Replication and Its Application to Weak Convergence

by

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Abstract

Herein, a new methodology is developed to replicate functions, measures and stochastic processes onto a compact metric space. Each replica object is a weak or strong modification of the original object, so many results are easily established for the replica objects and then transferred back to the original ones. Two problems are solved within to demonstrate the method: (1) Finite-dimensional convergence Herein, a new methodology is developed to replicate functions, measures and stochastic processes onto a compact metric space. Each replica object is a weak or strong modification of the original object, so many results are easily established for the replica objects and then transferred back to the original ones. Two problems are solved within to demonstrate the method: (1) Finite-dimensional convergence to possibly non-càdlàg limits is established for processes living on general topological spaces. (2) New tightness and relative compactness criteria are given for the Skorokhod space of Tychonoff-space-valued càdlàg mappings. The methods herein are also used in companion papers to establish the: (3) existence of, uniqueness of and convergence to martingale problem solutions, (4) classical FKK and DMZ filtering equations and stationary filters, (5) finite-dimensional convergence to stationary signal-filter pairs, (6) invariant measures of Markov processes, and (7) Ray-Knight theory, all in general settings.

Preface

The current thesis is an original work by Chi Dong under the supervision of Dr. Michael A. Kouritzin. This manuscript is based on the preprint of a submitted research paper with the same title and co-authored by Chi Dong and Michael A. Kouritzin. The research conducted for this thesis is the core component of several interrelated projects of Chi Dong and Dr. Kouritzin in various domains of mathematics. These projects were partly initiated by Dr. Kouritzin. The theory that interrelates them and generates this thesis is co-proposed by Chi Dong and Dr. Kouritzin. The formulation of the research problems and the concrete research works are done by Chi Dong with the advice of Dr. Kouritzin. The composition of this manuscript is responsible by Chi Dong.

To

My Fiancée
Ms. Lan Lin

My Parents
Mr. Yiding Zhang, Mrs. Chao Dong

My Grandparents
Mr. Yonggan Zhang† (1925 - 2015), Mrs. Chunmei Huang
Mr. Genling Dong, Mrs. Wanzhen Zhou

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At the same time, I would like to say special thanks to Dr. Hongwei Long, my master supervisor. Dr. Long was a postdoctoral fellow of and recommended me to Dr. Kouritzin. He gave me the first ticket for studying abroad, taught my first lession of advanced stochastic mathematics and launched my life as a researcher. If my Ph.D. thesis is a story of mathematics, then the classroom of Dr. Long is where the story begins.

As a student in the mathematical finance program, I thank Dr. Christoph Frei for his respectable efforts in our program as well as his generous help and wise advice to me in both academia and career. I thank all of my exam committee members, especially the external thesis reader Dr. Jie Xiong, for reading my long thesis and joining the last step of my graduate studies. I appreciate all the wonderful time spent with my lovely colleagues at University of Alberta. In particular, I would like to convey my wholehearted appreciation to Mr. Chenzhe Diao and Mr. Sile Tao, who offered me selfless support during my difficult time close to the defense. Special thanks also go to the Department of Mathematical and Statistical Sciences for providing financial support and excellent working atmosphere for my research. Moreover, I appreciate all the sharp mathematicians, too many to mention their names, who taught me mathematics in China, United States of America and Canada.

Finally, I wish to thank my family, especially my fiancée Lan Lin and my parents, for all their trust, patience, sacrifice and support. This thesis is a

tribute to my grandfather Prof. Genling Dong as a fulfillment of his wish. Nothing was, is and will be achieved by myself without these irreplaceable people. I have written a thesis about mathematics, but they wrote me.

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Chapter 1

Introduction

Traditionally, researchers have focused on stochastic processes living on "good" topological spaces which are metrizable, separable, completely metrizable and/or (locally) compact etc. However, there are many settings of interest that violate these convenient assumptions. For example, Holley and Stroock [1979] and Mitoma [1983] considered probability measures on the Skorokhod $\mathscr{J}_1$-space of tempered distributions. Szpirglas [1976] considered the nonlinear filtering problem for càdlàg signals living on Lusin spaces. Meyer and Zheng [1984] considered tightness in the space of all càdlàg functions from the non-negative real numbers $\mathbf{R}^+$ to the real line $\mathbf{R}$ equipped with the pseudo-path topology, which was further discussed by e.g. Stricker [1985] and Kurtz [1991]. Fitzsimmons [1988] considered the construction of Markov branching processes whose values are finite Borel measures on a Lusin space. Jakubowski [1997a] considered extending the Skorokhod Representation Theorem to tight sequences of probability measures on non-metrizable spaces. Jakubowski [1997b] considered a sequentially defined topology on the space of all càdlàg functions from the compact interval $[0,T]$ to $\mathbf{R}$. Dembo and Zeitouni [1998] considered the space of all Borel probability measures on a Polish space equipped with the strong topology. Jakubowski [1986] and Kouritzin [2016] considered probability measures on $D([a,b];E)$, the Skorokhod $\mathscr{J}_1$-space of all càdlàg mappings from a compact interval $[a,b]$ to a Tychonoff space E. Lyons [1994, 1998], Friz and Victoir [2010] and Friz and Hairer [2014] worked with non-separable Banach spaces of rough paths equipped with homogeneous p-variation or $1/p$-Hölder norms. None of these spaces are necessarily Polish nor compact, some even non-metrizable or non-separable.

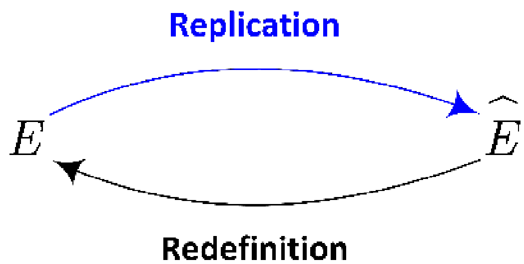

Figure 1: *The main idea of replication*

The point of our work, as illustrated in Figure 1 above, is that Borel measurable functions, finite measures and stochastic processes living on a *general topological space* E often can be *replicated* as *replica functions*, *replica measures* or *replica processes* living on some *compact metric space* $\widehat{E}$. In particular, even non-càdlàg processes can have càdlàg replicas. These replica objects are more easily analyzed on $\widehat{E}$ than the original objects on E, and many results about the replica objects are transferrable back to the original ones by proper redefinitions.

One could extend results one at a time to various generalized settings. However, replication is probably an easier and more unified approach of extending results from compact or Polish spaces to a large category of exotic spaces simultaneously. This approach is believed to have equal merit in many different areas such as weak convergence, martingale problems, nonlinear filtering, large deviations and Markov processes etc., where compactness or metric completeness can play a big role. Indeed, even a Polish space can be improved by adding compactness.

The contributions of this work are:

Theme 1 Methodology of replication (Chapter 3 - Chapter 6).

Theme 2 Criteria for finite-dimensional convergence of possibly non-càdlàg processes living on general spaces (§6.2 and Chapter 7).

Theme 3 Criteria for tightness and relative compactness in $D(\mathbf{R}^+; E)$, the Skorokhod $\mathscr{J}_1$-space of all càdlàg mappings from $\mathbf{R}^+$ to a Tychonoff space E (§6.4 and Chapter 8).

Theme 1, conceptualizing and concretizing the idea of replication, answers the question *when* and *how* one can perform replication and serves as the theoretical foundation of the current and several companion works (see Dong and Kouritzin [2017a,b,d]). Our developments were motivated in part by and especially benefit from the works of Ethier and Kurtz [1986], Bhatt and Karandikar [1993b], Blount and Kouritzin [2010] and Kouritzin [2016], which exploit the use of imbedding and compactification techniques in various aspects of probability theory.

The question *what* replication can do is partially answered by **Theme 2** and **Theme 3**. **Theme 2** grew out of investigating the convergence of a stochastic evolution system to its stationary distribution(s) or solution(s) over the long term, which has been the central topic of many classical works on both theory and application ends. For example, [Ethier and Kurtz, 1986, §10.2 and §10.4] considered the existence of stationary distributions for diffusion approximations of the Wright-Fisher model. For a Fleming-Viot process X, Ethier and Kurtz [1993] and Donnelly and Kurtz [1999] considered the existence of a stationary distribution μ, and Ethier and Kurtz [1998] established the pointwise ergodic theorem

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X_t)dt = \int_E f(x)\mu(dx),\ a.s. \tag{1.1}$$

Regarding nonlinear filtering, [Kunita, 1971, Theorem 4.1] considered the "asymptotic mean square filtering error"

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\left[\left(\pi_t^\mu(f)-f(X_t^\mu)\right)^2\right]dt, \tag{1.2}$$

where X^μ is the signal with initial distribution μ and $\pi^\mu(f)$ is the optimal filter for function f of X^μ. [Budhiraja and Kushner, 1999, (2.6)] considered weak limit points of the "pathwise average error"

$$\frac{1}{T}\int_0^T \left(\pi_t^\mu(f)-f(X_t^\mu)\right)^2 dt \text{ as } T\uparrow\infty. \tag{1.3}$$

[Budhiraja, 2001, (1.2)] studied the "(μ, μ')-stability"

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[\left(\pi_t^{\mu}(f) - \pi_t^{\mu'}(f)\right)^2\right] dt \tag{1.4}$$

of $\pi^{\mu}(f)$, where $\pi^{\mu'}$ represents an approximate filter with incorrect initiation μ'. In other areas, Cox et al. [2010] and Cox et al. [2013] considered non-trivial stationary solutions for the Lotka-Volterra model and those for perturbations of the voter model. Cox and Griffeath [1983] and Cox [1988] established (1.1) for a basic voter process X and an invariant measures μ of X. All the works (and many others) above were based on separable compact Hausdorff spaces, Polish spaces or compact metric spaces. The following question, considering a weak and abstract form of long-time-average limits like (1.1), (1.2), (1.3) and (1.4), still remains unanswered:

Q1 Let E be a non-Polish, non-compact or even non-metrizable space, and $X = \{X_t\}_{t\geq 0}$ be an E-valued, non-càdlàg, measurable process. Then, is there an E-valued stationary process X^{∞} such that the long-time-averaged distributions

$$\frac{1}{T_n} \int_0^{T_n} \mathbb{P} \circ \left(X_{\tau+t_1}, ..., X_{\tau+t_d}\right)^{-1} d\tau \tag{1.5}$$

converge weakly to the distribution of $(X_{t_1}^{\infty}, ..., X_{t_d}^{\infty})$ as $T_n \uparrow \infty$ for *almost all* finite subset $\{t_1, ..., t_d\}$ of $\mathbf{R}^+$?

