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UNIVERSITY OF ALBERTA

A DISCUSSION OF SURJECTIVE CELLULAR AUTOMATA

BY

CHRISTOPHER JAMES WANT ©

A thesis submitted to the Faculty of Graduate Studies And Research in partial fulfillment of the requirements of the degree of Master of Science.

DEPARTMENT OF MATHEMATICAL SCIENCES

Edmonton, Alberta
Spring 1995

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Christopher James Want

10811-126 street, Edmonton,

Alberta, Canada T5M 0P2

Date: DECEMBER 22, 1994

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **A Discussion of Surjective Cellular Automata** submitted by **Christopher James Want** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Abstract

The first chapter of this paper is concerned with finding necessary and sufficient conditions for the surjectivity of automata. This problem is reduced to the study of semigroups of matrices. Partition permutivity, a generalisation of the concept of permutivity is developed and analysed.

Chapter two is devoted to the study of surjective automata from the point of view of ergodic theory. Surjectivity is shown to be equivalent to measure preservation. Several types of automata are shown to be strongly mixing. Bipermutive automata are shown to be equivalent to one-sided Bernoulli shifts. Products of automata are shown to also be automata. An index of ergodicity is developed for local functions.

Preface

Cellular Automata (which are also known as tessellation structures or endomorphisms of the shift dynamical system) have become an important tool for modelling natural phenomena in the last three decades. A (one dimensional) cellular automaton is a continuous shift invariant mapping ϕ of bi-infinite sequence space $S^{\mathbf{Z}}$ into itself, where the sequence entries are from a finite set $S = \{0, 1, \ldots, N-1\}$ for some $N > 1$ (each point of the space can be regarded as an infinite lattice of cells, each cell holds a value from 0 to $N-1$). This gives ϕ a local component-wise structure in the form of a *local function* - that is, the components of future cells depend on a finite number of past cells. Cellular Automata are the ultimate discrete dynamical systems.

Cellular automata were first used by Von Neumann and Ulam in the 1940's to model the survival, reproduction and evolution of organisms. We might call this the "first wave" of automata theory.

The "second wave", whose two leaders would surely be Hedlund and Conway began in the late 1960's. Hedlund was concerned with studying the mathematical properties of automata themselves - as opposed to using them for modelling. Conway's contribution was a two dimensional automaton called "Life", which at the time attracted much attention from the mathematical biologists.

Stephen Wolfram is undoubtably the instigator of the "third wave" of automata theory. This wave started in the early 1980's and still continues to the present day. Wolfram applied automata to computation theory and statistical mechanics (which is the father of ergodic theory).

Today, cellular automata are being used for modelling in physics, biology, chemistry, information theory, computation theory, ordinary and partial differential equations, fractal geometry, and many other branches of mathematical science (two good sources for applied cellular automata are [CA] and [Wol]). They even have application in the arts: any one with a computer connection to the World Wide Web and a gif viewer can retrieve the document URL *gopher://life.anu.edu.au/i9/landscape_ecology /firenet/software/ignite/monalisa.gif* to see the face of the Mona Lisa burn up.

In this thesis, we will be looking at those automata which are surjective. These automata are important as they will be found to be measure preserving transformations - transformations which are the basis of ergodic theory. We divide our studies into two chapters.

The first of these chapters will concern finding conditions for an automaton to be surjective. We reduce the problem to the study of automata which have bivariate local functions. For such automata, we construct a matrix semigroup (called the family of preimage matrices) to aid our analysis, particularly in showing that a new class of automata - the partition permutive automata - are onto. We conclude by showing that the composition of two permutive automata is partition permutive.

In the second chapter we apply ergodic theory to analyse surjective automata. We show that the class of measure preserving automata is exactly the class of surjective automata. We show that automata with large (either positive or negative) left and right indices are mixing. Certain permutive automata are found to be mixing, and some bipermutive automata are shown to be equivalent to Bernoulli shifts. We next consider products of automata, and we show how such constructions yield other automata. We conclude by summarising our results in the form of an index on local functions.

There are several people whose contributions have been invaluable. I would like to thank my mother and father, Rhona and James Want, and my grandmothers, E. Muriel Milne and Winifred Want, for their never ending love and support over the years. I would also like to thank my supervisor Tom Rogers for several years of tutelage and encouragement. Special Thanks go to Kevin Charter for helping me survive UNIX, the C programming language, and the LaTeX typesetting language.

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Contents

Chapter 1

Surjective Automata

1.1 Preliminaries

In the past twenty-five years, most authors who have written about surjective automata have restricted their attention to permutive automata, and usually on a two letter alphabet. We shall spend this chapter exploring the rich structure of surjective automata in full generality.

In this chapter we analyse the local functions of surjective automata. We will develop a method to treat local functions with several variables as bivariate functions, and this will lead to an intriguing method of analysis using semigroups of matrices. We will exploit these matrix methods in the development and analysis of partition permutive automata.

We begin with some basic definitions and constructions.

Let $N \in \mathbb{Z}$, with $N > 1$ and let $\mathcal{S}_N = \{0, 1, \ldots, N-1\}$. We call $\mathcal{S}_N = \{0, 1, \ldots, N-1\}$ a *symbol set* or an N *letter alphabet* . Give $\mathcal{S}_N$ the discrete topology and give $\mathcal{S}_N^{\mathbb{Z}}$ the induced product topology.

A subbasis for this topology are the sets of the form $A_{a,i} = \{x \in \mathcal{S}_N^{\mathbb{Z}} : x_i = a\}$, where $a \in \mathcal{S}_N, i \in \mathbb{Z}$. We choose for a basis of $\mathcal{S}_N^{\mathbb{Z}}$ the collection of *cylinder sets* , i.e., sets of the form

$$A_{a_1,i_1} \cap A_{a_2,i_2} \cap \ldots \cap A_{a_n,i_n} = \{x \in \mathcal{S}_N^{\mathbb{Z}} : x_{i_1} = a_1, x_{i_2} = a_2, \ldots, x_{i_n} = a_n\},$$

where $a_1, a_2, \ldots, a_n \in \mathcal{S}_N$, $i_1, i_2, \ldots, i_n \in \mathbb{Z}$, with $i_1 < i_2 < \ldots < i_n$. Note: we also consider the empty set to be a cylinder set so that the collection of cylinder sets is closed under finite intersections.

By Tychonoff's Theorem, $\mathcal{S}_N^{\mathbb{Z}}$ is compact under this topology. We give $\mathcal{S}_N^{\mathbb{Z}}$ a metric, d, in the following way: If $x, y \in \mathcal{S}_N^{\mathbb{Z}}$ we let $d(x,y) = 0$ if $x = y$, and $d(x,y) = \frac{1}{2^{i^*}}$, where $i^* = \min\{|i| : x_i \neq y_i\}$, if $x \neq y$. This metric generates the product topology on $\mathcal{S}_N^{\mathbb{Z}}$.

Let $r, s \in \mathbb{Z}$, $r \leq s$. Let $f : \mathcal{S}_N^{s-r+1} \longrightarrow \mathcal{S}_N$ be given by $f = f(x_r, x_{r+1}, \ldots, x_s)$. Let $f_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ be given component-wise by $(f_\infty(x))_i = f(x_{i+r}, \ldots, x_{i+s})$, for all $i \in \mathbb{Z}$. f_∞ is called a (one-dimensional) *cellular automaton* with *local function* f. r and s are called the *left and right indices of* f_∞ respectively (note that in using the notation $f = f(x_r, \ldots, x_s)$ we have implicitly built the left and right indices into the local function f).

Let $\mathcal{F}_{r,s}(\mathcal{S}_N)$ denote the set of all local functions with left and right indices r and s respectively, i.e.,

$$\mathcal{F}_{r,s}(\mathcal{S}_N) = \{f : \mathcal{S}_N^{s-r+1} \longrightarrow \mathcal{S}_N : f = f(x_r, \ldots, x_s)\},$$

and define $\mathcal{F}(\mathcal{S}_N) = \bigcup_{r \leq s} \mathcal{F}_{r,s}(\mathcal{S}_N)$.

Note that the identity map $id : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ is a cellular automaton with local function $id(x_0) = x_0 \in \mathcal{F}_{0,0}(\mathcal{S}_N)$.

The *shift map* on $\mathcal{S}_N^{\mathbb{Z}}$ is the map $\sigma : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ which satisfies $(\sigma(x))_i = x_{i+1}$, for all $x \in \mathcal{S}_N^{\mathbb{Z}}$. σ is a cellular automaton (with local function $\sigma = \sigma(x_1) = x_1 \in \mathcal{F}_{1,1}(\mathcal{S}_N)$) which is a homoeomorphism of $\mathcal{S}_N^{\mathbb{Z}}$ (we drop the subscript ∞ for convenience).

One important property of σ is that σ commutes with all automata (i.e, $f_\infty \circ \sigma = \sigma \circ f_\infty$). It has been shown that this property may be used to formulate an equivalent definition of a cellular automaton: $\Phi : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ is a cellular automaton $\Longleftrightarrow$ Φ is continuous and commutes with the shift map [Hed].

Historically, the shift map plays an important role in ergodic theory, chaos theory, and dynamical systems in general. The shift is the prototypical chaotic map, and a map whose dynamical properties are transparent. Many kinds of different shifts are considered in these subjects such as Bernoulli shifts, Markov shifts, subshifts of finite type, and sofic systems, which are defined by the underlying spaces on which they act, and the measures endowed upon these spaces (The shift defined above, together with the measure we give $\mathcal{S}_N^{\mathbb{Z}}$ in the next chapter, is a two-sided *Bernoulli shift*). The other shift that we will consider is the *one-sided shift* . This is the map $\sigma : \mathcal{S}_N^{\mathbb{N}} \longrightarrow \mathcal{S}_N^{\mathbb{N}}$ defined component-wise by $[\sigma(x)]_i = x_{i+1}$, for all $i \in \mathbb{N}$ (in general, we can define one-sided automata on $\mathcal{S}_N^{\mathbb{N}}$ but of these we shall interest ourselves with the one-sided shift only). Notice that the one-sided shift on $\mathcal{S}_N^{\mathbb{N}}$ is an N-to-1 map, whereas the two-sided shift is a homeomorphism. The one-sided shift will play an important role in the discussion of bipermutive automata in chapter 2.

Example 1.1 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_p)$, where p is a prime number. If we give $\mathcal{S}_p$ a field structure (i.e., $\mathcal{S}_p = \mathbb{Z}/p\mathbb{Z}$) then we may express f as a polynomial in the indeterminates $x_r, x_{r+1}, \ldots, x_s$, i.e.,*

$$f = \sum_{\substack{0 \le m_1, m_2, \ldots, \\ m_{s-r+1} \le p-1}} a_{m_1, m_2, \ldots, m_{s-r+1}} x_r^{m_1} x_{r+1}^{m_2} \cdots x_s^{m_{s-r+1}}$$

where $a_{m_1, m_2, \ldots, m_{s-r+1}} \in \mathcal{S}_p$.

For $n \ge 1$, let $f_n : \mathcal{S}_N^{s-r+n} \longrightarrow \mathcal{S}_N^n$ be the n^{th} *block map of* f, defined by

$$\begin{aligned} & f_n(x_r, x_{r+1}, \ldots, x_s, x_{s+1}, \ldots, x_{s+n-1}) \\ = \;& (f(x_r, \ldots, x_s), f(x_{r+1}, \ldots, x_{s+1}), \ldots, f(x_{r+n-1}, \ldots, x_{s+n-1})) \end{aligned}$$

The following theorem is a combination of two theorems from [Hed]:

Theorem 1.2 (Hedlund) *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$. The following are equivalent:*

1) f_∞ is onto.

2) f_m is onto for all m.

3) card $f_m^{-1}(a_1, a_2, \ldots, a_m) = N^{s-r}$ for all $(a_1, a_2, \ldots, a_m) \in \mathcal{S}_N^m$.

Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$, with $r < s$ (i.e., f is not univariate). We say that f_∞ is *left permutive* if for each $(a_{r+1}, \ldots, a_s) \in \mathcal{S}_N^{s-r}$, $f(\cdot, a_{r+1}, \ldots, a_s)$ is a permutation of $\mathcal{S}_N$ (in this case, we also say that f is left permutive). Similarly, f_∞ is *right permutive* if for each $(a_r, \ldots, a_{s-1}) \in \mathcal{S}_N^{s-r}$, $f(a_r, \ldots, a_{s-1}, \cdot)$ is a permutation of $\mathcal{S}_N$. A *bipermutive* automaton is both left and right permutive.

It is a well known fact that permutive automata are onto (this will be proven later as a consequence of Theorem 1.29).

Example 1.3 *Let $f = f(x_r, \ldots, x_s) \in \mathcal{F}_{r,s}(\mathcal{S}_2)$ be left permutive. If we treat $\mathcal{S}_2$ as the field with two elements, we may express f in a particularly nice form: $f = x_r + g(x_{r+1}, \ldots, x_s)$, for some $g \in \mathcal{F}_{r+1,s}(\mathcal{S}_2)$. The reason for this is that there are only two permutations of $\mathcal{S}_2$, given by $i(x) = x$ and $p(x) = x + 1$. Thus we construct g as follows: For any $a_{r+1}, \ldots, a_s \in \mathcal{S}_2$ let*

$$g(a_{r+1}, \ldots, a_s) = \begin{cases} 0 & , \quad \text{if } f|_{\mathcal{S}_2 \times (a_{r+1}, \ldots, a_s)}(x_r, \ldots, x_s) = x_r, \\ 1 & , \quad \text{if } f|_{\mathcal{S}_2 \times (a_{r+1}, \ldots, a_s)}(x_r, \ldots, x_s) = x_r + 1. \end{cases}$$

Note that if $f \in \mathcal{F}_{r,s}(\mathcal{S}_p)$ is permutive, where p is a prime number, f need not be linear in either x_r or x_s when expressed as a polynomial in the indeterminates $x_r, \ldots, x_s$ over the field with p members. An example of this is the left permutive map $f \in \mathcal{F}_{0,1}(\mathcal{S}_3)$ given by $f = x_0 + 2x_0x_1^2 + x_0x_1 + 2x_1$ (notice that $f(x_0, 0) = x_0$, $f(x_0, 1) = x_0 + 2$, and $f(x_0, 2) = 2x_0 + 1$).

In an entirely analogous fashion, if $f \in \mathcal{F}_{r,s}(\mathcal{S}_2)$ is right permutive, we have $f = g(x_r, \ldots, x_{s-1}) + x_s$ for some $g \in \mathcal{F}_{r,s-1}(\mathcal{S}_2)$.

We give $\mathcal{F}(\mathcal{S}_N)$ a binary operation. For each $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$, let $fg \in \mathcal{F}_{r_1+r_2,s_1+s_2}(\mathcal{S}_N)$ be defined by

$$\begin{aligned} & fg(x_{r_1+r_2}, x_{r_1+r_2+1}, \ldots, x_{s_1+s_2}) \\ &= f(g(x_{r_1+r_2}, x_{r_1+r_2+1}, \ldots, x_{r_1+s_2}), g(x_{r_1+r_2+1}, x_{r_1+r_2+2}, \ldots, x_{r_1+s_2+1}), \ldots, \\ &\quad g(x_{s_1+r_2}, x_{s_1+r_2+1}, \ldots, x_{s_1+s_2})) \end{aligned}$$

We call this binary operation *local composition* .

This leads to the following:

Proposition 1.4 *Let $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$. Then $(fg)_\infty = f_\infty \circ g_\infty$.*

Proof. $(fg)_\infty = f_\infty \circ g_\infty$ if and only if $(f_\infty \circ g_\infty(x))_i = ((fg)_\infty(x))_i$, for all $x \in \mathcal{S}_N^{\mathbb{Z}}$ and all $i \in \mathbb{Z}$. We calculate

$$\begin{aligned} (f_\infty \circ g_\infty(x))_i &= (f_\infty(g_\infty(x)))_i = f((g_\infty(x))_{i+r_1}, \ldots, (g_\infty(x))_{i+s_1}) \\ &= f(g(x_{i+r_1+r_2}, \ldots, x_{i+r_1+s_2}), \ldots, g(x_{i+s_1+r_2}, \ldots, x_{i+s_1+s_2})) \\ &= fg(x_{i+r_1+r_2}, \ldots, x_{i+s_1+s_2}) \\ &= ((fg)_\infty(x))_i. \square \end{aligned}$$

One particular implication of the above proposition is that $\mathcal{F}(\mathcal{S}_N)$ is a semigroup under local composition. Indeed, if $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$, and $h \in \mathcal{F}_{r_3,s_3}(\mathcal{S}_N)$, then $(fg)h \in \mathcal{F}_{r_1+r_2+r_3,s_1+s_2+s_3}(\mathcal{S}_N)$ is the local function of $(f_\infty \circ g_\infty) \circ h_\infty = f_\infty \circ (g_\infty \circ h_\infty)$ which has local function $f(gh) \in \mathcal{F}_{r_1+r_2+r_3,s_1+s_2+s_3}(\mathcal{S}_N)$.

Remark 1.5 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r',s'}(\mathcal{S}_N)$. Then*
1) If f and g are both left [right] permutive, then fg is left [right] permutive.
2) We define $f^2 = ff \in \mathcal{F}_{2r,2s}(\mathcal{S}_N)$, and for any $n \in \mathbb{Z}$, $n > 2$ define $f^n = ff^{n-1} \in \mathcal{F}_{nr,ns}(\mathcal{S}_N)$. f^n is the local function of f^n_∞. We also have that if f is left [right,bi-]

permutive, then f^n is left [right,bi-] permutive for all n.
3) $\sigma \in \mathcal{F}_{1,1}(\mathcal{S}_N)$, $\sigma^k \in \mathcal{F}_{k,k}(\mathcal{S}_N)$, for all $k \in \mathbb{Z}$ and we have $f\sigma^k = \sigma^k f$. We have, furthermore, that f is left [right] permutive if and only if $\sigma^k f \in \mathcal{F}_{r+k,s+k}(\mathcal{S}_N)$ is left [right] permutive for all $k \in \mathbb{Z}$ (Note that f and $\sigma^k f$ are the same function when considered as mappings from $\mathcal{S}_N^{s-r+1}$ to $\mathcal{S}_N$, but their left and right indices differ).

An important theorem due to Hedlund should not be overlooked: If f_∞ is injective, then f_∞ is surjective. Hence all injective automata are homeomorphisms.

We will begin our analysis of surjective automata with a brief look at those automata which have univariate local functions.

1.2 Univariate Local Functions

The surjectivity of an automaton with univariate local function is the easiest to decide:

Theorem 1.6 (Hedlund) *Let $f = f(x_r) \in \mathcal{F}_{r,r}(\mathcal{S}_N)$. The following are equivalent:*
1) f is a permutation of $\mathcal{S}_N$,
2) $f_m : \mathcal{S}_N^m \longrightarrow \mathcal{S}_N^m$ is surjective,
3) card $f_m^{-1}(B) = 1$ for each $m \geq 1$, and each $B \in \mathcal{S}_N^m$,
4) f_∞ is surjective,
5) f_∞ is a homeomorphism.

Proposition 1.7 *Let $f = f(x_r) \in \mathcal{F}_{r,r}(\mathcal{S}_N)$ and let f_∞ be onto. Then there exists $n \in \mathbb{Z}$, $n \geq 1$ such that $f_\infty^n = \sigma^{nr}$.*

Proof. Let $f = f(x_r) \in \mathcal{F}_{r,r}(\mathcal{S}_N)$ with f_∞ onto. Then f is a permutation of $\mathcal{S}_N$, so there exists a $N \in \mathbb{Z}^+$ such that f^n is the identity on $\mathcal{S}_N$. We then have $f^n = x_{nr} \in \mathcal{F}_{nr,nr}(\mathcal{S}_N)$, so that for any $a \in \mathcal{S}_N^{\mathbb{Z}}$ and any $i \in \mathbb{Z}$ we have

$$[(f_\infty)^n(a)]_i = [(f^n)_\infty(a)]_i = a_{i+nr} = [\sigma^{nr}(a)]_i.\square$$

We will see in the next chapter that automata with univariate local functions are also the simplest to analyse as dynamical systems.

