Robust Filter Design in Networked Control Systems

by

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Abstract

In this thesis, we study the problem of robust filtering under network-induced errors. Our intention is to design a robust filter that provides stable estimates of the plant states when the plant model is uncertain, the states are disturbed with an unknown input, and the measurements are quantized and therefore erroneous. To this end, we tackle the problem by first studying the various problems caused by the network and their effects on the filtering process when there are no model uncertainties and unknown inputs. Once familiarized with the challenges encountered in the design process, an active approach is proposed to deal with the error caused by quantization and packet dropouts, which gives way to considerably better performance specially when a coarse quantizer is considered.

Since our final design needs to be robust to unknown disturbances, we will propose two novel unknown-input linear filters, which are free of some of the restrictive assumptions seen in the literature. Both of these filters are based on a modified plant model, however, one of them has more design parameters and comes with a heavier computational burden than the other, but in return it generates slightly smoother estimates of both the states and the unknown input.

Having two distinct classes of filters, one with the ability to estimate the network-induced errors and one capable of estimating and rejecting unknown disturbances, we next propose a two-zone robust filter, which estimates the states with limited information and under unknown disturbances. The two-zone idea is based on the fact that the error caused by a linear quantizer is significant only when the estimates are close to their real values. Taking advantage of this fact, the estimation space can be divided into two operating zones based on the reliability of the received information. Finally, the two-zone filter is adapted for a fault-tolerant filtering application where the measurements are assumed to undergo coarse quantization, and unknown disturbances and model uncertainties are employed to

model various fault scenarios.

Preface

Chapter 2 of this thesis has been accepted to be published as Charandabi, B.A., Marquez, H.J. "$\mathcal{H}_\infty$ Filtering of Nonlinear Plants over Networks" in the International Journal of Robust and Nonlinear Control. I was responsible for the mathematical derivation of the design approach, simulation of results and manuscript composition. H.J Marquez was the supervisory author and was involved with manuscript composition.

The first part of chapter 4 (Approach I) has also been accepted to be published as Charandabi, B.A., Marquez, H.J. "A Novel Approach to Unknown Input Filter Design for Discrete-Time Linear Systems" in Automatica. I was responsible for the mathematical derivation of the design approach, simulation of results and manuscript composition. H.J Marquez was the supervisory author and was involved with manuscript composition.

To my amazing parents,

Gholamreza and Fariba

without whom none of this would be possible.

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Abbreviations

A/D	Analog to Digital
LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
NCS	Networked-Control System
FDI	Fault Detection and Isolation
FTC	Fault-Tolerant Control

Notation

$\mathbb{R}$, $\mathbb{Z}$	The sets of real and integer numbers
$\mathbb{R}^+$, $\mathbb{Z}^+$	The sets of nonnegative real and integer numbers
$\mathbb{R}^n$	The set of real n-dimensional vector
$\mathbb{R}^{n\times m}$	The set of real $n \times m$ matrices
ℓ_p	Function space with well-defined p-norm
ℓ_{pe}	Extended ℓ_p space of truncated signals
$\exists$	Existential quantifier
$\forall$	Universal quantifier
$x \in X$	x is an element of set X
$X \subset Y$	X is a subset of Y
A^T	Transpose of matrix or vector A
A^{-1}	Inverse of matrix A
I	Identity matrix of appropriate dimension
$\\|\cdot\\|$	Euclidean norm of a vector
$*$	Block symmetric matrix in LMI's
$P > (<)0$	P is a positive (negative) definite matrix
$P \geq (\leq)0$	P is a positive (negative) semi-definite matrix

In addition, a function $\phi(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be locally Lipschitz in region $\mathcal{D} \subset \mathbb{R}^n$ with respect to x, uniformly in u, if there exists $l > 0$ satisfying

$$\|\phi(x_1, u^*) - \phi(x_2, u^*)\| \leq l\|x_1 - x_2\| \qquad \forall x \in \mathcal{D}.$$

The smallest $l > 0$ satisfying the above equation is known as the Lipschitz constant.

A sequence v is said to have *finite support* if there is an integer N such that $v(k) = 0$ for any $k > N$. A sequence v with support in the set $0, \leq k \leq s$ will be denoted $v_s(k)$;

i.e. $v_s(k) = \{v(0), v(1), \ldots, v(s), 0, 0, \ldots\}$ (see [1]). It is clear that any real sequence with finite support, belongs to the ℓ_2 space.

Chapter 1

Introduction

1.1 Background and Literature Review

Numerous applications of state estimation and observation have drawn great interest into the field since its early years. In a classical configuration, an observer receives information directly from the plant. In a modern configuration, however, the information between the plant and the observer is transmitted over a communication channel. "Networked control systems" have seen a great deal of attention over the last decade. Figure 1 shows the general schematic of a networked control system.

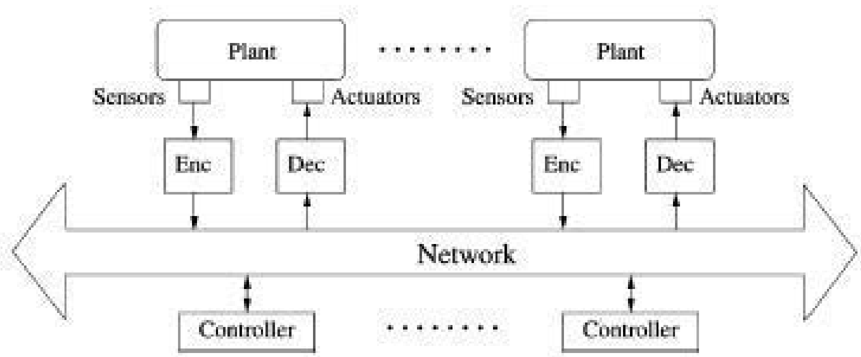


Figure 1.1: General schematic of networked control systems

Having multiple controllers, sensors and actuators connected through a communication network has several advantages such as low wiring cost, easy maintenance, reduced power requirements, and high reliability. However, these advantages become less noticeable as the number of the connected elements drop.

The insertion of the communication network in the feedback loop, on the other hand, makes the analysis and design of an NCS far more complex than a network-free control system. The main difficulties encountered are the following:

- Limited channel capacity and quantization effects: data rates used throughout networks are typically constrained, thus limiting the sampling rates used in control design. Quantization effects therefore become important because the number of quantization levels used in the transmission affects the communication flow and the capacity required to transmit information.

- Network-induced delays: time-varying delays originated and dependent upon network transmission delays.

- Packet-dropouts: information transmitted through the network is broken into a stream of packets. Depending on the characteristics of the network, packets can not only suffer delays but they can also be lost during transmission.

- Communication constraints: the presence of more than one sensor on the network asks for a network-access protocol. This leads to the unavailability of the sensor (or controller) data to the filter (or actuator) at every sampling instant.

A report published by Richard Murray in 2003, (see [2]) stated that control over communication networks would be one of the major challenges and future directions in control. Since those years, many important results have been obtained in this field. Pioneer work on NCS include, among others, [3], [4], [5], [6] and [7]. See also the recent survey [8] covering the subject up to 2007. In this thesis our interest is in the state estimation or *filtering* problem over communication networks. A filter is, in fact, a state observer with the capacity to limit the effects of exogenous inputs on the estimation error and is one of the classical problems encountered in systems theory. Perhaps the most celebrated result in estimation theory is the classical Kalman filter [9], which has been applied in numerous applications ranging from guidance, navigation and control, to biological systems [10], [11]. A standard assumption in Kalman filtering is that the process dynamics and measurements are affected by additive white noise with known covariance properties, which are not always easy to obtain. An alternative to the Kalman filtering problem is the $\mathcal{H}_\infty$ filtering which provides a guaranteed noise attenuation level in the presence of noise with unknown statistics. $\mathcal{H}_\infty$ filtering has received much attention. See, for example, [12], [13], [11] and the references therein.

The importance of networked control systems has also been officially acknowledged in industry. In 2007 the International Society of Automation (ISA) released the ISA100.11 protocol setting industrial standards for networked based control. In 2008 they also released the ISA100.11a, which focuses on process systems such as the petrochemical industry.

In this thesis our interest is in the robust filtering problem in a network setting. Our goal is to explicitly discuss the effects of the network on the filter stability and performance and to propose a design methodology that can limit those effects with an arbitrary attenuation level. Different approaches that deal with the filtering problem can be found in the literature. In [14] the problem of state estimation for discrete-time linear systems with quantized measurements is investigated. The quantization error is modeled as a multiplicative noise and a static-gain observer is designed to address the problem. [15] designs a filter for networked linear systems with communication constraints. In that work, a stochastic approach is taken to model the network access probability. In [16] filtering of discrete-time linear systems over wireless fading channels is discussed where a mobile sensor observes a dynamical system and sends its observation to a remote estimation unit. [17] considers the state estimation of continuous-time linear time varying uncertain systems via a limited capacity communication channel. [12] considers the problem of $\mathcal{H}_\infty$ estimation for continuous-time linear uncertain systems when network-induced problems such as quantization, delay and packet dropout are present. In [18] an stochastic approach is taken to estimate the states of a Lipschitz nonlinear system with time-varying delays in the states. [19] finds a minimum data transmission rate for the convergent estimation of a process with a specific distribution. [20] introduces a two-agent scheme, namely observer and estimator, under communication constraints. The former is located by the sensor and is responsible for evaluating the sensor data and transmitting them. The latter is by the user on the other end of the channel, and is responsible for using the received data to generate the state estimates. Another similar two-agent scheme is introduced in [21] under communication constraints, where the system is assumed to be uncertain and the estimator part is a Kalman filter. In [22] the problem of Kalman filtering over a network with packet dropouts is considered. In that work, authors model the information loss using a probability function and then find the minimum number of the packets that need to be transmitted for filter stability. More works on estimation and filtering over networks can be found in [23], [24], [25], [26] and the references therein.

Among the various problems induced by the network, we are more concentrated on the issues arose by quantization and packet dropouts. Most references in this area handle the quantization errors in one of two ways: (i) as model uncertainty when they deal with logarithmic quantization, or (ii) as an unknown disturbance when the quantizer is linear. In [22], the state estimation problem with packet dropouts is studied and an stochastic approach with Kalman filter is proposed. [14] considers a joint design procedure for both the estimator and the quantizer. In that work, both static and dynamic quantization schemes are studied and the trade-off between performance degradation and quantization density is investigated. In [17], the authors use the deterministic form of Kalman filter (see [27]) to design a state estimator in the presence of quantization. [20] investigates a two-agent estimation problem where the first agent observes the process and decides whether or not the current information should be disclosed to the second. The second agent then generates the state estimates based on the limited information received. In [24], the random packet dropout rate is modeled as an stochastic parameter and then a set of LMI's is derived to design an $\mathcal{H}_2$ optimal filter. References [28] and [29] consider the problem of $\mathcal{H}_\infty$ filtering in the presence of quantization and random sensor packet losses. In [15], channel accessing processes are modeled as Bernoulli processes and an optimal linear filter is designed using the orthogonal projection principle and the innovation analysis. [30] studies the filter design problem under uncertain delay for Lipschitz nonlinear systems.

Our focus in this thesis is on a specific class of filters known as Unknown Input Filters. Unknown disturbances can lead to significant deviations between the true plant states and those reconstructed by the observer and therefore much attention has been drawn to the solution of this problem. One important approach is by using the disturbance decoupling principle to render a state observer that is immune to those effects. The approach was first proposed by Wang, Davison and Dorato in 1975, [31], and since then it has been the subject of constant research. In [32] the authors design an unknown input observer with the assumption that the C matrix in the state space realization has a specific structure. The idea is to use a similarity transformation to partition the states into two groups such that only the second set of states is affected by the disturbance. A conventional observer is then designed for the first partition and the remaining states are obtained from them. In [33] the problem of unknown input observer design is discussed in detail for generalized state space models. [34] introduces an optimal unknown input filter with a form similar to the well-known Kalman filter. In [35] the authors assume bounded unknown inputs and propose a full-order observer, with the same state parameters as the original continuous-time linear system. This work was later extended in [36] to reduced order observers. [37] proposes a full-order observer for delay-free estimation of the system states by allowing very small errors between the original and estimated states, which account for the variations of the unknown input. In [38] the general structured observer framework is used to design an unknown input observer for discrete-time linear systems. [39] proposes a reduced order dynamic observer with an $\mathcal{H}_\infty$ performance measure. In [40] a new dynamical observer framework is used to design an $\mathcal{H}_\infty$ filter for Lipschitz nonlinear systems with unknown inputs. In [41] the authors combine an unknown input observer and a finite time observer to achieve finite time convergence of the estimation error dynamics. See also [42], [43], [44], [45], [33], [38], [39] , [40], and the references therein.

Unknown input observers rely on a key structural assumption on the state-space realization of the system; namely: $rank(CB_2) = rank(B_2)$, where C and B_2 are respectively the state space matrices corresponding to the measurements and the unknown inputs. This assumption was first introduced in [46] and places restrictive necessary and sufficient conditions in the solution of the decoupling problem and thus in the very existence of the unknown input observer. Several authors have attempted to circumvent this structural assumption and design unknown input observers under less restrictive conditions ([47], [48], [49], [50] , [51], [52] , and [53]).

These references overcome the structural limitations in [46] at the expense of alternative assumptions. In [47] and [48], Sundaram and Hadjicostis provide a characterization of observers with delay, which relaxes the well-known structural assumption. Their method provides a parameterization of the observer gain that decouples the unknown inputs from the estimation error and then uses the remaining freedom to ensure stability of the error system. [49] later extended this work by developing a design procedure that characterizes the set of all linear functionals of the system states that can be estimated by a linear observer with a given delay. In [50] an algorithm is proposed to design finite-time observers provided that the system is left-invertible with sampling delays. The algorithm computes some variables that are not affected by the unknown inputs and then performs a change of coordinates that makes the transformed system well-suited for designing the delayed estimator. In [51] the structural condition is circumvented by using a nonlinear sliding mode observer. [52] also employs a nonlinear sliding mode observer and provides finite-time error-free estimates assuming that the unknown inputs belong to $\mathcal{L}_2$. Unlike [51], their observer also generates estimates of the unknown inputs, a very important feature in fault detection applications. [53] replaces the structural condition in [46] with the assumption that the state space matrix A is invertible. In that work, the authors design an observer assuming that at each sampling instant, the unknown input can be approximated using a polynomial.

When both of the previously mentioned problems, namely filter design with limited information and unknown input filtering, are considered together, a joint problem is formed which is more complex to solve. The closest literature, which in a way addresses the joint problem can be found in the field of fault detection in networked systems where a very common approach for detecting a fault is by designing a residual generating filter. This filter is designed such that it is robust to the problems imposed by the network and yet sensitive to faults, which are usually modelled by an unknown signal affecting the states or the measurements. In [54] a fault detection filter is designed to detect ℓ_2-bounded faults when unknown disturbances and packet dropouts are present. [55] proposes an $\mathcal{H}_\infty$ approach to designing a residual generating filter which is robust to network-imposed delays and also packet dropouts. In [56] authors introduce a fault detection filter for a class of nonlinear systems when random delays and packet dropouts are present and the system is subject to ℓ_2 bounded faults and disturbances. Sensor fault detection through robust filter design under packet dropouts have also been studied in works such as [57], [58], and [59]. For more work in the area of fault detection filter design over networks, see for example [60], [61] and [62], among others.

One of the most important applications of robust filters is in the fault-related problems. The fault problem is usually dealt with from either a detection and isolation point of view or a tolerance and compensation perspective. In the field of fault detection and isolation (FDI), mainly a threshold analysis is done on the residuals which are generated by fault detection filters. These filters are designed in a way that they are very sensitive to specific user-defined faults. For example some works such as [63], [64], [65] and [66], model the fault as an unknown external disturbance and therefore design a filter which outputs a residual with high sensitivity to that external disturbance.

Fault accommodation, also known as fault-tolerant control, endeavours to maintain system stability as well as its performance in the presence of faults. In general, fault-tolerant control methods can fall under one of the following two main categories: I. Active and II. Passive. The approaches in the first category use an internal FDI scheme to either change the controller parameters (and even structure), or generate a compensation signal which is added to the main control signal, that is of course in case a fault is detected. Works such as [67], [68], [69], [70], [71], [72] and [73] are all examples of active methods. The approaches in the second category are robust control schemes which are designed in a way that they maintain system stability even in the presence of the worst expected scenario. Examples of this category can be found in, but not limited to, works such as [74], [75], [76], [77] and [78]. The passive approaches are usually simpler to design and implement compared to the active ones. However, they usually produce a considerably weaker performance and can cover a smaller range of faults.

With the growth of networked control systems in the past decade, fault-related problems in networked setups have attracted a lot of attention in the literature. Network-induced issues such as quantization, uncertain delays and packet dropouts make the fault detection and accommodation problems even more challenging. Various works have been done in the past few years in both the detection and compensation areas. [79] studies the fault detection problem for networked nonlinear systems when both the measurements and the control signal are subject to delays and packet losses. [60] proposes a fault detection filter for linear systems subject to Markovian packet losses. In [80] authors employ a Takagi-Sugeno fuzzy model to detect faults in a networked setup with delays. More works in this field can be found in [81], [82], [83], [55], [84], [85] and [86]. All of the works mentioned above investigate the detection problem, however, there are several works in the literature that tackle the accommodation aspect. [87] proposes a fault-tolerant control system for networked linear systems subject to access constraints and packet dropouts. In [88] authors use decoupling techniques to introduce a fault-tolerant controller for Lipschitz nonlinear systems undergoing delays and packet losses. More related works can be found in [89], [90], [91], [92] and [93].

1.2 Problem Formulation

Consider the following linear system:

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B_1 + \Delta B_1)u(k) + B_2 d(k) \\ y(k) &= (C + \Delta C)x(k) + Du(k) \\ z(k) &= Hx(k) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state vector; $y \in \mathbb{R}^p$ represents the measured outputs; $z \in \mathbb{R}^r$ is the vector to be estimated; $u \in \mathbb{R}^{m_1}$ is the known input; $d \in \mathbb{R}^{m_2}$ is the unknown input; $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m_1}$, $H \in \mathbb{R}^{r\times n}$ are the certain parameters of the model; and $\Delta A \in \mathbb{R}^{n\times n}$, $\Delta B_1 \in \mathbb{R}^{n\times m_1}$, $\Delta C \in \mathbb{R}^{p\times n}$ represent the uncertain parameters of the model. We assume that

- the measurements are quantized via a linear quantizer and transmitted through a communication network,
- (A, C) is an observable pair,
- B_2 has full column rank and C has full row rank with $p \geq m_2$,
- $rank(CB_2) = rank(B_2)$ is not necessarily satisfied.

The objective is to design a filter that produces estimates of the system states ib the presence of the model uncertainties and unknown disturbances when the measurements are transmitted through a communication channel.

1.3 Thesis Outline

The rest of this thesis is organized as follows:

Chapter 2: This chapter considers nonlinear discrete-time systems and proposse a filter design method in a network setting. The nonlinear system is modeled as a linear system plus a nonlinear function that satisfies a Lipschitz continuity condition and is disturbed by unwanted exogenous inputs. Lipschitz systems are important because they provide a mechanism that can account for the effects of, at least, mild nonlinearities and because of their generality most nonlinear system models can be represented as a linear system plus a Lipschitz nonlinearity, at least locally around an equilibrium point. In this work we assume that the data sent by the sensors is subject to finite-level quantization, uncertain delays and communication constraints. In order to understand the limitations introduced in the design by each of these effects, first the effects of quantization and uncertain delay are investigated separately, and distinct design procedures are introduced for each case. Next, these two issues along with the problem of communication constraints are assumed to be present simultaneously. Using a Lyapunov-Krasovskii function, an optimization problem with linear matrix inequalities is proposed to guarantee filter stability as well as an $\mathcal{H}_\infty$ bound on the error system. Although packet dropouts are not explicitly addressed, our modeling of communication constraints can easily include this case in the sense that when packet dropouts occur, we are forced to use the estimated measurement instead of the real one and therefore the effect can be modeled as the error between these two.

Chapter 3: In this chapter we propose a novel adaptive approach to design filters for discrete-time (i) linear and (ii) nonlinear Lipschitz systems, whose outputs are subject to quantization and limited channel capacity. We model network-induced effects as an unknown error between the real measurement and the one received by the filter, and employ an adaptive approach to estimate this error. The estimated error signal at every sampling instant is used to generate state estimates with desired attenuation bounds on the effects of state and measurement noise and also variations of the network-induced errors on the estimation error. A hysteresis quantizer is used for the quantization of the measurements. This quantizer reduces the noise-caused chattering between neighbouring quantization levels at the expense of a larger error margin between the real and quantized measurements. Since our approach is based on the estimation of this error regardless of its magnitude, the benefits of the hysteresis quantizer heavily outweigh its shortcomings. Our design process is first formulated as a linear matrix inequality (LMI) feasibility problem for linear systems and then extended to Lipschitz nonlinear systems.

Chapter 4: In this chapter we propose two new approaches to the design of unknown input filters that overcome the structural assumption given in [46], *i.e.* $rank(CB_2) = rank(B_2)$. Although not without limitations, we show that our assumptions are less restrictive than those in [46]. The main idea in both methods is to modify the plant model in a way that in the revised model the measurement is *directly* affected by the value of the unknown disturbance. The major distinction between the two proposed filters is the model of the filter on which the design is based. Throughout the chapter our focus is on the *filtering* problem. In other words, we consider the problem in which both states and measurements are disrupted by noise and design a filter to bound the effects of noise on the estimation error. The proposed filters provide both state and unknown input estimates, a very important property in fault detection and correction applications, with zero-delay for the states, and behave exactly as a conventional Luenberger observer in the absence of noise and unknown inputs. Necessary stability conditions are established using basic linear system theory for both methods, and then two LMI-based approaches are proposed to design the $\mathcal{H}_\infty$ filters. Finally, the proposed filters will be simulated for examples systems to show the effectiveness of the approach.

Chapter 5: This chapter studies the filter design problem when the system states are subject to unknown external disturbances and the measurements are transmitted through a network and therefore are erroneous. Our approach in this article is based on the concept of the reliability of the received information by the filter. We define a reliable packet of information as a piece of information which contains more information on the system behaviour rather than the network-induced errors. Similarly, an unreliable packet of information is a packet which contains more data about the network-induced errors than the system behaviour. Using this definition, the estimation space can be divided into two separate zones in which the received information needs to be treated differently. In one zone the information is considered reliable and therefore a filter is designed to estimate the states as well as the unknown disturbances, whereas in the other zone the information is unreliable and a filter is designed to take this into account when estimating the system states. Eventually an overall two-zone $\mathcal{H}_\infty$ filter is formulated using these two filters with optimized attenuation gains on the unwanted effects of the network and the unknown disturbance. Through simulation, it will be shown that the proposed two-zone filter is more effective than the predesigned single-zone filters.

Chapter 6: This chapter focuses on the design of a fault-tolerant filter under quantized measurements. In order to cover a wider range of potential faults, we model the internal faults as model uncertainties and common actuator faults such as offset and stuck as an unknown disturbance on the states. In addition to the existence of faults, we also assume that the measurements are quantized via a linear quantizer and therefore the measurements are subject to quantization errors in both presence and absence of faults. Our treatment for this problem is through a dual-zone robust filter which consists of two different sub-filters, each operating in one zone. The *dual-zone* idea is based on the fact that the effects of quantization become significant only when the estimation error is near the origin. Using this, we define zone 1 as the zone where quantization effects are insignificant and zone 2 as the zone where quantization effects are significant. A separate robust filter is designed for each zone and through a Lyapunov-based approach with $\mathcal{H}_\infty$ performance, necessary LMIs are derived. Finally, the effectiveness of the proposed filter is illustrated through simulation examples.

Chapter 7: This chapter discusses the concluding remarks and the suggested future research.

Chapter 2

$\mathcal{H}_\infty$ Filtering of Nonlinear Plants over Networks

In this chapter, we consider the filtering problem for Lipschitz systems in a networked environment. We assume that the measurements transmitted over the network are subject to quantization, uncertain delays and communication constraints. We first analytically demonstrate how each of the these issues affect the filtering problem. Second, we tackle the filter design as an optimization problem with LMI constraints. The optimization maximizes the Lipschitz constant and thus the region of attraction for which the filter is stable and an $\mathcal{H}_\infty$ bound is satisfied by the error system.

The rest of the chapter is organized as follows. In section 2.1, we introduce the notation used in this chapter along with the plant and filter models. section 2.2 discusses the effects of the network-imposed problems on the filter design and proposes design methodologies. In section 2.3 the proposed filters are tested via simulation and section 2.4 summarizes the results of this chapter.

2.1 Plant and Filter Models

Consider now the following discrete-time nonlinear plant model,

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) + \phi(x,u) \\ y(k) &= Cx(k) + v(k) \\ z(k) &= Hx(k) \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ represents the measured outputs, and u is the control input. $z \in \mathbb{R}^r$ is the signal to be estimated, and $w \in \mathbb{R}^m$ and $v \in \mathbb{R}^p$ denote state and measurement noises, respectively. Both w and v are assumed to be in ℓ_2. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are the state space matrices of the linear part of the model; and ϕ is a Lipschitz function with Lipschitz constant l.

We note that the model description (2.1) is very general and can provide an accurate description of a large number of systems of interest, at least locally in a neighbourhood of an equilibrium point. Before introducing the filter model, we define the static logarithmic quantizer as follows (see [14]).

Definition 2.1. *A static logarithmic quantizer is given by*

$$\bar{y} = Q(y) = \begin{cases} \rho^j \mu & if \quad \frac{\rho^j \mu}{1+\delta} \leq y < \frac{\rho^j \mu}{1-\delta} \\ 0 & if \quad y = 0 \\ -Q(-y) & if \quad y < 0 \end{cases} \tag{2.2}$$

where $j = 0, \pm 1, \pm 2, \ldots$, and $Q(.)$ is the quantization function, $0 < \rho < 1$ is the quantization density, μ is a scaling parameter, and

$$\delta = (1-\rho)/(1+\rho) \tag{2.3}$$

For a signal quantized by (2.2), the quantization error is given as

$$e_q = \bar{y} - y = \Lambda y \tag{2.4}$$

where Λ is an uncertain variable (see [94]), which depends on y and is bounded by δ, i.e.

$$-\delta \leq \Lambda \leq \delta \tag{2.5}$$

Figure 2.1 shows the quantization error e_q with respect to y.

The discrete-time nonlinear filter is given by

$$\begin{aligned} x_F(k+1) &= Ax_F(k) + L(\mathcal{Y}(k) - y_F(k)) + \phi(x_F, u) \\ y_F(k) &= Cx_F(k) \\ z_F(k) &= Hx_F(k) \end{aligned} \tag{2.6}$$

where $x_F \in \mathbb{R}^n$ and $y_F \in \mathbb{R}^p$ are, respectively, the state and output vectors, and $z_F \in \mathbb{R}^r$ is the output estimate of the filter. L is the filter parameter to be designed, and $\mathcal{Y}(k) \in \mathbb{R}^p$ is the feedback term, which is different for each of the three problems to be discussed; namely, quantization, delay, and communication constraints.

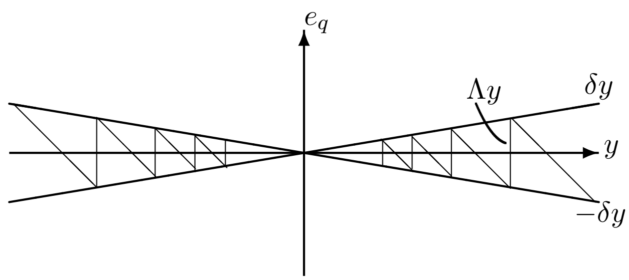


Figure 2.1: Quantization Error

2.2 Filter Design

In this section, we endeavour to design stable filters of the form (2.6) for the plant (2.1), when any or all of the aforementioned problems arise due to the presence of a communication network. More explicitly, we will pursue filter design under three different scenarios:

- measurements are quantized.
- measurements are transmitted with uncertain delay.
- quantized measurements are transmitted with uncertain delay and are subject to communication constraints.

We note that the delay is only present in the measurements not the states, and our proposed filter will be delay-independent in the sense that the structure of the filter itself is free of any delays.

Our interest is in designing a filter with the following properties:

- (Stability) In the absence of external disturbances the observer error converges to zero asymptotically.
- (Filtering) The region of attraction is maximized for an arbitrary attenuation level μ on the effects of exogenous disturbances on the estimation error; *i.e.* we find a maximized Lipschitz constant l such that

$$\|\varepsilon\| < \mu\|\omega\|$$

where ε is the estimation error, and $\omega \in \ell_2$ is the vector of exogenous disturbances defined as follows:

$$\varepsilon(k) = z(k) - z_F(k) \tag{2.7}$$

$$\omega = \begin{bmatrix} w \\ v \end{bmatrix} \tag{2.8}$$

Our solution is based on the use of Linear Matrix Inequalities (LMIs) and is therefore free of the stringent existence requirements encountered in the Riccati approach. Our design procedure is *effective* in the sense that it renders a stable observer, if one exists and can be solved efficiently using commercially available softwares. Some related results for discrete-time systems were recently presented in [95] and [96]. See also [30] for an earlier version of this work.

Remark 2.1. *Throughout this chapter, we will focus on the maximization of the Lipschitz constant assuming a given disturbance attenuation gain μ. However, the optimization problem can also be stated as the minimization of μ while assuming a given Lipschitz constant l. In our simulation section, we will illustrate a trade-off curve between the two optimization parameters.*

Before investigating the effects of network-imposed problems on the filter design process, the following lemmas need to be introduced.

Lemma 2.1. *([97])*
For any $x, y \in \mathbb{R}^n$ and any positive definite matrix $T \in \mathbb{R}^{n\times n}$, we have :

$$2x^T y \leq x^T T x + y^T T^{-1} y$$

Lemma 2.2. *([97])*
Let A, E, F, Λ and P be real matrices of appropriate dimensions with $P > 0$ and Λ satisfying $\Lambda^T \Lambda \leq I$. Then for any scalar $\epsilon > 0$ satisfying $P^{-1} - \epsilon^{-1} E E^T > 0$, we have:

$$\begin{aligned}(A + E\Lambda F)^T P (A + E\Lambda F) \leq A^T (P^{-1} - \epsilon^{-1} E E^T)^{-1} A \\ + \epsilon F^T F\end{aligned} \tag{2.9}$$

2.2.1 Filter Design with Quantized Measurements

In this subsection, we consider the filter design problem with quantized measurements. When the measurements are quantized, $\mathcal{Y}$ in (2.6) is given by $\mathcal{Y}(k) = \bar{y}(k)$, where $\bar{y}$ is as defined in (2.2). We define the state error as $e(k) = x(k) - x_F(k)$, which leads to the following error dynamics:

$$e(k+1) = (A - LC)e(k) - L\Lambda C x(k) + Bw(k)$$

$$
\begin{aligned}
&- Lv(k) - L\Lambda v(k) + \Delta\phi(x, x_F, u) \\
\varepsilon(k) &= z(k) - z_F(k) = He(k)
\end{aligned} \tag{2.10}
$$

where $\Delta\phi(x, x_F, u) = \phi(x, u) - \phi(x_F, u)$, and ε is known as the estimation error. Augmenting the plant and error models, we get

$$
\begin{aligned}
X(k+1) &= (\mathbb{A} + \Delta\mathbb{A})X(k) + (\mathbb{B} + \Delta\mathbb{B})\omega(k) + \Omega(X, u) \\
\varepsilon(k) &= \mathbb{C}X(k)
\end{aligned} \tag{2.11}
$$

where

$$
X = \begin{bmatrix} x \\ e \end{bmatrix} \quad \omega = \begin{bmatrix} w \\ v \end{bmatrix} \quad \Omega(X, u) = \begin{bmatrix} \phi(x, u) \\ \Delta\phi(x, x_F, u) \end{bmatrix} \tag{2.12}
$$

and

$$
\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \quad \Delta\mathbb{A} = \begin{bmatrix} 0 & 0 \\ -L\Lambda C & 0 \end{bmatrix} \\
\mathbb{B} &= \begin{bmatrix} B & 0 \\ B & -L \end{bmatrix} \quad \Delta\mathbb{B} = \begin{bmatrix} 0 & 0 \\ 0 & -L\Lambda \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix}.
\end{aligned}
$$

The augmented system is also Lipschitz and its Lipschitz constant is calculated as follows,

$$
\begin{aligned}
&\Omega^T\Omega = \phi^T\phi + \Delta\phi^T\Delta\phi \leq l^2(x^T x + e^T e) \\
&\Rightarrow \|\Omega\| \leq l\|X\|
\end{aligned} \tag{2.13}
$$

The following theorem establishes a filter design methodology for Lipschitz nonlinear systems with quantized measurements.

Theorem 2.1. *Consider the plant (2.1) with measurements quantized by (2.2). Then the filter given in (2.6) is optimal with an $\mathcal{H}_\infty$ bound μ on the effects of the unwanted external inputs on the estimation error, if there exist scalars α, η, $\epsilon > 0$ and matrices G, and $P = diag\{P_1, P_2\} > 0$ for which the following optimization problem has a solution:*

$$
\min \bar{w}\alpha + \eta \tag{2.14}
$$

s.t.

$$\begin{bmatrix} -P+\mathbb{C}^T\mathbb{C}+\epsilon F^TF & I & 0 & \mathbb{A}^TP & 0 & 0 \\ \star & -\alpha I & 0 & 0 & 0 & 0 \\ \star & \star & -\mu^2 I & \mathbb{B}^TP & 0 & 0 \\ \star & \star & \star & -P & P & E \\ \star & \star & \star & \star & -\eta I & 0 \\ \star & \star & \star & \star & \star & -\epsilon I \end{bmatrix} < 0 \tag{2.15}$$

where $\bar{w} > 0$ is an optimization weight for the Lipschitz constant, and

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ 0 & A-LC \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B & 0 \\ B & -L \end{bmatrix}, \quad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix}.$$

$$E = \begin{bmatrix} 0 & -G \end{bmatrix}^T, \quad F = \begin{bmatrix} C & 0 & 0 & I \end{bmatrix} \tag{2.16}$$

Also the optimal filter gain L and Lipschitz constant l can be calculated as follows,

$$L = P_2^{-1}G, \quad l = 1/\sqrt{\alpha\eta} \tag{2.17}$$

Proof. Consider the following discrete-time Lyapunov function:

$$V(k) = X(k)^TPX(k) \tag{2.18}$$

The forward difference of this Lyapunov function along the trajectories of the augmented system can be written as

$$\begin{aligned} \Delta V &= V(k+1) - V(k) \\ &= \xi^T(\Gamma_2' + \Delta\Gamma)^TP(\Gamma_2' + \Delta\Gamma)\xi + \Omega^TP\Omega \\ &- X^TPX + 2\xi^T(\Gamma_2' + \Delta\Gamma)^TP\Omega \end{aligned} \tag{2.19}$$

where $\xi = \begin{bmatrix} X^T & \omega^T \end{bmatrix}^T$ and $\Gamma_2' = \begin{bmatrix} \mathbb{A} & \mathbb{B} \end{bmatrix}$, $\Delta\Gamma = \begin{bmatrix} \Delta\mathbb{A} & \Delta\mathbb{B} \end{bmatrix}$.

Using lemma 2.1 we can write

$$2\xi^T\Gamma^TP\Omega \leq \xi^T\Gamma^TPT_1{}^{-1}P\Gamma\xi + \Omega^TT_1\Omega \tag{2.20}$$

where $\Gamma = \Gamma_2' + \Delta\Gamma$. We choose $T_1 = W$ where $W = \eta I - P > 0$ for some $\eta > 0$. Substituting (2.20) in (2.19), we get

$$\Delta V \leq \xi^T(\Gamma_2 + P\Delta\Gamma)^T(P^{-1} + W^{-1})(\Gamma_2 + P\Delta\Gamma)\xi$$

$$+ \xi^T \Gamma_1'' \xi \tag{2.21}$$

where $\Gamma_2 = P\Gamma_2'$, and $\Gamma_1'' = diag\{\eta l^2 I - P, 0\}$. Since P and W are codependent, we follow the procedure introduced in [95] to simplify the term $P^{-1} + W^{-1}$,

$$\begin{aligned}
P^{-1} + W^{-1} &= P^{-1} + (\eta I - P)^{-1} \\
&= (\eta I - P)^{-1}((\eta I - P)P^{-1} + I) \\
&= (\eta I - P)^{-1}\eta P^{-1} \\
&= (P - \eta^{-1}P^2)^{-1}
\end{aligned} \tag{2.22}$$

To handle the uncertain element $\Delta\Gamma$, we can write $P\Delta\Gamma = E\Lambda F$, where $E = P\begin{bmatrix}0 & -L^T\end{bmatrix}^T$, $F = \begin{bmatrix}C & 0 & 0 & I\end{bmatrix}$. Now, using lemma 2.2 and (2.22), the following inequalities are established:

$$\begin{aligned}
&(\Gamma_2 + P\Delta\Gamma)^T(P^{-1} + W^{-1})P(\Gamma_2 + P\Delta\Gamma_1) \\
&\quad \leq \Gamma_2^T(P - \eta^{-1}P^2 - \epsilon^{-1}EE^T)^{-1}\Gamma_1 + \epsilon F^T F
\end{aligned}$$

The simplified ΔV is given by

$$\Delta V \leq \xi^T \Gamma_2^T (P - \eta^{-1}P^2 - \epsilon^{-1}EE^T)^{-1}\Gamma_2 \xi + \xi^T \Gamma_1' \xi \tag{2.23}$$

where $\Gamma_1' = \Gamma_1'' + \epsilon F^T F$. Since our problem is filtering (not just observation), we need to bound the ratio of the estimation error to unwanted external inputs. To this end, we define

$$J \triangleq \sum_{k=0}^{\infty} \{\varepsilon(k)^T \varepsilon(k) - \mu^2 \omega(k)^T \omega(k)\} \tag{2.24}$$

Adding (2.18) to the right hand side of (2.24), we get $J \leq \sum_{k=0}^{\infty} \tilde{J}_k$, where

$$\tilde{J}_k = \varepsilon(k)^T \varepsilon(k) - \mu^2 \omega(k)^T \omega(k) + \Delta V_k \tag{2.25}$$

Now, if we design our filter such that $\tilde{J}_k \leq 0$, we can conclude that $J \leq 0$, which is equivalent to $\|\varepsilon\|^2 < \mu^2 \|\omega\|^2$. This implies that the second norm of the estimation error is bounded by a factor of the second norm of the exogenous input. In other words, this establishes an $\mathcal{H}_\infty$ bound on the estimation error system. Using (2.11) and (2.23), we have

$$\tilde{J}_k \leq \xi^T (\Gamma_1 + \Gamma_2^T \Gamma_3^{-1} \Gamma_2) \xi \tag{2.26}$$

where $\Gamma_2 = \Gamma_2' + diag\{\mathbb{C}^T\mathbb{C}, -\mu^2 I\}$, and $\Gamma_3 = P - \eta^{-1}P^2 - \epsilon^{-1}EE^T$.