In the E is a separable metric space setting, weak convergence of finite-dimensional distributions of E-valued càdlàg processes is implied by their weak convergence as $D(\mathbf{R}^+; E)$-valued random variables. Let $\mathbf{N}$ denote the positive integers, $\{X^n\}_{n\in\mathbf{N}}$ and X be E-valued càdlàg processes with paths in $D(\mathbf{R}^+; E)$ and "$\Rightarrow$" denote weak convergence of Borel probability measures. The weak convergence

$$X_n \Longrightarrow X \text{ as } n \uparrow \infty \tag{1.6}$$

on $D(\mathbf{R}^+; E)$ has two implications:

(1) The weak convergence

$$\left(X_{t_1}^n, ..., X_{t_d}^n\right) \Longrightarrow \left(X_{t_1}, ..., X_{t_d}\right) \text{ as } n \uparrow \infty \tag{1.7}$$

for all finite collection $\{t_1, ..., t_d\}$ in a dense subset of $\mathbf{R}^+$.

(2) $\{X^n\}_{n\in\mathbf{N}}$ is relatively compact in $D(\mathbf{R}^+; E)$.

It is often (2) that requires strong or difficult-to-verify conditions in practice. By contrast, (1) or a weaker form of it is believed to be establishable for possibly non-càdlàg processes under much milder conditions than those for weak convergence on $D(\mathbf{R}^+; E)$. For instance, Bhatt and Karandikar [1993b] discussed (1) with $\{X^n\}_{n\in\mathbf{N}}$ being progressive approximating processes and X being a progressive martingale problem solution, none of which is necessarily càdlàg. Their development was based on a Polish space E and furthered that of [Ethier and Kurtz, 1986, §4.8]. Herein, we work with a more general space E and answer the following more general questions:

Q2 When will a subsequence of E-valued processes $\{X_i\}_{i\in\mathbf{I}}$ converge finite-dimensionally to an E-valued process with general paths?

Q3 When will a subsequence of E-valued processes $\{X_i\}_{i\in\mathbf{I}}$ converge finite-dimensionally to an E-valued *progressive* process?

Either of these two questions may be answered in an individual way, but replication helps to handle **Q2**, **Q3** and the weak convergence of càdlàg processes on path spaces in one framework. We shall establish several relatively mild and explicitly verifiable criteria for uniqueness and existence of the limit processes in **Q2** and **Q3** above. These criteria will be used to deduce the finite-dimensional convergence in (1.5) and answer **Q1** that motivates **Theme 2**.

Theme 3 is concerned with two basic problems for càdlàg processes taking values in a Tychonoff space E:

Q5 When is a family of E-valued càdlàg processes bijectively indistinguishable from a tight family of $D(\mathbf{R}^+; E)$-valued random variables?

Q6 If the answer to **Q5** is uncertain, then what about relative compactness in lieu of tightness?

The main importance of tightness is that it implies relative compactness for Borel probability measures even on general Hausdorff spaces. However, the verification of tightness can be challenging. Kurtz [1975], Jakubowski [1986], Dawson [1993], Bhatt and Karandikar [1993b], Kallianpur and Xiong [1995], Perkins [2002] and Kouritzin [2016], to name just a few, all spent considerable efforts in establishing tightness of càdlàg processes on exotic spaces. In particular, Jakubowski [1986] developed systematic tightness criteria for probability measures on both $D([0,1];E)$ and $D(\mathbf{R}^+;E)$, which extended several results of [Ethier and Kurtz, 1986, §3.7 - 3.9] from the Polish to the possibly non-metrizable Tychonoff case. Kouritzin [2016] recently generalized the results of Jakubowski [1986] for $D([a,b];E)$ by loosening [Jakubowski, 1986, Theorem 3.1, (3.4)] to the milder Weak Modulus of Continuity Condition (see [Kouritzin, 2016, §6]). As a continuation of Kouritzin [2016] on infinite time horizon, we answer **Q5** by establishing the equivalence among:

- Indistinguishability from a tight family of $D(\mathbf{R}^+;E)$-valued random variables;
- Metrizable Compact Containment Condition plus Weak Modulus of Continuity Condition;
- Metrizable Compact Containment Condition plus Modulus of Continuity Condition; and
- Mild Pointwise Containment Condition plus Modulus of Continuity Condition for $\mathfrak{r}$ when $(E,\mathfrak{r})$ is a complete metric space.

Relative compactness is a weaker concept than tightness if the underlying space is non-Polish. With this difference in mind, we establish milder conditions for relative compactness than those for tightness. As aforementioned, weak convergnce on $D(\mathbf{R}^+;E)$ is commonly thought to be composed of finite-dimensional convergence along densely many times plus relative compactness in $D(\mathbf{R}^+;E)$. Herein, we give a more precise interpretation by showing that:

- Relative compactness in $D(\mathbf{R}^+;E)$ always implies Modulus of Continuity Condition.

- Weak convergence on $D(\mathbf{R}^+;E)$ implies finite-dimensional convergence along densely many times whenever E belongs to a much broader category than metrizable and separable spaces.
- When E is metrizable and separable, finite-dimensional convergence along densely many times plus Modulus of Continuity Condition (weaker than relative compactness) are sufficient for weak convergence on $D(\mathbf{R}^+;E)$.

Based on the results above, we answer **Q6** on metrizable spaces by showing that:

- When E is metrizable and separable, relative compactness in $D(\mathbf{R}^+;E)$ is equivalent to Modulus of Continuity Condition plus relative compactness with respect to finite-dimensional convergence to E-valued càdlàg processes along densely many times.
- When $(E,\mathfrak{r})$ is a metric space, the combination of Modulus of Continuity Condition for $\mathfrak{r}$, Mild Pointwise Containment Condition and relative compactness with respect to finite-dimensional convergence to $D(\mathbf{R}^+;E)$-valued random variables along densely many times is sufficient for relative compactness in $D(\mathbf{R}^+;E)$.

The results of **Theme 3** demonstrate the superfluity of Compact Containment Condition for relative compactness in Skorokohod $\mathscr{J}_1$-spaces. This is also why the second approach of Bhatt and Karandikar [1993b] was well received. While their work was restricted to a martingale problem setting, our results are general.

Besides **Theme 2** and **Theme 3**, several companion papers of this work also provide motivation for and impact of replication.

For martingale problems in general settings, Dong and Kouritzin [2017a] uses replication to establish existence of, uniqueness of and the finite-dimensional convergence of non-càdlàg approximating processes to non-càdlàg solutions. Previously, existence or uniqueness was mostly established on Polish spaces (see Ethier and Kurtz [1986] and Bhatt and Karandikar [1993a,b]). Convergence results were mostly done for the weak convergence of càdlàg approximating processes to càdlàg solutions on path space.

For the classical nonlinear filtering problem in general settings, Dong and Kouritzin [2017b] uses replication to establish the unique characterization of filters by classical filtering equations and the existence of stationary filter given a stationary signal. Previously, characterization of nonlinear filters by classical filtering equations were only known on Polish spaces (see Kurtz and Ocone [1988] and Kouritzin and Long [2008]), Lusin metric spaces (see Szpirglas [1976]) or separable compact Hausdorff spaces (see Kunita [1971]). Stationary filters have only been established on separable compact Hausdorff spaces (see Kunita [1971]) or Polish spaces (see Bhatt et al. [2000]).

The remainder of this manuscript is organized as follows. Chapter 2 serves a preliminary collection of notations, terminologies, facts and examples for the three themes of this work. **Theme 1** occupies four chapters: Chapter 3 develops the space change method of replication. Chapter 4 focuses on the replication of function and linear operator. Chapter 5 discusses weak convergence on general topological spaces, the replication of measure and their association. Chapter 6 is devoted to the replication of stochastic process and the associated convergence problems. Chapters 7 and 8 correspond to **Theme 2** and **Theme 3** respectively. We provide background content in Appendix A and miscellaneous results in Appendix B for self-containment and referral ease, especially for readers' convenience.

Chapter 2

Preliminaries

The current chapter makes the necessary preparation for our major developments in Chapter 3 - 8. For the sake of clarity and accuracy, §2.1 - §2.6 introduce our general notation system, specify relevant terminologies and reviews a few elementary facts. Relevant background materials are provided in Appendix A. Prior to introducing replication, §2.7 further motivates this approach by specifying several examples of "defective" settings or boosting a result by space chagne in probability theory.

2.1 Basic concepts

2.1.1 Numbers, sets and mappings

"$\varnothing$" denotes the empty set. $\mathbf{N}$ denotes the positive integers and $\mathbf{N}_0 \triangleq \mathbf{N} \cup \{0\}$[1]. $\mathbf{Q}$ denotes the rational numbers and $\mathbf{Q}^+ \triangleq \{q \in \mathbf{Q} : q \geq 0\}$. $\mathbf{R}$ denotes the real numbers and $\mathbf{R}^+ \triangleq \{x \in \mathbf{R} : x \geq 0\}$. "$\uparrow$" and "$\downarrow$" denote the non-decreasing and non-increasing convergence of real numbers (including convergence to $\pm\infty$) respectively.

"$\subset$" and "$\supset$" denote the containment of sets *including equalities*. Let E and $A \subset E$ be non-empty sets. $\aleph(E)$ denotes the cardinality of E. $\mathscr{P}_0(E)$ denotes the family of all finite *non-empty* subsets of E. A is a **cocountable** subset of E if $E \backslash A$ is a countable set. In this work, empty, finite and countably

[1] "$\triangleq$" means "being defined by".

infinite sets are all considered as countable sets[2].

"$\times$" denotes the Cartesian product of non-empty sets. Let $\mathbf{I}$ and $\{S_i\}_{i\in\mathbf{I}}$ be non-empty sets. $\prod_{i\in\mathbf{I}} S_i$ denotes the Cartesian product of $\{S_i\}_{i\in\mathbf{I}}$. When $S_i = E$ for all $i \in \mathbf{I}$, $\prod_{i\in\mathbf{I}} S_i$ is often denoted by $E^{\mathbf{I}}$ for general $\mathbf{I}$, or by E^∞ if $\aleph(\mathbf{I}) = \aleph(\mathbf{N})$, or by E^d if $\aleph(\mathbf{I}) = d \in \mathbf{N}$. The projection on $\prod_{i\in\mathbf{I}} S_i$ for non-empty sub-index-set $\mathbf{I}_0 \subset \mathbf{I}$ is defined by

$$\begin{aligned} \mathfrak{p}_{\mathbf{I}_0} : \prod_{i\in\mathbf{I}} S_i &\longrightarrow \prod_{i\in\mathbf{I}_0} S_i, \\ x &\longmapsto \prod_{i\in\mathbf{I}_0} \{x(i)\}. \end{aligned} \tag{2.1.1}$$

In particular, $\mathfrak{p}_i \mathrel{\mathring{=}} \mathfrak{p}_{\{i\}}$ is called the *one-dimensional projection* on $\Pi_{i\in\mathbf{I}} S_i$ for $i \in \mathbf{I}$.

