National Library of Canada
Bibliothèque nationale du Canada

Acquisitions and Bibliographic Services Branch
Direction des acquisitions et des services bibliographiques

395 Wellington Street
Ottawa, Ontario
K1A 0N4

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

Canada

UNIVERSITY OF ALBERTA

Equivariant Degree and Global Hopf Bifurcation for NFDEs with Symmetry

BY

HUAXING XIA ©

A Thesis
Submitted to the Faculty of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy

DEPARTMENT OF MATHEMATICS

Edmonton, Alberta
SPRING 1994

National Library of Canada

Bibliothèque nationale du Canada

Acquisitions and Bibliographic Services Branch

Direction des acquisitions et des services bibliographiques

395 Wellington Street
Ottawa, Ontario
K1A 0N4

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-612-11421-X

Canada

UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: Huaxing Xia

TITLE OF THESIS: **Equivariant Degree and Global Hopf Bifurcation for NFDEs with Symmetry**

DEGREE: Doctor of Philosophy

YEAR THIS DEGREE GRANTED: 1994

Permission is hereby granted to the UNIVERSITY OF ALBERTA LIBRARY to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research pruposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as hereinbefore provided neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

(Signed) ________________

Permanent Address:
Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: April 2?, 1994

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Equivariant Degree and Global Hopf Bifurcation for NFDEs with Symmetry** submitted by **Huaxing Xia** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Supervisor: W. Krawcewicz

L. H. Erbe

H. I. Freedman

J. So

A. N. Al-Hussaini

External Examiner: K. Schmitt

Date: April 8, 1994

Department of
Mathematics and Statistics

UNIVERSITÉ

UNIVERSITY

4700 KEELE STREET • NORTH YORK • ONTARIO • CANADA • M3J 1P3

April 20, 1994

To whom it may concern

Chapter 5 of H. Xia's PhD thesis is partially based on our joint work to which Xia has made significant contributions. This note is to permit the joint work to be included in his thesis.

Sincerely Yours

J. Wu

TELEPHONE (416) 736-5250 FAX (416) 736-5757 TELEX 065-24736 E-MAIL: MATHSTAT@YORKVM1.BITNET

TO MY WIFE XIAOXIA AND MY PARENTS

ABSTRACT

This thesis is concerned with the equivariant topological degree and its applications to global Hopf bifurcation theory for (neutral) functional differential equations (NFDEs) with symmetry. Firstly, an alternative new definition of the G-degree developed recently by K. Gęba, W. Krawcewicz and J. Wu is carried out by using equivariant generic approximations and an analytic formula for the computation of the degree is provided. Secondly, by extending the G-degree to that for equivariant condensing field in Banach G-spaces, a local symmetric bifurcation theorem of Krasnosel'skii type and a global symmetric bifurcation theorem of Rabinowitz type for a class of composite coincidence equations with symmetry are obtained via a G-degree approach. Thirdly, the above bifurcation results are applied to prove local and global Hopf bifurcation theorems for (neutral) functional differential equations with symmetry, which include several important theorems in bifurcation theory obtained by S. N. Chow, B. Fiedler, J. Ize, J. Mallet-Paret, R. D. Nussbaum and J. Yorke. Their applications to the Rashevsky-Turing theory are also considered. Finally, the discrete global bifurcation waves such as phase-locked oscillations and synchronous oscillations in a single-species ring patch model as well as in a ring array of coupled lossless transmission lines are investigated via global symmetric Hopf bifurcation theorems.

ACKNOWLEDGEMENT

First and foremost, I would like to thank my supervisor Dr. W. Krawcewicz for introducing me to the subject of Nonlinear Analysis during my studies at the University of Alberta. I am deeply indebted to him for his valuable guidance, kind help and incessant encouragement throughout my Ph.D. program. I would also like to express my gratitude to my friend and colleague Dr. J. Wu. His help in my research and comments on this thesis are greatly appreciated.

I owe a debt of gratitude to Dr. L. H. Erbe, Dr. H. I. Freedman and Dr. J. W.-H. So for their comments on this thesis and advice, support and encouragement throughout my stay at the University of Alberta. I am also grateful to Dr. A. N. Al-Hussaini and External Examiner Dr. K. Schmitt for their comments and suggestions on this thesis.

My sincere thanks also go to Mrs. E. Leonard for her formatting this thesis and my wife Xiaoxia for her understanding and assistance in typing this thesis.

Finally, I would like to acknowledge the financial support of the Province of Alberta Graduate Fellowship and the University of Alberta.

TABLE OF CONTENTS

LIST OF FIGURES

Page

INTRODUCTION

It has recently been recognized that many mathematical models from physics and engineering are dynamical systems with certain symmetry. The symmetry can either be intrinsic to the physical system or the idealization of the problem under investigation. Examples have been found in studying the dynamics of networks of coupled oscillators. Since identical (symmetrical) coupling has been used, as in the famous Turing's model of a ring of identical cells (see Turing [38]), three types of symmetry occur: the cyclic group $\mathbb{Z}_n$, the dihedral group D_n and the permutation group S_n (see Ashwin et al. [2]). Typical examples are also seen in the study of thermal convection in the molten inner layer of the Earth. By using a first approximation, one may consider the convective field flow between two spherical shells. If the rotation of the Earth is neglected one obtains a problem with $O(3)$-symmetry. If the rotation is included then the resulting mathematical model has $SO(2) \oplus \mathbb{Z}_2$-symmetry (see Golubitsky et al. [18]). Other examples include time-reversible systems (with a hidden $O(2)$-symmetry) treated by Vanderbauwhede [39], Hopfield models in neural networks with circulant symmetric matrices (with a $\mathbb{Z}_n$-symmetry) analyzed by Wu [40] and the translation and reflection invariant reaction difussion equations with periodic boundary conditions (with $O(2)$-symmetry) studied by Dangelmayr and Knobloch [6] and van Gils and Mallet-Paret [17]. The list of the examples goes on and we refer to Golubitsky et al. [18], Sattinger [36] and the references cited there for more examples of differential equations with symmetry which arise from various physical contexts.

When differential equations involve symmetry, the study of how the symmetry affects the dynamics is of interest. In particular, the Hopf bifurcation in the presence of symmetry has attracted considerable attention in recent years. Compared with systems without symmetry, the symmetric bifurcation patterns are much more complicated. Multiple eigenvalues are caused by the symmetry and the center manifolds (for ODEs) are thus high dimensional. As far as local Hopf bifurcation is concerned, there has been much progress toward a general theory of bifurcation with symmetry. First, the Liapunov-Schmidt reduction method can be extended to the equivariant case and the existence of periodic solutions is reduced to a symmetric static bifurcation problem. Second, by resorting to various results from equivariant singularity theory, the resulting symmetric bifurcation problem can be analyzed in detail with the help of equivariant normal forms. Due to the symmetry of the systems, calculations can often be simplified and conclusions on bifurcation patterns such as supcriticality/subcriticality and stability are finally drawn as in the non-symmetric case. We refer to Golubistky et al. [18] and Vanderbauwhede [39] for the detailed discussions of the theory and its applications to practical problems. For the use of center manifold reduction in analyzing local symmetric Hopf bifurcations, see also Swift [37] and van Gils and Mallet-Paret [17].

For a global theory of Hopf bifurcation with symmetry, topological methods seem more natural. In the nonequivariant case, i.e. when symmetries are ignored in the system, the first global result is due to Alexander and Yorke [1]. Their proofs involve generalized cohomology (framed cobordism) arguments and are later much simplified by Ize [24] using homotopy theory. A similar idea has also been employed by Nussbaum [34] to extend the Alexander-Yorke theorem to cover retarded functional differential equations (RFDEs). On the other hand, Chow and Mallet-Paret [4] have utilized the Fuller index for periodic solutions of an autonomous equation and given another proof of the global Hopf bifurcation result (for both ODEs and RFDEs), which is further improved in Chow, Mallet-Paret and Yorke [5]. Also, Ize [26] has attempted the approach of obstruction theory and considered multiparameter bifurcation and, as a special case, the global Hopf bifurcation theroem is recovered. In the meantime, the Alexander-Yorke theorem has also been obtained for parabolic systems by Ize [25] using homotopy arguments and by

Fiedler [9], resorting to the Hopf index refined from Mallet-Paret and Yorke's center index [33]. For the equivariant case, Fiedler [10] has extended his results in [9] to symmetric differential equations and several symmetric global Hopf bifurcation theorems are established by using generic approximations and an equivariant Hopf index. Since his arguments involve solution operators, it would be difficult and more complicated, if not impossible, to apply them to those differential equations (e.g. functional differential equations of mixed type) for which solution operators are not known. So are all other topological methods (except that of obstruction theory by Ize [26]) mentioned above, since they also depend upon the fact that a unique (local) solution exists for each Cauchy initial value problem and the solution is continuous with respect to the initial data. The idea of Ize [26], which uses the infinite dimensional space of periodic functions, instead of the Euclidean space $\mathbf{R}^N$, is exceptional and we believe that it could be extended to differential equations without solution operators. However, the arguments would involve certain calculations using heavy results from equivariant obstruction theory, which, to the best of our knowledge, are not established.

In looking for a more general and less involved topological proof for the global Hopf bifurcation theorem, a degree-theoretic approach has long been attempted. This is probably inspired by the work of Rabinowitz [35] on global static bifurcation for one-parameter nonlinear equations via Leray-Schauder degree. However, the idea of Rabinowitz can not directly be extended to the Hopf bifurcation problem, since the introduction of the unknown period as an additional parameter makes the problem into a two-parameter static bifurcation problem. Consequently, this has led to the generalized topological degrees defined by Gęba et al. [16] and Ize [24, 28] for continuous admissible maps with range space one dimension lower than the domain space. The values of their degrees are not integers, but are elements of the homotopy group $\pi_{n+1}(S^n)$. In consequence, the Hopf map is usually used to show the nontriviality of the degree and therefore they lose the power of the Brouwer degree in which an analytic computation may determine its nontriviality.

Moreover, as mentioned earlier, the proof of the global Hopf bifurcation theorem as simplified by Ize [24] uses the solution operator and the application of the degree of Gęba et al. [16] to Hopf bifurcation problems has not been found, to the best of our knowledge.

The first degree-theoretic proof for the global Hopf bifurcation theorem comes with the introduction of the S^1(-equivariant) degree and this is given by Gęba and Marzantowitz [15] for ordinary differential equations and is next extended to FDEs by Erbe et al. [8]. The S^1-degree , which is analytically constructed by Dylawerski et al. [7] for S^1-equivariant maps with range space one dimension lower than the domain space, possesses all the properties of Brouwer degree and, more importantly, it also takes integers as its values. Therefore, analytic formulas for the computation of the degree can be obtained. On the other hand, by using equivariant homotopy theory, a more general S^1-degree theory has also been carried out by Ize, Massabó and Vignoli [28, 29] and its applications to global Hopf bifurcation problems are extensively considered for ODEs (see [29]). By the S^1-degree approach, the treatment of the global Hopf bifurcation problem is very much the same as that for static bifurcation studied by Rabinowitz [35]. Since it does not use the solution operator, the analogs of the Alexander-Yorke theorem can also be obtained for (neutral) FDEs with advanced arguments. Moreover, due to the purely topological nature of the approach, it avoids the sophisticated decomposition and perturbation theory of linear FDEs as well as generic approximations, which, to the best of our knowledge, have not been developed to FDEs where delayed and advanced arguments are allowed to coexist. We refer to Erbe et al. [8] and Krawcewicz et al. [31] for more details.

Not only does the S^1-degree provide a new proof for the global Hopf bifurcation theorem, it also suggests constructions of more general G-degrees for any compact Lie group G. This has led to two G-equivariant degree theories developed recently by Gęba, Krawcewicz and Wu [11] and Ize, Massabó and Vignoli [28], which have offered useful approaches to global symmetric bifurcation problems. The computation of the degrees of course plays a central role in these theories and it relies heavily on the structure of the group G and the way it acts on representation spaces. By using an analytic computational formula in the case where G is abelian, applications to bifurcation problems have been made and symmetric global Hopf bifurcation theorems are proved for functional differential equations with symmetry (see Gęba, Krawcewicz and Wu [12, 13]). Moreover, Gęba , Krawcewicz and Wu [14] have also successfully computed the $O(2)$-degree and its application to time-reversable systems is considered.

In the present thesis, we shall develop further the equivariant degree theory of Gęba, Krawcewicz and Wu [11–14]. We are more interested here in computing the G-degree by analytic formulas as is the case in computing the classical Brouwer degree. Since neutral functional differential equations (NFDEs) are receving more and more attention, we would also like to extend the theory to cover a global Hopf bifurcation theorem for NFDEs with symmetry. Finally, it is of great interest to provide certain bifurcation analysis procedures and show how to apply the theory to specific biological, chemical and physical problems.

We have chosen the neutral functional differential equations as our main system of investigation for three reasons. First, neutral equations are a very general type of FDEs. They include retarded functional differential equations, integral equations and ordinary differential equations. Second, compared with RFDEs, neutral equations can display quite different and, in some cases, very interesting dynamics. Due to the fact that the characteristic equation of a neutral equation may have roots bifurcating from infinity, a phenomenon that does not occur in RFDEs, neutral delay systems are not structurely stable in the sense that the introduction of neutral terms may destabilize an asymptotically stable equilibrium (see Kuang [32]) and, in other cases, it may also stabilize an otherwise unstable equilibrium (see Gopalsamy and Zhang [20]). Since solutions to neutral equations are not necessarily differentiable, the Poincaré return map, a powerful tool in studying periodic solutions for ODEs and RFDEs, is therefore not generally differentiable (see Hale and Lunel [23]). This introduces extreme difficulties in studying neutral equations. Third, we have observed that there has recently been growing interest in neutral equations and more and more neutral equations are used as models to physical problems. As examples, a neutral dynamic model has been derived by Burns et al. [3] in elastic motions of a three-degree-of-freedom airfoil section in a two-dimensional imcompressible flow. In modelling population dynamics in a food-limited environment, Gopalsamy and Zhang [20] have proposed a neutral logistic equation. In the study of compartmental systems with pipes, a neutral equation with infinite delay has been obtained by Györi and Wu [22]. Other interesting examples can also been found in Kuang ([32], Chapter 9) for the use of neutral delay systems modelling two species interactions in a closed environment and in Gopalsamy ([19], Chapter 5) for various generalizations of neutral differential systems arising from population dynamics and neural networks. For more examples of neutral equations, we also refer to the recent monograph of Györi and Ladas [21].

We outline the contents of this thesis as follows. In Chapter 0, we prepare some preliminary facts from algebra and topology. Material on transformation groups and infinite dimensional representations, which is needed in subsequent chapters, is included. Chapter 1 is devoted to analytic computational formulas for the G-degree. As in the classical Brouwer degree case, an attempt has been made to present a more analytical approach to the construction of the G-degree. This is slightly different from the original definition by Gęba, Krawcewicz and Wu [11] and the problem has boiled down to finding generic approximations of equivariant maps.

In Chapter 2 and 3, we apply the G-degree to bifurcation problems. The static bifurcation with $G \times S^1$-symmetry is first treated in Chapter 2 and analogs of Krasnoselskii's local bifurcation theorem [30] and Rabinowitz's global bifurcation theorem [35] are proved for a class of nonlinear symmetric composite coincidence problem by using G-equivariant degree. Chapter 3 contains the applications of Chapter 2 to Hopf bifurcation problems. We present proofs of symmetric global Hopf bifurcation theorems, which are analogous to the Alexander-Yorke result, for neutral functional differential equations with symmetry. Hopf bifurcation for symmetric functional equations is also considered.

The last two chapters, Chapter 4 and Chapter 5, illustrate how the symmetric global Hopf bifurcation theorems obtained in the previous chapters can be applied to practical problems. We study in Chapter 4 a population model of a single-species distributed over a ring of identical patches. Our main concern is to explore the effect of the symmetry as well as the diffusion and neutral term on the occurence of bifurcating periodic waves. Chapter 5 focuses on global phase-locked oscillations in a ring array of symmetrically coupled lossless transmission lines. The model we derive is a system of neutral functional differential equations and may have other theoretical and practical interest. Finally, two theorems on lower bounds of periodic solutions to NFDEs are appended.

We mention that we will not consider in this thesis the G-degree theory developed by Ize, Mossabó and Vignoli [27–29], although it relates to that of Gęba, Krawcewicz and Wu [11–14]. For the construction and computation of this more general G-degree theory as well as its applications to Hopf bifurcation problems for ODEs, we refer to their original work.

References

[1] J. C. Alexander and J. A. Yorke, *Global bifurcation of periodic orbits*, Amer. J. Math. **100** (1978), 263–292.

[2] P. Ashwin, G. P. King and J. W. Swift, *Three identical oscillators with symmetric coupling,* Nonlinearity **3** (1990), 585–601.

[3] J. A. Burns, E. M. Cliff and T. L. Herdman, *A state-space model for an aeroelastic system,* 22nd IEEE Conference on Decision and Control **3** (1983), 1074–1077.

[4] S. N. Chow and J. Mallet-Paret, *The Fuller index and global Hopf bifurcation,* J. Differential Equations **29** (1978), 66–85.

[5] ——, —— and J. A. Yorke, *Global Hopf bifurcation from a multiple eigenvalue,* Nonlinear Analysis-TMA **2** (1978), 753–763.

[6] G. Dangelmayr and E. Knobloch, *On the Hopf bifurcation with broken $O(2)$ symmetry,* In The Physics of Structure Formation (W. Güttinger and D. Dangelmayr, eds.), Springer-Verlag, Heidelberg, 1987, 387–393.

[7] G. Dylawerski, K. Gęba, J. Jodel and Marzantowicz, *S^1-equivariant degree and the Fuller index,* Ann. Polon. Math. **52** (1991), 243–280.

[8] L. H. Erbe, K. Gęba, W. Krawcewicz and J. Wu, *S^1-degree and global Hopf bifurcation theory of functional differential equations,* J. Differential Equations **98** (1992), 277–298.

[9] B. Fielder, *An index for global Hopf bifurcation in parabolic systems,* J. Reine Angew. Math. **359** (1985), 1–36.

[10] ——, "Global Bifurcation of Periodic Solutions with Symmetry", Lecture Notes in Mathematics **1309**, Springer-Verlag, 1988.

[11] K. Gęba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems, I: Construction of the degree,* J. London Math. Soc., to appear.

[12] —, —— and ——, *An equivariant degree with applications to symmetric bifurcation problems, II: Abstract bifurcation theorems,* Preprint, 1993.

[13] ——, —— and ——, *An equivariant degree with applications to symmetric bifurcation problems, III: Bifurcation theorems of functional differential equations with symmetry,* Preprint, 1993.

[14] ——, —— and ——, *Hopf bifurcation in time-reversible systems,* Preprint, 1993.

[15] K. Gęba and W. Marzantowicz, *Global bifurcation of periodic orbits,* Topological Methods in Nonlinear Analysis **1** (1993), 67–93.

[16] ——, I. Massabó and A. Vignili, *Generalized topological degree and bifurcation,* in Nonlinear Functional Analysis and its Applications (S. P. Sinhg, ed.), Reidel, Dordrecht, 54–73, 1986.

[17] S. A. van Gils and J. Mallet-Paret, *Hopf bifurcation and symmetry: travelling and standing waves on the circle,* Proc. Royal Soc. Edinburg **104A** (1986), 279–307.

[18] ——, —— and I. N. Stewart, "Singularities and Groups in Bifurcation Theory", Vol. II, Springer-Verlag, New York, 1988.

[19] K. Gopalsamy, "Stability and Oscillations in Delay Differential Equations of Population Dynamics", Kluwer Academic Publishers, Dordrecht, 1992.

[20] —— and B. G. Zhang, *On a neutral delay logistic equation,* Dyn. Stab. Syst. **2** (1988), 183–195.

[21] I. Györi and Ladas, "Oscillation Theory of Delay Differential Equations with Applications", Oxford University Press, Oxford, 1992.

[22] I. Györi and J. Wu, *A neutral equation arising from compartmental systems with pipes,* J. Dyn. Diff. Equas **3** (1991), 289–311.

[23] J. K. Hale and S. Lunel, "Functional Differential Equations", Springer-Verlag, New York, 1993.

[24] J. Ize, *Bifurcation theory for Fredholm operators,* Mem. Amer. Math. Soc. **174**, 1976.

[25] ——, *Periodic solutions of nonlinear parabolic equations,* Comm. Partial Differential Equations **4** (1979), 1299–1387.

[26] ——, *Obstruction theory and multiparameter Hopf bifurcation,* Trans. Amer. Math. Soc. **289** (1985), 757–792.

[27] ——, "Topological Bifurcation", Reportes de investigación, Universidad Nacional Autonoma de Mexico **3**(4), 1993.

[28] ——, I. Massabó and A. Vignoli, *Degree theory for equivariant maps I,* Trans, Amer. Math. Soc. **315** (1989), 433–510.

[29] —, —— and ——, *Degree theory for equivariant maps: The general S^1-action,* Mem. Amer. Math. Soc. **481** (1992).

[30] M. A. Krasnosel'skii, "Topological Methods in the Theory of Nonlinear Integral Equations", Pergamon Press, New York, 1965.

[31] W. Krawcewicz, J. Wu and H. Xia, *Global Hopf bifurcation theory for condensing fields and neutral equations with applications to lossless transmission problems,* Can. Appli. Math. Quarterly **1** (1993), 167–220.

[32] Y. Kuang, "Delay Differential Equations With Applications in Population Dynamics", Academic Press, Boston, 1993.

[33] J. Mallet-Paret and J. A. Yorke, *Snakes: Oriented families of periodic orbits, their sources, sinks and continuation,* J. Differential Equations **43** (1982), 419–450.

[34] R. D. Nussbaum, *A Hopf global bifurcation theorem for retarded functional differential equations,* Trans. Amer. Math. Soc. **238** (1978), 139–164.

[35] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems,* J. Funct. Anal. **7** (1971), 487–513.

[36] D. H. Sattinger, "Group Theoretic Methods in Bifurcation Theory", Lecture Notes in Mathematics **762**, Springer-Verlag, Berlin, 1979.

[37] J. W. Swift, *Hopf bifurcation with symmetry of the square,* Nonlinearity **1** (1988), 333-377.

[38] A. Turing, *The chemical basis of morphogenesis,* Phil. Trans. Roy. Soc. **B237** (1952), 37–72.

[39] A. Vanderbauwhede, "Local Bifurcation and Symmetry", Res. Notes Math. **75**, Pitman, Boston, 1982.

[40] J. Wu, *Global bifurcation waves in neural networks,* FI93-DS02, The Fields Institute, 1993.

CHAPTER 0

PRELIMINARIES

0.1. G-actions and G-spaces

In this section, we shall recall some basic facts from transformation group theory. Most of them will be stated without proof. We refer to the books by Bredon [4] and Kawakubo [18] for more details.

Throughout this section, we assume G is a compact Lie group.

Definition 0.1.1. Let X be a Hausdorff topological space. By a ***topological transformation group*** we mean a triple (G, X, φ), where $\varphi : G \times X \longrightarrow X$ is a continuous map such that

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G$ and $x \in X$;

(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity of G.

For a topological transformation group (G, X, φ), we call the map φ an ***action*** of G on X and the space X, under the action of G, a ***G-space***. Whenever there is no confusion, we shall use the notation $g(x)$ or gx for $\varphi(g, x)$. Similarly, for $C \subset G$ and $A \subset X$, we put $C(A) = \{g(x);\ g \in C, x \in A\}$. A set $A \subset X$ is said to be ***G-invariant***, or simply ***invariant***, if $G(A) = A$. It is easy to see that $G(A)$ is compact if A is compact.

For any $x \in X$, the subgroup $G_x = \{g \in G;\ gx = x\}$ of G, which is closed in G, is called the ***isotropy group*** of x and the invariant subspace

$G(x) := \{gx;\ g \in G\}$ of X is called the *orbit* of x. We denote by X/G the set of all orbits in X. There is a canonical projection $\pi : X \to X/G$, $x \to G(x)$. We provide X/G with the quotient topology with respect to the projection π and call it the *orbit space*. Under this topology, X/G is also Hausdorff and the projection π is a closed map.

Let X and Y be two G-spaces. A continuous map $f : X \to Y$ is called a *G-equivariant map*, or simply a *G-map* , if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. An equivariant homeomorphism is called a *G-homeomorphism* . We say two G-spaces X and Y are *G-homeomorphic* if there exists a G-homeomorphism $f : X \to Y$. A continuous map $f : X \to Y$ satisfying $f(gx) = f(x)$ for all $g \in G$ and $x \in X$ is said to be an *invariant map* .

Remark 0.1.1. The above concepts can be carried over to the case where $X = M$ is a smooth $(C^{\infty}-)$ manifold. If the action $\varphi : G \times M \to M$ is smooth, it is called a *smooth G-action* and M is called a *smooth G-manifold* or simply *G-manifold.* A smooth G-map $f : M \to N$ between G-manifolds is a *G-diffeomorphism* if f is a diffeomorphism. M and N are G-diffeomorphic if there exists a G-diffeomorphism $f : M \to N$.

Example 0.1.1. Let V be a finite dimensional real vector space. A *representation* of G on V is a continuous homomorphism $\rho : G \to \mathrm{Aut}(V)$. This representation induces a *linear G-action* $\varphi : G \times V \to V$ defined by $\varphi(g, v) = \rho(g)v$. By choosing a basis of V, we may identify V with a Euclidean space $\mathbb{R}^n$ $(n = \dim V)$ and $\mathrm{Aut}(V) \cong GL(\mathbb{R}^n)$. Under this identification, the continuous homomorphism ρ becomes a Lie homomorphism and the linear G-action φ is thus a smooth G-action . In this way, every representation space is a G-manifold.

To desbribe an orbit $G(x)$, we need to know the space G/H of left cosets gH of H in G, where H is a closed subgroup of G. We provide the quotient topology induced by the canonical map $\pi : G \to G/H$ so that G/H, the ***homogeneous space***, becomes a Hausdorff space. Recall that $f : X \to Y$ has a ***local cross-section*** if for each $y \in Y$, there exist a neighbourhood U of y and a map $s : U \to X$ such that $f \circ s$ is the identity map on U.

The following result tells much more about G/H.

Proposition 0.1.1. *Let H be a closed subgroup of G and $\pi : G \to G/H$ be the canonical projection map. Then there exists a unique smooth structure on G/H such that*

(i) π is smooth;

(ii) π has a smooth local cross-section.

Moreover, if H is a closed normal subgroup of G, then G/H becomes a Lie group and π is a homomorphism of Lie groups.

Proof. See Bredon [14] or Kawakubo [18].

In what follows, whenever we say G/H is a smooth manifold, we mean G/H is endowed with the above unique smooth structure.

Example 0.1.2. Let H be a closed subgroup of G. We can define a G-action on the homogeneous space G/H by left translation, i.e. $\varphi(g, g'H) := L_g(g'H) :=$

$gg'H$. This action is smooth. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow{\varphi} & G/H \end{array}$$

where m denotes the group multiplication. By Proposition 0.1.1, π is smooth and has a local cross-section. This implies that φ is the composition of smooth functions and therefore is smooth. Under the action φ, G/H is a smooth G-manifold. φ will be called the ***natural action.***

Now suppose that X is a G-space and $x \in X$. We have a natural map $f : G/G_x \to G(x)$ defined by $f(gG_x) = gx$. The following proposition describes the orbit $G(x)$.

Proposition 0.1.2. *Under the natural action of* G *on* G/G_x,

(i) $f : G/G_x \to G(x)$ *is a* G*-homeomorphism;*

(ii) $f : G/G_x \to G(x)$ *is a* G*-diffeomorphism if* X *is a* G*-manifold. Therefore, the orbit* $G(x)$ *is a* G*-invariant submanifold.*

Proof. See Kawakubo [18].

Recall that two closed subgroups H and K are ***conjugate*** in G, denoted by $H \sim K$, if $H = gKg^{-1}$ for some $g \in G$. Clearly, $\sim$ is an equivalence relation. The equivalence class of H is called a ***conjugacy class*** of H in G and will be denoted by (H).

From Proposition 0.1.2, we see that the quotient space G/G_x describes the orbit $G(x)$. Notice that $G_{gx} = gG_xg^{-1}$, i.e. G_{gx} and G_x are conjugate. This leads us to consider all the conjugacy classes of closed subgroups of G.

Let $O(G)$ stand for the set of all conjugacy classes of closed subgroups of G. The set $O(G)$ is then partially ordered under the following relation $\leq$: $\alpha \leq \beta$ for $\alpha, \beta \in O(G)$ if and only if there exist closed subgroups H and K such that $\alpha = (H)$, $\beta = (K)$ and K is conjugate to a subgroup of H. Given a G-space and $x \in X$, the conjugacy class (G_x) is called the ***orbit type*** of x. It follows from Proposition 0.1.2 that the orbit type fully describes the orbit $G(x)$.

Definition 0.1.2. A transformation group (X, G, φ) is said to be

(i) ***trivial*** if $G_x = G$ for all $x \in X$;

(ii) ***free*** if $G_x = \{e\}$ for all $x \in X$.

Let X be a G-space and H a closed subgroup of G. We consider the following subsets of X

$$\begin{aligned} X^H &:= \{x \in X;\ G_x \supseteq H\} \\ X_H &:= \{x \in X;\ G_x = H\} \\ X^{(H)} &:= \{x \in X;\ (G_x) \leq (H)\} \\ X_{(H)} &:= \{x \in X;\ (G_x) = (H)\} \end{aligned}$$

Denote by $N(H)$ the ***normalizer*** of the closed subgroup H of G. Then $N(H)$ is closed and hence is a Lie group. Therefore, by Proposition 0.1.1, the ***Weyl group*** $W(H) := N(H)/H$ is a Lie group.

The following proposition sumarizes the elementary properties of the above subsets.

Proposition 0.1.3. *Let X and Y be G-spaces and H a closed subgroup of G. Then*

(i) X^H *is a closed* $N(H)$*-space as well as a* $W(H)$*-space, where the* $W(H)$*-action is given by* $\varphi(nH, x) = nx$, *for* $n \in N(H)$ *and* $x \in X^H$. *Moreover, if* X *is a* G*-representation,* X^H *is a linear subspace of* X;

(ii) X_H *is an open, dense and free* $W(H)$*-space in* X^H;

(iii) $X^{(H)} = GX^H$ *and* $X_{(H)} = GX_H$. *In particular,* $X^{(H)}$ *is closed;*

(iv) If (H) *is a minimal orbit type which occurs in* X, *then* $X_{(H)}$ *is closed in* X;

(v) Any G*-map* $f : X \to Y$ *induces* $W(H)$*-maps* $f^H : X^H \to Y^H$, *where* $f^H := f|_{X^H}$.

Proof. See Kawakubo [18].

Definition 0.1.3. Let E and X be topological (Hausdorff) spaces and $p : E \to X$ be a continuous map. (E, X, p) is said to be a *vector bundle* if

(i) for each $x \in X$, $E_x := p^{-1}(x)$ has the structure of a vector space;

(ii) for each $x \in X$, there exist an open neighbourhood U of x and a homeomorphism $\Psi : p^{-1}(x) \to U \times \mathbb{R}^n$ such that $p \circ \Psi = \pi$ and for each $y \in U$, $\Psi|_{p^{-1}(x)} : p^{-1}(x) \to \{y\} \times \mathbb{R}^n$ is a linear isomorphism, where $\pi : U \times \mathbb{R}^n \to U$ denotes the projection.

We call E_x the *fibre* of x. The condition (ii) is referred to as the *local triviality* of the vector bundle. A continuous map $s : X \to E$ such that $p \circ s = Id$ is called a *section* of the bundle.

Definition 0.1.4. A vector bundle (E, X, p) is called a *G-vector bundle* if E, X are G-spaces and p is a G-map, and for each $g \in G$ and $x \in X$, the map $g : p^{-1}(x) \to p^{-1}(gx)$ is a linear isomorphism. A G-vector bundle (E, X, p) is said to be smooth if X and E are G-manifolds and p is smooth. Two G-vector bundles (E, X, p) and (E', X, p') are called isomorphic if there exists a G-map $f : E \to E'$ such that $p' \circ f = p$ and $f|_{E_x} : E_x \to E'_x$ is a linear isomorphism for each $x \in X$. Such an f is called a *G-bundle isomorphism.*

By definition, any vector bundle (E, X, p) is a G-vector bundle where G acts on E and X trivially.

Example 0.1.3. (i) Let $p : E \to X$ be a G-vector bundle and $f : Y \to X$ be a G-map of G-spaces. Define the *pull-back*

$$f^*(E) := \{(y, v) \in Y \times E; f(y) = p(v)\}$$

and $p' : f^*(E) \to Y$ by $p'(y, v) = y$. Then $f^*(E)$ is G-invariant in $Y \times E$ and $(f^*(E), Y, p')$ is a G-vector bundle.

(ii) Let $p : E \to X$ be a G-vector bundle and $Y \subset X$ be an invariant subspace. Then $p : E|_Y := p^{-1}(Y) \to Y$ is a G-vector bundle. We call it the *restriction* of E to Y. It is straightforward to see that $(E|_Y, Y, p)$ is G-isomorphic to the pull-back $i^*(E)$ where $i : Y \hookrightarrow X$ is the inclusion.

(iii) Let $p : E \to X$ be a G-vector bundle. If a G-invariant subspace E' of E and the restriction $p' : p|'_E : E' \to X$ satisfy (1): $E'_x = E' \cap E_x$ is a vector subspace space of E_x for each $x \in X$; and (2): $p' : E' \to X$ is a G-vector bundle with respect to the structure of the vector space on E'_x in (1), then (E', X, p') is called the *G-vector sub-bundle* of E.

(iv) Let V be a representation space of G and X be any G-space. We can form the G-vector bundle $p: X \times V \to X$, where p is the natural projection onto X. We call $(X \times V, X, p)$ the ***trivial G-vector bundle.***

(v) Given a G-vector bundle $p: E \to X$ over a paracompact G-space X, one can find a ***G-invariant metric*** $\langle \cdot, \cdot \rangle$ on E, i.e.

$$\langle gu, gv \rangle_{gx} = \langle u, v \rangle_x, \quad x \in X$$

for all $g \in G$ and $u, v \in E_x$. We define for each $\varepsilon > 0$ two G-invariant subspaces of E

$$D(E) := \{v \in E;\ \langle v, v \rangle_{p(v)} \leq \varepsilon\}$$

$$S(E) := \{v \in E;\ \langle v, v \rangle_{p(v)} = \varepsilon\}.$$

They are called the ***ε-disc bundle*** and the ***ε-sphere bundle***, respectively.

(vi) Given a G-vector bundle $p: E \to X$ and its G-vector sub-bundle E'. Suppose that X is paracompact and $\langle \cdot, \cdot \rangle$ is a G-invariant metric on E. We put

$$E'^{\perp} := \bigcup_{x \in X} E_x'^{\perp}$$

where $E_x'^{\perp}$ denotes the orthogonal complement of E_x' in E_x with respect to the inner product $\langle \cdot, \cdot \rangle_x$. Then $E'^{\perp}$ is a G-vector bundle of E, which is called the ***orthogonal complement*** of E' in E.

Example 0.1.4. Suppose that M is a G-manifold. Let $x \in M$ and denote by $T_x(M)$ the tangent space of M at x. Then the ***tangent (vector) bundle***

$$T(M) := \bigcup_{x \in M} T_x(M)$$

with $p : T(M) \to M$, $p(T_x(M)) = x$, $x \in M$ can be endowed with a G-vector bundle structure by defining a G-action $\varphi : G \times T(M) \to T(M)$ by $\varphi(g, v) = g_* v$, $(g, v) \in G \times T(M)$, and $g_* : T(M) \to T(M)$ is the tangent map of the diffeomorphism $g : M \to M$. Under the action φ, $T(M)$ is a G-manifold and $g_* : T_x(M) \to T_{gx}(M)$ is a linear isomorphism. We call $(T(M), M, p)$ the *tangent G-vector bundle* of M.

Recall that a *Riemannian metric* $\langle \cdot, \cdot \rangle$ on M is a smooth metric on the tangent bundle $T(M)$. By using the Haar integral (see later in this section), one sees that on the tangent G-vector bundle $T(M)$ there exists a *smooth G-invariant Riemannian metric* $\langle \cdot, \cdot \rangle$.

Let $A \subset M$ be a G-invariant submanifold of M. Then we have the G-vector sub-bundle $T(A)$ of the restriction $T(M)|_A$ of the tangent G-vector bundle $T(M)$ of M. With respect to the above G-invariant metric, there exists an orthogonal complement of $T(A)$ in $T(M)|_A$

$$\nu(A) := T(A)^{\perp}$$

which is called the *normal G-vector bundle* of A in M. Notice that $\nu(A)$ is canonically endowed with a G-invariant metric.

Recall that two G-maps $f_1, f_2 : Y \to X$ are G-homotopic if there exists a G-map $H : Y \times [0,1] \to X$ such that $f_i = H(\cdot, i)$, $i = 0, 1$, where G acts on $[0,1]$ trivially.

The following result will be needed in section 0.4.

Proposition 0.1.4. *If $f_1, f_2 : Y \to X$ are G-homotopic G-maps, and Y is compact and E is a G-vector bundle on X, then $f_1^*(E)$ is G-isomorphic to $f_2^*(E)$.*

Proof. See Segal [30].

Definition 0.1.5. Let H be a closed subgroup of G and A an H-space. Define an H-action on $G \times A$ by $\varphi(h,(g,a)) = (gh^{-1}, ha)$ for $h \in H$, $a \in A$. The orbit space $G \times_H A := (G \times A)/H$ of this H-action is called the *twisted product* of G and A. We denote by $[g,a]$ the H-orbit of (g,a).

Remark 0.1.2. The twisted product $G \times_H A$ is again a G-space with the G-action

$$\widetilde{\varphi} : G \times (G \times_H A) \to G \times_H A$$

defined by $\widetilde{\varphi}(g', [g,a]) = [g'g, a]$. The following properties of $G \times_H A$ are straightforward:

(i) If A is a G-space, then $G \times_G A$ is G-homeomorphic to A;

(ii) If H is a subgroup of G and A is an H-space, then $(G \times_H A)/G$ is homeomorphic to A/H.

Using the twisted product, we can present the following important result, which is fundamental in the study of the structure of G-manifolds. For a proof, see Kawakubo [18].

Theorem 0.1.5. (THE SLICE THEOREM) *Let M be a G-manifold. For any $x \in M$ and $H = G_x$, there exists a unique H-representation V and a G-diffeomorphism $f : G \times_H V \to M$ onto an open neighbourhood of the orbit $G(x)$ such that $f([g,0]) = gx$. Moreover, the normal G-vector bundle ν of the G-invariant submanifold $G(x)$ in M is isomorphic to*

$$p : G \times_H V \to G/H$$

as smooth G-vector bundles, where $p([g,v]) = [g]$ and $[g]$ denotes the H-orbit of $g \in G$ and H acts on G by $h \cdot g = gh^{-1}$, for $h \in H$.

In view of the above theorem, the H-invariant image $S := f([e,v])$ of V under the G-diffeomorphism f above will be called a *slice* of $G(x)$ at x and $f(G \times_H V)$ in M will be called a *tube about the orbit* $G(x)$.

As a consequence of the slice theorem, we have the following corollary concerning the orbit types around the orbit $G(x)$.

Corollary 0.1.6. *Let M be a G-manifold and $x \in M$. Then there exists a neighbourhood U of the orbit $G(x)$ in M such that $(G_x) \leq (G_y)$ for all $y \in U$.*

Proof. Let S be a slice of $G(x)$ at x. Then $G(S)$ is an open neighbourhood of $G(x)$. For any $y \in G(S)$, $y = gs$ for some $g \in G$ and $s \in S$. Now S is a G_x-representation. One has $G_s \subset G_x$. This implies that $G_y = gG_sg^{-1}$, i.e. $(G_x) \leq (G_s) = (G_y)$. The corollary follows by letting $U = G(S)$.

Among other important consequences of the slice theorem, we introduce several theorems which will be needed in the construction of the G-degree.

The first result, describing the fixed point set of a G-action, improves Proposition 0.1.3.

Theorem 0.1.7. *Let M be a G-manifold and H be a closed subgroup of G. Then*

(i) the fixed point set M^G is a closed submanifold of M;

(ii) the H-fixed point set M^H is a $W(H)$-manifold.

Proof. See Kawakubo [18].

The following concept was first introduced by Peschke [29].

Definition 0.1.6. A compact Lie group G is said to be *bi-orientable* if it has an orientation which is invariant under left and right translations.

By a *left translation* L_g, $g \in G$, we mean a map $L_g : G \to G$ defined by $L_g(g') = gg'$ for $g' \in G$. Similarly, a *right translation* R_g, $g \in G$ is defined by $R_g(g') = g'g^{-1}$ for $g' \in G$. By definition, any compact abelian (resp. connected) Lie group is bi-orientable. For more information on bi-orientability of Lie groups, we refer to Peschke [29].

We have the following result considering the structure of orbit spaces.

Theorem 0.1.8. *Let M be a free G-manifold. Then*

(i) the orbit space M/G is a smooth manifold and the projection $\pi : M \to M/G$ is smooth and admits a smooth local cross-section. Moreover, for any $[x] \in M/G$, there exist a neighbourhood $[\mathcal{U}]$ of $[x]$ in M/G and a

G-diffeomorphism $\varphi : \pi^{-1}([\mathcal{U}]) \to G \times [\mathcal{U}]$ *such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}([\mathcal{U}]) & \xrightarrow{\varphi} & G \times [\mathcal{U}] \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p} & \\ [\mathcal{U}] & & \end{array}$$

where $p : G \times [\mathcal{U}] \to [\mathcal{U}]$ *is the projection.*

(ii) if G *is bi-orientable,* M *is orientable and the action of* G *preserves the orientation of* M*, i.e. for each* $g \in G$, $\varphi_g : M \to M$*, defined by* $\varphi_g(x) = gx$ *for* $x \in M$*, preserves the orientation of* M*, then the orbit space* M/G *is orientable. Moreover, the orientation of* M/G *can be determined by the choice of orientation of* G *and* M*.*

Remark 0.1.3. For any topological space X, we can always obtain a G-space $G \times X$ where G acts by $g'(g, x) = (g'g, x)$ for $g', g \in G$ and $x \in X$.

Proof of Theorem 0.1.8. (i) See Kawakubo [18].

(ii) First fix an orientation of the tangent space $T_e G$ and use the left translation $L_g : G \to G$ to choose an orientation of $T_g G$ for any given $g \in G$ such that the orientations are invariant under L_g. Since G is bi-orientable, the right translation $R_g : G \to G$ preserves these orientations.

Let $x \in M$. By Proposition 0.1.2, the map $\mathcal{L}_x : G \to G(x)$ defined by $\mathcal{L}_x(g) = gx$, $g \in G$, is a G-diffeomorphism. This allows us to choose an orientation on $G(x)$. Note that if $y \in G(x)$ then there exists $g \in G$ such that $y = gx$.

Since R_g preserves the orientation of G, from the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{L}_x} & G(x) \\ {\scriptstyle R_g}\downarrow & \nearrow & \\ G & \mathcal{L}_y & \end{array}$$

it follows that the orientation of $G(x)$ determined above does not depend on the choice of $y \in G(x)$. Also, by the condition in (ii), we can choose an orientation of T_yM, $y \in G(x)$, which does not depend on the choice of y.

Let $[x]$ denote the orbit $G(x)$ in M/G. Choose an orientation of M. Then T_xM and T_eG are oriented and from

$$T_xM \cong T_{[x]}(M/G) \oplus T_x(G(x)) \cong T_{[x]}(M/G) \oplus T_e(G)$$

an orientation of $T_{[x]}(M/G)$ can be determined in such a way that, together with the orientation of G, it provides the given orientation of M via the above isomorphism. Therefore, M/G is orientable and the proof is complete.

Now let M be a free G-manifold and V a representation vector space of G. Then the product $M \times V$ is also a free G-manifold with the diagonal action. Let $p : M \times V \to M$ be the natural projection. p then induces a map $q : (X \times V)/G \to X/G$ between orbit spaces so that the diagram

$$\begin{array}{ccc} M \times V & \xrightarrow{\pi_1} & (M \times V)/G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{\pi_2} & M/G \end{array}$$

commutes. By Theorem 0.1.8, $(X \times V)/G$ and X/G are manifolds, and π_1, π_2 admit smooth local cross-sections. It follows that q is smooth.

Theorem 0.1.9. *The above induced map* $q : (M \times V)/G \to M/G$ *is a smooth vector bundle with typical fibre* V.

Proof. Let $[x] \in M/G$. By Theorem 0.1.8, there exist a neighbourhood $[\mathcal{U}]$ of $[x]$ in M/G and a G-diffeomorphism $\varphi : \pi_2^{-1}([\mathcal{U}]) \to G \times [\mathcal{U}]$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_2^{-1}([\mathcal{U}]) & \xrightarrow{\varphi} & G \times [\mathcal{U}] \\ {\scriptstyle \pi_2}\downarrow & \swarrow {\scriptstyle p} & \\ [\mathcal{U}] & & \end{array}$$

This gives a G-diffeomorphism $\varphi^{-1} \times id : G \times [\mathcal{U}] \times V \to \pi_2^{-1}([\mathcal{U}]) \times V$ between free G-manifolds. Passing to their orbit spaces, one has a diffeomorphism $\psi : (G \times [\mathcal{U}] \times V)/G \to (\pi_2^{-1}([\mathcal{U}]) \times V)/G$.

Now define a map $j : [\mathcal{U}] \times V \to G \times [\mathcal{U}] \times V$ by $j([u], v) = (e, [u], v)$ and if $\pi : G \times [\mathcal{U}] \times V \to (G \times [\mathcal{U}] \times V)/G$ denotes the projection then the composition $k := \pi \circ j : [\mathcal{U}] \times V \to (G \times [\mathcal{U}] \times V)/G$ is a diffeomorphism. Note that $(\pi_2^{-1}([\mathcal{U}]) \times V)/G = \pi_1(\pi_2^{-1}([\mathcal{U}] \times V) = q^{-1}([\mathcal{U}])$. We arrive at the following commutative diagram

$$\begin{array}{ccc} q^{-1}([\mathcal{U}]) & \xrightarrow{k^{-1} \circ \psi^{-1}} & [\mathcal{U}] \times V \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle p} & \\ [\mathcal{U}] & & \end{array}$$

Therefore $k^{-1} \circ \psi^{-1}$ gives a (smooth) local trivialization with typical fibre V. This completes the proof.

For a G-manifold M, there exist generally more than one orbit type. Recall that $M_{(H)}$ denotes the union of all orbits of the type (H). By definition, $M_{(H)}$ is

G-invariant and M decomposes as $M = \bigcup_{(H)\in\mathcal{O}(G)} M_{(H)}$ with $M_{(H)} \cap M_{(K)} = \varnothing$ if $(H) \neq (K)$.

Theorem 0.1.10. *Let M be a G-manifold and H be a closed subgroup of G. Then $M_{(H)}$ is a G-invariant submanifold of M. Moreover, $M_{(H)}$ is closed if (H) is a minimal orbit type, and it is open and dense in M if M/G is connected and (H) is a maximal orbit type. Consequently, $M_{(H)}/G$ is a manifold and M_H is a free $W(H)$-submanifold of M.*

Proof. See Kawakubo [18].

When M is compact, as the following theorem shows, there are only a finite number of disjoint nonempty invariant subsets $M_{(H)}$ in the decomposition $M = \bigcup_{(H)\in\mathcal{O}(G)} M_{(H)}$. For a subset A (not necessarily invariant) of a G-manifold M, we denote by $\mathcal{J}(A)$ the number of orbit types occuring in A, i.e. $\mathcal{J}(A) := \#\{(G_x),\ x \in A\}$.

Theorem 0.1.11. *The following statements hold true:*

(i) If A is a compact set of a G-manifold M, then $\mathcal{J}(A) < \infty$. In particular, if M is a compact G-manifold, then $\mathcal{J}(M) < \infty$.

(ii) If V is a G-representation vector space, then $\mathcal{J}(A) < \infty$.

Proof. See Kawakubo [18].

Finally, we include some theorems from G-equivariant topology.

We begin with the Haar integral on a compact Lie group G.

Let $\mathbb{E}$ be a Banach space. We denote by $C_{\mathbb{E}}(G)$ (resp. $C(G)$) the set of continuous functions $f : G \to \mathbb{E}$ (resp. $\mathbb{R}$). It is well known (see, for example, Dinculeanu [9]) that there exists a unique invariant integral, called the *Haar integral*, on G, which we denote by $\int_G f(g)dg$ for $f \in C_{\mathbb{E}}(G)$ (resp. $C(G)$). Here dg denotes the *normalized Haar integral*, i.e. $\int_G x dg = x$ (resp. $\int_G dg = 1$) for all $x \in \mathbb{E}$. The Haar integral enjoys the following properties:

(i) For every $\alpha, \beta \in \mathbb{R}$, $f_1, f_2 \in C_{\mathbb{E}}(G)$

$$\int_G (\alpha f_1(g) + \beta f_2(g))dg = \alpha \int_G f_1(g)dg + \beta \int_G f_2(g)dg;$$

(ii) (Left and right invariance) For any $h \in G$ and $f \in C_{\mathbb{E}}(G)$

$$\int_G f(gh^{-1})dg = \int_G f(hg)dg = \int_G f(g)dg;$$

(iii) For any $f \in C_{\mathbb{E}}(G)$

$$\int_G f(g^{-1})dg = \int_G f(g)dg;$$

(iv) Let A be a topological space and $f : G \times A \to \mathbb{E}$ be continuous. Then the function $F : A \to \mathbb{E}$ defined by

$$F(a) = \int_G f(g, a)dg$$

is continuous;

(v) Let M be a smooth manifold and $f : G \times M \to \mathbb{E}$ be continuous. If $f(g, x)$ is C^r-differentiable with respect to x, then the function

$$F(x) = \int_G f(g, x)dg$$

is also C^r-differentiable;

(vi) Let $T : \mathbf{E} \to \mathbf{E}$ be a linear mapping. Then for every $f \in C_{\mathbf{E}}(G)$

$$\int_G T(f(g))dg = T(\int_G f(g)dg);$$

(vii) If $f \in C(G)$ is nonegative,

$$\int_G f(g)dg \geq 0.$$

Remark 0.1.4. The Haar integral is often used to produce an equivariant (resp. invariant) function from a nonequivariant one. More precisely, let X be a G-space and $\mathbb{E}$ be a Banach G-representation space (see definition in section 0.2). Given a continuous function $f : X \to \mathbb{E}$, by *averaging* f *on* G, we obtain an equivariant (resp. invariant) function $\widetilde{f} : X \to \mathbb{E}$ defined by

$$\widetilde{f}(x) = \int_G g^{-1} f(gx)dg, \quad x \in \mathbb{E}$$
$$(\text{resp. } \widetilde{f}(x) = \int_G f(gx)dg, \quad x \in \mathbb{E}).$$

The continuity and equivariance (resp. invariance) of $\widetilde{f}$ follows from the properties (ii) and (iv).

Definition 0.1.7. Let L be a topological linear space (over $\mathbb{R}$). We call L a *topological linear* G*-space* if G acts on L linearly. By a *metric* G*-space* we mean a G-space with a G-invariant metric.

We present the following equivariant version Dugundji extension theorem [10]. See Murayama [24] for its proof.

Theorem 0.1.12. *Let L be a locally convex topological linear G-space and $A \subset X$ be a closed invariant subspace of metric G-space X. If $f : A \to L$ is a G-map, then there exists a G-equivariant extension $f : X \to L$ of f such that the image $f(X)$ is contained in the convex hull of $f(A) \cup \{0\}$.*

If $L = \mathbb{R}^n$ is a Euclidean G-representation space, the above theorem reduces to the *equivariant Tietze-Gleason theorem* (see Bredon [4]).

When one has several pieces of smooth G-maps, to "glue" them to get one G-map, the *invariant Urysohn function* takes its place.

Theorem 0.1.13. *Let A and B be disjoint, closed and invariant subsets of a G-manifold M. There exists a smooth invariant function $\varphi : M \to [0,1]$ such that $\varphi|_A = 0$ and $\varphi|_B = 1$.*

Proof. Let ψ be a smooth Urysohn function with $\psi|_A = 0$ and $\psi|_B = 1$ (see Bröcker and Jänich [6]). Averaging ψ on G, we obtain a smooth invariant function $\varphi : M \to [0,1]$,

$$\varphi(x) = \int_G \psi(gx)dg, \quad x \in M,$$

as desired.

0.2. Isotypical decompositions of Banach G-spaces

In this section, we introduce some background material from the representation theory for compact Lie groups. For the purpose of applications to bifurcation theory, we shall lay particular stress on *real* representations in Banach spaces. For more details we refer to Adams [1], Bröcker and tom Dieck [5], Kirillov [19] and Lyubich [23].

We begin with representations in finite dimensional spaces. Let V be a finite dimensional real (resp. complex) vector space and G be a compact Lie group. By a *representation* of G on V, we mean a continuous homomorphism $\varphi : G \to GL(V)$ from the group G to the general linear group of automorphisms of V. V is called the *representation space* of φ. From section 0.1, φ induces a linear action on V so that V becomes a G-manilold. Two representations $\varphi_i : G \to GL(V_i)$, $i = 1, 2$, are said to be *equivalent*, denoted by $V_1 \simeq V_2$, if there is an isomorphism $f : V_1 \to V_2$, such that $f(\varphi_1(g)x) = \varphi_2(g)f(x)$, for all $g \in G$ and $x \in V_1$. By using the Haar integral, it follows that every real (resp. complex) representation of G is equivalent to an orthogonal (resp. unitary) representation. Therefore, without loss of generality, we shall assume that the space V is endowed with an inner product such that φ is an orthogonal (resp. unitary) representation, i.e. $\varphi : G \to O(V)$ (resp. $U(V)$), where $O(V)$ (resp. $U(V)$) denotes the group of all orthogonal (resp. unitary) operators on V.

With respect to a chosen basis, an orthogonal (resp. unitary) representation can be given by its matrix form $\varphi(g) = (\varphi_{ij}(g))$, where φ_{ij} are called the *representation functions* of φ. Define a function $\chi_V(g) := Tr(\varphi(g)) = \Sigma_i \varphi_{ii}(g)$, $g \in G$. $\chi_V(g)$ is called the *character* of φ. Clearly, χ_V does not depend on the choice of a basis and $\chi_V(ghg^{-1}) = \chi_V(h)$ for all $g, h \in G$.

A representation $\varphi : G \to GL(V)$ is said to be *irreducible* if $V \neq \{0\}$ and the only G-invariant subspaces of V are $\{0\}$ and V. For a real representation $\varphi : G \to GL_{\mathbb{R}}(V)$, by its *complexification* $\varphi_{\mathbb{C}} : G \to GL_{\mathbb{C}}(cV)$, we mean $\varphi_{\mathbb{C}}(g) = \varphi(g) \in GL_{\mathbb{C}}(cV)$ with $cV := \mathbb{C} \otimes_{\mathbb{R}} V$. By definition, $\chi_V(g) = \chi_{cV}(g)$ for all $g \in G$.

Let (φ_1, V_1) and (φ_2, V_2) be two representations of G. Set

$$\mathcal{D}(V_1, V_2) := \{f : V_1 \to V_2;\ f \text{ is linear and equivariant}\}.$$

The following result (see Kirillov [19]) classifies the set $\mathcal{D}(V_1, V_2)$.

Theorem 0.2.1. (Generalized Schur's Lemma) *The following statements hold true:*

(i) If V_1 *and* V_2 *are irreducible and* $f \in \mathcal{D}(V_1, V_2)$, *then* f *is either zero or an isomorphism;*

(ii) If V *is a complex irreducible representation and* $f \in \mathcal{D}(V, V)$, *then* $f(v) = \lambda v$ *for some* $\lambda \in \mathbb{C}$;

(iii) If V *is a real irreducible representation, then* $\mathcal{D}(V, V)$ *is a division algebra over* $\mathbb{R}$ *and is isomorphic to one of* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *where* $\mathbb{H}$ *is the four-dimensional algebra of quaternions.*

A real irreducible representation (φ, V) is called a representation of *real*, *complex*, or *quaternion type*, if $\mathcal{D}(V, V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively.

The following result (see Kirillov [19]) describes the relation between a real represention and its complexification.

Theorem 0.2.2. *(i) Let* V *be a real irreducible representation of real, complex, or quaternion type. Then its complexification* cV *is respectively irreducible, the direct sum of two inequivalent* $\dim_{\mathbb{R}} V$*-dimensional irreducible representations, or of two equivalent irreducible representations.*

(ii) Let V_1 *and* V_2 *be two real representations. If* cV_1 *is equivalent to* cV_2, *then* V_1 *is equivalent to* V_2.

Let $f_1, f_2 : G \to \mathbb{C}$ be two continuous functions. Define the *convolution* $f_1 * f_2 : G \to \mathbb{C}$ by

$$f_1 * f_2(g) := \int_G f_1(h) f_2(h^{-1}g) dh$$

where dh denotes the normalized Haar measure on G.

Remark 0.2.1. Let (φ, V) and (ψ, W) be two inequivalent irreducible unitary matrix representations of G with representation functions (φ_{ij}) and (ψ_{ij}), respectively. Then one has (see Bröcker and tom Dieck [5])

$$\chi_V * \chi_V = \frac{1}{d_V}\chi_V, \quad \chi_V * \chi_W = 0,$$
$$\int_G \varphi_{ij}(g)\varphi_{kl}(g)dg = \frac{1}{d_V}\delta_{ik}\delta_{jl},$$
$$\int_G \varphi_{ij}(g)\psi_{kl}(g)dg = 0$$

where d_V denotes the (complex) dimension of V.

Combining Theorem 0.2.2 and Remark 0.2.1 leads to the following result on real representations.

Proposition 0.2.3. *Let V be a real irreducible representation of G. Then*

$$\chi_V * \chi_V = \begin{cases} \frac{1}{d_V}\chi_V, & \text{if } V \text{ is of real type} \\ \frac{2}{d_V}\chi_V, & \text{if } V \text{ is of complex type} \\ \frac{4}{d_V}\chi_V, & \text{if } V \text{ is of quaternionic type.} \end{cases}$$

Moreover, if W is another irreducible real representation of G which is inequivalent to V, then $\chi_V * \chi_W = 0$.

Proof. If V is of real type, we have $\chi_V = \chi_{cV}$. Since $d_V = d_{cV}$, the conclusion follows from Remark 0.2.1. Suppose now that V is not of real type. By Theorem

0.2.2, $cV = V_1 \oplus V_2$, where V_1, V_2 are two complex irreducible representations of G. From the properties of characters, $\chi_V = \chi_{cV} = \chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ and $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} cV = 2\dim_{\mathbb{C}} V_1 = 2\dim_{\mathbb{C}} V_2 = d_V$. If V is of complex type, then by Remark 0.2.1,

$$\begin{aligned}\chi_V * \chi_V &= (\chi_{V_1} + \chi_{V_2}) * (\chi_{V_1} + \chi_{V_2}) \\ &= \chi_{V_1} * \chi_{V_1} + 2\chi_{V_2} * \chi_{V_1} + \chi_{V_2} * \chi_{V_2} \\ &= \chi_{V_1} * \chi_{V_1} + \chi_{V_2} * \chi_{V_2} \\ &= \tfrac{1}{\dim_{\mathbb{C}} V_1}\chi_{V_1} + \tfrac{1}{\dim_{\mathbb{C}} V_2}\chi_{V_2} \\ &= \tfrac{2}{d_V}(\chi_{V_1} + \chi_{V_2}) = \tfrac{2}{d_V}\chi_V.\end{aligned}$$

since, from Theorem 0.2.2, V_1 is inequivalent to V_2 and $\chi_{V_1} * \chi_{V_2} = 0$.

Similarly, if V is of quaternionic type, then $\chi_V = \chi_{V_1} + \chi_{V_2} = 2\chi_{V_1} = 2\chi_{V_2}$ and

$$\begin{aligned}\chi_V * \chi_V &= (2\chi_{V_1}) * (2\chi_{V_2}) = 4(\chi_{V_1} * \chi_{V_1}) \\ &= \tfrac{4}{\dim_{\mathbb{C}} V_1}\chi_{V_1} = \tfrac{8}{d_V}\chi_{V_1} = \tfrac{4}{d_V}\chi_V.\end{aligned}$$

Finally, if W is inequivalent to V, by (ii) of Theorem 0.2.2, cW is not equivalent to cV. Therefore, Remark 0.2.1 implies that $\chi_{cV} * \chi_{cW} = 0$. Consequently, $\chi_V * \chi_W = 0$.

This completes the proof.

Let V be a real irreducible representation. In view of Proposition 0.2.3, we define a number $n(V)$ as follows

$$n(V) = \begin{cases} \dim_{\mathbb{R}} V, & \text{if } V \text{ is of real type} \\ \frac{\dim_{\mathbb{R}} V}{2}, & \text{if } V \text{ is of complex type} \\ \frac{\dim_{\mathbb{R}} V}{4}, & \text{if } V \text{ is of quaternionic type.} \end{cases}$$

We call $n(V)$ the ***intrinsic dimension*** of V.

We will also need the following well-known theorem which plays an important role in the representation theory. See, for example, Bröcker and tom Dieck [5] for its proof.

Theorem 0.2.4. (PETER-WYEL THEOREM) *Let $L^2(G)$ denote the Hilbert space of square-integrable complex valued functions on G. Then the vector space generated by all representation functions is a dense and orthogonal subset of $L^2(G)$.*

We now consider representations of G in Banach spaces.

Let W be a real (resp. complex) Banach space with norm $|\cdot|$. *A representation of G in W* is a continuous map $\varphi : G \times W \to W$ such that the $\varphi(g, \cdot) : W \to W$ is a linear invertible operator for every $g \in G$. The notions of equivalence between representations and irreducibility can be carried over to Banach representations. It is a well-known fact that every complex (and therefore real) irreducible Banach representation of G is finite dimensional (see Bröcker and tom Dieck [5]). Consequently, it follows from Theorem 0.2.1 and 0.2.2 that every complex (and therefore real) irreducible representation of a compact *abelian* Lie group is one dimensional (resp. one or two dimensional).

Proposition 0.2.5. *For each Banach representation (φ, W), there exists an equivalent norm $\|\cdot\|$ on W relative to which the representation φ is isometric, i.e. $\varphi(g) : W \to W$ is a linear isometry for every $g \in G$.*

Proof. For every $x \in W$, we use the normalized Haar integral to set

$$\|x\| = \int_G |\varphi(g)x|dg.$$

The property of the Haar integral ensures $\|\varphi(g)x\| = \|x\|$ for all $g \in G$ and $x \in W$.

Due to Proposition 0.2.5, in what follows, we always assume that a Banach representation is isometric.

The following result is useful in decomposing the whole space W into invariant relatively simpler subspaces.

Theorem 0.2.6. *Let (φ, W) be a real Banach representation of G. Suppose that V is a real irreducible representation of G with the intrinsic dimension $n(V)$. Then the linear mapping $P_V : W \to W$ defined by*

$$P_V x := n(V) \int_G \chi_V(g)\varphi(g)x dg, \quad x \in W$$

is a G-equivariant projection in W such that

(i) If $x \in W$ belongs to a representation space of an irreducible subrepresentation of W which is equivalent to V, then $P_V x = x$;

(ii) If $x \in W$ belongs to a representation space of an irreducible subrepresentation of W which is not equivalent to V, then $P_V x = 0$.

Remark 0.2.2. Theorem 0.2.6 indicates that the image $P_V(W)$ is the ***isotypical*** (or ***primary***) component of W corresponding to V, i.e. $P_V(W)$ is generated by the irreducible subrepresentations in W that are equivalent to V. Since every irreducible representation of G is finite dimensional, for every $x \in P_V(W)$, the orbit $G(x)$ is contained in a finite dimensional subspace of W.

Proof of Theorem 0.2.6. The G-equivariance of P_V follows from the linearity of the Haar integral. To see that P_V is a projection, we show $P_V \circ P_V = P_V$.

Indeed,

$$\begin{aligned}
P_V \circ P_V(x) &= (n(V))^2 \int_G \chi_V(h) \int_G \chi_V(g)\varphi(h)\varphi(g)x dh\, dh \\
&= (n(V))^2 \int_G \chi_V(h) \int_G \chi(g)\varphi(hg)x dg\, dh \\
&= (n(V))^2 \int_G \chi_V(h) \int_G \chi_V(gh^{-1})\varphi(g)x dg\, dh \\
&= (n(V))^2 \int_G (\int_G \chi_V(gh^{-1})\chi_V(h)dh)\varphi(g)x dg \\
&= (n(V))^2 \int_G (\chi_V * \chi_V)\varphi(g)x dg \\
&= (n(V))^2 \int_G \frac{\chi_V(g)}{n(V)}\varphi(g)x dg \\
&= n(V) \int_G \chi_V(g)\varphi(g)x dg = P_V x
\end{aligned}$$

where we used Proposition 0.2.3.

On the other hand, let x belong to an irreducible subrepresentation space U of W. Since U is finite dimensional, we can identify it with a matrix representation $(u_{ij}(g))$. Then

$$\begin{aligned}
P_V x &= n(V) \int_G \chi_V(g)(u_{ij}(g))x dg \\
&= n(V) \int_G \sum_k v_{kk}(u_{ij}(g))x dg \\
&= n(V) \sum_k \int_G (v_{kk}(g)u_{ij}(g))x dg
\end{aligned}$$

where $(v_{ij}(g))$ is the matrix representation of V. If V and U are inequivalent, then by the orthogonal relations in Remark 0.2.1,

$$\int_G v_{kk}(g)u_{ij}(g)x dg = 0$$

and therefore $P_V x = 0$. If V and U are equivalent, again by Remark 0.2.1,

$$P_V x = n(V) \sum_k \int_G \frac{\delta_{ki}\delta_{kj}}{n(V)} x \, dg = \int_G (\delta_{ij}) x \, dg = x.$$

This completes the proof.

By Theorem 0.2.4, there exist only a countable number of real and mutually inequivalent irreducible representations of G, which we denote by (ρ_n, V_n), $n = 1, 2, \ldots$. Let χ_{V_n}, $n = 1, 2, \ldots$, denote their corresponding characters. For every integer n, we define a projection $P_n : W \to W$ as follows

$$P_n x := n(V_n) \int_G \chi_{V_n}(g) \varphi(g) x dg, \quad x \in W,$$

where (φ, W) is a real representation of G on W. Set

$$\begin{aligned} W_0 &= W^G, \\ W_n &= P_n(W), n \geq 1, \quad \text{and} \\ W^\infty &= \bigoplus_{n=0}^{\infty} W_n. \end{aligned}$$

We therefore obtain the ***isotypical decomposition*** of W, as shown in the following theorem.

Theorem 0.2.7. *The subspaces W_n are the isotypical components of W such that*

(i) the space $W^\infty = \bigoplus_{n=0}^{\infty} W_n$ is dense in W;

(ii) if $A : W \to W$ is an equivariant linear mapping, then $A(W_n) \subset W_n$ for all $n \geq 0$.

Proof. The conclusion (ii) follows directly from the Generalized Schur's Lemma. To see $\overline{W^\infty} = W$, we assume $\overline{W^\infty}$ is a proper subspace of W for a contradiction. Consider the following exact sequence of representations of G

$$\{0\} \longrightarrow \overline{W^\infty} \xrightarrow{\ i\ } W \xrightarrow{\ q\ } W/\overline{W^\infty} \longrightarrow \{0\}$$

where $W/\overline{W^\infty}$ is the factor representation of W, i is the inclusion and q is the canonical quotient map. Clearly, i and q are equivariant. Without loss of generality, we can assume that $W/\overline{W^\infty}$ is finite dimensional (otherwise. we may choose a finite dimensional subrepresentation V in $W/\overline{W^\infty}$ and replace W by $q^{-1}(V)$ and $W/\overline{W^\infty}$ by V in the above sequence). Hence, W^∞ is finite codimensional in W and the above short exact sequence splits. Therefore W is equivalent to the direct sum $\overline{W^\infty} \oplus W/\overline{W^\infty}$. Now $W/\overline{W^\infty}$ is a subrepresentation of W. Denote by (ρ, V) any irreducible subrepresentation of G in $W/\overline{W^\infty}$ which is also irreducible in W. By Theorem 0.2.6 and the definition of W^∞, (ρ, V) is inequivalent to any one of (ρ_n, V_n), $n = 1, 2, \ldots$. This is a contradiction since $\rho_1, \rho_2, \ldots$ are all irreducible representations of G. The proof is therefore completed.

Let W_{fin} denotes the set of all points $x \in W$ such that the orbit $G(x)$ is contained in a finite dimensional invariant subspace of W. Since $\oplus_{n=1}^{\infty} W_n \subseteq W_{fin}$, we have the following result.

Corollary 0.2.8. *W_{fin} is dense in W.*

We finally include two examples concerning the isotypical decompositions of the circle group S^1 and cyclic group $\mathbf{Z}_n$.

Example 0.2.1. Representations of S^1 and their isotypical decomposition.

We identify S^1 with the unit circle $\{\xi \in \mathbb{C}; |\xi| = 1\}$. Note that S^1 is abelian. The ***irreducible real representation*** of S^1 is either one dimensional or two dimensional. From Vanderbauwhede [34], we know that every one-dimensional real representation of S^1 is trivial, which we denote by $(\rho_0, \mathbf{R})$, and any two dimensional real representation of S^1 is equivalent to one of the following matrix representations on $\mathbf{R}^2$

$$\rho_n(\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$\theta \in [0, 2\pi]/\{0, 2\pi\} \cong S^1, x_1, x_2 \in \mathbf{R}, n = 1, 2, \ldots.$$

On the other hand, if we are considering the complex representations of S^1, then we obtain the following all irreducible one dimensional ***complex representations*** on $\mathbb{E}$

$$\varphi_n(\xi)z = \xi^n z, n \in \mathbf{Z}, \xi \in S^1, z \in \mathbb{C}.$$

Note that here n is allowed to take negative integers. By using the natural isomorphism $\mathbb{C} \cong \mathbf{R}^2$, we see that $(\varphi_{\pm n}, \mathbb{C})$ is equivalent to $(\rho_n, \mathbf{R}^2)$ for $n \geq 1$. In fact, $(\rho_0, \mathbf{R})$ is the only irreducible S^1-representation of real type and $(\rho_n, \mathbf{R}^2)$ are all the real irreducible S^1-representations of complex type. There exist no quaternionic type real representations of S^1.

