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Quantum Fields in Curved Space-time

By

David John Lamb ©

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Abstract

The question of how to define particles in a curved spacetime has seen much recent interest. Using a particle definition due to Capri and Roy [7] we calculate the particle creation due to the gravitational interaction in a number of model universes. In chapter 6 the trace anomaly is calculated for a general 1+1 dimensional spacetime. The regularization methods used involve only normal ordering and defining a fairly straightforward integral.

Contents

CHAPTER 1

Introduction

The expression Quantum Field Theory in Curved Spacetime describes the study of the interaction between quantum fields and classical curved backgrounds which are described using general relativity. The usual goal of work done in this area is to quantize a field and then calculate expectation values for observables associated with the field. Perhaps the most important of these observables is the expectation value of the stress-energy tensor. It would be this expectation value that one would place on the right hand side of the Einstein field equations if one wanted to calculate the back reaction. of the field. on the classical background. In most treatments of this subject the back reaction is not addressed and for the most part this thesis is no exception. The idea of eventually calculating the back reaction will, however, influence which models one chooses to investigate and in particular whether such models include particle detectors or not. The approach taken in this thesis is to not include the analysis of particle detector response functions. The reason for this is twofold. First a local object cannot detect particles as we are used to describing them in quantum field theory, particle states are nonlocal objects. Secondly, if one's ultimate goal is to calculate the backreaction of the field, one must have a universe full of particle detectors for the analysis to make physical sense.

The study of quantum fields interacting with gravitational potentials goes as far back as 1932 when Schrödinger published a paper involving the electron in a gravitational field "Diracsches Elektron im Schwerefeld" [1]. Seven years later a second paper was published [2] which found that a single particle interpretation was inconsistent with the mathematical behaviour of solutions to the generalised Klein-Gordon equation for an expanding universe.

There was little work done after this, in the area of quantum field theory in curved spacetime, until L. Parker published the results of his Ph.D. thesis "The creation of particles in an expanding Universe" in two papers [3] and [4]. These papers seem to have interested many people and there was a flurry of activity in the early 70's. This activity then increased even more when in 1974/75 Hawking [5] discovered that black holes are really not all that black but that a collapsing body will radiate particles. This was a fairly difficult calculation and for this reason people attempted to understand the so called Hawking radiation with simpler mathematical models such as eternal black holes [6]. At this time a mathematical connection was made between these gravitational effects and what might be observed by an accelerating observer. This connection was made by calculating the Bogolubov transformation relating the normal Minkowski modes to modes whose positive frequency is defined in terms of the boost parameter τ, the "Rindler time". These observers were soon replaced by explicit particle detectors which then gave a very intuitive explanation as to why one only considered the detector to respond to things within its past light cone. This line of reasoning is required because without it one has no reason to choose the Rindler coordinates which only cover part of Minkowski space over the usual Minkowski coordinates. Intuitively this particle creation can be understood as "Bremsstrahlung" radiation which impedes the detector's acceleration.

With this intuitive notion for the "clicking" of a particle detector it is not difficult to question whether an accelerating observer who is not carrying a detector would observe anything unusual. This is exactly what has happened recently with many authors discovering that an accelerated observer who is in Minkowski space *is* still in Minkowski space and hence still uses the usual Minkowski time with which to define particles, [7],[8] and [15]. Indeed any prescription for defining particles which depends only on the geometry and the observer's position and velocity will not find any particle creation as observed by an accelerating observer in flat space. The analysis cannot differ from the normal analysis of inertial observers in Minkowski space.

Unfortunately quantizing a field propagating on a curved background is not a trivial task. In section 1.1 the usual procedure for quantizing a free field in flat space is presented. This procedure relies heavily on the time translational invariance of the theory, which is normally seen through the Poincaré group. This invariance allows one to decompose a field into its positive and negative frequency parts easily. From here a natural particle interpretation of the theory is clear and the problems associated with what appears to be an infinite vacuum energy are solved in a straightforward manner using only a normal ordering procedure. In this section the concept of Bogolubov transformations which relate different sets of creation and annihilation operators will also be introduced.

In general relativity there isn't a natural means of performing a 3+1 split of a spacetime. Indeed the whole philosophy of general relativity is that of coordinate invariance. If one is going to perform some sort of frequency decomposition of a field it must therefore be done in a coordinate independent manner. Before addressing this issue, the curved space genalization of the Klein Gordon equation is discussed.

As stated earlier, it is really the presence of Poincaré invariance that allows one to perform a frequency decomposition in flat space. Unfortunately the Poincaré transformations do not generalise to curved backgrounds. This does not mean however that one must forget about the Poincaré group entirely. It was this philosophy that led Capri and Roy [7] to propose a procedure which uses the Poincaré group whenever possible. This procedure will be desribed in section 1.2 and is the approach to field quantization that will be used throughout this thesis. This procedure provides a direction of time which can be used to decompose the field into positive and negative frequency parts leading to a natural particle interpretation of the theory as well as a normal ordering procedure. As implied above this prescription is performed using a coordinate independent approach and makes use of the Poincaré group whenever possible. The time which is used to decompose the field is given as the direction normal to the spacelike hypersurface consisting of those spacelike geodesics which are orthogonal to the observer's 4-velocity. This prescription is therefore a coordinate independent prescription which depends only on the geometry and the observer's trajectory. An equivalent procedure was developed independently by Massacand and Schmid [8].

There have been many attempts to define the vacuum for a free field propagating in a nontrivial spacetime. It is a common feature of all these attempts that the choice of vacuum is determined by a particular choice of time coordinate. This is true even for such general quantization procedures as Deutsch and Najmi [17] although there the dependence is not explicit. Instead they require a foliation of spacetime by a family of spacelike hypersurfaces which, in essence, defines "instants of time" and the normals to these surfaces define the "direction of time". The choice of time coordinate, in most computations, has usually been based on calculational convenience and not on a local physical principle.

In chapter 2 we show explicitly that there can be particle creation in a static spacetime. This is done by calculating the Bogolubov transformation relating the creation and annihilation operators from two different spacelike surfaces. Because this transformation involves a non-zero $\beta(p, p')$ coefficient in the Bogolubov transformation an observer who moves from one of these surfaces to the other will observe particle creation. Unfortunately we are not able to calculate the actual spectrum of created particles but we are able to show that particles are created.

The particle creation produced in an anisotropic universe is calculated in the third chapter. The model is that of an anisotropic generalisation of 1+1 deSitter space where the expansion only occurs in 1 of the 3 spatial dimensions. It is found that the spectrum of created particles has a discrete shift. This discrete shift is associated with the one natural length scale of the geometry, which is the curvature. This length scale provides the energy scale by which the spectrum is shifted. The $\beta(p,q)$ coefficient for the Bogolubov transformation calculated is proportional to a series of delta functions whose argument contains $(p+q)$ and half multiples of the root of the curvature.

In chapter 4 we calculate the massive particle creation as seen by a stationary observer in a $1+1$ dimensional deSitter space. The Bogolubov transformation relating the annihilation and creation operators between two spacelike surfaces is calculated. The particle creation, as observed by a stationary observer who moves from the first spacelike surface to the second is then calculated, and shown to be finite, as is expected for a spacetime with finite spatial volume.

In the previous chapter it was shown that particle creation was finite for a model which was compact in space. It was therefore thought that a similar model in 3+1 dimensions may provide a similar result. Almost identical techniques are used to show that in 3+1 deSitter space the particle creation as observed by an observer moving from one spacelike surface to another is also finite. Unfortunately in both these models the spectrum of created particles is too complicated to compute explicitly.

In the final chapter the trace anomaly is calculated by comparing the different particle definitions for two different observers passing through the same point. By using the fact that the expectation value of the stress tensor should transform under rotations the same way as the tangents to the observer's wordlines one is able to calculate the trace anomaly. The reason one has to use two different observers is because one does not know what the normal ordered expectation value for the pressure T^{11} should be. By using two different observers one need only know what the expectation values for the energy and momentum on the preferred hypersurfaces for each observer are. If the state chosen for the expectation value is chosen as the vacuum for one of these observers then the trace anomaly follows much easier than the usual calculations. This procedure was originally suggested by Massacand and Schmid [8].

1.1 Free fields in flat space

To see how the quantization process is generalised to curved space it is beneficial to first review the canonical quantization of a massive free scalar field in flat space. There are basically two stages to the quantization procedure. The first stage is that of solving the field equations and imposing the equal time (anti)-commutation relations. The second stage is that of introducing the particle concept by specifying a representation of the algebra of the fields as operators on a Hilbert space. The field equation for a free scalar massive field propagating in Minkowski space is the Klein Gordon equation

$$\left(\Box + m^2\right)\phi(x) = 0. \tag{1.1}$$

One set of solutions to this differential equation is exponentials of the form

$$f_{\mathbf{k}}(x) \propto e^{-ik\cdot x} \tag{1.2}$$

where

$$\begin{aligned}
k \cdot x &= k_0 x_0 - \mathbf{k} \cdot \mathbf{x} \\
k_0 = \omega_k &= \left(\mathbf{k}^2 + m^2\right)^{\frac{1}{2}} \\
\mathbf{k}^2 &= k_1^2 + k_2^2 + k_3^2.
\end{aligned} \tag{1.3}$$

These modes are said to be positive frequency modes with respect to t as they are eigenfunctions of the operator $\frac{\partial}{\partial t}$

$$\frac{\partial}{\partial t} f_{\mathbf{k}}(x) = -i\omega_k f_{\mathbf{k}}(x). \tag{1.4}$$

The time independent scalar product is defined by:

$$(\phi_1, \phi_2) = -i \int d^3x \phi_1^*(x) \overleftrightarrow{\partial_t} \phi_2(x). \tag{1.5}$$

It is now convenient to limit ourselves to the interior of a box of side L and thus limit the values of k to k_n

$$k_n = \frac{2\pi n}{L} \quad \text{where} \quad n = 0, \pm 1, \pm 2. \ldots \tag{1.6}$$

Now chosing the normalization for the $f_{\mathbf{k}}(x)$ as

$$f_{\mathbf{k}}(x) = \frac{1}{\sqrt{2L^3\omega_k}} e^{-ik\cdot x} \tag{1.7}$$

the $f_{\mathbf{k}}(x)$ are now orthonormal

$$\begin{aligned}
(f_{\mathbf{k}}(x), f_{\mathbf{k}'}(x)) &= -i \int d^3x \left(f_{\mathbf{k}}^*(x) \partial_t f_{\mathbf{k}'}(x) - (\partial_t f_{\mathbf{k}}^*(x)) f_{\mathbf{k}'}(x)\right) \\
&= \delta_{kk'}.
\end{aligned} \tag{1.8}$$

To quantize the field we now treat it as an operator and impose the following equal time commutation relations,

$$\begin{aligned}
[\ \phi(t, \mathbf{x}), \phi(t, \mathbf{x}')\ ] &= 0 \\
[\ \pi(t, \mathbf{x}), \pi(t, \mathbf{x}')\ ] &= 0 \\
[\ \phi(t, \mathbf{x}), \pi(t, \mathbf{x}')\ ] &= i\delta^3(\mathbf{x} - \mathbf{x}')
\end{aligned} \tag{1.9}$$

where

$$\pi(t, \mathbf{x}) = \partial_t \phi(t, \mathbf{x}). \tag{1.10}$$

Expanding the field in terms of the complete orthonormal basis given by the $f_{\mathbf{k}}(x)$ we now write the field as,

$$\phi(x) = \sum_{\mathbf{k}} \left[a_{\mathbf{k}} f_{\mathbf{k}}(x) + a_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}^{*}(x) \right]. \tag{1.11}$$

The equal time commutation (1.9) relations can then be rewritten in terms of the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$

$$\begin{aligned} \left[a_{\mathbf{k}}, a_{\mathbf{k}'} \right] &= 0 \\ \left[a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger} \right] &= 0 \\ \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger} \right] &= \delta_{\mathbf{k}\mathbf{k}'}. \end{aligned} \tag{1.12}$$

We now use these operators to construct a Fock space which leads to a natural particle interpretation. In the Heisenberg picture the quantum states are time independent and are constructed using the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ which are called the annihilation and creation operators respectively. The vacuum state is defined as being that state which is annihilated by all the annihilation operators

$$a_{\mathbf{k}}|0\rangle = 0 \quad \forall\, \mathbf{k}. \tag{1.13}$$

Particle states are now constructed by acting on the vacuum state with the creation operators. A state consisting of a single particle with 3-momentum $\mathbf{k}$ is therefore

$$|\mathbf{k}\rangle = a_{\mathbf{k}}^{\dagger}|0\rangle. \tag{1.14}$$

Generalizing this to many particle states one constructs the state

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\ldots, n_{\mathbf{k}_j}\rangle = \frac{(a_{\mathbf{k}_1}^{\dagger})^{n_1}(a_{\mathbf{k}_2}^{\dagger})^{n_2}(a_{\mathbf{k}_j}^{\dagger})^{n_j}}{\sqrt{n_1! n_2! n_j!}}|0\rangle. \tag{1.15}$$

The factorial factors are required to accomodate the Bose statistics of identical particles. The operation on a given state by the annihilation and creation operators given above leads to

$$\begin{aligned} a^\dagger_{\mathbf{k}_j}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}..., n_{\mathbf{k}_j}, ...\rangle &= \sqrt{n_{\mathbf{k}_j}+1}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}..., n_{\mathbf{k}_j}+1, ...\rangle \\ a_{\mathbf{k}_j}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}..., n_{\mathbf{k}_j}, ...\rangle &= \sqrt{n_{\mathbf{k}_j}}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}..., n_{\mathbf{k}_j}-1, ...\rangle. \end{aligned} \tag{1.16}$$

In this way the normalisation of the states remains constant

$$\langle n_{\mathbf{k}_1}, n_{\mathbf{k}_2}..., n_{\mathbf{k}_j}|n_{\mathbf{k'}_1}, n_{\mathbf{k'}_2}..., n_{\mathbf{k'}_j}\rangle = \delta_{\mathbf{k}_1\mathbf{k}'_1}\delta_{\mathbf{k}_2\mathbf{k}'_2}...\delta_{\mathbf{k}_j\mathbf{k}'_j}. \tag{1.17}$$

It is now possible to introduce operators whose expectation values give us information about different states. One can introduce the operators $N_{\mathbf{k}}$ and N

$$\begin{aligned} N_{\mathbf{k}} &= a^\dagger_{\mathbf{k}}a_{\mathbf{k}} \\ N &= \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}}a_{\mathbf{k}}. \end{aligned} \tag{1.18}$$

From the definition of the vacuum we can see that the expectation value of the above operators in the vacuum state is zero,

$$\langle 0|N_{\mathbf{k}}|0\rangle = 0 \quad \forall \mathbf{k}, \tag{1.19}$$

and for the many particle state the number operator gives the number of particles with that particular momentum,

$$\langle n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ..., n_{\mathbf{k}_j}|N_{\mathbf{k}_j}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ..., n_{\mathbf{k}_j}\rangle = n_{\mathbf{k}_j}. \tag{1.20}$$

The expectation value of the operator N gives the total number of particles in a state. This combination of annihilation and creation operators will form the basis of many operators such as the Hamiltonian. To calculate the Hamiltonian we first calculate the Hamiltonian density by taking the appropriate part the the stress-energy tensor $T_{\mu\nu}(x)$

$$T_{\mu\nu}(x) = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \tfrac{1}{2}m^2\phi^2\eta_{\mu\nu} \tag{1.21}$$

the Hamiltonian density is then

$$T_{00}(x) = \frac{1}{2}\left[(\partial_t\phi)^2 + (\partial_{x_1}\phi)^2 + (\partial_{x_2}\phi)^2 + (\partial_{x_3}\phi)^2 + m^2\phi^2\right]. \tag{1.22}$$

Substituting into this expression ϕ from (1.11) and integrating over all space we get the total Hamiltonian H

$$H = \int d^3x T_{00} = \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \tfrac{1}{2}\right)\omega_{\mathbf{k}}. \tag{1.23}$$

The first term in this expression clearly gives the energy of the state by summing all the different particle energies. The second term which is state independent seems to correspond to the vacuum energy and must somehow be subtracted from the expectation values of the Hamiltonian to give the energy of a state. Unfortunately this vacuum energy is infinite. One means of dealing with this vacuum energy problem is to define a normal ordering presciption which we denote by : :. To put an operator in normal ordered form one simply writes all the annihilation operators to the right of the creation operators. Using this procedure we can write the Hamiltonian (1.23) in normal ordered form.

$$\begin{aligned} :H: &= \sum_{\mathbf{k}} \tfrac{1}{2}\omega_{\mathbf{k}}(:a_{\mathbf{k}}^\dagger a_{\mathbf{k}}: + :a_{\mathbf{k}} a_{\mathbf{k}}^\dagger:) \\ &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \end{aligned} \tag{1.24}$$

In this way the vacuum energy is no longer a problem.

The possibility of particle creation can be understood by examining (1.11) expressed in terms of a different set of modes. If one chooses to decompose the field using a different set of modes this may imply a different particle definition. To see this we suppose that such a decomposition is performed. For now we will not concern ourselves with why one would perform a different decomposition but just look at the consequences. Using a different choice of complete modes one decomposes the scalar field and writes out the equivalent of (1.11) as

$$\phi(x) = \sum_{\mathbf{k}} \left[b_{\mathbf{k}} g_{\mathbf{k}}(x) + b^{\dagger}_{\mathbf{k}} g^{*}_{\mathbf{k}}(x) \right]. \tag{1.25}$$

By now using the orthogonality of the original modes in terms of the inner product (1.8) one can calculate the Bogolubov transformation relating the two sets of creation and annihilation operators

$$a_{\mathbf{k}} = (f_{\mathbf{k}}(x), \phi(x)). \tag{1.26}$$

We now substitute in $\phi(x)$ written in terms of the $b_{\mathbf{k}}$ operators and are left with the Bogolubov transformation

$$\begin{aligned} a_{\mathbf{k}} &= \sum_{\mathbf{k}'} \left\{ (f_{\mathbf{k}}(x), g_{\mathbf{k}'}(x))\, b_{\mathbf{k}} + (f_{\mathbf{k}}(x), g^{*}_{\mathbf{k}'}(x))\, b^{\dagger}_{\mathbf{k}} \right\} \\ &= \sum_{\mathbf{k}'} \left\{ \alpha_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}} + \beta_{\mathbf{k}\mathbf{k}'} b^{\dagger}_{\mathbf{k}} \right\}. \end{aligned} \tag{1.27}$$

At this point it is clear that if the modes $f_{\mathbf{k}}(x)$ are the same as the modes $g_{\mathbf{k}}(x)$ then $\alpha_{\mathbf{k}\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'} = 0$. It is a nonzero β coefficient that implies there is mixing of the positive and negative frequency modes and hence a different definition of particles. If an observer was to move from one point to another and the particle definition changes so as to imply a non-zero β coefficient the observer will observe particle creation. To see this particle creation we examine the composition of the vacuum state which is defined in terms of the $b_{\mathbf{k}}$ operators with the number operator which "counts" the number of $a^{\dagger}_{\mathbf{k}}$ particles in this state

$$\sum_{\mathbf{k}} \left\langle {}_b0 \left| a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \right| {}_b0 \right\rangle = \sum_{\mathbf{k}\mathbf{k}'} |\beta_{\mathbf{k}\mathbf{k}'}|^2. \tag{1.28}$$

If the sum on the right-hand side of (1.28) is finite this implies that the Bogolubov transformation relating the two Fock spaces is unitarily implementable [9]. This is equivalent to saying that the new vacuum and the many particle states constructed on it are in the same Hilbert space as the old vacuum and its corresponding many particle states . A Bogolubov transformation which satisfies the above inequality is said to be Hilbert-Schmidt.

1.2 Coordinate independent time definition

It would be unphysical if one were free to choose any set of modes with which to decompose the field into positive and negative frequency parts and be left with a valid particle interpretation of the theory. In this section a procedure is presented which in a coordinate independent manner selects the positive frequency part of the field. This procedure is due to Capri and Roy [7] and is equivalent to a different procedure, which also uses spacelike geodesics to define the surface of instantaneity, which was developed independently by Massacand and Schmid [8].