"$\circ$" denotes the composition of mappings. $\mathbf{1}_A$ denotes the indicator function of A. For a mapping f defined on E, $f|_A$ denotes the restriction of f to A. For a family of mappings $\mathcal{D} = \{f_i \in S_i^E\}_{i\in\mathbf{I}}$, we define

$$\mathcal{D}|_A \mathrel{\mathring{=}} \{f|_A : f \in \mathcal{D}\} \tag{2.1.2}$$

and

$$\begin{aligned} \bigotimes \mathcal{D} = \bigotimes_{i\in\mathbf{I}} f_i : E &\longrightarrow \prod_{i\in\mathbf{I}} S_i, \\ x &\longmapsto \prod_{i\in\mathbf{I}} \{f_i(x)\}. \end{aligned} \tag{2.1.3}$$

2.1.2 Measurable space and measure space

Let $(E, \mathscr{U})$ and $(S, \mathscr{A})$ be measurable spaces and $A \subset E$ be non-empty. The **concentration of $\mathscr{U}$ on** A is defined by

$$\mathscr{U}|_A = \{B \cap A : B \in \mathscr{U}\}, \tag{2.1.4}$$

[2] So, "countable" is indifferent from "at most countable".

which is apparently a σ-algebra on A. For a family of mappings $\mathcal{D}$ from E to S, the **σ-algebra induced by $\mathcal{D}$** is defined by

$$\sigma(\mathcal{D}) \circeq \sigma\left(\left\{f^{-1}(B) : B \in \mathscr{A}, f \in \mathcal{D}\right\}\right). \tag{2.1.5}$$

δ_x denotes the *Dirac measure at* $x \in E$, i.e. $\delta_x(B) = 1$ precisely when $B \in \mathscr{U}$ contains x.

$\mathfrak{M}^+(E, \mathscr{U})$ (resp.[3] $\mathfrak{P}(E, \mathscr{U})$) denotes the family of all *non-trivial* finite measures (resp. probability measures) on $(E, \mathscr{U})$. Herein, non-triviality of a measure μ on $(E, \mathscr{U})$ means $\mu(E)$, the *total mass of* μ is non-zero. Also, we consider measures to be non-negative and countably additive as most probabilistic literature does.

Let $(E, \mathscr{U}, \mu)$ be a measure space (i.e. $\mu \in \mathfrak{M}^+(E, \mathscr{U})$). $\mathscr{N}(\mu)$ denotes the family of all *μ-null subsets* of E, i.e. each member of $\mathscr{N}(\mu)$ has zero measure under the *outer measure induced by* μ (see [Dudley, 2002, p.89]). Complements of the members of $\mathscr{N}(\mu)$ are called **μ-conull** sets. If $\mathscr{N}(\mu) \subset \mathscr{U}$, then $\mathscr{U}$ is called *μ-complete* and $(E, \mathscr{U}, \mu)$ is called *complete*.

A is a **support**[4] **of** μ (or μ is **supported on** A) if $E \backslash A \in \mathscr{N}(\mu)$. The **expansion of** $\nu \in \mathfrak{M}^+(A, \mathscr{U}|_A)$ **onto** E is defined by

$$\nu|^E(B) \circeq \nu(A \cap B), \ \forall B \in \mathscr{U}(E). \tag{2.1.6}$$

When $A \in \mathscr{U}$, the **concentration of** μ **on** A is defined by

$$\mu|_A(B) \circeq \mu(B), \ \forall B \in \mathscr{U}|_A \subset \mathscr{U}. \tag{2.1.7}$$

The following facts are well-known and we omit the proof for brevity.

Fact 2.1. *Let $(E, \mathscr{U})$ be a measurable space, $A \subset E$ be non-empty, $\mu \in \mathfrak{M}^+(E, \mathscr{U})$ and $\nu \in \mathfrak{M}^+(A, \mathscr{U}|_A)$. Then, the following statements are true:*

(a) If $A \in \mathscr{U}$, then (2.1.7) well defines $\mu|_A \in \mathfrak{M}^+(A, \mathscr{U}|_A)$. If, in addition, $\mu \in \mathfrak{P}(E, \mathscr{U})$, then $\mu|_A \in \mathfrak{P}(A, \mathscr{U}|_A)$ precisely when $\mu(A) = 1$.

[3] "resp." abbreviates "respectively".
[4] By our definition, a measure may have more than one supports.

(b) (2.1.6) *well defines* $\nu|^E \in \mathfrak{M}^+(E, \mathscr{U})$ *and* $\nu|^E \in \mathfrak{P}(E, \mathscr{U})$ *precisely when* $\nu(A) = 1$*.*

(c) If $A \in \mathscr{U}$*, then* $\nu = (\nu|^E)|_A$*. If, in addition,* $\mu(E\backslash A) = 0$*, then* $\mu = (\mu|_A)|^E$*.*

For a measurable mapping $f : E \to S$, the *push-forward measure of* μ *by* f is defined by

$$\mu \circ f^{-1}(B) \mathrel{\hat{=}} \mu\left(f^{-1}(B)\right), \ \forall B \in \mathscr{A} \tag{2.1.8}$$

and is well-known to be a member of $\mathfrak{M}^+(S, \mathscr{A})$.

Let $\mathscr{V}$ be another σ-algebra on E. If $\mathscr{V} \subset \mathscr{U}$ and ν is the restriction of μ as a set function to $\mathscr{V}$, then $\nu \in \mathfrak{M}^+(E, \mathscr{V})$ is called **the restriction of** μ **to** $\mathscr{V}$, μ is called **an extension of** ν **to** $\mathscr{U}$ and $(E, \mathscr{U}, \mu)$ is called **an extension of** $(E, \mathscr{V}, \nu)$. $(E, \mathscr{U}, \mu)$ is the **completion of** $(E, \mathscr{V}, \nu)$ if: (1) $(E, \mathscr{U}, \mu)$ is a complete extension of $(E, \mathscr{V}, \nu)$, and (2) any complete extension $(E, \mathscr{U}', \mu')$ of $(E, \mathscr{V}, \nu)$ is also an extension of $(E, \mathscr{U}, \mu)$.

2.1.3 Topological space

Hereafter, we will not always include the underlying σ-algebra (resp. topology) in the notation of a measurable (resp. topological) space for simplicity.

Let E be a topological space and $A \subset E$ be non-empty. By $\mathscr{O}(E)$, $\mathscr{C}(E)$, $\mathscr{K}(E)$, $\mathscr{K}^{\mathbf{m}}(E)$, $\mathscr{K}_\sigma(E)$, $\mathscr{K}_\sigma^{\mathbf{m}}(E)$ and $\mathscr{B}(E) \mathrel{\hat{=}} \sigma(\mathscr{O}(E))$ we denote the families of all open, closed, *compact* (see p.224), *metrizable* (see p.220) *compact*, σ-compact, σ*-metrizable compact* (i.e. countable union of metrizble compact) and Borel subsets of E, respectively.

$$\mathscr{O}_E(A) \mathrel{\hat{=}} \{O \cap A : O \in \mathscr{O}(E)\} \tag{2.1.9}$$

denotes the subspace topology of A induced from E.

$$\mathscr{B}_E(A) \mathrel{\hat{=}} \sigma\left(\mathscr{O}_E(A)\right) = \mathscr{B}(E)|_A \tag{2.1.10}$$

denotes the subspace Borel σ-algebra of A induced from E.

Let $\mathcal{R}$ be a family of *pseudometrics* (see [Dudley, 2002, §2.1, Definition, p.26]) on E. The topology induced by $\mathcal{R}$ is generated by the topological basis

$$\left\{\bigcap_{\mathfrak{r}\in\mathcal{R}_0}\left\{y\in E:\mathfrak{r}(x,y)<2^{-p}\right\}:x\in E,p\in\mathbf{N},\mathcal{R}_0\in\mathscr{P}_0(\mathcal{R})\right\}. \tag{2.1.11}$$

This topology is the metric topology of $(E,\mathfrak{r})$ when $\mathcal{R}$ is the singleton of a metric $\mathfrak{r}$ on E. When $(E,\mathfrak{r})$ is a metric space, we define

$$A^\epsilon \doteq \{x\in E:\mathfrak{r}(x,y)<\epsilon \text{ for some } y\in A\},\ \forall\epsilon\in(0,\infty). \tag{2.1.12}$$

Let S be a topological space and $\mathcal{D}$ be a family of mappings from E to S. The topology generated by the *topological basis* (see [Munkres, 2000, §13, Definition, p.78])

$$\left\{\bigcap_{f\in\mathcal{D}_0}\bigcap_{O\in\mathscr{U}_0}f^{-1}(O)\cap A:\mathscr{U}_0\in\mathscr{P}_0\left[\mathscr{O}(S)\right],\mathcal{D}_0\in\mathscr{P}_0(\mathcal{D})\right\} \tag{2.1.13}$$

is called the **topology induced by $\mathcal{D}$ on** A and is denoted by $\mathscr{O}_{\mathcal{D}}(A)$. The **Borel σ-algebra induced by $\mathcal{D}$ on** A refers to $\mathscr{B}_{\mathcal{D}}(A)\doteq\sigma[\mathscr{O}_{\mathcal{D}}(A)]$.

Let $\mathscr{U}$ be another topology on E. If $\mathscr{U}\subset\mathscr{O}(E)$, then the topological space $(E,\mathscr{U})$ is called a **topological coarsening of** E, or equivalently, E is called a **topological refinement of** $(E,\mathscr{U})$.

Hereafter, by "$x_n\to x$ as $n\uparrow\infty$ in E" we mean that: (1) $\{x_n\}_{n\in\mathbf{N}}$ and x are members of E, and (2) $\{x_n\}_{n\in\mathbf{N}}$ converges to x with respect to the topology of E.

2.1.4 Morphisms

Let E and S be topological spaces and $f:E\to S$ be a mapping. f is a *homeomorphism* between E and S if f is bijective and both f and f^{-1} are continuous. E and S are *homeomorphic* to and *homeomorphs of* each other if there exists a homeomorphism between them. f is an *imbedding* from E into S if f is a homeomorphism between E and $(f(E),\mathscr{O}_S(f(E)))$.

Compared to Borel subset and homeomorphism, two less common notions are standard Borel subset and Borel isomorphism.

Definition 2.2. Let E and S be topological spaces[5] and $A \subset E$ be non-empty.

- A mapping $f : E \to S$ is a **Borel isomorphism between** E **and** S if f is bijective and both f and f^{-1} are measurable with respect to $\mathscr{B}(E)$ and $\mathscr{B}(S)$.
- E and S are **Borel isomorphic to** and **Borel isomorphs of each other** if there exists a Borel isomorphism between them.
- $(A, \mathscr{O}_E(A))$ is a **Borel subspace of** E if $A \in \mathscr{B}(E)$.
- E is a **standard Borel space** if E is Borel isomorphic to a Borel subspace of some *Polish* (see p.222) space.
- A is a **standard Borel subset of** E if $(A, \mathscr{O}_E(A))$ is a standard Borel space. $\mathscr{B}^{\mathbf{s}}(E)$ denotes the families of all standard Borel subsets of E.