Therefore, if (φ, W) is an isometric real Banach representation of S^1, corresponding to the irreducible representations ρ_n, $n \geq 1$, W has the following isotypical decomposition

$$W^\infty = \bigoplus_{n=0}^{\infty} W_n \underset{\text{dense}}{\subseteq} W$$

where

$$W_0 = W^{S^1}, \qquad W_n = P_n(W),$$
$$P_n x = 2 \int_0^{2\pi} (\cos n\theta)\varphi(\theta) x d\theta, \quad x \in W.$$

Moreover, from the irreducible representations $\rho_n, n \geq 1$, it follows that the isotropy group $(S^1)_x = \mathbf{Z}_n$ for every $x \in W_n \setminus \{0\}, n = 1, 2, \ldots$.

Example 0.2.2. Representations of $\mathbf{Z}_n$ and their isotypical decomposition.

Regard $\mathbf{Z}_n = \{\xi \in \mathbf{C}; \xi^n = 1\}$ as a finite subgroup of S^1 and denote by γ its generator. All *complex* irreducible representations of $\mathbf{Z}_n$ are one-dimensional and are given by

$$\varphi_m(\gamma^k)z = e^{\frac{2\pi i m k}{n}} z, \quad z \in \mathbf{C},$$
$$k = 0, 1, 2, \ldots, n-1; \; m = 0, 1, 2, \ldots, n-1.$$

Identifying $\mathbf{C}$ with $\mathbf{R}^2$, we obtain the following *real* (not necessarily irreducible) representation of $\mathbf{Z}_n$ on $\mathbf{R}^2$

$$\psi_m(\tfrac{k}{n}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{2mk}{n} & -\sin \frac{2mk}{n} \\ \sin \frac{2mk}{n} & \cos \frac{2mk}{n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$x_1, x_2 \in \mathbf{R}^1, m, k = 0, 1, 2, \ldots, n-1,$$

where $\frac{k}{n} \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \cong \mathbf{Z}_n$. As in Example 0.2.1, for $m > 0$, $(\psi_m, \mathbf{R}^2)$ is equivalent to $(\psi_{-m}, \mathbf{R}^2)$. Notice that *real irreducible* representation of $\mathbf{Z}_n$ is either one-dimensional or two-dimensional. We have the following all the real irreducible representations of $\mathbf{Z}_n$

$$(\rho_0, \mathbf{R}), (\rho_m, \mathbf{R}^2), \; m = 1, 2, \ldots, \tfrac{n-1}{2}, \quad \text{if } n \text{ is odd;}$$
$$(\rho_0, \mathbf{R}), (\rho_{\frac{n}{2}}, \mathbf{R}), (\rho_m, \mathbf{R}^2), \; m = 1, 2, \ldots, \tfrac{n-2}{2}, \quad \text{if } n \text{ is even,}$$

where $(\rho_0, \mathbf{R})$ is the trivial one-dimensional real irreducible representation of $\mathbf{Z}_n$, $(\rho_{\frac{n}{2}}, \mathbf{R})$ is given by $\rho_{\frac{n}{2}}(\frac{k}{n})x = -x, \frac{k}{n} \in \mathbf{Z}_n, x \in \mathbf{R}$, and $\rho_m = \psi_m, m \neq 0, \frac{n}{2}$.

Assume now that $(\rho, \mathbf{R}^N)$ is a finite dimensional orthogonal representation of $\mathbf{Z}_n$. We look for its isotypical decomposition. By Theorem 0.2.7, corresponding to each real irreducible representation ρ_m of $\mathbf{Z}_n$, there is an isotypical component

$\mathbf{R}_m^N$ such that $\oplus_{m=0}^{[\frac{n}{2}]}\mathbf{R}_m^N = \mathbf{R}^N$. (Note that $\mathbf{R}^N$ is N-dimensional). Moreover, $\mathbf{R}_m^N$ can be explicitly given as follows

$$\mathbf{R}_m^N = \ker(\rho(\gamma) - e^{\frac{2\pi i m}{n}} Id), \; m = 0, 1, 2, \ldots, [\tfrac{n}{2}]$$

where Id denotes the $N \times N$ identity matrix and the complex space $\mathbb{C}$ is identified with $\mathbb{R}^2$.

Remark 0.2.3. The (real) isotypical decomposition $\mathbf{R}^N = \oplus_{m=0}^{[\frac{n}{2}]}\mathbf{R}_m^N$ of $\mathbf{R}^N$ has been used by Dylawerski et al. [11] for computing S^1-equivariant degree and by Fiedler [13] for Hopf bifurcation with $\mathbf{Z}_n$-symmetry. This decomposition is slightly different from the *canonical* one. See, for example, Serre [31] for the complex case. For notational convenience, we have allowed the isotypical component to take the trivial vector space $\{0\}$.

There is another real decomposition of $\mathbb{R}^N$ with respect to a representation $\rho : \mathbf{Z}_n \to O(\mathbb{R}^N)$. Let $\mathbb{R}_j^N = \Lambda(\rho(\gamma), \Pi_j)$, the generalized eigenvalue space corresponding to eigenvalues in Π_j, where $\Pi_j = \{\lambda \in \mathbb{C}; \lambda^j = 1 \text{ and } \lambda^r \neq 1 \text{ for } 0 < r < j\}, j \in J := \{j \in \mathbb{N} : j|n\}$. It follows that $\mathbb{R}^N = \bigoplus_{j\in J}\mathbb{R}_j^N$. This decomposition has been used by Erbe, Geba and Krawcewicz [12] for computing $\mathbf{Z}_n$-equivariant index. It is not difficult to see that this decomposition is different from the isotypical decomposition (the latter is finer). Two decompositions coincide if and only if every set Π_j, $j > \left[[\frac{n}{2}] - \frac{n}{2}\right] + 2$, includes at most one pair of eigenvalues of $\rho(\gamma)$.

0.3. An equivariant bijection theorem

In this section, we prove an equivariant version of the bijection theorem due to Nussbaum [25,26], which will be used to extend the G-degree to equivariant condensing fields.

Let $\mathbb{E}$ denote a real Banach space. Throughout this section, we assume that $\mathbb{E}$ is an *isometric representation* of a compact Lie group G. By $\mathbb{E}^{\infty+n}$, $n = 0, 1, 2, \ldots$, we shall denote the Banach space $\mathbb{E} \times \mathbb{R}^n$ equipped with the norm $\|(x, \lambda)\| = \max\{\|x\|, |\lambda|\}$, $(x, \lambda) \in \mathbb{E} \times \mathbb{R}^n$. The natural projection of $\mathbb{E}^{\infty+n}$ onto $\mathbb{E}$, $(x, \lambda) \mapsto x$, will be denoted by π. The action of G on the space $\mathbb{E}$ induces an action on $\mathbb{E}^{\infty+n}$ given by $g(x, \lambda) = (gx, \lambda)$, $g \in G$, $(x, \lambda) \in \mathbb{E}^{\infty+n}$, and the operator $\pi : \mathbb{E}^{\infty+n} \to \mathbb{E}$ is then G-equivariant.

To state an equivariant version of the Bijection Theorem of Nussbaum [25, 26] (see also [7, 20]), we recall the notion of an abstract measure of noncompactness on the spaces $\mathbb{E}^{\infty+n}$. By $\mathcal{M}_n$ we denote the class of all bounded subsets of $\mathbb{E}^{\infty+n}$, $n = 0, 1, 2, \ldots$, and we put $\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}_n$. A function $\mu : \mathcal{M} \to [0, \infty)$ is called a *measure of noncompactness* if the following conditions are satisfied.

(μ - 1) $\quad \mu(X) = 0 \iff \overline{X}$ is compact;

(μ - 2) $\quad \mu(X) = \mu(\overline{X})$;

(μ - 3) $\quad X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;

(μ - 4) $\quad \mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$;

(μ - 5) $\quad \mu(\operatorname{Conv} X) = \mu(X)$;

(μ - 6) $\quad \mu(\alpha X) = |\alpha|\mu(X), \quad \alpha \in \mathbb{R}$;

(μ - 7) $\quad \mu(X + Y) \leq \mu(X) + \mu(Y)$;

(μ - 8) $\quad \mu(\pi(X)) = \mu(X)$,

where, $X, Y \in \mathcal{M}$ and $\operatorname{Conv} X$ denotes the convex hull of X. We remark that the above set of axioms is not minimal. A classical example of a measure of noncompactness is the *Kuratowski measure of noncompactness* $\alpha : \mathcal{M} \to [0, \infty)$ which is defined by

$$\alpha(X) = \inf\{\epsilon > 0 : \text{there exists } X_1, X_2, \ldots, X_q \subset X \text{such that}$$
$$X = X_1 \cup \cdots \cup X_q, \text{ and } \operatorname{diam}(X_j) \leq \epsilon, \text{ for } j = 1, \ldots, q\}$$

For more properties of measure of noncompactness and other examples, we refer to Banaś and Goebel [3], Krawcewicz [20] and Nussbaum [25,26].

Let $X \subset \mathbb{E}^{\infty+n}$ and $F : X \to \mathbb{E}^{\infty+m}$ be a continuous map such that F sends bounded sets into bounded sets, where m and n are nonnegative integers. We say that F is a *μ-Lipschitzian map with a constant* $k \geq 0$ if $\mu(F(A)) \leq k\mu(A)$ for all bounded $A \subset X$. If $\mu(F(X)) = 0$, F is called *compact*; if $k = 0$, we call F a *completely continuous* map; and if $k < 1$, we say that F is a *Darbo* map (or *μ-set contraction*). We say that F is a *condensing* map if F is μ-Lipschitzian with $k = 1$ and $\mu(F(A)) < \mu(A)$ for all bounded subsets $A \subset X$ with $\mu(A) > 0$. In what follows, we will denote by Comp, Darbo and Cond the classes of compact, Darbo and condensing maps, respectively. Clearly, Comp $\subset$ Darbo $\subset$ Cond.

Let C be a fixed invariant closed convex subset of $\mathbb{E}$ and (A, X) be a pair of closed bounded invariant subsets of $\mathbb{E}^{\infty+n}$ with $A \subseteq X$. We use $\mathcal{A}^G(X, A)$ to denote the class of all G-maps $F : X \to C$ such that

(i) $F \in \mathcal{A}$,

(ii) $\pi(z) \neq F(z)$ for all $z \in A$,

where $\mathcal{A}$ stands for any of the classes: Comp, Darbo, Cond. Consequently, we get the following sequence of inclusions

$$\operatorname{Comp}^G(X, A) \subset \operatorname{Darbo}^G(X, A) \subset \operatorname{Cond}^G(X, A).$$

A G-map $H : X \times [0,1] \to C$ is called a ***homotopy*** in $\mathcal{A}^G(X,A)$ if $H \in \mathcal{A}^G(X \times [0,1], A \times [0,1])$. Evidently, if H is a homotopy in $\mathcal{A}^G(X,A)$, then $H_t := H(\cdot,t) \in \mathcal{A}^G(X,A)$ for all $t \in [0,1]$. We say that $F_0, F_1 \in \mathcal{A}^G(X,A)$ are ***homotopic*** in $\mathcal{A}^G(X,A)$, denoted by $F_0 \sim F_1$ in $\mathcal{A}^G(X,A)$, if there is a homotopy H in $\mathcal{A}^G(X,A)$ such that $F_i = H_i$, for $i = 0,1$. It can be verified that the homotopy relation " $\sim$ " is an equivalence relation in $\mathcal{A}^G(X,A)$ ([20]). In what follows, we will denote by $\mathcal{A}^G[X,A]$ the set of all homotopy classes in $\mathcal{A}^G(X,A)$. Notice that the following inclusions

$$\begin{array}{ccc} \mathrm{Comp}^G(X,A) & \xrightarrow{\ i\ } & \mathrm{Cond}^G(X,A) \\ & \searrow^{i_1} & \uparrow{\scriptstyle i_2} \\ & & \mathrm{Darbo}^G(X,A) \end{array}$$

induce the maps making the following diagram commutative

$$\begin{array}{ccc} \mathrm{Comp}^G[X,A] & \xrightarrow{\ i_*\ } & \mathrm{Cond}^G[X,A] \\ & \searrow^{(i_1)_*} & \uparrow{\scriptstyle (i_2)_*} \\ & & \mathrm{Darbo}^G[X,A] \end{array}$$

Moreover, we remark that if $F \in \mathcal{A}^G(X,A)$, then $\pi - F$ is a proper map ([20, 25, 26]), and we have that

$$\epsilon \triangleq \inf\{\|\pi(z) - F(z)\| : z \in A\} > 0,$$

Thus, for every G-map $F_1 : X \to C$, if $F_1 \in \mathcal{A}$ and $\|F_1(z) - F(z)\| < \epsilon$ for any $z \in A$, we have that $F_1 \in \mathcal{A}^G(X,A)$ and $F_1 \sim F$ in $\mathcal{A}^G(X,A)$.

Now we are in the position to state the main result of this section.

Theorem 0.3.1. (EQUIVARIANT BIJECTION THEOREM) *For any closed bounded invariant pair* (X, A) *with* $\emptyset \neq A \subset X \subseteq \mathbb{E}^{\infty+n}$, *the induced map*

$$i_*: Comp^G[X, A] \to Cond^G[X, A]$$

is bijective.

Proof. By the commutativity of the above diagrams, it suffices to show that $(i_1)_*$ and $(i_2)_*$ are both bijective. We begin with proving the surjectivity of $(i_1)_*$. Let $F \in \text{Darbo}^G(X, A)$ and $0 \leq k < 1$ be a constant such that F is a μ-Lipschitzian G-map with the constant k. We need to show that F is homotopic in $\text{Darbo}^G(X, A)$ to a compact G-map. For this purpose, we consider the following sequence of subsets of C

$$Q_1 \triangleq \overline{\text{Conv}\,\{F(X), \{0\}\}};$$
$$Q_{i+1} \triangleq \overline{\text{Conv}\,\{F(X \cap \pi^{-1}(Q_i), \{0\}\}}, \quad i = 1, 2, 3, \ldots$$

Obviously, $Q_1 \supset Q_2 \supset \ldots$, and every Q_i is closed bounded and invariant. Note that $\mu(Q_{i+1}) = \mu(F(X \cap \pi^{-1}(Q_i)) \leq k\mu(X \cap \pi^{-1}(Q_i)) \leq k\mu(Q_i)$ for all $i = 1, 2, \ldots$. It follows that $\mu(Q_i) \leq k^i\mu(X)$, $i = 1, 2, \ldots$, thus $\lim_{i\to\infty} \mu(Q_i) = 0$. Let $Q = \bigcap_{i=1}^{\infty} Q_i$. Then Q is invariant, $\mu(Q) = 0$ and $F(X \cap \pi^{-1}(Q)) \subset Q$.

Since $0 \in Q$, Q must be a nonempty convex compact set. By Theorem 0.1.12, there exists a compact G-map $R : C \to Q$ which extends $Id|_Q$. Define $H(z,t) = (1-t)F(z) + tR(F(z))$, $z \in X$, $t \in [0,1]$. Clearly, H is a Darbo G-map. To prove $F \sim R \circ F$ in $\text{Darbo}^G(X, A)$, we need to show that $\pi(z) \neq H(z,t)$ for all $z \in A$ and $t \in [0,1]$. Suppose, to the contrary, that for some $(z_0, t_0) \in A \times [0,1]$, $\pi(z_0) = H(z_0, t_0)$. Since $F(z_0) \in Q_1$, $R(F(z_0)) \in Q \subseteq Q_1$. The convexity of Q_1 implies that $\pi(z_0) \in Q_1$. Thus $z_0 \in X \cap \pi^{-1}(Q_1)$, which gives that $F(z_0) \in Q_2$ and $\pi(z_0) \in Q_2$. Inductively, we have $\pi(z_0) \in Q$ and $F(z_0) \in Q$. Since $R|_Q = Id|_Q$, we have

$$\begin{aligned}\pi(z_0) &= (1-t_0)F(z_0) + tR(F(z_0)) \\ &= (1-t_0)F(z_0) + t_0F(z_0) = F(z_0).\end{aligned}$$

This leads to a contradiction to the assumption $F \in \mathrm{Darbo}^G(X,A)$. So F is homotopic to the compact map $R \circ F$ via the homotopy H in $\mathrm{Darbo}^G(X,A)$. This proves that $(i_1)_*$ is surjective.

We next show that $(i_1)_*$ is injective. Let $F_0, F_1 \in \mathrm{Comp}^G(X,A)$ and $H \in \mathrm{Darbo}^G(X\times[0,1], A\times[0,1])$ be a G-homotopy from F_0 to F_1. We want to show that $F_0 \sim F_1$ in $\mathrm{Comp}^G(X,A)$. In fact, by the surjectivity of $(i_1)_*$, for the pair $A\times[0,1] \subseteq X\times[0,1]$, we have a compact G-map $\overline{H} : X\times[0,1] \to C$ and a G-homotopy $\widetilde{H} \in \mathrm{Darbo}^G(X\times[0,1]\times[0,1], A\times[0,1]\times[0,1])$ from H to $\overline{H}$ of the form

$$\widetilde{H}(z,s,t) = (1-t)H(z,t) + t\overline{H}(z,s),$$

where $(z,s,t) \in X\times[0,1]\times[0,1]$. Since $H|_{X\times\{0,1\}}$ are compact G-maps, $\widetilde{H}|_{X\times\{0,1\}\times[0,1]}$ are compact. Also, $\widetilde{H}|_{X\times[0,1]\times\{1\}} = \overline{H}$ is an equivariant compact map. We now define $H^* : X\times[0,1] \to C$ by

$$H^*(z,t) = \begin{cases} \widetilde{H}(z,0,3t) & \text{if } t \in [0,\frac{1}{3}],\ z \in X, \\ \widetilde{H}(z,3t-1,1) & \text{if } t \in [\frac{1}{3},\frac{2}{3}],\ z \in X, \\ \widetilde{H}(z,1,3-3t) & \text{if } t \in [\frac{2}{3},1],\ z \in X. \end{cases}$$

From the above construction, it follows that H^* is a G-homotopy from F_0 to F_1 in $\mathrm{Comp}^G(X,A)$. This proves the injectivity of $(i_1)_*$.

We now prove $(i_2)_* : \mathrm{Darbo}\,[X,A] \to \mathrm{Cond}\,[X,A]$ is surjective. Let $F \in \mathrm{Cond}^G(X,A)$ and $\varepsilon = \inf\{\|\pi(z) - F(z)\| : z \in A\} > 0$. Since F is bounded,

$\|F(A)\| \triangleq \sup\{\|F(z)\| : z \in A\} < \infty$. Choose a constant $k > 0$ such that $1 - \varepsilon / \|F(A)\| < k < 1$. Then kF is a Darbo G-map, and for all $z \in A$,

$$\|kF(z) - F(z)\| \leq (1-k)\,\|F(A)\| < \varepsilon.$$

Define $H : X \times [0,1] \to C$ by

$$H(z,t) = (1-t)F(z) + tkF(z), \quad (z,t) \in X \times [0,1].$$

Clearly, $H \in \mathrm{Cond}^G(X, A)$. Moreover, if $H(z_0, t_0) = \pi(z_0)$ for some $(z_0, t_0) \in A \times [0,1]$, then

$$\begin{aligned}\varepsilon &= \inf\{\|\pi(z) - F(z)\| : z \in A\} \leq \|\pi(z_0) - F(z_0)\| \\ &= \|H(z_0,t_0) - F(z_0)\| = (1-k)t_0\,\|F(z_0)\| \leq (1-k)\,\|F(A)\|,\end{aligned}$$

contradicting the choice of k. So, F is G-homotopic to kF in $\mathrm{Cond}^G(X, A)$. This shows the surjectivity of $(i_1)_*$.

The injectivity of $(i_2)_*$ can be proved by an argument similar to that for $(i_1)_*$, replacing "compact" and "Darbo" by "Darbo" and "condensing," respectively. This justifies the bijectivity of $(i_2)_*$ and completes the proof.

0.4. Intersection numbers of bundle sections

In this section, we recall the definitions and properties of the intersection number of a smooth vector bundle section. As in the orginal definition of the equivariant degree [15], we will also need the notion of intersection number in order to develop a simpler and more applicable formula for the computations of G-degree.

Throughout this section, we assume that $p : E \to M$ is a smooth n-dimensional vector bundle over a smooth n-dimensional manifold M. By a section of E we shall mean a continuous map $s : M \to E$ such that $p(s(x)) = x$ for all $x \in M$. The zero section of the bundle $p : E \to M$ will be denoted by $z : M \to E$. We will often identify M with $z(M)$. For a given section $s : M \to E$, we use $s(x) = 0$, $x \in M$, to mean $s(x) = z(x)$.

Recall that for a differentiable (C^1-)section $s : M \to E$ and $K \subset M$ a subset, s is called ***transversal to M along K***, denoted by $s \pitchfork_K M$, if

$$T_{s(x)}E = T_{s(x)}M + T_x s(T_x M) \quad \text{for all} \quad x \in K \cap s^{-1}(M).$$

Definition 0.4.1. Let $s : M \to E$ be a C^1-section. We say that $x \in M$ is a ***regular zero*** of s if $s(x) = 0$ and $D_v s(x) : T_x M \to E_x$ defined by the composition

$$T_x M \xrightarrow{T_x s} T_x E \xrightarrow{q} E_x$$

is an isomorphism of vector spaces, where q is the natural quotient map.

By definition, if $s \pitchfork_K M$, then every zero of s in K is regular.

Suppose now that $\Omega \subset M$ is an open subset and $s : \Omega \to E$ a section of the restricted bundle $E|_\Omega$. We say that s is an ***Ω-admissible section*** if $s^{-1}(M)$ is a compact subset of Ω. We first assume that E, as a manifold, is orientable and let $s : \Omega \to E$ be an Ω-admissible section such that $s \pitchfork_{\overline{\Omega}} M$. Then the set $s^{-1}(M)$ is finite and consists of regular zeros of s only. Since E is orientable, for every given $x \in s^{-1}(M)$ a chosen orientation of $T_x M$ determines an orientation of E_x so that the identification $T_{x(x)}E \cong T_xM \oplus E_x$ preserves the orientations. Define for each $x \in s^{-1}(M)$ an integer

$$n(s,x) = \begin{cases} +1, & \text{if } D_v s(x) \text{ preserves the orientations,} \\ -1, & \text{if } D_v s(x) \text{ reverses the orientations.} \end{cases}$$

Using the fact that an orientation of a vector space is a choice of an ordered basis, we can verify that the above definition of $n(s,x)$ does not depend on the choice of the orientation of T_xM.

Definition 0.4.2. A continuous map $h : \Omega \times [0,1] \to E$ is called an *Ω-admissible homotopy* if $h_t : \Omega \to E$ for every $t \in [0,1]$ is a section of $E|_\Omega$ and $h^{-1}(M)$ is a compact subset of $\Omega \times [0,1]$, where $h_t(x) = h(x,t)$ for all $(x,t) \in \Omega \times [0,1]$.

Let $s : \Omega \to E$ be an Ω-admissible (continuous) section. We choose an open subset $U \subset \Omega$ such that $U \supset s^{-1}(M)$ has compact closure. Note that the set of all sections of E transversal to M is dense in the set of all continuous sections equipped with the open-compact topology (see [15] and [17] for proofs). We can choose a differentiable section $\bar{s} : U \to E$ such that $\bar{s} \pitchfork_{\overline{U}} M$ and

$$\sup_{x \in \overline{U}} \|\bar{s}(x) - s(x)\|_x < \inf_{x \in \partial U} \|s(x)\|_x.$$

We will call such a section $\bar{s}$ the regular approximation of s with respect to U. It then follows that a number

$$\chi(s,\Omega) = \sum_{x \in \bar{s}^{-1}(M) \cap \Omega} n(\bar{s},x)$$

is well-defined. Moreover, it can be shown (see [15]) that $\chi(s,U)$ does not depend on the choice of the transversal section $\bar{s} : U \to E$ and it enjoys all the properties of a standard topological degree.

Definition 0.4.3. Let $s : \Omega \to E$ be an Ω-admissible section. Choose an open subset $U \subset \Omega$ such that $U \supset s^{-1}(M)$ has compact closure and put

$$\chi(s, \Omega) := \chi(s, U).$$

We call $\chi(s, \Omega)$ the *intersection number* of s with respect to Ω.

By using the excision property of $\chi(s, \Omega)$ (see [15]), it can be shown that the above intersection number $\chi(s, \Omega)$ does not depend cn the choice of the open subset U.

When E is not orientable, we give the definition of intersection number as follows.

Definition 0.4.4. Suppose that $s : \Omega \to E$ is an Ω-admissible section, where E is not orientable. We define the *modulo two intersection number* of s with respect to Ω as an element of the group $\mathbf{Z}_2 = \{0, 1\}$ given by

$$\chi_2(s, \Omega) = \#(\bar{s}^{-1}(M) \cap U)(\text{mod } 2),$$

where $\#(A)$ denotes the cardinality of a finite set A, U is the same as in Definition 0.4.3 and $\bar{s} : \Omega \to E$ is a regular approximation of s with respect to U.

It can also be verified, by using the excision and homotopy property (see [15]), that the above definition does not depend on the choice of the open subset U and the regular approximation $\bar{s}$.

We summarize the most important properties of the intersection number in the following theorem. Its proof follows directly from the definitions and the corresponding properties of the intersection number for open set Ω with compact closure.

Theorem 0.4.1. *The intersection number of bundle sections satisfies the following properties:*

(i) (Existence). If $s : \Omega \to E$ is an Ω-admissible section and $\chi(s, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $s(x) = 0$;

(ii) (Excision). If $\Omega_1 \subseteq \Omega$ is an open subset and $s : \Omega \to E$ is an Ω-admissible section such that $s(x) \neq 0$ for $x \in \Omega \backslash \Omega_1$, then $\chi(s, \Omega) = \chi(s, \Omega_1)$;

(iii) (Additivity). If Ω_1, Ω_2 are two disjoint open subsets of Ω and $s : \Omega \to E$ is an Ω-admissible section such that $s(x) \neq 0$ for $x \in \Omega \backslash (\Omega_1 \cup \Omega_2)$, then

$$\chi(s, \Omega) = \chi(s, \Omega_1) + \chi(s, \Omega_2);$$

(iv) (Homotopy Invariance). If $h : \Omega \times [0, 1] \to E$ is an Ω-admissible homotopy, then $\chi(h_t, \Omega)$ is a constant independent of $t \in [0, 1]$.

0.5. Homotopical properties of $GL^G_{\text{cond}}(\mathbb{E})$

Let $\mathbb{E}$ be a real or complex Banach representation of a compact Lie group G. We denote by $GL^G(\mathbb{E})$ (resp. $Pr^G(\mathbb{E})$) the set of all equivariant linear invertible operators (resp. all equivariant projections) on $\mathbb{E}$ and by $L^G_{\text{cond}}(\mathbb{E})$ the set of linear equivariant operators of the form $Id + A$, where A is a linear equivariant condensing operator on $\mathbb{E}$ and Id is the identity operator. Set

$$GL^G_{\text{cond}} := GL(\mathbb{E}) \cap L^G_{\text{cond}}(\mathbb{E}).$$

We are going to study homotopical properties of $GL^G_{\mathrm{cond}}(\mathbb{E})$ in this section.

We begin with several lemmas.

Lemma 0.5.1. *Let (X, A) be a pair of compact trivial G-spaces with $A \subset X$. Given a continuous map $a : (X, A) \to (L^G_{\mathrm{cond}}(\mathbb{E}),\ GL^G_{\mathrm{cond}}(\mathbb{E}))$, there exist a closed invariant subspace $\mathbb{E}^0$ of $\mathbb{E}$ and a continuous map $p : X \to Pr^G(\mathbb{E})$ such that*

(i) $\mathbb{E}^0$ *is of finite codimension;*

(ii) $\mathbb{E}^0 \cap \ker a(x) = \{0\}$ *for all* $x \in X$;

(iii) $p(x)\mathbb{E} = a(x)\mathbb{E}$ *for all* $x \in X$;

(iv) $p(x) = a(x) \circ Q \circ a^{-1}(x)$ *for all* $x \in A$, *where* Q *is an equivariant projection onto* $\mathbb{E}^0$;

(v) $E(X) = \{(v, x) \in \mathbb{E} \times X;\ v \in \ker p(x)\}$ *is equipped with a structure of a* G*-vector bundle over* X.

Proof. The proof of Lemma A2.2 in Krawcewicz [20] can be carried over to this equivariant case. Therefore we omit it.

Lemma 0.5.2. *Let X be a compact space and*

$$a : (X \times [0,1], X \times \{1\}) \to (L^G_{\mathrm{cond}}(\mathbb{E}),\ GL^G_{\mathrm{cond}}(\mathbb{E}))$$

be continuous. Then there exists a closed invariant subspace $\mathbb{E}^0 \subset \mathbb{E}$ of finite codimension such that $\mathbb{E}^0 \cap \ker a(x,t) = \{0\}$ for all $(x,t) \in X \times [0,1]$ together with a continuous map $b : X \times [0,1] \to GL^G_{\mathrm{cond}}(\mathbb{E})$ satisfying

(i) $b(x,t)|_{\mathbb{E}^0} = a(x,t)|_{\mathbb{E}^0}$ *for all* $(x,t) \in X \times [0,1]$;

(ii) $b|_{X\times\{1\}} = a|_{X\times\{1\}}$.

Proof. By Lemma 0.5.1, there exist an invariant closed subspace $\mathbb{E}^0$ of finite codimension and a continuous map $p : X \times [0,1] \to Pr^G(\mathbb{E})$ such that $p(x,t)\mathbb{E} = a(x,t)\mathbb{E}^0$ for all $(x,t) \in X \times [0,1]$ and $p(x,t) = a(x,t) \circ Q \circ a^{-1}(x,t)$ for all $(x,t) \in X \times \{1\}$. Consider the G-vector bundle

$$E(X \times [0,1]) = \{(v,x,t) \in \mathbb{E} \times X \times [0,1];\ v \in \ker p(x,t)\}.$$

We claim that the restricted bundle $E(X \times \{1\})$ over $X \times \{1\}$ is trivial. Indeed, let $\mathbb{L}_0 = (I-Q)\mathbb{E}$. It follows that for all $(x,t) \in X \times \{1\}$, $a(x,t)\mathbb{L}_0 = a(x,t)(I-Q)\mathbb{E} = (I - p(x,t))\mathbb{E} = \ker p(x,t)$. Therefore, $\Psi : E(X \times \{1\}) \to \mathbb{L}_0 \times X \times \{1\}$ defined by

$$\Psi(v,x,1) = (a^{-1}(x,1)v, x, 1), \quad v \in \ker p(x,1),\ (x,1) \in X \times \{1\}$$

gives a trivialization of $E(X \times \{1\})$.

In what follows, we show that the G-vector bundle $E(X \times [0,1])$ itself is also trivial. To see this, we consider the following commutative diagram of G-vector bundles

$$\begin{array}{ccccc}
\rho^* \circ i^*(E) & \longrightarrow & i^*(E) & \longrightarrow & E(X \times [0,1]) \\
\pi_1 \downarrow & & \pi_2 \downarrow & & \downarrow \pi_3 \\
X \times [0,1] & \xrightarrow{\ \rho\ } & X \times \{1\} & \xrightarrow{\ i\ } & X \times [0,1]
\end{array}$$

where $i : X \times \{1\} \to X \times [0,1]$ is the inclusion, $\rho : X \times [0,1] \to X \times \{1\}$ is the projection and $i^*(E)$ and $\rho^* \circ i^*(E)$ are the pull-backs of $E(X \times [0,1])$ and $i^*(E)$, respectively. Since $E(X \times \{1\})$ is trivial, $i^*(E)$ is also trivial, which implies further that $\rho^* \circ i^*(E)$ is a trivial G-vector bundle. Now the triviality of $E(X \times [0,1])$ follows from Proposition 0.1.4 by using the fact that the composition $\rho \circ i : X \times [0,1] \to X \times [0,1]$ is homotopic to the identity map.

Moreover, we can express explicitly $\rho^* \circ i^*(E)$ and $i^*(E)$. From the definition of pull-backs, we have

$$\rho^* \circ i^*(E) = \{(v,x,t) \in \mathbb{E} \times X \times [0,1];\ v \in \ker p(x,1)\},$$
$$i^*(E) = \{(v,x,t) \in \mathbb{E} \times X \times \{1\};\ v \in \ker p(x,1)\}.$$

Note that $\ker p(x,1) = a(x,1)\mathbb{L}_0$. We see that $i^*(E) = E(X \times \{1\})$. Now $\rho^* \circ i^*(E) \cong E(X \times [0,1])$ implies that there exists a G-map $f : \rho^* \circ i^*(E) \to E(X \times [0,1])$ such that the following diagram commutes

$$\begin{array}{ccc} \rho^* \circ i^*(E) & \xrightarrow{f} & E(X \times [0,1]) \\ \pi_1 \big\downarrow & & \big\downarrow \pi_3 \\ X \times [0,1] & \xrightarrow{\text{Id}} & X \times [0,1]. \end{array}$$

This gives rise to a G-isomorphism

$$A(x,t) := f(\cdot, x, t) : \pi_1^{-1}(x,t) \to \pi_3^{-1}(x,t)$$

for any $(x,t) \in X \times [0,1]$. A is continuous on $X \times [0,1]$ since f is. Moreover, from the expressions of $\rho^* \circ i^*(E)$ and $E(X \times [0,1])$, $A(x,1) = Id$, $\pi_1^{-1}(x,t) = \ker p(x,1) = a(x,1)\mathbb{L}_0$ and $\pi_3^{-1}(x,t) = \ker p(x,t)$. Therefore

$$A(x,t) : a(x,1)\mathbb{L}_0 \to \ker p(x,t)$$

which gives a G-map $S : \mathbb{L}_0 \times X \times [0,1] \to E(X \times [0,1])$ as follows

$$S(v,x,t) = (A(x,t)a(x,1)v, x, t), \quad (v,x,t) \in \mathbb{L}_0 \times X \times [0,1].$$

Clearly, S is an equivariant G-vector bundle isomorphism and

$$S|_{\mathbb{L}_0 \times X \times \{1\}} \circ \Psi = Id|_{E(X \times \{1\})}.$$

We now define a continuous map $\widetilde{S} : X \times [0,1] \to L^G_{\text{cond}}(\mathbb{L}_0, \mathbb{E})$ by

$$\widetilde{s}(x,t)v = S(v,x,t), \quad (v,x,t) \in \mathbb{L}_0 \times X \times [0,1].$$

It follows that for each $(x,t) \in X \times [0,1]$, $\widetilde{S}(x,t) : \mathbb{L}_0 \to \ker p(x,t)$ is an equivariant isomorphism. Let $b : X \times [0,1] \to L^G(\mathbb{E})$ be defined by

$$b(x,t) := a(x,t) \circ Q + \widetilde{S}(x,t) \circ (Id - Q), \; (x,t) \in X \times [0,1].$$

As composition of continuous maps, b is continuous. Since $a(x,t) \circ Q$ is invertible and $\widetilde{S}(x,t)$ is a finite dimensional invertible map, $b(x,t) \in GL^G_{\text{cond}}(\mathbb{E})$. Moreover, $b(x,t)|_{\mathbb{E}^0} = a(x,t)|_{\mathbb{E}^0}$ since $Q(\mathbb{E}) = \mathbb{E}^0$ and $(Id - Q)|_{\mathbb{E}^0} = 0$, and $b(x,1) = a(x,1)$ follows from $S|_{\mathbb{L}_0 \times X \times \{1\}} \circ \Psi = Id|_{E(X \times \{1\})}$. This completes the proof.

We now state and prove the main result of this section.

Theorem 0.5.3. *Let $A \subseteq X$ be a pair of compact trivial G-metric spaces. If $a : (X,\ A) \to (L^G_{\text{cond}}(\mathbb{E}),\ GL^G_{\text{cond}}(\mathbb{E})$ is continuous, then there exist a closed invariant subspace $\mathbb{E}^0$ and $\mathbb{E}_0$ of $\mathbb{E}$, and a continuous map $H : (X \times [0,1],\ A \times [0,1]) \to (L_{\text{cond}}(\mathbb{E}),\ GL^G_{\text{cond}}(\mathbb{E}))$ such that*

(i) $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}_0, \quad \dim \mathbb{E}_0 < \infty;$

(ii) $H(x,0)|_{\mathbb{E}^0} = Id|_{\mathbb{E}^0}, \quad x \in X;$

(iii) $H(x,1)|_{\mathbb{E}_0} : \mathbb{E}_0 \longrightarrow \mathbb{E}_0;$

(iv) $H(x,0) = a(x), \quad x \in X.$

Proof. Let $a(x) = Id - T(x)$ for $x \in X$, where $T(x)$ is a linear condensing G-map. We define a mapping $b : X \times [0,1] \to L^G_{\text{cond}}(\mathbb{E})$ by

$$b(x,t)v = v - tT(x)v, \quad (x,t) \in X \times [0,1], v \in \mathbb{E}.$$

Then $b : (A \times [0,1], A \times \{1\}) \to (L^G_{\text{cond}}(\mathbb{E}), GL^G_{\text{cond}}(\mathbb{E}))$ is continuous. From the proof of Lemma 0.5.2, there exist closed invariant subspaces $\mathbb{E}^0$ and $\mathbb{E}_0$ of $\mathbb{E}$ satisfying (i) together with a continuous map $h : A \times [0,1] \to GL^G_{\text{cond}}(\mathbb{E})$ such that

$$h(x,t) = b(x,t) \circ Q + \widetilde{S}(x,t) \circ (Id - Q), \quad (x,t) \in A \times [0,1]$$

with $h(x,1) = a(x)$, $x \in A$, where $\widetilde{S} : A \times [0,1] \to L^G(\mathbb{L}_0, \mathbb{E})$ and $\mathbb{L}_0 = (Id - Q)\mathbb{E}$. By Theorem 0.1.12, we extend $\widetilde{S}$ to $X \times [0,1]$ such that $\widetilde{S}(x,1) = a(x)$, for $x \in A$ and put

$$H(x,t) = b(x,t) \circ Q + \widetilde{S}(x,t))Id - Q), \ (x,t) \in X \times [0,1].$$

Note that $b(x,0) = Id$ and $Q(\mathbb{E}) = \mathbb{E}^0$. The conclusions (ii) and (iii) follow. Finally, since $(Id - Q)$ is compact, $H(x,t) \in L^G_{\text{cond}}(\mathbb{E}), (x,t) \in X \times [0,1]$. Therefore H maps $(X \times [0,1], \ A \times [0,1])$ into $(L^G_{\text{cond}}(\mathbb{E}), \ GL^G_{\text{cond}}(\mathbb{E}))$. This completes the proof.

In what follows, we consider the homotopy groups of $GL^G_{\text{cond}}(\mathbb{E})$. We first use all possible finite dimensional subrepresentations of G in $\mathbb{E}$ to obtain a ***direct set*** $\Lambda = \{\alpha\}$ such that each $\alpha \in \Lambda$ corresponds to a pair of invariant subspaces $\{(\mathbb{E}_\alpha, \mathbb{E}^\alpha)\}$ of $\mathbb{E}$ satisfying

(i) $\beta \geq \alpha; \ \beta, \alpha \in \Lambda \iff \mathbb{E}_\alpha \subseteq \mathbb{E}_\beta;$

(ii) $\dim \mathbb{E}_\alpha < \infty$,

(iii) $\mathbb{E}_\alpha \oplus \mathbb{E}^\alpha = \mathbb{E}$;

(iv) $\mathbb{E}^\beta \subseteq \mathbb{E}^\alpha$, for every $\beta \geq \alpha$.

(v) for every finite dimensional invariant subspace V of $\mathbb{E}$, there exists an $\alpha \in \Lambda$ such that $\mathbb{E}_\alpha \cong V$.

We now define imbeddings, for $\beta \geq \alpha$, $i_{\alpha,\beta} : GL^G(\mathbb{E}_\alpha) \to GL^G(\mathbb{E}_\beta)$ by

$$i_{\alpha,\beta}(T)(v_1, v_2) = (Tv_1, v_2),$$

where $T \in GL^G(\mathbb{E}_\alpha)$, $(v_1, v_2) \in \mathbb{E}_\alpha \oplus \mathbb{E}^\perp$ and $\mathbb{E}^\perp$ denotes the orthogonal complement of $\mathbb{E}_\alpha$ in $\mathbb{E}_\beta$. Let $i_\alpha : GL^G(\mathbb{E}_\alpha) \longrightarrow GL^G_{\text{cond}}(\mathbb{E})$ be defined by

$$i_\alpha(T)(x_1, x_2) = (Tx_1, x_2),$$

where $T \in GL^G(\mathbb{E}_\alpha)$ and $(x_1, x_2) \in \mathbb{E}_\alpha \oplus \mathbb{E}^\alpha$. Therefore, taking the direct limit, we have the following well-defined inclusion

$$i : \lim_\alpha GL^G(\mathbb{E}_\alpha) \longrightarrow GL^G_{\text{cond}}(\mathbb{E})$$

where $\lim_\alpha GL^G(\mathbb{E}_\alpha) = \bigcup_{\alpha \in \Lambda} GL^G(\mathbb{E}_\alpha)$ is a topological space with the finest topology such that every inclusion $GL^G(\mathbb{E}_\alpha) \hookrightarrow \lim_\alpha GL^G(\mathbb{E}_\alpha)$ is continuous (see [16, 32]).

Using Lemma 0.5.2, we can also prove the following important result.

Theorem 0.5.4. *Let* $A \subseteq X$ *be a pair of compact metric spaces. If* $a : (X,\ A) \to (GL^G_{\text{cond}}(\mathbb{E}),\ \lim_\alpha GL^G(\mathbb{E}_\alpha))$ *is continuous, then there exist subspaces* $\mathbb{E}^0$ *and* $\mathbb{E}_0$ *of* $\mathbb{E}$, *and a homotopy* $H : (X \times [0,1],\ A \times [0,1]) \to (GL^G_{\text{cond}}(\mathbb{E}),\ \lim_\alpha GL^G(\mathbb{E}_\alpha))$ *such that*

(i) $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}_0, \quad \dim \mathbb{E}_0 < \infty;$

(ii) $H(x,1)|_{\mathbb{E}^0} = Id|_{\mathbb{E}^0}, \quad x \in X;$

(iii) $H(x,0) = a(x), \quad x \in X;$

(iv) $H(x,1)|_{\mathbb{E}_0} : \mathbb{E}_0 \longrightarrow \mathbb{E}_0.$

Proof. Define a map $b : (X \times [0,1],\ X \times \{1\}) \to (L^G_{\text{cond}}(\mathbb{E}),\ GL^G_{\text{cond}}(\mathbb{E}))$ by

$$b(x,t)v = (1-t)v + ta(x)v, \quad (x,t,v) \in X \times [0,1] \times \mathbb{E}.$$

By Lemma 0.5.2, there exist a decomposition $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}_0$ and a homotopy $h : X \times [0,1] \to GL^G_{\text{cond}}(\mathbb{E})$ such that $h(x,1) = a(x)$, $x \in X$ and $h(x,t)|_{\mathbb{E}^0} = b(x,t)|_{\mathbb{E}^0}$ for $(x,t) \in X \times [0,1]$. Thus $h(x,0)|_{\mathbb{E}^0} = Id|_{\mathbb{E}^0}$ for $x \in X$. Moreover, h is of the form

$$h(x,t) = b(x,t) \circ Q + \widetilde{S}(x,t) \circ (Id - Q), \quad (x,t) \in X \times [0,1].$$

Note that $a(x) \in \lim_\alpha GL^G(\mathbb{E}_\alpha)$ for $x \in A$ and $\widetilde{S}(x,t)$ is a finite dimensional map. $h(x,t) \in \lim_\alpha GL^G(\mathbb{E}_\alpha)$ for $(x,t) \in A \times [0,1]$. In consequence, h maps $(X \times [0,1],\ A \times [0,1])$ into $(GL^G_{\text{cond}}(\mathbb{E}),\ \lim_\alpha GL^G(\mathbb{E}_\alpha))$ and there exist maps $S : X \to L^G(\mathbb{E}^0, \mathbb{E}_0)$ and $T : X \to GL^G(\mathbb{E}^0, \mathbb{E}_0)$ such that

$$h(x,0)(v^0, v_0) = v^0 + S(x)v^0 + T(x)v_0$$

where $(v^0, v_0) \in \mathbb{E}^0 \oplus \mathbb{E}_0$ and $x \in X$. Now define $g : (X \times [0,1], A \times [0,1]) \to (GL^G_{\text{cond}}(\mathbb{E}), \lim_\alpha GL^G(\mathbb{E}_\alpha))$ by

$$g(x,t)(v^0, v_0) = v^0 + tS(x)v^0 + T(x)v_0$$

for $(x,t) \in X \times [0,1]$ and $(v^0, v_0) \in \mathbb{E}^0 \oplus \mathbb{E}_0$. Then the homotopy $H : (X \times [0,1], A \times [0,1]) \to (GL^G_{\text{cond}}(\mathbb{E}), \lim_\alpha GL^G(\mathbb{E}_\alpha))$ defined by

$$H(x,t)v = \begin{cases} h(x, 1-2t)v, & 0 \le t \le \frac{1}{2} \\ g(x, 2-2t)v, & \frac{1}{2} \le t \le 1 \end{cases}$$

for $(x,t) \in X \times [0,1]$ and $v \in \mathbb{E}$ is the required homotopy.

The next theorem concerns an important property of the inclusion $i : \lim_\alpha GL^G(\mathbb{E}_\alpha) \hookrightarrow GL^G_{\text{cond}}(\mathbb{E})$.

Theorem 0.5.5. *The inclusion i defined above is a weak homotopy equivalence, i.e., the induced homomorphism i_* between their homotopy groups is an isomorphism. In particular, for all $k = 0, 1, 2, \dots$,*

$$\lim_\alpha \pi_k(GL^G(\mathbb{E}_\alpha)) \cong \pi_k(GL^G_{\text{cond}}(\mathbb{E})),$$

where $\pi_k(Y)$ denotes the k-th homotopy group of a topological space Y.

Proof. We first prove that i_* is a monomorphism. Suppose $\alpha : S^k \to \lim_\alpha GL^G(\mathbb{E}_\alpha)$ is a given map. We want to show that α is null-homotopic with a homotopy in $\lim_\alpha GL^G(\mathbb{E}_\alpha)$. Let $H_1 : S^k \times [0,1] \to GL^G_{\text{cond}}(\mathbb{E})$ be a homotopy between α and c, where $c : S^k \to \{Id\} \subseteq GL^G_{\text{cond}}(\mathbb{E})$ is the constant map with the value being the identity of $GL^G_{\text{cond}}(\mathbb{E})$. Observe that $H_1 : (S^k \times [0,1],\ S^k \times \{0,1\}) \to$

$(GL^G_{\mathrm{cond}}(\mathbb{E}), \lim_\alpha GL^G(\mathbb{E}_\alpha))$. By Theorem 0.5.4 there exist a decomposition $\mathbb{E} = E^0 \oplus \mathbb{E}_0$ with $\dim \mathbb{E}_0 < \infty$ and a homotopy

$$\widetilde{H} : (S^k \times [0,1] \times [0,1], S^k \times \{0,1\} \times [0,1]) \longrightarrow (GL^G_{\mathrm{cond}}(\mathbb{E}), \lim_\alpha GL^G(\mathbb{E}_\alpha))$$

such that $\widetilde{H}(x,t,0) = H_1(x,t)$, $\widetilde{H}(x,t,1)|_{\mathbb{E}_0} = Id|_{\mathbb{E}_0}$ and $\widetilde{H}(x,t,1)|_{\mathbb{E}_0} : \mathbb{E}_0 \to \mathbb{E}_0$ for all $(x,t) \in S^k \times [0,1]$. Define now a homotopy $H : S^k \times [0,1] \to \lim_\alpha GL^G(\mathbb{E}_\alpha)$ by

$$H(x,t) = \begin{cases} \widetilde{H}(x,0,3t), & \text{if } 0 \le t \le \frac{1}{3}, \\ \widetilde{H}(x,3t-1,1), & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \widetilde{H}(x,1,3-3t), & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

where $x \in S^k$. Then

$$H(x,0) = \widetilde{H}(x,0,0) = H_1(x,0) = \alpha(x)$$

and

$$H(x,1) = \widetilde{H}(x,1,0) = H_1(x,0) = Id.$$

Therefore, α is homotopic to the constant map c, and hence, i_* is a monomorphism.

Next, we show that i_* is an epimorphism. Let $\beta : S^k \to GL^G_{\mathrm{cond}}(\mathbb{E})$ be a map. We want to find a map $\alpha : S^k \to \lim_\alpha GJ^G(\mathbb{E}_\alpha)$ such that α is homotopic to β in $GL^G_{\mathrm{cond}}(\mathbb{E})$. By Theorem 0.5.4, there exist a decomposition $\mathbb{E} = E^0 \oplus \mathbb{E}_0$ with $\dim \mathbb{E}_0 < \infty$, and a homotopy $H : S^k \times [0,1] \to GL^G_{\mathrm{cond}}(\mathbb{E})$ such that for all $x \in S^k$, $H(x,0) = \beta(x)$, $H(x,1)|_{E_0} = Id|_{E_0}$ and $H(x,1)|_{\mathbb{E}_0} : \mathbb{E}_0 \to \mathbb{E}_0$. Put $\alpha = H(\cdot,1)$. Then $\alpha : S^k \to \lim_\alpha GL^G(\mathbb{E}_\alpha)$ and H is a homotopy between β and α. This completes the proof.

For a topological space Y, we denote by $[S^1, Y]$ the set of homotopy classes of continuous maps $\beta : S^1 \to Y$. When the group $G = \{e\}$, we denote $GL^G_{\text{cond}}(\mathbb{E})$ (resp. $GL^G(\mathbb{E}_\alpha)$) by $GL_{\text{cond}}(\mathbb{E})$ (resp. $GL(\mathbb{E}_\alpha)$).

Proposition 0.5.6. *Let* $\mathbb{E}$ *be a complex Banach space. There exists a bijection*

$$\Delta : [S^1, GL_{\text{cond}}(\mathbb{E})] \longrightarrow \mathbb{Z}.$$

Moreover, if $\dim_{\mathbb{C}} \mathbb{E} < \infty$, *then*

$$\Delta[\beta] = \deg_B(\det{}_{\mathbb{C}}(\beta)),$$

where $\beta : S^1 \to GL_{\mathbb{C}}(\mathbb{E})$, $\det : GL_{\mathbb{C}}(\mathbb{E}) \to C^* = C \setminus \{0\}$ *is the determinant homomorphism and* $\deg_B(\cdot)$ *denotes the Brouwer degree of functions from* S^1 *into* $\mathbb{C}^*$.

Proof. If $\dim_{\mathbb{C}} \mathbb{E} < \infty$, then the statement is a well-known fact. If $\dim_{\mathbb{C}} \mathbb{E} = \infty$, then by Theorem 0.5.5, $\lim_\alpha \pi_1(GL(\mathbb{E}_\alpha)) = \pi_1(GL_{\text{cond}}(\mathbb{E}))$. But $[S^1, GL(\mathbb{E}_\alpha)] \cong \mathbb{Z}$. So we have

$$[S^1, GL_{\text{cond}}(\mathbb{E})] \cong \lim_{\alpha}[S^1, GL(\mathbb{E}_\alpha)] \cong \mathbb{Z}.$$

This completes the proof.

We end this section with two remarks.

Remark 0.5.1. Note that for a real Banach space $\mathbb{E}$, there are two connected components of the space $GL(\mathbb{E}_\alpha)$ for each $\alpha \in \Lambda$. It follows from Theorem 0.5.5 that there are also two connected components for $GL_{\text{cond}}(\mathbb{E})$. We denote by $GL^+_{\text{cond}}(\mathbb{E})$ the component containing the identity Id. The other component will be denoted by $GL^-_{\text{cond}}(\mathbb{E})$.

Remark 0.5.2. All the results proved in this section for $GL^G_{\mathrm{cond}}(\mathbb{E})$ hold also true if we replace $GL^G_{\mathrm{cond}}(\mathbb{E})$ by $GL_c(\mathbb{E})$, where $GL_c(\mathbb{E})$ denotes the set of all invertible equivariant linear operators of the form $Id - A$ with A completely continuous. The proofs are the same. For more information about the homotopic properties of $GL_{\mathrm{cond}}(\mathbb{E})$ and $GL_c(\mathbb{E})$, we refer to [14, 20, 22, 27, 28, 33, 35].

References

[1] J. F. Adams, "Lectures on Lie Groups", Benjamin, New York, 1969.

[2] M. F. Atiyah, " K-Theory ", Benjamin, New York, 1967.

[3] J. Banaś and K. Goebel, "Measures of Noncompactness in Banach Spaces", Lecture Notes in Pure and Appl. Math. **60**, Marcel Dekker, New York, 1980.

[4] G. E. Bredon, "Introduction to Compact Transformation Groups", Academic Press, New York, 1972.

[5] Th. Bröker and T. tom Dieck, "Representations of Compact Lie Groups", Springer-Verlag, New York, 1985.

[6] Th. Bröker and K. Jänich, "Introduction to Differential Topology", Cambridge University Press, New York, 1982.

[7] K. Deimling, "Nonlinear Functional Analysis", Springer-Verlag, New York, 1984.

[8] T. tom Dieck, "Transformation Groups", de Gruyter, Berlin-New York, 1987.

[9] N. Dinculeanu, "Integration on Locally Compact Groups", Noordhoff International Publishing, 1974.

[10] J. Dugundji, "Topology", Allyn and Bacon, Boston, 1966.

[11] G. Dylawerski, K. Gęba, J. Jodel and W. Marzantowicz, *S^1-equivariant degree and the Fuller index*, Ann. Polon. Math. **52**(1991), 243–280.

[12] L. H. Erbe, K. Gęba and W. Krawcewicz, *Equivariant fixed point index and the period doubling cascades*, Canad. J. Math. **43**(1991), 738–747.

[13] B. Fiedler, "Global Bifurcation of Periodic Solutions with Symmetry", Lecture Notes in Mathematics **1309**, Springer-Verlag, New York, 1988.

[14] K. Gęba, *On the homotopy group of $GL_c(\mathbb{E})$*, Bull. Acad. Polon. Sci. **16**(1968), 699–702.

[15] ——, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems I: Construction of the degree*, J. Lond. Math. Soc. , to appear.

[16] G. Gray, "Homotopy Theory: An Introduction to Algebraic Topology", Academic Press, New York, 1975.

[17] M. W. Hirsch, " Differential Topology", Springer-Verlag, New York, 1976.

[18] K. Kawakubo, "The Theory of Transformation Groups", Oxford University Press, New York, 1991.

[19] A. A. Kirillov, "Elements of the Theory of Representations", Springer-Verlag, New York, 1976.

[20] W. Krawcewicz, *Contribution à la théorie des équations non linéaries dans les espoces de Banach*, Dissertations Math. **273**(1988), 1–83.

[21] ———, J. Wu and H. Xia, *Global Hopf bifurcation theory for condensing fields and neutral equations with applications to lossless tranmission problems*, Can. Appl. Math. Quarterly **1**(1993), 167–220.

[22] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology **3**(1965), 19–30.

[23] Y. I. Lyubich,"Introduction to the Theory of Banach Representations of Groups", Birkhäuser Verlag, Boston, 1988.

[24] M. Murayama, *On G-ANR's and their G-homotopy types*, Osaka J. Math. **20**(1983), 479–512.

[25] R. D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl. **89**(1971), 217–258.

[26] ———, *Degree theory for local condensing maps*, J. Math. Anal. Appl. **37**(1972), 741–766.

[27] R. S. Palais, *On the homotopy type of certain groups of operators*, Topology **3**(1965), 271–279.

[28] ———, *Homotopy theory of infinite dimensional manifolds*, Topology **5**(1966), 1–16.

[29] G. Peschke, Degree of certain equivariant maps into a representation sphere, to appear in Topology.

[30] G. Segal, *Equivariant K-theory*, Publ. Math. Inst. Hautes Études Scient (Paris) **34**(1968), 129–151.

[31] J.-P. Serre, " Linear Representations of Finite Groups", Springer-Verlag, New York, 1977.

[32] E. Spanier, "Algebraic Topology", McGraw-Hill, New York, 1966.

[33] A. S. Švarc, *The homotopy topology of Banach spaces*, Dokl. Akad. Nauk USSR **154**(1964), 61–63.

[34] A. L. Vanderbauwhede, *An abstract setting for the study of the Hopf bifurcation*, Nonlinear Analysis-TMA 4(1980), 547–566.

[35] P. Xu, *A remark to Kuiper's theorem*, Topology **32**(1993), 353–361.

CHAPTER 1

EQUIVARIANT DEGREE THEORY

IN FINITE DIMENSIONAL SPACES

1.1. Introduction

In this chapter, we describe the notion of the G-equivariant degree constructed recently by Gęba, Krawcewicz and Wu [16]. Based on generic approximations of equivariant maps, our approach, which is more appropriate for the computation of the degree, follows the main idea of [16] and complements [16].

Let us first briefly review various equivariant degrees which exist in the literature.

Suppose that V is a finite dimensional orthogonal representation of a compact Lie group G and $f: V \oplus \mathbb{R}^n \to V$ is an equivariant continuous mapping such that $f(x) \neq 0$ for all $x \in \partial\Omega$, where G acts on $\mathbb{R}^n$ trivially and Ω is an invariant bounded subset of $V \oplus \mathbb{R}^n$. The equivariant degrees or equivariant indices of f have been studied by a number of authors. From the algebraic topology point of view, any G-degree of f which has already been defined is closely related to the ***stable equivariant homotopy group of sphere*** ω_n^G (see tom Dieck [7] for notation). When $n = 0$, Segal [30] has been concerned with ω_0^G for finite group G and has shown that ω_0^G is isomorphic to the *Burnside ring* $A(G)$. This result is then extended by Rubinstein [29] to a general compact Lie group. Based on these studies, Dancer [4, 5] refines a degree in the case of symmetry invariance, which uses the ordinary mapping degree of the restriction of f to each isotropy set. The Leray-Schauder G-degree of Marzantowicz [26] is similar. Defined as a sequence of the ordinary Leray-Schauder degrees, it is also directly related to ω_0^G. Other related discussions in this direction can also be found in Tornehave [31] on mapping degrees with conjugate orthogonal actions and Lück [25] for more general treatment. We refer to tom Dieck [7] and the references therein for more information on ω_0^G.

On the other hand, the stable homotopy group of sphere ω_0^G has also been recovered in the study of equivariant fixed point indices of fibre-preserving maps by Dold [8-10], where an isomorphism between G-FIX(pt) and the Burnside ring $A(G)$ is constructed. The details and other more general results are published in Ulrich [32]. Following the work of Dold, in his interesting paper, Komiya [23] has found a family of integers indexed on the orbit types and a precise relation to the G-FIX(pt) (and hence to the Burnside ring $A(G)$) is obtained by using Euler characteristic and Möbius inversion. The equivariant index of Komiya appears to be convenient in applications. We refer to the paper of Erbe, Gęba and Krawcewicz [14] for the computation of the equivariant index for $G = \mathbb{Z}_n$ and its use in dynamical systems.

In the case $n \geq 1$, the *n-th stable homotopy group* ω_n^G is much more complicated and very little is known. However, attempts to get integer-valued G-degrees have been made. For $n = 1$, Dancer [6] has defined an S^1-degree for equivariant maps, which is then extended in Ize, Massabò and Vignoli [19] by using obstruction theory. An analytic construction for a general S^1-degree is conducted by Dylawerski et al. [13]. Again, this degree relates to the Dancer's degree (see Dylawerski [12]). The values of these S^1-degrees can be expressed in terms of integers and they possesses all standard properties of the classical Brouwer degree (see Lloyd [24]).

An "integer-valued" G-degree for a general $n \geq 0$ is first attempted by Gęba, Massabò and Vignoli [18] for equivariant gradient maps. The approach is remarkable and is now adopted in [16] to construct the G-degree $G\text{-Deg}(f,\Omega)$ of Gęba, Krawcewicz and Wu for any $n \geq 0$. In a recent paper, by extending the equivariant fixed point indices of Dold to the general case $n \geq 0$, Prieto and Ulrich [28] obtain a G-degree as an element of $G\text{-FIX}^n$ which turns out to be isomorphic to ω_n^G. On the other hand, Peschke [27] has used the splitting of ω_n^G (see tom Dieck [7]) and computed via obstruction theory those summands in ω_n^G with $\dim W(H) = n$ where H is a closed subgroup of G. They are either $\mathbf{Z}$ or $\mathbf{Z}_2$ depending on whether $W(H)$ is bi-orientable or not bi-orientable. Moreover, Peschke has found that the direct sum of these summands classifies the $W(H)$-homotopy classes of $W(H)$-maps between spheres (see also an earlier paper of Dylawerski [11] for the case $n = 1$). Although we will not pursue this in the thesis, we strongly believe that the G-degree of Gęba, Krawcewicz and Wu can be expressed (in a certain way) in terms of those summands computed by Peschke, which again coincides with the degree of Prieto and Ulrich. It then can be inferred that the equivariant degree $G\text{-Deg}(f,\Omega)$ [16] is the most general (in the sense that it classifies the homotopy classes of $W(H)$-maps of spheres as in the classical Brouwer degree case) and the best suited (in the sense that it is integer-valued as in the classical Brouwer degree case) G-equivariant degree one can define. Moreover, this G-degree yields naturally the Burnside ring $A(G)$ when $n = 0$ (see also above for the isomorphisms $G\text{-FIX} \cong A(G) \cong \omega_0^G$).

For the general $n \geq 0$, there is another G-equivariant degree theory carried out by Ize, Massabó and Vignoli [20, 21] where they define an equivariant degree $\deg_G(f,\Omega)$ as an element of a certain equivariant homotopy group of sphere $\pi_M^G(S^N)$. Here $f : (\Omega \subset \mathbb{R}^M, \partial\Omega) \to (\mathbb{R}^N, \mathbb{R}^N \setminus \{0\})$ is equivariant with respect to ***possibly different linear actions*** of G on $\mathbb{R}^M$ and $\mathbb{R}^N$, and there is not necessarily a component of $\mathbb{R}^N$ in $\mathbb{R}^M$ on which G acts trivially. This degree is much more general and it can be concluded from Remark 2.8 in [28] that $G\text{-Deg}(f,\Omega)$ lies in the ***stabilization*** of $\deg_G(f,\Omega)$. The degree $\deg_G(f,\Omega)$ has a universal property which implies that if $\deg_G(f,\Omega) = 0$ then $G\text{-Deg}(f,\Omega) = 0$.

Therefore, if they are both applied to nonlinear problems, $\deg_G(f,\Omega)$ could be helpful whenever G-Deg(f,Ω) fails to give any existence information. Moreover, $\deg_G(f,\Omega)$ has also recovered the Brouwer degree, Fuller's degree [15], generalized topological degree [17], $\deg_n^{S^1}(Id-f,\Omega)$ of Dancer [6], S^1-Deg(f,Ω) of Dylawerski et al. [13]. Finally, in the case of $G=S^1$, the computation of $\deg_{S^1}(f,\Omega)$ has been performed without appealing to the heavy machinery from algebraic topology (see [21]). However, as pointed out by the authors, $\deg_G(f,\Omega)$ is not fully additive in general (see the counterexample in Appendix [21]) and it is so up to one suspension. In addition, since $\deg_G(f,\Omega)$ is directly related to the homotopy group of the sphere, which is far from known, it may not be integer-valued and the ultimate computation of $\deg_G(f,\Omega)$ could be a task of considerable difficulties. In contrast to $\deg_G(f,\Omega)$, G-Deg(f,Ω) is fully additive and, due to its elementary construction, it is simpler to be computed as a sequence of integers.

In this chapter, we present a slightly modified approach to the equivariant degree G-Deg(f,Ω) of Gęba, Krawcewicz and Wu, which we will call simply *equivariant degree* or *G-degree* . This is achieved by so called *regular generic approximations.* The approach is different from the original construction of the equivariant degree which uses the induction over orbit types. We shall try to present a "practical" approach to the definition of the G-degree and develope a simple computational formula for the equivariant degree, which can be used as an alternative definition, i.e. a "practical" definition. However, in order not to complicate the presentation, we will not repeat the proof of the existence of the equivariant degree, which can be found in [16]. We hope this presentation is easier to access for specialists in Applied Mathematics who are interested in nonlinear problems with symmetry.

Let us now introduce the G-Deg(f,Ω) of Gęba, Krawcewicz and Wu and state its standard properties.

Recall that we use $O(G)$ to stand for the set of all conjugacy classes of closed subgroups of G. For a closed subgroup H of G, $N(H)$ denotes the *normalizer* of H in G and $W(H)$ is the *Weyl group* $N(H)/H$ of H in G. For every $n \in \mathbf{N}$, we put

$$\Phi_n(G) := \{(H) \in O(G);\ \dim W(H) = n\}.$$

Definition 1.1.1. We define $\mathbf{Z}\Phi_n(G)$ (resp. $\mathbf{Z}_2\Phi_n(G)$) as the free $\mathbf{Z}$-module (resp. $\mathbf{Z}_2$-module) generated by those $(H) \in \Phi_n(G)$ with $W(H)$ bi-orientable (resp. not bi-orientable) and define $A_n(G)$ as the direct sum $\mathbf{Z}\Phi_n(G) \oplus \mathbf{Z}_2\Phi_n(G)$.

An element of $A_n(G)$ will be written as $\gamma = \sum_{\alpha \in \Phi_n(G)} \gamma_\alpha \cdot \alpha$ where

$$\gamma_\alpha \in \begin{cases} \mathbf{Z} & \text{if } \alpha \in \Phi_n(G) \text{ with bi-orientable } W(H) \\ \mathbf{Z}_2 & \text{if } \alpha \in \Phi_n(G) \text{ with not bi-orientable } W(H). \end{cases}$$

Recall that V is a real orthogonal finite dimensional representation of the Lie group G. It induces by diagonal action a representation space $V \oplus \mathbb{R}^n$, where G acts trivially over $\mathbb{R}^n$.

Definition 1.1.2. For an open invariant set Ω of $V \oplus \mathbb{R}^n$, an equivariant continuous map $f : \Omega \to V$ is said to be *Ω-admissible* if $f^{-1}(0)$ is a compact subset of Ω. An equivariant continuous map $h : \Omega \times [0,1] \to V$, where G acts on $[0,1]$ trivially, is called an *equivariant homotopy.* An equivariant homotopy $h : \Omega \times [0,1] \to V$ is said to be *admissible,* if $h^{-1}(0)$ is compact in $\Omega \times [0,1]$. For an admissible homotopy $h : \Omega \times [0,1] \to V$, we say that h_0 and h_1 are *Ω-homotopic,* where $h_t : \Omega \to V$ for $t \in [0,1]$ is defined by $h_t(x) = h(x,t)$, $x \in \Omega$.

The following result is proved by K. Gęba, W. Krawcewicz and J. Wu [16].

Theorem 1.1.1. *For every Ω-admissible map $f : \Omega \to V$, where Ω is an open invariant subset of $V \oplus \mathbf{R}^n$, we can assign an element $G\text{-}Deg(f, \Omega) \in A_n(G)$ such that the following properties are satisfied:*

(P1) Existence. If $G\text{-}Deg(f, \Omega) \neq 0$, i.e. there is an $\alpha \in A_n(G)$ such that $\gamma_\alpha \neq 0$, then there exists $x \in f^{-1}(0)$ such that $(G_x) \leq \alpha$;

(P2) Homotopy Invariance. If $h : \Omega \times [0,1] \to V$ is an admissible homotopy, then $G\text{-}Deg(h_t, \Omega)$ does not depend on $t \in [0,1]$;

(P3) Excision. If $\Omega_0 \subseteq \Omega$ is an open and invariant subset and $f^{-1}(0) \subseteq \Omega_0$, then $G\text{-}Deg(f, \Omega) = G\text{-}Deg(f, \Omega_0)$;

(P4) Additivity. If Ω_1 and Ω_2 are two open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $f^{-1}(0) \subset \Omega_1 \cup \Omega_2$, then

$$G\text{-}Deg(f, \Omega) = G\text{-}Deg(f, \Omega_1) + G\text{-}Deg(f, \Omega_2);$$

(P5) Product Formula. If W is another orthogonal representation of G, $\mathcal{U}$ is an open invariant subset of W such that $0 \in \mathcal{U}$, and $g : \Omega \times \mathcal{U} \to V \times W$ is defined by $g(x, y) = \big(f(x), y\big)$ for $(x, y) \in \Omega \times \mathcal{U}$, then $G\text{-}Deg(g, \Omega \times \mathcal{U}) = G\text{-}Deg(f, \Omega)$.

The element $G\text{-Deg}(f, \Omega) \in A_n(G)$ will be called the *G-(equivariant) degree* of the map f with respect to the set Ω. By Theorem 1.1.1, the properties of this equivariant degree are completely parallel to those of the classical Brouwer degree. However, its value is not taken as an integer. Compare Bartsch [1] and Bartsch and Mawhin [2] for the Brouwer degree for equivariant maps.