To avoid running into bilinear matrix inequalities, the following variable changes need to be performed (see [95]):

$$P = diag\{P_1, P_2\}, \quad G = P_2 L, \quad \alpha^{-1} = \eta l^2 \tag{2.27}$$

Now for stability we should have $\tilde{J}_k < 0$, which by using Schur's complement, leads to the LMI given in (2.15). $\square$

Remark 2.2. *According to (2.17), maximization of the Lipschitz constant l can be accomplished by simultaneous minimization of α and η. A common way of solving this two-objective optimization problem is to linearly combine these two objective functions into a single one like the one given in (2.14).*

2.2.2 Filter Design with Delayed Measurements

In this subsection we focus on filter design in the presence of uncertain transmission delay. When the measurements are transmitted with an uncertain delay d_k satisfying $0 \leq d_k \leq d_M$, one can express $\mathcal{Y}$ in (2.6) as $\mathcal{Y}(k) = y(k - d_k)$. Defining the estimation error as $e(k) = x(k) - x_F(k)$ leads to the following error system:

$$\begin{aligned} e(k) &= (A - LC)e(k) + LCx(k) - LCx(k - d_k) \\ &\quad + Bw(k) + Lv(k - d_k) + \Delta\phi(x, x_F, u) \\ \varepsilon(k) &= z(k) - z_F(k) = He(k) \end{aligned} \tag{2.28}$$

where $\Delta\phi(x, x_F, u) = \phi(x, u) - \phi(x_F, u)$. The augmented system will be given as

$$\begin{aligned} X(k+1) &= \mathbb{A}X(k) + \mathbb{A}_d X(k - d_k) + \mathbb{B}\omega(k) + \Omega(X, u) \\ \varepsilon(k) &= \mathbb{C}X(k) \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} X(k) &= \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \quad \omega = \begin{bmatrix} w(k) \\ v(k - d_k) \end{bmatrix} \\ \Omega(X, u) &= \begin{bmatrix} \phi(x, u) \\ \Delta\phi(x, x_F, u) \end{bmatrix} \end{aligned} \tag{2.30}$$

and

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \quad \mathbb{A}_d = \begin{bmatrix} 0 & 0 \\ -LC & 0 \end{bmatrix}$$

$$\mathbb{B} = \begin{bmatrix} B & 0 \\ B & -L \end{bmatrix} \qquad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix} \tag{2.31}$$

Similar to the quantization case, the augmented system is also Lipschitz and its Lipschitz constant is l. The following theorem establishes a filter design methodology for Lipschitz nonlinear systems affected by variable delays.

Theorem 2.2. *Given the plant (2.1) with measurements transmitted with an uncertain delay d_k satisfying $0 \leq d_k \leq d_M$. Then the filter given in (2.6) is optimal with an $\mathcal{H}_\infty$ bound μ on the effects of the unwanted external inputs on the estimation error, if there exist scalars α, η_1, $\eta_2 > 0$ and matrices $P = P' + P''$, $P' = diag\{P_1', P_2'\}, P'' = diag\{P_1'', P_2''\} > 0$, and $Q, R \geq 0$, and G, and M, S, N for which the following optimization problem has a solution:*

$$\min \bar{w}\alpha + \eta_1 + d_M\eta_2 \tag{2.32}$$

s.t.

$$\begin{bmatrix} \Gamma_J + \Gamma_3 + \Gamma_4 + \Gamma_4^T & \Gamma_0 & \Gamma_1^T & 0 & \Gamma_2^T & 0 & \Gamma_5 \\ \star & -\alpha I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -P & P & 0 & 0 & 0 \\ \star & \star & \star & -\eta_1 I & 0 & 0 & 0 \\ \star & \star & \star & \star & -P & P & 0 \\ \star & \star & \star & \star & \star & -\eta_2 I & 0 \\ \star & \star & \star & \star & \star & \star & -\Gamma_6 \end{bmatrix} < 0 \tag{2.33}$$

where $\bar{w} > 0$ is an optimization weight for the Lipschitz constant, and

$$\Gamma_J = diag\{\mathbb{C}^T\mathbb{C}, 0, 0, -\mu^2 I\}$$

$$\Gamma_0 = diag\{I, 0, 0, 0\}$$

$$\Gamma_1 = \begin{bmatrix} P\mathbb{A} & P\mathbb{A}_d & 0 & P\mathbb{B} \end{bmatrix}$$

$$\Gamma_2 = \sqrt{d_M} \begin{bmatrix} P(\mathbb{A} - I) & P\mathbb{A}_d & 0 & P\mathbb{B} \end{bmatrix}$$

$$\Gamma_3 = diag\{-P + \mathbb{C}^T\mathbb{C} + (d_M + 1)Q + R, -Q, -R, 0\}$$

$$\Gamma_4 = \begin{bmatrix} M+N & S-M & -S-N & 0 \end{bmatrix}$$
$$\Gamma_5 = \begin{bmatrix} \sqrt{d_M}M & \sqrt{d_M}S & \sqrt{d_M}N \end{bmatrix}$$
$$\Gamma_6 = diag\{P', P', P''\} \tag{2.34}$$

with

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix}, \quad \mathbb{A}_d = \begin{bmatrix} 0 & 0 \\ -LC & 0 \end{bmatrix}$$
$$\mathbb{B} = \begin{bmatrix} B & 0 \\ B & -L \end{bmatrix}, \qquad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix}$$

Also the optimal filter gain L and Lipschitz constant l can be calculated as follows,

$$L = (P_2' + P_2'')^{-1}G, \quad l = 1/\sqrt{\alpha(\eta_1 + d_M\eta_2)} \tag{2.35}$$

Proof. Consider the following discrete-time Lyapunov-Krasovskii function ([98]):

$$\begin{aligned} V(k) &= X(k)^T P X(k) + \sum_{i=k-d_k}^{k-1} X(i)^T Q X(i) \\ &+ \sum_{i=k-d_M}^{k-1} X(i)^T R X(i) + \sum_{j=-d_M+1}^{0} \sum_{i=k+j}^{k-1} X(i)^T Q X(i) \\ &+ \sum_{j=-d_M}^{-1} \sum_{i=k+j}^{k-1} \tau(i)^T (P' + P'')\tau(i) \end{aligned} \tag{2.36}$$

where P', P'', $P = P' + P'' > 0$, Q, $R \geq 0$, and $\tau(k) = X(k+1) - X(k) = (\mathbb{A} - I)X(k) + \mathbb{A}_d X(k-d_k) + \mathbb{B}\omega(k) + \Omega(X,u)$. The forward difference of (2.36) can be written as

$$\begin{aligned} \Delta V &= \xi^T \Gamma_1'^T P \Gamma_1' \xi + 2\xi^T \Gamma_1'^T P \Omega + \Omega^T P \Omega + \xi^T \Gamma_3'' \xi \\ &- \sum_{i=k-d_M+1}^{k} X(i)^T Q X(i) + d_M \xi^T \Gamma_2'^T P \Gamma_2' \xi \\ &+ 2d_M \xi^T \Gamma_2'^T P \Omega + d_M \Omega^T P \Omega - \sum_{i=k-d_M}^{k-d_k-1} \tau(i)^T P' \tau(i) \\ &- \sum_{i=k-d_k}^{k-1} \tau(i)^T P' \tau(i) - \sum_{i=k-d_M}^{k-1} \tau(i)^T P'' \tau(i) \end{aligned} \tag{2.37}$$

where $\xi = \begin{bmatrix} X(k)^T & X(k-d_k)^T & X(k-d_M)^T & \omega(k)^T \end{bmatrix}^T$, $\Gamma_1' = \begin{bmatrix} \mathbb{A} & \mathbb{A}_d & 0 & \mathbb{B} \end{bmatrix}$, $\Gamma_3' = diag\{-P + (d_M+1)Q + R, -Q, -R, 0\}$, $\Gamma_2' = \begin{bmatrix} \mathbb{A} - I & \mathbb{A}_d & 0 & \mathbb{B} \end{bmatrix}$. Now, using lemma 2.1, we can establish the following inequalities:

$$2\Gamma_1'^T P\Omega \leq \Omega^T W_1 \Omega + \Gamma_1'^T P W_1^{-1} P \Gamma_1' \tag{2.38}$$

$$2\Gamma_2'^T P\Omega \leq \Omega^T W_2 \Omega + \Gamma_2'^T P W_2^{-1} P \Gamma_2' \tag{2.39}$$

$$\begin{aligned}
&2\xi^T M \left(X(k) - X(k-d_k) - \textstyle\sum_{i=k-d_k}^{k-1} \tau(i) \right) \\
&\quad \leq \sum_{i=k-d_k}^{k-1} \tau(i)^T P' \tau(i) + d_k \xi^T M P'^{-1} M^T \xi \\
&\quad + 2\xi^T M \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \xi
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
&2\xi^T S \left(X(k-d_k) - X(k-d_M) - \textstyle\sum_{i=k-d_M}^{k-d_k-1} \tau(i) \right) \\
&\quad \leq \sum_{i=k-d_M}^{k-d_k-1} \tau(i)^T P' \tau(i) + (d_M - d_k) \xi^T S P'^{-1} S^T \xi \\
&\quad + 2\xi^T S \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix} \xi
\end{aligned} \tag{2.41}$$

$$\begin{aligned}
&2\xi^T N \left(X(k) - X(k-d_M) - \textstyle\sum_{i=k-d_M}^{k-1} \tau(i) \right) \\
&\quad \leq \sum_{i=k-d_M}^{k-1} \tau(i)^T P'' \tau(i) + d_M \xi^T N P''^{-1} N^T \xi \\
&\quad + 2\xi^T N \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} \xi
\end{aligned} \tag{2.42}$$

where $W_i = \eta_i I - P > 0 \quad i = 1, 2$, and $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & 0 \end{bmatrix}^T$, $S = \begin{bmatrix} S_1^T & S_2^T & S_3^T & 0 \end{bmatrix}^T$, $N = \begin{bmatrix} N_1^T & N_2^T & N_3^T & 0 \end{bmatrix}^T$ are matrices with appropriate dimensions. The first two inequalities help us eliminate the terms involving the nonlinear function Ω, while the last three provide a less conservative approach to avoid the summation terms in (2.37). It should be noted that in the last three inequalities, the left-hand side of the inequality is equal to zero. Using the above inequalities and also the Lipschitz property of Ω, we get

$$\begin{aligned}
\Delta V &\leq \xi^T \Gamma_3 \xi + \xi^T (\Gamma_4 + \Gamma_4^T) \xi + \xi^T \Gamma_1^T (P^{-1} + W_1^{-1}) \Gamma_1 \xi \\
&+ \xi^T \Gamma_2^T (P^{-1} + W_2^{-1}) \Gamma_2 \xi + d_k \xi^T M P'^{-1} M^T \xi \\
&+ d_M \xi^T N P''^{-1} N^T \xi + (d_M - d_k) \xi^T S P'^{-1} S^T \xi
\end{aligned} \tag{2.43}$$

where

$$\Gamma_1 = P\Gamma_1' \quad , \quad \Gamma_2 = \sqrt{d_M} P \Gamma_2'$$

$$\Gamma_3 = \Gamma'_3 + diag\{(\eta_1 + d_M\eta_2)l^2 I, 0, 0, 0\}$$
$$\Gamma_4 = \begin{bmatrix} M+N & S-M & -S-N & 0 \end{bmatrix}$$

Now defining $\Gamma_5 = \begin{bmatrix} \sqrt{d_M}M & \sqrt{d_M}S & \sqrt{d_M}N \end{bmatrix}$, $\Gamma_6 = diag\{P', P', P''\}$, and using (2.22), one can simplify ΔV as given by

$$\begin{aligned} \Delta V \leq \xi^T(&\Gamma_3 + \Gamma_4 + \Gamma_4^T + \Gamma_1^T\Gamma_7^{-1}\Gamma_1 \\ &+ \Gamma_2^T\Gamma_8^{-1}\Gamma_2 + \Gamma_5\Gamma_6^{-1}\Gamma_5^T)\xi \end{aligned} \tag{2.44}$$

where $\Gamma_5 = P - \eta_1^{-1}P^2$, and $\Gamma_6 = P - \eta_2^{-1}P^2$. It should be noted that d_k is an uncertain variable with upper and lower bounds, and thus it cannot be used in the design formulation. As a result, it needs to be replaced with its bounds such that the inequality (2.58) still holds, as done in formulating (2.44). Now, for the $\mathcal{H}_\infty$ filtering problem, J and $\tilde{J}_k$ are defined as in (2.24) and (2.25), respectively. Using (2.44), we have

$$\begin{aligned} \tilde{J}_k \leq \xi^T(&\Gamma_3 + \Gamma_J + \Gamma_4 + \Gamma_4^T + \Gamma_1^T\Gamma_7^{-1}\Gamma_1 \\ &+ \Gamma_2^T\Gamma_8^{-1}\Gamma_2 + \Gamma_5\Gamma_6^{-1}\Gamma_5^T)\xi \end{aligned} \tag{2.45}$$

where $\Gamma_J = diag\{\mathbb{C}^T\mathbb{C}, 0, 0, -\mu^2 I\}$. Similar to the quantization case, in order to convert bilinear matrix inequalities into linear ones, the following variable changes need to be performed:

$$\begin{aligned} P' &= diag\{P'_1, P'_2\}, \quad P'' = diag\{P''_1, P''_2\} \\ G &= (P'_2 + P''_2)L, \quad \alpha^{-1} = (\eta_1 + d_M\eta_2)l^2 \end{aligned} \tag{2.46}$$

Now for stability we should have $\tilde{J}_k < 0$, which by using Schur's complement is the same as the LMI given in (2.33). □

Remark 2.3. *According to (2.35), maximization of the Lipschitz constant l can be accomplished by simultaneous minimization of α, η_1 and η_2. A common way of solving this multi-objective optimization problem is to linearly combine these objective functions into a single one like the one given in (2.32).*

2.2.3 Filter Design with Variably Delayed Quantized Measurements Subject to Communication Constraints

In this subsection, we assume that due to the presence of a communication channel, quantized measurements are transmitted with an uncertain delay and also communication constraints apply. When the measurements are quantized, transmitted with an uncertain delay and subject to communication constraints, $\mathcal{Y}$ in (2.6) is given as follows,

$$\mathcal{Y}(k) = \begin{cases} \bar{y}(k-d_k) & \text{if network access is granted} \\ y_F(k) & \text{otherwise} \end{cases} \tag{2.47}$$

where d_k is the uncertain delay satisfying $0 \leq d_k \leq d_M$. From equation (2.47), one can see that when a sensor is granted access to the network, our filter will use its data, which is delayed and quantized, for the correction of the estimates. However, when there is no data coming from the sensors, the estimates will not be corrected. In other words, the latter means that the filter will operate in open loop until it receives new data from the sensor. We can rewrite (2.47) as

$$\bar{y}_\nu(k) = y(k-d_k) + \Lambda y(k-d_k) + \Delta y_\nu(k) \tag{2.48}$$

where Λy is the error induced by quantization, and Δy_ν is the bounded error induced by communication constraints (unavailability of sensor data). It is important to note that Λy and d_k are both zero for those elements of y, for which we have no sensor data. On the other hand, for the ones that we have sensor data, $\Delta y_\nu = 0$.

The error system is given as:

$$\begin{aligned} e(k+1) &= (A-LC)e(k) + LCx(k) - LCx(k-d_k) \\ &\quad - L\Lambda Cx(k-d_k) + Bw(k) - Lv(k-d_k) \\ &\quad - L\Lambda v(k-d_k) - L\Delta y_\nu(k) + \Delta\phi(x, x_F, u) \\ \varepsilon(k) &= z(k) - z_F(k) = He(k) \end{aligned} \tag{2.49}$$

where $\Delta\phi(x, x_F, u) = \phi(x, u) - \phi(x_F, u)$. For this case, the augmented system is given by,

$$\begin{aligned} X(k+1) &= \mathbb{A}X(k) + (\mathbb{A}_d + \Delta\mathbb{A}_d)X(k-d_k) \\ &\quad + (\mathbb{B} + \Delta\mathbb{B})\omega(k) + \Omega(X, u) \\ \varepsilon(k) &= \mathbb{C}X(k) \end{aligned} \tag{2.50}$$

where

$$X(k) = \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \quad \omega(k) = \begin{bmatrix} w(k) \\ v(k-d_k) \\ \Delta y_\nu(k) \end{bmatrix}$$

$$\Omega(X,u) = \begin{bmatrix} \phi(x,u) \\ \Delta\phi(x,x_F,u) \end{bmatrix} \tag{2.51}$$

and

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} \quad \mathbb{A}_d = \begin{bmatrix} 0 & 0 \\ -LC & 0 \end{bmatrix}$$

$$\mathbb{B} = \begin{bmatrix} B & 0 & 0 \\ B & -L & -L \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix}$$

$$\Delta\mathbb{A}_d = \begin{bmatrix} 0 & 0 \\ -L\Lambda C & 0 \end{bmatrix} \quad \Delta\mathbb{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -L\Lambda & 0 \end{bmatrix} \tag{2.52}$$

Similar to previous cases, Ω is Lipschitz with the constant l.

Theorem 2.3. *Given the plant (2.1) with measurements quantized by (2.2) and transmitted with the uncertain delay d_k, where $0 \leq d_k \leq d_M$, and subject to communication constraints. Then the filter given in (2.6) is optimal with $\mathcal{H}_\infty$ bounds μ and ρ on the effects of the unwanted external inputs and communication constraints on the estimation error, respectively, if there exist scalars α, η_1, η_2, ϵ_1, $\epsilon_2 > 0$ and matrices $P = P' + P''$, $P' = diag\{P_1', P_2'\}, P'' = diag\{P_1'', P_2''\} > 0$, and $Q, R \geq 0$, and G, and M, S, N for which the following optimization problem has a solution:*

$$\min \bar{w}\alpha + \eta_1 + d_M\eta_2 \tag{2.53}$$

s.t.

$$\begin{bmatrix} \Gamma_J + \Gamma_3 + \Gamma_4 + \Gamma_4^T & \Gamma_0 & \Gamma_1^T & 0 & 0 & \Gamma_2^T & 0 & 0 & \Gamma_5 \\ \star & -\alpha I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -P & P & E & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\eta_1 I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\epsilon_1 I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P & P & E & 0 \\ \star & \star & \star & \star & \star & \star & -\eta_2 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\epsilon_2 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -\Gamma_6 \end{bmatrix} < 0 \tag{2.54}$$

where $\bar{w}$ is the optimization weight for Lipschitz constant, and

$$\Gamma_J = diag\{\mathbb{C}^T\mathbb{C}, 0, 0, diag\{-\mu^2 I, -\mu^2 I, -\rho^2 I\}\}$$

$$\Gamma_0 = diag\{I, 0, 0, 0\}$$
$$\Gamma_1 = \begin{bmatrix} P\mathbb{A} & P\mathbb{A}_d & 0 & P\mathbb{B} \end{bmatrix}$$
$$\Gamma_2 = \sqrt{d_M} \begin{bmatrix} P(\mathbb{A} - I) & P\mathbb{A}_d & 0 & P\mathbb{B} \end{bmatrix}$$
$$\Gamma_3 = diag\{-P + (\epsilon_1 + d_M\epsilon_2)F^T F + (d_M + 1)Q + R, -Q, -R, 0\}$$
$$\Gamma_4 = \begin{bmatrix} M + N & S - M & -S - N & 0 \end{bmatrix}$$
$$\Gamma_5 = \sqrt{d_M} \begin{bmatrix} M & S & N \end{bmatrix} \quad \Gamma_6 = diag\{P', P', P''\} \tag{2.55}$$

with

$$\mathbb{A} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix}, \quad \mathbb{A}_d = \begin{bmatrix} 0 & 0 \\ -LC & 0 \end{bmatrix}$$
$$\mathbb{B} = \begin{bmatrix} B & 0 & 0 \\ B & -L & -L \end{bmatrix}, \quad \mathbb{C} = \begin{bmatrix} 0 & H \end{bmatrix}$$
$$E = \begin{bmatrix} 0 & -G \end{bmatrix}^T, \quad F = \begin{bmatrix} 0 & 0 & C & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

Also the optimal filter gain L and Lipschitz constant l can be calculated as follows,

$$L = (P_2' + P_2'')^{-1}G, \quad l = 1/\sqrt{\alpha(\eta_1 + d_M\eta_2)} \tag{2.56}$$

Proof. Consider the discrete-time Lyapunov-Krasovskii function given in (2.36). The forward difference of this function can be written as

$$\begin{aligned} \Delta V = {} & \xi^T(\Gamma_1' + \Delta\Gamma)^T P(\Gamma_1' + \Delta\Gamma)\xi + 2\xi^T(\Gamma_1' + \Delta\Gamma)^T P\Omega \\ & + \Omega^T P\Omega + \xi^T\Gamma_3''\xi - \sum_{i=k-d_M+1}^{k} X(i)^T Q X(i) \\ & + d_M\xi^T(\Gamma_2' + \Delta\Gamma)^T P(\Gamma_2' + \Delta\Gamma)\xi \\ & + 2d_M\xi^T(\Gamma_2' + \Delta\Gamma)^T P\Omega + d_M\Omega^T P\Omega \\ & - \sum_{i=k-d_M}^{k-d_k-1} \tau(i)^T P'\tau(i) - \sum_{i=k-d_k}^{k-1} \tau(i)^T P'\tau(i) \\ & - \sum_{i=k-d_M}^{k-1} \tau(i)^T P''\tau(i) \end{aligned} \tag{2.57}$$

where $\Delta\Gamma = \begin{bmatrix} 0 & \Delta\mathbb{A}_d & 0 & \Delta\mathbb{B} \end{bmatrix}$, $\Gamma_3'' = diag\{-P + (d_M + 1)Q + R, -Q, -R, 0\}$ and ξ, Γ_1', Γ_2' are as given in section 2.2.2, only with the parameters defined in (2.51) and (2.52).

Now, using lemma 2.1, we can establish the following inequalities:

$$
\begin{aligned}
2(\Gamma_1' + \Delta\Gamma)^T P\Omega \leq\ & \Omega^T W_1 \Omega \\
& + (\Gamma_1' + \Delta\Gamma)^T P W_1^{-1} P (\Gamma_1' + \Delta\Gamma) \\
2(\Gamma_2' + \Delta\Gamma)^T P\Omega \leq\ & \Omega^T W_2 \Omega \\
& + (\Gamma_2' + \Delta\Gamma)^T P W_2^{-1} P (\Gamma_2' + \Delta\Gamma)
\end{aligned}
$$

where $W_i = \eta_i I - P > 0 \quad i = 1, 2$, and $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & 0 \end{bmatrix}^T$, $S = \begin{bmatrix} S_1^T & S_2^T & S_3^T & 0 \end{bmatrix}^T$, $N = \begin{bmatrix} N_1^T & N_2^T & N_3^T & 0 \end{bmatrix}^T$ are matrices with appropriate dimensions. Using these inequalities along with the Lipschitz property of Ω and inequalities given in (2.40-2.42), we will get

$$
\begin{aligned}
\Delta V \leq\ & \xi^T \Gamma_3' \xi + \xi^T (\Gamma_4 + \Gamma_4^T) \xi \\
& + \xi^T (\Gamma_1' + \Delta\Gamma)^T P (P^{-1} + W_1^{-1}) P (\Gamma_1' + \Delta\Gamma) \xi \\
& + d_M \xi^T (\Gamma_2' + \Delta\Gamma)^T P (P^{-1} + W_2^{-1}) P (\Gamma_2' + \Delta\Gamma) \xi \\
& + d_k \xi^T M P'^{-1} M^T \xi + d_M \xi^T N P''^{-1} N^T \xi \\
& + (d_M - d_k) \xi^T S P'^{-1} S^T \xi
\end{aligned} \tag{2.58}
$$

where $\Gamma_3' = \Gamma_3'' + diag\{(\eta_1 + d_M \eta_2) l^2 I, 0, 0, 0\}$, $\Gamma_4 = \begin{bmatrix} M + N & S - M & -S - N & 0 \end{bmatrix}$.

To handle the uncertain element $\Delta\Gamma$, we can write $P\Delta\Gamma = E\Lambda F$, where $E = P \begin{bmatrix} 0 & -L^T \end{bmatrix}^T$, $F = \begin{bmatrix} 0 & 0 & C & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$. Now, using lemma 2.2 and (2.22), the following inequality is established:

$$
\begin{aligned}
& (P\Gamma_i' + P\Delta\Gamma)^T (P^{-1} + W_i^{-1}) (P\Gamma_i' + P\Delta\Gamma) \\
& \quad \leq \Gamma_i'^T P (P - \eta_i^{-1} P^2 - \epsilon_i^{-1} E E^T)^{-1} P \Gamma_i' + \epsilon_i F^T F
\end{aligned}
$$

for $i = 1, 2$. The simplified ΔV is given by

$$
\begin{aligned}
\Delta V \leq\ & \xi^T (\Gamma_3 + \Gamma_4 + \Gamma_4^T + \Gamma_1^T \Gamma_7^{-1} \Gamma_1 \\
& + \Gamma_2^T \Gamma_8^{-1} \Gamma_2 + \Gamma_5 \Gamma_6^{-1} \Gamma_5^T) \xi
\end{aligned} \tag{2.59}
$$

where $\Gamma_1 = P\Gamma_1'$, $\Gamma_2 = \sqrt{d_M} P\Gamma_2'$, $\Gamma_3 = \Gamma_3' + (\epsilon_1 + d_M \epsilon_2) F^T F$, $\Gamma_7 = P - \eta_1^{-1} P^2 - \epsilon_1^{-1} E E^T$, $\Gamma_8 = P - \eta_2^{-1} P^2 - \epsilon_2^{-1} E E^T$, $\Gamma_5 = \begin{bmatrix} \sqrt{d_M} M & \sqrt{d_M} S & \sqrt{d_M} N \end{bmatrix}$, $\Gamma_6 = diag\{P', P', P''\}$. As explained in section 2.2.2, the uncertain d_k needed to be replaced with its known bounds i.e. $0, d_M$. For $\mathcal{H}_\infty$ filtering, we define

$$J \triangleq \sum_{k=0}^{\infty}\{\varepsilon(k)^T\varepsilon(k) - \mu^2 w(k)^T w(k) \quad (2.60)$$
$$- \mu^2 v(k-d_k)^T v(k-d_k) - \rho^2 \Delta y_\nu(k)^T \Delta y_\nu(k)\}$$

Adding (2.59) to the right hand side of (2.60), we get $J \leq \sum_{k=0}^{\infty} \tilde{J}_k$, where

$$\tilde{J}_k = \Delta V(k) + \varepsilon(k)^T\varepsilon(k) - \mu^2 w(k)^T w(k) \quad (2.61)$$
$$- \mu^2 v(k-d_k)^T v(k-d_k) - \rho^2 \Delta y_\nu(k)^T \Delta y_\nu(k)$$

Now, if the filter such that $\tilde{J}_k < 0$, we can conclude that $J < 0$, which is equivalent to

$$\|\varepsilon\|^2 < \mu^2\|w\|^2 + \mu^2\|v\|^2 + \rho^2\|\Delta y_\nu\|^2 \quad (2.62)$$

This implies that the effect of the unwanted external inputs on the second norm of the estimation error is bounded by μ, and the effect of the measurement error, caused by communication constraints, is bounded by ρ. Substituting (2.59) in (2.61), we get

$$\tilde{J}_k \leq \xi^T(\Gamma_3 + \Gamma_J + \Gamma_4 + \Gamma_4^T + \Gamma_1^T\Gamma_7^{-1}\Gamma_1$$
$$+ \Gamma_2^T\Gamma_8^{-1}\Gamma_2 + \Gamma_5\Gamma_6^{-1}\Gamma_5^T)\xi \quad (2.63)$$

where $\Gamma_J = diag\{0, H^T H, 0, 0, 0, 0, -\mu^2 I, -\mu^2 I, -\rho^2 I\}$. To avoid BMI's, the following variable changes need to be performed:

$$P' = diag\{P_1', P_2'\},\ P'' = diag\{P_1'', P_2''\}$$
$$G = (P_2' + P_2'')L,\ \alpha^{-1} = (\eta_1 + d_M\eta_2)l^2 \quad (2.64)$$

Now for stability we need to have $\tilde{J}_k < 0$, which by using Schur's complement is equivalent to the LMI given by (2.54). □

It should be noted that remark 2 is also true about the objective function given in (2.53).

2.3 Simulation Results

In this section, we will design stable filters using the results of section 2.2 for an example system. Our intention is to show how the network-imposed imperfections will affect the

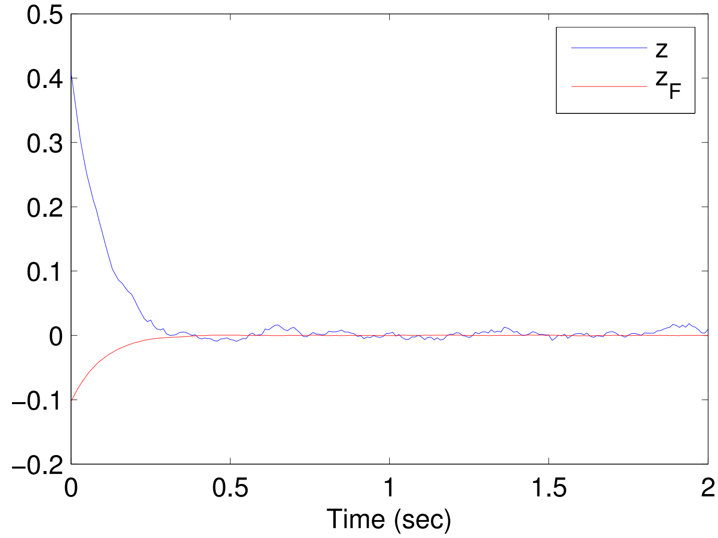


Figure 2.2: z and its estimate z_F

designed filter gains and also the region of attraction of the filters. We assume that the discrete model of the plant is obtained using a $t = 0.01sec$ sampling period.

Consider the system given in (2.1) with the following parameters,

$$A = \begin{bmatrix} 0.9323 & 0.0185 \\ -0.0092 & 0.9138 \end{bmatrix} \quad B = \begin{bmatrix} 0.0098 \\ 0.0095 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\phi(x, u) = \begin{bmatrix} 0 \\ -0.1(1 - cos(x_2^3)) \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Choosing $\mu = 0.5$, and $\rho = 0.1$, the maximum Lipschitz constant and the corresponding optimal observer gain are calculated to be as follows:

when the measurements are quantized:

$$L = \begin{bmatrix} 0.16 \times 10^{-4} & 3.9 \times 10^{-4} \end{bmatrix}^T \qquad l = 0.059$$

when the measurements are transmitted with an uncertain delay:

$$L = \begin{bmatrix} -0.12 \times 10^{-2} & -0.09 \times 10^{-2} \end{bmatrix}^T \qquad l = 0.0464$$

when the quantized measurements are transmitted with an uncertain delay on a network where every sensor is granted access to the communication channel once in every 3 samples:

$$L = \begin{bmatrix} -0.061 \times 10^{-5} & -0.11 \times 10^{-5} \end{bmatrix}^T \qquad l = 0.0464$$

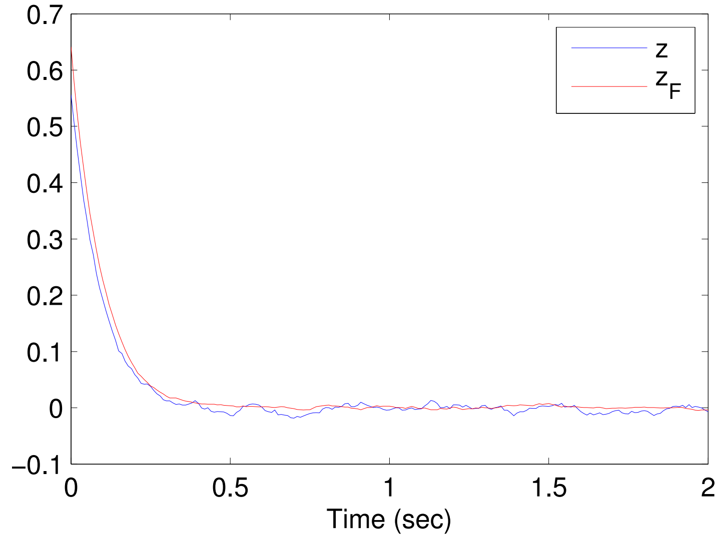


Figure 2.3: z and its estimate z_F

The value of l in each case determines a neighborhood of x_2, in which the filter stability is guaranteed. In other words, the filter with quantized measurements is locally stable as long as $|x_2| \leq 0.71$, and the filter with uncertain delay as well as the filter with all of the aforementioned issues are locally stable as long as $|x_2| \leq 0.69$. Figures 2.2, 2.3, 2.4 illustrate the estimated variable z_F for the case with quantized measurements, the case with uncertain delay, and the case with all the issues, respectively. It can be seen from these figures that the proposed filters output smooth estimates of z using the noisy measurements which undergo quantization and delay.

As mentioned in Remark 1, there's always a trade-off between the maximized Lipschitz constant l and the disturbance attenuation level μ. Figure (2.5) shows the trade-off curve between these two parameters in the third case. As one can see from this figure, tighter bounds on the effects of the external disturbances come at the expense of smaller regions of attraction, or in other words, with a smaller μ comes a smaller l. A similar trade-off was also examined between l and ρ and it was seen that different values of ρ have very small effect on the maximized Lipschitz constant l. However, the filter gain L was significantly affected by the changes in ρ such that smaller values of ρ led to smaller maximum singular values of L, *i.e.* $\overline{\sigma}(L)$. In order to see how the upper bound of the network-induced transmission delay d_M affects the maximized Lipschitz constant l, a trade-off curve between the two is illustrated in figure (2.6) for the third case. It can be

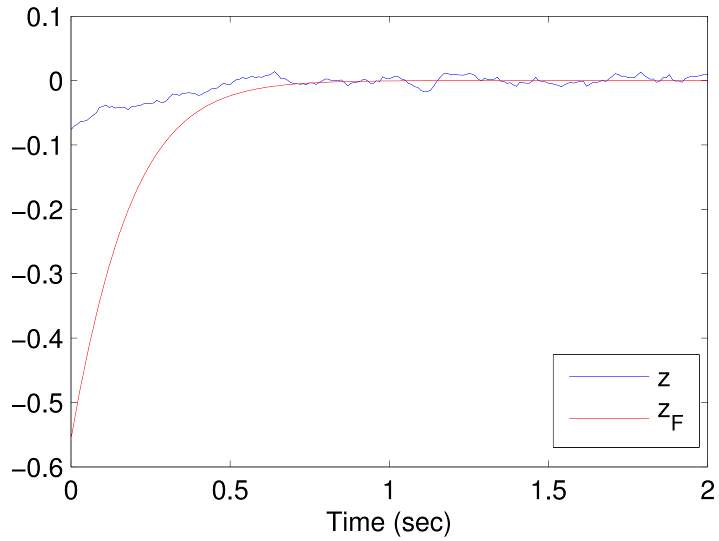


Figure 2.4: z and its estimate z_F

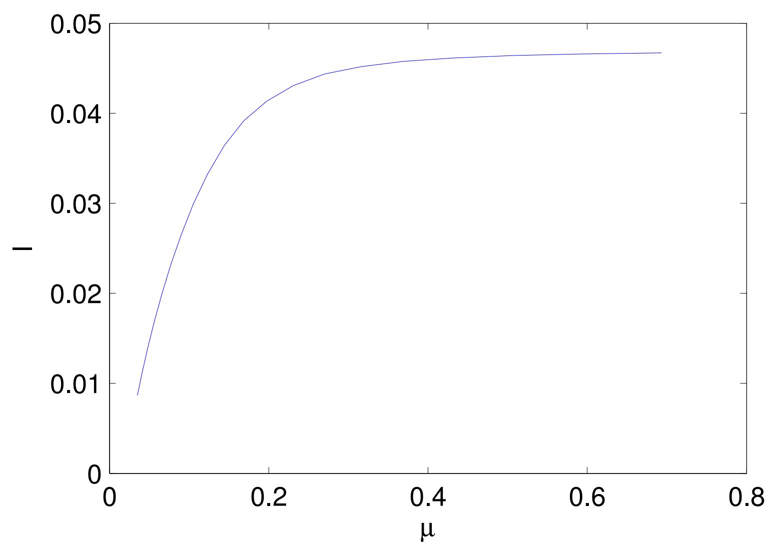


Figure 2.5: The disturbance attenuation gain μ and the maximized Lipschitz constant l

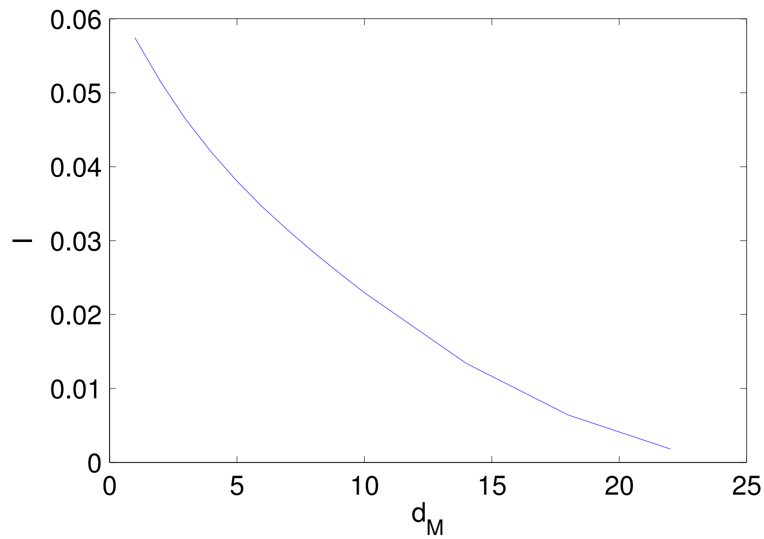


Figure 2.6: The upper bound on the delay d_M and the maximized Lipschitz constant l

seen from this figure that as delay grows larger, the maximized Lipschitz constant becomes smaller until the LMI's become infeasible at $d_M = 23$.

2.4 Summary

In this chapter, the filtering problem for Lipschitz nonlinear systems in a NCS setup was addressed. It was assumed that the data sent by sensors were quantized by finite-level quantizers, and also subject to uncertain delays. It was also assumed that the filter receives sensor data once in every few samples, which models the communication constraints imposed by the network. First the effects of quantization and delay on filter design were investigated separately, and then the general filtering problem, with delayed quantized measurements and subject to communication constraints, was formulated. Using a Lyapunov-Krasovskii function along with an $\mathcal{H}_\infty$ bound on the estimation error system, both stability and performance of the filter were investigated. It was seen that the associated LMI with the general filtering problem was very similar to that of the filtering problem with delay. Although quantization and communication constraints introduced some new variables into the resulting LMI, the structure remained intact. Simulation results were presented to show the effectiveness of the proposed approach.