The standard Borel property derives from Borel sets in Polish spaces and allows a topological (sub)space to acquire the nice properties of Borel σ-algebras of Polish spaces. A brief review of standard Borel spaces/subsets is provided in §A.5.

2.1.5 Product space

"$\otimes$" denotes the product of σ-algebras. Given measurable spaces $\{(S_i, \mathscr{A}_i)\}_{i \in \mathbf{I}}$, the product σ-algebra of $\{\mathscr{A}_i\}_{i \in \mathbf{I}}$ on $\prod_{i \in \mathbf{I}} S_i$ refers to

$$\bigotimes_{i \in \mathbf{I}} \mathscr{A}_i \doteq \sigma\left(\{\mathfrak{p}_i\}_{i \in \mathbf{I}}\right). \tag{2.1.14}$$

When $(S_i, \mathscr{A}_i) = (S, \mathscr{A})$ for all $i \in \mathbf{I}$, $\bigotimes_{i \in \mathbf{I}} \mathscr{A}_i$ is often denoted by $\mathscr{A}^{\otimes \mathbf{I}}$ for general $\mathbf{I}$, or by $\mathscr{A}^{\otimes \mathbf{N}}$ if $\aleph(\mathbf{I}) = \aleph(\mathbf{N})$, or by $\mathscr{A}^{\otimes d}$ if $\aleph(\mathbf{I}) = d \in \mathbf{N}$. The following facts are well-known and we omit the proofs for brevity.

Fact 2.3. *Let* $(E, \mathscr{U})$ *and* $\{(S_i, \mathscr{A}_i)\}_{i \in \mathbf{I}}$ *be measurable spaces,* $\mathbf{I}_0$ *be a countable subset of* $\mathbf{I}$*,* $S \doteq \prod_{i \in \mathbf{I}} S_i$*,* $\mathscr{A} \doteq \bigotimes_{i \in \mathbf{I}} \mathscr{A}_i$*,* $S_0 \doteq \prod_{i \in \mathbf{I}_0} S_i$*,* $\mathscr{A}_0 \doteq \bigotimes_{i \in \mathbf{I}_0} \mathscr{A}_i$ *and* $f : E \to S_i$ *be a mapping for each* $i \in \mathbf{I}$*. Then:*

[5] Standard Borel property can be defined for general measurable spaces. Herein, we focus on the topological space case.

(a) $\mathfrak{p}_{\mathbf{I}_0}$ *is a measurable mapping from* $(S, \mathscr{A})$ *to* $(S_0, \mathscr{A}_0)$.

(b) $\bigotimes_{i\in\mathbf{I}} f_i : (E, \mathscr{U}) \to (S, \mathscr{A})$ *is measurable if and only if* $f_i : (E, \mathscr{U}) \to (S_i, \mathscr{A}_i)$ *is measurable for all* $i \in \mathbf{I}$.

"$\otimes$" also denotes the product of topologies. Given topological spaces $\{S_i\}_{i\in\mathbf{I}}$, the product topology of $\{\mathscr{O}(S_i)\}_{i\in\mathbf{I}}$ on $\prod_{i\in\mathbf{I}} S_i$ refers to

$$\bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i) \doteq \mathscr{O}_{\{\mathfrak{p}_i\}_{i\in\mathbf{I}}}\left(\prod_{i\in\mathbf{I}} S_i\right). \tag{2.1.15}$$

When $S_i = E$ for all $i \in \mathbf{I}$, $\bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i)$ is often denoted by $\mathscr{O}(E)^{\mathbf{I}}$ for general $\mathbf{I}$, or by $\mathscr{O}(E)^\infty$ if $\aleph(\mathbf{I}) = \aleph(\mathbf{N})$, or by $\mathscr{O}(E)^d$ if $\aleph(\mathbf{I}) = d \in \mathbf{N}$. The following facts are well-known and we omit the proof for brevity.

Fact 2.4. *Let* E *and* $\{(S_i, \mathscr{A}_i)\}_{i\in\mathbf{I}}$ *be topological spaces,* $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$, $S \doteq \prod_{i\in\mathbf{I}} S_i$, $\mathscr{U} \doteq \bigotimes_{i\in\mathbf{I}} \mathscr{O}(E_i)$, $S_0 \doteq \prod_{i\in\mathbf{I}_0} S_i$, $\mathscr{U}_0 \doteq \bigotimes_{i\in\mathbf{I}_0} \mathscr{O}_i(E)$ *and* $f : E \to S_i$ *be a mapping for each* $i \in \mathbf{I}$. *Then:*

(a) $\mathfrak{p}_{\mathbf{I}_0}$ *is a continuous mapping from* $(S, \mathscr{U})$ *to* $(S_0, \mathscr{U}_0)$.

(b) $\bigotimes_{i\in\mathbf{I}} f_i : E \to (S, \mathscr{U})$ *is continuous if and only if* $f_i : E \to S_i$ *is continuous for all* $i \in \mathbf{I}$.

(c) $\bigotimes_{i\in\mathbf{I}} f_i : E \to (S, \mathscr{U})$ *is continuous at* $x \in E$ *(see [Munkres, 2000, p.104]) if and only if* $f_i : E \to S_i$ *is continuous at* $x \in E$ *for all* $i \in \mathbf{I}$.

Standard discussions about product topological spaces can be found in e.g. [Munkres, 2000, §15 and §19] and [Bogachev, 2007, Vol.II, §6.4]. Herein, we remind the readers of one basic but indispensable fact: For general topological space $\{S_i\}_{i\in\mathbf{I}}$, the Borel σ-algebra $\sigma[\bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i)]$ generated by their product topology is likely to differ from $\bigotimes_{i\in\mathbf{I}} \mathscr{B}(S_i)$, the product of their individual Borel σ-algebras. Such difference happens even in the two-dimensional case (see [Bogachev, 2007, Vol.II, Example 6.4.3]). This is why we use different notations for product σ-algebra and product topology. Avoidance of the difference above needs additional countability of the product topology $\bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i)$ (see Proposition B.46).

Hereafter, $\mathbf{R}^k$ (with $k \in \mathbf{N}$) denotes the k-dimensional Euclidean space equipped with the usual norm "$|\cdot|$" and the k-dimensional Lebesgue measure. *Conull subsets* of $\mathbf{R}^k$ are in the Lebesgue sense, which are well-known to be dense subsets. $|\cdot|$ also denotes the norm metric on $\mathbf{R}^k$.

2.2 Spaces of mappings

2.2.1 Spaces of general mappings

Let $\mathbf{I}$, E and S be non-empty sets. The Cartesian power $E^{\mathbf{I}}$ is the family of all mappings from $\mathbf{I}$ to E. When $\mathbf{I}$ has certain index meaning (e.g. time, order), a member of $E^{\mathbf{I}}$ is often considered as a "*path indexed by* $\mathbf{I}$". So, we define the associated **path mapping of** $f \in S^E$ by[6]

$$\varpi_{\mathbf{I}}(f) \mathrel{\mathring{=}} \bigotimes_{i\in\mathbf{I}} f \circ \mathfrak{p}_i \in (S^{\mathbf{I}})^{E^{\mathbf{I}}}. \tag{2.2.1}$$

This mapping sends every E-valued path x indexed by $\mathbf{I}$ to the S-valued path $f \circ x$ indexed by $\mathbf{I}$. We define the associated **joint path mapping of** $\mathcal{D} \subset S^E$ by

$$\varpi_{\mathbf{I}}(\mathcal{D}) \mathrel{\mathring{=}} \bigotimes \{\varpi_{\mathbf{I}}(f) : f \in \mathcal{D}\} \in \left[(S^{\mathbf{I}})^{\mathcal{D}}\right]^{E^{\mathbf{I}}}. \tag{2.2.2}$$

For simplicity, $\varpi_{\mathbf{I}}(f)$ is often denoted by $\varpi(f)$ if $\mathbf{I} = \mathbf{R}^+$, or by $\varpi_T(f)$ if $\mathbf{I} = [0,T]$, or by $\varpi_{a,b}(f)$ if $\mathbf{I} = [a,b]$. Similar notations apply to $\varpi_{\mathbf{I}}(\mathcal{D})$.

Remark 2.5. $\varpi_{\mathbf{I}}(\mathcal{D})$ and $\varpi_{\mathbf{I}}(\bigotimes \mathcal{D})$ are different. The latter is a mapping from $E^{\mathbf{I}}$ to $(S^{\mathcal{D}})^{\mathbf{I}}$.

Let $\delta, T \in (0,\infty)$, $[a,b] \subset \mathbf{R}^+$ and $\mathfrak{r}$ be a pseudometric on E. We define the $\mathfrak{r}$-modulus of continuity

$$\begin{aligned} w'_{\mathfrak{r},\delta,T}(x) \mathrel{\mathring{=}} \inf \Bigg\{ & \max_{1\le i\le n} \sup_{s,t\in[t_{i-1},t_i), s<t} \mathfrak{r}\left(x(t), x(s)\right) : 0 \le t_0 \\ & < ... < T < t_n, \inf_{i\le n}(t_i - t_{i-1}) > \delta, n \in \mathbf{N} \Bigg\} \end{aligned} \tag{2.2.3}$$

[6] ϖ is the calligraphical form of the greek letter π.

for each $x \in E^{\mathbf{R}^+}$, define

$$\mathfrak{r}_{[a,b]}(x,y) \mathrel{\hat{=}} \sup_{t\in[a,b]} 1 \wedge \mathfrak{r}\,(x(t),y(t)) \tag{2.2.4}$$

for each $x, y \in E^{[a,b]}$ or $E^{\mathbf{R}^+}$, and let

$$J(x) \mathrel{\hat{=}} \left\{t \in \mathbf{R}^+ : x(t) \neq \lim_{s\to t-} x(s) \in E\right\} \tag{2.2.5}$$

denotes the **set of left-jump times of** $x \in E^{\mathbf{R}^+}$.

2.2.2 Spaces of measurable, càdlàg and continuous mappings

When E and S are measurable spaces, $M(S;E)$ denotes the family of all measurable mappings from S to E. When E or S is a topological space, $M(S;E)$ abbreviates $M(S;E,\mathscr{B}(E))$ or $M(S,\mathscr{B}(S);E)$ respectively. When E and S are both topological spaces, $C(S;E)$, $\mathbf{hom}(S;E)$, $\mathbf{imb}(S;E)$ and $\mathbf{biso}(S;E)$ denote the families of all continuous mappings, homeomorphisms, imbeddings and Borel isomorphisms from S to E, respectively.

