hypotenuse and the building as the adjacent side. That student also stated that s/he would use the tangent ratio because the problem dealt with the adjacent side and the hypotenuse. The tangent ratio does not deal with the hypotenuse of a right-angle triangle.

Theme 3: Selection of an incorrect ratio.

Some students incorrectly selected cosine as the most appropriate ratio to find the height of the building. Different justifications were given for choosing the cosine ratio. *Figure 5.5* shows one inappropriate justification.

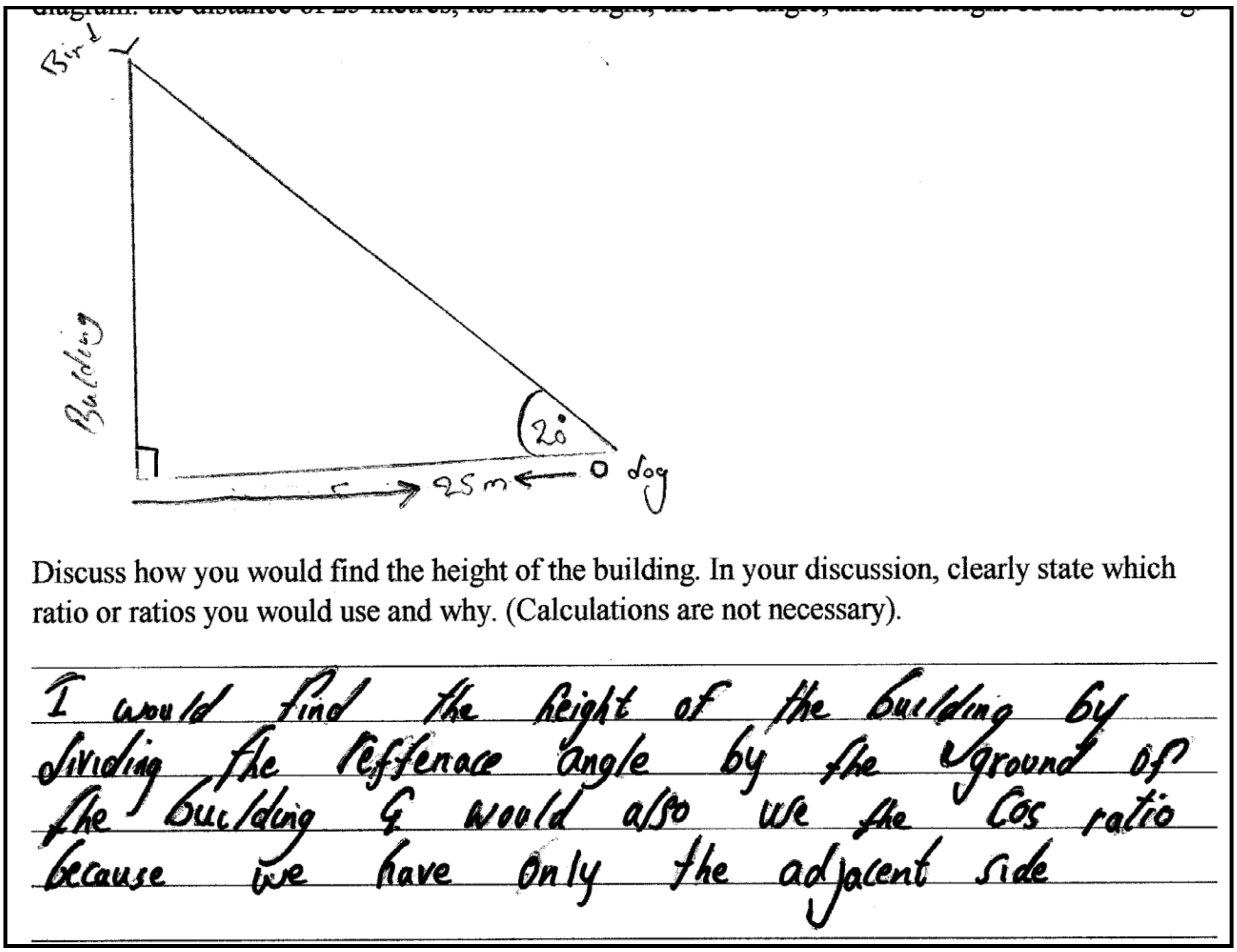


Figure 5.5: Participant I-15 justification for choosing the cosine ratio. The student's writing is written over for clarity.

In *Figure 5.5*, the student stated that they would use the cosine ratio because s/he had only the adjacent side. Although the measure of the adjacent side was the only side length given in the problem, two sides must be used to identify a ratio. The cosine ratio makes use of the adjacent side and the hypotenuse, but in this problem, the hypotenuse does not represent the height of the building.

Figure 5.6 shows another inappropriate justification for choosing the cosine ratio to find the height of the building.

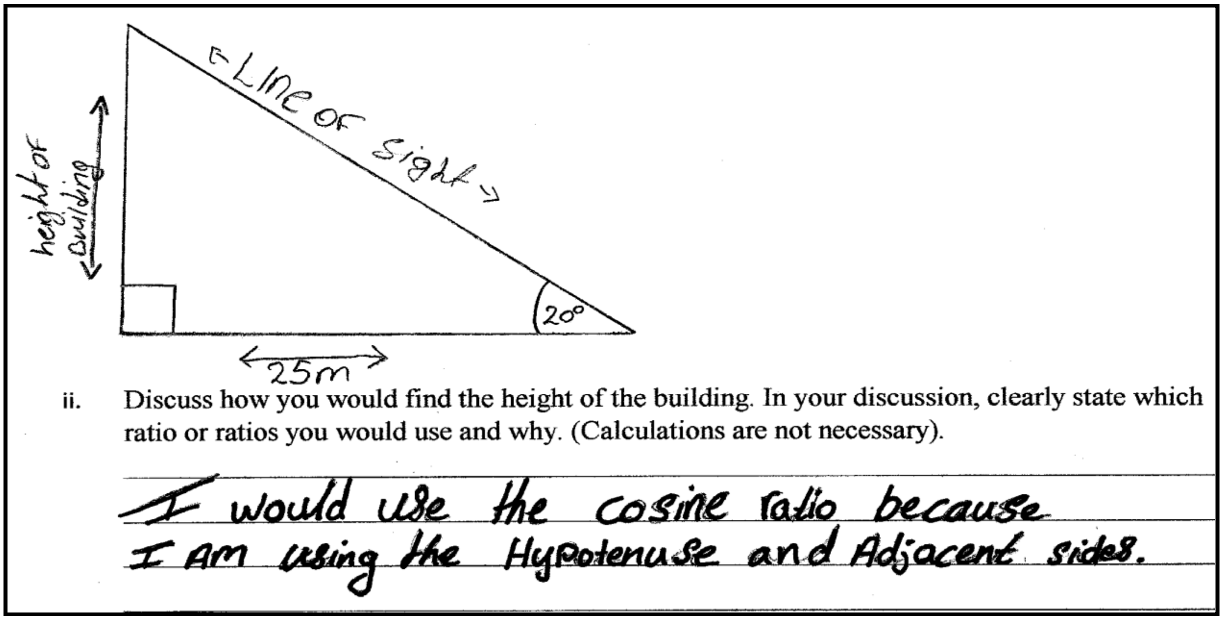


Figure 5.6: Participant I-9 justification for choosing the cosine ratio. The student's writing is written over for clarity.

In *Figure 5.6,* the student identified the cosine ratio as the most appropriate to determine the height of the building because s/he felt that they were dealing with the adjacent side and the hypotenuse. In the student's diagram (see *Figure 5.6*), the height of the building is correctly identified, but it is not the hypotenuse. Hence, with the given information, selecting the cosine ratio to find the height of the building was not appropriate.

Theme 4: Calculation procedure.

Students gave several incorrect procedures for calculating the height of the building. One incorrect procedure is shown in *Figure 5.5* above. The student whose work is shown in *Figure 5.5*, said that s/he would divide the angle (20^0) by the distance along the ground (25m). This calculation procedure is incorrect in two ways. One, the arithmetic operation identified was division instead of multiplication. The student may have chosen division because s/he incorrectly selected cosine as the most appropriate ratio; thus, using the building as the hypotenuse. Given that: $cos\theta = \frac{adjacent\ side}{hypotenuse}$, when transposed, $hypotenuse = \frac{adjacent\ side}{cos\theta}$. Two, the stated calculation procedure calls for carrying out the arithmetic operation (division) with the angle and not the ratio of the angle. This is incorrect because in every instance it is the ratio of the angle (*cos θ, sin θ, or tan θ*) that is equal to the division of two side lengths. As illustrated in the transposed formula above, arithmetic operations are carried out with the ratio of the angle (e.g., *cos θ*) and not the angle itself when finding a missing side of a right-angle triangle.

Another incorrect calculation procedure given was to multiply the angle by the distance along the ground (25m). *Figure 5.7* shows one such calculation procedure.

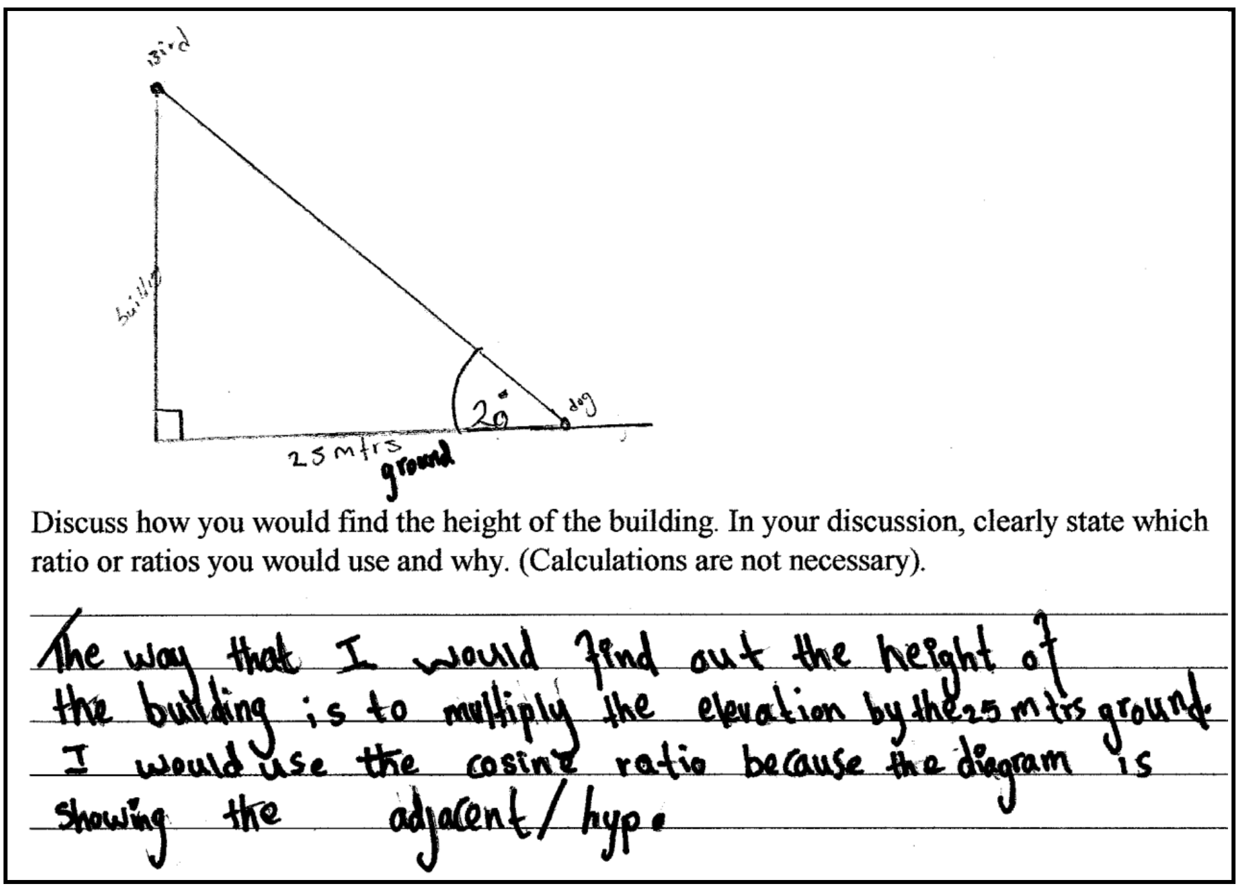


Figure 5.7: Participant I-12 procedure for finding the height of the building. The student's writing is written over for clarity.

In *Figure 5.7*, the students stated that they would multiply the elevation (the angle of 20^0) with the 25 meters along the ground. As was the case in *Figure 5.5* above, this student calculation procedure is incorrect because s/he would carry out the arithmetic operation with the angle (20^0) and not the ratio of the angle. Also, multiplication would have been the correct arithmetic operation if tangent was selected as the appropriate ratio. However, cosine was selected, and it is unclear which side the student thought is the hypotenuse.

Multiple representations of a single concept.

The prompt soliciting students' responses for this domain asked students to represent the cosine ratio using three different forms of representation and to justify their choices:

> This question requires you to use and discuss multiple representations of the cosine ratio.
> iii. Including the formula, show **<u>at least</u>** three (3) different representations of the cosine ratio.
> iv. Discuss how you know that each of these representation shows the cosine ratio.

The analysis of students' responses to this prompt yielded five themes: use of a formula, use of a graph, use of a table, use of a diagram, and justification based on sides. The contents of these themes comprised of both correct and incorrect responses.

Theme 1: Use of a formula.

Some students provided the correct formula for cosine with all elements: angle, adjacent side, and hypotenuse given in the proper positions. *Figure 5.8* shows one such response.

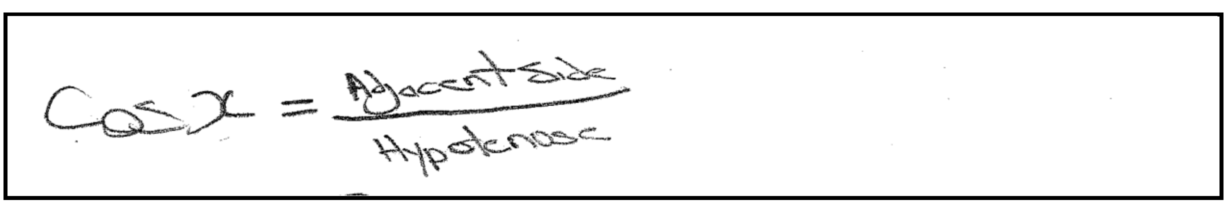


Figure 5.8: Participant I-1 representation of the cosine formula.

Figure 5.8 shows the cosine of the angle x ($\cos x$) being equal to the adjacent side divided by the hypotenuse. The division of the two identified sides is correct, and the cosine ratio is correctly abbreviated ($\cos x$). This is a correct representation of the formula for the cosine ratio.

In some cases, the adjacent side and the hypotenuse were correctly divided, but the formula was not given in terms of an angle. *Figure 5.9* shows one such response.

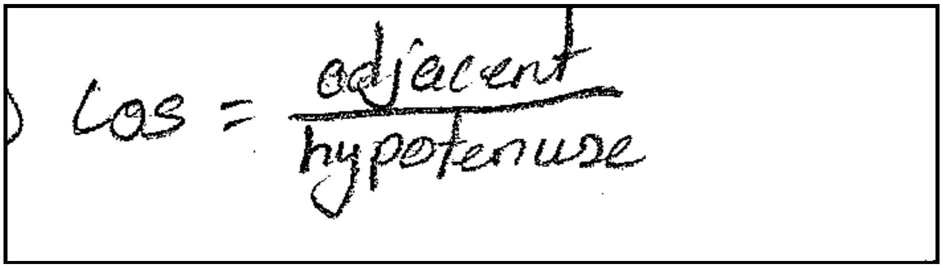


Figure 5.9: Participant I-4 representation of the cosine formula.

Figure 5.9 shows cosine without an angle (*cos*) is equal to the adjacent side divided by the hypotenuse. The angle is an essential element of all the trigonometric ratios because the trigonometric ratios are functions of angles. Hence, the angle must always be shown in the formulas of the trigonometric ratios. Hence, this representation of the formula for the cosine ratio is limited (not completely correct).

Several incorrect representations of the formula for the cosine ratio were presented. *Figure 5.10* shows one such incorrect representation.

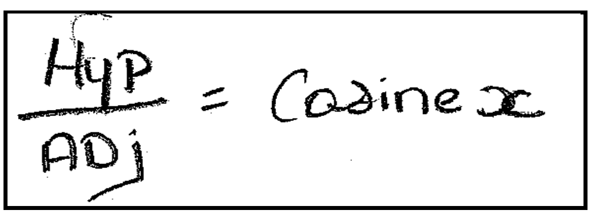


Figure 5.10: Participant I-13 representation of the cosine formula.

Figure 5.10 shows the cosine of the angle *x* (*cosine x*) is equal to the hypotenuse divided by the adjacent side. This is an incorrect representation of the formula for the cosine ratio because the sides, adjacent and hypotenuse, are not properly divided; it is the adjacent side divided by the hypotenuse. Also, the cosine ratio is not properly abbreviated (*cosine x* was given instead of *cos x*).

Figure 5.11 shows a second incorrect representation of the formula for the cosine ratio.

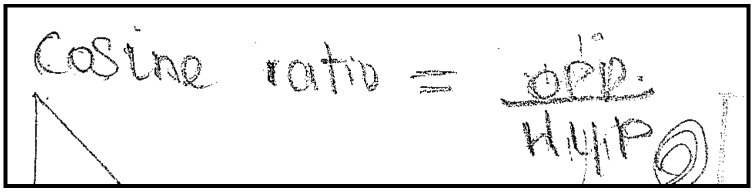


Figure 5.11: Participant I-6 representation of the cosine formula.

Figure 5.11 shows the cosine ratio is equal to the opposite side divided by the hypotenuse. This is an incorrect representation of the formula for the cosine ratio because the cosine ration does not make use of the opposite side. The opposite side divided by the hypotenuse is part of the formula for the sine ratio. Also, there was no mention of an angle.

Figure 5.12 shows a third incorrect representation of the formula for the cosine ratio.

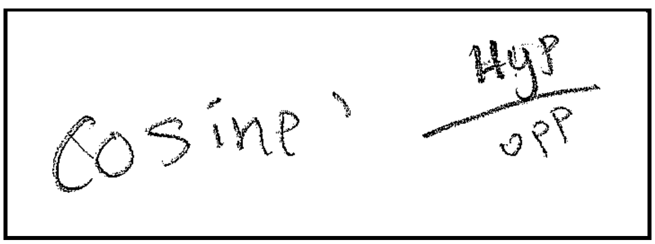


Figure 5.12: Participant I-12 representation of the cosine formula.

Figure 5.12 shows cosine is equal to the hypotenuse divided by the opposite side. This is an incorrect representation of the formula for the cosine ratio because the cosine ration does not make use of the opposite side. The hypotenuse divided by the opposite side is part of the formula for the cosecant ratio, which is not one of the three primary trigonometric ratios. Also, there was no mention of an angle.

Theme 2: Use of a graph.

Some students provided an appropriate graph to represent the cosine ratio. *Figure 5.13* shows one such graph.

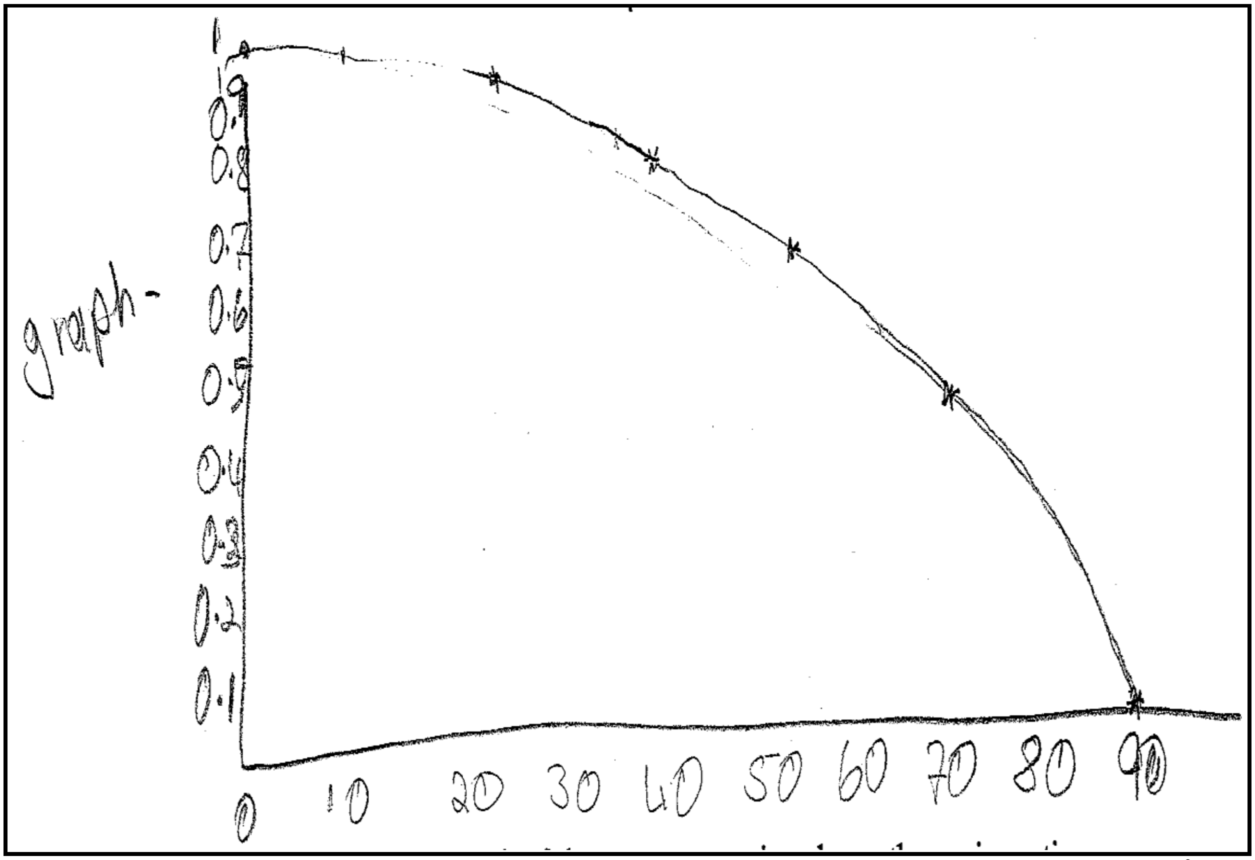


Figure 5.13: Participant I-10 graphical representation of cosine for angles between 0 and 90^0.

The graph in *Figure 5.13* is a curve showing the values of cosine decreasing from one to zero as the values of the angles increased from zero to 90^0. The cosine values are represented along the vertical axis (y-axis) and increase from zero to one. The values of the angles are represented along the horizontal axis (x-axis) and increase from zero to 90^0. Cosine is shown to have a value of one at zero degrees and a value of zero at 90^0.

Several inappropriate graphs were presented to represent the cosine ratio. *Figure 5.14* shows one such graph.

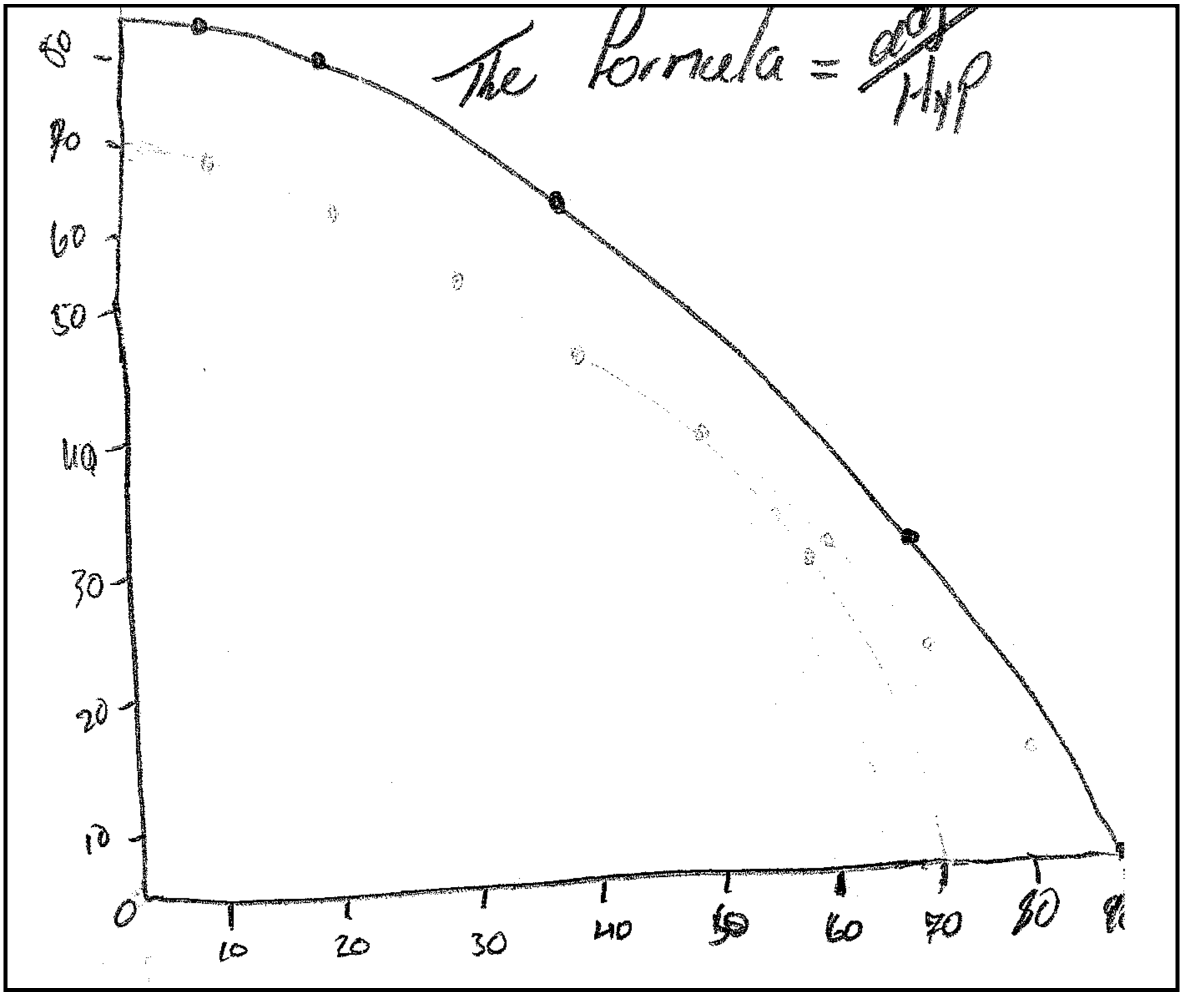


Figure 5.14: Participant I-11 graphical representation of cosine for angles between 0 and 90^0.

The graph in *Figure 5.14* is a curve showing the values of cosine decreasing from a value greater than 80 to zero as the values of the angles increased from zero to 90^0. The cosine values are represented along the vertical axis (y-axis) and increase from zero to more than 80. This calibration of the y-axis is inappropriate for the cosine ratio because values of cosine are between zero and one for all angles. The values of the angles are represented along the horizontal axis (x-axis) and increase from zero to 90^0. Cosine is shown to have a value of greater than 80 at zero degrees and a value of zero at 90^0. The value of cosine at zero degrees is one; thus, this graph is incorrect.

Figure 5.15 shows a second inappropriate graph of the cosine ratio.

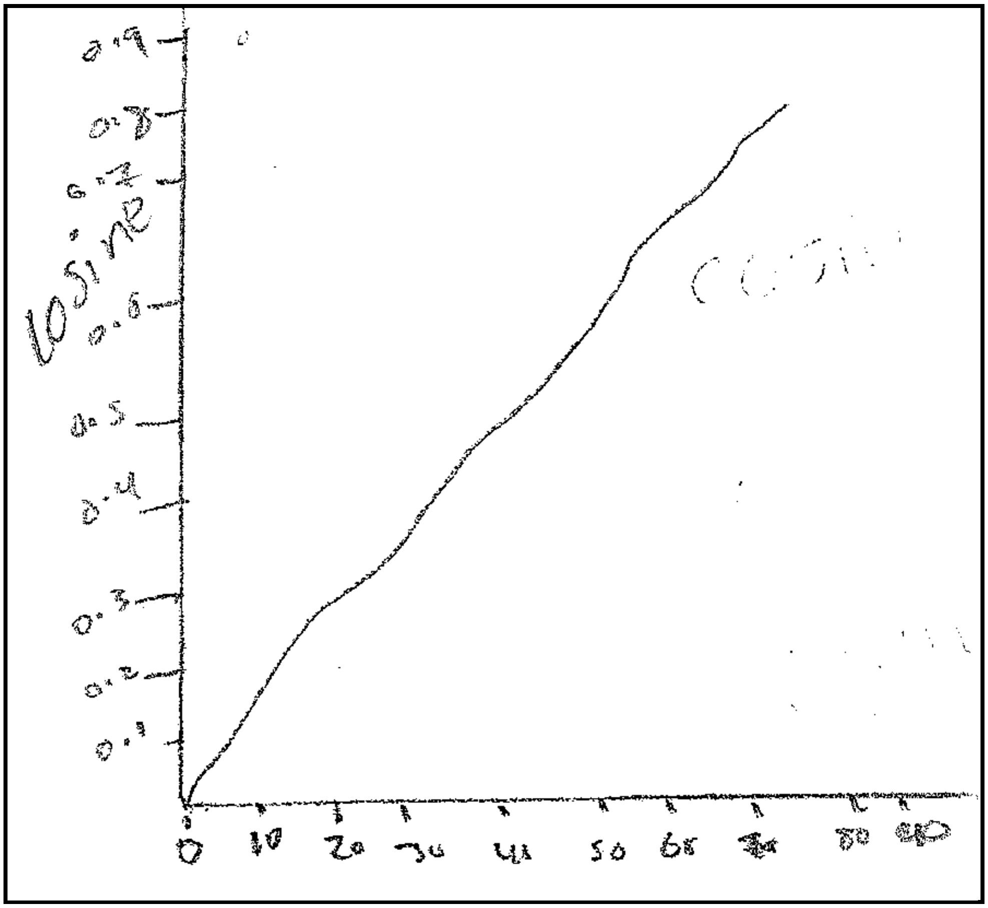


Figure 5.15: Participant I-12 graphical representation of cosine for angles between 0 and 90^0.

The graph in *Figure 5.15* is a ‘straight line’ showing the values of cosine increasing from zero towards one as the values of the angles increased from zero to 90^0. The cosine values are represented along the vertical axis (y-axis) and increase from zero towards one. The values of the angles are represented along the horizontal axis (x-axis) and increase from zero to 90^0. Cosine is shown to have a value of zero at zero degrees. This graph is incorrect in several ways. The cosine ratio does not produce a straight line graph because, as the angle changes, the length of the adjacent side also changes, but the length of the hypotenuse remains constant (see the unit circle on page xxx). Thus, the line that represents the cosine ratio has a different gradient (slope) at every angle. Also, the cosine of zero degrees is not zero, and the values of cosine decrease for increasing values of angles between zero degrees and 90^0.

Figure 5.16 shows a third inappropriate graph of the cosine ratio.