These procedures define the surface of instantaneity as being that surface which is orthogonal to the tangent to the observer's worldline and is constructed from spacelike geodesics. For convenience the coordinates of this surface are chosen to be Riemann coordinates based at the observer's position although this is not important as the frequency decomposition only depends on the choice of the time coordinate. The preferred time coordinate of a general point which does not lie on this surface of instantaneity is given by the proper distance along the timelike geodesic which intersects the surface of instantaneity orthogonally. The spatial coordinates of this general point are the Riemann coordinates of the point of intersection, based at the observer's position. As a consequence of this procedure the metric in two dimensions, when expressed in terms of these preferred coordinates, is of the form

$$ds^2 = dt^2 + g_{11}dx^2 \tag{1.29}$$

where $g_{11} < 0$ and $g_{11} = -1 + 0(t^2)$ near the origin of the coordinates.

The wave equation that must be solved in curved space is just a generalisation of the flat space wave equation (1.1). To see how this is accomplished we first rewrite equation (1.1) in terms of the Minkowski metric $\eta_{\mu\nu}$

$$(\partial_\mu(\eta^{\mu\nu}\partial_\nu) + m^2)\phi(x) = 0. \tag{1.30}$$

The generalisation of this equation is

$$(\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) + m^2 + \xi R(x))\phi(x) = 0 \tag{1.31}$$

where $g = -det|g|$. The first two terms are natural covariant generalisations of the flat space wave equation where the partial derivatives are replaced by covariant derivatives. The second term involves the scalar curvature $R(x)$ and a coupling constant ξ. There are two reasons why one includes the second term in this equation[16]. The first reason is that for massless fields and a specific choice of the constant ξ, which only depends on the number of dimensions, the action and equation of motion are conformally invariant. The second reason is that for interacting theories in curved space-time the renormalization of the theory involves a counterterm of the form $R\phi^2$ [9]. For $\xi = 0$ the field is said to be minimally coupled and for the choice $\xi = \frac{n-2}{4(n-1)}$, where n is the dimension of the spacetime, the field is said to be conformally coupled.

In these coordinates the minimally coupled massive Klein Gordon equation is

$$\partial_t^2\phi + \frac{1}{\sqrt{g}}\partial_t(\sqrt{g})\partial_t\phi + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij})\partial_j\phi + m^2\phi = 0. \tag{1.32}$$

To define the positive frequency modes one looks at the spatial part of this equation

$$(\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) + m^2)A_{\mathbf{k}}(t,\mathbf{x}) = \omega_{\mathbf{k}}(t)^2 A_{\mathbf{k}}(t,\mathbf{x}). \tag{1.33}$$

The positive frequency modes are defined as being the modes which satisfy the differential equation (1.32) and the initial conditions

$$\phi^+_{\mathbf{k}}(0,\mathbf{x}) = A_{\mathbf{k}}(0,\mathbf{x}) \qquad \partial_t\phi^+_{\mathbf{k}}(t,\mathbf{x})|_{t=0} = -i\omega_{\mathbf{k}}(0)A_{\mathbf{k}}(0,\mathbf{x}). \tag{1.34}$$

The field can now be written in quantized form as

$$\varphi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{4\pi\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} \varphi^{+}(x) + a^{\dagger}_{\mathbf{k}} \varphi^{+*}(x) \right) \tag{1.35}$$

where $\omega_{\mathbf{k}} = \omega_{\mathbf{k}}(0)$.

It can now be seen how the above particle creation analysis can be understood. Observers who perform the above procedure at different times may find that their definition of positive frequency changes. This then implies that they would have a different particle definition and the resultant particle creation could be calculated as was shown in (1.25) through (1.28).

CHAPTER 2

Particle creation in a static 1+1 dimensional universe

2.1 Introduction

In this chapter we calculate the particle creation that would be observed by an observer who is stationary in a static 1+1 dimensional universe. Although the metric appears static in the original coordinates, when one writes the metric in terms of the preferred coordinates the metric no longer appears static and the observer who moves from one spacelike surface to another observes particle creation. This particle creation is calculated but unfortunately the expressions are too complicated to examine exactly so an approximate form of the Bogolubov tansformations is derived. This approximate form is expressed as an expansion in a parameter which describes the separation of the two spacelike surfaces.

Before getting on with the specifics of the model which is investigated in this chapter we first make a few comments on static spacetimes in general. A metric is called static if in some coordinate system all the metric coefficients are independent of the timelike coordinate t and all the g_{0i} are zero. In such spacetimes there can still be a question as to which time coordinate one should use to decompose a quantum field if for example there are two such times for which the spacetime appears static. In fact if one develops a procedure for deciding which of the static times should be used one may find that in other models the time which makes the spacetime static is not the one chosen by the procedure. This is what occurs in the model investigated in this chapter. To illustrate the case where there are two different static times we first discuss the ramifications of this procedure in Rindler space. In two dimensions this can be seen quite easily. The Rindler coordinatization puts the metric in the form

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{2.1}$$

or by using the coordinate transformations

$$\begin{aligned} t &= \frac{e^{a\xi}}{a}\sinh(a\eta) \\ x &= \frac{e^{a\xi}}{a}\cosh(a\eta) \end{aligned}$$

one finds that Rindler space is just a wedge of Minkowski space with the metric now having the form

$$ds^2 = dt^2 - dx^2. \tag{2.2}$$

Here we actually have two reasons for the choosing the normal Minkowski coordinatization. As mentioned in the introduction, the procedure for defining the preferred coordinates will choose the geodesically complete coordinatization of the manifold if possible and therefore would choose the Minkowski coordinates which describe the entire manifold. One can also look at the commutation relations satisfied by the generators of translation in the two timelike directions and find that one is a generator of boosts and the other is a genuine generator of time translation.

Of further interest are metrics which have been coined ultrastatic. A static metric is ultrastatic if in the same set of coordinates which satisfy the static requirements the metric coefficient $g_{00}(x)$ is equal to unity for all x [16].

In a globally hyperbolic spacetime with one timelike and one spacelike dimension the surface of instantaneity, in this case a line, for a given observer is given by the particular spacelike geodesic which passes through the point at which the observer is located and is normal to the observer's timelike worldline. The direction of time on this surface is then defined to be everywhere normal to this spacelike geodesic. It has been shown that this definition of time is the unique one to obey the physical principle mentioned earlier [7].

The coordinates on the spacelike surface are chosen, for convenience, to be Riemann coordinates based at the observer's position, although any other coordinatization of the spacelike surface will do. To define the direction of time Gaussian Geodesic Normal coordinates are constructed on this spacelike surface so that the time coordinate of some point off the surface is just given by the proper distance from the point to the spacelike surface.

When one expresses the metric in terms of these new coordinates one finds that the metric in 1 + 1 dimensions has the form

$$ds^2 = dt^2 + g_{11}dx^2 \tag{2.3}$$

with $g_{11} < 0$ and $g_{11} = -1 + O(t^2)$ near the origin of coordinates.

To now decompose the field into positive and negative frequency modes we impose initial conditions that force the field to have the correct time dependence ($\exp(-i\omega t)$) in the neighbourhood of this spacelike surface. To ensure this correct time dependence we impose the initial conditions

$$\phi_n^+(t,x)\,|_{t=0} = A_n(0,x) \quad \text{and} \quad (\partial_t\phi_n^+(t,x))\,|_{t=0} = -i\omega_n(0)A_n(0,x) \tag{2.4}$$

where the $A_n(0,x)$ are the eigenmodes of the spatial part of the Laplace-Beltrami operator on the surface $t = 0$ where the decomposition is to be performed.

In section 2.2 we construct the metric in terms of these physically preferred coordinates and write out explicitly the boundary conditions which determine the positive frequency modes at the surface $t = 0$. In section 2.3 a complete set of modes for the Laplace-Beltrami operator for a massive scalar field is obtained and in section 2.4 the orthogonality relation satisfied by these modes is calculated. In section 2.5 we then use these orthogonality relations to impose the physically relevant boundary conditions which were calculated in section 2.2. In section 2.6 the actual particle creation due to the presence of the gravitational field is calculated for an observer who is stationary with respect to the original static coordinates.

2.2 The preferred coordinates

The static spacetime we are interested in is described by the metric [18]

$$ds^2 = \alpha(X)dT^2 - \frac{dX^2}{\alpha(X)} \tag{2.5}$$

where

$$\alpha(X) = 1 - \exp(-q(|X| - r)). \tag{2.6}$$

This spacetime was first investigated by Witten [18] as a 1 + 1 dimensional eternal black hole spacetime. The properties of this spacetime have been studied by R.B.Mann et al. [19] but quantum particle creation was not investigated. To simplify the technical discussion later on we choose r such that $\exp(qr) < 2$; then $-1 < \alpha(X) < 1$.

To construct the preferred coordinates we must solve the geodesic equations for this spacetime. The first integrals of the geodesic equations are

$$\frac{dT}{ds} = \frac{C_0}{\alpha(X)} \quad : \quad \frac{dX}{ds} = \epsilon_1(C_0^2 - \epsilon\alpha(X))^{\frac{1}{2}} \tag{2.7}$$

where $\epsilon_1 = \pm 1$ and $\epsilon = -1$ for spacelike geodesics and $\epsilon = 1$ for timelike geodesics. The spacelike geodesic which is perpendicular to the timelike vector $\frac{1}{\sqrt{\alpha_0}}(1, 0)$,

($\alpha_0 \equiv \alpha(X_0)$) and which can be treated as the tangent vector to the worldline of the observer at $P_0(T_0, X_0)$, is given by setting $C_0 = 0$ and $\epsilon = -1$. We can therefore see on this surface S_0 that $T_0 = T_1$ simply because along the geodesic connecting these points T doesn't change. T_1 is the T coordinate of the point $P_1(T_1, X_1)$ which is the point at which the geodesic from the general point $P(T, X)$ intersects this spacelike surface orthogonally. The preferred time coordinate t is given by the proper distance along the timelike geodesic connecting P_1 to the general point $P(T, X)$ which is normal to the surface S_0 at P_1. This timelike geodesic is given by (2.7) with $C_0^2 = \alpha_1 \equiv \alpha(X_1)$ and $\epsilon = 1$ so that

$$t = \int_{X_1}^{X} dX' \frac{ds}{dX'} = \int_{X_1}^{X} dX' \frac{\epsilon_1}{\sqrt{\alpha(X_1) - \alpha(X')}}. \tag{2.8}$$

One can also calculate the change in the coordinate T along this geodesic

$$T - T_1 = T - T_0 = \int_{P_1}^{P} dT = \int_{X_1}^{X} dX' \frac{\epsilon_1 \epsilon_2 \sqrt{\alpha(X_1)}}{\alpha(X')\sqrt{\alpha(X_1) - \alpha(X')}}. \tag{2.9}$$

These two equations allow us to express the metric in terms of the coordinates t and X_1. The preferred coordinate x on S_0 is now constructed using a 2-bein of orthogonal basis vectors at P_0, $e_0(P_0)$ and $e_1(P_0)$. With $e_0(P_0)$ given by $\frac{1}{\sqrt{\alpha_0}}(1, 0)$ the tangent to the observer's worldline and p^μ given by the tangent vector at P_0 to the geodesic connecting P_0 to P_1, the Riemann normal coordinates η^α of P_1 are given by

$$sp^\mu = \eta^\alpha e_\alpha^\mu(P_0) \tag{2.10}$$

where s is the distance along the geodesic $P_0 - P_1$. Using $e_\alpha^\mu e_{\beta\mu} = \eta_{\alpha\beta}$ (Minkowski metric), and the orthogonality of p^μ to $e_0(P_0)$ we have

$$\eta^0 = sp^\mu e_\mu^0(P_0) \qquad \eta^i = -sp^\mu e_\mu^i(P_0). \tag{2.11}$$

The surface S_0 is just the surface $\eta^0 = 0$ and the coordinate x is

$$x = \eta^1 = -sp^\mu e_\mu^1(P_0) = \int_{X_0}^{X_1} dX' \frac{1}{\sqrt{\alpha(X')}}. \tag{2.12}$$

The preferred coordinates (t, x) are then given by solving the above integrals for $X > 0$ or $X < 0$. After choosing $\epsilon_1 = -1$ one obtains for $X > 0$

$$T = \frac{-2\sqrt{-1+e^{q(-r+X_1)}}\tan^{-1}(\sqrt{-1+e^{q(-X+X_1)}})\,\epsilon_2}{q} + \frac{2\tanh^{-1}\left(\frac{\sqrt{-1+e^{q(-X+X_1)}}}{\sqrt{-1+e^{q(-r+X_1)}}}\right)\epsilon_2}{q} + T_0 \tag{2.13}$$

$$x = \frac{2}{q}\left\{\tanh^{-1}(\sqrt{\alpha_1}) - \tanh^{-1}(\sqrt{\alpha_0})\right\} \tag{2.14}$$

$$t = \frac{2\,e^{\frac{q(-r+X_1)}{2}}\tan^{-1}(\sqrt{-1+e^{q(-X+X_1)}})}{q} \tag{2.15}$$

and for $X < 0$

$$T = \frac{2\sqrt{-1+e^{-(q(r+X_1))}}\tan^{-1}(\sqrt{-1+e^{q(X-X_1)}})\,\epsilon_2}{q} - \frac{2\tanh^{-1}\left(\frac{\sqrt{-1+e^{q(X-X_1)}}}{\sqrt{-1+e^{-(q(r+X_1))}}}\right)\epsilon_2}{q} + T_0 \tag{2.16}$$

$$x = \frac{2}{q}\left\{-\tanh^{-1}(\sqrt{\alpha_1}) + \tanh^{-1}(\sqrt{\alpha_0})\right\} \tag{2.17}$$

$$t = \frac{-2\tan^{-1}(\sqrt{-1+e^{q(X-X_1)}})}{e^{\frac{q(r+X_1)}{2}}\,q} \tag{2.18}$$

where

$$\epsilon_2 = \pm 1. \tag{2.19}$$

From these coordinate transformations we can see that x and t both run from $-\infty$ to $+\infty$ and cover the region of the original space corresponding to $|X| > r$, the region outside the horizon. The region inside the horizon is shrunk to a point. Furthermore, the region between the observer and the horizon ($r < |X| < |X_0|$) is covered twice.

In terms of the coordinates (t, x) the metric is now

$$ds^2 = dt^2 - (1 + tp(x)\tan[tp(x)])^2\,dx^2 \tag{2.20}$$

where

$$p(x) = \frac{q}{2}\text{sech}\left[B(x)\right] \tag{2.21}$$

and

$$B(x) = \tanh^{-1}\left[\sqrt{\alpha_0}\right] + \frac{xq}{2}. \tag{2.22}$$

We can see now that in this coordinate system, which does have a physical basis, the metric no longer appears static.

In these new coordinates the Klein-Gordon equation for a massive scalar field is

$$\partial_t^2\phi + \frac{1}{2}\left(\partial_t \ln(|g|)\right)\partial_t\phi + \frac{1}{\sqrt{|g|}}\partial_x\left(\sqrt{|g|}g^{11}\partial_x\right)\phi + m^2\phi = 0. \tag{2.23}$$

We now define instantaneous eigenfunctions $A_n(t,x)$ of the spatial part of the Laplace-Beltrami operator, such that

$$\left[\frac{1}{\sqrt{|g|}}\partial_x\left(\sqrt{|g|}g^{11}\partial_x\right) + m^2\right]A_k(t,x) = \omega_k^2(t)A_k(t,x). \tag{2.24}$$

The positive frequency solutions of (2.23) are then defined as those which satisfy the initial conditions

$$\phi_k^+(t,x)\,|_{t=0} = A_k(0,x) \quad \text{and} \quad \left(\partial_t\phi_k^+(t,x)\right)|_{t=0} = -i\omega_k(0)A_k(0,x). \tag{2.25}$$

These initial conditions ensure that the positive frequency part of the field has the desired time dependence near the line $t = 0$. These positive frequency solutions form a vector space which is made into a Hilbert space using the standard Klein-Gordon inner product.

From the simple form of the metric at $t = 0$ we see that

$$A_k(0,x) = \sin(2\frac{k}{q}B(x)) \text{ or } \cos(2\frac{k}{q}B(x)) \quad \text{and} \quad \omega_k^2(0) = (k^2 + m^2). \tag{2.26}$$

With the positive frequency solutions defined in this way we can then write out the quantized field as

$$\Psi_1 = \int_0^\infty dk \frac{1}{\sqrt{2\omega_k}} \left\{ \phi_k^+(t,x) a_k + \phi_k^{(+)*}(t,x) a_k^\dagger \right\} \tag{2.27}$$

where the subscript 1 of the field simply denotes the surface on which the positive frequency modes have been defined. In this expression we have written $\omega_k(0)$ as ω_k and we will continue this practice. Unfortunately (2.20) is too complicated to obtain the general form of the modes in terms of the coordinates (t, x). This is, however, not really a problem as the point of this approach is to find out what boundary conditions should be imposed. It is therefore sufficient to solve the field equations in whatever coordinate system is convenient and then express these solutions in the preferred coordinate system to impose the boundary conditions.

2.3 Modes of the field equation

From the form of (2.5) we can see that in terms of the original coordinates (T, X) the field equations are separable. For this reason we solve for the modes in these coordinates and then express the solutions in terms of the preferred coordinates using the coordinate transformations given above (2.13-2.18). In terms of the coordinates (T, X) the Klein-Gordon operator has the form

$$\frac{1}{\alpha(X)} \partial_T^2 \phi - (\partial_X \alpha(X)) \partial_X \phi - \alpha(X) \partial_X^2 \phi + m^2 \phi = 0. \tag{2.28}$$

By assuming a T dependence for the field of the form $\exp(-i\omega_p T)$ we obtain the following differential equation.

$$\partial_X(\alpha(X) \partial_X \phi) + (\frac{\omega_p^2}{\alpha(X)} - m^2)\phi = 0. \tag{2.29}$$

To construct a self-adjoint extension for this operator we are required to construct solutions which vanish at the horizon where $\alpha(X) = 0$.

By making a change of variable to $z = 1 - \exp(-q(|X| - r)) = \alpha(X)$ we obtain the following differential equation in terms of z,

$$z(1-z)^2\Psi''(z) + (1-z)(1-2z)\Psi'(z) + (\frac{p^2}{z} - \mu^2)\Psi(z) = 0 \tag{2.30}$$

where

$$p^2 = \frac{\omega_p^2}{q^2} \quad \text{and} \quad \mu^2 = \frac{m^2}{q^2}. \tag{2.31}$$

We are interested in constructing solutions outside the horizon so that $|X| > r$ and $z > 0$. As mentioned above we also require that the solutions vanish at $z = 0$.

The two independent solutions to (2.29) are

$$\Psi_{1p}(z) = z^n(1-z)^l F(a,b,c,z) \tag{2.32}$$

where $F(a,b,c,z)$ is a hypergeometric function and

$$\begin{aligned} n &= ip \\ l &= i\sqrt{p^2 - \mu^2} \\ a &= n + l \\ b &= n + l + 1 \\ c &= 1 + 2n \end{aligned} \tag{2.33}$$

and

$$\Psi_{2p}(z) = z^n(1-z)^l F(a,b,c,z) \tag{2.34}$$

where

$$n = -ip$$

$$
\begin{aligned}
l &= -i\sqrt{p^2 - \mu^2} \\
a &= n + l \\
b &= n + l + 1 \\
c &= 1 + 2n. \qquad (2.35)
\end{aligned}
$$

We can now finally write out the desired solution to (2.30)

$$\Psi(p, X) = [\Psi_{2p}(0)\Psi_{1p}(z) - \Psi_{1p}(0)\Psi_{2p}(z)]\,\epsilon(X). \qquad (2.36)$$

The general solution to (2.28) can then be written,

$$\Psi(T, X) = \int_{\mu}^{\infty} dp\,\{(A(p)\Psi(p, X)\exp(-i\omega_p T) + B(p)\Psi(p, X)\exp(i\omega_p T))\}\,. \qquad (2.37)$$

We now impose the initial conditions (2.26) which then give some physical meaning to the decomposition of this field into positive and negative frequency parts. To explicitly impose these initial conditions it is first useful to find the orthogonality relation satisfied by the modes $\Psi(p, X)$.