$\mathbf{TC}(\mathbf{R}^+)$ (resp. $\mathbf{TC}([a,b])$) denotes the family of all *time-changes on* $\mathbf{R}^+$ (resp. $[a,b] \subset \mathbf{R}^+$). That is, each $\lambda \in \mathbf{TC}(\mathbf{R}^+)$ (resp. $\lambda \in \mathbf{TC}([a,b])$) is a strictly increasing homeomorphism from $\mathbf{R}^+$ (resp. $[a,b]$) to itself and satisfies

$$|||\lambda||| \mathrel{\hat{=}} \sup_{t>s}\left|\ln\frac{\lambda(t)-\lambda(s)}{t-s}\right| < \infty. \tag{2.2.6}$$

Then, we define

$$\varrho^{\mathfrak{r}}_{[a,b]}(x,y) \mathrel{\hat{=}} \inf_{\lambda\in\mathbf{TC}([a,b])} \big(|||\lambda||| \vee \mathfrak{r}_{[a,b]}(x\circ\lambda, y)\big) \tag{2.2.7}$$

for each $x, y \in E^{[a,b]}$, and define

$$\varrho^{\mathfrak{r}}(x,y) \mathrel{\hat{=}} \inf_{\lambda\in\mathbf{TC}(\mathbf{R}^+)} \left(|||\lambda||| \vee \int_0^\infty e^{-u}\mathfrak{r}_{[0,u]}(x\circ\lambda, y)du\right) \tag{2.2.8}$$

for each $x, y \in E^{\mathbf{R}^+}$.

When E is a topological space, $x \in E^{\mathbf{R}^+}$ is *càdlàg* (i.e. right-continuous and left-limited) if for every $t \in \mathbf{R}^+$, there exists a unique $y^t \in E$ such that $x(u_n) \to y^t$ as $n \uparrow \infty$ in E for all $u_n \uparrow t$ and $x(v_n) \to x(t)$ as $n \uparrow \infty$ in E for all $v_n \downarrow t$. When E is a *Tychonoff space*[7] (see p.231), Proposition A.25 to follow shows that the topology of E is induced by a family $\mathcal{R}$ of pseudometrics on E. Then, by $D(\mathbf{R}^+; E)$ (resp. $D([a,b]; E)$) we denote the space of all càdlàg members of $E^{\mathbf{R}^+}$ (resp. $E^{[a,b]}$) equipped with the Skorokhod $\mathscr{J}_1$-topology $\mathscr{J}(E)$ (resp. $\mathscr{J}_{a,b}(E)$), that is, the topology induced by pseudometrics $\{\varrho^{\mathfrak{r}}\}_{\mathfrak{r}\in\mathcal{R}}$ (resp. $\{\varrho^{\mathfrak{r}}_{[a,b]}\}_{\mathfrak{r}\in\mathcal{R}}$). It is worth noting that $\mathscr{J}(E)$ and $\mathscr{J}_{a,b}(E)$ turn out to be independent of the choice of the pseudometrics $\mathcal{R}$. We refer the readers to §A.6 for more information about Skorokhod $\mathscr{J}_1$-spaces.

2.2.3 Spaces of $\mathbf{R}^k$-valued functions

Let E be a non-empty set. Given $\{f, g\} \subset \mathbf{R}^E$, we define $f \vee g(x) \doteq \max\{f(x), g(x)), \ f \wedge g(x) \doteq \min\{f(x), g(x)\}, \ f^+(x) \doteq \max\{f(x), 0\}$ and $f^-(x) \doteq \max\{-f(x), 0\}$ for all $x \in E$. A subset of $\mathbf{R}^E$ is a *function lattice* if it is closed under the operations "$\wedge$" and "$\vee$".

Let $k \in \mathbf{N}$ and $\mathcal{D} \subset (\mathbf{R}^k)^E$. The **additive expansion of** $\mathcal{D}$ is defined by

$$\mathfrak{ae}(\mathcal{D}) \doteq \mathcal{D} \cup \{f + g : f, g \in \mathcal{D}\}, \tag{2.2.9}$$

and the **additive closure of** $\mathcal{D}$ is defined by

$$\mathfrak{ac}(\mathcal{D}) \doteq \left\{ \sum_{f \in \mathcal{D}_0} f : \mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D}) \right\}. \tag{2.2.10}$$

When $k = 1$, the **multiplicative closure of** $\mathcal{D}$ is defined by

$$\mathfrak{mc}(\mathcal{D}) \doteq \left\{ \prod_{f \in \mathcal{D}_0} f : \mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D}) \right\}, \tag{2.2.11}$$

[7] We use the terminologies "Tychonoff space" instead of "completely regular space" since the latter sometimes is used in a non-Hausdorff context.

the **Q-algebra generated by** $\mathcal{D}$ is defined by

$$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D}) \circeq \mathfrak{ac}\left(\{af : f \in \mathfrak{mc}(\mathcal{D}), a \in \mathbf{Q}\}\right), \tag{2.2.12}$$

the **algebra generated by** $\mathcal{D}$ is defined by

$$\mathfrak{ag}(\mathcal{D}) \circeq \mathfrak{ac}\left(\{af : f \in \mathfrak{mc}(\mathcal{D}), a \in \mathbf{R}\}\right), \tag{2.2.13}$$

and, for a *finite* index set $\mathbf{I}$, we define

$$\Pi^{\mathbf{I}}(\mathcal{D}) \circeq \left\{ g \in \mathbf{R}^{E^{\mathbf{I}}} : g = \prod_{i=1}^{j} f_{i,j} \circ \mathfrak{p}_i, f_{i,j} \in \mathcal{D}, 1 \leq j \leq \aleph(\mathbf{I}) \right\}, \tag{2.2.14}$$

which is formed in a similar way to the function class in [Ethier and Kurtz, 1986, §4.4, (4.15)]. Hereafter, $\Pi^{\mathbf{I}}(\mathcal{D})$ is often denoted by $\Pi^d(\mathcal{D})$ with $d \circeq \aleph(\mathbf{I})$. The enlargements of $\mathcal{D}$ above are often used to construct a rich but countable collection of functions that includes $\mathcal{D}$. Some of their basic properties are specified in §A.2 and §B.1 - §B.2.

"$\xrightarrow{u}$" denotes uniform convergence of $\mathbf{R}^k$-valued functions. When the members of $\mathcal{D} \subset (\mathbf{R}^k)^E$ are bounded[8], $\mathfrak{cl}(\mathcal{D})$ denotes the closure of $\mathcal{D}$ under the supremum norm $\|\cdot\|_\infty$ and, if $k = 1$, we define

$$\mathfrak{ca}(\mathcal{D}) \circeq \mathfrak{cl}\left[\mathfrak{ag}(\mathcal{D})\right] = \mathfrak{cl}\left[\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D})\right]. \tag{2.2.15}$$

The second equality above is immediate by the denseness of $\mathbf{Q}$ in $\mathbf{R}$ and properties of uniform convergence.

$M_b(E;\mathbf{R}^k)$ (resp. $C_b(E;\mathbf{R}^k)$) denotes the Banach space over scalar field $\mathbf{R}$ of all bounded members of $M(E;\mathbf{R}^k)$ (resp. $C(E;\mathbf{R}^k)$) equipped with $\|\cdot\|_\infty$. $C_c(E;\mathbf{R}^k)$ denotes the subspace of all members of $C(E;\mathbf{R}^k)$ that have *compact supports*, i.e. the closure of $E \backslash f^{-1}(\{0\})$ is compact for all $f \in C_c(E;\mathbf{R}^k)$. $C_0(E;\mathbf{R}^k)$ denotes the subspace of all members of $C(E;\mathbf{R}^k)$ that *vanish at infinity*, i.e. for any $\epsilon > 0$, there exists a $K_\epsilon \in \mathscr{K}(E)$ such that $\|f|_{E\backslash K_\epsilon}\|_\infty < \epsilon$.

[8] f is bounded if $\|f\|_\infty \in \mathbf{R}^+$.

2.2.4 Functions and separation of points

This work intensively uses the following point-separation properties of function classes. Let E and $A \subset E$ be non-empty sets and $\mathcal{D} \subset \mathbf{R}^E$. $\mathcal{D}$ **separates points on** A if $\bigotimes \mathcal{D}$ is injective, or equivalently, $f(x) = f(y)$ for all $f \in \mathcal{D}$ implies $x = y$ in A. Suppose E is a topological space. Then, $\mathcal{D}$ **strongly separates points on** A if $\mathscr{O}_E(A) \subset \mathscr{O}_{\mathcal{D}}(A)$. $\mathcal{D}$ **determines point convergence on** A if $\bigotimes \mathcal{D}(x_n) \to \bigotimes \mathcal{D}(x)$ as $n \uparrow \infty$ in $(\mathbf{R}^{\mathcal{D}}, \mathscr{O}(\mathbf{R})^{\mathcal{D}})$ implies $x_n \to x$ as $n \uparrow \infty$ in $(A, \mathscr{O}_E(A))$.

Remark 2.6. $\bigotimes \mathcal{D}(x_n) \to \bigotimes \mathcal{D}(x)$ as $n \uparrow \infty$ is equivalent to $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in \mathcal{D}$ by Fact B.11.

Note 2.7. The point separability, strong point separability or point convergence determining of $\mathcal{D} \subset \mathbf{R}^E$ on $A \subset E$ is apparently inherited by any $\mathcal{D}' \subset \mathbf{R}^E$ with $\mathcal{D} \subset \mathcal{D}'$.

The following are several simple examples of function classes with the aforementioned point-separation properties.

Example 2.8.

(I) Let $C([0,1];\mathbf{R})$ be alternatively equipped with the product topology $\mathscr{O}(\mathbf{R})^{[0,1]}$. The one-dimensional projection $\mathfrak{p}_x$ is continuous on $C([0,1];\mathbf{R})$ for all $x \in [0,1]$, since the convergence under product topology means pointwise convergence (see [Munkres, 2000, Theorem 46.1]). Note that

$$\bigotimes_{x\in\mathbf{Q}\cap[0,1]} \mathfrak{p}_x(f) = \bigotimes_{x\in\mathbf{Q}\cap[0,1]} \mathfrak{p}_x(g) \tag{2.2.16}$$

implies $f = g$ by the denseness of $\mathbf{Q}$ in $\mathbf{R}$ and the continuity of f and g. Hence, $\{\mathfrak{p}_x\}_{x\in\mathbf{Q}\cap[0,1]}$ is a countable collection of $\mathbf{R}$-valued continuous functions and separates points on $C([0,1];\mathbf{R})$.

(II) Let $(E, \mathfrak{r})$ be a metric space and define

$$g_{y,k}(x) \mathrel{\hat{=}} [1 - k\mathfrak{r}(x,y)] \vee 0, \ \forall x, y \in E, k \in \mathbf{N}. \tag{2.2.17}$$

[Blount and Kouritzin, 2010, (4)] showed that $\{g_{y,k}\}_{y\in E,k\in\mathbf{N}}$ strongly separates points on E. It separates points and determines point convergence on E by Proposition A.17 (a, b) to follow.