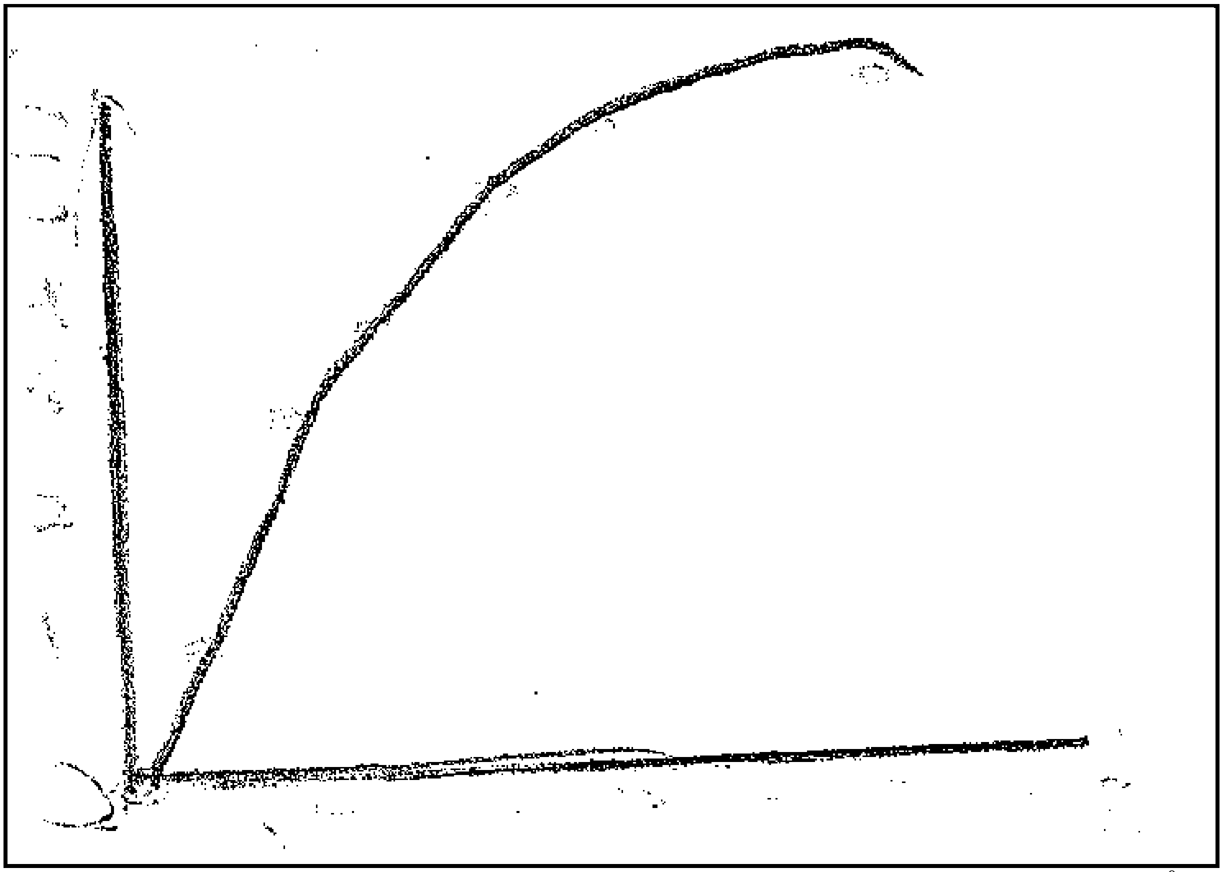


Figure 5.16: Participant I-1 graphical representation of cosine for angles between 0 and 90^0.

Figure 5.16 presents a curve that is typical of the behaviour of the sine ratio between zero and 90^0. In this curve, the values of the cosine ratio appear to be increasing for increasing values of angles between zero degrees and 90^0; whereas, the values of cosine should be decreasing for increasing values of angles between zero and 90^0. This evaluation is based on the shape of the curve because the values along the vertical and horizontal axes are indecipherable.

Figure 5.17 shows a fourth inappropriate graph of the cosine ratio.

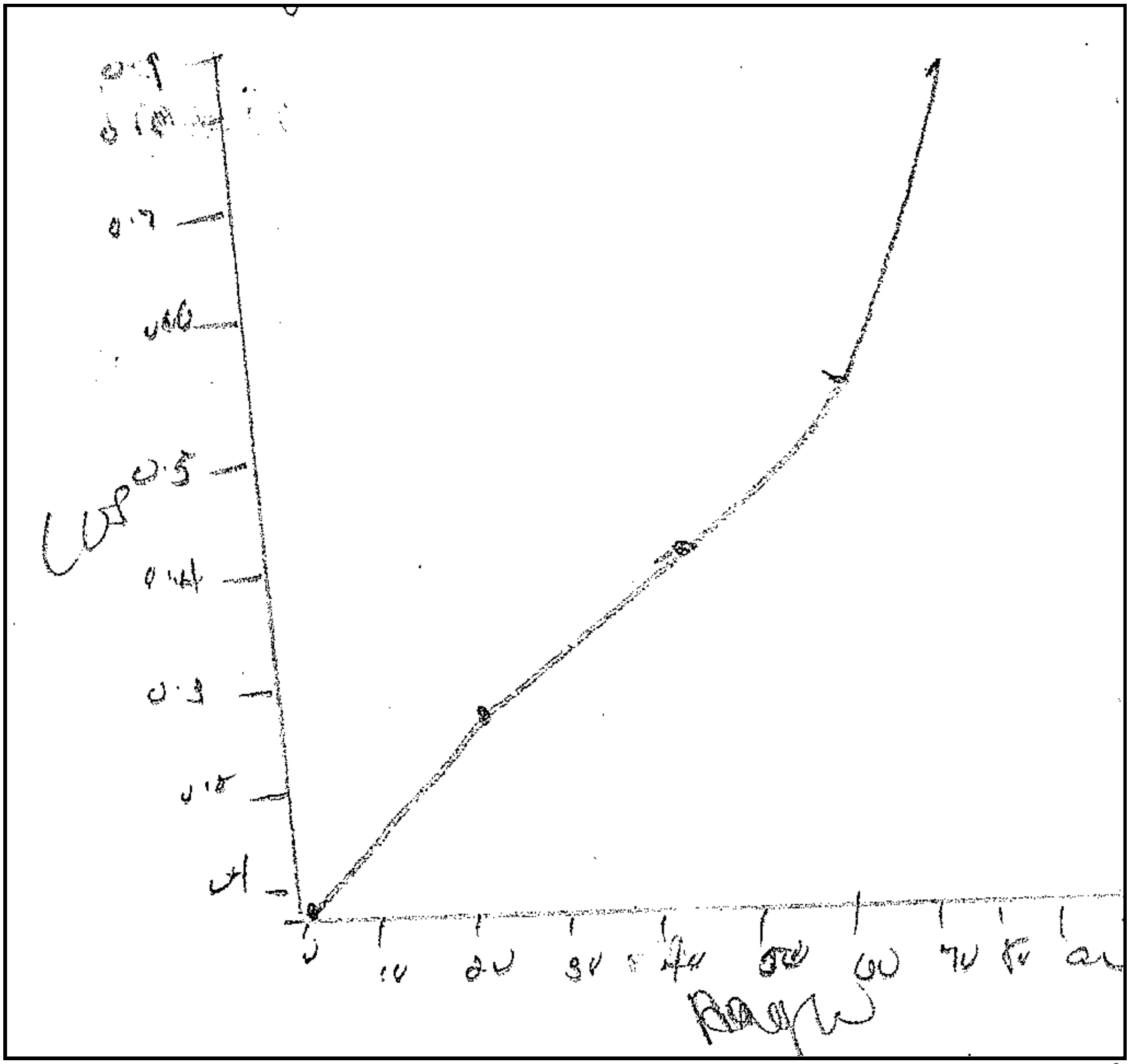
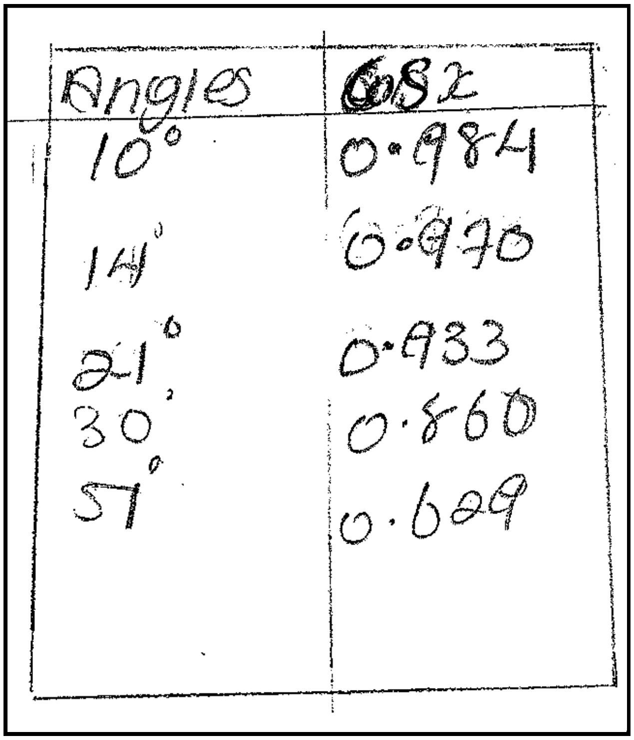


Figure 5.17: Participant I-13 graphical representation of cosine for angles between 0 and 90^0.

Figure 5.17 presents a curve that is typical of the behaviour of the tangent ratio between zero and 90^0. In this curve, the values of the cosine ratio are shown to be increasing for increasing values of angles between zero degrees and 90^0 going towards an asymptote at 90^0; whereas, the values of cosine should be decreasing for increasing values of angles between zero and 90^0. Unlike the tangent curve, the vertical axis (y-axis) shows only values between zero and one, which is appropriate for the cosine curve.

Theme 3: Use of a table.

Some students provided an appropriate table to represent the cosine ratio. *Figure 5.18* shows one such table.

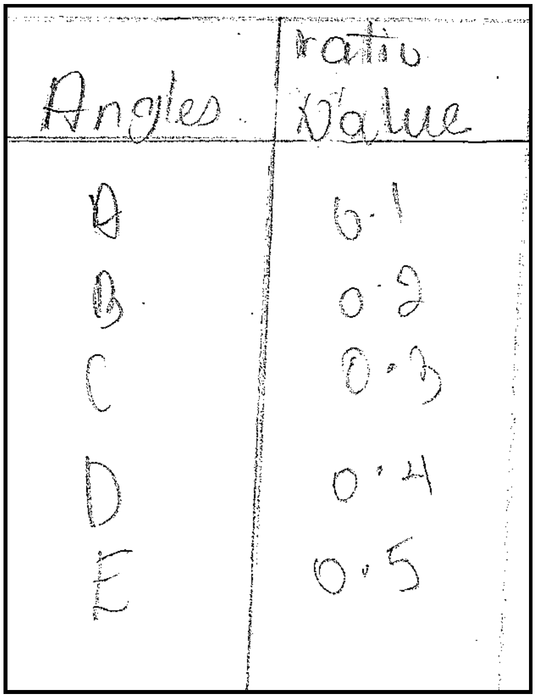


Angles	Cos x
10^0	0.984
14^0	0.970
21^0	0.933
30^0	0.860
51^0	0.629

Figure 5.18: Participant I-2 tabular representation of cosine for angles between 0 and 90^0.

The table in *Figure 5.18* shows the values of cosine decreasing as the values of the angles increased. For instance, the cosine of 10^0 is given as 0.984, and the cosine of 51^0 is given as 0.629. The values of the cosine ratio decrease for increasing values of angles between zero and 90^0.

Some students also produced an inappropriate table to represent the cosine ratio. *Figure 5.19* shows one such table.



Angles	ratio Value
A	0.1
B	0.2
C	0.3
D	0.4
E	0.5

Figure 5.19: Participant I-6 tabular representation of cosine for angles between 0 and 90^0.

The table in *Figure 5.19* shows the values of cosine increasing as the values of the angles increased. It is assumed that A, B, C, D, and E represent angles in increasing order of size. The cosine of angle A is given as 0.1, and the cosine of angle E is given as 0.5, where angle A is less than angle E. This table is inappropriate because the values of the cosine ratio decrease for increasing values of angles between zero and 90^0.

Theme 4: Use of a diagram.

A right-angle triangle (diagram) with a reference angle, the adjacent side, and the hypotenuse marked are the essential elements of the cosine ratio. Some students provided an appropriate diagram to represent the cosine ratio. *Figure 5.20* shows one such diagram.

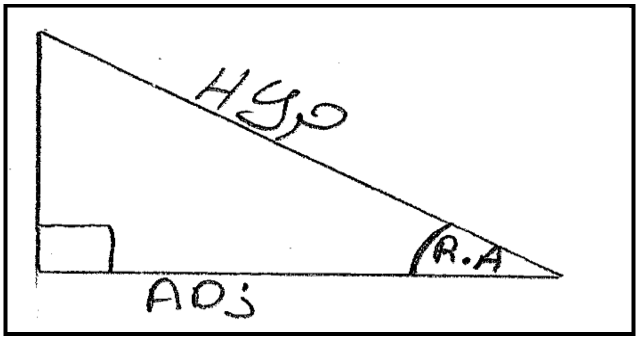


Figure 5.20: Participant I-9 diagram representing the cosine ratio.

Figure 5.20 is a right-angle triangle showing a reference angle (R.A), the adjacent side (Adj), and the hypotenuse (HYP) that are correctly located. A box is used to identify the right angle, and R.A is used to identify the reference angle. The hypotenuse is opposite the right angle and forms one arm of the reference angle. The adjacent side forms the other arm of the reference angle. All essential elements are identified and correctly located.

Some students produced a diagram that is limited in its appropriateness to represent the cosine ratio. *Figure 5.21* shows one such diagram.

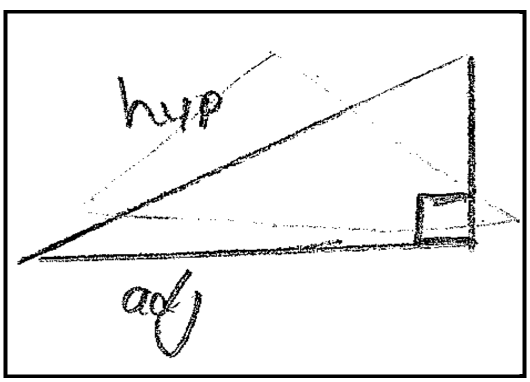


Figure 5.21: Participant I-15 diagram representing the cosine ratio.

Figure 5.21 is a right-angle triangle showing an adjacent side (adj) and the hypotenuse (hyp). A box is used to identify the right angle. The hypotenuse is correctly located opposite the right angle. No reference angle is shown; hence, the location of the adjacent side is questionable because its location depends on the reference angle. The absence of a marked reference angle limits the appropriateness of this diagram.

Some students also produced an inappropriate diagram to represent the cosine ratio. *Figure 5.22* shows one such diagram.

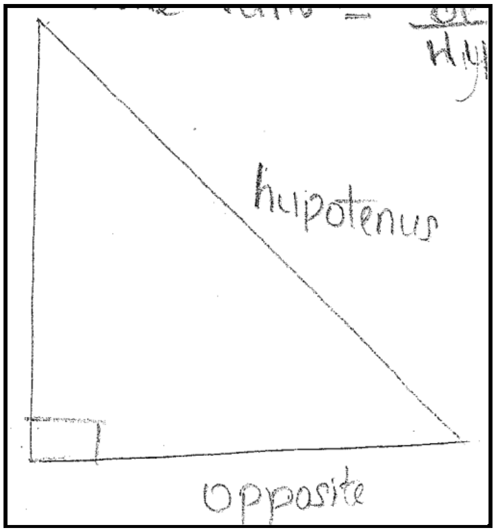


Figure 5.22: Participant I-6 diagram representing the cosine ratio.

Figure 5.22 is a right-angle triangle showing an opposite side (opposite) and the hypotenuse (hypotenus). A box is used to identify the right angle. The hypotenuse is correctly located opposite the right angle. No reference angle is shown; hence, the location of the opposite side is questionable because its location depends on the reference angle. Moreover, the opposite side is not relevant to the cosine ratio. The absence of a marked reference angle and the presence of an opposite side make this diagram an inappropriate representation of the cosine ratio.

Figure 5.23 shows a second inappropriate diagram used to represent the cosine ratio.

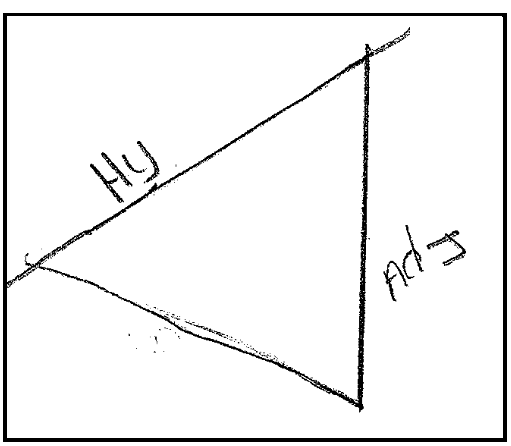


Figure 5.23: Participant I-14 diagram representing the cosine ratio.

Figure 5.23 is a non-right-angle triangle showing an adjacent side (Adj) and the hypotenuse (hy). The term hypotenuse relates to the longest side of only right-angle triangles; thus, the labelling of a hypotenuse in *Figure 5.23* is inappropriate. Furthermore, no reference angle is shown; hence, the location of the adjacent side is questionable because its location depends on the reference angle. Primarily, the use of a non-right-angle triangle makes this diagram an inappropriate representation of the cosine ratio.

Figure 5.24 shows a third inappropriate diagram used to represent the cosine ratio.

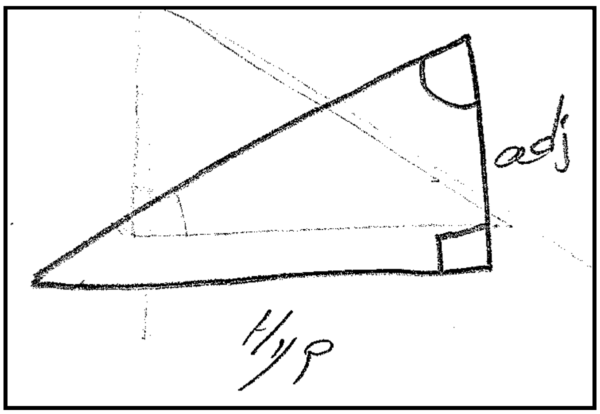
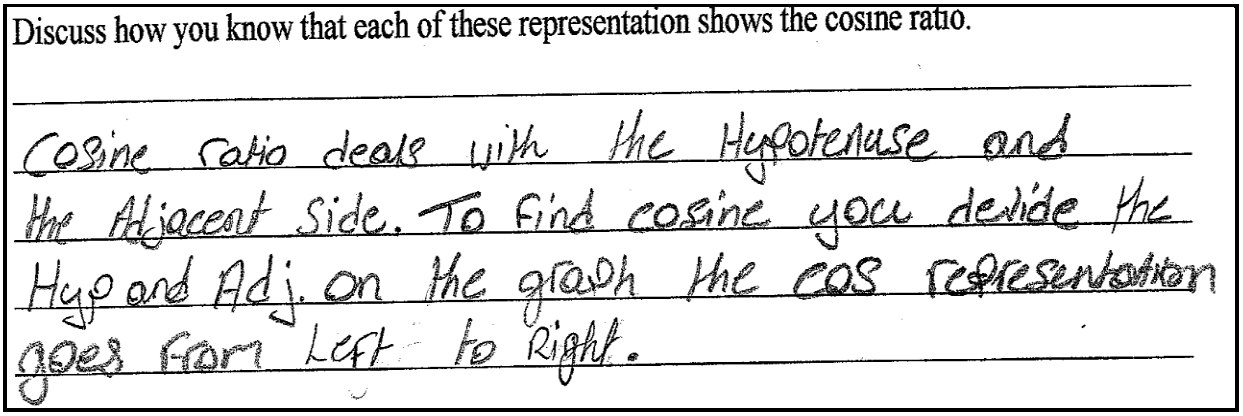


Figure 5.24: Participant I-11 diagram representing the cosine ratio.

Figure 5.24 is a right-angle triangle showing a reference angle, the adjacent side (adj), and a hypotenuse (Hyp). A box is used to identify the right angle, and a curved line is used to identify the reference angle. The hypotenuse is incorrectly located opposite the reference angle. The adjacent side is correctly located and forms one arm of the reference angle. The essential elements are identified; however, the incorrect location of the hypotenuse makes this diagram an inappropriate representation of the cosine ratio.

Theme 5: Justification based on sides.

Students, who provided an appropriate justification for their selection of representations, focused on the sides that form the cosine ratio. Part of the text presented in *Figure 5.25* shows one such justification.

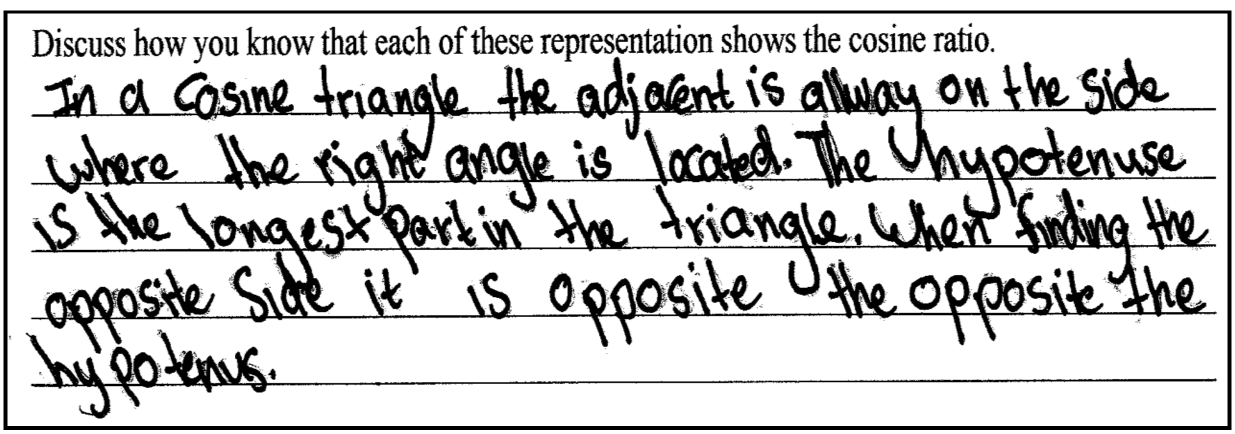

Discuss how you know that each of these representation shows the cosine ratio.

Cosine ratio deals with the Hypotenuse and the Adjacent Side. To find cosine you devide the Hyp and Adj. on the graph the cos representation goes from Left to Right.

Figure 5.25: Participant I-9 justification for the choice of representations.

In the first sentence of the text presented in *Figure 5.25*, the student correctly indicated that the cosine ratio deals with the adjacent side and the hypotenuse. The student also indicated that these sides are divided to form the cosine ratio. S/he did not state if it is the adjacent side that is divided by the hypotenuse or the other ways around. In referring to the graph, it is unclear what the students meant by “the cos representation goes from left to right.”

Some students’ discussions did not say how they knew that their representations showed the cosine ratio. *Figure 5.26* shows one such discussion.

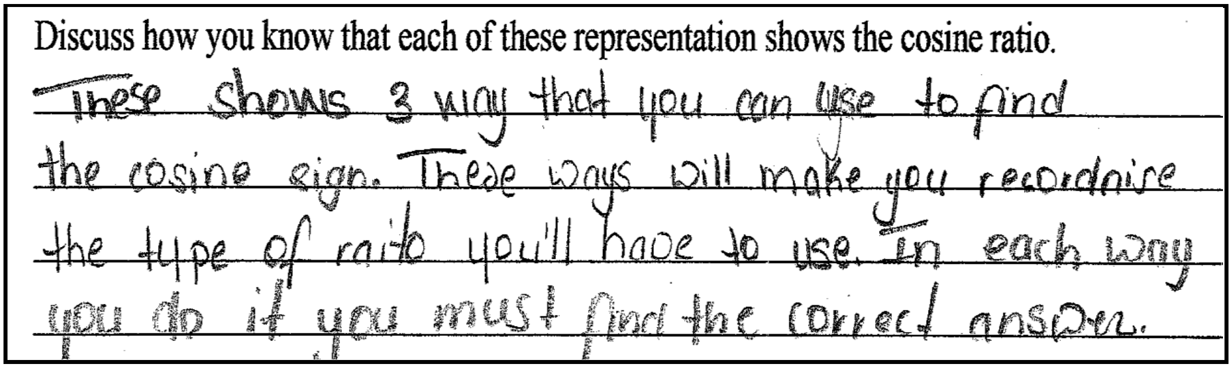

Discuss how you know that each of these representation shows the cosine ratio.

In a Cosine triangle the adjacent is allway on the side where the right angle is located. The hypotenuse is the longest part in the triangle. When finding the opposite Side it is Opposite the opposite the hypotenus.

Figure 5.26: Participant I-16 justification for the choice of representations. The student’s writing is written over for clarity.

In *Figure 5.26*, the student is attempting to identify the locations of the adjacent side, hypotenuse, and the opposite side. S/he locates the adjacent side next to the right angle, which could be any of the two shorter sides in a right-angle triangle. S/he describes the hypotenuse as the "longest part" of the triangle and incorrectly located the opposite side opposite the hypotenuse.

Some students' discussions were unintelligible. *Figure 5.27* shows one such discussion.



Discuss how you know that each of these representation shows the cosine ratio.

These shows 3 way that you can use to find the cosine sign. These ways will make you recordnise the type of raito you'll hace to use. In each way you do it you must find the correct answer.

Figure 5.27: Participant I-13 justification for the choice of representations.

The "3 ways" referred to in the first sentence of the text in *Figure 5.27* are the student's choices of representations of the cosine ratio for which s/he needed to provide justifications. The student wrote about using the representations to recognize the type of ratio to use and about finding the correct answer. At no point did this student say how s/he knew that these representations showed the cosine ratio.

Comparing related concepts.

The prompt soliciting students' responses for this domain asked students to represent the sine, cosine and tangent ratios using the same form of representation and to compare these representations:

> This question requires you to show the three trigonometric ratios using the **same form** of representation, then to compare and contrast these representations.
>
> iii. Use the **same form** of representation to show the sine, cosine, and tangent ratios.
>
> iv. Discuss the similarities and differences among these representations.

The analysis of students' responses to this prompt yielded three themes: comparisons based on representations, comparisons based on sides, and comparisons based on values. The contents of these themes comprised of both correct and incorrect responses.

Theme 1: Comparisons based on representations.

Students represented the sine, cosine, and tangent ratios using the different formulas. In some instances, all the essential elements are included in the formulas. *Figure 5.28* shows one instance when all essential elements are included in the formulas.

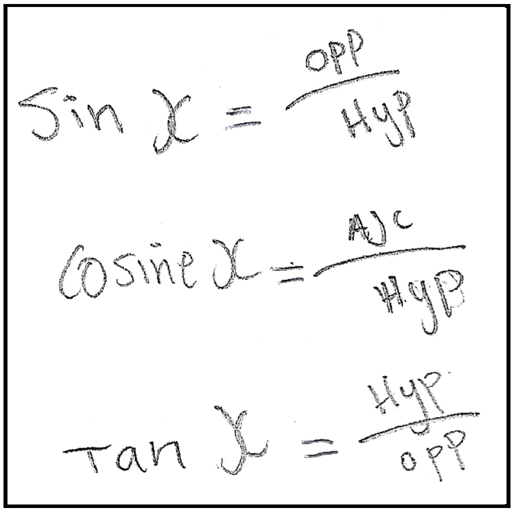


Figure 5.28: Participant I-12 formula representations of the three primary trig-ratios.

Figure 5.28 shows one student's representations of the sine, cosine, and tangent ratios. The sine ratio is correctly represented as the sine of the angle *x* (*sin x*) being equal to the opposite side (opp) divided by the hypotenuse (Hyp). The cosine ratio is correctly represented as the cosine of the angle *x* (*cosine x*) being equal to the adjacent side (Ajc) divided by the hypotenuse (Hyp). However, the cosine of angle *x* was not properly abbreviated in this formula representation: *cosine x* (incorrect) instead of *cos x* (correct). The tangent ratio is incorrectly represented as the tangent of the angle *x* (*tan x*) being equal to the hypotenuse (Hyp) divided by the opposite side (opp). The hypotenuse is not an element of the tangent ratio. Furthermore, the opposite side is in the numerator of the tangent formula, not in the denominator.

In some instances, all the essential elements were not included in the formulas. *Figure 5.29* shows one instance when all the essential elements were not included in the formulas.

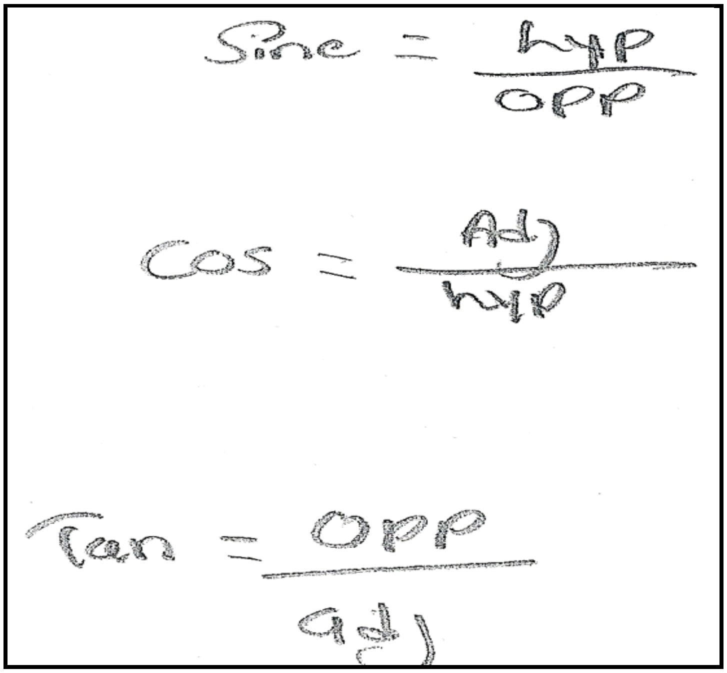
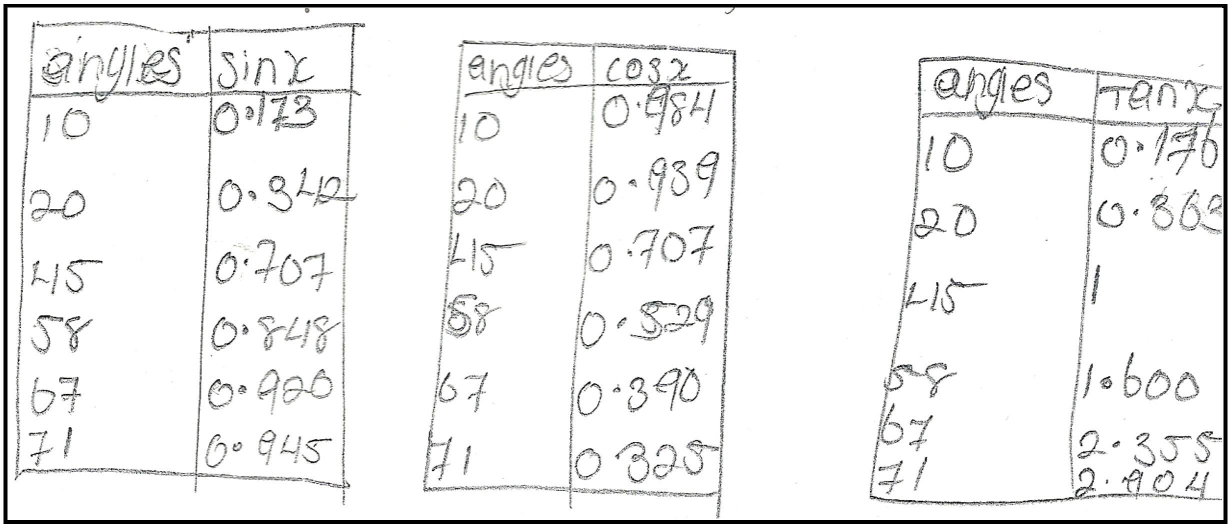


Figure 5.29: Participant I-15 formula representations of the three primary trig-ratios.