2.4 Orthogonality of the modes

To find the orthogonality relation satisfied by the mode $\Psi(p, X)$ we follow the standard Sturm-Liouville approach and recall that the modes satisfy

$$z(1 - z)^2 F''(p, z) + (1 - z)(1 - 2z)F'(p, z) + (\frac{p^2}{z} - \mu^2)F(p, z) = 0. \qquad (2.38)$$

We can also write out a similar equation which is satisfied by the modes $F^*(k, z)$. If one now multiplies the equation for $F(p, z)$ by $F^*(k, z)$ and the equation for $F^*(k, z)$ by $F(p, z)$ and looks at the difference of the two equations one can see that after integrating over the range $z = 0$ to $z = 1$ and integrating the two terms by parts once we are left with the relation

$$\int_0^1 dz \frac{F(p,z)F^*(k,z)}{z(1-z)} = \lim_{z\to 1} \frac{z(1-z)}{(p^2-k^2)} \left(F'(p,z)F^*(k,z) - F(p,z)F'^*(k,z)\right) - \frac{z(1-z)}{(p^2-k^2)} \left(F'(p,z)F^*(k,z) - F(p,z)F'^*(k,z)\right)\Big|_{z=0} \quad (2.39)$$

Because of the boundary conditions satisfied by $F(p,z)$ and $F^*(k,z)$ (they vanish at $z = 0$) the second term in this relation is identically zero. The first term, as we show, is proportional to a delta function. This shows that these modes are orthogonal. To see that this expression is indeed proportional to a delta function we first smear it with a smooth function of p and show that the result is proportional to that function evaluated at $p = k$. When one attempts to evaluate the limit in the first term one finds that all the various terms are proportional to a common factor which produces the delta function, this factor is

$$\lim_{z\to 1} \frac{(1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})}}{(p-k)}. \quad (2.40)$$

To proceed we introduce a regularization factor $(1-z)^\epsilon$ and write,

$$F(p,z) = \lim_{\epsilon\to 0} F(p,z)(1-z)^\epsilon. \quad (2.41)$$

The integral we must evaluate is

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dp \frac{f(p)}{p-k} \lim_{z\to 1} (1-z)^{2\epsilon} \left\{(1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})}\right\}. \quad (2.42)$$

This shows that there is no contribution to the integral from the regions where $|p-k| > R$. In these regions the pole at $p = k$ is not realized so one may interchange the order in which the limits are performed. These contributions then go to zero as the limit

$z \to 1$ is performed. We are then left with the integral

$$\lim_{\epsilon\to 0}\lim_{R\to 0}\int_{k-R}^{k+R} dp \frac{f(p)}{p-k}\lim_{z\to 1}(1-z)^{2\epsilon}$$
$$\left\{(1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})}\right\}. \tag{2.43}$$

It is now convenient to make a change of variable to the variable x where

$$x = \frac{k\ln(1-z)(p-k)}{\sqrt{k^2-\mu^2}}. \tag{2.44}$$

We next expand the integrand in powers of $(p-k)$ and find that as $R \to 0$ we are left with the smooth function $f(p)$ evaluated at the pole multiplied by a function of k.

Using the above analysis for the orthogonality relations in z one can then write the orthogonality relations in X,

$$\begin{aligned} \int_{|X|>r} dX \frac{\Psi(p,X)\Psi^*(k,X)}{\alpha(X)} &= \delta(p-k)\left|(A\Psi_{1k}(0) + B\Psi_{2k}(0))\right|^2 \\ &\equiv \delta(p-k)\left|F(k)\right|^2 \end{aligned} \tag{2.45}$$

where

$$\begin{aligned} A = \quad & \frac{\sqrt{2}(k-\sqrt{k^2-\mu^2})\Gamma(1-2ik)}{\pi\sqrt{kq\sinh(2\pi\sqrt{k^2-\mu^2})}}\Gamma^2(-i(k-\sqrt{k^2-\mu^2})) \\ & \times\sinh(\pi(k-\sqrt{k^2-\mu^2}))\sinh(\pi(k+\sqrt{k^2-\mu^2})) \end{aligned} \tag{2.46}$$

$$\begin{aligned} B = \quad & \frac{\sqrt{2}(k+\sqrt{k^2-\mu^2})\Gamma(1+2ik)}{\pi\sqrt{kq\sinh(2\pi\sqrt{k^2-\mu^2})}}\Gamma^2(-i(k+\sqrt{k^2-\mu^2})) \\ & \times\sinh(\pi(k-\sqrt{k^2-\mu^2}))\sinh(\pi(k+\sqrt{k^2-\mu^2})). \end{aligned} \tag{2.47}$$

2.5 Frequency decomposition and the vacuum

We now decompose the field into positive and negative frequency parts by picking out the positive frequency part of the field as that which satisfies the initial conditions in the preferred coordinates. This allows us to extract the annihilation operator for this field and thus define the vacuum state for this field on this particular spacelike hypersurface (i.e. the appropriate hypersurface which passes through the point (T_0, X_0)). The physically relevant question is, of course, how this decomposition depends on the point (T_0, X_0) which could represent the position of an observer. If this decomposition depends on the position of the observer, in an essential manner, then at some different position presumably the observer would observe some sort of particle density due to the change in composition of the vacuum state. To see this we must impose the initial conditions relevant to the quantization on this surface. Recall the initial conditions

$$\phi_k^+(t,x)\,|_{t=0} = A_k(0,x) \;\text{ and }\; (\partial_t\phi_k^+(t,x))\,|_{t=0} = -i\omega_k(0)A_k(0,x) \tag{2.48}$$

where

$$A_k(0,x) = \sin(2\frac{k}{q}B(x)) \;\text{ and }\; \omega_k(0) = (k^2+m^2)^{\frac{1}{2}}. \tag{2.49}$$

We can thus write the general form of the solution which satisfies these initial conditions for this particular mode k

$$\Psi_k(T,X) = \int_\mu^\infty dp\,\{(A_k(p)\Psi(p,X)\exp(-i\omega_p T) + B_k(p)\Psi(p,X)\exp(i\omega_p T))\} \tag{2.50}$$

where we now regard T,X and z as functions of (t,x). This can be easily done given the coordinate transformations of section II. Because the initial conditions are imposed at $t=0$ we need only be concerned with the form of this field and its derivative normal to $t=0$ for $z(t=0,x)$ in order to evaluate the expansion coefficients $A(p,n)$,$A^*(p,n)$,$B(p,n)$ and $B^*(p,n)$.

By using the orthogonality relations calculated in the last section we can write out the initial condition equations

$$
\begin{aligned}
|F(k)|^2 (A_k(k) \exp(-i\omega_k T_0) &+ B_k(k) \exp(i\omega_k T_0)) \\
&= \int_{|X|>r} dX \frac{\Psi^*(k,X)\sin(2\frac{k}{q}B(x))}{\alpha(X)}
\end{aligned}
\tag{2.51}
$$

$$
\begin{aligned}
|F(k)|^2 (B_k(k) \exp(-i\omega_k T_0) &- B_k(k) \exp(i\omega_k T_0)) \\
&= \int_{|X|>r} dX \frac{\Psi^*(k,X)\sin(2\frac{k}{q}B(x))}{\alpha(X)(\frac{\partial T}{\partial t})|_{t=0}}.
\end{aligned}
\tag{2.52}
$$

In taking the time derivative of (2.50) one does not pick up a $\frac{\partial X}{\partial t}$ because at $t = 0$ this is zero. Again in these expressions it can be seen that we are still regarding z as $z(0,x)$ and x is the inverse of this function in the integral. We have now determined $A(p,n)$,$A^*(p,n)$, $B(p,n)$ and $B^*(p,n)$ and can therefore decompose the field explicitly in terms of positive and negative frequency modes on this surface

$$
\Psi_1 = \int_0^\infty dk \frac{1}{\sqrt{2\omega_k}} \left\{ \phi^+_{k1}(t,x) a_1(k) + \phi^{(+)*}_{k1}(t,x) a_1(k)^\dagger \right\}
\tag{2.53}
$$

where the extra subscripts denote the surface on which the frequency decomposition has been performed and the modes $\phi^{(+)*}_{k1}(t,x)$ and $\phi^+_{k1}(t,x)$ are the ones constructed with the expansion coefficients which satisfy (2.51) and (2.52). One may now define the vacuum relevant to this field on the surface ($t = 0$) in the usual way

$$
a_1(k)\,|0_1\rangle = 0 \quad \forall \;\; k
\tag{2.54}
$$

where again the subscript denotes "when" this is the vacuum state for the field. To see whether particles are created by the gravitational field in this spacetime one must look closely at how this state depends on the surface chosen. At this stage all one needs is the normal derivative, with respect to the spacelike surface, of the field on the surface. In Section 2.6 the full transformation equations will be required.

2.6 Particle creation

To see whether particles are created by the gravitational field in the spacetime one must look closely at how the field decomposition depends on the surface chosen (i.e. the position of the observer). To obtain the spectrum of particles created one must calculate the Bogolubov transformation between the different annihilation and creation operators and look at the mixing of positive and negative frequency parts. To calculate the Bogolubov transformation we can just match the field from two different quantizations on a common surface. The easiest way to do this is to propagate one field to the surface on which the second is quantized. We can therefore write

$$\Psi_1(t,x) = \Psi_2(0,x') \quad \text{and} \quad \partial_{t'}\Psi_1(t,x) = \partial_{t'}(\Psi_2(t',x'))\,|_{t'=0} \tag{2.55}$$

where t is the proper distance between the two quantization surfaces. This distance will, in general, depend on where on the surface one calculates the distance. The calculation is made simpler by noticing that $X_0' = X_0$ because the observer is stationary with respect to the original coordinates where the metric is static.

Because of the simple form of the modes at $t = 0$ one can calculate the Bogolubov coefficients and write an expression of the form

$$a_2(k) = \int dp \left(\alpha(p,k)a_1(p) + \beta(p,k)a_1(p)^\dagger\right). \tag{2.56}$$

The particle density experienced by an observer travelling from surface 1 to surface 2 is then given by

$$|\beta(p,k)|^2 . \tag{2.57}$$

In $1+1$ dimensions $\beta(p,k)$ in general has the form

$$\beta(p,k) = \int_{-\infty}^{\infty} dx' \frac{q}{2} \frac{A_p(0,x')}{iq\pi\sqrt{\omega_p\omega_k}} \left\{ \dot{\phi}_{1k}^{+*}(t,x)\frac{\partial t}{\partial t'} + \left(\partial_x \phi_{1k}^{+*}(t,x)\right)\frac{\partial x}{\partial t'} - i\omega_p \phi_{1k}^{+*}(t,x) \right\}_{t'=0} . \quad (2.58)$$

In this equation the factors $\frac{\partial t}{\partial t'}$ and $\frac{\partial x}{\partial t'}$ are required because we are matching the field's normal derivative with respect to the second surface.

To calculate an approximate form of β, valid for short time intervals, we expand the integrand about $t = 0$. To $O(t^2)$ we obtain

$$\beta(p,k) = -\int_{-\infty}^{\infty} dy \frac{\sin(\frac{2p}{q}y)}{q\pi\sqrt{\omega_p\omega_k}} \tanh^2(y)\sin(\frac{2k}{q}y)p^2(x')\omega_k(T_0' - T_0)^2 \quad (2.59)$$

where $p(x')$ is given by (2.21) and we have changed variables from x' to $y = B(x')$. In getting from (2.58) to (2.59) the second term of (2.58) doesn't contribute to the integral because it is odd in y. It should be restated that this is particle creation observed by an observer stationary with respect to the original static coordinates. This can now be rewritten as

$$\beta(p,k) = -\frac{(T_0' - T_0)^2\omega_k q}{4\pi}\sqrt{\frac{1}{\omega_k\omega_p}} \int_{-\infty}^{\infty} dy \frac{\sin(\frac{2p}{q}y)}{\cosh^2(y)} \tanh^2(y)\sin(\frac{2k}{q}y). \quad (2.60)$$

Several comments are in order here. Firstly, $\beta(p,k)$ is clearly non-zero so that particles are produced in this short time interval $\delta t = T_0' - T_0$.

Secondly, our approximation only holds for $\omega_k < q$ since the expansion breaks down for $\omega_k\delta t > 1$. This means that we can only crudely estimate the number of particles produced in the time δt since an ultraviolet cutoff of $k = \sqrt{q^2 - m^2}$ is required.

Putting all this together we see that the momentum density of particles labelled by k produced in the time interval t is:

$$n_t(k) = \int_0^{\infty} dp\, |\beta(p,k)|^2$$

$$\simeq \frac{q^2\delta t^2}{4\pi}\frac{\sqrt{\omega_k}}{q}\int_0^\infty \frac{dp}{\sqrt{p^2+m^2}}\frac{\pi^2}{q^2}\left|\frac{(p+k)(\frac{(p+k)^2}{q^2}+3)}{\sinh\frac{\pi(p+k)}{q}} - \frac{(p-k)(\frac{(p-k)^2}{q^2}+3)}{\sinh\frac{\pi(p-k)}{q}}\right|^2 . \tag{2.61}$$

Within the spirit of the approximation, the total number of particles created in the time δt with $\omega_k < q$ is:

$$N_t = \int_0^{\sqrt{q^2-m^2}} n_t(k)dk. \tag{2.62}$$

This integral is finite, of course. If the upper limit is allowed to go to ∞ then the integral diverges linearly. This does not mean that the Bogolubov transformation is not unitarily implementable. Our approximations simply break down and our results are inconclusive. The difficulty arises from the fact that there are two time scales namely $T_1 = \frac{1}{q}$ and $T_2 = \frac{1}{\omega_k}$. For a fixed ω_k it is possible to expand in $\frac{\delta t}{T}$ where T is the smaller of T_1, T_2. However, if ω_k is unbounded no such expansion is possible.

2.7 Conclusions

We have shown that although a spacetime may be static this may not preclude particle creation which is a time dependent phenomenon [20], as the gaussian coordinatization may not be static. The only metrics which always lead to static Gaussian coordinates are those which have been dubbed "ultrastatic" by Fulling [16]. We have shown explicitly in this simple 1 + 1 dimensional case how the choice of which coordinates should be used leads to some interesting results. In particular, the coordinates which are chosen via a physical principle seem to suggest that although the spacetime may be manifestly static in one coordinate system these may not be the coordinates that one should use to quantize a field propagating in the spacetime.

Unfortunately the analysis to find out whether the Bogolubov transformation is unitarily implementable was inconclusive. This is due to the fact that the approximate form of β which was analysed was not valid for large k.

CHAPTER 3

Particle creation in an anisotropic universe

3.1 Introduction

The fact that the universe today is extremely isotropic is a fact that has puzzled cosmologists for many years. One suggestion to explain this observation is that quantum effects caused any early anisotropy to be almost entirely wiped out and thus led, through the back reaction due to particle creation, to the essentially isotropic universe we have today. This idea was first proposed by Zel'dovich and Starobinski [21] in a paper which investigated the particle creation in strong anisotropic gravitational fields.

To understand how most calculations have been done for anisotropic universes it is beneficial to first review what has been done in some isotropic models. In the next section 3.2 we review the use of what's called the conformal time. This conformal time has been used extensively as the time with which to decompose quantum fields. At the end of section 3.2 we also explain how generalizations to this conformal time have been used in models which are not conformally invariant and in particular how this relates to some anisotropic model calculations.

In the sections 3.3 through 3.6 we calculate the particle creation in an anisotropic universe using the same definition of time to decompose the field as we have in the preceding chapters. This anisotropic universe is basically an anisotropic generalization of 1 + 1 deSitter space. The results of this calculation show that the energy spectrum of the created particles is shifted by a discrete amount proportional to the one natural length scale of the geometry, that being the square root of the scalar curvature.

3.2 Isotropic models and conformal time

To intoduce what's called the conformal time which is quite popular in the literature we start with the isotropic model

$$ds^2 = dt^2 - a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{3.1}$$

we then introduce

$$\eta = \int^t \frac{1}{a(t')} dt \quad \text{and} \quad C(\eta) = a^2(t) \tag{3.2}$$

the metric in terms of the conformal time η is

$$ds^2 = C(\eta)\left(d\eta^2 - dx_1^2 - dx_2^2 - dx_3^2\right) \tag{3.3}$$

it is clear why this is called the conformal time as the metric is now conformal to the Minkowski metric. To see why this is useful we now look at how the field equations and solutions to these field equations change due to this conformal factor. Conformally flat spacetimes such as this one can always be described in terms of metric tensors which are conformal to Minkowski space

$$g_{\mu\nu}(x) = C^2(\eta, \mathbf{x})\eta_{\mu\nu}. \tag{3.4}$$

The conformally coupled massless field equation in 3 + 1 dimensions is

$$\left[\Box + \frac{1}{6}R\right]\varphi = 0 \tag{3.5}$$

where $\Box$ is short for the first term in 1.31, R is the scalar curvature and the factor $\frac{1}{6}$ depends on the number of dimensions which in this case is four. Under the conformal transformation

$$g_{\mu\nu} \to C^{-2}(\eta,\mathbf{x})g_{\mu\nu}(\eta,\mathbf{x}) = \eta_{\mu\nu} \tag{3.6}$$

the field equation transforms to

$$\Box\bar{\phi} = \eta^{\mu\nu}\partial_\mu\partial_\nu(C(\eta,\mathbf{x})\phi) = 0 \tag{3.7}$$

where $\bar{\phi} = C(\eta,\mathbf{x})\phi$ is the field which satisfies the wave equation in the new metric which in this case is Minkowski space. To now decompose the field in terms of positive and negative frequency parts we use the decomposition for Minkowski space and just . 'd the conformal factor to the field

$$\phi(\eta,\mathbf{x}) = C^{-1}(\eta)\sum_{\mathbf{k}}\left[a_{\mathbf{k}}\bar{u}_{\mathbf{k}}(\eta,\mathbf{x}) + a^\dagger_{\mathbf{k}}\bar{u}^*_{\mathbf{k}}(\eta,\mathbf{x})\right] \tag{3.8}$$

where everything is just as it was in the introduction to flat space quantum field theory except for the factor out in front in which we have left only the η dependence as that is the case for this example. The vacuum defined in terms of the operators $a_{\mathbf{k}}$ is called the conformal vacuum. We see from this expression that the spatial dependence of the field is just as it was in flat space and is made up of simple exponentials. However it is the time dependence of the field which is the important part of the whole procedure and is what is responsible for the physics resulting from a particular choice of Fock space representation for the Hilbert space. In this case the time dependence is fairly complicated and we seem to have abandoned any semblance of quantum field theory in flat space by the end of the procedure.

For the most part it has been anisotropic generalizations to this procedure which have been employed to study anisotropic effects in anisotropic models. To illustrate this we consider the Bianchi type I model

$$ds^2 = dt^2 - a_1(t)^2dx_1^2 - a_2(t)^2dx_2^2 - a_3(t)^2dx_3^2 \tag{3.9}$$

for this model the standard practice would be to introduce

$$C(t) = a^2(t) = (a_1 a_2 a_3)^{\frac{2}{3}} \quad \text{and} \quad \eta = \int a^{-1}(t')dt' \tag{3.10}$$

which reduces to the conformal time in the isotropic limit $a_1 = a_2 = a_3$.

Because of the difficulty in finding solutions to the wave equation for most of these models perturbative approaches have been taken with perturbations about isotropy. The model studied in this chapter unfortunately does not allow such an analysis as the anisotropy grows with time. It is however interesting to note that in our analysis, the surface on which the decomposition is performed the metric does appear isotropic.

3.3 The model

The model we investigate in this paper is an anisotropic $3+1$ generalization of $1+1$ de Sitter space, the simplest generalization being just the addition of a 2-plane. Specifically we are investigating particle creation due to the gravitational field which is described by the metric

$$ds^2 = dT^2 - e^{\lambda T}(dX^1)^2 - (dX^2)^2 - (dX^3)^2. \tag{3.11}$$

More precisely we investigate the particle creation as observed by an observer stationary with respect to the coordinates (T, X^1, X^2, X^3).