The rest of this chapter is now organized as follows. In Section 1.2, we present generic approximations for an Ω-admissible map $f : \Omega \to V$. By using a linear slice at each regular orbit of zeros, the equivariant degree of Gęba, Krawcewicz and Wu is discussed in Section 1.3 for a regular generic G-map. This has led to an alternative definition of $G\text{-Deg}(f,\Omega)$ in Section 1.4. Section 1.5 discusses the mutiplicativity property of $G\text{-Deg}(f,\Omega)$ in the case $n = 0$ and its relation to the Burnside ring is indicated. Finally in Section 1.6, we provide a computational formula for $G\text{-Deg}(f,\Omega)$ when the zero orbit is regular and the group G is abelian. Several specific examples illustrating how to compute the G-degree are also included.

1.2. Generic approximations

Let G be a compact Lie group. To simplify notations, we put $W := V \oplus \mathbb{R}^n$, where V is a real orthogonal finite-dimensional representation of G and $\mathbb{R}^n$ is a trivial G-space.

Given $\alpha \in O(G)$, it is known from Theorem 0.1.10 that $W_\alpha := \{x \in W;\ (G_x) = \alpha\}$ is a G-invariant submanifold of W. We consider the normal bundle $\nu(W_\alpha)$ to W_α in W, denoted by $\gamma_\alpha : N^\alpha \to W_\alpha$. (Here we have abused the terminology "normal bundle" for $\mu(W_\alpha)$ because it is not a normal bundle in the strict sense but is composed of possibly more than one normal bundle with different dimensions, as W_α may have components of different dimensions (see Bredon [3] and Kawakubo [22])). The normal bundle $\nu(W_\alpha)$ is a collection of G-vector bundles over W_α. Let D be a compact invariant subset of W_α. We put

$$\mathcal{N}(D,\varepsilon) := \{(v,w) \in N^\alpha;\ v \in D,\ w \in N^\alpha_v,\ \|w\| \leq \varepsilon\}$$

where $\|\cdot\|$ is an invariant norm induced by the standard invariant Euclidean metric in W. $\mathcal{N}(D,\varepsilon)$ is the total space of the ε-disk sub-bundles of the restricted vector bundles $\nu(D) := \nu(W_\alpha)|_D$. Consider the G-map $\mu : \mathcal{N}(D,\varepsilon) \to W$ given by $\mu(v,w) = v + w$. Then, for a sufficiently small $\varepsilon > 0$, the G-map μ is a diffeomorphism onto its image $\mu(\mathcal{N}(D,\varepsilon))$, which will be called an *α-normal neighbourhood of* D. Consequently, each element x of $\mu(\mathcal{N}(D,\varepsilon))$ can be written uniquely as $x = v + w$, where $v \in D$ and $w \in N_v^\alpha$ with $\|w\|_v < \varepsilon$.

Definition 1.2.1. Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and Ω_0 be an invariant compact subset of Ω. For an equivariant map $f : \Omega \to V$ and $\alpha \in O(G)$, we say that f is *α-normal* in Ω_0 if

(i) α is a minimal orbit type in Ω_0;

(ii) Ω_0 . n α-normal neighbourhood, i.e. $\Omega_0 = \mu(\mathcal{N}(D,\varepsilon))$ for some D and $\varepsilon > 0$;

(iii) $f(x) = f(v + w) = f(v) + w$, for all $x = v + w \in \Omega_0$, where $v \in D$, $w \in N_v^\alpha$.

Definition 1.2.2. Let Ω and Ω_0 be as above and $h : \Omega \times [0,1] \to V$ be a G-map, where G acts on $[0,1]$ trivially. The map h is s' .d to be an *α-normal homotopy* in Ω_0 if the conditions (i) and (ii) in Definition 1.2.1 are satisfied together with (iii): $h(x,t) = h(v + w, t) = h(v,t) + w$ for all $(x,t) \in \Omega_0 \times [0,1]$, where $x = v + w$, $v \in D$, $w \in N_v^\alpha$.

Remark 1.2.1. For simplicity, we have used the condition (iii) in Definition 1.2.1 and 1.2.2 for the α-normality of an equivariant map f. The conclusions in this section hold also true if we replace the condition (iii) by $f(v+w) = f(v) + N(w)$ for all $v \in D$ and $w \in N_v^\alpha$, where N is an equivariant (possibly nonlinear) map such that the derivative $D_w N(0)$ is homotopic to the identity $Id|_{N_v^o}$. This latter condition will be more convenient in computing the degree.

Definition 1.2.3. Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ be an Ω-admissible G-map. We say that f is (Ω)-*generic* if for every $\alpha \in \mathcal{J}(f^{-1}(0))$ there is an α-normal neighbourhood $\Omega(\alpha) \subset \Omega$ such that

(i) $\Omega(\alpha) \cap \Omega(\beta) = \varnothing$ for $\alpha \neq \beta$;

(ii) $\bigcup_{\alpha \in \mathcal{J}(f^{-1}(0))} \Omega(\alpha)$ is a compact neighbourfood of $f^{-1}(0)$;

(iii) For every $\alpha \in \mathcal{J}(f^{-1}(0))$ the G-map $f|_{\Omega(\alpha)}$ is α-normal in $\Omega(\alpha)$.

The following result says that Ω-generic maps are dense in the space of Ω-admissible G-maps.

Theorem 1.2.1. (GENERIC APPROXIMATION THEOREM) *Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ be an Ω-admissible equivariant map. Then for every $\eta > 0$ there exists a generic G-map $f_0 : \Omega \to V$ such that*

(i) f_0 is Ω-admissible;

(ii) $\sup_{x \in \Omega} \|f_0(x) - f(x)\| < \eta$.

Proof. Let D_1 be a compact invariant neighbourhood of $f^{-1}(0)$ in Ω, and $\delta > 0$ be such that $K := \overline{B(D_1, \delta)} \subset \Omega$, where

$$B(D_1, \delta) = \{x \in V \oplus \mathbb{R}^n : \operatorname{dist}(D_1, x) < \delta\}.$$

Set $m = |\mathcal{J}(K)|$. We extend the partial order in $\mathcal{J}(K)$ to a total order $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} < \cdots < \alpha_m$. Assume that $0 < \varepsilon < \delta$. We consider the minimal orbit type $\alpha := \alpha_1 \in \mathcal{J}(K)$ and let $D := D_1 \cap W_\alpha \subset \operatorname{Int} K$. Without loss of generality, we assume that $D \neq \emptyset$. It follows from Theorem 0.1.10 that D is an invariant compact set. This gives an α-normal neighbourhood $N_0 := \mu(\mathcal{N}(D,\varepsilon)) \subset \operatorname{Int} K$ of D, where $\mu : \mathcal{N}(D,\varepsilon) \to \Omega$ is a diffeomorphism onto its image N_0 for a suitable small $\varepsilon > 0$. By Theorem 0.1.13 there exists an invariant Urysohn function $\gamma : \Omega \to [0,1]$ such that $\gamma(x) = 1$ for $x \in \mu(\mathcal{N}(D, \frac{\varepsilon}{2}))$ and $\gamma(x) = 0$ for $x \in \Omega \setminus N_0$. Define now a G-map $f_1 : \Omega \to V$ as follows

$$f_1(x) := \begin{cases} f(x), & x \in \Omega \setminus N_0, \\ \gamma(x)[f(v) + w] + [1 - \gamma(x)]f(x), & x = v + w \in N_0. \end{cases}$$

Since we can take $\varepsilon > 0$ as small as we wish, we can assume that

$$\sup_{v+w \in N_0} \|f(v) - f(v+w)\| < \frac{\eta}{m} - \varepsilon.$$

Consequently,

$$\begin{aligned} \sup_{x\in\Omega} \|f_1(x) - f(x)\| &= \sup_{x \in N_0} \|f_1(x) - f(x)\| \\ &= \sup_{x=v+w\in N_0} \|\gamma(x)[f(v) - f(x)] + \gamma(x) w\| \\ &\leq \sup_{x=v+w\in N_0} \|f(v) - f(x)\| + \varepsilon < \frac{\eta}{m}. \end{aligned}$$

This implies that $f_1 : \Omega \to V$ is Ω-admissible and is α-normal in $\Omega(\alpha) := \mu(\mathcal{N}(D, \frac{\varepsilon}{2}))$ and $\Omega(\alpha) \supset f_1^{-1}(0) \cap W_\alpha$.

Assume now that for $k \geq 1$ we have already constructed a G-map $f_k : \Omega \to V$ satisfying the following conditions:

(a) For every $\alpha \leq \alpha_k$, there is a compact invariant set $\Omega_\alpha \subset \operatorname{Int} K$ such that the G-map f_k is α-normal in $\Omega(\alpha)$;

(b) For every α, $\beta \leq \alpha_k$, $\Omega(\alpha) \cap \Omega(\beta) = \emptyset$ for $\alpha \neq \beta$;

(c) Every $\alpha \in \mathcal{J}(f^{-1}(0) \setminus \bigcup_{i=1}^{k} \Omega(\alpha_i))$ is greater than α_k;

(d) $\sup_{x\in\Omega}\|f_k(x)-f(x)\|<k\frac{\eta}{m}$.

From the above, $A := f_k^{-1}(0)\setminus\bigcup_{i=1}^k\Omega(\alpha_i)$ is an invariant compact subset of $\operatorname{Int} K$. There is a compact G-neighbourhood D_{k+1} of A in $\Omega_{k+1} := \operatorname{Int} K\setminus\bigcup_{i=1}^k\Omega(\alpha_i)$ such that any $\alpha\in\mathcal{J}(D_{k+1})$ is greater than α_k. Assume that $\delta_{k+1}>0$ is such that $\overline{B(D_{k+1},\delta_{k+1})}\subset\Omega_{k+1}$, and consider $D := W_{\alpha_{k+1}}\cap D_{k+1}$. The set D is compact by Theorem 0.1.10 since α_{k+1} is a minimal orbit type in D_{k+1}. Then again, for a sufficiently small $0<\varepsilon<\delta_{k+1}$, the G-map $\mu:\mathcal{N}(D,\varepsilon)\to\Omega$ is a diffeomorphism onto its image $N_0 := \mu(\mathcal{N}(D,\varepsilon))\subset\Omega_{k+1}\subset\Omega$, i.e. $\Omega(\alpha_{k+1}) := N_0$ gives an α_{k+1}-normal neighbourhood. We choose again an invariant Urysohn function $\gamma:\Omega\to[0,1]$ such that $\gamma(x)=1$ for $x\in\mu(\mathcal{N}(D,\frac{\varepsilon}{2}))$ and $\gamma(x)=0$ for $x\in\Omega\setminus N_0$. Define the following G-map $f_{k+1}:\Omega\to V$ by

$$f_{k+1}(x) := \begin{cases} f_k(x), & x\in\Omega\setminus N_0,\\ \gamma(x)[f_k(v)+w]+[1-\gamma(x)]f_k(x), & x=v+w\in N_0.\end{cases}$$

Since we can take $\varepsilon>0$ as small as we wish, we can also assume that

$$\sup_{v+w\in N_0}\|f_k(v)-f_k(v+w)\|<\frac{\eta}{m}-\varepsilon,$$
$$\sup_{x\in\Omega}\|f_{k+1}(x)-f_k(x)\|<\frac{\eta}{m}.$$

Consequently,

$$\begin{aligned}\sup_{x\in\Omega}\|f_{k+1}(x)-f(x)\| &\le \sup_{x\in\Omega}\|f_{k+1}(x)-f_k(x)\|+\sup_{x\in\Omega}\|f_k(x)-f(x)\|\\ &<\frac{\eta}{m}+k\frac{\eta}{m}=(k+1)\frac{\eta}{m}.\end{aligned}$$

By definition, $f_{k+1}:\Omega\to V$ is Ω-admissible and is α-normal in $\Omega(\alpha) := \mu(\mathcal{N}(D,\frac{\varepsilon}{2}))$, $\alpha\le\alpha_{k+1}$. Therefore, by applying the induction principle we obtain a G-map $f_0 := f_m$ and the conclusion follows.

Definition 1.2.4. Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ be an Ω-admissible G-map. We say that f is a *regular* (Ω)*-generic map* if

(i) f is of class C^1;

(ii) f is an Ω-generic G-map;

(iii) for every $\alpha \in \mathcal{J}(f^{-1}(0))$, $\alpha = (H)$, zero is a regular value of

$$f_H := f|_{\Omega(\alpha)_H} : \Omega(\alpha)_H \to V^H.$$

Remark 1.2.2. Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ be an Ω-admissible G-map. Then for every sufficiently small $\varepsilon > 0$ there exists an Ω-admissible G-map $f_0 : \Omega \to V$ of class C^1 such that

$$\sup_{x \in \Omega} \|f(x) - f_0(x)\| < \varepsilon.$$

Indeed, we first find an Ω-admissible C^{-1}-map $F : \Omega \to V$ such that $\sup_{x\in\Omega} \|f(x) - F(x)\| < \varepsilon$. Let K be an invariant compact set such that $K \supset F^{-1}(0)$. By using smooth Urysohn function, we can find a C^1-map $\bar{f}_0 : \Omega \to V$ (not necessary equivariant) such that

$$\sup_{x \in K} \|f(x) - \bar{f}_0(x)\| < \varepsilon$$

and $F(x) = \bar{f}_0(x)$ outside K. Put

$$f_0(x) := \int_G g\bar{f}_0(g^{-1}x)dg, \quad x \in \Omega.$$

Then f_0 is of class C^1 and

$$\|f(x) - f_0(x)\| \le \int_G \|f(g^{-1}x) - f_0(g^{-1})\|dg < \varepsilon,$$

as desired.

Remark 1.2.3. Let $f : \Omega \to V$ be a map as in Remark 1.2.2. Then for every $\eta > 0$ there exists an Ω-generic G-map $f_0 : \Omega \to V$ of class C^1 such that

$$\sup_{x\in\Omega} \|f(x) - f_0(x)\| < \eta.$$

Indeed, it is sufficient to note that in the proof of Theorem 1.2.1, one can choose Urysohn functions to be smooth ones, and therefore if f is of class C^1, the resulting Ω-generic approximation f_0 is also of class C^1.

In what follows, we are interested in finding regular generic approximations for a given Ω-admissible map f. From Theorem 1.2.1 and Remark 1.2.2 and 1.2.3, the problem now is to find a C^1 Ω-generic approximation which also satisfies (iii) of Definition 1.2.4.

Theorem 1.2.2. (REGULAR GENERIC APPROXIMATION THEOREM) *Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ be an Ω-admissible G-map. Then for every $\eta > 0$ there exists a regular generic G-map $f_0 : \Omega \to V$ such that*

(i) $t \cdot f + (1-t)f_0$ *is Ω-admissible for every $t \in [0,1]$;*

(ii) $\sup_{x\in\Omega} \|f_0(x) - f(x)\| < \eta.$

Proof. By Theorem 1.2.1, there exists an Ω-generic G-map $\tilde{f} : \Omega \to V$ such that $\tilde{f}$ is Ω-admissible and $\sup_{x\in\Omega} \|\tilde{f}(x) - f(x)\| < \frac{\eta}{2}$. By Remark 1.2.2 and 1.2.3, we can assume that $\tilde{f}$ is of class C^1. Using the same argument as in Ulrich [32] (Proof of Theorem 3.2, Step 1) we can approximate

$\tilde{f}_H$, $\alpha = (H) \in \mathcal{J}(\tilde{f}^{-1}(0))$, by a $W(H)$-map $\hat{f}_H$ which comes transversally to zero. We then extend the $W(H)$-map $\hat{f}_H$ to a G-map $\hat{f}_{(H)}$ defined on the set $\Omega(\alpha)_{(H)}$. Next, by using the condition (iii) of Definition 1.2.1, we extend $\hat{f}_{(H)}$ (as an identity in normal directions) to a G-map $\hat{f}_\alpha$ on the set $\Omega(\alpha)$. Finally, with the help of smooth Urysohn function, we can extend the G-map $\hat{f}_\alpha$ to a new G-map, still denoted by $\tilde{f}$, on the set Ω, and the obtained G-map, which is an approximation of the original $\tilde{f}$, satisfies the condition (iii) of Definition 1.2.4 on the set $\Omega(\alpha)$. Since the same 'adjustment' of $\tilde{f}$ can be made on every $\Omega(\alpha')$, $\alpha' \in \mathcal{J}(\tilde{f}^{-1}(0))$, the above construction shows that $\tilde{f}$ can be approximated by a regular Ω-generic G-map with an arbitrary degree of accuracy. This completes the proof.

We finally remark that according to the definition, if f is a regular generic G-map, then f_H is transversal to $0 \in V^H$. But it does not imply that f is transversal to $0 \in V$. In fact, it may happen that the mapping $f_{(H)} := f|_{\Omega(\alpha)_{(H)}}$, which is the unique G-equivariant extension of f_H, already has zero as a critical value. Consequently, the above Regular Generic Approximation Theorem can not be considered as a version of the *Equivariant Transversality Theorem* (see Ulrich [32]).

1.3. G-degree for regular generic G-maps

Let $(H) \in \Phi_n(G)$ be such that $W(H)$ is bi-orientable. We fix an invariant orientation of $W(H)$. If $H_1 \sim H$, i.e. $H_1 = gHg^{-1}$ for some $g \in G$, then the natural isomorphism $t_g : W(H_1) \to W(H)$, defined by $t_g(hH_1) = g^{-1}hgH$, $h \in N(H_1)$, uniquely determines an invariant orientation of $W(H_1)$. Therefore, we can choose an invariant orientation of $W(H)$, which is preserved by isomorphisms t_g for every representative of the class $(H) \in \Phi_n(G)$. Such an orientation will be called an (H)-*orientation.* In what follows, we will assume that for every class $(H) \in \Phi_n(G)$ with $W(H)$ bi-orientable, there is fixed an (H)-orientation.

Let V be a finite dimensional orthogonal representation of G and $V \oplus \mathbb{R}^n$ be the induced representation of G with trivial G-action on $\mathbb{R}^n$. Given any closed subgroup H of G, its normalizer $N(H)$ acts on $V_H \times \mathbb{R}^n = (V \oplus \mathbb{R}^n)_H$, and consequently induces a free action of $W(H)$ on $V_H \times \mathbb{R}^n$. Similarly, we obtain a free diagonal action of $W(H)$ on the product space $V_H \times \mathbb{R}^n \times V^H$ (see Proposition 0.1.3). The projection $\pi : V_H \times \mathbb{R}^n \times V^H \to V_H \times \mathbb{R}^n$ onto the space $V_H \times \mathbb{R}^n$ is clearly equivariant with respect to the above actions and therefore, by Theorem 0.1.8, it induces a smooth map $p : E \to M$ between smooth manifolds $E := (V_H \times \mathbb{R}^n \times V^H)/W(H)$ and $M := (V_H \times \mathbb{R}^n)/W(H)$. Moreover, Theorem 0.1.9 implies that $p : E \to M$ is a smooth vector bundle with a typical fibre V^H. Note that an orientation of V^H determines the product orientation of $V^H \times V^H$ which does not depend on the orientation of V^H. Thus the product $V^H \times V^H$ has a natural preferred orientation. This preferred orientation of $V^H \times V^H$ together with the standard orientation of $\mathbb{R}^n$ determines an orientation of $V_H \times \mathbb{R}^n \times V^H$. If $W(H)$ is bi-orientable, then it follows from Theorem 0.1.8 that the manifold $E = (V_H \times \mathbb{R}^n \times V^H)/W(H)$ is orientable and the orientation of E is determined by the fixed orientation of $W(H)$.

Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ an Ω-admissible regular generic G-map such that $f^{-1}(0) \subset \bigcup_{\alpha \in \mathcal{J}(f^{-1}(0))} \Omega(\alpha)$, where the sets $\Omega(\alpha)$ are the same as in Definition 1.2.3. By the excision property (P3) and additivity property (P4) of the G-degree, we have

$$G\text{-Deg}\,(f, \Omega) = \sum_{\alpha \in \mathcal{J}(f^{-1}(0))} G\text{-Deg}\,(f, \Omega(\alpha)).$$

Consequently, it suffices to describe the value of $G\text{-Deg}\,(f, \Omega(\alpha))$. We can assume, by Gleason-Tietze Theorem, that f is a G-map defined on the whole space

$V \oplus \mathbb{R}^n$. For every $\alpha = (H)$, since f is G-equivariant, the restriction of f to the subspace $V_H \times \mathbb{R}^n$ induces a $W(H)$-equivariant map $f_H : V_H \times \mathbb{R}^n \to V^H$, $f_H(x) = f(x)$, $x \in V_H \times \mathbb{R}^n$. Clearly, f_H is $\Omega(\alpha)_H$-admissible. Define $F_H : V_H \times \mathbb{R}^n \to V_H \times \mathbb{R}^n \times V^H$ by

$$F_H(x) = (x, f_H(x)), \quad x \in V_H \times \mathbb{R}^n.$$

Then F_H is a $W(H)$-equivariant section of the bundle $\pi : V_H \times \mathbb{R}^n \times V^H \to V_H \times \mathbb{R}^n$. Passing to the orbit spaces, we obtain a section $s_{f,H} : M \to E$ of the vector bundle $p : E \to M$. This section is $\Omega(\alpha)_H/W(H)$-admissible. Moreover, since the vector bundle is of dimension $\dim V^H$ and $\dim W(H) = n$, we have

$$\dim M = \dim(V^H \times \mathbb{R}^n) - n = \dim V^H.$$

That is, the dimension of M is equal to the dimension of the fibre E. Therefore, from Section 0.4, the intersection number of $s_{f,H}$ with respect to $\Omega(\alpha)_H/WH$ is well defined.

Now we can write down the G-equivariant degree $G\text{-Deg}(f, \Omega(\alpha))$, namely

$$G\text{-Deg}(f, \Omega(\alpha)) = \sum_{\beta \in \Phi_n(G)} n_\beta \cdot \beta \in A_n(G)$$

where in the case $\alpha = (H) \in \mathcal{J}(f^{-1}(0)) \cap \Phi_n(G)$ and $W(H)$ is bi-orientable, we have

$$n_\beta = \begin{cases} \chi(s_{f,H}) & \text{if } \beta = \alpha \\ 0 & \text{otherwise,} \end{cases}$$

and in the case when $\alpha = (H) \in \mathcal{J}(f^{-1}(0)) \cap \Phi_n(G)$ and $W(H)$ is not bi-orientable,

$$n_\beta = \begin{cases} \chi_2(s_{f,H}) & \text{if } \beta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we are going to establish an explicit computation formula for the intersection numbers $\chi(s_{f,H})$ and $\chi_2(s_{f,H})$.

We consider first the case where $\alpha = (H) \in \mathcal{J}(f^{-1}(0)) \cap \Phi_n(G)$ is such that $W(H)$ is bi-orientable. By the assumption, the mapping $f_H : \Omega(\alpha)_H \to V^H$ has zero as a regular value, therefore $f_H^{-1}(0) \subset \Omega(\alpha)_H$ is composed of a finite number of $W(H)$ orbits, say $W(H)x_1, \ldots, W(H)x_k$. Fix an orientation (arbitrarily) of the space V^H. The invariant orientation of the group $W(H)$ determines an orientation of the orbit $W(H)x_i$ for every $i = 1, \ldots, k$. We denote by S_i the *linear slice* to the orbit $W(H)x_i$ at the point x_i, $i = 1, \ldots, k$, i.e.

$$S_i = \{v \in V^H \oplus \mathbb{R}^n : v - x_i \perp T_{x_i} W(H)x_i\}.$$

We choose an orientation of the slice S_i such that the orientation of $T_{x_i}W(H)x_i$ followed by the orientation of S_i gives the (fixed) orientation of $V^H \oplus \mathbb{R}^n = T_{x_i}W(H)x_i \oplus S_i$.

Assume that $A : S_i \to V^H$ is a linear isomorphism. We define $\operatorname{sign} A$ to be 1 if A preserves the above chosen orientations of S_i and V^H, and -1 otherwise. The explicit formula for $\chi(s_{f,H})$ now reads as follows

$$\chi(s_{f,H}) = \sum_{i=1}^{k} \operatorname{sign} Df(x_i)|_{S_i}$$

where $Df(x_i)|_{S_i} = Df_H(x_i)|_{S_i} : S_i \to V^H$ is an isomorphism.

In the case where $W(H)$ is not bi-orientable, we define $\operatorname{sign} A = 1$ for every linear isomorphism $A : S_i \to V^H$ and put

$$\chi_2(s_{f,H}) = \sum_{i=1}^{k} \operatorname{sign} Df(x_i)|_{S_i} \pmod{2}.$$

We summarize the above results in the following proposition.

Proposition 1.3.1. *Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ an Ω-admissible regular generic G-map. Then $G\text{-Deg}(f, \Omega) = \sum_{\beta \in \Phi_n(G)} n_\beta \cdot \beta$, where*

$$n_\beta = \begin{cases} 0 & \text{if } \beta \notin \mathcal{J}(f^{-1}(0)) \\ \displaystyle\sum_{W(H)x \subset f^{-1}(0)_H} \operatorname{sign} Df(x)|_{S_x} & \text{if } \beta = (H),\ W(H) \text{ bi-orientable} \\ \displaystyle\sum_{W(H)x \subset f^{-1}(0)_H} \operatorname{sign} Df(x)|_{S_x} \pmod 2 & \text{if } \beta = (H),\ W(H) \text{ not bi-orientable} \end{cases}$$

and S_x denotes the linear slice to the orbit $W(H)x$ in the space $V^H \oplus \mathbb{R}^n$ at $x \in f^{-1}(0)_H$ and x is an arbitrarily fixed zero of f.

1.4. An alternative definition of G-degree

In this section we present a new and completely analytical definition of the G-equivariant degree, based on the Regular Generic Approximation Theorem and the analytical formula for regular generic G-map stated in Proposition 1.3.1. This new definition, resulting from the standard properties of the G-degree, can be used for computational purposes.

Let $\Omega \subset V \oplus \mathbb{R}^n$ be an open invariant set and $f : \Omega \to V$ an Ω-admissible G-map. By the Regular Generic Approximation Theorem (Theorem 1.2.2), there exist a regular generic G-map $g : \Omega \to V$ and an admissible G-homotopy $h : \Omega \times [0, 1] \to V$ between f and g. Then the G-equivariant degree $G\text{-Deg}(f, \Omega) =$

$\sum_\alpha n_\alpha \cdot \alpha$ can be expressed by the following formula

$$n_\alpha = \begin{cases} 0 & \text{if } \alpha \notin \mathcal{J}(g^{-1}(0)) \\ \displaystyle\sum_{W(H)x \subset g^{-1}(0)_H} \operatorname{sign} Dg(x)|_{S_x} & \text{if } \alpha = (H),\ W(H) \text{ bi-orientable} \\ \displaystyle\sum_{W(H)x \subset g^{-1}(0)_H} \operatorname{sign} Dg(x)|_{S_x} \pmod 2 & \text{if } \alpha = (H),\ W(H) \text{ not bi-orientable} \end{cases} \qquad (*)$$

where S_x denotes the linear slice to the orbit $W(H)x$ in the space $V^H \oplus \mathbb{R}^n$. Note that by homotopy invariance of intersection number, n_α does not depend on the choice of the regular generic G-map g. This observation leads to the following alternative definition of the equivariant degree of Gęba, Krawcewicz and Wu [16].

Definition 1.4.1. Let $f : \Omega \to V$ be as above. We define the equivariant degree of f as follows

$$G\text{-Deg}\,(f, \Omega) := \sum_{\alpha \in \Phi_n(G)} n_\alpha \cdot \alpha$$

where n_α is given by $(*)$ and g is a regular Ω-generic appriximation of f such that f and g are Ω-homotopic.

The above new definition of the equivariant degree provides an analytical formula for the computation of the degree, and it can be viewed as a variant of the analytical formula of the Brouwer degree. The only difference is that in the equivariant case we 'count' the orbits of zeros instead of isolated zeros, and the computation of the sign of determinants is done with respect to the linear slice in the appropriate fixed-point subspace $V^H \oplus \mathbb{R}^n$. From this point of view, we may interpret the G-degree $G\text{-Deg}\,(f, \Omega)$ as a sequence of integers $\{n_\alpha\}$, indexed by appropriate orbit types α and showing for a regular generic G-map f the existence of at least n_α orbits of solutions to the equation $f(x) = 0$ with each of these solutions being of exactly the orbit type α. Similarly to the case of the Brouwer degree, where we consider approximations by regular mappings, the existence property, as it is expressed for a regular generic G-map, loses its power if we consider an arbitrary G-map instead of a regular generic G-map. As we are making a deformation of regular generic G-maps, some of the orbits of zeros may collapse onto orbits of smaller orbit types, and consequently we can only express the existence property as it is stated in Theorem 1.1.1. This can be seen in the following example.

Example 1.4.1. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, -1\} \subset \mathbb{C}$, and $V = \mathbb{R}^3$. We define the action of G on V as follows

$$(\gamma_1, \gamma_2) \cdot (x, y, z) := (\gamma_1 x, \gamma_2 y, \gamma_1 \gamma_2 z), \quad (\gamma_1, \gamma_2) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2, \ (x, y, z) \in \mathbb{R}^3.$$

We put $\Omega = \{v \in V : \|v\| < 1\}$ and let $f : \Omega \to V$ be given by $f(v) = -v$, $v \in \Omega$. It is clear that $f^{-1}(0) = \{0\}$. We compute $G\text{-Deg}(f, \Omega) \in A_0(G)$ below.

We begin with the description of the generators of $A_0(G)$, i.e. the description of $\Phi_0(G)$. Let $\alpha_0 = (G)$, $\alpha_1 = (\mathbb{Z}_2 \oplus \{1\}) = (\{(1,1),(-1,1)\})$, $\alpha_2 = (\{1\} \oplus \mathbb{Z}_2) = (\{1,1),(1,-1)\})$, $\alpha_3 = (\{(1,1),(-1,-1)\}$ and $\alpha_4 = (\{1,1\})$. Consequently, $\Phi_0(G) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. As an "almost" regular generic approximation we take the map $g : \Omega \to V$, given by

$$g(x, y, z) = (-x(x - \tfrac{1}{2})(x + \tfrac{1}{2}), -y(y - \tfrac{1}{2})(y + \tfrac{1}{2}), -z(z - \tfrac{1}{2})(z + \tfrac{1}{2}), \quad (x, y, z) \in \Omega.$$

The G-map g is not exactly a regular generic G-map in the sense of Definition 1.2.1. However, the only violation of genericity property is that g, instead of acting in normal directions as identity, acts as cId, where $c > 0$. Evidently, g can be adjusted to a regular generic G-map, but by the homotopy invariance it is not really necessary (see also Remark 1.2.1).

We have the following orbits of zeros in the set $g^{-1}(0)$:

Orbit of Zeros Gv	Orbit Type (G_v)	sign $Dg(v)\|_{S_v}$
$\{(0,0,0)\}$	α_0	$+1$
$\{(0,-\frac{1}{2},0),(0,\frac{1}{2},0)\}$	α_1	-1
$\{(-\frac{1}{2},0,0),(\frac{1}{2},0,0)\}$	α_2	-1
$\{(0,0,-\frac{1}{2}),(0,0,\frac{1}{2})\}$	α_3	-1
$\{(-\frac{1}{2},-\frac{1}{2},0),(\frac{1}{2},-\frac{1}{2},0),(-\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},0)\}$	α_4	$+1$
$\{(-\frac{1}{2},0,-\frac{1}{2}),(\frac{1}{2},0,-\frac{1}{2}),(-\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},0,\frac{1}{2})\}$	α_4	$+1$
$\{(0,-\frac{1}{2},-\frac{1}{2}),(0,\frac{1}{2},-\frac{1}{2}),(0,-\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2})\}$	α_4	$+1$
$\{(\frac{1}{2},\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})\}$	α_4	-1
$\{(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},-\frac{1}{2})\}$	α_4	-1

Consequently, we obtain $G\text{-Deg}\,(f,\Omega)=\sum_{i=1}^{4} n_{\alpha_i}\cdot\alpha_i$, where

α	n_α
α_0	1
α_1	-1
α_2	-1
α_3	-1
α_4	1

or, more clearly

$$G\text{-Deg}\,(f,\Omega)=\alpha_0-\alpha_1-\alpha_2-\alpha_3+\alpha_4.$$

The following example illustrates the computation of G-degree for a regular generic G-map with one orbit of zeros.

Example 1.4.2. Let $G = S^1$ act on $V = \mathbf{C}$ by the formula

$$\gamma \cdot z = \gamma^n z, \quad \gamma \in S^1 \subset \mathbf{C},\ z \in \mathbf{C}$$

where $n \in \{1, 2, 3, \dots\}$ is a fixed positive integer. Let $\Omega \subset W := V \oplus \mathbf{R}$ be the set

$$\Omega := \{(z,t) \in \mathbf{C} \times \mathbf{R} :\ \tfrac{1}{2} < |z| < 2,\ -1 < t < 1\}.$$

We fix the standard orientation of $G = S^1$ and assume that for each n the orientation of $S^1/\mathbf{Z}_n$ is such that the natural homomorphisms $S^1 \to S^1/\mathbf{Z}_n$ preserves the orientations.

We define $f : \Omega \to V$ by

$$f(z,t) = \frac{z}{|z|}(1 - |z| + it),\ \ (z,t) \in \Omega.$$

Identify $\mathbf{C} \cong \mathbf{R}^2$. We see that f is S^1-equivariant and $f^{-1}(0) = \{(z,0) : |z| = 1\} = S^1 \times \{0\}$, i.e. $f^{-1}(0)$ consists of only one orbit. Moreover, since Ω contains points of only one orbit type $(\mathbf{Z}_n)$, f is a generic G-map. We choose the point $v = (1,0) \in f^{-1}(0)$ and consider the linear slice S to the orbit $Gv = f^{-1}(0)$ at $(1,0)$, i.e. $S = \{(x,0,t) \in \mathbf{R}^3 : x, t \in \mathbf{R}\} \subset V \oplus \mathbf{R}$. Fix an orientation of S (i.e. an ordered basis of S) in such a way that the orientation vector of $f^{-1}(0)$ at $(1,0,0)$, i.e. the vector $< 0,1,0 > \in \mathbf{R}^3 = V \oplus \mathbf{R}$, followed by the chosen basis of S results in an ordered basis of $\mathbf{R}^3$ giving the same orientation as the standard basis $\vec{e}_1 =< 1,0,0 >$, $\vec{e}_2 =< 0,1,0 >$, $\vec{e}_3 =< 0,0,1 >$. For example, we can choose the basis of S consisting of vectors $\{< 0,0,1 >, < 1,0,0 >\}$. Next, we consider the restriction f_0 of f to a neighbourhood of $(1,0,0)$ in the slice S. The mapping f_0 is expressed by

$$f_0(t,x) = \frac{x}{x}(1 - x + it) = (1 - x, t).$$

Note that f is a regular generic G-map and $\det_{\mathbb{R}} f_0(0,1) = 1 > 0$, the G-degree of f is

$$G\text{-Deg}\,(f,\Omega) = (\mathbf{Z}_n).$$

1.5. The Burnside ring and mutiplicativity property

Let V be a finite dimensional orthogonal representation of G and $f : \Omega \subset V \to V$ be an Ω-admissible equivariant map. The G-degree $G\text{-Deg}\,(f,\Omega)$ defined in the previous sections is related to the Burnside ring of G. We will briefly explain their relation in this section.

Let us first recall a definition of the Burnside ring. Two compact G-manifolds X and Y are said to be ***equivalent***, which we denote by $X \sim Y$, if for all subgroups $H \subset G$ the spaces X^H and Y^H have the same Euler characteristic. We denote by $A(G)$ the set of all equivalence classes of this relation, and by $[X] \in A(G)$ we denote the class of X. Disjoint union and Cartesian product of G-manifolds are compatible with this equivalence relation and induce the ***addition*** and ***multiplication*** operations on $A(G)$. Together with these two operations $A(G)$ is a commutative ring with identity, which is called the ***Burnside ring*** of G. We refer to tom Dieck [7] for more details.

Let $\Phi(G)$ denote the set of conjugacy classes (H) such that $N(H)/H$ is finite. It is well known (see tom Dieck [7]) that $A(G)$ is the free abelian group on $[G/H]$, $(H) \in \Phi(G)$, and for each compact G-manifold X, the following relation holds

$$[X] = \sum_{(H)\in\Phi(G)} \chi_c(X_{(H)})[G/H] \tag{1.5.1}$$

where χ_c denotes the Euler characteristic using homology with compact support. The multiplication table of the generators $[G/H]$ is given by the relation

$$[G/H]\cdot[G/K] = \sum_{(L)\in\Phi(G)} n_L[G/L] \tag{1.5.2}$$

where $n_L = \chi_c((G/H \times G/K)_{(L)}/G)$. Note that the space $(G/H \times G/K)_{(L)}/G$ is finite. Indeed, we have

$$\begin{aligned}(G/H \times G/K)_{(L)} &\cong (G/H \times G/K)_L/N(L)\\ &\subset (G/H \times G/K)^L/N(L)\\ &= (G/H^L \times G/K^L)/(N(L)/L).\end{aligned}$$

Since the spaces G/H^L and G/K^L consist of finitely many $N(L)/L$-orbits (see Bredon [3] and tom Dieck [7]) and by assumption that $N(L)/L$ is finite, both G/H^L and G/K^L are finite. Consequently the set $(G/H\times G/K)_{(L)}/G$ is finite.

In the case where G is an abelian group the formula (1.5.2) simplifies to

$$[G/H]\cdot[G/K] = n_{H\cap K}[G/(H\cap K)] \tag{1.5.3}$$

where $n_{H\cap K}$ is equal to the number of all $(H\cap K)$-orbits in $G/H \times G/K$. From this we may say that the number n_L in the formula (1.5.2) represents the number of elements in $(G/H \times G/K)_{(L)}/G$, i.e. it is the *number of G-orbits in $G/H\times G/K$ of the orbit type (L)*.

It is clear that, as an abelian group, $A(G)$ is naturally isomorphic to $A_0(G) = \mathbf{Z}\Phi_0(G)$, by a transformation which identifies a generator (H) of $A_0(G)$ with $[G/H] \in A(G)$. Consequently, in the case where $n = 0$, the G-degree coincides with the equivariant degree associated with the equivariant fixed point index studied by Dold [9], Ulrich [32] and others. In this case G-degree takes values in $A(G)$, which has an additional multiplicative structure. The following property of G-degree corresponds to the well known *multiplicativity property* of the fixed point index (see Ulrich [32], III.1.12).

Theorem 1.5.1. *In the case where $n = 0$ the G-equivariant degree satisfies the following property*

(P6) Multiplicativity Property. Let V, W be two orthogonal representations of G, and $\Omega \subset V$, $\mathcal{U} \subset W$ be two invariant open bounded subsets. If $f : V \to V$ (resp. $g : W \to W$) is an Ω-admissible (resp. $\mathcal{U}$-admissible) G-map, then the G-map $F : V \times W \to V \times W$, defined by $F(x, y) = (f(x), g(y))$, $(x, y) \in V \times W$, is $\Omega \times \mathcal{U}$-admissible and

$$G\text{-}Deg(F, \Omega \times \mathcal{U}) = G\text{-}Deg(f, \Omega) \cdot G\text{-}Deg(g, \mathcal{U})$$

where the product is taken in $A(G)$.

Proof. The proof can be found in Ulrich [32] and Gęba et al. [16]. We therefore omit it.

Remark 1.5.1. When G is *abelian,* corresponding to each $(H) \in \Phi_n(G)$, there exists a natural homomorphism of abelian groups

$$\Im : A(H) \to A_n(G)$$

which can be defined on the generators $[H/K]$ as follows

$$\Im([H/K]) = (K), \tag{1.5.4}$$

where $[H/K] \in \Phi(H)$. Since H/K is a finite group, G/K has dimension n and thus $(K) \in A_n(G)$. This remark will be used in the next section.

Remark 1.5.2. When G is abelian, we can also define an $A_0(G)$-module structure on $A_n(G)$ as follows. For every $(H) \in \Phi_n(G)$ and $(K) \in \Phi_0(G)$, the G-space $G/H \times G/K$ has only a finite number of G-orbits. In fact, $\dim G/H \cap K = n$ and $G/H \cap K$ acts freely on the manifold $G/H \times G/K$ of dimension n. The G-orbit space $(G/H \times G/K)/G = (G/H \times G/K)/G/(h \cap K)$ has dimension 0, i.e. it is finite. We therefore define the action $A_0(G) \times A_n(G) \to A_n(G)$ by

$$(K) \cdot (H) = n_{H \cap K}(H \cap K)$$

where $n_{H \cap K}$ is the number of the G-orbits in $G/H \times G/K$.

We finally present the following example illustrating the mutiplicativity property of the G-degree.

Example 1.5.1. Let us consider again Example 1.4.1, i.e. $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and G acts on $V = \mathbf{R}^3$ by the formula

$$(\gamma_1, \gamma_2) \cdot (x, y, z) := (\gamma_1 x, \gamma_2 y, \gamma_1 \gamma_2 z), \quad (\gamma_1, \gamma_2) \in \mathbf{Z}_2 \oplus \mathbf{Z}_2, \ (x, y, z) \in \mathbf{R}^3,$$

$\Omega = \{v \in V : \|v\| < 1\}$ $f : \Omega \to V$, $f(v) = -v$, $v \in \Omega$. The orbit types in $\Phi_0(G)$ are denoted by $\alpha_0 = (G)$, $\alpha_1 = (\mathbf{Z}_2 \oplus \{1\}) = (\{(1,1),(-1,1)\})$,

$\alpha_2 = (\{1\} \oplus \mathbf{Z}_2) = (\{1,1),(1,-1)\})$, $\alpha_3 = (\{(1,1),(-1,-1)\}$ and $\alpha_4 = (\{$
According to (1.5.3), we have the following multiplication table in $A_0(G)$

	α_0	α_1	α_2	α_3	α_4
α_0	α_0	α_1	α_2	α_3	α_4
α_1	α_1	$2\alpha_1$	α_4	α_4	$2\alpha_4$
α_2	α_2	α_4	$2\alpha_2$	α_4	$2\alpha_4$
α_3	α_3	α_4	α_4	$2\alpha_3$	$2\alpha_4$
α_4	α_4	$2\alpha_4$	$2\alpha_4$	$2\alpha_4$	$4\alpha_4$

Since $V = V^{\alpha_1} \oplus V^{\alpha_2} \oplus V^{\alpha_3}$ and $f(V^{\alpha_i}) \subset V^{\alpha_i}$ for $i = 1,2,3$, by Multiplicativity Property

$$G\text{-Deg}\,(f,\Omega) = G\text{-Deg}\,(f_1,\Omega_1) \cdot G\text{-Deg}\,(f_2,\Omega_2) \cdot G\text{-Deg}\,(f_3,\Omega_3)$$

where $\Omega_i = \Omega \cap V^{\alpha_i}$, $f_i := f|_{\Omega_i}$, $i = 1,2,3$. Note that for every $i = 1,2,3$, $f_i(t) = -t$. By using an "almost" regular generic approximation $g_i(t) = -t(t - \frac{1}{2})(t + \frac{1}{2})$ we obtain that $G\text{-Deg}\,(f_i,\Omega_i) = G\text{-Deg}\,(g_i,\Omega_i) = \alpha_0 - \alpha_i$, $i = 1,2,3$. Consequently,

$$\begin{aligned} G\text{-Deg}\,(f,\Omega) &= (\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_3) \\ &= (\alpha_0 - \alpha_1 - \alpha_2 + \alpha_4)(\alpha_0 - \alpha_3) \\ &= \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4. \end{aligned}$$

This computation agrees with that in Example 1.4.1.

1.6. A regular value formula for abelian groups

Throughout this section, we assume that G is an *abelian* compact Lie group, V is a finite dimensional orthogonal representation of G, $\Omega \subset V \oplus \mathbb{R}^n$ is an open bounded G-invariant subset and $f : V \oplus \mathbb{R}^n \to V$ is a G-equivariant Ω-admissible C^1-mapping such that

(a) 0 is a regular value of $f|_\Omega$;

(b) $f^{-1}(0) \cap \Omega = Gx_0$ for some $x_0 \in \Omega$;

(c) $\dim G/G_{x_0} = n$.

Let $H_0 = G_{x_0}$. Denote by S the linear orthogonal slice of the orbit Gx_0 at x_0, i.e. $S = \{x \in V \oplus \mathbb{R}^n;\ x - x_o \perp T_{x_0}(Gx_0)\}$. By Theorem 0.1.5, S is an orthogonal representation of H_0.

Suppose that for every $(H) \in \Phi_n(G)$ there has been chosen an invariant orientation of G/H. Since for two subgroups H_1 and H_2 of G such that $(H_1), (H_2) \in \Phi_n(G)$ and $H_1 \subset H_2$, the natural homomorphism $\varphi : G/H_1 \to G/H_2$ is a local diffeomorphism, we can assume without loss of generality that the chosen orientations in $\mathcal{J}(\Omega) \cap \Phi_n(G)$ are preserved by φ.

The purpose of this section is to provide an explicit formula for $G\text{-Deg}(f, \Omega)$.

Let $L := Df(x_0)|_S : S \to V$. By assumption (a), L is an H_0-equivariant isomorphism. The spaces S and V can be decomposed into the following H_0-invariant orthogonal direct sums

$$S = S^{H_0} \oplus T, \quad \text{and} \quad V = V^{H_0} \oplus T'.$$

As L is an H_0-equivariant isomorphism, $L(S^{H_0}) = V^{H_0}$ and $L(T) = T'$. Moreover, since $\{0\} \times \mathbf{R}^n \subset S^{H_0}$, it implies that $T \subset V \times \{0\}$ and therefore T and T' are the same subspace of V, under the identification of V with $V \times \{0\}$, which we will denote simply by T.

Let T_α denote the direct sum of all irreducible subrepresentations of T equivalent to a fixed irreducible representation of H_0 such that the isotropy group of its nonzero elements is K_α, where $\alpha = (K_alpha)$. The subspace T_α is called the (nontrivial) *α-isotypical component of* T. It follows that $L(T_\alpha) = T_\alpha$. In view of Theorem 0.2.7, we write

$$T = \bigoplus_{\alpha \in \mathcal{J}(T) \setminus (H_0)} T_\alpha,$$

which is called the nontrivial *isotypical decomposition* of T.

We fix an orientation of V^{H_0}. The orientation of G/H_0 followed by the orientation of V^{H_0} determines a unique orientation of S^{H_0}. Define an integer η_0 by

$$\eta_0 := \begin{cases} 1 & \text{if } L|_{S^{H_0}} \text{ preserves the orientations,} \\ -1 & \text{otherwise.} \end{cases}$$

Definition 1.6.2. Let T_α be an isotypical component of T as above. We define the *local α-index* of f on the orbit Gx_0 as the following element of $A(H_0)$

$$\alpha\text{-}\mathbf{ind}\,(f, Gx_0) \triangleq H_0\text{-Deg}(L|_{T_\alpha}, B \cap T_\alpha)$$

where B denotes the unit ball in T.

Now we can formulate the main result of this section, the *regular value formula* for an isolated orbit of zeros.

Theorem 1.6.1. (REGULAR VALUE FORMULA) *Under the above assumptions, we have*

$$G\text{-}Deg(f,\Omega) = \eta_0 \Im\Big(\prod_{\alpha\in\mathcal{J}(T)\setminus(H_0)} \alpha\text{-}\mathbf{ind}(f,Gx_0)\Big),$$

where the homomorphism $\Im$ *is defined by (1.5.4).*

Proof. Let $\delta > 0$ be given and $D = \{x \in S;\ \|x\| < \delta\}$. From Theorem 0.1.5 and the excision property, by choosing a sufficiently small $\delta > 0$, we may assume that $\Omega = G\times_{H_0} D$, i.e. Ω is a *tube* around the orbit Gx_0. Define $\tilde{f} : \overline{D} \to V$ by $\tilde{f}(x) := f(x+x_0)$. Then $\tilde{f}$ is an H_0-equivariant D-admissible map. Therefore, the equivariant degree $H_0\text{-Deg}(\tilde{f},D) \in A(H_0)$ is well defined. It follows directly from the definition of the G-equivariant degree [16] and the assumption on the orientations of G/K, $(K) \in \Phi_n(G)\cap\mathcal{J}(\Omega)$, that $G\text{-Deg}(f,\Omega) = \Im(H_0\text{-Deg}(\tilde{f},D))$.

Since 0 is a regular value of $\tilde{f}$ such that $\tilde{f}^{-1}(0) = \{0\}$, by taking δ sufficiently small, we may assume that $\tilde{f}$ is D-homotopic to L. Therefore, $\Im(H_0\text{-Deg}(L,D)) = G\text{-Deg}(f,\Omega)$. Since $L|_T = \bigoplus L_\alpha : \bigoplus T_\alpha \to \bigoplus T_\alpha$, where $L_\alpha = L|_{T_\alpha} : T_\alpha \to T_\alpha$, it follows from the Multiplicativity Property (P6) that

$$\begin{aligned} H_0\text{-Deg}(L,D) &= \eta_0 \prod_{\alpha\in\mathcal{J}(T)\setminus(H_0)} H_0\text{-Deg}(L_\alpha, B\cap T_\alpha) \\ &= \eta_0 \prod_{\alpha\in\mathcal{J}(T)\setminus(H_0)} \alpha\text{-}\mathbf{ind}(f,Gx_0). \end{aligned}$$

This completes the proof.

We put

$$\mathcal{F}_{2,H_0} := \{\alpha \in \mathcal{J}(T);\ H_0/K_\alpha \cong \mathbb{Z}_2\}$$

and define the H_0-equivariant automorphism $L_\alpha : T_\alpha \to T_\alpha$ by $L_\alpha := L|_{T_\alpha}$. We also define for $\alpha = (K_\alpha)$ the following element of the Burnside ring $A(H_0)$

$$\eta_\alpha(L) := \begin{cases} 0, & \text{if} \quad \det L_\alpha > 0 \\ [H_0/H_0] - [H_0/K_\alpha], & \text{if} \quad \det L_\alpha < 0. \end{cases}$$

We have the following result.

Proposition 1.6.2. *(Gęba et al. [16]) Under the above assumptions we have*

$$\alpha\text{-}\mathbf{ind}\,(f, Gx_0) = \begin{cases} \eta_\alpha(L) & \textit{if } \alpha \in \mathcal{F}_{2,H_0} \\ [H_0/H_0] & \textit{otherwise.} \end{cases}$$

Proof. For the sake of completness we include the proof, which is taken from [16].

Let $\alpha \notin \mathcal{F}_{2,H_0}$. Then T_α can be equipped with a complex structure such that an automorphism of T_α is H_0-equivariant if and only if it is a complex automorphism. Therefore, by connectness of the groups $GL(n, \mathbb{C})$, L_α can be connected to the identity Id by a continuous path in the space of H_0-equivariant linear automorphisms of T_α. If we use this path as a $B \cap T_\alpha$-admissible homotopy between L_α and Id, we obtain that $\alpha\text{-}\mathbf{ind}\,(f, Gx_0) = 1$.

Assume now that $\alpha \in \mathcal{F}_{2,H_0}$. Since $H_0/K_\alpha \cong \mathbb{Z}_2$, every $\mathbb{R}$-linear automorphism of T_α is H_0-equivariant. The linear group $GL(T_\alpha)$ has two connected components. If $\det L_\alpha > 0$ then L_α can be connected by a continuous path in $GL(T_\alpha)$ to Id, and consequently, by the same argument as in the previous case, $\alpha\text{-}\mathbf{ind}\,(f, Gx_0) = 1$. If $\det L_\alpha < 0$, then L_α can be connected by a path in $GL(T_\alpha)$ to a linear map $\tilde{L}_\alpha$ which has the following representation with respect to a certain basis in T_α

$$\tilde{L}_\alpha(t_1, t_2, \ldots, t_m) = (-t_1, t_2, \ldots, t_m) \in T_\alpha.$$

We define a (nonlinear) map $B_\alpha : T_\alpha \to T_\alpha$ by

$$B_\alpha(t_1, t_2, \dots, t_m) = (-t_1(t_1 - \tfrac{\delta}{2})(t_1 + \tfrac{\delta}{2}), t_2, \dots, t_m).$$

Clearly, B_α is $D \cap T_\alpha$-homotopic to L_α , and the equation $B_\alpha(t) = 0$ has the following types of solutions

(i) the zero solution $t = 0$ of the orbit type (H_0);

(ii) one orbit of solutions $t = (\pm\frac{\delta}{2}, 0, \dots, 0)$ of the orbit type $\alpha = (K_\alpha)$ such that det $DB_\alpha(t) < 0$.

Consequently, we obtain that

$$\alpha\text{-}\mathbf{ind}\,(f, Gx_0) = [H_0/H_0] - [H_0/K_\alpha]$$

and the proof is complete.

Combining Theorem 1.6.1 and Proposition 1.6.2 leads to the following explicit computation formula for the G-degree.

Corollary 1.6.3. *Under the above assumptions we have*

$$G\text{-}Deg(f, \Omega) = \eta_0 \Im(\prod_{\alpha \in \mathcal{F}_{2,H_0}} \eta_\alpha(L)).$$

We end this section with a simple example.

Example 1.6.1. Let $V := \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^4$ and $G = S^1$ act on V by

$$\gamma \cdot (z_1, z_2) = (\gamma^n z_1, \gamma^{2n} z_2), \quad (z_1, z_2) \in V,\ \gamma \in S^1$$

where n is a positive integer. Let

$$\Omega = \{(z_1, z_2, t) \in V \oplus \mathbb{R} : |z_1| < 1,\ \tfrac{1}{2} < |z_2| < 2,\ |t| < 1\}$$

and define $f : \Omega \to V$ by

$$f(z_1, z_2, t) = (\bar{z}_1 z_2, \tfrac{z_2}{|z_2|}(1 - |z_2| + it)),\quad (z_1, z_2, t) \in \Omega.$$

The mapping f is S^1-equivariant. Indeed,

$$\begin{aligned} f(\gamma \cdot (z_1, z_2, t)) &= f(\gamma^n z_1, \gamma^{2n} z_2, t) \\ &= (\gamma^{-n} \bar{z}_1 \gamma^{2n} z_2, \tfrac{\gamma^{2n} z_2}{|z_2|}(1 - |z_2| + it)) \\ &= (\gamma^n (\bar{z}_1 z_2), \tfrac{\gamma^{2n} z_2}{|z_2|}(1 - |z_2| + it)) \\ &= \gamma \cdot (\bar{z}_1 z_2, \tfrac{z_2}{|z_2|}(1 - |z_2| + it)) \\ &= \gamma \cdot f(z_1, z_2, t). \end{aligned}$$

Note that $f^{-1}(0) = \{(0, z_2, 0) : |z_2| = 1\} = G(0,1,0)$ is exactly the orbit of $v_0 = (0,1,0) \in \Omega$. The isotropy group of v_0 is $G_{v_0} = \mathbb{Z}_{2n} =: H$. The linear slice S to the orbit Gv_0 at $(0,1,0)$ reads

$$S = \{(z_1, z_2, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R} : \operatorname{Im} z_2 = 0\} = \mathbb{C} \times \mathbb{R} \times \mathbb{R}.$$

The restriction f_0 of f to the slice $S \cap \Omega$ can be expressed by

$$f_0(z_1, t, x_2) = (\bar{z}_1 x_2, 1 - x_2 + it),\quad (z_1, t, x_2) \in S \cap \Omega$$

where we have chosen the ordered basis $\vec{e}_1 = (1,0,0,0,0)$, $\vec{e}_2 = (0,1,0,0,0)$, $\vec{e}_3 = (0,0,0,0,1)$, $\vec{e}_4 = (0,0,1,0,0)$ in the slice S such that the orientation of the orbit followed by the orientation of the slice gives us the standard orientation of the space $V \oplus \mathbb{R}$. The isotypical decomposition of S is

$$S = T_{\alpha_n} \oplus T_{\alpha_{2n}}$$

where $\alpha_n = (\mathbb{Z}_n)$, $\alpha_{2n} = (\mathbb{Z}_{2n})$, and $T_{\alpha_n} \cong \mathbb{C}$ is exactly the first $\mathbb{C}$-component of V, and $T_{\alpha_{2n}} = \{(t, x_2) \in \mathbb{R}^2 : t, x_2 \in \mathbb{R}\}$ is the space of the stationary points of the action of $H_0 = \mathbb{Z}_{2n}$ on S, i.e. $T_{\alpha_{2n}} = S^{H_0}$. The derivative $L = Df_0(0, 1, 0)$ has the following matrix form

$$L = \begin{array}{c} \\ x_1 \\ y_1 \\ t \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & y_1 & t & x_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Thus $\eta_0 = 1$ and $\eta_{\alpha_n}(L) = [H_0/H_0] - [H_0/\mathbb{Z}_n]$. Note that $\mathcal{F}_{2,H_0} = \{\alpha_n\}$. It follows that

$$\begin{aligned} H_0\text{-Deg}(L, D) &= [H_0/H_0] \cdot ([H_0/H_0] - [H_0/\mathbb{Z}_n]) \\ &= [H_0/H_0] - [H_0/\mathbb{Z}_n]. \end{aligned}$$

Consequently,

$$G\text{-Deg}(f, \Omega) = (\mathbb{Z}_{2n}) - (\mathbb{Z}_n).$$

References

[1] Th. Bartsch, *A simple proof of the degree formula for $\mathbb{Z}/p$-equivariant maps*, Math. Z. **212** (1993), 285–292.

[2] ——— and J. Mawhin, *The Leray-Schauder degree of S^1-equivariant operators associated to autonomous neutral equations in spaces of periodic functions*, J. Differential Equations **92** (1991), 90–99.

[3] G. E. Bredon, "Introduction to Compact Transformation Groups", Academic Press, New York, 1972.

[4] E. N. Dancer, *Symmetries, degree, homotopy indices and asymptotically homogeneous problems*, Nonlinear Analysis-TMA **6** (1982), 667–686.

[5] ———, *Perturbation of zeros in the presence of symmetries*, J. Austral. Math. Soc. **36A** (1984), 106–125.

[6] ———, *A new degree for S^1-invariant gradient mappings and applications*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **2** (1985), 329–370.

[7] T. tom Dieck, "Transformation Groups", de Gruyter, Berlin-New York, 1987.

[8] A. Dold, *The fixed point index of fibre preserving maps*, Invent. Math. **25** (1974), 281–297.

[9] ——, *Fixed point indices and homotopy theory*, Contem. Math. **12** (1982), 105–115.

[10] ——, *Fixed point indices of iterated maps*, Invent. Math. **74**(1983), 419–435.

[11] G. Dylawerski, *An S^1-degree and S^1-maps between representation sphere*, Algebraic Topology and Transformation Groups (T. tom Dieck, ed.), Lecture Notes in Mathematics **1361**, Springer-Verlag, 1988, 14–28.

[12] ——, *A relation between the Dancer S^1-degree and the generlized S^1-degree*, Preprint.

[13] ——, K. Gęba, J. Jodel and W. Marzantowicz, *S^1-equivariant degree and the Fuller index*, Ann. Pol. Math. **52** (1991), 243–280.

[14] L. H. Erbe, K. Gęba and W. Krawcewicz, *Equivariant fixed point index and the period doubling cascades*, Can. J. Math. **43** (1991), 738–747.

[15] F. B. Fuller, *An index of fixed point type for periodic orbits*, Amer. J. Math. **89** (1967), 133–148.

[16] K. Gęba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems* I: *Construction of the degree*, to appear in J. Lond. Math. Soc..

[17] ———, I. Massabó and A. Vignoli, *Generalized topological degree and bifurcation*, in Nonlinear Functional Analysis and its Applications (S. P. Singh, ed.), Mathematical and Physical Sciences **173**, 54–73, Reidel, Dordrecht, 1986.

[18] ——, —— and ——, *On the Euler characteristic of equivariant gradient vector fields*, Bollettino Unione Mat. Italiana **4-A** (1990), 243–251.

[19] J. Ize, I. Massabó and A. Vignoli, *Global results on continuation and bifurcation for equivariant maps*, in Nonlinear Functional Analysis and its Applications (S. P. Singh, ed.), Mathematical and Physical Sciences **173**, 74–111, Reidel, Dordrecht, 1986.

[20] ——, —— and ——, *Degree theory for equivariant maps* I, Trans. Amer. Math. Soc. **315** (1989), 433–510.

[21] ——, —— and ——, *Degree theory for equivariant maps, the general S^1 - action* II, Memoirs of Amer. Math. Soc., Volume **100**, Number 481, 1992.

[22] K. Kawakubo, "The Theory of Transformation Groups", Oxford University Press, Oxford, 1991.

[23] K. Komiya, *Fixed point indices of equivariant maps and Möbius inversion*, Invent. Math. **91** (1988), 129–135.

[24] N. Lloyd, "Degree Theory", Cambridge University Press, Cambridge, 1978.

[25] W. Lück, *The equivariant degree*, In Algebraic Topology and Transformation Groups (T. tom Dieck, ed.), Lecture Notes in Mathematics **1361**, Springer-Verlag, 1988, 123–167.

[26] W. Marzantowicz, *On the nonlinear elliptic equations with symmetry*, J. Math. Anal. Appl. **91** (1981), 156–181.

[27] G. Peschke, *Degree of certain equivariant maps into a representation sphere*, to appear in Topology.

[28] C. Prieto and H. Ulrich, *Equivariant fixed point index and fixed point transfer in nonzero dimensions*, Trans. Amer. Math. Soc. **328** (1991), 731–745.

[29] R. Rubinsztein, *On the equivariant homotopy of spheres*, Dissert. Math. **134** (1976), 1–48.

[30] G. B. Segal, *Equivariant stable homotopy theory*, Actes Congr. Internat. Mat. **2** (1970), 59–63.

[31] J. Tornehave, *Equivariant maps of spheres with conjugate orthogonal actions*, Can. Math. Soc. Conf. Proc. **2-2** (1982), 275–301.

[32] H. Ulrich, *Fixed point theory of parametrized equivariant maps*, Lecture Notes in Mathematics **1343**, Springer-Verlag, Berlin, 1988.

CHAPTER 2

SYMMETRIC BIFURCATION THEORY IN BANACH G-SPACES

2.1. Introduction

This chapter studies the bifurcation problem for the following composite coincidence equation

$$L_0[\pi_0(x) - B_0(x)] = N_0(x), \quad \pi_0(x) - B_0(x) \in \mathrm{Dom}\,(L_0), \tag{BP}$$

where V and W are two isometric Banach representations of $G := \Gamma \times S^1$, Γ is a compact Lie group, $L_0 : \mathrm{Dom}\,(L_0) \subset V \to W$ is a given equivariant closed Fredholm operator of index zero, $\pi_0 : V \times \mathbb{R}^2 \to V$ is the natural projection, $B_0 : V \times \mathbb{R}^2 \to V$ and $N_0 : V \times \mathbb{R}^2 \to W$ are G-maps of class C^1 such that (B_0, N_0) is an L_0-condensing G-pair. We shall apply the G-equivariant degree together with Ize's complementary function method to obtain equivariant analogs of the local bifurcation theorem of Krasnosel'skii [28] and the global bifurcation theorem of Rabinowitz [40].

Our motivation to study such a *two-parameter* bifurcation problem is two-fold. First, we try to provide a *degree theoretic proof* of the local and global symmetric *Hopf bifurcation* theorem for a class of one-parameter functional differential equations of neutral type (NFDEs). After an appropriate reformulation, a Hopf bifurcation problem of NFDEs is abstracted to a bifurcation problem of a composite coincidence equation of type (BP) with two parameters, where the additional parameter comes from the period of periodic solutions, the action of S^1 is introduced from the usual shifting of the temporal argument and the action of Γ represents the spatial symmetry of NFDEs (see Chapter 3 and the references there). Second, the bifurcation problem (BP) is also important in its own right in applications. For examples in this direction, we refer to Chossat et al. [6], Dancer [9-11], Golubitsky et al. [22], Sattinger [46] and Vanderbauwhede [49].

In the nonequivariant case, i.e. when $\Gamma \times S^1$ acts on V and W trivially, the bifurcation problem (BP) with *one parameter* has been studied extensively (see Chow and Hale [7], Deimling [12], Gaines and Mawhin [19] and the references therein). Among numerous bifurcation results the general existence theorem of Krasnosel'skii [28] for *local* bifurcation points and a two-fold alternative theorem of Rabinowitz [40-42] are most well-known. We refer to Chiappinelli [5], Deimling [12], Gaines and Mawhin [19], Stuart [47] and Thomas [48] for various extensions of Krasnosel'skii type bifurcation theorem, and to Alexander [1], Alexander et al. [2-4], , Erbe et al. [14], Fitzpatrick et al. [16–18], Gęba et al. [20], Gęba and Marzantowicz [21], Hetzer [23], Ize [25], Krawcewicz et al. [29], MacBain [30], Makhmudov and Aliev [31], Pesachowicz [38], Webb and Welsh [50] and Welsh [51] for global bifurcation considerations. Since these two types of bifurcation theorems are proved by topological degree arguments, it is natural to use the equivariant degree to study the symmetric bifurcation problem (BP). However, the local and global bifurcation theorems with symmetry we shall provide in this chapter are analogues, rather than generalizations, of those in the nonequivariant case, since the action of S^1, the temporal argument shifting, is *not trivial* on V. Moreover, in the case where Γ is *abelian*, the nontriviality of the action of S^1 is essential for allowing us to include two parameters in the bifurcation problem (BP).

The effect of the presence of symmetry on the existence of bifurcaton points has been studied by many authors, among whom we mention Golubitsky, Stewart and Schaeffer [23] and Vanderbauwhede [49] where analytic methods are employed in dealing with bifurcation problems with symmetry. For the use of equivariant degree in symmetric bifurcation, we refer to Dancer [9-11], Erbe et al. [14], Gęba et al. [20], Gęba and Marzantowicz [21], Ize et al. [26] and Krawcewicz et al. [29].

The work of Gicogna [8] and Sattinger [45] is also interesting, where by restricting the equation to a certain invariant fixed point subspace the bifurcation problem is reduced to that in the nonequivariant case.

Our approach in this chapter follows the main lines of [20], which we explain as follows. In order to study the bifurcation problem (BP), we need a G-equivariant degree in infinite dimensional space. We shall use the equivariant bijection theorem (Theorem 0.3.1) to extend the G-degree of Gęba, Krawcewicz and Wu for compact fields to condensing fields and obtain an equivariant version of composite coincidence degree for (BP) (see Section 2.3). In Section 2.4, we apply the equivariant composite coincidence degree in conjunction with Ize's complementary function method and an equivariant analog of Krasnosel'skii local bifurcation theorem is obtained. Since the nontriviality of a G-degree is involved in this theorem, we compute the value of G-degree for condensing fields by reducing it to that in finite dimensional case where an explicit computational formula for an ***abelian*** Γ is available (see Section 2.2). It turns out that when Γ is abelian the G-degree for condensing field related to (BP) is compleletely determined by the information of its linear approximation on certain isotypical components of V with respect to the $\Gamma \times S^1$ action. Finally, by using the computation formula in Section 2.4, we prove in Section 2.5 global bifurcation theorems of Rabinowitz type.

This chapter extends the results of Gęba, Krawcewicz and Wu [20] for compact fields to those for condensing fields. It also generalizes sections 2–4 of Krawcewicz, Wu and Xia [29], where Γ is assumed to be a trivial group.

2.2. A computation formula

In this section, we present a computation formula for $G\text{-Deg}(f, \Omega)$. For its proof, we refer to Gęba, Krawcewicz and Wu [20].

Let Γ be a compact *abelian group* and $G = \Gamma \times S^1$. We choose the natural orientation of S^1 and assume that for every $(H) \in \Phi_1(G)$ there has been chosen an invariant *concordant orientation* of G/H, i.e. for $(H_1), (H_2) \in \Phi_1(G)$ with $H_1 \subset H_2$, the natural homomorphism $\varphi : G/H_1 \to G/H_2$, which is a local diffeomorphism, preserves the chosen orientations. This concordant orientation will be called *natural* if for $H = \Gamma \times \{1\}$, $G/H = S^1$ has been chosen its natural orientation.

Let V be a finite dimensional real orthogonal representation of G and let

$$V = V^G \oplus \bigoplus_{\beta \in \mathcal{B}} V_\beta$$

be the isotypical decomposition of V. It follows that every non-zero element in V_β has the same orbit type H_β. Put

$$\mathcal{B}_0 = \{ \beta \in \mathcal{B};\ \dim G/H_\beta = 0 \};$$

$$\mathcal{B}_1 = \{ \beta \in \mathcal{B};\ \dim G/H_\beta = 1 \};$$

$$\mathcal{B}_{02} = \{ \beta \in \mathcal{B};\ \dim H/H_\beta \cong \mathbb{Z}_2 \}.$$

Let us denote by $GL^G(V)$ (resp. $GL^G(V_\beta)$) the group of all equivariant linear automorphisms of V (resp. V_β) and let

$$\omega : S^1 \to GL^G(V)$$

be a continuous mapping. Since $A \in GL^G(V)$ implies $A(V_\beta) = V_\beta$ for all β, there are defined the restrictions

$$\omega_\beta : S^1 \to GL^G(V_\beta), \quad \omega_\beta(\lambda) := \omega(\lambda)|_{V_\beta},\ \lambda \in S^1.$$

For every $\beta \in B_1$, we can find the natural inclusion $S^1 \subset G/H_\beta$, and the restricted action of S^1 on the component V_β induces a complex structure on V_β. It then follows that an $\mathbf{R}$-linear operator $A : V_\beta \to V_\beta$ is G-equivariant if and only if it is $\mathbf{C}$-linear with respect to this complex structure. Consequently, the group $GL^G(V_\beta)$ is the group of $\mathbf{C}$-linear automorphisms of V_β and we may define an integer μ_β by setting

$$\mu_\beta := \deg(\det_{\mathbf{C}}(\omega_\beta))$$

where deg denotes the Brouwer degree. We call μ_β the *winding number* of ω_β. This leads to a well-defined element

$$\mu(\omega) := \sum_{\beta \in B_1} \mu_\beta (H_\beta) \in A_1(G)$$

for every continuous map $\omega : S^1 \to GL^G(V)$. We call $\mu(\beta)$ the *winding degree* of ω.