Figure 5.29 shows one student's representations of the sine, cosine, and tangent ratios where the angle is missing from the formulas. Besides the angle that is missing from the formulas, the sine ratio is incorrectly represented as sine being equal to the hypotenuse (hyp) divided by the opposite side (opp). Both the opposite side and the hypotenuse are elements of the sine ratio; however, they are wrongly divided in this student's representation of the formula for sine ratio. The correct operation is the opposite side divided by the hypotenuse.

Some students represented the sine, cosine, and tangent ratios using appropriate tables. These tables show how the values of each ratio change as the values of the angles increase from zero to 90 degrees. *Figure 5.30* shows one such set of tables.



angles	sin x
10	0.173
20	0.342
45	0.707
58	0.848
67	0.920
71	0.945

angles	cos x
10	0.984
20	0.939
45	0.707
58	0.529
67	0.390
71	0.325

angles	tan x
10	0.176
20	0.36
45	1
58	1.600
67	2.355
71	2.904

Figure 5.30: Participant I-2 tabular representations of the three primary trig-ratios.

In *Figure 5.30*, the first table shows the values of sine increasing but remaining less than one as the values of the angles increase. The second table shows the values of cosine decreasing from approximately one going towards zero as the values of the angles increase. The third table shows the values of tangent increasing beyond one as the value of the angle increases. This set of tables correctly represents the sine, cosine, and tangent ratios.

Some students represented the sine, cosine, and tangent ratios using graphs with appropriately shaped curves. These graphs show how the curve of each ratio behaves. *Figure 5.31* shows one such set of graphs.

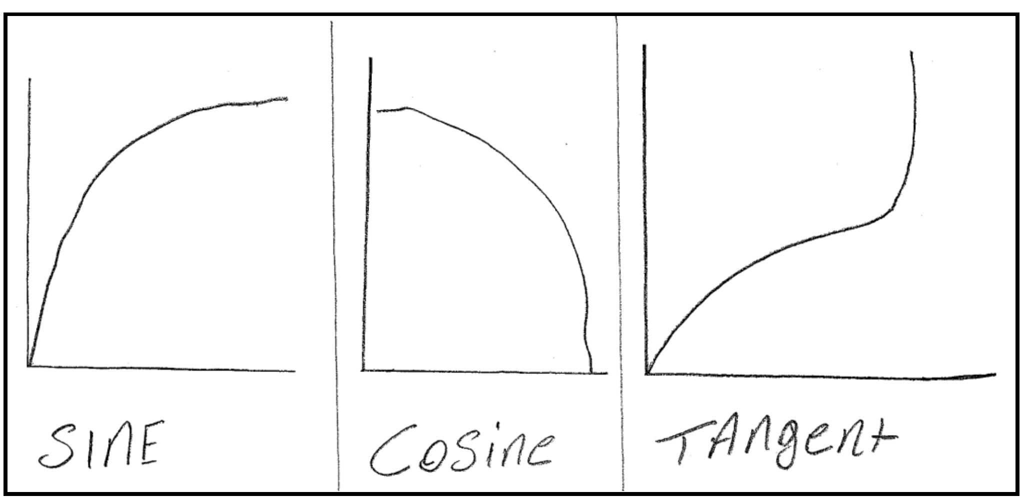


Figure 5.31: Participant I-9 graphical representations of the three primary trig-ratios.

In *Figure 5.31*, the first curve correctly shows the values of sine increasing as the values of the angles increase. The second curve also correctly shows the values of cosine decreasing as the values of the angles increase. The third curve also correctly shows the values of tangent increasing, differently from the sine curve, as the value of the angle increase. These graphs demonstrate the general behaviour of the three primary trigonometric ratios. However, critical values such as zero and one on the vertical axis and zero and 90^0 on the horizontal axis were not given. The non-inclusion of these critical values somewhat limits the effectiveness of these graphs in representing the sine, cosine, and tangent ratios.

Students represented the sine, cosine, and tangent ratios using the different diagrams. All the essential elements of the ratios were not given in the diagrams at any instance; thus, making them inappropriate as representations of the trigonometric ratios. *Figure 5.32* shows one such inappropriate set of diagrams.

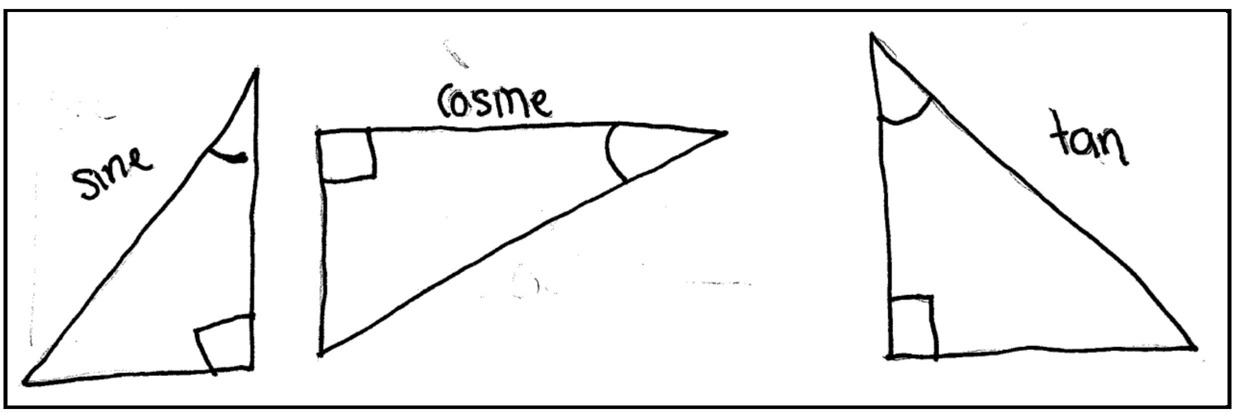


Figure 5.32: Participant I-16 diagram representations of the three primary trig-ratios. The student's writing is written over for clarity.

Figure 5.32 shows diagrams that are all right-angle triangles, which is appropriate. Each right-angle triangle also shows a reference angle (marked by a curve in the triangle), which is also appropriate. However, none of these triangles have any of their sides—opposite side, adjacent side, or hypotenuse—marked. Hence, they do not represent any specific trigonometric ratio.

Figure 5.33 shows a second set of diagrams that do not appropriately represent the sine, cosine, and tangent ratios.

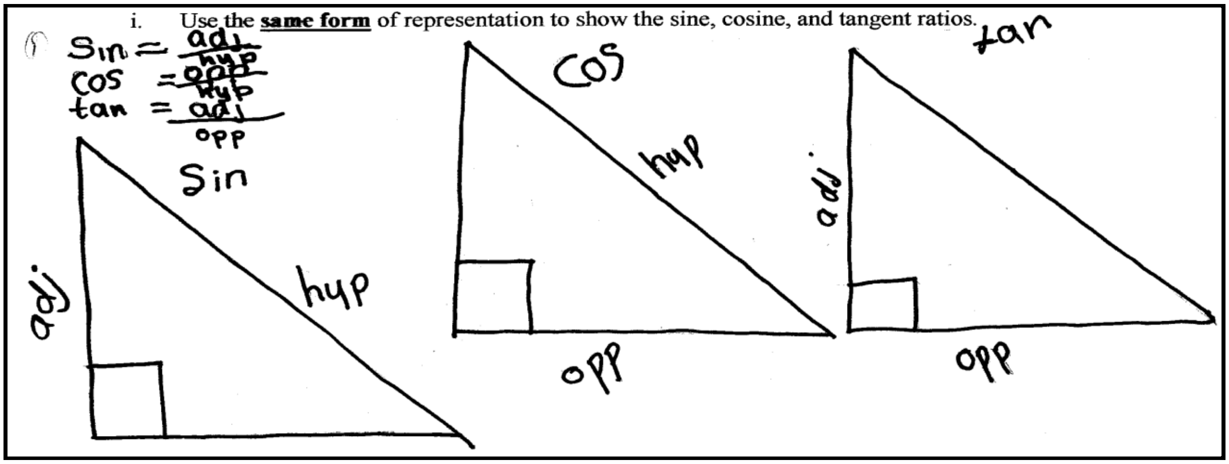


Figure 5.33: Participant I-6 diagram representations of the three primary trig-ratios. The student's writing is written over for clarity.

Figure 5.33 shows diagrams that are all right-angle triangles, which is appropriate. A reference angle is not shown in any of the triangles. The non-inclusion of a reference angle is inappropriate because an angle is an essential element in all the trigonometric ratios. Without a marked reference angle, the location of the opposite and adjacent sides shown on some of the diagrams are questionable, because their location depends on the location of the reference angle. Furthermore, the first diagram is marked as a representation of the sine ratio (sin) but shows the adjacent side (adj). This representation is inappropriate because the sine ratio does not use the adjacent side. Also, the second diagram is marked as a representation of the cosine ratio (cos) but shows the opposite side (opp). This representation is inappropriate because the cosine ratio does not use the opposite side.

Theme 2: Comparisons based on sides.

Some students provided appropriate comparisons of the three primary trigonometric ratios based on the sides that form these ratios. *Figure 5.34* shows one student's comparisons that were appropriate and based on the sides that form the ratios.

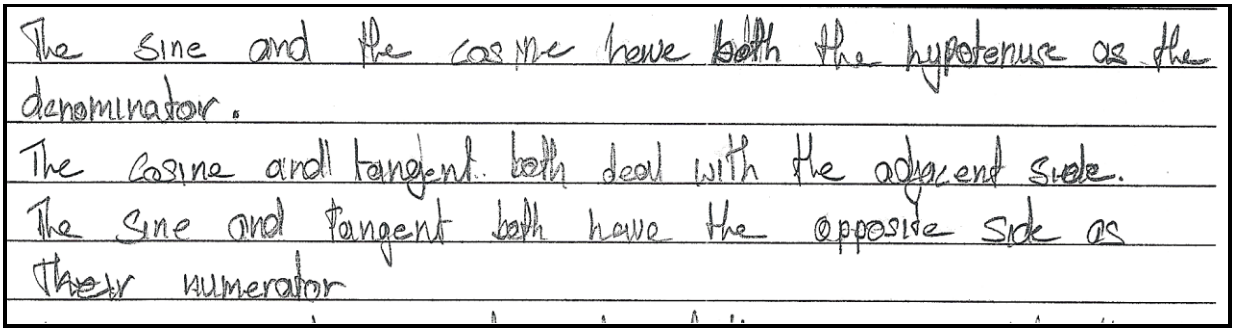

The sine and the cosine have both the hypotenuse as the denominator.
The cosine and tangent both deal with the adjacent side.
The sine and tangent both have the opposite side as their numerator

Figure 5.34: Participant I-10 comparison of the sides of the trig-ratios.

In *Figure 5.34*, the students identified a similarity between the sine ratio and the cosine ratio—s/he correctly stated that both the sine and cosine ratios have the hypotenuse as the denominator. The student also identified a similarity between the sine ratio and the tangent ratio—s/he correctly stated that both sine and tangent have the opposite side as the numerator. A third similarity was identified; it was between the cosine ratio and the tangent ratio—s/he correctly stated that both cosine and tangent deal with the adjacent side.

Figure 5.35 shows another student's comparisons that were appropriate and based on the sides that form the ratios.

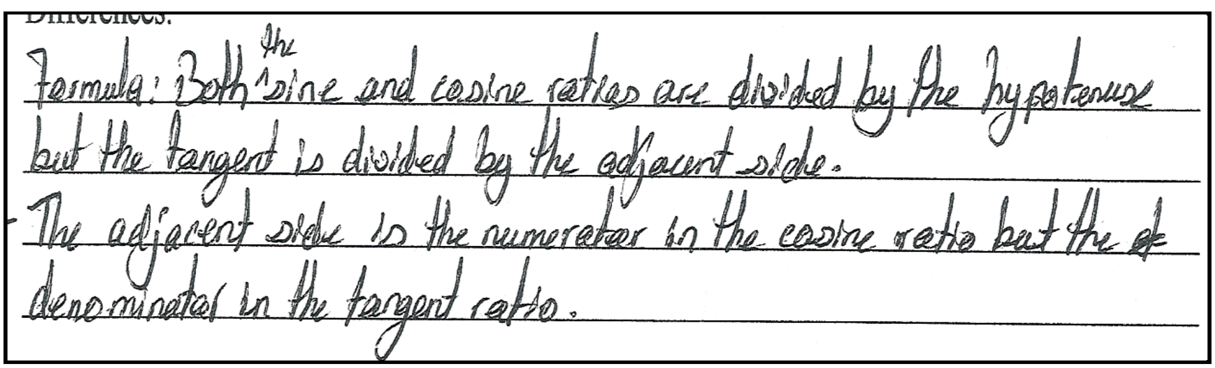

Differences:
Formula: Both the sine and cosine ratios are divided by the hypotenuse but the tangent is divided by the adjacent side.
The adjacent side is the numerator in the cosine ratio but the denominator in the tangent ratio.

Figure 5.35: Participant I-4 comparison of the sides of the trig-ratios.

In *Figure 5.35*, the students first compared all three ratios based on the sides in the different formulas. S/he stated that in both sine and cosine, the hypotenuse is the dividing side (denominator)—a similarity between the sine ratio and the cosine ratio—while the adjacent side was the dividing side in the tangent ratio—a difference between the tangent ratio and the sine and cosine ratios. Also, the student stated that the adjacent side is located differently in the formulas for the cosine and tangent ratios—it is in the numerator in the cosine ratio, but it is in the denominator in the tangent ratio.

However, some inappropriate comparisons were presented by students. In these comparisons, students incorrectly stated the position of at least one side in a ratio. *Figure 5.36* shows one such comparison.

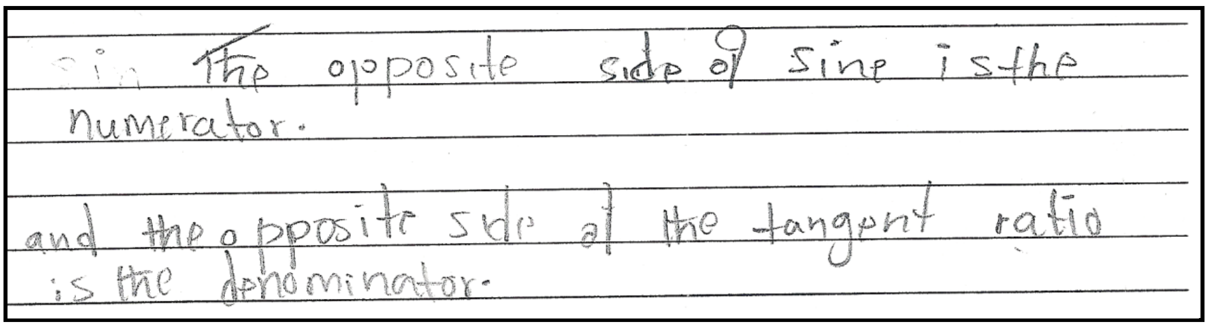


Figure 5.36: Participant I-12 comparison of the sides of the trig-ratios.

In *Figure 5.36*, the student is comparing the location of the opposite side in the sine and tangent ratios. S/he correctly places the opposite side in the numerator of the formula for the sine ratio but incorrectly places it in the denominator of the formula for the tangent ratio. The formula for the tangent ratio does not use the opposite side.

Theme 3: Comparisons based on values.

Some students provided appropriate comparisons of the three primary trigonometric ratios based on values. These students described the similarities and differences in the behaviour of the values of the ratios as the values of the angles change between zero and 90^0. *Figure 5.37* shows one such comparison.

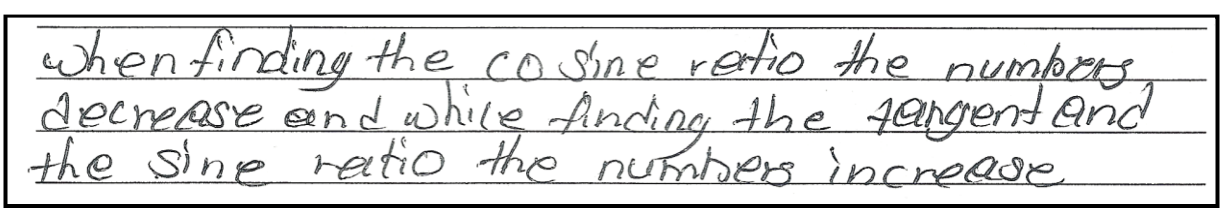


Figure 5.37: Participant I-2 comparison of the values of the trig-ratios.

In *Figure 5.37*, the student correctly stated that for both sine and tangent, the ratio values increase as the values of the angles increase, but for cosine, the ratio values decrease as the angle values increase. These students were exposed only to angle values between zero and 90^0 for the three primary trigonometric ratios.

Some students provided inappropriate comparisons of the three primary trigonometric ratios based on values. *Figure 5.38* shows one such comparison.

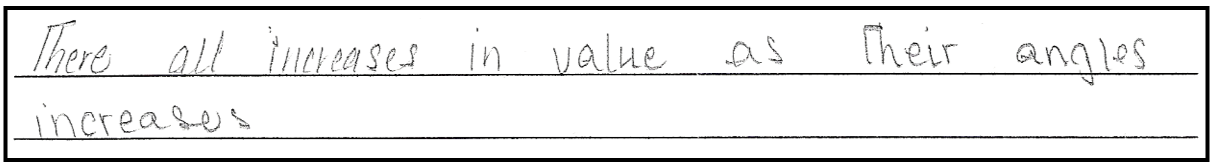

There all increases in value as their angles increases

Figure 5.38: Participant I-3 comparison of the values of the trig-ratios.

In *Figure 5.38*, the students incorrectly stated that the values of all the ratios increase as the values of the angles increase from zero to 90 degrees. This behaviour of the values is not the case for the cosine ratio, where the values decrease for increasing angles between zero and 90 degrees.

It is important to note that these students studied the three primary trigonometric ratios for angles from zero degrees to 90 degrees. Thus, they were not expected to know the behaviour of these trigonometric ratios for angles beyond 90 degrees.

Group profile.

This profile highlights the strengths and weaknesses in the post-test written responses of students taught by Investigation. It does so by listing and discussing their error-free responses and their erroneous responses as they worked with the different forms of representations—formulas, graphs, tables, diagrams—and their discussions surrounding these representations. The strengths and weaknesses are given under three domains: representing a contextual problem, multiple representations of a single concept, and comparing related concepts, which draws from Kilpatrick et al. (2001) work on conceptual understanding. There are strengths and weaknesses associated with each domain. A statement about the group's conceptual understanding is presented at the end of this profile.

Representing a contextual problem.

This section draws from the four themes presented under this domain—diagram used to represent the problem, selection of the correct ratio, selection of an incorrect ratio, calculation procedure. These themes were derived from students' responses to a prompt that asked them to draw a diagram to represent a contextual situation and discuss how the trigonometric ratios could be used to solve a problem embedded in the situation. This group of students:

Strengths.

- Presented an appropriate diagram representation of the problem.
- Selected the appropriate trigonometrical ratio to solve the problem and provided an appropriate justification for choosing that ratio.

Weaknesses.

- Selected the appropriate trigonometrical ratio for solving the problem but gave an inappropriate justification for choosing that ratio.
- Selected an inappropriate trigonometrical ratio for solving the problem.
- Presented an inappropriate calculation procedure for finding the height of the building.

Multiple representations of a single concept.

This section draws from the five themes presented under this domain—use of a formula, use of a graph, use of a table, use of a diagram, and justification based on sides. These themes were derived from students' responses to a prompt that asked them to use and discuss multiple representations of the cosine ratio. This group of students:

Strengths.

- Provided the correct formula for the cosine ratio.

- Provided an appropriate representation of the graph for the cosine ratio.
- Provided an appropriate representation of a table of values for the cosine ratio.
- Provided an appropriate representation of a diagram for the cosine ratio.
- Provided an appropriate justification for the choice of representations for the cosine ratio.

Weaknesses.

- Provided an incorrect formula for the cosine ratio.
- Provided an incomplete formula for the cosine ratio.
- Provided an inappropriate representation of the graph for the cosine ratio.
- Provided an inappropriate representation of a table of values for the cosine ratio.
- Provided an inappropriate representation of a diagram for the cosine ratio.
- Provided an inappropriate justification for the choice of representations for the cosine ratio.

Comparing related concepts.

This section draws from the three themes presented under this domain—comparisons based on representations, comparisons based on sides, and comparisons based on values. These themes were derived from students' responses to a prompt that asked them to show the three trigonometric ratios using the same form of representation, then to compare these representations. This group of students:

Strengths.

- Presented the same form of representation for all three primary trigonometric ratios that were appropriate.
- Presented appropriate comparisons of the three primary trigonometric ratios based on their sides.

- Presented appropriate comparisons of the three primary trigonometric ratios based on their values.

Weaknesses.

- Presented the same form of representation that did not correctly depict all three ratios.
- Presented inappropriate comparisons of the three primary trigonometric ratios based on their sides.
- Presented inappropriate comparisons of the three primary trigonometric ratios based on their values.

Conceptual understanding, in this study, is defined as students' ability to identify and discuss different representations of the three primary trigonometry ratios, produce and discuss a diagram representation of a contextual problem, and compare the same form of representation of these ratios. This group of students taught by Investigation showed a mixture of abilities in all three domains by presenting both appropriate and inappropriate answers. Hence, based on the above group profile, the group taught by Investigation did not appear to attain a full conceptual understanding of the three primary trigonometric ratios.

Group Taught by Exemplification

Sixteen fourth-form (grade 10) students participated and completed the post-test for this group. Hence, this analysis shows how teaching through Exemplification affected these 16 participants' conceptual understanding of the three primary trigonometric ratios. Of these 16 fourth-form students, approximately 31% ($N = 5$) were male and approximately 69% ($N = 11$) were female. The ages of the students in this group ranged from 15 to 17, with the mean age of 16.3 years and a standard deviation of 1.65 (M = 16.3, SD = 1.65). The following are the themes generated based on the analysis of their written responses on the post-test.

Representing a contextual problem.

The result of the analysis of students' responses to the following prompt is presented in this section.

> A dog is lying on the ground 25 metres away from the foot of a building. It observes a bird on top of the building at an angle of elevation of 20^0.
>
> v. Draw a diagram (triangle) to represent the situation described above. Mark clearly on your diagram: the distance of 25 metres, its line of sight, the 20^0 angle, and the height of the building.
>
> vi. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

The analysis yield four themes: diagram used to represent the problem, selection of the correct ratio, selection of an incorrect ratio, and calculation procedure. The contents of these themes comprised of both correct and incorrect responses and are supported by extracts from students' responses. Extracts are cropped sections of students' written responses that were selected to support the relevant theme.

Theme 1: Diagram used to represent the problem.

Some students who responded to section one of this prompt correctly represented the problem with a right-angle triangle. In some cases, the most relevant information was accurately placed on that diagram. *Figure 5.39* presents one such response.

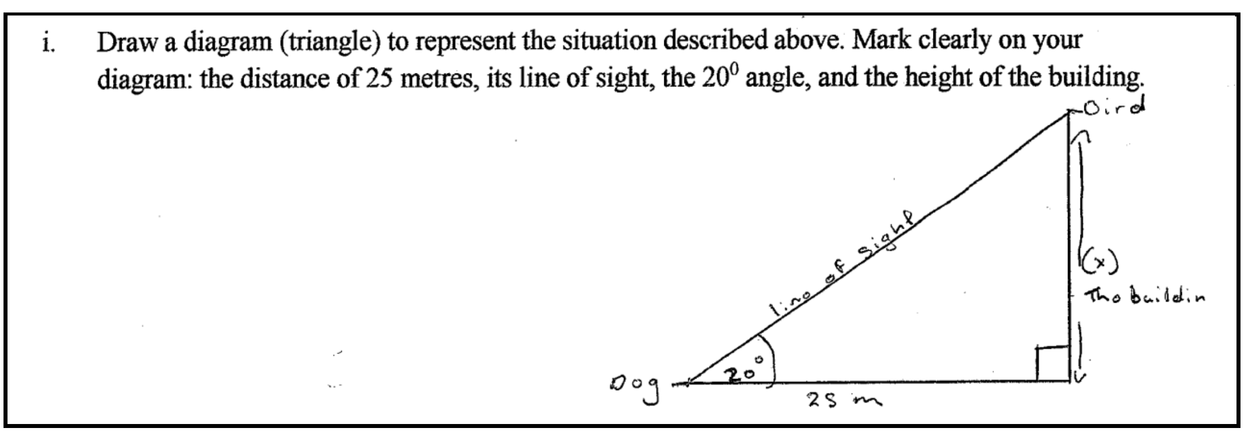


Figure 5.39: Participant E-4 representation of the problem.

Figure 5.39 shows the position of the dog on the ground (horizontal line) 25 meters away from the foot of the building, and the bird positioned at the top of the building. It shows a vertical line representing the building with a right angle between the building and the ground. A diagonal line connects the positions of the dog and the bird; this line represents the dog's line of sight (the direction in which the dog is looking). The 20^0 angle between the diagonal line and the ground represents the angle of elevation. An upward arrow should have been placed on the diagonal line to show that it is the dog looking upwards and not the bird looking downwards; thus, giving significance to the concept of angle of elevation.

In some instances, some relevant information was not placed on the diagram, but the information placed on the diagram were accurate and essential. *Figure 5.40* shows one such response.

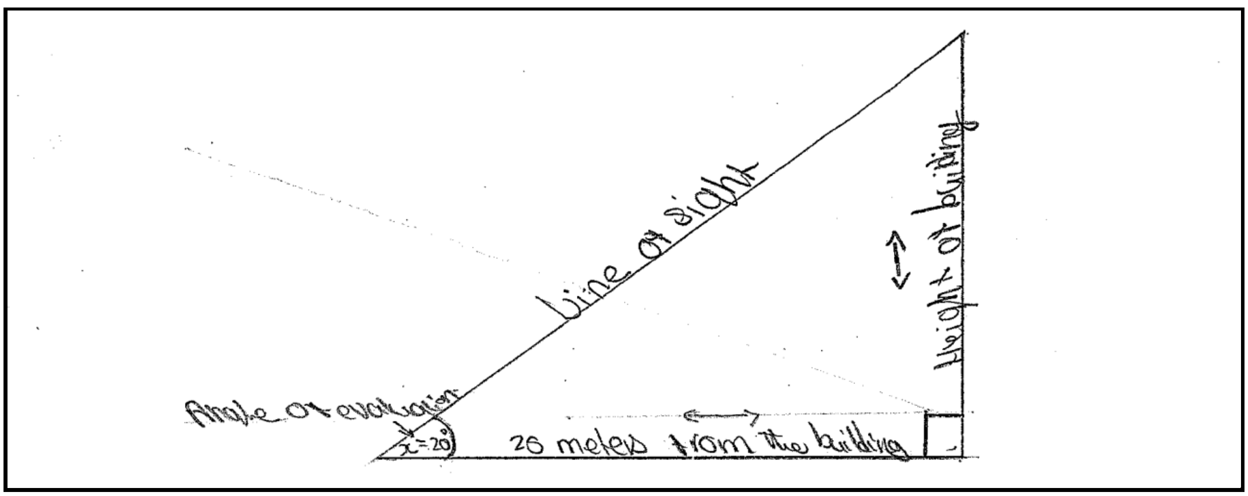


Figure 5.40: Participant E-8 representation of the problem.

Figure 5.40 shows several essential and accurately placed pieces of information. A right-angle triangle is used to represent the problem. The 20^0 angle, which is the angle of elevation, is between the diagonal line and the horizontal. The line of sight, the building, and the 25 meters are all accurately presented. Missing is the position of the dog on the ground and the position of the bird at the top of the building. Also missing are upward arrows on the diagonal line to show that it is the dog that is looking upwards; thus, an angle of elevation.

Some students who responded to section one of this prompt incorrectly represented the situation. *Figure 5.41* presents one such response.

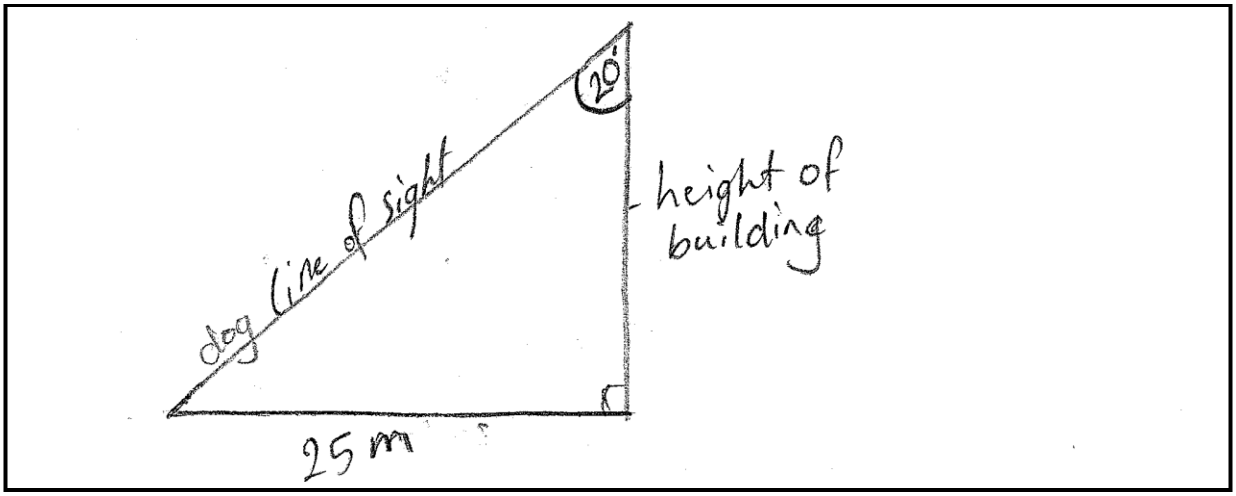


Figure 5.41: Participant E-2 representation of the problem.

Figure 5.41 represents the situation with a right-angle triangle, which is appropriate. The student also appropriately identified the building, the 25 meters, and the line of sight. However, the 20^0 angle of elevation is positioned between the vertical line and the diagonal line, which is incorrect. An angle of elevation is always formed with the horizontal; thus, this diagram representation of the contextual situation is inappropriate.

Theme 2: Selection of the correct ratio.

Some students correctly selected tangent as the most appropriate ratio to solve the problem but provided a somewhat limited justification for that selection. *Figure 5.42* shows one such justification.