To follow the prescription as outlined in the introduction and also used in the preceding chapters we first find the geodesics in this spacetime. The first integrals of the geodesics are:

$$\frac{dX^1}{ds} = \frac{c_1}{e^{\lambda T}}\,, \quad \frac{dX^2}{ds} = c_2\,, \quad \frac{dX_3}{ds} = c_3\,, \quad \frac{dT}{ds} = \sqrt{\epsilon + \frac{c_1^2}{e^{\lambda T}} + c_2^2 + c_3^2} \tag{3.12}$$

where $\epsilon = \pm 1$ depending on whether the geodesic is timelike or spacelike respectively.

The preferred coordinates on the surface are constructed using a 4-bein of orthogonal basis vectors at P_0, the observer's position. We choose these vectors to be,

$$\begin{aligned} e_0(P_0) &= (1,0,0,0) \;\; e_1(P_0) = (0, e^{-\lambda\frac{T_0}{2}}, 0, 0) \\ e_2(P_0) &= (0,0,1,0) \;\; e_3(P_0) = (0,0,0,1). \end{aligned} \tag{3.13}$$

In this way the tangent to the chosen observer's worldline at P_0 corresponds to $e_0(P_0)$.

To construct the spacelike surface orthogonal to the tangent of the observer's worldline we therefore require that

$$\frac{dT}{ds}|_{P_0} = 0 \quad \text{which implies} \quad \frac{c_1^2}{e^{\lambda T_0}} + c_2^2 + c_3^2 = 1. \tag{3.14}$$

The preferred coordinates on the spacelike hypersurface are chosen to be Riemann coordinates based on the observer's position $P_0 = (T_0, X_0^1, X_0^2, X_0^3)$. With p^μ given by the tangent vector, at P_0, to the geodesic connecting P_0 to P_1. The point P_1 is the point at which the timelike geodesic "dropped" from an arbitrary point $P = (T, X^1, X^2, X^3)$ intersects the spacelike surface orthogonally. The Riemann coordinates η^α of the point P_1 are given by

$$sp^\mu = \eta^\alpha e^\mu_\alpha(P_0) \tag{3.15}$$

where s is the distance along the geodesic $P_0 - P_1$. Using $e^\mu_\alpha e_{\beta\mu} = \eta_{\alpha\beta}$ (Minkowski metric), and the orthogonality of p^μ to $e_0(P_0)$ we have

$$\eta^0 = sp^\mu e^0_\mu(P_0) \qquad \eta^i = -sp^\mu e^i_\mu(P_0). \tag{3.16}$$

The surface S_0 is just the surface $\eta^0 = 0$ and the coordinates x^i are

$$x^1 = s\frac{c_1}{\sqrt{e^{\lambda T_0}}} \quad x^2 = sc_2 \quad x^3 = sc_3 \tag{3.17}$$

where s is the geodesic distance between the points P_0 and P_1.

The direction of time is given by the normal to the spacelike hypersurface and the preferred time t for the arbitrary point P is given by the proper distance along the timelike geodesic connecting P to P_1. The timelike geodesic is also determined by (3.12) except with $\epsilon = -1$ and a different choice of the constants which we denote by b_i. The condition that the geodesic connecting P to P_1 is normal to the spacelike hypersurface requires that

$$\sqrt{\left(1+\frac{b_1^2}{e^{\lambda T_1}}+b_2^2+b_3^2\right)}\sqrt{\left(\frac{c_1^2}{e^{\lambda T_1}}-\frac{c_1^2}{e^{\lambda T_0}}\right)}=\frac{b_1c_1}{e^{\lambda T_1}}+b_2c_2+b_3c_3. \tag{3.18}$$

We can now calculate the dependence of (T, X^1, X^2, X^3) on the preferred coordinates (t, x^1, x^2, x^3) and then calculate the metric in its preferred form. To calculate this dependence we must use the above equations for x^i (3.17) and also calculate the change in the coordinates X^i along the spacelike and timelike geodesics which ultimately connect P_0 to P

$$\begin{aligned}
X^1 &= X_0^1+\int_{T_0}^{T_1} dT\frac{c_1}{e^{\lambda T_1}}\left(\frac{c_1^2}{e^{\lambda T}}-\frac{c_1^2}{e^{\lambda T_0}}\right)^{-\frac{1}{2}}+\int_{T_1}^{T} dT'\frac{b_1}{e^{\lambda T'}}\left(1+\frac{b_1^2}{e^{\lambda T'}}+b_2^2+b_3^2\right)^{-\frac{1}{2}}\\
X^2 &= X_0^2+\int_{T_0}^{T_1} dT\frac{c_2}{e^{\lambda T_1}}\left(\frac{c_1^2}{e^{\lambda T}}-\frac{c_1^2}{e^{\lambda T_0}}\right)^{-\frac{1}{2}}+\int_{T_1}^{T} dT'\frac{b_2}{e^{\lambda T'}}\left(1+\frac{b_1^2}{e^{\lambda T'}}+b_2^2+b_3^2\right)^{-\frac{1}{2}}\\
X^3 &= X_0^3+\int_{T_0}^{T_1} dT\frac{c_3}{e^{\lambda T_1}}\left(\frac{c_1^2}{e^{\lambda T}}-\frac{c_1^2}{e^{\lambda T_0}}\right)^{-\frac{1}{2}}+\int_{T_1}^{T} dT'\frac{b_3}{e^{\lambda T'}}\left(1+\frac{b_1^2}{e^{\lambda T'}}+b_2^2+b_3^2\right)^{-\frac{1}{2}}
\end{aligned} \tag{3.19}$$

and t the proper distance along $P - P_1$

$$t=\int_{T_1}^{T} dT'\left(1+\frac{b_1^2}{e^{\lambda T'}}+b_2^2+b_3^2\right)^{-\frac{1}{2}}. \tag{3.20}$$

At this point we can see that if we choose $b_2 = b_3 = 0$ this just corresponds to aligning the spacelike hypersurfaces so that $X^2 = X_1^2$ and $X^3 = X_1^3$. This simplifies the analysis considerably and gives the expected result that

$$X^2 = X_0^2 + x^2 \quad \text{and} \quad X^3 = X_0^3 + x^3. \tag{3.21}$$

The only non-trivial part of the transformation therefore involves (T, X^1) and (t, x^1). By performing the above integral for X^1 and inverting the t integral one is left with the coordinate transformations

$$\begin{aligned} e^{\frac{\lambda}{2}(T-T_0)} &= \sinh(\frac{\lambda t}{2}) + \cosh(\frac{\lambda t}{2})\cos(\frac{\lambda x^1}{2}) \\ \frac{\lambda}{2}(X^1 - X_0^1)e^{\lambda\frac{t}{2}} &= -\cosh(\frac{\lambda t}{2})\sin(\frac{\lambda x^1}{2}). \end{aligned} \tag{3.22}$$

In terms of the preferred coordinates (t, x^i) the metric now has the form

$$ds^2 = dt^2 - \cosh^2(\frac{\lambda t}{2})(dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{3.23}$$

where the range of x_1 is $-\infty < x_1 < \infty$.

This result is of course not a surprise to anyone familiar with the different forms of de Sitter space in 1 + 1 dimensions. Unfortunately the usual analysis does not deal with the observer dependent nature of the coordinate transformations. We will see that this is in fact where the interesting physics comes from. Indeed if one proceeds to quantize the field on t =constant surfaces it is easy to see that all these surfaces can be made to look like Minkowski space. The point is that they cannot be made to all look like Minkowski space simultaneously. It would therefore seem obvious that the physics is going to be determined not by the form of the metric on a particular surface but by the transformations relating one surface's preferred coordinates to another surface's preferred coordinates.

3.4 Modes and quantization

In the coordinates constructed in the last section the non-minimally coupled massive Klein Gordon equation is

$$\partial_t^2\phi + \frac{1}{\sqrt{g}}\partial_t(\sqrt{g})\partial_t\phi + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\phi) + (m^2+\xi R)\phi = 0. \tag{3.24}$$

This equation is strictly hyperbolic so long as g^{ij} is negative definite. The solutions are therefore uniquely determined by the initial data.

To quantize a scalar field on the $t = 0$ surface we now define the positive frequency modes in the neighbourhood of this surface. The positive frequency modes are defined as those which satisfy the initial conditions

$$\phi_k^+(t,\mathbf{x})\,|_{t=0} = A_k(0,\mathbf{x}) \quad \text{and} \quad \partial_t(\phi_k^+(t,\mathbf{x}))\,|_{t=0} = -i\omega_k(0)A_k(0,\mathbf{x}) \tag{3.25}$$

where $A_k(t,\mathbf{x})$ are the instantaneous eigenmodes of the spatial part of the Laplace-Beltrami operator, and $\omega_k(t)^2$ are the corresponding eigenvalues:

$$\left[\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + m^2 + \xi R\right] A_k(t,\mathbf{x}) = \omega_k^2(t)A_k(t,\mathbf{x}). \tag{3.26}$$

Henceforth we just write ω_k for $\omega_k(0)$. Due to the simple form of $g_{\mu\nu}$ at $t = 0$ the eigenmodes and values take on the simple form

$$\begin{aligned} A_k(0,\mathbf{x}) &= e^{i\mathbf{k}\cdot\mathbf{x}} \\ \omega_k^2(0) &= \mathbf{k}^2 + m^2 + \xi R. \end{aligned} \tag{3.27}$$

Near the surface $t = 0$ the second term of (3.24) vanishes to $O(t^2)$, this implies that the initial conditions for the time dependence of the field are also good to $O(t^2)$.

To impose these initial conditions we must find a complete set of modes for the entire wave operator. Because the differential equation is separable we look for solutions of the form $f_k(t)e^{i\mathbf{k}\cdot\mathbf{x}}$. The differential equation satisfied by the $f_k(t)$ is then

$$\partial_t^2 f_k(t) + \frac{\lambda}{2}\tanh(\frac{\lambda t}{2})\partial_t f_k(t) + \left(k_1^2\text{sech}^2(\frac{\lambda t}{2}) + k_2^2 + k_3^2 + m^2 + \xi R\right) f_k(t) = 0. \quad (3.28)$$

The positive frequency modes are those whose "time" part satisfies the above differential equation and the initial conditions

$$f_k(0) = 1 \quad \text{and} \quad \dot{f}_k(0) = -i\omega_k. \quad (3.29)$$

The positive frequency modes are given in terms of hypergeometric functions $H(a, b, c, x)$ by

$$\begin{aligned} \phi_k^+(t, \mathbf{x}) &= e^{i\mathbf{k}\cdot\mathbf{x}}\text{sech}(\frac{\lambda t}{2})^{2n}\left\{H(\alpha, \beta, \frac{1}{2}, \tanh^2(\frac{\lambda t}{2}))\right. \\ &- \left. i\frac{2\omega_k}{\lambda}\tanh(\frac{\lambda t}{2})H(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \frac{3}{2}, \tanh^2(\frac{\lambda t}{2}))\right\} \end{aligned} \quad (3.30)$$

where

$$\begin{aligned} n &= \frac{1}{4} - \frac{i}{\lambda}\sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}} \\ \alpha &= -\frac{k_1}{\lambda} + \frac{1}{4} - \frac{i}{\lambda}\sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}} \\ \beta &= \frac{k_1}{\lambda} + \frac{1}{4} - \frac{i}{\lambda}\sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}}. \end{aligned} \quad (3.31)$$

We can now write out the field which has been quantized on surface 1 as

$$\Psi_1 = \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{2\omega_k}}\left\{\phi_k^+(t, \mathbf{x})a_1(k) + \phi_k^{+*}(t, \mathbf{x})a_1^\dagger(k)\right\}. \quad (3.32)$$

3.5 Particle creation

To investigate particle creation in the model universe as observed by an observer stationary with respect to the original coordinates (T, X^1, X^2, X^3) we calculate the

Bogolubov transformation relating the annihilation and creation operators from two different surfaces of quantization that the observer passes through. To calculate the coefficients of this transformation we equate the same field from two different quantizations on a common surface

$$\Psi_1(t,\mathbf{x}) = \Psi_2(t'(t,x),\mathbf{x}'(t,x)). \tag{3.33}$$

Here $\Psi_1(t,x)$ is the field written out explicitly in (3.32) and $\Psi_2(t',x')$ is the same field which has been quantized on a second surface $t'=0$. The "second" field is therefore quantized for the same observer as the first but at some later time T_0'. At this time the remark made at the end of the third section becomes clearer. All the physics of the observations made by this observer are determined by the functions $t'(t,x)$, $x'(t,x)$ and the derivatives of these functions with respect to t. In this way the geometry of the spacetime via the coordinate independent prescription we have used, determines the spectrum of created particles.

For simplicity we calculate the Bogolubov transformation by "matching" the field and its first derivative with respect to t at $t=0$,

$$\begin{aligned} a_1(k) &= \frac{i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\left\{-i\omega_k\Psi_1(0,x) + (\partial_t\Psi_1(t,x))\,|_{t=0}\right\} \\ &= \frac{i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\left\{-i\omega_k\Psi_2(t'(0,x),x'(0,x))\right. \\ &+ \left.(\partial_t\Psi_2(t'(t,x),x'(t,x)))\,|_{t=0}\right\}. \end{aligned} \tag{3.34}$$

Using this equation, we can write out the Bogolubov transformation in the form

$$a_1(k) = \int d^3p\alpha(k,p)a_2(p) + \int d^3p\beta(k,p)a_2^\dagger(p). \tag{3.35}$$

The spectrum of created particles is determined by $|\beta(k,p)|^2$. Writing out $\beta(k,p)$ explicitly we find it has some interesting properties due to its dependence on the inverse relations $t'(t,x_1), x'_1(t,x_1)$.

$$\beta(k,p) = \frac{-i}{2\pi}\delta(p_2+k_2)\delta(p_3+k_3)\int dx_1 \frac{e^{ik_1x_1}}{\sqrt{4\omega_p\omega_k}} \left\{ i\omega_k f_p^{+*}(t'(0,x_1))e^{ip_1x'_1(0,x_1)} - \partial_t \left\{ f_p^{+*}(t'(t,x_1))e^{ip_1x'_1(t,x_1)} \right\}|_{t=0} \right\} \quad (3.36)$$

where

$$x'_1(t,x_1) = \frac{2}{\lambda}\tan^{-1}\left(\frac{\cosh(\frac{\lambda t}{2})\sin(\frac{\lambda x_1}{2})}{\cosh(\frac{\lambda t}{2})\cos(\frac{\lambda x_1}{2})\cosh(\frac{\lambda}{2}(T'_0-T_0)) - \sinh(\frac{\lambda t}{2})\sinh(\frac{\lambda}{2}(T'_0-T_0))}\right)$$

$$t'(t,x_1) = \frac{2}{\lambda}\sinh^{-1}\left(\sinh(\frac{\lambda t}{2})\cosh(\frac{\lambda}{2}(T'_0-T_0)) - \cosh(\frac{\lambda t}{2})\cos(\frac{\lambda x_1}{2})\sinh(\frac{\lambda}{2}(T'_0-T_0))\right). \quad (3.37)$$

3.6 Discrete shift of energy spectrum

Unfortunately, due to the complicated nature of the expression for $\beta(k,p)$ we cannot write it out in a more transparent form which is still exact. We can however discover some interesting facts about the spectrum of created particles by investigating the integrand of the integral for $\beta(k,p)$. In fact it is not difficult to see that the particles observed by our stationary observer possess a discrete energy spectrum shift. To see this we rewrite (3.36) as

$$\beta(k,p) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} dx_1 \frac{e^{i(p_1+q_1)x_1}}{\sqrt{4\omega_p\omega_k}} F(k,p,x_1)\delta(p_2+q_2)\delta(p_3+q_3) \quad (3.38)$$

where

$$F(k,p,x_1) = e^{ip_1(x'_1(0,x_1)-x_1)}\left\{ i\omega_k f_p^{+*}(t'(0,x)) - ik_1\frac{\partial x'_1}{\partial t} f_p^{+*}(t'(t,x_1)) - \frac{\partial t'}{\partial t}\partial_{t'}\left(f_p^{+*}(t'(t,x_1))\right)\right\}|_{t=0}. \quad (3.39)$$

By inspection of the inverse relations (3.37) one sees that $F(k,p,x_1)$ is a well behaved periodic function in x_1, with a period of $\frac{4\pi}{\lambda}$. The only difficulty arises with the exponential factor. This factor is also periodic in x_1 if one is careful to ensure that in the analysis both x_1' and x_1 retain their range of $-\infty$ to ∞. This can be seen by writing x_1 in terms of an inverse tan function and then observing that $(x_1' - x_1)$ can be simplified as it involves the difference of two inverse tan functions.

Because of these properties we can expand $F(p,k,x)$ as a Fourier series,

$$F(p,k,x) = \sum_{n=-\infty}^{\infty} C_n(p,k) e^{in\frac{\lambda x}{2}} \tag{3.40}$$

which implies that,

$$\beta(p,k) = \frac{-i}{\sqrt{4\omega_p \omega_k}} \sum_{n=-\infty}^{\infty} C_n(p,k)\delta(p_1 + k_1 + \frac{n\lambda}{2})\delta(p_2 + k_2)\delta(p_3 + k_3). \tag{3.41}$$

Unfortunately we cannot evaluate the $C_n(p,k)$ analytically but we can evaluate them numerically for some specific values of $(T_0' - T_0)$,λ, **p** and **k**. This numerical analysis suggests that the particle creation drops off rapidly for large **p** and **k**. Nevertheless, it is expected that the total particle creation, as in all such problems in infinite spaces, is infinite. The reason for this seems to be that the external field can pump in an infinite amount of energy in a finite time [25]. In this particular model the energy density of the classical matter field giving rise to the geometry of the model is constant. If one calculates the total energy of the classical matter field it is therefore infinite.

3.7 Conclusions

We see from the above analysis that the particles created due to the gravitational field as seen by a stationary observer in the model universe, $ds^2 = dT^2 - e^{\lambda T}(dX^1)^2 - (dX^2)^2 - (dX^3)^2$, contain a spectrum of particles shifted by a discrete amount. It appears that the one length scale of the geometry namely $\sqrt{R}$ plays a role similar to the role the length of a box plays for modes in a box. In this sense the discrete energy spectrum shift may almost be expected.

CHAPTER 4

Particle creation in 1+1 deSitter space

4.1 Introduction

As was seen in the last chapter if one is dealing with an infinite spacetime it can be expected that the total number of particles created by the gravitational field be infinite. This can still be physically reasonable as it does not imply that the particle production per unit volume is infinite. In this chapter we investigate the particle creation in 1+1 deSitter space. As the preffered coordinatization of the space is compact one expects that the particle creation will be finite in agreement with Fulling's analysis for isotropic universes of finite spatial volume [16].

In this analysis we start with the universe defined in terms of the metric

$$ds^2 = dT^2 - e^{\lambda T} dX^2. \tag{4.1}$$

This metric is just a standard parametrization of a portion of deSitter space. When one expresses this spacetime in terms of the preferred coordinates, introduced in the introduction and used in the preceding chapters, one finds that the geodesically complete description of the spacetime is just the entire $1 + 1$ deSitter manifold.

To show that the particle creation as observed by an observer who is stationary with respect to the original coordinates (4.1) is finite we examine some properties of the square of the Bogolubov $\beta(m, n)$ coefficient. Because the integral expression for this object has a range which is compact we are able to argue that the square of the Bogolubov $\beta(m,n)$ coefficient drops off faster than any inverse power of m or n.

4.2 The model

The model we investigate is that of a compact 1 + 1 dimensional spacetime described by the metric

$$ds^2 = dT^2 - e^{\lambda T}dX^2, \qquad -\infty \leq X < \infty. \tag{4.2}$$

To follow the prescription mentioned above we first must calculate the geodesics. The first integrals of the geodesic equations are:

$$\frac{dX}{ds} = \frac{c_1}{e^{\lambda T}} \qquad \frac{dT}{ds} = \sqrt{\epsilon + \frac{c_1^2}{e^{\lambda T}}} \tag{4.3}$$

where $\epsilon = \pm 1$ depending on whether the geodesic is timelike or spacelike respectively.