Chapter 3

An Adaptive Approach to Filter Design with Limited Information

In this chapter, a novel adaptive approach is proposed to account for network-imposed imperfections such as quantization and limited channel capacity on the filter design. The main idea is to estimate the error caused by quantization and communication constraints instead of treating it as a bounded noise input or a norm-bounded uncertainty. Using an LMI based approach, the filter design procedure is formulated for both linear and Lipschitz nonlinear systems with desired attenuation bounds on the effects of the state and measurement noise inputs and also variations of the network-imposed errors. Simulation results are given to illustrate the effectiveness of the proposed filter.

The rest of the chapter is organized as follows. In section 3.1, we introduce the plant and filter models. Section **??** and **??** discuss the proposed design approach for linear and nonlinear systems, respectively. In section **??** the proposed filters are tested via simulation and section 3.5 summarizes the results of this chapter.

3.1 Plant and Filter Models

We now introduce the plant and filter models as well as the quantizer to be used in later sections.

3.1.1 Linear Models

Consider the following discrete-time linear plant model,

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 u(k) + B_2 w(k) \\
y(k) &= Cx(k) + Du(k) + v(k) \\
z(k) &= Hx(k)
\end{aligned}
\tag{3.1}
$$

where $x \in \mathbb{R}^n$ is the state vector; $y \in \mathbb{R}^p$ represents the measured outputs; $u \in \mathbb{R}^{m_1}$ is the known input; $z \in \mathbb{R}^r$ is the signal to be estimated, and $w, v \in \ell_2$ denote state and measurement noises, respectively, and

$$
\begin{aligned}
&A \in \mathbb{R}^{n\times n},\ B_1 \in \mathbb{R}^{n\times m_1},\ \ B_2 \in \mathbb{R}^{n\times m_2} \\
&C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m_1},\ \ H \in \mathbb{R}^{r\times n}.
\end{aligned}
$$

The discrete-time linear filter is given by

$$
\begin{aligned}
x_F(k+1) &= Ax_F(k) + B_1 u(k) + L(y_{rec}(k) \\
&\quad - y_F(k) - \theta_F(k)) \\
y_F(k) &= Cx_F(k) + Du(k) \\
z_F(k) &= Hx_F(k)
\end{aligned}
\tag{3.2}
$$

where $x_F \in \mathbb{R}^n$, $y_F \in \mathbb{R}^p$ and $z_F \in \mathbb{R}^r$ are, respectively, the state vector, output vector, and estimate vector of the filter. L is the filter gain to be designed, $\theta_F \in \mathbb{R}^p$ is the adaptive parameter of the filter, and y_{rec} represents the measurement signal available to the filter and is defined as follows:

$$
y_{rec}(k) = \begin{cases} y_F(k) + \theta_F(k) & \text{if packet is lost} \\ y_q(k) & \text{otherwise} \end{cases}
\tag{3.3}
$$

where $y_q(k) \in \mathbb{R}^p$ represents the quantized measurement received by the filter and can be written as

$$
y_q(k) = y + \theta(k)
\tag{3.4}
$$

where $\theta = y_q - y$ is the error induced by quantization and will be estimated by θ_F.

Remark I. Please note that according to (3.3), when a packet dropout occurs, the filter will run in open loop, and therefore no corrections will be made to the state estimation trajectories.

3.1.2 Quantization

The quantization is assumed to be done via a dynamic linear hysteresis quantizer, which is defined as follows,

$$y_{qi}(k) = \begin{cases} y_{qi}(k-1) \ \textbf{if} \ |y_i(k) - y_{qi}(k-1)| \leq \tau_i/2 + h_i\tau_i/2 \\ \qquad\qquad\quad \textbf{otherwise} \\ f\tau_i \qquad\qquad \textbf{if} \ f\tau_i - \tau_i/2 < y_i(k) \leq f\tau_i + \tau_i/2 \\ -f\tau_i \qquad\quad \textbf{if} \ -f\tau_i - \tau_i/2 \leq y_i(k) < -f\tau_i + \tau_i/2 \end{cases} \tag{3.5}$$

where $f \in \mathbb{N}$, and $\ldots, -2\tau_i, -\tau_i, 0, \tau_i, 2\tau_i, \ldots$ represent the quantization levels for the ith measurement; and $0 < h_i < 1$ determines the size of the hysteresis region for that measurement. It should be noted that the initial conditions of the quantizer, *i.e.* $y_{qi}(-1)$ cannot be arbitrarily chosen and they should all belong to $Y_{q0} = \{\ldots, -2\tau_i, -\tau_i, 0, \tau_i, 2\tau_i, \ldots\}$. In the sequel, a brief description of the operation of this quantizer will be given, however, for more detailed information on hysteretic quantizers, the reader is referred to the work done by Ceragioli *et. al* in [99].

In order to understand how the hysteresis quantizer given in (3.5) works, one first needs to understand that the main purpose of this quantizer is reducing the noise-induced chattering between two adjacent quantization levels. To do so, this quantizer compares the current non-quantized measurement, *i.e.* $y_i(k)$, with the last quantized one, *i.e.* $y_{qi}(k-1)$, and if the absolute value of the error is less than or equal to $\tau_i/2 + h_i\tau_i/2$, then it carries on with the last quantized measurement. This dynamic behavior helps reduce the noise-induced back and forth switching between adjacent levels. Furthermore, if the absolute value of the error is greater than $\tau_i/2 + h_i\tau_i/2$, then the quantizer acts like a conventional linear quantizer to find the new quantized value. Figure 3.1 shows the quantization error $e_{qi} = y_i - y_{qi}$ versus y_i.

Remark II. In the hysteresis quantizer given by (3.5), there are two design parameters. The first one, which is common among all linear quantizers, is τ_i. This parameter is the main design parameter in any linear quantizer and determines how coarse or fine the quantization levels are. A bigger τ_i corresponds to coarser quantization levels and therefore a larger bound on the quantization error. The second parameter is h_i, which is exclusive to hysteresis quantizers, and determines how sensitive the quantizer is to noise when switching from one quantization level to another. A larger h_i implies a less sensitive

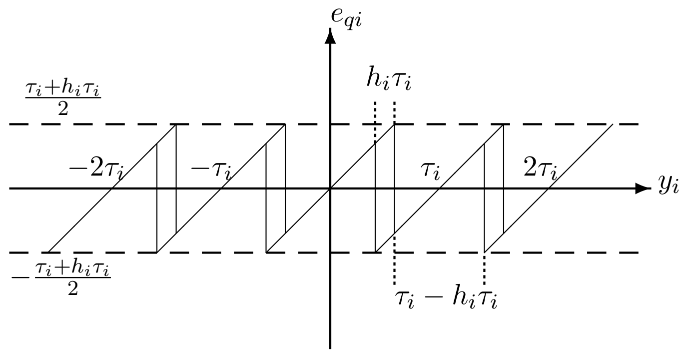


Figure 3.1: Quantization Error

quantizer. However, the reduced sensitivity to noise comes at the price of a larger bound on the quantization error. *i.e.* $|e_{qi}| \leq \tau_i/2 + h_i/2$.

We note that the hysteresis bands can also be applied to logarithmic quantizers. in this chapter, however, we restrict our attention to linear quantizers.

3.1.3 Nonlinear Models

Consider the following nonlinear plant model,

$$\begin{aligned}
x(k+1) &= Ax(k) + B_1u(k) + B_2w(k) + \phi(x(k), u(k)) \\
y(k) &= Cx(k) + Du(k) + v(k) \\
z(k) &= Hx(k)
\end{aligned} \tag{3.6}$$

where all the variables and parameters are as defined for the linear model, and $\phi(x, u)$ is a Lipschitz function with the Lipschitz constant l.

The dynamic model of our adaptive filter is defined as follows

$$\begin{aligned}
x_F(k+1) &= Ax_F(k) + B_1u(k) + \phi(x_F, u) \\
&\quad + L(y_{rec}(k) - y_F(k) - \theta_F(k)) \\
y_F(k) &= Cx_F(k) + Du(k) \\
z_F(k) &= Hx_F(k)
\end{aligned} \tag{3.7}$$

where all of the variables and parameters are as defined in (3.2). Similar to the linear case, quantization is assumed to be done via the linear hysteresis quantizer given in (3.5).

3.2 Linear Filter Design

In this section we introduce a novel adaptive approach to handle the filtering problem of linear systems subject to quantization and packet dropouts. As discussed in the previous section, θ_F is an adaptive parameter of the filter which estimates the network-induced quantization error modelled by θ. Consequently, for our filter to function as desired, we first need to derive an stable adaptive law for θ_F.

3.2.1 Adaptive Law Extraction

In order to extract an stable adaptive law for θ_F, we first need to investigate how the estimation error of the network-induced quantization error, *i.e.* $\tilde{\theta} = \theta - \theta_F$, is related to the measurement estimation error, *i.e.* $\epsilon(k)$, defined as follows:

$$\epsilon(k) = y_{rec}(k) - y_F(k) - \theta_F(k). \tag{3.8}$$

To this end, we use (3.1), (3.2) and (3.3) to simplify the above equation into the following form:

$$\epsilon(k) = \begin{cases} 0 & \text{if packet is lost} \\ Ce(k) + \tilde{\theta}(k) + v(k) & \text{otherwise} \end{cases} \tag{3.9}$$

where $e(k) \triangleq x(k) - x_F(k)$ represents the state estimation error. The idea is to use the Gradient optimization rule (see [100]) to find the desired adaptation law. Define the minimization cost function as follows:

$$J(\tilde{\theta}) = \frac{1}{2}\epsilon^T \epsilon. \tag{3.10}$$

Based on the Gradient optimization rule, for J to be minimized with respect to $\tilde{\theta}$, the adaptive law for $\tilde{\theta}$ should be in the following format:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \Gamma \frac{\partial J(\tilde{\theta}(k))}{\partial \tilde{\theta}(k)} \tag{3.11}$$

where $\Gamma = diag\{\gamma_1, \gamma_2, \ldots, \gamma_p\} > 0$ is the adaptation gain, and $\partial J / \partial \tilde{\theta}$ can be calculated as follows

$$\frac{\partial J(\tilde{\theta}(k))}{\partial \tilde{\theta}(k)} = \frac{dJ(\tilde{\theta}(k))}{d\epsilon(k)} \frac{\partial \epsilon(k)}{\partial \tilde{\theta}(k)} = \epsilon(k). \tag{3.12}$$

As a result, the minimizing adaptive law can be written as below

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \Gamma\epsilon(k). \tag{3.13}$$

Substituting $\tilde{\theta} = \theta - \theta_F$ into (3.13), we have

$$\theta(k+1) - \theta_F(k+1) = \theta(k) - \theta_F(k) - \Gamma\epsilon(k)$$
$$\Rightarrow \theta_F(k+1) = \theta_F(k) + \Gamma\epsilon(k) + \Delta\theta(k+1)$$

where $\Delta\theta(k+1) = \theta(k+1) - \theta(k)$ represents the variation of θ between the sampling instants k and $k+1$. Since $\Delta\theta$ is unknown, alternatively we choose the following adaptation law for θ_F:

$$\theta_F(k+1) = \theta_F(k) + \Gamma\epsilon(k). \tag{3.14}$$

Remark III. Note that as long as the quantizer is not saturated, $\Delta\theta(k)$ is bounded for all k, and its truncation, *i.e.* $\Delta\theta_s(k) = \{\Delta\theta(0), \Delta\theta(1), \ldots, \Delta\theta(s), 0, 0, \ldots\}$ has finite support, which implies $\Delta\theta_s \in \ell_2$.

3.2.2 The Proposed Design Approach

Based on (3.9), $\epsilon(k)$ can have one of the two values:

- $\epsilon(k) = 0$ which implies an open loop filtering error system whose stability properties are the same as the plant.
- $\epsilon(k) = Ce(k) + \tilde{\theta}(k) + v(k)$ which closes the loop and stability of the corresponding closed-loop filtering error system depends on the how L and Γ are chosen.

Since our goal is to design a closed-loop filter, in the remainder of this article we will only focus on $\epsilon(k) = Ce(k) + \tilde{\theta}(k) + v(k)$, having in mind that when a packet dropout occurs, the loop is opened and the closed loop filter will resume its operation with new initial conditions as soon as the next measurement is available.

Now using (3.1), (3.2) and (3.14), the closed-loop filtering error system can be written as

$$e(k+1) = (A - LC)e(k) + B_2 w(k) - Lv(k) - L\tilde{\theta}(k)$$
$$\tilde{\theta}(k+1) = -\Gamma Ce(k) + (I - \Gamma)\tilde{\theta}(k) - \Gamma v(k) + \Delta\theta(k)$$

$$\varepsilon(k) = z(k) - z_F(k) = He(k) \tag{3.15}$$

where ε is the estimation error.

Our interest is in designing a filter that can attenuate the effects of state and measurement noise as well as network induced errors on the estimation error ε with bounds $\mu_{wv}, \mu_\theta > 0$ defined as follows:

$$||\varepsilon_s||^2 \leq \mu_{wv}^2(||w_s||^2 + ||v_s||^2) + \mu_\theta^2||\Delta\theta_s||^2 \quad \forall w_s,\ v_s,\ \Delta\theta_s \in \ell_2 \tag{3.16}$$

Remark IV. The above inequality establishes $\mathcal{H}_\infty$ bounds on the effects of the truncated signals $w_s, v_s, \Delta\theta_s$ on the truncated signal ε_s. Note that we do not claim that the inequality is satisfied in the space ℓ_2 because, in general, $\Delta\theta$ is not ℓ_2 bounded.

The following theorem formulates the proposed adaptive approach for linear systems.

Theorem 3.1. *Consider the linear system (3.1), with measurements quantized via a hysteresis linear quantizer, and the linear filter in (3.2) with θ_F updated by (3.14). Then the filtering error system satisfies ℓ_2 bounds μ_{wv}, μ_θ on the effects of the noise and network-induced errors, if there are matrices $P > 0$, G_p, and diagonal matrices $Q, G_q > 0$ satisfying the following LMI:*

$$\begin{bmatrix} \Xi_1 & \Xi_2^T & \Xi_3^T \\ \star & -P & 0 \\ \star & \star & -Q \end{bmatrix} < 0 \tag{3.17}$$

where

$$\begin{aligned} \Xi_1 &= diag\{H^T H - P, -Q, -\mu_{wv}^2 I, -\mu_{wv}^2 I, -\mu_\theta^2 I\} \\ \Xi_2 &= \begin{bmatrix} PA - G_p C & -G_p & PB_2 & -G_p & 0 \end{bmatrix} \\ \Xi_3 &= \begin{bmatrix} -G_q C & Q - G_q & 0 & -G_q & Q \end{bmatrix} \end{aligned} \tag{3.18}$$

The filter parameters L and Γ can be calculated via $L = P^{-1}G_p$, and $\Gamma = Q^{-1}G_q$, respectively.

Proof. *In order to analyze the stability of the error dynamics, we introduce the following Lyapunov function candidate:*

$$V(k) = e(k)^T Pe(k) + \tilde{\theta}(k)^T Q\tilde{\theta}(k) \tag{3.19}$$

where $P > 0$ *and* $Q = diag\{Q_1, Q_2, \dots, Q_p\} > 0$. *The forward difference of this Lyapunov function can be written as*

$$\Delta V(k) = \xi(k)^T(\hat{\Xi}_1 + \hat{\Xi}_2^T P\hat{\Xi}_2 + \hat{\Xi}_3^T Q\hat{\Xi}_3)\xi(k) \tag{3.20}$$

where $\xi(k) = \begin{bmatrix} e(k)^T & \tilde{\theta}(k)^T & w(k)^T & v(k)^T & \Delta\theta(k)^T \end{bmatrix}^T$, *and*

$$\begin{aligned}
\hat{\Xi}_1 &= diag\{-P, -Q, 0, 0, 0\} \\
\hat{\Xi}_2 &= \begin{bmatrix} A - LC & -L & B_2 & -L & 0 \end{bmatrix} \\
\hat{\Xi}_3 &= \begin{bmatrix} -\Gamma C & I - \Gamma & 0 & -\Gamma & I \end{bmatrix}
\end{aligned} \tag{3.21}$$

Substituting $G_p = PL$ *and* $G_q = Q\Gamma$ *in (3.20) yields*

$$\Delta V(k) = \xi^T(\hat{\Xi}_1 + \Xi_2^T P^{-1}\Xi_2 + \Xi_3^T Q^{-1}\Xi_3)\xi \tag{3.22}$$

where Ξ_2, Ξ_3 *are as defined in (3.18).*

To show that the error system satisfies the desired $\mathcal{H}_\infty$ *performance, we need to establish attenuation levels on the effects of the noise signals and network-induced errors in any finite time interval, under zero initial conditions. To this end,* J *is defined as follows:*

$$\begin{aligned}
J \triangleq& \sum_{i=0}^{s}\{\varepsilon(i)^T\varepsilon(i) - \mu_{wv}^2(w(i)^T w(i) + v(i)^T v(i)) \\
&- \mu_\theta^2\Delta\theta(i)^T\Delta\theta(i)\} \\
=& \sum_{i=0}^{\infty}\{\varepsilon_s(i)^T\varepsilon_s(i) - \mu_{wv}^2(w_s(i)^T w_s(i) + v_s(i)^T v_s(i)) \\
&- \mu_\theta^2\Delta\theta_s(i)^T\Delta\theta_s(i)\}
\end{aligned} \tag{3.23}$$

where $s > 0$ *is any finite integer, and* $0 < \mu_{wv}, \mu_\theta \leq 1$ *are, respectively, upper bounds on the effects of the noise, and network-induced errors on the estimation error. Since* $w_s, v_s, \Delta\theta_s$ *all have finite supports,* J *is bounded. A negative* J *implies limited effects of these unwanted signals on the estimation error in any finite time interval* $[0, s]$. *Since* $V(k)$ *is a positive definite function, under zero initial conditions,* $V(s) - V(0) = \sum_{i=0}^{s}\Delta V(i)$ *is positive semi-definite and therefore adding it to the right hand side of (3.23) results in* $J \leq \sum_{i=0}^{s}\tilde{J}(k)$ *where*

$$\tilde{J}(k) = \varepsilon(k)^T\varepsilon(k) - \mu_{wv}^2(w(k)^T w(k) + v(k)^T v(k))$$

$$- \mu_\theta^2 \Delta\theta(k)^T \Delta\theta(k) + \Delta V(k). \tag{3.24}$$

Substituting (3.22) in (3.24), we obtain

$$\tilde{J} = \xi^T(\Xi_1 + \Xi_2^T P^{-1}\Xi_2 + \Xi_3^T Q^{-1}\Xi_3)\xi \tag{3.25}$$

where $\Xi_1 = \hat{\Xi}_1 + diag\{H^T H, 0, -\mu_{wv}^2 I, -\mu_{wv}^2 I, -\mu_\theta^2 I\}$*. Therefore, the estimation error* ε *is bounded and satisfies the* $\mathcal{H}_\infty$ *performance inequalities given in (3.16) if* $\tilde{J} < 0$ *which is guaranteed if the LMI given in (3.17) holds true.*

Remark V. Note that although the proposed linear filter was obtained solving the feasibility problem in theorem 3.1, the problem can also be stated as an optimization problem with an objective to minimize the $\mathcal{H}_\infty$ bounding parameters μ_θ and μ_{wv}. To this end, one can define $min \quad \bar{\mu}_{wv} + W\bar{\mu}_\theta$ as the objective function with W as a weighting parameter and $\bar{\mu}_{wv} = \mu_{wv}^2$ and $\bar{\mu}_\theta = \mu_\theta^2$ as simple variable replacements to avoid bilinear terms in the LMI given in (3.17).

3.3 Nonlinear Filter Design

In this section, we extend the adaptive approach of section 3 to nonlinear Lipschitz systems. Similar to the linear case, we define the state error as $e = x - x_F$ and the estimation error of the adaptive parameter as $\tilde{\theta} = \theta - \theta_F$. The closed-loop filtering error system can be written as follows:

$$\begin{aligned} e(k+1) &= (A - LC)e(k) + B_2 w(k) - Lv(k) \\ &\quad - L\tilde{\theta}(k) + \Delta\phi(x, x_F, u) \\ \tilde{\theta}(k+1) &= -\Gamma C e(k) + (I - \Gamma)\tilde{\theta}(k) - \Gamma v(k) + \Delta\theta(k) \\ \varepsilon(k) &= He(k) \end{aligned} \tag{3.26}$$

where $\Delta\theta(k) = \theta(k+1) - \theta(k)$ and $\Delta\phi(x, x_F, u) = \phi(x, u) - \phi(x_F, u)$. Before formulating the proposed approach, we introduce the following lemma that will be used in the proof of the main theorem:

Lemma 3.1. *([97]) For any* $x, y \in \mathbb{R}^n$ *and any positive definite matrix* $T \in \mathbb{R}^{n\times n}$*, we have:*

$$2x^T y \leq x^T T x + y^T T^{-1} y$$

The following theorem formulates the proposed adaptive approach for Lipschitz nonlinear systems.

Theorem 3.2. *Consider the nonlinear system (3.6), with measurements quantized via a hysteresis linear quantizer, and the nonlinear filter in (3.7) with θ_F updated by (3.14). Then the filtering error system satisfies ℓ_2 attenuation levels μ_{wv} and μ_θ on the effects of the noise and network-induced errors, if there exist matrices $P > 0$, G, and diagonal matrices $Q, G_q > 0$, and a scalar $\eta > 0$ such that the following LMI is feasible:*

$$\begin{bmatrix} \Xi_1 & \Xi_2^T & \Xi_3^T \\ \star & -\Xi_4 & 0 \\ \star & \star & -Q \end{bmatrix} < 0 \tag{3.27}$$

where

$$\begin{aligned}
\Xi_1 &= diag\{H^T H - P + \eta l^2 I, -Q, -\mu_{wv}^2 I, -\mu_{wv}^2 I, -\mu_\theta^2 I\} \\
\Xi_2 &= \begin{bmatrix} PA - G_p C & -G_p & PB_2 & -G_p & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\Xi_3 &= \begin{bmatrix} -G_q C & Q - G_q & 0 & -G_q & Q \end{bmatrix} \\
\Xi_4 &= \begin{bmatrix} P & -P \\ -P & \eta I \end{bmatrix}
\end{aligned} \tag{3.28}$$

The filter parameter L, and the adaptation gain Γ can be calculated as $L = P^{-1}G$ and $\Gamma = Q^{-1}G_q$, respectively.

Proof. *In order to analyze the stability properties of the error system given in (3.26), we use the Lyapunov function (3.19). The forward difference of this Lyapunov function can be written as*

$$\begin{aligned}
\Delta V(k) = \xi(k)^T(\Omega_1^T P \Omega_1 + \Omega_2^T Q \Omega_2)\xi(k) + 2\xi(k)^T \Omega_1^T P \Delta\phi \\
+ \Delta\phi^T P \Delta\phi - e(k)^T P e(k) - \tilde{\theta}(k)^T Q \tilde{\theta}(k)
\end{aligned} \tag{3.29}$$

where $\xi(k) = \begin{bmatrix} e(k)^T & \tilde{\theta}(k)^T & w(k)^T & v(k)^T & \Delta\theta(k)^T \end{bmatrix}^T$, *and*

$$\begin{aligned}
\Omega_1 &= \begin{bmatrix} (A - LC) & 0 & B_2 & -L & -L \end{bmatrix} \\
\Omega_2 &= \begin{bmatrix} -\Gamma C & I - \Gamma & 0 & -\Gamma & I \end{bmatrix}.
\end{aligned}$$

Using lemma 3.1 we can write

$$2\xi^T \Omega_p^T \Delta\phi \leq \xi^T \Omega_p T_1^{-1} \Omega_p^T \xi + \Delta\phi^T T_1 \Delta\phi \tag{3.30}$$

where $\Omega_p = P\Omega_1$. *Choosing* $T_1 = W$ *where* $W = \eta I - P > 0$ *for some* $\eta > 0$ *and then substitute (3.30) in (3.29), we get*

$$\begin{aligned}\Delta V(k) \leq\ & \xi(k)^T \Omega_p (P^{-1} + W^{-1}) \Omega_p^T \xi(k) + \xi(k)^T \Omega_2^T Q \Omega_2 \xi(k) \\ & + e(k)^T (\eta l^2 I - P) e(k) - \tilde{\theta}(k)^T Q \tilde{\theta}(k)\end{aligned} \tag{3.31}$$

where P *and* W *are codependent. We have:*

$$\begin{aligned}P^{-1} + W^{-1} &= P^{-1} + (\eta I - P)^{-1} \\ &= (\eta I - P)^{-1}((\eta I - P)P^{-1} + I) \\ &= (\eta I - P)^{-1} \eta P^{-1} = (P - \eta^{-1} P^2)^{-1}\end{aligned} \tag{3.32}$$

Using (3.32), we can simplify (3.31) into the following inequality,

$$\Delta V(k) \leq \xi^T [\hat{\Xi}_1 + \hat{\Xi}_2^T (P - \eta^{-1} P^2)^{-1} \hat{\Xi}_2 + \Xi_3^T Q^{-1} \Xi_3] \xi \tag{3.33}$$

where

$$\begin{aligned}\hat{\Xi}_1 &= diag\{\eta l^2 I - P, -Q, 0, 0, 0\} \\ \hat{\Xi}_2 &= \begin{bmatrix} PA - G_p C & -G_p & PB_2 & -G_p & 0 \end{bmatrix} \\ \Xi_3 &= \begin{bmatrix} -G_q C & Q - G_q & 0 & -G_q & Q \end{bmatrix}\end{aligned} \tag{3.34}$$

with $G_p = PL$, *and* $G_q = Q\Gamma$.

Proceeding as in Theorem 3.1, we consider J *and* $\tilde{J}$ *defined in (3.23) and (3.24), respectively. Substituting (3.33) in (3.24) and using Schur's complement, we obtain*

$$\tilde{J} \leq \xi^T (\Xi_1 + \Xi_2^T \Xi_4^{-1} \Xi_2 + \Xi_3^T Q^{-1} \Xi_3) \xi \tag{3.35}$$

where $\Xi_1, \Xi_2, \Xi_3, \Xi_4$ *are all as defined in (3.28). Therefore, the estimation error* ε *is bounded and satisfies the performance inequalities given in (3.16) if* $\tilde{J} < 0$, *which is guaranteed if the LMI given in (3.27) holds true.*

3.4 Simulation Results

In this section we consider two illustrative examples and compare the performance of the adaptive filters of sections 3 and 4 to those obtained using conventional (non-adaptive) filters and linear quantizers. We assume that the discrete model of the plant is obtained using a sampling period $t = 0.01sec$.

Example 1: Linear Case Consider the following linear system,

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.9323 & 0.0185 \\ -0.0092 & 0.9138 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} w(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + 0.1u(k) + v(k) \\ z(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned} \tag{3.36}$$

where $u(k) = sin(5k)$ is an exogenous input, and $w, v \in \ell_2$ are noise signals. It is assumed that both of the quantizers (linear and hysteresis) cover the range $[-1.5, 1.5]$ with $\tau = 0.5$ and therefore have 7 levels, which leads to 3-bit data packets. For the hysteresis quantizer, we choose the hysteresis parameter $h = 0.1$. We assume that there are no communication constraints *i.e.* the sensor can access the network at every sampling instant.

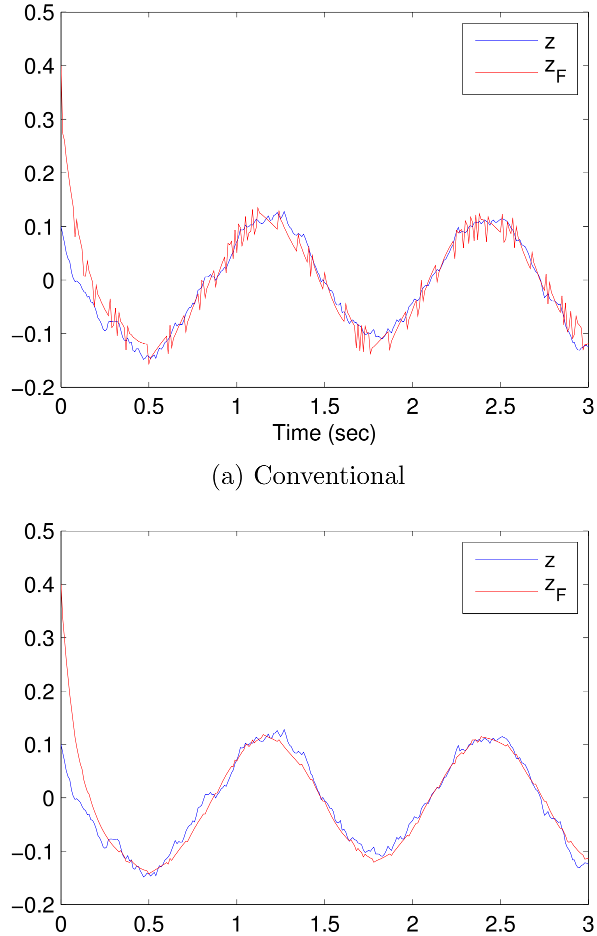
The Conventional $\mathcal{H}_\infty$ Filter: In designing this filter, the network-induced error, *i.e.* θ, is modeled as noise input, and the $\mathcal{H}_\infty$ filter tries to bound the unwanted effects of θ, whereas, in the proposed approach, θ is estimated by θ_F and the $\mathcal{H}_\infty$ filter tries to bound the unwanted effects of the variations of this signal, *i.e.* $\Delta\theta$. Choosing the $\mu_{wv} = \mu_\theta = 0.5$, the resulting filter gain is $L_{conv} = \begin{bmatrix} 1.1072 & 0.0732 \end{bmatrix}^T$.

The Proposed Filter: Choosing $\mu_{wv} = \mu_\theta = 0.5$, the filter parameters are $\Gamma = 0.5331$ and $L_{adapt} = \begin{bmatrix} 0.5061 & 0.0101 \end{bmatrix}^T$. Figures 3.2, and 3.3 show the operation of both the conventional and proposed filters. Comparing figures 3.2, we see that the proposed filter is less sensitive to noise and network-induced errors than the conventional counterpart.

Example 2: Nonlinear Case Assume the linear system given in (3.36) plus the following Lipschitz nonlinear function in the state equation:

$\phi(x) = 0.05(1 - cos(x_2(k))^2)$,

which is globally Lipschitz with $l = 0.05$. The quantizer characteristics are also assumed to be the same as those in the linear case. Furthermore, we assume that 3 agents use the network and therefore our sensor is granted access to the network once in every 3 samples.



(a) Conventional

(b) Adaptive

Figure 3.2: z and its estimate z_F

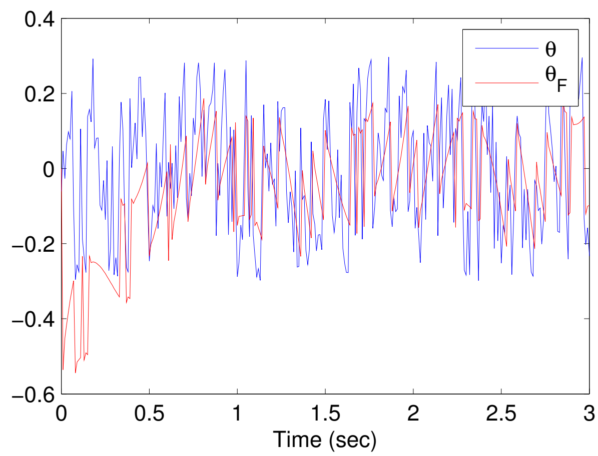


Figure 3.3: θ and its estimate θ_F

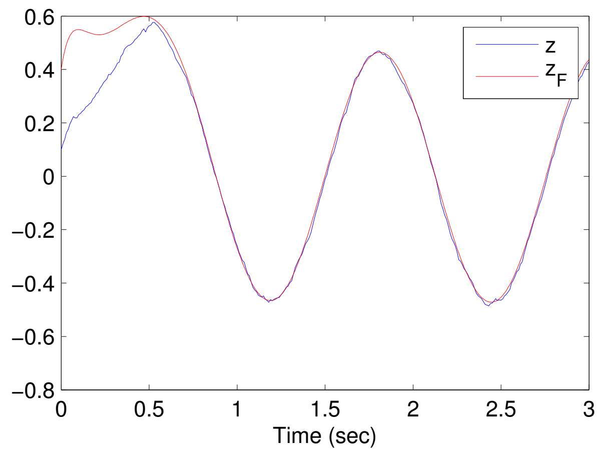


Figure 3.4: z and its estimate z_F

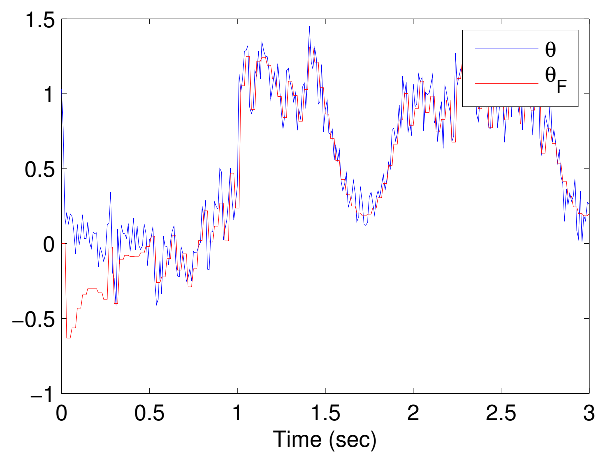


Figure 3.5: θ and its estimate θ_F

We assume that at $t = 1$ sec a fault occurs in the sensor, causing a constant offset with the amplitude 1 in the measurement.

Choosing the gains $\mu_{wv} = 0.5, \mu_\theta = 1$, the filter parameters are calculated to be $L_{adapt} = \begin{bmatrix} 0.21 & 0.11 \end{bmatrix}^T \times 10^{-3}$, $\Gamma = 0.998$. Figures 3.4, 3.5, and 3.6 show the operation of the proposed filter. Due to the offset added to the measurement at $t = 1$ sec, the quantizer becomes saturated when $y > 1.5$, which can be seen in figure 3.6. In the proposed approach, however, this error is identified and estimated by the adaptive parameter θ_F (figure 3.5). As a result, as shown in figure 3.4, the generated signal z_F follows z without any significant deviation.

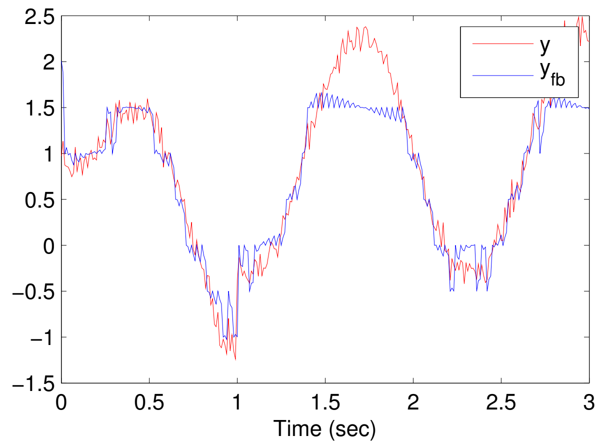


Figure 3.6: y and y_{fb}

3.5 Summary

This chapter considered the filtering problem for discrete-time linear and Lipschitz nonlinear systems over communication networks. Our formulation assumes that the sensor data is first quantized and then transmitted to the filter after access to the network is granted. The error imposed by the network was modeled as an unknown disturbance of the measurements and then an adaptive law was proposed to estimate this error. Using a Lyapunov-based approach, it was shown that the estimation error is bounded with arbitrary attenuation gains on the undesired effects of the network and noise inputs if certain LMI's were feasible. Finally, the effectiveness of the proposed approach was illustrated through simulation.

Chapter 4

Unknown Input Filter Design for Discrete-Time Linear Systems

In this chapter, a new $\mathcal{H}_\infty$ filter design approach is proposed for discrete-time linear systems with unknown inputs. The proposed Lyapunov-based approach, free of any similarity transformations, designs a linear filter for the modified model of the plant, and then extracts the original states of the system. The designed filter estimates both the system states and the unknown input simultaneously and does not have many of the restrictive assumptions and restrictions that the existing unknown input filters do. In the end, simulation results are used to illustrate the effectiveness of the proposed filter.

The rest of the chapter is organized as follows. Section II introduces the plant model. In sections III and IV the two design approaches are discussed. In section V the corresponding simulation results are given for both methods and finally section 4.5 summarizes the results of this chapter.

4.1 Plant Model

Consider the following linear system:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2d(k) + B_3w(k) \\ y(k) &= Cx(k) + v(k) \\ z(k) &= Hx(k) \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state vector; $y \in \mathbb{R}^p$ represents the measured outputs; $z \in \mathbb{R}^r$ is the vector to be estimated; $u \in \mathbb{R}^{m_1}$ is the known input; $d \in \mathbb{R}^{m_2}$ is the unknown low-frequency disturbance input; $w \in \mathbb{R}^{m_3}$ and $v \in \mathbb{R}^p$ are the state and measurement noise inputs, respectively, and A, B_1, B_2, B_3, C, H are the state space matrices of the model. We assume that (A, C) is an observable pair and also B_2, C are full rank with $rank(B_2) = m_2$, $rank(C) = p$ and $p \geq m_2$.

Now we define the new state variable $\bar{x}$ as

$$\bar{x}(k+1) = x(k+1) - \sum_{i=0}^{\sigma} A^i B_2 d(k-i) \tag{4.2}$$

where σ can be calculated through either

$$\begin{aligned} & min \;\; \sigma \\ & s.t. \quad rank\left(CA^\sigma B_2\right) = rank(B_2) \end{aligned} \tag{4.3}$$

or

$$\begin{aligned} & min \;\; \sigma \\ & s.t. \quad rank\left(\textstyle\sum_{i=0}^{\sigma} CA^i B_2\right) = rank(B_2) \end{aligned} \tag{4.4}$$

These two conditions are extracted through the stability analyses of two correlated yet distinct approaches which are based on two different filter models. The details of each approach will be discussed in the next two sections of this chapter.

Before introducing these approaches, we first rewrite the plant model by substituting (4.2) in (4.1), which leads to

$$\begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + B_1 u(k) + A^{\sigma+1} B_2 d(k-\sigma-1) \\ &\quad + B_3 w(k) \\ y(k) &= C\bar{x}(k) + \sum_{i=0}^{\sigma} CA^i B_2 d(k-i-1) + v(k) \\ z(k) &= H\bar{x}(k) + \sum_{i=0}^{\sigma} HA^i B_2 d(k-i-1) \end{aligned} \tag{4.5}$$

Based on the above revised model we will propose two distinct filter models and the corresponding design approach.