On the other hand, for every $\beta \in B_{02}$, we can define an integer ν_β by

$$\nu_\beta := \frac{1}{2}(1 - \operatorname{sign} \det \omega_\beta(\lambda))$$

and an element $\nu(\omega) \in A_0(G)$ by

$$\nu(\omega) := \prod_{\beta \in B_{02}} ((G) - \nu_\beta (H_\beta)),$$

where the product is taken in the Burnside ring $A_0(G)$. Note that $A_1(G)$ is an $A_0(G)$-module (see Remark 1.5.2). The product $\nu(\omega) \cdot \mu(\omega)$ defines an element in $A_1(G)$.

Assume now that $f : V \times \mathbf{R}^2 \to V \times \mathbf{R}$ is a G-equivariant C^1-map. We denote by $P : V \times \mathbf{R} \to V$ the natural projection onto V. We make the following hypotheses:

(A) There is an open bounded invariant set $\Omega \subset V \times \mathbf{R}^2$ such that f is Ω-admissible, 0 is a regular value for $f|_\Omega$ and $\Sigma := f^{-1}(0) \cap \Omega \subset V^G \times \mathbf{R}^2$ is diffeomorphic to the unit circle S^1.

By the assumption (A), G-Deg(f, Ω) is well defined. Our goal in this section is to give an explicit formula for the computation of G-Deg(f, Ω).

Fix an orientation of $V^G \times \mathbf{R}$ and orient $V^G \times \mathbf{R}^2 = (V^G \times \mathbf{R}) \times \mathbf{R}$ with the product orientation. For $x \in \Sigma$ the derivative $Df(x)$ maps $(T_x\Sigma)^\perp$ isomorphically onto $V^G \times \mathbf{R}$ and thus it induces an orientation of $(T_x\Sigma)^\perp$. Let $\eta : S^1 \to \Sigma$ be a diffeomorphism such that the chosen orientation of $(T_x\Sigma)^\perp$ followed by the orientation of $T_x\Sigma$, induced from that of S^1 by η, yields the orientation of $V^G \times \mathbf{R}^2$. Define $\omega : S^1 \to GL^G(V)$ by $\omega(\lambda) := PDf(\eta(\lambda))|_V \in GL^G(V)$, $\lambda \in S^1$, and consequently we arrive at the winding degree $\mu(\omega)$ of ω.

The following theorem is a particular case of the main result proved in [20].

Theorem 2.1.1. *Suppose that $f : V \times \mathbf{R}^2 \to V \times \mathbf{R}$ satisfies the assumption (A). Then, with the same notation as above, we have*

$$G\text{-Deg}(f, \Omega) = \nu(\omega) \cdot \mu(\omega) = \left(\prod_{\beta \in \mathcal{B}_{02}} ((G) - \nu_\beta(H_\beta)) \right) \left(\sum_{\beta \in \mathcal{B}_1} \mu_\beta(H_\beta) \right)$$

where $\omega(\lambda) = PDf(\eta(\lambda))|_V$, $\lambda \in S^1$.

Note that $\{1\} \times S^1$ is a subgroup of G. We have

$$V = V_\infty \times V_1 \times V_2 \times \cdots \times V_k,$$

the isotypical decomposition of V with respect to the restricted action of S^1, i.e. for $x \in V_j \setminus \{0\}$, we have $G_x \cap (\{1\} \times S^1) = \mathbf{Z}_j$, $j = 1, 2, \ldots, k, \infty$, $\mathbf{Z}_\infty := S^1$. Note that we do not exclude the case where $V_j = \{0\}$ for some indices j. Identify $S^1 \cong \{e^{i\theta},\ 0 \leq \theta < 2\pi\}$. The S^1-action induces a complex structure on each V_j, $j = 1, 2, \ldots, k$, which can be defined as follows

$$(a + ib)x := ax + b\exp(i\tfrac{\pi}{2j})x, \quad a + ib \in \mathbb{C},\ x \in V_j.$$

Let $\beta \in \mathcal{B}$. By definition, there is a positive integer k such that $V_\beta \subset V_j$. It can be verified that there is a homomorphism $\rho_\beta : \Gamma \to S^1/\mathbf{Z}_j$ such that

$$H_\beta = \{(\gamma, z) \in \Gamma \times S^1 : z \in \rho_\beta(\gamma)\}.$$

An orbit type of this form will be called a *basic orbit type*. It follows that $\dim G/H_\beta = 1$ and $\mathcal{B}_1 = \mathcal{B}_{11} := \{\beta \in \mathcal{B};\ V_\beta \subset V_j \text{ for some } j > 0\}$. As examples, for $\Gamma = \mathbf{Z}_n$, where n is a positive integer, all possible basic orbit types of $\Gamma \times S^1$ are $(H_{k,r})$, $0 \leq r \leq n-1$, $k = 0, 1, 2, \ldots$, where

$$H_{k,r} := \left\{ (e^{i \cdot \frac{2\pi j}{n}}, e^{i \cdot \frac{2\pi}{k}(m - \frac{rj}{n})}) \in \mathbf{Z}_n \times S^1 : \begin{matrix} 0 \leq j \leq n-1 \\ 0 \leq m \leq k-1 \end{matrix} \right\}$$

In the case where $\Gamma = S^1 = \mathbf{Z}_\infty$, all possible basic orbit types are $(K_{k,r})$, where $k, r \geq 0$ are two integers, and

$$K_{k,r} := \{(e^{2\pi\tau}, e^{i \cdot \frac{2\pi}{k}(m - r\tau)}) \in S^1 \times S^1 : \tau \in [0,1], 0 \leq m \leq k-1\}.$$

For notational convenience, let $B_1(G)$ (resp. $\tilde{B}_1(G)$) denote the $\mathbf{Z}$-submodule of $A_1(G)$ generated by all basic (resp. non-basic) orbit types of G. We have the following corollary.

Corollary 2.1.2. *Under the same assumptions as in Theorem 2.1.1,*

$$G\text{-}Deg(f,\Omega) = \mu(\omega) + \Lambda(\omega)$$

where $\mu(\omega) = \sum_{\beta\in\mathcal{B}_{11}} \mu_\beta(H_\beta) \in B_1(G)$ *is the winding degree of* $\omega : S^1 \to GL^G(V)$ *defined by* $\omega(\lambda) = PDf(\eta(\lambda))|_V$ *and* $\Lambda(\omega) \in \tilde{B}_1(G)$.

2.3. G-degree for equivariant condensing fields

In this section, we extend the G-degree in finite dimensional spaces to that for condensing fields in Banach G-spaces. This will allow us to study equivariant bifurcation problems via G-equivariant degree.

We begin by defining the G-degree for compact fields.

Let W be a real infinite dimensional isometric Banach representation of G. We define the G-degree for compact fields by applying the standard method of finite dimensional approximations. The following result is already known (see also [13, 20]). For the sake of convenience, we will present its proof.

Theorem 2.3.1. *Let X be a G-space and $F : X \to W$ a G-equivariant compact mapping. Then for any $\varepsilon > 0$ there exists an equivariant finite-dimensional map $F_\varepsilon : X \to W$ such that*

$$\|F_\varepsilon(x) - F(x)\| < \varepsilon \quad \text{for all} \quad x \in X.$$

Proof. Since $F(X)$ is relatively compact, there exists a finite set $N = \{w_1, \ldots, w_n\}$ W_{fin} such that $F(X) \subseteq N_\varepsilon := N + B_\varepsilon(0)$, where $B_\varepsilon(0) = \{w \in W;\ \|w\| < \varepsilon\}$. Let $\mu_i : N_\varepsilon \to \mathbf{R}$ denote the mapping defined by

$$\mu_i(w) = \max\{0, \varepsilon - \|w - w_i\|\}, \quad i = 1, \ldots, n,\ w \in N_\varepsilon$$

and put

$$P_\varepsilon(w) = \frac{1}{\sum_{i=1}^{n} \mu_i(w)} \sum_{i=1}^{n} \mu_i(w) w_i, \quad w \in N_\varepsilon.$$

This leads to a map $\widetilde{F}_\varepsilon : X \to W$ below

$$\widetilde{F}_\varepsilon(x) = P_\varepsilon(F(x)), \quad x \in X.$$

Then $\widetilde{F}_\varepsilon$ is an ε-approximation of F with $\widetilde{F}_\varepsilon(X) \subseteq \operatorname{span}\{w_1, \ldots, w_n\}$. Averaging $\widetilde{F}_\varepsilon$ over G, we get an equivariant map $F_\varepsilon : X \to W$, i.e.

$$F_\varepsilon(x) = \int_G g\widetilde{F}_\varepsilon(g^{-1}x)dg, \quad x \in X.$$

The property of the Haar integral ensures that F_ε is an equivariant ε-approximation of F. Moreover, $N \subseteq W_{fin}$ implies that GN is contained in a finite-dimensional invariant subspace, and so is $F_\varepsilon(X)$. This completes the proof.

Suppose now that n is a non-negative integer. Then the action of G on W induces a diagonal isometric action on $W \times \mathbf{R}^n$ with the trivial action on $\mathbf{R}^n$.

Definition 2.3.1. Assume that Ω is an open bounded invariant subset of $W \times \mathbf{R}^n$. An equivariant continuous map $f : W \times \mathbf{R}^n \to W$ is called an Ω-*admissible* compact (resp. condensing) field if

(i) $f(x,\lambda) \neq 0$ for all $(x,\lambda) \in \partial\Omega$.

(ii) $f = \pi - F$, where $\pi : W \times \mathbb{R}^n \to W$ denotes the natural projection on W and $F : W \times \mathbb{R}^n \to W$ is an equivariant compact (resp. condensing) mapping.

By definition, if f is an Ω-admissble compact (resp. condensing) field, then $F \in \operatorname{Comp}^G(\overline{\Omega}, \partial\Omega)$ (resp. $\operatorname{Cond}^G(\overline{\Omega}, \partial\Omega)$).

Let $f : W \times \mathbb{R}^n \to W$ be a given Ω-admissible map. By Theorem 2.3.1, we can find an equivariant finite-dimensional map $F_\epsilon : W \times \mathbb{R}^n \to W$ such that

$$\|F_\epsilon(x) - F(x)\| < \inf\{\|y - F(y,\lambda)\|;\ (y,\lambda) \in \partial\Omega\} \tag{P1}$$

and

$$F_\epsilon(W \times \mathbb{R}^n) \subseteq W_0 \subseteq W \tag{P2}$$

where W_0 is a finite dimensional G-invariant subspace of W. We define

$$G\text{-Deg}(f,\Omega) := G\text{-Deg}\big((\pi - F_\epsilon)|_{W_0 \times \mathbb{R}^n}, \Omega \cap W_0 \times \mathbb{R}^n\big).$$

By applying the same arguments as in the nonequivariant case, one can verify that the above definition of G-Deg(f,Ω) does not depend on the choice of the equivariant approximation F_ϵ as well as the invariant subspace W_0 satisfying (P1) and (P2). Moreover, the defined G-Deg(f,Ω) possesses the standard existence, excision, homotopy invariance, additivity and product properties.

We now illustrate how to apply the equivariant bijection theorem (Theorem 0.3.1) to define G-degree for equivariant condensing fields. Our approach represents an analog in the equivariant case of that introduced by Nussbaum ([34-36])

in the non-equivariant case. From now on, let Ω be an invariant open bounded subset of $W \times \mathbf{R}^n$ and $F \in \mathrm{Cond}^G(\overline{\Omega}, \partial\Omega)$. By Theorem 0.3.1, there exists $F_1 \in \mathrm{Comp}^G(\overline{\Omega}, \partial\Omega)$ such that $F \sim F_1$ in $\mathrm{Cond}^G(\overline{\Omega}, \partial\Omega)$. Moreover, if $F_2 \in \mathrm{Comp}^G(\overline{\Omega}, \partial\Omega)$ is another compact map such that $F \sim F_2$ in $\mathrm{Cond}^G(\overline{\Omega}, \partial\Omega)$, then $F_1 \sim F_2$ in $\mathrm{Comp}^G(\overline{\Omega}, \partial\Omega)$. Therefore, by the homotopy invariance of G-degree for equivariant compact fields, $G\text{-Deg}(\pi - F_1, \Omega) = G\text{-Deg}(\pi - F_2, \Omega)$. This justifies the following definition.

Definition 2.3.2. Let $F \in \mathrm{Cond}^G(\overline{\Omega}, \partial\Omega)$. Then the *G-equivariant degree* of the Ω-admissible condensing field $\pi - F$ is defined by the formula

$$G\text{-Deg}\,(\pi - F, \Omega) := G\text{-Deg}\,(\pi - F_1, \Omega),$$

where $F_1 \in \mathrm{Comp}^G(\overline{\Omega}, \partial\Omega)$ is a compact map such that $F \sim F_1$ in $\mathrm{Cond}^G(\overline{\Omega}, \partial\Omega)$.

We have observed that by the homotopy invariance, the above definition does not depend on the choice of $F_1 \in \mathrm{Comp}^G(\overline{\Omega}, \partial\Omega)$. Moreover, it can be verified, by a similar argument to that in the nonequivariant case, that the G-degree for condensing fields has also the same standard properties as in the case of finite dimensional spaces. For the convenience of references, we formulate these standard properties below.

Theorem 2.3.2. *The above well-defined G-Deg$(\pi - F, \Omega)$ for $F \in \mathrm{Cond}^G(\Omega, \partial\Omega)$ satisfies the following properties:*

(i) Existence *If G-Deg$(\pi - F, \Omega) \neq 0$, then there exists an $\alpha \in \Phi_n(G)$ and $x \in \Omega \cap (\pi - F)^{-1}(0)$ such that $(G_x) \leq \alpha$;*

(ii) *Excision* If $\Omega_0 \subset \Omega$ is an open invariant subset and $F \in Cond^G(\overline{\Omega}, \overline{\Omega \setminus \Omega_0})$, then

$$G\text{-}Deg(\pi - F, \Omega) = G\text{-}Deg(\pi - F, \Omega_0)$$

(iii) *Additivity* If Ω_1 and Ω_2 are two open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $F \in Cond^G(\overline{\Omega}, \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)})$, then

$$G\text{-}Deg(\pi - F, \Omega) = G\text{-}Deg(\pi - F, \Omega_1) + G\text{-}Deg(\pi - F, \Omega_2).$$

(iv) *Homotopy Invariance* If $H : \overline{\Omega} \times [0,1] \to W$ is a homotopy in $Cond^G(\overline{\Omega}, \partial\Omega)$, then

$$G\text{-}Deg(\pi - H(\cdot, 0), \Omega) = G\text{-}Deg(\pi - H(\cdot, 1), \Omega).$$

(v) *Product Formula* Suppose that $F \in Cond^G(\overline{\Omega}, \partial\Omega)$, W_1 is another isometric Banach representation of G and U is an open bounded invariant subset of W_1 with $0 \in U$. Define $\Phi : \overline{\Omega} \times \overline{U} \to W \times W_1$ by $\Phi(x, y, t) = (F(x,t), y)$ for $y \in \overline{U}$ and $(x,t) \in \Omega$. Then

$$G\text{-}Deg(\Phi, U \times \Omega) = G\text{-}Deg(F, \Omega).$$

To apply the G-degree to nonlinear problems which we will describe below, we need to discuss the notion of *equivariant composite coincidence degree.*

Let $\mathbb{E}$ and $\mathbb{F}$ be two isometric representations of G and assume that $L : \mathrm{Dom}(L) \subset \mathbb{E} \to \mathbb{F}$ is a closed equivariant Fredholm operator of index zero, where $\mathrm{Dom}(L)$ is an invariant subspace of $\mathbb{E}$. A *compact resolvent* of L is a compact linear operator $K : \mathbb{E} \to \mathbb{F}$ such that $L + K : \mathrm{Dom}(L) \to \mathbb{F}$ is a bijection, and hence the inverse $R_K := [L+K]^{-1} : \mathbf{F} \to \mathbb{E}$ is well defined and continuous. We use $CR(L)$ to denote the set of all compact resolvents of L. It is well known that $CR(L) \neq \varnothing$, but $CR^G(L) \triangleq \{K \in CR(L) : K \text{ is } G\text{-equivariant}\}$ may be empty. However, in many cases, and in particular, in the application to Hopf bifurcation problems of neutral equations, the assumption $CR^G(L) \neq \varnothing$ is always satisfied.

Let $X \subset \mathbb{E} \times \mathbf{R}^n$ be a bounded invariant closed subset and let $B : X \to \mathbb{E}$ and $N : X \to \mathbf{F}$ be two equivariant maps. We consider the following nonlinear problem

$$\begin{cases} \textit{Find} & x \in X \ \ \textit{such that} \ \ \pi(x) - B(x) \in \mathrm{Dom}\,(L) \\ \textit{and} & L[\pi(x) - B(x)] = N(x). \end{cases} \tag{P}$$

Following [15], we call this nonlinear problem an *equivariant composite coincidence problem*. Let $K \in CR^G(L)$ be a fixed compact equivariant resolvent of L, and define

$$\Theta_K(B,N) \triangleq B + R_K[N + K(\pi - B)] : X \to \mathbb{E}.$$

As compositions of equivariant maps, $\Theta_K(B,N)$ is equivariant. It can be verified (see [15]) that the problem (P) is equivalent to the following fixed point problem

$$\begin{cases} \text{Find} \ \ x \in X \ \ \text{such that} \\ \pi(x) = \Theta_K(B,N)(x). \end{cases}$$

Assuming that the space $\mathbb{E}$ is equipped with a measure of noncompactness μ, we now introduce the following definition.

Definition 2.3.3. Let $B : X \to \mathbb{E}$ and $N : X \to \mathbb{F}$ be two equivariant maps. The pair (B,N) is said to be an *L-condensing G-pair* if the map $\Theta_K(B,N) : X \to \mathbf{E}$, defined above, is a condensing G-map.

The above definition does not depend on the choice of $K \in CR^G(L)$. Indeed, if $K^1 \in CR^G(L)$ is another equivariant resolvent, then we have

$$\Theta_K(B,N) = \Theta_{K^1}(B,N) + R_K \circ (K^1 - K)\Theta_{K^1}(B,N) + R_{K^1}(K - K^1),$$

and thus $\Theta_K(B,N)$ is condensing if and only if $\Theta_{K^1}(B,N)$ is condensing.

Let (X,A) be a pair of closed bounded invariant subsets of $\mathbb{E} \times \mathbb{R}^n$. We denote by $\mathrm{Cond}_L^G(X,A)$ the class of all L-condensing G-pairs (B,N), $B : X \to \mathbb{E}$ and $N : X \to \mathbb{F}$ such that (P) has no solution in A, i.e. $\pi(x) \neq \Theta_K(B,N)(x)$ for all $x \in A$.

Definition 2.3.4. Let $\Omega \subset \mathbb{E}\times\mathbb{R}^n$ be an open bounded invariant subset. For every L-condensing G-pair $(B,N) \in \mathrm{Cond}_L^G(\overline{\Omega}, \partial\Omega)$, we define the G*-composite coincidence degree* of (L,B,N) on Ω as an element of $A_n(G)$ given by

$$G\text{-Deg}\,[(L,B,N),\Omega] \triangleq G\text{-Deg}\,(\pi - \Theta_K(B,N),\Omega),$$

where $K \in CR^G(L)$ is fixed.

We note that this definition may depend on the choice of the equivariant resolvent K. However every component of the degree is unique up to sign. See [37]. Furthermore, it follows from Theorem 2.3.2 that the standard properties such as existence, excision, additivity and homotopy invariance hold also true for G-composite coincidence degree.

We finally remark that composite coincidence problems of type (P) in the non-equivariant case have been studied in Gaines and Mawhin [19]. The special case of L-condensing G-pair $(0,N)$ with N being L-compact (i.e., R_KN is compact) can be found in [14, 20, 21] in conjunction with the Hopf bifurcation problem of retarded functional differential equations and parabolic partial differential equations. A typical example of L-condensing pair is (B, N) with B being condensing and N being L-compact. It will be clear in the next chapter that this example arises naturally from the study of Hopf bifurcation problem for neutral equations.

2.4. A local bifurcation theorem with symmetry

In this section, we apply the G-composite coincidence degree to obtain a variant of the local bifurcation theorem of Krasnosel'skii for a class of G-equivariant nonlinear problem.

Let k be a positive integer. We assume that V and W are two isometric Banach representations of G, $\mathbb{E} = V \times \mathbb{R}$ and $\mathbb{F} = W \times \mathbb{R}$. Suppose that

$$L_0 : \ \mathrm{Dom}\,(L_0) \subset V \to W$$

is a given equivariant closed Fredholm operator of index zero such that $CR^G(L_0) \neq \varnothing$. We extend L_0 to a Fredholm operator of index zero $L : \mathrm{Dom}\,(L) \subset \mathbb{E} \to \mathbb{F}$, where $\mathrm{Dom}\,(L) = \mathrm{Dom}\,(L_0) \times \mathbb{R}$ and $L(v, r) = (L_0 v, 0)$ for $(v, r) \in \mathrm{Dom}\,(L_0) \times \mathbb{R}$. It follows from $CR^G(L_0) \neq \varnothing$ that $CR^G(L) \neq \varnothing$.

Consider now the following nonlinear problem

$$L_0[\pi_0(x) - B_0(x)] = N_0(x), \quad \pi_0(x) - B_0(x) \in \mathrm{Dom}\,(L_0), \qquad \text{(B.P)}$$

where $\pi_0 : V \times \mathbb{R}^k \to V$ is the natural projection, $B_0 : V \times \mathbb{R}^k \to V$, and $N_0 : V \times \mathbb{R}^k \to W$ are two equivariant mappings of class C^1 such that (B_0, N_0) and their derivatives $(D_v B_0, D_v N_0)$ are L_0-condensing G-pairs.

To describe our bifurcation problem, we assume that there exists a k-dimensional submanifold $M \subset V^G \times \mathbf{R}^k$ satisfying the following conditions:

(A) For every $x \in M$, $\pi_0(x) - B_0(x) \in \mathrm{Dom}\,(L_0)$ and $L_0[\pi_0(x) - B_0(x)] = N_0(x)$, i.e. x is a solution to the equation (B.P).

(B) If $(v_0, \lambda_0) \in M$, $v_0 \in V^G$, $\lambda_0 \in \mathbf{R}^k$, then there exist an open neighborhood $\mathcal{U}_{\lambda_0}$ of λ_0 in $\mathbf{R}^k$, an open neighborhood $\mathcal{U}_{v_0}$ of v_0 in V^G and a C^1-map $\eta : \mathcal{U}_{\lambda_0} \to V^G$ such that

$$M \cap (\mathcal{U}_{v_0} \times \mathcal{U}_{\lambda_0}) = \{(\eta(\lambda), \lambda) : \lambda \in \mathcal{U}_{\lambda_0}\}.$$

Since all points $(v, \lambda) \in M$ are solutions of (B.P), we call those points *trivial solutions*. All other solutions of (B.P) will be called *nontrivial*. A point $(v_0, \lambda_0) \in M$ is called a *bifurcation point* if in any neighborhood of (v_0, λ_0) there exists a nontrivial solutions for (B.P).

The problem (B.P) is equivalent to the equation

$$v = \Theta_{K_0}(B_0, N_0)(v, \lambda), \quad (v, \lambda) \in V \times \mathbf{R}^k,$$

where $\Theta_{K_0}(B_0, N_0) : V \times \mathbf{R}^k \to V$ is given by $\Theta_{K_0}(B_0, N_0) = B_0 + R_{K_0}[N_0 + K_0(\pi_0 - B_0)]$. We define a mapping $f : V \times \mathbf{R}^k \to V$ by $f(v, \lambda) = v - \Theta_{K_0}(B_0, N_0)(v, \lambda)$. By assumption,

$$D_v f(v, \lambda) = Id - \Theta_{K_0}(D_v B_0(v, \lambda), D_v N_0(v, \lambda)) : V \to V,$$

where D_v denotes the derivative with respect to $v \in V$, is a condensing linear field, and hence is a Fredholm operator of index zero on the space V. Furthermore, if $(v, \lambda) \in M$, then $D_v f(v, \lambda)$ is also an equivariant operator. It follows from the implicit function theorem that if $(v_0, \lambda_0) \in M$ is a bifurcation point then the derivative $D_v f(v_0, \lambda_0)$ is not an isomorphism of V, which means that all bifurcation points of (B.P) are contained in the set

$$\Lambda \triangleq \{(v, \lambda) \in M : D_v f(v, \lambda) \text{ is not an isomorphism}\}.$$

In what follows, we call a point (v_0, λ_0) in Λ a *V-singular point*.

Our goal is to find non-trivial solutions, or more precisely, a bifurcation point for the equation (B.P), which is equivalent to the following equation

$$f(v, \lambda) = 0, \quad (v, \lambda) \in V \times \mathbf{R}^k. \tag{2.4.1}$$

Our following idea of finding non-trivial solutions to (2.4.1) in a given open bounded invariant neighborhood $\mathcal{U} \subset V \times \mathbf{R}^k$ of a V-singular point $(v_0, \lambda_0) \in M$ is based on the notion of the *complementary function* for the equation (2.4.1). This method has been developed in the non-equivariant case by Ize [25] and recently has been employed in [14, 20, 21] for the study of G-equivariant bifurcation problems. A "complementary function" for the equation (2.4.1) on $\mathcal{U}$ is an invariant function $\varphi : \overline{\mathcal{U}} \to \mathbf{R}$ satisfying the condition $\varphi(v, \lambda) < 0$ for all $(v, \lambda) \in \overline{\mathcal{U}} \cap M$. Therefore, every solution to the system

$$\begin{cases} f(v, \lambda) = 0, & \\ \varphi(v, \lambda) = 0, & (v, \lambda) \in \overline{\mathcal{U}}, \end{cases} \tag{2.4.2}$$

is a nontrivial solution to (2.4.1). This leads to the equivariant map $F_\varphi : \overline{\mathcal{U}} \to V \times \mathbf{R} = \mathbf{E}$, $F_\varphi(v, \lambda) = (f(v, \lambda), \varphi(v, \lambda))$, and we can replace the problem of finding a nontrivial solution to (B.P) in $\mathcal{U}$ by the problem of solving the equation $F_\varphi(v, \lambda) = 0$ for $(v, \lambda) \in \mathcal{U}$. On the other hand, we can easily verify that (2.4.2) is equivalent to the following coincidence problem

$$L[\pi(x) - B(x)] = N_\varphi(x), \quad x \in \mathcal{U}, \tag{2.4.3}$$

where π is the natural projection of $\mathbf{E} \times \mathbf{R}^{k-1}$ onto $\mathbf{E}$, $B(x) \triangleq (B_0(x), 0) \in \mathbf{E}$ for $x \in \mathbf{E} \times \mathbf{R}^{k-1} = V \times \mathbf{R}^k$ and $N_\varphi(x) \triangleq (N_0(x), \varphi(x)) \in \mathbf{F}$. Therefore, if we assume that (2.4.3) has no solution in $\partial\mathcal{U}$, then the G-composite coincidence degree $G\text{-Deg}\,[(L, B, N_\varphi), \mathcal{U}]$ is well defined and it can be used as a tool to investigate the existence of solutions of (2.4.3). In particular, the nontriviality of $G\text{-Deg}\,[(L, B, N_\varphi), \mathcal{U}]$ will imply the existence of a non-trivial solution of (B.P) in $\mathcal{U}$.

In the rest of this section, we will show that for an isolated V-singular point $(v_0, \lambda_0) \in M$ it is possible to compute the $G\text{-Deg}\,[(L, B, N_\varphi), \mathcal{U}]$ in terms of derivatives $D_v B$ and $D_v N$ near (v_0, λ_0). As we will see, this computation plays a crucial role in developing a bifurcation theory for (2.4.1) and its application to the global Hopf bifurcation problem of neutral equations.

In order to compute $G\text{-Deg}\,[(L, B, N_\varphi)]$ for a certain complementary function φ, we take as a neighborhood $\mathcal{U}$ of the isolated V-singular point $(v_0, \lambda_0) \in M$ the invariant set

$$B_M(v_0, \lambda_0; r, \rho) \triangleq \{(v, \lambda) \in V \times \mathbf{R}^k : |\lambda - \lambda_0| < \rho,\ \|v - \eta(\lambda)\| < r\}$$

where ρ, $r > 0$ are sufficiently small numbers such that

(i) if $(v, \lambda) \in B_M(v_0, \lambda_0; r, \rho)$ and $v \neq \eta(\lambda)$ at $|\lambda - \lambda_0| = \rho$, then $f(v, \lambda) \neq 0$;

(ii) (v_0, λ_0) is the only V-singular point in $B_M(v_0, \lambda_0; r, \rho)$;

(iii) $B(\lambda_0, \rho) := \{\lambda \in \mathbf{R}^{k-1};\ |\lambda - \lambda_0| < \rho\} \subset \mathcal{U}_{\lambda_0}$, where $\mathcal{U}_{\lambda_0}$ is the open neighborhood of λ_0 in Assumption (B).

Such a neighborhood $\mathcal{U}$ is called a ***special neighborhood*** of (v_0, λ_0) determined by (r, ρ). The existence of a special neighborhood follows from the implicit function theorem. Moreover, for a special neighborhood $\mathcal{U}$, we say that a continuous invariant function $\theta : \overline{\mathcal{U}} \to \mathbf{R}$ is an *almost complementary function* if

(i) $\theta(\eta(\lambda), \lambda) = -|\lambda - \lambda_0|$ for all $\lambda \in B(\lambda_0, \rho)$;

(ii) $\theta(v, \lambda) = r$ if $\|v - \eta(\lambda)\| = r$ and $\lambda \in B(\lambda_0, \rho)$;

(iii) $\theta(v, \lambda_0) = \|v - \eta(\lambda_0)\|$ if $\|v - \eta(\lambda_0)\| \leq r$.

The existence of such a function θ follows from the Gleason-Tietze theorem. Note that if θ is an almost complementary function, then for each $\delta > 0$, $\varphi(v, \lambda) \triangleq \theta(v, \lambda) - \delta$ is negative on the set of trivial solutions $\overline{\mathcal{U}} \cap M$. For a sufficiently small $\delta > 0$, F_φ and F_θ are homotopic in $\mathrm{Cond}^G(\overline{\mathcal{U}}, \partial U)$. Therefore, by the Homotopy Invariance Property $G\text{-Deg}(F_\varphi, \mathcal{U}) = G\text{-Deg}(F_\theta, \mathcal{U})$. Consequently the nontriviality of the degree $G\text{-Deg}(F_\theta, \mathcal{U})$ implies the existence of a nontrivial solution of (2.4.1) in $\mathcal{U}$.

The proposition below follows directly from the existence property of G-degree.

Proposition 2.4.1. *Let $(v_0, \lambda_0) \in M$ be an isolated V-singular point and $\mathcal{U} = B_M(v_0, \lambda_0; r, \rho)$ a special neighbourhood of (v_0, λ_0). If $G\text{-Deg}(F_\theta, \mathcal{U}) \neq 0$ for some almost complementary function $\theta : \overline{\mathcal{U}} \to \mathbf{R}$, then (v_0, λ_0) is a bifurcation point for (2.4.1). More precisely, if $G\text{-Deg}(F_\theta, \mathcal{U}) = \sum \gamma_\alpha \cdot \alpha$ and $\gamma_\alpha \neq 0$ for some $\alpha \in \mathcal{J}(\mathcal{U}) \cap \Phi_{k-1}(G)$, then (2.4.1) has a sequence of nontrivial solutions (v_n, λ_n) such that $\lim_{n\to\infty}(v_n, \lambda_n) = (v_0, \lambda_0)$ and $(G_{v_n}) \leq \alpha$ for $n = 1, 2, \ldots$.*

Proof. Choose $\varepsilon > 0$ sufficiently small so that the function $\varphi_\varepsilon : \overline{\mathcal{U}} \to \mathbf{R}$, defined by

$$\varphi_\varepsilon(v, \lambda) = \theta(v, \lambda) - \varepsilon, \quad (v, \lambda) \in \overline{\mathcal{U}},$$

is a well defined complementary function. By the definition of composite coincidence degree and the homotopy invariance of G-degree we have

$$G\text{-Deg}\,[(L, B, N_{\varphi_\epsilon}, \mathcal{U}] = G\text{-Deg}\,(F_{\varphi_\epsilon}, \mathcal{U}) = G\text{-Deg}\,(F_\theta, \mathcal{U}).$$

Therefore, $G\text{-Deg}\,(F_\theta, \mathcal{U}) \neq 0$ implies that the equation $F_{\varphi_\epsilon}(v, \lambda) = 0$ has a solution in $\mathcal{U}$, and hence the equation (2.4.1) has a nontrivial solution in $\mathcal{U}$. Consequently, the bifurcation result follows.

From Proposition 2.4.1, it is important to compute $G\text{-Deg}\,(F_\theta, \mathcal{U})$. To do this, we first linearize the map. It follows that, by using a linear homotopy, for sufficiently small $r > 0$ and $\rho > 0$,

$$G\text{-Deg}\,(F_\theta, \mathcal{U}) = G\text{-Deg}\,(DF_\theta, \mathcal{U}),$$

where

$$DF_\theta(v, \lambda) = ((\mathrm{Id} - D_v f(\eta(\lambda), \lambda))(v - \eta(\lambda)), \theta(v, \lambda)), \quad (v, \lambda) \in \overline{\mathcal{U}}.$$

By the excision property of G-degree, we know that $G\text{-Deg}\,(F_\theta, \mathcal{U})$ is independent of the choice of r and ρ.

In what follows we give an explicit computational formula for $G\text{-Deg}\,(F_\theta, \mathcal{U})$ in the case when $k = 2$ and $G = \Gamma \times S^1$, where Γ is *a compact abelian Lie group*. This implies that the bifurcation problem (2.4.1) has a two-dimensional parameter space. We will use this particular setting to study one-parameter Hopf bifurcation problem in the subsequent chapters.

Let $(v_0, \lambda_0) \in M$ be an isolated V-singular point and let $\eta : \mathcal{U}_{\lambda_0} \to V^G$ be the map defined in Assumption (B) and let $\mathcal{U}$ be a special neighbourhood of

(v_0, λ_0). Set $B(\lambda_0; \rho) := \{\lambda \in \mathbf{R}^2;\ |\lambda - \lambda_0| \le \rho\}$. We define

$$a(\lambda) := Id - D_v f(\eta(\lambda), \lambda), \quad \lambda \in B(\lambda_0; \rho),$$

where $f(v, \lambda) = v - \Theta_{K_0}(B_0, N_0)(v, \lambda)$. By assumption,

$$a(\lambda) \in \begin{cases} L^G_{\text{cond}}(V) & \text{if } \lambda \in B(\lambda_0; \rho), \\ GL^G_{\text{cond}}(V) & \text{if } \lambda \in \partial B(\lambda_0; \rho). \end{cases} \tag{2.4.4}$$

This gives a map $a : (B(\lambda_0; \rho), \partial B(\lambda_0; \rho)) \to (L^G_{\text{cond}}(V),\ GL^G_{\text{cond}}(V))$. We will show below that under the above assumptions the G-degree $G\text{-Deg}(F_\theta, \mathcal{U})$ can be computed from the homotopical properties of the map $a(\cdot)$ defined by (2.4.4).

First we consider the irreducible representations of S^1. There are a total of countable number of irreducible representations $\rho_1, \rho_2, \dots, \rho_n, \dots$ of S^1 on $\mathbf{C}$ which are given by

$$\rho_n(\xi) := \xi^n z; \quad n = 1, 2, 3, \dots, \xi \in S^1,\ z \in \mathbf{C}.$$

Therefore, by Example 2.2.1, we have the following isotypical decomposition $V_\infty = \bigoplus_{n=0}^{\infty} V_n$, $\overline{V_\infty} = V$, $V_0 = V^{S^1}$, of the space V with respect to the restricted action of S^1 on V. For every $n > 0$ the subspace V_n has a natural complex structure such that an $\mathbf{R}$-linear operator on V_n is S^1-equivariant if and only if it is $\mathbf{C}$-linear with respect to this complex structure. Since the actions of Γ and S^1 commute, the S^1-isotypical components V_n, $n = 0, 1, 2, \dots$, are Γ-invariant, and therefore, V_0 is a real isometric representation of Γ and for $n = 1, 2, \dots$, V_n is a complex isometric representation of Γ. Let $\theta_{n1}, \theta_{n2}, \dots, \theta_{ni}, \dots : \Gamma \to S^1 \cong S^1/\mathbf{Z}_n$ denote the sequence of all irreducible complex representations of Γ. Then for $n > 0$, by Theorem 0.2.7, corresponding to the sequence $\{\theta_{ni}\}$, we have the following isotypical decomposition

$$V_{n\alpha} = \bigoplus_{i=0}^{\infty} V_{ni}; \quad V_{n0} := V_n^{\Gamma}, \tag{2.4.5}$$

of the subspace V_n. Notice that Γ is abelian. The invariant space V_{ni}, $n > 0$, $i \geq 0$, is a G-isotypical component of V corresponding to the irreducible representation $\varphi_{ni} : \Gamma \times S^1 \to S^1$ given by

$$\varphi_{ni}(\gamma, \xi)z := \theta_{ni}(\gamma)\xi^n z, \quad (\gamma, \xi) \in \Gamma \times S^1, \ z \in \mathbb{C}.$$

This implies that any $\mathbb{C}$-linear operator on V_{ni} is G-equivariant, and therefore $GL^G_{\text{cond}}(V_{ni}) = GL^{\mathbb{C}}_{\text{cond}}(V_{ni})$. Moreover, for $v \in V_{ni} \setminus \{0\}$, $n > 0$, $i \geq 0$, we have that the isotropy group G_v is exactly the subgroup $G_{ni} := \{(\gamma, \xi);\ (\gamma, \xi^{-n}) \in \text{Graph}(\theta_{ni})\} = \{(\gamma, \xi);\ \xi \in \theta_{ni}(\gamma)\}$, where Graph (θ_{ni}) denotes the graph of the homomorphism θ_{ni}.

Let $\vartheta_1, \ldots, \vartheta_m : \Gamma \to \mathbb{C}$ be the sequence of all (continuous) homomorphisms, characterizing all non-trivial real one-dimensional subrepresentations of Γ in V_0. We will denote by V_{0i}, $i = 0, 1, \ldots, m$, the isotypical components of V_0, such that for $x \in V_{0i} \setminus \{0\}$, $i > 0$, the isotropy group G_x is exactly $G_{0i} = \text{Ker}\,\vartheta_i \times S^1$, and $V_{00} = V^G$. For any $A \in GL_{\text{cond}}(V_{0i})$ we define

$$\text{sign}\, A = \begin{cases} 1 & \text{if } A \in GL^+_{\text{cond}}(V_{0i}), \\ -1 & \text{if } A \in GL^-_{\text{cond}}(V_{0i}). \end{cases}$$

Let $(v_0, \lambda_0) \in M$ be an isolated V-singular point and $a : (B_\rho(\lambda_0), \partial B_\rho(\lambda_0)) \to (L^G_{\text{cond}}(V),\ GL^G_{\text{cond}}(V))$ be defined by (2.4.4). For all integers $n \geq 0$ and $i \geq 0$ there are well-defined restrictions

$$a_{ni}(\lambda) := a(\lambda)|_{V_{ni}} : V_{ni} \to V_{ni}, \quad \lambda \in B(\lambda_0; \rho).$$

Note that

$$a_{ni} : S^1 \cong \partial B(\lambda_0; \rho) \to GL^{\mathbf{C}}_{\text{cond}}(V_{ni}).$$

By Proposition 0.5.6, we have the following integers μ_{ni}, for each $n > 0$, $i \geq 0$,

$$\mu_{ni} = \mu_{ni}(v_0, \lambda_0) := \Delta\left([a_{ni}]\right),$$

where Δ is the homomorphism defined in Propostion 0.5.6 and $[a_{ni}]$ denotes the homotopy class of a_{ni}. For $n = 0$, we set

$$\nu_{0i} = \nu_{0i}(v_0, \lambda_0)\frac{1}{2}(1 - \operatorname{sign} a_{0i}(\lambda)), \quad i > 0,$$
$$\varepsilon(v_0, \lambda_0) := \text{sign } a_{00}(\lambda), \quad \lambda \in \partial B(\lambda_0; \rho).$$

In the following theorem, we set $\mu(v_0, \lambda_0) = \sum_{n>0, i\geq 0} \mu_{ni} \cdot (G_{ni}) \in A_1(G)$, where (G_{ni}) denotes the orbit type of non-zero elements in V_{ni}, and $\nu(v_0, \lambda_0) = \prod_{i=1}^{m}((G) - \nu_{0i}(G_{0i}))$.

Theorem 2.4.2. *Under the above assumptions, if* $(v_0, \lambda_0) \in M$ *is an isolated* V*-singular point such that* $a_{00}(\lambda_0) \in GL(V^G)$, *then*

$$G\text{-}Deg(F_\theta, \mathcal{U}) = \varepsilon(v_0, \lambda_0)\nu(v_0, \lambda_0) \cdot \mu(v_0, \lambda_0).$$

Proof. First, by applying the standard linear homotopy argument, we get

$$G\text{-Deg}\,[(L, B, N_\theta), \mathcal{U}] = G\text{-Deg}\,(F_\theta, \mathcal{U}) = G\text{-Deg}\,(\tilde{F}_\theta, \mathcal{U})$$

where

$$\tilde{F}_\theta(v, \lambda) = (a(\lambda)(x - \eta(\lambda)), \theta(v, \lambda)), \quad (v, \lambda) \in \overline{\mathcal{U}},$$
$$a(\lambda) := D_v f(\eta(\lambda), \lambda), \quad \lambda \in B(\lambda_0; \rho).$$

We next use Theorem 0.5.3 to obtain a direct invariant decomposition $V = V_0 \oplus V^0$ such that V^0 is an invariant subspace of finite dimension and

$$a : (B(\lambda_0;\rho), \partial B(\lambda_0;\rho)) \to (L^G_{\text{cond}}(V),\ GL^G_{\text{cond}}(V))$$

is homotopic to some

$$b : (B(\lambda_0;\rho), \partial B(\lambda_0;\rho)) \to (L^G_{\text{cond}}(V),\ GL^G_{\text{cond}}(V))$$

such that $b(\lambda)|_{V_0} = Id|_{V_0}$ and $b(\lambda)|_{V^0} : V^0 \to V^0$ for $\lambda \in B(\lambda_0;\rho)$. Consequently, by the homotopy invariance and the definition of G-degree for compact vector fields

$$G\text{-Deg}\,(F_\theta, \mathcal{U}) = G\text{-Deg}\,(T_\psi, \mathcal{U}) = G\text{-Deg}\,(T_\psi|_{V^0\times\mathbb{R}^2}, \mathcal{U} \cap V^0 \times \mathbb{R}^2), \qquad (2.4.7)$$

where $\psi(v,\lambda)$ is a new almost complementary function defined by $\psi(v,\lambda) = \varepsilon + \theta(x,\lambda)$, $\varepsilon > 0$ is a sufficiently small number and

$$T_\psi(v,\lambda) = (b(\lambda)(x - \eta(\lambda)), \psi(v,\lambda)), \quad (v,\lambda) \in \overline{\mathcal{U}}.$$

From the definition of an almost complementary function, we see that $T_\psi|_{V^0\times\mathbb{R}^2}$ satisfies (A) in Theorem 2.2.1. Therefore we have

$$G\text{-Deg}\,(T_\psi|_{V^0\times\mathbb{R}^2}, \mathcal{U} \cap V^0 \times \mathbb{R}^2) = \varepsilon^*\nu^*(v_0,\lambda_0)\cdot\mu(v_0,\lambda_0),$$

where

$$\nu^*(v_0,\lambda_0) = \prod_{i=1}^{m}((G) - \nu^*_{0i}(G_{0i})),$$
$$\mu^*(v_0,\lambda_0) = \sum_{n>0,i\geq 0} \mu^*_{ni}(G_{ni})$$

and

$$\nu_{0i}^* = \frac{1}{2}(1 - \operatorname{sign} b_{0i}(\lambda)), \quad i > 0,$$
$$\varepsilon^* = \operatorname{sign}\, \det_{\mathbf{R}} b_G(\lambda), \; b_G(\lambda) := b(\lambda)|_{(V^0)^G},$$
$$\mu_{ni}^* = \Delta\left([b_{ni}(\lambda)]\right) \quad b_{ni}(\lambda) := b(\lambda)|_{V_{ni}^0}, \quad \lambda \in B(\lambda_0;\rho),$$

in which Δ denotes the bijection in Proposition 0.5.6 and V_{ni}^0 is the isotypical component of V^0 defined in the same way as in (2.4.5).

Recall that

$$a : (B(\lambda_0;\rho), \partial B(\lambda_0;\rho)) \to (L_{\mathrm{cond}}^G(V),\; GL_{\mathrm{cond}}^G(V))$$

is homotopic to

$$b : (B(\lambda_0;\rho), \partial B(\lambda_0;\rho) \to (L_{\mathrm{cond}}^G(V),\; GL_{\mathrm{cond}}^G(V))$$

and $b(\lambda)|_{V_0} = Id|_{V_0}$ for $\lambda \in B(\lambda_0;\rho)$. By homotopy invariance, we have $\varepsilon^* = \varepsilon(v_0, \lambda_0)$, $\mu_{ni}^* = \mu_{ni}$, $\nu_{0i}^* = \nu_{0i}$ and (2.4.6) follows from (2.4.7) and (2.4.8).

This completes the proof.

Assume that $n > 0$, $i \geq 0$. There is a homomorphism $\theta_{ni} : \Gamma \to S^1/\mathbf{Z}_n$ such that $G_{ni} = \{(\gamma, z) \in \Gamma \times S^1;\;\; z \in \theta_{ni}(\gamma)\}$. Recall that such an orbit type (G_{ni}) is called a *basic orbit type.* and by $B_1(G)$ (resp. $\tilde{B}_1(G)$) we denote the $\mathbf{Z}$-submodule of $A_1(G)$ generated by all basic (resp. non-basic) orbit types. Theorem 2.4.2 now simplifies to the following result.

Corollary 2.4.3. *Under the same assumptions as in Theorem 2.4.2,*

$$G\text{-}Deg\,[(L,B,N_\theta),\mathcal{U}] = \varepsilon(v_0,\lambda_0)\Big(\mu(v_0,\lambda_0) + \zeta(v_0,\lambda_0)\Big)$$

where $\mu(v_0,\lambda_0) = \sum_{n>0,i\geq 0} \mu_{ni}(G_{ni}) \in B_1(G)$, *and* $\zeta(v_0,\lambda_0) \in \tilde{B}_1(G)$.

The following theorem represents an analog of the Krasnosel'skii local bifurcation theorem [28] and is a convenient form in the applications to Hopf bifurcation for differential equations with symmetry.

Theorem 2.4.4. *Suppose that* $L_0 : Dom(L_0) \subset V \to W$ *is a given* G*-equivariant closed Fredholm operator of index zero,* $B_0 : V \times \mathbb{E}^2 \to V$, $N_0 : V \times \mathbb{R}^2 \to W$ *are two* G*-maps of class* C^1 *such that* (B_0, N_0) *and* $(D_v B_0,\ D_v N_0)$ *are* L_0*-condensing* G*-pairs. Assume further that* $M \subset V^G \times \mathbb{R}^2$ *is a 2-dimensional submanifold satisfying the conditions (A) and (B). If* $(v_0, \lambda_0) \in M$ *is an isolated* V*-singular point and there exist* $n > 0$, $i \geq 0$ *such that* $\mu_{ni} \neq 0$, *then* (v_0, λ_0) *is a bifurcation point for the problem (B.P). More precisely, there exists a sequence* (v_k, λ_k) *of nontrivial solutions to (B.P) such that* $(G_{v_k}) \leq (G_{ni})$ *and* $\lim_{k\to\infty}(v_k, \lambda_k) = (v_0, \lambda_0)$.

Proof. Notice that $\mu_{ni} \neq 0$ implies that $\mu(v_0,\lambda_0) \neq 0$. Therefore, by Corollary 2.4.3, G-Deg$\,[(L,B,N_\varphi),\mathcal{U}] \neq 0$ for some complementary function φ and special neighborhood $\mathcal{U}$ of (v_0,λ_0). The theorem then follows from Proposition 2.4.1.

2.5. A global bifurcation theorem with symmetry

Let Γ be a compact Lie group, $G = \Gamma \times S^1$ and V be a real isometric Banach representation of G. For a moment we do not assume that Γ is an abelian group. However the main result of this section is valid only in this particular case.

Recall that we use V_0 to denote the set of all fixed points of V with respect to the restricted S^1-action, i.e. $V_0 = \{v \in V;\ \xi v = v \text{ for all } \xi \in S^1\}$. We call V_0 the set of ***stationary points.***

Assume that W is another isometric Banach representation of G and $L_0 : \mathrm{Dom}(L_0) \subset V \to W$ is a given equivariant closed Fredholm operator of index zero such that $CR^G(L_0) \neq \varnothing$. Let $\pi_0 : V \times \mathbb{R}^2 \to V$ be the natural projection, $B_0 : V \times \mathbb{R}^2 \to V$ and $N_0 : V \times \mathbb{R}^2 \to W$ be G-maps of C^1 such that (B_0, N_0) and $(D_v B_0, D_v N_0)$ are L_0-condensing G-pairs. We are going to consider the nonlinear problem

$$L_0[\pi_0(x) - B_0(x)] = N_0(x), \quad \pi_0(x) - B_0(x) \in \mathrm{Dom}(L_0),\ x \in V \times \mathbb{R}^2 \quad \text{(GBP)}$$

subject to the following condition:

(H) There exists a 2-dimensional G-invariant submanifold $M \subset V^G \times \mathbb{R}^2$ such that for every $(v, \lambda) \in M$,

(i) $L_0[\pi(v, \lambda) - B_0(v, \lambda)] = N_0(v, \lambda)$, and

(ii) $D_v f(v, \lambda)|_{V_0} \in GL(V_0)$, where $f(v, \lambda) = v - \Theta_{K_0}(B_0, N_0)(v, \lambda)$ for a fixed $K_0 \in CR^G(L_0)$. Each point in M is called a ***trivial solution*** to (GBP). It follows from the implicit function theorem that for every $(v_0, \lambda_0) \in M$ there exists an open neighbourhood $\mathcal{U}_{v_0}$ of v_0 in V^G, an open neighbourhood $\mathcal{U}_{\lambda_0}$ of λ_0 in $\mathbb{R}^2$ and a C^1-map $\eta : \mathcal{U}_{\lambda_0} \to V_0$ such that

$$M \cap (\mathcal{U}_{v_0} \times \mathcal{U}_{\lambda_0}) = \{(\eta(\lambda), \lambda);\ \lambda \in \mathcal{U}_{\lambda_0}\}. \qquad (2.5.2)$$

Throughout this section we assume that (H) holds. Note also that the assumption (H) excludes the bifurcation of stationary solutions.

Let L, B and N_θ denote the extensions of L_0, B_0 and N_0 as in Section 2.4, where $\theta : V \times \mathbf{R}^2 \to \mathbf{R}$ is an invariant function. Our general global bifurcation theorem reads as follows.

Theorem 2.5.1. *Suppose that every V-singular point in M is isolated and M is complete. Let S denote the closure of the set of all nontrivial solutions to (GBP). Then for each bounded connected component C of S the set $GC \cap M$ is finite and consists of a finite number of disjoint Γ-orbits*

$$GC \cap M = \bigcup_{i=1}^{q} \Gamma(v_i, \lambda_i).$$

Moreover, we have

$$\sum_{i=1}^{q} G\text{-}Deg\,[(L, B, N_{\theta_i}, \mathcal{U}_i) = 0, \tag{2.5.2}$$

where $\mathcal{U}_i$ denotes a special neighbourhood of (v_i, λ_i) and θ_i is an almost complementary function on $\mathcal{U}_i$.

Proof. Note that every point of $GC \cap M$ is a bifurcation point and the V-singular points of M are isolated. It follows that the set $GC \cap M$ is finite. Write $GC \cap M = \Gamma(v_1, \lambda_1) \cup \cdots \cup \Gamma(v_q, \lambda_q)\}$ for some integer $q > 0$. Choose $r > \rho > 0$ sufficiently small so that for each $i = 1, 2, \ldots, q$, we can choose a special neighbourhood $\mathcal{U}_i$ of (v_i, λ_i) with $\overline{\mathcal{U}}_i \cap \overline{\mathcal{U}}_j = \varnothing$ if $i \neq j$. Let $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_q$. The set $\mathcal{U}$ is open an G-invariant and we can find $\Omega_1 \subset V \times \mathbf{R}^2$, an open bounded G-invariant subset such that $\overline{\Omega}_1 \cap M = \varnothing$, $GC \setminus \mathcal{U} \subset \Omega_1$ and $(\partial\Omega_1 \setminus \mathcal{U}) \cap S = \varnothing$.

Put $\Omega = \mathcal{U} \cup \Omega_1$. We construct a complementary function $\theta : \overline{\Omega} \to \mathbf{R}$ such that

(i) $\theta(v, \lambda) = -|\lambda - \lambda_i|$ if $(v, \lambda) \in \mathcal{U}_i \cap M$,

(ii) $\theta(v,\lambda) = r$ if $(v,\lambda) \in \overline{\Omega}\backslash\mathcal{U}$.

Let $F_\theta : \overline{\Omega} \to V \times \mathbf{R}$ be defined by

$$F_\theta(v,\lambda) \triangleq \big(v - f(v,\lambda), \theta(v,\lambda)\big), \quad (v,\lambda) \in \overline{\Omega}.$$

Then $F_\theta^{-1}(0) \subseteq GC$ and hence $G\text{-Deg}\,(F_\theta, \Omega)$ is well defined.

We now consider the following homotopy $H : \overline{\Omega} \times [0,1] \to V \times \mathbf{R}$,

$$H(v,\lambda,t) = \big(F(v,\lambda), (1-t)\theta(v,\lambda) - t\rho\big), \quad (v,\lambda,t) \in \overline{\Omega} \times [0,1].$$

By (i)–(ii), $H(v,\lambda,t) \neq 0$ for all $(v,\lambda,t) \in \partial\Omega\times[0,1]$, thus H is an Ω-admissible homotopy. Since $H(v,\lambda,0) = F_\theta(v,\lambda)$ and $H(v,\lambda,1) = \big(f(v,\lambda),\rho\big) \neq 0$ for all $(v,\lambda) \in \overline{\Omega}$, it follows that $G\text{-Deg}\,(F_\theta,\Omega) = 0$. By (ii), $F_\theta^{-1}(0) \subset GC \cap \mathcal{U}$. Therefore, by the excision and additivity properties of G-degree

$$0 = G\text{-Deg}\,(F_\theta,\Omega) = \sum_{i=1}^{q} G\text{-Deg}\,(F_{\theta_i},\mathcal{U}_i) = \sum_{i=1}^{q} G\text{-Deg}\,[(L,B,N_{\theta_i}),\mathcal{U}_i].$$

where $\theta_i = \theta|_{\mathcal{U}_i}$. The proof is completed.

We now assume that Γ is an *abelian group*. In this particular case, the local invariant $G\text{-Deg}\,(F_{\theta_i},\mathcal{U}_i)$ can be computed from Theorem 2.2.1. Notice that $G\text{-Deg}\,(F_{\theta_k},\mathcal{U}_k) \neq 0$ if and only if there is a non-zero winding number $\mu_{ni}(v_k,\lambda_k)$. We have the following corollary.

Corollary 2.5.2. *Suppose that $M \subset V^G \times \mathbb{R}^2$ is such that all the V-singular points in M are isolated and M is complete. Let S denotes the closure of* *the set of all non-trivial solutions to (GBP). Then for each bounded connected component $\mathcal{C}$ of $\mathcal{S}$ we have $G\mathcal{C} = \mathcal{C}$, $\mathcal{C} \cap M$ is finite and if*

$$\mathcal{C} \cap M = \{(v_1, \lambda_1), \ldots, (v_q, \lambda_q)\},$$

then for every $n > 0$, $i \geq 0$ we have

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k) \cdot \mu_{ni}(v_k, \lambda_k) = 0.$$

Proof. By Theorem 2.5.1, we have that

$$\sum_{k=1}^{q} G\text{-Deg}\,(F_{\theta_k}, \mathcal{U}_k) = 0$$

where $\mathcal{U}_k$ is a special neighbourhood of (v_k, λ_k) and θ_k is an almost complementary function on $\mathcal{U}_k$. It follows from Corollary 2.4.3 that

$$0 = \sum_{k=1}^{q} G\text{-Deg}\,(F_{\theta_k}, \mathcal{U}_k) = \sum_{k=1}^{q} \varepsilon(v_k, \lambda_k)\Big(\mu(v_i, \lambda_k) + \zeta(v_k, \lambda_k)\Big).$$

Consequently,

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k)\mu(v_k, \lambda_k) = 0$$

and the conclusion follows.

Recall that for each $n > 0$, $i \geq 0$, V_{ni} denotes the G-isotypical component of V with respect to the irreducible representation $\varphi_{ni} : \Gamma \times S^1 \to S^1$ defined by

$$\varphi_{ni}(\gamma, \xi)z := \theta_{ni}(\gamma)\xi^n z, \quad (\gamma, \xi) \in \Gamma \times S^1,\ z \in \mathbf{C}$$

where θ_{ni} is an irreducible complex representation of Γ. Recall also that for any $v \in V_{ni} \setminus \{0\}$, the isotropy group $G_v = G_{ni} = \{(\gamma, \xi);\ (\gamma, \xi^n) \in \text{Graph}(\theta_{ni})\}$.

We refine the corollary 2.5.3 into the following form for the convenience of applications.

Theorem 2.5.4. *Suppose that $M \subset V^G \times \mathbb{R}^2$ is complete such that all V-singular points in M are isolated. For any $n > 0$, $i \geq 0$, let S^{ni} denote the closure of the set of all non-trivial solutions in $V^{G_{ni}}$. Then for each bounded connected component C^{ni} of S^{ni} we have $GC^{ni} = C^{ni}$, $C^{ni} \cap M$ is a finite set and if*

$$C^{ni} \cap M = \{(v_1, \lambda_1), (v_2, \lambda_2), \ldots, (v_q, \lambda_q)\},$$

then

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k) \cdot \mu_{ni}(v_k, \lambda_k) = 0.$$

Proof. The proof for $GC^{ni} = C^{ni}$ is obvious and $C^{ni} \cap M$ must be finite.

Consider $f_\theta^{G_{ni}} = (f, \theta)|_{V^{G_{ni}} \times \mathbb{R}^2}$, where $f(v, \lambda) = v - \Theta_{K_0}(B_0, N_0)(v, \lambda)$ and θ is an almost complementary function. Then $f_\theta^{G_{ni}} : V^{G_{ni}} \times \mathbb{R}^2 \to V^{G_{ni} \times \mathbb{R}}$. Moreover, $V^{G_{ni}}$ is G-invariant and $M \subset (V^{G_{ni}})^G \times \mathbb{R}^2 = V^G \times \mathbb{R}^2$. By Theorem 2.5.2,

$$\sum_{k=1}^{q} G\text{-Deg}\,(f_{\theta_k}^{G_{ni}}, \mathcal{U}_k^{ni}) = 0$$

where $\mathcal{U}_k^{ni}$ is a special neighborhood in $V^{G_{ni}} \times \mathbb{R}^2$ of (v_k, λ_k) and $\theta_k = \theta|_{\mathcal{U}_k^{ni}}$. Let $\mathcal{U}_k$ be a special neighborhood in $V \times \mathbb{R}^2$ of (v_k, λ_k). It follows from Theorem 2.4.2 that the (G_{ni})-component of G-Deg$(f_\theta, \mathcal{U}_k)$ is equal to $\varepsilon(v_k, \lambda_k)\mu_{ni}$. On the other hand, from the construction of the G-degree, the (G_{ni})-component of $G\text{-Deg}(f_{\theta_k}^{G_{ni}}, \mathcal{U}_k^{ni})$ is also $\varepsilon(v_k, \lambda_k)\mu_{ni}$. Consequently,

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k)\mu_{ni}(v_k, \lambda_k) = 0,$$

as desired.

References

[1] J. C. Alexander, *Bifurcation of zeros of parametrized functions*, J. Funct. Anal. **29** (1978), 37–53.

[2] ——— and J. A. Yorke, *Calculating bifurcation invariants as elements in the homotopy of the general linear group*, J. Pure Appl. Algebra **13** (1978), 1–8.

[3] —— and P. M. Fitzpatrick, *The homotopy of certain space of nonlinear operators, and its relation to global bifurcation of the fixed points of parametrized condensing operators*, J. Funct. Anal. **34** (1979), 87–106.

[4] —— and ———, *Global bifurcations of equations involving several parameter multivalued condensing mappings*, in Fixed Point Theory (B. V. Loginov, ed.), Lecture Notes in Mathematics **886**, Springer-Verlag, New York, 1981, 1-19.

[5] R. Chiappinelli, *Remarks on Krasnoselskii bifurcation theorem*, Comment. Math. Univ. Carolinae **30** (1989), 235–241.

[6] P. Chossat, R. Lauterbach and I. Melbourne, *Steady-state bifurcation with $O(3)$-symmetry*, Arch. Rational Mech. Anal. **113** (1990), 313–376.

[7] S. N. Chow and J. K. Hale, "Methods of Bifurcation Theory", Springer-Verlag, New York, 1982.

[8] G. Cicogna, *Bifurcation from topology and symmetry arguments*, Boll. U. M. I. **(6)3-A** (1984), 131–138.

[9] E. N. Dancer, *Symmetries, degree, homotopy indices and asymptotically homogeneous problem*, Nonlinear Analysis–TMA **6** (1982), 667–686.

[10] ———, *Perturbation of zeros in the presence of symmetries*, J. Austral. Math. Soc. **(A)36** (1984), 106–125.

[11] ———, *A new degree for S^1-invariant gradient mappings and applications*, Ann. Inst. Henri Poincaré **2** (1985), 329–370.

[12] K. Deimling, "Nonlinear Functional Analysis", Springer-Verlag, New York, 1984.

[13] G. Dylawerskii, K. Gęba, J. Jodel and W. Marzantowicz, *An S^1-equivariant degree and the Fuller index*, Ann. Pol. Math. **52** (1991), 243–280.

[14] L. H. Erbe, K. Gęba, W. Krawcewicz and J. Wu, *S^1-degree and global Hopf bifurcation theory of functional differential equations*, J. Differential Equations **98** (1992), 277–298.

[15] ——, W. Krawcewicz and J. Wu, *A composite coincidence degree with applications to boundary value problems of neutral equations*, Trans. Amer. Math. Soc. **335** (1993), 459–478.

[16] P. M. Fitzpatrick, *Homotopy, linearization and bifurcation*, Nonlinear Analysis–TMA **12** (1988), 171–184.

[17] ——— and J. Pejsachowicz, *The degree of proper C^2 Fredholm mappings I*, J. reine angew. Math **427** (1992), 1–33.

[18] —— and ———, *Orientation and the Leray-Schauder theory for fully nonlinear elliptic boundary value problems*, Mem. Amer. Math. Soc. **483**, 1993.

[19] R. E. Gaines and J. Mawhin, "Coincidence Degree and Nonlinear Differential Equations", Lecture Notes in Mathematics **568**, Springer-Verlag, 1977.

[20] K. Gęba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems II: Abstract Hopf bifurcation theorems*, Preprint, 1993.

[21] —— and W. Marzantowicz, *Global bifurcation of periodic solutions*, Topological Methods in Nonlinear Analysis **1** (1993), 67-93.

[22] M. Golubitsky, I. N. Stewart and D. G. Schaeffer, " Singularities and Groups in Bifurcation Theory", Vol. 2, Springer-Verlag, New York, 1988.

[23] G. Hetzer, *Bifurcation theorems of Rabinowitz type*, Nonlinear Analysis-TMA **1** (1977), 471–479.

[24] ——, *Some remarks on Φ_+-operators and the coincidence degree for a Fredholm equation with noncompact nonlinear perturbations*, Ann. Soc. Sci. Bruxelles Ser.I, **89** (1975), 497–508.

[25] J. Ize, *Bifurcation theory for Fredholm operators*, Mem. Amer. Math. Soc. **174**, Amer. Math. Soc., Providence. R. I. , 1976.

[26] ——, I. Massabó and A. Vignoli, *Global results on continuation and bifurcation for equivariant maps*, in Nonlinear Functional Analysis and its Applications (S. P. Singh, ed.), Mathematical and Physical Sciences **173**, Reidel, Dordrecht, 1986, 75–111.

[27] ——, —— and ——, *Degree theory for equivariant maps, the general S^1-action* II, Mem. Amer. Math. Soc. **481**, 1992.

[28] M. A. Krasnosel'skii, "Topological Methods in the Theory of Nonlinear Integral Equations", Pergamon, New York, 1965.

[29] W. Krawcewicz, J. Wu and H. Xia, *Global Hopf bifurcation theory for condensing fields and neutral equations with applications to lossless transmission problems,* Canad. Appl. Math. Quarterly 1 (1993), 167–220.

[30] J. MacBain, *Global bifurcation theorems for noncompact operators*, Bull. Amer. Math. Soc. **80** (1974), 1005–1009.

[31] A. P. Makhmudov and Z. S. Aliev, *Global bifurcation of solutions of certain nonlinear eigenvalue problems*, Differential Equations (English Translation) **25** (1989), 71–76.

[32] —— and ——, *Nondifferentiable perturbations of spectral problems for a pair of selfadjoint operators and global bifurcation*, Soviet Math. Dokl. **38** (1989), 122– 126.

[33] J. Mawhin, "Topological degree methods in nonlinear boundary value problems", CBMS **40**, Amer. Math. Soc., Providence, R. I., 1979.

[34] R. D. Nussbaum, *The fixed point index and fixed point theorems for k-set contractions*, Ph. D. Thesis, Univ. of Chicago, 1969.

[35] ———, *The fixed point index for locally condensing maps*, Ann. Mat. Pura Appl. **89** (1971), 216–258.

[36] ———, *Degree theory for local condensing maps*, J. Math. Anal. Appl. **37** (1972), 741–766.

[37] J. Pajsachowicz and A. Vignoli, *On the topological coincidence degree for perturbation of Fredholm operators*, Boll. Un. Math. Ital. **B17** (1980), 1457–1466.

[38] ———, *Degree theory and bifurcation of Fredholm maps*, Rev. Union Math. Argentina **37** (1993), 288–293.

[39] P. J. Rabier, *New aspects in bifurcation with symmetry*, Differnetial and Integral Equations **3** (1990), 1015–1034.

[40] P. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[41] ———, *Some aspects of nonlinear eigenvalue problems*, Rocky Mtn. J. Math. **3** (1973), 161–202.

[42] ———, *On bifurcation from infinity*, J. Differential Equations **3** (1973), 462–475.

[43] D. Ruelle, *Bifurcation in the presence of a symmetry group*, Arch. Rational Mech. Anal. **51** (1973), 136–152.

[44] B. N. Sadovski, *Limit compact and condensing operators*, Uspehi Math. Nauk **163** (1972), 81–146, Russian Math. Surveys **27** (1972), 85–146.

[45] D. H. Sattinger, *Transformation groups and bifurcation at multiple eigenvalues*, Bull. Amer. Math. Soc. **79** (1973), 709–711.

[46] ———, "Group Theoretic Methods in Bifurcation Theory", Lecture Notes in Mathematics **762**, Springer-Verlag, Berlin, 1979.

[47] C. A. Stuart, *Some bifurcation theory for k-set contractions*, Proc. London Math. Soc. **27** (1973), 531–550.

[48] J. W. Thomas, *A bifurcation theorem for k-set contractions*, Pac. J. Math. **44** (1973), 749–756.

[49] A. Vanderbauwhede, "Local Bifurcation and Symmetry", Research Notes in Mathematics **75**, Pitman, Boston, 1982.

[50] J. R. L. Webb and S. C. Welsh, *Topological degree and global bifurcation*, Proc. Symp. Pure Math. **45**, Part 2, Amer. Math. Soc., 1986.

[51] S. C. Welsh, *Global results concerning bifurcation for Fredholm maps of index zero with a transversality condition*, Nonlinear Analysis–TMA **12** (1988), 1137–1148.

CHAPTER 3

GLOBAL HOPF BIFURCATION OF NFDEs WITH SYMMETRY

3.1. Introduction

In this chapter, we apply the equivariant bifurcation theory of Chapter 2 to establish an analog of the Alexander-Yorke's global Hopf bifurcation theory [8, 33, 34, 130] and develop a powerful tool in order to obtain the existence and multiplicity of ***symmetric periodic solutions*** for the following one parameter family of ***equivariant*** neutral functional differential equations (NFDEs)

$$\frac{d}{dt}[x(t) - b(x_t, \alpha)] = F(x_t, \alpha), \quad \alpha \in \mathbb{R} \tag{3.1.1}$$

when the parameter α is ***far away*** from those values at which the linearization of (3.1.1) has a pair of pure imaginary characteristic values, where $x \in \mathbb{R}^n$, $\tau \geq 0$, $x_t(\theta) = x(t+\theta)$ for $\theta \in (-\infty, \tau]$, $C_\tau := C((-\infty, \tau]; \mathbb{R}^n)$, F and $b : C_\tau \times \mathbb{R} \to \mathbb{R}^n$ are continuously differentiable and for a representation $\rho : \Gamma \to GL(\mathbb{R}^n)$ of a compact Lie group Γ on $\mathbb{R}^n$

$$\begin{aligned} F(\rho(\gamma)\varphi, \alpha) &= \rho(\gamma)F(\varphi, \alpha) \\ b(\rho(\gamma)\varphi, \alpha) &= \rho(\gamma)b(\varphi, \alpha), \quad \gamma \in \Gamma, \ \varphi \in \tau \end{aligned} \tag{3.1.2}$$

where $(\rho(\gamma)\varphi)(\theta) = \rho(\gamma)\varphi(\theta)$ for $\theta \in [-\infty, \tau]$, and b satisfies the Lipschitz condition

$$|b(\varphi, \alpha) - b(\psi, \alpha)| \leq k \sup_{s \in (-\infty, \tau]} \|\varphi(s) - \psi(s)\|, \quad \varphi, \ \psi \in C_\tau \tag{3.1.3}$$

for a constant $k \in [0, 1)$.