1.3 Multivariate Local Functions

In this section we will look at those automata whose local functions depend on more than two variables. We will show that such automata may be treated as automata whose local functions depend on only two variables (we will study such automata in the next section).

Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ and suppose $s - r + 1 > 2$ (f depends on more than two variables). We wish to associate f_∞ with a new automaton with bivariate local function $\tilde{f} : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ (where $\mathcal{T} = \mathcal{S}_{N^{s-r}}$) such that $\tilde{f}_\infty = h_1 \circ f_\infty \circ h_2$ for some homeomorphisms $h_1 : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{T}^{\mathbb{Z}}$, $h_2 : \mathcal{T}^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$.

Let $\mathcal{T} = \{0, 1, \ldots, N^{s-r} - 1\}$,and let $\psi : \mathcal{S}_N^{s-r} \longrightarrow \mathcal{T}$ be given by $\psi(x_0, \ldots, x_{s-r-1}) = \sum_{i=0}^{s-r-1} x_{s-r-1-i} N^i$. ψ is a bijection of sets,which induces the homeomorphism $\psi_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{T}^{\mathbb{Z}}$ given component-wise by

$$(\psi_\infty(x))_i = \psi(x_{(s-r)i}, \ldots, x_{(s-r)(i+1)-1}),$$

for all $x \in \mathcal{S}_N^{\mathbb{Z}}$, $i \in \mathbb{Z}$

Now, consider the $(s-r)^{th}$-block map of f, f_{s-r}, with domain $\mathcal{S}_N^{2(s-r)}$ and range $\mathcal{S}_N^{s-r}$. We can treat this map as $f_{s-r} : \mathcal{S}_N^{s-r} \times \mathcal{S}_N^{s-r} \longrightarrow \mathcal{S}_N^{s-r}$, and in doing so we will define $\tilde{f} : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ by $\tilde{f}(z_0, z_1) = \psi(f_{s-r}(\psi^{-1}(z_0), \psi^{-1}(z_1)))$, for all $(z_0, z_1) \in \mathcal{T} \times \mathcal{T}$. We then similarly define $\tilde{f}_\infty : \mathcal{T}^{\mathbb{Z}} \longrightarrow \mathcal{T}^{\mathbb{Z}}$ component-wise by $(\tilde{f}_\infty(z))_i = \tilde{f}(z_i, z_{i+1})$, for all $z \in \mathcal{T}^{\mathbb{Z}}, i \in \mathbb{Z}$. If $z \in \mathcal{T}^{\mathbb{Z}}$ and $x = \psi_\infty^{-1}(z) \in \mathcal{S}_N^{\mathbb{Z}}$, we have

$$\begin{aligned}(\tilde{f}_\infty(z))_i &= \tilde{f}(z_i, z_{i+1}) = \psi(f_{s-r}(\psi^{-1}(z_i), \psi^{-1}(z_{i+1}))) \\ &= \psi(f_{s-r}(x_{(s-r)i}, \ldots, x_{(s-r)(i+1)-1}, x_{(s-r)(i+1)}, \ldots, x_{(s-r)(i+2)-1}))\end{aligned}$$

$$
\begin{aligned}
&= \psi(f(x_{(s-r)i}, \ldots, x_{(s-r)(i+1)}), f(x_{(s-r)i+1}, \ldots, x_{(s-r)(i+1)+1}), \\
&\quad \ldots, f(x_{(s-r)(i+1)-1}, \ldots, x_{(s-r)(i+2)-1})) \\
&= \psi([f_\infty(x)]_{(s-r)i-r}, \ldots, [f_\infty(x)]_{(s-r)(i+1)-1-r}) \\
&= \psi([\sigma^{-r} \circ f_\infty(x)]_{(s-r)i}, \ldots, [\sigma^{-r} \circ f_\infty(x)]_{(s-r)(i+1)-1}) \\
&= [\psi_\infty(\sigma^{-r} \circ f_\infty(x))]_i \\
&= [\psi_\infty \circ \sigma^{-r} \circ f_\infty \circ \psi_\infty^{-1}(z)]_i.
\end{aligned}
$$

Thus $\tilde{f}_\infty = \psi_\infty \circ \sigma^{-r} \circ f_\infty \circ \psi_\infty^{-1} = h_1 \circ f_\infty \circ h_2$, where $h_1 : \mathcal{S}_N^{\mathbf{Z}} \longrightarrow \mathcal{T}^{\mathbf{Z}}$, $h_2 : \mathcal{T}^{\mathbf{Z}} \longrightarrow \mathcal{S}_N^{\mathbf{Z}}$, $h_1 = \psi_\infty \circ \sigma^{-r}$, $h_2 = \psi_\infty^{-1}$. We call $\tilde{f}$ the *tuba* map of f (tuba is an acronym - two unknowns, bigger alphabet).

Remark 1.8 *$\tilde{f}_\infty$ is topologically conjugate to $\sigma^{-r} f_\infty$ via ψ_∞. If $r \neq 0$, then in general, the dynamical properties of f_∞ and $\tilde{f}_\infty$ will be significantly different.*

Example 1.9 *Let $f = f(x_{-1}, x_0, x_1) \in \mathcal{F}_{-1,1}(\mathcal{S}_2)$ be given by* $f(0,0,0) = 1$, $f(0,0,1) = 0, f(0,1,0) = 0, f(0,1,1) = 0, f(1,0,0) = 1, f(1,0,1) = 1, f(1,1,0) = 0, f(1,1,1) = 1$. *We calculate $\tilde{f} \in \mathcal{F}_{0,1}(\mathcal{S}_4)$:*

$$
\begin{aligned}
\tilde{f}(0,0) &= \psi(f_2(\psi^{-1}(0), \psi^{-1}(0))) &= \psi(f_2(0,0,0,0)) &= \psi(1,1) &= 3, \\
\tilde{f}(0,1) &= \psi(f_2(\psi^{-1}(0), \psi^{-1}(1))) &= \psi(f_2(0,0,0,1)) &= \psi(1,0) &= 2,
\end{aligned}
$$

and similarly

$$
\begin{array}{llll}
\tilde{f}(0,2) = 0, & \tilde{f}(0,3) = 0, & \tilde{f}(1,0) = 1, & \tilde{f}(1,1) = 1, \\
\tilde{f}(1,2) = 0, & \tilde{f}(1,3) = 1, & \tilde{f}(2,0) = 3, & \tilde{f}(2,1) = 2, \\
\tilde{f}(2,2) = 2, & \tilde{f}(2,3) = 2, & \tilde{f}(3,0) = 1, & \tilde{f}(3,1) = 1, \\
\tilde{f}(3,2) = 2, & \tilde{f}(3,3) = 3. & &
\end{array}
$$

Proposition 1.10 *Let $f : \mathcal{S}_N^{s-r+1} \longrightarrow \mathcal{S}_N$, $f = f(x_r, x_{r+1}, \ldots, x_s)$, and let $\tilde{f} : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$, $\tilde{f} = \tilde{f}(z_0, z_1)$ be the tuba map of f. Then*

1) f_∞ is onto if and only if $\tilde{f}_\infty$ is onto;

2) f_∞ is k-to-1 if and only if $\tilde{f}_\infty$ is k-to-1;

3) f_∞ is left(right,bi-)permutive if and only if $\tilde{f}_\infty$ is left(right,bi-) permutive.

Proof. 1) and 2) are obvious and follow from basic set theory. For 3) we direct the reader to proposition 1.1 of [Elo].□

Remark 1.11 *Thus many aspects of the surjectivity of cellular automata may be solved by studying those automata with local function $f = f(x_0, x_1) \in \mathcal{F}_{0,1}(\mathcal{S}_N)$. This is good news, since for such automata we may introduce the notion of preimage matrices, as will be done in the next section.*

1.4 Bivariate Local Functions and Preimage Matrices

Let $f = f(x_0, x_1) \in \mathcal{F}_{0,1}(\mathcal{S})$. To f we assign an $N \times N$ *image matrix* F (with indices $i, j = 0, 1, \ldots, N-1$) defined by $(F)_{ij} = f(i,j)$. We also define for f, and for each $a \in \mathcal{S}_N$ an $N \times N$ *(primary) preimage matrix* A_a (again, with indices in $\mathcal{S}_N$), defined by

$$(A_a)_{ij} = \begin{cases} 1 & , \text{ if } f(i,j) = a, \\ 0 & , \text{ if } f(i,j) \neq a. \end{cases}$$

Example 1.12 *Let $f = f(x_0, x_1) \in \mathcal{F}_{0,1}(S_4)$ be given by*

$$\begin{array}{llll} f(0,0) = 1, & f(0,1) = 0, & f(0,2) = 3, & f(0,3) = 1, \\ f(1,0) = 2, & f(1,1) = 2, & f(1,2) = 0, & f(1,3) = 3, \\ f(2,0) = 3, & f(2,1) = 1, & f(2,2) = 1, & f(2,3) = 0, \\ f(3,0) = 0, & f(3,1) = 2, & f(3,2) = 2, & f(3,3) = 3. \end{array}$$

The image matrix, F, of f is given by

$$F = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 0 & 3 \\ 3 & 1 & 1 & 0 \\ 0 & 2 & 2 & 3 \end{pmatrix}.$$

We calculate the primary preimage matrices of f to be

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ 1 & & 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\textit{and } A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.13 *The following are obvious:*

$$\sum_{a \in \mathcal{S}_N} (A_a)_{ij} = 1, \text{ for all } i,j \in \mathcal{S}_N, \tag{1.1}$$

$$\sum_{a \in \mathcal{S}_N} \sum_{i \in \mathcal{S}_N} (A_a)_{ij_1} = \sum_{a \in \mathcal{S}_N} \sum_{j \in \mathcal{S}_N} (A_a)_{i_1 j} = N, \text{ for all } i_1, j_1 \in \mathcal{S}_N, \tag{1.2}$$

$$\sum_{a \in \mathcal{S}_N} \sum_{i \in \mathcal{S}_N} \sum_{j \in \mathcal{S}_N} (A_a)_{ij} = N^2, \tag{1.3}$$

$$F = \sum_{a \in \mathcal{S}_N} a \cdot A_a, \tag{1.4}$$

$$(A_{f(i,j)})_{ij} = 1, \text{ for all } i,j \in \mathcal{S}_N. \tag{1.5}$$

These formula are just translations of properties of f. For example, 1.1 states that for all $(i,j) \in \mathcal{S}_N^2$, there exists exactly one $a \in \mathcal{S}_N$ such that $f(i,j) = a$, and 1.3 states that card [domain of f] = card $[\mathcal{S}_N^2] = N^2$.

For $a_1, a_2, \ldots, a_n \in \mathcal{S}_N$ let $A_{a_1,a_2,\ldots,a_n} = A_{a_1} \cdot A_{a_2} \cdot \ldots \cdot A_{a_n}$. For any $n \geq 1$ let $\mathcal{A}_n = \{A_{a_1,a_2,\ldots,a_k} : a_i \in \mathcal{S}_N, 1 \leq i \leq k, k \leq n\}$, and define

$$\mathcal{A}_f = \bigcup_{n=1}^{\infty} \mathcal{A}_n.$$

We call $\mathcal{A}_f$ the *family of preimage matrices of f*.

Remark 1.14 *The family $\mathcal{A}_f$ is a matrix semigroup, with generators $\{A_a\}_{a \in \mathcal{S}_n}$.*

Define the matrix norm $|\cdot|$ for any non-negative matrix A by

$$\begin{aligned} |A| &= \text{sum of entries of } A \\ &= \sum_{j=0}^{N-1}\sum_{i=0}^{N-1}(A)_{ij} \end{aligned}$$

This norm has some nice properties, in particular if A, B are non-negative matrices and if $c \geq 0$ [illegible] real number then $|A+B| = |A| + |B|$, and $|c \cdot A| = c \cdot |A|$. The matrix norm $|\cdot|$ has a special relationship with the family of preimage matrices:

Lemma 1.15 *Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$, and let $(a_1, \ldots, a_n) \in \mathcal{S}_N^n$ for some $n \geq 1$. Then*

$$(A_{a_1,a_2,\ldots,a_n})_{ij} = card\Big\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\Big[\{i\}\times\mathcal{S}_N^{n-1}\times\{j\}\Big]\Big\},$$

for all $i, j \in \mathcal{S}_N$.

Proof. (by induction on n). For $n = 1$, we have

$$\begin{aligned} \text{card}\Big\{f^{-1}(a_1)\bigcap\{(i,j)\}\Big\} &= \begin{cases} \text{card}\{(i,j)\} = 1, \text{ if } f(i,j) = a_1, \\ \text{card } \emptyset = 0, \text{ if } f(i,j) \neq a_1. \end{cases} \\ &= (A_{a_1})_{ij}. \end{aligned}$$

Suppose the property holds for some $n \geq 1$ and consider $(A_{a_1,a_2,\ldots,a_n,a_{n+1}})_{ij}$ for some $a_1, a_2, \ldots, a_n, a_{n+1} \in \mathcal{S}_N$. Then,

$$\begin{aligned} &(A_{a_1,a_2,\ldots,a_n,a_{n+1}})_{ij} = (A_{a_1,a_2,\ldots,a_n}\cdot A_{a_{n+1}})_{ij} \\ &= \sum_{k=0}^{N-1}(A_{a_1,a_2,\ldots,a_n})_{ik}\cdot(A_{a_{n+1}})_{kj} \\ &= \sum_{k=0}^{N-1}\text{card}\Big\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\Big[\{i\}\times\mathcal{S}_N^{n-1}\times\{k\}\Big]\Big\}\cdot\text{card}\Big\{f^{-1}(a_{n+1})\bigcap\{(k,j)\}\Big\}. \end{aligned}$$

We have, for any $i, j, k \in \mathcal{S}_N$,

$$\begin{aligned} &\text{card}\Big\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\Big[\{i\}\times\mathcal{S}_N^{n-1}\times\{k\}\Big]\Big\}\cdot\text{card}\Big\{f^{-1}(a_{n+1})\bigcap\{(k,j)\}\Big\} \\ &= \begin{cases} \text{card}\Big\{f_n^{-1}(a_1,\ldots,a_n)\cap\Big[\{i\}\times\mathcal{S}_N^{n-1}\times\{k\}\Big]\Big\}, \text{ if } f(k,j) = a_{n+1}, \\ 0, \text{ if } f(k,j) \neq a_{n+1}. \end{cases} \\ &= \text{card}\Big\{f_{n+1}^{-1}(a_1,\ldots,a_n,a_{n+1})\bigcap\Big[\{i\}\times\mathcal{S}_N^{n-1}\times\{k\}\times\{j\}\Big]\Big\}. \end{aligned}$$

Now, noticing that $\left\{\{i\} \times \mathcal{S}_N^{n-1} \times \{k\} \times \{j\}\right\}_{k\in\mathcal{S}_N}$ is a partition of $\left\{\{i\} \times \mathcal{S}_N^{n-1} \times \mathcal{S}_N \times \{j\}\right\} = \{\{i\} \times \mathcal{S}_N^{n} \times \{j\}\}$, we have

$$\begin{aligned}
&(A_{a_1,a_2,\ldots,a_n,a_{n+1}})_{ij} \\
&= \sum_{k=0}^{N-1} \text{card}\left\{f_{n+1}^{-1}(a_1,\ldots,a_n,a_{n+1})\bigcap\left[\{i\} \times \mathcal{S}_N^{n-1} \times \{k\} \times \{j\}\right]\right\} \\
&= \text{card} \bigcup_{k=0}^{N-1} \left\{f_{n+1}^{-1}(a_1,\ldots,a_n,a_{n+1})\bigcap\left[\{i\} \times \mathcal{S}_N^{n-1} \times \{k\} \times \{j\}\right]\right\} \\
&= \text{card} \left\{f_{n+1}^{-1}(a_1,\ldots,a_n,a_{n+1})\bigcap[\{i\} \times \mathcal{S}_N^{n} \times \{j\}]\right\}. \square
\end{aligned}$$

Theorem 1.16 *Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$, and let $(a_1,\ldots,a_n) \in \mathcal{S}_N^n$ for some $n \geq 1$. Then*

$$|A_{a_1,\ldots,a_n}| = card\{f_n^{-1}(a_1,\ldots,a_n)\}$$

Proof. We use Lemma 1.15 and the fact that $\left\{\{i\} \times \mathcal{S}_N^{n-1} \times \{j\}\right\}_{i,j\in\mathcal{S}_N}$ is a partition of $\mathcal{S}_N^{n+1}$:

$$\begin{aligned}
|A_{a_1,\ldots,a_n}| &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(A_{a_1,\ldots,a_n})_{ij} \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \text{card}\left\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\left[\{i\} \times \mathcal{S}_N^{n-1} \times \{j\}\right]\right\} \\
&= \text{card} \bigcup_{i=0}^{N-1}\bigcup_{j=0}^{N-1} \left\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\left[\{i\} \times \mathcal{S}_N^{n-1} \times \{j\}\right]\right\} \\
&= \text{card}\left\{f_n^{-1}(a_1,\ldots,a_n)\bigcap\mathcal{S}_N^{n+1}\right\} = \text{card}\left\{f_n^{-1}(a_1,\ldots,a_n)\right\}. \square
\end{aligned}$$

We translate theorem 1.2 into the language of preimage matrices:

Theorem 1.17 *Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$. The following are equivalent:*

1) f_∞ is onto.

2) $0 \notin \mathcal{A}_f$.

3) $|A| = N$ for all $A \in \mathcal{A}_f$.

Proof.

This follows easily from Theorem 1.2 and Theorem 1.16.□

Theorem 1.18 *Let $f \in \mathcal{F}_{0,1}(S_N)$. f_∞ is onto if and only if $\mathcal{A}_f$ is a finite set.*

Proof. Ontoness implies $|A| = N$ for all $A \in \mathcal{A}_f$. Let $\mathcal{B}$ be the set of all nonnegative $N \times N$ matrices with integer entries which have norm N. This set is finite (if $B \in \mathcal{B}$ then $0 \leq (B)_{ij} \leq N$ for all $i, j \in S_N$, so card $\mathcal{B} \leq N^{N^2}$). $\mathcal{A}_f \subset \mathcal{B}$ so $\mathcal{A}_f$ is finite.

To prove the converse statement, suppose f_∞ is not onto. Lemma 5.8 of [Hed] states that if $f \in \mathcal{F}_{r,s}(S_N)$ and if f_∞ is not onto then for every $k, t \geq 1$ there exists $(a_1, \ldots, a_m) \in S_N^m$, for some $m \geq t$ such that card$f_m^{-1}(a_1, \ldots, a_m) > k$. Thus we have if $f \in \mathcal{F}_{0,1}(S_N)$ and if f_∞ is not onto then for every $k, t \geq 1$ there exists $(a_1, \ldots, a_m) \in S_N^m$, for some $m \geq t$ such that $|A_{a_1,\ldots,a_m}| > k$. Thus $\mathcal{A}_f$ must be infinite, as it contains matrices of arbitrarily large norm. □

Proposition 1.19 *Let $f \in \mathcal{F}_{0,1}(S_N)$. Let f_∞ be onto. Then:*

1) For any real $b_0, b_1, \ldots, b_{N-1} \geq 0$ let $B = \sum_{a=0}^{N-1} b_a \cdot A_a$. Then

$$|B^n| = \left(\sum_{a=0}^{N-1} b_a\right)^n \cdot N$$

for all $n \in \mathbb{Z}, n \geq 1$ (in particular, the image matrix of f, F, satisfies $|F^n| = \frac{N^{n+1}(N-1)^n}{2^n}$).

2) If λ is a characteristic root of some $A \in \mathcal{A}_f$ then λ is either zero or a root of unity.

Proof.