4.2 Approach I

In this approach we assume that σ is calculated by (4.3). Consider the following filter model,

$$\begin{aligned} \bar{x}_F(k+1) &= A\bar{x}_F(k) + A^{\sigma+1}B_2 d_F(k-\sigma-1|k-1) \\ &\quad + B_1 u(k) + L(y(k) - y_{Fc}(k)) \\ y_{Fp}(k) &= C\bar{x}_F(k) + \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-i-1|k-1) \\ y_{Fc}(k) &= C\bar{x}_F(k) + \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-i-1|k) \\ z_F(k) &= H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 d_F(k-i-1|k) \end{aligned} \tag{4.6}$$

where $\bar{x}_F \in \mathbb{R}^n$ is the state vector of the filter; $d_F(k-i|k-j) \in \mathbb{R}^{m_2}$ is the estimated disturbance vector $d_F(k-i)$, which is calculated using the measurements up to the sampling instant $k-j$; $y_{Fp} \in \mathbb{R}^p$ is the predicted measurement vector which is calculated using the estimated d_F up to the sampling instant $k-1$; $y_{Fc} \in \mathbb{R}^p$ is the corrected measurement vector which is calculated using the estimated d_F up to the sampling instant k; $z_F \in \mathbb{R}^r$ is the estimated vector; and L is the static filter parameter to be designed. It should be noted that this filter has exactly the same structure and same state space matrices as the original plant if the disturbance is zero.

4.2.1 Stability Analysis

The first step in the stability analysis is to find the unknown input estimate d_F. As explained before, to calculate the value of d_F at every sampling instant we propose an stable adaptive law, which uses the plant measurements to update d_F. In this section, we will first extract this stable adaptive law and then discuss the necessary conditions for stability.

Adaptive Law Extraction

In order to extract an stable adaptive law for d_F, we first need to find how the unknown input estimation error $\tilde{d} = d - d_F$, is related to the measurement y received by the filter.

To this end, we define the predicted measurement error ϵ_p as

$$\begin{aligned}\epsilon_p(k) &= y(k) - y_{Fp}(k)\\ &= C\bar{e}(k) + \sum_{i=0}^{\sigma} CA^i B_2 \tilde{d}(k-i-1|k-1)\\ &+ v(k)\end{aligned} \tag{4.7}$$

where $\bar{e} = \bar{x} - \bar{x}_F$ is the state error, v is the unknown error introduced by the measurement noise. Similarly, the corrected measurement error ϵ_c can be defined as

$$\begin{aligned}\epsilon_c(k) &= y(k) - y_{Fc}(k)\\ &= C\bar{e}(k) + \sum_{i=0}^{\sigma} CA^i B_2 \tilde{d}(k-i-1|k)\\ &+ v(k)\end{aligned} \tag{4.8}$$

We now construct our adaptation law to ensure that ϵ is exponentially decreasing. More explicitly; the adaptive law ensures that $\|\epsilon_p(k+1)\|$, the predicted measurement error at $k+1$, is smaller than $\|\epsilon_c(k)\|$, the corrected error at k, and also that $\epsilon_c(k+1)$,the corrected error at $k+1$, is smaller than $\epsilon_p(k+1)$, the predicted error at $k+1$. If those conditions are satisfied we can conclude that $\|\epsilon\|$ is decreasing with respect to $\tilde{d}$ in accordance with the *Gradient Optimization Rule*. Therefore, using (4.7) and (4.8) we can write

$$\begin{aligned}\epsilon_p(k+1) - \epsilon_c(k) = &\; C\bar{e}(k+1) - C\bar{e}(k) + \delta\\ &+ \sum_{i=0}^{\sigma} CA^i B_2 (\tilde{d}(k-i|k)\\ &- \tilde{d}(k-i-1|k))\end{aligned} \tag{4.9}$$

where $\delta = v(k+1) - v(k)$ is an unknown error term introduced via the measurement noise. We now choose $\tilde{d}(k-i|k)$ as

$$\tilde{d}(k-i|k) = \tilde{d}(k-i-1|k) - \Gamma B_2^T A^{i^T} C^T \epsilon_c(k) \tag{4.10}$$

for $i = 0, \ldots, \sigma$ where Γ is a diagonal positive definite matrix. Substituting (4.10) in (4.9), we get

$$\epsilon_p(k+1) = (I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T A^{i^T} C^T)\epsilon_c(k)$$

$$+ C(\bar{e}(k+1) - \bar{e}(k)) + \delta \tag{4.11}$$

From the above equation, we can conclude that for a stable $\bar{e}$ and a bounded error term δ, the predicted output error at $k+1$, *i.e.* $\epsilon_p(k+1)$, is decreasing with respect to the corrected output error at k, *i.e.* $\epsilon_c(k)$, provided that Γ is chosen in such a way that the following inequality is satisfied:

$$|\lambda(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T)| \leq 1 \tag{4.12}$$

where $\lambda(\ M\ )$ represents the eigenvalue of M. We claim that there always exists a small enough $\Gamma_0 > 0$, that for all $\Gamma < \Gamma_0$ the condition given in (4.12) is satisfied. To show this more clearly we first break down this condition into the following two inequalities:

$$\lambda_{max}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T) \leq 1 \tag{4.13}$$

$$\lambda_{min}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T) \geq -1 \tag{4.14}$$

The first inequality is always satisfied due to the fact that for a positive definite Γ, the matrix given by $\sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T$ is always positive semi-definite. To investigate the second inequality we can write

$$\begin{aligned}
&\lambda_{min}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T) \\
&\quad = 1 - \lambda_{max}(\sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T) \\
&\quad \geq 1 - \| \sum_{i=0}^{\sigma} CA^i B_2 \Gamma B_2^T {A^i}^T C^T \| \\
&\quad \geq 1 - \|\Gamma\| \| \sum_{i=0}^{\sigma} CA^i B_2 B_2^T {A^i}^T C^T \|
\end{aligned}$$

Therefore, for (4.14) to be satisfied we need to have

$$\|\Gamma\| \| \sum_{i=0}^{\sigma} CA^i B_2 B_2^T {A^i}^T C^T \| \leq 2$$

Based on the assumption given in (4.3), $\| \sum_{i=0}^{\sigma} CA^i B_2 B_2^T {A^i}^T C^T \|$ is non-zero and therefore there always exists a small enough $\Gamma > 0$ for which (4.14) is satisfied.

Next step is to show that $\epsilon_c(k)$ is also decreasing with respect to $\epsilon_p(k)$. Going back to (4.10), it is easy to see that at every sampling instant k, $\sigma + 1$ consecutive values of

$\tilde{d}$ starting from $\tilde{d}(k-\sigma|k)$ to $\tilde{d}(k|k)$ are updated. Solving (4.10) in a recursive manner, links all of the updated variables to $\tilde{d}(k-\sigma-1|k)$, which was last updated at $k-1$, *i.e.*

$$\tilde{d}(k-\sigma-1|k+j) = \tilde{d}(k-\sigma-1|k-1) \quad j = 0, 1, \ldots \tag{4.15}$$

As a result, (4.10) can be rewritten as follows:

$$\tilde{d}(k-i|k) = \tilde{d}(k-\sigma-1|k-1) - \Gamma \sum_{j=i}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k) \tag{4.16}$$

for $i = 0, \ldots, \sigma$. It should be noted that the adaptive law given in (4.16) is not realizable due to the existence of the unknown disturbance d in the equation. To handle this problem, we rewrite (4.16) as

$$\begin{aligned} d(k-i) - d_F(k-i|k) & \\ &= d(k-\sigma-1) - d_F(k-\sigma-1|k-1) \\ &- \Gamma \sum_{j=i}^{\sigma} B_2^T {A^i}^T C^T \epsilon_c(k) \\ \Rightarrow d_F(k-i|k) &= d_F(k-\sigma-1|k-1) \\ &+ \Gamma \sum_{j=i}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k) + \Delta d(k-i) \end{aligned} \tag{4.17}$$

where $\Delta d(k-i) = d(k-i) - d(k-\sigma-1)$ is an unknown term, which is bounded for any continuous signal and cannot be realized. As an alternative, we choose the adaptation law to be

$$\begin{aligned} d_F(k-i|k) = d_F(k-\sigma-1|k-1) & \\ + \Gamma \sum_{j=i}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k) & \end{aligned} \tag{4.18}$$

for $i = 0, \ldots, \sigma$. Now in order to calculate ϵ_c we first need to calculate the corrected filter measurement, *i.e.* y_{Fc}. To this end, we substitute (4.18) in the third equation of (4.6), which results in

$$\begin{aligned} y_{Fc}(k, l+1) = C\bar{x}_F(k) + \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-\sigma-1|k-1) & \\ + \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k, l) & \end{aligned} \tag{4.19}$$

where $l = 0, 1, 2, \ldots$ is a variable introduced to solve the above equation in an iterative manner. The initial values of this equation are given as

$$y_{Fc}(k, 0) = y_{Fp}(k), \quad \epsilon_c(k, 0) = \epsilon_p(k)$$

By substituting (4.19) in $\epsilon_c = y - y_{Fc}$, we get

$$\begin{aligned}\epsilon_c(k, l+1) = y(k) &- C\bar{x}_F(k) \\ &- \sum_{i=0}^{\sigma} CA^i B_2 d_F(k - \sigma - 1|k-1) \\ &- \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k, l)\end{aligned}$$

Using this equation, we can conclude that at every iteration, $\|\epsilon_c(k, l+1)\|$ becomes smaller with respect to $\|\epsilon_c(k, l)\|$ if Γ is chosen small enough so that the following inequality holds:

$$\lambda_{max}(\sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T) \leq 1 \tag{4.20}$$

Based on (4.20), at every iteration, $\epsilon_c(k, l)$ will become smaller until it reaches its steady state value, *i.e.* $\epsilon_c(k)$. As a result we can state the following corollary:

Corollary 4.1. *$\epsilon_c(k+1)$ and $\epsilon_p(k+1)$ are both decreasing with respect to $\epsilon_c(k)$ and $\epsilon_p(k)$ if the inequalities given in (4.14) and (4.20) are satisfied.*

The steady state value of $\epsilon_c(k, l)$, *i.e.* $\epsilon_c(k)$ can be calculated via the following equation:

$$\begin{aligned}\epsilon_c(k) &= \lim_{l \to \infty} \epsilon_c(k, l+1) = \lim_{l \to \infty} \epsilon_c(k, l) \\ &= \lim_{l \to \infty} \{y(k) - C\bar{x}_F(k) \\ &- \sum_{i=0}^{\sigma} CA^i B_2 d_F(k - \sigma - 1|k-1) \\ &- \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k, l)\}\end{aligned} \tag{4.21}$$

Simplifying this equation will result in:

$$\begin{aligned}\epsilon_c(k) = \Phi^{-1} y(k) &- \Phi^{-1} C\bar{x}_F(k) \\ &- \Phi^{-1} \sum_{i=0}^{\sigma} CA^i B_2 d_F(k - \sigma - 1|k-1)\end{aligned} \tag{4.22}$$

where

$$\Phi = I + \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \tag{4.23}$$

It should be noted that for a small enough Γ, Φ is positive-definite and therefore invertible.

Note that among the $\sigma + 1$ consecutive values of d_F, which are updated at every sampling instant, two stand out. The first one is $d_F(k-\sigma)$ which is being corrected for one last time at this instant and is therefore of great importance in our mathematical analysis and design. The second one is $d_F(k)$ which is being predicted for the first time and is therefore important in the analysis of our simulation results. We will refer to the former as the corrected disturbance estimate or d_{Fc} and to the latter as the predicted disturbance or d_{Fp}.

Necessary Stability Conditions

In order to obtain the necessary stability conditions, we first need to calculate ϵ_c with respect to $\bar{e}$ and $\tilde{d}$, which can be accomplished by substituting the second equation of (4.5) in (4.22), *i.e.*

$$\begin{aligned}\epsilon_c(k) = \Phi^{-1} C \bar{e}(k) + \Phi^{-1} \sum_{i=0}^{\sigma} CA^i B_2 \tilde{d}(k-\sigma-1|k-1) \\ + \Phi^{-1} \sum_{i=0}^{\sigma-1} CA^i B_2 \Delta d(k-i-1) + \Phi^{-1} v(k)\end{aligned} \tag{4.24}$$

Now we use this equation along with (4.5), (4.6) and (4.18) to write the error dynamics as follows:

$$\begin{aligned}\bar{e}(k+1) &= A_S^{11} \bar{e}(k) + A_S^{12} \tilde{d}(k-\sigma-1|k-1) \\ &\quad + B_3 w(k) + B_S^{11} v(k) + B_S^{12} \overline{\Delta d}_{k-\sigma}^{k-1} \\ \tilde{d}(k-\sigma|k) &= A_S^{21} \bar{e}(k) + A_S^{22} \tilde{d}(k-\sigma-1|k-1) \\ &\quad + B_S^{21} v(k) + B_S^{22} \overline{\Delta d}_{k-\sigma}^{k-1}\end{aligned} \tag{4.25}$$

where

$$A_S^{11} = A - L\Phi^{-1} C$$

$$
\begin{aligned}
A_S^{12} &= A^{\sigma+1}B_2 - L\Phi^{-1}CA_\Sigma B_2 \\
A_S^{21} &= -\Gamma(CA^\sigma B_2)^T\Phi^{-1}C \\
A_S^{22} &= I - \Gamma(CA^\sigma B_2)^T\Phi^{-1}CA_\Sigma B_2 \\
B_S^{11} &= -L\Phi^{-1} \\
B_S^{12} &= -L\Phi^{-1}C\begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
B_S^{21} &= -\Gamma(CA^\sigma B_2)^T\Phi^{-1} \\
B_S^{22} &= \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix} \\
&\quad - \Gamma(CA^\sigma B_2)^T\Phi^{-1}C\begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
A_\Sigma &= \sum_{i=0}^{\sigma} A^i
\end{aligned}
$$

and

$$
\overline{\Delta d}_{k-\sigma}^{k-1} = \begin{bmatrix} \Delta d(k-1) \\ \Delta d(k-2) \\ \vdots \\ \Delta d(k-\sigma) \end{bmatrix}
$$

It should be noted that our stability analysis is done for $i = \sigma$ due to the fact that at sampling instant k, the only updated $\tilde{d}$, that won't be corrected in the next sampling instants, is $\tilde{d}(k-\sigma)$. Now using (4.25) the error stability matrix can be written as

$$
\mathcal{S} = \begin{bmatrix} A_S^{11} & A_S^{12} \\ A_S^{21} & A_S^{22} \end{bmatrix} \tag{4.26}
$$

Assume that B_2 is partitioned as $B_2 = \begin{bmatrix} B_2' & B_2'' \end{bmatrix}$ and

$$
rank(CA^\sigma B_2) = rank(B_2') < rank(B_2). \tag{4.27}
$$

Defining

$$
\begin{aligned}
M' &= (A^{\sigma+1} - L\Phi^{-1}CA_\Sigma)B_2' \\
M'' &= (A^{\sigma+1} - L\Phi^{-1}CA_\Sigma)B_2'' \\
N' &= (CA^\sigma B_2')^T\Phi^{-1}C \\
N'' &= (CA^\sigma B_2'')^T\Phi^{-1}C
\end{aligned}
$$

$$\Gamma = diag\{\Gamma_1, \Gamma_2\} \tag{4.28}$$

then (4.26) can be rewritten as

$$\mathcal{S} = \begin{bmatrix} A_S^{11} & M' & M'' \\ -\Gamma_1 N' & I - \Gamma_1 N' A_\Sigma B_2' & -\Gamma_1 N' A_\Sigma B_2'' \\ -\Gamma_2 N'' & -\Gamma_2 N'' A_\Sigma B_2' & I - \Gamma_2 N'' A_\Sigma B_2'' \end{bmatrix} \tag{4.29}$$

According to (4.27), the columns of $CA^\sigma B_2''$ are the linear combination of the columns of $CA^\sigma B_2'$, or in other words $CA^\sigma B_2'' = CA^\sigma B_2' R$. As a result $N'' = R^T N'$. To analyze the stability of this matrix, we introduce the following similarity transformation matrix:

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Gamma_2 R^T \Gamma_1^{-1} & I \end{bmatrix} \tag{4.30}$$

Applying this transformation to $\mathcal{S}$, we get

$$\begin{aligned} \mathcal{S}_T &= T\mathcal{S}T^{-1} \\ &= \begin{bmatrix} A_S^{11} & M' & M'' \\ -\Gamma_1 N' & E & -\Gamma_1 N' A_\Sigma B_2'' \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{4.31}$$

where $E = I - \Gamma_1 N' A_\Sigma B_2' - \Gamma_1 N' A_\Sigma B_2'' \Gamma_2 R^T \Gamma_1^{-1}$. It is easy to show that $\mathcal{S}_T$ has h eigenvalues at 1, where $h = rank(B_2) - rank(B_2')$.

Corollary 4.2. *The filter (4.6) with the adaptive law given in (4.18) can produce stable estimates if the following inequality holds true:*

$$rank(CA^\sigma B_2) \geq rank(B_2) \tag{4.32}$$

It is crucial to note that the necessary stability condition given in (4.32) is less restrictive than the well-known necessary condition $rank(CB_2) \geq rank(B_2)$, which is very common in unknown input observer design. The following example illustrates this point:

Example: Assume

$$x(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -0.5 & 0 \\ 0.25 & 0.25 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} d(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} x(k)$$

For this system, we have $rank(B_2) = 2$, $rank(CB_2) = 1$ and $rank(CAB_2) = 2$.

4.2.2 The Proposed Filter

In this section, we will propose an LMI-based filter design procedure for discrete-time linear systems subject to unknown state disturbances.

Consider the revised linear model given in (4.5) and the filter model given in (4.6). The following theorem, formulates the proposed adaptive approach.

Theorem 4.1. *Consider the linear system (4.1) along with the linear filter (4.6) with d_F being updated as follows:*

$$d_F(k-i) = d_F(k-\sigma-1) + \Gamma \sum_{j=i}^{\sigma} B_2^T {A^j}^T C^T \epsilon_c(k)$$

where $\epsilon_c = y - y_{Fc}$ is the measurement estimation error and $\Gamma > 0$ is the adaptation gain. Let μ_w, μ_v, μ_d, be the ℓ_2 attenuation gains bounding the effects of state noise, measurement noise, and disturbance variations, on the estimation error. Then the $\mathcal{H}_\infty$ filtering problem with disturbance estimator has a solution if there exist matrices $P_e, P_d > 0$ and G_e satisfying the following LMI:

$$\begin{bmatrix} \Xi_1 & \Xi_2 \\ \star & -\mathbb{P} \end{bmatrix} < 0 \tag{4.33}$$

where

$$\mathbb{P} = diag\{P_e, P_d\}$$

$$\Xi_1 = \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix}$$

$$\Xi_1^{11} = \begin{bmatrix} {C_S^1}^T C_S^1 - P_e & {C_S^1}^T C_S^2 \\ {C_S^2}^T C_S^1 & {C_S^2}^T C_S^2 - P_d \end{bmatrix}$$

$$\Xi_1^{12} = \begin{bmatrix} 0 & {C_S^1}^T D_S^1 & {C_S^1}^T D_S^2 \\ 0 & {C_S^2}^T D_S^1 & {C_S^2}^T D_S^2 \end{bmatrix}$$

$$\Xi_1^{22} = \begin{bmatrix} -\mu_w^2 I & 0 & 0 \\ 0 & {D_S^1}^T D_S^1 - \mu_v^2 I & {D_S^1}^T D_S^2 \\ 0 & {D_S^2}^T D_S^1 & {D_S^2}^T D_S^2 - \mu_d^2 I \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} \Xi_2^1 & \Xi_2^2 \end{bmatrix}$$

$$\Xi_2^1 = \begin{bmatrix} A^T P_e - C^T \Phi^{-T} G_e^T \\ B_2^T ({A^{\sigma+1}}^T P_e - \sum_{i=0}^{\sigma} {A^i}^T C^T \Phi^{-T} G_e^T) \\ B_3^T P_e \\ -\Phi^{-T} G_e^T \\ -B_2^T C^T \Phi^{-T} G_e^T \\ -B_2^T A^T C^T \Phi^{-T} G_e^T \\ \vdots \\ -B_2^T {A^{\sigma-1}}^T C^T \Phi^{-T} G_e^T \end{bmatrix}$$

$$\Xi_2^2 = \begin{bmatrix} -C^T \Phi^{-T} C A^\sigma B_2 \Gamma P_d \\ (I - B_2^T \sum_{i=0}^{\sigma} {A^i}^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma) P_d \\ 0 \\ -\Phi^{-T} C A^\sigma B_2 \Gamma P_d \\ -B_2^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma P_d \\ -B_2^T A^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma P_d \\ \vdots \\ P_d - B_2^T {A^{\sigma-1}}^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma P_d \end{bmatrix}$$

$$C_S^1 = H(I - \sum_{i=0}^{\sigma} A^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \Phi^{-1} C)$$

$$C_S^2 = \sum_{i=0}^{\sigma} H A^i B_2 (I - \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \Phi^{-1} C A_\Sigma B_2)$$

$$D_S^1 = -\sum_{i=0}^{\sigma} H A^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \Phi^{-1}$$

$$\begin{aligned} D_S^2 = {} & H \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \\ & - \sum_{i=0}^{\sigma} H A^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \Phi^{-1} C \\ & \times \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \end{aligned}$$

$$\Phi = I + \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \tag{4.34}$$

Then the filter parameter is calculated as $L = P_e^{-1} G_e$, and the disturbance estimate d_F is updated as given in (4.18).

Proof. Using (4.25), (4.5) and (4.6), the estimation error system can be written as

$$\begin{aligned}
\bar{e}(k+1) &= A_S^{11}\bar{e}(k) + A_S^{12}\tilde{d}(k-\sigma-1|k-1) \\
&\quad + B_3 w(k) + B_S^{11} v(k) + B_S^{12}\overline{\Delta d}_{k-\sigma}^{k} \\
\tilde{d}(k-\sigma) &= A_S^{21}\bar{e}(k) + A_S^{22}\tilde{d}(k-\sigma-1|k-1) \\
&\quad + B_S^{21} v(k) + B_S^{22}\overline{\Delta d}_{k-\sigma}^{k} \\
\varepsilon(k) &= z(k) - z_F(k) \\
&= C_S^1 \bar{e}(k) + C_S^2 \tilde{d}(k-\sigma-1|k-1) \\
&\quad + D_S^1 v(k) + D_S^2 \overline{\Delta d}_{k-\sigma}^{k}
\end{aligned} \tag{4.35}$$

where C_S^1, C_S^2, D_S are as defined in (4.34); and A_S^{ij}, B_S^{ij} are as given in (4.25). Augmenting $\bar{e}$ and $\tilde{d}$ as $X(k) = \begin{bmatrix} \bar{e}(k)^T & \tilde{d}(k-\sigma-1|k-1)^T \end{bmatrix}^T$ and defining $\omega = \begin{bmatrix} w^T & v^T & \Delta d^T \end{bmatrix}^T$, the augmented error model will be given as:

$$\begin{aligned}
X(k+1) &= \mathbb{A}X(k) + \mathbb{B}\omega(k) \\
\varepsilon(k) &= \mathbb{C}X(k) + \mathbb{D}\omega(k)
\end{aligned} \tag{4.36}$$

where

$$\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A_S^{11} & A_S^{12} \\ A_S^{21} & A_S^{22} \end{bmatrix} \\
\mathbb{B} &= \begin{bmatrix} B_3 & B_S^{11} & B_S^{12} \\ 0 & B_S^{21} & B_S^{22} \end{bmatrix} \\
\mathbb{C} &= \begin{bmatrix} C_S^1 & C_S^2 \end{bmatrix} \\
\mathbb{D} &= \begin{bmatrix} 0 & D_S^1 & D_S^2 \end{bmatrix}
\end{aligned} \tag{4.37}$$

To analyze the stability of the augmented system, the following Lyapunov function is used,

$$V(k) = X(k)^T \mathbb{P} X(k) \tag{4.38}$$

The forward difference of this Lyapunov function can be written as

$$
\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&= X(k)^T \mathbb{A}^T \mathbb{P} \mathbb{A} X(k) + \omega(k)^T \mathbb{B}^T \mathbb{P} \mathbb{B} \omega(k) \\
&+ X(k)^T \mathbb{A}^T \mathbb{P} \mathbb{B} \omega(k) + \omega(k)^T \mathbb{B}^T \mathbb{P} \mathbb{A} X(k) \\
&- X(k)^T \mathbb{P} X(k)
\end{aligned} \tag{4.39}
$$

Defining $\xi = \begin{bmatrix} X^T & \omega^T \end{bmatrix}^T$, (4.39) can be simplified as follows:

$$
\Delta V(k) = \xi(k)^T \{ \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix} \mathbb{P} \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix}^T + \begin{bmatrix} -\mathbb{P} & 0 \\ 0 & 0 \end{bmatrix} \} \xi(k) \tag{4.40}
$$

In order to establish an $\mathcal{H}_\infty$ bound on the effects of the unwanted noise inputs and also the effects of the variations of the unknown disturbance *i.e.* Δd, we define

$$
J \triangleq \sum_{k=0}^{h} \{ \varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu \omega(k) \} \tag{4.41}
$$

where $h > 0$ is a finite integer and $\mu = diag\{\mu_w, \mu_v, \mu_d\}$. Adding (4.38) to the right hand side of (4.41), we get

$$
J < \sum_{k=0}^{h} \{ \varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu \omega(k) + \Delta V(k) \} = \sum_{k=0}^{h} \tilde{J}_k \tag{4.42}
$$

Now, if we design our filter such that $\tilde{J}_k \leq 0$, we can conclude that $J \leq 0$, which implies that in the time interval $[0, h]$, the second norm of the estimation error is bounded by factors of the second norms of the noise inputs and disturbance variations. In other words, it establishes an $\mathcal{H}_\infty$ bound on the estimation error system. Using (4.36) $\varepsilon^T \varepsilon$ can be simplified as follows:

$$
\varepsilon^T \varepsilon = \xi(k)^T \begin{bmatrix} \mathbb{C}^T \mathbb{C} & \mathbb{C}^T \mathbb{D} \\ \mathbb{D}^T \mathbb{C} & \mathbb{D}^T \mathbb{D} \end{bmatrix} \xi(k) \tag{4.43}
$$

Substituting (4.43) and (4.40) in (4.42), we have

$$
\tilde{J}_k \leq \xi^T (\Omega_1 + \Omega_2 \mathbb{P}^{-1} \Omega_2^T) \xi \tag{4.44}
$$

where

$$
\Omega_1 = \begin{bmatrix} \mathbb{C}^T \mathbb{C} - \mathbb{P} & \mathbb{C}^T \mathbb{D} \\ \mathbb{D}^T \mathbb{C} & \mathbb{D}^T \mathbb{D} - \mu^T \mu \end{bmatrix}
$$

$$\Omega_2 = \begin{bmatrix} \mathbb{A}^T\mathbb{P} \\ \mathbb{B}^T\mathbb{P} \end{bmatrix} \tag{4.45}$$

Now if we define $G_e = P_e L$, we can rewrite (4.44) as follows:

$$\tilde{J}_k \leq \xi^T(\Xi_1 + \Xi_2\mathbb{P}^{-1}\Xi_2^T)\xi \tag{4.46}$$

where Ξ_1, Ξ_2, Ξ_3 are as given in (4.34). Using Schur's Complement one can show that (4.46) holds true if the LMI given in (4.33) is satisfied. □

4.3 Approach II

Consider the following filter model,

$$\begin{aligned} \bar{x}_F(k+1) &= A\bar{x}_F(k) + A^{\sigma+1}B_2 d_F(k-\sigma-1) \\ &\quad + B_1 u(k) + L(y(k) - y_F(k)) \\ y_F(k) &= C\bar{x}_F(k) + \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-\sigma-1) \\ z_F(k) &= H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 d_F(k-\sigma-1) \end{aligned} \tag{4.47}$$

where $\bar{x}_F$ is the $n \times 1$ state vector of the filter; d_F is the $m_2 \times 1$ estimated disturbance vector, which will follow an stable adaptive law to track the unknown disturbance; y_F represents the $p \times 1$ estimated measurement vector; z_F is the $r \times 1$ estimated vector; and L is the static filter parameter to be designed.

4.3.1 Stability Analysis

Similar to Approach I, in this section we will first derive a stable adaptive law for estimating the unknown input and then we'll move on to the necessary stability conditions for this approach.

Adaptive Law Extraction

In order to extract an stable adaptive law for d_F, we first need to find how the unknown input estimation error $\tilde{d} = d - d_F$, is related to the measurement vector. To this end, we define the measurement estimation error as

$$\epsilon(k) = y(k) - y_F(k)$$

$$= C\bar{e}(k) + \sum_{i=0}^{\sigma} CA^i B_2 \tilde{d}(k-\sigma-1)$$

$$+ \sum_{i=0}^{\sigma-1} CA^i B_2 \Delta d(k-i-1) + v(k) \tag{4.48}$$

where $\bar{e} = \bar{x} - \bar{x}_F$ represents the state estimation error; $\Delta d(k-i) = d(k-i) - d(k-\sigma-1)$ represents the variation of the unknown disturbance; and v is the measurement noise. We now construct our adaptation law to ensure that ϵ is exponentially decreasing. Using (4.48) we can write

$$\epsilon(k+1) - \epsilon(k) = C\bar{e}(k+1) - C\bar{e}(k) + \delta$$

$$+ \sum_{i=0}^{\sigma} CA^i B_2 (\tilde{d}(k-\sigma) - \tilde{d}(k-\sigma-1)) \tag{4.49}$$

where

$$\delta = \sum_{i=0}^{\sigma} CA^i B_2 (d(k-i) - d(k-i-1))$$

$$+ v(k+1) - v(k)$$

We now choose $\tilde{d}(k-\sigma)$ as

$$\tilde{d}(k-\sigma) = \tilde{d}(k-\sigma-1) - \Gamma \sum_{i=0}^{\sigma} B_2^T A^{i^T} C^T \epsilon(k) \tag{4.50}$$

for $i = 0, \ldots, \sigma$ where Γ is a diagonal positive definite matrix which represents the adaptation gain. Substituting (4.50) in (4.49), we get

$$\epsilon(k+1) = (I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T A^{i^T} C^T)\epsilon(k)$$

$$+ C(\bar{e}(k+1) - \bar{e}(k)) + \delta \tag{4.51}$$

From the above equation, we can conclude that for a stable $\bar{e}$ and a bounded error term δ, the measurement estimation error, *i.e.* ϵ, is decreasing in time, provided that Γ is chosen in such a way that the following inequality is satisfied:

$$|\lambda(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T A^{i^T} C^T)| \le 1 \tag{4.52}$$

where $\lambda(\ M\ )$ represents the eigenvalues of M. We claim that there always exists a small enough $\Gamma_0 > 0$, that for all $\Gamma < \Gamma_0$ the condition given in (4.52) is satisfied. To show this more clearly we first break down this condition into the following two inequalities:

$$\lambda_{max}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T) \leq 1 \tag{4.53}$$

$$\lambda_{min}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T) \geq -1 \tag{4.54}$$

The first inequality is always satisfied due to the fact that for a positive definite Γ, the matrix given by $\sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T$ is always positive definite and therefore the largest eigenvalue will always be smaller than 1. To investigate the second inequality we can write

$$\begin{aligned}
&\lambda_{min}(I - \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T) \\
&\quad = 1 - \lambda_{max}(\sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T) \\
&\quad \geq 1 - \| \sum_{i=0}^{\sigma} CA^i B_2 \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \| \\
&\quad \geq 1 - \|\Gamma\| \| \sum_{i=0}^{\sigma} CA^i B_2 \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \|
\end{aligned}$$

Therefore, for (4.54) to be satisfied we need to have

$$\|\Gamma\| \| \sum_{i=0}^{\sigma} CA^i B_2 \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \| \leq 2$$

Based on the assumption given in (4.4), $\| \sum_{i=0}^{\sigma} CA^i B_2 \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \|$ is non-zero and therefore there always exists a small enough $\Gamma > 0$ for which (4.54) is satisfied.

It should be noted that the adaptive law given in (4.50) is not realizable due to the existence of the unknown disturbance d in the equation. To handle this problem, we rewrite (4.50) as

$$\begin{aligned}
d(k-\sigma) - d_F(k-\sigma) &\\
&= d(k-\sigma-1) - d_F(k-\sigma-1) \\
&- \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \epsilon(k) \\
\Rightarrow d_F(k-\sigma) &= d_F(k-\sigma-1) \\
&+ \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \epsilon(k) + \Delta d(k-\sigma)
\end{aligned} \tag{4.55}$$

where $\Delta d(k-\sigma) = d(k-\sigma) - d(k-\sigma-1)$ is an unknown term, which is bounded for any continuous signal and cannot be realized. As an alternative, we choose the adaptation law to be

$$d_F(k-\sigma) = d_F(k-\sigma-1) + \Gamma \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \epsilon(k) \tag{4.56}$$

Necessary Stability Conditions

In order to establish necessary stability conditions, we first form the filtering error system by using (4.5), (4.47) and (4.56) as follows:

$$\begin{aligned}
\bar{e}(k+1) &= \mathbb{A}_{11}\bar{e}(k) + \mathbb{A}_{12}\tilde{d}(k-\sigma-1) + B_3 w(k) + \mathbb{B}_{11}v(k) + \mathbb{B}_{12}\overline{\Delta d}_{k-\sigma}^{k-1} \\
\tilde{d}(k-\sigma) &= \mathbb{A}_{21}\bar{e}(k) + \mathbb{A}_{22}\tilde{d}(k-\sigma-1) + \mathbb{B}_{21}v(k) + \mathbb{B}_{22}\overline{\Delta d}_{k-\sigma}^{k-1} \\
\varepsilon(k) &= z(k) - z_F(k) \\
&= \mathbb{C}_1\bar{e}(k) + \mathbb{C}_2\tilde{d}(k-\sigma-1) + \mathbb{D}_1 v(k) + \mathbb{D}_2\overline{\Delta d}_{k-\sigma}^{k-1}
\end{aligned} \tag{4.57}$$

where

$$\begin{aligned}
\mathbb{A}_{11} &= A - LC \\
\mathbb{A}_{12} &= A^{\sigma+1}B_2 - LCA_\Sigma B_2 \\
\mathbb{A}_{21} &= -\Gamma(CA_\Sigma B_2)^T C \\
\mathbb{A}_{22} &= I - \Gamma(CA_\Sigma B_2)^T CA_\Sigma B_2 \\
\mathbb{B}_{11} &= -L \\
\mathbb{B}_{12} &= -LC\begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
\mathbb{B}_{21} &= -\Gamma(CA_\Sigma B_2)^T \\
\mathbb{B}_{22} &= \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix} \\
&\quad - \Gamma(CA_\Sigma B_2)^T C\begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
\mathbb{C}_1 &= H \\
\mathbb{C}_2 &= HA_\Sigma B_2 \\
\mathbb{D}_2 &= H\begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
A_\Sigma &= \sum_{i=0}^{\sigma} A^i
\end{aligned}$$

and

$$\overline{\Delta d}_{k-\sigma}^{k-1} = \begin{bmatrix} \Delta d(k-1) \\ \Delta d(k-2) \\ \vdots \\ \Delta d(k-\sigma) \end{bmatrix}$$

Now using the above model, we can write the error stability matrix as follows,

$$\mathcal{S} = \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix} \tag{4.58}$$

Assume that B_2 is partitioned as $B_2 = \begin{bmatrix} B_2' & B_2'' \end{bmatrix}$ and

$$rank(\sum_{i=0}^{\sigma} CA^i B_2) = rank(B_2') < rank(B_2). \tag{4.59}$$

Defining

$$\begin{aligned} M' &= (A^{\sigma+1} - LCA_\Sigma)B_2' \\ M'' &= (A^{\sigma+1} - LCA_\Sigma)B_2'' \\ N' &= (CA_\Sigma B_2')^T C \\ N'' &= (CA_\Sigma B_2'')^T C \\ \Gamma &= diag\{\Gamma_1, \Gamma_2\} \end{aligned} \tag{4.60}$$

then (4.58) can be rewritten as

$$\mathcal{S} = \begin{bmatrix} \mathbb{A}_{11} & M' & M'' \\ -\Gamma_1 N' & I - \Gamma_1 N' A_\Sigma B_2' & -\Gamma_1 N' A_\Sigma B_2'' \\ -\Gamma_2 N'' & -\Gamma_2 N'' A_\Sigma B_2' & I - \Gamma_2 N'' A_\Sigma B_2'' \end{bmatrix} \tag{4.61}$$

According to (4.59), columns of $CA_\Sigma B_2''$ are linear combinations of the columns of $CA_\Sigma B_2'$, or in other words $CA_\Sigma B_2'' = CA_\Sigma B_2' R$. As a result $N'' = R^T N'$. To analyze the stability of this matrix, we introduce the following similarity transformation matrix:

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\Gamma_2 R^T \Gamma_1^{-1} & I \end{bmatrix} \tag{4.62}$$

Applying this transformation to $\mathcal{S}$, we get

$$\begin{aligned}\mathcal{S}_T &= T\mathcal{S}T^{-1} \\ &= \begin{bmatrix} \mathbb{A}_{11} & M' & M'' \\ -\Gamma_1 N' & E & -\Gamma_1 N' A_\Sigma B_2'' \\ 0 & 0 & I \end{bmatrix}\end{aligned} \tag{4.63}$$

where $E = I - \Gamma_1 N' A_\Sigma B_2' - \Gamma_1 N' A_\Sigma B_2'' \Gamma_2 R^T \Gamma_1^{-1}$. It is easy to show that $\mathcal{S}_T$ has h eigenvalues at 1, where $h = rank(B_2) - rank(B_2')$.

Corollary 4.3. *The filter (4.47) with the adaptive law given in (4.56) can produce stable estimates if the following inequality holds true:*

$$rank(\sum_{i=0}^{\sigma} CA^i B_2) \geq rank(B_2) \tag{4.64}$$

Similar to the necessary stability condition of Approach I, the above condition is less restrictive than the well-known necessary condition $rank(CB_2) \geq rank(B_2)$.

4.3.2 The Proposed Filter

In this section, we will explain the design procedure for the proposed filter in the second approach.

Augmenting $\bar{e}$ and $\tilde{d}$ as $X(k) = \begin{bmatrix} \bar{e}(k)^T & \tilde{d}(k-\sigma-1)^T \end{bmatrix}^T$ and defining $\omega = \begin{bmatrix} w^T & v^T & \overline{\Delta d}^T \end{bmatrix}^T$, the augmented error model can be written as:

$$\begin{aligned} X(k+1) &= \mathbb{A}X(k) + \mathbb{B}\omega(k) \\ \varepsilon(k) &= \mathbb{C}X(k) + \mathbb{D}\omega(k) \end{aligned} \tag{4.65}$$

where

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix} \\ \mathbb{B} &= \begin{bmatrix} B_3 & \mathbb{B}_{11} & \mathbb{B}_{12} \\ 0 & \mathbb{B}_{21} & \mathbb{B}_{22} \end{bmatrix} \\ \mathbb{C} &= \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 \end{bmatrix} \\ \mathbb{D} &= \begin{bmatrix} 0 & 0 & \mathbb{D}_2 \end{bmatrix} \end{aligned} \tag{4.66}$$

The following theorem formulates the design approach for the zone 1 filter.