(III) Let $(E,\mathfrak{r})$ be a metric space. For each $x\in E$,

$$g_x(y) \mathrel{\hat{=}} \mathfrak{r}(y,x),\ \forall y\in E \tag{2.2.18}$$

is a Lipschitz function by triangular inequality. If $x\neq y$ in E, then

$$g_x(y)-g_x(x)=\mathfrak{r}(x,y)>0. \tag{2.2.19}$$

If $g_x(x_n)\to g_x(x)$ as $n\uparrow\infty$ in $\mathbf{R}$, then

$$\lim_{n\to\infty}\mathfrak{r}(x_n,x)=\lim_{n\to\infty}|g_x(x_n)|=\lim_{n\to\infty}|g_x(x_n)-g_x(x)|=0 \tag{2.2.20}$$

and so $x_n\to x$ as $n\uparrow\infty$ in E. $\mathcal{D}\mathrel{\hat{=}}\{g_x\}_{x\in E}$ separates points, determines point convergence and strongly separates points on E by (2.2.19), (2.2.20) and Proposition A.17 (b) to follow. The family of all Lipschitz functions on E has the same point-separation properties as $\mathcal{D}$ by Note 2.7.

(IV) For each $n\in\mathbf{N}$,

$$f_n(x)\mathrel{\hat{=}}\begin{cases}1, & \text{if } x\in\left[0,\frac{1}{n}\right],\\ -\frac{n}{n^2-1}x+\frac{n^2}{n^2-1}, & \text{if } x\in\left(\frac{1}{n},n\right),\\ 0, & \text{if } x\in[n,\infty)\end{cases} \tag{2.2.21}$$

defines a bounded continuous function on $\mathbf{R}^+$ which is strictly decreasing on its compact support $[0,n]$. One immediately observes that $\mathcal{D}\mathrel{\hat{=}}\{f_n\}_{n\in\mathbf{N}}$ separates points and determines point convergence on $\mathbf{R}^+$. $\mathcal{D}$ strongly separates points on E by Proposition A.17 (b) to follow. $C_c(\mathbf{R}^+;\mathbf{R})$ has the same point-separation properties as $\mathcal{D}$ by Note 2.7.

Note 2.9. $C(E;\mathbf{R})$ and $C_b(E;\mathbf{R})$ separate points and strongly separate points on E when E is a Tychonoff space (see Proposition A.25). For more general E, however, even $C(E;\mathbf{R})$ does not necessarily separate points on E (see Example A.39).

For $\mathcal{D} = \{f_j\}_{j\in\mathbf{N}} \subset \mathbf{R}^E$ and $d \in \mathbf{N}$, the pseudometric

$$\rho_{\mathcal{D}}(x_1,x_2) \mathrel{\hat{=}} \sum_{j=1}^{\infty} 2^{-j+1}\left(|f_j(x_1)-f_j(x_2)| \wedge 1\right), \ \forall x_1,x_2 \in E \tag{2.2.22}$$

on E induces $\mathscr{O}_{\mathcal{D}}(E)$ (see Proposition A.17 (d)), and the pseudometric

$$\rho_{\mathcal{D}}^d(y_1,y_2) \mathrel{\hat{=}} \max_{1\le i\le d} \rho_{\mathcal{D}}\left(\mathfrak{p}_i(y_1),\mathfrak{p}_i(y_2)\right), \ \forall y_1,y_2 \in E^d \tag{2.2.23}$$

on E^d induces $\mathscr{O}_{\mathcal{D}}(E)^d$ (see Corollary A.18).

2.2.5 Linear Operators

We review several basic notions about linear operators which are involved in §4.2. Let $(S,\|\cdot\|)$ be a Banach space over scalar field $\mathbf{R}$ (like $(C_b(E;\mathbf{R}),\|\cdot\|_\infty)$). By a **single-valued linear operator** $\mathcal{L}$ **on** S we refer to a linear subspace $\mathcal{L} \subset S\times S$ such that for each $f\in S$, $\{g\in S : (f,g)\in\mathcal{L}\}$ is either $\varnothing$ or a singleton denoted by $\{\mathcal{L}f\}$. The domain $\mathfrak{D}(\mathcal{L})$ and range $\mathfrak{R}(\mathcal{L})$ of $\mathcal{L}$ are defined respectively by

$$\mathfrak{D}(\mathcal{L}) \mathrel{\hat{=}} \{f\in S : \mathcal{L}\cap(\{f\}\times S) \neq \emptyset\} \tag{2.2.24}$$

and

$$\mathfrak{R}(\mathcal{L}) \mathrel{\hat{=}} \{g\in S : \mathcal{L}\cap(S\times\{g\}) \neq \emptyset\}, \tag{2.2.25}$$

which are well-known to be linear subspaces of S.

$\mathcal{L}$ is **closed** if it is a closed subspace of $S\times S$. The **restriction of** $\mathcal{L}$ **to** (subdomain) $\mathcal{D}\subset\mathfrak{D}(\mathcal{L})$ is defined by

$$\mathcal{L}|_{\mathcal{D}} \mathrel{\hat{=}} \{(f,\mathcal{L}f)\in\mathcal{L} : f\in\mathcal{D}\}. \tag{2.2.26}$$

$\mathcal{L}$ is *dissipative* if

$$\beta\|f\| \le \|\beta f - \mathcal{L}f\|, \ \forall f\in\mathfrak{D}(\mathcal{L}), \beta\in(0,\infty). \tag{2.2.27}$$

$\mathcal{L}$ satisfies *positive maximum principle* if $\sup_{x\in E} f(x) = f(x_0) \geq 0$ implies $\mathcal{L}f(x_0) \leq 0$ for all $f \in \mathfrak{D}(\mathcal{L})$. $\mathcal{L}$ is a **strong generator on** S if the closure of $\mathcal{L}$ under $\|\cdot\|$ is the *infinitesimal generator* (see [Yosida, 1980, p.231]) of a *strongly continuous contraction semigroup* (see [Yosida, 1980, p.232]) on S. When E is a *locally compact* (see p.224) *Hausdorff* (see p.217) space and $S = (C_0(E;\mathbf{R}), \|\cdot\|_\infty)$, $\mathcal{L}$ is a **Feller generator on** S if $\mathfrak{cl}(\mathcal{L})$ is the infinitesimal generator of a *Feller semigroup*[9] (see [Ethier and Kurtz, 1986, §4.2, p.166]) on S.

If needed, more details about operators on Banach spaces can be found in standard texts like [Yosida, 1980, Chapter VIII and Chapter IX] and [Ethier and Kurtz, 1986, Chapter 1].

2.3 Spaces of non-negative finite Borel measures

Let E be topological space. The members of $\mathfrak{M}^+(E,\mathscr{B}(E))$ and $\mathfrak{P}(E,\mathscr{B}(E))$ are called *finite Borel measures* and *Borel probability measures* respectively.

Definition 2.10. Let E be a topological space and $\mathscr{U}$ be a sub-σ-algebra of $\mathscr{B}(E)$. Then, any extension of $\mu \in \mathfrak{M}^+(E,\mathscr{U})$ to $\mathscr{B}(E)$ is said to be a **Borel extension of** μ.

Hereafter, $\mathfrak{be}(\mu)$ denotes the family of all Borel extension(s) of μ (if any). If μ' is the unique member of $\mathfrak{be}(\mu)$, then we specially denote $\mu' = \mathfrak{be}(\mu)$.

Note 2.11. Any $\mu' \in \mathfrak{be}(\mu)$ has the same total mass as μ since the full space lies in and μ is indifferent from μ' restricted to the domain of μ.

$\mathcal{M}^+(E)$ denotes the space of all finite Borel measures on E equipped with the weak topology. $\mathcal{P}(E)$ specially denotes the subspace of all probabilistic members of $\mathcal{M}^+(E)$. To be specific, the weak topology of $\mathcal{M}^+(E)$ is defined by

$$\mathscr{O}\left[\mathcal{M}^+(E)\right] \circeq \mathscr{O}_{C_b(E;\mathbf{R})^*}\left[\mathcal{M}^+(E)\right], \tag{2.3.1}$$

[9]There are multiple definitions of Feller semigroup in literature. Herein, we choose the traditional definition based on locally compact spaces.

where f^* denotes the linear functional

$$\begin{aligned} f^* :&\mathcal{M}^+(E) \longrightarrow \mathbf{R}, \\ &\mu \longmapsto \int_E f(x)\mu(dx) \end{aligned} \tag{2.3.2}$$

for each $f \in M_b(E;\mathbf{R})$ and

$$\mathcal{D}^* \mathrel{\hat{=}} \{f^* : f \in \mathcal{D}\} \tag{2.3.3}$$

for each $\mathcal{D} \subset M_b(E;\mathbf{R})$. Given $f \in M_b(E;\mathbf{R}^k)$ (with $k \in \mathbf{N}$), we define $f^* \mathrel{\hat{=}} \bigotimes_{i=1}^k (\mathfrak{p}_i \circ f)^*$.

Remark 2.12. The weak topology of $\mathcal{M}^+(E)$ is sometimes called "narrow topology", since it is generally *different from* the standard notion of *weak-$*$ topology induced by dual space*. This confusion is avoided when E is a locally compact Hausdorff space (see [Malliavin, 1995, Chapter II, §6.5 - 6.7]). Hereafter, we conventionally use the notation "f^*" to denote the linear functional in (2.3.2), but when $\mathcal{D}$ is a linear space, $\mathcal{D}^*$ does not mean any dual space of $\mathcal{D}$.

Weak convergence is one of the central interests of this work. As specified in §2.1.3 for general topological spaces, the statement

$$\mu_n \Longrightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E) \tag{2.3.4}$$

means that: (1) $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\}$ are members of $\mathcal{M}^+(E)$, and (2) $\{\mu_n\}_{n\in\mathbf{N}}$ **converges to μ with respect to the weak topology of $\mathcal{M}^+(E)$** (converge weakly to μ for short). Similar terminology and notation apply to $\mathcal{P}(E)$.

Remark 2.13. Weak convergence in $\mathcal{M}^+(E)$ can always be rescaled to that in $\mathcal{P}(E)$ (see Fact B.21 (b)).

$\mu \in \mathcal{M}^+(E)$ is a **weak limit point of** $\Gamma \subset \mathcal{M}^+(E)$ if there exist $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma$ satisfying (2.3.4). By

$$\text{w-}\lim_{n\to\infty} \mu_n = \mu \tag{2.3.5}$$

we denote μ is the **weak limit of** $\{\mu_n\}_{n\in\mathbf{N}} \subset \mathcal{M}^+(E)$, that is, (2.3.4) holds and μ is the *unique* weak limit point of $\{\mu_n\}_{n\in\mathbf{N}}$. $\Gamma \subset \mathcal{M}^+(E)$ is **rela-**

tively compact if any infinite subset of Γ has at least one weak limit point in $\mathcal{M}^+(E)$[10].