1) We recall that if A and B are nonnegative $N \times N$ matrices and if $c \geq 0$ is a real number, then $|A + B| = |A| + |B|$ and $|c \cdot A| = c \cdot |A|$. Now since f_∞ is onto, we have

$|A| = N$ for all $A \in \mathcal{A}_f$. The distributive law then gives us

$$\begin{aligned}
|B^n| &= \left|\left(\sum_{a=0}^{N-1} b_a \cdot A_a\right)^n\right| \\
&= \left|\sum_{a_1=0}^{N-1} \cdots \sum_{a_n=0}^{N-1} b_{a_1} \cdot \ldots \cdot b_{a_n} \cdot A_{a_1} \cdot \ldots \cdot A_{a_n}\right| \\
&= \left|\sum_{a_1=0}^{N-1} \cdots \sum_{a_n=0}^{N-1} b_{a_1} \cdot \ldots \cdot b_{a_n} \cdot A_{a_1,a_2,\ldots,a_n}\right| \\
&= \sum_{a_1=0}^{N-1} \cdots \sum_{a_n=0}^{N-1} b_{a_1} \cdot \ldots \cdot b_{a_n} \cdot |A_{a_1,a_2,\ldots,a_n}| \\
&= \sum_{a_1=0}^{N-1} \cdots \sum_{a_n=0}^{N-1} b_{a_1} \cdot \ldots \cdot b_{a_n} \cdot N \\
&= \left(\sum_{a=0}^{N-1} b_a\right)^n \cdot N.
\end{aligned}$$

Now from equation 1.4 and the fact that $\sum_{a=0}^{N-1} a = N(N-1)/2$, we have

$$|F^n| = |(\sum_{a=0}^{N-1} a \cdot A_a)^N| = \left(\sum_{a=0}^{N-1} a\right)^n \cdot N = \left(\frac{N(N-1)}{2}\right)^n \cdot N = \frac{N^{n+1}(N-1)^n}{2^n}$$

2) If f_∞ is onto, then $\mathcal{A}_f$ is a finite semigroup. Thus if $A \in \mathcal{A}_f$, then there must exist $k_1, k_2 \in \mathbb{Z}, k_2 > k_1 \geq 0$, such that $A^{k_1} = A^{k_2}$ (for if no such k_1, k_2 exist, then $\{A^n\}_{n=1}^{\infty}$ is an infinite set contained in finite $\mathcal{A}_f$). We have $A^{k_2} - A^{k_1} = 0$, so the minimal polynomial of A divides the polynomial $p(x) = x^{k_2} - x^{k_1} = x^{k_1}(x^{k_2-k_1} - 1)$, whose roots are either zero or roots of unity. Thus the eigenvalues of A are either zero or roots of unity.□

Remark 1.20 *Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$ with image matrix F. Although the above proposition tells us that when f_∞ is onto, $|F^n| = \frac{N^{n+1}(N-1)^n}{2^n}$, for all $n \in \mathbb{Z}, n \geq 1$, the converse statement is not generally true. A counter example is $f \in \mathcal{F}_{0,1}(\mathcal{S}_4)$ with* *image matrix*

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix},$$

which satisfies $|F^n| = \frac{4^{n+1}3^n}{2^n} = 4 \cdot 6^n$ *for all* $n \geq 1$, *yet* f_∞ *is not onto (notice that* $y \notin f_\infty(S_4^{\mathbb{Z}})$, *where* $y \in S_N^{\mathbb{Z}}$ *satisfies* $y_i = 1$ *for all* $i \in \mathbb{Z}$*). Although the equation is fallible, it is however useful as a quick test in deciding which automata are* not *onto (particularly when analysing a large number of automata).*

Proposition 1.21 *Let* $f \in \mathcal{F}_{0,1}(S_N)$. *Let* $A \in \mathcal{A}_f$. *If* $(A)_{ij} = k > 1$ *for any* $i, j \in S_N$ *then* f_∞ *is not onto.*

Proof. Suppose $(A)_{ij} = k > 1$. Let $B = A \cdot A_{f(j,i)} \in \mathcal{A}_f$. Then equation 1.5 gives us

$$\begin{aligned}(B)_{ii} = \sum_{k=0}^{N-1} (A)_{ik} \cdot (A_{f(j,i)})_{kj} &= (A)_{ij} \cdot (A_{f(j,i)})_{ji} + \sum_{k \neq j} (A)_{ik} \cdot (A_{f(j,i)})_{kj} \\ &= k + \sum_{k \neq j} (A)_{ik} \cdot (A_{f(j,i)})_{kj} \geq k.\end{aligned}$$

Now, $B^n \in \mathcal{A}_f$ for all n, and $(B^n)_{ii} \geq k^n$ for all n (since $(B)_{ii} \geq k$ and B is a nonnegative matrix), so we must have $|B^n| \geq k^n$ for all n. We must then have $|B^n| > N$ for some n, and so by Theorem 1.17, f_∞ is not onto.□

Remark 1.22 *Proposition 1.21 implies that if* f_∞ *is onto, then* $(A)_{ij} \in \{0, 1\}$ *for all* $A \in \mathcal{A}_f$, $i, j \in S_N$.

Proposition 1.23 *Let* $f \in \mathcal{F}_{0,1}(S_N)$. *Then*

1) f *is left permutive if and only if* A_a *has exactly one 1 in every column (all other entries are 0), for all* $a \in S_N$, *if and only if each column of* F *contains every member of* S_N.

2) f *is right permutive if and only if* A_a *has exactly one 1 in every row (all other*

entries are 0), for all $a \in \mathcal{S}_N$, if and only if each row of F contains every member of $\mathcal{S}_N$.

3) f is bipermutive if and only if A_a is invertible for all $a \in \mathcal{S}_N$; if and only if each row and each column of F contains every member of $\mathcal{S}_N$

Proof.

1) f is left permutive $\iff$ $f(\cdot, j)$ is a permutation for all $j \in \mathcal{S}_N$ $\iff$ $f(i,j) = a$ has exactly one solution $i \in \mathcal{S}_N$ for all $a, j \in \mathcal{S}_N$ $\iff$ $(A_a)_{ij} = 1$ for exactly one $i \in \mathcal{S}_N$ for all $a, j \in \mathcal{S}_N$ $\iff$ A_a has exactly one 1 in every column for all $a \in \mathcal{S}_N$.

The statement about F follows from equation 1.4.

2) Similar to proof of (1)

3) If f is bipermutive it is left permutive and right permutive so (1) and (2) imply that each A_a is a permutation matrix, hence invertible for all $a \in \mathcal{S}_N$.

If A_a is invertible for each $a \in \mathcal{S}_N$, then A_a has no zero rows and no zero columns. Thus each A_a has at least N nonzero entries. Equation 1.3 ensures that each A_a has exactly N nonzero entries which are equal to 1. Since A_a has no zero rows or columns, A_a must have exactly one 1 in every row and every column. (1) and (2) then imply that f is bipermutive.

Again, the statement about F follows from equation 1.4.□

The next theorem connects the surjectivity question to number theory.

Theorem 1.24 ***Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_p)$, where p is a prime number. Then f_∞ is onto if and only if f_∞ is permutive.***

Proof. This is a simple mixing of theorems from [Hed] which unfortunately requires a number of tools. Although it is impractical to cover all these tools in much detail, we shall use what we need and refer to the appropriate Theorems in [Hed].

$x \in \mathcal{S}_N^{\mathbb{Z}}$ is said to be a *bilaterally transitive point (of σ)* if for every $n \geq 1$ and for every $(b_1, b_2, \ldots, b_n) \in \mathcal{S}_N^n$ there exists $i_1, i_2 \in \mathbb{Z}$ with $i_1 < 0 < i_2$ such that

$x_{i_1} = b_1 = x_{i_2-n+1}, x_{i_1+1} = b_2 = x_{i_2-n+2}, \ldots, x_{i_1+n-1} = b_n = x_{i_2}$ This means that every $y \in \mathcal{S}_N^{\mathbf{Z}}$ is a limit point of both the forward orbit of x under σ and the backward orbit of x under σ, i.e.,

$$y \in \overline{\{\sigma^k(x)\}_{k=0}^{k=\infty}} \text{ and } y \in \overline{\{\sigma^k(x)\}_{k=-\infty}^{k=0}},$$

where the overline denotes closure.

Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ with f_∞ onto. From Theorems 11.1 and 11.2 of [Hed], there exists a positive integer $M(f)$ such that $\mathrm{card} f_\infty^{-1}(x) \geq M(f)$ for all $x \in \mathcal{S}_N^{\mathbf{Z}}$ and such that $\mathrm{card} f_\infty^{-1}(x) = M(f)$ if x is bilaterally transitive.

Let $A \in \mathcal{S}_N^n$ for some $n \geq s - r$, and let $\mathcal{B} \subset \mathcal{S}_N^m$ for some $m \geq 1$. let $q = n + m - (s - r)$. The set $\mathcal{B}$ is said to be right [left] compatible with A and f if $f_q(AB) = f_q(AC)$ $[f_q(BA) = f_q(CA)]$ for all $B, C \in \mathcal{B}$. Let $RC(A, f)_m = \{\mathcal{B} \subset \mathcal{S}_N^m :$ $\mathcal{B}$ is right compatible with A and$f\}$. Define

$$R(A, f) = \sup_{m \geq 1}\{\max_{\mathcal{B} \in RC(A,f)_m}\{\mathrm{card}\mathcal{B}\}\}$$

Remark 14.3 of [Hed] states that $R(A, f) = R(C, f)$ for all $A \in \mathcal{S}_N^{n_1}, C \in \mathcal{S}_N^{n_2}$, with $n_1, n_2 \geq s - r$. Let $R(f)$ be this common value. $L(f)$ is defined analogously.

Theorem 14.9, pg. 362 of [Hed] states that $L(F) \cdot M(f) \cdot R(f) = N^{s-r}$.

Lemma 17.1,page 370 of [Hed] states that f is left [right] permutive if and only if $L(f) = 1$ $[R(f) = 1]$.

Now, let $f \in \mathcal{F}_{0,1}(\mathcal{S}_p)$, where p is a prime number. If f_∞ is onto, we have $L(f) \cdot M(f) \cdot R(f) = p^{1-0} = p$. If the product of three positive integers is a prime p then two of those integers are 1 and the other is p. Thus at least one of $L(f)$ and $R(f)$ is one, hence f must be permutive.□

We will conclude this section by developing a generalisation of the concept of permutivity.

Let $\mathcal{P}$ be a partition of $\mathcal{S}_N$ into equal parts, i.e., there exists $n, m \in \mathbb{Z}$, with $n, m > 0$ and $n \cdot m = N$, such that card $\mathcal{P} = n$, and card $P = m$ for all $P \in \mathcal{P}$. We call $Q \subset \mathcal{S}_N$ a *slice* of $\mathcal{P}$ if Q contains exactly one member from each $P \in \mathcal{P}$. Let $\mathcal{Q}_\mathcal{P}$ be the collection of all slices of $\mathcal{P}$. We call $\mathcal{Q}_\mathcal{P}$ the *cross section* of $\mathcal{P}$.

Remark 1.25 *Let $\mathcal{P}$ and $\mathcal{Q}_\mathcal{P}$ be as above. Then:*
1) If $Q \in \mathcal{Q}_\mathcal{P}$ and $y, z \in Q$ then we must have either $y = z$ or $y \in P_1, z \in P_2$, for some $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$. Conversely, if $y \in P_1, z \in P_2$ with $P_1 \neq P_2$, then there exists $Q \in \mathcal{Q}_\mathcal{P}$ such that $y, z \in Q$.
2) We can write $\mathcal{P} = \{P_1, \ldots, P_n\}$, where $P_i = \{p_{i1}, p_{i2}, \ldots, p_{im}\}$, for $i = 1, \ldots, n$. If we let $Q_{j_1,\ldots,j_n} = \{p_{1j_1}, p_{2j_2}, \ldots, p_{nj_n}\}$, for $1 \leq j_1, \ldots, j_n \leq m$ then $\mathcal{Q}_\mathcal{P} = \{Q_{j_1,\ldots,j_n} : 1 \leq j_1, \ldots, j_n \leq m\}$ (notice that $Q_{j_1,\ldots,j_n} \cap P_i = p_{ij_i}$) and card $\mathcal{Q}_\mathcal{P} = m^n$).

Definition 1.26 *Let $\mathcal{P}$ be a partition of $\mathcal{S}_N$ into equal parts and let $\mathcal{Q}_\mathcal{P}$ be the cross section of $\mathcal{P}$. Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$. f is* left $\mathcal{P}$-permutive *if $f|_{P \times Q} : P \times Q \longrightarrow \mathcal{S}_N$ is a bijection for all $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$.*

f is right $\mathcal{P}$-permutive *if $f|_{Q \times P} : Q \times P \longrightarrow \mathcal{S}_N$ is a bijection for all $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$.*

If f is left (right) $\mathcal{P}$-permutive for some partition $\mathcal{P}$, we say that f is left (right) partition permutive .

Example 1.27 *Let $\mathcal{P} = \{\{0,2\},\{1,3\}\}$ be a partition of $\mathcal{S}_4$ into equal parts. The cross section of $\mathcal{P}$ is then*

$$\mathcal{Q}_\mathcal{P} = \{\{0,1\},\{0,3\},\{2,1\},\{2,3\}\}.$$

If we take $f \in \mathcal{F}_{0,1}(\mathcal{S}_4)$ from example 1.12, we see that f is right $\mathcal{P}$-permutive since

$$\begin{aligned} & f(\{0,1\} \times \{0,2\}) = f(\{0,3\} \times \{0,2\}) = f(\{2,1\} \times \{0,2\}) \\ = & f(\{2,3\} \times \{0,2\}) = f(\{0,1\} \times \{1,3\}) = f(\{0,3\} \times \{1,3\}) \\ = & f(\{2,1\} \times \{1,3\}) = f(\{2,3\} \times \{1,3\}) = \{0,1,2,3\} \end{aligned}$$

Remark 1.28 *Let f be left $\mathcal{P}$-permutive. Then we have the following:*
1) Let $a \in \mathcal{S}_N$ and let $y \in P_1$, $z \in P_2$, with $P_1, P_2 \in \mathcal{P}$, $P_1 \neq P_2$. Then $f(a,y) \neq f(a,z)$.
2) Let $a \in \mathcal{S}_N$ and let $y, z \in P$, for some $P \in \mathcal{P}$. Then $f(y,a) = f(z,a)$ if and only if $y = z$.

If $\mathcal{P} = \{\mathcal{S}_N\}$ then $\mathcal{Q}_{\mathcal{P}} = \{\{0\},\{1\},\ldots,\{N-1\}\}$. In this case, if f is left $\mathcal{P}$-permutive, then $f|_{\mathcal{S}_N \times \{a\}}$ is a bijection for all $a \in \mathcal{S}_N$, i.e., f is left permutive.

If $\mathcal{P} = \{\{0\},\{1\},\ldots,\{N-1\}\}$ then $\mathcal{Q}_{\mathcal{P}} = \{\mathcal{S}_N\}$. Right $\mathcal{P}$-permutivity in this case also implies left permutivity.

Analogous formulations of right permutivity exist.

Thus left and right permutivity are special sub cases of left and right partition permutivity. Permutive automata are onto, but what can we say about partition permutive automata?

Theorem 1.29 *Let f be partition permutive. Then f_∞ is onto.*

Before we prove this result we need some definitions and lemmas.

Definition 1.30 *Let A be an $N \times N$ nonnegative, integer matrix (with indices in $\mathcal{S}_N$, as usual). Let $\mathcal{P}$ be a partition of $\mathcal{S}_N$ into equal parts with cross section $\mathcal{Q}_{\mathcal{P}}$. A is* left $\mathcal{P}$-compatible *if for every $P \in \mathcal{P}$, $Q \in \mathcal{Q}_{\mathcal{P}}$ we have*

$$\sum_{p \in P} \sum_{q \in Q} (A)_{p,q} = 1.$$

Note that $\sum_{p\in P}\sum_{q\in Q}(A)_{p,q} = 1$ if and only if $(A)_{p,q} = 1$ for exactly one couple $(p,q) = (\acute{p},\acute{q}) \in P \times Q$, and $(A)_{p,q} = 0$ if $(p,q) \in P \times Q \setminus \{(\acute{p},\acute{q})\}$ (since $(A)_{p,q} \in \mathbb{Z}$ and $(A)_{p,q} \geq 0$ for all p,q).

Lemma 1.31 *f is left $\mathcal{P}$-permutive if and only if A_a is left $\mathcal{P}$-compatible for all $a \in \mathcal{S}_N$.*

Proof. Let $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$ and suppose $f|_{P\times Q}$ is a bijection. This is equivalent to saying that for every $a \in \mathcal{S}_N$, the equation $f(p,q) = a$ has a unique solution $(\acute{p},\acute{q})$ in $P \times Q$. This will occur if and only if $(A_a)_{p,q} = 1$ for exactly one couple $(\acute{p},\acute{q})$ in $P \times Q$, if and only if $\sum_{p\in P}\sum_{q\in Q}(A_a)_{p,q} = 1$. Thus $f|_{P\times Q}$ is a bijection for all $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$ if and only if $\sum_{p\in P}\sum_{q\in Q}(A_a)_{p,q} = 1$ for all $a \in \mathcal{S}_N$, $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$. □

Lemma 1.32 *Let A be an $N \times N$ left $\mathcal{P}$-compatible matrix, and let $P_1, P_2 \in \mathcal{P}$. Let $p_1, p_2 \in P_2$. Then*

$$\sum_{p\in P_1}(A)_{p,p_1} = \sum_{p\in P_1}(A)_{p,p_2}.$$

Proof. Consider $Q_1 \in \mathcal{Q}_\mathcal{P}$ with $p_1 \in Q_1$. Then the set $Q_2 = (Q_1 \setminus \{p_1\}) \cup \{p_2\}$ is in $\mathcal{Q}_\mathcal{P}$ also, since it contains exactly one member of each $P \in \mathcal{P}$. Le $Q^* = Q_1 \setminus \{p_1\} = Q_2 \setminus \{p_2\}$. Left $\mathcal{P}$-compatibility of A gives $\sum_{p\in P_1}\sum_{q\in Q_1}(A)_{p,q} = \sum_{p\in P_1}\sum_{q\in Q_2}(A)_{p,q} = 1$. Thus we have

$$\begin{aligned}
0 &= 1-1 = \sum_{p\in P_1}\sum_{q\in Q_1}(A)_{p,q} - \sum_{p\in P_1}\sum_{q\in Q_2}(A)_{p,q} \\
&= \sum_{p\in P_1}(A)_{p,p_1} + \sum_{p\in P_1}\sum_{q\in Q_1\setminus\{p_1\}}(A)_{p,q} \\
&- \sum_{p\in P_1}(A)_{p,p_2} - \sum_{p\in P_1}\sum_{q\in Q_2\setminus\{p_2\}}(A)_{p,q} \\
&= \sum_{p\in P_1}(A)_{p,p_1} + \sum_{p\in P_1}\sum_{q\in Q^*}(A)_{p,q} \\
&- \sum_{p\in P_1}(A)_{p,p_2} - \sum_{p\in P_1}\sum_{q\in Q^*}(A)_{p,q} \\
&= \sum_{p\in P_1}(A)_{p,p_1} - \sum_{p\in P_1}(A)_{p,p_2}.
\end{aligned}$$

This implies the lemma.□

Remark 1.33 *Lemma 1.32 and lemma 1.31 imply that if f is left $\mathcal{P}$-permutive and if $P_1, P_2 \in \mathcal{P}$, then $f(P_1 \times \{p_1\}) = f(P_1 \times \{p_2\})$ for all $p_1, p_2 \in P_2$.*

Lemma 1.34 *Let A, B be $N \times N$ matrices which are left $\mathcal{P}$-compatible. Then $A \cdot B$ is left $\mathcal{P}$-compatible.*