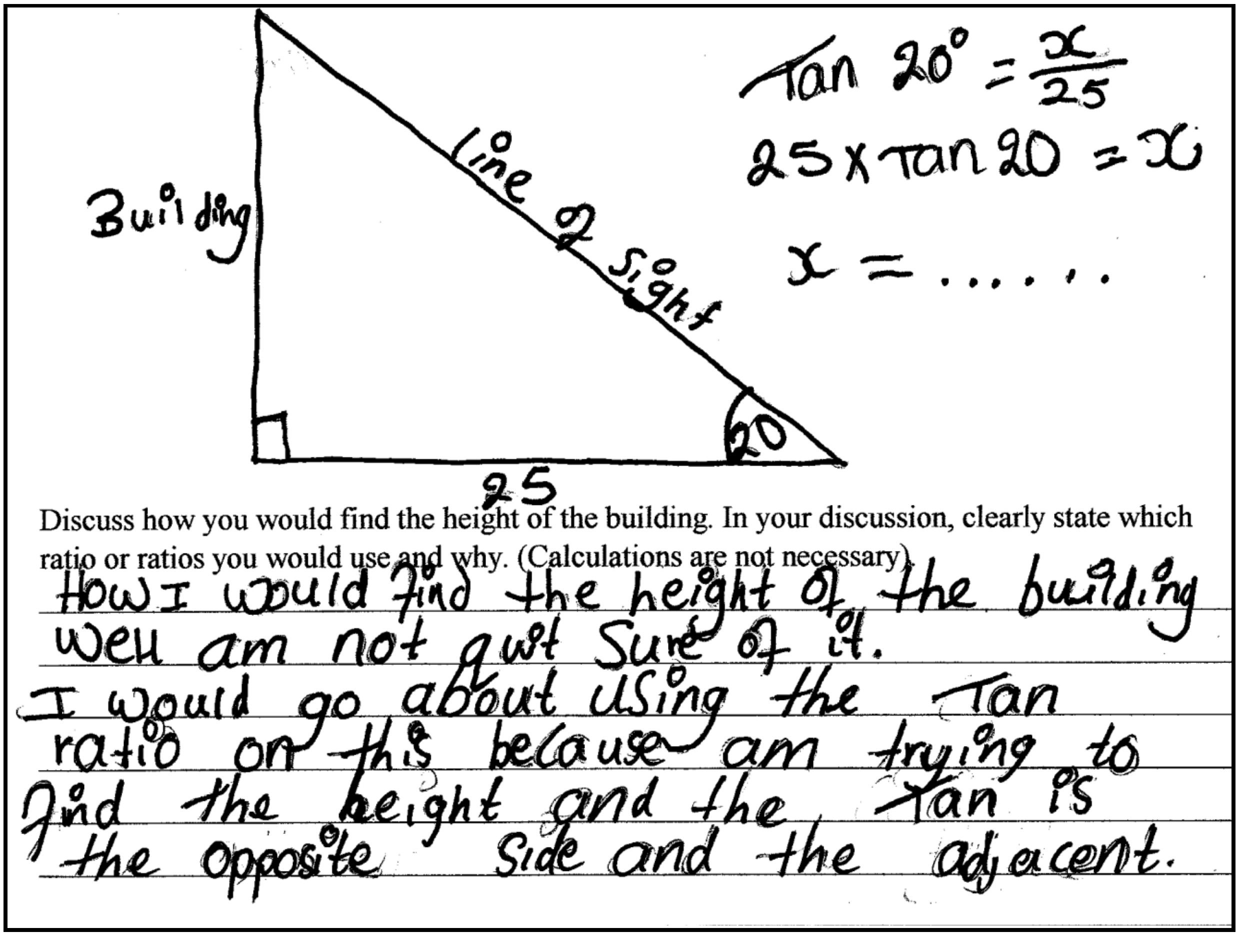
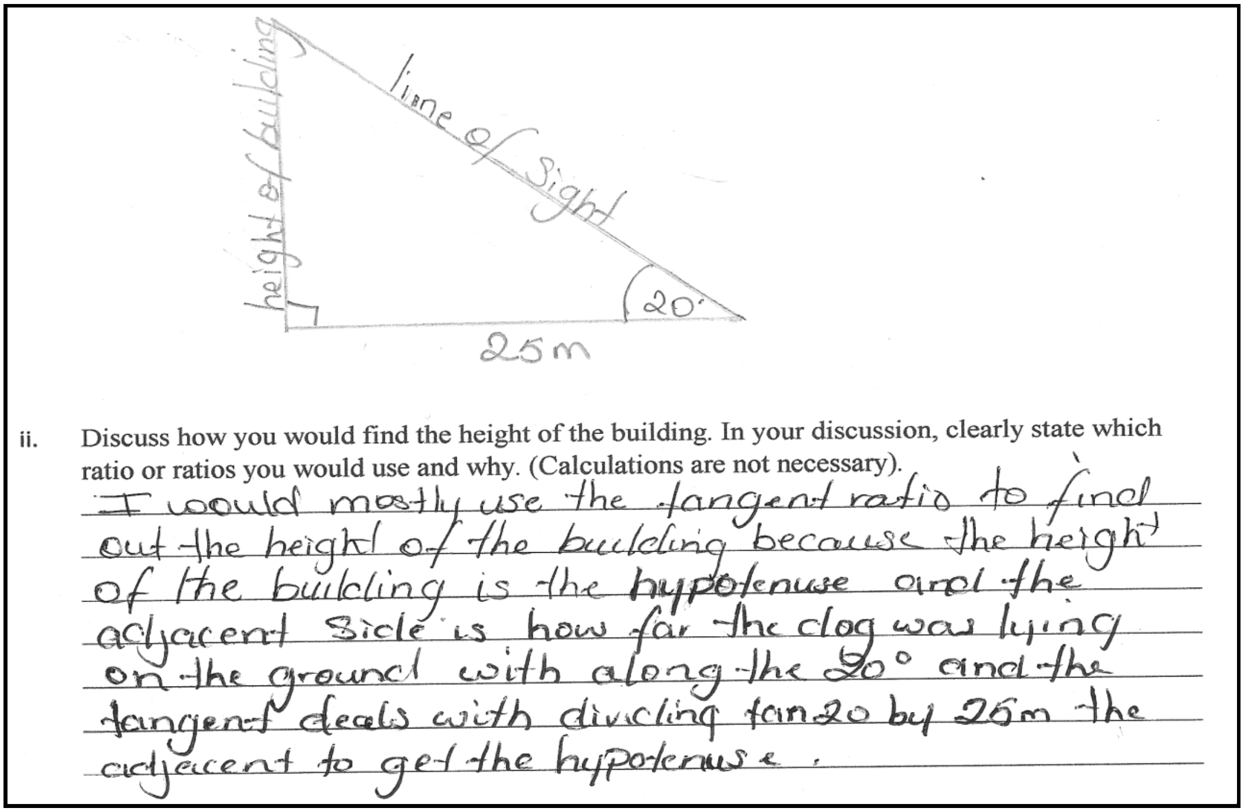


Figure 5.42: Participant E-17 justification for choosing the tangent ratios. The student's writing is written over for clarity.

In the justification presented in *Figure 5.42*, the student relates the tangent ratio to the appropriate side lengths—the adjacent and opposite sides. However, this student did not specify the aspect of the diagram that represents the opposite side (building) or the adjacent side (25 meters). The student did provide a procedure that shows how the tangent ratio could be used, but s/he did not say what the x in the procedure represents. The student's use of tangent is accurate if the x represents the height of the building.

Some students gave a flawed justification for selecting the tangent ratio. In these instances, they did not correctly identify at least one side on the diagram representing the problem or they stated at least one incorrect side that formed the tangent ratio. *Figure 5.43* shows one such justification.



ii. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

I would mostly use the tangent ratio to find out the height of the bulding because the height of the building is the hypotenuse and the adjacent Side is how far the dog was lying on the ground with along the 20° and the tangent deals with dividing tan20 by 25m the adjacent to get the hypotenuse.

Figure 5.43: Participant E-14 justification for choosing the tangent ratio.

In the justification presented in *Figure 5.43*, the student incorrectly identified the height of the building as the hypotenuse and correctly identified the distance along the ground as the adjacent side. The tangent ratio was inappropriately associated with the hypotenuse by this student. The tangent ratio does not deal with the hypotenuse of a right-angle triangle.

Theme 3: Selection of an incorrect ratio.

Some students incorrectly selected cosine as the most appropriate ratio to find the height of the building. Different justifications were given for choosing the cosine ratio. *Figure 5.44* shows one inappropriate justification.

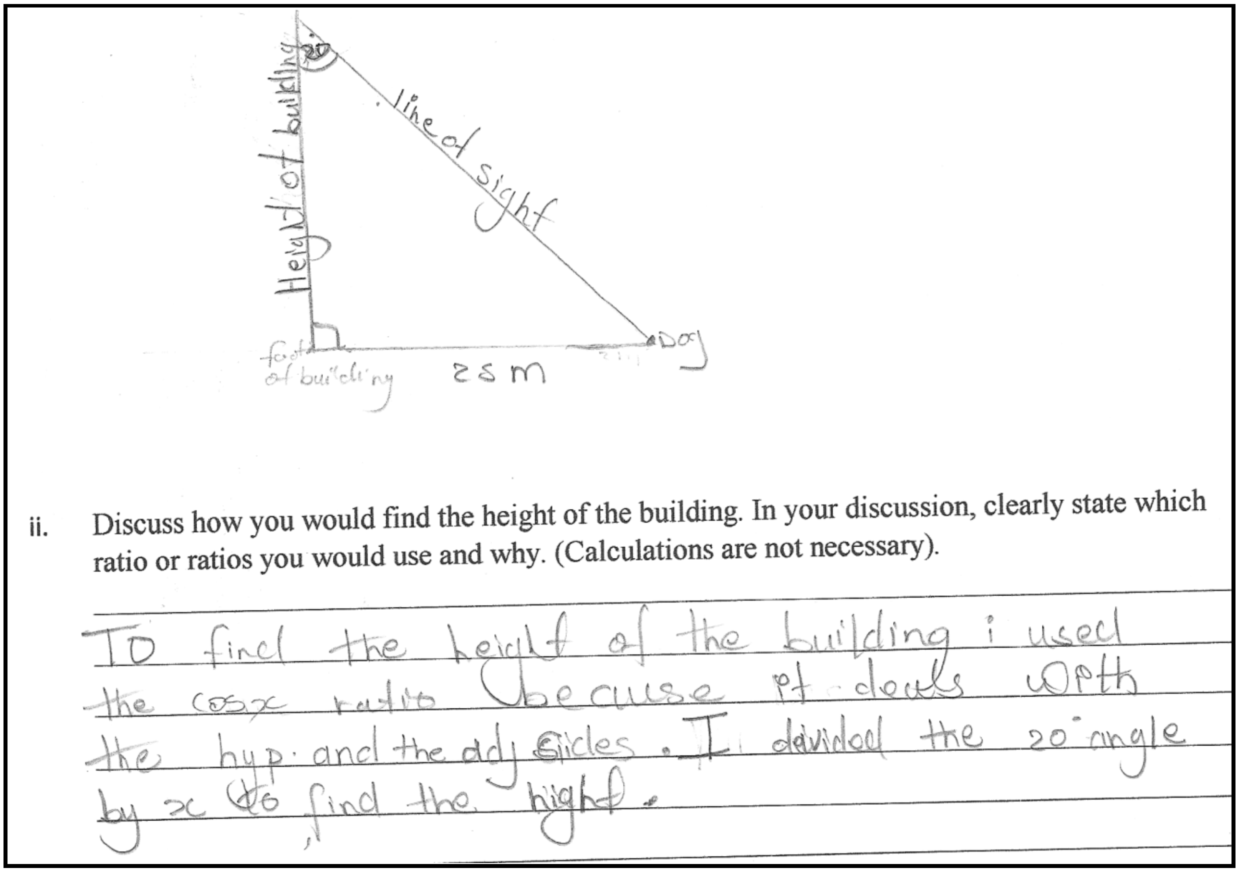
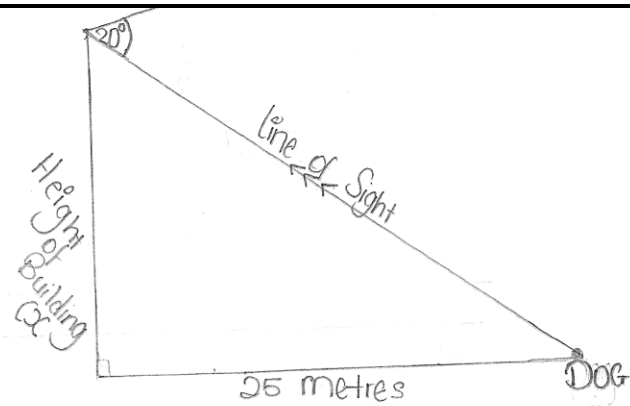


Figure 5.44: Participant E-6 justification for choosing the cosine ratio.

In *Figure 5.44*, the student stated that they would use the cosine ratio because cosine deals with the hypotenuse and the adjacent side. It is unclear which side of the diagram that the student is referring to as the hypotenuse or the adjacent side. Based on the student's diagram, the building is the adjacent side, but the 25-meter distance is not the hypotenuse. Since the hypotenuse (line of sight) is not known, the selection of cosine is flawed.

Figure 5.45 shows another inappropriate justification for choosing the cosine ratio to find the height of the building.



ii. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

To find the height of the building, I would use the cosine ratio because you are told to find the Height of the building which is the opposite Side of a triangle and the Cosine formula is adjacent Side divided by the Hypotenuse to find the to find the opposite Side.

Figure 5.45: Participant E-7 justification for choosing the cosine ratio.

In *Figure 5.45*, the student's justification claims that the opposite side is the height of the building that could be found by dividing the adjacent side by the hypotenuse. This justification ignores the fact that the opposite side is not included in the formula for the cosine ratio; therefore, it cannot be found using the cosine ratio. Also, the student identified the building as the opposite side of the triangle despite there being no reference angle in the triangle diagram because the angle of elevation was wrongly located.

Some students also inappropriately selected sine as the most appropriate ratio to solve the problem. *Figure 5.46* shows one such inappropriate justification.

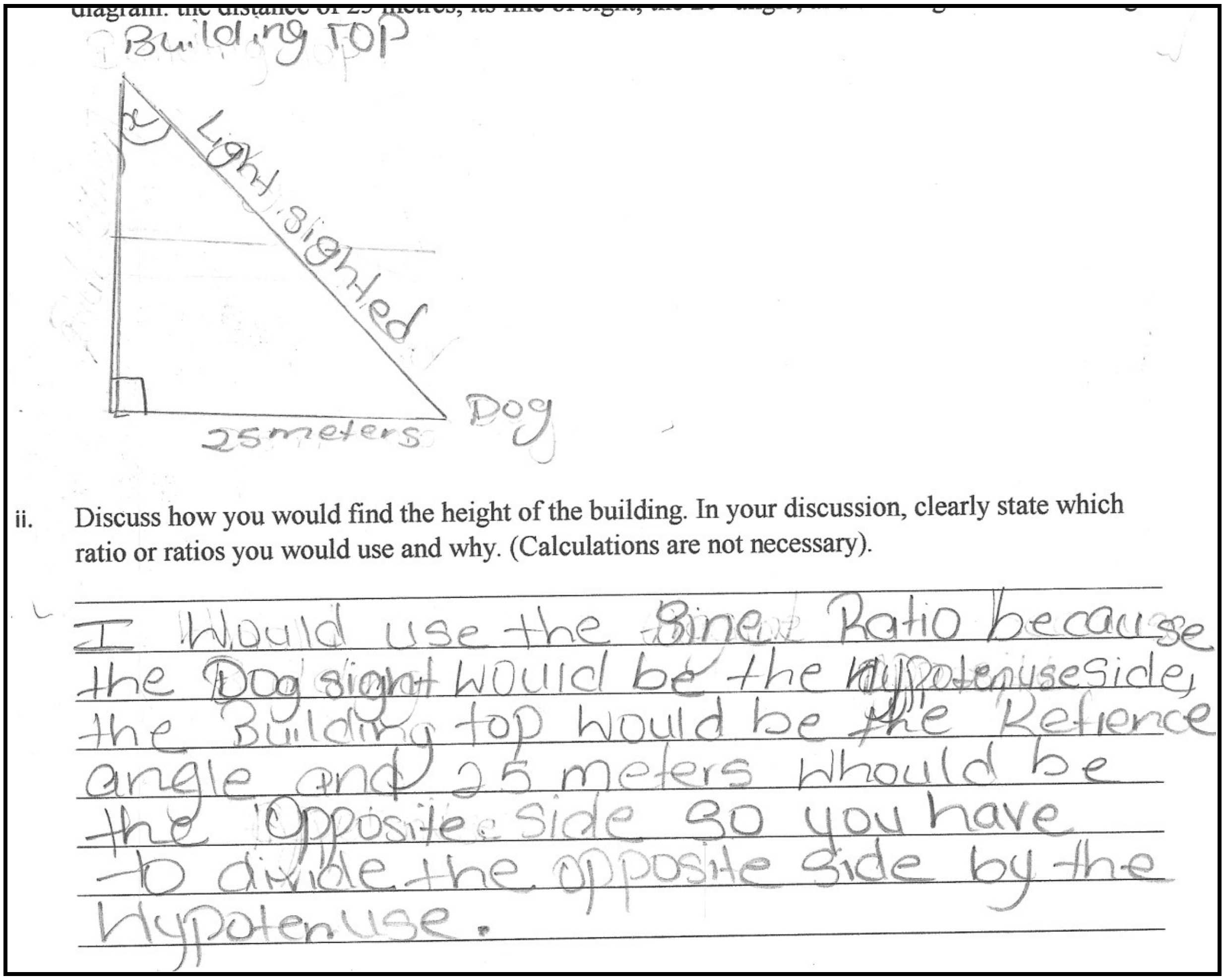


Figure 5.46: Participant E-16 justification for choosing the sine ratio.

In *Figure 5.46*, the student correctly identified the line of sight as the hypotenuse and identified the distance along the ground as the opposite side. Based on the location of the reference angle (x) in the diagram, the distance along the ground is the opposite side. However, the angle x was incorrectly placed if it represents the angle of elevation. In any case, the height of the building was not featured in the student's choice of a trigonometric ratio (sine); thus, sine is not an appropriate choice of trigonometric ratio to find the height of the building.

Theme 4: Calculation procedure.

Some students gave an appropriate calculation procedure for finding the height of the building. *Figure 5.42* above presents one such procedure. This student showed that the tangent of the angle (*tan* 20^0) is equal to *x* divided by 25. This equation is correct if *x* represents the height of the building. S/he then correctly transpose this equation to show that *x* (the height of the building) is equal to 25 times tan 20^0.

Students gave several incorrect procedures for calculating the height of the building. *Figure 5.47* shows one such calculation procedure.

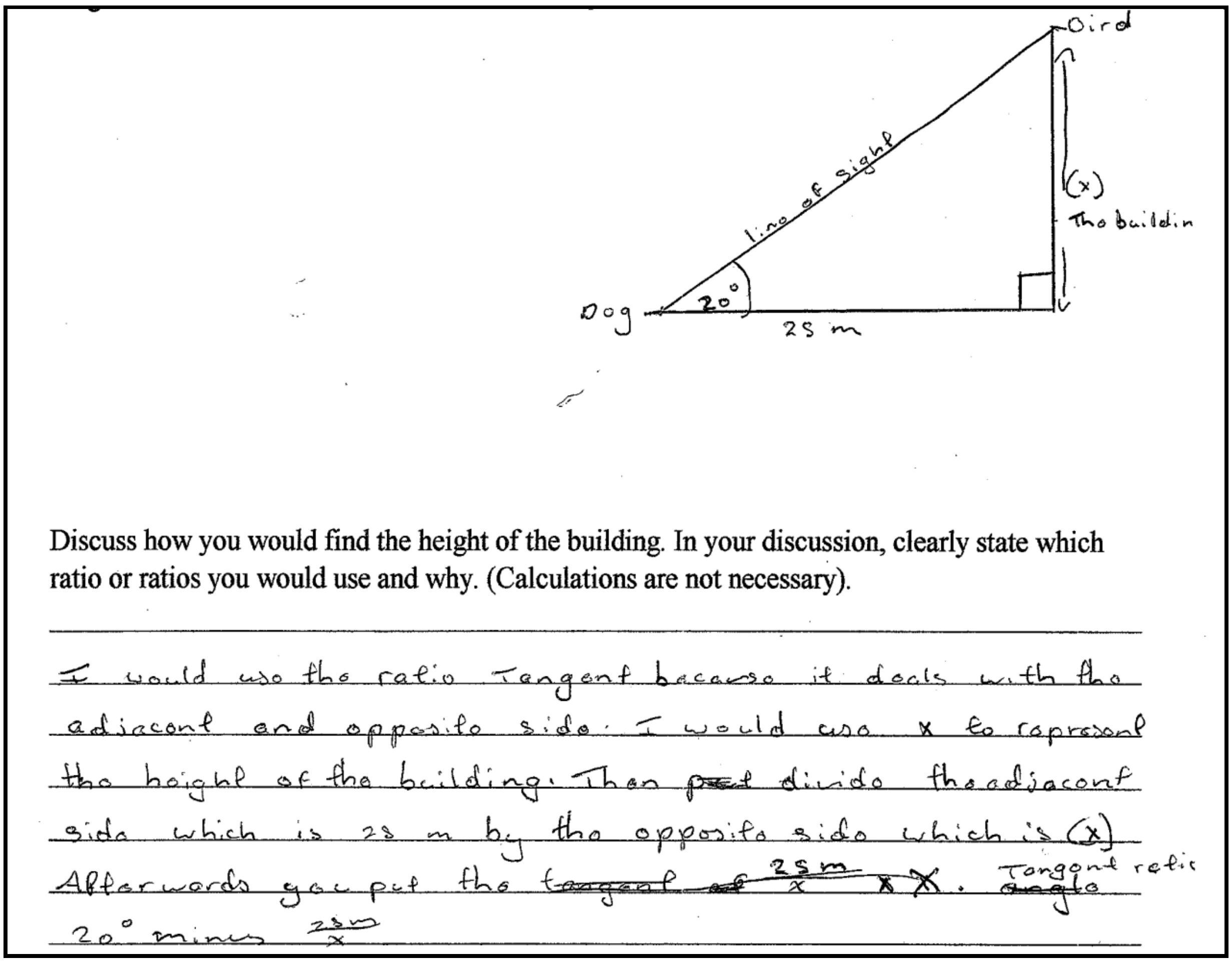


Figure 5.47: Participant E-4 procedure for finding the height of the building.

In *Figure 5.47*, the student appropriately identified the tangent ratio and the appropriate side lengths—the opposite and adjacent sides—to be used. However, s/he would incorrectly divide the adjacent side (25 m) by the opposite side (the height of the building x). In the tangent ratio, it is the opposite side that is divided by the adjacent side ($\tan\theta = \frac{opposite\ side}{adjacent\ side}$). As a further error, the student would subtract the result of the division from the tangent of 20^0.

Figure 5.48 shows a second inappropriate calculation procedure for finding the height of the building.

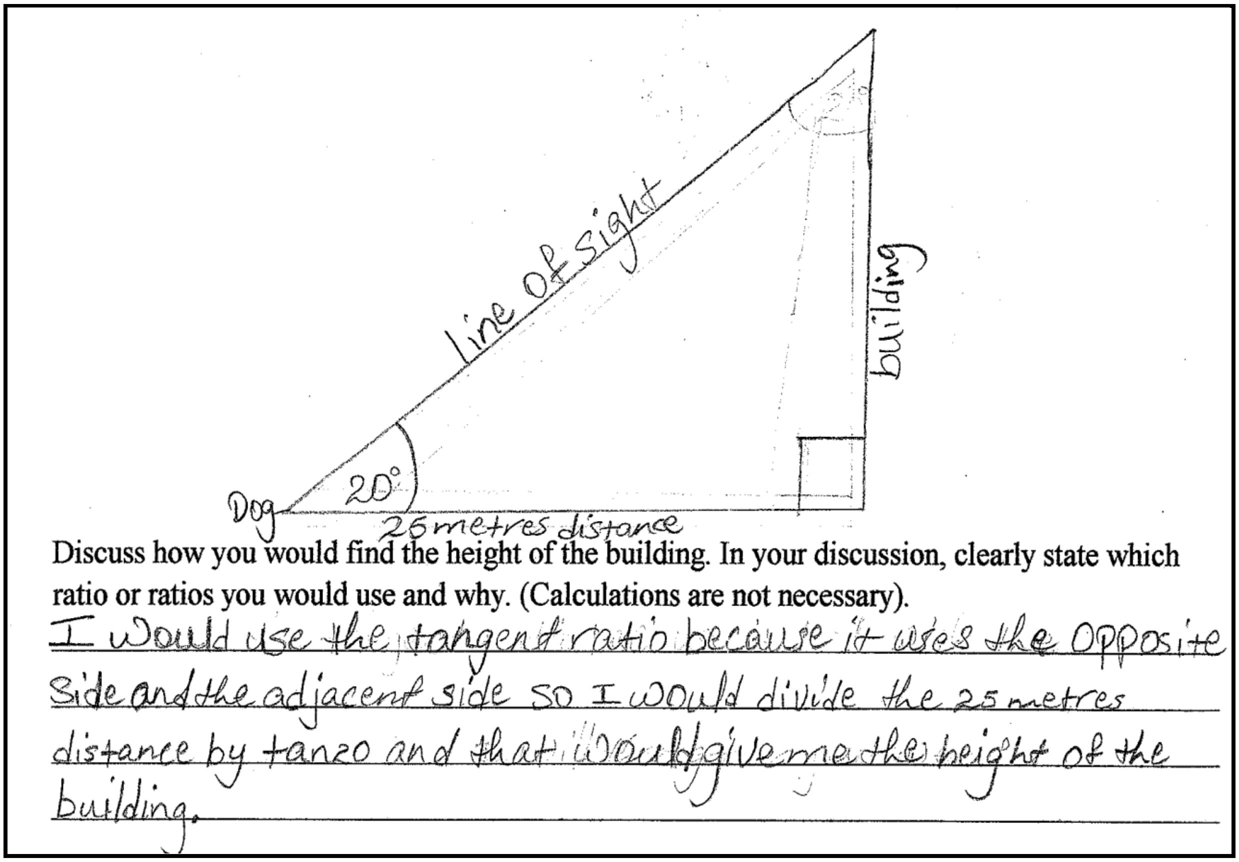


Figure 5.48: Participant E-9 procedure for finding the height of the building.

In *Figure 5.48,* the student appropriately identified the tangent ratio and the appropriate side lengths—the opposite and adjacent sides—to be used. However, s/he would incorrectly divide the adjacent side (25m) by the tangent of the angle (*tan* 20^0). Give that

$\tan\theta = \frac{opposite\ side}{adjacent\ side}$, the correct arithmetic operation to find the height of the building (the opposite side) is to multiple the adjacent side by the tangent of the angle.

Multiple representations of a single concept.

The prompt soliciting students' responses for this domain asked students to represent the cosine ratio using three different forms of representation and to justify their choices:

> This question requires you to use and discuss multiple representations of the cosine ratio.
>
> i. Including the formula, show **at least** three (3) different representations of the cosine ratio.
>
> ii. Discuss how you know that each of these representation shows the cosine ratio.

The analysis of students' responses to this prompt yielded six themes: use of a formula, use of a graph, use of a table, use of a diagram, justification based on sides, and justification based on values. The contents of these themes comprised of both correct and incorrect responses.

Theme 1: Use of a formula.

Some students provided the correct formula for cosine with all elements: angle, adjacent side, and hypotenuse given in the proper positions. *Figure 5.49* shows one such response.

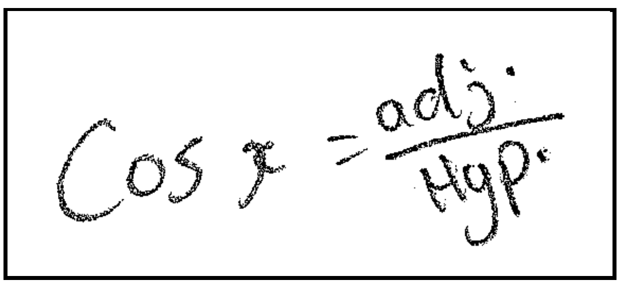


Figure 5.49: Participant E-1 representation of the cosine formula.

Figure 5.49 shows the cosine of the angle *x* (*cos x*) being equal to the adjacent side (adj.) divided by the hypotenuse (Hyp.). The division of the two identified sides is correct, and the cosine ratio is correctly abbreviated (*cos x*). This is a correct representation of the formula for the cosine ratio.

In some cases, the adjacent side and the hypotenuse were correctly divided, but the formula was not given in terms of an angle. *Figure 5.50* shows one such response.

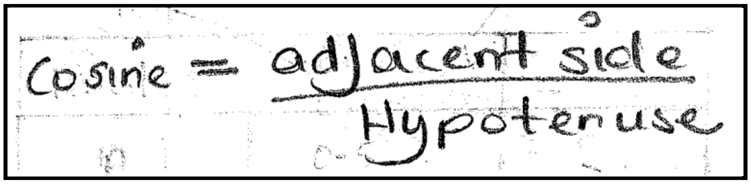


Figure 5.50: Participant E-5 representation of the cosine formula.

Figure 5.50 shows cosine, without an angle, being equal to the adjacent side divided by the hypotenuse. The angle is an essential element of all the trigonometric ratios because the trigonometric ratios are functions of angles. Hence, the angle must always be shown in the formulas of the trigonometric ratios. Hence, this representation of the formula for the cosine ratio is limited (not completely correct).

Theme 2: Use of a graph.

Some students provided an appropriate graph to represent the cosine ratio. *Figure 5.51* shows one such graph.

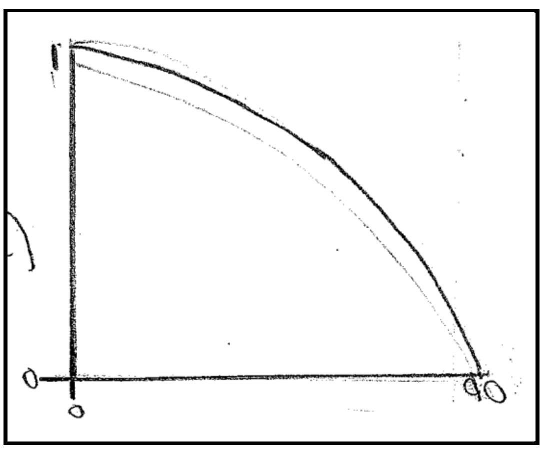


Figure 5.51: Participant E-8 graphical representation of the cosine ratio.

In *Figure 5.51*, the student's graph is a curve for which the values of cosine decreased from one to zero as the values of the angles increased from zero degrees to 90^0. Two critical cosine values (zero and one) are represented along the vertical axis (y-axis), and their corresponding angle values (one and zero) are represented along the horizontal axis (x-axis). Cosine is correctly shown to have a value of one at zero degrees and a value of zero at 90^0.

In some instances, students drew a curve going in the correct direction but with no values for neither the angles nor the cosine ratio. *Figure 5.52* shows one such graph.

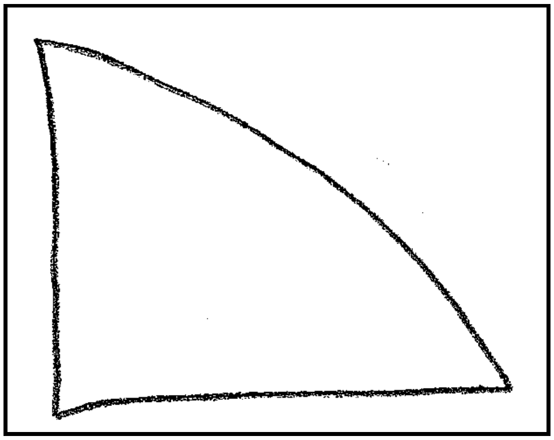


Figure 5.52: Participant E-2 graphical representation of the cosine ratio.

Figure 5.52 shows a typical cosine curve for angles between zero and 90^0. However, the appropriateness of this graph is limited because it presents no numerical values along neither the vertical axis (y-axis) or the horizontal axis (x-axis). Without values, it is impossible to tell if the curve is touching the y-axis at one (when the angle is zero degrees) and if it is touching the x-axis at 90^0 (when the ratio value is zero).

Theme 3: Use of a table.

Some students provided an appropriate table to represent the cosine ratio. *Figure 5.53* shows one such table.

Angle (°)	Cosine x
10	0.985
20	0.940
40	0.766
90°	0

Figure 5.53: Participant E-10 tabular representation of the cosine ratio.

The table in *Figure 5.53* shows the values of cosine decreasing as the values of the angles increased. For instance, the cosine of 10^0 is given as 0.985, and the cosine of 40^0 is given as

0.766. The values of the cosine ratio decrease for increasing values of angles between zero and 90^0.

Theme 4: use of a diagram.

A right-angle triangle with a reference angle, the adjacent side, and the hypotenuse marked are the essential elements of a diagram demonstrating the cosine ratio. Some students provided an appropriate diagram to represent the cosine ratio. *Figure 5.54* shows one such diagram.