Remark 2.14. (2.3.4) does not necessarily imply (2.3.5) since $\mathcal{M}^+(E)$ in general is not guaranteed to be a Hausdorff space so μ in (2.3.4) might not be unique.

2.3.1 Weak topology in sequential view

In this work, we consider weak convergence as a type of **topological convergence**, i.e. convergence induced from a topology. Apparently, (2.3.4) is equivalent to the integral test

$$\lim_{n\to\infty}\int_E f(x)\mu_n(dx) = \int_E f(x)\mu(dx),\ \forall f \in C_b(E;\mathbf{R}). \tag{2.3.6}$$

In literature, one more often defines weak convergence by (2.3.6) and then defines weak topology sequentially by weak convergence. Below we briefly explain the connection between our and the sequential definitions of weak topology.

The relationship between topology and convergence are bidirectional. In one direction, topological convergence comes after defining open or closed sets. In the other direction, convergence (topological or non-topological) can induce a sequential topology. Indeed, convergence is definable without any topological structure or even be non-topologizable. Weak convergence defined by (2.3.6) is one example since it does not formally involve any topology of $\mathcal{M}^+(E)$. Two other examples are almost sure convergence and *bounded pointwise convergence* (see [Ethier and Kurtz, 1986, §3.4]). Suppose a sense of convergence is given, which is often called **convergence a priori**. Then, one defines closedness of a set to be the containment of limits of all convergent a priori sequences. A sequential topology is generated by these "sequentially closed" sets, which claims its own topological convergence called **convergnce a posteriori**.

Convergence a priori and convergence a posteriori need not be the same in general (see [Jakubowski, 2012, §5] for further details). One example is the $\mathcal{S}$-topology introduced by Jakubowski [1997b] (see §2.7.2 for a short glance). Below is another elementary example.

[10] Of course, this weak limit point need not belong to Γ.

Example 2.15. We say $\{x_n\}_{n\in\mathbf{N}}$ $\star$-converges to x in $\mathbf{R}$ if

$$|x_n - x| \leq 2^{-n},\ \forall n \in \mathbf{N}. \tag{2.3.7}$$

The specific control of convergence rate makes $\star$-convergence strictly stronger than the Euclidean convergence in $\mathbf{R}$ ($\{n^{-1}\}_{n\in\mathbf{N}}$ is not $\star$-convergent). However, if one considers $\star$-convergence as convergence a priori, then the induced sequential topology turns out to be the Euclidean topology (see Fact B.13). In this case, convergence a posteriori is the Euclidean convergence and differs from convergence a priori.

The property of a point to be the limit of some subsequence, and that of a set to have some convergent subsequence are sequential rather than topological concepts, which can be defined for any convergence a priori. Indeed, "weak limit point" and "relative compactness" of $\Gamma \subset \mathcal{M}^+(E)$ are such sequential concepts with weak convergence being convergence a priori. Their independence of the weak topology of $\mathcal{M}^+(E)$ may cause ambiguity in general settings:

- Weak limit point is stronger than *limit point* (see [Munkres, 2000, p.97]) with respect to weak topology, since $\mathcal{M}^+(E)$ is not guaranteed to be *first-countable*[11] (see p.217).

- The topological interpretation of "relative compactness" is usually the existence of compact closure (see p.224). However, relative compactness of $\Gamma \subset \mathcal{M}^+(E)$ is generically different from the compactness of Γ's closure in $\mathcal{M}^+(E)$.

- Relative compactness of $\Gamma \subset \mathcal{M}^+(E)$ might also be different from Γ having a *limit point compact* (see p.224) or *sequentially compact* (see p.224) closure with respect to weak topology.

The ambiguity above is due to that weak topology is generally coarser than the sequential topology induced by weak convergence as convergence a priori. This also causes the possible difference of weak convergence and the associated

[11] The absence of first-countability results in that a limit point with respect to weak topology does not necessarily imply a subsequence that converges weakly to this limit point.

convergence a posteriori. Similar issue is also incurred by the $\mathcal{S}$-topology (see [Jakubowski, 2012, §2 - 3]).

When E is a metrizable space as in the majority of probabilistic literature, $\mathcal{M}^+(E)$ is also metrizable (see Proposition A.45) and does not incur the ambiguity above. This explains why the sequential definition of weak topology by (2.3.6) and the ambiguous use of "weak limit point" and "relative compactness" are conventional nowadays.

2.3.2 Separation of measures by functions

The measure-separation properties of $\mathcal{D} \subset M_b(E; \mathbf{R})$ (i.e. point-separation properties of $\mathcal{D}^*$) are vital for studying weak convergence and $\mathcal{M}^+(E)$-valued or $\mathcal{P}(E)$-valued processes (e.g. filters, measure-valued diffusions, non-Markov branching particle systems). The following terminologies are conventionally adapted from [Ethier and Kurtz, 1986, §3.4]: $\mathcal{D} \subset M_b(E; \mathbf{R})$ is **separating** or **convergence determining on** E if $\mathcal{D}^*$ separates points or determines point convergence on $\mathcal{M}^+(E)$ respectively.

Note 2.16. $C_b(E; \mathbf{R})^*$ by definition strongly separates points and so determines point convergence on $\mathcal{M}^+(E)$ (see Proposition A.17 (b)). Hence, $C_b(E; \mathbf{R})$ is convergence determining on $\mathcal{M}^+(E)$.

We refer the readers to §A.4, [Topsøe, 1970, Part II] and [Bogachev, 2007, Vol. II, Chapter 8] for more details about the topological properties of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$.

2.3.3 Portmanteau's Theorem

One way of establishing (2.3.4) is testing the integral convergence in (2.3.6) for all f from a convergence determining collection. A useful alternative is the Portamenteau's Theorem. This useful tool was commonly established on metric spaces (see [Kallianpur and Xiong, 1995, Lamma 2.2.2]). [Topsøe, 1970, p.XII and p.40 - 41] gave the following partial generalization on Hausdorff spaces.

Theorem 2.17 (**Portmanteau's Theorem**, [Topsøe, 1970, Theorem 8.1]). *Let E be a Hausdorff space. Consider the following statements:*

(a) (2.3.4) holds.

(b) $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ *for all* $F \in \mathscr{C}(E)$*.*

(c) $\liminf_{n\to\infty} \mu_n(O) \geq \mu(O)$ *for all* $O \in \mathscr{O}(E)$*.*

Then, (b) and (c) are equivalent and each of them implies (a). If, in addition, E is a Tychonoff space, then (a) - (c) are equivalent.

2.3.4 Tightness

Tightness is often more explicitly verifiable than relative compactness. Compact subsets are not necessarily Borel subsets in non-Hausdorff spaces. At the same time, they can lie in the domain of possibly non-Borel measures (see §3.3.4). So, we slightly adjust the ordinary definition of and extend tightness to general finite measures.

Definition 2.18. Let $(E, \mathscr{U})$ be a measurable space, S be a topological space and $\mathscr{A}$ be a σ-algebra on S.

- When $S \subset E$, $\Gamma \subset \mathfrak{M}^+(E, \mathscr{U})$ is **tight in** S (resp. **m-tight in** S) if for any $\epsilon \in (0, \infty)$, there exists a $K_\epsilon \in \mathscr{K}(S)$ (resp. $K_\epsilon \in \mathscr{K}^{\mathbf{m}}(S)$) such that $K_\epsilon \in \mathscr{U}$ and $\sup_{\mu\in\Gamma} \mu(E \backslash K_\epsilon) \leq \epsilon$.
- $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$ is **tight in** $A \subset S$ (resp. **m-tight in** A) if A is non-empty and Γ is tight (resp. **m**-tight) in $(A, \mathscr{O}_S(A))$.
- $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$ is **tight** (resp. **m-tight**) if it is tight (resp. **m**-tight) in S.

Note 2.19. Hereafter, any type of tightness of a measure μ refers to that of the singleton $\{\mu\}$.

Remark 2.20. **m**-tightness is stronger than tightness, and they are the same if every compact subset of the underlying space is metrizable. We refer the readers to §3.3.4 for specific discussion about metrizable compact subsets.

The classical Ulam's Theorem (see [Billingsley, 1968, Theorem 1.4]), showing tightness of every finite set of Borel probability measures on a Polish space E, has the following stronger form about **m**-tightness.

Theorem 2.21 (**Ulam's Theorem**, [Bogachev, 2007, Vol.II, Theorem 7.4.3])**.** *If E is a Souslin space (see p.222), especially if E is a Lusin (see p.222) or Polish space, then any finite subset of $\mathcal{M}^+(E)$ is* $\mathbf{m}$*-tight.*

The Prokhorov's Theorem is a fundamental result that connects relative compactness and tightness of finite Borel measures. Part (a) below is adapted from [Bogachev, 2007, Vol.II, Theorem 8.6.2] which gives one direction of the classical Prokhorov's Theorem. Part (b), extending the other direction from Polish to Hausdorff spaces, is adapted from [Kallianpur and Xiong, 1995, Theorem 2.2.1].

Theorem 2.22 (**Prokhorov's Theorem**)**.**

(a) If E is a Polish space, then relative compactness implies tighteness for any subset of $\mathcal{M}^+(E)$.

(b) If E is a Hausdorff space, then tightness implies relative compactness for any subset of $\mathcal{P}(E)$.

2.3.5 Finite Borel measures on $D(\mathbf{R}^+; E)$

When E is a Tychonoff space, the Skorokhod $\mathscr{J}_1$-space $D(\mathbf{R}^+; E)$ always satisfies

$$\mathscr{B}\left[D(\mathbf{R}^+; E)\right] = \sigma\left[\mathscr{J}(E)\right] \supset \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{D(\mathbf{R}^+;E)} \tag{2.3.8}$$

and

$$\mathcal{M}^+\left[D(\mathbf{R}^+; E)\right] \subset \mathfrak{M}^+\left(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{D(\mathbf{R}^+;E)}\right). \tag{2.3.9}$$

However, one may not have the equality of (2.3.8) or (2.3.9) in general. The **set of fixed left-jump times of** $\mu \in \mathfrak{M}^+(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$ refers to

$$J(\mu) \mathrel{\hat=} \left\{t \in \mathbf{R}^+ : \mu\left(\left\{x \in D(\mathbf{R}^+; E) : t \in J(x)\right\}\right) > 0\right\} \tag{2.3.10}$$

if it is well-defined. These measurability issues about $D(\mathbf{R}^+; E)$ are further discussed in §A.6.