Proof. Let A, B be left $\mathcal{P}$-compatible and let $P \in \mathcal{P}$, $Q \in \mathcal{Q}_{\mathcal{P}}$. Then

$$\begin{aligned}\sum_{p\in P}\sum_{q\in Q}(A\cdot B)_{p,q} &= \sum_{p\in P}\sum_{q\in Q}(\sum_{k=0}^{N-1}(A)_{p,k}\cdot(B)_{k,q}) \\ &= \sum_{k=0}^{N-1}(\sum_{p\in P}(A)_{p,k})\cdot(\sum_{q\in Q}(B)_{k,q})\end{aligned}$$

Now we can write $\mathcal{P} = \{P_1, \ldots, P_n\}$, where $P_i = \{p_{i1}, p_{i2}, \ldots, p_{im}\}$, for $i = 1, \ldots, n$, as in remark 1.25 (2), so that $\mathcal{S}_N = \{k : k \in \mathcal{S}_N\} = \{p_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$, and we have

$$\sum_{p\in P}\sum_{q\in Q}(A\cdot B)_{p,q} = \sum_{i=1}^{n}\sum_{j=1}^{m}(\sum_{p\in P}(A)_{p,p_{ij}})\cdot(\sum_{q\in Q}(B)_{p_{ij},q})$$

For each i fix $p_i^* \in P_i$. From lemma 1.32 we have $\sum_{p\in P}(A)_{p,p_{ij}} = \sum_{p\in P}(A)_{p,p_i^*}$, for each i, j, thus

$$\begin{aligned}\sum_{p\in P}\sum_{q\in Q}(A\cdot B)_{p,q} &= \sum_{i=1}^{n}\sum_{j=1}^{m}(\sum_{p\in P}(A)_{p,p_i^*})\cdot(\sum_{q\in Q}(B)_{p_{ij},q}) \\ &= \sum_{i=1}^{n}(\sum_{p\in P}(A)_{p,p_i^*})\cdot\underbrace{(\sum_{j=1}^{m}\sum_{q\in Q}(B)_{p_{ij},q})}_{1} \\ &= \sum_{i=1}\sum_{p\in P}(A)_{p,p_i^*} = 1,\end{aligned}$$

since A and B are left $\mathcal{P}$-compatible, $P_i = \{p_{ij}\}_{j=1}^{m}$, and $\{p_i^*\}_{i=1}^{n} \in \mathcal{Q}_{\mathcal{P}}$. Thus $A \cdot B$ is left $\mathcal{P}$-compatible.□

We are finally ready to tackle Theorem 1.29:

Proof (of theorem 1.29). Let f be left $\mathcal{P}$-permutive. Theorem 1.31 states that $A_a \in A_f$ is left $\mathcal{P}$-compatible for all $a \in \mathcal{S}_N$. The set $\{A_a\}_{a \in \mathcal{S}_N}$ generates A_f, and products of left $\mathcal{P}$-compatible matrices are left $\mathcal{P}$-compatible, thus every member of A_f is left $\mathcal{P}$-compatible. Now let A be an arbitrary $\mathcal{P}$-compatible matrix. If we write $\mathcal{P} = \{P_1, \ldots, P_n\}$, $\mathcal{Q}_\mathcal{P} = \{Q_{j_1,\ldots,j_n} : 1 \leq j_1, \ldots, j_n \leq m\}$, as in remark 1.25(2) then the set $\{Q_{\underbrace{j,\ldots,j}_{\text{n times}}}\}_{j=1}^{m}$ is a partition of $\mathcal{S}_N$. Since A is left $\mathcal{P}$-permutive, we have $\sum_{p \in P_i} \sum_{q \in Q_{j,\ldots,j}} (A)_{p,q} = 1$, for all $1 \leq i \leq n, 1 \leq j \leq m$. We then have

$$
\begin{aligned}
|A| &= \sum_{k \in \mathcal{S}_N} \sum_{l \in \mathcal{S}_N} (A)_{kl} \\
&= \sum_{i=1}^{n} \sum_{p \in P_i} \sum_{j=1}^{m} \sum_{q \in Q_{j,\ldots,j}} (A)_{p,q} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{\left(\sum_{p \in P_i} \sum_{q \in Q_{j,\ldots,j}} (A)_{p,q} \right)}_{1} \\
&= n \cdot m = N
\end{aligned}
$$

Thus we have $|A| = N$ for all $A \in \mathcal{A}_f$, so f_∞ is onto.□

The following proposition describes the block maps of partition permutive automata.

Corollary 1.35 *If $f \in \mathcal{F}_{0,1}(\mathcal{S}_N)$ is left $\mathcal{P}$-permutive then for every $n \geq 1$, $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$*

$$f_n|_{P \times \mathcal{S}_N^{n-1} \times Q} : P \times \mathcal{S}_N^{n-1} \times Q \longrightarrow \mathcal{S}_N^n$$

is a bijection.

Proof. Let $P \in \mathcal{P}$, $Q \in \mathcal{Q}_\mathcal{P}$. For each $(a_1, \ldots, a_n) \in \mathcal{S}_N^n$, $A_{a_1,\ldots,a_n}$ is a $\mathcal{P}$-compatible matrix, so $\sum_{p \in P} \sum_{q \in Q} (A_{a_1,\ldots,a_n})_{p,q} = 1$. From lemma 1.15 we have

$$\begin{aligned}
&\operatorname{card}\{f_n^{-1}(a_1,\ldots,a_n)\cap P\times \mathcal{S}_N^{n-1}\times Q\}\\
&= \sum_{p\in P}\sum_{q\in Q}\operatorname{card}\{f_n^{-1}(a_1,\ldots,a_n)\cap\{p\}\times \mathcal{S}_N^{n-1}\times\{q\}\}\\
&= \sum_{p\in P}\sum_{q\in Q}(A_{a_1,\ldots,a_n})_{p,q} = 1.
\end{aligned}$$

Thus every $(a_1,\ldots,a_n)\in\mathcal{S}_N^n$ has exactly one preimage in $P\times\mathcal{S}_N^{n-1}\times Q$ under f_n, hence $f_n|_{P\times\mathcal{S}_N^{n-1}\times Q}$ is a bijection.□

We will generalise the concept of partition permutivity to multivariate local functions in the final section of this chapter.

1.5 Partition Permutive Automata

Definition 1.36 *Let $f\in\mathcal{F}_{r,s}(\mathcal{S}_N)$ with $s>r$ and let $\tilde{f}\in\mathcal{F}_{0,1}(\mathcal{S}_{N^{s-r}})$ be the tuba map of f. We say that f is* left [right] partition permutive *if and only if $\tilde{f}$ is left [right] partition permutive.*

The following proposition should be clear, given the definition of partition permutivity for bivariate local functions, and the derivation of the tuba map of a local function.

Proposition 1.37 *Let $f\in\mathcal{F}_{r,s}(\mathcal{S}_N)$. f is left partition permutive if and only if there exists a partition $\mathcal{P}$ of $\mathcal{S}_N^{s-r}$ into equal parts with associated cross section $\mathcal{Q}_{\mathcal{P}}$ such that*

$$f_{s-r}|_{P\times Q}: P\times Q\longrightarrow \mathcal{S}_N^{s-r}$$

is a bijection for all $P\in\mathcal{P}$, $Q\in\mathcal{Q}_{\mathcal{P}}$.

Let $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$. Remark 1.5 states that if f and g are left [right] permutive, then fg is left [right] permutive. What can be said about fg when f is right permutive and g is left permutive? (or if f is left permutive and g is right permutive?) We will show that such maps are partition permutive. To simplify the analysis, we may assume that $r_1 = r_2 = 0$ (since $\sigma^{-r_1} f \in \mathcal{F}_{0,s_1-r_1}(\mathcal{S}_N)$ is right permutive if and only if f is right permutive, $\sigma^{-r_2} g \in \mathcal{F}_{0,s_2-r_2}(\mathcal{S}_N)$ is left permutive if and only if g is left permutive, and $\sigma^{-r_1} f \sigma^{-r_2} g = \sigma^{-r_1-r_2} fg$ is partition permutive if and only if fg is partition permutive).

Proposition 1.38 *Let $f \in \mathcal{F}_{0,s_1}(\mathcal{S}_N)$ be right permutive and let $g \in \mathcal{F}_{0,s_2}(\mathcal{S}_N)$ be left permutive. Then $fg \in \mathcal{F}_{0,s_1+s_2}(\mathcal{S}_N)$ is left $\mathcal{P}$-permutive, where*

$$\mathcal{P} = \{P_{a_0,\ldots,a_{s_1-1}}\}_{a_0,\ldots,a_{s_1-1} \in \mathcal{S}_N}$$

is a partition of $\mathcal{S}_N^{s_1+s_2}$ with

$$\begin{aligned} P_{a_0,\ldots,a_{s_1-1}} &= \{(x_0,\ldots,x_{s_1+s_2-1}) \in \mathcal{S}_N^{s_1+s_2} : g_{s_1}(x_0,\ldots,x_{s_1+s_2+1}) = (a_0,\ldots,a_{s_1-1})\} \\ &= g_{s_1}^{-1}(a_0,\ldots,a_{s_1-1}), \end{aligned}$$

for all $a_0,\ldots,a_{s_1-1} \in \mathcal{S}_N$.

Proof. Let $P_{a_0,\ldots,a_{s_1-1}} \in \mathcal{P}$ for some $a_0,\ldots,a_{s_1-1} \in \mathcal{S}_N$ and let $Q \in \mathcal{Q}_{\mathcal{P}}$. We must show that

$$(fg)_{s_1+s_2}|_{P_{a_0,\ldots,a_{s_1-1}} \times Q} : P_{a_0,\ldots,a_{s_1-1}} \times Q \longrightarrow \mathcal{S}_N^{s_1+s_2}$$

is a bijection. By theorem 1.2 and since g_∞ is onto, we have $\text{card}P_{a_0,\ldots,a_{s_1-1}} = \text{card}g_{s_1}^{-1}(a_0,\ldots,a_{s_1-1}) = N^{s_2}$, and $\text{card}Q = \text{card}\mathcal{P} = N^{s_1}$ so $\text{card}(P_{a_0,\ldots,a_{s_1-1}} \times Q) = N^{s_1} \cdot N^{s_2} = \text{card}\mathcal{S}_N^{s_1+s_2}$, so we need only show that $(fg)_{s_1+s_2}|_{P_{a_0,\ldots,a_{s_1-1}} \times Q}$ is one-to-one.

Let $y = (y_0,\ldots,y_{2(s_1+s_2)-1})$, $z = (z_0,\ldots,z_{2(s_1+s_2)-1}) \in P_{a_0,\ldots,a_{s_1-1}} \times Q$, with $(fg)_{s_1+s_2}(y) = (fg)_{s_1+s_2}(z)$. We must show that $y = z$.

Now, since $(y_0, \ldots, y_{s_1+s_2-1}), (z_0, \ldots, z_{s_1+s_2-1}) \in P_{a_0,\ldots,a_{s_1-1}}$, we must have

$$g(y_i, \ldots, y_{i+s_2}) = g(z_0, \ldots, z_{i+s_2}) = a_i, \text{ for } i = 0, \ldots, s_1 - 1. \tag{1.6}$$

$(fg)_{s_1+s_2}(y) = (fg)_{s_1+s_2}(z)$ implies that

$$\begin{aligned} &f(g(y_i, \ldots, y_{i+s_2}), g(y_{i+1}, \ldots, y_{i+1+s_2}), \ldots, g(y_{i+s_1}, \ldots, y_{i+s_1+s_2})) \\ &= f(g(z_i, \ldots, z_{i+s_2}), g(z_{i+1}, \ldots, z_{i+1+s_2}), \ldots, g(z_{i+s_1}, \ldots, z_{i+s_1+s_2})), \end{aligned} \tag{1.7}$$

for $i = 0, \ldots, s_1 + s_2 - 1$. If we let $i = 0$ in equation 1.7 and substitute equation 1.6, we have

$$f(a_0, a_1, \ldots, a_{s_1-1}, g(y_{s_1}, \ldots, y_{s_1+s_2})) = f(a_0, a_1, \ldots, a_{s_1-1}, g(z_{s_1}, \ldots, z_{s_1+s_2})).$$

f is right permutive, so $f(a_0, \ldots, a_{s_1-1}, \cdot)$ is a permutation of $\mathcal{S}_N$, hence there must exist an $a_{s_1} \in \mathcal{S}_N$ such that

$$g(y_{s_1}, \ldots, y_{s_1+s_2}) = g(z_{s_1}, \ldots, z_{s_1+s_2}) = a_{s_1}$$

Now, if we let $i = 1$ in equation 1.7, we have

$$f(a_1, \ldots, a_{s_1}, g(y_{s_1+1}, \ldots, y_{s_1+s_2+1})) = f(a_1, \ldots, a_{s_1}, g(z_{s_1+1}, \ldots, z_{s_1+s_2+1})),$$

and again, the right permutivity of f ensures the existence of an $a_{s_1+1} \in \mathcal{S}_N$ such that $g(y_{s_1+1}, \ldots, y_{s_1+s_2+1}) = g(z_{s_1+1}, \ldots, z_{s_1+s_2+1}) = a_{s_1+1}$. Inductively, for $i = s_1, \ldots, 2s_1 + s_2 - 1$, we find $a_i \in \mathcal{S}_N$ such that

$$g(y_i, \ldots, y_{i+s_2}) = g(z_i, \ldots, z_{i+s_2}) = a_i. \tag{1.8}$$

In particular, we have

$$g_{s_1}(y_{s_1+s_2}, \ldots, y_{2(s_1+s_2)-1}) = g_{s_1}(z_{s_1+s_2}, \ldots, z_{2(s_1+s_2)+1}) = (a_{s_1+s_2}, \ldots, a_{2s_1+s_2-1}),$$

thus

$$(y_{s_1+s_2},\ldots,y_{2(s_1+s_2)-1}),(z_{s_1+s_2},\ldots,z_{2(s_1+s_2)+1}) \in P_{a_{s_1+s_2},\ldots,a_{2s_1+s_2-1}}. \tag{1.9}$$

Now, $y,z \in P_{a_0,\ldots,a_{s_1-1}} \times Q$ implies that

$$(y_{s_1+s_2},\ldots,y_{2(s_1+s_2)-1}),(z_{s_1+s_2},\ldots,z_{2(s_1+s_2)+1}) \in Q.$$

By remark 1.25 we must have either $(y_{s_1+s_2},\ldots,y_{2(s_1+s_2)-1}) = (z_{s_1+s_2},\ldots,z_{2(s_1+s_2)+1})$, or $(y_{s_1+s_2},\ldots,y_{2(s_1+s_2)-1}) \in P_1, (z_{s_1+s_2},\ldots,z_{2(s_1+s_2)+1}) \in P_2$, where $P_1,P_2 \in \mathcal{P}$ with $P_1 \neq P_2$. The latter case contradicts equation 1.9, so we must conclude that the first case holds. If we apply equation 1.8 with $i = s_1 + s_2 - 1$, we have

$$\begin{aligned} g(y_{s_1+s_2-1},y_{s_1+s_2},\ldots,y_{s_1+2s_2-1}) &= g(z_{s_1+s_2-1},z_{s_1+s_2},\ldots,z_{s_1+2s_2-1}) \\ &= g(z_{s_1+s_2-1},y_{s_1+s_2},\ldots,y_{s_1+2s_2-1}). \end{aligned}$$

Now, g is left permutive, so $g(\cdot,y_{s_1+s_2},\ldots,y_{s_1+2s_2-1})$ is a bijection, so we find that $y_{s_1+s_2-1} = z_{s_1+s_2-1}$. We similarly use the left permutivity of g to conclude that $y_i = z_i$ for $i = 0,\ldots,s_1+s_2-2$, hence $y = z$ and fg is left $\mathcal{P}$-permutive.□

Remark 1.39 *Note that in the above proposition, gf is right $\mathcal{P}$-permutive, where*

$$\mathcal{P} = \{P_{a_0,\ldots,a_{s_2-1}}\}_{a_0,\ldots,a_{s_2-1}\in S_N}$$

is a partition of $S_N^{s_1+s_2}$ with

$$\begin{aligned} P_{a_0,\ldots,a_{s_2-1}} &= \{(x_0,\ldots,x_{s_1+s_2-1}) \in S_N^{s_1+s_2} : f_{s_2}(x_0,\ldots,x_{s_1+s_2+1}) = (a_0,\ldots,a_{s_2-1})\} \\ &= f_{s_2}^{-1}(a_0,\ldots,a_{s_2-1}), \end{aligned}$$

for all $a_0,\ldots,a_{s_2-1} \in S_N$. A similar method to that above is used to prove this statement.

We will end this chapter by discussing some of the drawbacks and strengths of the concept of partition permutivity, and suggest some lines of further research.

The first drawback is that not all surjective automata are partition permutive. In fact if we consider $f, g, h \in \mathcal{F}_{0,1}(\mathcal{S}_3)$ with image matrices

$$F = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \; G = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \text{ and } H = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

respectively, then it can be shown using rather exhaustive methods that $fgh \in \mathcal{F}_{0,3}(\mathcal{S}_3)$ is not partition permutive. This means that theorem 1.38 cannot be generalised to include arbitrary compositions of permutive automata, as we have f is left permutive, g is right permutive, and h is left permutive.

Another problem is that for automata on large alphabets and for automata with local functions with many variables it is very difficult to verify whether or not the automaton is partition permutive: The "rather exhaustive methods" mentioned above involved analysing the 27×27 image matrix of the tuba map of fgh.

The main reason why partition permutivity is a worthwhile concept is this: All surjective automata with local functions in $\mathcal{F}_{0,1}(\mathcal{S}_4)$ are partition permutive. This statement was proven by a computer program which checked the surjectivity of every automaton with local function in $\mathcal{F}_{0,1}(\mathcal{S}_4)$, but it has not been proven with rigourous methods. We pose the following conjecture, which may perhaps be related to theorem 1.24:

Conjecture 1 *Let $f \in \mathcal{F}_{0,1}(\mathcal{S}_{p \cdot q})$, where p, q are prime numbers. Then f_∞ is onto if and only if f_∞ is partition permutive.*

A possible weakening of this conjecture is to demand that $p = q$. One possible approach to this problem may be to use the numbers $L(f)$, $M(f)$, and $R(f)$ from the proof of proposition 1.24, since for $f \in \mathcal{F}_{0,1}(\mathcal{S}_{p \cdot q})$ we have $L(f) \cdot M(f) \cdot R(f) = p \cdot q$, so at least one of $L(f)$, $M(f)$, or $R(f)$ must be equal to one (in the cases $L(f) = 1$ and $R(f) = 1$, we have f is permutive, hence partition permutive). At any rate, a proof or disproof to this conjecture is not obvious.

Chapter 2

Ergodic Theory

General Results

Now that we have discussed the structure of surjective automata in terms of local functions, we are more than ready to discuss the dynamical properties of surjective automata. We will see that another sufficient and necessary condition for surjectivity is the preservation of measure, and this brings us into the realm of ergodic theory. Many of the results of the next two sections are generalisations of results from [SR]. Their paper dealt with automata on two symbols, and hence such algebraic niceties as those discussed in examples 1.1 and 1.3 were exploited. In discussing these results in a more general setting, we must be a little more careful in our set up and analysis. Lets begin.

To describe the ergodic properties of cellular automata, it is of course first necessary to introduce a measure on $\mathcal{S}_N^{\mathbb{Z}}$.

We first introduce a probability measure p on $\mathcal{S}_N$, by demanding that for any $A \subset \mathcal{S}_N$, $p(A) = \frac{\text{card}A}{N}$. We now give $\mathcal{S}_N^{\mathbb{Z}}$ the probability measure μ, the product measure corresponding to the measure p on $\mathcal{S}_N$. If $i_1, \ldots, i_n \in \mathbb{Z}$ with $i_1 < i_2 < \ldots < i_n$, if

$a_1, \ldots, a_n \in \mathcal{S}_N$, and if A is the cylinder set defined by

$$A = \{x \in \mathcal{S}_N^{\mathbf{Z}} : x_{i_k} = a_k, \text{for } k = 1, \ldots, n\},$$

we then have $\mu[A] = (1/N)^n = N^{-n}$. μ acts on the smallest σ-algebra containing the cylinder sets, and is hence a Borel measure, as the cylinder sets are a basis for the topology of $\mathcal{S}_N^{\mathbf{Z}}$.