Our first idea is standard. By introducing the unknown period as an additional parameter, we reformulate the NFDEs (3.1.1) into an equivariant coincidence problem of the following type

$$L[\pi(x) - B(x)] = N(x), \quad x \in \mathbb{E} \times \mathbb{R}^2 \tag{3.1.4}$$

where $L : \mathrm{Dom}(L) \subseteq \mathbb{E} \to \mathbb{F}$ is an equivariant Fredholm operator of index zero between $\mathbb{E}$ and $\mathbb{F}$ which are isometric Banach representations of $\Gamma \times S^1$, π is the natural projection of $\mathbb{E} \times \mathbb{R}^2$ onto $\mathbb{E}$ and $B : \mathbb{E} \times \mathbb{R}^2 \to \mathbb{E}$, $N : \mathbb{E} \times \mathbb{R}^2 \to \mathbb{F}$ are such that (B, N) is an L-condensing $\Gamma \times S^1$-pair due to (3.1.2) and (3.1.3). In consequence, the Hopf bifurcation problem is treated as a static bifurcation of (3.1.4) and the application of a local bifurcation theorem of Krasnosel'skii type and the global bifurcation theorem of Rabinowitz type we presented in Chapter 2 facilitates the proofs of symmetric (equivariant) Hopf bifurcation theorems for NFDEs (3.1.1).

The main ingredient of the equivariant Hopf bifurcation theorems is the *symmetric crossing number* which enters as follows. First, by a group-theoretic argument, the characteristic equation is decomposed (decoupled) with respect to the induced representation of Γ on certain complex space. To each isolated center and each decomposed characteristic equation, a crossing number is then assigned by using the classical Brouwer degree of some analytic function, which, in turn, appears to be a component of certain equivariant degree. The local symmetric Hopf bifurcation theorem says that the nontriviality of this crossing number implies the existence of a family of periodic solutions with prescribed symmetry bifurcating from the origin, while by applying the additivity property of the equivariant composite coincidence degree, the global symmetric Hopf bifurcation theorem claims that any maximal symmetric continuum of local symmetric Hopf bifurcation points is either unbounded (in the Fuller space) or contains a finite number of isolated centers such that the sum of all crossing numbers at these centers is identical to zero. This presents a generalization to neutral equations with symmetry of the Alexander-Yorke's global bifurcation theorem [8] in the absence of symmetry. Moreover, the crossing number we defined is intimately related to the index, Hopf index and center index, which are introduced respectively by Nussbaum [130] for retarded equations, Fiedler [59] for parabolic equations and Mallet-Paret and Yorke [117] for ordinary differential equations. It appears that our global equivariant Hopf bifurcation theorem provides a new insight into neutral equations with symmetry and includes the above mentioned global results as special cases.

Our approach in this chapter is an enrichment of that of Gęba, Krawcewicz and Wu [70] for retarded equations. In comparison to the general theory of equivariant Hopf bifurcation for ordinary differential equations and some parabolic partial differential equations in Golubitsky and Stewart [76], Sattinger [146], Ruelle [142], Schecter [150] and Fiedler [61, 62], our study provides an alternative to the investigation of the symmetric Hopf bifurcation for functional differential equations. Due to the topological nature of our approach, we do not need the genericity conditions on vector fields [62], dimension restrictions on invariant fixed point subspace and maximality assumptions on certain isotropy groups [76]. Moreover, we avoid the solution operator, the sophisticated decomposition and perturbation theory of linear functional differential equations and certain generic approximations as well as generic bifurcation pictures (see, for example, [8, 33, 34, 117, 130, 143]) which, to the best of our knowledge, have not been developed to NFDEs (3.1.1) where delayed and advanced arguments are allowed to coexist (i.e. neutral equations of mixed type) and symmetries are presented to complicate the dynamics.

It should be mentioned that a considerable amount of research has been devoted to the existence of periodic solutions for *autonomous retarded* functional differential equations (see [27, 31, 33, 46, 55, 83, 84, 87, 103, 127–131, 161–163] and the references cited there), but little has been done for neutral functional differential equations despite the fact that more and more neutral equations arise from population dynamics, electrodynamics and control theory (see, for example, the references cited in Chapter 4 and Chapter 5). Although the existence of periodic solutions for ***nonautonomous*** neutral equations has been considered in [23, 56, 86, 90, 107, 114–116, 136, 144, 145, 152, 154], that of nontrivial periodic solutions for ***autonomous*** neutral equations is more difficult to study. Earlier trials along this line are [2, 24, 25, 82, 107, 124, 141, 156] obtained by using asymptotic power series expansion method and some later results can be found in [127] by a global bifurcation technique and [134, 160] via local Hopf bifurcation theorems. More recently, we have established in [110] a global bifurcation theory for general neutral equations and multiplicity results on the existence of nontrivial periodic solutions of a specific neutral equation have been carefuly studied. However, a global Hopf bifurcation theory for NFDEs (3.1.1) in the presence of symmetry and the existence of symmetric periodic solutions for NFDEs (3.1.1) are still void, to the best of our knowledge.

We have included a rather extensive bibliography at the end of this chapter. Since the literature on the subject of Hopf bifurcation is vast, the list of references we give does not pretend to be exhaustive. Instead, it is selective and the choice has been made to the references that deal with Hopf bifurcation from a general point of view and the equations under consideration involving certain symmetries. Some attempts have also been made to include the most recent literature on symmetric Hopf bifurcation theory. We refer to [1, 26, 27, 31, 43–45, 47, 79, 87, 103, 105, 119, 135, 153, 161–163, 173, 180] for a more complete literature on (local) Hopf bifurcation in various cases in the absence of symmetry. For the treatment of equivariant Hopf bifurcation theory, we refer to [12–16, 50–52, 61, 62, 70, 73–76, 96, 97, 106, 121–123, 142, 146, 150, 157, 164, 166–170, 172–174] and the references cited there. The papers of Chossat et al. [28, 29], Cicogna et al. [36–42], Dancer [48], Field [63], Gaeta [68, 69], Golubitsky et al. [72], Renardy [139], Sattinger [147–149], which deal with equivariant bifurcation theory (not necessarily Hopf bifurcation), are also related to our topics. Finally, the global bifurcation results are preferred and, therefore, the work of Alexander et al. [3–8], Alligood et al. [9], Auchmuty [17], Chow et al. [30–35], Dylawerski et al. [53], Erbe et al. [55], Fiedler [58–62], Gęba et al. [70, 71], Healey [89], Ize [94–97], Krawcewicz et al. [110], Mallet-Paret et al. [117], Nussbaum [127–131] and Wu [179] will be frequently cited.

The chapter is organized as follows. In Section 3.2, we discuss briefly the symmetry of periodic solutions and show how the symmetry of a periodic solution of NFDEs (3.1.1) equivariant with respect to a representation of a compact Lie group Γ can be related to an isotropy group of this solution in a Banach space of periodic functions and explain why we assume $\Gamma = \mathbf{Z}_n$ in our discussion. Section 3.3 is devoted to the proofs of local and global symmetric Hopf bifurcation theorems with $\Gamma = \mathbf{Z}_n$. These bifurcation results are then applied, in Section 3.4, to the Rashevsky-Turing theory and bifurcation of a ring of identical cells governed by neutral equations and coupled by delayed diffussion are considered. Finally, in Section 3.5, we extend the results to functional equations and neutral equations of general type.

3.2. Symmetry of periodic solutions

Let $M > 0$ be an integer and $C(\mathbb{R}, \mathbb{R}^M)$ denote the Banach space of continuous bounded function from $\mathbb{R}$ to $\mathbb{R}^M$ equipped with the usual supremum norm $\|\varphi\| = \sup_{v\in\mathbb{R}} |\varphi(v)|$ for $\varphi \in C(\mathbb{R}; \mathbb{R}^n)$, where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^n$. If $\varphi \in C(\mathbb{R}; \mathbb{R}^n)$ and $t \in \mathbb{R}$, then $\varphi_t \in C(\mathbb{R}; \mathbb{R}^n)$ is defined as $\varphi_t(v) = \varphi(t+v)$ for $v \in \mathbb{R}$.

Consider the following *neutral functional differential equations* (NFDEs)

$$\frac{d}{dt}[x(t) - b(x_t)] = F(x_t) \tag{3.2.1}$$

where F, $b : C(\mathbb{R}; \mathbb{R}^M) \to \mathbb{R}^M$ are continuously differentiable mappings satisfying the following assumptions:

(A1) $F : C(\mathbb{R}; \mathbb{R}^M) \to \mathbb{R}^M$ is completely continuous and there exists a constant $k \in [0, 1)$ such that

$$|b(\varphi) - b(\psi)| \leq k\|\varphi - \psi\|, \quad \varphi,\ \psi \in C(\mathbb{R}; \mathbb{R}^M).$$

(A2) There exist a compact Lie group Γ and an orthogonal real representation $\rho : \Gamma \to O(\mathbb{R}^M)$ such that

$$b(\rho(\gamma)\varphi) = \rho(\gamma)b(\varphi),$$
$$F(\rho(\gamma)\varphi) = \rho(\gamma)F(\varphi),$$

for all $\varphi \in C(\mathbb{R}; \mathbb{R}^M)$ and $\gamma \in \Gamma$, where $\rho(\gamma)\varphi \in C(\mathbb{R}; \mathbb{R}^M)$ is defined by

$$(\rho(\gamma)\varphi)(v) = \rho(\gamma)\varphi(v), \quad v \in \mathbb{R}.$$

In what follows, a system (3.2.1) possessing (A2) will be said to be *equivariant* with respect to the linear action of Γ on $\mathbb{R}^M$.

Suppose now that $x = x(t)$ is a periodic solution of (3.2.1) with minimal period $p > 0$. Let O_x denote the trajectory of x, i.e.

$$O_x := \{x_t; \quad t \in \mathbb{R}\} \subset C(\mathbb{R}; \mathbb{R}^M).$$

Define

$$H := \{\gamma \in \Gamma;\ \rho(\gamma)O_x = O_x\} \tag{3.2.2}$$

$$K := \{\gamma \in \Gamma;\ \rho(\gamma)x_0 = x_0\} \tag{3.2.3}$$

By the continuity of representation ρ, H and K are closed subgroups of Γ. Moreover, from (3.2.2), for every $h \in H$, there exists a unique $\Theta(h) \in \mathbb{R}/\mathbb{Z}$ such that $\rho(h)x_0 = x_{\Theta(h)p}$. Note that, under assumption (A1), solutions of (3.2.1) with the same initial values are unique (see also [83]). We must have

$$\rho(h)x_t = x_{t+\Theta(h)p}, \quad t \in \mathbb{R}. \tag{3.2.4}$$

By definition, the map $\Theta : H \to \mathbb{R}/\mathbb{Z}$ is a continuous homomorphism since p is the minimal period of $x(t)$, where the (additive) group $\mathbb{R}/\mathbb{Z}$ gets the induced topology from $\mathbb{R}$. (It follows directly from the Weyl Theorem [171] that H, K and $\mathbb{R}/\mathbb{Z}$ are moreover Lie subgroups and Θ is a Lie homomorphism).

Notice that $\Gamma_{x_t} = \Gamma_{x_0}$ for all $t \in \mathbb{R}$, where for any $\varphi \in C(\mathbb{R}; \mathbb{R}^M)$, $\Gamma_\varphi := \{\gamma \in \Gamma;\ \rho(\gamma)\varphi = \varphi\}$. We have $K = \ker \Theta$. By the homomorphism theorem [171], it follows that K is a closed normal subgroup of H and

$$H/K \cong Im\,\Theta \cong \begin{cases} \mathbb{Z}_n := \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\} \subset \mathbb{R}/\mathbb{Z} \\ \mathbb{Z}_\infty := \mathbb{R}/\mathbb{Z} \cong S^1. \end{cases} \tag{3.2.5}$$

Following Fiedler [62] and Gęba et al. [70], we give the following definitions.

Definition 3.2.1. Let $x(t)$ be a periodic sulotion of (3.2.1) with minimal period $p > 0$. We call the triple (H, K, Θ), defined by (3.2.2)–(3.2.4), a *symmetry* of x and the subgroup K a *spatial symmetry* of x.

Referring to (3.2.5), we specify the symmetry of x below.

Definition 3.2.2. Let (H, K, Θ) be a symmetry of x. The periodic solution $x(t)$ is called

a concentric wave if $H = K$;

a discrete wave if $H/K \cong \mathbf{Z}_n$, $1 \leq n < \infty$;

a rotating wave if $H/K \cong \mathbf{Z}_\infty$.

In applying the G-degree to detect the existence of nontrivial solutions, one usually tells a part of their symmetry, which means that the full symmetry of those nontrivial solutions could be "bigger". This observation gives rise to the following definition.

Definition 3.2.3. Let $(\hat{H}, \hat{K}, \hat{\Theta})$ be a triple and (H, K, Θ) be a symmetry of periodic solution x. We say that x has *symmetry at least* $(\hat{H}, \hat{K}, \hat{\Theta})$ if $H \geq \hat{H}$ and $\Theta|_{\hat{H}} = \hat{\Theta}$.

The symmetry of a periodic solution can be reinterpreted as an isotropy group of this periodic function in the Banach G-space of periodic functions, where $G := \Gamma \times S^1$ acts on $W := L^2(S^1; \mathbf{R}^M)$ by

$$(\gamma, \theta)y(t) = \rho(\gamma)y(t + \theta), \quad t \in \mathbf{R},\ \gamma \in \Gamma,\ \theta \in S^1 \quad \text{and} \quad y \in W.$$

Recall that for a subgroup $H \subset \Gamma$ and a group homomorphism $\Theta : H \to \mathbf{R}/\mathbf{Z} \cong S^1$, the *twisted subgroup* $H^\Theta := \{(h, \Theta(h)) \in \Gamma \times S^1;\ h \in H\}$.

Analogous to [62, 70], we have the following reinterpretation of the symmetry of a periodic solution of (3.2.1).

Proposition 3.2.1. *Let $x(t)$ be a periodic solution of (3.2.1) with minimal period $p > 0$ and $y \in W$ is defined by $y(t) = x(pt)$ for $t \in \mathbf{R}$. Then*

(i) x has a symmetry (H, K, Θ) iff the twisted subgroup H^{Θ} is the isotropy group of y;

(ii) x has symmetry at least $(\hat{H}, \hat{K}, \hat{\Theta})$ iff the twisted subgroup $\hat{H}^{\hat{\Theta}}$ is a subgroup of the isotropy group of y.

Proof. (i) Let (h, K, Θ) be the symmetry of x and G_y be the isotropy group of y. If $(\gamma, \theta) \in G_y$, then $(\gamma, \theta)y(t) = y(t)$. By the definition of the action G and $y(t)$, $\rho(\gamma)x(pt + p\theta) = x(pt)$, which implies that $\rho(\gamma)x(t + p\theta) = x(t)$ for $t \in \mathbf{R}$. Note that p is the minimal period. From (3.2.2), $\gamma \in H$ and $\theta = \Theta(\gamma)$, i.e. $(\gamma, \theta) \in H^{\Theta}$. This proves $G_y \leq H^{\Theta}$. Similarly, $H^{\Theta} \leq G_y$. Hence $G_y = H^{\Theta}$.

The converse can be proved analogously.

(ii) is a direct consequence of Definition 3.2.3 and the statement (i).

Let (H_0, K_0) be a given pair with K_0 a normal subgroup of $H_0 \leq \Gamma$ and

$$H_0/K_0 \cong \mathbf{Z}_n, \quad 1 \leq n \leq \infty.$$

We look for periodic solutions of (3.2.1) with a symmetry (H_0, K, Θ), where $K_0 \leq K$ is a normal subgroup of H_0. To this end, we consider Eq. (3.2.1) restricted to the invariant subspace $X := (\mathbf{R}^M)^{K_0}$ of $\mathbf{R}^M$. The action Γ on $\mathbf{R}^M$ induces an H_0/K action on X by

$$[h_0]x = \rho(h_0)x, \quad x \in X, \quad h_0 \in [h_0] \in H_0/K$$

and $F,\ b : C^{K_0}(\mathbf{R};\mathbf{R}^M) \times \mathbf{R} \to X$ is equivariant under the induced action, i.e.

$$\begin{aligned}
&F([h_0]\varphi, \alpha) = [h_0]F(\varphi, \alpha),\\
&b([h_0]\varphi, \alpha) = [h_0]b(\varphi, \alpha),\\
&h_0 \in [h_0] \in H_0/K, \quad \varphi \in C^{K_0}(\mathbf{R}, \mathbf{R}^M),
\end{aligned}$$

where

$$C^{K_0}(\mathbf{R}, \mathbf{R}^M) = \{\varphi \in C(\mathbf{R}, \mathbf{R}^M);\ \varphi(\theta) \in X \quad \text{for} \quad \theta \in \mathbf{R}\}$$

Consequently, it suffices to consider the system (3.2.1) on the H_0/K-representation space X.

Note that H_0/K is a subgroup of $\mathbf{Z}_n$, $1 \le n \le \infty$. H_0/K is either cyclic or $\mathbf{Z}_\infty$. Due to this fact, to find periodic solutions of (3.2.1) with prescribed symmetry (H_0, K, Θ), we shall restrict our discussion to the case where $\Gamma = \mathbf{Z}_n$, $1 \le n \le \infty$, in the subsequent sections.

3.3. Local and global Hopf bifurcation with symmetry

Let $\tau \ge 0$ be a given constant. We denote by C_τ the Banach space of bounded continuous functions from $(-\infty, \tau]$ to $\mathbf{R}^N$ equipped with the usual supremum norm $\|\varphi\| = \sup_{\theta\in(-\infty,\tau]} |\varphi(\theta)|$ for $\varphi \in C_\tau$, where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^N$. For $x \in C(\mathbf{R}, \mathbf{R}^N)$ and $t \in \mathbf{R}$, $x_t \in C_\tau$ is defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in (-\infty, \tau]$.

We consider the following one parameter family of *neutral functional differential equations*

$$\frac{d}{dt}[x(t) - b(x_t, \alpha)] = F(x_t, \alpha) \tag{3.3.1}$$

where $x \in \mathbf{R}^N$, $\alpha \in \mathbf{R}$, $b, f : C_\tau \times \mathbf{R} \to \mathbf{R}^N$ are continuously differentiable mappings satisfying the following assumptions:

(A1) $F : C_\tau \times \mathbf{R} \to \mathbf{R}^N$ is completely continuous and there exists a constant $k \in [0,1)$ such that

$$|b(\varphi,\alpha) - b(\psi,\alpha)| \le k\|\varphi - \psi\|, \quad \varphi,\ \psi \in C_\tau, \quad \alpha \in \mathbf{R}.$$

(A2) There exists a real orthogonal representation $\rho : \mathbf{Z}_n \to O(\mathbf{R}^N)$ of the abelian group $\mathbf{Z}_n$, $1 \le n \le \infty$, on $\mathbf{R}^N$ such that

$$b(\rho(\gamma)\varphi,\alpha) = \rho(\gamma)b(\varphi,\alpha),$$
$$F(\rho(\gamma)\varphi,\alpha) = \rho(\gamma)F(\varphi,\alpha),$$

for all $\varphi \in C_\tau$, $\alpha \in \mathbf{R}$ and $\gamma \in \mathbf{Z}_n$ where $\rho(\gamma)\varphi \in C_\tau$ is defined as $(\rho(\gamma)\varphi)(\theta) = \rho(\gamma)\varphi(\theta)$ for $\theta \in (-\infty,\tau]$.

(A3) $F(0,\alpha) = 0$ for all $\alpha \in \mathbf{R}$ and there exists an $\alpha_0 \in \mathbf{R}$ such that $D_x\bar{F}(0,\alpha_0) : \mathbf{R}^N \to \mathbf{R}^N$ is an isomorphism, where $\bar{F} : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}^N$, the restriction of F on $\mathbf{R}^N$, is defined by

$$\bar{F}(x,\alpha) = F(\bar{x},\alpha), \quad x \in \mathbf{R}^N, \quad \alpha \in \mathbf{R},$$

$\bar{x}$ is the constant map from $(-\infty,\tau]$ into $\mathbf{R}^N$ with the value $x \in \mathbf{R}^N$, and $D_x\bar{F}(0,\alpha_0)$ denotes the derivative of $\bar{F}$ with respect to x at $(0,\alpha_0)$.

We call all solutions $(0,\alpha)$ of (3.3.1) the *stationary solutions* and $(0,\alpha_0)$ a *nonsigular* stationary solution. By linearizing Eq. (3.3.1) at the stationary solution $(0,\alpha)$, we obtain the following *characteristic equation*

$$\det{}_{\mathbf{C}}\, \Delta_\alpha(\lambda) = 0 \tag{3.3.2}$$

where $\Delta_\alpha(\lambda)$ is an $N \times N$ complex matrix defined as follows:

$$\Delta_\alpha(\lambda) := \lambda[Id - D_\varphi b(0,\alpha)(e^{\lambda \cdot} Id)] - D_\varphi F(0,\alpha)(e^{\lambda \cdot} Id)] : \mathbf{C}^N \to \mathbf{C}^N$$
$$D_\varphi b(0,\alpha)(e^{\lambda \cdot} Id) = (D_\varphi b(0,\alpha)(e^{\lambda \cdot}\epsilon_1), \ldots, D_\varphi b(0,\alpha)(e^{\lambda \cdot}\epsilon_N))$$
$$D_\varphi F(0,\alpha)(e^{\lambda} Id) = (D_\varphi F(0,\alpha)(e^{\lambda \cdot}\epsilon_1), \ldots, D_\varphi F(0,\alpha)(e^{\lambda \cdot}\epsilon_N))$$
$$e^{\lambda \cdot}\epsilon_j(\theta) = e^{\lambda\theta}\epsilon_j, \quad \theta \in (-\infty, \tau],$$

and $\{\epsilon_1, \ldots, \epsilon_N\}$ is the standard basis of $\mathbf{R}^N$ and $\mathbf{C}^N := \mathbf{R}^N + i\mathbf{R}^N$.

A solution $\lambda \in \mathbf{C}$ to the equation (3.3.2) is called a *characteristic value* of the stationary solution $(0,\alpha)$. $(0,\alpha)$ is said to be a *center* of (3.3.1) if (3.3.2) has a pure imaginary characteristic value, and it is said to be an *isolated center* if there is no other center in some neighborhood of $(0,\alpha)$ in $\mathbf{R}^N \times \mathbf{R}$.

We also make the following assumption.

(A4) $(0,\alpha_0)$ is an isolated center of (3.3.2).

By (A4), there exist constants $\beta_0 > 0$ and $\delta > 0$ such that $\det_{\mathbf{C}} \Delta_{\alpha_0}(i\beta_0) = 0$ and if $0 < |\alpha - \alpha_0| < \delta$, then $i\mathbf{R} \cap \{\lambda \in \mathbf{C};\ \det_C \Delta_\alpha(\lambda) = 0\} = \varnothing$.

Now choose constants $b = b(\alpha_0, \beta_0) > 0$ and $c = c(\alpha_0, \beta_0) > 0$ such that the closure of $\Omega := (0,b) \times (\beta_0 - c, \beta_0 + c) \subset \mathbf{R}^2 \cong \mathbf{C}$ contains no other zero of $\det_C \Delta_{\alpha_0}(\lambda)$. Note that $\det_{\mathbf{C}} \Delta_\alpha(\lambda)$ is analytic in $\lambda \in \Omega$ and continuous in $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, $\det_{\mathbf{C}} \Delta_{\alpha_0 \pm \delta}(\lambda) \neq 0$ for $\lambda \in \partial\Omega$.

We first consider the case where $\mathbf{Z}_n$ is finite, i.e, $n < \infty$.

To begin with, identify $\mathbf{C}^N$ with $\mathbf{R}^N + i\mathbf{R}^N$. The real orthogonal representation ρ of $\mathbf{Z}_n$ by (A2) thus induces a unitary representation, again denoted by ρ, of $\mathbf{Z}_n$ on $\mathbf{C}^N$ as follows

$$\rho(\gamma)(x+iy)=\rho(\gamma)x+i\rho(\gamma)y, \quad x+iy\in\mathbf{C}^N, \quad \gamma\in\mathbf{Z}.$$

Let us identify $\mathbf{Z}_n \cong \{\gamma\in C:\gamma^n=1\}$ and let $\gamma_n\in\mathbf{Z}_n$ denote the generator of $\mathbf{Z}_n$, i.e. $\gamma_n=e^{i\frac{2\pi}{n}}$. Put $T_n=\rho(\gamma_n):\mathbf{R}^N\to\mathbf{R}^N$ and denote by $\sigma(T_n)\in\mathbf{C}$ the spectrum of T_n.

Define a subset of integers $J=\{j\in\{0,1,2,\ldots,n-1\};\ e^{i\frac{2\pi j}{n}}\in\sigma(T_n)\}$. We have the following ***isotypical decomposition*** of $\mathbf{C}^N$ (see also [53, 62, 74, 155])

$$\mathbf{C}^N=\bigoplus_{j\in J}\mathbf{C}_j^N$$

where $\mathbf{C}_j^N$, $j\in J$, is the direct sum of all one-dimensional $\mathbf{Z}_n$-irreducible subrepresentation spaces V of $\mathbf{C}^N$ such that each restricted representation $\rho|_V$ is isomorphic to the irreducible representation of $\mathbf{Z}_n$ on $\mathbf{C}$ given by

$$\rho(e^{i\frac{2\pi j}{n}})z=e^{i\frac{2\pi j}{n}}z, \quad z\in\mathbf{C}, \quad j\in J.$$

Note that b and F are $\mathbf{Z}_n$-equivariant by (A2). $\Delta_\alpha\,(\lambda):\mathbf{C}^N\to\mathbf{C}^N$ is $\mathbf{Z}_n$-equivariant for all $\alpha\in\mathbf{R}$ and $\lambda\in C$ with Re $\lambda\geq 0$. Therefore $\Delta_\alpha\,(\lambda)\mathbf{C}_j^N\subseteq\mathbf{C}_j^N$ for each $j\in J$. This gives a map for each $j\in J$ below

$$\Delta_{\alpha,j}\,(\lambda):=\Delta_\alpha\,(\lambda)|_{\mathbf{C}_j^N}, \quad j\in J.$$

Recall that $\det_{\mathbf{C}}\Delta_{\alpha_0\pm\delta}\,(\lambda)\neq 0$ for $\lambda\in\partial\Omega$. We have $\det_{\mathbf{C}}\Delta_{\alpha_0\pm\delta,j}\,(\lambda)\neq 0$ for $\lambda\in\partial\Omega$ and $j\in J$. Consequently, there gives rise to a well-defined number

$$c_j(\alpha_0,\beta_0):=\deg_B(\det_{\mathbf{C}}\Delta_{\alpha_0-\delta,j}\,(\cdot),\Omega)-\deg_B(\det_{\mathbf{C}}\Delta_{\alpha_0+\delta,j}\,(\cdot),\Omega)$$

for each $j \in J$, where $\deg_B$ denotes the classical Brouwer degree. We will call $\Delta_{\alpha,j}(\lambda) = 0$ the *j-th characteristic equation* and $c_j(\alpha_0, \beta_0)$ the *j-th crossing number* of (α_0, β_0).

In order to apply the abstract bifurcation theorems from Chapter 2, we introduce some Banach spaces and linear and nonlinear mappings between them.

First, let $S^1 := \mathbf{R}/\mathbf{Z}$ be the compact Lie group of the unit circle and consider Banach spaces $V = L^2(S^1; \mathbf{R}^N)$, $W = C(S^1; \mathbf{R}^N)$ and the Sobolev space $H^1(S^1; \mathbf{R}^N)$. Define a linear action of $G := \mathbf{Z}_n \times S^1$ on V (resp. W) below

$$\begin{aligned} &(\eta(\gamma,\theta)x)(t) = \rho(\gamma)x(t+\theta), \\ &(\gamma,\theta) \in \mathbf{Z}_n \times S^1, \quad t \in S^1 \quad \text{and} \quad x \in V \ (\text{resp. } W). \end{aligned} \tag{3.3.3}$$

Under the action (3.3.3), V and W (and hence $H^1(S^1; \mathbf{R}^N)$) are isometric Banach representations of G, where the representation is given by $\eta : G \to GL(V)$ (resp. $GL(W)$). Associated with these spaces and NFDEs (3.3.1) are the following two linear maps

$$\begin{aligned} &L_0 : H^1(S^1; \mathbf{R}^N) \to V, \quad (L_0 z)(t) := z'(t), \ z \in H^1(S^1; \mathbf{R}^N), \ t \in S^1, \\ &K_0 : H^1(S^1; \mathbf{R}^N) \to V, \quad (K_0 z)(t) := \int_0^1 z(s)ds, \ z \in H^1(S^1; \mathbf{R}^N), \ t \in S^1. \end{aligned} \tag{3.3.4}$$

Note that differentiation and integration are linear operations. L_0 and K_0 are G-equivariant with respect to the action (3.3.3).

The properties of L_0 and K_0 are summarized in the lemma below.

Lemma 3.3.1. *$(L_0 + K_0)^{-1} : V \to W$, the inverse of $L_0 + K_0$, exists and is compact. Moreover, $(L_0 + K_0)$ can be explicitly given by*

$$[(L_0 + K_0)^{-1}x](t) = \int_0^t x(s)ds + \int_0^1 (\frac{1}{2} - t + s)x(s)ds, \quad x \in V, \quad t \in S^1. \tag{3.3.5}$$

Proof. The compactness of $(L_0 + K_0)^{-1}$ follows from the Sobolev inequality. Other statements are straightforward.

Remark 3.3.1. Using (3.3.5), we have that

$$(L_0 + K_0)^{-1} \sin 2\pi t \cdot Id = -\frac{1}{2\pi} \cos 2\pi t \cdot Id,$$
$$(L_0 + K_0)^{-1} \cos 2\pi t \cdot Id = \frac{1}{2\pi} \sin 2\pi t \cdot Id, \quad t \in S^1.$$

We now consider two nonlinear maps defined below.

$$\begin{aligned} &B_0 : W \times \mathbb{R}^2 \to W,\ B_0(z,\alpha,\beta)(t) = b(z_{t,\beta},\alpha),\ z \in W,\ (\alpha,\beta) \in \mathbb{R}^2, \\ &N_0 : W \times \mathbb{R}^2 \to V,\ N_0(z,\alpha,\beta)(t) = \frac{2\pi}{\beta} F(z_{t,\beta},\alpha),\ z \in W,\ (\alpha,\beta) \in \mathbb{R}^2, \end{aligned} \tag{3.3.6}$$

where $z_{t,\beta}(\theta) = z(t + \frac{\beta}{2\pi}\theta)$ for $\theta \in (-\infty, \tau]$. By definition, B_0 and N_0 are G-equivariant with respect to the action (3.3.3).

The following lemma shows that we can reduce the periodic problem to an equivalent coincidence problem.

Lemma 3.3.2. *Finding a $\frac{2\pi}{\beta}$-periodic solution of (3.3.1) is equivalent to finding a solution to the G-equivariant composite coincidence problem*

$$L_0(\pi_0 - B_0)(z,\alpha,\beta) = N_0(z,\alpha,\beta), \quad (z,\alpha,\beta) \in W \times \mathbb{R}^2, \tag{3.3.7}$$

where $\pi_0 : W \times \mathbb{R}^2 \to W$ is the projection. Moreover, (z,α,β) is a solution of (3.3.7) iff $z = f(z,\alpha,\beta)$, where the map $f : W \times \mathbb{R}^2 \to W$ is defined by

$$f(z,\alpha,\beta) = B_0(z,\alpha,\beta) + (L_0 + K_0)^{-1}[N_0 + K_0(\pi_0 - B_0)](z,\alpha,\beta). \tag{3.3.8}$$

Proof. Suppose $x(t)$ is a $\frac{2\pi}{\beta}$-periodic solution of (3.3.1). Let $z(t) = x(\frac{2\pi}{\beta}t)$ for $t \in \mathbf{R}$. Then $z \in W$ and (z, α, β) is a solution of (3.3.7) by the definitions of L_0, K_0 in (3.3.4) and B_0, N_0 in (3.3.6). Conversely, if $z \in W$ is a solution to (3.3.7), then $x(t) = z(\frac{\beta}{2\pi}t)$ is a $\frac{2\pi}{\beta}$-periodic solution of (3.3.1). This proves the first statement. The second statement follows directly from the formulation of a coincidence problem in Chapter 2 into a fixed point problem.

This completes the proof.

Remark 3.3.2. By Lipschitz condition (A1), we see that B_0 is a G-equivariant condensing map. Moreover, by (3.3.8) and the compactness of $(L_0 + K_0)^{-1}$ from Lemma 3.3.1, (B_0, N_0) is an L_0-condensing G-pair.

Following Chapter 2, we now give an explicit decomposition of W. First, by restricting to the subgroup S^1 of $G = \mathbf{Z}_n \times S^1$, we have an orthogonal representation of S^1 on W which decomposes isotypically as follows

$$W = W_0 \oplus W_1 \oplus \cdots \oplus W_k \oplus \cdots$$

where W_0 is the space of all constant maps in W and W_k, $k \geq 1$, is the vector space of all functions of the form $x \sin 2k\pi t + y \cos 2k\pi t$, $t \in S^1$, $x + iy \in \mathbf{C}^N$.

We complexify W_1 by defining a complex structure below

$$\begin{aligned}&(a + bi) \cdot (x \sin 2\pi t + y \cos 2\pi t)\\&= (bx + ay) \cos 2\pi t + (ax - by) \sin 2\pi t,\\&\quad a + bi \in \mathbf{C} \quad \text{and} \quad t \in S^1.\end{aligned} \tag{3.3.9}$$

Consequently, W_1 becomes a unitary representation space of $\mathbf{Z}_n$, whose isotypical decomposition reads

$$W_1 = \bigoplus_{j \in J'} W_{1,j}, \quad J' \subseteq \{0, 1, 2, \ldots, n\}.$$

We identify W_1 with $\mathbf{C}^N$. Then the complex structure (3.3.9) on W_1 becomes the natural complex structure on $\mathbf{C}^N$ and the $\mathbf{Z}_n$-representation on W_1 is isomorphic to the $\mathbf{Z}_n$-representation on $\mathbf{C}^N$. This implies that $J' = J$ and for each $j \in J$

$$W_{1,j} = \{x \sin 2\pi t + y \cos 2\pi t,\ t \in S^1,\ x + yi \in \mathbf{C}_j^N\}.$$

Let

$$\begin{aligned} a_{1,j}(\alpha, \beta) :=& Id - D_z B_0(0, \alpha, \beta) \\ & - (L_0 + K_0)^{-1}[D_z N_0(0, \alpha, \beta) + K_0(\pi_0 - D_z B_0)(0, \alpha, \beta)]|_{W_{1,j}} \end{aligned}$$

for $j \in J$, $(\alpha, \beta) \in \mathbb{R}^2$. Note that $H^1(S^1; \mathbb{R}^N) \subset W$. We have

$$a_{1,j}(\alpha, \beta) = (L_0 + K_0)^{-1}[L_0(Id - D_z B_0(0, \alpha, \beta)) - D_z N_0(0, \alpha, \beta)]|_{W_{1,j}}. \quad (3.3.10)$$

Lemma 3.3.3. $a_{1,j}(\alpha, \beta) = \frac{1}{i\beta}\ \Delta_\alpha\ (i\beta)|_{W_{1,j}}$.

Proof. Let $z(t) = x \sin 2\pi t + y \cos 2\pi t \in W_{1,j}$, $t \in S^1$, $x + yi \in \mathbf{C}_j^N$. By (3.3.10) and (3.3.5), it follows that

$$\begin{aligned} a_{1,j}(\alpha, \beta) z(t) &= (L_0 + K_0)^{-1}[(z(t) - D_z B_0(0, \alpha, \beta) z(t))' - D_z N_0(0, \alpha, \beta) z(t)] \\ &= (L_0 + K_0)^{-1}[\dot{z}(t) - D_\varphi b(0, \alpha) \dot{z}_{t,\beta} - \frac{2\pi}{\beta} D_\varphi F(0, \alpha) z_{t,\beta}] \\ &= (L_0 + K_0)^{-1}[\dot{z}(t) - 2\pi \cos 2\pi t D_\varphi b(0, \alpha) x \cos_\beta + 2\pi \sin 2\pi t D_\varphi b(0, \alpha) x \sin_\beta) \end{aligned}$$

$$
\begin{aligned}
&+ 2\pi \sin 2\pi t D_\varphi b(0,\alpha) y \cos_\beta + 2\pi \cos 2\pi t D_\varphi b(0,\alpha) y \sin_\beta) \\
&- \frac{2\pi}{\beta}(\sin 2\pi t D_\varphi F(0,\alpha) x \cos_\beta + \cos 2\pi t D_\varphi F(0,\alpha) x \sin_\beta) \\
&- \frac{2\pi}{\beta}(\cos 2\pi t D_\varphi F(0,\alpha) y \cos_\beta - \sin 2\pi t D_\varphi F(0,\alpha) y \sin_\beta)]
\end{aligned}
$$

where $\cos_\beta$ and $\sin_\beta$, being elements of $C((-\infty,\tau],\mathbb{R})$, are defined by

$$
\cos_\beta \theta = \cos \beta\theta, \quad \sin_\beta \theta = \sin \beta\theta, \quad \theta \in (-\infty, \tau].
$$

By Remark 3.3.1 and the complex structure (3.3.9), it follows that for $t \in S^1$,

$$
\begin{aligned}
a_{1,j} z(t) =& x \sin 2\pi t + y \cos 2\pi t - \sin 2\pi t D_\varphi b(0,\alpha) x \cos_\beta \\
& - \cos 2\pi t D_\varphi b(0,\alpha) x \sin_\beta - \cos 2\pi t D_\varphi b(0,\alpha) y \cos_\beta \\
& + \sin 2\pi t D_\varphi b(0,\alpha) y \sin_\beta \\
& + \frac{1}{\beta}(\cos 2\pi t D_\varphi F(0,\alpha) x \cos_\beta - \sin 2\pi t D_\varphi(0,\alpha) x \sin_\beta) \\
& + \frac{1}{\beta}(-\sin 2\pi t D_\varphi F(0,\alpha) y \cos_\beta - \cos 2\pi t D_\varphi F(0,\alpha) y \sin_\beta) \\
=& x \sin 2\pi t + y \cos 2\pi t \\
& - (\sin 2\pi t D_\varphi b(0,\alpha) x \cos_\beta + \sin 2\pi t D_\varphi b(0,\alpha) x i \sin_\beta) \\
& - (\cos 2\pi t D_\varphi b(0,\alpha) y \cos_\beta + \cos 2\pi t D_\varphi b(0,\alpha) x i \sin_\beta) \\
& - \frac{1}{\beta i}(\sin 2\pi t D_\varphi F(0,\alpha) x \cos_\beta + \sin 2\pi t D_\varphi F(0,\alpha) x i \sin_\beta) \\
& - \frac{1}{\beta i}(\cos 2\pi t D_\varphi F(0,\alpha) y \cos_\beta + \cos 2\pi t D_\varphi F(0,\alpha) y i \sin_\beta) \\
=& x \sin 2\pi t + y \cos 2\pi t - (D_\varphi b(0,\alpha)(\cos_\beta + i \sin_\beta) Id)(x \sin 2\pi t + y \cos 2\pi t) \\
& - \frac{1}{\beta i}(D_\varphi F(0,\alpha)(\cos_\beta + i \sin_\beta) Id)(x \sin 2\pi t + y \cos 2\pi t) \\
=& \frac{1}{\beta i}[i\beta Id - i\beta D_\varphi(0,\alpha)(e^{i\beta} Id) - D_\varphi F(0,\alpha)(e^{i\beta} Id)][x \sin 2\pi t + y \cos 2\pi t] \\
=& \frac{1}{\beta i}[i\beta(Id - D_\varphi b(0,\alpha)(e^{i\beta} Id)) - D_\varphi F(0,\alpha)(e^{i\beta} Id)] z(t) \\
=& \frac{1}{i\beta}\ \Delta_\alpha\ (i\beta) z(t).
\end{aligned}
$$

This proves the lemma.

Combining now Lemmas 3.3.1–3.3.3, we can prove the following (local) Hopf bifurcation theorem with $\mathbf{Z}_n$-symmetry.

Theorem 3.3.4. *Assume (A1)–(A4) hold. If there exists a $j \in J$ such that $c_j(\alpha_0, \beta_0) \neq 0$, then (α_0, β_0) is a bifurcation point. More precisely, there is a sequence of triples $\{(x_k, \alpha_k, \beta_k)\}_{k=1}^{\infty}$ such that*

(i) $(x_k, \alpha_k, \beta_k) \to (0, \alpha_0, \beta_0)$ uniformly for $t \in \mathbb{R}$ as $k \to \infty$;

(ii) $x_k(t)$ is a $\frac{2\pi}{\beta_k}$-periodic solution of (3.3.1) with $\alpha = \alpha_k$, $k = 1, 2, \ldots$;

(iii) $\rho(e^{\frac{2\pi}{n}i})x_k(t) = x_k(t + \frac{2\pi}{\beta_k}\frac{j}{n})$ for $t \in \mathbb{R}$, $k = 1, 2, \ldots$.

Proof. By Lemmas 3.3.1–3.3.3, we need only to show that $(0, \alpha_0, \beta_0)$ is a bifurcation point to the following $G := \mathbf{Z}_n \times S^1$-equivariant composite coincidence problem

$$\begin{cases} L_0(\pi_0 - B_0)(z, \alpha, \beta) = N_0(z, \alpha, \beta) \\ (z, \alpha, \beta) \in W \times \mathbb{R}^2, \end{cases}$$

where (B_0, N_0) is an L_0-condensing G-pair by Remark (3.3.2). Let $\mathcal{D}(\alpha_0, \beta_0) = (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c) \subset \mathbb{R}^2$ and put

$$M = \{(0, \alpha, \beta);\ (\alpha, \beta) \in \mathcal{D}(\alpha_0, \beta_0)\} \subset \{0\} \times \mathbb{R}^2.$$

By the way of the G-action, $M \subseteq W^G \times \mathcal{D}(\alpha_0, \beta_0)$. Moreover, M is a 2-dimensional submanifold of $W^G \times \mathcal{D}(\alpha_0, \beta_0)$ satisfying (A) and (B) in Section 2.4 and, by assumption (A4) together with the implicit function theorem, $(0, \alpha_0, \beta_0)$ is an isolated singular point.

To apply Theorem 2.4.4, we compute $\mu_{1,j}(0, \alpha_0, \beta_0)$. First, by definition of $\mu_{1,j}(0, \alpha_0, \beta_0)$ and Lemma 3.3.3, we have

$$\begin{aligned}\mu_{1,j} :&= \deg_B(\det_{\mathbf{C}} a_{1,j}(\cdot), \mathcal{D}(\alpha_0, \beta_0)) \\ &= \deg_B(\det_{\mathbf{C}}\ \Delta_{1,j}\ (\cdot), \mathcal{D}(\alpha_0, \beta_0)).\end{aligned}$$

On the other hand, a lemma of Erbe et al. [55] implies

$$\begin{aligned}&\deg_B(\det_{\mathbf{C}}\ \Delta_{1,j}\ (\cdot), \mathcal{D}(\alpha_0, \beta_0)) \\ &\quad = \deg_B(\det_{\mathbf{C}}\ \Delta_{\alpha_0-\delta,j}\ (\cdot), \Omega) - \deg_B(\det_{\mathbf{C}}\ \Delta_{\alpha_0+\delta,j}\ (\cdot), \Omega).\end{aligned}$$

Therefore $\mu_{1,j}(0, \alpha_0, \beta_0) = c_j(\alpha_0, \beta_0)$ and $(0, \alpha_0, \beta_0)$ is a bifurcation point if $c_j(\alpha_0, \beta_0) \neq 0$, by Theorem 2.4.4. This shows (i) and (ii).

To see (iii), let $z(t) = x \sin 2\pi t + y \cos 2\pi t \in W_{1,j}$. We have

$$\begin{aligned}\rho(e^{\frac{2\pi i}{n}})z(t) =& e^{i\frac{2\pi j}{n}}(x \sin 2\pi t + y \cos 2\pi t) \\ =&(\cos \tfrac{2\pi j}{n} + i \sin \tfrac{2\pi j}{n})(x \sin 2\pi t + y \cos 2\pi t) \\ =&[x \cos \tfrac{2\pi j}{n} \sin 2\pi t + y \cos \tfrac{2\pi j}{n} \cos 2\pi t \\ &+ \sin \tfrac{2\pi j}{n}(x \cos 2\pi t - y \sin 2\pi t)] \\ =&[x \sin(2\pi t + \tfrac{2\pi j}{n}) + y \cos(2\pi t + \tfrac{2\pi j}{n})] \\ =&z(t + \frac{j}{n})\end{aligned}$$

where we used (3.3.9). Note that $z(t) = x(\frac{2\pi}{\beta}t)$. (iii) follows. This completes the proof.

We now consider the case where $\Gamma = S^1$. The action of S^1 on $\mathbf{R}^N$ induces an action on $\mathbf{C}^N$. With respect to this action (representation), we have the following isotypical decomposition of $\mathbb{C}^N$ (see also [155])

$$\mathbb{C}^N = \bigoplus_{j=1}^{m} \mathbb{C}_j^N$$

where m is a certain positive integer, $\mathbb{C}_j^N$, $1 \leq j \leq m$, is the direct sum of all one-dimensional S^1-irreducible representation spaces V of $\mathbb{C}^N$ such that the representation of S^1 restricted to V is isomorphic to

$$\rho_j(e^{i2\pi\theta})z = e^{i2\pi n_j \theta}z, \quad z \in \mathbb{C}, \quad e^{i2\pi\theta} \in S^1,$$

for some integer $n_j \geq 0$. Putting

$$\begin{aligned}
\Delta_\alpha(\lambda) &:= \lambda[Id - D_\varphi b(0,\alpha)(e^{\lambda\cdot}Id)] - D_\varphi F(0,\alpha)(e^{\lambda\cdot}Id)], \\
\Delta_{\alpha,j}(\lambda) &= \Delta_\alpha(\lambda)|_{\mathbb{C}_j^N}, \quad 1 \leq j \leq m,
\end{aligned}$$

under the assumptions (A1)–(A4), we can also define a *j-th crossing number* for the *j-th characteristic equation* $\Delta_{\alpha,j}(\lambda) = 0$ as follows

$$c_j(\alpha_0, \beta_0) := \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0-\delta,j}(\cdot), \Omega) - \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0+\delta,j}(\cdot), \Omega)$$

for each $j \in \{1, 2, \ldots, m\}$, where δ and Ω are the same as before.

Similarly, one obtains the local Hopf bifurcation theorem for the S^1-symmetry.

Theorem 3.3.5. *Let $\mathbb{Z}_n = S^1$ and (A1)–(A4) hold. If $c_j(\alpha_0, \beta_0) \neq 0$ for some $j \in \{1, 2, \ldots, m\}$, then there exists a sequence of triples $\{(x_k(t), \alpha_k, \beta_k)\}_{k=1}^{\infty}$ such that*

(i) $(x_k(t), \alpha_k, \beta_k) \to (0, \alpha_0, \beta_0)$ *uniformly for* $t \in \mathbb{R}$ *as* $k \to \infty$;

(ii) $x_k(t)$ *is a* $\frac{2\pi}{\beta_k}$*-periodic solution of (3.3.1) with* $\alpha = \alpha_k$ *for* $k = 1, 2, \ldots$;

(iii) $\rho(e^{i2\pi\theta})x_k(t) = x_k(t + \frac{2\pi}{\beta_k}n_j\theta)$ *for* $e^{i2\pi\theta} \in S^1$ *and* $t \in \mathbf{R}$.

Proof. Proof is essentially the same as that of Theorem 3.3.4. We therefore omit it.

Remark 3.3.3. From (iii) of Theorems 3.3.4 and 3.3.5, we see that the spatial symmetry of the bifurcating periodic solutions $x_k(t)$ has decreased to the subgroup $\mathbf{Z}_d$ with $d = \gcd(j, n)$ when $n < \infty$ or $\mathbf{Z}_{n_j}$ when $n = \infty$. This type of periodic solution has been refered to as *rotating waves* in the literature and has been extensively studied by Alexander and Auchmuty [6], Aronson et al. [12-14], Ashwin et al. [15-16], Fiedler [62], Golubitsky and Stewart [75], Golubitsky, Schaeffer and Stewart [74] and Smith [159] for ordinary differential equations and Gęba, Krawcewicz and Wu [70] for retarded functional differential equations.

The periodic solutions obtained by Theorem 3.3.4 and 3.3.5 are only of small amplitude. To see its global continuum, we apply now the global bifurcation theorem from Chapter 2. For the sake of convenience, we introduce the period p of periodic solution as an additional parameter and rewrite (3.3.1) as

$$\frac{d}{dt}[z(t) - b(z_t, \tfrac{2\pi}{p}, \alpha)] = pF(z_t, \tfrac{2\pi}{p}, \alpha), \tag{3.3.11}$$

where $z_{t, \frac{2\pi}{p}}(\theta) = z(t + \frac{2\pi}{p}\theta)$, $\theta \in (-\infty, \tau]$. Using the same notation as in (3.3.6), we can define

$$\widetilde{B}_0(z, \alpha, p) \triangleq B_0(z, \alpha, \tfrac{2\pi}{p})$$
$$\widetilde{N}_0(z, \alpha, p) \triangleq N_0(z, \alpha, \tfrac{2\pi}{p}), \quad z \in C(S^1; \mathbb{R}^N),$$

and, consequently, we obtain the following composite coincidence problem which is equivalent to (3.3.11)

$$L_0[\pi_0 - \widetilde{B}_0](z,\alpha,p) = \widetilde{N}_0(z,\alpha,p), \quad p > 0. \tag{3.3.12}$$

We can further reduce (3.3.12) to a fixed point problem below

$$z = f(z,\alpha,p) \tag{3.3.13}$$

where $f : C(S^1;\mathbb{R}^N) \times \mathbb{R} \times (0,\infty) \to C(S^1;\mathbb{R}^N)$ is defined by

$$f(z,\alpha,p) = \widetilde{B}_0(z,\alpha,p) + (L_0 + K_0)^{-1}[\widetilde{N}_0 + K_0(\pi_0 - \widetilde{B}_0)](z,\alpha,p), \tag{3.3.14}$$

and L_0, K_0 and π_0 are the same as before. Consequently, $z \in C(S^1;\mathbb{R}^N)$ is a 1-periodic solution of (3.3.11) if and only if z solves (3.3.13) for some $\alpha \in \mathbb{R}$, $p > 0$. Moreover, $(\widetilde{B}_0, \widetilde{N}_0)$ is also an L_0-condensing G-pair.

We need the following "global" assumptions.

(GA3) $\bar{F}(x,\alpha) = 0$ with $x \in (\mathbb{R}^N)^{\mathbb{Z}_n}$ if and only if $x = 0$. Moreover $D_x\bar{F}(0,\alpha) \in GL(\mathbb{R}^N)$ for every $\alpha \in \mathbb{R}$.

(GA4) The set $\mathcal{A} := \{\alpha \in \mathbb{R};$ the stationary solution $(0,\alpha)$ has pure imaginary characteristic values $\}$ is discrete.

For every $j \in J$ (when $n < \infty$) or $1 \le j \le m$ (when $n = \infty$), define now a subset of $C(S^1;\mathbb{R}^N) \times \mathbb{R}^2$ by

$$\begin{aligned} \mathcal{S}^j = Cl\{(z,\alpha,p);\ & z \text{ is a 1-periodic solutionof (3.3.11) such that} \\ & \rho(e^{i\frac{2\pi}{N}})z(t) = z(t+\tfrac{j}{N}) \quad \text{if} \quad n<\infty \quad \text{or} \\ & \rho(e^{i2\pi\theta})z(t) = z(t+N_j\theta) \quad \text{if} \quad n = \infty\}. \end{aligned}$$

Put

$$M := \{(0, \alpha, p);\ \alpha \in \mathbf{R},\ p > 0\} \subset C(S^1; \mathbf{R}^N) \times \mathbf{R}^2.$$

We obtain the following global symmetric Hopf bifurcation theorem.

Theorem 3.3.6. Let (A1) (A2) and (GA3) (GA4) hold. If there exists an integer j such that $\mathcal{S}^j$ has a bounded connected component $\mathcal{C}^j$, then $\mathcal{C}^j \cap M$ is a finite set and

$$\sum_{(0,\alpha,p) \in \mathcal{C}^j \cap M} \mathcal{C}_j(\alpha, \tfrac{2\pi}{p}) = 0. \tag{3.3.15}$$

Proof. First note that, under the assumption (GA3) and (GA4), zero is a regular value of the restriction $f_0 := f|_{W^G \times \mathbb{R} \times \mathbb{R}_+} : W^G \times \mathbb{R} \times \mathbb{R}_+ \to W^G$, where f is given by (3.3.14) and $W = C(S^1; \mathbb{R}^N)$, $G := \mathbf{Z}_n \times S^1$. Consequently, $f_0^{-1}(0) = M$ is a 2-dimensional submanifold of $W^G \times \mathbb{R}^2$ such that $M \subset f^{-1}(0)$. This implies that M verifies the conditions (A) and (B) from Section 2.4. The set M is referred to as a *trivial (stationary)* periodic solution set of (3.3.11).

By definition of f in (3.3.14) and the calculations in Lemma 3.3.3, we see that $(0, \alpha, p) \in M$ is a singular point if and only if $\alpha \in \mathcal{A}$ and $\frac{2\pi}{p} i$ is a characteristic value of $(0, \alpha)$. Therefore, (GA4) implies that every singular point is isolated. Moreover, (GA3) implies the condition (H) in Section 2.5 and for the bounded component $\mathcal{C}^j$ of $\mathcal{S}^j$, it follows from Appendix that

$$\inf\{p :\ (z, \alpha, p) \in \mathcal{C}^j\} > 0.$$

Consequently, we can assume the problem (3.3.12) is well posed on the whole space $W \times \mathbb{R}^2$ (see also [33, 34, 55, 110]). Note that every singular point $(0, \alpha, p)$

is fixed under the action of G. The theorem and (3.3.15) follow directly from Theorem 2.5.4. This completes the proof.

Remark 3.3.4. Global Hopf bifurcation, disregarding the symmetry aspects, has been considered by Alexander and Yorke [8], Chow, Mallet-Paret and Yorke [34], Ize [94, 95] and Mallet-Paret and Yorke [117] for ordinary differential equations, Chow and Mallet-Paret [33] and Nussbaum [130] for retarded equations and Fiedler [59, 60] for Volterra integral equations and parabolic equations. More recently, an S^1-degree approach has been provided by Gęba and Marzantowicz [71] and Erbe, Gęba, Krawcewicz and Wu [55] to deal with global Hopf bifurcation for ordinary and functional differential equations (with infinite delay and possibly of anticipatory type), which is extended next in [110] to neutral equations. By ignoring the symmetry in the equations, Theorem 3.3.6 thus provides an analog, for *neutral* equations, of the above mentioned global Hopf bifurcation theorems. For a detailed comparison of the S^1-degree approach with earlier ideas by Mallet-Paret and Yorke, Nussbaum and Fiedler, we refer to [55, 110].

Remark 3.3.5. For a symmetric global Hopf bifurcation, we mention the book by Fiedler [62], where a general global Hopf bifurcation theory with $\mathbb{Z}_n$, $1 \leq n \leq \infty$, has been developed for *ordinary differential equations*. The proofs are by generic approximation and rather analytic. For a specific $\mathbb{Z}_n$-symmetry of ring coupled oscillators, Alexander and Auchmuty [6] have considered phase-locked global (Hopf) bifurcations. By using their G-degree, Gęba, Krawcewicz and Wu [70] obtained a global symmetric Hopf bifurcation theorem for *retarded functional differential equations* with symmetry. Our results in this section extend those of [70] to *neutral equations*. We postone a brief comparison of Theorem 3.3.6 with similar ideas by Fiedler [62] to Section 3.5 and discuss phase-locked oscillations in neutral equations in Section 3.4, where most of the results of Alexander and Auchmuty [6] are extended to neutral equations via Theorem 3.3.6.

Remark 3.3.6. As far as the local Hopf bifurcation theorem is concerned, the advantage of the topological approach is obvious. It allows for multiple and/or integer multiple characteristic values to exist. Also, it is permitted that the purely imaginary charateristic value goes across the imaginary axis with a zero speed. More precisely, let $\lambda(\alpha)$ denote the characteristic value such that $\lambda(\alpha_0) = \beta_0 i$. It follows from the property of Brouwer degree that if

$$\frac{d^n}{d\alpha^n}[\mathrm{Re}\lambda(\alpha)]|_{\alpha=\alpha_0} = 0 \quad \text{for} \quad n = 1, 2, \ldots, k-1, \quad \text{and}$$
$$\frac{d^k}{d\alpha^k}[\mathrm{Re}\lambda(\alpha)]|_{\alpha=\alpha_0} \neq 0,$$

where $k > 0$ is an integer, then the crossing number $c(\alpha_0, \beta_0) \neq 0$ if k is odd, implying $(0, \alpha_0, \beta_0)$ is a Hopf bifurcation point. When k is even, $c(\alpha_0, \beta_0) = 0$ and we have no information on the existence of bifurcating periodic solutions. In fact, in the case when k is even, higher order terms are involved in determining the existence (see Schmidt [151]). Such a violation of transversality condition in the original theorem of Hopf (see [119] for the English translation) is also considered by Chaffee [26, 27] and Freedman [66], to name just two.

Unfortunately, another aspect of this topological approach comes out as a defect. It can not determine the stability of bifurcating periodic solutions, although stability is an important issue in many applications. A plausible explaination of this fact may be as follows. In our local Hopf bifurcation theorem, only linear terms are used in determining whether or not a point is a bifurcation point (in the definition of isolated center and the calculation of G-degree via *linear* approximations). However, it is known from many calculations that stability of periodic solutions in Hopf bifurcation relies heavily on the higher order terms. See, for example, Hassard et al. [87] and Marsden et al. [119] for more details.

3.4. An application to the Rashevsky-Turing Theory

In attempting to construct a metaphor for morphogenetic behavior in biological systems, Rashevsky and, independently, Turing have developed a reaction-diffusion theory of morphogenesis [138, 140, 165]. In this theory, a mechanism of coupling and diffusing among cells is suggested as a possible basis for spatial organization and temporal oscillations in morphogenetic processes. Among various geometrical arrangement of cells, a simple and illuminating configuration of a *ring of identical cells* has been proposed and studied by Turing [165]. This ring, as it is now called *Turing ring*, provides a great variety of models for many situations in biology, chemistry and electrical engineering. Moreover, Turing ring brings about mathematically tractable systems which exhibit a symmetry of a group $\mathbf{Z}_n$, $2 \leq n \leq \infty$, and a rather extensive literature now exists on the study of discrete waves and oscillations within groups of cells–the genesis of form and rhythm. We refer to [3–6, 15, 16, 20–22, 62, 70, 74, 75, 77, 78, 93, 125] for the Hopf bifurcations in a ring array of coupled oscillators and [19, 49, 88, 99–102, 104, 108, 109, 118, 120, 125, 131, 132, 137, 158, 165, 169, 170, 175–178] for many other considerations.

In this section, we continue the above mentioned studies by applying our symmetric Hopf bifurcation theorem to a Turing ring of coupled identical cells. We propose models of functional differential equations of neutral type as the kinetics and consider the delayed coupling and diffusion in the system. We will show how the temporal delay (both in kinetics and coupling) affects the type of oscillations that may be observed in the system. In particular, we shall prove the existence of phase-locked and synchronous periodic solutions in these ring-structured neutral systems.

We emphasize the significance of temporal delays in the coupling between cells, since in many chemical and biological oscillators (cells coupled via membrane transport of ions, etc) the time needed for transport or processing of chemical components or signals may be of considerable length. While such delay equations in mathematical biology have been extensively studied in the literature (see [10, 11, 46, 60, 77, 98, 112, 113] for just a few), the study of the effect of temporal delays on oscillations of *coupled* oscillators is not much known, to the best of our knowledge.

Let N be a positive integer. We consider now a ring of N identical cells that are coupled by diffusion along the sides of an N-gon. We assume the *state* (or morphogenetic state) of the k-th cell at an instant of time is completely specified by the value of one state variable at that instant. This state variable, denoted by u^k, will be called *morphogen*; this may be regarded, as it was by Turing [165], as the concentrations of specific chemical substances. Assume also the k-th morphogen $u^k(t)$ of the k-th cell obeys the following kinetic equation

$$\frac{d}{dt}[u^k(t) - b(u_t^k, \alpha)] = f(u_t^k, \alpha), \quad 1 \leq k \leq N, \tag{3.4.1}$$

where $t \in \mathbb{R}$ denotes the instant of time, $\alpha \in \mathbb{R}$ is a parameter and $b,\ f : C_\tau := C((-\infty, \tau], \mathbb{R}) \to \mathbb{R}$ are continuously differentiable functionals which represent the kinetics within each cell.

Note that if coupling between cells occurs, the morphogen transformations follow and thus cells interact. Note also that a ring coupling is nearest-neighbor so that interaction occurs only between any neighboring pair. Assume now the linearity of the coupling (diffusion) between adjacent cells and take into account the effect of diffusion taking place after a certain amount of time. We arrive at a system of neutral functional differential equations

$$\begin{aligned} \frac{d}{dt}[u^k(t) - b(u_t^k, \alpha)] = f(u_t^k, \alpha) + K(\alpha)(u_t^{k+1} - 2u_t^k + u_t^{k-1}) \\ k = 1, 2, \ldots, N, \ (\mathrm{mod}\, N) \end{aligned} \tag{3.4.2}$$

where $K(\alpha) : C_r \to \mathbf{R}$ is a bounded linear functional and the mapping $\alpha \in \mathbf{R} \to K(\alpha) \in L(C_r; \mathbf{R})$ is continuously differentiable. $K(\alpha)$ represents the *coupling rate functional* and the coupling term

$$K(\alpha)(u_t^{k+1} - u_t^k) + K(\alpha)(u_t^{k-1} - u_t^k) \tag{3.4.3}$$

in (3.4.2) is assumed to obey the ordinary *law of diffusion*, i.e. the diffusion of a morphogen between the two cells is proportional to the difference in morhogen concentration.

Suppose now that $f(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$. So $(0, \ldots, 0, \alpha)$ is a *homogeneous* stationary solution of (3.4.3) and the linearized equation of (3.4.3) at $(0, \ldots, 0, \alpha)$ reads

$$\frac{d}{dt}(x^k(t) - D_\varphi b(0, \alpha)x_t^k) = D_\varphi f(0, \alpha)x_t^k + K(\alpha)[x_t^{k+1} - 2x_t^k + x_t^{k-1}]$$
$$1 \le i \le N, \ \ (\text{mod } N), \tag{3.4.4}$$

where $D_\varphi b(0, \alpha)$ and $D_\varphi f(0, \alpha)$ denote their respective derivatives with respect to φ. Consequently, Eq. (3.4.4) gives the *characteristic equation* of Eq. (3.4.3) as follows

$$\det \Delta_\alpha (\lambda) = 0 \tag{3.4.5}$$

where for each $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\Delta_\alpha (\lambda) : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$\Delta_\alpha (\lambda) := \text{diag}(\lambda[1 - D_\varphi b(0, \alpha)(e^{\lambda \cdot})]) - D_\varphi f(0, \alpha)(e^{\lambda \cdot}) - \delta(\lambda, \alpha) \tag{3.4.6}$$

in which the *discretized Laplacian* $\delta(\lambda, \alpha) : \mathbb{C}^N \to \mathbb{C}^N$ is defined by

$$\{\delta(\lambda, \alpha)z\}_k = K(\alpha)[e^{\lambda \cdot}(z^{k+1} - 2z^k + z^{k-1})]$$
$$1 \le k \le N, \ \ (\text{mod } N),$$

for $z = (z^1, z^2, \dots, z^N) \in \mathbb{C}^N$.

Let $\xi = e^{i\frac{2\pi}{N}} \in \mathbb{C}$. Then $\xi^{-j} = \overline{\xi^j} = \xi^{N-j}$ for any $0 \le j \le N-1$. Define for each $0 \le j \le N-1$ a complex vector space

$$\mathbb{C}_j^N := \{(\xi^{(N-1)j}, \xi^{(n-2)j}, \dots, \xi^{2j}, \xi^j, 1)^T x, \quad x \in \mathbb{R}\}. \tag{3.4.7}$$

It follows that

$$\mathbb{C}^N = \mathbb{C}_0^N \oplus + \mathbb{C}_1^N \oplus + \cdots \oplus + \mathbb{C}_{N-1}^N. \tag{3.4.8}$$

Put

$$g(\lambda, \alpha) := \lambda[1 - D_\varphi b(0, \alpha)(e^{\lambda \cdot})] - D_\varphi f(0, \alpha)(e^{\lambda \cdot}). \tag{3.4.9}$$

For any $x \in \mathbb{R}$, $j \in \{0, 1, \dots, N-1\}$ and $k \in \{1, 2, \dots, N\}$, we have

$$\begin{aligned}
&(\Delta_\alpha\,(\lambda)(\xi^{(N-1)j}, \xi^{(N-2)j}, \dots, \xi^{2j}, \xi^j, 1)^T x)_k \\
&= [g(\lambda, \alpha)\xi^{(N-k)j} + k(\alpha)e^{\lambda \cdot}(\xi^{(N-k+1)j} - 2\xi^{(N-k)j} + \xi^{(N-k-1)j})]x \\
&= [g(\lambda, \alpha) + K(\alpha)e^{\lambda \cdot}(\xi^j + \xi^{-j} - 2)]\xi^{(N-k)j}x \\
&= [g(\lambda, \alpha) + 2K(\alpha)e^{\lambda \cdot}(\cos \tfrac{2\pi j}{N} - 1)]\xi^{(N-k)j}x \\
&= [g(\lambda, \alpha) - 4\sin^2 \tfrac{\pi j}{N} K(\alpha)e^{\lambda \cdot}]\xi^{(N-k)j}x, \quad k \pmod{N}.
\end{aligned} \tag{3.4.10}$$

This implies that $\Delta_\alpha\,(\lambda)|_{\mathbb{C}_j^N} \subseteq \mathbb{C}_j^N$ and, under the decomposition (4.4.8) of $\mathbb{C}^N$, $\Delta_\alpha\,(\lambda)$ is a diagonal complex matrix, i.e. $\Delta_\alpha\,(\lambda)$ is similar to the diagonal matrix

$$\operatorname{diag}[g(\lambda, \alpha) - 4a \sin^2 \frac{\pi j}{N} K(\alpha)e^{\lambda \cdot}]_{j=0}^{N-1}.$$

Thus we have reached the following proposition.

Proposition 3.4.1. *Let* $g(\lambda,\alpha)$ *be given by (3.4.9). Then*

$$\det \Delta_\alpha(\lambda) = \Pi_{j=1}^{N-1}[g(\lambda,\alpha) - 4a\sin^2\frac{\pi j}{N}K(\alpha)e^{\lambda\cdot}]$$

and, consequently, $\lambda \in \mathbf{C}$ *is a characteristic value of* $(0,\dots,0,\alpha)$ *iff there exists a* $j \in \{0,1,\dots,N-1\}$ *such that*

$$p_j(\lambda,\alpha) := g(\lambda,\alpha) - 4\sin^2\frac{\pi j}{N}K(\alpha)e^{\lambda\cdot} = 0. \tag{3.4.11}$$

Remark 3.4.1. We call (3.4.11) the *j-th characteristic equation* of (3.4.2). Note that $\sin^2\frac{\pi j}{N} = \sin^2\frac{\pi(N-j)}{N}$ for any $j \in \{0,1,\dots,N-1\}$. It follows from Proposition 3.4.1 that every zero of $p_j(\lambda,\alpha)$, $j \neq 0, \frac{N}{2}$, is of *even multiplicity*. This is due to the *symmetry* in the system, which forces characteristic values to be multiple. See also [16, 74–76, 164].

Let us now consider the symmetry aspect of (3.4.2). We claim that Eq. (3.4.2) has a symmetry of the *dihedral group* D_N. To see it more clearly, note that D_N is generated by a rotation r and a reflection s such that

$$\langle r\rangle \cong \mathbf{Z}_N, \quad \langle s\rangle \cong \mathbf{Z}_2 \quad \text{and} \quad srs = r^{-1}$$

(see [155]). Define now a real orthogonal representation $\rho : D_N \to O(\mathbf{R}^N)$ by

$$\begin{aligned}(\rho(r)x)_k &= x_{k+1},\\ (\rho(s)x)_k &= x_{N-k}, \quad k = 1,2,\dots,N, \ (\text{mod } N), \ x \in \mathbf{R}^N.\end{aligned} \tag{3.4.12}$$

Note that all N cells are identical and the diffusions are the same. Under the representation (3.4.12), Eq. (3.4.2) is D_N-equivariant with respect to ρ.

We now apply Theorem 3.3.4 to find *discrete waves* in Eq. (3.4.2). To this end, we take $H_0 := \langle r\rangle \cong \mathbf{Z}_N$ and $K_0 = \{id\}$. Therefore $H_0/K_0 \cong \mathbf{Z}_N$ is a cyclic factor of the dihedral group D_N. Now $(\mathbb{R}^N)^{K_0} = \mathbb{R}^N$. The representation ρ induces a subrepresentation, again denoted by ρ, of $H_0 \cong \mathbb{Z}_N$ on $\mathbb{R}^N$. We use identification $\mathbb{C}^N \cong \mathbb{R}^N + i\mathbb{R}^N$ and thus ρ gives a unitary representation of $\mathbb{Z}_N$ on $\mathbb{C}^N$.