The preferred coordinates on the hypersurface of instantaneity are constructed using a 2-bein of orthogonal basis vectors based at P_0, the observer's position We choose these vectors to be.

$$e_0(P_0) = (1,0) \quad e_1(P_0) = (0, \frac{1}{e^{\frac{\lambda T_0}{2}}}); \tag{4.4}$$

in this way $e_0(P_0)$ is tangent to the observer's worldline at P_0.

To construct a spacelike geodesic which is orthogonal to the observer's world line it is required that

$$\frac{dT}{ds}|_{P_0} = 0 \quad \text{which} \quad \text{implies} \quad \frac{c_1^2}{e^{\lambda T_0}} = 1. \tag{4.5}$$

The preferred coordinates on the spacelike hypersurface are chosen to be Riemann coordinates based on the observer's position $P_0 = (T_0, X_0)$. The point $P_1 = (T_1, X_1)$ is the point at which a timelike geodesic "dropped" from an arbitrary point $P = (T, X)$ intersects the spacelike hypersurface orthogonally. The Riemann coordinates η^α of the point P_1 are given by

$$sp^\mu = \eta^\alpha e^\mu_\alpha(P_0) \tag{4.6}$$

where s is the distance along the geodesic $P_0 - P_1$ and p^μ is the tangent vector, at P_0, to the geodesic connecting P_0 to P_1. These equations can be solved for the η^α using the orthogonality of p^μ to $e_0(P_0)$ and the identity $e^\mu_\alpha e_{\beta\mu} = \eta_{\alpha\beta}$ (Minkowski metric) to give

$$\eta^0 = sp^\mu e^0_\mu(P_0) \qquad \eta^1 = -sp^\mu e^1_\mu(P_0). \tag{4.7}$$

The surface of instantaneity S_0 is just the surface $\eta^0 = 0$ and the preferred spatial coordinate $x^1 = \eta^1$ is

$$x^1 = s\frac{c_1}{\sqrt{e^{\lambda T_0}}}, \tag{4.8}$$

where s is the geodesic distance between the points P_0 and P_1. The direction of time is given by the normal to this spacelike hypersurface. The preferred time coordinate t for the point P is given by the proper distance along the timelike geodesic connecting P to P_1. This timelike geodesic is also determined by (4.3) with $\epsilon = -1$ and a different choice of integration constant, b_1. The condition that the geodesic connecting P to P_1 is normal to the spacelike hypersurface requires that

$$\sqrt{1 + \frac{b_1^2}{e^{\lambda T_1}}}\sqrt{\frac{c_1^2}{e^{\lambda T_1}} - \frac{c_1^2}{e^{\lambda T_0}}} = \frac{b_1 c_1}{e^{\lambda T_1}}. \tag{4.9}$$

The metric can now be calculated in terms of the preferred coordinates (t, x^1) by calculating $(T(t, x^1), X(t, x^1))$. To calculate these dependances we use the above equation for x^1 (4.8) and also calculate the change in the coordinate X along the spacelike and timelike geodesics which connect P_0 to P

$$X = X_0 + \int_{T_0}^{T_1} dT \frac{c_1}{e^{\lambda T_1}} \left(\frac{c_1^2}{e^{\lambda T}} - \frac{c_1^2}{e^{\lambda T_0}}\right)^{-\frac{1}{2}} + \int_{T_1}^{T} dT' \frac{b_1}{e^{\lambda T'}} \left(1 + \frac{b_1^2}{e^{\lambda T'}}\right)^{-\frac{1}{2}} \tag{4.10}$$

and

$$t = \int_{T_1}^{T} dT' \left(1 + \frac{b_1^2}{e^{\lambda T'}}\right)^{-\frac{1}{2}}. \tag{4.11}$$

By performing the above integral for X_1 and inverting the t integral one is left with the coordinate transformations

$$\begin{aligned} e^{\frac{\lambda}{2}(T-T_0)} &= \sinh(\frac{\lambda t}{2}) + \cosh(\frac{\lambda t}{2})\cos(\frac{\lambda x^1}{2}) \\ \frac{\lambda}{2}(X - X_0)e^{\lambda\frac{T}{2}} &= -\cosh(\frac{\lambda t}{2})\sin(\frac{\lambda x^1}{2}). \end{aligned} \tag{4.12}$$

In terms of the preferred coordinates (t, x^1) the metric now has the form

$$ds^2 = dt^2 - \cosh^2(\frac{\lambda t}{2})(dx^1)^2. \tag{4.13}$$

The range of x^1 is $0 \leq x^1 < \frac{4\pi}{\lambda}$. To write this in a more convenient form we introduce the angular coordinate $\alpha = \frac{\lambda x^1}{2}$ which covers the range $0 \leq \alpha < 2\pi$. In terms of this angular coordinate the metric takes the form

$$ds^2 = dt^2 - \frac{4}{\lambda^2}\cosh^2(\frac{\lambda t}{2})d\alpha^2. \tag{4.14}$$

4.3 Modes and initial conditions

In the coordinates constructed above the minimally coupled massive Klein Gordon equation is

$$\partial_t^2\phi + \frac{1}{\sqrt{g}}\partial_t(\sqrt{g})\partial_t\phi + \frac{1}{\sqrt{g}}\partial_1(\sqrt{g}g^{11})\partial_1\phi + m^2\phi = 0. \tag{4.15}$$

To quantize a scalar field on the $t = 0$ surface we now define the positive frequency modes. The positive frequency modes are defined as those which satisfy the initial conditions

$$\phi_k^+(t, \mathbf{x})\,|_{t=0} = A_k(0, \alpha) \quad \text{and} \quad \partial_t(\phi_k^+(t, \alpha))\,|_{t=0} = -i\omega_k(0)A_k(0, \alpha), \tag{4.16}$$

where $A_k(t,\alpha)$ are the instantaneous eigenmodes of the spatial part of the Laplace-Beltrami operator, and $\omega_k(t)^2$ are the corresponding eigenvalues:

$$\left[\frac{1}{\sqrt{g}}\partial_1\left(\sqrt{g}g^{11}\partial_1\right)+m^2\right]A_k(t,\alpha)=\omega_k^2(t)A_k(t,\alpha). \tag{4.17}$$

Henceforth we just write ω_k for $\omega_k(0)$. Due to the simple form of $g_{\mu\nu}$ at $t=0$ the eigenmodes and eigenvalues take on the simple form

$$\begin{aligned} A_k(0,\alpha) &= e^{i\frac{2k}{\lambda}\alpha} \\ \omega_k^2(0) &= k^2+m^2. \end{aligned} \tag{4.18}$$

Near the surface $t=0$ the second term of (4.15) vanishes to $O(t^2)$, this implies that the initial conditions for the time dependence of the field are also good to $O(t^2)$. We impose periodic boundary conditions on $A_k(0,\alpha)$ to choose a self adjoint extension for the differential operator on the left hand side of (4.17). This requires that $\frac{2k}{\lambda}=s$ where s is an integer.

To impose the initial conditions we need a complete set of modes for the entire wave operator. Because the wave equation is separable we look for solutions of the form $f_s(t)e^{is\alpha}$. The differential equation satisfied by the $f_s(t)$ is then

$$\partial_t^2 f_s(t)+\frac{\lambda}{2}\tanh(\frac{\lambda t}{2})\partial_t f_s(t)+\left(\frac{s^2\lambda^2}{4}\mathrm{sech}^2(\frac{\lambda t}{2})+m^2\right)f_s(t)=0. \tag{4.19}$$

The positive frequency modes are those whose time part satisfies the above differential equation and the initial conditions

$$f_s(0)=1 \quad \text{and} \quad \dot{f}_s(0)=-i\omega_k. \tag{4.20}$$

The positive frequency modes are given in terms of hypergeometric functions $H(a,b,c,g(t))$ by

$$o_s^+(t,\alpha) \;=\; e^{is\alpha}\cosh(\frac{\lambda t}{2})^s\{\; H(\alpha,\beta,\frac{1}{2},-\sinh^2(\frac{\lambda t}{2}))$$

$$- \quad i\frac{2\omega_s}{\lambda}\sinh(\frac{\lambda t}{2})H(\alpha+\frac{1}{2},\beta+\frac{1}{2},\frac{3}{2},-\sinh^2(\frac{\lambda t}{2}))\Big\} \tag{4.21}$$

where

$$\begin{aligned} \alpha &= \frac{s}{2}+\frac{1}{4}+\frac{i}{4\lambda}\sqrt{16m^2-\lambda^2} \\ \beta &= \frac{s}{2}+\frac{1}{4}-\frac{i}{4\lambda}\sqrt{16m^2-\lambda^2} \\ \omega_s &= \sqrt{(\frac{\lambda s}{2})^2+m^2}. \end{aligned} \tag{4.22}$$

We can now write out the field which has been quantized on surface 1 as

$$\Psi_1 = \sum_{s=-\infty}^{s=\infty} \frac{1}{\sqrt{2\omega_s}} \left\{ \phi_s^+(t,\alpha)a_1(s) + \phi_s^{+*}(t,\alpha)a_1^\dagger(s) \right\}. \tag{4.23}$$

4.4 Particle creation

To investigate particle creation in the model universe as observed by an observer stationary with respect to the original coordinates (T, X) we calculate the Bogolubov transformation relating the annihilation and creation operators from two different surfaces of quantization that the observer passes through. To calculate the coefficients of this transformation we equate the same field from two different quantizations on a common surface

$$\Psi_1(t,\alpha) = \Psi_2(t'(t,\alpha),\alpha'(t,\alpha)). \tag{4.24}$$

Here $\Psi_1(t,\alpha)$ is the field written out explicitly in (4.23) and $\Psi_2(t',\alpha')$ is the same field which has been quantized on a second surface $t' = 0$. The "second" field is therefore quantized for the same observer as the first but at some later time T_0' with $\theta_0 = \theta_0'$. All the physics of the observations made by this observer are determined by the functions $t'(t,\alpha)$, $x'(t,\alpha)$ and the derivatives of these functions with respect to t. In this way the geometry of the spacetime via the coordinate independent prescription we have used determines the spectrum of created particles.

We calculate the Bogolubov transformation by "matching" the field and its first derivative with respect to t at $t = 0$:

$$\begin{aligned} a_1(s) &= \frac{i}{(2\pi)}\frac{1}{\sqrt{2\omega_s}}\int_0^{2\pi} d\alpha e^{-is\alpha}\left\{i\omega_s\Psi_1(0,\alpha) - (\partial_t\Psi_1(t,\alpha))\,|_{t=0}\right\} \\ &= \frac{i}{(2\pi)}\frac{1}{\sqrt{2\omega_s}}\int d\alpha e^{-is\alpha}\left\{i\omega_s\Psi_2(t'(0,\alpha),\alpha'(0,\alpha))\right. \\ &- \left.(\partial_t\Psi_2(t'(t,\alpha),\alpha'(t,\alpha)))\,|_{t=0}\right\}. \end{aligned} \tag{4.25}$$

Using this equation, we can write out the Bogolubov transformation in the form

$$a_2(n) = \sum_s \alpha(n,s)a_1(s) + \beta(n,s)a_1^\dagger(s). \tag{4.26}$$

The spectrum of created particles is determined by $|\beta(n,s)|^2$.

Writing out $\beta(n,s)$ explicitly we find it has some interesting properties due to its dependence on the inverse relations $t'(t,x)$,$x'(t,x)$,

$$\begin{aligned} \beta(n,s) &= \frac{-i}{2\pi}\int d\alpha\frac{e^{-in\alpha}}{\sqrt{4\omega_n\omega_s}}\left\{i\omega_n f_s^{+*}(t'(0,\alpha))e^{-is\alpha'(0,\alpha)}\right. \\ &- \left.\partial_t\left\{f_s^{+*}(t'(t,\alpha))e^{-is\alpha'(t,\alpha)}\right\}|_{t=0}\right\}, \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} \alpha'(t,\alpha) &= \tan^{-1}\left(\frac{\cosh(\frac{\lambda t}{2})\sin(\alpha)}{\cosh(\frac{\lambda t}{2})\cos(\alpha)\cosh(\tau) - \sinh(\frac{\lambda t}{2})\sinh(\tau)}\right) \\ t'(t,\alpha) &= \frac{2}{\lambda}\sinh^{-1}\left(\sinh(\frac{\lambda t}{2})\cosh(\tau) - \cosh(\frac{\lambda t}{2})\cos(\alpha)\sinh(\tau)\right) \end{aligned} \tag{4.28}$$

and

$$\tau = \frac{\lambda}{2}(T_0' - T_0). \tag{4.29}$$

4.5 Total number of particles created

To find out whether the total number of particles created is finite we must find out if $\beta(n,s)$ is Hilbert-Schmidt, namely

$$\sum_s \sum_n |\beta(n,s)|^2 < \infty. \tag{4.30}$$

If this inequality holds it means that the total number of created particles is finite and the Bogolubov transformation is unitarily implementable. To calculate the total number of particles created we write $\beta(n,s)$ in a slightly different form,

$$\beta(n,s) = \frac{-i}{4\pi\sqrt{\omega_s\omega_n}} \int d\alpha e^{-i(n+s)\alpha} e^{-is(\alpha'(0,\alpha)-\alpha)} g(n,s,\alpha) \tag{4.31}$$

where

$$g(n,s,\alpha) = \left\{ i\omega_n f_s^{+*}(t'(0,\alpha)) - is\frac{\partial\alpha'}{\partial t} f_s^{+*}(t'(t,\alpha)) - \frac{\partial t'}{\partial t}\partial_{t'}\left(f_s^{+*}(t'(t,\alpha))\right) \right\}|_{t=0} . \tag{4.32}$$

We have written $\beta(n,s)$ in this form to allow us to write $\alpha'(0,\alpha)$ in a form which takes care of the problem of which branch of the $\tan^{-1}(y)$ in (4.28) to take.

To investigate the asymptotic form of $g(n,s,\alpha)$ we have to find the asymptotic behaviour of the hypergeometric functions involved. The first simplification that can be made is due to the fact that the first two arguments of the hypergeometric functions are complex conjugates of each other ($\beta = \alpha^*$). By writing $\alpha = a + ib$ we see directly from the series for the hypergeometric functions that for large a one can drop the imaginary part of α

$$H(\alpha,\beta,c,z) = 1 + \frac{\alpha\beta}{c}z + \frac{\alpha\beta(\alpha+1)(\beta+1)}{c(c+1)}\frac{z^2}{2} + \ldots$$

$$
\begin{aligned}
&= 1 + \frac{(a^2+b^2)}{c} z + \frac{(a^2+b^2)((a+1)^2+b^2)}{c(c+1)} \frac{z^2}{2} + \ldots \qquad (4.33)\\
&\approx 1 + \frac{(a^2)}{c} z + \frac{(a^2)(a+1)^2}{c(c+1)} \frac{z^2}{2} + \ldots \\
&= H(a,a,c,z).
\end{aligned}
$$

From (4.21) we see that we need asymptotic forms for hypergeometric functions of the form $H(a,a,\frac{1}{2},-x^2)$ and $xH(a+\frac{1}{2},b+\frac{1}{2},\frac{3}{2},-x^2)$. For the first form we can write the hypergeometric function in terms of a Legendre function using the identity [27]

$$
\begin{aligned}
H(a,a,1/2,-x) &= \frac{\pi^{\frac{-1}{2}}}{2}\Gamma(a+\frac{1}{2})\Gamma(1-a)(1+x)^{-a} \times \\
&\left(P_{2a-1}[\frac{x^{\frac{1}{2}}}{\sqrt{(1+x)}}] + P_{2a-1}[\frac{-x^{\frac{1}{2}}}{\sqrt{(1+x)}}]\right). \qquad (4.34)
\end{aligned}
$$

To obtain the asymptotic form for $xH(a+\frac{1}{2},b+\frac{1}{2},3/2,-x^2)$ we notice that we can write it in terms of the derivative of the first hypergeometric function

$$
xH(a+\frac{1}{2},a+\frac{1}{2},\frac{3}{2},-x^2) = \frac{-1}{2(a-\frac{1}{2})^2}\frac{d}{dx}H(a-\frac{1}{2},a-\frac{1}{2},1/2,-x^2). \qquad (4.35)
$$

We now use an expression for the Legendre functions valid for large ν [28],

$$
P_\nu[\cos(\theta)] \approx \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\sqrt{\frac{2}{\pi\sin(\theta)}}\cos((\nu+\frac{1}{2})\theta-\frac{\pi}{4}). \qquad (4.36)
$$

By using the reflection formula

$$
\Gamma(1-x) = \frac{\pi}{\Gamma(x)\sin(\pi z)}, \qquad (4.37)
$$

and taking the asymptotic form for the gamma functions which is valid for large argument.

$$
\Gamma(ax+b) \approx \sqrt{2\pi}e^{-ax}(ax)^{ax+b-\frac{1}{2}}, \qquad (4.38)
$$

we find that the gamma functions from (4.34) and (4.36) combine in such a way as to cancel their s dependence leaving

$$\beta(n,s) = \frac{1}{4\pi\sqrt{\omega_s\omega_n}} \int d\alpha e^{-i(n+s)\alpha} e^{-is(\alpha'(0,\alpha)-\alpha)} \left(\cos[s\cos^{-1}[p(\alpha)]] + \sin[s\cos^{-1}[p(\alpha)]]\right)$$
$$\times \left(A(\alpha)\left(\cos[\frac{\pi s}{2}] + \sin[\frac{\pi s}{2}]\right) + B(\alpha)\left(\cos[\frac{\pi s}{2}] - \sin[\frac{\pi s}{2}]\right)\right) \tag{4.39}$$

where

$$p(\alpha) = -\frac{\cos(\alpha)\sinh(\tau)}{\sqrt{1+\cos^2(\alpha)\sinh^2(\tau)}} \tag{4.40}$$

$$\begin{aligned} A(\alpha) &= \frac{\lambda|s|}{f(\alpha)^2(2|s|-1)^2}\left\{f(\alpha)^2|n|(1-(-1)^s)(|s|-1)\right. \\ &+ i\cos[\alpha]\sinh[\tau](1-(-1)^s) \\ &+ i\cosh[\tau](1+(-1)^s)(2-f(\alpha)+f(\alpha)|s|-f(\alpha)s^2) \\ &+ \left.\sin[\alpha]\sinh[\tau](1-(-1)^s)(f(\alpha)s|s|-f(\alpha)s)\right\} \end{aligned} \tag{4.41}$$

$$\begin{aligned} B(\alpha) &= \frac{\lambda}{4f(\alpha)}\{if(\alpha)|n|(1+(-1)^s)) + |s|\cosh[\tau](1-(-1)^s) \\ &+ is\sin[\alpha]\sinh[\tau](1+(-1)^s)\} \end{aligned} \tag{4.42}$$

and

$$f(\alpha) = 1+\cos^2[\alpha]\sinh^2[\tau]. \tag{4.43}$$

The entire point of writing $\beta(n,s)$ in this way is to allow us to integrate the above expression by parts. After expanding the $\sin[s\cos^{-1}[p(\alpha)]]$ and $\cos[s\cos^{-1}[p(\alpha)]]$ in terms of exponentials, each term making up $\beta(n,s)$ can be written in the form,

$$\beta(n,s) \propto \int d\alpha e^{-i(n+s)\alpha} e^{isg(\alpha)} K_{n,s}(\alpha), \tag{4.44}$$

where $K_{n,s}(\alpha)$ incorporates the last term in (4.39) which contains $A(\alpha)$ and $B(\alpha)$ and

$$g(\alpha) = -(\alpha'(0,\alpha) - \alpha) \pm \cos^{-1}(p(\alpha)), \tag{4.45}$$

where the $\pm$ depends on which of the two terms one is dealing with. The important point is that the behaviour of $K_{n,s}(\alpha)$ in terms of n, s does not change when one differentiates with respect to α because the dependence on n, s is decoupled from α. One can now integrate by parts indefinitely to observe that the expression must drop off faster than any inverse power of n, s. For example after integrating by parts twice one is left with

$$\beta(n,s) \propto \int d\alpha e^{-i(n+s)\alpha} e^{\pm isg(\alpha)} \frac{d}{d\alpha}\left(\frac{1}{-i(n+s) \pm isg'(\alpha)} \frac{d}{d\alpha}\left(\frac{K_{n,s}(\alpha)}{-i(n+s) \pm isg'(\alpha)}\right)\right). \tag{4.46}$$

The only problem that could arise is if $-(n+s) \pm sg'(\alpha)$ ever vanished. This is not a problem however because the function $g'(\alpha)$ is always less than one.