We will require some basic definitions from ergodic theory.

Definition 2.1 *Let $(\mathcal{X}, \beta, \mu)$ be a measure space, i.e., $\mathcal{X}$ is a nonempty set, β is a σ-algebra of subsets of $\mathcal{X}$, and μ is a normalised measure on $\mathcal{X}$ (i.e., $\mu[\mathcal{X}] = 1$. Let $\phi : \mathcal{X} \longrightarrow \mathcal{X}$ be a measureable function. Then*
1) ϕ is measure preserving *if $\mu[\phi^{-1}(B)] = \mu[B]$ for all $B \in \beta$.*
2) ϕ is ergodic *if ϕ is measure preserving and for every $B \in \beta$ satisfying $\phi^{-1}(B) = B$ we have either $\mu[B] = 0$ or $\mu[B] = 1$.*
3) ϕ is (strongly) mixing *if ϕ is measure preserving and for every $A, B \in \beta$ we have $\lim_{n\to\infty} \mu[A \cap \phi^{-n}(B)] = \mu[A] \cdot \mu[B]$.*

It is clear that mixing transformations are ergodic (for if $\phi^{-1}(B) = B$, taking $A = \mathcal{X} \setminus B$ in (3) above yields $(1 - \mu[B])\mu[B] = 0$).

Intuitively:

1) measure preserving systems are analogous to volume preserving flows from differential equations;

2) an ergodic system is one that cannot be decomposed into two systems on sets of statistically significant size;

3) a mixing transformation is one that exhibits a form of asymptotic statistical independence, in particular, if ϕ is mixing, we have

$$\lim_{n\to\infty} \frac{\mu[A \cap \phi^{-n}(B)]}{\mu[A]} = \mu[B],$$

which is a statement about conditional probabilities. Roughly, we have that the proportion of $\phi^{-n}(B)$ that is in A approaches the proportion of B in the whole space $\mathcal{X}$ (this almost suggests that $\phi^{-n}(B)$ becomes like a fractal, as n increases).

An equivalent formulation of ergodicity is ϕ is ergodic if and only if for all $A, B \in \beta$ we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu[A \cap \phi^{-k}(B)] = \mu[A] \cdot \mu[B]$.

In general, given two measure spaces $(\mathcal{X}_1, \beta_1, \mu_1)$, $(\mathcal{X}_2, \beta_2, \mu_2)$ and a measureable function $\phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, we say that ϕ is *measure preserving* if $\mu_1[\phi^{-1}(U)] = \mu_2[U]$, for all $U \in \beta_2$.

Consider $(\mathcal{X}_1, \beta_1, \mu_1)$, and $(\mathcal{X}_2, \beta_2, \mu_2)$ where $\mathcal{X}_1$, $\mathcal{X}_2$, are topological spaces, and β_1, β_2 are the σ-algebras of Borel sets on these spaces respectively. If $\phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ is

1) a homeomorphism with inverse $\phi^{-1} : \mathcal{X}_2 \longrightarrow \mathcal{X}_1$,

2) measure preserving,

and if

3) ϕ^{-1} is measure preserving,

then we call ϕ a *measure theoretical homeomorphism*.

If furthermore we have two continuous functions (dynamical systems) $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_1$ and $g : \mathcal{X}_2 \longrightarrow \mathcal{X}_2$ which satisfy $\phi \circ f = g \circ \phi$, then we say that f and g are *measure theoretically equivalent* or *measure theoretically conjugate*, and we call ϕ a *measure theoretical conjugacy* in this case. (Note: this terminology may conflict with the standard nomenclature which is used for more abstract settings - such as non-topological measure spaces - but the definitions above more closely resemble the scenarios we will face).

If the dynamical systems f and g are measure theoretically conjugate they are essentially the same system when viewed from topological and measure theoretical standpoints. This is partly because the homeomorphism ϕ preserves Borel sets, i.e., $U \in \beta_1$ if and only if $\phi(U) \in \beta_2$, and $V \in \beta_2$ if and only if $\phi^{-1}(V) \in \beta_1$. A consequence of this is that f is measure preserving if and only if g is measure preserving. Indeed, if f is measure preserving and $U \in \beta_2$, we have $V = \phi^{-1}(U) \in \beta_1$ so $\mu_2[g^{-1}(U)] = \mu_2[g^{-1}(\phi(V))] = \mu_2[(\phi^{-1} \circ g)^{-1}(V)] = \mu_2[(f \circ \phi^{-1})^{-1}(V)] = \mu_2[\phi(f^{-1}(V))] = \mu_1[f^{-1}(V)] = \mu_1[V] = \mu_2[U]$, since ϕ and ϕ^{-1} are measure preserving. We similarly have that f is ergodic [mixing] if and only if g is ergodic [mixing].

We will now apply some of these concepts to cellular automata.

Let $\mathcal{C}_N$ denote the set of cylinder sets on $S_N^{\mathbb{Z}}$. $\mathcal{C}_N$ has a very useful property: if $A, B \in \mathcal{C}_N$ then $A \setminus B = \cup_{i=1}^n C_i$ for some $C_1, \ldots, C_n \in \mathcal{C}_N$, all pairwise disjoint. This follows from the fact that $\mathcal{C}_N$ is closed under intersections, and from the fact that

$$S_N^{\mathbb{Z}} \setminus \{x \in S_N^{\mathbb{Z}} : x_{i_1} = a_1, \ldots, x_{i_n} = a_n\} = \bigcup_{\substack{(b_1,\ldots,b_n) \in S_N^n, \\ (b_1,\ldots,b_n) \neq (a_1,\ldots,a_n)}} \{x \in S_N^{\mathbb{Z}} : x_{i_1} = b_1, \ldots, x_{i_n} = b_n\},$$

for any $n \in \mathbb{Z}^+$, $a_1, \ldots, a_n \in S_N$, and $i_1, \ldots, i_n \in \mathbb{Z}$ with $i_1 < \ldots < i_n$. This next theorem illustrates the value of this property:

Theorem 2.2 *Let $(\mathcal{X}, \beta, \mu)$ be a measure space, and let $\phi : \mathcal{X} \longrightarrow \mathcal{X}$. Suppose $\mathcal{C} \subset \beta$ generates β (i.e., β is the smallest σ-algebra containing $\mathcal{C}$). Suppose that for all $A, B \in \mathcal{C}$ we have $A \setminus B$ is a finite union of pairwise disjoint sets in $\mathcal{C}$. Then*
1) if $\mu[\phi^{-1}(C)] = \mu[C]$ for all $C \in \mathcal{C}$, then ϕ is measure preserving.
2) if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu[A \cap \phi^{-k}(B)] = \mu[A] \cdot \mu[B]$ for all $A, B \in \mathcal{C}$, then ϕ is ergodic.
3) if $\lim_{n\to\infty} \mu[A \cap \phi^{-n}(B)] = \mu[A] \cdot \mu[B]$ for all $A, B \in \mathcal{C}$, then ϕ is mixing.

Remark 2.3 *If we give S_N an (additive) group structure which induces a group structure on $S_N^{\mathbb{Z}}$ via component wise addition, then μ is the (unique) Haar measure on $S_N^{\mathbb{Z}}$, i.e., for any measurable set $U \subset S_N^{\mathbb{Z}}$, and for any $a \in S_N^{\mathbb{Z}}$, we have $\mu[a + U] = \mu[U]$, where*

$$a + U = \{x \in S_N^{\mathbb{Z}} : x = a + u \text{ for some } u \in U\}.$$

The proof goes as follows: For any $a \in S_N^{\mathbf{Z}}$ *we let* $\phi_a : S_N^{\mathbf{Z}} \longrightarrow S_N^{\mathbf{Z}}$ *be defined by* $\phi_a(x) = x - a$, *for all* $x \in S_N^{\mathbf{Z}}$ *(and hence* $\phi_a^{-1}(x) = x + a$*). We then have for any cylinder set* $U = \{x \in S_N^{\mathbf{Z}} : x_{i_1} = c_1, \ldots, x_{i_n} = c_n\}$ *that* $\phi_a^{-1}(U) = \{x \in S_N^{\mathbf{Z}} : x_{i_1} = c_1 + a_{i_1}, \ldots, x_{i_n} = c_n + a_{i_n}\} = a + U$, *and hence* $\mu[\phi_a^{-1}(U)] = \mu[U]$ *for cylinders. Theorem 2.2 (1) then ensures that* $\mu[a + U] = \mu[U]$ *for all measurable sets* U.

We will now develop a useful way to describe the open sets of $S_N^{\mathbf{Z}}$.

Define $\mathcal{F}^2_{r,s}(S_N) \subset \mathcal{F}_{r,s}(S_N)$ to be the set of all local functions (with left and right indices r and s respectively) which have image set $\{0,1\}$, i.e.,

$$\mathcal{F}^2_{r,s}(S_N) = \{f : S_N^{s-r+1} \longrightarrow \{0,1\}\},$$

and let $\mathcal{F}^2(S_N) = \bigcup \mathcal{F}^2_{r,s}(S_N)$. If $p \in \mathcal{F}^2_{i,j}(S_N)$, we define $M(p) = \{x \in S_N^{\mathbf{Z}} : p(x_i, \ldots, x_j) = 1\}$. $M(p)$ is the finite union of cylinder sets, i.e,

$$M(p) = \bigcup_{(a_i,\ldots,a_j)\in p^{-1}(1)} \{x \in S_N^{\mathbf{Z}} : x_i = a_i, x_{i+1} = a_{i+1}, \ldots, x_j = a_j\}. \tag{2.1}$$

Proposition 2.4 *Let* $f \in \mathcal{F}_{r,s}(S_N)$, *and let* $p \in \mathcal{F}^2_{i,j}(S_N)$. *Then* $pf \in \mathcal{F}^2_{i+r,j+s}(S_N)$ *and* $M(pf) = f_\infty^{-1}(M(p))$.

Proof.

$$\begin{aligned}
x \in M(pf) &\iff pf(x_{i+r}, \ldots, x_{j+s}) = 1 \\
&\iff p(f(x_{i+r}, \ldots, x_{i+s}), \ldots, f(x_{j+r}, \ldots, x_{j+s})) = 1 \\
&\iff p((f_\infty(x))_i, \ldots, (f_\infty(x))_j) = 1 \\
&\iff f_\infty(x) \in M(p) \\
&\iff x \in f_\infty^{-1}(M(p)).\square
\end{aligned}$$

We give $\mathcal{F}^2(S_N)$ a binary operation. If $p \in \mathcal{F}^2_{i_1,j_1}(S_N)$ and if $q \in \mathcal{F}^2_{i_2,j_2}(S_N)$ then we define $p \cap q \in \mathcal{F}^2_{min\{i_1,i_2\},max\{j_1,j_2\}}(S_N)$ by

$$p \cap q(x_{min\{i_1,i_2\}}, \ldots, x_{max\{j_1,j_2\}}) = \begin{cases} 1 & , \text{ if } p(x_{i_1}, \ldots, x_{j_1}) = q(x_{i_2}, \ldots, x_{j_2}) = 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

We call this operation on $\mathcal{F}^2(\mathcal{S}_N)$ *local intersection* .

Proposition 2.5 *Let $p \in \mathcal{F}^2_{i_1,j_1}(\mathcal{S}_N)$, $q \in \mathcal{F}^2_{i_2,j_2}(\mathcal{S}_N)$. Then $M(p \cap q) = M(p) \cap M(q)$.*

Proof. We have

$$\begin{aligned} M(p \cap q) &= \{x \in \mathcal{S}_N^{\mathbb{Z}} : p \cap q(x_{min\{i_1,i_2\}}, \ldots, x_{max\{j_1,j_2\}}) = 1\} \\ &= \{x \in \mathcal{S}_N^{\mathbb{Z}} : p(x_{i_1}, \ldots, x_{j_1}) = q(x_{i_2}, \ldots, x_{j_2}) = 1\} \\ &= \{x \in \mathcal{S}_N^{\mathbb{Z}} : p(x_{i_1}, \ldots, x_{j_1}) = 1\} \bigcap \{x \in \mathcal{S}_N^{\mathbb{Z}} : q(x_{i_2}, \ldots, x_{j_2}) = 1\} \\ &= M(p) \cap M(q). \square \end{aligned}$$

For any $a \in \mathcal{S}_N$, $i \in \mathbb{Z}$ define $\chi_{a,i} \in \mathcal{F}^2_{i,i}(\mathcal{S}_N)$ by $\chi_{a,i}(x_i) = 1 \iff x_i = a$. This provides a useful representation of cylinder sets, i.e., if $U = \{x \in \mathcal{S}_N^{\mathbb{Z}} : x_{i_k} = a_k, \text{for } k = 1, \ldots, n\}$, for some $n > 0, i_1, i_2, \ldots, i_n \in \mathbb{Z}$, with $i_1 < i_2 \quad \ldots < i_n$, and $a_1, a_2, \ldots, a_n \in \mathcal{S}_N$, then $U = M(\chi_{a_1,i_1}(x_{i_1}) \cap \ldots \cap \chi_{a_n,i_n}(x_{i_n}))$.

Remark 2.6 *We now have an efficient way to characterise the properties of measure preservation, ergodicity, and mixing for cellular automata: Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$.*

1) f_∞ is measure preserving if and only if $\mu[M(pf)] = \mu[M(p)]$, for all $p = \chi_{a_1,i_1}(x_{i_1}) \cap \ldots \cap \chi_{a_m,i_m}(x_{i_m}) \in \mathcal{F}^2_{i_1,i_m}(\mathcal{S}_N)$, with $a_1, a_2, \ldots, a_m \in \mathcal{S}_N$, $i_1, i_2, \ldots, i_m \in \mathbb{Z}$, and $i_1 < i_2 < \ldots < i_m$.

2) f_∞ is ergodic if and only if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu[M(pf^n \cap q)] = \mu[M(p)] \cdot \mu[M(q)]$ for all $p = \chi_{a_1,i_1}(x_{i_1}) \cap \ldots \cap \chi_{a_m,i_m}(x_{i_m}) \in \mathcal{F}^2_{i_1,i_m}(\mathcal{S}_N)$, $q = \chi_{b_1,j_1}(x_{j_1}) \cap \ldots \cap \chi_{b_l,j_l}(x_{j_l}) \in \mathcal{F}^2_{j_1,j_l}(\mathcal{S}_N)$, with $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_l \in \mathcal{S}_N$, $i_1, i_2, \ldots, i_m$, $j_1, j_2, \ldots, j_l \in \mathbb{Z}$, $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_l$.

3) f_∞ is mixing if and only if $\lim_{n\to\infty} \mu[M(pf^n \cap q)] = \mu[M(p)] \cdot \mu[M(q)]$ for all $p = \chi_{a_1,i_1}(x_{i_1}) \cap \ldots \cap \chi_{a_m,i_m}(x_{i_m}) \in \mathcal{F}^2_{i_1,i_m}(\mathcal{S}_N)$, $q = \chi_{b_1,j_1}(x_{j_1}) \cap \ldots \cap \chi_{b_l,j_l}(x_{j_l}) \in \mathcal{F}^2_{j_1,j_l}(\mathcal{S}_N)$, with $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_l \in \mathcal{S}_N$, $i_1, i_2, \ldots, i_m$, $j_1, j_2, \ldots, j_l \in \mathbb{Z}$, $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_l$.

If $f \in \mathcal{F}^2_{r,s}(\mathcal{S}_N)$, we define $\text{supp}(f) = \{r, r+1, \ldots, s\}$. We let $|f| = s - r + 1$, and let $c(f) = \text{card} f^{-1}\{1\}$

Proposition 2.7 *Let $p, q \in \mathcal{F}^2(\mathcal{S}_N)$. Then*
1) $\mu[M(p)] = c(p)N^{-|p|}$,
2) If $supp(p) \cap supp(q) = \emptyset$, then $\mu(M(p) \cap M(q)) = \mu(M(p)) \cdot \mu(M(q))$.

Proof.
1) Let $p \in \mathcal{F}^2_{i,j}(\mathcal{S}_N)$, and for any $(a_i, \ldots, a_j) \in p^{-1}(1)$, let $U_{a_i,\ldots,a_j} = \{x \in \mathcal{S}_N^{\mathbf{Z}} : x_i = a_i, x_{i+1} = a_{i+1}, \ldots, x_j = a_j\}$ be a cylinder set. We then have $\mu[U_{a_i,\ldots,a_j}] = N^{-(j-i+1)} = N^{-|p|}$, and $U_{a_i,\ldots,a_j} \cap U_{b_i,\ldots,b_j} = \emptyset$, whenever $(a_i, \ldots, a_j), (b_i, \ldots, b_j) \in p^{-1}(1)$, with $(a_i, \ldots, a_j) \neq (b_i, \ldots, b_j)$. Equation 2.1 then gives us

$$\begin{aligned}
\mu[M(p)] &= \mu\left[\bigcup_{(a_i,\ldots,a_j)\in p^{-1}(1)} U_{a_i,\ldots,a_j}\right] \\
&= \sum_{(a_i,\ldots,a_j)\in p^{-1}(1)} \mu[U_{a_i,\ldots,a_j}] \\
&= \sum_{(a_i,\ldots,a_j)\in p^{-1}(1)} N^{-|p|} \\
&= c(p)N^{-|p|}.
\end{aligned}$$

2) Let $p \in \mathcal{F}^2_{i_1,i_2}(\mathcal{S}_N), q \in \mathcal{F}^2_{j_1,j_2}(\mathcal{S}_N)$. $\text{supp}(p) \cap \text{supp}(q) = \{i_1, \ldots, i_2\} \cap \{j_1, \ldots, j_2\} = \emptyset$ implies that either $j_1 > i_2$ or $i_1 > j_2$. Suppose $j_1 > i_2$. We then have $|p \cap q| = j_2 - i_1 + 1$. Let $(a_{i_1}, \ldots, a_{i_2}) \in p^{-1}(1)$ and let $(a_{j_1}, \ldots, a_{j_2}) \in q^{-1}(1)$. Then for every $(a_{i_2+1}, \ldots, a_{j_1-1}) \in \mathcal{S}_N^{j_1-i_2-1}$ we have $(a_{i_1}, \ldots, a_{i_2}, a_{i_2+1}, \ldots, a_{j_1-1}, a_{j_1}, \ldots, a_{j_2}) \in (p \cap q)^{-1}(1)$. We then have $c(p \cap q) = \text{card } (p \cap q)^{-1}(1) = \text{card } p^{-1}(1) \cdot \text{card } q^{-1}(1) \cdot \text{card } \mathcal{S}_N^{j_1-i_2-1} = c(p) \cdot c(q) \cdot N^{j_1-i_2-1}$. Thus $\mu[M(p) \cap M(q)] = \mu[M(p \cap q)] = c(p \cap q) \cdot N^{-|p \cap q|} = c(p) \cdot c(q) \cdot N^{j_1-i_2-1} \cdot N^{-(j_2-i_1+1)} = c(p) \cdot N^{-(i_2-i_1+1)} \cdot c(q) \cdot N^{-(j_2-j_1+1)} = c(p) \cdot N^{-|p|} \cdot c(q) \cdot N^{-|q|} = \mu[M(p)] \cdot \mu[M(q)]$.