Recalling the representation ρ in (3.4.12) and the definition of the complex space $\mathbb{C}_j^N$ in (3.4.7), we have for any $x \in \mathbb{R}$

$$\begin{aligned}
&\rho(r)(\xi^{(N-1)j}, \xi^{(N-2)j}, \ldots, \xi^{2j}, \xi^j, 1)^T x \\
&= (\xi^{(N-2)j}, \xi^{(N-3)j}, \ldots, \xi^j, 1, \xi^{(N-1)j})^T x \\
&= \xi^{-1}(\xi^{(N-1)j}, \xi^{(N-2)j}, \ldots, \xi^{2j}, \xi^j, 1)^T x.
\end{aligned}$$

That is,

$$\rho(r)z = \xi^{-j}z = e^{-i\frac{2\pi j}{N}}z, \quad z \in \mathbb{C}_j^N.$$

Consequently, each $\mathbb{C}_j^N$ defined by (3.4.7) is an irreducible representation space of ρ and the decomposition (3.4.8) is the isotypical decomposition of $\mathbb{C}^N$ with respect to the subrepresentation ρ of $\mathbb{Z}_N$. If we follow the same notation as in Section 3.3, this implies that $J = \{0, 1, 2, \ldots, N-1\}$ and, by (3.4.11), the *j-th characteristic equation* is of the form

$$\Delta_{\alpha_1, j}(\lambda) = \Delta_\alpha(\lambda)|_{\mathbb{C}_j^N} = p_j(\lambda, \alpha) = 0, \quad j \in \{0, 1, 2, \ldots, N-1\}.$$

By applying Theorem 3.3.4, we have the following theorem.

Theorem 3.4.2. *Let b and f satisfy (A1). Suppose that there exist positive numbers $\alpha_0, \beta_0, \varepsilon, \delta$ and $j \in \{0, 1, 2, \ldots, N-1\}$ such that*

(i) $g(0, \alpha) - 4\sin^2 \frac{\pi j}{N} \bar{K}(\alpha_0) \neq 0$;

(ii) $p_j(i\beta, \alpha_0) = 0$ *for some* $(\alpha, \beta) \in [\alpha_0 - \varepsilon, \alpha + \varepsilon] \times [\beta_0 - \delta, \beta_0 + \delta]$ *iff* $\alpha = \alpha_0$ *and* $\beta = \beta_0$;

(iii) $p_j(u + iv, \alpha_0) = 0$ *for some* $(u, v) \in \partial\Omega$ *with* $\Omega := (0, \varepsilon) \times (\beta_0 - \delta, \beta_0 + \delta)$ *iff* $u = 0$ *and* $v = \beta_0$;

(iv) $deg_B(p_j(\cdot, \alpha_0 - \varepsilon), \Omega) \neq deg_B(p_j(\cdot, \alpha + \varepsilon), \Omega)$.

Then there exists a sequence of triples $\{(x_l, \alpha_l, \beta_l)\}_{l=1}^{\infty}$ *such that*

(a) $(x_l(t), \alpha_l, \beta_l) \to (0, \alpha_0, \beta_0)$ *uniformly with* $t \in \mathbb{R}$ *as* $l \to \infty$;

(b) $x_l(t)$ *is a* $\frac{2\pi}{\beta_l}$*-periodic solution of (3.4.2) with* $\alpha = \alpha_l$, $l = 1, 2, \ldots$;

(c) $x_l^k(t) = x_l^{k-1}(t + \frac{2\pi}{\beta_l} \cdot \frac{j}{N})$ *for all* $t \in \mathbb{R}$ *and* $l = 1, 2, \ldots$.

Proof. Note that (i) implies the assumption (A2) and (ii) and (iii) gives (A4). The theorem follows by appealing to Theorem 3.3.4.

For every $j \in \{0, 1, 2, \ldots, N - 1\}$ we now define a subset

$$\begin{aligned} S^j := Cl\{(z, \alpha, p);\ & x(t) = z(\tfrac{t}{p}) \text{ is a } p-\text{periodic solution of (3.4.2) with} \\ & x^{k-1}(t) = x^k(t - \tfrac{j}{N}p), \quad t \in \mathbb{R}, \quad k \ (\mathrm{mod}\ N)\} \\ & \subseteq C(S^1; \mathbb{R}^N) \times \mathbb{R}^2. \end{aligned}$$

Put

$$M := \{(0, \alpha, p);\ \alpha \in \mathbb{R},\ p > 0\} \subset C(S^1; \mathbb{R}^N) \times \mathbb{R}^2.$$

Theorem 3.4.6 implies the following global $\mathbb{Z}_N$-symmetric Hopf bifurcation theorem.

Theorem 3.4.3. *Let b and f satisfy (A1). Suppose that there exists $j \in \{0, 1, \ldots, N-1\}$ such that*

(i) $g(0, \alpha) \neq 4 \sin^2 \frac{\pi j}{N} \bar{K}(\alpha)$ *for all* $\alpha \in \mathbb{R}$;

(ii) the set $\mathcal{B} := \{\alpha \in \mathbb{R};\ p_j(\lambda, \alpha) = 0$ *has pure imaginary roots* $\}$ *is discrete;*

(iii) $\sum_{(0,\alpha,p) \in M^*} (deg_B(p_j(\cdot, \alpha - \varepsilon), \Omega) - deg_B(p_j(\cdot, \alpha + \varepsilon), \Omega)) \neq 0$ *for any finite subset* M^* *of* $\{(0, \alpha, p);\ \alpha, \frac{2\pi}{p}, \varepsilon, \delta$ *satisfy (ii) and (iii) for* $p_j(\lambda, \alpha)$ *in Theorem 3.4.2.*

Then there must exist an unbounded connected component C^j *of* S^j *such that* $C^j \cap M \neq \varnothing$.

We end this section with several remarks.

Remark 3.4.2. Following Alexander and Auchmuty [6], we call the periodic solution obtained in Theorem 3.4.2 *synchronous oscillations* in the case where $j = 0$ and *phase-locked oscillations* otherwise. Intuitively, synchronous oscillations occur when all the morphogens oscillate in phase and phase-locked oscillations are those where each morphogen oscillates just like the others except not necessarily in phase with each other. We refer to [6, 14–16, 62, 74, 80, 81, 159] for more delailed discussion in the case of ordinary differential equations.

Remark 3.4.3. We are only concerned with *discrete* diffusion on the Turing ring in the Rashevsky-Turing theory. For the *continuous* reaction-diffusion case, there exists an extensive literature. We refer to [61, 62, 64, 79, 100, 108, 120, 125, 133, 159, 165, 169, 170] for more details. On the other hand, several authors have also considered the Turing ring for maps, where the kinetics equations are described by discrete dynamical systems. See [99–102, 175] for the applications of Turing ring in chemical reactions.

Remark 3.4.4. For simplicity, we have assumed the *isotropy* of coupling between N cells on a ring, i.e. the forward and backward "pulls" are the same. Similar results can also be obtained for the *anisotropic coupling,* in which the diffusion term (3.4.3) is replaced by

$$K(\alpha)[x_t^{k+1} - x_t^k] + a^2 K(\alpha)[x_t^{k-1} - x_t^k] \tag{3.4.13}$$

where $a - 1$ is a measure of the anisotropy. The coupling term (3.4.13), where the temporal delay is neglected, has been used in modelling the electrical activity in small intestines (see Kopell [108], Kopell and Ermentrout [109]).

Remark 3.4.5. In many applications Turing ring seems very rarely seen. However, if we just use *part of symmetry* of the system, Turing ring appears quite often. In the work of Hadley et al.[80, 81] on phase-locking of Josephson-junction arrays, we believe our theory can be applied (by using only the cylic group $\mathbf{Z}_n$ as a symmetry, instead of the full symmetry of the symmetric group S_n). See also Aronson et al. [13, 14]. Similarly, in many compartment systems [98] or coupled oscillators (not necessarily in nearest-neighbour), one can find the ring structure if we consider only part of the coupling or connections (see Smith [159] and Swift [164] for such typical examples).

3.5. Extensions and discussions

We extend the symmetric Hopf bifurcation results obtained in previous sections to two variants of NFDEs (3.3.1), i.e. to functional equations and neutral functional differential equations of general type. We will use the same notation and terminology as in Sections 3.3 and 3.4. Many details will be omitted to avoid repetitions.

We first consider the one parameter family of *functional equations*

$$x(t) = F(x_t, \alpha) \tag{3.5.1}$$

where $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$, $F : C_\tau \times \mathbb{R} \to \mathbb{R}^N$ is a continuously differentiable mapping satisfying the following conditions:

(B2) There exists a real orthogonal representation $\rho : \mathbb{Z} \to O(\mathbb{R}^N)$ such that F is $\mathbb{Z}_N$-equivariant;

(B3) $F(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$.

Any $(0, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ such that $Id - D_x \bar{F}(0, \alpha)$ is an isomorphism will be called a *nonsigular stationary point*. By linearizing Eq. (3.5.1) at $(0, \alpha)$, we obtain the following *characteristic equation*

$$\det_{\mathbb{C}} \Delta_\alpha (\lambda) = 0 \tag{3.5.2}$$

where

$$\Delta_\alpha (\lambda) :\triangleq Id - D_\varphi F(0, \alpha)(e^{\lambda \cdot} Id) : \mathbb{C}^N \to \mathbb{C}^N. \tag{3.5.3}$$

With (3.5.2)–(3.5.3) in mind, we can also make the following assumption.

(B4) There exists an $\alpha_0 \in \mathbb{R}$ such that $(0, \alpha_0)$ is an isolated center of (3.5.1).

Similarly, we can find β_0, b, c, $\delta > 0$ such that for each $j \in J$

$$\bar{c}_j(\alpha_0, \beta_0) := \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0 - \delta, j} (\cdot), \Omega) - \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0 \delta, j} (\cdot), \Omega) \tag{3.5.4}$$

is well-defined, where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c)$.

Let $W := C(S^1; \mathbf{R}^N)$ and define a nonlinear map $N_0 : W \times \mathbb{R}^2 \to W$ by

$$N_0(z, \alpha, \beta)(t) = F(z_{t,\beta}, \alpha, \beta) \in W \times \mathbb{R}^2. \tag{3.5.5}$$

Then N_0 is $\mathbf{Z}_N \times S^1$-equivariant with respect to the action (3.3.3) of $\mathbf{Z}_N \times S^1$. We have the following lemma.

Lemma 3.5.1. *$x(t)$ is a $\frac{2\pi}{\beta}$-periodic solution of (3.5.1) if and only if $z(t) = x(\frac{2\pi}{\beta}t)$ is a solution to the $\mathbf{Z}_N \times S^1$-equivariant fixed-point problem below*

$$z = N_0(z, \alpha, \beta), \quad (z, \alpha, \beta) \in W \times \mathbb{R}^2. \tag{3.5.6}$$

Proof. It is straightforward.

Let $W_{1,j}$ denote the same component in the isotypical decomposition of W for each $j \in J$. We define

$$a_{1,j}(\alpha, \beta) := [Id - D_z N_0(0, \alpha, \beta)]|_{W_{1,j}}. \tag{3.5.7}$$

Lemma 3.5.2. $a_{1,j}(\alpha, \beta) = \Delta_\alpha\, (i\beta)|_{W_{1,j}}$.

Proof. Let $z(t) = x \sin 2\pi t + y \cos 2\pi t \in W_{1,j}$, where $t \in S^1$, $x + iy \in \mathbb{C}^N$. With the complex structure (3.3.9) in mind, we have

$$\begin{aligned} a_{1,j}(\alpha, \beta) z(t) &= z(t) - D_\varphi F(0, \alpha) z_{t,\beta} \\ &= z(t) - \sin 2\pi t D_\varphi F(0, \alpha) y \cos_\beta + \cos 2\pi t D_\varphi F(0, \alpha) x \sin_\beta \end{aligned}$$

$$
\begin{aligned}
&- \cos 2\pi t D_\varphi F(0,\alpha) y \cos_\beta + \sin 2\pi t D_\varphi F(0,\alpha) y \sin_\beta \\
=& z(t) - [\sin 2\pi t D_\varphi F(0,\alpha) x \cos_\beta + \sin 2\pi t D_\varphi F(0,\alpha) x i \sin_\beta] \\
&- [\cos 2\pi t D_\varphi F(0,\alpha) y \cos_\beta + \cos 2\pi t D_\varphi F(0,\alpha) y i \sin_\beta] \\
=& z(t) - D_\varphi F(0,\alpha)(e^{i\beta} Id)(x \sin 2\pi t + y \cos 2\pi t) \\
=& [Id - D_\varphi F(0,\alpha)(e^{i\beta} Id)] z(t) \\
=& \Delta_\alpha\ (i\beta) z(t).
\end{aligned}
$$

This establishes the lemma.

We therefore obtain the following local Hopf bifurcation theorem with $\mathbf{Z}_n$-symmetry for functional equation (3.5.1).

Theorem 3.5.3. *Assume (B2)–(B4) hold. If* $\bar{c}_j(\alpha_0, \beta_0) \neq 0$ *for some* $j \in J$, *then* (α_0, β_0) *is a bifurcation point. More precisely, there is a sequence of triples* $\{(x_k, \alpha_k, \beta_k)\}_{k=1}^{\infty}$ *such that*

(i) $(x_k(t), \alpha_k, \beta_k) \to (0, \alpha_0, \beta_0)$ *uniformly for* $t \in \mathbb{R}$ *as* $k \to \infty$;

(ii) $x_k(t)$ *is a* $\frac{2\pi}{\beta_k}$*-periodic solution of (3.5.1) with* $\alpha = \alpha_k$, $k = 1, 2, \ldots$;

(iii) $\rho(e^{\frac{2\pi}{N} i}) x_k(t) = x_k(t + \frac{2\pi}{\beta_k}\frac{j}{N})$ *for* $t \in \mathbb{R}$, $k = 1, 2, \ldots$.

Proof. By Lemmas 5.3.1 and 5.3.2, it follows from the abstract symmetric bifurcation theorem (by identifying (3.5.6) as an equivariant coincidence problem with $L_0 = Id$, $B_0 \equiv 0$ and (B_0, N_0) is an L_0-condensing $\mathbf{Z}_N \times S^1$-pair in a neighbourhood of zero), as in the proof of Theorem 3.3.4.

Next, we consider the neutral functional differential equation of general type

$$x'(t) = F(x_t, \dot{x}_t, \alpha) \tag{3.5.8}$$

where $F : C_\tau \times C_\tau \times \mathbb{R} \to \mathbb{R}^N$ is continuously differentiable with $F(0,0,\alpha) \equiv 0$. We assume

(C1) there exist a constant $k \in [0,1)$ such that

$$|F(\varphi,\psi_1,\alpha) - F(\varphi,\psi_2,\alpha)| \le k \parallel \varphi - \psi \parallel, \quad \varphi,\ \psi_1,\ \psi_2 \in C_\tau, \quad \alpha \in \mathbb{R};$$

(C2) there is a real orthogonal representation $\rho : \mathbf{Z} \to 0(\mathbb{R}^N)$ such that

$$F(\rho(\gamma)\varphi, \rho(\gamma)\psi, \alpha) = \rho(\gamma)F(\varphi,\psi,\alpha)$$

for all $\varphi, \psi \in C_\tau$, $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{Z}_n$;

(C3) there exists an $\alpha_0 \in \mathbb{R}$ such that $D_x\bar{F}(0,0,\alpha)$ is an isomorphism, where $\bar{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is the restriction of F on $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$.

The *characteristic equation* of (3.5.8) now reads as follows

$$\det\nolimits_{\mathbb{C}} \Delta_\alpha(\lambda) = 0 \tag{3.5.9}$$

where

$$\Delta_\alpha(\lambda) := \lambda[Id - D_\psi F(0,0,\alpha)(e^{\lambda\cdot}Id)] - D_\varphi F(0,0,\alpha)(e^{\lambda\cdot}Id) : \mathbb{C}^N \to \mathbb{C}^N. \tag{3.5.10}$$

Referring to the characteristic equation (3.5.9), we also assume

(C4) $(0,\alpha_0)$ is an isolated center of (3.5.8);

(GC4) $\bar{F}(x,0,\alpha) = 0$ with $x \in (\mathbb{R}^N)^{\mathbf{Z}_n}$ iff $x = 0$. Moreover, $D_x\bar{F}(0,0,\alpha) \in GL(\mathbb{R}^N)$ for every $\alpha \in \mathbb{R}$;

(GC5) The set $\mathcal{A} := \{\alpha \in \mathbb{R};$ the stationary point $(0,\alpha)$ has pure imaginary characteristic values with respect to (3.5.9)–(3.5.10) $\}$ is discrete.

Let $j \in J$. We put

$$\Delta_{\alpha,j}(\lambda) := \Delta_\alpha(\lambda)|_{C_j^N}.$$

Similarly, the j-*th crossing number* of each $(\alpha,\beta) \in \mathcal{A}$ is defined as

$$\tilde{c}_j(\alpha_0,\beta_0) = \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0-\delta,j}(\cdot),\Omega) - \deg_B(\det_{\mathbb{C}} \Delta_{\alpha_0+\delta,j}(\cdot),\Omega)$$

for some $\beta_0, \delta, b, c > 0$ and $\Omega := (0,b) \times (\beta_0 - c, \beta_0 + c)$.

Theorem 3.5.4. *Assume (C1)–(C4) hold. If $\tilde{c}_j(\alpha_0,\beta_0) \neq 0$ for some $j \in J$, then (α_0,β_0) is a bifurcation point of (3.5.8). Moreover, the bifurcating $\frac{2\pi}{\beta_k}$-periodic solution $x_k(t)$ has the symmetry $\rho(e^{i\frac{2\pi}{N}})x(t) = x(t + \frac{2\pi}{\beta_k}\frac{i}{N})$, where $(x_k(t),\alpha_k,\beta_k) \to (0,\alpha_0,\beta_0)$ uniformly for $t \in \mathbb{R}$ as $k \to \infty$.*

Proof. We only give a sketch of the proof.

Let $x(t) = z(\frac{\beta}{2\pi}t)$ for $t \in \mathbb{R}$. Then finding a $\frac{2\pi}{\beta}$-periodic solution of Eq. (3.5.8) is equivalent to finding a solution to

$$\dot{z}(t) = \frac{2\pi}{\beta}F(z_{t,\beta}, \frac{\beta}{2\pi}z'_{t,\beta}, \alpha) \tag{3.5.11}$$

where $z_{t,\beta}$ is defined as before and $z'_{t,\beta}(\theta) = z'(t + \frac{\beta}{2\pi}\theta)$ for $\theta \in (-\infty, \tau]$.

Let $V = C(S^1;\mathbb{R}^N)$ and $W = C^1(S^1;\mathbb{R}^N)$. We define

$$L_0 : W \to V,\ L_0 z(t) = z'(t),\ z \in H^1(S^1;\mathbb{R}^N),\ t \in S^1$$
$$K_0 : W \to V,\ K_0 z(t) = \int_0^1 z(s)ds,\ z \in V,\ t \in S^1.$$
$$N_0 : W \times \mathbb{R}^2 \to V,\ N_0(z,\alpha,\beta)(t) := \frac{2\pi}{\beta}F(z_{t,\beta}, \frac{\beta}{2\pi}z'_{t,\beta}, \alpha).$$

As before, V and W are $\mathbf{Z}_N \times S^1$-representations and L_0, K_0 and N_0 are all $\mathbf{Z}_N \times S^1$-equvariant with respect to (3.3.3). Moreover, Eq. (3.5.11) reduces to the following $\mathbf{Z}_N \times S^1$-equivariant coincidence equation

$$L_0 z = N_0(z, \alpha, \beta), \quad (z, \alpha, \beta) \in W \times \mathbb{R}^2. \tag{3.5.12}$$

It follows from [90, 144, 145] that $(0, N_0)$ is an L_0-condensing $\mathbf{Z}_N \times S^1$-pair under the assumption (C1) and (C2).

Now formulating (3.5.12) into a fixed-point problem gives a map

$$a(\alpha, \beta) = Id - (L_0 + K_0)^{-1}[K_0 + D_z N_0(0, \alpha, \beta)]$$

and considering the isotypical decomposition of W leads to the map

$$\begin{aligned} a_{1,j}(\alpha, \beta) &= a(\alpha, \beta)|_{W_{1,j}} \\ &= (L_0 + K_0)^{-1}[L_0 - D_z N_0(0, \alpha, \beta)]|_{W_{1,j}}. \end{aligned}$$

Note that

$$D_z N_0(0, \alpha, \beta) = \tfrac{2\pi}{\beta} D_\varphi F(0, 0, \alpha) z_{t,\beta} + D_\psi F(0, 0, \alpha) z'_{t,\beta}.$$

It follows that for $z = x \sin 2\pi t + y \cos 2\pi t \in W$ and $j \in J$

$$\begin{aligned} (a_{1,j}) z(t) &= (L_0 + K_0)^{-1}[z'(t) - \frac{2\pi}{\beta} D_\varphi F(0, 0, \alpha) z_{t,\beta} - D_\psi F(0, 0, \alpha) z'_{t,\beta}] \\ &= \frac{1}{\beta i} \, \Delta_\alpha \, (\beta i) z(t). \end{aligned}$$

Therefore $a_{1,j}|_{W_{1,j}} = \frac{1}{\beta i} \, \Delta_\alpha \, (i, \beta)|_{W_{1,j}}$. A similar argument to that of Theorem 3.3.4 implies (α_0, β_0) is a bifurcation point.

This completes the proof.

Under additional global assumptions (GC4) and (GC5), a global $\mathbf{Z}_n$-symmetric Hopf bifurcation result for Eq. (3.5.8) can also be proved analogously.

Theorem 3.5.5. *Let (C1)–(C3) and (GC4) (GC5) hold. If there exists an integer $j \in J$ such that S^j has a bounded connected component C^j, then $C^j \cap M$ is a finite set and*

$$\sum_{(0,\alpha,p)\in C^j\cap M} \widetilde{c}_j(\alpha, \tfrac{2\pi}{p}) = 0$$

where S^j is defined by

$$\begin{aligned} S^j := Cl\{(z,\alpha,p);\ x(t) = z(\tfrac{t}{p}) \text{ is a } p\text{-periodic solution} \\ \text{of (3.5.8) such that } \rho(e^{i\frac{2\pi}{N}}z(t) = z(t+\tfrac{i}{N}))\} \\ \subset C^1(S^1;\mathbb{R}^N)\times\mathbb{R}^2 \end{aligned}$$

and

$$M := \{(0,\alpha,p);\ \alpha\in\mathbb{R},\ p>0\} \subset C^1(S^1;\mathbb{R}^N)\times\mathbb{R}^2.$$

Proof. The proof is similar to that of Theorem 3.3.5. We omit it.

We finally give some discussions.

Remark 3.5.1. Our extension of local Hopf bifurcation results to functional equations is motivated by a paper of Hale and Oliveira [85], where a class of multiparameter functional equations (without symmetry) is considered by using Fourier series together with a Liapunov-Schmidt reduction method (see also Staffans [160]). Our conditions are essentially the same as those of [85,160] except that our transversality condition is weaker(compare Remark 3.3.6). For the *global* Hopf bifurcation, we are unable to obtain any general result at this stage. This is due to the fact that we have no control on the lower bound of the periods of any possible periodic solutions to (3.5.1).

For global Hopf bifurcation results for certain special integral equations, we mention the paper by Fiedler [60], where a global Hopf bifurcation theorem (without symmetry) is obtained for a class of *Volterral integral equations*, since Fiedler uses an ODE-approximation appproach and the integral equation is equivalent, under certain conditions, to an ordinary differential equation. This allows the author to prove global results for integral equations. For general functional equations, a global Hopf bifurcation theorem seems still not available, to the best of our knowledge.

Since functional equations include integral equation as a special case, our local theory applies to the various models of integral equations. See [11, 30, 46, 77, 84, 85, 87, 112] for examples. In particular, a model of van der Heiden [11, 60] for circular neural nets can be analyzed by our *symmetric* Hopf bifurcation theorem.

Remark 3.5.2. The neutral equations of general type (3.5.8) have been studied by many authors. We refer to Gopalsamy [77], Kuang [112] and Sadovski [144, 145] and the references cited there. Although the existence of periodic solutions has been studied by Hetzer [90], Sadovski [144, 145] and others (see [77]), the Hopf bifurcation of Eq. (3.5.8) has not been found, as far as is known to us. Theorem 3.5.5 thus gives an answer to the question proposed by Kuang [113] in the absence of symmetry.

We shall consider an explicit neutral equation of the type (3.5.8) in Chapter 4 to illustrate our Theorem 3.5.4 and 3.5.5.

Remark 3.5.3. Similarly, we can apply the results in this section to the *Rashevsky-Turing theory* in the same fassion as in Section 3.4. The kinetics equations for each cell now are described by Eq. (3.5.1) or Eq. (3.5.8). We will illustrate this application in Chapter 4, where we study a neutral delay equation of the type (3.5.8).

Remark 3.5.4. For the sake of convenience, we have assumed throughout this chapter that the functional defining the equations (3.3.1), (3.4.1), (3.5.1) and (3.5.8) are well-defined for all φ, $\psi \in C_r$ and $\alpha \in \mathbb{R}$. By modifying the assumptions naturally, the results in this chapter also hold true if they are only defined on an open subset of $C_r \times \mathbb{R}$ or $C_r \times C_r \times \mathbb{R}$. The "unbounded" (resp. "bounded") must be understood as "approaching the boundary of the open subset" (resp. "uniformly bounded away from the boundary of the open subset"). This remark will be implicitly used in Chapter 4.

We finish this chapter with a brief discussion on the *binary orbits* from Fiedler [62].

Remark 3.5.5. For simplicity, let us take the Turing ring as an example. As it is shown in Section 3.4, $\mathbf{Z}_N$ is a symmetry of the ring and $\mathbf{Z}_N$ acting on $\mathbb{R}^N$ gives a $\mathbf{Z}_N$-representation on $\mathbb{C}^N$. It is also seen that every charactor of the representation appears and, consequently, there are a total of N well-defined crossing numbers, $c_j(\alpha_0, \beta_0)$, $i \leq j \leq N$, at each center $(0, \alpha_0)$. By (3.4.11), however, these crossing numbers are not different from each other. Let $C(N)$ denotes the number of different crossing numbers. We have obviously from (3.4.11) that

$$C(N) = \begin{cases} \frac{N}{2} + 1, & \text{if } N \text{ is even} \\ \frac{N+1}{2}, & \text{if } N \text{ is odd.} \end{cases}$$

Next, to take into the account *periodic doubling,* we can also consider the *equivalent classes* $D(N)$ on the set $\mathbf{Z}(n) := \{0, 1, 2, \ldots, N-1\}$, which is defined by

$j_1 \sim j_2$ iff there exist integers n_1, $n_2 \geq 0$ such that $2^{n_1} j_1 = 2^{n_2} j_2 \pmod{N}$.

Any element $d \in D(N)$ is called *an binary orbit* (see [62]). Obviously, the cardinality $|D(N)| \leq C(N)$ and $|D(N)|$ gives *different* crossing numbers. In this sense, we have also 24, possibly different, crossing numbers for $N = 1986$.

To look at the *global* aspect of the above consideration, we shall assume also the *generic* conditions on the $\mathbf{Z}_n$-centers (see Fiedler [62]). For any $d \in D(N)$, we define the generic *global equivariant Hopf index* $\mathcal{H}_N^d$ as

$$\mathcal{H}_N^d := \sum_{j \in d} c_j$$

where the sum ranges over all $\mathbf{Z}_n$-centers. In consequence, if $\mathcal{H}_N^d \neq 0$, then $\sum_{\mathbf{Z}_n\text{-centers}} c_j \neq 0$ for at least one $j \in d$. By our global Hopf bifurcation theorem, these exists an unbounded *global continuum* $\mathcal{Z}^j \subseteq H^1(S^1; \mathbb{R}) \times \mathbb{R}^2$ such that the periodic solutions in $\mathcal{S}^j$ are all phase-locked (see Theorem 3.3.6). This conclusion is analogous to that of Fiedler [62].

Admittedly, we point out that in our conclusion, the period is not minimal. That defect has been commonly seen in the topological approach to Hopf bifurcation theory. See also Alexander and Yorke [8] for an approach via generalized homology theory, Chow and Mallet-Parret [33] utilizing Fuller index ([18, 57, 65, 67]), Ize [94, 95] and Nussbaum [130] using homotopy arguments. However, in some situations, one can control the period by showing the nonexistence of periodic solutions of certain periods (see [33, 110, 179] and also Chapter 5 how to sidestep this difficulty).

Finally, it is worthwile to note that besides the Turing ring, the $\mathbf{Z}_n$-symmetry is also seen in other situations. We refer to Eigen and Schuster [54] and Hofbauer et al. [91, 92] for examples of the hypercyle system which may exhibit a symmetry of $\mathbf{Z}_n$.

References

[1] M. Adimy, *Bifurcation de Hopf locale par semi-groupes intégrés*, C. R. Acad. Sci. Paris, **t. 311, Série I** (1990), 423-428.

[2] R. R. Akhmerov, M. I. Kamenskii, V. S. Kozyakin and A. V. Sobolev, *Periodic solutions of systems of autonomous functional-differential equations of neutral type with small delays,* Diff. Urav. **10** (1974), 1479–1485.

[3] J. C. Alexander, *A period-doubling bubble in the dynamics of two coupled oscillators,* in "Nonlinear Oscillations in Biology and Chemistry" (H. G. Othmer, ed.), Lecture Notes in Biomathematics **66**, Springer-Verlag, New York, 1986, 208–229.

[4] ———, *Patterns at primary Hopf bifurcations of a plexus of identical oscillations,* SIAM J. Appl. Math. **46** (1986), 199–221.

[5] ———, *Spontaneous oscillations in two 2-component cells coupled by diffusion,* J. Math. Biol. **23** (1986), 205–219.

[6] ——— and J. F. G. Auchmuty, *Global bifurcations of phase-locked oscillators,* Arch. Rational Mech. Anal. **93** (1986), 253–270.

[7] ——— and ———, *Global bifurcation of waves,* Manuscripta Math. **27** (1979), 159–166.

[8] ——— and J. A. Yorke, *Global bifurcation of periodic orbits,* Amer. J. Math. **100** (1978), 263-292.

[9] K. T. Alligood, J. Mallet-Paret and J. A. Yorke, *Families of periodic orbits: local continuability does not imply global continuability,* J. Differential Geometry **16** (1981), 483–492.

[10] U. an der Heiden, *Delays in physiological systems,* J. Math. Biology **8** (1978), 345–364.

[11] ———, "Analysis of Neural Networks", Lecture Notes in Biomathematics **35**, Springer-Verlag, New York, 1980.

[12] D. G. Aronson, G. B. Ermentrout and N. Kopell, *Amplitude response of coupled oscillations,* Physica D **41** (1990), 403–449.

[13] ———, M. Golubitsky and M. Krupa, *Coupled arrays of Josephson junctions and bifurcation of maps with S_N symmetry,* Nonlinearity **4** (1991), 861–902.

[14] ——, —— and J. Mallet-Paret, *Ponies on a merry-go-round in large arrays of Josephson junctions,* Nonlinearity 4 (1991), 903–910.

[15] P. Ashwin, *Symmetric chaos in systems of three and four forced oscillators,* Nonlinearity **3** (1990), 603–617.

[16] ——, G. P. King and J. W. Swift, *Three identical oscillators with symmetric coupling,* Nonlinearity **3** (1990), 585–601.

[17] J. F. G. Auchmuty, *Bifurcating wave,* in [79], 263–278.

[18] I. K. Babenko and S. A. Bogatyi, *The behavior of the index of periodic points under iterations of a mapping,* Math. USSR Izvestiya **38** (1992), 1–26.

[19] J. Bard and I. Lauder, *How well does Turing's theory of morphogenesis work?* J. Theor. Biol. **45** (1974), 501–531.

[20] K. Bar-Eli, *The dynamics of coupled chemical oscillators,* J. Phys. Chem. **88** (1984), 6174–6177.

[21] ——, *Coupling of chemical oscillators,* J. Phys. Chem. **88** (1984), 3616–3622.

[22] ——, *On the stability of coupled chemical oscillators,* Physica **14D** (1985), 242–252.

[23] T. Bartsch and J. Mawhin, *The Leray-Schauder degree of S^1-equivariant operators associated to autonomous neutral equations in spaces of periodic functions,* J. Differential Equations **92** (1991), 90–99.

[24] R. K. Brayton, *Bifurcation of periodic solutions in a nonlinear difference-diferential equations of neutral type,* Quarterly Appl. Math. **24** (1966), 215–224.

[25] ——, *Nonlinear oscillations in a distributed network,* Quarterly Appl. Math. **24** (1967), 289–301.

[26] N. Chafee, *The bifurcation of one or more closed orbits from an equilibrium point of an autonomous differential equation,* J. Differential Equations **4** (1968), 661–679.

[27] ——, *A bifurcation problem for a functional differential equation of finitely retarded type,* J. Math. Anal. Appl. **35** (1971), 312–348.

[28] P. Chossat and M. Golubitsky, *Iterates of maps with symmetry,* SIAM J. Math. Anal. **19** (1988), 1259–1270.

[29] ——, R. Lauterbach and I. Melbourne, *Steady-state bifurcation with $O(3)$ - symmetry*, Arch. Rational. Mech. Anal. **113** (1990), 313–376.

[30] S. N. Chow, O. Diekmann and J. Mallet-Paret, *Stability, multiplicity and global continuation of symmetric periodic solutions of a nonlinear Volterra integral equations,* Japan J. Appl. Math. **2** (1985), 433–469.

[31] —— and J. K. Hale, "Methods of Bifurcation Theory", Springer-Verlag, New York, 1982.

[32] —— and J. Mallet-Paret, *Integral averaging and bifurcation,* J. Differential Equations **26** (1977), 112–159.

[33] —— and ——, *The Fuller index and global Hopf bifurcation,* J. Differential Equations **29** (1978), 66–85.

[34] ——, —— and J. A. Yorke, *Global Hopf bifurcation from a multiple eigenvalue,* Nonlinear Analysis–TMA **2** (1978), 753–763.

[35] ——, —— and ——, *A periodic orbit index which is a bifurcation invariant*, in Lecture Notes in Math. **1007** (J. Palis Jr., ed.), Springer-Verlag, 1988, 109–131.

[36] G. Cicoga, *Symmetry breakdown from bifurcation*, Lettere Nuovo Cimento **31** (1981), 600–602.

[37] ——, *Periodic and bifurcating solutions of equivariant ODE's*, Proc. Inter. Conf. Theory and Appl. Diff. Equations, Ohio, 1988, 181–185.

[38] ———, *Hopf-type bifurcations in the presence of linear and nonlinear symmetries*, in "Bifurcation and Symmetry" (E. Allgower, K. Böhmer and M. Golubitsky, eds.), ISNM **104**, 1992, 75–84.

[39] ——— and G. Gaeta, *Periodic solutions from quaternionic bifurcation*, Lettere Nuovo Cimento **44** (1985), 65–68.

[40] ——— and ———, *Quaternionic-like bifurcation in the absence of symmetry*, J. Phys. A: Math. Gen. **20** (1987), 79–89.

[41] ——— and ———, *Bifurcations, symmetries and maximal isotropy subgroups*, Nuovo Cimento **102B** (1988), 451–470.

[42] ——— and ———, *Periodic solutions from symmetry*, Nonlinear Analysis–TMA **13** (1989), 475–488.

[43] J. R. Claeyssen, *The integral-averaging bifurcation method and the general one-delay equation*, J. Math. Anal. Appl. **78** (1980), 428–439.

[44] M. G. Crandall and P. H. Rabinowitz, "The Hopf Bifurcation Theorem", Mathematics Research Center, University of Wisconsin, Madison, TSR **1604**, 1976.

[45] ——— and ———, *The Hopf bifurcation theorem in infinite dimensions*, Arch. Rational Mech. Anal. **67** (1977), 53–72.

[46] J. M. Cushing, "Integrodifferential Equations and Delay Models in Population Dynamics", Lecture Notes in Biomathematics **20**, Springer-Verlag, New York, 1977.

[47] G. Da Prato and A. Lunardi, *Hopf bifurcation for nonlinear integrodifferential equations in Banach spaces with infinite delay*, Indiana Univ. Math. J. **36** (1987), 241–255.

[48] E. N. Dancer, *A new degree for S^1-invariant gradient mappings and application*, Ann. Inst. Henri Poincaré **2** (1985), 329–370.

[49] G. Dangelmayr and M. Kirby, *On diffusively coupled oscillators*, in "Bifurcation and Symmetry" (E. Allgower, K. Böhmer and M. Golubistky, eds.), ISNM **104**, 1992, 75–84.

[50] ——— and E. Knobloch, *On the Hopf bifurcation with broken $O(2)$-symmetry*, in "Physics of Structure Formation: Theory and Simulation" (W. Güttinger and G. Dangelmayr, eds.), Springer-Verlag, 1987.

[51] ——— and ———, *Hopf bifurcation in reaction-diffusion equations with broken translation symmetry*, Proc. Inter. Conf. on Bifurcation Theory and its Numer. Anal. (Li Kaitai, J. E. Marsden, M. Golubitsky and G. Iooss, eds.), Xián Jiaotong Press, Xián (China), 1989, 62–170.

[52] N. Dellnitz and B. Werner, *Computational methods for bifurcation problems with symmetries–with special attention to steady state and Hopf bifurcation points*, J. Computational and Appl. Math. **26** (1989), 97–123.

[53] G. Dylawerski, K. Gęba, J. Jodel and W. Marzantowicz, *S^1-equivariant degree and the Fuller index*, Ann. Pol. Math. **52** (1991), 243–280.

[54] M. Eigen and P. Schuster, "The Hypercycle: A Principle of Natural Self-Organization", Springer-Verlag, Berlin, 1979.

[55] L. H. Erbe, K. Gęba, W. Krawcewicz and J. Wu, *S^1-degree and global Hopf bifurcation theory of functional differential equations*, J. Differential Equations **98** (1992), 277–298.

[56] ———, W. Krawcewicz and J. Wu, *Leray-Schauder degree for semilinear Fredholm maps and periodic boundary value problems of neutral equations*, Nonlinear Analysis-TAM **15** (1990), 747–764.

[57] C. C. Fenske, *An index for periodic orbits of functional differential equations*, Math. Ann. **285** (1989), 381–392.

[58] B. Fiedler, *Global Hopf bifurcation in porous catalysts*, Equadiff **82** (H. W. Knobloch and K. Schmitt, eds.), Lecture Notes in Mathematics **1017**, Spriner-Verlag, Heidelberg, 1983, 177–183.

[59] ———, *An index for global Hopf bifurcation in parabolic systems*, J. reine u. angew. Math. **359** (1985), 1–36.

[60] ———, *Global Hopf bifurcation for Volterra integral equations*, SIAM J. Math. Anal. **17** (1986), 911–932.

[61] ———, *Global Hopf bifurcation in reaction diffussion systems with symmetry*, in "Dynamics of Infinite Dimensional Systems" (S. N. Chow and J. K. Hale, eds.), Springer-Verlag, New York, 1987, 81–104.

[62] ———, "Global Hopf Bifurcation of Periodic Solutions with Symmetry", Leture Notes in Mathematics **1309**, Springer-Verlag, 1988.

[63] M. Field, *Equivariant bifurcation theory and symmetry breaking*, J. Dynamics Diff. Eqns **1** (1989), 369–421.

[64] R. J. Field and M. Burger (eds.), "Oscillations and Travelling Waves in Chemical Systems", Wiley & Sons, New York, 1985.

[65] R. D. Franzosa, *An homology index generalizing Fuller's index for periodic orbits*, J. Differential Equations **84** (1990), 1–14.

[66] H. I. Freedman, *On a bifurcation theorem of Hopf and Friedrichs*, Canad. Math. Bull. **20** (1977), 95–102.

[67] F. B. Fuller, *An index of fixed point type for periodic orbits*, Amer. J. Math. **89** (1967), 133–148.

[68] G. Gaeta, *Bifurcation and symmetry breaking*, Phys. Reports **189** (1990), 1–87.

[69] ——, *Bifurcation theory and nonlinear symmetries*, Nonlinear Analysis-TMA **17** (1991), 825–831.

[70] K. Gęba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems, III: Bifurcation theorems of functional differential equations with symmetries*, Preprint, 1993.

[71] —— and W. Marzantowicz, *Global bifurcation of periodic solutions*, Topological Methods in Nonlinear Analysis **1** (1993), 67–93.

[72] M. Golubitsky and D. G. Schaeffer, *A disscussion of symmetry and symmetry breaking*, Proc. Symposia in Pure Math. **40** (1983), Part I, 499–515.

[73] —— and ——, "Singularities and Groups in Bifurcation Theory I", Appl. Math. Sci. **51**, Springer-Verlag, New York, 1986.

[74] ——, —— and I. N. Stewart, "Singularities and Groups in Bifurcation Theory II", Appl. Math. Sci. **69**, Springer-Verlag, New York, 1988.

[75] —— and I. N. Stewart, *Hopf bifurcation with dihedral group symmetry: coupled nonlinear oscillations*, in " Multiparameter Bifucation Theory"(M. Golubitsky and J. Guckenheimer, eds.), Comtemporary Math. **56** (1986), 131–137.

[76] —— and ——, *Hopf bifurcation in the presence of symmetry*, Arch. Rational Mech. Anal. **87** (1985), 107–165.

[77] K. Gopalsamy, "Stability and Oscillations in Delay Differential Equations of Population Dynamics", Kluwer Academic Publishers, Dordrecht, 1992.

[78] J. Grasman and M. J. W. Jansen, *Mutually synchronized relaxation oscillators as prototypes of oscillating systems in biology*, J. Math. Biology **7** (1979), 171–197.

[79] O. Gurel and O. E. Rössler (eds.), "Bifurcation Theory and Applications in Scientific Disciplines", Ann. New York Acad. Sci. **316**, 1979.

[80] P. Haley, M. R. Beasley and K. Wiesenfeld, *Phase locking of Josephson-junction series arrays*, Phys. Rev. **B(38)**, 1988, 8712–8719.

[81] ——, —— and ——, *Phase locking of Josephson-junction arrays*, Appl. Phys. Lett. **52** (1988), 1619–1621.

[82] J. K. Hale, *Critical cases for neutral functional differential equations*, J. Differential Equations **10** (1971), 59–82.

[83] ——, "Theory of Functional Differential Equations", Springer-Verlag, New York, 1977.

[84] ——, *Nonlinear oscillations in equations with delays*, in "Oscillations in Biology" (F. R. Hoppensteadt, eds.), Lectures in Applied Mathematics **17**, 1979, 157–185.

[85] —— and J. C. F. De Oliverira, *Hopf bifurcation for functional equations*, J. Math. Anal. Appl. **74** (1980), 41–59.

[86] J. K. Hale and J. Mawhin, *Coincidence degree and periodic solutions of neutral equations*, J. Differential Equations **15** (1974), 295–307.

[87] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, "Theory and Applications of Hopf Bifurcation", Cambridge University Press, Cambridge, 1981.

[88] B. Hasslacher, R. Kapral and A. Lawniczak, *Molecular Turing structures in the biochemistry of the cell*, Priprint, 1993.

[89] T. J. Healey, *Global bifurcation and continuation in the presence of symmetry with an application to solid mechanics*, SIAM J. Math. Anal. **19** (1988), 825–840.

[90] G. Hetzer, *Some applications of the coincidence degree for set-contractions to functional differential equations of neutral type*, Comment. Math. Univ. Carolina **16** (1975), 121–138.

[91] J. Hofbauer and K. Sigmund, "The Theory of Evolution and Dynamical Systems", Cambridge University Press, Cambridge, 1988.

[92] ——, J. Mallet-Paret and H. L. Smith, *Stable periodic solutions for the hypercycle system*, J. Dynamics and Diff. Eqns **3** (1991), 423–436.

[93] L. N. Howard, *Nonlinear oscillations*, in "Nonlinear Oscillations in Biology" (F. R. Hoppensteadt, ed.), AMS Lecture Notes in Applied Mathematics **17**, 1979, 1–69.

[94] J. Ize, *Global bifurcation of periodic orbits*, Communications Technicas CIMAS **5**, 1974.

[95] ——, *Bifurcation theory for Fredholm operators*, Mem. Amer. Math. Soc. **174**, 1976.

[96] ——, I. Massabó and A. Vignoli, *Degree theory for equivariant maps I*, Trans. Amer. Math. Soc. **315** (1989), 433–510.

[97] ——, —— and ——, *Degree theory for equivariant maps II: The general S^1-action* , Mem. Amer. Math. Soc. **100**, 1992.

[98] J. A. Jacqnez and C. P. Simon, *Qualitative theory of compartmental systems*, Preprint, 1992.

[99] R. Kapral, *Pattern formation in two-dimensional arrags of coupled, discrete-time oscillators*, Physical Review A, **31** (1985), 3868–3879.

[100] ——, *Discrete models for chemically reacting systems*, J. Math. Chem. **6** (1991), 113–163.

[101] ——, *Chemical waves and coupled lattices*, in "Theory and Applications of Coupled Map Laltices" (K. Kaneko, ed.), John Wiley & Sons, 1993.

[102] ——, A. Lawniczak and P. Masiar, *Reactive dynamics in a multispecies lattice-gas automation*, J. Chem. Phys. **96** (1992), 2762–2776.

[103] N. D. Kazarinoff, P. van den Driessche and Y. H. Wan, *Hopf bifurcation and stability of periodic solutions of differential-difference and integro-differential equations*, J. Inst. Math. Appl. **21** (1978), 461–477.

[104] A. I. Khibnik, R. M. Borisyuk and D. Roose, *Numerical bifurcation analysis of a model coupled neural oscillators*, in "Bifurcation and Symmetry" (E. Allgower, K. Böhmer and M. Golubitsky, eds.), ISNM **104**, 1992, 215–228.

[105] H. Kielhöfer, *Hopf bifurcation at multiple eigenvalue*, Arch. Rational Mech. Anal. **69** (1979), 53–83.

[106] E. Knobloch and M. Silber, *Hopf bifurcation with* $\mathbb{Z}_4 \times \mathbb{T}^2$ *symmetry*, ISMN **104** in [104], 241–252.

[107] V. B. Kolmanovskii and V. R. Nosov, *Neutral-type systems with after effects*, Automation and Remote Contral **45** (1984), 1–28.

[108] N. Kopell, *Forced and coupled oscillators in biological applications*, Proc. Int. Congr. Math., Warsaw, 1983, 1645–1659.

[109] ——— and G. B. Ermentrout, *Coupled oscillators and mammalian small intestines*, in "Oscillations in Mathematical Biology" (J. D. E. Hodgson, ed.), Lecture Notes in Biomathematics **51**, 1983, 24–36.

[110] W. Krawcewicz, J. Wu and H. Xia, *Global Hopf bifurcation for condensing fields and neutral equations with applications to lossless transmission problems*, Canad. Appli. Math. Quarterly **1** (1993), 167–220.

[111] ——, T. Spanily and J. Wu, *Hopf bifurcation for parametrized equivariant coincidence problems and parabalic equations with delays*, Funkc. Ekvac., to appear.

[112] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics", Mathematics in Science and Engineering **91**, Academic Press, Boston, 1993.

[113] —— and A. Feldstein, *Boundness of solutions of a nonlinear nonautonomous neutral delay equation*, J. Math. Anal Appl. **156** (1991), 293–304.

[114] O. Lopes, *Periodic solutions of perturbed neutral differential equations*, J. Differential Equations **15** (1974), 70–76.

[115] ——, *Forced oscillation in nonlinear neutral differential equations*, SIAM J. Appl. Math. **29** (1975), 196–201.

[116] ——, *Stability and forced oscillations*, J. Math. Anal. Appl. **55** (1976), 686–698.

[117] J. Mallet-Paret and J. A. Yorke, *Snakes: oriented families of orbits, their sources, sinks and continuations*, J. Differential Equations **43** (1982), 419–450.

[118] M. Marek, *Turing structures, periodic and chaotic regions in coupled cells*, in "Modelling of Patterns in Space and Time"(W. Jäger and J. D. Murray, eds.), Lecture Notes in Biomathematics **55**, 1984, 214–227.

[119] J. E. Marsden and M. McCracken, "The Hopf Bifurcation and its Applications", Springer–Verlag, New York, 1976.

[120] H. M. Martinez, *Morphogenesis and chemical dissipative structures: A computer simulated case study*, J. Theor. Biol. **36** (1972), 479–501.

[121] W. Nagatta, *Perturbations of a symmetric Hopf bifurcation*, Canad. Math. Soc. Conf. Proceedings **8**, 1987, 545–560.

[122] ———, *Symmetric Hopf bifurcation and magnetoconvection*, in "Multiparameter Bifurcation Theory" (M. Golubitsky and J. Guckenheimer, eds.), Comtemporary Mathematics **56**, 1986, 237–266.

[123] ———, *Dynamics near a symmetric Hopf bifurcation*, Proc. Roy. Soc. Edinburgh **108A** (1988), 249–267.

[124] J. Nagumo and M. Shimura, *Self-oscillation in a transmission line with a tunnel diode*, Proc. IRE **49** (1961), 1281–1291.

[125] G. Nicolis and I. Prigogine, "Self-organization in Nonequilibrium Systems", Wiley & Sons, New York, 1977.

[126] Y. Niikura and Y. Yamada, *Bifurcation of periodic solutions for nonlinear parabolic equations with infinite delays*, Funkc. Ekvac. **29** (1986), 309–333.

[127] R. D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura. Appl. **101** (1974), 263–306.

[128] ———, *A global bifurcaton theorem with application to functional differential equations*, J. Funct. Anal. **19** (1975), 319–338.

[129] ———, *Global bifurcation of periodic solutions of some autonomous functional diferential equations*, J. Math. Anal. Appl. **55** (1976), 699–725.

[130] ———, *A Hopf global bifurcaton theorem for retarded functional differential equations*, Trans. Amer. Math. Soc. **238** (1978), 139–164.

[131] ———, *Differential-delay equations with two time lags*, Mem. Amer. Math. Soc. **205**, 1978.

[132] H. G. Othmer (ed.), *Nonlinear Oscillations in Biology and Chemistry*, Lect. Notes Biomath. **66**, Springer-Verlag, New York, 1986.

[133] ——— and L. E. Scriven, *Instability and dynamic pattern formation in cellular networks*, J. Theor. Biol. **32** (1971), 507–537.

[134] J. C. F. De Oliverira. *Hopf bifurcation for functional differential equations*, Nonlinear Anal–TMA **4** (1980), 217–229.

[135] H. Petzeltová, *The Hopf bifurcation theorem for parabulic equations with infinite delay*, Math. Bohemica **116** (1991), 181–190.

[136] M. Preisenberger, *Periodische losungen neutraler differenzen-differential gleichungen*, Ph. D. Thesis, Ludwig Maximilians Universitat, 1984.

[137] R. H. Rand, D. W. Storti, S. K. Upadhyaya and J. R. Cooke, *Dynamics of coupled stomatal oscillators*, J. Math. Biology **15** (1982), 131–149.

[138] N. Rashevsky, "Mathematical Biophysics", 3rd ed., Vol. I, Dover, 1960.

[139] M. Renardy, *Bifurcation from rotating waves*, Arch. Rational. Mech. Anal. **79** (1982), 49–84.

[140] R. Rosen, "Dynamical System Theory in Biology", Volume 1: Stability Theory and Its Applications, Wiley-Interscience, New York, 1970.

[141] V. I. Rozhkov, *Asymptotic expansion in powers of the lag of the periodic solution of an equation of neutral type with a small delay*, Diff. Urav. **7** (1988), 1250–1257.

[142] D. Ruelle, *Bifurcation in the presence of a symmetry group*, Arch. Rational Mech. Anal. **51** (1973), 136–152.

[143] V. Rustichini, *Hopf bifurcation for functional differential equations of mixed type*, J. Dynamics and Differential Equations **1** (1989), 145–177.

[144] B. N. Sadovski, *Applications to topological methods in the theory of periodic solutions of nonlinear differential operator equations of neutral type*, Soviet Math. Dokl. **12** (1971), 1543–1547.

[145] ———, *Limit-compact and condensing operators*, Russian Math. Surveys **27** (1972), 85–155.

[146] D. H. Sattinger, "Group Theoretic Methods in Bifurcation Theory", Lecture Notes in Mathematics **762**, Springer-Verlag, Heidelberg, 1979.

[147] ——, *Bifurcation and symmetry breaking in applied mathematics*, Bull. Amer. Math. Soc. **3** (1980), 177–819.

[148] ——, "Branching in the Presence of Symmetry", CBMS-NSF Reg. Conf. Ser. Appl. Math. **40**, SIAM, Philadelphia, 1983.

[149] ——, *Bifurcation from spherical symmetry*, Lecture Notes in Applied Mathematics **20**, 1983, 207–224.

[150] S. Schecter, *Bifurcations with symmetry*, in [119], 224–249.

[151] D. S. Schmidt, *Hopf's bifurcation theorem and the center theorem of Liapunov*, in [119], 95–103.

[152] K. Schmidt, *Fixed point and coincidence theorems with applications to nonlinear differential and integral equations*, Sém. Math. Appl. et Mécan. Univ. Cathol. Louvain 97.

[153] K. R. Schneider and G. Heinz, *Branch inclusion in generic Hopf bifurcation: general approach*, Z. angew. Math. Mech. **71** (1991), 241–250.

[154] E. Serra, *Periodic solutions for some nonlinear differential equations of neutral type*, Nonlinear Analysis-TMA **17** (1991), 139–151.

[155] J. P. Serre, "Linear Representation of Finite Group", Springer-Verlag, New York, 1977.

[156] M. Shimura, *Analysis of some nonlinear phenomena in a transmission line*, IEEE Trans. on Circuit Theory **14** (1967), 60–68.

[157] M. Silber and E. Knobloch, *Hopf bifurcation in a square lattice*, IMA Preprint **675**, University of Minnesota, 1990.

[158] S. Smale, *A mathematics model of two cells via Turing's equation*, in "Some Mathematical Quastions in Biology V" (J. D. Cowan, ed.), AMS Lect. Math. Life Sci. **6**, 1974, 15–26.

[159] H. L. Smith, *Periodic rotating waves in a model of microbial competition in a circular gradostat*, Canad. Appl. Math. Quarterly **1** (1993), 83–113.

[160] O. J. Staffans, *Hopf bifurcation for functional and functional differential equations with infinite delay*, J. Differential Equations **70** (1987), 114–151.

[161] H. W. Stech, *Hopf bifurcation calculations for functional differential equations*, J. Math. Anal. Appl. **109** (1985), 472–491.

[162] ——, *Nongeneric Hopf bifurcation in functional differential equations*, SIAM J. Appl. Math. **16** (1985), 1134–1151.

[163] ——, *Generic Hopf bifurcation in a class of integro-differential equations*, J. Integral Equations **3** (1991), 175–193.

[164] J. W. Swifft, *Hopf bifurcation with symmetry of the square*, Nonlinearity **1** (1988), 333–377.

[165] A. M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. **B237** (1952), 37–72.

[166] J. C. van der Meer, *Hamiltonian Hopf bifurcation with symmetry*, Nonlinearity **3** (1990), 1041–1056.

[167] S. A. van Gils and M. Golubitsky, *A torus bifurcation theorem with symmetry*, J. Dynamics and Differential Equations **2** (1990), 133–1562.

[168] ——, M. Krupa and W. F. Langford, *Hopf bifurcation with non-semisimple* 1 : 1 *resonance*, Nonlinearity **3** (1990), 825–850.

[169] —— and J. Mallet-Paret, *Hopf bifurcation and symmetry: travelling and standing waves on the circle*, Proc. Roy. Soc. Edinburgh **104 A** (1986), 279–307.

[170] —— and T. Valkering, *Hopf bifurcation and symmetry: standing and travelling waves in a circular-chain*, Japan J. Appl. Math. **3** (1986), 207–222.

[171] B. L. van der Waerden, "Algebra I", Springer-Verlag, Berlin, 1971.

[172] A. L. Vanderbauwhede, *An abstract setting for the study of the Hopf bifurcation*, Nonlinear Analysis-TMA **4** (1980), 547–566.

[173] ——, "Local Bifurcation and Symmetry", Research Notes in Mathematics **75**, Pitman, Boston, 1982.

[174] ———, *Bifurcation of periodic solutions in a rotationally symmetric oscillation system*, J. reine u. angew. Math. **360** (1985), 1–18.

[175] I. Waller and R. Kapral, *Spatial and temporal structure in systems of coupled nonlinear oscillators*, Physical Review A, **30** (1984), 2047–2055.

[176] C. W. Wardlaw, *A commentary on Turing's diffusion-reaction theory of morphogenesis*, New Phytologist **52** (1953), 40–47.

[177] ———, *Evidence relating to the diffussion-reaction theory of morphogenesis*, New Phytologist **55** (1955), 39–48.

[178] A. T. Winfree, "The Geometry of Biological Time", Springer-Verlag, New York, 1980.

[179] J. Wu, *Global continua of periodic solutions to some differential equations of neutral type*, Tôhoku Math. J. **45** (1993), 67–88.

[180] K. Yoshida, *The Hopf bifurcation and its stability for semilinear diffusion equation with time delay arising in ecology*, Hiroshima Math. J. **12** (1982), 321–348.

CHAPTER 4

PHASE-LOCKED OSCILLATIONS IN A SINGLE-SPECIES RING PATCH MODEL

4.1. Introduction

In this chapter we illustrate our symmetric Hopf bifurcation results obtained in Chapter 3 by considering a specific differential equation modelling the dynamics of a single-species population distributed over a ring of identical patches (islands or habitats). Our model equation is a neutral delay system which is continuous in time and discrete in space. It allows for population dispersing from one patch to its nearest neighbors. We shall study the phase-locked oscillations in the model and draw some consequences of the effect of dispersion as well as the delay and neutral term on the population dynamics.

The role of space and dispersal in interactions among biological populations has been the subject of much theoretical and experimental work [6, 20, 27-30, 34, 37 and references therein]. It is widely recognized that the spatial heterogeneity of environment, which leads to ecological interactions, operates in general to increase species diversity. For example, it has been asserted that in some cases dispersal can lend stability to interactions [18, 19, 34, 37, 43] while in other cases dispersal can also give rise to instability [27, 34, 37, 43]. For the references related to this subject, we refer to [18, 26, 27, 33, 34, 42] for the study of Lotka-Volterra models in a spatially heterogeneous environment on the persistence and stability, and to [5, 7, 8, 11, 12, 17, 19, 22, 34, 37, 39, 43, 45] for similar discussions on single-species models in a patchy environment.

Our point of departure in this chapter is the classical single-species delay logistic model. By introducing a spatially heterogeneous environment to this model, we arrive at a system which describes a population that grows and disperses in two different phases. The growth phase (or the local growth rate) is modeled by a neutral logistic equation that arises in the study of "food limited" population. The dispersal phase is modeled by a linear operator that accounts for the redistribution (or migration) of the population in its spatial habitat. There are a neutral term and time delays in the local growth rate, which affect the stability of a positive equilibrium and can give rise to Hopf bifurcations of symmetric periodic solutions which exhibit the phase-locked oscillations and synchronous oscillations (see [2, 4, 14, 15, 26, 31] for the effect of delay on dynamics in other cases). We will show that (i) in the case of instantaneous dispersion feedback, the dispersal in the local growth rate as well as the neutral term have a stabilizing effect on the population dynamics; (ii) increasing the delay in the growth phase changes the stability of the positive equilibrium and leads to a Hopf bifurcation of synchronous as well as phase-locked oscillations if the dispersions are small; (iii) the neutral term may bring about several global branches of phase-locked oscillations which would not occur in the absence of neutral term. In this situation, the neutral term has a destablizing influence.

We have chosen the single species logistic equation as a begining to an investigation of spatial heterogeneity and phase-locked oscillations for two reasons. First, it is the simplest single-species population model and contains no complex regulatory mechanisms that might obscure the effects of environmental variation. Second, there has been considerable literature , both mathematical and biological, available on the study of the logistic equation, and its dynamics are well-known, so any change of its behavior due to environmental heterogeneity will be apparent.

We emphasize that our study on single-species population dynamics in a patchy environment is limited to a theoretical aspect and we have not tried to find any experimental (or laboratory) data to fit the theory. We treat spaces as discrete ones, so only patch models are considered and dispersal is thus viewed as a between-habitat phenomenon. The continuous space diffusion model is left for a future investigation.

This chaper is organized as follows. In Section 4.2, we present the model equation by introducing the discrete diffusion to a neutral logistic equation [15] which models the single-species population dynamics in a food-limited environment. The Hopf bifurcation of phase-locked oscillations as well as synchronous oscillations are considered in Section 4.3 in the case where the diffusion feedback in local dynamics is instantaneous. We draw some consequences of the effect of the delay and diffusion on the stability of a positive equilibrium. Section 4.4 is devoted to the analysis of phase-locked oscillations when the feedback in the local dynamics is delayed. Finally, in Section 4.5, we deal with the global bifurcation of phase-locked oscillations in the appearance of the neutral term. In some special cases, several global branches of phase-locked and synchronous periodic solutions are obtained.

Our study in this chapter is a continuation of that initiated in [22].

4.2. The model equation

Let $N(t)$ denote the numerical size of a single-species population growing in a constant homogeneous environment closed to immigration and emigration. The classical Verhulst-Pearl logistic equation, which models the dynamics of the population growth, takes the following form

$$\frac{dN}{dt} = rN(t)[1 - \frac{N(t)}{K}] \tag{4.2.1}$$

where r is the intrinsic growth rate, K is the saturation level or the carrying capacity of the environment. The basic assumption in Eq. (4.2.1) is that the per capita growth rate $\frac{1}{N}\frac{dN}{dt}$ is a linear function of the population size N. Due to its mathematical simplicity and biological clarity, this model has been widely used not only in ecology but also in biology and chemical engineering. For more details, we refer to [9, 16, 32, 35, 38] and the references therein.

In his studies, however, Nicholson [36] observed that population sizes (or densities) usually have a tendency to fluctuate around an equilibrium and in cases of convergence to a positive equilibrium, such a convergence is rarely monotonic. This obviously does not agree with the dynamics of Eq. (4.2.1). To incorporate such oscillations in population model system, Hutchinson [21] therefore suggested the following modification of (4.2.1)

$$\frac{dN}{dt} = rN(t)[1 - \frac{N(t-\tau)}{K}], \quad \tau \in (0, \infty). \tag{4.2.2}$$

This equation is commonly known as the "delay-logistic" equation. The delay τ comprises various factors causing delayed growth rate response such as slow replacement of food supplies, maturation and gestation periods. Eq. (4.4.2) has been extensively investigated and the validity of this model has been seen in several different practical situations (cf. [35]). It is proved that if $r\tau \le \frac{3}{2}$, then the unique positive equilibrium K is globally stable and the (local) asymptotic stability continues for $r\tau < \frac{\pi}{2}$. $r\tau = \frac{\pi}{2}$ is a critical value which gives rise to a Hopf bifurcation and for every $r\tau > \frac{\pi}{2}$, Eq. (4.4.2) has a nonconstant periodic solution. For details, see [4, 31, 46].

Of course, due to the complexity of biological systems and the diversity of environments in the real world, Eq. (4.2.1) and Eq. (4.2.2) are often unrealistic. In his experiments on the population dynamics of ***Daphnia magna***, Smith [41] observed that the per capita growth rate $\frac{1}{N}\frac{dN}{dt}$ is not a linear function of the density but rather a concave function. For a food-limited population, Smith argued that the term $(1 - N/K)$ should then be replaced with a term r ›resenting the proportion of "the rate of food supply not momentarily being used by the population." Therefore

$$\frac{dN}{dt} = rN(t)[1 - \frac{F}{T}] \tag{4.2.3}$$

where F is the rate at which a population of density N uses food and T is the corresponding rate when the population reaches saturation level. The ratio F/T is not the same as N/K. Clearly, a growing population will use "food" faster than a saturated population. This is due to the fact that F/T, during the growth phase of a population, food is consumed both for maintenance and growth whereas when the population reaches saturation level, food is used mainly for maintenance only. Thus it is reasonable to assume that F depends on N (the size of the population being maintained) and $\frac{dN}{dt}$ (the rate at which the population is growing). As a first approximation, Smith then suggested a linear function F as follows

$$F = c_1 N + c_2 \frac{dN}{dt}, \quad c_1 > 0, \quad c_2 \geq 0.$$

When saturation is attained, $dN/dt = 0$, $N = F$ and $T = K$. Thus Eq. (4.2.1) becomes

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t) + c\frac{dN(t)}{dt}}{K}\right] \tag{4.2.4}$$

where $c = c_2/c_1 \geq 0$. Again, it is realistic to incorporate the delayed growth rate reponse by putting a discrete delay τ in the per capita growth rate. This has led Gopalsamy and Zhang [15] to consider the following ***neutral logistic equation*** as a generalization of Hutchinson's equation (4.2.2)

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t-\tau) + cN'(t-\tau)}{K}\right] \tag{4.2.5}$$

in which c is a real number and r, τ, K are as in (4.2.2). $cN'(t-\tau)$ is called the ***neutral term***. Eq. (4.2.5) has been studied by several authors. It is proved that the positive steady state $N(t) = K$ is stable if $0 < |cr| < 1$ and $0 < r\tau < \beta_o(1 - c^2r^2)$ where $\beta_o = \beta_o(c, \tau) \in (\frac{1}{2}\pi, \pi)$. Consequently, the presence of the neutral term has brought about a stabilizing influence in the systerm (cf. [15]). Eq. (4.2.5) and its modifications are also studied by other authors. We refer to [10, 12, 23–25] for the asymptotic behaviour of the solutions and [13] for the existence of $m\tau$ periodic solutions, where m is an integer.

Eq. (4.2.5) and all others above are modeled in a constant homogeneous environment and the spatial heterogeneity is therefore neglected. However, since all ecological systems of varying complexity exist on landscapes or seascapes, the dynamics of population and processes can not be divorced from these spatial contexts. Following Levin [27–30], we therefore consider a single-species population distributed over a ring of n patches. Assume, for simplicity, that the growth of the species in each patch can be described by the model equation (4.2.5) and that the dispersion from one patch to the other occurs only in nearest neighbors and is proportional to the difference of population sizes between two patches. Since a portion of the population in one patch affects a portion of the population in another patch through movement of population members or transmission of signals through space, and since the physical environment varies from point to point in space, rates of population growth and interspecific interactions also vary, and, as a consequence, population density varies through space, too. Therefore, we arrive at the following system of neutral delay equations

$$\begin{aligned}\frac{dN_i(t)}{dt} =& rN_i(t)\left[1 - \frac{N_i(t-\tau) + cN_i'(t-\tau)}{K}\right] \\ &- rd_1N_i(t)\left[\frac{N_{i+1}(t-\sigma) - 2N_i(t-\sigma) + N_{i-1}(t-\sigma)}{K}\right], \\ &+ d_2(N_{i+1}(t) - 2N_i(t) + N_{i-1}(t)), \\ &\quad 1 \le i \le n, \quad (\text{mod } n)\end{aligned} \tag{4.2.6}$$

where $N_i(t)$ denotes the population size in the i-th patch, $N_{n+1}(t) = N_1(t)$, $N_0(t) = N_n(t)$, d_1 is the transfer rate at which the dispersion serves as a feedback in the localized per capita growth rate and d_2 is the transfer rate at which the dispersion affects the growth rate in each path. d_1 may be negative and, if that occurs, the dispersion is a positive feedback to the system. The feedback can be delayed and $\sigma \geq 0$ is incorporated to reflect this delay, $d_2 \geq 0$. In Eq. (4.2.6), we have assumed the forword and backward dispersion are the same and the anisotropy of the dispersion is neglected.

Admittdedly, the model (4.2.6) represents a vast simplification of ecological reality. In particular, it still ignores the consequences of structure other than space within the population modelled. This can be age structure, physiological structure, genetic or phenotypic structure. We incorporate a diffusion term in the localized per capita growth rate (i.e. d_1 may not be zero), which is not seen in the literature, by assuming that the dispersion may make a contribution to the local dynamics, at least in the "food-limited" environment situation. This is motivated by a similar consideration in [40] where the population's per capita growth rate is assumed to be a function of a linear combination of the densities of the individual population (called the *weighted total density*) as in the predator-prey or competitive systems. Even though the system (4.2.6) is a much simplified model, as we will see in the subsequent sections, the mathematical analysis of the dynamics of the model is still a problem of formidable complexity. However, the tractable analysis does give some of the implications of the dispersion and delay effect on the oscillation of population growth.

Clearly, if $N(t)$ is a solution of the neutral delayed logistic equation (4.2.5), then $(N(t), N(t), \ldots, N(t))$ is a solution of the system (4.2.6). In particular, $(K, K, \ldots, K)$ is a positive equilibrium of (4.2.6).

Let $x_i(t) = N_i(t) - K$. Eq. (4.2.6) becomes

$$\begin{aligned}\frac{dx_i(t)}{dt} =& - r(x_i(t) + K)\frac{x_i(t-\tau) + cx_i'(t-\tau)}{K} \\ & - \frac{rd_1}{K}(x_i(t) + K)[x_{i+1}(t-\sigma) - 2x_i(t-\sigma) + x_{i-1}(t-\sigma)] \\ & + d_2[x_{i+1}(t) - 2x_i(t) + x_{i-1}], \quad 1 \le i \le n, \quad (\text{mod } n).\end{aligned}$$

Its linearized equation at the origin reads

$$\begin{aligned}\frac{dx_i(t)}{dt} =& - rx_i(t-\tau) - rcx_i'(t-\tau) \\ & - rd_1[x_{i+1}(t-\sigma) - 2x_i(t-\sigma) + x_{i-1}(t-\sigma)] \\ & + d_2[x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)] \\ & \quad 1 \le i \le n, \quad (\text{mod } n).\end{aligned} \tag{4.2.7}$$

From Chapter 3, the characteristic equation of (4.2.7) reads below

$$p(\lambda, \tau, \sigma, c) = \prod_{j=0}^{n-1} p_j(\lambda, \tau, \sigma, c) = 0 \tag{4.2.8}$$

where

$$\begin{aligned}& p_j(\lambda, \tau, \sigma, c) = \lambda + r(1 + \lambda c)e^{-\lambda\tau} - rd_1 a_j e^{-\lambda\sigma} + d_2 a_j \\ & a_j = 4\sin^2\frac{\pi j}{n}, \quad 0 \le j \le n-1.\end{aligned} \tag{4.2.9}$$

We end this section with the following observation.