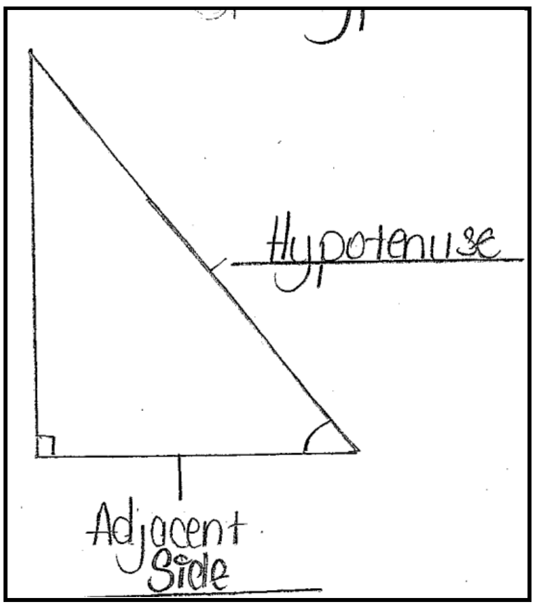


Figure 5.54: Participant E-7 diagram representation of the cosine ratio.

Figure 5.54 is a right-angle triangle showing a reference angle (marked by a curve in the triangle), the adjacent side, and the hypotenuse that are correctly located. A box is used to identify the right angle. The hypotenuse is opposite the right angle and forms one arm of the reference angle. The adjacent side forms the other arm of the reference angle. All essential elements are identified and correctly located.

Some students produced an inappropriate diagram to represent the cosine ratio. *Figure 5.55* shows one such diagram.

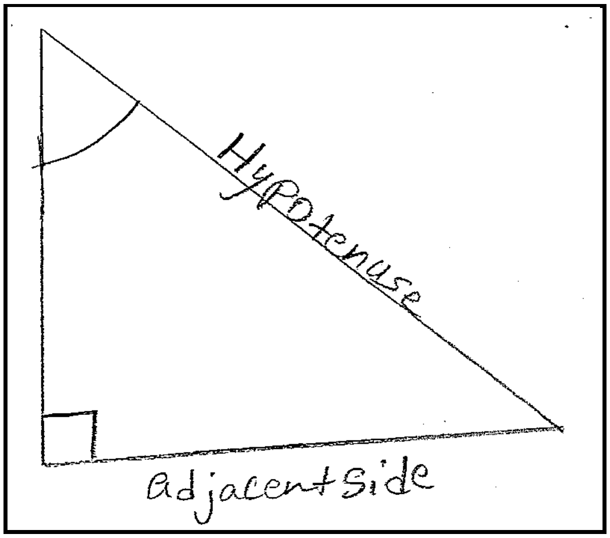
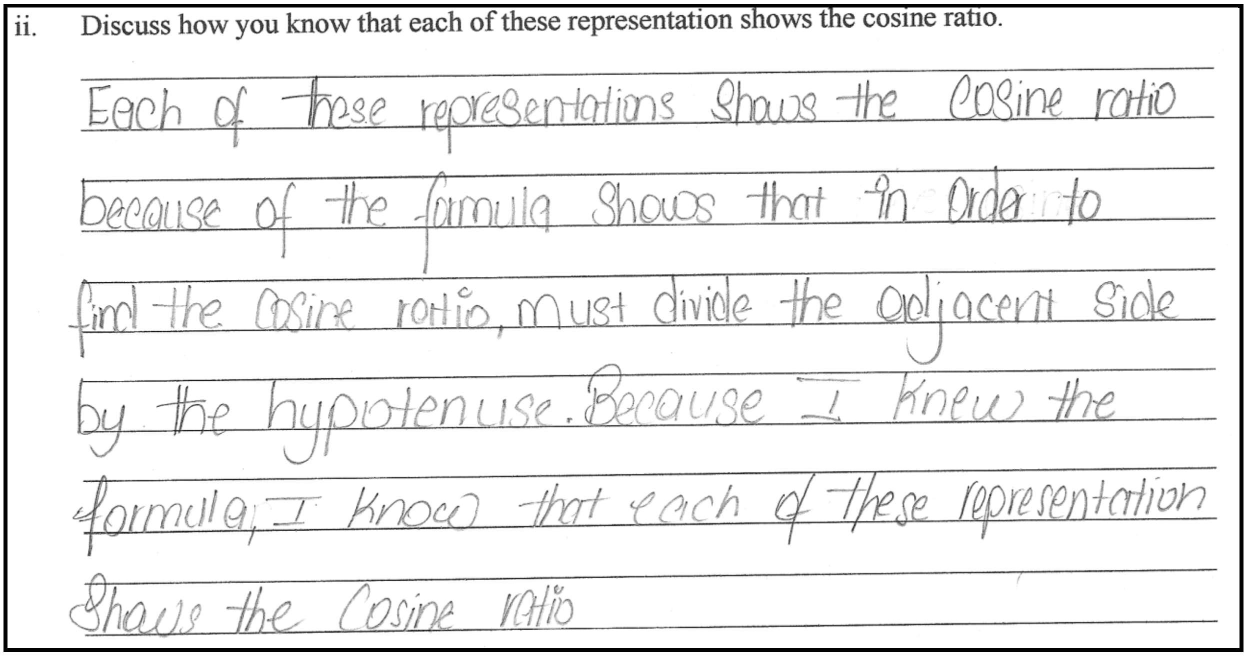


Figure 5.55: Participant E-9 diagram representation of the cosine ratio.

Figure 5.55 is a right-angle triangle showing a reference angle, the adjacent side, and a hypotenuse. A box is used to identify the right angle, and a curved line is used to identify the reference angle. The adjacent side is incorrectly located opposite the reference angle. The hypotenuse is correctly located opposite the right angle. The essential elements are identified; however, the incorrect location of the adjacent side makes this diagram an inappropriate representation of the cosine ratio.

Theme 5: Justification based on sides.

Some students provided an appropriate justification for their selection of representations based on side lengths that form the cosine ratio. *Figure 5.56* shows one such justification.


ii. Discuss how you know that each of these representation shows the cosine ratio.

Each of these representations shows the Cosine ratio because of the formula shows that in order to find the Cosine rattio, must divide the adjacent side by the hypotenuse. Because I knew the formula, I know that each of these representation shows the Cosine ratio

Figure 5.56: Participant E-7 justification for the choice of formula.

In *Figure 5.56*, the student correctly indicated that the cosine ratio deals with the adjacent side and the hypotenuse. The student indicated that the adjacent side is divided by the hypotenuse to form the cosine ratio and claimed that s/he knew that the representations show the cosine ratio because s/he knew the cosine formula.

Theme 6: Justification based on values.

Some students provided an appropriate justification for their selection of representations for the cosine ratio based on how the ratio values change with changing angle values. *Figure 5.57* shows one such justification.

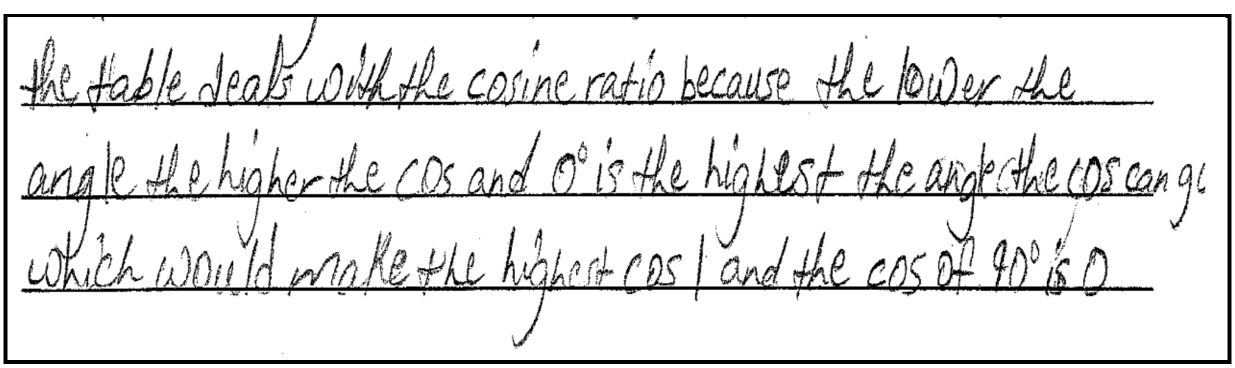

the table deals with the cosine ratio because the lower the angle the higher the cos and 0° is the highest the angle the cos can go which would make the highest cos 1 and the cos of 90° is 0

Figure 5.57: Participant E-9 justification for the choice of diagram and table.

In *Figure 5.57*, the student justified their choices of representation by correctly stating that the lower the angle, the higher the cosine value. That is, the cosine values decrease as the values of the angles increase from zero to 90^0. The student also correctly stated that the highest value of cosine is one at zero degrees, and at 90^0 the cosine value is zero.

Comparing related concepts.

The prompt soliciting students' responses for this domain asked students to represent the sine, cosine and tangent ratios using the same form of representation and to compare these representations.

> This question requires you to show the three trigonometric ratios using the **same form** of representation, then to compare and contrast these representations.
>
> i. Use the **same form** of representation to show the sine, cosine, and tangent ratios.
> ii. Discuss the similarities and differences among these representations.

The analysis of students' responses to this prompt yielded three themes: comparisons based on representations, comparisons based on sides, and comparisons based on values. The contents of these themes comprised of both correct and incorrect responses.

Theme 1: Comparisons based on representations.

Some students represented the sine, cosine, and tangent ratios using the different formulas, but without all the essential elements included. *Figure 5.58* shows one such set of formulas.

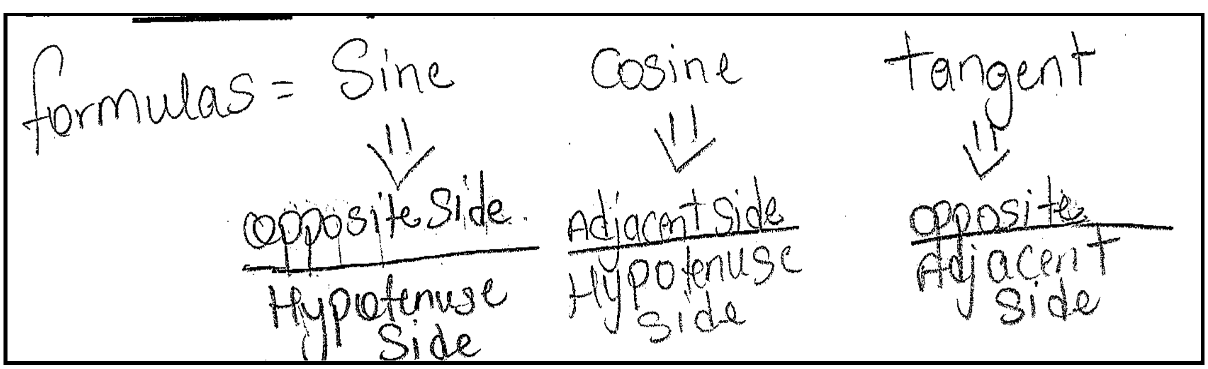


Figure 5.58: Participant E-11 formula representations of the trig-ratios.

In *Figure 5.58,* the formula for the sine ratio is given as the opposite side divided by hypotenuse, the formula for the cosine ratio is given as the adjacent side divided by hypotenuse, and the formula for the tangent ratio is given as the opposite side divided by the adjacent side. The relevant sides are correctly divided for the three stated ratios; however, these formulas were written without an angle, which is an essential element of the trigonometric ratios.

Some students represented the sine, cosine, and tangent ratios using graphs with appropriately shaped curves. These graphs show how the curve of each ratio behaves for angles between zero and 90^0. *Figure 5.59* shows one such set of graphs.

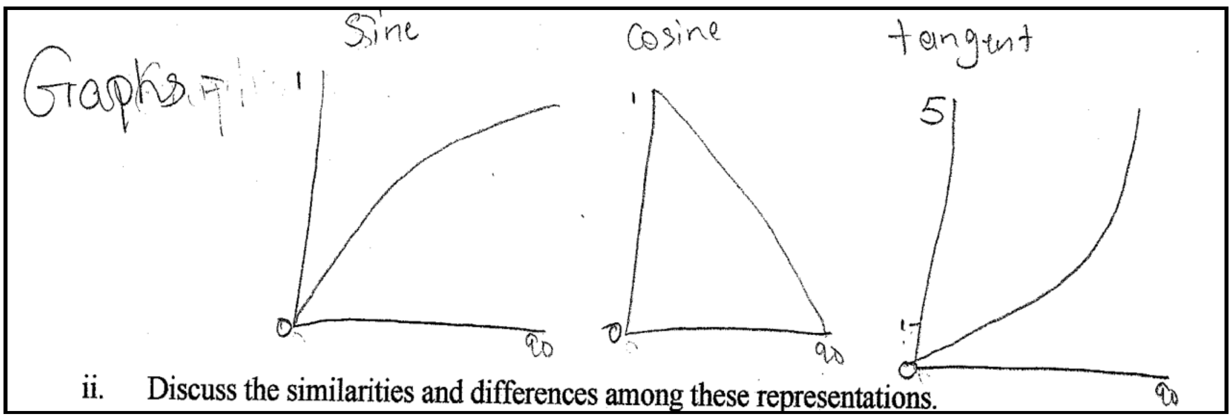


Figure 5.59: Participant E-11graphical representations of the trig-ratios.

In *Figure 5.59*, the student correctly showed the sine curve increasing from zero to one as the angles increase from zero to 90^0, the cosine curve decreasing from one to zero as the angles increase from zero to 90^0, and the tangent curve increasing from zero to values greater than one as the angles increase from zero and approach 90^0. These graphs typify the behaviours of the sine, cosine, and tangent curves.

Some students represented the sine, cosine, and tangent ratios using different diagrams. All the essential elements were given and correctly located in the respective diagrams; thus, making them appropriate representations of the trigonometric ratios. *Figure 5.60* shows one such set of diagrams.

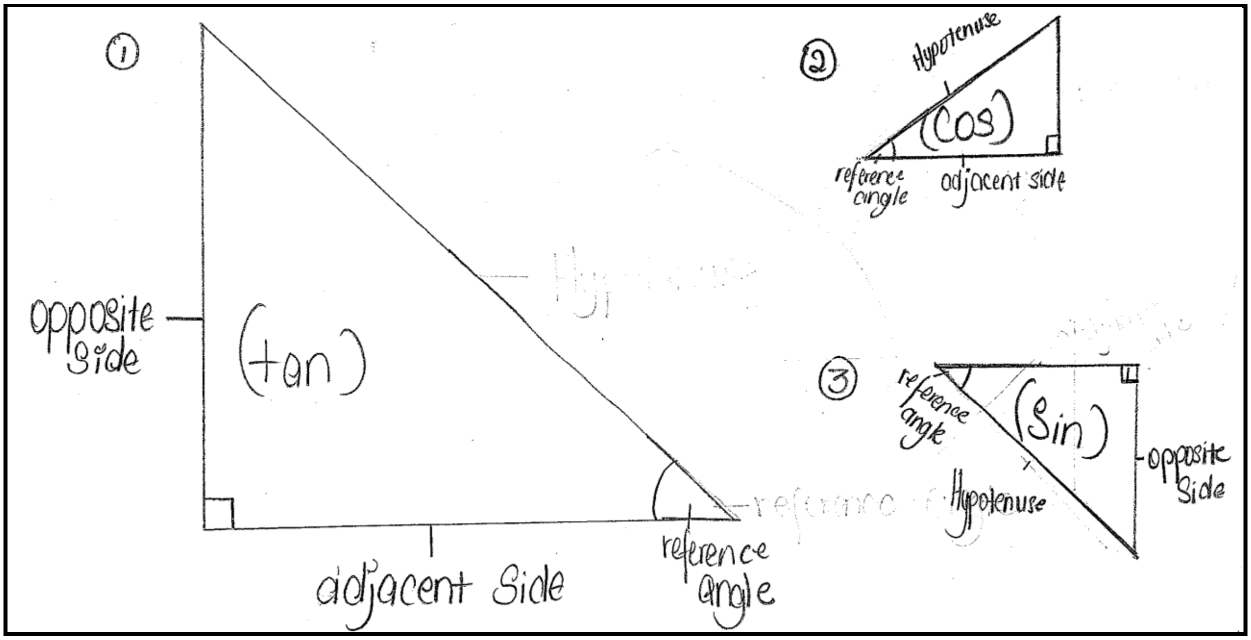


Figure 5.60: Participant E-7 diagram representation of the trig-ratios.

In *Figure 5.60*, the diagrams are right-angle triangles marked with a reference angle and the relevant sides that form either the sine, cosine, or tangent ratios. The diagram representing the sine ratio (diagram 3) shows a reference angle marked with a curve and labelled, the opposite side correctly located opposite the reference angle, and the hypotenuse correctly located opposite the right angle. The diagram representing the cosine ratio (diagram 2) shows a reference angle marked with a curve and labelled, the adjacent side correctly located as one arm of the reference angle, and the hypotenuse correctly located opposite the right angle. The diagram representing the tangent ratio (diagram 1) shows a reference angle marked with a curve and labelled, the opposite side correctly located opposite the reference angle, and the adjacent side correctly located as one arm of the reference angle.

Theme 2: Comparisons based on sides.

Some students provided appropriate comparisons of the three primary trigonometric ratios based on the sides that form these ratios. *Figure 5.61* shows one student's comparisons that were appropriate and based on the sides that form the ratios.

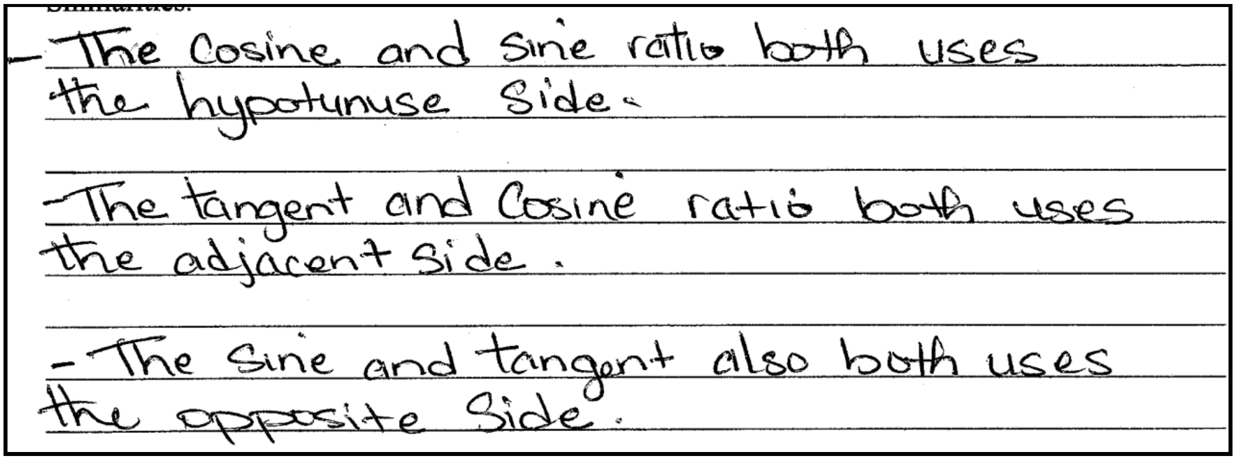

- The Cosine and Sine ratio both uses the hypotunuse Side.
- The tangent and Cosine ratio both uses the adjacent Side.
- The Sine and tangent also both uses the opposite Side.

Figure 5.61: Participant E-12 comparison of sides for the trig-ratios.

In Figure 5.61, the student correctly stated that the hypotenuse is present in both the sine and cosine ratios. S/he also correctly stated that the adjacent side is used by both the tangent and cosine ratios, and the opposite side is used by both the sine and tangent ratios. These comparisons highlight only similarities among the three primary trigonometric ratios.

Figure 5.62 shows a second set of comparisons based on the sides that form the trigonometric ratios.

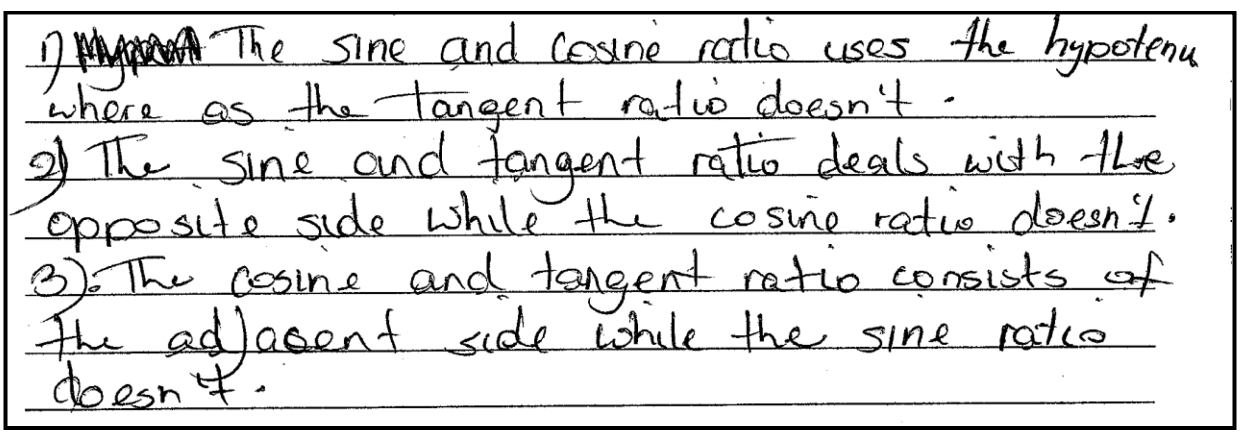

1) The sine and cosine ratio uses the hypotenu where as the tangent ratio doesn't.
2) The sine and tangent ratio deals with the opposite side while the cosine ratio doesn't.
3). The cosine and tangent ratio consists of the adjacent side while the sine ratio doesn't.

Figure 5.62: Participant E-5 comparison of sides for the trig-ratios.

In *Figure 5.62*, the student correctly stated that both the sine and cosine ratios make use of the hypotenuse, but the tangent ratio does not. S/he also correctly stated that the sine and tangent ratios deal with the opposite side while the cosine ratio does not, and the cosine and tangent ratio consist of the adjacent side while the sine ratio does not. These comparisons highlight both similarities and differences among the three primary trigonometric ratios.

Theme 3: Comparisons based on values.

Some students provided appropriate comparisons of the three primary trigonometric ratios based on values. Several appropriate comparisons were presented. *Figure 5.63* shows one such comparison.

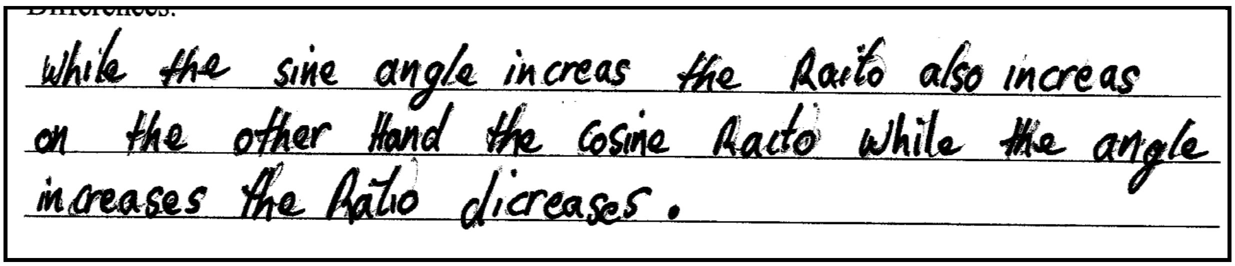

Differences:
while the sine angle increas the Raito also increas on the other Hand the cosine Racto while the angle increases the Ratio dicreases.

Figure 5.63: Participant E-13 comparison of values for the sine and cosine ratios. The student's writing is written over for clarity.

In *Figure 5.63*, the student correctly stated that for sine, the value of the ratio increases as the angle value increases, but the ratio value decreases for cosine as the angle value increases. This comparison highlights a difference between the sine and cosine ratios.

Figure 5.64 shows a second appropriate comparison based on values.

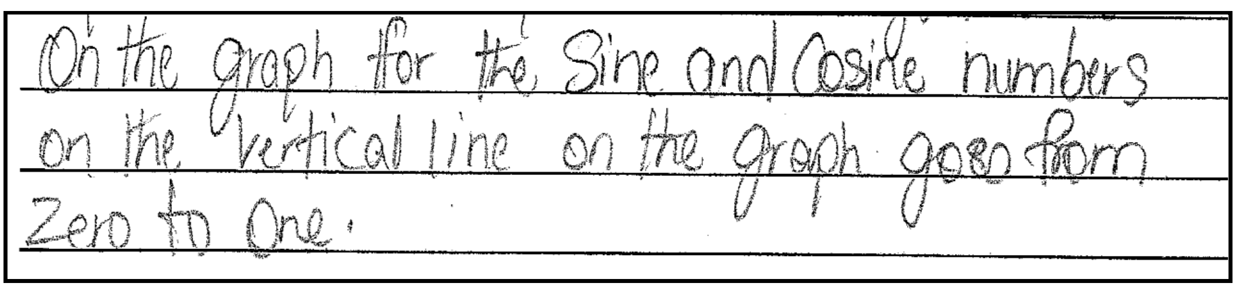

On the graph for the Sine and Cosine numbers on the vertical line on the graph goes from Zero to One.

Figure 5.64: Participant E-11 comparison of values for the sine and cosine ratios.

In *Figure 5.64*, the student correctly stated that the ratio values for both the sine and cosine ratios range from zero to one on the graph. This comparison highlights a similarity between the sine and cosine ratios.

Figure 5.65 shows a third appropriate comparison based on values.

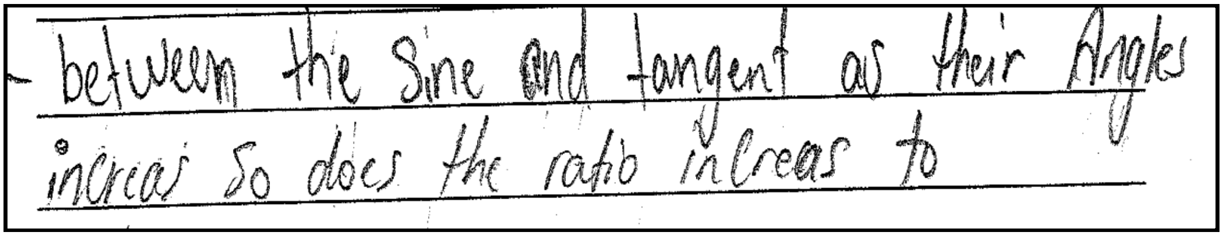


Figure 5.65: Participant E-2 comparison of values for the sine and tangent ratios.

In *Figure 5.65*, the student correctly stated that the ratio value increases as the angle value increases for the sine and tangent ratios. This comparison highlights a similarity between the sine and tangent ratios.

Figure 5.66 shows a fourth appropriate comparison based on values.

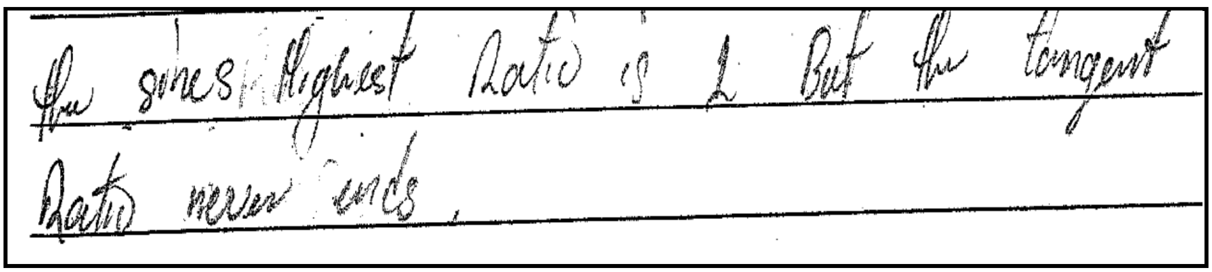


Figure 5.66: Participant E-13 comparison of values for the sine and tangent ratios.

In *Figure 5.66*, the student correctly stated that the sine ratio has the highest value of one, but the tangent ratio value never ends. This comparison highlights a difference between the sine and tangent ratios.

Figure 5.67 shows a fifth appropriate comparison based on values.

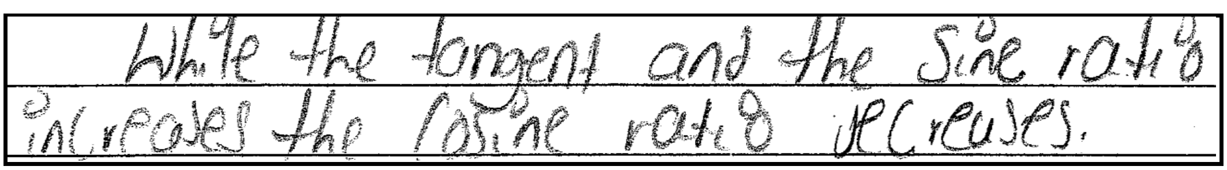


Figure 5.67: Participant E-17 comparison of values for the three trig-ratios.

In *Figure 5.67*, the student correctly stated that while the sine and tangent ratios increase, the cosine ratio decrease. It is assumed that these changes take place for angle values between zero and 90^0. This comparison highlights both a similarity and a difference among the three primary trigonometric ratios.

Figure 5.68 shows a sixth appropriate comparison based on values.

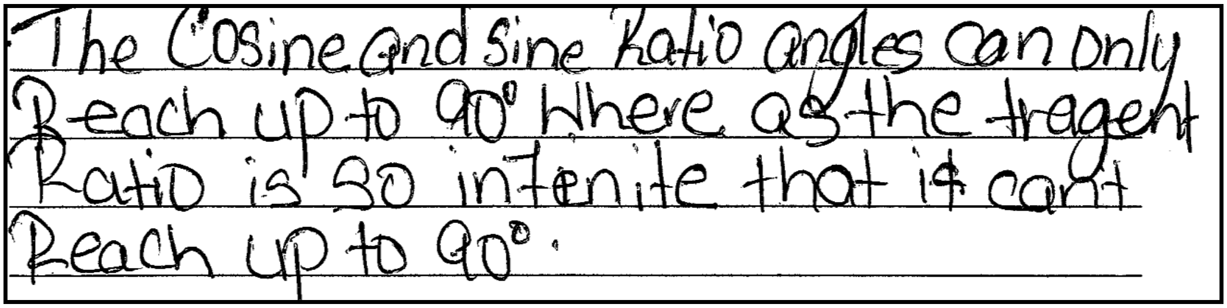

The Cosine and Sine Ratio angles can only Reach up to 90° Where as the tragent Ratio is so intenite that it can't Reach up to 90°.

Figure 5.68: Participant E-16 comparison of values for the three trig-ratios.

In *Figure 5.68*, the student correctly stated that the cosine and sine ratios have values at 90^0, but tangent does not. This comparison highlights both a similarity and a difference among the three primary trigonometric ratios.

Some students provided inappropriate comparisons of the three primary trigonometric ratios based on values. *Figure 5.69* shows on such comparison.

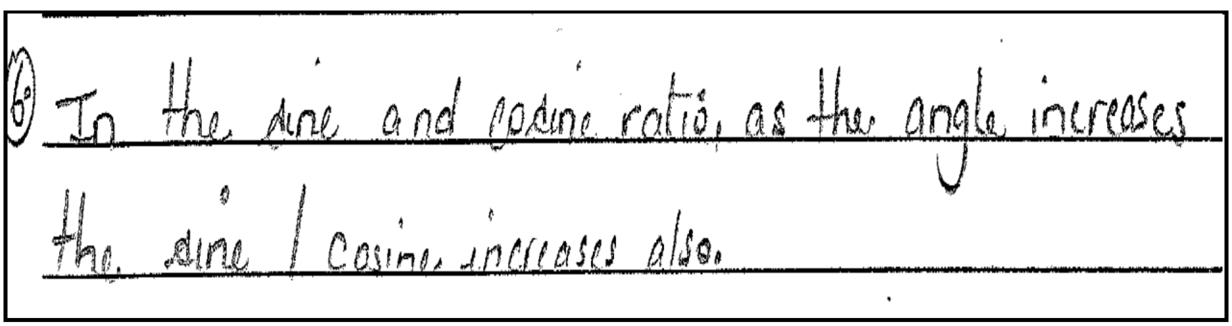

6) In the sine and cosine ratio, as the angle increases the sine / cosine increases also.