The proof when $i_1 > j_2$ is similar.□

Theorem 2.8 *f_∞ is onto if and only if f_∞ is measure preserving.*

Proof. Let f_∞ be onto and let $p(x_{i_1}, x_{i_1+1}, \ldots, x_{i_m}) = \chi_{c_1,i_1}(x_{i_1}) \cap \chi_{c_2,i_2}(x_{i_2}) \cap \ldots \cap \chi_{c_m,i_m}(x_{i_m})$ so that $M(p)$ is a cylinder set. We have $\mu[M(p)] = N^{-m}$ and, by remark 2.6, we must show that $\mu[M(pf)] = N^{-m}$.

Now, $pf \in \mathcal{F}^2_{i_1+r,i_m+s}(\mathcal{S}_N)$, so we have $|pf| = (i_m + s) - (i_1 + r) + 1$. We calculate $c(hf)$. Let $(a_{i_1+r}, \ldots, a_{i_m+s}) \in \mathcal{S}_N^{(i_m+s)-(i_1+r)+1}$. Then $pf(a_{i_1+r}, \ldots, a_{i_m+s}) = 1 \iff p(f(a_{i_1+r}, \ldots, a_{i_1+s}), \ldots, f(a_{i_m+r}, \ldots, a_{i_m+s})) = 1 \iff \chi_{c_1,i_1}(f(a_{i_1+r}, \ldots, a_{i_1+s})) \cap \ldots \cap \chi_{c_m,i_m}(f(a_{i_m+r}, \ldots, a_{i_m+s})) = 1 \iff f_{i_m-i_1+1}(a_{i_1+r}, \ldots, a_{i_m+s}) \in \{c_1\} \times \mathcal{S}_N^{i_2-i_1-1} \times \{c_2\} \times \mathcal{S}_N^{i_3-i_2-1} \times \ldots \times \mathcal{S}_N^{i_m-i_{m-1}-1} \times \{c_m\} = T \iff (a_{i_1+r}, \ldots, a_{i_m+s}) \in f^{-1}_{i_m-i_1+1}(T)$. Thus $c(pf) = \text{card} f^{-1}_{i_m-i_1+1}(T)$. $\text{card}T = N^{i_2-i_1-1} \cdot N^{i_3-i_2-1} \cdot \ldots \cdot N^{i_m-i_{m-1}-1} = N^{i_m-i_1-(m-1)}$, and since f_∞ is onto, we have by Theorem 1.2 that $\text{card} f^{-1}_{i_m-i_1+1}(b) = N^{s-r}$ for all $b \in T$, so $\text{card} f^{-1}_{i_m-i_1+1}(T) = \text{card} \bigcup_{b\in T} f^{-1}_{i_m-i_1+1}(b) = \sum_{b\in T} \text{card} f^{-1}_{i_m-i_1+1}(b) = \sum_{b\in T} N^{s-r} = \text{card}T \cdot N^{s-r} = N^{i_m-i_1-(m-1)+s-r}$. Thus we have by proposition 2.7, $\mu[M(pf)] = c(pf)N^{-|pf|} = N^{i_m-i_1-(m-1)+s-r} \cdot N^{-[(i_m+s)-(i_1+r)+1]} = N^{-m} = \mu[M(p)]$, so f_∞ is measure preserving.

To prove the converse statement, suppose that f_∞ is measure preserving. Then we have $1 = \mu[\mathcal{S}_N^{\mathbb{Z}}] = \mu[f_\infty^{-1}(f_\infty(\mathcal{S}_N^{\mathbb{Z}}))] = \mu[f_\infty(\mathcal{S}_N^{\mathbb{Z}})]$, so $f_\infty(\mathcal{S}_N^{\mathbb{Z}})$ is dense in $\mathcal{S}_N^{\mathbb{Z}}$. $\mathcal{S}_N^{\mathbb{Z}}$ is compact, so $f_\infty(\mathcal{S}_N^{\mathbb{Z}})$ is also compact, hence closed, so we must have $f_\infty(\mathcal{S}_N^{\mathbb{Z}}) = \mathcal{S}_N^{\mathbb{Z}}$, and f_∞ is onto.□

Theorem 2.9 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ with either $r > 0$ or $s < 0$ and let f_∞ be onto. Then f_∞ is strongly mixing.*

Proof. We prove the case $r > 0$. Let $p \in \mathcal{F}^2_{i_1,i_2}(\mathcal{S}_N), q \in \mathcal{F}^2_{j_1,j_2}(\mathcal{S}_N)$. We will show that $\mu[M(pf^n \cap q)] = \mu[M(p)] \cdot \mu[M(q)]$ for all sufficiently large n. We have $pf^n \in \mathcal{F}^2_{i_1+nr,i_2+ns}(\mathcal{S}_N)$, for each $n \in \mathbb{Z}$. There exists n^* such that $i_1 + nr > j_2$ for all $n \geq n^*$. $\text{supp}(q) = \{j_1, j_1 + 1, \ldots, j_2\}$ and $\text{supp}(pf^n) = \{i_1 + nr, i_1 + nr + 1, \ldots, i_2 +$

$ns\}$, so if $n \geq n^*$, we have $\text{supp}(q) \cap \text{supp}(pf^n) = \emptyset$. By proposition 2.7, we have $\mu[M(pf \cap q)] = \mu[M(pf^n) \cap M(q)] = \mu[M(pf^n)] \cdot \mu[M(q)] = \mu[f_\infty^{-n}(M(p))] \cdot \mu[M(q)] = \mu[M(p)] \cdot \mu[M(q)]$ for all $n \geq n^*$ (the last equality follows since f_∞ is onto, so f_∞^n is measure preserving). Thus f_∞ is strongly mixing.□

Corollary 2.10 *Let $f \in \mathcal{F}_{r,r}(S_N)$ be univariate, and let f_∞ be onto. Then f_∞ is mixing if and only if $r \neq 0$.*

Proof. Theorem 2.9 ensures that f_∞ is mixing if $r \neq 0$.

If $r = 0$, we show that f_∞ is not ergodic by presenting a measurable set V with $0 < \mu[V] < 1$ which satisfies $f_\infty^{-1}(V) = V$.

Now, proposition 1.7 states that there exists $n > 0$ such that $f_\infty^n = \sigma^0 = id$. If we let $U \subset S_N$ be such that $0 < \mu[U] < 1/n$, and let

$$V = U \cup f_\infty^{-1}(U) \cup \ldots \cup f_\infty^{-(n-1)}(U),$$

then

$$\begin{aligned} f_\infty^{-1}(V) &= f_\infty^{-1}(U \cup f_\infty^{-1}(U) \cup \ldots \cup f_\infty^{-(n-2)}(U) \cup f_\infty^{-(n-1)}(U)) \\ &= f_\infty^{-1}(U) \cup f_\infty^{-2}(U) \cup \ldots \cup f_\infty^{-(n-1)}(U) \cup f_\infty^{-n}(U) \\ &= f_\infty^{-1}(U) \cup f_\infty^{-2}(U) \cup \ldots \cup f_\infty^{-(n-1)}(U) \cup U = V. \end{aligned}$$

We then have $0 < \mu[V] = \mu[\cup_{k=0}^{n-1} f_\infty^{-k}(U)] \leq \sum_{k=0}^{n-1} \mu[f_\infty^{-k}(U)] = \sum_{k=0}^{n-1} \mu[U] < n \cdot (1/n) = 1$. Thus f_∞ is not ergodic, hence f_∞ is not mixing. □

2.2 Permutive Automata

In this section we will continue to generalize the results of [SR]. We will look at left permutive automata with negative left indices and right permutive automata with positive right indices and show that such automata are mixing. In view of theorem 2.9, this means that left permutive automata with nonzero left indices and right permutive automata with nonzero right indices are mixing, and we will conclude the section with a brief look at some permutive automata which do not satisfy these conditions.

We start with a technical lemma which describes certain independence conditions of cylinder sets under the influence of permutive automata.

Lemma 2.11 *Let $p = p(x_\alpha, \ldots, x_\beta) \in \mathcal{F}^2_{\alpha,\beta}(\mathcal{S}_N)$. Let $g = g(x_r, \ldots, x_s) \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ be left permutive. Let $i \in \mathbb{Z}$ with $i + r < \alpha$. Then for any $a \in \mathcal{S}_N$, $\chi_{i,a}g \in \mathcal{F}^2_{i+r,i+s}(\mathcal{S}_N)$, and*

1)

$$c(\chi_{i,a}g \cap p) = \begin{cases} c(p)N^{\alpha-(i+r)-1} & , \ \text{if}\ i+s \leq \beta, \\ c(p)N^{\alpha-r+s-\beta-1} & , \ \text{if}\ i+s \geq \beta. \end{cases},$$

2)

$$\mu[M(\chi_{i,a}g \cap p)] = N^{-1} \cdot \mu[M(p)].$$

Proof.

1) Let $h = \chi_{i,a}g \cap p$.

Suppose $i + s \leq \beta$ and let $(u_{i+r}, \ldots, u_\alpha, \ldots, u_\beta) \in h^{-1}(1)$. Then $p(u_\alpha, \ldots, u_\beta) = 1$. There are $c(p)$ choices for $(u_\alpha, \ldots, u_\beta)$. For any choice of $u_{i+r+1}, \ldots, u_{\alpha-1}$ we have $g(\cdot, u_{i+r+1}, \ldots, u_{i+s})$ is a bijection of $\mathcal{S}_N$ so there is exactly one $u_{i+r} \in \mathcal{S}_N$ such that $g(u_{i+r}, \ldots, u_{i+s}) = a$, and hence $\chi_{i,a}g(u_{i+r}, \ldots, u_{i+s}) = 1$. Thus there are $c(p) \cdot N^{\alpha-1-(i+r+1)+1} = c(p) \cdot N^{\alpha-(i+r)-1}$ choices for $(u_{i+r}, \ldots, u_\beta)$.

Now, let $\beta \leq i+s$ and let $(u_{i+r}, \ldots, u_\alpha, \ldots, u_\beta, \ldots, u_{i+s}) \in h^{-1}(1)$. We must have $p(u_\alpha, \ldots, u_\beta) = 1$ (there are again $c(p)$ possibilities). For any choice of $u_{i+r+1}, \ldots, u_{\alpha-1}$, $u_{\beta+1}, \ldots, u_{i+s}$ there is exactly one $u_{i+r} \in \mathcal{S}_N$ such that $g(u_{i+r}, \ldots, u_{i+s}) = a$. Thus $c(h) = c(p) \cdot N^{\alpha-1-(i+r+1)+1} \cdot N^{i+s-(\beta+1)+1} = c(p) \cdot N^{\alpha-\beta+s-r-1}$.

2) If $i+s \leq \beta$, then $|\chi_{i,a}g \cap p| = \beta - (i+r) + 1$. We then have

$$\begin{aligned}
\mu[M(\chi_{i,a}g \cap p)] &= c(\chi_{i,a}g \cap p) \cdot N^{-|\chi_{i,a}g \cap p|} \\
&= c(p)N^{\alpha-(i+r)-1} \cdot N^{-\beta+(i+r)-1} \\
&= N^{-1}c(p)N^{-(\beta-\alpha+1)} = N^{-1}c(p)N^{-|p|} \\
&= N^{-1} \cdot \mu[M(p)].
\end{aligned}$$

If $i+s \geq \beta$, then $|\chi_{i,a}g \cap p| = (i+s) - (i+r) + 1 = s - r + 1$. We then have

$$\begin{aligned}
\mu[M(\chi_{i,a}g \cap p)] &= c(\chi_{i,a}g \cap p) \cdot N^{-|\chi_{i,a}g \cap p|} \\
&= c(p)N^{\alpha-r+s-\beta-1} \cdot N^{-s+r-1} \\
&= N^{-1}c(p)N^{-(\beta-\alpha+1)} = N^{-1} \cdot \mu[M(p)]. \square
\end{aligned}$$

Notice that in the above we have $\mu[M(\chi_{i,a}g \cap p)] = N^{-1} \cdot \mu[M(p)] = \mu[M(\chi_{i,a})] \cdot \mu[M(p)]$. In general, if $g = g(x_r, \ldots, x_s) \in \mathcal{F}_{r,s}(S_N)$ is left permutive, if $p = p(x_\alpha, \ldots, x_\beta) \in \mathcal{F}^2_{\alpha,\beta}(S_N)$, and if $q = q(x_i, \ldots, x_j) \in \mathcal{F}^2_{i,j}(S_N)$, with $j + r < \alpha$ we have $\mu[M(qg \cap p)] = \mu[M(q)] \cdot \mu[M(p)]$. Lemma 2.11 is sufficiently strong to help prove the main theorem of this section:

Theorem 2.12 *Let $f \in \mathcal{F}_{r,s}(S_N)$ be left [right] permutive with $r < 0$ [$s > 0$]. Then f_∞ is strongly mixing.*

Proof. We prove the first reading.

If $s < 0$, then f_∞ is mixing by Theorem 2.9, so assume that $s \geq 0$.

Let $p = \chi_{a_1,i_1}(x_{i_1}) \cap \ldots \cap \chi_{a_m,i_m}(x_{i_m}) \in \mathcal{F}^2_{i_1,i_m}(S_N)$ and let $q = \chi_{b_1,j_1}(x_{j_1}) \cap \ldots \cap \chi_{b_l,j_l}(x_{j_l}) \in \mathcal{F}^2_{j_1,j_l}(S_N)$, for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_l \in S_N$, $i_1, i_2, \ldots, i_m$, $j_1, j_2, \ldots, j_l \in \mathbb{Z}$, with $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_l$.

To show that f_∞ is mixing, it is sufficient to show that $\mu[M(pf^n \cap q)] = \mu[M(p)] \cdot \mu[M(q)] = N^{-m} \cdot N^{-l}$, for all sufficiently large n.

Now, there exists an $n^* \in \mathbb{Z}$ such that if $n \geq n^*$ we have $i_m + nr < j_1$. We have

$$\begin{aligned} pf^n \cap q &= \chi_{a_1,i_1} f^n(x_{i_1+nr}, \ldots, x_{i_1+ns}) \cap \chi_{a_2,i_2} f^n(x_{i_2+nr}, \ldots, x_{i_2+ns}) \\ &\quad \cap \ldots \cap \chi_{a_m,i_m} f^n(x_{i_m+nr}, \ldots, x_{i_m+ns}) \cap q(x_{j_1} \ldots, x_{j_m}) \end{aligned}$$

If we let

$$h_1 = \chi_{a_m,i_m} f^n(x_{i_m+nr}, \ldots, x_{i_m+ns}) \cap q(x_{j_1} \ldots, x_{j_m}),$$

and for $k = 2, \ldots, m$ let

$$h_k = \chi_{a_{m-k+1},i_{m-k+1}} f^n(x_{i_{m-k+1}+nr}, \ldots, x_{i_{m-k+1}+ns}) \cap h_{k-1},$$

then we clearly have $pf^n \cap q = h_m$. Now, f^n is left permutive and $i_m + nr < j_1$ for $n \geq n^*$, so lemma 2.11 gives

$$\mu[M(h_1)] = N^{-1} \cdot \mu[M(q)] = N^{-1} \cdot N^{-l}.$$

We also have for $k = 2, \ldots, m$, that $i_{m-k+1} + nr < i_{m-k+2} + nr$ so lemma 2.11 gives

$$\mu(h_k) = N^{-1} \cdot \mu[M(h_{k-1})]$$

We thus have

$$\begin{aligned} \mu[M(pf^n \cap q)] &= \mu[M(h_m)] = N^{-1} \cdot \mu[M(h_{m-1})] \\ &= N^{-2} \mu[M(h_{m-2})] \\ &\vdots \\ &= N^{-(m-1)} \mu[M(h_1)] \\ &= N^{-(m-1)} \cdot N^{-1} \cdot N^{-l} \\ &= N^{-m-l} = \mu[M(p)] \cdot \mu[M(q)]. \square \end{aligned}$$

Remark 2.13 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ be left permutive with $r = 0$. Is f_∞ strongly mixing? The answer is maybe.*

If $s > 0$ and f is actually bipermutive, then the second reading of theorem 2.12 ensures that f_∞ is mixing.

On the other hand, consider $f \in \mathcal{F}_{0,1}(\mathcal{S}_3)$ with image matrix given by

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

We then have $f^{-1}(0) = \{0\} \times \mathcal{S}_3$, and so if $U = \{x \in \mathcal{S}_3^{\mathbb{Z}} : x_0 = 0\}$ we have $f_\infty^{-1}(U) = U$. $\mu[U] = 1/3$, so f_∞ is not ergodic, hence not mixing.

2.3 Bipermutive Automata

The main theorem of this chapter states that certain bipermutive automata (i.e., those with local functions $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$, with $r < 0 < s$) are identical to one-sided Bernoulli shifts as far as topology and measure theory are concerned. The case $s = -r$ was proven in [[illegible]A] by showing that f_∞ is expansive and by constructing a Markov partition, [illegible] providing a topological conjugacy. They then went on to show that the measure on the space which the bipermutive automaton acted upon was that of maximal entropy. In the more accessable proof provided here we will construct a measure theoretical conjugacy.

The driving force behind this theorem, in layman's terms, is that every $x \in \mathcal{S}_N^{\mathbb{Z}}$ is completely determined by a "thin strip" through its orbit under f_∞. Being a little less vague, what we mean is that we only need look at the evolution of $[f_\infty^n(x)]_i$ for a few $i \in \mathbb{Z}$ to completely know what the orbit of x looks like dynamically, and such "strips" are unique to x. The following lemma may help to clarify this point somewhat.

Lemma 2.14 *Let $r < 0 < s$ and let $f = f(x_r, \ldots, x_s) \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ be bipermutive. Let $x, y \in \mathcal{S}_N^{\mathbb{Z}}$. Then $x = y$ if and only if $[f_\infty^n(x)]_i = [f_\infty^n(y)]_i$ for all $n \geq 0, 0 \leq i \leq s-r-1$.*

Proof. It is obvious that if $x = y$ then $[f_\infty^n(x)]_i = [f_\infty^n(y)]_i$ for all $n \geq 0, 0 \leq i \leq s-r-1$.

Let $i < 0$ and suppose that $[f_\infty^n(x)]_k = [f_\infty^n(y)]_k$ for all $n \geq 0, i < k \leq s-r-1$. Then for each n we have $[f_\infty^{n+1}(x)]_{i-r} = f([f_\infty^n(x)]_i, [f_\infty^n(x)]_{i+1}, \ldots, [f_\infty^n(x)]_{i+s-r})$. Now we have $[f_\infty^{n+1}(x)]_{i-r} = [f_\infty^{n+1}(y)]_{i-r}$ since $i < i-r \leq s-r-1$. We also have $[f_\infty^n(x)]_j = [f_\infty^n(y)]_j$ for $j = i+1, i+2, \ldots, i+s-r$ since $i < i+1 < i+2 < \ldots < i+s-r \leq s-r-1$. f is left permutive so we must have $[f_\infty^n(x)]_i = [f_\infty^n(y)]_i$, since

$$\begin{aligned} f([f_\infty^n(x)]_i, [f_\infty^n(x)]_{i+1}, \ldots, [f_\infty^n(x)]_{i+s-r}) &= f([f_\infty^n(y)]_i, [f_\infty^n(y)]_{i+1}, \ldots, [f_\infty^n(y)]_{i+s-r}) \\ &= f([f_\infty^n(y)]_i, [f_\infty^n(x)]_{i+1}, \ldots, [f_\infty^n(x)]_{i+s-r}) \end{aligned}$$

and $f(\cdot, [f_\infty^n(x)]_{i+1}, \ldots, [f_\infty^n(x)]_{i+s-r})$ is a bijection.

Thus inductively we have $[f_\infty^n(x)]_i = [f_\infty^n(y)]_i$ for all $n \geq 0, i \leq s-r-1$, in particular $x_i = y_i$ for all $i \leq s-r-1$.

The proof that $x_i = y_i$ for all $i > s-r-1$ uses the fact that f is right permutive, and is entirely similar.□

The following lemma is purely technical, but will serve us well later.