Theorem 4.2. *Consider the linear system in (4.1) . Then the linear filter given in (4.47) with the disturbance estimator given in (4.56) generate stable estimates of* z *and* d *with arbitrary* $\mathcal{H}_\infty$ *gains* $\mu_w, \mu_v, \mu_d > 0$*, which respectively bound the effects of the state noise, measurement noise and disturbance variations, if there exist matrices* $P_e, P_d, G_d > 0$ *and* G_e *satisfying the following LMI:*

$$\begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 \\ \star & -P_e & 0 \\ \star & \star & -P_d \end{bmatrix} < 0 \tag{4.67}$$

where

$$\Xi_1 = \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix}$$

$$\Xi_1^{11} = \begin{bmatrix} H^T H - P_e & H^T H A_\Sigma B_2 \\ B_2^T A_\Sigma^T H^T H & B_2^T A_\Sigma^T H^T H A_\Sigma B_2 - P_d \end{bmatrix}$$

$$\Xi_1^{12} = \begin{bmatrix} 0 & 0 & H^T \mathbb{D}_2 \\ 0 & 0 & B_2^T A_\Sigma^T H^T \mathbb{D}_2 \end{bmatrix}$$

$$\Xi_1^{22} = \begin{bmatrix} -\mu_w^2 I & 0 & 0 \\ 0 & -\mu_v^2 I & 0 \\ 0 & 0 & \mathbb{D}_2^T \mathbb{D}_2 - \mu_d^2 I \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} A^T P_e - C^T G_e^T \\ B_2^T ({A^{\sigma+1}}^T P_e - A_\Sigma^T C^T G_e^T) \\ B_3^T P_e \\ -G_e^T \\ -B_2^T C^T G_e^T \\ -B_2^T A^T C^T G_e^T \\ \vdots \\ -B_2^T {A^{\sigma-1}}^T C^T G_e^T \end{bmatrix}$$

$$\Xi_3 = \begin{bmatrix} -C^T C A_\Sigma B_2 G_d \\ P_d - B_2^T A_\Sigma^T C^T C A_\Sigma B_2 G_d \\ 0 \\ -C A_\Sigma B_2 G_d \\ -B_2^T C^T C A_\Sigma B_2 G_d \\ -B_2^T A^T C^T C A_\Sigma B_2 G_d \\ \vdots \\ P_d - B_2^T {A^{\sigma-1}}^T C^T C A_\Sigma B_2 G_d \end{bmatrix}$$

$$\mathbb{D}_2 = H \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \tag{4.68}$$

Once solved, the filter parameters are calculated as $L = P_e^{-1} G_e$ *and* $\Gamma = P_d^{-1} G_d$.

Proof. To analyze the stability of the filtering error system given in (4.65), the following Lyapunov function is used,

$$V(k) = X(k)^T \mathbb{P}_1 X(k) \tag{4.69}$$

where $\mathbb{P} = diag\{P_e, P_d\}$. The forward difference of this Lyapunov function can be written as

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= X(k)^T \mathbb{A}^T \mathbb{P} \mathbb{A} X(k) + \omega(k)^T \mathbb{B}^T \mathbb{P} \mathbb{B} \omega(k) \\ &+ X(k)^T \mathbb{A}^T \mathbb{P} \mathbb{B} \omega(k) + \omega(k)^T \mathbb{B}^T \mathbb{P} \mathbb{A} X(k) \\ &- X(k)^T \mathbb{P} X(k) \end{aligned} \tag{4.70}$$

Defining $\xi = \begin{bmatrix} X^T & \omega^T \end{bmatrix}^T$, (4.70) can be simplified as follows:

$$\Delta V(k) = \xi(k)^T \{ \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix} \mathbb{P} \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix}^T + \begin{bmatrix} -\mathbb{P} & 0 \\ 0 & 0 \end{bmatrix} \} \xi(k) \tag{4.71}$$

In order to establish attenuation bounds on the effects of the network-induced error Δy_q and also effects of the variations of the unknown disturbance *i.e.* Δd, we define

$$J \triangleq \sum_{k=0}^{s} \{ \varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu \omega(k) \} \tag{4.72}$$

where $s > 0$ is any bounded integer and $\mu = diag\{\mu_w, \mu_v, \mu_d\}$. Adding (4.69) to the right hand side of (4.72), we get

$$J < \sum_{k=0}^{s} \{ \varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu \omega(k) + \Delta V(k) \} = \sum_{k=0}^{s} \tilde{J}_k \tag{4.73}$$

Now, if we design our filter such that $\tilde{J}_k \leq 0$, we can conclude that $J \leq 0$, which implies that in the time interval $[0, s]$, the second norm of the estimation error is bounded by factors of the second norms of noise inputs and disturbance variations. In other words, it establishes an $\mathcal{H}_\infty$ bound on the filtering error system. Using (4.65) $\varepsilon^T\varepsilon$ can be simplified as follows:

$$\varepsilon^T\varepsilon = \xi(k)^T \begin{bmatrix} \mathbb{C}^T\mathbb{C} & \mathbb{C}^T\mathbb{D} \\ \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} \end{bmatrix} \xi(k) \tag{4.74}$$

Substituting (4.74) and (4.71) in (4.73), we have

$$\tilde{J}_k \leq \xi^T(\Omega_1 + \Omega_2\mathbb{P}^{-1}\Omega_2^T)\xi \tag{4.75}$$

where

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} \mathbb{C}^T\mathbb{C} - \mathbb{P} & \mathbb{C}^T\mathbb{D} \\ \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} - \mu^T\mu \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} \mathbb{A}^T\mathbb{P} \\ \mathbb{B}^T\mathbb{P} \end{bmatrix} \end{aligned} \tag{4.76}$$

Now if we define $G_e = P_eL$ and $G_d = P_d\Gamma$, we can rewrite (4.75) as follows:

$$\tilde{J}_k \leq \xi^T(\Xi_1 + \Xi_2P_e^{-1}\Xi_2^T + \Xi_3P_d^{-1}\Xi_3^T)\xi \tag{4.77}$$

where Ξ_1, Ξ_2, Ξ_3 are as given in (4.68). Using Schur's Complement one can show that (4.77) holds true if the LMI given in (4.67) is satisfied. □

4.4 Simulation Results

In this section, we simulate the proposed unknown input filter for three examples. In all of these examples we assume the following:

- the sampling time is assumed to be $T_s = 0.1$ *sec*,
- w and v are white noise inputs with the standard deviations of 0.5 and 0.1, respectively,
- the unknown disturbance d is applied to the system at $t = 3$ *sec* as a step input with the amplitude of 2 and then it grows linearly till $t = 4$ *sec* where it reaches the amplitude of 4,

- $\mathcal{H}_\infty$ bounds are chosen as $\mu_w = \mu_v = \mu_d = 0.5$.

In all of the figures, z_{F1}, d_{F1} represent the estimated signals by Approach I and z_{F2}, d_{F2} represent the estimated signals by Approach II.

Example 4.1. *In this example a stable system with an invertible A matrix is considered and the results of both of the proposed approaches are compared with those generated by the approach introduced in [53].*

Consider the following stable linear system,

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.8 & 0 & 0 \\ 2.1 & -1.3 & -0.6481 \\ 0 & 0.6481 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k) \\
&+ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d(k) + \begin{bmatrix} 0.01 \\ 0.01 \\ 0.01 \end{bmatrix} w(k) \\
y(k) &= \begin{bmatrix} 0 & 0 & 1.543 \end{bmatrix} x(k) + v(k) \\
z(k) &= \begin{bmatrix} 0 & 0.1 & 0.1 \end{bmatrix} x(k)
\end{aligned} \tag{4.78}$$

For this system we have $CB_2 = CAB_2 = 0$ and therefore majority of the approaches in the literature will fail to produce stable results. For both of our approaches we get $\sigma = 2$. Choosing now the adaptation gain as $\Gamma = 0.2$, the filter gain derived through Approach I is

$$L = \begin{bmatrix} 0.1968 & 0.7734 & -0.3164 \end{bmatrix}^T$$

For Approach II, the filter and adaptation gains are calculated as

$$\begin{aligned}
L &= \begin{bmatrix} 0.2425 & 0.8284 & -0.2962 \end{bmatrix}^T \\
\Gamma &= 0.2314
\end{aligned}$$

Using the above parameters, we simulate the filter in two scenarios:

Scenario 1. *we assume that the known input u is zero,*

Scenario 2. *we assume $u(k) = sin(5kT_s)$.*

Figures 4.1, 4.2, 4.3 and 4.4 compare the estimated signal z_F and d_F produced by both of our approaches to those produced by [53] in scenario 1. In all of the figures of this *example z_{F0}, d_{F0} represent the estimated signals by the method given in [53]. As seen in these figures, the proposed filters illustrate better performance in tracking of both z and d.*

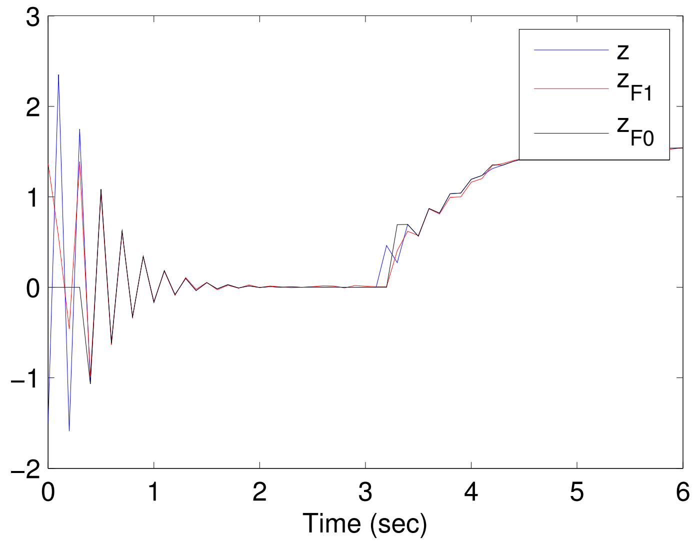


Figure 4.1: Approach I: Real signal z and its estimate z_F for the proposed filter in the first scenario of example 1

Figures 4.5, 4.6, 4.7 and 4.8 compare the estimated signal z_F and d_F produced by both of our approaches to those produced by [53] in scenario 2. The supremacy of the proposed approaches is very obvious in these figures as the approach in [53] does not take the effects of the known input u into account.

Example 4.2. *In this example we will consider a stable system with noninvertible A matrix. This example is to demonstrate that our approaches do not require the invertibility assumption of the matrix A encountered in [53].*

Consider the following stable linear system,

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ &+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(k) + \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} w(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) + v(k) \\ z(k) &= \begin{bmatrix} 0.1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{4.79}$$

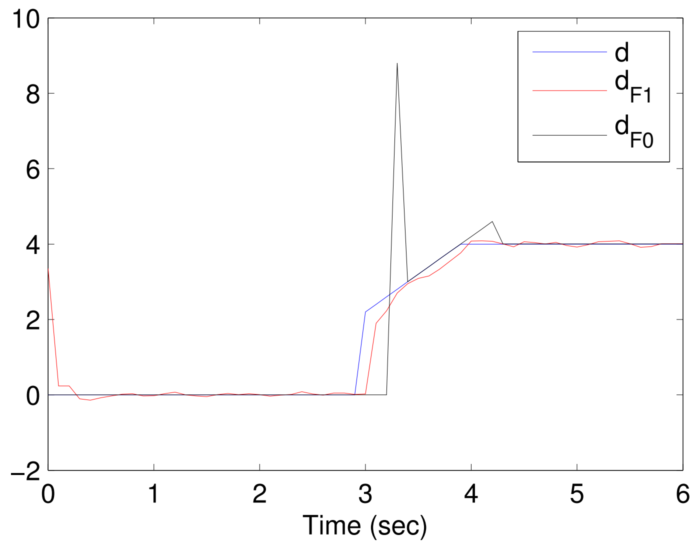


Figure 4.2: Approach I: Disturbance d and its estimate d_F for the proposed filter in the first scenario of example 1

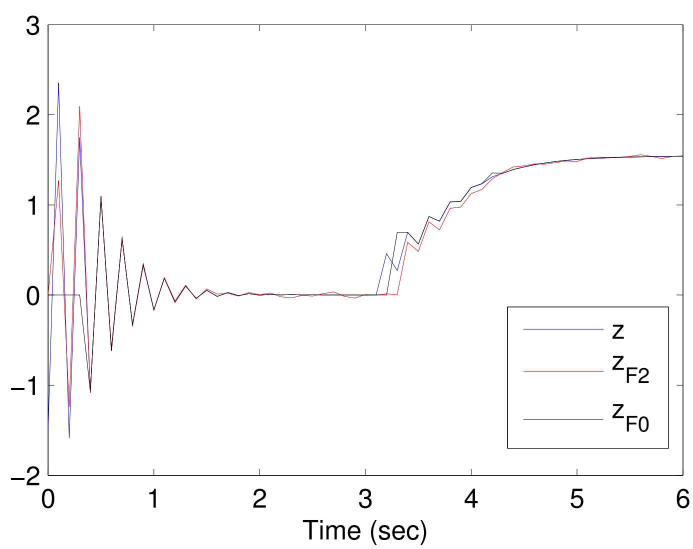


Figure 4.3: Approach II: Real signal z and its estimate z_F for the proposed filter in the first scenario of example 1

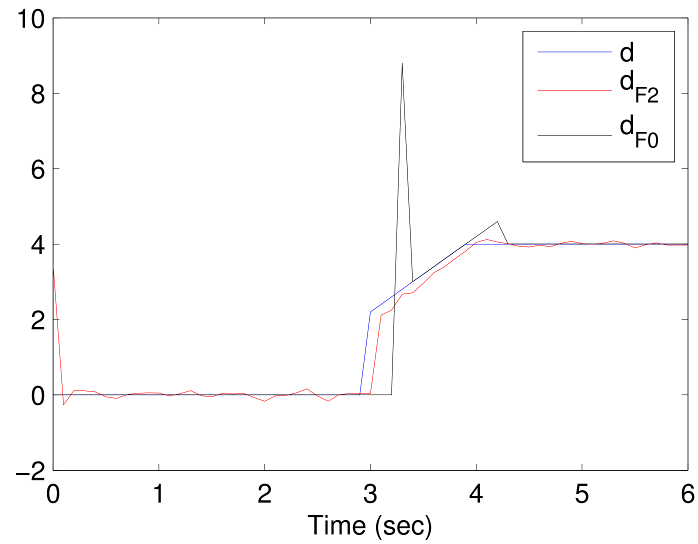


Figure 4.4: Approach II: Disturbance d and its estimate d_F for the proposed filter in the first scenario of example 1

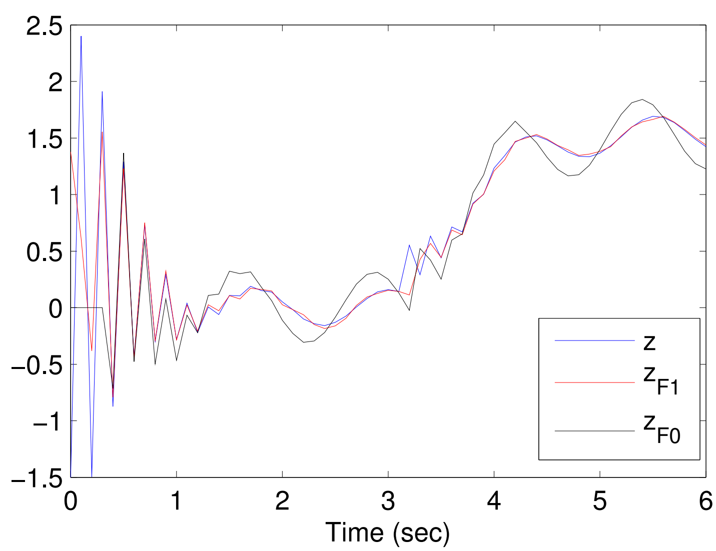


Figure 4.5: Approach I: Real signal z and its estimate z_F for the proposed filter in the second scenario of example 1

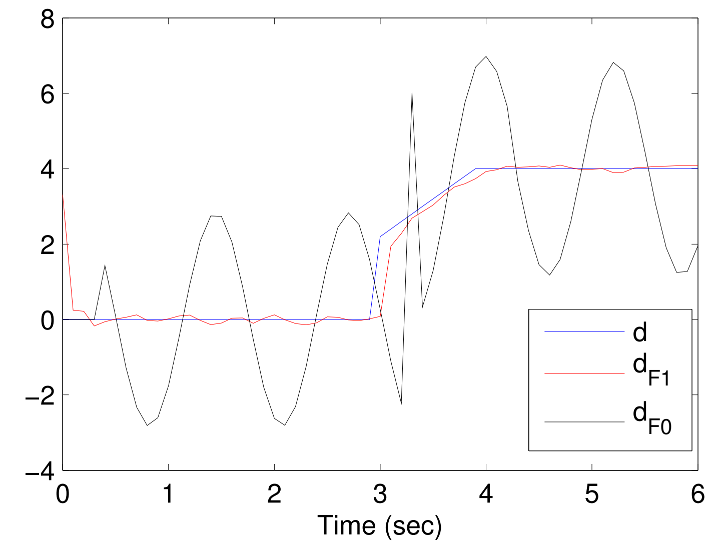


Figure 4.6: Approach I: Disturbance d and its estimate d_F for the proposed filter in the second scenario of example 1

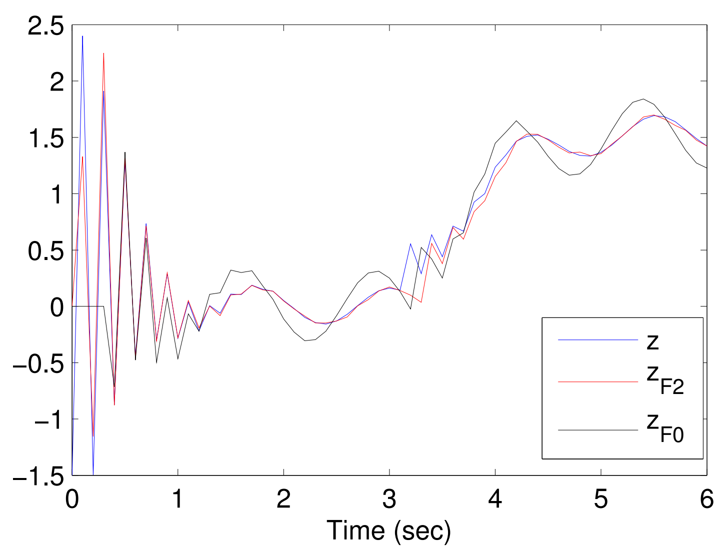


Figure 4.7: Approach II: Real signal z and its estimate z_F for the proposed filter in the second scenario of example 1

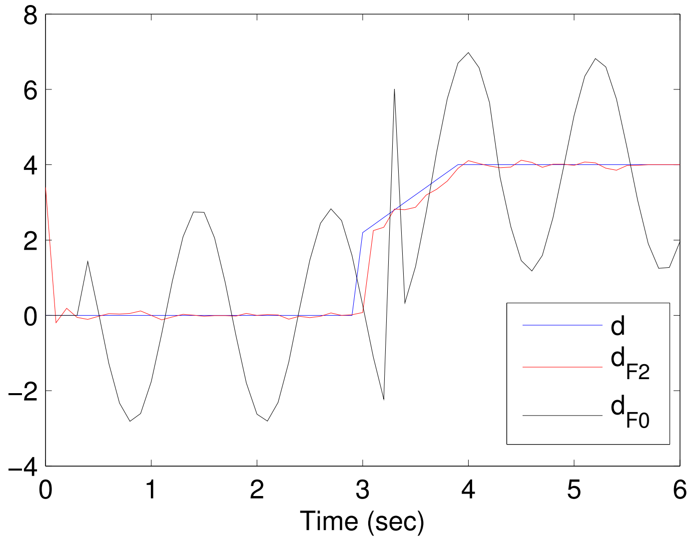


Figure 4.8: Approach II: Disturbance d and its estimate d_F for the proposed filter in the second scenario of example 1

For this system we have $CB_2 = 0$ and σ for both of our approaches is calculated as $\sigma = 1$. Choosing now the adaptation gain as $\Gamma = 0.5$, the filter gain derived through Approach I is

$$L = \begin{bmatrix} 0.1363 & 0.2758 \end{bmatrix}^T$$

For Approach II, the filter and adaptation gains are calculated as

$$L = \begin{bmatrix} 0.218 & 0.4199 \end{bmatrix}^T$$
$$\Gamma = 0.7748$$

Figures 4.9, 4.10, 4.11 and 4.12 show the estimated signal z_F and d_F for both of the proposed approaches. As seen in these figures, the proposed filters estimates both z_F and d_F regardless of the invertiblity of A.

Example 4.3. *In this example we consider an unstable system to show that system stability is not a requirement for our proposed filters.*

Consider the following unstable linear system,

$$x(k+1) = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 1.01 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

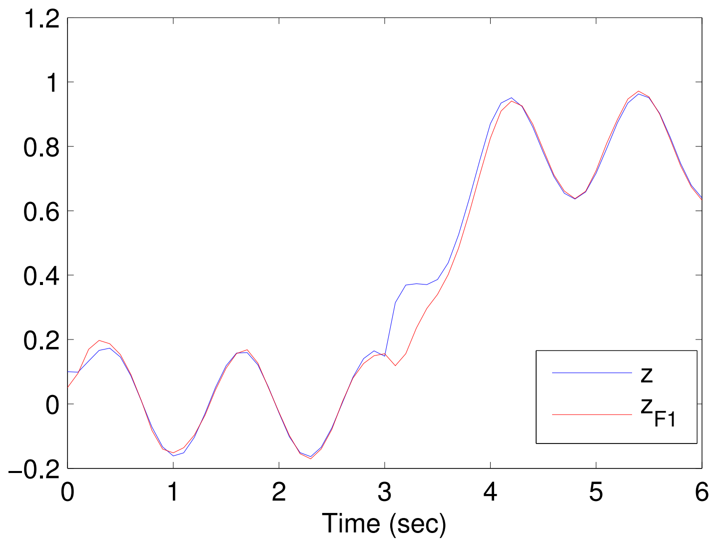


Figure 4.9: Approach I: Real signal z and its estimate z_F for the proposed filter in example 2

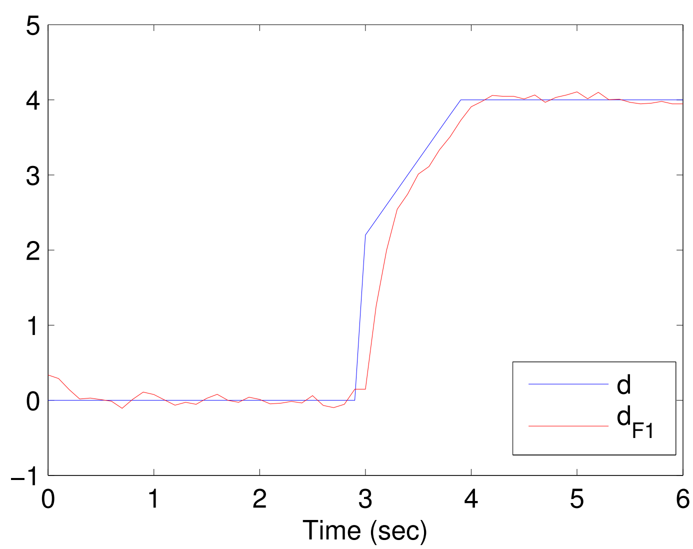


Figure 4.10: Approach I: Disturbance d and its estimate d_F for the proposed filter in example 2

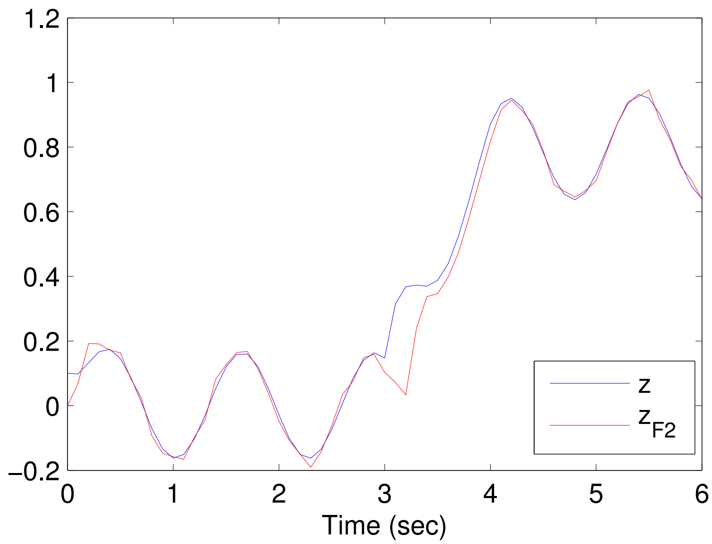


Figure 4.11: Approach II: Real signal z and its estimate z_F for the proposed filter in example 2

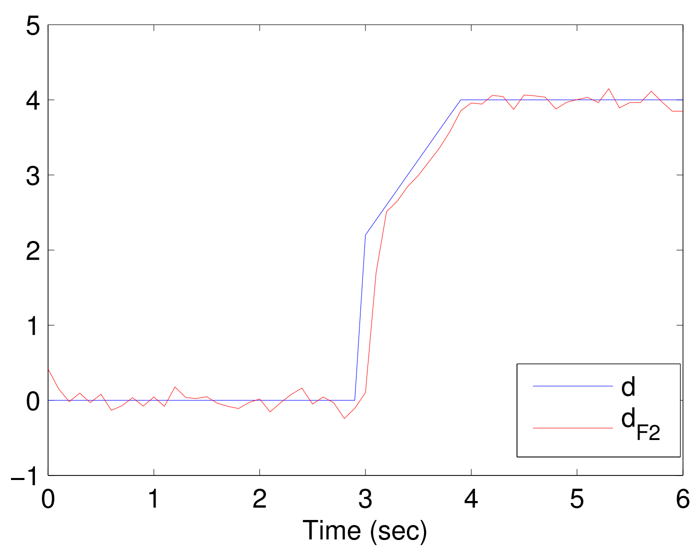


Figure 4.12: Approach II: Disturbance d and its estimate d_F for the proposed filter in example 2

$$+\begin{bmatrix}1\\0\end{bmatrix}d(k)+\begin{bmatrix}0.01\\0.01\end{bmatrix}w(k)$$

$$y(k)=\begin{bmatrix}0 & 1\end{bmatrix}x(k)+v(k)$$

$$z(k)=\begin{bmatrix}0.02 & 0\end{bmatrix}x(k) \tag{4.80}$$

For this system too, $CB_2 = 0$ *and* $\sigma = 1$ *for both approaches. Choosing now the adaptation gain as* $\Gamma = 5$*, the filter gain derived through Approach I is*

$$L=\begin{bmatrix}0.2901 & 0.6425\end{bmatrix}^T$$

For Approach II, the filter and adaptation gains are calculated as

$$L=\begin{bmatrix}0.7795 & 1.1422\end{bmatrix}^T$$

$$\Gamma = 6.5152$$

Figures 4.13, 4.14, 4.15 and 4.16 show the estimated signal z_F *and* d_F *for both of the proposed approaches. These figures illustrate the effectiveness of the proposed filters even with unstable systems.*

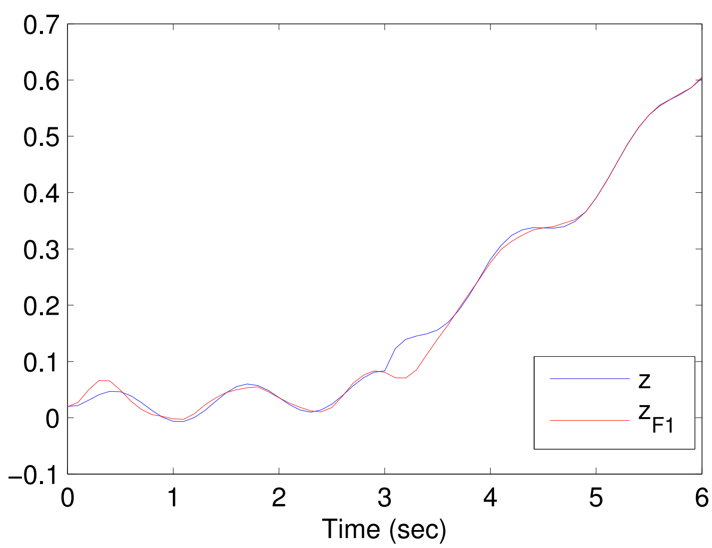


Figure 4.13: Approach I: Real signal z and its estimate z_F for the proposed filter in example 3

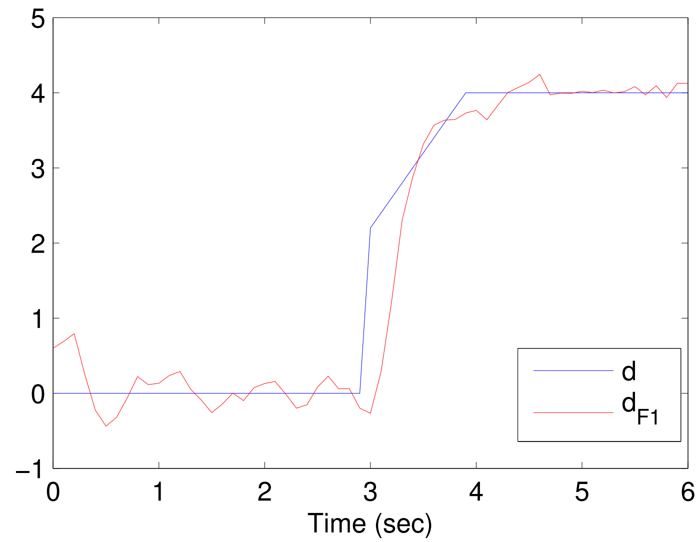


Figure 4.14: Approach I: Disturbance d and its estimate d_F for the proposed filter in example 3

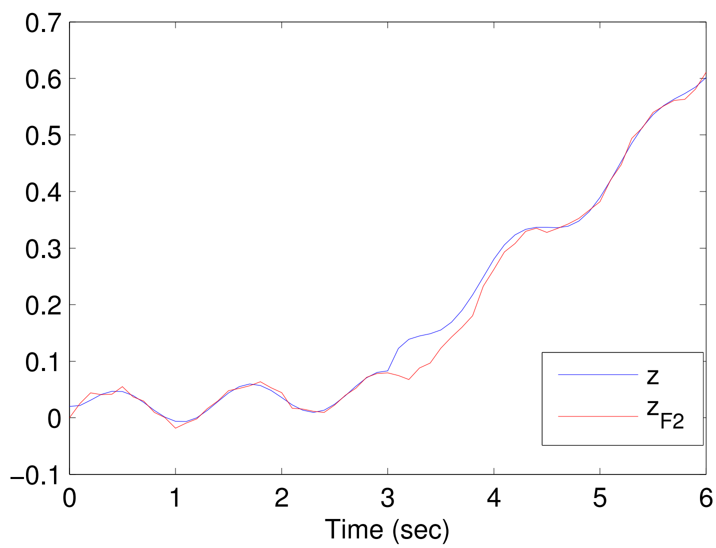


Figure 4.15: Approach II: Real signal z and its estimate z_F for the proposed filter in example 3

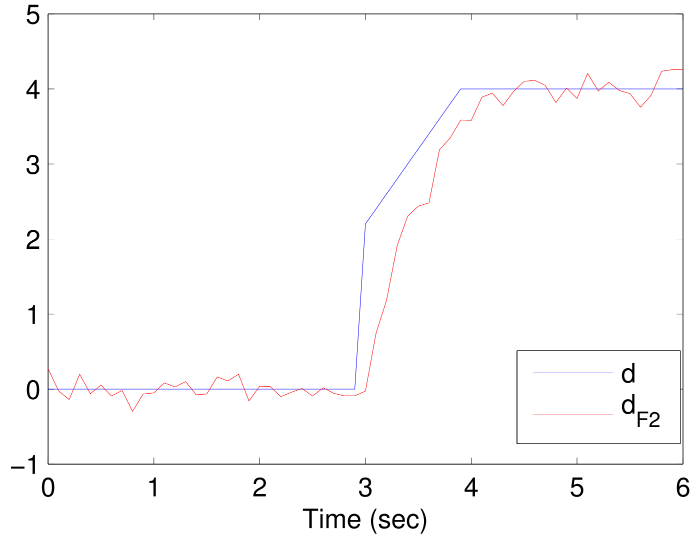


Figure 4.16: Approach II: Disturbance d and its estimate d_F for the proposed filter in example 3

4.5 Summary

In this chapter, two approaches for designing discrete-time unknown input linear filters were introduced. The proposed filters had similar state space parameters to the plant with minor changes, and directly estimated the unknown disturbances using the measurement error. The proposed design approaches were similar in nature, however, they were based on two different filter models and consequently led to two different filters. Based on the Lyapunov theory, the $\mathcal{H}_\infty$ filter design problems were transformed into LMI feasibility problems, and finally simulation results were employed to verify the applicability of the proposed approaches.

Chapter 5

Robust Filter Design with Limited Information

In this chapter, we propose a novel approach for robust filter design in the presence of an unknown state disturbance when the sensor-provided measurements are transmitted over a network and therefore are subject to network-induced errors. The main idea is to divide the estimation space into two zones based on whether the received information is reliable or not. This dual-zone approach enables us to treat the reliable packets differently from the unreliable ones. In the reliable zone, the received information is used to estimate the states as well as the unknown disturbance, whereas in the unreliable zone, estimation of the unknown input stops and a secondary filter kicks in to estimate the unreliable part of the information and the states accordingly. Using an LMI based approach, the filter design procedure is formulated in both zones and finally an overall dual-zone filter is proposed. Simulation results are given to illustrate the effectiveness of the approach.

The rest of the chapter is organized as follows. In section 5.1, the main problem is formulated. Sections 5.2 and 5.3 discuss the different design approaches in the two zones and section 5.4 introduces the new dual-zone filter. In section 5.5 the proposed filters are tested via simulation and section 5.6 summarizes the results of this chapter.

5.1 Problem Formulation

Consider the following linear system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 u(k) + B_2 d(k) \\
y(k) &= Cx(k) + Du(k) \\
z(k) &= Hx(k)
\end{aligned} \tag{5.1}
$$

where $x_{n\times 1}$ is the state vector; $y_{p\times 1} = [y_1, \ldots, y_p]^T$ represents the measured outputs; $z_{r\times 1}$ is the vector to be estimated; $u_{m_1\times 1}$ is the known input; $d_{m_2\times 1}$ is the unknown disturbance input; and A, B_1, B_2, C, D, H are the state space matrices of the model. We assume that (A, C) is an observable pair and also B_2, C are full rank with $rank(B_2) = m_2$, $rank(C) = p$ and $p \geq m_2$. Now assume that the measurements are quantized via the following coarse linear quantizer:

$$
y_{qi}(k) = Q_{lin}(y_i) = \begin{cases} r_i\tau_i & \textbf{if } |y_i(k) - r_i\tau_i| \leq \tau_i/2 \\ -Q_{lin}(-y_i) & \textbf{if } y_i(k) < 0 \end{cases} \tag{5.2}
$$

where $r_i = 0, 1, 2, \ldots$, and τ_i is the range of each quantization level for the ith measurement. For this quantizer, the real measurement is related to its quantized version by the following equation,

$$
y_{qi} = y_i + \Delta y_{qi} \qquad |\Delta y_{qi}| \leq \frac{\tau_i}{2} \tag{5.3}
$$

We also assume that the quantized measurements may not reach the filter at every sampling instant due to possible packet dropouts. Consequently when a packet is lost, our filter will not make corrections to the state trajectories, or in other words, the filter temporarily operates in open-loop. From a mathematical point of view, the measurement signal received by the filter, *i.e.* y_{rec}, can be written as

$$
y_{rec} = \begin{cases} y_F & \text{if the packet is lost} \\ y_q & \text{otherwise} \end{cases} \tag{5.4}
$$

where y_F is the estimated measurement by the filter and $y_q = [y_{q1}, \ldots, y_{qp}]^T$ represents the quantized measurements. Based on (5.4) when a packet is lost, the filter will use its own estimated measurement instead which will lead to a zero residual error and therefore no trajectory corrections.

Our intention is to propose a design methodology which guarantees robustness to unknown external disturbances as well as the network-induced errors such as packet dropouts and quantization. To this end we first analyze the residual error $\epsilon = y_{rec} - y_F$. Using (5.4), we can write ϵ as follows,

$$\epsilon = \begin{cases} 0 & \text{if the packet is lost} \\ y - y_F + \Delta y_q & \text{otherwise} \end{cases} \tag{5.5}$$

where $\Delta y_q = [\Delta y_{q1}, \ldots, \Delta y_{qp}]^T$ represents the quantization error vector. Using the inequality given in (5.3), we can claim that $\|\Delta y_q\| \leq \tau_{bnd}/2$ where

$$\tau_{bnd}^2 = \sum_{i=1}^{p} |\tau_i|^2$$

As seen in (5.5), the residual error ϵ is either zero, which is due to the loss of information, or the summation of two signals: The first signal, *i.e.* $y - y_F$, is the assurable component of ϵ and represents the error between the real measurement and its estimate; the second, *i.e.* Δy_q, is the uncertain component of ϵ and represents the bounded error introduced by quantization. Between these two, Δy_q is an unwanted error term and any corrections made in the estimates based on this signal can be erroneous. Since the error introduced by quantization is always bounded, *i.e.* $\|\Delta y_q\| \leq \tau_{bnd}/2$, we can claim that when $\|y - y_F\| \geq \tau_{bnd}/2$, the information in ϵ is more certain than uncertain and vice versa. As a result, when the information is not lost, the value of $y - y_F$ can divide the estimation space into two zones:

- *Zone 1:* ϵ is a reliable signal with a bounded small error.
- *Zone 2:* ϵ is unreliable.

Since $y - y_F$ is not available, we need to translate $\|y - y_F\| \geq \tau_{bnd}/2$ into a condition on $\|\epsilon\|$. To this end, we can rewrite (5.5) by applying the norm operator to both sides as follows,

$$\|\epsilon\| = \begin{cases} 0 & \text{if the packet is lost} \\ \|y - y_F + \Delta y_q\| & \text{otherwise} \end{cases} \tag{5.6}$$

Since $\|\epsilon\| \geq 0$, based on (5.6) the following inequality holds true:

$$\|\epsilon\| \leq \|y - y_F\| + \|\Delta y_q\|$$

$$\leq \|y - y_F\| + \frac{\tau_{bnd}}{2}$$

It is easy to see that if $\|\epsilon\| \geq \tau_{bnd}$ then $\|y - y_F\| \geq \tau_{bnd}/2$. Therefore we can redefine the aforementioned zones as follows,

- if $\|\epsilon\| \geq \tau_{bnd}$ then we're in zone 1.
- if $\|\epsilon\| < \tau_{bnd}$ then we're in zone 2.