Figure 5.69: Participant E-10 comparison of values for the sine and cosine ratios.

In *Figure 5.69*, the student incorrectly stated that values for the sine and cosine ratios increase as the angle values increase. This change in values is correct for the sine ratio but incorrect for the cosine ratio. For cosine, the ratio values decrease as the angle values increase.

Figure 5.70 shows a second inappropriate comparison based on values.

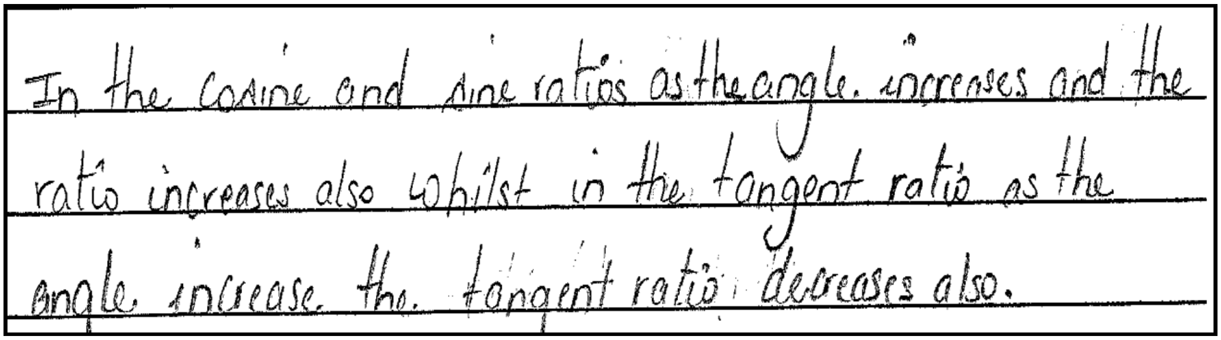


Figure 5.70: Participant E-10 comparison of values for the three trig-ratios.

In *Figure 5.70*, the student incorrectly stated that as the values of the angles increase, the ratio values increase in sine and cosine but decrease in tangent. The changes in values described are correct for the sine ratio but incorrect for both the cosine and tangent ratios. For the cosine ratio, the ratio values decrease as the angle values increase and increase as the angle values increase for the tangent ratio.

It is important to note that these students who were taught through Exemplification, studied the trigonometric ratios for angles from zero to 90^0. Thus, they were not expected to know the behaviour of the trigonometric functions beyond 90 degrees.

Group profile.

This profile highlights the strengths and weaknesses in the post-test written responses of students taught by Exemplification. It does so by listing and discussing their error-free responses and their erroneous responses as they worked with the different forms of representations—formulas, graphs, tables, diagrams—and their discussions surrounding these representations. The strengths and weaknesses are given under three domains: representing a contextual problem, multiple representations of a single concept, and comparing related concepts, which draws from Kilpatrick et al. (2001) work on conceptual understanding. There are strengths and weaknesses associated with each domain. A statement about the group's conceptual understanding is presented at the end of this profile.

Representing a contextual problem.

This section draws from the four themes presented under this domain—diagram used to represent the problem, selection of the correct ratio, selection of an incorrect ratio, and calculation procedure. These themes were derived from students' responses to a prompt that asked them to draw a diagram to represent a contextual situation and discuss how the trigonometric ratios could be used to solve a problem embedded in the situation. This group of students:

Strengths.

- Presented a correct representation of the contextual situation.
- Selected the appropriate ratio and provided an appropriate justification for their selection of that ratio.
- Provided an appropriate calculation procedure for finding the height of the building.

Weaknesses.

- Presented an inappropriate representation of the contextual situation.
- Selected the appropriate ratio to find the height of the building but provided a flawed justification for selecting that ratio.
- Selected an inappropriate ratio to find the height of the building.
- Presented an inappropriate calculation procedure for finding the height of the building.

Multiple representations of a single concept.

This section draws from the six themes presented under this domain—use of a formula, use of a graph, use of a table, use of a diagram, justification based on sides, and justification based on values. These themes were derived from students' responses to a prompt that asked them to use and discuss multiple representations of the cosine ratio. This group of students:

Strengths.

- Provided the correct formula for the cosine ratio.
- Provided an appropriate representation of the graph for the cosine ratio.
- Provided an appropriate representation of a table of values for the cosine ratio.
- Provided an appropriate representation of a diagram for the cosine ratio.
- Provided appropriate justifications for the choice of representations for the cosine ratio.

Weaknesses.

- Provided an incomplete formula for the cosine ratio.
- Provided an inappropriate representation of a diagram for the cosine ratio.

Comparing related concepts.

This section draws from the three themes presented under this domain—comparisons based on representations, comparisons based on sides, and comparisons based on values. These themes were derived from students' responses to a prompt that asked them to show the three trigonometric ratios using the same form of representation, then to compare these representations. This group of students:

Strengths.

- Presented the same form of representation for all three primary trigonometric ratios that were appropriate.
- Presented appropriate comparisons of the three primary trigonometric ratios based on their sides.
- Presented appropriate comparisons of the three primary trigonometric ratios based on their values.

Weaknesses.

- Presented incorrect comparisons of the three primary trigonometric ratios based on their values.

Conceptual understanding, in this study, is defined as students' ability to identify and discuss different representations of a primary trigonometry ratio, produce and discuss a diagram representation of a contextual problem, and compare the same form of representation of these ratios. This group of students taught by Exemplification showed a mixture of abilities in all three domains by presenting both appropriate and inappropriate responses. Hence, based on the above group profile, the group taught by Exemplification did not appear to attain a full conceptual understanding of the three primary trigonometric ratios.

Conceptual Understanding: A Comparison Across Groups

Both groups of students, those taught by Investigation and those taught by Exemplification, responded to all sections of the three prompts soliciting their written answers on the post-test. Similar responses were given in many instances. Furthermore, both appropriate and inappropriate responses were given by both groups. Their profile showed that neither group attained a full conceptual understanding of the three primary trigonometric ratios. However, there are differences in the groups' profiles, thus, suggesting differences in the group's conceptual understanding.

For the domain: representing a contextual problem, both groups presented an appropriate representation of the problem and selected the correct ratio to find the height of the building with appropriate justification. Both groups also selected an inappropriate ratio or gave an inappropriate justification for a selected ratio. Furthermore, both groups presented an incorrect procedure for finding the height of the building. In contrast, the group taught by Exemplification presented a correct calculation procedure for finding the height of the building while the group taught by Investigation did not provide a correct calculation procedure. On the other hand, the group taught by Exemplification presented an incorrect representation of the problem, while all representations of the problem presented by the group taught by Investigation were adequate.

For the domain: multiple representations of a single concept, both groups presented several appropriate representations of the cosine ratio. For the cosine ratio, both groups presented: the correct formula, an appropriate graph, an appropriate table of values, an appropriate diagram, and an appropriate justification for their choices of representations. Both groups also presented an incomplete formula and an inappropriate diagram to represent the cosine ratio. In contrast, the group taught by Investigation presented several errors that were not presented by the group taught by Exemplification. In representing the cosine ratio, students in the group taught by Investigation presented: an incorrect formula, an inappropriate graph, an inappropriate table of values, and an inappropriate justification for their choices of representations.

For the domain: comparing related concepts, both groups presented several appropriate comparisons of the three primary trigonometric ratios. Both groups represented the three primary trigonometric ratios using the same form of representation that was appropriate. Also, both groups presented appropriate comparisons of the ratios based on their sides and their values. Furthermore, both groups presented inappropriate comparisons of the three primary trigonometric ratios based on their values. In contrast, the group taught by Investigation presented comparison errors that were not made by the group taught by Exemplification. Students in the group taught by Investigation presented representations that did not adequately depict all three ratios, and they presented incorrect comparisons of the ratios based on their sides.

Summary

This chapter started with an introduction that stated the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* It briefly discussed the representation domains under which data were collected and analyzed to produce results. Results were given in the form of themes that were supported with extracts from students' work. A wide range of extracts was used within each group to show that the thematic results presented are reflective of an entire group. These themes were used to develop profiles for each group.

Each group profile provided a list of appropriate responses and a list of inappropriate responses related to students' use of representations of the three primary trigonometric ratios. These profiles were also used to make statements about the groups' conceptual understanding of the three primary trigonometric ratios. A comparison of the groups' profiles was also presented. In chapter six, this across group comparison is mixed with quantitative results from chapter four, to help interpret the scope of the conceptual understanding attained by each group of students—the group taught by Investigation and the group taught by Exemplification.

Chapter 6: Integration, Interpretation, Discussion

This chapter presents the interpretation of and discussion on the qualitative results obtained in Chapter five and the quantitative results obtained in Chapter four that were used to answer the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* In this chapter, the themes of Chapter five are mixed with the tabular results of Chapter four to compare the conceptual understanding the group of students taught by Investigation to the group taught by Exemplification. The chapter also provides further insights into the results of the ANOVA used to compare the achievements of both groups of students. The full scope of the study is presented in various sections to make these interpretations and discussions more meaningful.

The first section presents a summary of the results where the general and research problems and the significance of the study are restated. This section also briefly summarises the literature reviewed and the methodology used in this study. It also provides a concise recapitulation of the findings from chapters four and five. The second section presents the discussion of the results with a focus on the strengths and weaknesses of the study. It looks at how well the results answer the research questions of the study and support its hypotheses. It also discusses possible reasons for the results that were obtained. The third section discusses the results in light of the present literature. It positions this study in the literature by examining findings from previous studies that agree and disagree with its findings. It also looks at the significance of the findings to Dominican stakeholders and the mathematics education community.

Summary of the Study

The primary purpose of this study was to determine the effects that Exemplification compared to Investigation have on students' achievement and conceptual understanding of mathematical concepts at the secondary level. The study was one step in searching for solutions to the prolong dismal performances on CSEC mathematics examinations obtained by secondary school leavers in Dominica. At least one study (Charles, 2015) concluded that the instructional practices of Dominican secondary mathematics teachers might be one factor contributing to this run of poor performances. Therefore, the researcher aimed to find out how two inquiry-based strategies affected students learning. Moreover, he wanted to find out if the effects of Exemplification, which is not widely investigated at the secondary school level, were different from the effects of Investigation, which is widely investigated at all levels of education.

The findings of this study will contribute to the literature and could be of benefit to the Dominican society in which it was conducted. Dominican mathematics teachers, principals, and other stakeholders might use the findings of this study to rethink and reshape how mathematics is taught at secondary schools. Furthermore, other researchers might use these findings to inform their studies, in the same way that this study used the findings from previous research to lay its foundation.

This study reviewed the work of several scholars to lay its foundation. Leading among these are the works of Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996), which were used to provide a theoretical framework for the teaching aspects of the study. Teaching focused on the works of Jaworski (1986), who provided an approach to teaching through Investigation, and Watson and Mason (2005), who provided an approach to teaching through Exemplification. The work of Kilpatrick et al. (2001) and Hiebert and Carpenter (1992) provided the framework for analyzing and interpreting the data on students' conceptual understanding. The works of several others were reviewed to show how learners' achievement and conceptual understanding were affected by Investigations and Exemplification. The findings of these works, among several others, were presented in Chapter two. Among these are the works of several scholars who argued for the mixed-methods approach to research.

A pre-test–post-test independent measures experimental design was used in this study. Thirty-two students were randomly assigned to two groups; 16 students per group. The researcher taught the first group using Exemplification for approximately three weeks. The pre-test was given in the first session of those three weeks, and the post-test was given in the last session. Both tests were parallel and contained multiple-choice items and prompts that solicited students' written responses. In the subsequent three-weeks period, the same pre-test–teaching–post-test sequence was completed with the second group of students who were taught by Investigation. They completed the same tests as the previous group under similar conditions. The results of these tests were analyzed using both qualitative and quantitative means.

Pre-test and post-test scores were analyzed using a two-way, 2 x 2 mixed-factorial ANOVA, and students' written responses on the post-test were analyzed by themes. The responses within these themes were further analyzed to show the numbers of students who responded appropriately or inappropriately to the different prompts. The quantitative analysis was mixed with the qualitative themes to provide a measure of the effects that both teaching approaches had on students' conceptual understanding. The results of the mixed-factorial ANOVA provided the measure of the effects that both teaching approaches had on students' achievement.

Empirical Findings

The major empirical findings can be found in Chapter four, where the results of the quantitative analyses were presented and in Chapter five, where the results of the qualitative analysis were presented. This section synthesizes these findings to answer the four sub-questions:

1. How was students' level of achievement affected after being taught by Investigation?
2. How was students' level of achievement affected after being taught by Exemplification?
3. How did students' levels of achievement differ after being taught by Investigation compared with Exemplification?
4. How did students' conceptual understanding differ after being taught using Investigation compared with Exemplification?

Effects of Investigation on achievement.

The results of the ANOVA presented in Chapter four showed that the group of students taught by Investigation had a significant increase ($p<0.001$) in their achievement after the teaching. This increase is reflected in the rate at which members of this group responded to the prompts soliciting their written answers, the analysis of the scores obtained for those responses, and the number of correct answers they received on the multiple-choice items. No student from this group responded to a single prompt in the section of the pre-test that solicited their written responses. However, there was a response rate of approximately 76% on that section of the post-test for this group of students (see Table 4.8). Moreover, Table 4.9 shows that ten (21%) of these students' responses received three or more marks (75.0% or higher), 14 (29%) of their responses received between two and three marks (50.0% - 74.9%), and 24 (50%) of their responses received less than two marks (49.9% and lower). Furthermore, this group of students increased their rates of correct answers on the multiple-choice sections of the tests from 20% on the pre-test to 48% on the post-test (see Table 4.13).

Effects of Exemplification on achievement.

The results of the ANOVA presented in Chapter four showed that the group of students taught by Exemplification had a significant increase ($p<0.001$) in their achievement after the teaching. This increase is reflected in the rate at which members of this group responded to the prompts soliciting their written answers, the analysis of the scores obtained for those responses, and the number of correct answers they received on the multiple-choice items. No student from this group responded to a single prompt in the section of the pre-test that solicited their written responses. However, there was a response rate of approximately 98% on that section of the post-test for this group of students (see Table 4.8). Moreover, Table 4.9 shows that 20 (41.5%) of these students' responses received three or more marks (75.0% or higher), ten (21%) of their responses received between two and three marks (50.0% - 74.9%), and 18 (37.5%) of their responses received less than two marks (49.9% and lower). Furthermore, this group of students increased their rates of correct answers on the multiple-choice sections of the tests from 23% on the pre-test to 58% on the post-test (see Table 4.13).

Differences in effects on achievement.

The results of the ANOVA shows that the achievement of students taught by Exemplification was significantly higher ($p=0.025$) than that of students taught by Investigation. This higher achievement is reflected in the higher rate at which members of the Exemplification group responded to the prompts soliciting their written responses compared to members of the Investigation group, the comparison of the scores obtained for those responses, and the comparison of the number of correct answers they received on the multiple-choice items. There was a response rate of 98% on the written responses for the group taught by Exemplification compared to 76% for the group taught by Investigation on the post-test. Additionally, approximately 42% of these responses received a score of three (75%) or higher for the

Exemplification group compared to approximately 21% for the Investigation group. Furthermore, approximately 63% of the written responses of students taught by Exemplification received a score of at least half the maximum attainable score (4 marks) compare to 50% for the group taught by Investigation.

Effects of Investigation on conceptual understanding.

A group profile was presented in Chapter five for students taught by Investigation. This group profile highlighted the strengths and weaknesses of the group: a strength being an appropriate response, and weakness being an inappropriate response. These strengths and weaknesses were presented according to the three representational domains—representing a contextual problem, multiple representations of a single concept, and comparing related concepts—and provided a measure of the group's conceptual understanding of the three primary trigonometric ratios. Based on this profile, the group of students taught by Investigation appeared not to have attained a full understanding of the concepts of the sine, cosine, and tangent ratios.

This was evident in the domain called representing a contextual problem, where some students failed to select an appropriate ratio with proper justification for carrying out the required calculation, and no student demonstrated the ability to carry out the correct calculation procedure. Furthermore, only one student scored three marks (75%) or higher, which indicated a good conceptual understanding; ten students scored between two and three marks (50% - 74.9%), which indicated an average conceptual understanding; and five students scored less than two marks (below 50%), which indicated a low conceptual understanding. Twenty-seven out of a possible 32 responses (84%) were submitted for this domain, which indicates that some students had no conceptual understanding of some aspects of the trigonometric ratios assessed in this domain.

Further evidence was found in the domain called multiple representations of a single concept, where some students presented incorrect formulas, graphs, tables, and diagrams of the cosine ratio along with inappropriate justifications for these representations. Furthermore, only two students scored three marks (75%) or higher, which indicated a good conceptual understanding; three students scored between two and three marks (50% - 74.9%), which indicated an average conceptual understanding; and 11 students scored less than two marks (below 50%), which indicated a low conceptual understanding. Twenty-three out of a possible 32 responses (72%) were submitted for this domain, which indicates that some students had no conceptual understanding of some aspects of the trigonometric ratios assessed in this domain.

Also, in the domain called comparing related concepts, some students presented formulas, tables, graphs, or diagrams that did not correctly depict all three ratios, and they gave incorrect comparisons of the ratios based on their sides and values. Furthermore, seven students scored three marks (75%) or higher, which indicated a good conceptual understanding; one student scored between two and three marks (50% - 74.9%), which indicated an average conceptual understanding; and eight students scored less than two marks (below 50%), which indicated a low conceptual understanding. Twenty-three out of a possible 32 responses (72%) were submitted for this domain, which indicates that some students had no conceptual understanding of some aspects of the trigonometric ratios assessed in this domain.

This group of students demonstrated a level of conceptual understanding of the three primary trigonometric ratios that allowed them to perform most of the required tasks solicited by the given prompts. However, the group did not provide an appropriate calculation procedure for finding the height of the building. This failure represents a gap in this group of students conceptual understanding of the trigonometric ratios after being taught through the method of

Investigation. Furthermore, 50% of the group's responses obtained a score that was less than half of the maximum attainable score, which indicated that approximately half of this group of students attained a low conceptual understanding of the three primary trigonometric ratios.

Effects of Exemplification on conceptual understanding.

A group profile was presented in Chapter five for students taught by Exemplification. This group profile highlighted the strengths and weaknesses of the group: a strength being an appropriate response, and weakness being an inappropriate response. These strengths and weaknesses were presented according to the three representational domains—representing a contextual problem, multiple representations of a single concept, and comparing related concepts—and provided a measure of the group's conceptual understanding of the three primary trigonometric ratios. Based on this profile, the group of students taught by Exemplification appeared not to have attained a full understanding of the concepts of the sine, cosine, and tangent ratios.

This was evident in the domain called representing a contextual problem, where some students presented an incorrect representation of the problem, selected an inappropriate ratio for carrying out the required calculation, and demonstrated an incorrect procedure for that calculation. Furthermore, two students scored three marks (75%) or higher, which indicated a good conceptual understanding; eight students scored between two and three marks (50% - 74.9%), which indicated an average conceptual understanding; and six students scored less than two marks (below 50%), which indicated a low conceptual understanding. Thirty-two out of a possible 32 responses (100%) were submitted for this domain, which indicates that all students had gained a measure of conceptual understanding of aspects of the trigonometric ratios assessed in this domain.

Further evidence was found in the domain called multiple representations of a single concept, where some students presented an incomplete formula and incorrect diagrams of the cosine ratio. Furthermore, eight students scored three marks (75%) or higher, which indicated a good conceptual understanding; two students scored between two and three marks (50% - 74.9%), which indicated an average conceptual understanding; and six students scored less than two marks (below 50%), which indicated a low conceptual understanding. Thirty-one out of a possible 32 responses (97%) were submitted for this domain, which indicates that most students had gained a measure of conceptual understanding of aspects of the trigonometric ratios assessed in this domain.

Also, in the domain called comparing related concepts, some students gave incorrect comparisons of the ratios based on their values. Furthermore, ten students scored three marks (75%) or higher, which indicated a good conceptual understanding; no student scored between two and three marks (50% - 74.9%), which indicated an everage conceptual understanding; and six students scored less than two marks (below 50%), which indicated a low conceptual understanding. Thirty-one out of a possible 32 responses (97%) were submitted for this domain, which indicates that most students had gained a measure of conceptual understanding of aspects of the trigonometric ratios assessed in this domain.

This group of students demonstrated a level of conceptual understanding of the three primary trigonometric ratios that allowed them to perform all the required tasks solicited by the given prompts. Approximately 38% of their responses obtained a score that was less than half of the maximum attainable score, which indicated that the majority of this group of students attained an average–good level of conceptual understanding of the three primary trigonometric ratios.

Differences in effects on conceptual understanding.

The groups' profiles presented in Chapter five suggest that there were differences in the conceptual understanding of the two groups of students: the group taught by Investigation and the group taught by Exemplification. This suggestion is supported by differences in the analysis of scores for the written-responses (see Table 4.9), and the analysis of the number of responses submitted (see Table 4.8). However, the result of a Chi-square test of homogeneity, conducted with an alpha level of 0.05, showed no statistically significant difference in the conceptual understanding of both groups [X^2 (2, N = 96) = 4.86, p = .088]. Therefore, the comparison in conceptual understanding between the groups discussed here, focuses on the groups' profiles presented in Chapter five and the percentages analyses presented in Chapter four.

From the groups' profiles, both groups of students demonstrated a level conceptual understanding of the three primary trigonometric ratios by providing a correct representation of a contextual problem and by selecting an appropriate ratio to carry out the required calculations. However, only the group taught by Exemplification provided a correct calculation procedure to find the height of the building; thus, demonstrating a higher level of conceptual understanding in that domain than the group taught by Investigation. Additionally, both groups of students provided the correct formula, graph, table, and diagram to represent the three primary trigonometric ratios for angles between zero and 90^0. They also gave meaningful reasons for the suitability of their representations. Furthermore, both groups of students provided meaningful similarities and differences among the sine, cosine, and tangent ratios. What sets these two groups of students apart in these domains in their conceptual understanding, is the number and type of errors (see group profiles in Chapter five) made by each group of students. The group taught by Exemplification made fewer errors than the group taught by Investigation, indicating that this group of students had attained a higher level of conceptual understanding than the group taught by Investigation.

This apparent difference in conceptual understanding is further accentuated by the results for the analysis of score for the written responses (see Table 4.9), and the results for the analysis of the number of written responses (see Table 4.8). For instance, 50% of the written responses for the group taught by Investigation were scored at less than half of the maximum attainable score, indicating a low level of conceptual understanding, while 62% of the written responses for the group taught by Exemplification were scored at two or more, indicating an average–good conceptual understanding. Furthermore, the group taught by Investigation submitted 76% of all possible responses, indicating that many students of that group did not attained a measure of conceptual understanding of some aspects of the three primary trigonometric ratios. On the other hand, the group taught by Exemplification submitted 98% of all possible responses, indicating that most students of that group had attained a measure of conceptual understanding of all aspects of the three primary trigonometric ratios. The researcher assumed that, in most instances, written responses to prompts were not submitted because students did not know how to respond; thus, suggesting a lack of conceptual understanding.

Discussion of the Results

This section discusses how well the results answered the research questions. The research posits that the quantitative results obtained in Chapter four and the qualitative results obtained in Chapter five provided credible evidence to answer the study research questions because of the scientific approach used, the validity measures taken, and the solid foundation laid for the study. The scientific approach used refers to the use of a mixed-methods approach to collect and analyze data, and the experimental design used to structure the study. The validity measures taken refer to the validating of the data collection instruments by two experts and the use of these experts in the grading of students' written responses on the post-test using an assessment rubric. The use of tried and tested theories to develop and use the frameworks for implementing the teaching and for analyzing the data also lend to the validity of these results. Also, the foundation was laid for the use of achievement scores to compare the two groups of students, given its widespread use in many published research studies.

Many scholars, in their work, have advocated for the use of a mixed-methods approach to research. Among them are the works of Johnson and Onwuegbuzie (2004) and Ercikan and Roth (2006) that argued that a mixed-method approach, which involves both qualitative and quantitative data collection and analysis, iron out the deficits of one approach. This researcher believes, that if the study had only analyzed participants' test scores, questions about their areas of conceptual understanding would have been left unanswered. Likewise, if only their written responses were analyzed to judge their conceptual understanding, then the results would not have reflected the improved achievements of the groups and the number of students who have achieved a range of scores. Having information in both areas was essential to this study, given the practical nature of the study. Ling (2017) situated the mixed-methods approach in the pragmatic paradigm where the practical needs drive the choice of methodology.

Furthermore, several recent studies have employed a mixed-methods approach to conduct similar studies. Among them are Bal (2016), who investigated the effects of differentiated instruction on students' achievement, and Prince and van Jaarsveld (2017), who evaluated students' conceptual understanding from their written responses on trigonometric ideas. Hence, this research study was well positioned in terms of its use of a mixed-methods approach. Thus, giving credence to its results.

Several studies (Akhter, Akhtar, & Abaidullah, 2015; Kesan, & Caliskan, 2013; Xie, Cofflan, & Yang, 2012) have used an experimental design to investigate the effects of instructional practices on students' learning of mathematics ideas. For example, like this study, Xie, Cofflan, and Yang (2012) used a randomized experimental design to investigate the effects of instructor's feedback on the performance of first-year undergraduate students in calculus. The randomized experimental design is purported as one of the most effective approaches to research that compares the effects of different conditions on participants (Polit & Beck, 2016). Such effectiveness is due to the random assignment of participants into groups to account for pre-existing differences among members of established groups (Gamst, Meyers, & Guarino, 2008).

The use of other experts' professional judgment also enhanced the credibility of the results obtained in this study. The validation of the instruments for the pre-test and post-test and the assessment rubric used in the grading of the written responses was done on the advice of several scholars. Danielson and Marquez (2016) advocated for several experts to be involved in the development of an instrument and to judge students' work. Polite and Beck (2016) sanctioned the use of experts to check the validity of data collection instruments.

Several studies (Jonsson & Svingby, 2007; Meier et al., 2006; Smit & Birri, 2014) argued that assessment rubrics improve the reliability in scoring complex tasks. This study used two teachers, both with over ten years of experience teaching and grading students' work in mathematics and with at least a B.Ed in mathematics education, to help develop the study data collection instruments, and to grade students' work. Furthermore, the two experts who used the rubric were trained to use it before grading began. This action was recommended by studies such as Jonsson and Svingby (2007) and Brown, Glasswell, and Harland (2004).

The study also drew from the theories of several prominent scholars in the field of mathematics education. It combined the works of Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996) to build a framework for conducting the teaching to both groups of students. This framework highlighted the importance of the roles of the student, the teacher, and the environment. The study also positioned Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005) within this framework. Furthermore, the work of Kilpatrick et al. (2001) on representation was used to develop a framework for evaluating students' conceptual understanding based on their written responses. The analysis based on this framework yield qualitative thematic results. These results were integrated with quantitative results to obtain more complete results to answer the research questions. This integration drew from the work of Hiebert and Carpenter (1992). Thus, the teaching, data collection, and data analyses were well-grounded in the theories and work of several prominent scholars. This foundation gives credence to the results of this study.

There was also a sound basis established for using test scores to compare the effects of Investigation and Exemplification on students' achievements and conceptual understanding in this study. A synthesis of several studies showed that test scores had been used to compare the effects of different teaching practices on students' achievement and understanding of mathematical ideas. For example, Galeshi (2014) used 2007 test scores from TIMSS to compare the mathematics programs of Taiwan and the United States. Another example came from Parke and Keener (2011), who used test scores to investigate the effects that mobility, course content, and sequencing have on students' mathematics performance. A third example came from Akinsolaab and Awofalab (2009), who also used test scores to investigate the effects of classroom groupings on students' mathematics achievement. The critical factor is ensuring that these test instruments measure what they were developed to measure and that they were reliably scored. Steps were to ensure these validity and reliability measures, as discussed above, were employed in this study.

However, the credibility of the results could have been compromised by collaboration among students of the different groups since both groups of students were at the same school, and one group completed the intervention before the other. The researcher took two steps to reduce if not eliminate the effects of such collaboration. One, the pre-test and post-test were done immediately before and after the teaching with each group. In doing so, the pre-test reflected the knowledge that each group of students took into the teaching phase, including any knowledge students from the second group would have acquired from students from the first group. Two, the Exemplification group, which was the group of most interest to the researcher, was taught first. This action ensured that students from the Investigation group could not share their knowledge acquired in the teaching period with students from the Exemplification group; thus, confounding the results obtained from the group taught by Exemplification.

Therefore, it is reasonable to conclude that the results obtained from the data analyses provided strong evidence for answering the research questions and that the findings provided by these answers are credible.

Discussion of the Findings

This section discusses the findings of this study, considering the literature. It does so by comparing the answer to each research question to previous research, focusing on whether there is agreement or disagreement among the findings. The section also posits reasons for any agreement or disagreement between this study and others in the literature by discussing the similarities and differences among its methodology and context, and theirs. Furthermore, the section looks briefly at what the findings in this study mean for the mathematics education community and teaching and learning of mathematics in Dominican secondary schools.

For sub-question one, the result shows that students who were taught by Investigation performed significantly better in the post-test than they did in the pre-test. A level of significance of $p<0.001$ was recorded in the ANOVA, indicating a statistically significant increase in the achievement of this group of students regarding the three primary trigonometry ratios. That is, this result suggests that teaching mathematics by Investigation is a practical approach for increasing Dominican secondary students' achievement in mathematics. This finding is consistent with that of several recent studies (Budak, 2015; Ekwueme et al., 2015; Erbas & Yenmez, 2011; Mainali & Heck, 2017). Like this study, all of these studies investigated the effect of Investigation on students' test scores and measured this effect by comparing their pre-test achievement to at least one post-test achievement. Furthermore, some of these studies (Ekwueme et al., 2015; Mainali & Heck, 2017) were conducted within the secondary school context.

This result is also in agreement with several other studies (Ponte & Matos, 1992; Staples, 2011; Yanik et al., 2014) that did not use a pre-test, post-test approach, but collected and analyze data through other means. Unlike this study, Ponte and Matos (1992) came to their conclusion through the analysis of data collected via videotape while Yanik et al. (2014) and Staples (2011) reported on classroom observations. The finding from this study that Investigation increased students' achievement in mathematics is consistent with those found in the literature that were reviewed in this study.

One plausible reason for this level of consistency can be gleaned from the four meta-analyses (Carbonneau et al., 2013; Johnson et al., 2000; Li & Ma, 2010; Rosli et al., 2014) discussed in chapter two. Johnson et al. (2000), Carbonneau et al. (2013), and Rosli et al. (2014) showed that the use of cooperative learning and concrete materials have significant positive effects on students learning. Students taught by Investigation in this study were exposed to these two elements during teaching. Throughout the teaching, students worked in small groups (cooperative learning) to measure (use of concrete materials) and determine representations of the three primary trigonometric ratios, and to discuss the appropriateness of these representations.

For sub-question two, the result shows that students who were taught by Exemplification performed significantly better in the post-test than they did in the pre-test. A level of significance of $p<0.001$ was recorded in the ANOVA, indicating a statistically significant increase in the achievement of this group of students regarding the three primary trigonometry ratios. That is, this result suggests that teaching mathematics by Exemplification is a practical approach for increasing Dominican secondary students' achievements in mathematics.

This finding is consistent with that of several studies (Abdul-Rahman, 2005; Dahlberg & Housman, 1997; Dinkelman & Cavey, 2015; Furinghetti et al., 2011; Iannone et al., 2011; Rawson & Dunlosky, 2016; Sandefur et al., 2012; Scataglini-Belghitar & Mason, 2012; Watson & Mason, 2002; Watson & Shipman, 2008) reviewed in this study. However, unlike this study, none of the studies reviewed investigated the effect of Exemplification on students' achievement by comparing pre-test and post-test results. This difference is significant because secondary students' achievements on high-stakes tests, like the CSEC examinations, is a critical factor in determining opportunities for success in many of their future endeavours: work, study, and scholarship opportunities, to name a few.