Lemma 2.15 *Let $r < 0 < s$ and let $f = f(x_r, \ldots, x_s) \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ be bipermutive. Let $i \in \mathbb{Z}$, $i > 0$, and let $p \in \mathcal{F}^2_{0,s-r-1}(\mathcal{S}_N)$, $q \in \mathcal{F}^2_{\alpha,\beta}(\mathcal{S}_N)$, where $\alpha \geq ri$, $\beta \leq si+s-r-1$, and $\beta - \alpha + 1 < (s-r)i + 1$ (i.e., $|q| < |f^i|$). Then*
1) $c(pf^i \cap q) = c(p)c(q)N^{i(s-r)-(\beta-\alpha+1)}$,
2) $\mu[M(pf^i \cap q)] = \mu[M(p)] \cdot \mu[M(q)]$.

Proof.

1) $(u_{ri},\ldots,u_{si+s-r-1}) \in (pf^i \cap q)^{-1}(1)$ if and only if $q(u_\alpha,\ldots,u_\beta) = 1$, and $pf^i(u_{ri},\ldots,u_{si+s-r-1}) = 1$ (i.e., there must exist $(v_0,\ldots,v_{s-r-1}) \in p^{-1}(1)$ such that $f^i(u_{ri},\ldots,u_{si}) = v_0$, $f^i(u_{ri+1},\ldots,u_{si+1}) = v_1$, $\ldots$, $f^i(u_{ri+s-r-1},\ldots,u_{si+s-r-1}) = v_{s-r-1}$).

Now, let $(u_\alpha,\ldots,u_\beta) \in q^{-1}(1)$, and $(v_0,\ldots,v_{s-r-1}) \in p^{-1}(1)$ (there are $c(p)\cdot c(q)$ ways in which $(u_\alpha,\ldots,u_\beta)$ and $(v_0,\ldots,v_{s-r-1})$ may be chosen). We will show that there are exactly $N^{i(s-r)-(\beta-\alpha+1)}$ choices for $u_{ri},\ldots,u_\alpha,u_{\beta+1},\ldots,u_{si+s-r-1}$ such that $f^i(u_{ri},\ldots,u_{si}) = v_0$, $f^i(u_{ri+1},\ldots,u_{si+1}) = v_1$, $\ldots$, $f^i(u_{ri+s-r-1},\ldots,u_{si+s-r-1}) = v_{s-r-1}$ (and thus $c(pf^i\cap q) = c(p)c(q)N^{i(s-r)-(\beta-\alpha+1)}$).

Now, there exists $k \in \mathbb{Z}$ with $0 \le k \le s-r-1$, which satisfies either 1) $ri+k = \alpha$ and $\beta < si+k$, or, 2) $ri+k < \alpha$ and $\beta = si+k$ (since we have $\beta - \alpha < (s-r)i = (si+k)-(ri+k)$).

Now, if k satisfies 1) then if we choose $u_\beta + 1,\ldots,u_{si+k-1}$ arbitrarily (there are $N^{si+k-1-(\beta+1)+1} = N^{si+(\alpha-ri)-\beta-1} = N^{(s-r)i-(\beta-\alpha+1)}$ ways to do this), then

$f^i(u_{ri+k},\ldots,u_{si+k-1},\cdot)$ is a permutation of $\mathcal{S}_N$, so there exists exactly one $u_{ri+k} \in \mathcal{S}_N$ such that $f^i(u_{ri+k},\ldots,u_{si+k-1},u_{si+k}) = v_k$.

If k satisfies 2), and if we choose $u_{ri+k+1},\ldots,u_{\alpha-1}$ arbitrarily (again, there are $N^{(s-r)i-(\beta-\alpha+1)}$ ways to do this), the left permutivity of f^i ensures that there exists exactly one $u_{ri+k} \in \mathcal{S}_N$ such that $f^i(u_{ri+k},u_{ri+k+1},\ldots,u_{si+k}) = v_k$.

If $k > 0$, then we have $f^i(\cdot,u_{ri+k},\ldots,u_{si+k-1})$ is a bijection, so there is exactly one u_{ri+k-1} such that $f^i(u_{ri+k-1},u_{ri+k},\ldots,u_{si+k-1}) = v_{k-1}$, and similarly we prove that there is exactly one choice of $u_{ri},\ldots,u_{ri+k-2}$ such that $f^i(u_{ri},\ldots,u_{si}) = v_0$, $f^i(u_{ri+1},\ldots,u_{si+1}) = v_1$, $\ldots$,$f^i(u_{ri+k-2},\ldots,u_{si+k-2}) = v_{k-2}$.

If $k < s-r-1$, using the right permutive nature of f^i, we also conclude that there is exactly one choice of $u_{si+k+1},\ldots,u_{si+s-r-1}$ such that

$f^i(u_{ri+k+1},\dots,u_{si+k+1}) = v_{k+1},\dots,f^i(u_{ri+s-r-1},\dots,u_{si+s-r-1}) = v_{s-r-1}.$

2) We have $\mu[M(p)] = c(p)\cdot N^{-(s-r)}$, and $\mu[M(q)] = c(q)\cdot N^{-(\beta-\alpha+1)}$, thus

$$\begin{aligned}
\mu[M(pf^i \cap q)] &= c(pf^i \cap q)\cdot N^{-|pf^i\cap q|}\\
&= c(pf^i \cap q)\cdot N^{si+s-r-1-ri+1}\\
&= c(pf^i \cap q)\cdot N^{-(s-r)i-(s-r)}\\
&= c(p)\cdot c(q)\cdot N^{(s-r)i-(\beta-\alpha+1)}\cdot N^{si+s-r-1-ri+1}\\
&= c(p)\cdot N^{-(s-r)}\cdot c(q)\cdot N^{-(\beta-\alpha+1)}\\
&= \mu[M(p)]\cdot\mu[M(q)].\square
\end{aligned}$$

We finally have:

Theorem 2.16 *Let $r < 0 < s$ and let $f = f(x_r,\dots,x_s) \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ be bipermutiv Then f_∞ is measure-theoretically equivalent to a one-sided Bernoulli shift on N^{s-} symbols.*

Proof. We denote the measures on $\mathcal{S}_N^{\mathbb{Z}}$ and $\mathcal{S}_{N^{s-r}}^{\mathbb{N}}$ by μ_1 and μ_2 respectively (in the case with two-sided sequence space, the measure μ_2 on $\mathcal{S}_{N^{s-r}}^{\mathbb{N}}$ is the produ measure induced by the uniform measure on $\mathcal{S}_{N^{s-r}}$).

Let $x \in \mathcal{S}_N^{\mathbb{Z}}$. Define $\eta : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_{N^{s-r}}$ by $\eta(x) = \sum_{i=0}^{s-r-1} x_i \cdot N^i$ (note th $\eta(x) = \eta(y)$ if and only if $x_i = y_i$ for $i = 0,1,\dots,s-r-1$). Let $\eta_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_{N^{s}}^{\mathbb{N}}$ be given by $[\eta_\infty(x)]_i = \eta(f_\infty^i(x))$ for all $i \in \mathbb{N}$ (η_∞ serves as an encoding of the th strips mentioned earlier into one-sided sequences).

It is easy to see that if $x \in \mathcal{S}_N^{\mathbb{Z}}$, then we have $[\eta_\infty(f_\infty(x))]_i = \eta(f_\infty^i(f_\infty(x)))$ $\eta(f_\infty^{i+1}(x)) = [\eta_\infty(x)]_{i+1} = [\sigma \circ \eta_\infty(x)]_i$, for all $i \in \mathbb{N}$, so the following diagram co mutes:

$$\begin{array}{ccc} \mathcal{S}_N^{\mathbb{Z}} & \xrightarrow{f_\infty} & \mathcal{S}_N^{\mathbb{Z}} \\ \downarrow \eta_\infty & & \downarrow \eta_\infty \\ \mathcal{S}_{N^{s-r}}^{\mathbb{N}} & \xrightarrow[\sigma]{} & \mathcal{S}_{N^{s-r}}^{\mathbb{N}} \end{array}$$

What needs to be shown is that η_∞ is a measure preserving homeomorphism. Let $x, y \in \mathcal{S}_N^{\mathbb{Z}}$ with $\eta_\infty(x) = \eta_\infty(y)$. Then we must have

$$[\eta_\infty(x)]_n = \sum_{i=0}^{s-r-1} [f_\infty^n(x)]_i N^i = \sum_{i=0}^{s-r-1} [f_\infty^n(y)]_i N^i = [\eta_\infty(y)]_n,$$

for all $n \geq 0$. This holds if and only if $[f_\infty^n(x)]_i = [f_\infty^n(y)]_i$ for all $n \geq 0, 0 \leq i \leq s-r-1$. By lemma 2.14, we must have $x = y$.

We prove that η_∞ is continuous and measure preserving. Let $a_1, \ldots, a_n \in \mathcal{S}_{N^{s-r}}$ for some $n > 0$ and let $i_1, \ldots, i_n \in \mathbb{Z}$ with $0 \leq i_1 < i_2 < \ldots < i_n$. Let $U = \{x \in \mathcal{S}_{N^{s-r}}^{\mathbb{N}} : x_{i_1} = a_1, \ldots, x_{i_n} = a_n\}$ be a cylinder set of $\mathcal{S}_{N^{s-r}}^{\mathbb{N}}$. We then have $\mu_2[U] = (N^{s-r})^{-n}$. For $k = 1, 2, \ldots, n$ let $a_{k,0}, a_{k,1}, \ldots, a_{k,s-r-1} \in \mathcal{S}_N$ be the unique members of $\mathcal{S}_N$ such that $\sum_{l=0}^{s-r-1} a_{k,l} \cdot N^l = a_k$. We then have

$$\begin{aligned}
y \in \eta_\infty^{-1}(U) \quad &\Longleftrightarrow \quad \eta(f_\infty^{i_k}(y)) = a_k, \text{ for } k = 1, \ldots, n \\
&\Longleftrightarrow \quad \sum_{l=0}^{s-r-1} [f_\infty^{i_k}(y)]_l \cdot N^l = a_k, \text{ for } k = 1, \ldots, n \\
&\Longleftrightarrow \quad [f_\infty^{i_k}(y)]_l = a_{k,l}, \text{ for } k = 1, \ldots, n, l = 1, \ldots, s-r-1 \\
&\Longleftrightarrow \quad f_\infty^{i_k}(y) \in M(\chi_{a_{k,l},l}), \text{ for } k = 1, \ldots, n, l = 1, \ldots, s-r-1 \\
&\Longleftrightarrow \quad y \in f_\infty^{-i_k}(M(\chi_{a_{k,l},l})), \text{ for } k = 1, \ldots, n, l = 1, \ldots, s-r-1 \\
&\Longleftrightarrow \quad y \in M(\chi_{a_{k,l},l} f^{i_k}), \text{ for } k = 1, \ldots, n, l = 1, \ldots, s-r-1 \\
&\Longleftrightarrow \quad y \in \bigcap_{k=1}^{n} \bigcap_{l=0}^{s-r-1} M(\chi_{a_{k,l},l} f^{i_k})
\end{aligned}$$

Thus we have

$$\begin{aligned}\eta_\infty^{-1}(U) &= \bigcap_{k=1}^{n}\bigcap_{l=0}^{s-r-1} M(\chi_{a_{k,l},l}f^{i_k}) \\ &= \bigcap_{k=1}^{n} M(\chi_{a_{k,0},0}f^{i_k}\cap\ldots\cap\chi_{a_{k,s-r-1},s-r-1}f^{i_k}) \\ &= \bigcap_{k=1}^{n} M((\chi_{a_{k,0},0}\cap\ldots\cap\chi_{a_{k,s-r-1},s-r-1})f^{i_k}).\end{aligned}$$

It is clear that η_∞ is continuous as $\eta_\infty^{-1}(U)$ is the finite intersection of open sets.

For $k = 1, 2, \ldots, n$, let $p_k = \chi_{a_{k,0},0}\cap\ldots\cap\chi_{a_{k,s-r-1},s-r-1} \in \mathcal{F}^2_{0,s-r-1}(S_N)$. We clearly have $\mu_1[M(p_k)] = N^{-(s-r)}$, for each k. We then have $\eta_\infty^{-1}(U) = \cap_{k=1}^{n} M(p_k f^{i_k}) = M(p_1 f^{i_1}\cap\ldots\cap p_n f^{i_n})$, where $p_k f^{i_k} \in \mathcal{F}^2_{ri_k, si_k+s-r-1}(S_N)$. Let $h = p_1 f^{i_1}\cap\ldots\cap p_n f^{i_n}$.

Now, let $q_1 = p_1 f^{i_1}$ and for $k = 2, \ldots, n$ let $q_k = p_k f^{i_k} \cap q_{k-1}$. We then have $h = q_n$.

For $k = 2, \ldots, n$ we have $q_{k-1} \in \mathcal{F}^2_{ri_{k-1}, si_{k-1}+s-r-1}(S_N)$, and $ri_{k-1} \geq ri_k$, $si_{k-1} + s - r - 1 \geq si_k + s - r - 1$, and $|q_{k-1}| = (s-r)(i_{k-1}+1) < (s-r)i_k + 1 = |f^{i_k}|$, thus we can apply lemma 2.15,

$$\begin{aligned}\mu_1[M(q_k)] = \mu_1[M(p_k f^{i_k}\cap q_{k-1})] &= \mu_1[M(p_k)]\cdot\mu_1[M(q_{k-1})] \\ &= N^{-(s-r)}\mu_1[M(q_{k-1})].\end{aligned}$$

We thus have

$$\begin{aligned}\mu_1[\eta_\infty^{-1}(U)] &= \mu_1[M(q_n)] \\ &= N^{-(s-r)}\mu_1[M(q_{n-1})] \\ &= (N^{-(s-r)})^2\mu_1[M(q_{n-2})] \\ &\vdots \\ &= (N^{-(s-r)})^{n-1}\mu_1[M(q_1)] = (N^{-(s-r)})^{n-1}\mu_1[M(p_1)] \\ &= (N^{-(s-r)})^{n} = \mu_2[U].\end{aligned}$$

Thus η_∞ is continuous and measure preserving. The compactness of both $\mathcal{S}_N^{\mathbb{Z}}$ and $\mathcal{S}_{N^{s-r}}^{\mathbb{N}}$ ensures that η_∞ is onto, and that η_∞^{-1} is both continuous and measure preserving.□

It is perhaps interesting to note that this theorem may have been anticipated in [Hed] via the theorem: Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$. Then card $f_\infty^{-1}(x) = N^{s-r}$ for all $x \in \mathcal{S}_N^{\mathbb{Z}}$ if and only if f is bipermutive.

2.4 Product Automata

Give any two dynamical systems to a mathematician and he will give you a third by taking the product.

Products of dynamical systems play an important role in ergodic theory. In this section we will see th products of cellular automata are essentially cellular automata themselves (via me e preserving conjugacies).

Let $N, p, q \in \mathbb{Z}$, with $N = p \cdot q$ and $p, q > 1$. Consider the bijection $\psi : \mathcal{S}_p \times \mathcal{S}_q \longrightarrow \mathcal{S}_N$ given by $\psi(y, z) = y \cdot q + z$, for all $(y, z) \in \mathcal{S}_p \times \mathcal{S}_q$ (where addition and multiplication are taken over the integers). Let $\gamma = \psi^{-1}$. We may write $\gamma = \alpha \times \beta$, $\alpha : \mathcal{S}_N \longrightarrow \mathcal{S}_p$, $\beta : \mathcal{S}_N \longrightarrow \mathcal{S}_q$, where $\beta(x) = x \bmod q$, and $\alpha(x) = \frac{x - \beta(x)}{q}$, for all $x \in \mathcal{S}_N$ ($\alpha(x)$ is the quotient upon dividing x by q, so we have $x = \alpha(x) \cdot q + \beta(x)$).

ψ induces a bijection $\psi_\infty : \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ given component-wise by $[\psi_\infty(y, z)]_i = \psi(y_i, z_i)$, for all $i \in \mathbb{Z}$ and all $y \in \mathcal{S}_p^{\mathbb{Z}}$, $z \in \mathcal{S}_q^{\mathbb{Z}}$. The inverse of ψ_∞ is $\gamma_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$, where $\gamma_\infty = \alpha_\infty \times \beta_\infty$ and $\alpha_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_p^{\mathbb{Z}}$, $\beta_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_q^{\mathbb{Z}}$, are given by $[\alpha_\infty(x)]_i = \alpha(x_i)$, $[\beta_\infty(x)]_i = \beta(x_i)$, for all $i \in \mathbb{Z}$ and all $x \in \mathcal{S}_N^{\mathbb{Z}}$.

Give $\mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$ the product topology. Denote the measures on $\mathcal{S}_N^{\mathbb{Z}}$, $\mathcal{S}_p^{\mathbb{Z}}$, and $\mathcal{S}_q^{\mathbb{Z}}$ by μ_N, μ_p, and μ_q respectively. We give $\mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$ the product measure μ_{pq}, i.e., the σ-algebra of measurable sets on $\mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$ is the smallest σ-algebra containing the sets of the form $U \times V$, where U is a measurable set of $\mathcal{S}_p^{\mathbb{Z}}$, and V is a measurable set of

$\mathcal{S}_q^{\mathbb{Z}}$, and we require that $\mu_{pq}[U \times V] = \mu_p[U] \cdot \mu_q[V]$.

Lemma 2.17 *$\psi_\infty : \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ is a measure preserving homeomorphism.*

Proof. Let $U = \{x \in \mathcal{S}_N^{\mathbb{Z}} : x_{i_1} = a_1, \ldots, x_{i_n} = a_n\}$, for some $n, i_1, \ldots, i_n \in \mathbb{Z}$, $a_1, \ldots, a_n \in \mathcal{S}_N$, where $n > 0$ and $i_1 < i_2 < \ldots < i_n$. We then have $\mu_N[U] = N^{-n}$, and

$$\begin{aligned}\psi_\infty^{-1}(U) &= \gamma_\infty(U) = \alpha_\infty(U) \times \beta_\infty(U) \\ &= \{y \in \mathcal{S}_p^{\mathbb{Z}} : y_{i_1} = \alpha(a_1), \ldots, y_{i_n} = \alpha(a_n)\} \\ &\quad \times \{z \in \mathcal{S}_q^{\mathbb{Z}} : z_{i_1} = \beta(a_1), \ldots, z_{i_n} = \beta(a_n)\}.\end{aligned}$$

$\psi_\infty^{-1}(U)$ is clearly open (it is the product of two open sets), so ψ_∞ is continuous (and compactness ensures that $\psi_\infty^{-1}$is also continuous). It is also clear that $\mu_p[\alpha_\infty(U)] = p^{-n}$, and $\mu_q[\beta_\infty(U)] = q^{-n}$, thus $\mu_{pq}[\psi_\infty^{-1}(U)] = \mu_{pq}[\alpha_\infty(U) \times \beta_\infty(U)] = \mu_p[\alpha_\infty(U)] \cdot \mu_q[\beta_\infty(U)] = p^{-n} \cdot q^{-n} = N^{-n} = \mu_N[U]$, thus ψ_∞ is measure preserving. □

ψ_∞ is "shift invariant" in the following way: Let σ_N, σ_p, and σ_q be the shift maps on the spaces $\mathcal{S}_N^{\mathbb{Z}}$, $\mathcal{S}_p^{\mathbb{Z}}$, and $\mathcal{S}_q^{\mathbb{Z}}$ respectively. We then have

$$\psi_\infty \circ (\sigma_p \times \sigma_q) = \sigma_N \circ \psi_\infty.$$

Thus the product dynamical system

$$\sigma_p \times \sigma_q : \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}} \longrightarrow \mathcal{S}_p^{\mathbb{Z}} \times \mathcal{S}_q^{\mathbb{Z}}$$

is measure theoretically conjugate to the automaton $\sigma_N : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$ via $\psi_\infty^{-1} = \gamma_\infty$. This motivates the following definition:

Definition 2.18 *Let $f \in \mathcal{F}_{r_1,s_1}(S_p)$, $g \in \mathcal{F}_{r_2,s_2}(S_q)$. Let $r = min\{r_1, r_2\}$ and $s = max\{s_1, s_2\}$. We define $f \times g \in \mathcal{F}_{r,s}(S_N)$ by*

$$(f \times g)(x_r, \ldots, x_s) = \psi(f(\alpha(x_{r_1}), \ldots, \alpha(x_{s_1})), g(\beta(x_{r_2}), \ldots, \beta(x_{s_2}))).$$

We call $f \times g$ the product *of f and g.*

Theorem 2.19 *$(f \times g)_\infty : S_N^{\mathbb{Z}} \longrightarrow S_N^{\mathbb{Z}}$ is measure theoretically conjugate to $f_\infty \times g_\infty : S_p^{\mathbb{Z}} \times S_q^{\mathbb{Z}} \longrightarrow S_p^{\mathbb{Z}} \times S_q^{\mathbb{Z}}$ via γ_∞.*

Proof. We know that γ_∞ is a measure preserving homeomorphism, so what we need to prove is that $(f \times g)_\infty = \gamma_\infty^{-1} \circ (f_\infty \times g_\infty) \circ \gamma_\infty$.