Proposition 4.2.1. *Let $\phi \in C([-\max(\sigma, \tau), 0], \mathbb{R}^n)$. If $\phi(s) \in \mathbb{R}^n_+$, the positive cone in $\mathbb{R}^n$, for every $s \in [-\max(\sigma, \tau), 0]$, then the solution $(N_i(\phi)(t))$ through $(0, \phi)$ of Eq. (4.2.6) remains in $\mathbb{R}^n_+$ for all $t > 0$.*

Proof. Suppose to the contrary that $(N_i(\phi)(t)) \notin \mathbb{R}^n_+$ for some $t > 0$. Then $\bar{t} = \min\{t > 0 \,|\, N_i(\phi)(t) = 0, 1 \le i \le n\}$ and $i_0 \in \{1, 2, \ldots, n\}$ exist such that

$N_{i_0}(\phi)(\bar{t}) = 0$ and $N'_{i_0}(\phi)(\bar{t}) < 0$ with $N_i(\phi)(\bar{t}) \geq 0$ for all $1 \leq i \leq n$. It follows from (4.2.6) that

$$\frac{dN_{i_0}(\phi)(\bar{t})}{dt} = d_2(N_{i_0+1}(\phi)(t) + N_{i_0-1}(\phi)(\bar{t}) \geq 0,$$

a contradiction to $N'_{i_0}(\phi)(\bar{t}) < 0$. This completes the proof.

4.3. Stability and Hopf bifurcation: Instantaneous feedback

Throughout this section, we assume that the dispersion feedback is instantaneous, i.e. $\sigma = 0$ in the equation (4.2.6).

Recall that in this case, we have the characteristic equation below

$$p(\lambda, \tau, c) = \prod_{j=0}^{n-1} p_j(\lambda, \tau, c) = 0 \tag{4.3.1}$$

where

$$\begin{aligned} p_j(\lambda, \tau, c) &= \lambda + r(1 + \lambda c)e^{-\lambda\tau} - (rd_1 - d_2)a_j \\ a_j &= 4\sin^2\frac{\pi j}{n}, \quad 0 \leq j \leq n-1. \end{aligned} \tag{4.3.2}$$

We first present a result on the local asymptotic stability of the positive equilibrium $(K, K, \ldots, K)$.

Theorem 4.3.1. *The following statements are true:*

(i) If $|rc| > 1$, the positive equilibrium $(K, \ldots, K)$ of (4.2.6) is not stable for all $\tau > 0$;

(ii) If $|rc| < 1$ and there exist two disjoint subsets J_1 and J_2 of $\{0, 1, \ldots, [\frac{n}{2}]\}$ such that $r \leq |(rd_1 - d_2)a_j|$ for all $j \in J_1$ and $r > |(rd_1 - d_2)a_j|$ for

all $j \in J_2$, then $(K,\ldots,K)$ is stable when $\tau < \tau^* = \min_{j\in J_2}\tau_j$, where $\tau_j = \theta_j/w_j$ and

$$\begin{aligned} w_j &= \sqrt{\frac{r^2-(rd_1-d_2)^2a_j^2}{1-r^2c^2}} \\ \theta_j &= \cot^{-1}\left(\frac{(rd_1-d_2)a_j-cw_j^2}{w_j(1+c(rd_1-d_2)a_j)}\right), \quad 0\le j\le [\frac{n}{2}]; \end{aligned} \tag{4.3.3}$$

(iii) If $|rc| < 1$ and $r > |(rd_1-d_2)a_j|$ for some $j \in \{0,1,\ldots,[\frac{n}{2}]\}$, then $(K,\ldots,K)$ is not stable if $\tau > \tau_j$, where τ_j is given in (ii).

Proof. Note that the characteristic equation at $(K,\ldots,K)$ has the forms (4.3.1)-(4.3.2). The conclusions follow directly from Theorem 4.2 of Kuang [24, Chapter 1] and Theorem 3.1 of Freedman and Kuang [10].

Remark 4.3.1. If $d_1 = 0$ (i.e. there is no feedback in the local dynamics) and $c = 0$ (i.e. no neutral term), then we have from (4.3.3)

$$w_j = \sqrt{r^2-d_2^2a_j^2}, \qquad \theta_j = \cot^{-1}\left[-\frac{d_2a_j}{\sqrt{r^2-d_2^2a_j^2}}\right]$$

and hence $\theta_j/w_j \ge \frac{\pi}{2r}$. By (ii), $(K,\ldots,K)$ is stable if $r\tau < \frac{\pi}{2}$. This implies that for the delay logistic system with discrete diffusion, dispersion can not change the stability of the local dynamics. This generalizes a result in [22].

Remark 4.3.2. Let $b_j = (rd_1-d_2)a_j$, $J_1 = \{1,2,\ldots,[\frac{n}{2}]\}$ and $J_2 = \{0\}$. It follows from (ii) that if

$$r \le |b_j| \quad \text{for} \quad 1\le j\le [\frac{n}{2}], \tag{4.3.4}$$

$$\tau \le \tau_0 = \frac{\sqrt{1-r^2c^2}}{r} \cot^{-1}\left(-\frac{rc}{\sqrt{1-r^2c^2}}\right), \tag{4.3.5}$$

then $(K,\ldots,K)$ is stable. Note that (4.3.5) allows us to choose $r,\ c > 0$ so that we still have the stability in case $r\tau > \frac{\pi}{2}$. We know this is impossible when $c = 0$ (i.e. the neutral term does not appear). (4.3.4) can be satisfied by increasing the dispersals. Therefore, the dispersals as well as the neutral term here exhibit a stabilizing influence on the population dynamics (see also Gopalsamy and Zhang [15] and Kuang [24]).

From Theorem 4.3.1, if $|rc| > 1$, $(K,\ldots,K)$ is always unstable. In what follows, we therefore assume $rc < 1$.

We fix a, r, d_1 and d_2, regard the delay τ as a parameter and consider the Hopf bifurcation in the equation (4.2.6). We find that when the dispersion is small, there are phase-locked oscillations on the population growth, as the following theorem shows.

Theorem 4.3.2. *Assume that* $\sigma = 0$, $rc < 1$ *and* $|(rd_1 - d_2)a_j| < r$ *for some* $j \in \{0, 1, \ldots, [\frac{n}{2}]\}$. *Let*

$$\beta_j = \sqrt{\frac{r^2 - (rd_1 - d_2)^2 a_j^2}{1 - r^2c^2}}, \tag{4.3.6}$$

$$\tau_j = \beta_j^{-1} \cot^{-1} \left[\frac{(rd_1 - d_2)a_j - c\beta_j^2}{\beta_j(1 + c(rd_1 - d_2)a_j)} \right]. \tag{4.3.7}$$

Then $\tau = \tau_j$ *is a Hopf bifurcation point of phase-locked osciallations for equation (4.2.6). More precisely, there exist a sequence of* p_k*-periodic solutions* $N^k(t)$ *of (4.2.6) with* $\tau = \tau^k$ *such that*

$$\tau^k \to \tau_j, \quad p_k \to 2\pi\beta_j^{-1}, \quad N^k(t) \to (K,\ldots,K)$$

uniformly for $t \in \mathbf{R}$ *as* $k \to \infty$ *and*

$$N_{i-1}{}^k(t) = N_i{}^k(t - p_k \frac{j}{n}), \quad 1 \leq i \leq n, \quad (\text{mod } n), \quad k \geq 1, \quad t \in \mathbf{R}.$$

Proof. Recall that $p_j(\lambda) = \lambda + (r + rc\lambda)e^{-\lambda\tau} - (rd_1 - d_2)a_j$. Let $\lambda = i\beta$, $\beta > 0$, and set $p_j(i\beta) = 0$. Separating the real parts and imaginary parts gives

$$\begin{cases} r\cos\beta\tau + rc\beta\sin\beta\tau = (rd_1 - d_2)a_j \\ rc\beta\cos\beta\tau - r\sin\beta\tau = -\beta. \end{cases} \tag{4.3.8}$$

Squaring them and solving for $\cos\beta\tau$ in (4.3.8) yeild

$$\begin{cases} r^2 + r^2c^2\beta^2 = (rd_1 - d_2)^2 a_j{}^2 + \beta^2 \\ \sin\beta\tau = \dfrac{\beta(r + rc(rd_1 - d_2)a_j)}{r^2(1 + c^2\beta^2)} \\ \cos\beta\tau = -\dfrac{rc\beta^2 - r(rd_1 - d_2)a_j}{r^2(1 + c^2\beta^2)}. \end{cases} \tag{4.3.9}$$

Therefore, by (4.3.6)

$$\beta = \beta_j = \sqrt{\frac{r^2 - (rd_1 - d_2)^2 a_j{}^2}{1 - r^2c^2}}$$

is such that $p_j(i\beta_j) = 0$ and solving for τ in (4.3.9) gives τ_j in (4.3.7).

On the other hand, differentiating $p_j(\lambda) = 0$ with respect to τ, we get

$$\frac{d\lambda}{d\tau} = \frac{\lambda(rc\lambda + r)e^{-\lambda\tau}}{1 + [rc - \tau(rc\lambda + r)]e^{-\lambda\tau}}. \tag{4.3.10}$$

Note that $(r + rc\lambda)e^{-\lambda\tau} = (rd_1 - d_2)a_j - \lambda$. It follows form (4.3.10) and (4.3.9) that

$$\begin{aligned}
&\operatorname{Sign}\left\{\frac{d}{d\tau}(\operatorname{Re}\lambda)\right\}\Big|_{\substack{\lambda=i\beta_j\\ \tau=\tau_j}} \\
&= \operatorname{Sign}\left\{\operatorname{Re}\Big(\frac{d\lambda}{d\tau}\Big)^{-1}\right\}\Big|_{\substack{\lambda=i\beta_j\\ \tau=\tau_j}} \\
&= \operatorname{Sign}\left\{\operatorname{Re}\Big[\frac{e^{\lambda\tau}+rc}{\lambda(rc\lambda+r)} - \frac{\tau}{\lambda}\Big]\right\}\Big|_{\substack{\lambda=i\beta_j\\ \tau=\tau_j}} \\
&= \operatorname{Sign}\left\{\operatorname{Re}\frac{e^{\lambda\tau}}{\lambda(rc\lambda+r)} + \operatorname{Re}\frac{rc}{\lambda(rc\lambda+r)}\right\}\Big|_{\lambda=i\beta_j} \\
&= \operatorname{Sign}\left\{\operatorname{Re}\frac{1}{\lambda((rd_1-d_2)a_j-\lambda)} + \operatorname{Re}\frac{rc}{\lambda(rc\lambda+r)}\right\}\Big|_{\lambda=i\beta_j} \\
&= \operatorname{Sign}\left\{\operatorname{Re}\frac{-i[(rd_1-d_2)a_j+i\beta_j]}{\beta_j[(rd_1-d_2)^2a_j{}^2+\beta_j{}^2]} + \operatorname{Re}\frac{-rci(r-irc\beta_j)}{\beta_j[r^2+r^2c^2\beta_j{}^2]}\right\} \\
&= \operatorname{Sign}\left\{\frac{1}{(rd_1-d_2)^2a_j{}^2+\beta_j{}^2} - \frac{r^2c^2}{r^2+r^2c^2\beta_j{}^2}\right\} \\
&= \operatorname{Sign}\left\{\frac{1-r^2c^2}{r^2+r^2c^2\beta_j{}^2}\right\} = 1 > 0.
\end{aligned} \tag{4.3.11}$$

This implies that $\deg_B(p_j(\cdot,\tau_j+\varepsilon),\Omega) \neq \deg_B(p_j(\cdot,\tau_j-\varepsilon),\Omega)$ for some small $\varepsilon > 0$, where $\Omega = \{x+iy;\ 0 < x < \delta,\ \beta_j - \delta < y < \beta_j + \delta\}$ and $\delta > 0$ is a sufficiently small number. Consequently, the theorem follows from Theorem 3.5.4.

Let $d_1 = d_2 = 0$ in (4.2.6). Then all local dynamics are identical, and are described by the neutral delay logistic equation (4.2.5). Recall that

$$\tau_0 = \frac{\sqrt{1-r^2c^2}}{r}\cot^{-1}\left(-\frac{cr}{\sqrt{1-r^2c^2}}\right).$$

It follows from Throrem 4.3.2 that $\tau = \tau_0$ is a local Hopf bifurcation point for equation (4.2.5). This leads to a branch of *synchronous oscillations* in (4.2.6).

Corollary 4.3.3. *Assume that* $\sigma = 0$ *and* $0 \leq rc < 1$. *Then* $\tau = \tau_0$ *is a Hopf bifurcation point of synchronous oscillations for Eq. (4.2.6). More precisely, there exsits a sequence of* p_k*-periodic solutions* $N^k(t) = (n^k(t), \ldots, n^k(t))$ *of (4.2.6) with* $\tau = \tau^k$ *such that* $n^k(t)$ *is a* p_k*-periodic solution for (4.2.5) and*

$$\tau^k \to \tau_0, \quad p_k \to \frac{2\pi\sqrt{1 - r^2c^2}}{r}, \quad n^k(t) \to K$$

uniformly for $t \in \mathbb{R}$ *as* $k \to \infty$.

Proof. Note that $\beta_0 = r/\sqrt{1 - r^2c^2}$. It directly follows from Theorem 4.3.2 by letting $d_1 = d_2 = 0$.

Note that $a_j = 4\sin^2 \frac{\pi j}{n} \geq 0$. If $d_1/d_2 > \frac{1}{r}$, then $(rd_1 - d_2)a_j$ increases with j. This implies that if $(rd_1 - d_2)a_j < r$ for some $j \in \{1, \ldots, [\frac{n}{2}]\}$, then $(rd_1 - d_2)a_l < r$ for all $0 \leq l \leq j$. By Theorem 4.3.2, we have also bifurcation points τ_l other than τ_j, where $1 \leq l < j$, τ_j is given by (4.3.6)-(4.3.7) with j replaced by l. Moreover, the crossing number at each bifurcation point is always -1. This implies the following simple observation.

Theorem 4.3.4. *Assume* $\sigma = 0$ *and* $0 \leq rc < 1$. *If* $|(rd_1 - d_2)a_j| < r$ *for some* $j \in \{1, \ldots, [\frac{n}{2}]\}$. *Then there undergoes a global Hopf bifurcation at each* $\tau = \tau_l$, $1 \leq l \leq j$, *such that every branch of phase-locked solutions of (4.2.6) in* $C^1(S^1; \mathbb{R}^n) \times (0, \infty) \times (0, \infty)$, *bifurcating from each* $(0, \tau_l, w_l)$ *with* τ_l *defined by (4.3.7) and* $w_l = \frac{2\pi}{\beta_l}$, *does not terminate at* $(0, \tau_m, \beta_m)$ *with* $m \neq l$.

Proof. Since all crossing numbers are of the same sign, from Theorem 3.5.5, our conclusion follows.

4.4. Stability and Hopf bifurcation: Delayed feedback

In this section we consider the case where the dispersion feedback in the local dynamics is delayed, i.e. $\sigma > 0$. We regard σ as a parameter.

Note that in this case, we have

$$p_j(\lambda,\sigma) = \lambda + r(1+\lambda c)e^{-\lambda\tau} - rd_1a_je^{-\lambda\sigma} + d_2a_j \tag{4.4.1}$$

where a_j is as in (4.3.2).

We obtain the following result on the asymptotic stability of $(K,\ldots,K)$.

Theorem 4.4.1. *Let (iii) in Theorem 4.3.1 hold. If $0 < |rd_1| < d_2$ and the equation*

$$\begin{aligned}(r^2d_1^2 - d_2^2)a_j{}^2 - r^2 =& y[(1+r^2c^2)y + 2cry\cos y\tau - 2r\sin\tau y] \\ &+ 2d_2a_j(r\cos y\tau + yrc\sin y\tau)\end{aligned} \tag{4.4.2}$$

has no positive solution y for every $j \in \{1,\ldots,[\frac{n}{2}]\}$, then $(K,\ldots,K)$ is asymptotically stable for (4.2.6) with any $\sigma \geq 0$ and $\tau < \tau^*$, where τ^* is given by (ii) of Theorem 4.3.1.

Proof. We show that for every $\sigma \geq 0$, all roots of $p_j(\lambda,\tau)$ have negative real parts. To see this, let $P(\lambda) = \lambda + r(1+\lambda c)e^{-\lambda\tau} + d_2a_j$ and $Q(\lambda) = -rd_1a_j$, $1 \leq j \leq [\frac{n}{2}]$. We have

(i) $\overline{P(-iy)} = P(iy)$ and $\overline{Q(-iy)} = Q(iy)$ for every real y;

(ii) $P(0) + Q(0) = r - (rd_1 - d_2)a_j > 0$ since $|rd_1| < d_2$;

(iii) $p_j(\lambda, 0) = \lambda + r(1+\lambda c)e^{-\lambda\tau} - (rd_1 - d_2)a_j$ has all roots of negative real parts for $\tau < \tau^*$ by Theorem 4.3.1;

(iv) $F(y) = |P(iy)|^2 - |Q(iy)|^2$ has neither positive nor negative zeros, since the right-hand side of (4.4.2) is an even function of y.

Note that $Q(\lambda)$ is a non-zero constant funtion. $Q(\lambda)$ and $P(\lambda)$ have no common imaginary zeros. By a result of Cooke and van den Driessche [3] (see also Freedman and Kuang [10]), $p_j(\lambda, \tau)$ has only roots with negative real parts when $\tau < \tau^*$ and the asymptotic stability of $(K, \dots, K)$ follows.

Remark 4.4.1. The assumption that the equation (4.4.2) has no positive solutions seems a complicated condition. However, we can show that if

$$\tau d_2 a_j(\tau - 2c) + 2\tau \leq \frac{(1-rc)^2}{r}, \quad cd_2 a_j < 1, \tag{4.4.3}$$

then (4.4.2) has no positive solutions. Indeed, under (4.4.3) and $|rd_1| < d_2$, for all $y > 0$

$$\begin{aligned} y[(1+c^2r^2)y &+ 2ycr\cos y\tau + (2d_2a_jrc(-2r)\sin \tau y] + 2d_2a_jr\cos y\tau \\ &\geq y^2[1 + c^2r^2 - 2rc + (2d_2a_jrc - 2r)\tau] + 2rd_2a_j\cos y\tau \\ &= y^2[(1-rc)^2 + 2r\tau(cd_2a_j - 1)] + 2rd_2a_j\cos y\tau \\ &> 2rd_2a_j \geq 0, \end{aligned}$$

while the left hand side of (4.4.2) is negative, implying that (4.4.2) has no positive solutions. Theorem 4.4.1 tells us that the delay in the dispersion feedback has no destablizing influence under certain conditions as shown in Theorem 4.4.1.

To obtain Hopf bifurcation, we now assume the equation (4.4.2) has at least one positive solution for some $j \in \{1, \dots, [\frac{n}{2}]\}$. Let us denote this solution by y_j. Then

$$(r\cos y_j\tau + y_jrc\sin y_j\tau + d_2a_j)^2 + (y_j - r\sin y_j\tau + y_jrc\cos y_j\tau)^2 = r^2{d_1}^2{a_j}^2$$

This implies that there exists a unique $\theta_j \in (0, 2\pi]$ such that

$$\begin{cases} d_2 a_j + r\cos y_j\tau + y_j rc\sin y_j\tau = rd_1 a_j \cos\theta_j \\ y_j - r\sin y_j\tau + y_j rc\cos y_j\tau = -rd_1 a_j \sin\theta_j. \end{cases} \tag{4.4.4}$$

Set

$$\sigma_j = \frac{\theta_j}{y_j}. \tag{4.4.5}$$

It follows that from (4.4.4), $p_j(iy_j, \sigma_j) = 0$, i.e. iy_j is a purely imaginary root of $p_j(\lambda, \sigma)$ with $\sigma = \sigma_j$. This leads us to the following Hopf bifurcation of phase-locked oscillations.

Theorem 4.4.2. *Let there exist* $j \in \{1, \ldots, [\frac{n}{2}]\}$ *such that (4.4.2) has a positive solution* y_j. *Let* y_j *satisfy*

$$(1+r^2c^2)y_j - r(1-cy_j{}^2 + (c-1)d_2a_j)\sin y_j\tau \neq r[1-2c-cd_2a_j]y_j\cos y_j\tau. \tag{4.4.6}$$

Then σ_j *is a Hopf bifurcation point of phase-locked oscillations, where* σ_j *is defined by (4.4.5).*

Proof. By Theorem 3.5.4 and the discussion preceeding this theorem, we need only to check that

$$\frac{d}{d\sigma}(\text{Re}\lambda)\bigg|_{\substack{\lambda=iy_j \\ \sigma=\sigma_j}} \neq 0$$

where λ is a root of $p_j(\lambda, \sigma) = 0$.

To see this, let us differentiate $p_j(\lambda, \sigma) = 0$ with respect to σ (by viewing λ as a function of σ). It follows that

$$\frac{d\lambda}{d\sigma} = \frac{-\lambda rd_1a_je^{-\lambda\sigma}}{1 + rce^{-\lambda\tau} - r(1+\lambda c)e^{-\lambda\tau} + \sigma rd_1a_je^{-\lambda\sigma}}. \tag{4.4.7}$$

Note that $p_j(\lambda, \sigma) = 0$ is equivalent to

$$r(1 + \lambda c)e^{-\lambda\tau} = -(\lambda + d_2 a_j) + r d_1 a_j e^{-\lambda\sigma}. \qquad (4.4.8)$$

Combining (4.4.7) and (4.4.8), one obtains that

$$\begin{aligned}(\frac{d\lambda}{d\sigma})^{-1} &= \frac{1 + rce^{-\lambda\tau} + (\lambda + d_2 a_j) - r d_1 a_j e^{-\lambda\sigma} + \sigma r d_1 a_j e^{-\lambda\sigma}}{-\lambda r d_1 a_j e^{-\lambda\sigma}} \\ &= \frac{(1 + d_2 a_j + \lambda) + rce^{-\lambda\tau}}{-\lambda r d_1 a_j e^{-\lambda\sigma}} + \frac{1 - \sigma}{\lambda}.\end{aligned} \qquad (4.4.9)$$

Therefore, from (4.4.9),

$$\begin{aligned}&\operatorname{Sign}\left(\frac{d}{d\sigma}(\mathrm{Re}\lambda)\right)\Big|_{\substack{\lambda=iy_j\\ \sigma=\sigma_j}} = \operatorname{Sign}\left(\mathrm{Re}\frac{d\lambda}{d\sigma}\right)\Big|_{\substack{\lambda=iy_j\\ \sigma=\sigma_j}} \\ &= \operatorname{Sign}\left(\mathrm{Re}(\frac{d\lambda}{d\sigma})^{-1}\right)\Big|_{\substack{\lambda=iy_j\\ \sigma=\sigma_j}} \\ &= \operatorname{Sign}\left(\mathrm{Re}\left\{\frac{(1 + d_2 a_j + \lambda) + rce^{-\lambda\tau}}{-\lambda r d_2 a_j e^{-\lambda\sigma}} + \frac{1-\sigma}{\lambda}\right\}\right)\Big|_{\substack{\lambda=iy_j\\ \sigma=\sigma_j}} \\ &= \operatorname{Sign}\left(\mathrm{Re}\frac{1 + d_2 a_j + \lambda + rce^{-\lambda\tau}}{-\lambda r d_2 a_j e^{-\lambda\tau}}\right)\Big|_{\substack{\lambda=iy_j\\ \sigma=\sigma_j}} \\ &= \operatorname{Sign}\left(\mathrm{Re}\frac{1 + d_2 a_j + iy_j + rce^{-iy_j\tau}}{-iy_j r d_1 a_j e^{-i\theta_j}}\right) \\ &= \operatorname{Sign}\left\{-\mathrm{Re}\frac{(1 + d_2 a_j)e^{i\theta_j}}{iy_j r d_1 a_j} - \mathrm{Re}\frac{rce^{(\theta_j - y_j\tau)i}}{iy_j r d_1 a_j e^{i\theta_j}} - \mathrm{Re}\frac{1}{r d_1 a_j}e^{i\theta_j}\right\} \\ &= \operatorname{Sign}\left\{\mathrm{Re}\frac{i(1 + d_2 a_j)e^{i\theta_j}}{y_j r d_1 a_j} + \mathrm{Re}\frac{irce^{(\theta_j - y_j\tau)i}}{y_j r d_1 a_j} - \frac{\cos\theta_j}{r d_1 a_j}\right\} \\ &= \operatorname{Sign}\left\{-\frac{(1 + d_2 a_j)\sin\theta_j - rc\sin(\theta_j - y_j\tau) - y_j\cos\theta_j}{y_j r d_1 a_j}\right\} \neq 0,\end{aligned}$$

whenever

$$(1 + d_2 a_j)\sin\theta_j \neq rc\sin(\theta_j - y_j\tau) - y_j\cos\theta_j. \qquad (4.4.10)$$

A direct calculation, by noting that θ_j satisfies (4.4.4), shows that (4.4.6) and (4.4.10) are equivalent. This proves the theorem.

Remark 4.4.2. The condition given by (4.4.2) and (4.4.6) are usually difficult to verify. However, we do have some solutions $y_j > 0$ to (4.4.2) and (4.4.6) in some special cases. Take $c = 0$ and $d_2 = 0$, for example. Then (4.4.2) and (4.4.6) simplify to

$$r^2({d_1}^2{a_j}^2 - 1) = y(y - 2r\sin y\tau) \tag{4.4.11}$$

$$y_j - r\sin y_j\tau \neq ry_j\cos y_j\tau. \tag{4.4.12}$$

Define $f(y) = \frac{r^2({d_1}^2{a_j}^2-1)}{y}$ and $g(y) = y - 2r\sin y\tau$, $y > 0$. If ${d_1}^2{a_j}^2 > 1$ and $2r\tau \leq 1$. Then $f(y)$ is decreasing and $g(y)$ is increasing. It follows that there exists a unique $y_j > 0$ such that $f(y_j) = g(y_j)$. This $y_j > 0$ gives a positive solution to (4.4.11).

On the other hand, note that if $r \leq \frac{1}{2}$,

$$\frac{r}{1 - r\cos y_j\tau} \leq \frac{r}{1-r} \leq 1 \quad \text{and} \quad \frac{y_j}{\sin y_j} > 1.$$

So y_j satisfies (4.4.12). Therefore, if

$${d_1}^2{a_j}^2 > 1, \quad 2r\tau \leq 1 \quad \text{and} \quad r \leq \frac{1}{2}$$

then $y_i > 0$ exists such that both (4.4.11) and (4.4.12) are verified. For the general case, we may use the computer to verify (4.4.2) and (4.4.6).

4.5. Global Hopf bifurcation: Neutral term effect

We now consider the global aspects of phase-locked as well as synchronous oscillations in the system (4.2.6). For simplicity, we only deal with the case where $\sigma = 0$.

In order to examine local bifurcation points, we shall regard $\alpha = rc < 1$ as a parameter. Recall that we have the j-th characteristic equation as follows

$$\begin{aligned} p_j(\lambda, \alpha) &= \lambda + (r + \alpha\lambda)e^{-\lambda\tau} - b_j = 0, \\ b_j &= (rd_1 - d_2)a_j, \quad j \in \{0, 1, 2, \dots, [\frac{n}{2}]\}. \end{aligned} \tag{4.5.1}$$

We look for purely imaginary roots of (4.5.1) for a fixed $1 \le j \le [\frac{n}{2}]$.

Let $\lambda = i\beta$, $\beta > 0$, be a root of $p_j(\lambda, \alpha)$, i.e. $p_j(i\beta, \alpha) = 0$. Separating the real and imaginary parts, respectively, we get

$$\begin{aligned} b_j \cos\beta\tau + \beta \sin\beta\tau &= r \\ \beta\cos\beta\tau - b_j \sin\beta\tau &= -\alpha\beta. \end{aligned} \tag{4.5.2}$$

Squaring both sides of (4.5.2) and adding them yield

$$\alpha = \frac{\sqrt{\beta^2 + b_j{}^2 - r^2}}{\beta} < 1 \tag{4.5.3}$$

if $b_j{}^2 < r^2$. Also, solving for $\cos\beta\tau$ and $\sin\beta\tau$ in (4.5.2) gives us

$$\tan\beta\tau = -\frac{\beta(r + \alpha b_j)}{\alpha\beta^2 - rb_j}. \tag{4.5.4}$$

Substituting (4.5.3) into (4.5.4), we then obtain

$$\tan\beta\tau = \frac{\beta r + b_j\sqrt{\beta^2 + b_j{}^2 - r^2}}{rb_j - \beta\sqrt{\beta^2 + b_j{}^2 - r^2}}. \tag{4.5.5}$$

Let

$$f(\beta) = \frac{\beta r + b_j\sqrt{\beta^2 + b_j{}^2 - r^2}}{rb_j - \beta\sqrt{\beta^2 + b_j{}^2 - r^2}}, \quad \beta \ge \sqrt{r^2 - b_j{}^2}.$$

Assume $b_j > 0$. It follows that $f'(\beta) > 0$ for all $\beta \ge \sqrt{r^2 - b_j{}^2}$ and hence $z = f(\beta)$ is an increasing function. Note that $f(\sqrt{r^2 - b_j{}^2}) = \sqrt{r^2 - b_j{}^2}/b_j > 0$ and $\lim_{\beta\to\infty} f(\beta) = 0$, $\lim_{\beta\to r^-} f(\beta) = +\infty$ and $\lim_{\beta\to r^+} f(\beta) = -\infty$. We have infinitely many solutions for β to the equation (4.5.5), which correspond to the β-coordinates of the intersection points of two graphs $z = \tan \beta\tau$ and $z = f(\beta)$.

Let $0 < r\tau < \frac{\pi}{2}$. Then we have sulotions β_m to Eq. (4.5.5) as follows

$$\frac{(2m-1)\pi}{2\tau} < \beta_m < \frac{m\pi}{\tau}, \quad m = 1, 2, 3, \dots. \tag{4.5.6}$$

More generally, if there is a positive integer N such that $\frac{(N-1)\pi}{\tau} < r \leq \frac{(2N-1)\pi}{2\tau}$, then

$$\frac{[2(N+m)-3]\pi}{2\tau} < \beta_m < \frac{[N+m-2]\pi}{\tau}, \quad m = 1, 2, \dots \tag{4.5.7}$$

exist as solutions to Eq. (4.5.4) and if $\frac{(N-q-1)\pi}{\tau} < \sqrt{r^2 - {b_j}^2} \leq \frac{(N-q)\pi}{\tau}$ for some positive integer q, then

$$\frac{(N-l-1)\pi}{\tau} < \beta_{-l} < \frac{(2N-2l-1)\pi}{2\tau}, \quad 1 \leq l \leq q \tag{4.5.8}$$

also exsit as solutions other than β_m to the equation (4.5.6). Thus, we get α values from (4.5.3) as follows

$$\alpha_m = \frac{\sqrt{{\beta_m}^2 + {b_j}^2 - r^2}}{\beta_m}, \quad m = -q, -q+1, \dots, -1, 1, 2, \dots. \tag{4.5.9}$$

This leads us to the following local bifurcation result.

Theorem 4.5.1. *Assume* $0 < b_j < r$ *for some* $j \in \{1, 2, \dots, [\frac{n}{2}]\}$. *If*

$$(N-1)\pi < r\tau \leq \frac{(2N-1)\pi}{2} \tag{4.5.10}$$

for some integer $N > 0$, *then* $(0, \alpha_m, \beta_m)$, $m = -q, -q+1, \dots, -1, 1, 2, \dots,$ *with* $\alpha_m < 1$ *are all local bifurcation points of phase-locked oscillations for (4.2.6).*

Proof. It suffices to show that

$$\frac{d}{d\sigma}\text{Re}\lambda\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} \neq 0$$

for each m. To see this, we differentiate two sides of $(\lambda - b_j)e^{\lambda\tau} + r + \alpha\lambda = 0$ with respect to α and get

$$\frac{d\lambda}{d\alpha} = -\frac{\lambda}{e^{\lambda\tau} + (\lambda - b_j)\tau e^{\lambda\tau} + \alpha}. \tag{4.5.11}$$

Setting $p_j(\lambda, \alpha) = 0$ implies that $(\lambda - b_j)e^{\lambda\tau} = -r - \alpha\lambda$. Substituting this into (4.5.11) gives

$$(\frac{d\lambda}{d\alpha})^{-1} = -\frac{e^{\lambda\tau}}{\lambda} - \frac{\alpha - r\tau}{\lambda} + \alpha\tau. \tag{4.5.12}$$

Consequently, using (4.5.2) in the last step, we have

$$\begin{aligned}
&\text{Sign}\,\frac{d}{d\alpha}(\text{Re}\,\lambda)\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} \\
&= \text{Sign}\,\text{Re}\frac{d\lambda}{d\alpha}\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} = \text{Sign}\,\text{Re}((\frac{d\lambda}{d\alpha})^{-1})\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} \\
&= \text{Sign}\,\text{Re}\left(-\frac{e^{\lambda\tau}}{\lambda} + \alpha\tau - \frac{\alpha - r\tau}{\lambda}\right)\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} = \text{Sign}\left(\alpha\tau - \text{Re}\frac{e^{\lambda\tau}}{\lambda}\right)\Big|_{\substack{\lambda=i\beta_m\\ \alpha=\alpha_m}} \\
&= \text{Sign}\left(\alpha_m\tau - \frac{\sin\beta_m\tau}{\beta_m}\right) = \text{Sign}\left(\frac{\alpha_m b_j + r}{b_j{}^2 + \beta_m{}^2} + \alpha_m\tau\right) = 1 \neq 0,
\end{aligned} \tag{4.5.13}$$

as desired. This completes the proof.

To study the global Hopf bifurcation, we choose any $0 < k < 1$ and let $|\alpha| < k$ and investigate the equation on the region $D := \{x \in \mathbb{R}^n_+;\ 0 < |x| < \frac{K}{k}\}$, where $|x| = \max_{1\leq i\leq n}\{|x_i|\}$ for $x \in \mathbb{R}^n$.

We need the following lemma concerning periods of periodic solutions to Eq. (4.2.6).

Lemma 4.5.2. *For any integer* $m > 0$, *the equation (4.2.6) has no nonconstant* $\frac{2\tau}{m}$*-periodic positive solution* $\{x_i(t)\}_{i=1}^n$ *with* $x_{i-1}(t) = x_i(t - \frac{\tau}{m})$ *in* D.

Proof. It suffices to show that the lemma holds for $m = 1, 2$. In the following we only give the proof for $m = 1$. The case $m = 2$ can be treated analogously.

By the way of contradiction, we suppose that $x(t) = \{x_i(t)\}_{i=1}^n$ is a nonconstant 2τ-periodic positive solution of (4.2.6) with $x_{i-1}(t) = x_i(t - \tau)$. Then $x_{i+1}(t) = x_i(t - \tau)$, $x_{i+1}(t - \tau) = x_i(t)$ and $x_{i-1}(t - \tau) = x_i(t)$. Let $y_i(t) = x_i(t - \tau)$. We have

$$x_{i+1}(t) - 2x_i(t) + x_{i-1}(t) = 2(y_i(t) - x_i(t))$$

and

$$\begin{cases} x_i' = rx_i\left[1 - \dfrac{y_i + 2d_1(y_i - x_i)}{K} + c(1 - \dfrac{y_i}{K})'\right] + 2d_2(y_i - x_i) \\ y_i' = ry_i\left[1 - \dfrac{x_i + 2d_1(x_i - y_i)}{K} + c(1 - \dfrac{x_i}{K})'\right] + 2d_2(x_i - y_i). \end{cases} \tag{4.5.14}$$

Put

$$u(t) = 1 - \frac{x_i(t)}{K} \qquad \text{and} \qquad v(t) = 1 - \frac{y_i(t)}{K}$$

and $\delta_i = 2d_i$, $i = 1, 2$. The equation (4.5.14) becomes an implicit differential equation of u and v

$$\begin{cases} u' = r(u-1)[v - \delta_1(u - v) + cv'] + \delta_2 K(u - v) \\ v' = r(v-1)[u - \delta_1(v - u) + cu'] + \delta_2 K(v - u) \end{cases}$$

which, by solving for u' and v', leads to an ordinary differential system below

$$\begin{cases} u' = \dfrac{r(u-1)[f(u,v) + cr(v-1)f(v,u) + g(v,u)] + g(u,v)}{1 - r^2c^2(u-1)(v-1)} \\ v' = \dfrac{r(v-1)[f(v,u) + cr(u-1)f(u,v) + g(u,v)] + g(v,u)}{1 - r^2c^2(u-1)(v-1)} \end{cases} \tag{4.5.15}$$

where

$$f(u,v) = v - \delta_1(u-v), \quad g(u,v) = \delta_2 K(u-v),$$
$$(u,v) \in \{(u,v) \in \mathbb{R}^2;\ |x-1| < 1/k,\ |y-1| < 1/k\}.$$

Note that (4.5.15) is symmetric about u and v. The diagonal $\Delta \cong \{(u,v) \in \mathbb{R}^2;\ |u-1| < 1/k, u = v\}$ is invariant under the system (4.5.15) of ordinary differntial equations. Since any autonomous one-dimensional ordinary differential equation has no nonconstant periodic solution, $(u(t), v(t)) \notin \Delta$ for all t. Without the loss of generality, we assume that

$$u(t) < v(t) \qquad \text{for all } t \in \mathbb{R}. \tag{4.5.16}$$

Replacing t by $t-\tau$ in (4.5.16), we get

$$u(t-\tau) < v(t-\tau) \qquad \text{for all } t \in \mathbb{R}. \tag{4.5.17}$$

On the other hand, we have

$$v(t-\tau) = 1 - \frac{y_i(t-\tau)}{k} = 1 - \frac{x_i(t)}{k} = u(t),$$
$$u(t-\tau) = 1 - \frac{x_i(t-\tau)}{k} = 1 - \frac{y_i(t)}{k} = v(t).$$

Consequently, (4.5.17) implies that $v(t) < u(t)$ for all $t \in \mathbb{R}$, which contradicts to (4.5.16). The proof is therefore completed.

We now state and prove the following global bifurcation theorem.

Theorem 4.5.3. *Assume n is even and $0 < (rd_1 - d_2) < \frac{r}{4}$. Suppose that there exists a positive integer N satisfying (4.5.10). Let q be an integer such that*

$$\frac{(N-q-1)\pi}{\tau} < \sqrt{r^2 - b_{\frac{n}{2}}{}^2} \le \frac{(N-q)\pi}{\tau}.$$

Then for each integer $-q \le m \le 1 - N + \frac{\tau}{\pi}\sqrt{\frac{r^2 - b_{\frac{n}{2}}^2}{1-k^2}}$, *at least one of the following conclusions holds:*

(i) *For any* $c \in (0, \frac{\alpha_m}{r})$, *the system (4.2.6) has a p-periodic solution* $\{N_i(t)\}_{i=1}^n$ *with period* $p \in \left[\frac{\tau}{N+m-1}, \frac{2\tau}{2(N+m)-3}\right]$ *and satisfying*

$$N_{i-1}(t) = N_i(t - \tfrac{p}{2}), \quad i = 1, 2, \ldots, n;$$

(ii) *For any* $c \in (\frac{\alpha_m}{r}, \frac{k}{r})$, *the conclusion in (i) holds;*

(iii) *For any* $A \in (0, K)$, *there exists a* $c_A > 0$ *and a p-periodic solution* $\{N_i(t)\}_{i=1}^n$ *to (4.2.6) with* $c = c_A$, *with period* $p > 0$ *as in (i) and satisfying*

$$N_{i-1}(t) = N_i(t - \tfrac{p}{2}), \quad \max_{\substack{1 \le i \le n \\ t \in \mathbb{R}}} |N_i(t)| = A, \quad i = 1, 2, \ldots, n;$$

(iv) *For any* $A \in (K, \frac{K}{k})$, *the conclusion in (iii) holds;*

where α_m *is given by (4.5.9).*

Proof. We choose $j = \frac{n}{2}$. Then by (4.3.2), $a_{\frac{n}{2}} = 4$ and $b_j = 4(rd_1 - d_2) < r$. It follows that, from (4.5.3) and (4.5.7)–(4.5.8), β_m and β_{-l} exist to Eq. (4.5.5), where $1 \le l \le q$ and

$$m \le 1 - N\frac{\tau}{\pi}\sqrt{\frac{r^2 - b_{\frac{n}{2}}^2}{1 - k^2}}$$

and the locations of β_m, β_{-l} are estimated by (4.5.7) and (4.5.8). Let α_m be any α_m given by (4.5.9). By Theorem 4.5.1, $(0, \alpha_m, \beta_m)$ is a bifurcation point with $\frac{n}{2}$-th crossing number $\gamma_{\frac{n}{2}}(\alpha_m, \beta_m) < 0$. Consequently, the assertion of Theorem

For any $c \in (0, \frac{\alpha_m}{r})$, *the system (4.2.6) has a p-periodic solution* $\{N_i(t)\}_{i=1}^n$ *with period* $p \in \left[\frac{\tau}{N+m-1}, \frac{2\tau}{2(N+m)-3}\right]$ *and satisfying*

$$N_{i-1}(t) = N_i(t - \tfrac{p}{2}), \quad i = 1, 2, \ldots, n;$$

For any $c \in (\frac{\alpha_m}{r}, \frac{k}{r})$, *the conclusion in (i) holds;*

For any $A \in (0, K)$, *there exists a* $c_A > 0$ *and a p-periodic solution* $\{N_i(t)\}_{i=1}^n$ *to (4.2.6) with* $c = c_A$, *with period* $p > 0$ *as in (i) and satisfying*

$$N_{i-1}(t) = N_i(t - \tfrac{p}{2}), \quad \max_{\substack{1 \le i \le n \\ t \in \mathbb{R}}} |N_i(t)| = A, \quad i = 1, 2, \ldots, n;$$

For any $A \in (K, \frac{K}{k})$, *the conclusion in (iii) holds;*

α_m *is given by (4.5.9).*

We choose $j = \frac{n}{2}$. Then by (4.3.2), $a_{\frac{n}{2}} = 4$ and $b_j = 4(rd_1 - d_2) < r$. ws that, from (4.5.3) and (4.5.7)–(4.5.8), β_m and β_{-l} exist to Eq. (4.5.5), $1 \le l \le q$ and

$$m \le 1 - N\frac{\tau}{\pi}\sqrt{\frac{r^2 - b_{\frac{n}{2}}^2}{1 - k^2}}$$

locations of β_m, β_{-l} are estimated by (4.5.7) and (4.5.8). Let α_m be any ven by (4.5.9). By Theorem 4.5.1, $(0, \alpha_m, \beta_m)$ is a bifurcation point with rossing number $\gamma_{\frac{n}{2}}(\alpha_m, \beta_m) < 0$. Consequently, the assertion of Theorem

As before, we first look at local bifurcation points. It follows that $p(\lambda, \alpha) = 0$ has purely imaginary roots $i\beta$ where each β is a solution of the equation

$$\tan \beta\tau = -\frac{r}{\sqrt{\beta^2 - r^2}}. \tag{4.5.19}$$

We can also estimate the location of β by viewing the solution β of (4.5.19) as the intersection points of two graphs $z = \tan \beta\tau$ and $z = -r/\sqrt{\beta^2 - r^2}$. If $r < \frac{\pi}{2\tau}$, then we have

$$\frac{(2m-1)\pi}{2\tau} < \beta_m < \frac{m\pi}{\tau}, \quad m = 1, 2, \ldots \tag{4.5.20}$$

and

$$\alpha_m = \frac{\sqrt{\beta_m^2 - r^2}}{\beta_m}, \qquad m = 1, 2, \ldots. \tag{4.5.21}$$

A similar calculation to that of (4.5.13) shows that each $(0, \alpha_m, \beta_m)$ is a local bifurcation point and their crossing numbers are all of same sign.

We need a similar result concerning the periods of periodic solutions to (4.2.5).

Lemma 4.5.4. *For each integer* $m > 0$ *and constant* $0 < k < 1$, *the equation (4.2.5) has no nonconstant* $\frac{2\tau}{m}$*-periodic solution* $N(t) \in (0, \frac{K}{k})$.

Proof. It suffices to show the lemma for $m = 1$. Suppose that $N(t)$ is a nonconstant 2τ-periodic solution of (4.2.5) with $N(t) \in (0, \frac{K}{k})$. Let $M(t) = N(t - \tau)$. We have

$$\begin{cases} N'(t) = rN(t)\left[1 - \dfrac{M(t) + cM'(t)}{K}\right] \\ M'(t) = rM(t)\left[1 - \dfrac{N(t) + cN'(t)}{K}\right]. \end{cases} \tag{4.5.22}$$

Put

$$u(t) = 1 - \frac{N(t)}{K} \quad \text{and} \quad v(t) = 1 - \frac{M(t)}{K}.$$

Then (4.5.21) simplifies to

$$\begin{cases} u'(t) = r(u-1)[v + cv'] \\ v'(t) = r(v-1)[u + cu']. \end{cases}$$

A similar argument to that in the proof of Lemma 4.5.2 now leads to a contradiction. This completes the proof.

We now obtain the following global result for Eq. (4.2.5).

Theorem 4.5.5. *Let* $\frac{\sqrt{3}}{2} < k < 1$ *be given. Assume that* $\pi\sqrt{1-k^2} \le r\tau < \frac{\pi}{2}$. *Then there exist* (α_m, β_m) *given by (4.5.20) and (4.5.21),* $m = 1, 2, \ldots, q$, *such that at least one of (i)–(iv) below holds for the equation (4.2.5):*

(i) For any $c \in (0, \frac{\alpha_m}{r})$, *(4.2.5) has a p-periodic positive solution* $N(t)$ *with period* $p \in [\frac{\tau}{m}, \frac{2\tau}{2m-1}]$;

(ii) For any $c \in (\frac{\alpha_m}{r}, \frac{k}{r})$, *the conclusion in (i) holds;*

(iii) For any $A \in (0, K)$, *there is a* $c_A > 0$ *such that a positive p-periodic solution* $N(t)$ *to (4.2.5) with* $c = c_A$ *exists, with period p as in (i) and* $\max_{t\in\mathbb{R}} N(t) = A$;

(iv) For any $A \in (K, \frac{K}{k})$, *the conclusion in (iii) holds;*

where q is an integer satisfying

$$\frac{r\tau}{\pi\sqrt{1-k^2}} - 1 < q \le \frac{r\tau}{\pi\sqrt{1-k^2}}.$$

$$u(t) = 1 - \frac{N(t)}{K} \quad \text{and} \quad v(t) = 1 - \frac{M(t)}{K}.$$

hen (4.5.21) simplifies to

$$\begin{cases} u'(t) = r(u-1)[v + cv'] \\ v'(t) = r(v-1)[u + cu']. \end{cases}$$

similar argument to that in the proof of Lemma 4.5.2 now leads to a contradic-
ɔn. This completes the proof.

We now obtain the following global result for Eq. (4.2.5).

heorem 4.5.5. *Let* $\frac{\sqrt{3}}{2} < k < 1$ *be given. Assume that* $\pi\sqrt{1-k^2} \le r\tau < \frac{\pi}{2}$. *hen there exist* (α_m, β_m) *given by (4.5.20) and (4.5.21),* $m = 1, 2, \ldots, q$, *such at at least one of (i)–(iv) below holds for the equation (4.2.5):*

(i) For any $c \in (0, \frac{\alpha_m}{r})$, *(4.2.5) has a* p*-periodic positive solution* $N(t)$ *with period* $p \in [\frac{\tau}{m}, \frac{2\tau}{2m-1}]$;

'ii) For any $c \in (\frac{\alpha_m}{r}, \frac{k}{r})$, *the conclusion in (i) holds;*

iii) For any $A \in (0, K)$, *there is a* $c_A > 0$ *such that a positive* p*-periodic solution* $N(t)$ *to (4.2.5) with* $c = c_A$ *exists, with period* p *as in (i) and* $\max_{t\in\mathbb{R}} N(t) = A$;

iv) For any $A \in (K, \frac{K}{k})$, *the conclusion in (iii) holds;*

ıere q *is an integer satisfying*

$$\frac{r\tau}{\pi\sqrt{1-k^2}} - 1 < q \le \frac{r\tau}{\pi\sqrt{1-k^2}}.$$

and β_m is given by (4.5.19)-(4.5.20).

Proof. Note that the equation (4.2.5) reduces to (4.2.2) when $c = 0$ and by Remark 4.5.1, Eq. (4.2.2) has no nonconstant periodic solutions. This excludes the alternative (i) in Theorem 4.5.5 and the conclusion follows.

References

[1] J. C. Alexander and G. Auchmuty, *Global bifurcation of phase-locked oscillations*, Arch. Rational Mech. Anal. **93** (1986), 253–270.

[2] J. Bélair, *Life span in population models: using time delays*, Lecture Notes in Biomathematics **92**, Proceedings of Differential Equations Models in Biology, Epidemiology and Ecology (S. Busenberg and M. Martelli Eds.), Claremont, 1990, 16–27.

[3] K. L. Cooke and P. Van den Driessche, *On zeros of some transcendental equations*, Funkcialaj Ekvacioj **29** (1986), 77–90.

[4] J. M. Cushing, "Integrodifferential Equations and Delay Models in Population Dynamics", Lecture Notes in Biomathematics **20**, Springer-Verlag, New York, 1977.

[5] D. F. DeArgelis, C. C. Travis and W. M. Post, *Persistence and stability of seed-dispersal species in a patchy environment*, Theoret. Population Biol. **16** (1979), 107–125.

[6] ——, W. M. Post and C. C. Travis, "Positive Feedback in Natural Systems", Biomathematics **15**, Springer-Verlag, New York, 1986.

[7] H. I. Freedman, *Single species migration in two habitats: persistence and extinction*, Math. Model **8** (1987), 778–780.

[8] ——, "Deterministic Mathematical Models in Population Ecology", Marcel Dekker, New York, 1988.

[9] ——, *Persistence and extinction in models of two-habitat migration*, Math. Comput. Modelling **12** (1989), 105–112.

[10] —— and Y. Kuang, *Stability switches in linear scalar neutral delay equations*, Funkcialaj Ekvacioj **34** (1991), 187–209.

[11] —— and J. Wu, *Persistence and global asymptotic stability of single species dispersal models with stage structure*, Quarterly of Applied Math. **49** (1991), 551–571.

[12] K. Gopalsamy, "Stability and Oscillations in Delay Differential Equations of Population Dynamics", Mathematics and Its Applications **74**, Kluwer Academic Publishers, Dordrecht, 1992.

[13] ———, X. Z. He and L. Z. Wen *On a periodic neutral logistic equation*, Glasgow Math. J. **33** (1991), 281–286.

[14] ——, M. R. S. Kulenović and G. Ladas, *Time delays in a "food-limited" population model*, Applicable Analysis **31** (1988), 225–237.

[15] —— and B. G. Zhang, *On a neutral delay logistic equation*, Dynamics Stability Systems **2** (1988), 183–195.

[16] T. G. Hallam, *Population dynamics in a homogeneous environment*, in Mathematical Ecology (T. G. Hallam and S. A. Levin Eds.), Springer-Verlag, New York, 1986, 61–94.

[17] D. P. Hardin, P. Takáč and G. F. Webb, *Dispersion population models discrete in time and continuous in space*, J. Math. Biol. **28** (1990), 1–20.

[18] A. Hastings, *Spatial heterogeneity and the stability of predator-prey systems*, Theoret. Population Biology **12** (1977), 37–48.

[19] ——, *Dynamics of a single species in a spatially varying environment: the stability role of high dispersal rates*, J. Math. Biology **16** (1982), 49–55.

[20] ——, *Mckendrick Von Foerster models for patchy dynamics*, Lecture Notes in Biomathematics **92**, Proceedings of Differential Equations Models in Biology, Epidemialogy and Ecology (S. Busenberg and M. Martelli Eds.), Clarement, 1990, 189–199.

[21] G. E. Hutchinson, *Circular causal systems in ecology*, Ann. New York Acad. Sci. **50** (1948), 221–246.

[22] W. Krawcewicz and J. Wu, *Discrete waves and phase-locked oscillations in the growth of a single-species population over a patchy environment*, Open Systems & Information Dynamics **1** (1992), 127–147.

[23] Y. Kuang, *On neutral delay two-species Lotka-Volterra competitive systems*, J. Austral. Math. Soc. Series B **32** (1991), 311–326.

[24] ——, "Delay Differential Equations with Applications in Population Dynamics", Mathematics in Science and Engineering **191**, Academic Press, New York, 1993.

[25] —— and A. Feldstein, *Boundedness of solutions of a nonlinear nonautonomous neutral delay equation*, J. Math. Anal. Appl. **158** (1991), 193–204.

[26] ——, R. H. Martin and H. L. Smith, *Global stability for infinite delay dispersive Lotka-Volterra systems: weakly interacting populations in nearly identical patches*, J. Dynamics and Differential Equations **3** (1991), 339–360.

[27] S. A. Levin, *Dispersion and population interaction*, Amer. Natur. **108** (1974), 207–228.

[28] ———, *Population dynamic models in heterogeneous environments*, Ann. Rev. Ecol. Syst. **7** (1976), 287–301.

[29] ———, *Spatial patterning and the structure of ecological communities*, in Some Mathematical Questions in Biology **VII**, Amer. Math. Soc., Providence, R. I., 1976, 1–36.

[30] ———, *Population models and community structure in heterogeneous environments*, in Mathematical Ecology (T. G. Hallam and S. A. Levin Eds.), Biomathematics **17**, Springer-Verlag, New York, 1986, 295–321.

[31] N. MacDonald, "Time Lags in Biological Models", Lecture Notes in Biomathematics **28**, Springer-Verlag, New York, 1979.

[32] R. M. May, *Models for single populations*, in Theoretical Ecology (R. M. May ed.), Saunders, Philadelphia, 1976, 4–25.

[33] J. F. McLaughlin, *Pattern and stability in predater-prey communities: how diffusion in spatially variable environments affects the Lokta-Volterra model*, Theoret. Population Biology **40** (1991), 148–172.

[34] R. McMurtrie, *Persistence and stability of single-species and prey-predator systems in spatially heterogeneous environments*, Math. Biosci. **39** (1978), 11–51.

[35] J. D. Murray, "Mathematical Biology", Biomathematics **19**, Springer-Verlag, New York, 1989.

[36] A. J. Nicholson, *Compensatory reactions of populations to stresses and their evolutionary significance*, Austral. J. Zool. **2** (1954), 9–65.

[37] A. Okubo, "Diffusion and Ecological Problems: Mathematical Models", Biomathematics **10**, Springer-Verlag, New York, 1980.

[38] E. C. Pielou, "Mathematical Ecology", Willy-Interscience, New York, 1977.

[39] D. A. Roff, *Spatial heterogeneity and the persistence of populations*, Oecologia **15** (1974), 245–258.

[40] J. Selgrade and Gene Namkuong, *Population interactions with growth rates dependent on weighted density*, Lecture Notes in Biomathematics **92**, Proceedings of Differential Equations Model in Biology, Epidemiology and Ecology (S. Busenberg and M. Martelli Eds.), Clarement, 1990, 247–255.

[41] F. E. Smith, *Population dynamics in Daphnia Magna*, Ecology **44** (1963), 651–663.

[42] Y. Takeuchi, *Diffusion effect on stability of Lotka-Volterra models*, Bull. Math. Biol. **48** (1986), 585–601.

[43] R. R. Vance, *The effect of dispersal on population stability in one-species, discrete-space population gowth models*, Amer. Natur. **123** (1984), 230–254.

[44] A. T. Winfree, "The Geomety of Biological Time", Springer-Verlag, Berlin, 1980.

[45] J. H. Wu and H. I. Freedman, *Global stability in a population model with dispersal and stage structure*, Lecture Notes in Biomathematics **92**, Proceedings of Differential Equations Models in Biology, Epidemiology and Ecology (S. Busenberg and M. Martelli Eds.), Clarement, 1990, 247–255.

[46] E. M. Wright, *A nonlinear difference-differential equation*, J. Reine Angew. Math. **194** (1955), 66–87.

.ou, "Mathematical Ecology", Willy-Interscience, New York, 1977.

:, *Spatial heterogeneity and the persistence of populations*, Oecologia , 245–258.

.e and Gene Namkuong, *Population interactions with growth rates : on weighted density*, Lecture Notes in Biomathematics **92**, Pro- ıf Differential Equations Model in Biology, Epidemiology and Ecol- ısenberg and M. Martelli Eds.), Clarement, 1990, 247–255.

th, *Population dynamics in Daphnia Magna*, Ecology **44** (1963),

:hi, *Diffusion effect on stability of Lotka-Volterra models*, Bull. ıl. **48** (1986), 585–601.

ıce, *The effect of dispersal on population stability in one-species, ıace population gowth models*, Amer. Natur. **123** (1984), 230–254.

ıfree, "The Geomety of Biological Time", Springer-Verlag, Berlin,

ınd H. I. Freedman, *Global stability in a population model with dis- stage structure*, Lecture Notes in Biomathematics **92**, Proceedings ıtial Equations Models in Biology, Epidemiology and Ecology (S. ; and M. Martelli Eds.), Clarement, 1990, 247–255.

ght, *A nonlinear difference-differential equation*, J. Reine Angew. **1** (1955), 66–87.

several generalizations of the LLTL equation were also presented in [13, 14, 43, 46]. In particular, Cooke and Krumme [13] gave a systematic procedure for reducing transimission line problems, which are described by linear partial differential equations subject to certain nonlinear initial-boundary conditions, to initial value problems for differential-difference or integral differential-difference equations. It should be emphasized that it is the LLTL equation that has motivated the theory, which was first systematically disscussed by Hale [28], for the D-type neutral functional differential equations (NFDEs).

Strictly speaking, a single transmission line as considered above is less usual than a multiconductor line in applications. As an electric circuit, a self-contained single transmission line is assumed to be removed far enough from other lines so that it is not affected by any electrical changes occuring in the latter. As soon as a second transmission line is placed close to the first one the fields of the first line induce a voltage and a current on the second. *Capacitive coupling* is then produced by the electric field and *inductive coupling* results from the magnetic field. The classical applications of telephone (or telegraph) lines and high-voltage power transmission line are often examples of coupling. The coupling phenomenon is also utilized in practice to realize directional couplers and interdigital filters. Moreover, in the modern high-speed integrated circuit (IC) technology, coupling among a group of physically close transimission lines is very common and interconnects in high-density IC are usually treated as transmission lines. We refer to [12, 23, 45, 52, 56, 60] and the references therein for detailed discussions on coupled electric circuits and transmission lines.

Motivated by Endo and Mori [17] and Winnerl et al. [64] we consider in this chapter a *ring array* of mutually coupled lossless transmission lines. For simplicity, we assume the transmission lines are *resistively coupled* and the capacitive and inductive couplings among the systems are neglected. We also assume that each linked transmission line is identical and terminates at each end by a lumped linear or nonlinear circuit element. By employing the telegrapher's equation at each line together with a coupling term in the initial-boundary condition, we derive a *symmetric difference-differential system of neutral type*, which is equivalent to the original partial differential equations governing the coupled lines. To study such a symmetric neutral system we use the local and *global* bifurcation theory from Chapter 2 and 3. We shall prove the existence and multiplicity of self-sustained phase-locked and synchronous periodic solutions. Due to the global nature of the bifurcation theorem, comparing our results with those of Shimura [55] and Brayton [8, 9] (for single transmission line equation) the periodic solutions we shall present are of large amplitude, in the sense that the parameter can be *far away* from the local bifurcation value.

Since the self-sustained oscillation occurs in the lossless transmission line, we may regard it as an *electric oscillator*. It should be noted that there recently has been great interest in the study of coupled nonlinear oscillators. For example, Alexander and Auchmuty [3] have considered the global bifurcation of phase-locked oscillations in the coupled brusselators and van del Pol oscillators. In their series of papers, Endo and Mori [17–19] have discussed the mode analysis of one-dimensional and two-dimensional multimode oscillators. As a mathematical model for slow-wave electrical activity of the gastro-intestinal tract of humans and animals, Allian and Likens [4] have proposed a tubular structure which comprises one-dimensional rings and two-dimensional arrays of interconnected nonlinear oscillators with third-power conductance characteristics. Similar mathematical models for the electrical activity in humans and animals are also postulated by Linkens et al. [42] and Sarna et al. [54], where a series of simulated relaxation oscillators are resistively coupled as a chain. Other problems related to the coupled electric oscillators are addressed by Gollab et al. [26] on periodicity and chaos and are systematically reviewed by Grasman [25] on various applications.

This chapter is now organized as follows. In Section 5.2, we use the standard reduction procedure developed in [9,, 13, 57] to derive the governing neutral equations for the resistively coupled lossless transmission lines of a ring structure. To investigate the global bifurcation of the neutral equations, three lemmas concerning the periods and upper and lower bounds are prepared in Section 5.3. Section 5.4 is devoted to the global Hopf bifurcation analysis and the existence of self-sustained phase-locked and synchronous periodic solutions of large amplitudes is proved. Finally, in Section 5.5, we draw some consequences and discuss briefly some of the implications of the lossless transmission line problem.

5.2. Coupled LLTL neutral equations

Let N be a positive integer. We consider a ring of N mutually coupled lossless transmission line (LLTL) networks which are interconnected by a common resistor R. We assume all coupled LLTL networks are identical, each of which is a uniformly distributed lossless transmission line with the series inductance L_s and parallel capacitance C_s per unit length of the line. To derive the network equations, let us take an x-axis in the direction of the line, with two ends of the normalized line at $x = 0$ and $x = 1$. See Fig. 5.2.1 and 5.2.2.

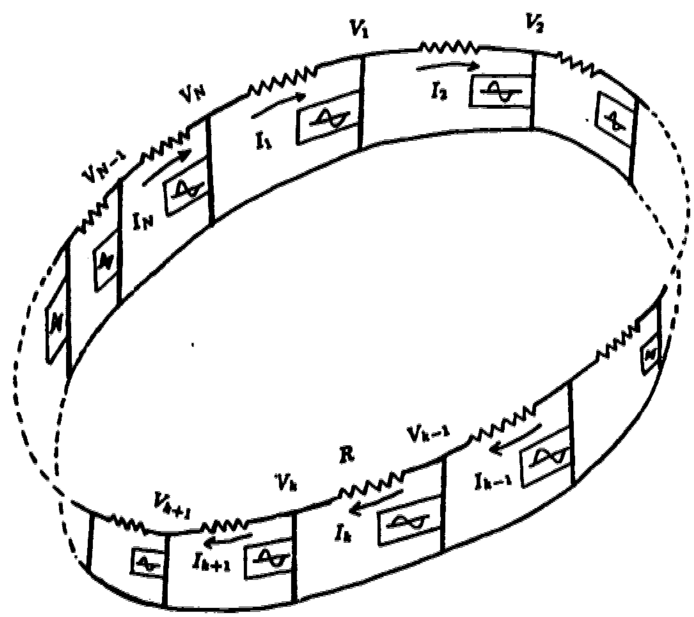

Fig. 5.2.1

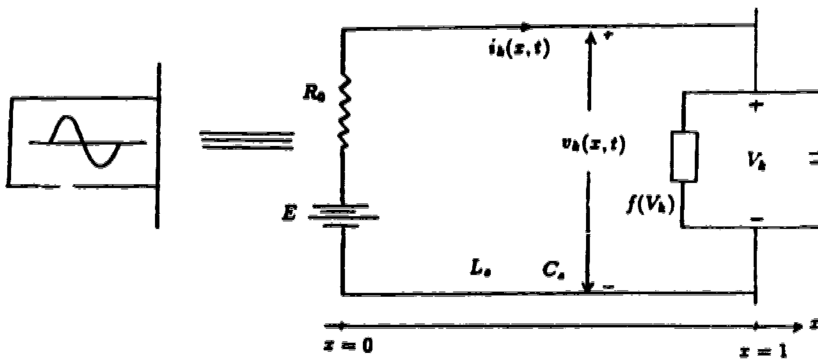

Fig. 5.2.2

Let $i_k(x,t)$ denote the current flowing in the k-th line at time t and distance x down the line and $v_k(x,t)$ denote the voltage across the line at t and x. It is well-known [38, 45, 59] that the functions i_k and v_k obey the following partial differential equations (*Telegrapher's equation*)

$$L_s \frac{\partial i_k}{\partial t} = -\frac{\partial v_k}{\partial x}$$
$$C_s \frac{\partial v_k}{\partial t} = -\frac{\partial i_k}{\partial x}, \qquad k = 1, 2, \ldots, N. \tag{5.2.1}$$

When these N networks are interconnected resistively in the way as shown in Fig. 5.2.1, the middle lines have coupling terms from the preceeding and succeeding lines, and at two ends $x = 0$ and $x = 1$, the line gives rise to the boundary conditions

$$0 = E - v_k(0,t) - R_0 i_k(0,t)$$
$$-C\frac{d}{dt}v_k(1,t) = -i_k(1,t) + f(v_k(1,t)) - (I_{k-1} - I_k) \tag{5.2.2}$$
$$v_k(1,t) - v_{k+1}(1,t) = RI_k(t)$$

where E is the constant DC dias voltage, $f(v_k(1,t))$ is the current ($V - I$ characteristic) through the nonlinear resistor in the direction shown in Fig. 5.2.1 and I_k is the network current coupling term.

Under equilibrium conditions, $\frac{\partial i_k}{\partial x} = \frac{\partial v_k}{\partial x} = 0$. We have $i_k(0,t) = i_k(1,t)$ and $v_k(0,t) = v_k(1,t)$. Thus, Eqs. (5.2.1) and (5.2.2) have the following equilibrium equations

$$\begin{aligned} &E - v_k - R_0 i_k = 0 \\ &i_k = f(v_k) - \tfrac{1}{R}(v_{k+1} - 2v_k + v_{k+1}). \end{aligned} \tag{5.2.3}$$

We assume that (5.2.3) has a uniqne homogeneous solution $(v_k, i_k) = (v^*, i^*)$, for all $1 \le k \le N$. By changing variables, the equilibrium can be shifted from (v^*, i^*) to $(0,0)$ and Eqs. (5.2.1) and (5.2.2) reduce to

$$\begin{cases} L_s \dfrac{\partial i_k}{\partial t} = -\dfrac{\partial v_k}{\partial x} \\ C_s \dfrac{\partial v_k}{\partial t} = \dfrac{\partial i_k}{\partial x} \\ 0 = v_k(0,t) + R_0 i_k(0,t) \\ -C\dfrac{d}{dt} v_k(1,t) = -i_k(1,t) + \tilde{g}(v_k(1,t)) - \tfrac{1}{R}(v_{k+1} - 2v_k + v_{k-1})(1,t) \end{cases} \tag{5.2.4}$$

where

$$\tilde{g}(v_k) = f(v_k + v^*) - f(v^*).$$

We now solve the partial differential equation (5.2.4). It is known [57, 59] that there exist unique solutions (*d'Alembert solution*) $i_k(x,t)$ and $v_k(x,t)$ which are of the form

$$\begin{aligned} v_k(x,t) &= \tfrac{1}{2}[\phi_k(x - \sigma t) + \psi_k(x + \sigma t)] \\ i_k(x,t) &= \tfrac{1}{2Z}[\phi_k(x - \sigma t) + \psi_k(x + \sigma t)] \end{aligned} \tag{5.2.5}$$

where

$$\sigma = \frac{1}{\sqrt{L_s C_s}}, \qquad Z = \sqrt{\frac{L_s}{C_s}} \tag{5.2.6}$$

are respectively the propagation velocity of waves and the characteristic impedance of the line, and

$$\phi_k \in C^1(-\infty, 1], \qquad \psi_k \in C^1[0, \infty).$$

Let

$$\phi_{k_1}(t) = \phi(1 - \sigma t), \qquad \phi_{k_0}(t) = \phi_k(-\sigma t)$$

$$\psi_{k_1}(t) = \psi(1 + \sigma t), \qquad \psi_{k_0}(t) = \psi_k(\sigma t)$$

and $V_k(t) = v_k(1, t)$. We have from (5.2.5) that

$$\begin{aligned} \phi_{k_1}(t) &= V_k(t) + Zi_k(1,t), \quad \phi_{k_0}(t) = V_k(0,t) + Zi_k(0,t) \\ \psi_{k_1}(t) &= V_k(t) - Zi_k(1,t), \quad \psi_{k_0}(t) = V_k(0,t) - Zi_k(0,t). \end{aligned} \tag{5.2.7}$$

Note that $\phi_{k_1}(t) = \phi_{k_0}(t - \frac{1}{\sigma})$ and $\psi_{k_1}(t) = \psi_{k_0}(t + \frac{1}{\sigma})$. By (5.2.7) and the boundary condition in (5.2.4)

$$\begin{aligned} V_k(t) + Zi_k(1,t) &= -q\psi_{k_1}(t - r) \\ V_k(t) - Zi_k(1,t) &= \psi_{k_1}(t) \end{aligned} \tag{5.2.8}$$

where

$$r = \frac{2}{\sigma} \qquad \text{and} \qquad q = \frac{Z - R_0}{Z + R_0}. \tag{5.2.9}$$

Now, the second boundary condition in (5.2.4) gives

$$i_k(1,t) = CV_k'(t) + \tilde{g}(V_k(t)) - \tfrac{1}{R}(V_{k+1}(t) - 2V_k(t) + V_{k-1}(t)). \tag{5.2.10}$$

Substituting (5.2.10) into (5.2.8) and eliminating $\psi_{k_1}(t - r)$ lead to

$$\begin{aligned} V_k(t) + Z[CV_k' + \tilde{g}(V_k) - \tfrac{1}{R}(V_{k+1}(t) - 2V_k(t) + V_{k-1}(t))] \\ = -qV_k(t-r) + qZ[CV_k'(t-r) + \tilde{g}(V_k(t-r))] \\ - \tfrac{qZ}{R}[V_{k+1}(t-r) - 2V_k(t-r) + V_{k-1}(t-r)]. \end{aligned}$$

This simplifies to

$$\begin{aligned}\frac{d}{dt}[V_k(t) - qV_k(t-r)] = &- \tfrac{1}{ZC}V_k(t) - \tfrac{q}{ZC}V_k(t-r) - \bar{g}(V_k) + q\bar{g}(V_k(t-r)) \\ &+ \tfrac{1}{RC}[V_{k+1}(t) - qV_{k+1}(t-r) - 2(V_k(t) - qV_k(t-r)) \\ &+ V_{k-1}(t) - qV_{k-1}(t-r)]\end{aligned}$$

where $\bar{g}(V_k) = \frac{1}{C}\tilde{g}(V_k)$. Define for each $\alpha \in \mathbb{R}$ the operator $D(\alpha) : C([-r,0];\mathbb{R}) \to \mathbb{R}$ by

$$D(\alpha)\varphi = \varphi(0) - \alpha\varphi(-r), \quad \varphi \in C([-r,0];\mathbb{R}). \tag{5.2.12}$$

Following [8, 9], we assume

$$\bar{g}(v) = -\gamma v + g(v), \quad v \in \mathbb{R}, \quad \gamma > 0 \tag{5.2.13}$$

where g is a continuous function. Using (5.2.12) and (5.2.13), we obtain from (5.2.11) the following LLTL-network coupling equations

$$\begin{aligned}\frac{d}{dt}D(q)V_t^k = &- (\tfrac{1}{ZC} - r)V^k(t) - q(\tfrac{1}{ZC} + r)V^k(t-r) - g(V^k) \\ &+ qg(V^k(t-r)) + \tfrac{1}{RC}D(q)(V_t^{k+1} - 2V_t^k + V_t^{k-1}) \\ &k = 1, 2, \ldots, N, \quad (\operatorname{mod} N)\end{aligned} \tag{5.2.14}$$

where for each $1 \leq k \leq N$, $t \in \mathbb{R}$, $V_t^k \in C([-r,0];\mathbb{R})$ is defined by $V_t^k(\theta) = V_k(t+\theta)$ for all $\theta \in [-r,0]$.