Furthermore, Dinkelman and Cavey (2015), Watson and Shipman (2008), and Watson and Mason (2002) were the only studies found that were conducted within the secondary school context. All other studies reviewed that investigated the effects of Exemplification were conducted at the post-secondary level. Of the three, Dinkelman and Cavey (2015) was the only one that was developed with a specific purpose, the other two were described as being opportunistic by the researchers (Watson & Shipman, 2008; Watson & Mason, 2002). Hence, this study, *Comparing the effects of two inquiry-based teaching strategies on secondary students' conceptual understanding and achievement in mathematics: A mixed-methods approach,* joins Dinkelman and Cavey (2015) in using a structured approach to demonstrate that Exemplification has a positive effect on secondary students' learning. Unlike Dinkelman and Cavey, however, it does so by comparing their performances using pre-test and post-test scores.

The researcher argues that both the cognitive activities—generating examples—performed by individuals and the cooperative activities—discussing generated examples—performed by students during teaching contributed to students' increased achievements. These activities were fashioned after Watson and Mason (2005) and drew on the theories of several constructivists: Piaget (1977), Vygotsky (1978), von Glasersfeld (1995), and Fosnot and Perry (1996). Also, like Investigation, Exemplification is an inquiry-based learning strategy that takes students beyond rote memorization and helps them to think about the mathematics while manipulating materials, their generated examples, in an enabling environment.

For sub-question three, the result showed that students who were taught by Exemplification performed significantly better on the post-test than the group taught by Investigation. A level of significance of $p=0.025$ was recorded in the ANOVA, indicating a statistically significant difference in the achievement of the Exemplification group compared to the Investigative group.

For sub-question four, a mixture of quantitative analysis and the qualitative analysis also showed that students who were taught by Exemplification attained a higher level of conceptual understanding compared to the group taught by Investigation. These results suggest that Exemplification might be a more effective approach to teaching aspects of mathematics to secondary students than Investigation.

Unfortunately, no study was found in the literature that compares the effects of Investigation and Exemplification on students' learning of mathematics. However, several studies (e.g., Dahlberg & Housman, 1997; Iannone et al., 2011; Rawson & Dunlosky, 2016) have compared Exemplification with other teaching and learning strategies and reported mixed findings. For instance, Dahlberg & Housman (1997) compared students' generating examples of a function with other learning strategies—redefining the function, memorizing the function, and decomposing the function—and found that Exemplification had more positive effects on students' learning of the function than the other strategies. In a more recent study, Rawson and Dunlosky (2016) compared test scores of one group of students who generated examples of a concept after reading about it, to the test scores of another group of students who only read about the concept. Rawson and Dunlosky found that the example generating group performed better than their counterparts. On the other hand, Iannone et al. (2011) found no significant difference in students' production of proofs between a group of students who generated examples of a function and their counterparts who only read about the function.

This studyadds to the list of studies that found Exemplification to be more effective than another strategy used to teach the same mathematical concept. More specifically, it found

Exemplification to be a more effective constructivist approach to teaching the three primary trigonometric ratios than Investigation.

The researcher argues that this difference in the effectiveness of Investigation and Exemplification may have been because the three essential factors (learners, the teacher, and an enabling environment) in a constructivist approach to teaching might have been better attended to during teaching by Exemplification compared to teaching by Investigative. During Exemplification teaching, both the learners and the teacher were genuinely actively engaged throughout lesson sessions. Learners were engaged both cognitively by generating examples and socially through meaningful discussions with their peers and a more knowledgeable other (Vygotsky, 1978)–their teacher. The teacher also played a central role in guiding students' cognitive and social actions by providing prompts and facilitating discussions. These actions are part of the nature of teaching through Exemplification.

On the other hand, students' cognitive and social activities may not have always been "meaningfully" enacted during Investigation teaching sessions in this study. Part of the nature of an Investigation is that students work collaboratively in groups doing hands-on activities. During such activities, students are often left to initiate and maintain cognitive and social activities such as conjecturing, testing and modifying, and discussing within groups. Aspects of these activities can easily be neglected or abused because the teacher cannot attend to every group at the same time. In this study, the researcher-teacher worked towards getting students to stay on task within group activities by initially helping them to conjecture, test and modify, and discuss mathematics ideas. However, as the researcher-teacher worked with one group during hands-on activities, others were left unattended. Furthermore, some groups might have benefited from the teachers' guidance but may not have requested it. These arguments are in keeping with Mayer (2004), who argued that discovery lessons in which the teacher provides clear guidance are more effective than discovery lessons with minimal guidance. Also, helping to make, test, and modify conjectures while teaching by Investigation might have reduced the congitive demands of some tasks (Stein, et al., 2009). According to Stein, et al. (2009), reducing the cognitive demands of a task leads to lower levels of understanding.

However, discussions of the findings related to the comparisons of the groups' performances must consider the limitations of the study. Four possible confounding factors were identified: the difference in the ratios of boys to girls across groups, the reverse sequencing of algebra and the three primary trigonometric ratios, the possible collaboration between students from both groups, and the possible effects of Hurricane Maria on students' performances. These limitations are thoroughly discussed in Chapter one. Hurricane Maria might have affected the overall performances of both groups of students; hence, its effects were not an issue in the comparison of the groups. The possible collaboration between students from the different groups and the reverse sequencing of algebra and the three primary trigonometric ratios were also not considered issues considering the results of the comparisons. The results of all comparisons showed that the group taught by Exemplification outperformed the group taught by Investigation, despite being taught first.

The group taught by Investigation (they were taught second) could not have helped the group taught by Exemplification (they were taught first). Evidence of their inability to assist the Exemplification group can be gleaned from the Investigation group's low performance ($M = 2.69$, $S.D = 2.27$) on their pre-test, which was taken after the teaching of the Exemplification group had ended. Also, the students from both groups used basic algebra during their study of the trigonometric ratios. Thus, the sequencing of the content was likely to have advantaged the group taught by Investigation because they studied algebra before the ratios. The limitation to be considered when interpreting the results of the comparisons, which showed that the group taught by Exemplification outperformed the group taught by Investigation, is the difference in the ratios of boys to girls across groups.

The better performance of the group taught by Exemplification could be attributed to its high percentage of female students compared to that of the group taught by Investigation. There were ten males (62.5%) and six females (37.5%) in the group taught by Investigation, and five males (31%) and 11 females (69%) in the group taught by Exemplification. In Dominica, girls have been performing better than boys in mathematics and other subject areas in both the elementary and secondary school levels (Dominica Ministry of Education, 2019). Therefore, the better performance of the group taught by Exemplification over the group taught by Investigation could have reflected the gender performance differences in Dominica and not the effects of the different strategies since these gender differences were not accounted for in the study.

With the above discussion in mind, this study advocates for every secondary mathematics teacher to become comfortable in and willing to use both Investigation and Exemplification in their teaching. The study shows that both strategies can increase students' achievement and improve their conceptual understanding of mathematics ideas at that level. Furthermore, Like Watson and Mason (2002), Watson and Shipman (2008), and Dinkelman and Cavey (2015), it presents Exemplification as a viable addition to the list of recommended strategies for teaching mathematics at the secondary level. The researcher believes that the frequent and purposeful use of both Exemplification and Investigation can help improve the success of Dominican secondary school leavers on CSEC mathematics examination.

Summary

Discussions in this chapter were presented in four sections after the introduction to the chapter. The first section provided a summary of the study to include its purpose, methodology and design, procedures, and the literature reviewed to lay its foundation. The summary reoriented readers to the study. The second section discussed the integration of the qualitative and quantitative results to give the findings of the study. Findings were given as answers to the four research sub-questions. The findings revealed that both Investigations and Exemplification significantly increase students' achievement and conceptual understanding of the three primary trigonometric ratios and that students taught by Exemplification appeared to have attained higher levels of achievement and conceptual understanding than those taught by Investigation.

The third section discussed the credibility of the results in light of the strengths and limitations of the study. The section portrays the results to be credible, citing many more strengths than limitations. The final section positioned the findings in the literature. Discussions in this section showed that the effects of both Investigation and Exemplification on students' achievement and conceptual understanding were in agreement with most studies reported in the literature. It also used the literature to put forth plausible reasons for these findings. Moreover, the discussions positioned Exemplification as a viable strategy for teaching mathematics at the secondary school level and encourage its use in Dominican secondary schools.

Chapter 7: Conclusions, Implications, Recommendations

This study set out to investigate the effects of two constructivist's teaching approaches on students' learning. The researcher, a Dominican mathematics educator, drew from his previous work (Charles, 2015) within that community that concluded that Dominican secondary students were not sufficiently exposed to inquiry-based approaches to teaching mathematics. Two such inquiry-based approaches are Investigation as proposed by Jaworski (1986) and learners' generating example—Exemplification—as proposed by Watson and Mason (2005). The study was designed to answer the research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* What follows is the conclusions to the findings regarding the research question, implications of these conclusions to the teaching and learning of mathematics, recommendations for further studies to explore the effects of Exemplification, and a conclusion to the study that includes the researcher's reflection.

Conclusion to the Findings

The results discussed in Chapter six showed that both Investigation and Exemplification led to an increase in students' achievement and conceptual understanding of the three primary trigonometric ratios, with Exemplification leading to a greater increase in both. The increases and differences in achievement were highlighted by the ANOVA of students' test scores. However, while both groups of students appear to have increased their conceptual understanding of the three primary trigonometric ratios after teaching, neither group appeared to have attained a full conceptual understanding of these concepts.

The results of the qualitative analysis presented in Chapter five and the several quantitative analyses presented in Chapter four suggest that the group of students taught by Exemplification attained a higher level of conceptual understanding compared to the group taught by Investigation. In terms of conceptual understanding, these groups are set apart by the number of errors made, the quality of scores obtained, and the number of written responses submitted on the post-test. Kilpatrick et al. (2001) referred to conceptual understanding as an “integrated and functional grasp of mathematical ideas” (p. 118) and advocated for it to be demonstrated through works dealing with various representations of these ideas. The prompts soliciting students’ responses in this study drew from Kilpatrick et al. Hiebert and Carpenter (1992) referred to conceptual understanding as the quality and quantity of representations that students produce of a mathematical concept. The integration of the qualitative and quantitative analyses drew from the work of Hiebert and Carpenter. The results of these analyses show that the group taught by Exemplification had more representational responses, with higher scores, and with fewer errors than the group taught by Investigation.

Setting aside these differences, this study concluded that both Investigation, as proposed by Jaworski (1986) and Exemplification as proposed by Watson and Mason (2005), had significant positive effects on students’ achievement and conceptual understanding of the three primary trigonometric ratios. The study also argues for both Investigation and Exemplification to be part of the Dominican secondary schools’ curriculum.

Implications of the Findings

Mathematics teachers, particularly Dominican secondary mathematics teachers, should adopt a more constructivist’s approach to teaching by exposing students to more inquiry-based approaches to teaching mathematics. To provide variety and to cater to different learning styles and interests, they must learn and become comfortable in using various types of constructivist approaches. Two of these approaches are Investigation and Exemplification. The results of this study suggest that both Investigation and Exemplification are effective in increasing students’ achievement and conceptual understanding of mathematics. Furthermore, this study presents several lessons that show how both Investigation and Exemplification can be implemented in the secondary mathematics classroom. Teachers may wish to adopt and adapt these lessons to suit their needs and the needs of their students.

To achieve such a movement, from a traditional approach to a more constructivist approach, teacher trainers need to expose both in-service and pre-service teachers to inquiry-based teaching approaches such as Investigation and Exemplification. To do so, they must become knowledgeable about these approaches and comfortable in helping teachers to use them. This study provides a wealth of information that could help to increase teacher trainers' knowledge about Investigation and Exemplification, and lessons that they could draw on as they expose teachers to these approaches to teaching mathematics.

Recommendations for Further Study

The current study focused on how the use of Exemplification compared to Investigation affected Dominican secondary students' achievement and conceptual understanding of the three primary trigonometric ratios. Further studies should be conducted to investigate the effects of Investigation and Exemplification on other areas of mathematics taught by secondary teachers, particularly in Dominica and the wider Caribbean region. This extension could take the form of several similar studies or a longitudinal study with several pre-tests and post-tests data collection points combined with data collected from observations and interviews. These studies could help paint a clearer picture of the effectiveness of Exemplification, which is not yet widely investigated at the secondary school level.

It would also be useful to know how a combination of Investigation and Exemplification would affect students' achievement and conceptual understanding of mathematical ideas compared to only one of these approaches. Two separate studies could be done: one comparing the effects of the combination to that of Investigation alone and the other comparing the effects of the combination to that of Exemplification alone. However, a researcher could undertake one study comparing the effects of all three conditions: a combination of Investigation and Exemplification, Investigation alone, and Exemplification alone.

A delayed post-test was left out in this study. However, high-stakes tests like the CSEC mathematics examination are not usually given right after teaching has taken place. Hence, students' retention of mathematics concepts and procedures is essential if they are to perform well on these tests. A delayed post-test is a good way of judging such retention. The literature reports studies (Erbas & Yenmez, 2011) that included a delayed post-test for the effects of Investigation. No study was found that included a delayed post-test for the effects of Exemplification. Therefore, research that includes a delayed post-test for the effects of Exemplification is needed to help judge the effects of this approach on students' retention.

The ways that students and teachers experience an instructional strategy can determine how frequently it is used, thus, affecting its effectiveness in the classroom. Neither this study or any reviewed examined the experiences of students who were taught by Exemplification. This study recommends that research is undertaken that examine students' and teachers' experiences of Exemplification.

Conclusion

Students need to be exposed to a multitude of teaching approaches when learning mathematics. Strategies that adhere to the constructivist theory of learning appears to be most effective. Both Investigation and Exemplification are aligned with the constructivist theory of learning and have been shown in this study to increase students' achievement and conceptual understanding of mathematics. Moreover, this study has added to the literature in showing that Exemplification may be more effective than Investigation in raising students' achievement and conceptual understanding of mathematical ideas at the secondary level.

In conducting this study, the researcher gained several vital pieces of knowledge. The study has helped consolidate his knowledge of the constructivist theory of learning, the major theorists who were involved in the early development of this theory, and the main factors to be considered in developing teaching instructions based on this theory. He has also learned to better plan and execute instructions using Investigation and Exemplification. The horning of these skills has better position him to help teachers understand and implement these instructional approaches in their mathematics classroom. Moreover, the researcher has further his skills in scholarly research and plans to use those skills to continue contributing to the field of mathematics education.

However, conducting this study came with some challenges. It was a challenge for the researcher to work with students that he did not know and who did not know him. As a teacher, he had to gain the students' trust for teaching to be meaningful and he had to do it within a short space of time. This issue was compounded by teaching these students using methods that they were not familiar with and which put the burden of thinking upon them. In the initial stages, the first week or so, he could feel their resistance and sense their mistrust, and he had to do all in his power (e.g., use jokes and tell stories about his time teaching at the school) to quickly gain their trust. He had decided to conduct his doctoral study in Dominica because he was familiar with its education system and schools' culture. Nevertheless, the students' trust and willingness to work with him did not come easily. This experience left him wondering what it would have been like to conduct such a study in an education jurisdiction where the system and schools' culture are foreign to him.

Another challenge came in having to teach the same content within a short space of time using two teaching approaches that bore some similarities while ensuring that mixing of these two approaches did not happen. Investigation and Exemplification are both inquiry-based teaching approaches, and many times during preparation and teaching, the researcher questioned whether there were any real differences between the two approaches. He had to reflect on the theoretical differences (see Chapter two, pp. 58–61) constantly to ensure that one approach did not degenerate into the other during teaching, thus, confounding the results of the study. This struggle highlighted the differences between a research study and an actual teaching practice. As a teacher, mixing these two approaches would not have mattered; perhaps, it would have been encouraged to add variety, thus catering to the diverse needs of students. However, the mixing of these two approaches would have negatively affected the validity of the results in this study.

The researcher wishes to part with this question: what now? This question is not for the readers of this dissertation to try answering; neither is it a rhetorical question. Rather, it speaks to the space within which the researcher is now immersed. Or could he refer to this feeling as being in-between spaces: an emerging academic, an emerging researcher, a highly experienced secondary mathematics educator? As to whichever direction his education career takes, one thing is certain; he will continue to investigate the effects of Exemplification on students' learning.

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Appendices

Appendix A: Information Letter to Chief Education Officer (CEO)

Study Title: Generating Examples Verses Investigations: Differences in students understanding of Trigonometry.

Research Investigator:
Christopher Charles
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Dear Chief Education Officer,

To learn more about the way in which the investigative and the exemplification approaches to teaching affect students' understanding of mathematics concepts, I would like your permission to conduct a study in two secondary schools: North East Comprehensive School (NECS) and Castle Bruce Secondary School (CBSS). The information gathered will be used in the preparation of my Ph.D. dissertation at the University of Alberta. The analyzed results of this research study may be published in an academic journal in Canada, and it may also be made available to policymakers and ministry officials in Dominica.

Purpose
My research has the potential to benefit teachers, students, and the Dominican community, by identifying ways in which the investigative and exemplification approaches can enhance students' achievements. In a recent study, I discovered that most Dominican secondary mathematics teachers use a direct instruction approach to teaching which may be negatively affecting students' achievements in CSEC and other examinations. The results of this study may provide an opportunity to show teachers what may be done differently in their classrooms.

Study Procedures
This study will make use of two (2) groups of third formers from different schools. I will teach one group using the investigative approach and the other group using the exemplification approach. Both methods espouse a constructivism approach to learning and come highly recommended by several scholars in the field of mathematics education. The intervention with each group of students will last for five teaching sessions and one day for testing. A pre-test will be done in both schools among all third formers to select the most appropriate group. Once your permission is granted, I will work with the principal and teachers to minimize interruptions to the work plan of the teachers and selected students.

Benefits
The intent of this research is to uncover the extent to which the two teaching method identified help students to develop conceptual understanding of mathematical ideas. Once uncovered, teachers, principals, and ministry officials will be made aware of the findings. If both methods have a significant positive impact on students' understanding, then their use will be recommended to Dominican mathematics teachers. Exemplification is a relatively new method that has not yet taken root at the secondary or primary school level. My hope is to recommend it

as an addition to the repertoire of practices of Dominican mathematics teachers. Moreover, I will be well positioned to assist these teachers in developing the skills needed to use this teaching method adequately.

Risk
There is little to no risk to any students, teacher, or school participating in this research. Students may feel some anxiety in having an unfamiliar teacher in their classroom. This is normal, and I will use my 20 plus years of experience teaching mathematics at the secondary level to ease students' fears. Teachers will not be directly involved in the study and the only threat to them is a slight disruption in their teaching schedule. However, this disruption can easily be accounted for since the contents that will be covered in the intervention has to be taught within the school year.

Confidentiality & Anonymity
As previously stated, I will use information collected in this study to prepare my Ph.D. dissertation which may be published in part and/or given to policymakers and ministry officials in Dominica. At no point during the study or in its preparation and presentation will any individual or groups of individuals be identified.

Because complete anonymity cannot be guaranteed because of the small number of students involved in each group, I will take all necessary precautions to keep students' information in strict confidence. The data will be analyzed by me and only my academic advisor at the University of Alberta, apart from me, may have access to students' work. However, in the event that others may have to access these documents, students work will be anonymized. The ethics research board at the University of Alberta may also have access if it so desires.

It is a policy of the ethics board that data be stored for a minimum of five years after a research study. In keeping to this requirement, I will be securing students work in a locked cupboard in my apartment/at my home. I will burn these document five years after the study is completed.

If you have any further questions regarding this study, please do not hesitate to contact Christopher Charles at email: ccharles@ualberta.ca, Tel.: 587 938 3757 (Alberta), Tel.: 767 276 5113 (Dominica).

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at (780) 492-2615.

Appendix B: Information Letter to Principal

Study Title: Generating Examples Verses Investigations: Differences in students understanding of Trigonometry.

Research Investigator:
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Dear Principal,

To learn more about the way in which the investigative and the exemplification approaches to teaching affect students' understanding of mathematics concepts, I would like your cooperation and assistance to conduct a study at your school. Your school will be one of two sites where research will be conducted. The information gathered will be used in the preparation of my Ph.D. dissertation at the University of Alberta. The analyzed results of this research study may be published in an academic journal in Canada, and it may be made available to ministry officials in Dominica.

Purpose
My research has the potential to benefit teachers and students, by identifying ways in which the investigative and exemplification approaches can enhance students' achievements. The results of this study may provide an opportunity to show teachers what may be done differently in their mathematics classrooms.

Study Procedures
This study will make use of one group of third formers at your school. I will teach that group of students using the investigative/exemplification method which espouses a constructivism approach to learning and comes highly recommended by several scholars in the field of mathematics education. The intervention with this group of students will last for five teaching sessions and one day for testing. A pre-test will be done among all third formers at your school to select the most appropriate group. Hence, I will need the cooperation of the third form teachers to get this completed in the most efficient manner. My aim is to work with you and the teachers to minimize interruptions to the work plan of the class in question.

Benefits
The intent of this research is to uncover the extent to which two teaching methods help students to develop conceptual understanding of mathematical ideas. Once uncovered, teachers and principals will be made aware of the findings. If both methods have a significant positive impact on students' understanding, then their use will be recommended to Dominican mathematics teachers. Exemplification is a relatively new method that has not yet taken root at the secondary or primary school level. My hope is to recommend it as an addition to the repertoire of practices of Dominican mathematics teachers. Moreover, I will be well positioned to assist these teachers in developing the skills needed to use this teaching method adequately.

Risk

There is little to no risk to any students, teacher, or the school. Students may feel some anxiety in having an unfamiliar teacher in their classroom. This is normal, and I will use my 20 plus years of experience teaching mathematics at the secondary level to ease students' fears. Teachers will not be directly involved in the study and the only threat to them is a slight disruption in their teaching schedule. However, this disruption can easily be accounted for since the contents that will be covered in the intervention has to be taught within the school year.

Confidentiality & Anonymity

As previously stated, I will use information collected in this study to prepare my Ph.D. dissertation which may be published in part and/or given to policymakers and ministry officials in Dominica. At no point during the study or in its preparation and presentation will any individual or groups of individuals be identified.

Because complete anonymity cannot be guaranteed because of the small number of students involved, I will take all necessary precautions to keep students' information in strict confidence. The data will be analyzed by me and only my academic advisor at the University of Alberta, apart from me, may have access to students' work. However, in the event that others may have to access these documents, students work will be anonymized. The ethics research board at the University of Alberta may also have access if it so desires.

It is a policy of the ethics board that data be stored for a minimum of five years after a research study. In keeping to this requirement, I will be securing students work in a locked cupboard in my apartment/at my home. I will burn these document five years after the study is completed.

If you have any further questions regarding this study, please do not hesitate to contact Christopher Charles at email: ccharles@ualberta.ca, Tel.: 587 938 3757 (Alberta), Tel.: 767 276 5113 (Dominica).

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at (780) 492-2615.

Appendix C: Information Letter to Participants and Parents

Study Title: "Learners Generating Examples and Investigations": Impacts on Students' Achievement and Conceptual Understanding of Trigonometry.

Research Investigator:
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Background
To learn more about the ways in which the Exemplification strategy in teaching affect students' understanding of mathematics concepts, fourth form students from the North East Comprehensive School are being asked to have their grades on a pre-test and a post-test used as data in a research study. The information gathered will be utilized in the preparation of my Ph.D. dissertation at the University of Alberta. The analyzed results of this research study may be published in academic journals, reported through conference presentations, and it may also be made available to policymakers and ministry officials in Dominica.

Purpose
To identify ways in which the Exemplification strategy can enhance students' achievement. The results of this study may also provide an opportunity for teachers to do things differently in their classrooms.

Study Procedures

1) Two identified classes of fourth form students will be randomly assigned to two different groups.
2) The students will be asked to take a pre-test based on the mathematics content to be taught by the researcher. This will be a paper and pencil test and will take no more than 90 minutes. All students in this class will complete this pre-test. However, only the results of students who gave their consent will be used in the study. The pre-test grades of all students, however, will be given to their classroom mathematics teacher. After the intervention, these grades in conjunction with students' post-test grades may be used by the classroom teacher and/or the researcher to plan remedial work for students if this is necessary.
3) The researcher will teach students for two consecutive weeks. All students (participants and non-participants) will be exposed to the same classroom activities. The classroom mathematics teacher is expected to be present in all of these sessions. They will assist the researcher in matters of discipline which may arise, and in some classroom teaching and learning activities.
4) After the teaching, students will be asked to take a post-test based on the mathematics content taught by the researcher. This will be a paper and pencil test and will take no more than 90 minutes. All students in this class will complete this post-test. However, only the results of students who gave their consent will be used in the study. The post-test grades of all students, however, will be given to their classroom mathematics teacher who

will decide whether or not to use them as school grades. These post-test grades may also be used in conjunction with the pre-test grades to plan remedial work for students if necessary.

Benefits

Potential benefits include the opportunity to help students learn mathematics more deeply and improve their performance on mathematics tests. It may also provide an opportunity for teachers to add to the strategies they use for teaching mathematics.

Risk

There is little to no risk to any student participating in this research. Students may feel some anxiety in having an unfamiliar teacher in their classroom. This reaction is normal, and I will use my 20 plus years of experience teaching mathematics at the secondary level to ease students' fears.

Voluntary Participation and Withdrawal

Students are under no obligation to participate in this study. Participation is completely voluntary. Even if students and their parents/guardians agree to participation in the study, students and their parents can change their mind and withdraw at any time up to one week after the post-test which is the last set of data that will be collected from students. Students (or their parents/guardians) may refuse to participate or withdraw from the research activity without penalty or jeopardy to his/her class standing.

Confidentiality & Anonymity

- Participating students will be given special identifiers (e.g. Student A), and their names will be removed from all document and replaced with these identifiers. The document showing these links will be kept in a locked cupboard (to which only I have access) until the complete grading of the post-test after which it will be destroyed.
- Working papers will be kept in files in a locked cupboard, and digital information will be stored on a computer with password protection. Records will be stored for five years.
- This research will be used primarily to develop my doctoral dissertation, but it will also inform research articles, presentations, and teaching. Students will not be personally identified in any of these.
- You may inquire about a report of the research findings by contacting Christopher Charles by email at ccharles@ualberta.ca

Further Information

If you have any further questions regarding this study, please do not hesitate to contact Christopher Charles at email: ccharles@ualberta.ca, Tel.: (767) 276-5113 or Dr. Florence Glanfield (my supervisor) at email: glanfiel@ualberta.ca, Tel.: 1 (780) 492 0743.

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at (780) 492-2615.

Consent

To participate in this study, the consent form attached to this document must be completed by both the student and his/her parent/guardian.

Please keep this letter for your records.

Appendix D: Consent Form to Participants and Parents

CONSENT FORM

Please complete the following form.

****NOTE: For a student to participate in this study, a parent/guardian *AND* student must indicate their consent/assent by signing this form.****

---

(1) ***For Parents/Guardians*: Please circle ONE of the following options:**

a) **YES**, I consent (or, agree) to my child's participation in the research study, "*Learners Generating Examples": Impacts on Students' Achievement and Conceptual Understanding of Trigonometry*."

or

b) **NO**, I do not consent to my child's participation in the research study, "*Learners Generating Examples": Impacts on Students' Achievement and Conceptual Understanding of Trigonometry*."

____________________________________ ____________
Parent or Guardian signature Date

____
Printed name of Parent or Guardian

---

(2) *For Students*: Please circle ONE of the following options:

a) **YES**, I consent (or, agree) to my participation in the research study, "*Learners Generating Examples": Impacts on Students' Achievement and Conceptual Understanding of Trigonometry*."

or

b) **NO**, I do not consent to my participation in the research study, "*Learners Generating Examples": Impacts on Students' Achievement and Conceptual Understanding of Trigonometry*."

____________________________________ ____________
Student signature Date

____
Printed name of Student

Appendix E: Letter to Raters

Christopher Charles
Windy Haven
Wesley, Dominica
April 13, 2018.
Dear …,

As a validity measure, I am using two subject (Mathematics Education) specialists to validate my instrument for measuring students' understanding of the three primary trigonometric ratios. The instrument is enclosed.

Using your professional judgment, please comment (briefly) on how effective you believe this instrument is in assessing students' understanding of the three primary trigonometric ratios. Also, the twelve multiple-choice items are designed to assess four mathematical constructs (***structure of the ratios, the relationship between sine and cosine, representations of the ratios, calculating with the ratios***) related to the ratios.

- Questions 1, 2, and 3 - the ***structure of the ratios***
- Questions 4, 5, and 6 - the ***relationship between sine and cosine***
- Questions 7, 8, and 9 - ***representations of the ratios***
- Questions 10, 11, and 12 – ***calculating with the ratios.***

Please comment (briefly) on how well you believe these questions are assessing these constructs.

Along with your brief comments, please provide a brief Bio of yourself:

- ✓ Your qualifications
- ✓ Years of experience teaching mathematics at the secondary level
- ✓ Your position (HOD, teacher, etc.)
- ✓ Any other detail that speaks to the credibility of your judgment (if your mark for CXC, etc.).

Thanks for your assistance.
Sincerely

………………………………….
Christopher Charles
Ph.D. Candidate
University of Alberta

Appendix F: Raters' Confidentiality Agreement

Learners Generating Examples and Investigations: Impacts on Students' Achievement and Conceptual Understanding of Trigonometry

I, ______________________________ , a rater for the research project *"Generating Examples vs. Investigations: Impacts on Students' Achievement and Conceptual Understanding of Trigonometry"*, agree to:

1. keep all the research information shared with me confidentially by not discussing or sharing the research information in any form or format (e.g., disks, tapes, transcripts) with anyone other than the *Researcher*.

2. keep all research information in any form or format (e.g., disks, tapes, transcripts) secure while it is in my possession.

3. return all research information in any form or format (e.g., disks, tapes, transcripts) to the *Researcher* when I have completed the research tasks.

4. after consulting with the *Researcher,* erase or destroy all research information in any form or format regarding this research project that is not returnable to the *Researcher* (e.g., information stored on computer hard drive).

5. Keep the identity of any participant that I may be exposed to confidential by not discussing or sharing their name and work with anyone other than the Researcher.

______________________	______________________	______________________
(Print Name)	(Signature)	(Date)

Researcher

______________________	______________________	______________________
(Print Name)	(Signature)	(Date)

The plan for this study has been reviewed for its adherence to ethical guidelines and approved by Research Ethics Board #2 at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at 1 (780) 492-2615.

Appendix G: Raters' Comments on Instruments Validity

Rater A

Validation of Instrument

Section A: Multiple Choice Items

Questions 1, 2, and 3: These test students' fluency in the fundamentals of the structure of trigonometric ratios. The questions aim to seek clarification on whether or not the students can define the three trigonometric ratios and can identify them in any given context.

Questions 4, 5, and 6: These questions delve into the relationship between sine and cosine. These questions test students' deeper understanding of the relationship that exists between sine and cosine.

Questions 1- 6 show that through these increasingly complex problems students will be able to showcase how they have developed a conceptual understanding of trigonometric ratios and utilize this ability by being able to recall and apply information accurately.

Questions 7, 8, and 9: These test the representation of the trigonometric ratios. The students are tested on their ability to identify the ratios diagrammatically, graphically and in tabular form. These questions are used to test whether the students can utilize models or representations to assist in their approach to a solution. Students' ability to use mathematical modeling and representations to solve problems is an important step in the development of geometric and spatial reasoning.

Questions 10 -12 are based on students' ability to use calculations involving trigonometric ratios. These tests both students' knowledge and deeper understanding of trigonometric ratios. The students are applying the knowledge attained and mathematical skills developed, to solve problems with increasing sophistication.

Section B: Written-response Questions

These questions test an increase in the level of sophistication of students' conceptual understanding of the topic by their ability to represent a trigonometric ratio by utilization of its properties (question 1), by use deductive arguments to show that the relationships exist between the trigonometric ratios (question 2) or by solving real-life problems based on these deductions (question 3).