4.6 Conclusions

We have shown that $\beta(n,s)$ drops off faster than any inverse power of n, s, for large n, s. This implies that the total number of particles created is finite and therefore the Bogolubov transformation is unitarily implementable. The fact that the total number of particles created is finite is in agreement with Fulling's analysis for an isotropic universe of finite spatial volume [16].

If in fact $\beta(n,s)$ drops off like an exponential then after performing one of the sums in $|\beta(n,s)|^2$ one will be left with a Planck spectrum. This is to be expected as for large momenta our analysis should be similar in nature to the analysis of massless particle creation.

It should be emphasized that this calculation does not involve calculating the Bogolubov transformation relating in essence to different spacetimes. This calculation involves comparing an observer's particle definition at two different times in the same spacetime. In this way one is not misinterpreting boundary effects by comparing fields quantized in overlapping but different spacetimes [15] as is the case in the standard Rindler analysis and many other calculations.

CHAPTER 5

Finite particle creation in 3+1 de Sitter space

5.1 Introduction

In this chapter we calculate the particle creation as seen by a stationary observer in 3+1 de Sitter space. This particle creation is calculated by looking at the Bogolubov transformation relating the observer's different definitions of particle states on two different spacelike hypersurfaces. The observer dependent nature of this calculation agrees with Gibbons and Hawking's [10] idea that what an observer measures is dependent not only on the spacetime and the quantum state of the system but also on the observer's worldline. In this calculation we do not include the analysis of what a detector carried by an observer would register. As a detector can only be influenced by events in its past light cone it cannot be expected to illuminate any of the non-local nature of quantum particle states.

There has been a number of papers published which study quantum field theory in de Sitter space. As mentioned in the Gibbons and Hawking [10] paper many of these studies [11] involve particle definitions which are observer-independent and de Sitter group invariant. This is clearly not a physically reasonable approach to take as it leads to particle creation rates which are either infinite or zero because if there is particle creation, the same particle creation rate must occur for all energies due to the de Sitter group invariance. Many other studies have followed the example of Lapedes [12] and chosen the static coordinatization of de Sitter space as the preferred set of coordinates with which to define particle states for an observer at the origin. These papers then calculate particle creation by calculating the Bogolubov transformation relating the static coordinatization to the geodesically complete coordinatization which will be used in the analysis of this chapter. The reason that the static coordinatization is chosen is because it has been shown to be the one that agrees with Unruh's [6] analysis of what a particle detector carried by the observer will measure. Although one can understand the nature of the horizon which is present in the static coordinatization as being a causal boundary for a detector by using an incomplete coordinatization of the spacetime one is not allowing for non-local effects from the rest of the spacetime to influence the construction of particle states. In the usual treatment of quantum field theory in Minkowski space such non-local effects are allowed for as states are constructed in terms of operators which are isolated by integrating a field and its time derivative over an entire spacelike surface.

The definition of particle states used here is that proposed by Capri and Roy [7] and is equivalent to the definition proposed by Massacand and Schmid [8]. This definition of particle states uses a coordinate independent and observer dependent definition of time which one uses to decompose the field into positive and negative frequency parts. This time is defined as being normal to the spacelike geodesic hypersurface which intersects the observer's worldline orthogonally. In this way the spacetime is spanned by geodesics. If there is a geodesically complete coordinatization for the spacetime this is the coordinatization that will be picked out by this procedure. In de Sitter space this implies that the radial coordinate is compact even though the coordinatization we start with would not suggest this. It is this compact coordinatization that allows us to eventually integrate by parts the expression for the total particle production and show that it is finite.

It is hoped that if there is a correct means of defining particles in curved backgrounds it will enable objects such as the expectation value of the stress tensor to be renormalized by using a normal ordering procedure similar to that used in flat space.

The particle production is shown to be finite as the Bogolubov $\beta(N, N', l)$ coefficient drops off faster than any inverse power of N or N'. If this drop off is actually an exponential then the particle production would be consistent with a thermal distribution which is what is expected for the large momenta limit. This finite particle creation agrees with the analysis presented in Fulling's book for expanding isotropic universes [16].

5.2 The model

We start with the following coordinatization of de Sitter space,

$$ds^2 = dT^2 - e^{\lambda T}\left((dX^1)^2 + (dX^2)^2 + (dX^3)^2\right). \tag{5.1}$$

To calculate the coordinates which provide the foliation mentioned in the introduction we must first calculate the geodesic equations. The first integrals of the geodesic equations are

$$\frac{dX^i}{ds} = c^i e^{-\lambda T} \quad \text{and} \quad \frac{dT}{ds} = \sqrt{\epsilon + e^{-\lambda T}\mathbf{c}^2} \tag{5.2}$$

where $i = 1$ to 3 and $\epsilon = \pm 1$ depending on whether the geodesic is timelike or spacelike respectively. The preferred coordinates on the hypersurface of instantaneity are constructed using a 4-bein of orthonormal basis vectors based at P_0, the observer's position. These vectors are chosen to be

$$\begin{aligned} e^0_\mu(P_0) &= (1,0,0,0) \quad e^1_\mu(P_0) = (0, e^{-\frac{\lambda T_0}{2}}, 0, 0) \\ e^2_\mu(P_0) &= (0,0,e^{-\frac{\lambda T_0}{2}},0) \quad e^3_\mu(P_0) = (0,0,0,e^{-\frac{\lambda T_0}{2}}). \end{aligned} \tag{5.3}$$

In this way $e^0_\mu(P_0)$ is tangent to the worldline of an observer which is stationary with respect to the coordinates c^i (1). To construct a spacelike geodesic which is orthogonal to the observer's worldline it is required that

$$\frac{dT}{ds}|_{P_0} = 0 \quad \text{which} \quad \text{implies} \quad \mathbf{c}^2 = e^{\lambda T_0}. \tag{5.4}$$

The preferred coordinates on the spacelike hypersurface are chosen to be Riemann coordinates based on the observer's position $P_0 = (T_0, X_0^1, X_0^2, X_0^3)$. The coordinates are constructed using the point $P_1 = (T_1, X_1^1, X_1^2, X_1^3)$ which is the point at which a timelike geodesic "dropped" from an arbitrary point $P = (T, X^1, X^2, X^3)$ intersects the spacelike hypersurface orthogonally. The Riemann coordinates η^α of the point P_1 are given by

$$s_s p^\mu = \eta^\alpha e^\mu_\alpha(P_0). \tag{5.5}$$

where s_s is the distance along the geodesic $P_0 - P_1$ and p^μ is the vector tangent to the geodesic connecting P_0 to P_1, at P_0. These equations can be solved for the coordinates η^α using the orthogonality of p^μ to $e_0(P_0)$ and the identity $e^\mu_\alpha e_{\beta\mu} = \eta_{\alpha\beta}$ (Minkowski metric) to give

$$\eta^0 = s_s p^\mu e^0_\mu(P_0) \qquad \eta^i \quad -s_s p^\mu e^i_\mu(P_0). \tag{5.6}$$

The surface of instantaneity is then just the surface $\eta^0 = 0$ and the preferred spatial coordinates are given by

$$x^i = s_s c^i e^{\frac{\lambda T_0}{2}}. \tag{5.7}$$

The preferred time coordinate t of an arbitrary point P is given by the proper distance along the timelike geodesic connecting P to P_1. This timelike geodesic is also determined by (2) with a different set of constants b^i and $\epsilon = 1$. The condition that this timelike geodesic is orthogonal to the spacelike hypersurface is

$$\sqrt{e^{\lambda(T_0 - T_1)} - 1}\sqrt{\mathbf{c}^2 e^{-\lambda T_1} + 1} = \mathbf{c} \cdot \mathbf{b} e^{-\lambda T_1}. \tag{5.8}$$

There is an arbitrary choice involved in how one solves these two equations for the constants **b** and **c**. This freedom can be understood as the ability to rotate the hypersurface of instantaneity through a reparametrization of the surface. The choice which we make for reasons of calculational simplicity is that

$$b^i = \sqrt{1 - e^{-\lambda(T_0 - T_1)}}c^i. \tag{5.9}$$

At this point it convenient to to introduce the variable r,

$$r^2 = \mathbf{x} \cdot \mathbf{x} = s_s^2 \frac{\mathbf{c} \cdot \mathbf{c}}{e^{\lambda T_0}} = s_s^2. \tag{5.10}$$

We can now calculate the metric in terms of the preferred coordinates $(t, \mathbf{x})$ by calculating $(T(t, x^i), \mathbf{X}(t, \mathbf{x}))$:

$$X^i = X_0^i + \int_{T_0}^{T_1} dT \frac{c^i}{\sqrt{e^{\lambda(T_0 - T)} - 1}} + \int_{T_1}^{T} dT \frac{b^i e^{-\lambda T}}{\sqrt{1 + \mathbf{b}^2 e^{-\lambda T}}}. \tag{5.11}$$

We also need to calculate t and s_s,

$$s_s = \int_{T_0}^{T_1} \frac{dT}{\sqrt{e^{\lambda(T_0 - T)} - 1}} \tag{5.12}$$

$$t = \int_{T_1}^{T} \frac{dT}{\sqrt{1 + \mathbf{b}^2 e^{-\lambda T}}}. \tag{5.13}$$

One can now obtain the coordinate transformations

$$\begin{aligned} e^{\frac{\lambda}{2}(T - T_0)} &= \cosh(\frac{\lambda t}{2})\cos(\frac{\lambda r}{2}) + \sinh(\frac{\lambda t}{2}) \\ \frac{\lambda}{2}(X^i - X_0^i)e^{\frac{\lambda}{2}T} &= \frac{x^i}{r}\cosh(\frac{\lambda t}{2})\sin(\frac{\lambda r}{2}). \end{aligned} \tag{5.14}$$

We can see here by looking at a particular t =constant surface that the range of r is now compact and the range $0 \leq \frac{\lambda r}{2} < \pi$ covers the entire manifold which was covered by the original coordinates $(T, \mathbf{X})$. It is now easy to put the preferred coordinates into polar form.

$$\begin{aligned} x^1 &= r\sin(\theta)\sin(\phi) \\ x^2 &= r\sin(\theta)\cos(\phi) \\ x^3 &= r\cos(\theta). \end{aligned} \tag{5.15}$$

In terms of these preferred coordinates the metric is

$$ds^2 = dt^2 - \cosh^2(\frac{\lambda t}{2})\left(dr^2 + \frac{4}{\lambda^2}\sin^2(\frac{\lambda r}{2})\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right). \tag{5.16}$$

This result is of course no surprise to anyone familiar with different coordinatizations of de Sitter space, given that the space was being coordinatized in terms of geodesics. The point here is not what the final form of the metric is as much as how these transformations will change as our observer moves to a different point and the entire construction is repeated.

5.3 Modes and initial conditions

In the coordinates constructed above, the minimally coupled massless Klein Gordon equation is

$$\partial_t^2\phi + \frac{1}{\sqrt{g}}\partial_t(\sqrt{g})\,\partial_t\phi + \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\right)\partial_j\phi = 0, \tag{5.17}$$

where $|g|$ and the g^{ij} can be read off from (5.16). To quantize a scalar field on the $t = 0$ surface we now define the positive frequency modes as those which satisfy the initial conditions

$$\phi^+_{Nln} = A_{Nln}(0,r,\theta,\phi) \quad \text{and} \quad \partial_t\phi^+_{Nln}|_{t=0} = -i\omega_N(0)A_{Nln}(0,r,\theta,\phi), \tag{5.18}$$

where $A_{Nln}(0,r,\theta,\phi)$ are the instantaneous eigenmodes of the spatial part of the Laplace-Beltrami operator, and $\omega_N(t)^2$ are the corresponding eigenvalues:

$$\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)A_{Nln}(t,r,\theta,\phi) = \omega_N(t)^2 A_{Nln}(t,r,\theta,\phi). \tag{5.19}$$

Henceforth we write ω_N for $\omega_N(0)$:

$$\omega_N \equiv \omega_N(0) = \sqrt{\frac{\lambda^2}{4}N(N+2)}. \tag{5.20}$$

The differential equations (5.17) and (5.19) must now be solved and the appropriate initial conditions imposed. The positive frequency solution to these differential equations which satisfies the correct initial conditions as just stated is

$$\phi^{+}_{Nln}(t,r,\theta,\phi) = F_{Nl} Y_{ln}(\theta,\phi) \sin^{l}(\frac{\lambda r}{2}) C^{l+1}_{N-l}\left[\cos(\frac{\lambda r}{2})\right] \operatorname{sech}^{\frac{3}{2}}(\frac{\lambda t}{2}) \times$$
$$\left(L P^{\frac{3}{2}}_{\frac{1}{2}+N}\left[\tanh(\frac{\lambda t}{2})\right] + M Q^{\frac{3}{2}}_{\frac{1}{2}+N}\left[\tanh(\frac{\lambda t}{2})\right],\right) \tag{5.21}$$

where

$$\begin{aligned}
L &= -\frac{(2+N)\,\lambda\, Q^{\frac{3}{2}}_{-\frac{1}{2}+N}(0) - 2\,i\, Q^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\,\omega_N}{\lambda\,(2+N)\left(-P^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\, Q^{\frac{3}{2}}_{-\frac{1}{2}+N}(0) + P^{\frac{3}{2}}_{-\frac{1}{2}+N}(0)\, Q^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\right)} \\
M &= \frac{(2+N)\,\lambda\, P^{\frac{3}{2}}_{-\frac{1}{2}+N}(0) - 2\,i\, P^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\,\omega_N}{\lambda\,(2+N)\left(-P^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\, Q^{\frac{3}{2}}_{-\frac{1}{2}+N}(0) + P^{\frac{3}{2}}_{-\frac{1}{2}+N}(0)\, Q^{\frac{3}{2}}_{\frac{1}{2}+N}(0)\right)} \\
F_{Nl} &= \frac{2^{\frac{1}{2}+l}\,\sqrt{1+N}\,\Gamma(1+l)\,\sqrt{\Gamma(1-l+N)}}{\sqrt{\pi}\,\sqrt{\Gamma(2+l+N)}}.
\end{aligned} \tag{5.22}$$

$C^{n}_{m}[x]$ are Gegenbauer polynomials and $P^{m}_{n}[x]$ and $Q^{m}_{n}[x]$ are associated Legendre functions. We can now write out the field which has been quantized on the $t = 0$ surface which corresponds to the geodesic surface passing through the point $(T_0, \mathbf{X}_0)$

$$\Psi_1 = \sum_{N=0}^{\infty} \sum_{l=0}^{N} \sum_{n=-l}^{l} \left\{ {}_1a_{Nln}\phi^{(+)}_{Nln}(t,r,\theta,\phi) + {}_1a^{\dagger}_{Nln}\phi^{*(+)}_{Nln}(t,r,\theta,\phi) \right\}. \tag{5.23}$$

Although these modes are not as simple as the modes resulting from some different coordinatizations these are the modes which are chosen by the particle definition we have chosen. Normally one would introduce the conformal time and define positive frequency modes in terms of natural conditions satisfied by the modes as the conformal time goes to past and future infinity [14]. According to the procedure adopted here this means of defining positive frequency would not be appropriate for the observer presently being studied.

5.4 Particle creation

To investigate the particle creation in this universe, as observed by an observer stationary with respect to the original coordinates $(T, \mathbf{X})$, we calculate the Bogolubov transformation relating the annihilation and creation operators from two different surfaces of quantization that the observer passes through. To calculate the coefficients of this transformation we equate the same field from two different quantizations on a common surface

$$\Psi_1(t,r,\theta,\phi) = \Psi_2(t'(t,r,\theta,\phi), r'(t,r,\theta,\phi), \theta'(t,r,\theta,\phi), \phi'(t,r,\theta,\phi)). \tag{5.24}$$

Here $\Psi_1(t,r,\theta,\phi)$ is the field written out in (23) and $\Psi_2(t',r',\theta',\phi')$ is the same field which has been quantized on a second surface $t'=0$. The "second" field is therefore quantized for the same observer as the first but at some later time T_0' with $\mathbf{X}_0 = \mathbf{X'}_0$. All the physics of the observations made by this observer are determined by the functions $t'(t,r,\theta,\phi)$, $r'(t,r,\theta,\phi)$, $\theta'(t,r,\theta,\phi)$, $\phi'(t,r,\theta,\phi)$, and the derivatives of these functions with respect to t. In this way the geometry of the spacetime via the coordinate independent prescription we have used, determines the spectrum of created particles. This is the reason for the comment at the end of section 5.2 about the form of the metric not being as important as the transformations that gave that form of the metric. These functions take on a fairly simple form for the stationary observer

$$\begin{aligned}
t' &= \frac{2}{\lambda}\sinh^{-1}\left[\sinh(\frac{\lambda t}{2})\cosh(\tau) - \cosh(\frac{\lambda t}{2})\cos(\frac{\lambda r}{2})\sinh(\tau)\right] \\
r' &= \frac{2}{\lambda}\tan^{-1}\left[\frac{\cosh(\frac{\lambda t}{2})\sin(\frac{\lambda r}{2})}{\cosh(\frac{\lambda t}{2})\cos(\frac{\lambda r}{2})\cosh(\tau) - \sinh(\frac{\lambda t}{2})\sinh(\tau)}\right] \\
\theta' &= \theta \\
\phi' &= \phi \qquad \text{where} \quad \tau = \frac{\lambda}{2}(T_0' - T_0)
\end{aligned} \tag{5.25}$$

We calculate the Bogolubov transformation by "matching" the field and its first derivative with respect to t at $t = 0$. This allows us to calculate the β coefficient of the Bogolubov transformation which gives rise to the particle creation. In calculating the Bogolubov β coefficient we are able to perform the θ and ϕ integrals of the spherical harmonics because of the simplicity of the coordinate transformations (5.25) leaving

$$\begin{aligned}\beta(N,N',l) &= \frac{i}{2\omega_N}\int_0^{\pi} d\chi \sin^2(\chi) R_{Nl}(\chi) \\ &\left(-i\omega_N f_{N'}^{*\,(+)}(t') R_{N'l}(\chi') + \partial_t \left(f_{N'}^{*\,(+)}(t') R_{N'l}(\chi')\right)\right)\Big|_{t=0}, \end{aligned} \tag{5.26}$$

where $\chi = \frac{\lambda r}{2}$ and $\chi' = \frac{\lambda r'}{2}$. For notational convenience we have split up the radial and time functions as

$$\begin{aligned} R_{Nl}(\chi) &= F_{Nl}\sin^l(\chi) C_{N-l}^{l+1}(\cos(\chi)) \\ f_N^{*\,(+)}(t) &= \operatorname{sech}^{\frac{1}{2}}(\frac{\lambda t}{2})\left(LP^{\frac{3}{2}}_{\frac{1}{2}+N}(\tanh(\frac{\lambda t}{2})) + MQ^{\frac{3}{2}}_{\frac{1}{2}+N}(\tanh(\frac{\lambda t}{2}))\right). \end{aligned} \tag{5.27}$$

In the next section we examine the structure of β in detail; this examination is simplified by first noting that the expression for β can be written in terms of an integral running from $0 \to 2\pi$

$$\begin{aligned}\beta(N,N',l) &= \frac{i}{4\omega_N}\int_0^{2\pi} d\chi \sin^2(\chi) R_{Nl}(\chi) \\ &\left(-i\omega_N f_{N'}^{*\,(+)}(t') R_{N'l}(\chi') + \partial_t \left(f_{N'}^{*\,(+)}(t') R_{N'l}(\chi')\right)\right)\Big|_{t=0}. \end{aligned} \tag{5.28}$$

It is not difficult to show that this integral is symmetric about $\chi = \pi$ which is why we can write the integral in this way. This change will make our final integration by parts more transparent.