Our approach is to design a separate filter in every zone. In the first zone, where the network-induced uncertainty of ϵ is minor, we design a filter to estimate the states as well as the unknown disturbance d. In the second zone, where the effects of the network are major, we will design a filter to estimate the states as well as the error signal introduced via the network.

Remark 5.1. *Note that the significance of the dual-zone treatment for the problem of robust filter design with unknown disturbances and limited information lies in the* active compensation *of the effects of the unknown disturbances as well as the network-induced errors.*

5.2 Filter Design in Zone 1

In this section we intend to employ the Approach II design method, introduced in the last chapter, to design our zone 1 filter. We assume that we enter zone 1 at $k = h_s^1$ and exit it towards zone 2 at $k = h_s^2$, where s is a counter solely defined to distinguish between the multiple entrances into the zones. All the results of this section are assumed to be valid in $h_s^1 \leq k < h_s^2$. We define the new state variable $\bar{x}$ as

$$\bar{x}(k+1) = x(k+1) - \sum_{i=0}^{\sigma} A^i B_2 d(k-i) \tag{5.7}$$

where σ can be easily calculated as follows:

$$\begin{aligned} &\sigma = min \;\; j \\ &s.t. \quad rank\left(\textstyle\sum_{i=0}^{j} CA^i B_2\right) = rank(B_2) \quad j = 0, 1, \dots \end{aligned} \tag{5.8}$$

Using (5.7), we can rewrite (5.1) as

$$\bar{x}(k+1) = A\bar{x}(k) + B_1 u(k) + A^{\sigma+1} B_2 d(k - \sigma - 1)$$

$$y(k) = C\bar{x}(k) + Du(k) + \sum_{i=0}^{\sigma} CA^i B_2 d(k-i-1)$$
$$z(k) = H\bar{x}(k) + \sum_{i=0}^{\sigma} HA^i B_2 d(k-i-1). \tag{5.9}$$

Based on the revised plant model given in (5.9), we introduce the following filter model,

$$\begin{aligned}
\bar{x}_F(k+1) &= A\bar{x}_F(k) + A^{\sigma+1}B_2 d_F(k-\sigma-1) \\
&\quad + B_1 u(k) + L_1(y_{rec}(k) - y_F(k)) \\
d_F(k-\sigma) &= ? \\
y_F(k) &= C\bar{x}_F(k) + Du(k) \\
&\quad + \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-\sigma-1) \\
z_F(k) &= H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 d_F(k-\sigma-1)
\end{aligned} \tag{5.10}$$

where $\bar{x}_F$ is the $n\times 1$ state vector of the filter; d_F is the $m_2\times 1$ estimated disturbance vector, which will follow an stable adaptive law to track the unknown disturbance; y_F represents the $p\times 1$ estimated measurement vector; y_{rec} represents the received measurement and is given by (5.4); z_F is the $r\times 1$ estimated vector; and L_1 is the static filter parameter to be designed.

As discussed earlier, when a packet dropout occurs our filter will operate in open-loop. Consequently, for the remainder of this chapter we only discuss the closed-loop characteristics of our filters when $y_{rec} = y_q$, which is when the packet reaches the filter. From last chapter, we choose the adaptation law as follows,

$$d_F(k-\sigma) = d_F(k-\sigma-1) + \Gamma_d \sum_{i=0}^{\sigma} B_2^T A^{i^T} C^T \epsilon(k) \tag{5.11}$$

Now, using the above equation along with the revised plant model in (5.9) and the filter model (5.10), the filtering error system can be written as follows,

$$\begin{aligned}
\bar{e}(k+1) &= \mathbb{A}_{11}\bar{e}(k) + \mathbb{A}_{12}\tilde{d}(k-\sigma-1) \\
&\quad + \mathbb{B}_{11}\Delta y_q(k) + \mathbb{B}_{12}\overline{\Delta d}_{k-\sigma}^{k-1} \\
\tilde{d}(k-\sigma) &= \mathbb{A}_{21}\bar{e}(k) + \mathbb{A}_{22}\tilde{d}(k-\sigma-1) \\
&\quad + \mathbb{B}_{21}\Delta y_q(k) + \mathbb{B}_{22}\overline{\Delta d}_{k-\sigma}^{k-1}
\end{aligned}$$

$$\begin{aligned}\varepsilon(k) &= z(k) - z_F(k) \\ &= \mathbb{C}_1\bar{e}(k) + \mathbb{C}_2\tilde{d}(k-\sigma-1) \\ &+ \mathbb{D}_1\Delta y_q(k) + \mathbb{D}_2\overline{\Delta d}_{k-\sigma}^{k-1}\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}\mathbb{A}_{11} &= A - L_1 C \\ \mathbb{A}_{12} &= A^{\sigma+1}B_2 - L_1 C A_\Sigma B_2 \\ \mathbb{A}_{21} &= -\Gamma_d (C A_\Sigma B_2)^T C \\ \mathbb{A}_{22} &= I - \Gamma_d (C A_\Sigma B_2)^T C A_\Sigma B_2 \\ \mathbb{B}_{11} &= -L_1 \\ \mathbb{B}_{12} &= -L_1 C \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\ \mathbb{B}_{21} &= -\Gamma_d (C A_\Sigma B_2)^T \\ \mathbb{B}_{22} &= \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix} \\ &- \Gamma_d (C A_\Sigma B_2)^T C \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\ \mathbb{C}_1 &= H \\ \mathbb{C}_2 &= H A_\Sigma B_2 \\ \mathbb{D}_2 &= H \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\ A_\Sigma &= \sum_{i=0}^{\sigma} A^i\end{aligned}$$

and

$$\overline{\Delta d}_{k-\sigma}^{k-1} = \begin{bmatrix} \Delta d(k-1) \\ \Delta d(k-2) \\ \vdots \\ \Delta d(k-\sigma) \end{bmatrix}.$$

Augmenting $\bar{e}$ and $\tilde{d}$ as $X(k) = \begin{bmatrix} \bar{e}(k)^T & \tilde{d}(k-\sigma-1)^T \end{bmatrix}^T$ and defining $\omega = \begin{bmatrix} \Delta y_q^T & \overline{\Delta d}^T \end{bmatrix}^T$, the augmented error model can be written as:

$$\begin{aligned}X(k+1) &= \mathbb{A}X(k) + \mathbb{B}\omega(k) \\ \varepsilon(k) &= \mathbb{C}X(k) + \mathbb{D}\omega(k)\end{aligned} \tag{5.13}$$

where

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{bmatrix}$$
$$\mathbb{B} = \begin{bmatrix} \mathbb{B}_{11} & \mathbb{B}_{12} \\ \mathbb{B}_{21} & \mathbb{B}_{22} \end{bmatrix}$$
$$\mathbb{C} = \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 \end{bmatrix}$$
$$\mathbb{D} = \begin{bmatrix} 0 & \mathbb{D}_2 \end{bmatrix} \tag{5.14}$$

The following theorem formulates the design approach for the zone 1 filter.

Theorem 5.1. *Consider the linear system in (5.1) with its measurements quantized by the linear quantizer given in (5.2). Then the linear filter given in (5.10) with the disturbance estimator given in (5.11) generate stable estimates of* z *and* d *with minimum* $\mathcal{H}_\infty$ *gains* μ_q *and* μ_d*, which respectively bound the effects of networked-induced errors and disturbance variations, if there exist matrices* $P_{e1}, P_d, G_d > 0$ *and* G_{e1}*, and scalars* $\bar{\mu}_q, \bar{\mu}_d > 0$ *which can solve the following optimization problem:*

$$min \quad \bar{\mu}_q + w_1 \bar{\mu}_d \tag{5.15}$$
$$s.t. \quad \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 \\ \star & -P_{e1} & 0 \\ \star & \star & -P_d \end{bmatrix} < 0 \tag{5.16}$$

where $w_1 \geq 0$ *is an arbitrary weighting coefficient and*

$$\Xi_1 = \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix}$$
$$\Xi_1^{11} = \begin{bmatrix} H^T H - P_{e1} & H^T H A_\Sigma B_2 \\ B_2^T A_\Sigma^T H^T H & B_2^T A_\Sigma^T H^T H A_\Sigma B_2 - P_d \end{bmatrix}$$
$$\Xi_1^{12} = \begin{bmatrix} 0 & H^T \mathbb{D}_2 \\ 0 & B_2^T A_\Sigma^T H^T \mathbb{D}_2 \end{bmatrix}$$
$$\Xi_1^{22} = \begin{bmatrix} -\bar{\mu}_q I & 0 \\ 0 & \mathbb{D}_2^T \mathbb{D}_2 - \bar{\mu}_d I \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} A^T P_{e1} - C^T G_{e1}^T \\ B_2^T ({A^{\sigma+1}}^T P_{e1} - A_\Sigma^T C^T G_{e1}^T) \\ -G_{e1}^T \\ -B_2^T C^T G_{e1}^T \\ -B_2^T A^T C^T G_{e1}^T \\ \vdots \\ -B_2^T {A^{\sigma-1}}^T C^T G_{e1}^T \end{bmatrix}$$

$$\Xi_3 = \begin{bmatrix} -C^T C A_\Sigma B_2 G_d \\ P_d - B_2^T A_\Sigma^T C^T C A_\Sigma B_2 G_d \\ -C A_\Sigma B_2 G_d \\ -B_2^T C^T C A_\Sigma B_2 G_d \\ -B_2^T A^T C^T C A_\Sigma B_2 G_d \\ \vdots \\ P_d - B_2^T {A^{\sigma-1}}^T C^T C A_\Sigma B_2 G_d \end{bmatrix}$$

$$\mathbb{D}_2 = H \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \tag{5.17}$$

Once solved, $\mu_q = \sqrt{\bar{\mu}_q}$*,* $\mu_d = \sqrt{\bar{\mu}_d}$*, and the filter parameters are calculated as* $L_1 = P_{e1}^{-1} G_{e1}$ *and* $\Gamma_d = P_d^{-1} G_d$*.*

Proof. To analyze the stability of the filtering error system given in (5.13), the following Lyapunov function is used,

$$V(k) = X(k)^T \mathbb{P}_1 X(k) \tag{5.18}$$

where $\mathbb{P}_1 = diag\{P_{e1}, P_d\}$. The forward difference of this Lyapunov function can be written as

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= X(k)^T \mathbb{A}^T \mathbb{P}_1 \mathbb{A} X(k) + \omega(k)^T \mathbb{B}^T \mathbb{P}_1 \mathbb{B} \omega(k) \\ &+ X(k)^T \mathbb{A}^T \mathbb{P}_1 \mathbb{B} \omega(k) + \omega(k)^T \mathbb{B}^T \mathbb{P}_1 \mathbb{A} X(k) \\ &- X(k)^T \mathbb{P}_1 X(k). \end{aligned} \tag{5.19}$$

Defining $\xi = \begin{bmatrix} X^T & \omega^T \end{bmatrix}^T$, (5.19) can be simplified as follows:

$$\Delta V(k) = \xi(k)^T \{ \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix} \mathbb{P}_1 \begin{bmatrix} \mathbb{A}^T \\ \mathbb{B}^T \end{bmatrix}^T + \begin{bmatrix} -\mathbb{P}_1 & 0 \\ 0 & 0 \end{bmatrix} \} \xi(k). \tag{5.20}$$

In order to establish attenuation bounds on the effects of the network-induced error Δy_q and also effects of the variations of the unknown disturbance *i.e.* Δd, we define

$$J \triangleq \sum_{k=h_s^1}^{h_s^2-1} \{\varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu\omega(k)\} \tag{5.21}$$

where $\mu = diag\{\mu_q, \mu_d\}$. Adding (5.18) to the right hand side of (5.21), we get

$$J < \sum_{k=h_s^1}^{h_s^2-1} \{\varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu\omega(k) + \Delta V(k)\} = \sum_{k=h_s^1}^{h_s^2-1} \tilde{J}_k \tag{5.22}$$

Designing our filter such that $\tilde{J}_k \leq 0$, we conclude that $J \leq 0$, which implies that in the time interval $[h_s^1, h_s^2 - 1]$, the second norm of the estimation error is bounded by factors of the second norms of quantization errors and disturbance variations. In other words, (5.22) establishes an $\mathcal{H}_\infty$ bound on the filtering error system. Using (5.13) $\varepsilon^T \varepsilon$ can be simplified as follows:

$$\varepsilon^T \varepsilon = \xi(k)^T \begin{bmatrix} \mathbb{C}^T\mathbb{C} & \mathbb{C}^T\mathbb{D} \\ \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} \end{bmatrix} \xi(k) \tag{5.23}$$

Substituting (5.23) and (5.20) in (5.22), we have

$$\tilde{J}_k \leq \xi^T (\Omega_1 + \Omega_2 \mathbb{P}_1^{-1} \Omega_2^T)\xi \tag{5.24}$$

where

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} \mathbb{C}^T\mathbb{C} - \mathbb{P}_1 & \mathbb{C}^T\mathbb{D} \\ \mathbb{D}^T\mathbb{C} & \mathbb{D}^T\mathbb{D} - \mu^T\mu \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} \mathbb{A}^T\mathbb{P}_1 \\ \mathbb{B}^T\mathbb{P}_1 \end{bmatrix} \end{aligned} \tag{5.25}$$

Defining $\bar{\mu}_q = \mu_q^2$, $\bar{\mu}_d = \mu_d^2$, $G_{e1} = P_{e1}L_1$ and $G_d = P_d\Gamma_d$, we can rewrite (5.24) as follows:

$$\tilde{J}_k \leq \xi^T (\Xi_1 + \Xi_2 P_{e1}^{-1} \Xi_2^T + \Xi_3 P_d^{-1} \Xi_3^T)\xi \tag{5.26}$$

where Ξ_1, Ξ_2, Ξ_3 are as given in (5.17). Using Schur's Complement one can show that (5.26) holds true if the LMI given in (5.1) is satisfied. $\square$

5.3 Filter Design in Zone 2

In this section, we employ the design approach introduced in chapter 3 to design a filter that is robust to errors induced by the network. We assume that we enter zone 2 at $k = h_s^2$ and exit it towards zone 1 at $k = h_{s+1}^1$. All the results of this section are assumed to be valid in $h_s^2 \leq k < h_{s+1}^1$. Consider the following discrete-time linear filter:

$$\begin{aligned} x_F(k+1) &= Ax_F(k) + B_1u(k) + B_2\mathcal{D}_F \\ &\quad + L_2(y_q(k) - y_F(k) - \theta_F(k)) \\ y_F(k) &= Cx_F(k) + Du(k) \\ z_F(k) &= Hx_F(k) \end{aligned} \tag{5.27}$$

where $x_F \in \mathbb{R}^n$, $y_F \in \mathbb{R}^p$ and $z_F \in \mathbb{R}^r$ are, respectively, the state vector, measurement vector, and estimate vector of the filter; $\mathcal{D}_F = cte.$ is an estimate of the unknown disturbance d; L_2 is the filter gain to be designed, and $\theta_F \in \mathbb{R}^p$ is an adaptive parameter for estimation of the quantization error.

Define the parameter $\theta \triangleq \Delta y_q$. Substituting this into (5.3) we can write:

$$y_q(k) = y(k) + \theta(k) \tag{5.28}$$

The goal of this filter is simultaneous estimation of the states and θ when the quantization error is fairly large. The following theorem formulates the design approach for the filter.

Theorem 5.2. *Consider the linear system (5.1), with its measurements quantized by the linear quantizer given in (5.2), and the linear filter in (5.27) with θ_F updated as follows:*

$$\theta_F(k+1) = \theta_F(k) + \Gamma_\theta \epsilon(k) \tag{5.29}$$

where $\epsilon(k) = y_q(k) - y_F(k) - \theta_F(k)$. Then the filtering error system is stable with minimized $\mathcal{H}_\infty$ bounds μ_D, μ_θ, which respectively bound the effects of the disturbance input and network-induced errors, if there exist matrices $P_{e2}, P_\theta, G_\theta > 0$ and G_{e2}, and scalars $\bar{\mu}_D, \bar{\mu}_\theta > 0$ that can solve the following optimization problem:

$$min \quad \bar{\mu}_D + w_2\bar{\mu}_\theta \tag{5.30}$$

$$s.t. \quad \begin{bmatrix} \Xi_4 & \Xi_5^T & \Xi_6^T \\ \star & -P_{e2} & 0 \\ \star & \star & -P_\theta \end{bmatrix} < 0 \tag{5.31}$$

where $w_2 \geq 0$ is an arbitrary weighting coefficient and

$$
\begin{aligned}
\Xi_4 &= diag\{H^T H - P_{e2}, -P_\theta, -\bar{\mu}_D I, -\bar{\mu}_\theta I\} \\
\Xi_5 &= \begin{bmatrix} P_{e2}A - G_{e2}C & -G_{e2} & P_{e2}B_2 & 0 \end{bmatrix} \\
\Xi_6 &= \begin{bmatrix} -G_\theta C & P_\theta - G_\theta & 0 & P_\theta \end{bmatrix}
\end{aligned} \tag{5.32}
$$

Once solved, $\mu_D = \sqrt{\bar{\mu}_D}$, $\mu_\theta = \sqrt{\bar{\mu}_\theta}$, and the filter parameters L_2 and Γ_θ can be calculated via $L_2 = P_{e2}^{-1} G_{e2}$, and $\Gamma_\theta = P_\theta^{-1} G_\theta$, respectively.

Proof. Defining the state error as $e = x - x_F$ and the estimation error of the adaptive parameter as $\tilde{\theta} = \theta - \theta_F$, the filtering error system can be written as follows

$$
\begin{aligned}
e(k+1) &= (A - L_2 C)e(k) + B_2 \Delta\mathcal{D}(k) - L_2 \tilde{\theta}(k) \\
\tilde{\theta}(k+1) &= -\Gamma_\theta C e(k) + (I - \Gamma_\theta)\tilde{\theta}(k) + \Delta\theta(k) \\
\varepsilon(k) &= z(k) - z_F(k) = He(k)
\end{aligned} \tag{5.33}
$$

where $\Gamma_\theta > 0$ is the diagonal adaptation gain; $\Delta\theta(k+1) = \theta(k+1) - \theta(k)$ is the variation of θ between the sampling instants k and $k+1$; and $\Delta\mathcal{D}(k) = d(k) - \mathcal{D}_F$ represents the error between the unknown disturbance d and its last estimated value (provided by the filter of zone 1). In the rest of the chapter, we will refer to $\Delta\mathcal{D}$ as the *unknown disturbance offset*. Both $\Delta\theta$ and $\Delta\mathcal{D}$ are bounded and therefore belong to ℓ_{2e}. To analyze the properties of the error system, we introduce the following Lyapunov function candidate:

$$
V(k) = e(k)^T P_{e2} e(k) + \tilde{\theta}(k)^T P_\theta \tilde{\theta}(k) \tag{5.34}
$$

where $P_{e2}, P_\theta > 0$. The forward difference of this Lyapunov function can be written as

$$
\Delta V(k) = \xi(k)^T (\Xi'_4 + \Xi'^T_5 P_{e2} \Xi'_5 + \Xi'^T_6 P_\theta \Xi'_6)\xi(k) \tag{5.35}
$$

where $\xi(k) = \begin{bmatrix} e(k)^T & \tilde{\theta}(k)^T & \Delta\bar{d}(k)^T & \Delta\theta(k+1)^T \end{bmatrix}^T$, and

$$
\begin{aligned}
\Xi'_4 &= diag\{-P_{e2}, -P_\theta, 0, 0\} \\
\Xi'_5 &= \begin{bmatrix} A - L_2 C & -L_2 & B_2 & 0 \end{bmatrix} \\
\Xi'_6 &= \begin{bmatrix} -\Gamma_\theta C & I - \Gamma_\theta & 0 & I \end{bmatrix}.
\end{aligned} \tag{5.36}
$$

Substituting $G_{e2} = P_{e2} L_2$ and $G_\theta = P_\theta \Gamma_\theta$ in (5.35) yields

$$
\Delta V(k) = \xi^T (\Xi'_4 + \Xi_5^T P^{-1} \Xi_5 + \Xi_6^T Q^{-1} \Xi_6)\xi \tag{5.37}
$$

where

$$\Xi_5 = \begin{bmatrix} P_{e2}A - G_{e2}C & -G_{e2} & P_{e2}B_2 & 0 \end{bmatrix}$$
$$\Xi_6 = \begin{bmatrix} -G_\theta C & P_\theta - G_\theta & 0 & P_\theta \end{bmatrix}. \tag{5.38}$$

To show that the error system satisfies the desired $\mathcal{H}_\infty$ performance, we need to establish attenuation bounds on the effects of the unknown disturbances and network-induced errors in any finite time interval. To this end, J is defined as follows:

$$J \triangleq \sum_{i=h_2^s}^{h_1^{s+1}} \{\varepsilon(i)^T\varepsilon(i) - \bar{\mu}_d^2 \Delta\bar{d}(i)^T\Delta\bar{d}(i) - \mu_\theta^2\Delta\theta(i)^T\Delta\theta(i)\} \tag{5.39}$$

where $\mu_D, \mu_\theta > 0$ are, respectively, upper bounds on the effects of unknown disturbance offset, and variations of the quantization error on the estimation error. Since $\Delta\mathcal{D}, \Delta\theta$ are both ℓ_{2e}, negative J implies limited effects of these unwanted signals on the estimation error in any finite time interval $[h_2^s, h_1^{s+1}]$. Since $V(k)$ is a positive definite function, under zero initial conditions, $V(h_1^{s+1}) - V(h_2^s) = \sum_{i=h_2^s}^{h_1^{s+1}} \Delta V(i)$ is positive semi-definite and therefore adding it to the right hand side of (5.39) results in $J \leq \sum_{i=h_2^s}^{h_1^{s+1}} \tilde{J}(k)$ where

$$\tilde{J}(k) = \varepsilon(k)^T\varepsilon(k) - \mu_d^2\Delta\bar{d}(k)^T\Delta\bar{d}(k) - \mu_\theta^2\Delta\theta(k)^T\Delta\theta(k)$$
$$+ \Delta V(k) \tag{5.40}$$

Defining now $\bar{\mu}_D = \mu_D^2$, $\bar{\mu}_\theta = \mu_\theta^2$, and substituting (5.37) in (5.40), we get

$$\tilde{J} = \xi^T(\Xi_4 + \Xi_5^T P^{-1}\Xi_5 + \Xi_6^T Q^{-1}\Xi_6)\xi \tag{5.41}$$

where $\Xi_4 = \Xi_4' + diag\{H^TH, 0, -\bar{\mu}_D I, -\bar{\mu}_\theta I\}$. Therefore, the estimation error ε is bounded with $\mathcal{H}_\infty$ performance if $\tilde{J} < 0$, which is guaranteed if the LMI given in (5.31) holds true. □

5.4 The Dual-Zone Filter

In this section we propose our dual-zone filter by summarizing the results of the last two sections into the following theorem.

Theorem 5.3. *Consider the following linear system:*

$$x(k+1) = Ax(k) + B_1u(k) + B_2d(k)$$

$$
\begin{aligned}
y(k) &= Cx(k) + Du(k) \\
z(k) &= Hx(k)
\end{aligned}
\tag{5.42}
$$

and the following two-zone filter:

- *Zone 1:* $(\epsilon_z(k) \geq \tau \Rightarrow h_1^s \leq k < h_2^s)$

$$
\begin{aligned}
\bar{x}_F(k+1) &= A\bar{x}_F(k) + A^{\sigma+1}B_2 d_F(k-\sigma-1) + B_1 u(k) + L_1\epsilon_1(k) \\
d_F(k-\sigma) &= d_F(k-\sigma-1) + \Gamma_d \sum_{i=0}^{\sigma} B_2^T {A^i}^T C^T \epsilon_1(k) \\
z_F(k) &= H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 d_F(k-\sigma-1) \\
\epsilon_1(k) &= y_q(k) - C\bar{x}_F(k) - \sum_{i=0}^{\sigma} CA^i B_2 d_F(k-\sigma-1) \\
\epsilon_z(k) &= \epsilon_1(k)
\end{aligned}
\tag{5.43}
$$

with the initial condition: $\bar{x}_F(h_1^s) = x_F(h_1^s) - \sum_{i=0}^{\sigma} A^i B_2 d_F(h_1^s - 1 - i)$.

- *Zone 2:* $(\epsilon_z(k) < \tau \Rightarrow h_2^s \leq k < h_1^{s+1})$

$$
\begin{aligned}
x_F(k+1) &= Ax_F(k) + B_1 u(k) + B_2\mathcal{D}_F + L_2\epsilon_2(k) \\
\theta_F(k+1) &= \theta_F(k) + \Gamma_\theta \epsilon_2(k) \\
y_F(k) &= Cx_F(k) + Du(k) \\
z_F(k) &= Hx_F(k) \\
\epsilon_2(k) &= y_q(k) - y_F(k) - \theta_F(k) \\
\epsilon_z(k) &= y_q(k) - y_F(k) \\
d_F(k) &= \mathcal{D}_F
\end{aligned}
\tag{5.44}
$$

with the initial conditions: $x_F(h_2^s) = \bar{x}_F(h_2^s) + \sum_{i=0}^{\sigma} A^i B_2 d_F(h_2^s - \sigma - 1)$ *and* $\mathcal{D}_F = d_F(h_2^s - 1)$.

where σ *can be calculated by*

$$
\begin{aligned}
\sigma &= min \;\; j, \quad j = 0, 1, \ldots \\
s.t. \quad & rank\left(\textstyle\sum_{i=0}^{j} CA^i B_2\right) = rank(B_2)
\end{aligned}
$$

Then the two-zone filtering error system is stable and satisfies the following $\mathcal{H}_\infty$ performance inequality:

$$\begin{aligned}\|z - z_F\|^2 \leq \mu_d^2\|\Delta d\|^2 + \mu_q^2\|\Delta y_q\|^2 \\ + \mu_D^2\|\Delta \mathcal{D}\|^2 + \mu_\theta^2\|\Delta\theta\|^2\end{aligned} \tag{5.45}$$

if there exist matrices $P_{e1}, P_{e2}, P_d, P_\theta, G_\theta, G_d > 0$ and G_{e1}, G_{e2}, and scalars $\bar{\mu}_q, \bar{\mu}_d, \bar{\mu}_D, \bar{\mu}_\theta > 0$ which can solve the following multi-objective optimization problem:

$$min \quad \bar{\mu}_q + w_1\bar{\mu}_d$$

$$min \quad \bar{\mu}_D + w_2\bar{\mu}_\theta \tag{5.46}$$

$$s.t. \quad \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 \\ \star & -P_{e1} & 0 \\ \star & \star & -P_d \end{bmatrix} < 0 \tag{5.47}$$

$$\begin{bmatrix} \Xi_4 & \Xi_5^T & \Xi_6^T \\ \star & -P_{e2} & 0 \\ \star & \star & -P_\theta \end{bmatrix} < 0 \tag{5.48}$$

where $w_1 \geq 0$ and $w_2 \geq 0$ are arbitrary weighting coefficients and

$$\Xi_1 = \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix}$$

$$\Xi_1^{11} = \begin{bmatrix} H^T H - P_{e1} & H^T H A_\Sigma B_2 \\ B_2^T A_\Sigma^T H^T H & B_2^T A_\Sigma^T H^T H A_\Sigma B_2 - P_d \end{bmatrix}$$

$$\Xi_1^{12} = \begin{bmatrix} 0 & H^T \mathbb{D}_2 \\ 0 & B_2^T A_\Sigma^T H^T \mathbb{D}_2 \end{bmatrix}$$

$$\Xi_1^{22} = \begin{bmatrix} -\bar{\mu}_q I & 0 \\ 0 & \mathbb{D}_2^T \mathbb{D}_2 - \bar{\mu}_d I \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} A^T P_{e1} - C^T G_{e1}^T \\ B_2^T (A^{\sigma+1^T} P_{e1} - A_\Sigma^T C^T G_{e1}^T) \\ -G_{e1}^T \\ -B_2^T C^T G_{e1}^T \\ -B_2^T A^T C^T G_{e1}^T \\ \vdots \\ -B_2^T A^{\sigma-1^T} C^T G_{e1}^T \end{bmatrix}$$

$$\Xi_3 = \begin{bmatrix} -C^T C A_\Sigma B_2 G_d \\ P_d - B_2^T A_\Sigma^T C^T C A_\Sigma B_2 G_d \\ -C A_\Sigma B_2 G_d \\ -B_2^T C^T C A_\Sigma B_2 G_d \\ -B_2^T A^T C^T C A_\Sigma B_2 G_d \\ \vdots \\ P_d - B_2^T A^{\sigma-1^T} C^T C A_\Sigma B_2 G_d \end{bmatrix}$$

$$\mathbb{D}_2 = H \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \tag{5.49}$$

and

$$\Xi_4 = diag\{H^T H - P_{e2}, -P_\theta, -\bar{\mu}_D I, -\bar{\mu}_\theta I\}$$
$$\Xi_5 = \begin{bmatrix} P_{e2} A - G_{e2} C & -G_{e2} & P_{e2} B_2 & 0 \end{bmatrix}$$
$$\Xi_6 = \begin{bmatrix} -G_\theta C & P_\theta - G_\theta & 0 & P_\theta \end{bmatrix} \tag{5.50}$$

Once solved, $\mu_q = \sqrt{\bar{\mu}_q}$, $\mu_d = \sqrt{\bar{\mu}_d}$, $\mu_D = \sqrt{\bar{\mu}_D}$, $\mu_\theta = \sqrt{\bar{\mu}_\theta}$, *and the filter parameters can be calculated as*

$$L_1 = P_{e1}^{-1} G_{e1}$$
$$L_2 = P_{e2}^{-1} G_{e2}$$
$$\Gamma_d = P_d^{-1} G_d$$
$$\Gamma_\theta = P_\theta^{-1} G_\theta$$

Proof. The proof is easy to establish using the proofs of theorems 5.1 and 5.2. □

5.5 Simulation Results

In this section, we present an illustrative example of the dual-zone filter and compare the results to those of the following filters:

Filter 1: a filter which is designed for zone 1 but operates in the whole space.

Filter 2: a filter which is designed for zone 2 but operates in the whole space.

Consider the following linear system,

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.8 & 0 & 0 \\ 2.1 & -1.3 & -0.6481 \\ 0 & 0.6481 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1.543 \end{bmatrix} x(k) \\ z(k) &= \begin{bmatrix} 0 & 0.1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{5.51}$$

where $u(k) = sin(5k)$ is an exogenous input, and d is an unknown disturbance. This example does not satisfy $rank(CB_2) = rank(B_2)$, which is a very common assumption in the majority of the unknown-input filters.

We set the quantizer in a way that it covers the range of $[-20, 20]$ with $\tau = 2$. We assume that the unknown disturbance d is zero until $t = 3$ sec and then it jumps to 3 where it remains steady.

The Designed Filter: Choosing the weighting parameters as $w_1 = w_2 = 0.1$, the filter parameters are calculated as

$$\begin{aligned} L_1 &= \begin{bmatrix} 0.2097 & 0.8085 & -0.3301 \end{bmatrix}^T \\ L_2 &= \begin{bmatrix} 0.0006 & 0.0087 & -0.0065 \end{bmatrix}^T \\ \Gamma_d &= 0.142 \\ \Gamma_\theta &= 0.9864 \end{aligned}$$

and the corresponding optimized attenuation bounds are $\mu_q = 0.2183$, $\mu_d = 0.488$, $\mu_D = 0.9725$ and $\mu_\theta = 0.1242$.

Figures 5.1 and 5.2 illustrate the estimated signals z_F, d_F by filter 1. This filter shows good capability in tracking the system states in the presence of the unknown disturbance d. However, since it treats all the received information as reliable packets, the estimates are

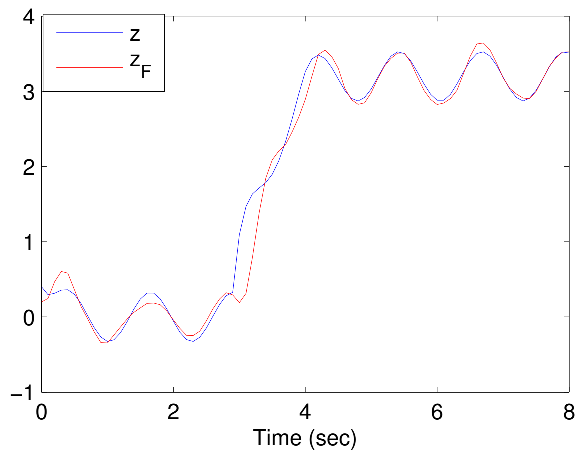


Figure 5.1: z and z_F for Filter 1

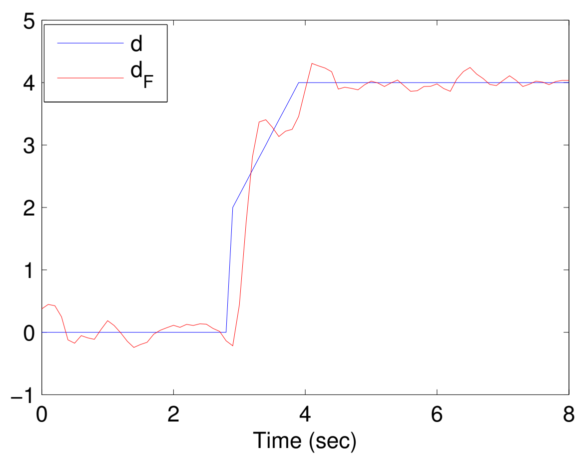


Figure 5.2: d and d_F for Filter 1

affected by the errors induced via quantization. These effects can be seen in the estimates as a considerably large additive noise signal. Figures 5.3 and 5.4 illustrate the estimated signals z_F, θ_F by filter 2. This filter, despite its capability to handle unreliable information, has poor performance in the presence of the unknown disturbance. The reason for this performance is the fact that this filter treats any changes in the received information as an error caused by the network and therefore tries to compensate it through θ_F.

Figures 5.5, 5.6 and 5.7 illustrate the estimated signals z_F, d_F, θ_F by the dual-zone filter. As seen from these figures, this filter shows good tracking performance in the presence of the unknown disturbance as well as network-induced errors. Due to the dual zone nature of this filter, the small noise-like errors which are imposed by the network do

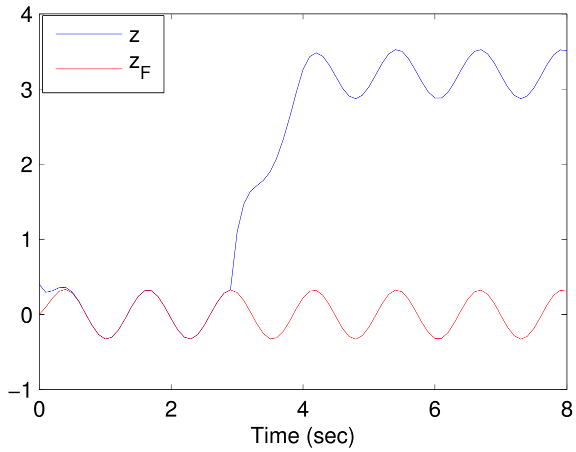


Figure 5.3: z and z_F for Filter 2

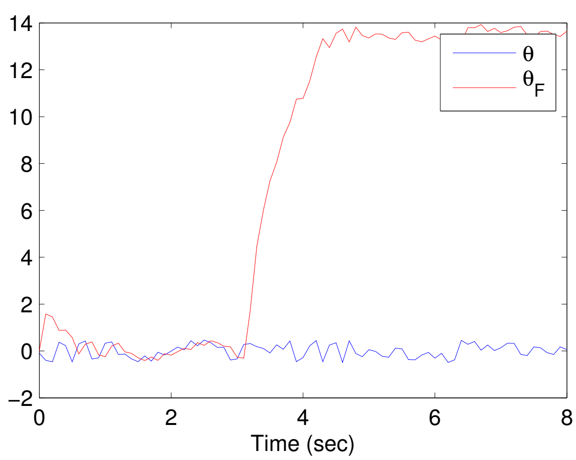


Figure 5.4: θ and θ_F for Filter 2

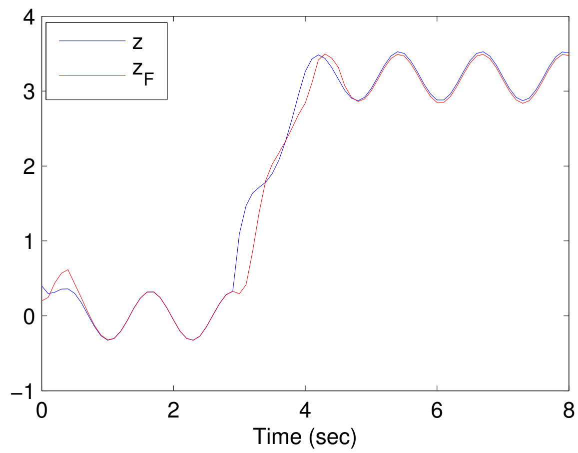


Figure 5.5: z and z_F for the proposed filter

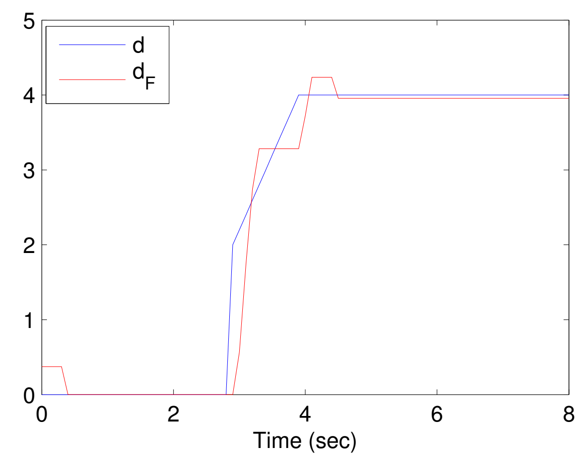


Figure 5.6: d and d_F for the proposed filter

not accumulate on the estimates provided by this filter, and at the same time, any changes in the states can be recognized and tracked.