Let $x \in S_N^{\mathbb{Z}}$. For any $i \in \mathbb{Z}$ we have

$$\begin{aligned}
[\gamma_\infty^{-1} \circ (f_\infty \times g_\infty) \circ \gamma_\infty(x)]_i &= [\psi_\infty \circ (f_\infty \times g_\infty) \circ \gamma_\infty(x)]_i \\
&= [\psi_\infty \circ (f_\infty(\alpha_\infty(x)), g_\infty(\beta_\infty(x))]_i \\
&= \psi([f_\infty(\alpha_\infty(x))]_i, [g_\infty(\beta_\infty(x))]_i) \\
&= \psi(f([\alpha_\infty(x)]_{i+r_1}, \ldots, [\alpha_\infty(x)]_{i+s_1}), g([\beta_\infty(x)]_{i+r_2}, \ldots, [\beta_\infty(x)]_{i+s_2})) \\
&= \psi(f(\alpha(x_{i+r_1}), \ldots, \alpha(x_{i+s_1})), g(\beta(x_{i+r_2}), \ldots, \beta(x_{i+s_2}))) \\
&= (f \times g)(x_{i+r}, \ldots, x_{i+s}) \\
&= [(f \times g)_\infty(x)]_i. \square
\end{aligned}$$

Give any three dynamical systems to a mathematician and he will desperately want to give you a well defined forth.

In order to define arbitrary products of automata in any meaningful manner, we must first endure the next proposition.

Proposition 2.20 *Let $p, q, w \in \mathbb{Z}$, with $p, q, w > 1$. Let $N = p \cdot q \cdot w$. Let $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_p)$, $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_q)$, $h \in \mathcal{F}_{r_3,s_3}(\mathcal{S}_w)$, and let $r = min\ \{r_1, r_2, r_3\}$, $s = min\ \{s_1, s_2, s_3\}$. Then*

$$(f \times g) \times h = f \times (g \times h) \in \mathcal{F}_{r,s}(\mathcal{S}_N).$$

Proof. Let $r' = \min\{r_1, r_2\}$, $s' = \min\{s_1, s_2\}$, $\tilde{r} = \min\{r_2, r_3\}$, $\tilde{s} = \min\{s_2, s_3\}$. For any $n_1, n_2 \in \mathbb{Z}$ with $n_1, n_2 > 1$ we define $\psi_{n_2} : \mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \longrightarrow \mathcal{S}_{n_1 \cdot n_2}$, (in the manner above) by $\psi_{n_2}(y, z) = y \cdot n_2 + z$, for all $(y, z) \in \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}$, with inverse $\gamma_{n_2} = \alpha_{n_2} \times \beta_{n_2}$, where $\beta_{n_2}(x) = x \bmod n_2$, and $\alpha_{n_2} = \frac{x - \beta_{n_2}(x)}{n_2}$ (so that $x = \alpha_{n_2}(x) \cdot n_2 + \beta_{n_2}(x)$) for all $x \in \mathcal{S}_{n_1 \cdot n_2}$.

We have $(f \times g)(x_{r'}, \cdots, x_{s'}) = f(\alpha_q(x_{r_1}), \ldots, \alpha_q(x_{s_1})) \cdot q + g(\beta_q(x_{r_2}), \ldots, \beta_q(x_{s_2}))$, and so

$$\begin{aligned}
&(f \times g) \times h(x_r, \cdots, x_s) \\
&= (f \times g)(\alpha_w(x_{r'}), \cdots, \alpha_w(x_{s'})) \cdot w + h(\beta_w(x_{r_3}), \ldots, \beta_w(x_{s_3})) \\
&= f(\alpha_q \circ \alpha_w(x_{r_1}), \ldots, \alpha_q \circ \alpha_w(x_{s_1})) \cdot q \cdot w \\
&\quad + g(\beta_q \circ \alpha_w(x_{r_2}), \ldots, \beta_q \circ \alpha_w(x_{s_2})) \cdot w \\
&\quad + h(\beta_w(x_{r_3}), \ldots, \beta_w(x_{s_3})).
\end{aligned} \tag{2.2}$$

Similarly, $(g \times h)(x_{\tilde{r}}, \cdots, x_{\tilde{s}}) = g(\alpha_w(x_{r_2}), \ldots, \alpha_w(x_{s_2})) \cdot w + h(\beta_w(x_{r_3}), \ldots, \beta_w(x_{s_4}))$, so it is easily computed that

$$\begin{aligned}
&f \times (g \times h)(x_r, \cdots, x_s) \\
&= f(\alpha_{qw}(x_{r_1}), \ldots, \alpha_{qw}(x_{s_1})) \cdot q \cdot w \\
&\quad + g(\alpha_w \circ \beta_{qw}(x_{r_2}), \ldots, \alpha_w \circ \beta_{qw}(x_{s_2})) \cdot w \\
&\quad + h(\beta_w \circ \beta_{qw}(x_{r_3}), \ldots, \beta_w \circ \beta_{qw}(x_{s_3})).
\end{aligned} \tag{2.3}$$

Comparing equations 2.2 and 2.3 we see that we must verify that

1) $\alpha_q \circ \alpha_w = \alpha_{qw}$,

2) $\beta_q \circ \alpha_w = \alpha_w \circ \beta_{qw}$, and

3) $\beta_w = \beta_w \circ \beta_{qw}$,

to show that $(f \times g) \times h = f \times (g \times h)$.

1) Let $x \in S_N$. We may write $x = \alpha_{qw}(x) \cdot qw + \beta_{qw}(x)$. We then have

$$\begin{aligned}
\alpha_w(x) &= \frac{(\alpha_{qw}(x) \cdot qw + \beta_{qw}(x)) - \beta_w(\alpha_{qw}(x) \cdot qw + \beta_{qw}(x))}{w} \\
&= \frac{\alpha_{qw}(x) \cdot qw}{w} + \frac{(\beta_{qw}(x)) - \beta_w(\beta_{qw}(x))}{w} \\
&= \alpha_{qw}(x) \cdot q + \alpha_w \circ \beta_{qw}(x), \qquad (2.4)
\end{aligned}$$

since $(\alpha_{qw}(x) \cdot qw) \bmod w = 0$, and

$$\begin{aligned}
\alpha_q \circ \alpha_w(x) &= \frac{(\alpha_{qw}(x) \cdot q + \alpha_w \circ \beta_{qw}(x)) - \beta_q(\alpha_{qw}(x) \cdot q + \alpha_w \circ \beta_{qw}(x))}{q} \\
&= \frac{\alpha_{qw}(x) \cdot q}{q} + \frac{(\alpha_w \circ \beta_{qw}(x)) - \beta_q(\alpha_w \circ \beta_{qw}(x))}{q} \\
&= \alpha_{qw}(x) + \alpha_q \circ \alpha_w \circ \beta_{qw}(x)
\end{aligned}$$

since $(\alpha_{qw}(x) \cdot q) \bmod q = 0$. We must show that $\alpha_q \circ \alpha_w \circ \beta_{qw}(x) = 0$. Now $\beta_{qw}(x) = x \bmod qw$, so $0 \leq \beta_{qw}(x) \leq qw - 1$. $\alpha_w(\beta_{qw}(x))$ is the quotient upon dividing $\beta_{qw}(x)$ by w, so

$$0 \leq \alpha_w \circ \beta_{qw}(x) \leq \frac{\beta_{qw}(x)}{w} \leq \frac{qw-1}{w},$$

and $\alpha_q(\alpha_w \circ \beta_{qw}(x))$ is the quotient upon dividing $\alpha_w \circ \beta_{qw}(x)$ by q, so

$$0 \leq \alpha_q \circ \alpha_w \circ \beta_{qw}(x) \leq \frac{\alpha_w \circ \beta_{qw}(x)}{q} \leq \frac{qw-1}{qw} < 1,$$

so we must have $\alpha_q \circ \alpha_w \circ \beta_{qw}(x) = 0$.

2) We have $\alpha_w(x) = \alpha_q(\alpha_w(x)) \cdot q + \beta_q(\alpha_w(x))$, so $\beta_q \circ \alpha_w(x) = \alpha_w(x) - \alpha_q \circ \alpha_w(x) \cdot q$. From equation 2.4 we have

$$\begin{aligned} \alpha_w \circ \beta_{qw}(x) &= \alpha_w(x) - \alpha_{qw}(x) \cdot q \\ &= \alpha_w(x) - \alpha_q \circ \alpha_w(x) \cdot q \\ &= \beta_q \circ \alpha_w(x). \end{aligned}$$

3) $x = \alpha_{qw}(x) \cdot qw + \beta_{qw}(x)$, so $\beta_q(x) = \beta_q(\alpha_{qw}(x) \cdot qw + \beta_{qw}(x)) = \beta_q \circ \beta_{qw}(x)$ since $(\alpha_{qw}(x) \cdot qw) \bmod q = 0$.

We thus have $f \times (g \times h) = (f \times g) \times h$.□

If we let $f \times g \times h = (f \times g) \times h = f \times (g \times h)$ from the above theorem, we may now define arbitrary products inductively: if $n_1, \ldots, n_k \in \mathbb{Z}$, with $n_1, \ldots, n_k > 1$ and $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$, and if $f_i \in \mathcal{F}_{r_i, s_i}(\mathcal{S}_{n_i})$, for $i = 1, \ldots, k$, and if $r = \min\{r_1, \ldots, r_k\}$, $s = \max\{s_1, \ldots, s_k\}$, then define $f_1 \times f_2 \times \ldots \times f_k \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ by $f_1 \times \ldots \times f_k = (f_1 \times \ldots \times f_{k-1}) \times f_k$. With this definition it is clear that proposition 2.19 implies that

$$(f_1 \times \ldots \times f_k)_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$$

is measure theoretically equivalent to

$$(f_1)_\infty \times \ldots \times (f_k)_\infty : \mathcal{S}_{n_1}^{\mathbb{Z}} \times \ldots \times \mathcal{S}_{n_k}^{\mathbb{Z}} \longrightarrow \mathcal{S}_{n_1}^{\mathbb{Z}} \times \ldots \times \mathcal{S}_{n_k}^{\mathbb{Z}}.$$

Remark 2.21 *If we let $\mathcal{F} = \bigcup_{N=2}^{\infty} \mathcal{F}(\mathcal{S}_N)$, then $\mathcal{F}$ becomes a semigroup under the binary operation $\times$.*

We now present some standard results from ergodic theory:

Theorem 2.22 *Let $(\mathcal{X}_1, \beta_1, \mu_1)$ and $(\mathcal{X}_2, \beta_2, \mu_2)$ be measure spaces. Let $\phi_1 : \mathcal{X}_1 \longrightarrow \mathcal{X}_1$ and $\phi_2 : \mathcal{X}_2 \longrightarrow \mathcal{X}_2$. Then*
1) $\phi_1 \times \phi_2$ is mixing if and only if both ϕ_1 and ϕ_2 are mixing.
2) If $\phi_1 \times \phi_2$ is ergodic then both ϕ_1 and ϕ_2 are ergodic.

We then have for product automata

Corollary 2.23 *If $n_1, \ldots, n_k \in \mathbb{Z}$, with $n_1, \ldots, n_k > 1$ and $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$, and if $f_i \in \mathcal{F}_{r_i, s_i}(\mathcal{S}_{n_i})$, for $i = 1, \ldots, k$, and if $r = min\ \{r_1, \ldots, r_k\}$, $s = max\ \{s_1, \ldots, s_k\}$ then*
1)

$$(f_1 \times \ldots \times f_k)_\infty : \mathcal{S}_N^{\mathbb{Z}} \longrightarrow \mathcal{S}_N^{\mathbb{Z}}$$

is mixing if and only if $(f_i)_\infty$ is mixing for $i = 1, \ldots, k$.
2) If $(f_1 \times \ldots \times f_k)_\infty$ is ergodic, then $(f_i)_\infty$ is ergodic for each $i = 1, \ldots, k$.

Example 2.24 *Consider $id \times \sigma \in \mathcal{F}_{0,1}(\mathcal{S}_4)$, where $id(x_0) = x_0 \in \mathcal{F}_{0,0}(\mathcal{S}_2)$, and $\sigma(x_1) = x_1 \in \mathcal{F}_{1,1}(\mathcal{S}_2)$. $(id \times \sigma)_\infty$ is clearly not ergodic, as id is not ergodic, and so $id \times \sigma : \mathcal{S}_2^{\mathbb{Z}} \times \mathcal{S}_2^{\mathbb{Z}} \longrightarrow \mathcal{S}_2^{\mathbb{Z}} \times \mathcal{S}_2^{\mathbb{Z}}$ is not ergodic (notice that we have $(id \times \sigma)^{-1}(U \times \mathcal{S}_2^{\mathbb{Z}}) = U \times \mathcal{S}_2^{\mathbb{Z}}$ for every measurable set $U \subset \mathcal{S}_2^{\mathbb{Z}}$). The image matrix of $id \times \sigma$ is*

$$F = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \\ 2 & 3 & 2 & 3 \end{pmatrix}.$$

Note that $id \times \sigma$ is left partition permutive with partition $\{\{0,2\},\{1,3\}\}$ and right partition permutive with partition $\{\{0,1\},\{2,3\}\}$ (it is suspected that all products of permutive automata are partition permutive).

2.5 The Non-Ergodic Index and Concluding Remarks

It is clear that two factors influence whether or not a surjective automaton f_∞ with local function $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ is ergodic:

1) The structure of the local function $f : \mathcal{S}_N^{s-r+1} \longrightarrow \mathcal{S}_N$.

2) The values of the left and right indices, r and s, of f_∞.

In this section we will provide an interesting way to analyse the first of these factors by constructing a "measurement" of ergodicity for local functions (or to be more precise, a measure of "non-ergodicity"), and in doing so, provide an elegant way to summarise some of the results of this chapter.

Definition 2.25 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ and let f_∞ be onto. Let*

$$NE(f) = card\ \{k \in \mathbb{Z} : \sigma^k \circ f_\infty \text{ is not ergodic}\}.$$

We call $NE(f)$ the non-ergodic index *of f.*

Remark 2.26 *Two things should be made clear:*

1) Theorem 2.9 ensures that $NE(F)$ is finite for all $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ with f_∞ onto, for we have $\sigma^k f \in \mathcal{F}_{r+k,s+k}(\mathcal{S}_N)$ and so $\sigma^k \circ f_\infty$ is mixing (and hence ergodic) if $k < -s$ and if $k > -r$.

2) The non-ergodic index eliminates the influence of the second factor above in the sense that $NE(f) = NE(\sigma^k f)$ for all $k \in \mathbb{Z}$ (f and $\sigma^k f$ are the same function, when regarded as functions from $\mathcal{S}_N^{s-r+1}$ to $\mathcal{S}_N$, but have different left and right indices).

We summarise some of the results of this chapter in the following proposition:

Proposition 2.27 *Let $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ and let f_∞ be onto. We have the following:*
1) $0 \leq NE(f) \leq s - r + 1$.
2) If f is permutive, then $NE(f) \leq 1$.
3) If f is bipermutive, then $NE(f) = 0$.
4) If f is univariate (i.e., $s = r$) then $NE(f) = 1$.
5) If $g \in \mathcal{F}_{r',s'}(\mathcal{S}'_N)$ and g_∞ is onto then $NE(f), NE(g) \leq NE(f \times g)$.
6) If N has n prime divisors, then there exists $f \in \mathcal{F}(\mathcal{S}_N)$ with $NE(f) = n$.

Proof.
1) This follows from the first remark above.
2) If $f \in \mathcal{F}_{r,s}(\mathcal{S}_N)$ is left permutive, then $\sigma^k \circ f_\infty$ is mixing if $k > -r$ by theorem 2.9 and if $k < -r$ by theorem 2.12.
3) If f is bipermutive, then $\sigma^k \circ f_\infty$ is mixing for all $k \in \mathbb{Z}$ by theorem 2.12.
4) This follows from corollary 2.10.
5) This is a consequence of corollary 2.23.
6) Let $p_1, \ldots, p_n$ be the prime divisors of N. For $k = 1, \ldots, n$ let $h_k = \sigma^k \in \mathcal{F}(\mathcal{S}_{p_k})$. Then $f = h_1 \times h_2 \times \ldots \times h_n$ is the desired f ($\sigma^k f$ is not ergodic if and only if $k = -1, -2, \ldots, -n$.□

We will now take a moment to pose some unanswered questions

In [SR] the following question was posed: Are all onto automata $f_\infty : \mathcal{S}_2^{\mathbb{Z}} \longrightarrow \mathcal{S}_2^{\mathbb{Z}}$ strongly mixing (with the exception of id and p_∞ where $p(x_0) = x_0 + 1 \in \mathcal{F}_{0,0}(\mathcal{S}_2)$). This is found to be false. The counter example is h_∞, where $h(x_{-1}, x_0, x_1, x_2) = x_0 + x_1 + x_{-1}x_1 + x_1x_2 + x_{-1}x_1x_2 \in \mathcal{F}_{-1,2}(\mathcal{S}_2)$ (this is σ_1 from [AP], Appendix A). One may verify that $h_\infty^2 = id$.

A less strong version of the question shall be posed here: Are all onto automata $f_\infty : \mathcal{S}_2^{\mathbb{Z}} \longrightarrow \mathcal{S}_2^{\mathbb{Z}}$ strongly mixing with the exception of those automata f_∞ which satisfy $f_\infty^k = id$ for some $k > 0$? This is not true for automata on $\mathcal{S}_N^{\mathbf{Z}}$ with $N > 2$ (see remark 2.13 and example 2.24 for examples).

Each time an automaton was shown to be ergodic in this paper, it was through the virtue of being mixing. This leads to the next question: Are there any ergodic automata which are not mixing? It is strongly suspected that there are not.

Now, let $f \in \mathcal{F}_{r_1,r_1}(\mathcal{S}_N)$, $g \in \mathcal{F}_{r_2,r_2}(\mathcal{S}_N)$ be univariate local functions, with f_∞, g_∞ onto. We then have $f_\infty \circ g_\infty$ is mixing if and only if $r_1 + r_2 \neq 0$. A similar result is: let $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$ and $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$ be left permutive. Then $f_\infty \circ g_\infty$ is mixing if $r_1 + r_2 \neq 0$.

Now, what can we say about $f_\infty \circ g_\infty$ if $f \in \mathcal{F}_{r_1,s_1}(\mathcal{S}_N)$ is right permutive and $g \in \mathcal{F}_{r_2,s_2}(\mathcal{S}_N)$ is left permutive? It is believed that $f_\infty \circ g_\infty$ is mixing if $s_1 + r_2 \neq 0$. The method developed in proving theorem 2.12 does not adapt well to this problem. Now fg is partition permutive. Are there any generalised conditions we can impose on the left and right indices of partition permutive automata (in the flavour of theorem 2.12) to ensure mixing?

These are all worthy problems with no easy answers.

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