As a result, these questions test the students' ability to reason mathematically and use the language of mathematics to communicate how they use their mathematical knowledge to make conjectures or generalizations, develop an argument for their solution of the problem and justify their solutions.

Biography of Mathematics Education Specialist

Qualifications: MSc in Instructional Design and Technology (UWI)
B.Ed. in Mathematics Education (UWI)
Years of experience teaching mathematics at the secondary level: 18 years
Position: Head of Mathematics Department
Other: CXC marker, assisted in writing the Key Stage 2 National Curriculum for mathematics in Dominica

Rater B

Validity of Pre-test Assessing some Constructs of the Trigonometric Ratios

- **Structure of the Ratios**
  Questions 1, 2 and 3 clearly assess candidates ability to identify the features of a given trigonometric ratio. Also, the features of the right-angled triangle are expressed in mixed forms; numerically and algebraically, thus exposing candidates to varied structures of writing the ratio.
  Suggestion: one of these questions could have included a diagram in reinforcing that these ratios only apply to right-angled triangles.

- **Relationship between sine and cosine**
  The introduction of diagrams in questions 4,5 and 6 plays a pivotal role in showing the relationship between sine and cosine. This assesses candidates' spatial sense and their ability to use a diagram from one question and apply it to another.

- **Representations the ratios**
  Questions 7, 8 and 9 test other objectives from other strands in Mathematics such as Relations, Function and Graphs, and Computation. This is critical in assessing candidates' transfer of knowledge.

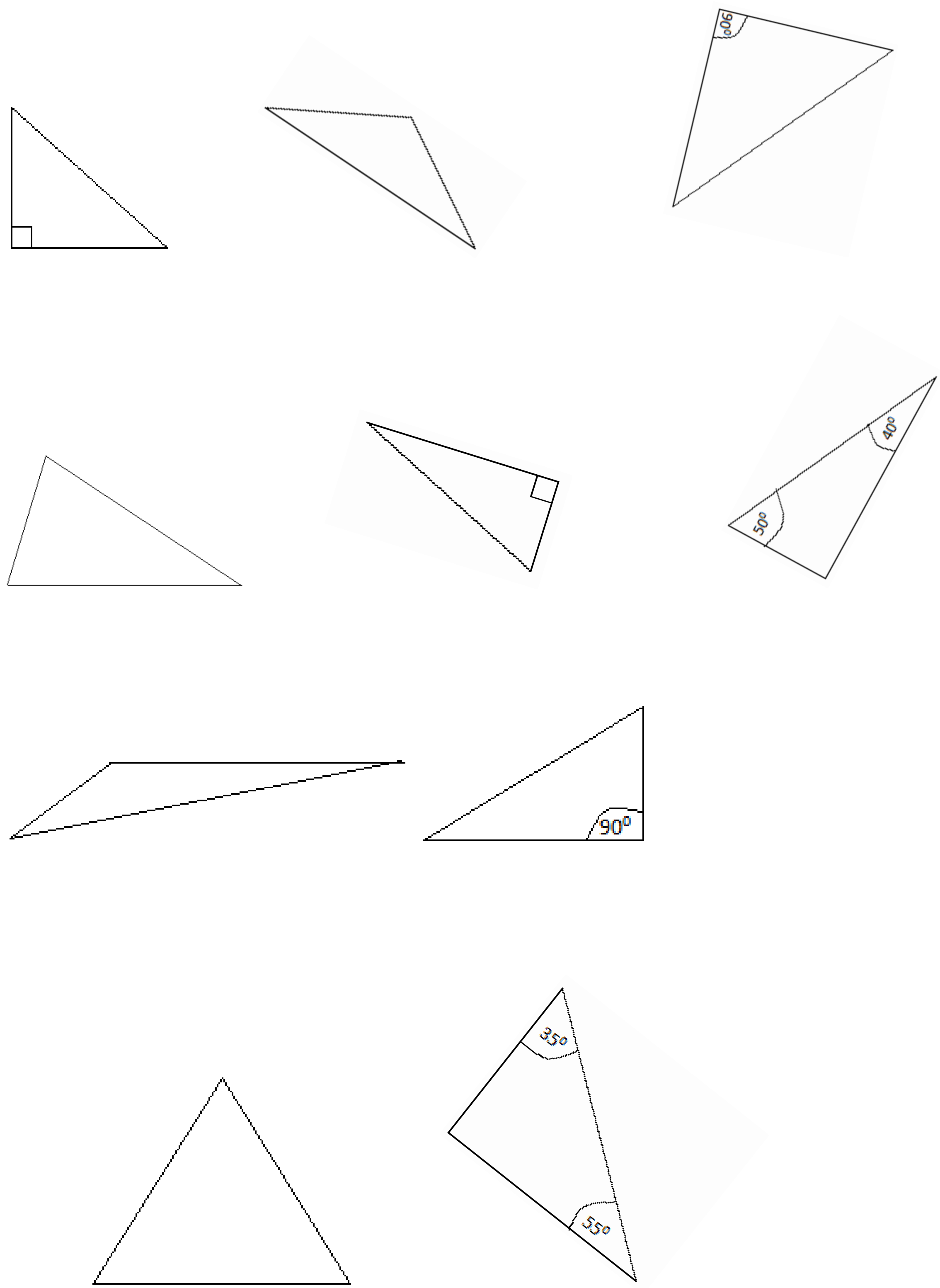
- **Calculating with the Ratios**
  Candidates are presented with multiple orientations of the right-angled triangle in questions 10, 11 and 12. This assesses their ability to work with any given right-angled triangle, in calculating a missing side or angle. Furthermore, these questions incorporate the above constructs which are pre-requisites for calculating with the ratios.

Biography

Developing students' competency in Mathematics was and still is a worldwide challenge, hence my research project for my Bachelor's Degree. Moreover, my educational and professional backgrounds have been mostly centered around the learning and teaching of Mathematics. I am [redacted], a Mathematics secondary school teacher for the past 16 years and Form Level Supervisor. My credentials include an Associates Degree in Secondary Education from the Dominica State College and a B. Ed. in Mathematics (Secondary) from the University of the West Indies. I am also a validated assistant examiner for Mathematics for the Caribbean Examination Council (CXC).

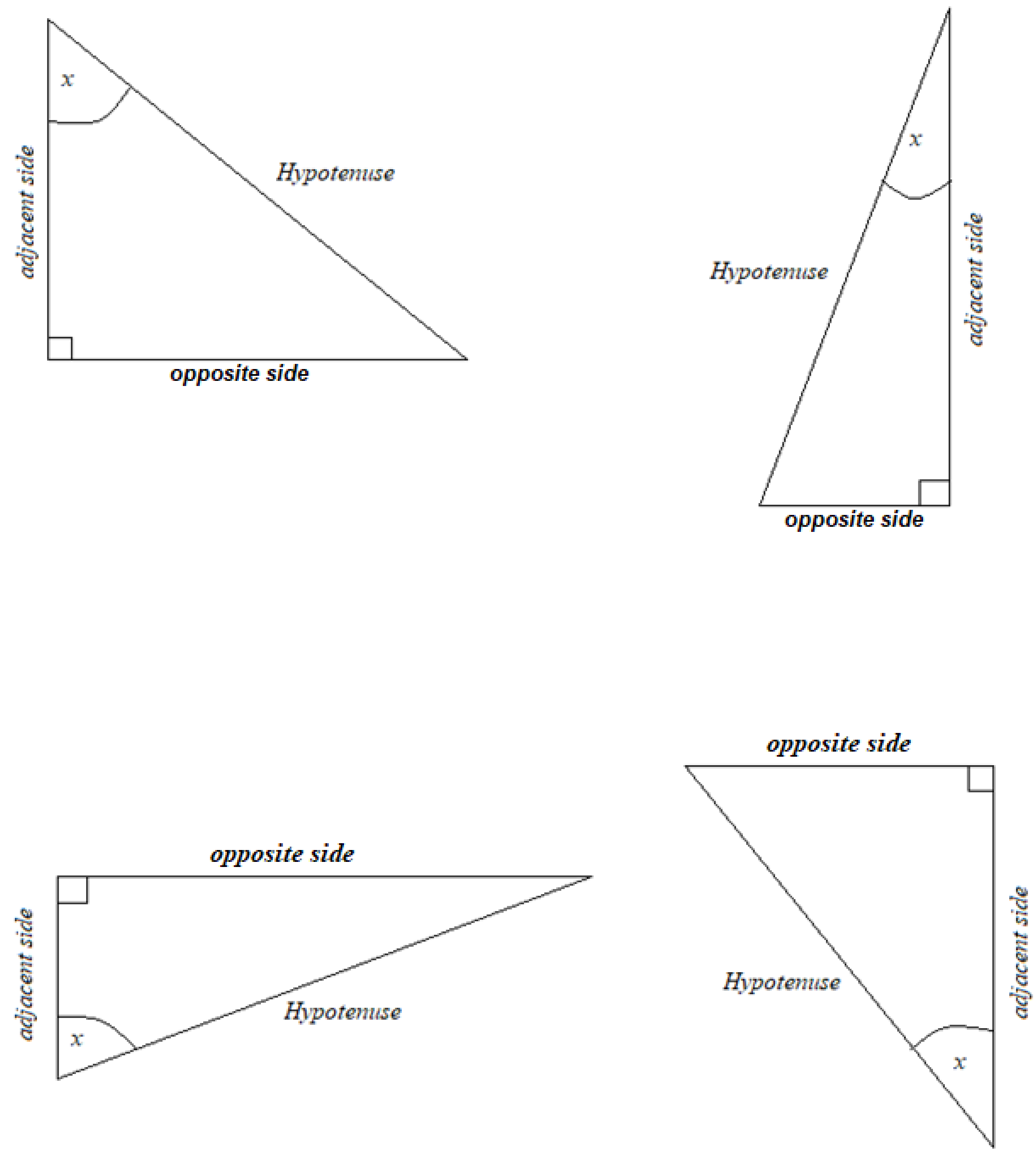
Appendix H: Worksheet Used in Session One for Both Groups

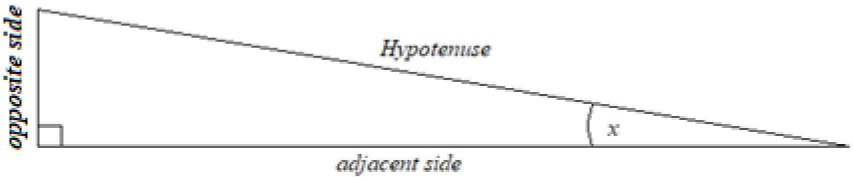
Put the letter "**R**" in all the triangles you think are right-angle triangles

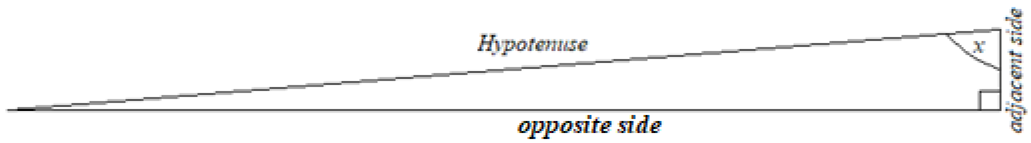


Appendix I: Triangles Used for Measurements

Measure the marked angle and sides in each of the following triangles and use them to complete the given table.




opposite side
Hypotenuse
x
adjacent side

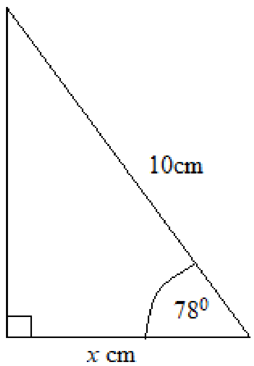

Hypotenuse
x
adjacent side
opposite side

Appendix J: Worksheet used in Session Six for Both Groups

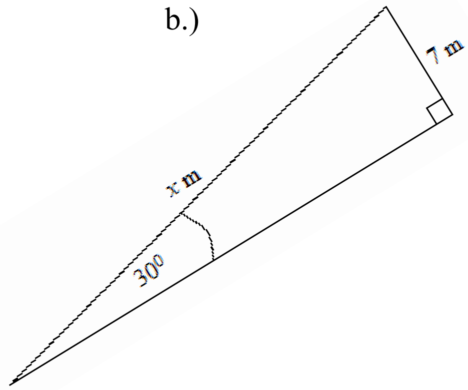
Worksheet

1. Determine the value of the missing side marked *x* in each diagram using the sine, cosine, or tangent ratio.

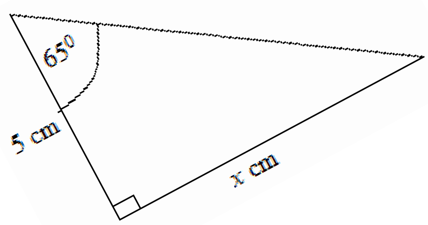
a.)



b.)

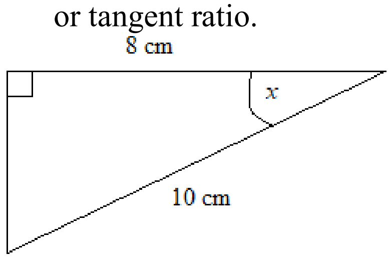


c.)

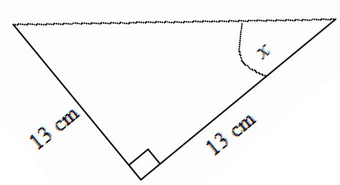


2. Determine the value of missing angles marked *x* in each diagram using the sine, cosine, or tangent ratio.

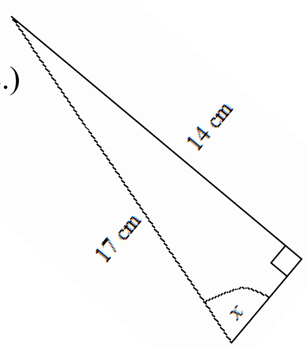
a.)



b.)



c.)



3. The angle of elevation of the top of a vertical tree from a man standing on level ground 25 metres from the base of the tree is 40^0.

   a) Draw a diagram to represent this information. Ensure that the 25m distance and the 40^0 angle are clearly represented on your diagram.
   b) Using the appropriate trigonometric ratio, determine the height of the tree.

4. From a coastal lookout point, 100 m above sea level, a sailor sights a boat in the distance. The angle of depression of the boat from the sailor is 25^0.

   a) Draw a diagram to represent this information showing all relevant information given.
   b) Explain in your own words how you would determine the distance the boat is from the lookout point. Be sure to specify which ratio you will use and why.
   c) Determine this distance.

Appendix K: Pre-test

Pre-test

Section A: 30 minutes

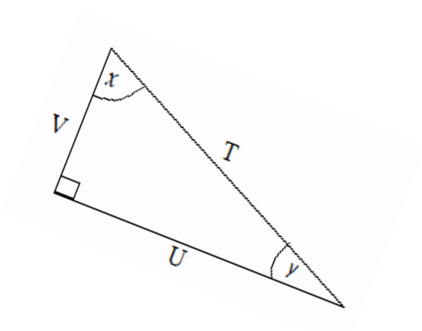
For each question in this section, **shade** the letter which corresponds to the best answer.

1. Angle, adjacent side, hypotenuse. Which ratio or ratios make use of these three features of a right-angled triangle?
(a) cosine only (b) sine and cosine only (c) sine only (d) sine, cosine, and tangent.

2. If the three sides of a right-angled triangle are 10cm, 24cm, and 26cm, which of the following is a possible tangent ratio?
(a) $\frac{10}{26}$ (b) $\frac{24}{26}$ (c) $\frac{24}{10}$ (d) $\frac{26}{10}$

3. A, B, and C are three sides of a right-angle triangle where A is the longest side. $C/_A$ could be:
(i) the sine ratio
(ii) the cosine ratio
(iii) the tangent ratio
(a) (i) only (b) (ii) only (c) (i) and (ii) only (d) (i), (ii), and (iii).

4. The diagram below is the drawing of a right-angled triangle where ***x*** and y are angles and T, U, and V represent its sides. Which two ratios are equal?

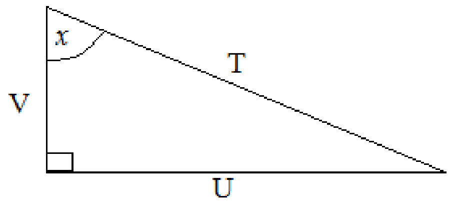


(a) sin(***x***) and sin(y) (b) sin(***x***) and cos(y) (c) sin(***x***) and tan(y) (d) cos(*x*) and tan(y)

5. If the cosine of x = A, the sine of which angle is also equal to A?
(a) $90^0 - x$ (b) $90^0 + x$ (c) $45^0 - x$ (d) $45^0 + x$

6. If cos(x) = A and sin(y) = B where x and y are two acute angles. If A is greater than B, then:
(a) x is sometimes greater than y (b) x is always greater than y
(c) x is always less than y (d) you cannot tell without the actual values of x and y.

7. The figure below is the drawing of a right-angled triangle where ***x*** is the reference angle and T, U, and V represent its sides. Which ratio does $^{v}/_{t}$ represent?



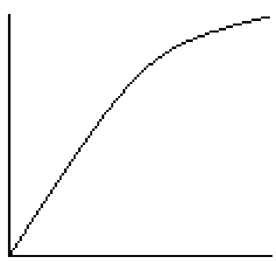
(a) sine (b) cosine (c) tangent (d) none

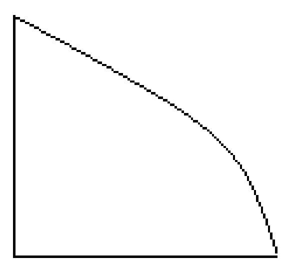
8. In the table below, the letters represent angles in **increasing** order of value (for example, B is greater than A). Which of the ratios could this table represent?

Angles	Ratio Value
A	0.1
B	0.2
C	0.3
D	0.4
E	0.5

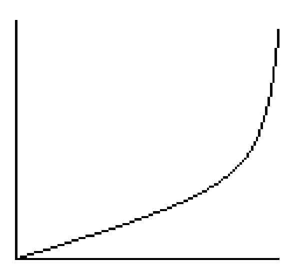
(a) sine and cosine only (b) cosine and tangent only
(c) sine and tangent only (d) sine, cosine and tangent

9. Select the graph which best represents the tangent ratio.

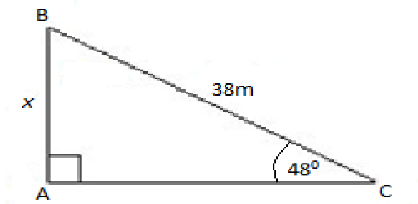
(a)


(b)


(c)

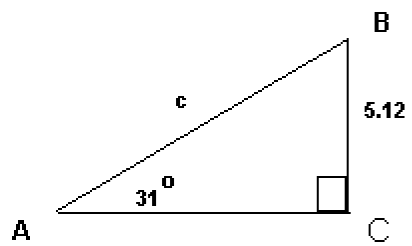

(d)


10. x is the length of side AB in the triangle below. You need to find the value of $\boldsymbol{x}$ by using a one-step procedure, which ratio is best suited to do this?



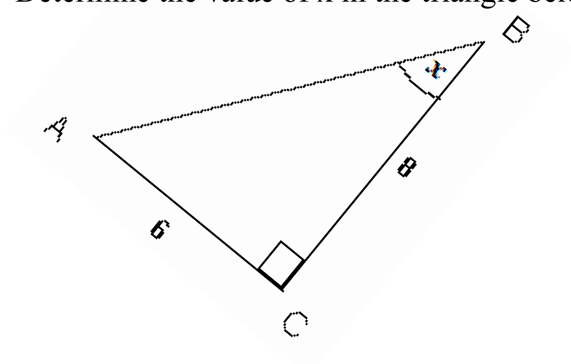
(a) sine (b) cosine (c) tangent (d) all

11. Determine the length of the missing side c in the triangle below.



(a) $5.12\sin(31^0)$ (b) $5.12\cos(31^0)$ (c) $\frac{5.12}{\cos(31)}$ (d) $\frac{5.12}{\sin(31)}$

12. Determine the value of x in the triangle below.



(a) $\frac{6}{8}$ (b) $\tan(\frac{6}{8})$ (c) $\tan^{-1}(\frac{6}{8})$ (d) $\tan^{-1}(\frac{8}{6})$

Section B: Written-response questions

Answer the following questions in the space provided.

1. This question requires you to use and discuss multiple representations of the cosine ratio.
   v. Including the formula, show **at least** three (3) different representations of the cosine ratio.

   vi. Discuss how you know that each of these representation shows the cosine ratio.

2. This question requires you to show the three trigonometric ratios using the **<u>same form</u>** of representation, then to compare and contrast these representations.

vii. Apart from the formula, use the **<u>same form</u>** of representation to show the sine, cosine, and tangent ratios.

viii. Discuss the similarities and differences among these representations.

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3. For this question, you will draw a diagram to representing a contextual situation, and discuss how that diagram can be used to solve a problem. **<u>NO</u>** calculations are necessary.

A boy is lying on the ground 5 metres away from the foot of a building. He observes a bird on top of the building. His line of sight (the direction he is looking) makes an angle of 30^0 with the ground. The ground is horizontal.

v. Draw a diagram (triangle) to represent the situation described above. Mark clearly on your diagram: the distance of 5 metres, his line of sight, the 30^0 angle, and the height of the building.

vi. Without doing any calculations, discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why.

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Appendix L: Post-test

Post-test

Section A

For each question in this section, **shade** the letter which corresponds to the best answer.

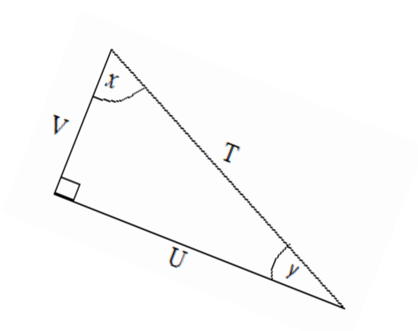
1. If the three sides of a right-angled triangle are 6cm, 8cm, and 10cm, which of the following is a possible tangent ratio?
   (a) $\frac{10}{6}$ (b) $\frac{8}{6}$ (c) $\frac{8}{10}$ (d) $\frac{6}{10}$

2. A, B, and C are three sides of a right-angle triangle where C is the longest side. B/C could be:
   (i) the sine ratio
   (ii) the cosine ratio
   (iii) the tangent ratio
   (a) (i), (ii), and (iii) (b) (i) and (ii) only (c) (ii) only (d) (i) only

3. Angle, adjacent side, hypotenuse. Which ratio or ratios make use of these three features of a right-angled triangle?
   (a) cosine only (b) sine and cosine only (c) sine only (d) sine, cosine, and tangent.

4. If cos(x) = A and sin(y) = B where x and y are two acute angles. If A is greater than B, then:
   (a) x is sometimes greater than y (b) y x is always less than y
   (c) x is always greater than (d) you cannot tell without the actual values of x and y.

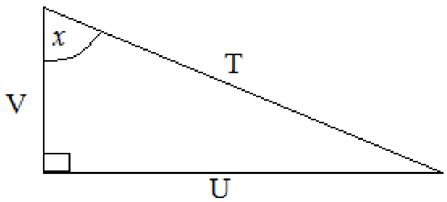
5. The diagram below is the drawing of a right-angled triangle where ***x*** and y are angles and T, U, and V represent its sides. Which two ratios are equal?



   (a) sin($\boldsymbol{x}$) and sin(y) (b) cos(x) and tan(y) (c) sin($\boldsymbol{x}$) and tan(y) (d) sin($\boldsymbol{x}$) and cos(y)

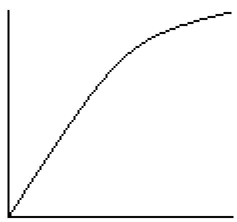
6. If the sine of x = A, the cosine of which angle is also equal to A?
   (a) $45^0 - x$ (b) $90^0 + x$ (c) $90^0 - x$ (d) $45^0 + x$

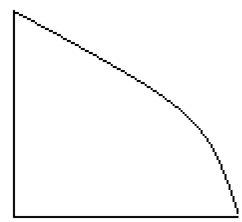
7. The figure below is the drawing of a right-angled triangle where ***x*** is the reference angle and *t, u,* and *v* represent its sides. Which ratio does $^{v}/_{t}$ represent?



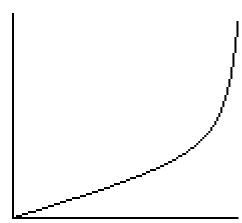
(a) sine (b) cosine (c) tangent (d) none

8. Select the graph which best represent the tangent ratio.

(a)


(b)


(c)

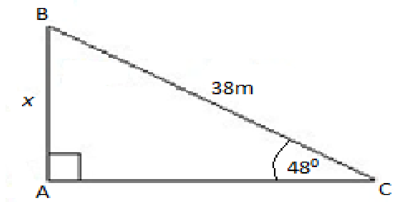

(d)


9. In the table below, the letters represent angles in **increasing** order of value (for example, B is greater than A). Which of the ratios could this table represent?

Angles	Ratio Value
A	0.1
B	0.2
C	0.3
D	0.4
E	0.5

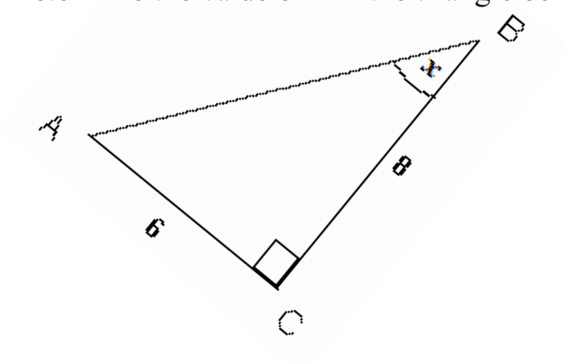
(a) cosine and tangent only (b) sine and cosine only
(c) sine, cosine, and tangent (d) sine and tangent only

10. x is the length of side AB in the triangle below. You need to find the value of $\boldsymbol{x}$ by using a one-step procedure, which ratio is best suited to do this?



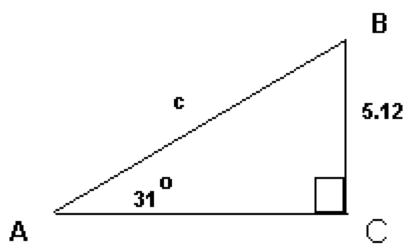
(a) sine (b) cosine (c) tangent (d) all

11. Determine the value of x in the triangle below.



(a) $\frac{6}{8}$ (b) $\tan(\frac{6}{8})$ (c) $\tan^{-1}(\frac{8}{6})$ (d) $\tan^{-1}(\frac{6}{8})$

12. Determine the length of the missing side "c" in the triangle below.



(a) $5.12\sin(31^0)$ (b) $5.12\cos(31^0)$ (c) $\frac{5.12}{\cos(31)}$ (d) $\frac{5.12}{\sin(31)}$

Section B: Written-response questions

Answer the following questions in the space provided. Use additional paper if needed.

1. For this question, you will draw a diagram to representing a contextual situation, and discuss how that diagram can be used to solve a problem. **NO** calculations are necessary.

   A dog is lying on the ground 25 metres away from the foot of a building. It observes a bird on top of the building at an angle of elevation of 20^0.

   vii. Draw a diagram (triangle) to represent the situation described above. Mark clearly on your diagram: the distance of 25 metres, its line of sight, the 20^0 angle, and the height of the building.

   viii. Discuss how you would find the height of the building. In your discussion, clearly state which ratio or ratios you would use and why. (Calculations are not necessary).

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2. This question requires you to use and discuss multiple representations of the cosine ratio.

vii. Including the formula, show **at least** three (3) different representations of the cosine ratio.

viii. Discuss how you know that each of these representation shows the cosine ratio.

3. This question requires you to show the three trigonometric ratios using the **<u>same form</u>** of representation, then to compare and contrast these representations.

ix. Use the **<u>same form</u>** of representation to show the sine, cosine, and tangent ratios.

x. Discuss the similarities and differences among these representations.

Similarities:

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Differences:

Appendix M: Assessment Rubric

Assessment Rubric

Question # and Themes	4 – Full Representational Knowledge	3 – Good Representational Knowledge	2 – Partial Representational Knowledge	1 – Very limited Representational Knowledge	0 – No Representational Knowledge	Rater's Grade and Comments
#1 Representing a contextual problem.	Uses a representation that is accurate in its aesthetics or mathematical precision. A clear and succinct strategy using all relevant information is presented.	Uses a representation that clearly depicts the problem. A clear strategy which will lead to the solution is presented.	Uses a representation that gives some important information about the problem. The strategy presented shows the "essence" of the problem but may not lead to a correct solution.	Uses a representation that gives little or no significant information about the problem. The strategy is inconsistent or unrelated to the problem.	No attempt is made to construct a mathematical representation or discuss a strategy for its solution.	
#2 Use of multiple representations	Three or more accurate representation with meaningful explanations is given.	Two accurate representation with meaningful explanations is given. Or, three accurate representation with some errors in explanation is given.	One accurate representation with a meaningful explanation is given. Or, two accurate representation with some errors in explanation is given.	At least one accurate representation is given. The explanation is erroneous.	No representation and accompanying explanation are given.	
#3 Comparing and contrasting representations	All three ratios are accurately represented in the same way (formulas, or tables, or graphs). At least one meaningful similarity and one meaningful difference among the representations are identified.	All three ratios are accurately represented in the same way. However, only meaningful similarity(ies) or meaningful difference(s) among the representations is/are identified.	All three ratios are accurately represented in the same way. However, no comparison is made, or comparison is erroneous.	Ratios are shown using different forms of representations.	No representation and comparison are given.	

Appendix N: Students' Pre-test and Post-test Scores

Participants	Pre-test scores				Post-test scores				
	M. C	Rater A	Rater B	Raw score	M. C	Rater A	Rater B	Mean	Raw score
I1	1	0	0	1	8	8.5	8.5	8.5	16.5
I2	4	0	0	4	7	6	5.5	5.75	12.75
I3	2	0	0	2	5	6	6.5	6.25	11.25
I4	2	0	0	2	10	9	8.5	8.75	18.75
I5	7	0	0	7	2	1	1.5	1.25	3.25
I6	1	0	0	1	5	4.5	3	3.75	8.75
I7	1	0	0	1	7	4	3	3.5	10.5
I8	2	0	0	2	1	1	1.5	1.25	2.25
I9	1	0	0	1	6	5.5	6.5	6	12
I10	3	0	0	3	8	11	9.5	10.25	18.25
I11	2	0	0	2	7	9	6.5	7.75	14.75
I12	1	0	0	1	3	5	3	4	7
I13	2	0	0	2	5	4.5	4.5	4.5	9.5
I14	3	0	0	3	6	5.5	6	5.75	11.75
I15	9	0	0	9	6	5.5	2.5	4	10
I16	2	0	0	2	3	3	2	2.5	5.5
I17	2	0	0	2	-	-	-	-	-
E1	1	0	0	1	-	-	-	-	-
E2	5	0	0	5	8	4	2.5	3.25	11.25
E3	4	0	0	4	4	4.5	3	3.75	7.75
E4	2	0	0	2	5	5.5	3	4.25	9.25
E5	4	0	0	4	9	10	10	10	19
E6	2	0	0	2	9	9	9.5	9.25	18.25
E7	5	0	0	5	8	8.5	7	7.75	15.75
E8	1	0	0	1	5	10	9	9.5	14.5
E9	1	0	0	1	5	9.25	5.5	7.375	12.375
E10	2	0	0	2	8	8.75	5.5	7.125	15.125
E11	4	0	0	4	8	9	8.5	8.75	16.75
E12	2	0	0	2	9	8.5	9.5	9	18
E13	1	0	0	1	4	5.5	2.5	4	8
E14	6	0	0	6	6	6.75	8.5	7.625	13.625
E15	3	0	0	3	7	11.5	11.5	11.5	18.5
E16	2	0	0	2	9	7.5	8	7.75	16.75
E17	1	0	0	1	8	8	9.5	8.75	16.75
E18	0	0	0	0	-	-	-	-	-