5.5 Total number of particles created

To show that the total number of particles created is finite we must show that the Bogolubov transformation is Hilbert-Schmidt, namely,

$$\sum_{NN'l} |\beta(N,N',l)|^2 < \infty. \tag{5.29}$$

Since the sum on the left hand side of this inequality gives the number of particles created, this inequality, if it holds, implies that the total number of particles created is finite and that the Bogolubov transformation is unitarily implementable. To show this one need only be concerned with the large N,N' and l behaviour. As the sum over l is a finite sum and $\beta(N,N',l)$ decreases with l when l is large then one only need be concerned with the large N and N' behaviour of $\beta(N,N',l)$. By looking at this asymptotic behaviour one is left with simpler functions that may be integrated exactly. We now show that indeed when looking at the large N and N' behaviour the integrals defining β may be bounded by terms implying that $|\beta(N,N',l)|^2$ drops off faster than any inverse power of N and N'. This also implies that the finite sum over l does not change this result as it only introduces a simple power of N. Using the following relations [27] for the functions that the modes are constructed from, we are able to obtain an approximate form of $\beta(N,N',l)$ valid for large N and N',

$$\begin{aligned}
C_n^m[x] &= \frac{\Gamma(2m+n)\Gamma(m+\frac{1}{2})}{\Gamma(2m)\Gamma(n+1)}\left\{\frac{1}{4}(x^2-1)\right\}^{\frac{1}{4}-\frac{m}{2}} P^{\frac{1}{2}-m}_{m+n-\frac{1}{2}}(x) \\
P_\nu^\mu[\cos(x)] &\approx \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\left(\frac{1}{2}\pi\sin(x)\right)^{-\frac{1}{2}}\cos\left((\nu+\frac{1}{2})x-\frac{\pi}{4}+\frac{\mu\pi}{2}\right) \quad \text{for large } \nu \\
Q_n^m[\cos(x)] &\approx \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\left(\frac{\pi}{2\sin(x)}\right)^{\frac{1}{2}}\cos\left((\nu+\frac{1}{2})x+\frac{\pi}{4}+\frac{\mu\pi}{2}\right) \quad \text{for large } \nu \\
\Gamma(ax+b) &\approx \sqrt{2\pi}e^{-ax}(ax)^{ax+b-\frac{1}{2}} \quad \text{for large a and x} > 0.
\end{aligned} \tag{5.30}$$

The expression for β now involves many terms but is still simple enough to see what is required

$$\beta(N,N',l) = \int_0^{2\pi} d\chi\ K((A(A_1 \times L + A_2 \times M) + B(B_1 \times L + B_2 \times M))M_1 + (A(C_1 \times L + C_2 \times M) + B(D_1 \times L + D_2 \times M))N_1) \qquad (5.31)$$

where

$$A = \cos\left(\frac{l\pi}{2} - \cos^{-1}(\frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \mathrm{sech}(\tau)^2\sin(\chi)^2}}) - N'\cos^{-1}(\frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \mathrm{sech}(\tau)^2\sin(\chi)^2}})\right)$$

$$B = \sin\left(\frac{l\pi}{2} - \cos^{-1}(\frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \mathrm{sech}(\tau)^2\sin(\chi)^2}}) - N'\cos^{-1}(\frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \mathrm{sech}(\tau)^2\sin(\chi)^2}})\right)$$

$$M_1 = \cos\left(N'\pi - N'\cos^{-1}(\frac{\cos(\chi)\sinh(\tau)}{\sqrt{1+\cos(\chi)^2\sinh(\tau)^2}})\right)\sin\left(\frac{l\pi}{2} - \chi - \mathrm{N}\chi\right)$$

$$N_1 = \sin\left(\frac{l\pi}{2} - \chi - \mathrm{N}\chi\right)\sin\left(N'\pi - N'\cos^{-1}(\frac{\cos(\chi)\sinh(\tau)}{\sqrt{1+\cos(\chi)^2\sinh(\tau)^2}})\right)$$

$$A_1 = 16(1+l)\lambda N'\Gamma(2(1+l))\Gamma(\frac{5}{2}+l)\sin(\chi)\sinh(\tau)$$

$$A_2 = -8(1+l)\lambda N'\pi\cos(\chi)\Gamma(2(1+l))\Gamma(\frac{5}{2}+l)\sin(\chi)\sinh(\tau)^2$$

$$B_1 = -4\cosh(\tau)^2\Gamma(\frac{3}{2}+l)\Gamma(2(2+l))\left(-2i\omega_N\mathrm{sech}(\tau)^2 + 2\lambda\cos(\chi)\tanh(\tau) + l\lambda\cos(\chi)\tanh(\tau) + \lambda N'\cos(\chi)\tanh(\tau) - 2i\cos(\chi)^2\omega_N\tanh(\tau)^2\right)$$

$$B_2 = 2\pi\Gamma(\frac{3}{2}+l)\Gamma(2(2+l))(-2\lambda\cosh(\tau) - \lambda N'\cosh(\tau) - 2i\cos(\chi)\omega_N\sinh(\tau) + l\lambda\cos(\chi)^2\cosh(\tau)\sinh(\tau)^2 - 2i\cos(\chi)^3\omega_N\sinh(\tau)^3)$$

$$C_1 = -16(1+l)\lambda N'\cos(\chi)\Gamma(2(1+l))\Gamma(\frac{5}{2}+l)\sin(\chi)\sinh(\tau)^2$$

$$C_2 = -8(1+l)\lambda N'\pi\Gamma(2(1+l))\Gamma(\frac{5}{2}+l)\sin(\chi)\sinh(\tau)$$

$$
\begin{aligned}
D_1 &= 4\Gamma(\frac{3}{2}+l)\Gamma(2(2+l))(-2\lambda\cosh(\tau) - \lambda N'\cosh(\tau) - 2i\cos(\chi)\omega_N\sinh(\tau) \\
&+ l\lambda\cos(\chi)^2\cosh(\tau)\sinh(\tau)^2 - 2i\cos(\chi)^3\omega_N\sinh(\tau)^3\Big) \\
D_2 &= \pi\Gamma(\frac{3}{2}+l)\Gamma(2(2+l))(-3i\omega_N + i\cos(2\chi)\omega_N - i\cosh(2\tau)\omega_N \\
&- i\cos(2\chi)\cosh(2\tau)\omega_N + 2\lambda\cos(\chi)\sinh(2\tau) \\
&+ l\lambda\cos(\chi)\sinh(2\tau) + \lambda N'\cos(\chi)\sinh(2\tau)) \\
K &= \frac{2^{2l-1}\sqrt{1+\mathrm{N}}\sqrt{N'}\sqrt{1+N'}\sqrt{\frac{2}{\pi^3}}\Gamma(1+l)^2\Gamma(\frac{3}{2}+l)}{\frac{1}{4}^l\sqrt{\mathrm{N}}\Gamma(2(1+l))^2\Gamma(2(2+l))\left(1+\cos(\chi)^2\sinh(\tau)^2\right)^{\frac{3}{2}}}.
\end{aligned}
\tag{5.32}
$$

The exact form of the above expressions are not important to understand the large N and N' behaviour of $|\beta(N,N',l)|^2$. What is important is to notice that the expressions $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, K$ do not change as far as their N and N' behaviour is concerned when differentiated with respect to χ. This implies that one can integrate the expression by parts indefinitely to observe that the expression must drop off faster than any inverse power of N and N'. A typical term, after writing out the trigonometric functions in terms of exponentials, reads

$$
\int_0^{2\pi} d\chi\, e^{\pm iN\chi} e^{\pm iN'\cos^{-1}(p(\chi))} e^{\pm iN'\cos^{-1}(q(\chi))} F(N,N',\chi). \tag{5.33}
$$

Here

$$
\begin{aligned}
p(\chi) &= \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \mathrm{sech}(\tau)^2\sin(\chi)^2}} \\
q(\chi) &= \frac{\cos(\chi)\sinh(\tau)}{\sqrt{1+\cos(\chi)^2\sinh(\tau)^2}}.
\end{aligned}
\tag{5.34}
$$

In the above expression the exponentials represent the contributions from the combinations of A, B, M_1, N_1 and $F(N,N',\chi)$ represents the contribution from the functions

$A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, K$. Equation (33) can be rewritten

$$\int_0^{2\pi} \frac{d(e^{\pm iN\chi} e^{\pm iN'(\cos^{-1}(p)\pm\cos^{-1}(q))})}{\pm iN \mp iN'(\frac{1}{\sqrt{1-p^2}}\frac{dp}{d\chi} \pm \frac{1}{\sqrt{1-q^2}}\frac{dq}{d\chi})} F(N, N', \chi). \tag{5.35}$$

Thus an integration by parts produces a term which drops off like,

$$\frac{d}{d\chi}\left(\frac{F(N, N', \chi)}{\pm iN \mp iN'(\frac{1}{\sqrt{1-p^2}}\frac{dp}{d\chi} \pm \frac{1}{\sqrt{1-q^2}}\frac{dq}{d\chi})}\right). \tag{5.36}$$

Because the behaviour of $F'(N, N', \chi)$ for large N and N' is no worse than $F(N, N', \chi)$ this procedure can be repeated indefinitely showing that $\beta(N, N', l)$ drops off faster than any inverse power of N and N' for large N, N'. We can then conclude that the particle creation is finite and that the Bogolubov transformation is unitarily implementable.

Concerning the l dependence in $\beta(N, N', l)$ we only have a finite sum for the total particle creation. It is easy to show that if one uses the same approximations (5.30) for the gamma functions involving the l's which are valid for large l, $\beta(N, N', l)$ drops off for large l as l increases. Thus, the probability of finding particles created with angular momentum l decreases as l increases. This means that when one does the finite sum over l the result will not grow any quicker than N. Therefore because the particle density in N and N' drops off faster than any inverse power of N and N' the total particle creation remains finite.

5.6 Conclusions

We have calculated explicitly the particle creation observed by an observer which is stationary in 3 + 1 de Sitter space. We calculate this particle creation by calculating the Bogolubov transformation relating the annihilation and creation operators from two different quantizations. These different quantizations are constructed using the same procedure on two different spacelike surfaces. Physically this particle creation can be understood as the particle creation seen by an observer moving from one of these surfaces to the next. By looking at the large momenta behaviour for the Bogolubov transformations we are able to show that the transformation is unitarily implementable and therefore the particle creation is finite. Because $\beta(N, N', l)$ drops off faster than any inverse power of N and N' it may be that it drops as an exponential suggesting a thermal spectrum.

It should be emphasized what this calculation is not. Many calculations have been done [11] calculating the Bogolubov transformations relating the creation and annihilation operators due to two different coordinatizations of similar spacetimes. One coordinatization usually covering the entire spacetime and the other only covering a portion of the spacetime. These calculations often use the static coordinatization to define particles relevant to an observer who is at the origin of the coordinate system. As the static coordinate system has a horizon this horizon is understood physically as being the causal boundary beyond which the detector carried by the observer cannot be influenced. This unfortunately does not allow for true particle states such as those which we understand from quantum field theory in Minkowski space to be analysed in a manner which can appreciate their non-local nature.

The procedure advocated in this thesis requires that one use the geodesically complete coordinatization as the spacetime is spanned by geodesics in the preferred coordinates. In this particular example this means that the preferred coordinatization is compact. It is this compactness that allows us to integrate by parts the expression for the total particle creation and show it is finite.

In spacetimes where there is a real boundary present such as a horizon one may have to impose boundary conditions at the horizon [29]. Comparing coordinatizations where one coordinatization implies a boundary and therefore does not cover the entire manifold has been investigated in a clear paper by Salaev and Krustalev [15]. In this paper the authors conclude that either one has a boundary in the spacetime or one does not, there is no in between. The alternative to this is that the observer somehow moves from one spacetime to the other, an issue that has been addressed earlier by Massacand and Schmid [8] and argued to be unreasonable.

CHAPTER 6

The trace anomaly

6.1 Introduction

One of the more interesting results of the study of quantum field theory in curved spacetime is the fact that the expectation value of the trace of the stress tensor of a conformally coupled field does not vanish. It has an anomaly. This trace or conformal anomaly, as it is known, was first noticed by Capper and Duff [30] using a dimensional regularization scheme. Since then many other regularization procedures have been used and when used correctly lead to same result [31],[9],[14]. Unfortunately, as anyone who has ever calculated this trace anomaly knows, the computations required are rather lengthy and certainly less than illuminating. On the other hand, if one has a particle interpretation the problem can be handled more simply. This fact was first exploited by Massacand and Schmid [8]. In this chapter we adapt their method to a computation in 1+1 dimensions using only the following two input.

1) The frame components of the stress tensor at a given point are, for two frames based at this point, related by a Lorentz transformation.

2) The vacuum expectation value of the energy momentum density (relative to a given frame) should vanish. Thus, the vacuum can have pressure, but no energy or momentum.

In general there would remain the vexing question, "Which vacuum?" The answer we propose is to use the coordinate independent definition of Capri and Roy that has been used throughout this thesis. In section 6.2 we give a brief review of this construction in a general 1 + 1 dimensional spacetime and apply the result to a calculation of the vacuum expectation value of the trace of the stress tensor in section 6.3. Our conclusions are set out in section 6.4.

6.2 Coordinate independent definition of time and vacuum

In a globally hyperbolic spacetime one can choose a foliation based solely on geodesics. Thus, given a timelike (unit) vector $N_\mu(P_0)$ at the point P_0 one establishes a frame (zweibein) at P_0 with components:

$$\begin{aligned} e^{\mu\hat{0}} &= N^\mu(P_0) \\ e^{\mu\hat{1}} &= p^\mu(P_0), \end{aligned} \tag{6.1}$$

where $p^\mu(P_0)$ is a unit vector orthogonal to $N^\mu(P_0)$ at P_0. The spacelike hypersurface (line) consiting of the geodesic through P_0 with tangent vector $p^\mu(P_0)$ defines the surface $t = 0$. The "time" t corresponding to an arbitrary point P is the distance along a geodesic $P_1 - P$ which intersects the line $t = 0$ orthogonally at some point P_1. The geodesic distance $P_0 - P_1$ along the line $t = 0$ yields the space coordinate x. These geodesic normal coordinates prove to be very useful since in these coordinates the metric becomes

$$ds^2 = dt^2 - a^2(t,x)dx^2 \tag{6.2}$$

where

$$a(0,0) = 1$$

$$\frac{\partial \alpha}{\partial t}|_{t=0} = \frac{\partial \alpha}{dx}|_{t=0} = 0 = \frac{\partial^2 \alpha}{\partial t \partial x}|_{P_0} = \frac{\partial^2 \alpha}{\partial x^2}|_{P_0}. \tag{6.3}$$

Also,

$$\frac{2}{\alpha}\frac{\partial^2 \alpha}{\partial t^2} = R \tag{6.4}$$

where R is the curvature scalar. We assume that the range of x is over the whole real line.

The field equations in these coordinates, for a massless scalar field read:

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g} g^{\mu\nu}\partial_\nu)\phi &= 0 \\ \frac{\partial^2 \phi}{\partial t^2} + \frac{\dot{\alpha}}{\alpha}\frac{\partial \phi}{\partial t} + \frac{\alpha'}{\alpha^3}\frac{\partial \phi}{\partial x} - \frac{1}{\alpha^2}\frac{\partial^2 \phi}{\partial x^2} &= 0. \end{aligned} \tag{6.5}$$

The positive frequency modes ϕ of this field are obtained by solving these field equations with the two initial conditions

$$1) \qquad \phi_{p,\epsilon}(0,x) = \frac{1}{\sqrt{4\pi p}}\exp(-ip\epsilon x) \qquad p > 0. \tag{6.6}$$

Here we have

$$\begin{aligned} \epsilon &= +1 \quad \text{corresponds to left travelling waves,} \\ \epsilon &= -1 \quad \text{corresponds to right travelling waves,} \end{aligned} \tag{6.7}$$

and

$$2) \qquad i\frac{\partial \phi_{p,\epsilon}}{\partial t}|_{t=0} = p\phi_{p,\epsilon}|_{t=0}. \tag{6.8}$$

A useful Ansatz to implement these initial conditions is:

$$\phi_{p,\epsilon}(t,x) = \frac{1}{\sqrt{4\pi p}}\exp(-ipf_\epsilon(t,x)) \tag{6.9}$$

where f_ϵ is real. Equation (6.5) then yields that

$$\frac{\partial f_\epsilon}{\partial t} = \frac{\epsilon}{\alpha}\frac{\partial f_\epsilon}{\partial x}. \tag{6.10}$$

The initial conditions become

$$f_\epsilon(0,x) = \epsilon x \tag{6.11}$$

and near $t = 0$

$$f_\epsilon(t,x) \approx t + \epsilon x. \tag{6.12}$$

The quantized field is now given by

$$\Psi(t,x) = \sum_{\epsilon=\pm 1} \int_0^\infty d(\epsilon p) \left(\phi_{p,\epsilon}(t,x) a_{p,\epsilon} + \phi^*_{p,\epsilon}(t,x) a^\dagger_{p,\epsilon} \right) \tag{6.13}$$

with the vacuum defined by

$$a_{p,\epsilon}|0>= 0. \tag{6.14}$$

These modes have been normalized such that

$$\begin{aligned}
(\phi_{p,\epsilon}, \phi_{q,\epsilon}) &= i\epsilon \int_{-\infty}^\infty dx \sqrt{g} \left(\phi^*_{p,\epsilon}(t,x) \overleftrightarrow{\partial_t} \phi_{q,\epsilon}(t,x) \right) \\
&= \frac{p+q}{4\pi\sqrt{pq}} \int_{-\infty}^\infty dx \epsilon \alpha \frac{\partial f_\epsilon}{\partial t} \exp(i(p-q) f_\epsilon(t,x)) \\
&= \frac{p+q}{4\pi\sqrt{pq}} \int_{-\infty}^\infty dx \frac{\partial f_\epsilon}{\partial x} \exp(i(p-q) f_\epsilon(t,x)) \\
&= \frac{p+q}{4\pi\sqrt{pq}} 2\pi\delta(p-q) \\
&= \delta(p-q).
\end{aligned} \tag{6.15}$$

6.3 The trace anomaly

We begin with two "observers" with tangents to their world lines given by

$$N^\mu(P_0) = (1,0) \quad \text{and} \quad \bar{N}^\mu(P_0) = (\cosh(\chi), \frac{\sinh(\chi)}{\bar{\alpha}}) \tag{6.16}$$

The corresponding frames are:

$$e^{\mu\hat{0}} = (1,0) \qquad e^{\mu\hat{1}} = (0, \frac{1}{\alpha}) \tag{6.17}$$

$$\bar{e}^{\mu\hat{0}} = (\cosh(\chi), \frac{\sinh(\chi)}{\bar{\alpha}}), \qquad \bar{e}^{\mu\hat{1}} = (\sinh(\chi), \frac{\cosh(\chi)}{\bar{\alpha}}). \tag{6.18}$$

Corresponding to this the metric has the two forms

$$ds^2 = dt^2 - a^2(t, x)dx^2 = d\bar{t}^2 - \bar{a}^2(\bar{t}, \bar{x}).d\bar{x}^2 \tag{6.19}$$

We can solve for the positive frequency modes in the barred as well as in the unbarred coordinates to obtain the corresponding quantized fields $\bar{\Psi}(\bar{t}, \bar{x})$ and $\Psi(t, x)$. Their respective sets of annihilation and creation operators are $(\bar{a}_{p,\epsilon}, \bar{a}^{\dagger}_{p,\epsilon})$ and $(a_{p,\epsilon}, a^{\dagger}_{p,\epsilon})$.