5.6 Summary

In this chapter, the robust filtering problem of discrete-time linear systems in a networked setup was addressed. It was assumed that the system states were disturbed by an unknown input, which could be time-varying and unbounded. We also assumed that the sensory information are transmitted via a communication channel and therefore are subject to network-imposed imperfections such as quantization and packet dropouts. To solve the problem, a two-zone filter design approach was proposed. Based on the reliability of the

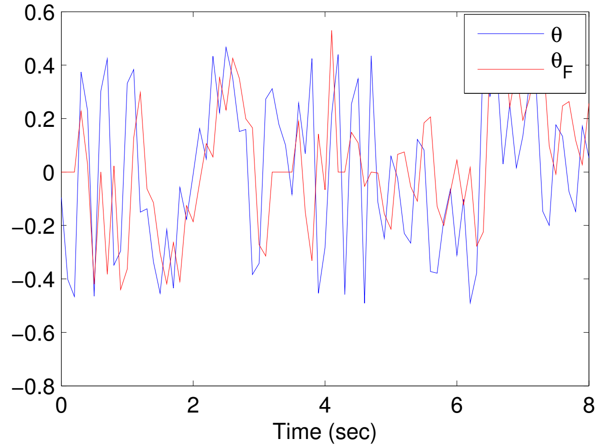


Figure 5.7: θ and θ_F for the proposed filter

residual error, two zones were defined and two different filters with two different approaches were designed. Then the two designed filters were combined to form the overall two-zone filter. In the end it was shown through simulation that the proposed filter provides good tracking of the system states, the unknown disturbance and the network-induced errors.

Chapter 6

Fault-Tolerant Filter Design with Quantized Measurements

In this chapter, a novel approach is proposed for designing fault-tolerant filters for linear discrete-time systems whose measurements undergo quantization. Two types of faults are considered: *internal* faults which are modelled by uncertainties in state space parameters; and *actuator* faults such as offset and stuck which are modelled by an unknown disturbance on the states. The proposed filter consists of two robust filters which operate in two different zones. These zones are defined based on the effects of the quantization on the residual signal between the quantized measurement and the one generated by the filter. Using an LMI-based approach with prescribed $\mathcal{H}_\infty$ performance, stability LMIs are derived for the proposed two-zone filter and finally the results are validated through simulation examples.

The rest of the chapter is organized as follows. Section 6.1 discusses the system and fault models. In sections 6.2 and 6.3 separate fault-tolerant filters are designed for each zone. Section 6.4 discusses the fault-tolerant two-zone filter design. In section 6.5, simulation results are given and finally section 6.6 summarizes the results of this chapter.

6.1 System and Fault Models

Consider the following linear fault-free system:

$$x(k+1) = Ax(k) + B_1u(k)$$

$$
\begin{aligned}
y(k) &= Cx(k) + Du(k) \\
z(k) &= Hx(k)
\end{aligned} \tag{6.1}
$$

where $x \in \mathbb{R}^n$ is the state vector; $y \in \mathbb{R}^p$ represents the measured outputs; $z \in \mathbb{R}^r$ is the vector to be estimated; $u \in \mathbb{R}^{m_1}$ is the known input; and A, B_1, C, D, H are the state space matrices of the model. We assume that (A, C) is an observable pair and the measurements are quantized via a linear quantizer as given below

$$
y_q(k) = Q_{lin}(y) = \begin{cases} s\tau & \textbf{if } |y(k) - s\tau| \leq \tau/2 \\ -Q_{lin}(-y) & \textbf{if } y(k) < 0 \end{cases} \tag{6.2}
$$

where $s = 0, 1, 2, \ldots$, and τ is the range of each quantization level. For this quantizer, the real measurement is related to its quantized version by the following equation,

$$
y_q = y + \Delta y_q \qquad |\Delta y_q| \leq \frac{\tau}{2} \tag{6.3}
$$

Now consider the following linear faulty system:

$$
\begin{aligned}
x(k+1) &= (A + \Delta A)x(k) + (B_1 + \Delta B_1)u(k) + B_2 f(k) \\
y(k) &= (C + \Delta C)x(k) + Du(k) \\
z(k) &= Hx(k)
\end{aligned} \tag{6.4}
$$

where $B_2 \in \mathbb{R}^{n \times m_2}$; and $\Delta A \in \mathbb{R}^{n \times n}$, $\Delta B_1 \in \mathbb{R}^{n \times m_1}$, $\Delta C \in \mathbb{R}^{p \times n}$ model the internal faults of the system such as changes in gains or movement of poles and zeros. Other faults such as actuator offset or stuck faults can be modelled using a signal $f \in \mathbb{R}^{m_2}$, where $m_2 \leq p$.

Our goal is to design a fault-tolerant filter which guarantees stability in the presence of system faults, and is also robust to quantization errors.

Based on the dual-zone idea introduced in the last chapter, we need to design different filters for our two operational zones, which will be discussed in the next two sections.

6.2 Fault-Tolerant Filter in Zone 1

In this section we design a filter which estimates both the states and the actuator offset or stuck fault and establishes an $\mathcal{H}_\infty$ bound on the effects of the quantization error. The basic idea of this filter parallels with the Approach I design method introduced in chapter 4. However, in this chapter the errors caused by quantization as well as the uncertainties introduced through internal faults are taken into account, which in the end lead to a different design approach. We assume that the results of this section are valid in $h_s^1 \leq k < h_s^2$, where h_s^1, h_s^2 represent the entrance and exit instants to and from zone 1, respectively.

6.2.1 Filter Model

We define the new state variable $\bar{x}$ as

$$\bar{x}(k+1) = x(k+1) - \sum_{i=0}^{\sigma} A^i B_2 f(k-i) \tag{6.5}$$

where σ can be easily calculated as follows:

$$\begin{aligned} \sigma &= min \;\; j \\ s.t. \quad rank &\left(CA^j B_2\right) = rank(B_2) \quad j = 0, 1, \ldots \end{aligned} \tag{6.6}$$

Substituting (6.5) in (6.4), we rewrite the faulty system model as:

$$\begin{aligned} \bar{x}(k+1) &= (A + \Delta A)\bar{x}(k) + (B_1 + \Delta B_1)u(k) \\ &+ A^{\sigma+1} B_2 f(k - \sigma - 1) \\ &+ (\Delta A) \sum_{i=0}^{\sigma} A^i B_2 f(k-i-1) \\ y(k) &= (C + \Delta C)\bar{x}(k) + Du(k) \\ &+ (C + \Delta C) \sum_{i=0}^{\sigma} A^i B_2 f(k-i-1) \\ z(k) &= H\bar{x}(k) + \sum_{i=0}^{\sigma} HA^i B_2 f(k-i-1) \end{aligned} \tag{6.7}$$

Based on the revised system model given in (6.7), the following filter model is introduced:

$$\begin{aligned} \bar{x}_F(k+1) &= A\bar{x}_F(k) + A^{\sigma+1} B_2 f_F(k - \sigma - 1) \\ &+ B_1 u(k) + L_1(y_q(k) - y_{Fc}(k)) \\ y_F(k) &= C\bar{x}_F(k) + Du(k) \\ &+ \sum_{i=0}^{\sigma} CA^i B_2 f_F(k-i-1) \end{aligned}$$

$$z_F(k) = H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 f_F(k-i-1) \tag{6.8}$$

where $\bar{x}_F$ is the $n \times 1$ state vector of the filter; f_F is the $m_2 \times 1$ estimated fault signal, which will follow a stable adaptive law to track the fault signal f; y_F is the $p \times 1$ estimated measurement vector; z_F is the $r \times 1$ estimated vector; and L_1 is the static filter parameter to be designed.

6.2.2 The Design Approach

In this section, we propose an LMI-based filter design procedure for discrete-time linear systems with quantized measurements which are subject to internal and actuator faults. Before stating the main theorem of this section, we first introduce the following lemma which will later be used in the proof of the theorem.

Lemma 6.1. *Let A, E, F, Λ and P be real matrices of appropriate dimensions with $P > 0$ and Λ satisfying $\Lambda^T\Lambda \leq I$. Then for any scalar $\eta > 0$ satisfying $P^{-1} - \eta^{-1}EE^T > 0$, we have ([97]):*

$$\begin{aligned}(A + E\Lambda F)^T P(A + E\Lambda F) \leq\ & A^T(P^{-1} - \eta^{-1}EE^T)^{-1}A \\ & + \eta F^T F.\end{aligned} \tag{6.9}$$

The following theorem, formulates the proposed robust approach.

Theorem 6.1. *Consider*

- *the linear system in (6.4),*
- *the actuator fault estimator (for $i = 0, \ldots, \sigma$) given by*

  $$f_F(k-i) = f_F(k-\sigma-1) + \Gamma_f \sum_{j=i}^{\sigma} B_2^T A^{j^T} C^T \epsilon(k) \tag{6.10}$$

  where $\epsilon = y - y_q$ is the measurement estimation error and $\Gamma_f > 0$ is the fault adaptation gain,
- *the ℓ_2 gains $\mu_q, \mu_{\delta f}, \mu_f, \mu_u$, which respectively bound the effects of networked-induced errors, actuator fault variations, actuator fault and the known input on the estimation error, and*

- *the weighting matrices* $W_{a1}, W_{a2}, W_{b1}, W_{b2}, W_{c1}, W_{c2} > 0$ *which satisfy* $W_{a1}^{-1}\Delta A W_{a2}^{-1} \leq I$, $W_{b1}^{-1}\Delta B_1 W_{b2}^{-1} \leq I$ *and* $W_{c1}^{-1}\Delta C W_{c2}^{-1} \leq I$.

Then the linear filter given in (6.8) will generate stable estimates of z *with guaranteed* $\mathcal{H}_\infty$ *performance if there exist matrices* $P_{e1}, P_f, P_{x1} > 0$ *and* G_{e1}*, and also scalars* $\eta_1, \eta_2 > 0$ *satisfying the following LMI:*

$$\begin{bmatrix} \Xi_1 & \Xi_2 & 0 & \Xi_3 & 0 \\ \star & -\mathbb{P}_1 & E_{1w} & 0 & 0 \\ \star & \star & -\eta_1 I & 0 & 0 \\ \star & \star & \star & -I & \Omega W_{c1} \\ \star & \star & \star & \star & -\eta_2 I \end{bmatrix} < 0 \tag{6.11}$$

where

$$\begin{aligned}
\mathbb{P}_1 &= diag\{P_{e1}, P_f, P_{x1}\} \\
\Xi_1 &= \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix} \\
\Xi_1^{11} &= diag\{-P_{e1}, -P_f, -P_{x1} + \eta_1 W_{a2}^T W_{a2} \\
&\qquad + (\eta_1 + \eta_2) W_{c2}^T W_{c2}\} \\
\Xi_1^{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & [\eta_1 W_{a2}^T W_{a2} + (\eta_1 + \eta_2) W_{c2}^T W_{c2}]\Psi_f & 0 \end{bmatrix} \\
\Xi_1^{22} &= diag\{-\mu_q^2 I, -\mu_{\delta f}^2 I, -\mu_f^2 I + \Psi_f^T[\eta_1 W_{a2}^T W_{a2} + (\eta_1 + \eta_2) W_{c2}^T W_{c2}]\Psi_f, \\
&\quad - \mu_u^2 I + W_{b2}^T W_{b2}\} \\
\Xi_2 &= [\Xi_2^1, \Xi_2^2] \\
\Xi_2^1 &= \begin{bmatrix} A^T P_{e1} - C^T \Phi^{-T} G_{e1}^T \\ B_2^T (A^{\sigma+1^T} P_{e1} - A_\Sigma^T C^T \Phi^{-T} G_{e1}^T) \\ 0 \\ -\Phi^{-T} G_{e1}^T \\ -\Psi_{\delta f}^T C^T \Phi^{-T} G_{e1}^T \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\Xi_2^2 = \begin{bmatrix} -C^T\Phi^{-T}CA^\sigma B_2\Gamma_f P_f & 0 \\ (I - B_2^T A_\Sigma^T C^T\Phi^{-T}CA^\sigma B_2\Gamma_f)P_f & 0 \\ 0 & A^T P_{x1} \\ -\Phi^{-T}CA^\sigma B_2\Gamma_f P_f & 0 \\ -\Psi_{\delta f}^T C^T\Phi^{-T}CA^\sigma B_2\Gamma_f P_f & 0 \\ 0 & \begin{bmatrix} 0 \\ B_2^T {A^{\sigma+1}}^T P_{x1} \end{bmatrix} \\ 0 & B_1^T P_{x1} \end{bmatrix}$$

$$\Xi_3 = \begin{bmatrix} H^T - C^T\Phi^{-T}\Omega^T \\ B_2^T A_\Sigma^T (H^T - C^T\Phi^{-T}\Omega^T) \\ 0 \\ -\Phi^{-T}\Omega^T \\ \Psi_{\delta f}^T (H^T - C^T\Phi^{-T}\Omega^T) \\ 0 \\ 0 \end{bmatrix}$$

$$E_{1w} = \begin{bmatrix} P_{e1}W_{a1} & P_{e1}W_{b1} & -G_{e1}\Phi^{-1}W_{c1} \\ 0 & 0 & -P_f\Gamma_f(CA^\sigma B_2)^T\Phi^{-1}W_{c1} \\ P_{x1}W_{a1} & P_{x1}W_{b1} & 0 \end{bmatrix}$$

$$\Omega = \sum_{i=0}^{\sigma} HA^i B_2\Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T$$

$$A_\Sigma = \sum_{i=0}^{\sigma} A^i$$

$$\Phi = I + \sum_{i=0}^{\sigma} CA^i B_2\Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T$$

$$\Psi_{\delta f} = \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix}$$

$$\Psi_f = \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma}B_2 \end{bmatrix} \tag{6.12}$$

Once solved, the filter parameter is calculated as $L_1 = P_{e1}^{-1}G_{e1}$.

Proof. The first step in the proof is calculating the measurement estimation error ϵ. By substituting (6.10) in the second equation of (6.8) we get

$$y_F(k) = C\bar{x}_F(k) + \sum_{i=0}^{\sigma} CA^i B_2 f_F(k-\sigma-1)$$

$$+\sum_{i=0}^{\sigma} CA^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T A^{j^T} C^T \epsilon_c(k). \tag{6.13}$$

Using this equation in $\epsilon = y - y_F$, we get

$$\begin{aligned}
\epsilon(k) = y(k) - C\bar{x}_F(k) - \sum_{i=0}^{\sigma} CA^i B_2 f_F(k-\sigma-1) \\
- \sum_{i=0}^{\sigma} CA^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T A^{j^T} C^T \epsilon(k).
\end{aligned}$$

Solving this equation for ϵ, we obtain

$$\epsilon(k) = \Phi^{-1} y(k) - \Phi^{-1} C\bar{x}_F(k) - \Phi^{-1} \sum_{i=0}^{\sigma} CA^i B_2 f_F(k-\sigma-1) \tag{6.14}$$

where

$$\Phi = I + \sum_{i=0}^{\sigma} CA^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T A^{j^T} C^T. \tag{6.15}$$

Note that for a small enough $\Gamma_f > 0$, Φ is always positive-definite and thus invertible.

If we define $\bar{e} = \bar{x} - \bar{x}_F$, $\tilde{f} = f - f_F$, $\varepsilon = z - z_F$ and use the second equation of (6.4), we can rewrite (6.14) as follows:

$$\begin{aligned}
\epsilon(k) = \Phi^{-1} C\bar{e}(k) + \Phi^{-1} \sum_{i=0}^{\sigma} CA^i B_2 \tilde{f}(k-\sigma-1) \\
+ \Phi^{-1} \sum_{i=0}^{\sigma-1} CA^i B_2 \Delta f(k-i-1) + \Phi^{-1} \Delta y_q(k) \\
+ \Phi^{-1} \Delta C \bar{x}(k) + \Phi^{-1} \Delta C \sum_{i=0}^{\sigma} A^i B_2 f(k-i-1)
\end{aligned} \tag{6.16}$$

where $\Delta f(k-i) = f(k-i) - f(k-\sigma-1)$. Now we use this equation along with (6.7), (6.8) and (6.10) to write the filtering error system as follows:

$$\begin{aligned}
\bar{e}(k+1) &= \mathbb{A}_{11}\bar{e}(k) + \mathbb{A}_{12}\tilde{f}(k-\sigma-1) \\
&\quad + \Delta\mathbb{A}_{13}\bar{x}(k) + \mathbb{B}_{11}\Delta y_q(k) + \mathbb{B}_{12}\overline{\Delta f}_{k-\sigma}^{k} \\
&\quad + \Delta\mathbb{B}_{13}\overline{f}_{k-\sigma-1}^{k-1} + \Delta B_1 u(k) \\
\tilde{f}(k-\sigma) &= \mathbb{A}_{21}\bar{e}(k) + \mathbb{A}_{22}\tilde{f}(k-\sigma-1) \\
&\quad + \Delta\mathbb{A}_{23}\bar{x}(k) + \mathbb{B}_{21}\Delta y_q(k) + \mathbb{B}_{22}\overline{\Delta f}_{k-\sigma}^{k-1} \\
&\quad + \Delta\mathbb{B}_{23}\overline{f}_{k-\sigma-1}^{k-1}
\end{aligned}$$

$$
\begin{aligned}
\varepsilon(k) &= z(k) - z_F(k) \\
&= \mathbb{C}_1 \bar{e}(k) + \mathbb{C}_2 \tilde{f}(k-\sigma-1) \\
&+ \Delta\mathbb{C}_3 \bar{x}(k) + \mathbb{D}_1 \Delta y_q(k) + \mathbb{D}_2 \overline{\Delta f}_{k-\sigma}^{k-1} \\
&+ \Delta\mathbb{D}_3 \overline{f}_{k-\sigma-1}^{k-1}
\end{aligned} \tag{6.17}
$$

where

$$
\begin{aligned}
\mathbb{A}_{11} &= A - L_1 \Phi^{-1} C \\
\mathbb{A}_{12} &= A^{\sigma+1} B_2 - L_1 \Phi^{-1} C A_\Sigma B_2 \\
\mathbb{A}_{21} &= -\Gamma_f (C A^\sigma B_2)^T \Phi^{-1} C \\
\mathbb{A}_{22} &= I - \Gamma_f (C A^\sigma B_2)^T \Phi^{-1} C A_\Sigma B_2 \\
\Delta\mathbb{A}_{13} &= \Delta A - L_1 \Phi^{-1} \Delta C \\
\Delta\mathbb{A}_{23} &= -\Gamma_f (C A^\sigma B_2)^T \Phi^{-1} \Delta C \\
\mathbb{B}_{11} &= -L_1 \Phi^{-1} \\
\mathbb{B}_{12} &= -L_1 \Phi^{-1} C \Psi_{\delta f} \\
\mathbb{B}_{21} &= -\Gamma_f (C A^\sigma B_2)^T \Phi^{-1} \\
\mathbb{B}_{22} &= \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix} - \Gamma_f (C A^\sigma B_2)^T \Phi^{-1} C \Psi_{\delta f} \\
\Delta\mathbb{B}_{13} &= \Delta A \Psi_f - L_1 \Phi^{-1} \Delta C \Psi_f \\
\Delta\mathbb{B}_{23} &= -\Gamma_f (C A^\sigma B_2)^T \Phi^{-1} \Delta C \Psi_f \\
\mathbb{C}_1 &= H - \Omega \Phi^{-1} C \\
\mathbb{C}_2 &= (H - \Omega \Phi^{-1} C) A_\Sigma B_2 \\
\Delta\mathbb{C}_3 &= -\Omega \Phi^{-1} \Delta C \\
\mathbb{D}_1 &= -\Omega \Phi^{-1} \\
\mathbb{D}_2 &= (H - \Omega \Phi^{-1} C) \Psi_{\delta f} \\
\Delta\mathbb{D}_3 &= -\Omega \Phi^{-1} \Delta C \Psi_f \\
\Psi_{\delta f} &= \begin{bmatrix} B_2 & A B_2 & \dots & A^{\sigma-1} B_2 \end{bmatrix} \\
\Psi_f &= \begin{bmatrix} B_2 & A B_2 & \dots & A^\sigma B_2 \end{bmatrix} \\
\Omega &= \sum_{i=0}^{\sigma} H A^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T
\end{aligned}
$$

$$A_\Sigma = \sum_{i=0}^{\sigma} A^i$$

and

$$\overline{\Delta f}_{k-\sigma}^{k-1} = \begin{bmatrix} \Delta f(k-1) \\ \Delta f(k-2) \\ \vdots \\ \Delta f(k-\sigma) \end{bmatrix}, \quad \overline{f}_{k-\sigma-1}^{k-1} = \begin{bmatrix} f(k-1) \\ f(k-2) \\ \vdots \\ f(k-\sigma-1) \end{bmatrix}.$$

Augmenting $\bar{e}$, $\tilde{f}$ and $\bar{x}$ as $X(k) = \begin{bmatrix} \bar{e}(k)^T & \tilde{f}(k-\sigma-1)^T & \bar{x}(k)^T \end{bmatrix}^T$ and defining $\mathbb{B}_{33} = [0, A^{\sigma+1}B_2]$ and $\omega = \begin{bmatrix} \Delta y_q^T & \overline{\Delta f}^T & \overline{f}^T & u^T \end{bmatrix}^T$, the augmented error model can be written as:

$$\begin{aligned} X(k+1) &= (\mathbb{A} + \Delta\mathbb{A})X(k) + (\mathbb{B} + \Delta\mathbb{B})\omega(k) \\ \varepsilon(k) &= (\mathbb{C} + \Delta\mathbb{C})X(k) + (\mathbb{D} + \Delta\mathbb{D})\omega(k) \end{aligned} \tag{6.18}$$

where

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} & 0 \\ \mathbb{A}_{21} & \mathbb{A}_{22} & 0 \\ 0 & 0 & A \end{bmatrix}, \quad \Delta\mathbb{A} = \begin{bmatrix} 0 & 0 & \Delta\mathbb{A}_{13} \\ 0 & 0 & \Delta\mathbb{A}_{23} \\ 0 & 0 & \Delta A \end{bmatrix} \\ \mathbb{B} &= \begin{bmatrix} \mathbb{B}_{11} & \mathbb{B}_{12} & 0 & 0 \\ \mathbb{B}_{21} & \mathbb{B}_{22} & 0 & 0 \\ 0 & 0 & \mathbb{B}_{33} & B_1 \end{bmatrix}, \quad \Delta\mathbb{B} = \begin{bmatrix} 0 & 0 & \Delta\mathbb{B}_{13} & \Delta B_1 \\ 0 & 0 & \Delta\mathbb{B}_{23} & 0 \\ 0 & 0 & \Delta A\Psi_f & \Delta B_1 \end{bmatrix} \\ \mathbb{C} &= \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 & 0 \end{bmatrix}, \quad \Delta\mathbb{C} = \begin{bmatrix} 0 & 0 & \Delta\mathbb{C}_3 \end{bmatrix} \\ \mathbb{D} &= \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 & 0 & 0 \end{bmatrix}, \quad \Delta\mathbb{D} = \begin{bmatrix} 0 & 0 & \Delta\mathbb{D}_3 & 0 \end{bmatrix}. \end{aligned} \tag{6.19}$$

To analyze the stability of the augmented system, we consider the following Lyapunov function candidate:

$$V(k) = X(k)^T \mathbb{P}_1 X(k) \tag{6.20}$$

where $\mathbb{P}_1 > 0 = diag\{P_{e1}, P_f, P_{x1}\}$. The forward difference of this Lyapunov function can be written as

$$\Delta V(k) = V(k+1) - V(k)$$

$$
\begin{aligned}
&= X(k)^T(\mathbb{A}+\Delta\mathbb{A})^T\mathbb{P}_1(\mathbb{A}+\Delta\mathbb{A})X(k)\\
&+ \omega(k)^T(\mathbb{B}+\Delta\mathbb{B})^T\mathbb{P}_1(\mathbb{B}+\Delta\mathbb{B})\omega(k)\\
&+ X(k)^T(\mathbb{A}+\Delta\mathbb{A})^T\mathbb{P}_1(\mathbb{B}+\Delta\mathbb{B})\omega(k)\\
&+ \omega(k)^T(\mathbb{B}+\Delta\mathbb{B})^T\mathbb{P}_1(\mathbb{A}+\Delta\mathbb{A})X(k)\\
&- X(k)^T\mathbb{P}_1X(k).
\end{aligned} \tag{6.21}
$$

Defining $\xi = \begin{bmatrix} X^T & \omega^T \end{bmatrix}^T$, (6.21) can be simplified as follows:

$$
\begin{aligned}
\Delta V(k) = \xi(k)^T\{ &\begin{bmatrix} (\mathbb{A}+\Delta\mathbb{A})^T \\ (\mathbb{B}+\Delta\mathbb{B})^T \end{bmatrix} \mathbb{P}_1 \begin{bmatrix} (\mathbb{A}+\Delta\mathbb{A})^T \\ (\mathbb{B}+\Delta\mathbb{B})^T \end{bmatrix}^T \\
&+ \begin{bmatrix} -\mathbb{P}_1 & 0 \\ 0 & 0 \end{bmatrix} \}\xi(k).
\end{aligned} \tag{6.22}
$$

In order to establish an ℓ_2 gain on the effects of the network-induced error Δy_q, variations of the actuator fault Δf, the actuator fault f and the known input u, we define

$$
J \triangleq \sum_{k=h_s^1}^{h_s^2-1} \{\varepsilon(k)^T\varepsilon(k) - \omega(k)^T\mu^T\mu\omega(k)\} \tag{6.23}
$$

where $\mu = diag\{\mu_q, \mu_{\delta f}, \mu_f, \mu_u\}$. Adding (6.20) to the right hand side of (6.23), we get

$$
\begin{aligned}
J &< \sum_{k=h_s^1}^{h_s^2-1} \{\varepsilon(k)^T\varepsilon(k) - \omega(k)^T\mu^T\mu\omega(k) + \Delta V(k)\} \\
&= \sum_{k=h_s^1}^{h_s^2-1} \tilde{J}_k.
\end{aligned} \tag{6.24}
$$

Now, if we design our filter such that $\tilde{J}_k \le 0$, we conclude that $J \le 0$, which implies that in the time interval $[h_s^1, h_s^2 - 1]$, the second norm of the estimation error is bounded by factors of the second norms of quantization errors and actuator fault variations. In other words, (6.24) establishes an $\mathcal{H}_\infty$ bound on the filtering error system. Using (6.18) $\varepsilon^T\varepsilon$ can be simplified as follows:

$$
\varepsilon^T\varepsilon = \xi(k)^T \begin{bmatrix} (\mathbb{C}+\Delta\mathbb{C})^T \\ (\mathbb{D}+\Delta\mathbb{D})^T \end{bmatrix} \begin{bmatrix} (\mathbb{C}+\Delta\mathbb{C})^T \\ (\mathbb{D}+\Delta\mathbb{D})^T \end{bmatrix}^T \xi(k). \tag{6.25}
$$

Substituting (6.25) and (6.22) in (6.24), we have

$$
\tilde{J}_k = \xi^T\{ \begin{bmatrix} (\mathbb{A}+\Delta\mathbb{A})^T\mathbb{P}_1 \\ (\mathbb{B}+\Delta\mathbb{B})^T\mathbb{P}_1 \end{bmatrix} \mathbb{P}_1^{-1} \begin{bmatrix} (\mathbb{A}+\Delta\mathbb{A})^T\mathbb{P}_1 \\ (\mathbb{B}+\Delta\mathbb{B})^T\mathbb{P}_1 \end{bmatrix}^T
$$

$$+\begin{bmatrix}(\mathbb{C}+\Delta\mathbb{C})^T\\(\mathbb{D}+\Delta\mathbb{D})^T\end{bmatrix}\begin{bmatrix}(\mathbb{C}+\Delta\mathbb{C})^T\\(\mathbb{D}+\Delta\mathbb{D})^T\end{bmatrix}^T+\begin{bmatrix}-\mathbb{P}_1 & 0\\0 & -\mu^T\mu\end{bmatrix}\}\xi. \tag{6.26}$$

In order to be able to use the results of lemma 6.1, we need to factorize the uncertain terms in (6.26) as follows,

$$\begin{aligned}\begin{bmatrix}\mathbb{P}_1\Delta\mathbb{A} & \mathbb{P}_1\Delta\mathbb{B}\end{bmatrix}&=E_1\Lambda F_1^1\\ \begin{bmatrix}\Delta\mathbb{C} & \Delta\mathbb{D}\end{bmatrix}&=\Omega\Delta C F_1^2\end{aligned} \tag{6.27}$$

where

$$\begin{aligned}\Lambda&=diag\{\Delta A,\Delta B_1,\Delta C\}\\ E_1&=\begin{bmatrix}P_{e1} & P_{e1} & -P_{e1}L_1\Phi^{-1}\\0 & 0 & -P_f\Gamma_f(CA^\sigma B_2)^T\Phi^{-1}\\P_{x1} & P_{x1} & 0\end{bmatrix}\\ F_1^1&=\begin{bmatrix}0&0&I&0&0&\Psi_f&0\\0&0&0&0&0&0&I\\0&0&I&0&0&\Psi_f&0\end{bmatrix}\\ F_1^2&=\begin{bmatrix}0&0&I&0&0&\Psi_f&0\end{bmatrix}.\end{aligned} \tag{6.28}$$

Since Λ may not satisfy $\Lambda^T\Lambda\leq I$, which is a necessary condition in lemma 6.1, we define the waiting matrices $W_{a1},W_{a2},W_{b1},W_{b2},W_{c1},W_{c2}$ such that

$$\begin{aligned}\Delta A&=W_{a1}\Lambda_a W_{a2}\\ \Delta B_1&=W_{b1}\Lambda_b W_{b2}\\ \Delta C&=W_{c1}\Lambda_c W_{c2}\end{aligned} \tag{6.29}$$

where $\Lambda_a,\Lambda_b,\Lambda_c$ are the normalized uncertain terms which satisfy the following inequalities:

$$\begin{aligned}\Lambda_a^T\Lambda_a&\leq I\\ \Lambda_b^T\Lambda_b&\leq I\\ \Lambda_c^T\Lambda_c&\leq I.\end{aligned}$$

Substituting (6.29) into (6.27) and defining $G_{e1}=P_{e1}L_1$, we can rewrite (6.27) as follows:

$$\begin{bmatrix}\mathbb{P}_1\Delta\mathbb{A} & \mathbb{P}_1\Delta\mathbb{B}\end{bmatrix}=E_{1w}\Lambda_w F_{1w}^1$$

$$\begin{bmatrix} \Delta\mathbb{C} & \Delta\mathbb{D} \end{bmatrix} = \Omega W_{c1} \Lambda_c F_{1w}^2 \tag{6.30}$$

where

$$\begin{aligned}
\Lambda_w &= diag\{\Lambda_a, \Lambda_b, \Lambda_c\} \\
E_{1w} &= \begin{bmatrix} P_{e1}W_{a1} & P_{e1}W_{b1} & -G_{e1}\Phi^{-1}W_{c1} \\ 0 & 0 & -P_f\Gamma_f(CA^\sigma B_2)^T\Phi^{-1}W_{c1} \\ P_{x1}W_{a1} & P_{x1}W_{b1} & 0 \end{bmatrix} \\
F_{1w}^1 &= \begin{bmatrix} 0 & 0 & W_{a2} & 0 & 0 & W_{a2}\Psi_f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_{b2} \\ 0 & 0 & W_{c2} & 0 & 0 & W_{c2}\Psi_f & 0 \end{bmatrix} \\
F_{1w}^2 &= \begin{bmatrix} 0 & 0 & W_{c2} & 0 & 0 & W_{c2}\Psi_f & 0 \end{bmatrix}.
\end{aligned} \tag{6.31}$$

Using the results of lemma 6.1 we can rewrite (6.26) as follows,

$$\begin{aligned}
\tilde{J}_k \leq \xi^T \{ & \begin{bmatrix} \mathbb{A}^T\mathbb{P}_1 \\ \mathbb{B}^T\mathbb{P}_1 \end{bmatrix} (\mathbb{P}_1 - \eta_1^{-1}E_{1w}E_{1w}^T)^{-1} \begin{bmatrix} \mathbb{A}^T\mathbb{P}_1 \\ \mathbb{B}^T\mathbb{P}_1 \end{bmatrix}^T \\
& + \begin{bmatrix} \mathbb{C}^T \\ \mathbb{D}^T \end{bmatrix} (I - \eta_2^{-1}\Omega W_{c1}W_{c1}^T\Omega^T)^{-1} \begin{bmatrix} \mathbb{C}^T \\ \mathbb{D}^T \end{bmatrix}^T \\
& + \eta_1 {F_{1w}^1}^T F_{1w}^1 + \eta_2 {F_{1w}^2}^T F_{1w}^2 + \begin{bmatrix} -\mathbb{P}_1 & 0 \\ 0 & -\mu^T\mu \end{bmatrix} \}\xi
\end{aligned} \tag{6.32}$$

Finally, using Schur's Complement it follows that (6.32) holds true if the LMI given in (6.11) is satisfied. □

6.3 Fault-Tolerant Filter in Zone 2

In this section, we intend to design a filter which actively reacts to quantization errors while being robust to system faults. Taking advantage of the design approach introduced in chapter 3, we will propose a fault-tolerant filter that will provide robust and stable estimates of the system states in the presence of both internal and actuator faults. We assume that we enter zone 2 at $k = h_s^2$ and exit it towards zone 1 at $k = h_{s+1}^1$. All the results of this section are valid in $h_s^2 \leq k < h_{s+1}^1$.

6.3.1 Filter Model

Consider the following discrete-time linear filter:

$$
\begin{aligned}
x_F(k+1) &= Ax_F(k) + B_1u(k) + B_2\bar{f}_F \\
&\quad + L_2(y_q(k) - y_F(k) - \theta_F(k)) \\
y_F(k) &= Cx_F(k) + Du(k) \\
z_F(k) &= Hx_F(k)
\end{aligned} \tag{6.33}
$$

where $x_F \in \mathbb{R}^n$, $y_F \in \mathbb{R}^p$ and $z_F \in \mathbb{R}^r$ are, respectively, the state vector, output vector, and estimate vector of the filter; $\bar{f}_F = cte.$ is an estimate of the actuator fault f; L_2 is the filter gain to be designed, and $\theta_F \in \mathbb{R}^p$ is an adaptive parameter for estimation of the network-induced errors.

Define the parameter $\theta \triangleq \Delta y_q$. Substituting this into (6.3) we can write:

$$
y_q(k) = y(k) + \theta(k) \tag{6.34}
$$

Defining now the state estimation error $e(k) \triangleq x(k) - x_F(k)$ and using (6.4), (6.33) and (6.34), we can express the filtering error system as follows,

$$
\begin{aligned}
e(k+1) &= (A - L_2C)e(k) + (\Delta A - L_2\Delta C)x(k) \\
&\quad - L_2\tilde{\theta}(k) + \Delta B_1u(k) + B_2\Delta F(k) \\
\varepsilon(k) &= z(k) - z_F(k) = He(k)
\end{aligned} \tag{6.35}
$$

where ε is the estimation error, $\Delta F(k) = f(k) - \bar{f}_F$, and $\tilde{\theta} = \theta - \theta_F$.

6.3.2 The Design Approach

In this section we introduce an adaptive approach to handle the filtering problem of linear systems subject to quantization.

The following theorem formulates the proposed adaptive approach for linear systems.