Note that Eq. (5.2.14) is a functional differential equation of neutral type with one time delay $r > 0$ (see [28, 29]). It is easily seen that this equation is equivalent to the system (5.2.4), which can be viewed as a neutral system with diffusion. Therefore, it may be considered as a special example of the Rashevsky-Turing theory [53, 61] (see also Section 3.4). If there is no coupling between these N networks, then (5.2.14) reduces to a single LLTL-network equation

$$\frac{d}{dt}D(q)V_t^k = -(\tfrac{1}{ZC}-\gamma)V^k(t) - q(\tfrac{1}{ZC}+\gamma)V^k(t-r) - g(V^k) + qg(V^k(t-r)) \tag{5.2.15}$$

This equation was first obtained by Naguma and Shimura [49], Shimura [55] (for $R_0 = 0$) and Brayton [9] (for any $R_0 \geq 0$) and was extensively investigated. See [8, 9, 16, 20, 28–36, 39, 43, 44, 49, 55, 57, 65] and the references therein.

5.3. Periods and a priori bounds

In this section, we prove three lemmas which will be needed in the study of global Hopf bifurcation of phase-locked oscillations. The first two lemmas concern the periods of periodic solutions to Eq. (5.2.14). In the third lemma we give a priori bounds on the amplitude of possible periodic solutions of Eq. (5.2.14).

We consider the following NFDEs

$$\begin{aligned}\frac{d}{dt}D(q)x_t^k =& - ax^k(t) - bqx^k(t-r) - g(x^k(t)) + qg(x^k(t-r)) \\ &+ dD(q)[x_t^{k+1} - 2x_t^k + x_t^{k-1}] \\ &k = 1, 2, \dots, N, \quad (\text{mod}\, N)\end{aligned} \tag{5.3.1}$$

where constants $d \geq 0$, $q \in [0, 1)$, $D(q) : C([-r, 0]; \mathbb{R}) \to \mathbb{R}$ is defined by

$$D(q)\varphi = \varphi(0) - q\varphi(-r), \qquad \varphi \in C([-r, 0]; \mathbb{R}), \tag{5.3.2}$$

r, a and b are positive constants, g is a differentiable function with $g(0) = g'(0) = 0$. Note that Eq. (5.3.1) is a condensed form of Eq. (5.2.14). The parameters r, a, b, d and q are of physical meanings (see (5.2.6) and (5.2.9)). Note also that Eq. (5.3.1) is a special case of the following more general NFDEs

$$\frac{d}{dt}D(q)x_t^k = F(q, x^k(t), x^k(t-r)) + dD(q)[x_t^{k+1} - 2x_t^k + x_t^{k-1}] \tag{5.3.3}$$

where q, d, r are constants as before and $F : \mathbb{R}^3 \to \mathbb{R}$ is locally Lipschitzan.

In analysing the global branch of phase-locked oscillations, we need the foloeing information on the periods of possible periodic solutions to Eq. (5.3.1).

Lemma 5.3.1. *For every integer $m > 0$, Eq. (5.3.3) has no nonconstant $\frac{2r}{m}$-periodic solution $x(t) := \{x^k(t)\}_{k=1}^N$ with $x^{k-1}(t) = x^k(t - \frac{r}{m})$ for all $t \in \mathbb{R}$ and $k = 1, 2, \ldots, N$, $(mod\, N)$.*

Proof. We consider two cases sepatately.

Case (I): m is odd. Note that if $x(t)$ is a nonconstant $\frac{2r}{m}$-periodic solution with $x^{k-1}(t) = x^k(t - \frac{r}{m})$, then

$$x^{k-1}(t) = x^k(t - \frac{r}{m}) = x^k(t - r + \frac{m-1}{2}\frac{2r}{m}) = x^k(t - r).$$

It suffices to show that the lemma is true for $m = 1$.

Suppose to the contrary that Eq. (5.3.3) has a nonconstant $2r$-periodic solution $x(t)$ with $x^{k-1}(t) = x^k(t-r)$. Let $y^k(t) = x^k(t-r)$. We have $x^{k+1}(t) = x^k(t+r) = x^k(t-r) = y^k(t)$. Similarly, $y^{k+1}(t) = x^k(t), x^{k-1}(t) = y^k(t)$ and $y^{k-1}(t) = x^k(t)$. Therefore, by (5.3.2)

$$\begin{aligned} D(q)x_t^{k+1} &= x^{k+1}(t) - qx^{k+1}(t-r) \\ &= y^k(t) - qy^{k+1}(t-r) \\ &= y^k(t) - qx^k(t) = D(q)x_t^{k-1} \\ & \quad k = 1, 2, 3, \ldots, N, \quad (\mathrm{mod}\, N) \end{aligned}$$

and $(x^k(t), y^k(t))$ satisfies the following ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}[x^k(t) - qy^k(t)] = F(q, x^k(t), y^k(t)) + 2d(1+q)(y^k(t) - x^k(t)) \\ \dfrac{d}{dt}[y^k(t) - qx^k(t)] = F(q, y^k(t), x^k(t)) + 2d(1+q)(x^k(t) - y^k(t)). \end{cases} \tag{5.3.4}$$

Put

$$\begin{cases} u(t) = x^k(t) - qy^k(t) \\ v(t) = y^k(t) - qx^k(t). \end{cases} \tag{5.3.5}$$

Then

$$\begin{cases} x(t) = \dfrac{u(t) + qv(t)}{1 - q^2} \\ y(t) = \dfrac{qu(t) + v(t)}{1 - q^2}. \end{cases} \tag{5.3.6}$$

Substituting (5.3.5) and (5.3.6) into (5.3.4), we see that $(u(t), v(t))$ is a solution to the following system of ordinary differential equations

$$\begin{cases} u'(t) = F(q, \dfrac{u + qv}{1 - q^2}, \dfrac{qu + v}{1 - q^2}) + 2d(1 + q)(v - u) \\ v'(t) = F(q, \dfrac{qu + v}{1 - q^2}, \dfrac{u + qv}{1 - q^2}) + 2d(1 + q)(u - v). \end{cases} \tag{5.3.7}$$

Eq. (5.3.7) is symmetric about $u(t)$ and $v(t)$. Therefore, the diagonal $\Delta \cong \{(u, v) \in \mathbb{R} : u = v\}$ is invariant under (5.3.7). Since any vector field on $\Delta \cong \mathbb{R}$ cannot have nonconstant periodic solutions, $(u(t), v(t) \notin \Delta$ for all $t \in \mathbb{R}$. So, without loss of generality, we may assume that

$$u(t) < v(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{5.3.8}$$

Replaceing t by $t - r$ in (5.3.8) gives

$$u(t - r) < v(t - r) \quad \text{for all} \quad t \in \mathbb{R}. \tag{5.3.9}$$

On the other hand, we have

$$\begin{aligned} v(t - r) &= y^k(t - r) - qx^k(t - r) \\ &= x^k(t) - qy^k(t) = u(t) \end{aligned}$$

and

$$\begin{aligned} u(t - r) &= x^k(t - \tau) - qy^k(t - r) \\ &= y^k(t) - qx^k(t) = v(t). \end{aligned}$$

Therefore, it follows from (5.3.9) that

$$v(t) < u(t) \quad \text{for all} \quad t \in \mathbb{R}$$

which contradicts to (5.3.8). This completes the proof for Case (I).

Case (II): m is even. Similarly, we need only to show the lemma for $m =$ 5.2.

By the way of contradiction, suppose $x(t)$ is an r-periodic solution to Eq. (5.3.3) with $x^{k-1}(t) = x^k(t - \frac{r}{2})$. Set $y^k(t) = x^k(t - \frac{r}{2})$. As in Case (I), $(x^k(t), y^k(t))$ satisfies the equations

$$\begin{cases} \dfrac{d}{dt}x^k(t) = \dfrac{F(q, x^k, x^k)}{1-q} + 2d(y^k - x^k) \\ \dfrac{d}{dt}y^k(t) = \dfrac{F(q, y^k, y^k)}{1-q} + 2d(x^k - y^k) \end{cases}$$

A similar argument to that in Case (I) leads also to a contradiction.

This completes the proof.

Remark 5.3.1. An analog of Lemma 5.3.1 for the single scalar NFDE (5.2.15) has been established in [39] for the case where no coupling occurs.

We will also need the following simple result.

lemma 5.3.2. *Assume $a > 0$, $d \geq 0$ and $xg(x) > 0$ for all $x \neq 0$. Then the system of ordinary differential equations*

$$\begin{aligned} \frac{d}{dt}x^k(t) = &- ax^k(t) - g(x^k(t)) + d(x^{k+1}(t) - 2x^k(t) + x^{k-1}(t)) \\ &k = 1, 2, \ldots, N, \quad (mod\, N) \end{aligned} \tag{5.3.10}$$

has no nonconstant periodic solutions.

Proof. Suppose that $x(t) = (x^1(t), \ldots, x^N(t))$ is a nonconstant periodic sulotion of (5.3.10). Set

$$V(x(t)) = \frac{1}{2}\sum_{k=1}^{N}(x^k(t))^2.$$

We have

$$\begin{aligned} V'_{(5.3.10)}(x(t)) &= \sum_{k=1}^{N} x^k[-ax^k - g(x^k) + d(x^{k+1} - 2x^k + x^{k-1})] \\ &= -a\sum_{k=1}^{N}(x^k)^2 - \sum_{k=1}^{N} x^k g(x^k) + d\sum_{k=1}^{N} x^k(x^{k+1} - 2x^k + x^{k-1}) \\ &\le -2aV(x) + 2d\sum_{k=1}^{N}(x^k x^{k+1} - x^k x^k) \\ &\le -2aV(x). \end{aligned}$$

This implies that

$$V(x(t)) \le V(x(0))e^{-2at} \to 0 \quad \text{as} \quad t \to +\infty.$$

It follows then that $\lim_{t\to\infty} x(t) = 0$. This is impossible since $x(t)$ is a nonconstant periodic solution. This proves the lemma.

In what follows, we provide a priori bounds on periodic solutions of Eq. (5.3.1).

Lemma 5.3.3. *Assume that* $0 < a < b$ *and*

(i) $xg(x) > 0$ *for all* $x \neq 0$;

(ii) $g(x)$ *is nondecreasing;*

(iii) $\lim_{x\to\pm\infty} \frac{g(x)}{x} = +\infty$;

(iv) for any $q_0 \in [\frac{a}{b}, 1)$,

$$\sup_{\frac{a}{b} \le q \le q_0} \overline{\lim}_{x \to \pm\infty} \frac{g(qx) - qg(x)}{qx} < -(a+b).$$

Then for any $\delta \in [\frac{a}{b}, 1)$, *there exists* $M = M(\delta) > 0$ *such that if* $q \in [\frac{a}{b}, \delta]$ *and* $x(t)$ *is a periodic solution of Eq. (5.3.1) with period* $p > 0$ *which satisfies* $x^{k-1}(t) = x^k(t - \frac{p}{2})$, *then* $|x(t)| \le M$ *for all* $t \in \mathbb{R}$.

Proof. We prove the existence of M such that $x^k(t) < M$ for any $k \in \{1, 2, \ldots, N\}$. The existence of M such that $x^k(t) \ge -M$ can be treated similarly.

Let $x(t)$ be a periodic solution of Eq. (5.3.1) with period $p > 0$ and $x^{k-1}(t) = x^k(t - \frac{p}{2})$ for $k = 1, 2, \ldots, N$, mod(N). Then $t \in \mathbb{R}$ exists such that

$$x^k(t) - qx^k(t - r) = \max_{s \in \mathbb{R}} [x^k(s) - qx^k(s - r)]. \tag{5.3.11}$$

Therefore, for each fixed $s \in \mathbb{R}$,

$$x^k(s) \le qx^k(s - r) + [x^k(t) - qx^k(t - r)]. \tag{5.3.12}$$

Or, equivalently,

$$D(q)x_s^k \le D(q)x_t^k. \tag{5.3.13}$$

Replacing s by $s - r$ in (5.3.12) yields

$$x^k(s) \le q^2 x^k(s - 2r) + (q + 1)D(q)x_t^k.$$

Repeating the above process m-times, we get

$$x^k(s) \le q^m x^k(s - mr) + \frac{1 - q^m}{1 - q} D(q)x_t^k.$$

Therefore, letting $m \to \infty$, we have

$$x^k(s) \le \frac{D(q)x_t^k}{1-q} \quad \text{for all} \quad s \in \mathbb{R}. \tag{5.3.14}$$

In particular,

$$x^k(t) \le \frac{x^k(t) - qx^k(t-r)}{1-q}$$

which gives

$$x^k(t) \ge x^k(t-r). \tag{5.3.15}$$

On the other hand, by (5.3.11), we see that $\frac{d}{dt}[x^k(t) - qx^k(t-r)] = 0$. Recall that $x(t)$ is a solution to Eq. (5.3.1). It follows that

$$\begin{aligned} ax^k(t) + bqx^k(t-r) = &- g(x^k(t)) + qg(x^k(t-r)) \\ &+ d[D(q)x_t^{k+1} - 2D(q)x_t^k + D(q)x_t^{k-1}]. \end{aligned} \tag{5.3.16}$$

Notice that $x^k(t-p) = x^k(t)$ and $x^{k-1}(t) = x^k(t-\frac{p}{2})$ for any $k = 1, 2, \ldots, N, \pmod N$. we have

$$\begin{aligned} D(q)x_t^{k+1} &= x^{k+1}(t) - qx^{k+1}(t-r) \\ &= x^k(t-\tfrac{p}{2}) - qx^k(t-\tfrac{p}{2}-r) = D(q)x^k_{t-\frac{p}{2}}. \end{aligned} \tag{5.3.17}$$

Similarly

$$D(q)x_t^{k-1} = D(q)x^k_{t-\frac{p}{2}}. \tag{5.3.18}$$

Substituting (5.3.17) and (5.3.18) into (5.3.16), we obtain

$$\begin{aligned} ax^k(t) + bqx^k(t-r) = &-g(x^k(t)) + qg(x^k(t-r)) \\ &+ 2d[D(q)x^k_{t-\frac{p}{2}} - D(q)x_t^k]. \end{aligned} \tag{5.3.19}$$

We now distinguish two cases:

Case (i): $x^k(t) > 0$. In this case, $x^k(t-r) < 0$. For otherwise, if $x(t-r) \geq 0$, then the left hand side of (5.3.19) is positive, but the right hand side

$$\begin{aligned}-g(x^k(t)) + qg(x^k(t-r)) &+ 2d[D(q)x^k_{t-\frac{r}{2}} - D(q)x^k_t] \\ &\leq -g(x^k(t)) + qg(x^k(t-r)) \\ &= g(x^k(t))[-1 + q\frac{g(x^k(t-r))}{g(x^k(t))}] < 0\end{aligned}$$

by (5.3.13) and the assumptions (i)-(ii) on g.

Now from (5.3.13) and (5.3.19), we see that

$$ax^k(t) + bqx^k(t-r) \leq -g(x^k(t)) + qg(x^k(t-r)) \tag{5.3.20}$$

which implies that

$$0 < ax^k(t) + g(x^k(t)) \leq qx^k(t-r)[\frac{g(x^k(t-r))}{x^k(t-r)} - b]. \tag{5.3.21}$$

Since $x^k(t-r) < 0$, (5.3.20) gives further that

$$\frac{g(x^k(t-r))}{x^k(t-r)} < b.$$

By assumption (iii), there must be a constant $M_1 > 0$ (independent of k) such that $x^k(t-r) \geq -M_1$. Substituting this into (5.3.21), we get

$$\begin{aligned}0 < ax^k(t) + g(x^k(t)) &\leq qg(x^k(t-r)) - bqx^k(t-r) \\ &\leq \max_{-M_1 \leq z \leq 0} \delta[g(z) - bz]\end{aligned} \tag{5.3.15}$$

from which another constant $M_2 > 0$ (independent of k) exists such that $x(t) \leq M_2$, due to assumption (iii). Therefore,

$$x^k(t) - qx^k(t-r) \leq M_2 + \delta M_1.$$

This, together with (5.3.14), implies that $x^k(s) \leq \frac{M_2+\delta M_1}{1-\delta}$ for all $s \in \mathbb{R}$.

Case (ii): $x(t) < 0$. In this case, $x^k(t-r) \le x^k(t) \le 0$. If $x^k(t) - qx^k(t-r) \le 0$, then, by (5.3.14), we are done. If $x^k(t) - qx^k(t-r) > 0$, then $x^k(t) \ge qx^k(t-r)$. From (5.3.20), we get

$$qg(x^k(t-r)) - bqx^k(t-r) \ge aqx^k(t-r) + g(qx^k(t-r)).$$

This implies

$$\frac{g(qx^k(t-r)) - qg(x^k(t-r))}{qx^k(t-r)} \ge -(a+b).$$

Therefore, by the assumption (iv), there exists $M_3 > 0$ such that $x^k(t-r) \ge -M_3$. Repeating the argument in the last part of Case (i), we can find a constant $M > 0$ (independent of k) such that $x^k(s) \le M$ for all $s \in \mathbb{R}$.

This completes the proof.

Remark 5.3.2. One can easily verify that all conditions (i)-(iv) are satisfied for the function $g(x) = cx^3$, $c > 0$. Physically, such a function g describes a ***cubic nonlinear conductance*** which can be realized with a tunnel diode or an operational amplifier (see [33, 34]). More generally, one can prove that every function $g(x) = \sum_{i=1}^{n} g_i x^{2i+1}$ with $g_1 > 0$, $g_i \ge 0$, $i \ne 1$ also verifies the condition (i)–(iv). For the use of higher order nonlinear conductance, we refer to [19].

5.4. Self-sustained periodic solutions

In this section, we apply the global symmetric Hopf bifurcation theorem to study the existence of phase-locked oscillations in Eq. (5.3.1).

We begin with the consideration of local Hopf bifurcations. Clearly, $x(t) \equiv 0$ is a solution of Eq. (5.3.1) for any $q \in [0,1)$. The characteristic equation of the stationary point $(q, 0)$ is

$$p(\lambda, q) = \prod_{j=0}^{N-1} p_j(\lambda, q) = 0 \tag{5.4.1}$$

where

$$\begin{aligned} &p_j(\lambda, q) = (\lambda + a_j)e^{\lambda r} - q(\lambda - b_j) \\ &a_j = a + dc_j, \qquad b_j = b - dc_j \\ &c_j = 4\sin^2 \tfrac{\pi j}{N}, \quad j = 0, 1, 2, \dots, N-1. \end{aligned} \tag{5.4.2}$$

The following lemma summarizes useful information about the characteristic equation (5.4.1).

Lemma 5.4.1. *If* $0 < a_j < b_j$, *for some* $j \in \{0, 1, 2, \dots, [\frac{N}{2}]\}$, *then*

(i) the equation

$$\tan \beta r = \frac{(a+b)\beta}{\beta^2 - a_j b_j} \tag{5.4.3}$$

has infinitely many positive solutions $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots \to \infty$ *as* $n \to \infty$, *such that*

(a) if $\sqrt{a_j b_j} = \frac{\pi}{2r}$, *then* $\frac{2r}{n+1} < \frac{2\pi}{\beta_n} < \frac{2r}{n} \le 2r$ *for* $n \ge 1$;

(b) if $\sqrt{a_j b_j} = \frac{\pi}{2r} + \frac{m\pi}{r}$ *for some positive integer* m, *then* $2r < \frac{2\pi}{\beta_1} < 4r$, $\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1} \le 2r$ *for* $2 \le n \le m$ *(when* $m \ge 2$, *)* $\frac{2r}{n+1} < \frac{2\pi}{\beta_n} < \frac{2r}{n} \le 2r$ *for* $n \ge m+1$;

(c) if $\frac{r\sqrt{a_j b_j}}{\pi} - \frac{1}{2}$ *is not an integer, then* $\frac{2\pi}{\beta_1} > 2r$ *and* $\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1} \le 2r$ *for* $n \ge 2$;

(ii) $\pm i\beta_n$ *are characteristic values of the stationary points* $(q_{\pm n}, 0)$, *where*

$$q_{\pm n} = \pm\sqrt{\frac{\beta_n^2 + a_j^2}{\beta_n^2 + b_j^2}}$$

Moreover, if $q > 0$ *and* $q \neq q_n$, $n = 1, 2, \ldots$, *there exists no purely imaginary characteristic value of* $p_j(\lambda, q)$ *of the stationary point* $(q, 0)$;

(iii) *Let* $\lambda_n(q) = u_n(q) + iv_n(q)$ *be the root of (5.4.2), where* q *is close to* q_n *such that* $u_n(q_n) = 0$ *and* $v_n(q_n) = \beta_n$. *Then* $\frac{d}{dq}u_n(q)|_{q=q_n} > 0$.

Proof. (i) We consider the graphs Γ_1 and Γ_2 in the region $\{(\beta, z) : \beta > 0\}$ of the $\beta - z$ plane for the function $z = \frac{(a+b)\beta}{\beta^2 - \sqrt{a_j b_j}}$ and $z = \tan\beta r$, respectively. If $\sqrt{a_j b_j} = \frac{\pi}{2r}$, then, as Figure 5.4.1 shows, Γ_1 and Γ_2 have infinitely many intersections (β_n, z_n) such that

$$\frac{n\pi}{r} < \beta_n < \frac{(2n+1)\pi}{2r}, \quad n = 1, 2, 3, \ldots.$$

This gives

$$\frac{2r}{n+1} < \frac{2\pi}{\beta_n} < \frac{2r}{n} \leq 2r, \quad n = 1, 2, 3, \ldots.$$

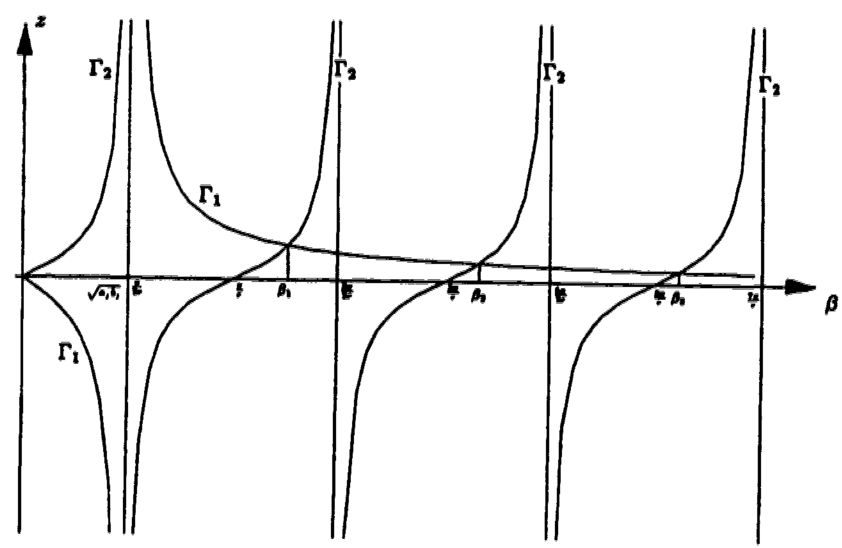

Fig. 5.4.1

If $\sqrt{a_j b_j} = \frac{\pi}{2r} + \frac{m\pi}{r}$ for a positive integer m, then Γ_1 and Γ_2 have infinitely many intersection points (β_n, z_n) such that

$$\frac{(2n-1)\pi}{2r} < \beta_n < \frac{n\pi}{r}, \quad n = 1, 2, \ldots, m$$

and

$$\frac{(m+k)\pi}{r} < \beta_{m+k} < \frac{2(m+k)+1}{2r}\pi, \quad k = 1, 2, \ldots.$$

See Figure 5.4.2.

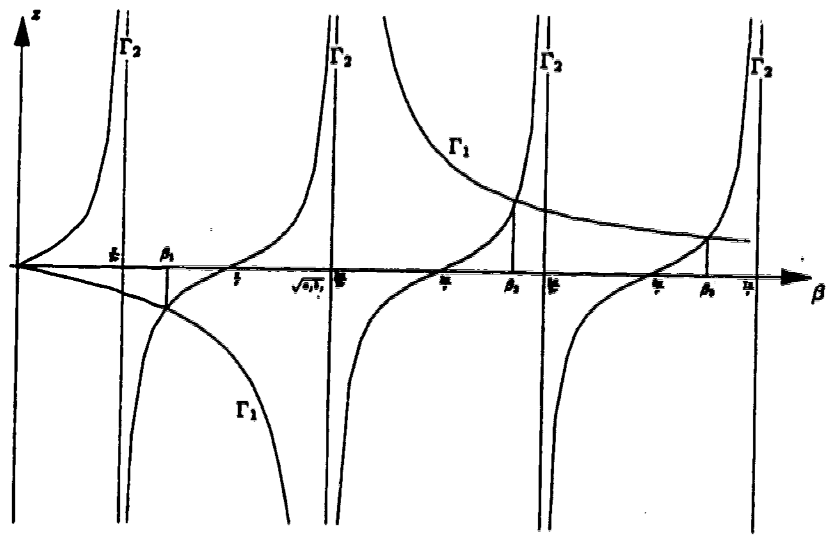

Figure 5.4.2

Therefore,

$$2r < \frac{2\pi}{\beta_1} < 4r,$$

$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n - \frac{1}{2}} < \frac{2r}{n-1}, \quad n = 2, 3, \ldots, m, \quad \text{when} \quad m \geq 2, \quad \text{and}$$

$$\frac{2r}{m+k+1} < \frac{2\pi}{\beta_{m+k}} < \frac{2r}{m+k}, \quad k = 1, 2, 3, \ldots.$$

If $\frac{\pi}{2r} + \frac{(m-1)\pi}{r} < \sqrt{a_j b_j} < \frac{\pi}{2r} + \frac{m\pi}{r}$ for some nonnegative integer m, then Γ_1 and Γ_2 have infinitely many intersection points (β_n, z_n) such that

$$\frac{(n-1)\pi}{r} < \beta_n < \frac{(2n-1)\pi}{2r}, \quad n = 1, 2, \ldots$$

in the case $m = 0$ (see Figure 5.4.3),

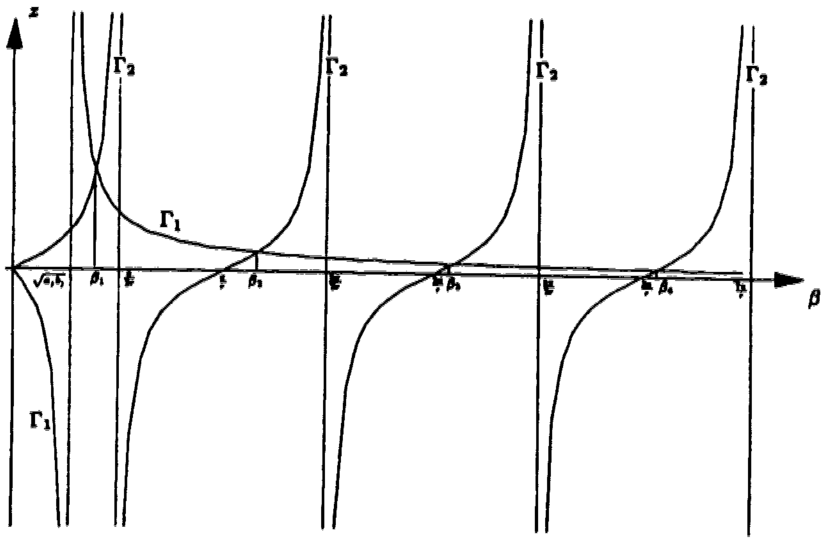

Figure 5.4.3

and

$$\frac{(2n-1)\pi}{2r} < \beta_n < \frac{n\pi}{r}, \quad n = 1, 2, \ldots, m$$
$$\frac{m+k+1}{r}\pi < \beta_{m+k} < \frac{2(m+k)-1}{2r}\pi, \quad k = 1, 2, 3, \ldots$$

in the case $m \geq 1$ (see Figure 5.4.4).

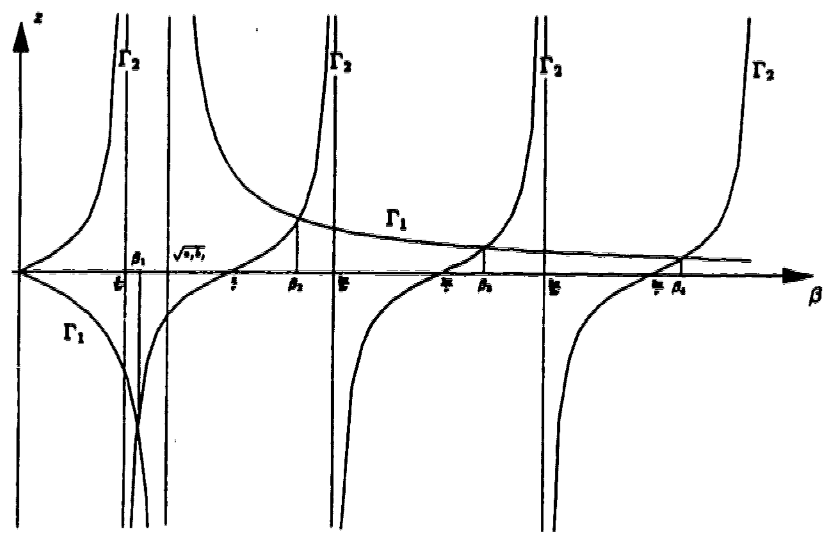

Figure 5.4.4

Therefore, we have

$$\frac{2\pi}{\beta_1} > 4r,$$
$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-1} \le 2r, \quad \text{for} \quad n \ge 2$$

if $m = 0$ and

$$2r < \frac{2\pi}{\beta_1} < 4r,$$
$$\frac{2r}{n} < \frac{2\pi}{\beta_n} < \frac{2r}{n-\frac{1}{2}} < \frac{2r}{n-1} \le 2r, \quad n = 2, 3, \ldots, m, \quad \text{when} \quad m \ge 2$$
$$\frac{2r}{m+k} < \frac{2r}{m+k-\frac{1}{2}} < \frac{2\pi}{\beta_m+k} < \frac{2r}{m+k-1} \le 2r, \quad k = 1, 2, \ldots.$$

if $m \ge 1$. This completes the proof of (i).

To prove (ii), we let $\lambda = i\beta$ in the j-th characteristic equation $p_j(\lambda, q) = 0$ of (5.4.1). By (5.4.2), this gives

$$(i\beta + a_j)e^{i\beta r} = q(i\beta - b_j)$$

which is equivalent to

$$\begin{cases} -a_j\cos\beta r+\beta\sin\beta r=qb_j \\ \beta\cos\beta r+a_j\sin\beta r=q\beta. \end{cases}$$

Thus

$$\begin{cases} \tan\beta r=\beta\frac{a+b}{\beta^2-a_jb_j} \\ \beta^2=\frac{q^2b_j^2-a_j^2}{1-q} \end{cases}$$

from which (ii) follows immediately.

Finally, we prove (iii). By viewing λ as a function of q, we differentiate both sides of $p_j(\lambda,q)=0$. It follows that

$$\frac{d\lambda}{dq}=\frac{\lambda-b_j}{[1+r(\lambda+a_j)]e^{\lambda r}-q}. \tag{5.4.4}$$

Note that $p_j(\lambda,q)=0$ implies

$$\lambda-b_j=\frac{(\lambda+a_j)e^{\lambda r}}{q}. \tag{5.4.5}$$

Substituting (5.4.5) into (5.4.4), we obtain

$$\begin{aligned}(\frac{d\lambda}{dq})^{-1}&=\frac{[1+r(\lambda+a_j)]e^{\lambda r}-q}{(\lambda+a_j)e^{\lambda r}}\\ &=\frac{1}{\lambda+a_j}+r-\frac{q}{\lambda-b_j}.\end{aligned} \tag{5.4.6}$$

Therefore, with (5.4.6) in mind, we have

$$\begin{aligned}\text{Sign}\{\tfrac{d}{dq}u_n(q)\}\Big|_{q=q_n}&=\text{Sign}\{\tfrac{d}{dq}\text{Re}\,\lambda\}\Big|_{q=q_n}\\ &=\text{Sign}\{\text{Re}\,(\tfrac{d\lambda}{dq})\}\Big|_{q=q_n}=\text{Sign}\{\text{Re}\,(\tfrac{d\lambda}{dq})^{-1}\}\Big|_{q=q_n}\\ &=\text{Sign}\{\text{Re}\,(\tfrac{1}{\lambda+a_j}+r-\tfrac{q}{\lambda-b_j})\}\Big|_{\substack{\lambda=i\beta_n\\ q=q_n}}\\ &=\text{Sign}\{r+\tfrac{a_j}{a_j^2+b_j^2}+\tfrac{q_nb_j}{b_j^2+\beta_n^2}\}=1>0.\end{aligned}$$

This proves (iii).

The proof of Lemma 5.4.1 now is completed.

According to Lemma 5.4.1, $(q_n, 0)$ is an isolated center of (5.3.1) and its j-th crossing number $\gamma_j(q_n, 0, \beta_n) = -1$. By the local symmetric Hopf bifurcation theorem (Theorem 3.3.4), $(q_n, 0)$ is a (local) bifurcation point and hence a branch of nonconstant phase-locked periodic solutions $\{x^k(t)\}_{k=1}^N$, with period $p > 0$ close to $\frac{2\pi}{\beta_n}$ and $x^{k-1}(t) = x^k(t - \frac{j}{n}p)$, $k = 1, 2, \ldots, N$, $(\operatorname{mod} N)$, bifurcates from the stationary point $(q_n, 0)$.

To investigate the maximal continua of the above (local) branch of phase-locked periodic solutions, we now apply the global Hopf bifurcation theorem (Theortem 3.3.6) in conjuction with the two lemmas in section 5.3. The results we obtain are formulated in the following theorem.

Theorem 5.4.2. *Suppose* $0 < a_j < b_j$, *where* N *is even and* $j = \frac{N}{2}$. *Assume that* g *satisfies the conditions (i)–(iv) in Lemma 5.5.3.*

(i) If $\sqrt{a_j b_j} = \frac{\pi}{2r}$, *then for any* $n \geq 1$ *and* $q \in (q_n, 1)$, *Eq. (5.3.1) has* n *phase-locked periodic solutions* $\{x_{l,q}^k(t)\}_{k=1}^N$ *whose periods* $p_{l,q}$ *satisfy* $\frac{2r}{l+1} < p_{l,q} < \frac{2r}{l}$, $l = 1, 2, \ldots, n$ *and* $x_{l,q}^{k-1}(t) = x_{l,q}^k(t - \frac{j}{N}p_{l,q})$, $k = 1, 2, \ldots, N$, $(\operatorname{mod} N)$;

(ii) If $\sqrt{a_j b_j} = \frac{\pi}{2r} + \frac{m\pi}{r}$ *for some positive integer* m , *then for any* $n \geq 2$ *and* $q \in (q_n, 1)$, *Eq. (5.3.1) has* $n-1$ *phase-locked periodic solutions* $\{x_{l,q}^k(t)\}_{k=1}^N$ *whose periods* $p_{l,q}$ *satisfy* $\frac{2r}{l} < p_{l,q} < \frac{2r}{l-1}$ *for* $2 \leq l \leq m$ *(when* $m \geq 2$ *),* $\frac{2r}{l+1} < p_{l,q} < \frac{2r}{l}$ *for* $m+1 \leq l \leq n$, *and* $x_{l,q}^{k-1}(t) = x_{l,q}^k(t - \frac{j}{N}p_{l,q}), k = 1, 2, \ldots, N$, $(\operatorname{mod} N)$;

(iii) If $\frac{r\sqrt{a_j b_j}}{\pi} - \frac{1}{2}$ *is not an integer, then for any* $n \geq 2$ *and* $q \in (q_n, 1)$, *Eq. (5.3.1) has* $n-1$ *phase-locked periodic solutions* $\{x_{l,q}^k(t)\}_{k=1}^N$ *where*

periods $p_{l,q}$ *satisfy* $\frac{2r}{l} < p_{l,q} < \frac{2r}{l-1}$ *for* $l = 2, 3, \ldots, n$ *and* $x_{l,q}^{k-1}(t) = x_{l,q}^{k}(t - \frac{j}{N}p_{l,q}), k = 1, 2, \ldots, N, \quad (mod\, N)$.

Proof. We only give the proof for (iii). Other cases can be proved analogously.

For any fixed positive integer n, we consider the following neutral equations

$$\begin{aligned}\frac{d}{dt}D(Q_n(\alpha))x_t^k &= -ax^k(t) - bQ(\alpha)x^k(t-r) - g(x^k(t)) + Q_n(\alpha)g(x^k(t-r)) \\ &\quad + dD(Q_n(\alpha))[x_t^{k+1} - 2x_t^k + x_t^{k-1}] \\ &\quad k = 1, 2, 3, \ldots, N, \quad (\mathrm{mod}\, N)\end{aligned} \tag{5.4.7}$$

where

$$Q_n(\alpha) = \frac{q_{n+1} + \frac{a}{b}}{\pi}(\arctan\alpha + \frac{\pi}{2}) - \frac{a}{b}.$$

Note that $Q_n(\alpha)$ is an increasing function with $\lim_{\alpha \to -\infty} Q_n(\alpha) = -\frac{a}{b}$. The map $B : \mathbb{R} \times C([-r, 0]; \mathbb{R}) \to \mathbb{R}$ defined by

$$B(\alpha, \varphi) = Q_n(\alpha)\varphi(-r)$$

for $(\alpha, \varphi) \in \mathbb{R} \times C([-r, 0]; \mathbb{R})$ satisfies a Lipschitz condition with Lipschitz constant $k = q_{n+1} < 1$.

Under the assumption (i) of Lemma 5.3.3, we can easily show that for any $\alpha \in (-\infty, \infty)$, if (α, x_0) is a homogeneous stationary point of Eq. (5.3.1), i.e. $x_0^{k-1} = x_0^k$, $k = 1, 2, 3, \ldots, N$, $(\mathrm{mod}\, N)$, where $x_0 = (x_0^1, x_0^2, \ldots, x_0^N)$, then $x_0 = 0$. Moreover, using (5.4.2), we can show that 0 is never a characteristic value since $Q_n(\alpha) > -\frac{a}{b}$. Therefore, all stationary points of (5.4.7) are nonsingular. Let $\alpha_l = Q_n^{-1}(q_l)$ for $l = 2, 3, \ldots, n$, where Q_n^{-1} denotes the inverse function of Q_n. Then $(0, \alpha_l)$ are isolated centers of (5.4.7) for each $2 \le l \le n$ by Lemma 5.4.1. Except at these isolated centers, there are no other purely imaginary characteristic values of $(0,\alpha,\beta_l)$ which satisfy $\gamma_j(0,\alpha,\beta_l)=-1$ for each $2\le l\le n$. Fix now $n\ge 2$ and consider the set

$$\begin{aligned}\mathcal{S}=Cl\{\,&(z,\alpha,p);\ x(t)=z(\tfrac{t}{p})\quad\text{is a}\quad p\text{-periodic solution of (5.4.7)}\\&\text{with } x^{k-1}(t)=x^k(t-\tfrac{p}{2}),\ t\in\mathbb{R},\ k=1,2,\dots,N,\ (\operatorname{mod} N)\,\}\\&\subset C(S^1;\mathbb{R}^N)\times\mathbb{R}^2.\end{aligned}$$

Let $\mathcal{C}(0,\alpha,\frac{2\pi}{\beta_l})$ denote the connected component of $\mathcal{S}$ containing $(0,\alpha,\frac{2\pi}{\beta_l})$. By the statements after Lemma 5.4.1, $\mathcal{C}(0,\alpha,\frac{2\pi}{\beta_l})$ is nonempty. Moreover, the Global Hopf Bifucation Theorem (Theorem 3.3.4) implies that $\mathcal{C}(0,\alpha,\frac{2\pi}{\beta_l})$ must be unbounded.

Recall that $\frac{2r}{l}<\frac{2\pi}{\beta_l}<\frac{2r}{l-1}$. By Lemma 5.3.1 and 5.3.3, there exists a constant $M_n=M_n(q_{n+1})>0$ such that

$$\mathcal{C}(0,\alpha,\tfrac{2\pi}{\beta_l})\subset BC(M_n)\times\mathbb{R}\times[\tfrac{2r}{l},\tfrac{2r}{l-1}],$$

where

$$BC(M_n)=\{\,y\in BC(\mathbb{R};\mathbb{R}^N);\ \sup_{t\in\mathbb{R}}|y(t)|<M_n\,\}.$$

On the other hand, since $Q_n(\alpha)$ increases from $-\frac{a}{b}$ to q_{n+1}, there is z_n such that $Q_n(z_n)=0$. At $\alpha=\alpha_n$, Eq. (5.4.7) reduces to the following ordinary differential equations

$$\begin{aligned}\frac{d}{dt}x^k(t)=&-ax^k(t)-g(x^k(t))+d(x^{k+1}(t)-2x^k(t)+x^{k-1}(t))\\&k=1,2,\dots,N,\quad(\operatorname{mod} N).\end{aligned}\tag{5.4.8}$$

By Lemma 5.3.2, Eq. (5.4.8) has no nonconstant periodic solutions. So we further conclude that

$$\mathcal{C}(0,\alpha,\tfrac{2\pi}{\beta_l})\subset BC(M_n)\times(z_n,\infty)\times[\tfrac{2r}{l},\tfrac{2r}{l-1}].$$

Therefore, since $C(0,\alpha,\frac{2\pi}{\beta_l})$ is unbounded, the projection of $C(0,\alpha,\frac{2\pi}{\beta_l})$ onto the parameter (α)-space must be unbounded above. This implies that for every $\alpha > \alpha_l$, Eq. (5.4.7) has a nonconstant phase-locked periodic solution $\{x_{l,\alpha}^k(t)\}_{k=1}^N$ with period $p_{l,\alpha} \in (\frac{2r}{l}, \frac{2r}{l-1})$ and $x_{l,\alpha}^{k-1}(t) = x_{l,\alpha}^k(t - \frac{p_{l,\alpha}}{2}), k = 1, 2, \ldots, N, \pmod N$. This, in turn, implies that for all $q \in (q_l, q_{n+1})$, Eq. (5.3.1) has a nonconstant phase-locked periodic solution $\{x_{l,q}^k(t)\}_{k=1}^N$ with period $p_{l,q} \in (\frac{2r}{l}, \frac{2r}{l-1})$ such that $x_{l,q}^{k-1}(t) = x_{l,q}^k(t - \frac{p_{l,q}}{2}), k = 1, 2, \ldots, N, \pmod N$. This completes the proof.

We end this section with several remarks.

Remark 5.4.1. We note that the existence of phase-locked periodic solutions of periods less than $2r$ for Eq. (5.3.1) with $q \in (q_2, 1)$ has been guaranteed by Theorem 5.4.2 in all cases. We call these solutions *rapidly oscillating solutions*. It has been observed, both numerically and theoretically, that rapidly oscillating periodic solutions appear to be unstable for many retarded equations. It is still a question whether or not the same phenomenon happens to the neutral equations.

Remark 5.4.2. For the existence of phase-locked periodic solutions with period greater than $2r$, we are unable to obtain the global results. A possible reason for this is that phase-locked periodic solution with period equal to nr, where $n \geq 2$ is an integer, still may exist. However, combining the local Hopf bifucation theorem (Theorem 3.3.4) and Lemma 5.4.1, we can conclude that phase-locked periodic solutions with period greater than $2r$ do exist for q *near* q_1 in the case (ii) and (iii). We call these periodic solutions *slowly oscillating solutions*. It is also an interesting question whether or not these periodic solutions are stable.

Remark 5.4.3. If $d = 0$ in Eq. (5.3.1), i.e. there is no coupling between lines, then Theorem 5.4.2 gives also a global branch of synchronous oscillating solutions. Physically, this can be interpreted as each terminal voltage osclllating in the same way (each voltage is of identically the same amplitude at any time) so that there is no current flowing through the coupling resistor R (in this case, we can take $R = \infty$ and $d = 0$).

Remark 5.4.4. It follows from Theorem 5.4.2 that if $0 < a_j < b_j$ for $j = \frac{N}{2}$ (when N is even), then the system (5.3.1) has large amplitude periodic solutions. It is not difficult to see that $0 < a_j < b_j$ for $j = \frac{N}{2}$ is equivalent to

$$0 < \gamma RC - 4 < R/Z. \tag{5.4.9}$$

It follows that, if $ZC > \frac{1}{\gamma}$, choosing a large coupling resistance R will guarantee (5.4.9). This also implies that if the lumped parallel capacitance C or the characteristic impedance Z is large, there likely exist phase-locked oscillations. Further, the synchronous oscillations always exist in the system (taking $R = \infty$, see Remark 5.4.3). This analysis seems in agreement with that obtained by Shimura [55].

5.5. Conclusions and discussions

In this chapter, we have studied the ring structured, resistively-coupled lossless transmission lines. The telegrapher's equation is reduced to a symmetric neutral system. We have proved, under fairly general conditions, the existence of *large amplitude* phase-locked and synchronous periodic solutions. To the best of our knowledge, it is the first global result on the existence of periodic solutions for the n-dimensional autonomous neutral functional differential systems. This is due to the symmetry of the equations in question and our global Hopf bifurcation theorem.

As electric circuits, the transmission lines can be coupled *resistively*, *inductively* (magnetically) or *capacitively* (electrostatically). In this chapter, only resistive coupling (by a common resistor R) is discussed. The same problem for inductive coupling or capacitive coupling should also be addressed. But the differential equations governing the transmission lines will be more complicated. This is beyond the scope of this thesis and we shall consider them elsewhere.

Note also that the inductive coupling may not be electrically connected at all. In this case, the coupling is affected through the *mutual inductance* of the lines in nearest neighbours. Moreover, a combination of the above couplings is also possible. We refer to [7, 25, 56, 58] for more details on circuit couplings.

We are only concerned with the *existence* of symmetric periodic solutions which describe phase-locked or synchronous oscillations. The *stability* of these periodic solutions is an important issue and remains unsolved. We will address this problem in our future investigations. (We have recently received a preprint of Hale [30] where an idea of how to determine the stability of periodic solutions to *neutral equations* is presented.)

A natural question also arises here. Since electric circuits are widely used to simulate biological rhythms, it is plausible to question the applicability of the lossless transmission line equations presented in this chapter to problems of oscillations in biology. The simulations of the classical van del Pol relaxation oscillator in various disciplines are well-known [3–7, 17–19, 25, 34, 40–42, 47, 62, 63, 66]. In neuron electro-physiology, numerous electric circuits have been built to model nerve activities [15, 22, 30–34]. In particular, Hodgkin-Huxley theory [31] on nerve conduction has represented a non-myelinated axon membrance as a one-dimensional transmission line. Although the electrical characterization of the membrance is very different from the (lossless) transmission line we described (the membrance has a distributed constant resistance but has no inductance), it is still interesting to construct a specific realization of the dynamical system (5.3.1) in mathematical biology.

Let us pursue this line of thought somewhat further before we leave this discussion. Actually, we have been led to a well-developed theory of ***dynamical analogies*** [7, 51, 53]. It is well known that ***any electrical system*** can be replaced by an analogous *mechanical system*, and conversely. Under this analogy, we conclude that the electric circuit considered in this chapter can be a simulator of almost all stringed instruments, where the tunnel diode in the transmission line corresponds to the Coulomb friction in the string, the voltage–current corresponds to frictional force–relative velocity and the inductance corresponds to the mass (see, for example, [55]). It is the main purpose of the theory of dynamical analogies to study those systems which are utterly diverse in character, yet there is a precise sense in which certain pairs of diverse systems may be considered dynamically equivalent.

References

[1] V. E. Abolinia and A. D. Mishkis, *A mixed problem for a linear hyperbolic system on the plane*, Latvijas Valsts Univ. Zinatn. Raksti **20** (1958), 87–104.

[2] V. E. Abolinia and A. D. Mishkis, *Mixed problems for quasi-linear hyperbolic systems in the plane*, Mat. Sb. (N.S.) **50** (1960), 423–442.

[3] J. C. Alexander and J. F. G. Auchmuty, *Global bifurcation of phase-locked oscillators*, Arch. Rational Mech. Anal. **93** (1986), 253–270.

[4] S. E. D. Alian and D. A. Linkens, *Mode analysis of a tubular structure of coupled non-linear oscillators for digestive-tract modelling*, Bull. Math. Biol. **471** (1985), 71–110.

[5] A. A. Andronov, A. A. Vitt and S. E. Khaikin, "Theory of oscillators", Pergamon press, Oxford, 1966

[6] J. Bélair and P. Holmes, *On linearly coupled relaxation oscillations*, Quart. Appl. Math. **42** (1984), 193–219.

[7] W. G. Bickley and A. Talbot, "An Introduction to the Theory of Vibrating Systems", Oxford University Press, 1961

[8] R. K. Brayton, *Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type*, Quart. Appl. Math. **24** (1966), 215–224.

[9] ———, *Nonlinear oscillations in a distributed network*, Quart. Appl. Math. **24** (1967), 289–301.

[10] ——— and W. L. Miranker, *A stability theory for nonlinear mixed initial boundary value problems*, Arch. Rational Mech. Anal. **17** (1964), 358–376.

[11] ——— and J. K. Moser, *A theory of nonlinear networks–I*, Quart. Appl. Math. **22** (1964), 1–33.

[12] S. Chowdhury, J. S. Barkatullah, D. Zhou, E. W. Bai and K. E. Lonngren, *A transmission line simulator for high-speed interconnects*, IEEE Trans. on Circuits and Systems, **CAS-39** (1992), 201–211.

[13] K. L. Cooke and D. W. Krumme, *Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations*, J. Math. Anal. Appl. **24** (1968), 372–387.

[14] K. L. Cooke, *A linear mixed problem with derivarive boundary conditions*, Lecture Series **15**, Institute for Fulid Dynamics and Applied Mathematics, University of Maryland, 1970.

[15] J. Cronin,"Mathematical Aspects of Hodgkin-Huxley Neural Theory", Cambridge University Press, New York, 1987.

[16] M. A. Cruz and J. K. Hale, *Stability of functional differential equations of neutral type*, J. Differential Equations **7** (1979), 334–355.

[17] T. Endo and S. Mori, *Mode analysis of a muitimode ladder osciallator*, IEEE Trans. on Circuit and System, **CAS-23** (1976), 100–113.

[18] ——— and ———, *Mode analysis of a two-dimensional low-pass multimode oscillator*, IEEE Trans. on Circuit and System, **CAS-23** (1976), 517–530.

[19] ——— and ———, *Mode analysis of a ring of a large number of mutually coupled van del Pol oscillators*, IEEE Trans. on Circuit and System, **CAS-25** (1978), 7–18.

[20] J. M. Ferreira, *On the stability of a distributed network*, SIAM J. Math. Anal. **17** (1986), 38–45.

[21] B. Fiedler, "Global Bifurcation of Periodic Solutions with Symmetry", Lecture Notes in Mathematics **1309**, Springer-Verlag, New York, 1988.

[22] R. FitzHugh, *Mathematical models of excitation and propagation in nerve*, in Biological Engineering (H. P. Schwan, ed.), Mc Graw-Hill, New York, 1969.

[23] F. E. Gardiol, "Lossy Transmission Lines", Artech House, MA, 1987.

[24] K. Geba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems, III: Bifurcation theorems of functional differential equations with symmetries*, Preprint, 1993.

[25] J. Grasman, "Asymptotic Methods for Relaxation Oscillations and Applications", Springer-Verlag, New York, 1987.

[26] J. P. Gollub, T. O. Brunner and B. G. Danly, *Periodicity and chaos in coupled nonlinear oscillators*, Science **200** (1978), 48–50.

[27] M. Golubitsky , I. Stewart and D. G. Schaeffer, "Singularities and Groups in Bifurcation Theory", Volume II, Springer-Verlag, New York, 1988.

[28] J. K. Hale, "Theory of Functional Differential Equations", Springer-Verlag, New York, 1977.

[29] ——— and S. Lunel, "Functional Differential Equations", Springer-Verlag, New York, 1993.

[30] ———, *Partial neutral functional differential equations*, Preprint, 1993.

[31] A. L. Hodgkin and A. F. Huxley, *A qualitative description of membrance current and its application to conduction and excitation in nerve*, J. Physiol. London **117** (1952), 500–544.

[32] F. C. Hoppensteadt, *Electrical models of neurons*, Lectures in Applied Mathematics (F. C. Hoppensteadt, ed.), Volume **19**, 1981, 327–344.

[33] ———, "An Introduction to the Mathematics of Neurons", Cambridge Studies in Mathematical Biology **6**, Cambridge University Press, New York, 1986.

[34] J. P. Keener, *Analog circuitry for the van der Pol and Fitztlugh-Nagumo equations*, IEEE Trans. on Systems, Man and Cybernetics, **SMC-13** (1983), 1010–1014.

[35] V. B. Kolmanovskii and V. R. Nosov, *Instability of systems with aftereffect*, Automation and Remote Control **44** (1983), 24–32.

[36] ——— and ———, *Neutral-type systems with aftereffect*, Automation and Remote Control **45** (1984), 1–28.

[37] ——— and ———, "Stability of Functional Differential Equations", Academic Press, London, 1986.

[38] N. S. Koshlyakov, M. M. Smirnov and E. B. Gliner, "Differential Equations of Mathematical Physics", North-Holand Publishing Company, Amsterdam, 1964.

[39] W. Krawcewicz, J. Wu and H. Xia, *Global Hopf bifurcation theory for condensing fields and neutral equations with applications to lossless transmission problems*, Canad. Appl. Math. Quartly **1** (1993), 167–220.

[40] D. A. Linkens, *Analytical solution of large number of mutually coupled near-sinusoidal oscillators*, IEEE Trans. on Circuit and System, **CAS-21** (1974), 294–300.

[41] ———, *The stability of entrainment conditions for RLC coupled van der Pol oscillators*, Bull. Math. Biol. **39** (1977), 359–372.

[42] ———, I. Taylor and H. L. Duthie, *Mathematical modelling of the colorectal myelectrical activity in humans*, IEEE Trans. Biomedical Engineering, **BME-23** (1976), 101–110.

[43] O. Lopes, *Forced oscillations in nonlinear neutral differential equations*, SIAM. J. Appl. Math. **29** (1975), 196–201.

[44] ———, *Stability and forced oscillations*, J. Math. Anal. Appl. **55** (1976), 686–698.

[45] P. C. Magnusson, G. C. Alexander and V. K. Tripathi, "Transmission Lines and Wave Propagation", 3rd edition, CRC Press, Boca Raton, 1992.

[46] S. I. Minkin and Yu. S. Sklyarov, *Analysis of transients in long dc lines by methods of the theory of differential-difference equations*, Izv. Vuzov. Elektromekhanika **7** (1955), 687–694.

[47] N. Minorsky, "Nonlinear Oscillations", D. Van Nostrand Company, New York, 1962.

[48] W. L. Miranker, *Periodic solution of the wave equation with a nonlinear interface condition*, IBM J. Res. Develop. **5** (1961), 2–24.

[49] J. Nagumo and M. Shimura, *Self-oscillation in a transmission line with a tunnel diode*, Proc. IRE **49** (1961), 1281–1291.

[50] Y. Nishio and S. Mori, *Mutually coupled oscillators with an extremely large number of steady states*, IEEE International Symposium on Circuit and Systems, **ISCAS** (1992), 819–822.

[51] H. F. Olson, "Dynamical Analogies", D. Van Nostrand Company, New York, 1958.

[52] F. Romeo and M. Santomauro, *Time-domain simulation of n coupled transmission lines*, IEEE Trans. on Microwave Theory Tech. **MTT-35** (1987), 131–136.

[53] R. Rosen, "Dynamical System Theory in Biology, Volume 1: Stability Theory and Its Applications", Wiley-Interscience, New-York, 1970.

[54] S. K. Sarna, E. E. Daniel and Y. T. Kingman, *Simulation of slow-wave electrical activity of small intestine*, Amer. J. Physiol. **221** (1971), 166–175.

[55] M. Shimura, *Analysis of some nonlinear phenomena in a transmission line*, IEEE Trans. on Circuit Theory **14** (1967), 60–68.

[56] H. H. Skilling, "Electrical Engineering Circuits", 2nd edition, John Wiley & Sons, New York, 1985.

[57] M. Slemrod, *Nonexistence of oscillations in a distributed network*, J. Math. Anal. Appl. **36** (1971), 22–40.

[58] S. Smale, *On the mathematical foundations of electrical circuit theory*, J. Diff. Geom. **7** (1972), 193–210.

[59] A. N. Tikhonov and A. A. Samarskii, "Equations of Mathematical Physics", Pergamn Press, The Macmillan Company, 1963.

[60] V. K. Tripathi and J. B. Rettig, *A SPICE model for multiple coupled microstrips and other transmission lines*, IEEE MTT-S Digest, 1985, 703–706.

[61] A. M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. London **B 237** (1952), 37–72.

[62] B. Van der Pol, *The nonlinear theory of electric oscillatons*, Proc. IRE **10** (1934), 1051–1086.

[63] ——— and J. van del Mark, *The heartbeat considered as a relaxation oscillation and an electrical model of the heart*, Phil. Mag. **6** (1928), 763–775.

[64] J. Winnerl, F. Neppl, A. Lill, G. Röska and W. Zatsch, *Ring oscillator structure for realistic dynamic stress of mosfets and interconnects*, IEEE Proc. on Mircroelectronic Test Structures, **1** (1988), 56–60.

[65] J. Wu, *Global continua of periodic solutions to some difference-differential equations of neutral type*, Tohuku Math. J. **45** (1993), 67–88.

[66] E. C. Zeeman, *Differential equations for the heartbeat and nerve impulse*, Dynamical Systems (M. M. Peixoto, ed.), Academic Press, New York, 1973, 683–741.

APPENDIX

Bounds for the Period of Periodic Solution to NFDEs

We consider the following two types of neutral equations

$$\frac{d}{dt}\left[x(t) - b(x_t, \alpha)\right] = F(x_t, \alpha) \tag{DE}$$

and

$$x'(t) = f(x_t, \dot{x}_t, \alpha) \tag{GE}$$

where $0 \leq a \leq \infty$, b and F are continuous maps from $S \times I$ into $\mathbb{R}^n$ and $f : S \times T \times I \to \mathbb{R}^n$ is continuous, S and T are subsets of $BC((-\infty, a]; \mathbb{R}^n)$ of continuous bounded functions, I is an open interval of $\mathbb{R}$, x_t (resp. $\dot{x}_t$) $\in C_a := BC((-\infty, a]; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t+\theta)$ (resp. $\dot{x}_t(\theta) = x'(t+\theta)$) for $\theta \in (-\infty, a]$. The norm of C_a, denoted by $\|\cdot\|_\infty$, is given by $\|\varphi\|_\infty = \sup_{\theta \in (-\infty, a]} |\varphi(\theta)|$.

We say b, F and f satisfy the Lipschitzian conditions uniformly with respect to α if the following inequalities hold

$$|F(\varphi, \alpha) - F(\psi, \alpha)| \leq L\|\varphi - \psi\|_\infty,$$
$$|b(\varphi, \alpha) - b(\psi, \alpha)| \leq k\|\varphi - \psi\|_\infty,$$

for all φ, $\psi \in S$, $\alpha \in I$ and

$$|f(\varphi_1, \psi, \alpha) - f(\varphi_2, \psi, \alpha)| \leq L\|\varphi_1 - \varphi_2\|,$$
$$|f(\varphi, \psi_1, \alpha) - f(\varphi, \psi_2, \alpha)| \leq k\|\psi_1 - \psi_2\|,$$

for all $\varphi_1, \varphi_2, \varphi \in S$ and $\psi_1, \psi_2, \psi \in T$, $\alpha \in I$, where L and k are nonnegative constants independent of α.

Let $x(t)$ be a nonconstant periodic solution of (DE) (resp. (GE)) with period $p > 0$. We shall show in this appendix that there is a close relationship between L, k and p. To this end, we need a lemma of Vidossich [7].

Lemma 1. *Let X be a Banach space, $V : \mathbb{R} \to X$ a p-periodic function with the following properties:*

(i) V is integrable and $\int_0^p V(t)\,dt = 0$;

(ii) there exists $U \in L^1([0, p/2]; \mathbb{R}^+)$ such that

$$|V(t) - V(s)| \le U(t-s)$$

for almost all s, t with $0 \le s \le t \le p$, $t - s \le \frac{p}{2}$. Then

$$p \sup_{t \in \mathbb{R}} |V(t)| \le 2 \int_0^{\frac{p}{2}} U(t)\,dt.$$

By using the above Lemma, we can prove the following result which provides an analogue, for neutral equations, of similar ones in [5–8].

Theorem 2. *If F and b satisfy the Lipschitzian conditions with constant L and k, $k < 1$, and $x(t)$ is a p-periodic solution of (DE), then $p \ge 4(1-k)/L$.*

Proof. Let $s \le t$ and $D(\alpha, x_t) = x(t) - b(x_t, \alpha)$. We have

$$\begin{aligned} |D(x_t, \alpha) - D(x_s, \alpha)| &= |x(t) - x(s) + b(x_s, \alpha) - b(x_t, \alpha)| \\ &\ge |x(t) - x(s)| - k\|x_t - x_s\|_\infty \end{aligned}$$

which gives that for all τ

$$|D(x_{t+\tau},\alpha) - D(x_{s+\tau},\alpha)| \geq |x(t+\tau) - x(s+\tau)| - k\|x_{t+\tau} - x_{s+\tau}\|_\infty.$$

By Mean Value Theorem, the above inequality yields

$$|x(t+\tau) - x(s+\tau)| \leq k\|x_{t+\tau} - x_{s+\tau}\|_\infty + |D'(x_\xi,\alpha)|(t-s)$$

for some $\xi \in [s+\tau, t+\tau]$. By taking the supremum with respect to τ on two sides of the above inequality and using the periodicity of $x(t)$ one obtains

$$\|x_t - x_s\|_\infty \leq k\|x_t - x_s\|_\infty + \sup_{\xi\in\mathbb{R}} |D'(x_\xi,\alpha)|(t-s).$$

It follows that

$$\|x_t - x_s\|_\infty \leq \frac{1}{1-k} \sup_{\xi\in\mathbb{R}} |D'(x_\xi,\alpha)|(t-s).$$

On the other hand,

$$\begin{aligned} |D'(x_t,\alpha) - D'(x_s,\alpha)| &= |F(x_t,\alpha) - F(x_s,\alpha)| \\ &\leq L\|x_t - x_s\|_\infty. \end{aligned}$$

Therefore,

$$|D'(x_t,\alpha) - D'(x_s,\alpha)| \leq \frac{L}{1-k} \sup_{\xi\in\mathbb{R}} |D'(x_\xi,\alpha)|(t-s).$$

Let $V(t) = D'(x_t,\alpha)$ and $U(t) = L/(1-k)\sup_{\xi\in\mathbb{R}} |D'(x_\xi,\alpha)|t$. Then applying Lemma 1 we get

$$p \sup_{t\in\mathbb{R}} |D'(x_t,\alpha)| \leq 2\int_0^{\frac{p}{2}} \frac{L}{1-k} \sup_{\xi\in\mathbb{R}} |D'(\xi,\alpha)| s\, ds.$$

Hence $p \geq 4(1-k)/L$ and the proof is complete.

For the general neutral differential equations (GE), we can also prove a similar result on positive lower bounds for the periods of periodic solutions to (GE).

Theorem 3. *If* f *satisfies the Lipschitzian conditions with constants* L *and* k, $k < 1$, *and* $x(t)$ *is a* p*-periodic solution, then* $p \geq 4(1-k)/L$.

Proof. Let $s, t \in I$. We have

$$\begin{aligned} |x'(t) - x'(s)| &= |f(x_t, \dot{x}_t, \alpha) - f(x_s, \dot{x}_s, \alpha)| \\ &\leq k\|\dot{x}_t - \dot{x}_s\|_\infty + L\|x_t - x_s\|_\infty. \end{aligned} \tag{1}$$

This gives that for all $\tau \in \mathbb{R}$,

$$\begin{aligned} |x'(t+\tau) - x'(s+\tau)| &\leq k\|\dot{x}_{t+\tau} - \dot{x}_{s+\tau}\| + L\|x_{t+\tau} - x_{s+\tau}\|_\infty \\ &= k\|\dot{x}_t - \dot{x}_s\|_\infty + L\|x_t - x_s\|_\infty \end{aligned}$$

by the periodicity of $x(t)$. Therefore,

$$\|\dot{x}_t - \dot{x}_s\|_\infty = \sup_{\tau \leq a} |x'(t+\tau) - x'(s+\tau)| \leq \|\dot{x}_t - \dot{x}_s\|_\infty + L\|x_t - x_s\|_\infty$$

and thus

$$\|\dot{x}_t - \dot{x}_s\|_\infty \leq \frac{L}{1-k}\|x_t - x_s\|_\infty. \tag{2}$$

On the other hand, Mean Value Theorem gives

$$|x(t) - x(s)| = |x'(\xi)|(t-s) \leq \|x'\|_\infty(t-s).$$

Consequently,

$$\|x_t - x_s\| \leq \|x'\|_\infty(t-s). \tag{3}$$

Substituting (2) and (3) into (1), one has

$$\begin{aligned}|x'(t) - x'(s)| &\le k \cdot \tfrac{L}{1-k}\|x_t - x_s\|_\infty + L\|x_t - x_s\|_\infty \\ &= \tfrac{L}{1-k}\|x_t - x_s\|_\infty \\ &= \tfrac{L}{1-k}\|x'\|_\infty (t-s).\end{aligned}$$

Choose now $V(t) = x'(t)$ and $U(t) = \frac{L}{1-k}\|x'\|_\infty t$. Applying Lemma 1 leads to

$$p \sup_{t\in\mathbb{R}} |x'(t)| \le 2\int_0^{\frac{p}{2}} \tfrac{L}{1-k}\|x'\|_\infty \, s\, ds = \frac{L}{4(1-k)}\|x'\|_\infty \, p^2$$

which implies that

$$p \ge \frac{4(1-k)}{L}.$$

This completes the proof.

Note that the above results can be improved if the neutral equations (DE) and (GE) reduce to ODEs (see [5–8]). For other improvement and similar results for discrete dynamical systems, we refer to [1–4].

References

[1] S. Busenberg, D. Fisher and M. Martelli, *Better bounds for periodic solutions of differential equations in Banach spaces,* Proc. Amer. Math. Soc. **98** (1986), 376–378.

[2] ——, —— and ——, *Minimal periods of discrete and smooth orbits,* Amer. Math. Monthly **96** (1987), 5–7.

[3] —— and M. Martelli, *Bounds for the period of periodic orbits of dynamical systems,* J. Differential Equations **26** (1987), 359–371.

[4] —— and ——, *Geometric inequalities in normed spaces*, Rocky Mountain J. Math. **22** (1992), 477–492.

[5] H. Lasota and J.Yorke, *Bounds for periodic solutions of differential equations in Banach spaces,* **10** (1971), 83–91.

[6] T. Y. Li, *Bounds for the periodic solutions of differential equations*, J. Math. Anal. Appli. **49** (1975), 124–129.

[7] G. Vidossich, *On the structure of periodic solutions of differential equations*, J. Differential Equations **21** (1976), 263–278.

[8] J. A. Yorke, *Periods of periodic solutions and the Lipschitz constant*, Proc. Amer. Math. Soc. **22** (1969), 509–512.