At P_0, the point with coordinates $(0, 0)$ in both coordinate systems, the two fields coincide, as do their first time derivatives. Corresponding to these two fields we have their respective Fock space vacuums $|\bar{0}>$, $|0>$ defined by

$$\bar{a}_{p,\epsilon}|\bar{0}>= 0 \quad , \quad a_{p,\epsilon}|0>= 0. \tag{6.20}$$

Any bilinear expression in the field operators which, for physical reasons, should have vanishing vacuum expectation value is defined by normal ordering with respect to its own vacuum. Thus since we expect the vacuum to be the state of zero energy and momentum density we require that

$$<\bar{0}| : \bar{T}^{\hat{0}\hat{\mu}} : |\bar{0}>= 0 \tag{6.21}$$

and

$$<0| : T^{\hat{0}\hat{\mu}} : |0>= 0, \tag{6.22}$$

where,

$$\begin{aligned} T^{\hat{\alpha}\hat{\beta}} &= e^{\mu\hat{\alpha}} e^{\nu\hat{\beta}} T_{\mu\nu} \\ \bar{T}^{\hat{\alpha}\hat{\beta}} &= \bar{e}^{\mu\hat{\alpha}} \bar{e}^{\nu\hat{\beta}} \bar{T}_{\mu\nu}. \end{aligned} \tag{6.23}$$

Furthermore, since the barred and unbarred frames $\bar{e}^{\mu\hat{\alpha}}$, $e^{\mu\hat{\alpha}}$ are related by a Lorentz transformation

$$\Lambda^{\hat{\beta}}_{\hat{\alpha}} = \begin{pmatrix} \cosh(\chi) & \sinh(\chi) \\ \sinh(\chi) & \cosh(\chi) \end{pmatrix} \tag{6.24}$$

we have that at P_0

$$: T^{\hat{\alpha}\hat{\beta}} : |_{P_0} = \Lambda^{\hat{\alpha}}{}_{\hat{\gamma}}\Lambda^{\hat{\beta}}{}_{\hat{\delta}} : \bar{T}^{\hat{\gamma}\hat{\delta}} : |_{P_0} \tag{6.25}$$

so that in particular

$$: T^{\hat{0}\hat{0}} : |_{P_0} = \cosh^2(\chi) : \bar{T}^{\hat{0}\hat{0}}|_{P_0} + 2\cosh(\chi)\sinh(\chi) : \bar{T}^{\hat{0}\hat{1}} : |_{P_0} + \sinh^2(\chi) : \bar{T}^{\hat{1}\hat{1}} : |_{P_0}. \tag{6.26}$$

Taking the vacuum expectation value with respect to the barred vacuum of this equation, and using (6.21) we have

$$< \bar{0}| : T^{\hat{0}\hat{0}} : |_{P_0}|\bar{0} >= \sinh^2(\chi) < \bar{0}| : \bar{T}^{\hat{1}\hat{1}} : |_{P_0}|\bar{0} > \tag{6.27}$$

Since $< \bar{0}| : \bar{T}^{\hat{0}\hat{0}} : |\bar{0} >= 0$ we find that the vacuum expectation value of the trace is:

$$< \bar{0}|\eta_{\hat{\alpha}\hat{\beta}} : \bar{T}^{\hat{\alpha}\hat{\beta}} : |_{P_0}|\bar{0} >= - < \bar{0}| : \bar{T}^{\hat{1}\hat{1}} : |_{P_0}|\bar{0} >= -\frac{1}{\sinh^2(\chi)} < \bar{0}| : T^{\hat{0}\hat{0}} : |_{P_0}|\bar{0} > \tag{6.28}$$

To evaluate this expression we have to take the term $: T^{\hat{0}\hat{0}} : |_{P_0}$ which has been normal ordered with respect to the vacuum $|0>$, rewrite it in terms of the operators $(\bar{a}_{p,\epsilon}, \bar{a}^{\dagger}_{p,\epsilon})$ and commute the terms so that the resulting expression is normal ordered with respect to the vacuum $|\bar{0}>$. To do this we write out the term $: T^{\hat{0}\hat{0}} : |_{P_0}$ explicitly. A simplification due to the use of equation (6.10) occurs so that only time derivatives of the field operators appear. Also since the fields Ψ and $\bar{\Psi}$ are just different ways of writing the same field we may write

$$: T^{\hat{0}\hat{0}} : |_{P_0} =: \frac{\partial\Psi}{\partial t}\frac{\partial\Psi}{\partial t} : |_{P_0} =: \frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial t} : |_{P_0} = \sum_{\epsilon=\pm1}\int d(\epsilon p)[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\phi_{p,\epsilon}}{\partial t}a_{p,\epsilon} + \frac{\partial\phi^*_{p,\epsilon}}{\partial t}a^{\dagger}_{p,\epsilon}\frac{\partial\bar{\Psi}}{\partial t}]|_{P_0}. \tag{6.29}$$

To simplify the notation we drop the $|_{P_0}$, but keep in mind that these equations only apply at the point P_0. Also we only evaluate this expression for a fixed ϵ. Thus,

$$: T^{\hat{0}\hat{0}}_{\epsilon} := \int_0^{\infty} d(\epsilon p)\int_0^{\infty} d(\epsilon q)\Big[\quad (\frac{\partial\bar{\phi}_{q,\epsilon}}{\partial t}\bar{a}_{q,\epsilon} + \frac{\partial\bar{\phi}^*_{q,\epsilon}}{\partial t}\bar{a}^{\dagger}_{q,\epsilon})a_{p,\epsilon}\frac{\partial\phi_{p,\epsilon}}{\partial t} + \frac{\partial\phi^*_{p,\epsilon}}{\partial t}a^{\dagger}_{p,\epsilon}(\frac{\partial\bar{\phi}_{q,\epsilon}}{\partial t}\bar{a}_{q,\epsilon} + \frac{\partial\bar{\phi}^*_{q,\epsilon}}{\partial t}\bar{a}^{\dagger}_{q,\epsilon})\Big]. \tag{6.30}$$

The operators $(a_{p,\epsilon}, a^{\dagger}_{p,\epsilon})$ are related to the barred operators $(\bar{a}_{k,\epsilon}, \bar{a}^{\dagger}_{k,\epsilon})$ by a Bogolubov transformation

$$a_{k,\epsilon} = \int d(\epsilon q)(\alpha_{k,q}\bar{a}_{q,\epsilon} + \beta^{*}_{k,q}\bar{a}^{\dagger}_{q,\epsilon}) \tag{6.31}$$

where

$$\begin{aligned} \alpha_{k,q} &= (\phi_{k,\epsilon}, \bar{\phi}_{q,\epsilon}) \\ \beta_{k,q} &= (\phi^{*}_{k,\epsilon}, \bar{\phi}_{q,\epsilon}). \end{aligned} \tag{6.32}$$

In our evaluation of the vacuum expectation value, the only term of interest is the c-number term that results from the commutator

$$\bar{a}_{q,\epsilon'}\bar{a}^{\dagger}_{k,\epsilon} = \bar{a}^{\dagger}_{k,\epsilon}\bar{a}_{q,\epsilon'} + \delta_{\epsilon,\epsilon'}\delta(k-q). \tag{6.33}$$

Thus, we get

$$< \bar{0}| : T^{\bar{0}\bar{0}}_{\epsilon} : |\bar{0}> = \int d(\epsilon p)d(\epsilon q)[\frac{\partial\bar{\phi}_{q,\epsilon}}{\partial t}\frac{\partial\phi_{p,\epsilon}}{\partial t}\beta^{*}_{p,q} + c.c.]. \tag{6.34}$$

These terms are evaluated by replacing β by its expression (6.32) and interchanging the order of integration to first do the momentum integrals. In doing so the only regularization required is to define an integral of the form

$$\int_0^{\infty} dx\, x \exp(ixp). \tag{6.35}$$

This is accomplished by replacing p by $p + i\delta$. No further regularizations are needed. We first write out β as

$$\beta^{*}_{p,q} = -i\epsilon\int_{-\infty}^{\infty} dy \left(\phi^{*}_{p,\epsilon}(y)\partial_t\bar{\phi}^{*}_{q,\epsilon}(\bar{y}) - \partial_t\phi^{*}_{p,\epsilon}(y)\bar{\phi}^{*}_{q,\epsilon}(\bar{y})\right) \tag{6.36}$$

We now perform the momenta integrals first and use the identity

$$i\pi\delta''(y-x) = \frac{1}{(x-y-i\delta)^3} - \frac{1}{(x-y+i\delta)^3}. \tag{6.37}$$

Now using the relations

$$\lim_{y\to x} \partial_y^2 \frac{x-y}{f(x)-f(y)} = \frac{(f'')^2}{2(f')^3} - \frac{f'''}{3(f')^2} \tag{6.38}$$

$$\lim_{y\to x} \partial_y^2 \left(g(y) \frac{(x-y)^2}{(f(x)-f(y))^2} \right) = -\frac{2g'f''}{(f')^3} + \frac{3g(f'')^2}{2(f')^4} + \frac{g''}{(f')^2} - \frac{2gf'''}{3(f')^3} \tag{6.39}$$

and evaluating everything at P_0 the final result is:

$$< \bar{0}| : T_\epsilon^{\hat{0}\hat{0}} : |_{P_0}|\bar{0} >= \frac{1}{24\pi} \frac{\frac{\partial^3 \bar{f}_\epsilon}{\partial x^3}|_{P_0}}{\frac{\partial \bar{f}_\epsilon}{\partial x}|_{P_0}} \tag{6.40}$$

So we only have to evaluate these terms. Now,

$$\frac{\partial \bar{f}_\epsilon}{\partial x} = \frac{\partial \bar{f}_\epsilon}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{f}_\epsilon}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial x} \tag{6.41}$$

and as initial conditions at P_0 we have

$$\frac{\partial \bar{x}}{\partial x}|_{P_0} = \cosh(\chi) \quad , \quad \frac{\partial \bar{t}}{\partial x}|_{P_0} = \sinh(\chi). \tag{6.42}$$

Furthermore, we also have that

$$\bar{f}_\epsilon(0,\bar{x}) = \epsilon \bar{x} \quad , \quad \frac{\partial \bar{f}_\epsilon}{\partial \bar{t}}|_{P_0} = 1 \quad , \quad \frac{\partial \bar{f}_\epsilon}{\partial \bar{x}}|_{P_0} = \epsilon \bar{\alpha} \frac{\partial \bar{f}_\epsilon}{\partial \bar{t}}|_{P_0} = \epsilon \tag{6.43}$$

since $\bar{\alpha}|_{P_0} = 1$. Also, as we stated earlier,

$$\frac{\partial \bar{\alpha}}{\partial \bar{t}}|_{P_0} = \frac{\partial \bar{\alpha}}{\partial \bar{x}}|_{P_0} = 0 \tag{6.44}$$

$$\frac{\partial^2 \bar{\alpha}}{\partial \bar{t} \partial \bar{x}}|_{P_0} = \frac{\partial^2 \bar{\alpha}}{\partial \bar{x}^2}|_{P_0} = 0 \tag{6.45}$$

and

$$\frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}|_{P_0} = \frac{R}{2}. \tag{6.46}$$

By repeatedly using the barred version of equation (6.10) , namely

$$\frac{\partial \bar{f}_\epsilon}{\partial \bar{x}} = \epsilon \bar{\alpha} \frac{\partial \bar{f}_\epsilon}{\partial \bar{t}} \tag{6.47}$$

as well as (6.41), (6.42) and (6.44) we find:

$$\frac{\partial^2 \bar{f}_\epsilon}{\partial \bar{t}^2}|_{P_0} = \frac{\partial^2 \bar{f}_\epsilon}{\partial \bar{t}\partial \bar{x}}|_{P_0} = \frac{\partial^2 \bar{f}_\epsilon}{\partial \bar{x}^2}|_{P_0} = 0 \tag{6.48}$$

as well as

$$\frac{\partial^3 \bar{f}_\epsilon}{\partial \bar{x}^3}|_{P_0} = \epsilon \frac{\partial^3 \bar{f}_\epsilon}{\partial \bar{t}^3}|_{P_0} + \epsilon \frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}\frac{\partial \bar{f}_\epsilon}{\partial \bar{t}}|_{P_0} = 0 \tag{6.49}$$

Thus we arrive at the result that

$$\frac{\partial^3 \bar{f}_\epsilon}{\partial \bar{t}^3}|_{P_0} = -\frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}|_{P_0} = -\frac{R}{2}|_{P_0} \tag{6.50}$$

This result now allows us to obtain that

$$\begin{aligned} \frac{\partial^3 \bar{f}_\epsilon}{\partial x^3}|_{P_0} &= [\frac{\partial^3 \bar{t}}{\partial x^3} + \epsilon\bar{\alpha}\frac{\partial^3 \bar{x}}{\partial x^3} + \epsilon\frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}\frac{\partial \bar{x}}{\partial x}(\frac{\partial \bar{t}}{\partial x})^2]\frac{\partial \bar{f}_\epsilon}{\partial \bar{t}}|_{P_0} + (\frac{\partial \bar{t}}{\partial x} + \epsilon\frac{\partial \bar{x}}{\partial x})\frac{\partial^3 \bar{f}_\epsilon}{\partial \bar{t}^3}(\frac{\partial \bar{t}}{\partial x})^2|_{P_0} \\ &= -\frac{R}{2}\sinh^3(\chi) + \frac{\partial^3 \bar{t}}{\partial x^3} + \epsilon\frac{\partial^3 \bar{x}}{\partial x^3} \end{aligned} \tag{6.51}$$

To evaluate the last two terms in this expression we use the fact that (t, x) as well as $(\bar{t}, \bar{x})$ satisfy the geodesic equations, but have different initial data on the spacelike geodesic that passes through P_0. These initial data are:

$$\frac{dx}{ds}|_{P_0} = 1 \quad , \quad \frac{dt}{ds}|_{P_0} = 0 \tag{6.52}$$

$$\frac{d\bar{x}}{ds}|_{P_0} = \cosh(\chi) \quad , \quad \frac{d\bar{t}}{ds}|_{P_0} = \sinh(\chi) \tag{6.53}$$

The geodesic equations read:

$$\begin{aligned} \frac{d^2 t}{ds^2} &= -\alpha\frac{\partial \alpha}{\partial t}(\frac{dx}{ds})^2, \\ \frac{d^2 x}{ds^2} &= -\frac{2}{\alpha}\frac{\partial \alpha}{\partial t}\frac{dx}{ds}\frac{dt}{ds} - \frac{1}{\alpha}\frac{\partial \alpha}{\partial x}(\frac{dx}{ds})^2 \end{aligned} \tag{6.54}$$

$$\begin{aligned} \frac{d^2 \bar{t}}{ds^2} &= -\bar{\alpha}\frac{\partial \bar{\alpha}}{\partial \bar{t}}(\frac{d\bar{x}}{ds})^2, \\ \frac{d^2 \bar{x}}{ds^2} &= -\frac{2}{\bar{\alpha}}\frac{\partial \bar{\alpha}}{\partial \bar{t}}\frac{d\bar{x}}{ds}\frac{d\bar{t}}{ds} - \frac{1}{\bar{\alpha}}\frac{\partial \bar{\alpha}}{\partial \bar{x}}(\frac{d\bar{x}}{ds})^2 \end{aligned} \tag{6.55}$$

By differentiating these equations as well as using (6.42) we find that

$$\frac{\partial^2 \bar{t}}{\partial x^2}|_{P_0} = \frac{\partial^2 \bar{x}}{\partial x^2}|_{P_0} = 0 \tag{6.56}$$

and

$$\frac{\partial^3 \bar{t}}{\partial x^3}|_{P_0} = -\frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}|_{P_0} \sinh(\chi)\cosh^2(\chi) = -\frac{R}{2}\sinh(\chi)\cosh^2(\chi) \tag{6.57}$$

$$\frac{\partial^2 \bar{x}}{\partial x^3}|_{P_0} = -2\frac{\partial^2 \bar{\alpha}}{\partial \bar{t}^2}|_{P_0} \cosh(\chi)\sinh^2(\chi) = -R\cosh(\chi)\sinh^2(\chi). \tag{6.58}$$

Combining these results we obtain that

$$< \bar{0}| : T_{\epsilon}^{\hat{0}\hat{0}} : |_{P_0}|\bar{0} >= -\frac{R}{48\pi}\epsilon \exp(\epsilon\chi)\sinh(\chi) \tag{6.59}$$

Adding the results for both values of ϵ we obtain

$$< \bar{0}| : T^{\hat{0}\hat{0}} : |_{P_0}|\bar{0} >= -\frac{R}{24\pi}\sinh^2(\chi) \tag{6.60}$$

Inserting this into equation (6.27) we finally obtain the vacuum expectation value of the trace of the stress-energy tensor, namely $\frac{R}{24\pi}$.

6.4 Conclusion

For the case of a conformally coupled massless scalar field in 1+1 dimensions it is much simpler to evaluate the trace anomaly using a particle picture than to avoid it. The only regularization required is very simple, but it must be this very simple regularization that suffices to break the conformal symmetry and thus give a non-zero result for the vacuum expectation value of the trace.

CHAPTER 7

Conclusions

In the preceding chapters we have presented a number of calculations, some with results which were to be expected, others that some people may have trouble believing. In chapter 2 a static model universe was examined for particle creation as observed by an observer static with respect to the static coordinates. It was found that by using the prescription for decomposing a field into its positive and negative frequency parts, as advocated by this thesis and introduced by Capri and Roy [7], there is particle creation. Although many calculations have been performed using the static time with which to define particles, this approach can lead to ambiguities. One example of this ambiguity was illustrated for the case of Rindler space where there exist two static coordinatizations, one based on the Rindler coordinates and the other based on the usual Minkowski coordinates. In this case the geodesically complete coordinatization, which is the one chosen by this procedure, chooses the Minkowski coordinatization. The interesting ramification of using this prescription in other models is that the static coordinatization may not be the preferred coordinatization chosen by the prescription. This is exactly what happened with the model investigated in the second chapter and it is shown that the observer in question does indeed observe particles being created. Unfortunately due to the complicated nature of the Bogolubov coefficient calculated we were unable to calculate the exact spectrum of created particles and the analysis to determine whether the particle creation was finite or not was inconclusive.

In hopes of understanding what effects anisotropy has on cosmological particle creation an anisotropic model was investigated in chapter 3. It is thought that anisotropic effects may lead to particle creation which has a damping effect on the level of anisotropy. These effects could then possibly explain the large degree of isotropy observed in the present universe. The anisotropic model which was investigated in this chapter was just a model generated by the addition of a 2-plane to 1+1 deSitter space. This model is therefore a universe of constant curvature which expands in one of the three spatial directions. Particle creation in this model was also investigated for an observer who was stationary with respect to the original coordinates. The β Bogolubov coefficient which relates the two different sets of modes chosen on the two different spacelike surfaces was analysed. We found that the square of this coefficient which describes the number of particles created could be expressed as an infinite series. Each term of this series represents a different discrete shift in the energy spectrum of created particles. The discrete shift present in each term is made up of integer multiples of the one natural length scale of the geometry, that being the scalar curvature. It was found in this analysis that the addition of the 2-plane played almost no role in the results calculated.

Having not been able to calculate conclusively the total particle creation in a particular model universe we then investigated the 1+1 deSitter space model in hopes of showing that the total particle creation is finite. The particle creation as observed by an observer who was again static with respect to the original coordinates is calculated and argued to be finite. This argument is based on the compact nature of the variable of integration in the integral expression for the square of the Bogolubov $\beta(m,n)$ coefficient. Due to this compact nature one is able to integrate by parts indefinitely and one finds that the expression for this object drops off faster than any inverse power of m or n. This then implies that the double sum of the square of the

β Bogolubov coefficient will be finite and therefore the total particle creation will be finite. If in fact the drop off of this object is exponential in nature the spectrum of created particles could be thermal in nature.

Calculating the particle creation for a stationary observer in 3+1 deSitter space was only slightly more complicated that the calculation in 1+1 deSitter space after a convenient choice of some integration constants was made. As in the 1+1 dimensional case the compact nature of the radial coordinate, when the metric is expressed in polar coordinates, allows one to show that the particle creation also drops off faster than any inverse power of the variables in question. Once again if this actually drops off as an exponential it could be consistent with a thermal spectrum.

In the last chapter the trace anomaly is calculated by modifying a method of calculation first suggested by Massacand and Schmid [8] to the particle definition that has been used throughout the thesis. This calculation finds the usual result for the trace in a general $1 + 1$ dimensional spacetime. This calculation is considerably more straightforward than the usual calculations and beyond normal ordering only a simple regularization is required

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