Theorem 6.2. *Given*

- *the linear system in (6.4) whose measurements are quantized via a linear quantizer,*
- *the quantization error estimator given by*

$$
\theta_F(k+1) = \theta_F(k) + \Gamma_\theta \epsilon(k) \tag{6.36}
$$

where $\epsilon = y - y_q - \theta_F$ *is the measurement estimation error, and* $\Gamma_\theta > 0$ *is an adaptation gain to be calculated.*

- *the* ℓ_2 *gains* $\mu_u, \mu_F, \mu_\theta, \bar{\mu}_f$*, which respectively bound the effects of the known input, actuator fault variations, variations of the quantization errors, and the actuator fault on the estimation error, and*
- *the weighting matrices* $W_{a1}, W_{a2}, W_{b1}, W_{b2}, W_{c1}, W_{c2} > 0$ *which satisfy* $W_{a1}^{-1}\Delta A W_{a2}^{-1} \leq I$*,* $W_{b1}^{-1}\Delta B_1 W_{b2}^{-1} \leq I$ *and* $W_{c1}^{-1}\Delta C W_{c2}^{-1} \leq I$*.*

Then the linear filter given in (6.33) with the quantization error estimator given in (6.36) generate stable estimates of z *and* θ *with guaranteed* $\mathcal{H}_\infty$ *performance if there exist matrices* $P_{e2}, P_\theta, P_{x2}.G_\theta > 0$ *and* G_{e2}*, and also a scalar* $\eta_3 > 0$ *satisfying the following LMI:*

$$\begin{bmatrix} \Xi_4 & \Xi_5 & 0 \\ \star & -\mathbb{P}_2 & E_{2w} \\ \star & \star & -\eta_3 I \end{bmatrix} < 0 \tag{6.37}$$

where

$$\begin{aligned}
\mathbb{P}_2 &= diag\{P_{e2}, P_\theta, P_{x2}\} \\
\Xi_4 &= diag\{H^T H - P_{e2}, -P_\theta, -P_{x2}, \\
&\quad - \mu_u^2 I, -\mu_F^2 I, -\mu_\theta^2 I, -\bar{\mu}_f^2 I\} \\
\Xi_5 &= \begin{bmatrix} A^T P_{e2} - C^T G_{e2}^T & -G_\theta & 0 \\ -G_{e2}^T & P_\theta - G_\theta & 0 \\ 0 & 0 & A^T P_{x2} \\ 0 & 0 & B_1^T P_{x2} \\ B_2^T P_{e2} & 0 & 0 \\ 0 & P_\theta & 0 \\ 0 & 0 & B_2^T P_{x2} \end{bmatrix} \\
E_{2w} &= \begin{bmatrix} P_{e2} W_{a1} & P_{e2} W_{b1} & -G_{e2} W_{c1} \\ 0 & 0 & -G_\theta W_{c1} \\ P_{x2} W_{a1} & P_{x2} W_{b1} & 0 \end{bmatrix}.
\end{aligned} \tag{6.38}$$

Once solved, the filter parameters L_2 *and* Γ_θ *can be calculated via* $L_2 = P_{e2}^{-1} G_{e2}$*, and* $\Gamma_\theta = P_\theta^{-1} G_\theta$*, respectively.*

Proof. using (6.35) and (6.36), the filtering error system can be written as follows

$$
\begin{aligned}
e(k+1) &= (A - L_2C)e(k) + (\Delta A - L_2\Delta C)x(k) \\
&\quad - L_2\tilde{\theta}(k) + \Delta B_1 u(k) + B_2\Delta F(k) \\
\tilde{\theta}(k+1) &= -\Gamma_\theta Ce(k) + (I - \Gamma_\theta)\tilde{\theta}(k) - \Gamma_\theta \Delta Cx(k) \\
&\quad + \Delta\theta(k+1) \\
\varepsilon(k) &= He(k)
\end{aligned} \tag{6.39}
$$

where Γ_θ is the diagonal adaptation gain; and $\Delta\theta(k+1) = \theta(k+1) - \theta(k)$ is the variation of θ between the sampling instants k and $k+1$. Since $\Delta\theta(k)$ is bounded for all k, then $\Delta\theta(k) \in \ell_{2e}$. Now if we augment the plant states and the error system states as $X(k) = [e(k)^T, \theta(k)^T, x(k)^T]^T$, and also define the input vector as $\omega(k) = [u(k)^T, \Delta F(k)^T, \Delta\theta(k+1)^T, f(k)^T]^T$, the augmented model can be written as follows,

$$
\begin{aligned}
X(k+1) &= (\mathbb{A} + \Delta\mathbb{A})X(k) + (\mathbb{B} + \Delta\mathbb{B})\omega(k) \\
\varepsilon(k) &= \mathbb{C}X(k)
\end{aligned} \tag{6.40}
$$

where

$$
\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A - L_2C & -L_2 & 0 \\ -\Gamma_\theta & I - \Gamma_\theta & 0 \\ 0 & 0 & A \end{bmatrix} \\
\Delta\mathbb{A} &= \begin{bmatrix} 0 & 0 & \Delta A - L_2\Delta C \\ 0 & 0 & -\Gamma_\theta \Delta C \\ 0 & 0 & \Delta A \end{bmatrix} \\
\mathbb{B} &= \begin{bmatrix} 0 & B_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ B_1 & 0 & 0 & B_2 \end{bmatrix} \\
\Delta\mathbb{B} &= \begin{bmatrix} \Delta B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta B_1 & 0 & 0 & 0 \end{bmatrix} \\
\mathbb{C} &= diag\{H, 0, 0\}.
\end{aligned} \tag{6.41}
$$

To analyze the stability of the augmented system, we introduce the following Lyapunov function candidate:

$$V(k) = X(k)^T \mathbb{P}_2 X(k) \tag{6.42}$$

where $\mathbb{P}_2 = diag\{P_{e2}, P_\theta, P_{x2}\}$ with $P_{e2}, P_\theta, P_{x2} > 0$. The forward difference of this Lyapunov function can be written as

$$\begin{aligned}\Delta V(k) = \xi(k)^T \{ & \begin{bmatrix} \mathbb{A}^T + \Delta\mathbb{A}^T \\ \mathbb{B}^T + \Delta\mathbb{B}^T \end{bmatrix} \mathbb{P}_2 \begin{bmatrix} \mathbb{A}^T + \Delta\mathbb{A}^T \\ \mathbb{B}^T + \Delta\mathbb{B}^T \end{bmatrix}^T \\ & + \begin{bmatrix} -\mathbb{P}_2 & 0 \\ 0 & 0 \end{bmatrix} \} \xi(k) \end{aligned} \tag{6.43}$$

where $\xi(k) = \begin{bmatrix} X(k)^T & \omega(k)^T \end{bmatrix}^T$. In order to use the results of lemma 6.1, we need to factorize the uncertain terms in (6.43) as follows,

$$\begin{bmatrix} \mathbb{P}_2\Delta\mathbb{A} & \mathbb{P}_2\Delta\mathbb{B} \end{bmatrix} = E_2 \Lambda F_2 \tag{6.44}$$

where

$$\begin{aligned} \Lambda &= diag\{\Delta A, \Delta B_1, \Delta C\} \\ E_2 &= \begin{bmatrix} P_{e2} & P_{e2} & -P_{e2}L_2 \\ 0 & 0 & -P_\theta \Gamma_\theta \\ P_{x2} & P_{x2} & 0 \end{bmatrix} \\ F_2 &= \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{6.45}$$

Since Λ may not satisfy $\Lambda^T \Lambda \le I$, which is a necessary condition in lemma 6.1, we define the waiting matrices $W_{a1}, W_{a2}, W_{b1}, W_{b2}, W_{c1}, W_{c2}$ such that

$$\begin{aligned} \Delta A &= W_{a1} \Lambda_a W_{a2} \\ \Delta B_1 &= W_{b1} \Lambda_b W_{b2} \\ \Delta C &= W_{c1} \Lambda_c W_{c2} \end{aligned} \tag{6.46}$$

where $\Lambda_a, \Lambda_b, \Lambda_c$ are the normalized uncertain terms which satisfy the following inequalities:

$$\Lambda_a^T \Lambda_a \le I$$

$$\Lambda_b^T \Lambda_b \leq I$$

$$\Lambda_c^T \Lambda_c \leq I.$$

Substituting (6.46) into (6.44)and also defining $G_{e2} = P_{e2}L_2$ and $G_\theta = P_\theta \Gamma_\theta$, we can rewrite (6.44) as follows:

$$\begin{bmatrix} \mathbb{P}_2 \Delta \mathbb{A} & \mathbb{P}_2 \Delta \mathbb{B} \end{bmatrix} = E_{2w} \Lambda_w F_{2w} \tag{6.47}$$

where

$$\begin{aligned} \Lambda_w &= diag\{\Lambda_a, \Lambda_b, \Lambda_c\} \\ E_{2w} &= \begin{bmatrix} P_{e2}W_{a1} & P_{e2}W_{b1} & -G_{e2}W_{c1} \\ 0 & 0 & -G_\theta W_{c1} \\ P_{x2}W_{a1} & P_{x2}W_{b1} & 0 \end{bmatrix} \\ F_{2w} &= \begin{bmatrix} 0 & 0 & W_{a2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_{b2} & 0 & 0 & 0 \\ 0 & 0 & W_{c2} & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{6.48}$$

Using the results of lemma 6.1 we can rewrite (6.43) as follows,

$$\begin{aligned} \Delta V(k) \leq \xi(k)^T \{ & \begin{bmatrix} \mathbb{A}^T \mathbb{P}_2 \\ \mathbb{B}^T \mathbb{P}_2 \end{bmatrix} (\mathbb{P}_2 - \eta_3^{-1} E_{2w} E_{2w}^T)^{-1} \begin{bmatrix} \mathbb{A}^T \mathbb{P}_2 \\ \mathbb{B}^T \mathbb{P}_2 \end{bmatrix}^T \\ & + \begin{bmatrix} -\mathbb{P}_2 & 0 \\ 0 & 0 \end{bmatrix} + \eta_3 F_{2w}^T F_{2w} \} \xi(k). \end{aligned} \tag{6.49}$$

To show that the error system satisfies the desired $\mathcal{H}_\infty$ performance, we need to establish attenuation gains on the effects of the noise signals and network-induced errors in any finite time interval, under zero initial conditions. To this end, J is defined as follows:

$$J \triangleq \sum_{i=h_2^s}^{h_1^{s+1}} \{\varepsilon(i)^T \varepsilon(i) - \omega(i)^T \mu^T \mu \omega(i)\} \tag{6.50}$$

where $\mu = diag\{\mu_u, \mu_F, \mu_\theta, \bar{\mu}_f\}$. Since $u, \Delta F, \Delta\theta, f$ are all assumed to be bounded and therefore belong to ℓ_{2e}, negative J implies limited effects of these signals on the estimation error in any finite time interval $[h_2^s, h_1^{s+1}]$. Since $V(k)$ is a positive definite function, under zero initial conditions, $V(h_1^{s+1}) - V(h_2^s) = \sum_{i=h_2^s}^{h_1^{s+1}} \Delta V(i)$ is positive semi-definite and therefore adding it to the right hand side of (6.50) results in $J \leq \sum_{i=h_2^s}^{h_1^{s+1}} \tilde{J}(k)$ where

$$\tilde{J}(k) = \varepsilon(k)^T \varepsilon(k) - \omega(k)^T \mu^T \mu \omega(k) + \Delta V(k) \tag{6.51}$$

Now, substituting (6.49) in (6.51), we get

$$\begin{aligned}\tilde{J}(k) \leq \xi(k)^T \{ & \begin{bmatrix} \mathbb{A}^T \mathbb{P}_2 \\ \mathbb{B}^T \mathbb{P}_2 \end{bmatrix} (\mathbb{P}_2 - \eta_3^{-1} E_{2w} E_{2w}^T)^{-1} \begin{bmatrix} \mathbb{A}^T \mathbb{P}_2 \\ \mathbb{B}^T \mathbb{P}_2 \end{bmatrix}^T \\ & + \begin{bmatrix} diag\{H^T H, 0, 0\} - \mathbb{P}_2 & 0 \\ 0 & -\mu^T \mu \end{bmatrix} \\ & + \eta_3 F_{2w}^T F_{2w} \} \xi(k). \end{aligned} \tag{6.52}$$

Finally, using Schur's Complement, it follows that the above inequality holds if the LMI given in (6.37) is satisfied. □

6.4 The Proposed Two-Zone Fault-Tolerant Filter

In this section we first propose the design approach, and then investigate the behaviour of the filter in the absence of faults.

6.4.1 The Design Approach

The following theorem lays out the design platform for our proposed two-zone fault-tolerant filter.

Theorem 6.3. *Consider the following faulty linear system:*

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B_1 + \Delta B_1)u(k) + B_2 f(k) \\ y(k) &= (C + \Delta C)x(k) + Du(k) \\ z(k) &= Hx(k) \end{aligned} \tag{6.53}$$

whose measurements are quantized by (6.2), and the following two-zone filter:

- *Zone 1:* $(\epsilon_z(k) \geq \tau \Rightarrow h_1^s \leq k < h_2^s)$

$$\bar{x}_F(k+1) = A\bar{x}_F(k) + A^{\sigma+1} B_2 f_F(k - \sigma - 1) + B_1 u(k) + L_1 \epsilon_1(k)$$

$$
\begin{aligned}
f_F(k-i) &= f_F(k-\sigma-1) + \Gamma_f \sum_{j=i}^{\sigma} B_2^T {A^j}^T C^T \epsilon_1(k) \\
z_F(k) &= H\bar{x}_F(k) + \sum_{i=0}^{\sigma} HA^i B_2 f_F(k-i-1) \\
\epsilon_1(k) &= \Phi^{-1} y_q(k) - \Phi^{-1} C\bar{x}_F(k) - \Phi^{-1} \sum_{i=0}^{\sigma} CA^i B_2 f_F(k-\sigma-1) \\
\epsilon_z(k) &= \Phi \epsilon_1(k)
\end{aligned}
\tag{6.54}
$$

with the initial condition: $\bar{x}_F(h_1^s) = x_F(h_1^s) - \sum_{i=0}^{\sigma} A^i B_2 f_F(h_1^s - 1 - i)$.

- *Zone 2:* $(\epsilon_z(k) < \tau \Rightarrow h_2^s \le k < h_1^{s+1})$

$$
\begin{aligned}
x_F(k+1) &= Ax_F(k) + B_1 u(k) + B_2 \bar{f}_F + L_2 \epsilon_2(k) \\
\theta_F(k+1) &= \theta_F(k) + \Gamma_\theta \epsilon_2(k) \\
y_F(k) &= Cx_F(k) + Du(k) \\
z_F(k) &= Hx_F(k) \\
\epsilon_2(k) &= y_q(k) - y_F(k) - \theta_F(k) \\
\epsilon_z(k) &= y_q(k) - y_F(k) \\
f_F(k) &= \bar{f}_F
\end{aligned}
\tag{6.55}
$$

with the initial conditions: $x_F(h_2^s) = \bar{x}_F(h_2^s) + \sum_{i=0}^{\sigma} A^i B_2 f_F(h_2^s - 1 - i)$ *and* $\bar{f}_F = f_F(h_2^s - 1)$.

where Φ *and* σ *can be calculated by*

$$
\begin{aligned}
\Phi &= I + \sum_{i=0}^{\sigma} CA^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T \\
\sigma &= min \;\; j, \quad j = 0, 1, \ldots \\
&\;\; s.t. \quad rank\left(CA^j B_2\right) = rank(B_2)
\end{aligned}
$$

Then for given

- $\Gamma_f > 0$,
- ℓ_2 *gains* $\mu_{\delta f}, \mu_f, \bar{\mu}_f, \mu_F, \mu_q, \mu_\theta, \mu_u > 0$, *and*
- *the weighting matrices* $W_{a1}, W_{a2}, W_{b1}, W_{b2}, W_{c1}, W_{c2} > 0$ *which satisfy* $W_{a1}^{-1} \Delta A W_{a2}^{-1} \le I$, $W_{b1}^{-1} \Delta B_1 W_{b2}^{-1} \le I$ *and* $W_{c1}^{-1} \Delta C W_{c2}^{-1} \le I$.

the two-zone filtering error system is stable and satisfies the prescribed $\mathcal{H}_\infty$ performance if there exist matrices $P_{e1}, P_{e2}, P_f, P_\theta, P_{x1}, P_{x2}, G_\theta > 0$, G_{e1}, G_{e2}, and scalars $\eta_1, \eta_2, \eta_3 > 0$ such that the following LMIs are feasible.

$$\begin{bmatrix} \Xi_1 & \Xi_2 & 0 & \Xi_3 & 0 \\ \star & -\mathbb{P}_1 & E_{1w} & 0 & 0 \\ \star & \star & -\eta_1 I & 0 & 0 \\ \star & \star & \star & -I & \Omega W_{c1} \\ \star & \star & \star & \star & -\eta_2 I \end{bmatrix} < 0 \tag{6.56}$$

$$\begin{bmatrix} \Xi_4 & \Xi_5 & 0 \\ \star & -\mathbb{P}_2 & E_{2w} \\ \star & \star & -\eta_3 I \end{bmatrix} < 0 \tag{6.57}$$

where

$$\begin{aligned}
\mathbb{P}_1 &= diag\{P_{e1}, P_f, P_{x1}\} \\
\Xi_1 &= \begin{bmatrix} \Xi_1^{11} & \Xi_1^{12} \\ \star & \Xi_1^{22} \end{bmatrix} \\
\Xi_1^{11} &= diag\{-P_{e1}, -P_f, -P_{x1} + \eta_1 W_{a2}^T W_{a2} \\
&\quad + (\eta_1 + \eta_2) W_{c2}^T W_{c2}\} \\
\Xi_1^{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & [\eta_1 W_{a2}^T W_{a2} + (\eta_1 + \eta_2) W_{c2}^T W_{c2}]\Psi_f & 0 \end{bmatrix} \\
\Xi_1^{22} &= diag\{-\mu_q^2 I, -\mu_{\delta f}^2 I, \\
&\quad - \mu_f^2 I + \Psi_f^T [\eta_1 W_{a2}^T W_{a2} + (\eta_1 + \eta_2) W_{c2}^T W_{c2}]\Psi_f, \\
&\quad - \mu_u^2 I + W_{b2}^T W_{b2}\} \\
\Xi_2 &= [\Xi_2^1, \Xi_2^2]
\end{aligned}$$

$$
\Xi_2^1 = \begin{bmatrix} A^T P_{e1} - C^T \Phi^{-T} G_{e1}^T \\ B_2^T ({A^{\sigma+1}}^T P_{e1} - A_\Sigma^T C^T \Phi^{-T} G_{e1}^T) \\ 0 \\ -\Phi^{-T} G_{e1}^T \\ -\Psi_{\delta f}^T C^T \Phi^{-T} G_{e1}^T \\ 0 \\ 0 \end{bmatrix}
$$

$$
\Xi_2^2 = \begin{bmatrix} -C^T \Phi^{-T} C A^\sigma B_2 \Gamma_f P_f & 0 \\ (I - B_2^T A_\Sigma^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma_f) P_f & 0 \\ 0 & A^T P_{x1} \\ -\Phi^{-T} C A^\sigma B_2 \Gamma_f P_f & 0 \\ -\Psi_{\delta f}^T C^T \Phi^{-T} C A^\sigma B_2 \Gamma_f P_f & 0 \\ 0 & \begin{bmatrix} 0 \\ B_2^T {A^{\sigma+1}}^T P_{x1} \end{bmatrix} \\ 0 & B_1^T P_{x1} \end{bmatrix}
$$

$$
\Xi_3 = \begin{bmatrix} H^T - C^T \Phi^{-T} \Omega^T \\ B_2^T A_\Sigma^T (H^T - C^T \Phi^{-T} \Omega^T) \\ 0 \\ -\Phi^{-T} \Omega^T \\ \Psi_{\delta f}^T (H^T - C^T \Phi^{-T} \Omega^T) \\ 0 \\ 0 \end{bmatrix}
$$

$$
E_{1w} = \begin{bmatrix} P_{e1} W_{a1} & P_{e1} W_{b1} & -G_{e1} \Phi^{-1} W_{c1} \\ 0 & 0 & -P_f \Gamma_f (C A^\sigma B_2)^T \Phi^{-1} W_{c1} \\ P_{x1} W_{a1} & P_{x1} W_{b1} & 0 \end{bmatrix}
$$

$$
\Omega = \sum_{i=0}^{\sigma} H A^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T
$$

$$
A_\Sigma = \sum_{i=0}^{\sigma} A^i
$$

$$
\Phi = I + \sum_{i=0}^{\sigma} C A^i B_2 \Gamma_f \sum_{j=i+1}^{\sigma} B_2^T {A^j}^T C^T
$$

$$
\begin{aligned}
\Psi_{\delta f} &= \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma-1}B_2 \end{bmatrix} \\
\Psi_f &= \begin{bmatrix} B_2 & AB_2 & \dots & A^{\sigma}B_2 \end{bmatrix}
\end{aligned} \tag{6.58}
$$

and

$$
\begin{aligned}
\mathbb{P}_2 &= diag\{P_{e2}, P_\theta, P_{x2}\} \\
\Xi_4 &= diag\{H^T H - P_{e2}, -P_\theta, -P_{x2}, \\
&\quad -\mu_u^2 I, -\mu_F^2 I, -\mu_\theta^2 I, -\bar{m}u_f^2 I\} \\
\Xi_5 &= \begin{bmatrix} A^T P_{e2} - C^T G_{e2}^T & -G_\theta & 0 \\ -G_{e2}^T & P_\theta - G_\theta & 0 \\ 0 & 0 & A^T P_{x2} \\ 0 & 0 & B_1^T P_{x2} \\ B_2^T P_{e2} & 0 & 0 \\ 0 & P_\theta & 0 \\ 0 & 0 & B_2^T P_{x2} \end{bmatrix} \\
E_{2w} &= \begin{bmatrix} P_{e2} W_{a1} & P_{e2} W_{b1} & -G_{e2} W_{c1} \\ 0 & 0 & -G_\theta W_{c1} \\ P_{x2} W_{a1} & P_{x2} W_{b1} & 0 \end{bmatrix}
\end{aligned} \tag{6.59}
$$

Once solved, the filter parameters can be calculated as

$$
\begin{aligned}
L_1 &= P_{e1}^{-1} G_{e1} \\
L_2 &= P_{e2}^{-1} G_{e2} \\
\Gamma_\theta &= P_\theta^{-1} G_\theta
\end{aligned}
$$

6.4.2 Filter Performance in the Absence of Faults

Under fault-free conditions, the system can be modelled by (6.1) and therefore it is ideally expected that f_F would converge to zero in the filter. However, in the proposed two zone filter f_F is only updated in zone 1 and as soon as ϵ_z enters zone 2, $\bar{f}_F = f_F$ will remain constant. This means that even under fault-free conditions, $\bar{f}_F$ can be nonzero and cause steady state error in our estimates. For better performance in the absence of faults, we need to reset $\bar{f}_F$ to zero and to do so, we have to find the largest $\bar{f}_F$ whose effects on the residual error ϵ_z *will* be mistaken for the effects of quantization. We already know that in zone 2 ϵ_z is given as follows,

$$\epsilon_z = Ce(k) + \theta(k) \tag{6.60}$$

where e is the state estimation error in zone 2 and θ is the quantization error. ϵ_z is affected by $\bar{f}_F$ through Ce and since the quantization error is bounded by $\tau_{bnd}/2$, any $\bar{f}_F$ which satisfies the following inequality, will have an effect smaller or equal to the quantization error.

$$\|Ce_{\bar{f}}(k)\| \leq 0.5\tau_{bnd} \tag{6.61}$$

where $e_{\bar{f}}$ is the part of e which is affected by $\bar{f}_F$. In order to calculate the effect of $\bar{f}_F$ on $e(k)$, we need to obtain the time response of $Ce(k)$ in the absence of faults. Using (6.39) and basic linear systems theory, we can write

$$\begin{aligned} Ce(k) = \mathcal{C}\mathcal{A}^{k-h_s^2}[e(h_s^2)^T \quad \tilde{\theta}(h_s^2)^T]^T \\ + \mathcal{C}\sum_{i=0}^{k-h_s^2-1} \mathcal{A}^i(\mathcal{B}_1\bar{f}_F + \mathcal{B}_2\Delta\theta) \end{aligned} \tag{6.62}$$

where $k \geq h_s^2$ and

$$\mathcal{A} = \begin{bmatrix} A - L_2C & -L2 \\ -\Gamma_\theta & I - \Gamma_\theta \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}$$
$$\mathcal{B}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}.$$

Using (6.62), $Ce_{\bar{f}}(k)$ can be written as

$$Ce_{\bar{f}}(k) = \mathcal{C}\sum_{i=0}^{k-h_s^2-1} \mathcal{A}^i\mathcal{B}_1\bar{f}_F. \tag{6.63}$$

Substituting (6.63) into (6.61), we get,

$$\|\mathcal{C}\sum_{i=0}^{k-h_s^2-1} \mathcal{A}^i\mathcal{B}_1\bar{f}_F\| \leq 0.5\tau_{bnd}. \tag{6.64}$$

Then we can claim that the above inequality is satisfied if

$$\|\bar{f}_F\| \leq \frac{1}{2}\|\mathcal{C}\sum_{i=0}^{k-h_s^2-1}\mathcal{A}^i\mathcal{B}_1\|^{-1}\tau_{bnd}. \tag{6.65}$$

As a result we can say that any $\bar{f}_F$ which satisfies the above inequality has an effect, smaller than or equal to the maximum quantization error and therefore it can indicate a practically fault-free system.

Definition: A system in a networked setting is called *practically fault-free* if the effects of the occurred faults (if any) are small enough to be mistaken for the effects of the quantization.

Using the results of this section, the two-zone filter introduced in theorem 6.3 can be modified in a way that the steady state error, which is induced by a nonzero $\bar{f}_F$ in a practically fault-free case, is eliminated. To this end, one only needs to add the following rule to the filter operating in zone 2:

- If $\|\bar{f}_F\| \leq \frac{1}{2}\|\mathcal{C}\sum_{i=0}^{k-h_s^2-1}\mathcal{A}^i\mathcal{B}_1\|^{-1}\tau_{bnd}$, then $\bar{f}_F = 0$.

6.5 Simulation Results

In this section, we simulate the proposed fault-tolerant filter for an example system under two fault scenarios. In the first scenario we assume that the system experiences different internal faults plus actuator offset fault, both independently and simultaneously, and in the second scenario we simulate the system with the actuator stuck fault.

Example. Consider the following linear system with $x(0) = [1, 1, 1]^T$:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.8 & 0 & 0 \\ 2.1 & -1.3 & -0.6481 \\ 0 & 0.6481 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ &+ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1.543 \end{bmatrix} x(k) \\ z(k) &= \begin{bmatrix} 0 & 0.1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{6.66}$$

where $u(k) = 5 + sin(0.2kT_s)$ with $T_s = 0.1$ representing the sampling time. We assume that the quantizer parameter is chosen as $\tau = 1$. Now if we choose the weighting matrices as $W_{a1} = W_{a2} = W_{b1} = W_{b2} = W_{c1} = W_{c2} = 0.1$, the actuator fault adaptation gain as $\Gamma_f = 0.15$, and the ℓ_2 gains as $\mu_f = \mu_{\delta f} = \mu_F = \mu_u = 1$, $\mu_q = 0.8$, $\mu_\theta = 0.5$, $\bar{\mu}_f = 0.9$, then the filter parameters are calculated as follows

$$
\begin{aligned}
L_1 &= \begin{bmatrix} 0.1132 & 0.6162 & -0.3225 \end{bmatrix}^T \\
L_2 &= \begin{bmatrix} 0.0121 & 0.1765 & -0.1314 \end{bmatrix}^T \\
\Gamma_\theta &= 0.7319
\end{aligned}
$$

The considered fault scenarios are as follows,

Fault Scenario I.

- System is fault-free in $t \in [0, 10] \bigcup [15, 20] \bigcup [25, 30] \bigcup [35, 40] \bigcup [45, 50]$, where $\bigcup$ represents the union of the intervals,
- $f = 3$ in $t \in (10, 15)$,
- $\Delta A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$ in $t \in (20, 25)$,
- $\Delta B_1 = \begin{bmatrix} -0.1 & -0.1 & -0.1 \end{bmatrix}^T$ in $t \in (30, 35)$,
- $\Delta C = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix}$ in $t \in (40, 45)$,
- all of the above faults are present in $t \in (50, 60]$.

This hypothetical fault scenario helps us see how the proposed filter works in the presence of individual faults as well as all of them together.

Fault Scenario II. Actuator gets stuck, *i.e.* $u = 0$, after $t = 20$ *sec.* This can be easily modelled by $f(k) = -5 - sin(0.2kT_s)$ for $k > 200$.

Figures 6.1 and 6.2 show the estimated signals z_F and f_F in the first scenario, respectively. As seen in these figures, the proposed dual-zone filter demonstrates smoother and better results than the individual filters designed for each zone. The designed filter for zone 2 shows barely any sensitivity to the changes in the measurement which are caused by the faults and as a result its operation is only desired when there are either no faults present or only sensor faults exist (nonzero ΔC). The exact opposite of this filter is the filter designed for zone 1 which is highly sensitive to any changes in the measurement. This filter reflects these changes onto its estimates and as a result appearance and disappearance of any fault can easily be seen in both z_F and θ_F. However, this also leads to the appearance of the quantization error in the estimates. Our proposed dual-zone filter takes the best of the two filters and stays sensitive to the actual faults without showing any sensitivity to the errors caused by quantization.

Figures 6.3 and 6.4 illustrate the operation of the proposed filter in the second scenario.

6.6 Summary

In this chapter, a dual-zone fault-tolerant filter was proposed for discrete-time linear systems with quantized measurements. The considered faults were modelled as either model uncertainties representing the internal faults or an unknown disturbance representing actuator faults. Based on the significance of the quantization effects on the residual between the quantized measurements and the ones estimated by the filter, two different zones were defined and a robust filter was designed for each zone. Using a Lyapunov-based approach with $\mathcal{H}_\infty$ performance, the design problem was transformed into solving two LMIs feasibility problem, and finally simulation results were employed to verify the applicability of the proposed approach.

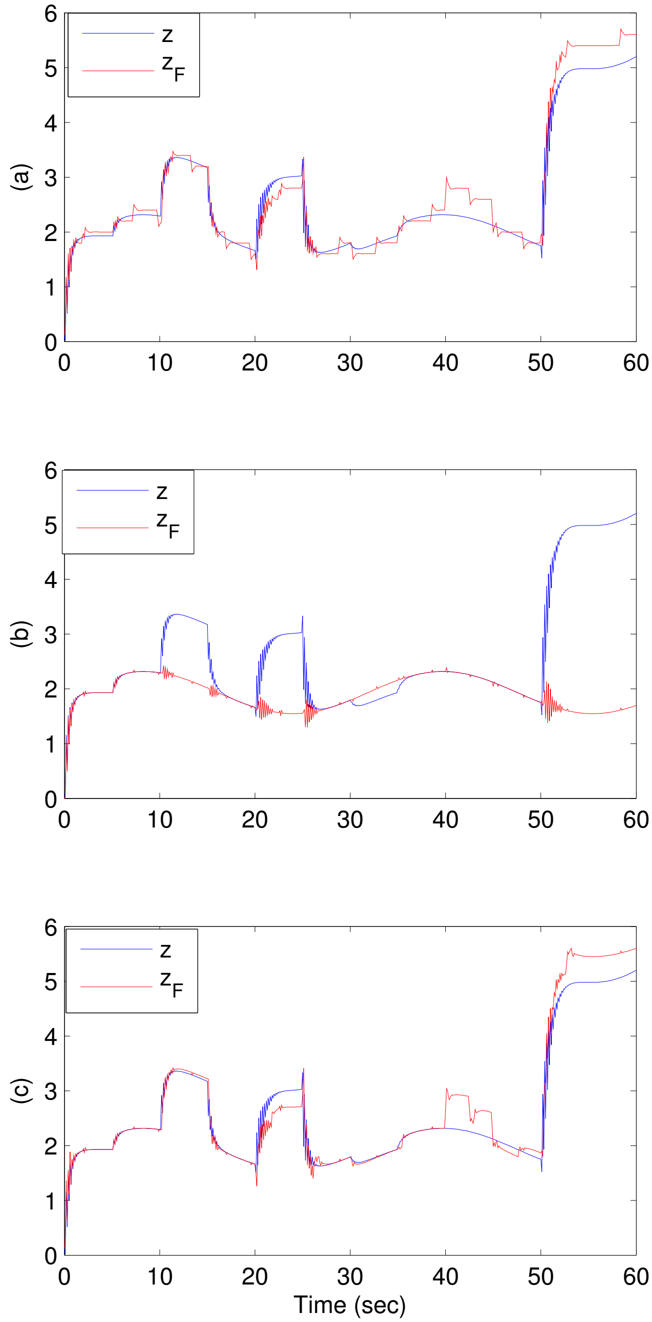


Figure 6.1: Scenario I: (a) z_F estimated by the filter designed in zone 1, (b) z_F estimated by the filter designed in zone 2, (c) z_F estimated by the proposed dual-zone filter

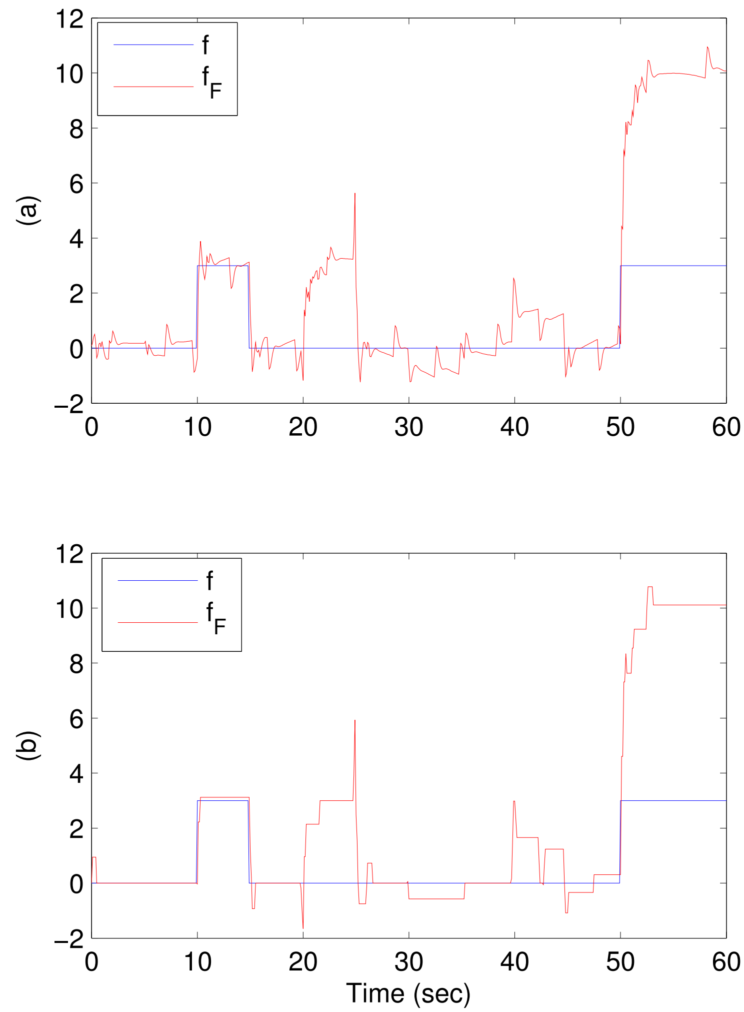


Figure 6.2: Scenario I: (a) f_F estimated by the filter designed in zone 1, (b) f_F estimated by the proposed dual-zone filter

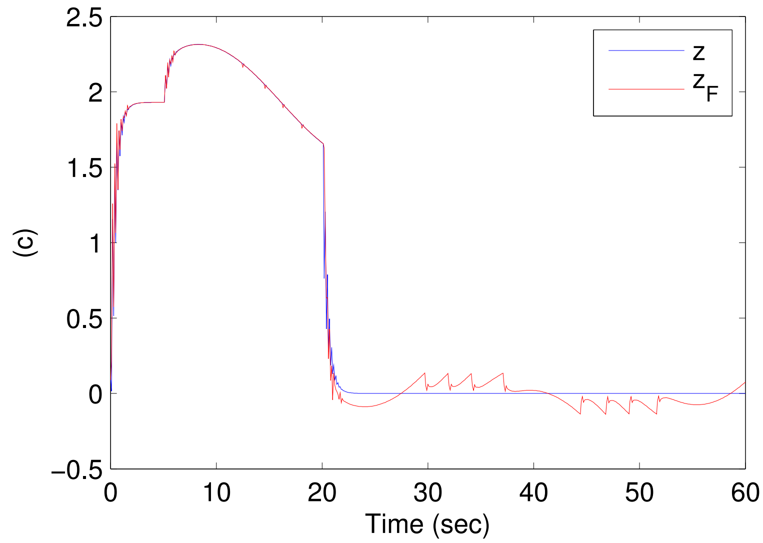


Figure 6.3: Scenario II: z_F estimated by the proposed dual-zone filter

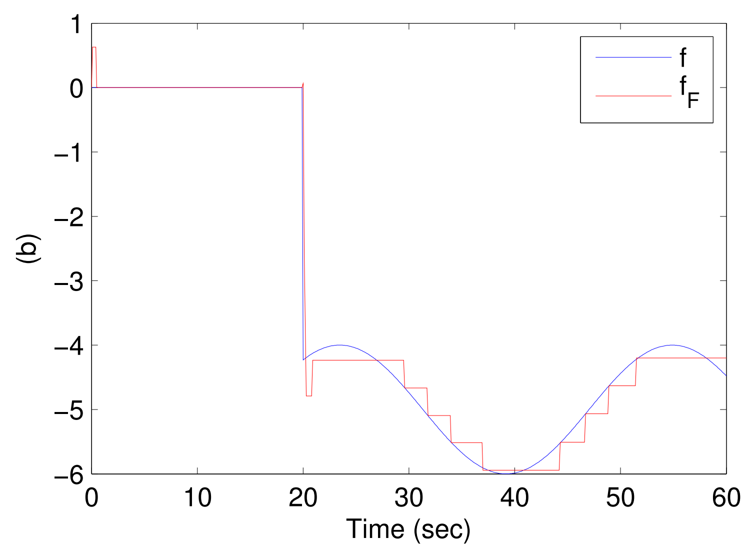


Figure 6.4: Scenario II: f_F estimated by the proposed dual-zone filter

Chapter 7

Conclusions and Future Works

In this thesis, the filter design problem was investigated in the presence of unknown disturbances, model uncertainties and network-induced errors. The problem was motivated by the growing interest and practical application of communication networks in control and monitorng. In real-life systems, presence of unknown disturbances, noise inputs and model uncertainties are quite common and proposing a solution without considering the significant effects of these imperfections is not very helpful from a practical point of view. Throughout this research, we looked into the different problems that can be caused by communication over networks, and then focusing on the quantization errors and packet dropouts, we proposed an active approach to robust filter design.

7.1 Concluding Remarks

The main contributions of this research can be listed as follows:

- A filter design methodology was proposed for discrete-time Lipschitz nonlinear systems under network-induced problems such as quantization effects, uncertain delays and communication constraints. To this end, first the filter design problem was studied under the errors caused by a logarithmic quantizer for which the quantization error was modeled as a bounded uncertainty multiplied by the actual measurement. This uncertainty was later translated into uncertainties in the plant parameters and then a Lyapunov-based approach was employed to derive the stability LMI's for the filter. The design process was repeated for filtering under uncertain delays, with the difference that there were no uncertainties present and the stability LMI's were extracted using a Lyapunov-Krasovskii function. Having an individual solution for each problem, the filter design problem was then tackled considering all network-induced issues and an LMI-based solution was proposed in the form of an optimization problem to maximize the Lipschitz constant while ensuring both stability and $\mathcal{H}_\infty$ performance.

- Unlike the existing approaches, which deal with the network-induced issues such as quantization from a passive point of view, an active method was proposed as a better alternative to filter design with limited information. In the static point of view, the error caused by quantization is usually modeled as either measurement noise or model uncertainty. This in return, gives way to conservative designs and poor performance, specially in the case of coarse quantization. In order to avoid all of this, the proposed approach employed an estimator for the errors caused by quantization and used the estimated errors for producing better estimates of the system states.

- Motivated by the fact that unknown disturbances can considerably influence the effectivity of any state observer, two novel unknown input filters were introduced. Both of the proposed approaches employed a modified version of the plant model, which was derived by finding out how many samples later, the effects of the unknown disturbance would appear in the measurements. Based on this modification, a very important and restrictive assumption which is very common in the literature, was circumvented and in return replaced by more relaxed assumptions. The first design approach used an exact equivalent of the modified plant model as the filter model and consequently involved an internal prediction-correction loop for producing its estimates. The second approach, however, employed a practical equivalent of the modified model as its filter model and therefore provided a simlpler and more straight forward solution. Through simulation it was shown that the first approach provided slightly smoother results which came at the expense of heavier computational burden.

- Trying to propose a novel approach to robust filter design with limited information, a unified filtering approach was introduced assuming that the measurements undergo quantization and packet dropouts, and an unknown disturbance is affecting the state trajectories. The unified approach brought together two distinct filters: the one designed for the rejection of unknown disturbances and the one designed for eliminating the effects of quantization. Taking advantage of the fact that the errors caused by quantization are only effective when the estimation error is small, a two-zone filter, which consisted of the two aforementioned filters, was proposed. In zone 1, assuming that the received measurements are reliable, the proposed filter tracked the states as well as the unknown disturbances, and bounded the minor effects of the quantization errors. In zone 2, it produced estimates of the states and the network-induced errors knowing that the received measurements could be considerably erroneous.

- With the intention of designing a fault-tolerant filter under quantized measurements, the proposed two-zone approach was extended to systems with model uncertainties. The unknown disturbance was used to account for the actuator faults such as offset and stuck, and the uncertainties in the model parameters were employed to model the internal faults. Entering the uncertainties into the design equations, new sets of LMI's were derived and then a performance-improving modification was added for the fault-free case.

7.2 Future Research

The results of this research can further be extended in the following areas:

- As most of the contributions of this work in the network area are focused on quantization and packet dropouts and issues such as delay haven't been considered, the active approach given in chapter 3 can be extended by taking the network-induced delay into account.

- Bringing the uncertain delay into the equations, the two-zone filter given in chapter 5 can be modified and redesigned.

- The proposed unknown-input filters can be extended to different classes of nonlinear systems such as Lipschitz nonlinear systems.

- The unknown-input filters can be integrated with different control schemes to form active fault-tolerant control systems.
- The fault-tolerant two-zone filter can be integrated with different networked control systems to result in active fault-tolerant control architectures with robustness to network-induced errors.

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