2.4 Random variable

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space (i.e. $\mathbb{P} \in \mathfrak{P}(\Omega, \mathscr{F})$), $(E, \mathscr{U})$ be a measurable space and S be a topological space. Any $X \in M(\Omega, \mathscr{F}; E, \mathscr{U})$ is said to be an $(E, \mathscr{U})$-valued random variable. $\mathbb{P} \circ X^{-1} \in \mathfrak{P}(E, \mathscr{U})$, the push-forward measure of $\mathbb{P}$ by X is called the *distribution of* X. S-valued random variables refer to $(S, \mathscr{B}(S))$-valued random variables if not otherwise specified. Hereafter, by $(\Omega, \mathscr{F}, \mathbb{P}; X)$ we abbreviate a random variable X defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Any type of tightness in Definition 2.18 is defined for random variables by referring to the corresponding property of their distributions. "$X_n \Rightarrow X$ as $n \uparrow \infty$ on S" means the distributions of S-valued random variables $\{X_n\}_{n\in\mathbf{N}}$ converge weakly to that of S-valued random variable X as $n \uparrow \infty$ in $\mathcal{P}(S)$. Similar interpretations apply to the statements "X is a the weak limit of $\{X_n\}_{n\in\mathbf{N}}$ on S", " X is a weak limit point of $\{X_i\}_{i\in\mathbf{I}}$ on S" and "$\{X_i\}_{i\in\mathbf{I}}$ is relatively compact in S".

2.5 Stochastic process

The stochastic processes treated in this work are indexed by time horizon $\mathbf{R}^+$ and take values in topological spaces[12]. Throughout this section, we let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, E be a topological space and $X \in (E^{\mathbf{R}^+})^\Omega$.

2.5.1 Definition

X is an E**-valued** (**stochastic**) **process** if $\mathscr{B}(E)^{\otimes\mathbf{R}^+}$ is a sub-σ-algebra of

$$\mathscr{U}_X \mathrel{\hat{=}} \left\{B \subset E^{\mathbf{R}^+} : X^{-1} \in \mathscr{F}\right\}, \tag{2.5.1}$$

or equivalently,

$$X \in M(\Omega, \mathscr{F}; E^{\mathbf{R}^+}, \mathscr{B}(E)^{\otimes\mathbf{R}^+}). \tag{2.5.2}$$

Remark 2.23. The $\mathscr{U}_X$ in (2.5.1) is often called the "push-forward σ-algebra of X". In any case, $X \in M(\Omega, \mathscr{F}; E^{\mathbf{R}^+}, \mathscr{U}_X)$.

[12]Stochastic processes are definable on general measurable spaces without any topological structure.

Let $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued process[13]. For each $\omega\in\Omega$, $X(\omega)\in E^{\mathbf{R}^+}$ is called a (realization) **path of** X. The **process distribution of** an E-valued process X refers to the push-forward measure of $\mathbb{P}$ by $X:(\Omega,\mathscr{F})\to(E^{\mathbf{R}^+},\mathscr{U}_X)$ and is denoted by $\mathrm{pd}(X)\in\mathfrak{P}(E^{\mathbf{R}^+},\mathscr{U}_X)$. For each $t\in\mathbf{R}^+$, $X_t\doteq\mathfrak{p}_t\circ X$ denotes the (one-dimensional) **section of** X **for** t. For each $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$, the **section of** X **for** $\mathbf{T}_0$ refers to the $E^{\mathbf{T}_0}$-valued mapping $X_{\mathbf{T}_0}\doteq\mathfrak{p}_{\mathbf{T}_0}\circ X$, and the **finite-dimensional distribution of** X **for** $\mathbf{T}_0$ refers to $\mathrm{pd}(X)\circ\mathfrak{p}_{\mathbf{T}_0}^{-1}$. From Fact 2.3 we immediately observe that:

Fact 2.24. *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and E be a topological space. Then, the following statements are true:*

(a) $X\in(E^{\mathbf{R}^+})^{\Omega}$ is an E-valued process if and only if $\mathfrak{p}_t\circ X\in M(\Omega,\mathscr{F};E)$ for all $t\in\mathbf{R}^+$.

(b) If $\zeta^t\in M(\Omega,\mathscr{F};E)$ for all $t\in\mathbf{R}^+$, then

$$X(\omega)(t)\doteq\zeta^t(\omega),\ \forall t\in\mathbf{R}^+,\omega\in\Omega \tag{2.5.3}$$

well defines an E-valued process X satisfying $\zeta^t=\mathfrak{p}_t\circ X$ for all $t\in\mathbf{R}^+$.

(c) The section of an E-valued process $(\Omega,\mathscr{F},\mathbb{P};X)$ for each $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$ is a member of $M(\Omega,\mathscr{F},E^{\mathbf{T}_0},\mathscr{B}(E)^{\otimes\mathbf{T}_0})$.

(d) The finite-dimensional distribution of an E-valued process X for each $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$ is the distribution of $X_{\mathbf{T}_0}$ and belongs to $\mathfrak{P}(E^{\mathbf{T}_0},\mathscr{B}(E)^{\otimes\mathbf{T}_0})$. In particular, the finite-dimensional distribution of X for each $t\in\mathbf{R}^+$ is a member of $\mathcal{P}(E)$.

Remark 2.25. Given an E-valued process X and a general $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$, X (resp. $X_{\mathbf{T}_0}$) need not be an $(E^{\mathbf{R}^+},\mathscr{B}(E^{\mathbf{R}^+}))$-valued (resp. $(E^{\mathbf{T}_0},\mathscr{B}(E^{\mathbf{T}_0}))$-valued) random variable, nor is the process distribution of X (resp. the finite-dimensional distribution of X for $\mathbf{T}_0$) necessarily a Borel measure. This is due to the possible difference between the Borel σ-algebra generated by product topology and product of Borel σ-algebras on each individual dimension, which was mentioned in §2.1.5.

[13] An E-valued process is an $(E^{\mathbf{R}^+},\mathscr{B}(E)^{\otimes\mathbf{R}^+})$-valued random variable, hence it is consistent for $(\Omega,\mathscr{F},\mathbb{P};X)$ to denote an E-valued process X defined on $(\Omega,\mathscr{F},\mathbb{P})$.

According to Fact 2.24 above, an E-valued process is indifferent from a family of E-valued random variables indexed by $\mathbf{R}^+$. Hereafter, an E-valued process X is also denoted by $X = \{X_t\}_{t\geq 0}$ or just by $\{X_t\}_{t\geq 0}$; its section $X_{\mathbf{T}_0}$ for $\mathbf{T}_0 = \{t_1, ..., t_d\}$ is also denoted by $(X_{t_1}, ..., X_{t_d})$.

Let S be a topological space, X be an E-valued process and $f \in M(E; S)$. The process $\{f \circ X_t\}_{t\geq 0}$ is exactly the mapping $\varpi(f) \circ X$ that sends every $\omega \in \Omega$ to the S-valued path $\varpi(f)[X(\omega)]$. A popular notation of this process is $f \circ X$. Herein, we prefer to treat processes as path-valued mappings and so we stick to the more precise notation $\varpi(f) \circ X$.

2.5.2 Càdlàg process

X is an E-**valued càdlàg process** if

$$\left\{\omega \in \Omega : X(\omega) \text{ is not a càdlàg member of } E^{\mathbf{R}^+}\right\} \in \mathscr{F} \cap \mathscr{N}(\mathbb{P}), \tag{2.5.4}$$

which is apparently an E-valued process. When E is a Tychonoff space, the **path space of** an E-valued càdlàg process refers to $D(\mathbf{R}^+; E)$. From (2.3.8) we immediately have that:

Fact 2.26. *Let E be a Tychonoff space. Then:*

(a) Every E-valued càdlàg process $(\Omega, \mathscr{F}, \mathbb{P}; X)$ satisfies that

$$\Omega \backslash X^{-1}\left[D(\mathbf{R}^+; E)\right] \in \mathscr{F} \cap \mathscr{N}(\mathbb{P}), \tag{2.5.5}$$

that

$$X^{-1}(A) \in \mathscr{F}, \ \forall A \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{D(\mathbf{R}^+;E)}, \tag{2.5.6}$$

and that

$$\mathrm{pd}(X)|_{D(\mathbf{R}^+;E)} \in \mathfrak{P}\left(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{D(\mathbf{R}^+;E)}\right). \tag{2.5.7}$$

(b) Every member of $M(\Omega, \mathscr{F}; D(\mathbf{R}^+; E))$[14] *is an E-valued càdlàg process defined on $(\Omega, \mathscr{F}, \mathbb{P})$.*

[14] $M(\Omega, \mathscr{F}; D(\mathbf{R}^+; E))$ denotes the family of all $D(\mathbf{R}^+; E)$-valued random variable defined on $(\Omega, \mathscr{F}, \mathbb{P})$.

Remark 2.27. As $\mathscr{B}[D(\mathbf{R}^+;E)]$ is generically larger than $\mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)}$, an E-valued càdlàg process is not necessarily a $D(\mathbf{R}^+;E)$-valued random variable. More details about càdlàg processes are presented in §A.7.

2.5.3 Relevant terminologies

The following are several common terminologies about stochastic processes. Let X and Y be two E-valued processes defined on $(\Omega, \mathscr{F}, \mathbb{P})$. The **set of fixed left-jump times of** X refers to

$$J(X) \circeq \left\{ t \in \mathbf{R}^+ : \mathbb{P}\left(\lim_{s \to t-} X_s = X_t \right) < 1 \right\} \tag{2.5.8}$$

if it is well-defined. X is a *stationary process* if

$$\mathbb{P} \circ X_{\mathbf{T}_0}^{-1} = \mathbb{P} \circ X_{\mathbf{T}_0 + c}^{-1}, \ \forall \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+), c \in (0, \infty), \tag{2.5.9}$$

where

$$\mathbf{T}_0 + c \circeq \{t + c : t \in \mathbf{T}_0\}. \tag{2.5.10}$$

X and Y are (*pathwisely*) *indistinguishable* if $\{X \neq Y\} \in \mathscr{N}(\mathbb{P}) \cap \mathscr{F}$. X and Y are *modifications of* each other if $\{X_t \neq Y_t\} \in \mathscr{N}(\mathbb{P}) \cap \mathscr{F}$ for all $t \in \mathbf{R}^+$.

A *filtration* (see [Dudley, 2002, p.453]) $\{\mathscr{G}_t\}_{t \geq 0}$ *on* $(\Omega, \mathscr{F}, \mathbb{P})$ is $\mathbb{P}$-*complete* if $\mathscr{G}_t$ is $\mathbb{P}$-complete for all $t \geq 0$. We call $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t \geq 0}, \mathbb{P})$ a *stochastic basis* if both $\mathscr{F}$ and $\{\mathscr{G}_t\}_{t \geq 0}$ are $\mathbb{P}$-complete. X is $\mathscr{G}_t$-*adapted* if $\mathscr{F}_t^X \subset \mathscr{G}_t$ for all $t \geq 0$, where

$$\mathscr{F}_t^X \circeq \sigma\left[\sigma\left(\{X_u : u \in [0,t]\}\right) \cup \mathscr{N}(\mathbb{P})\right], \ \forall t \geq 0. \tag{2.5.11}$$

$\mathscr{F}^X \circeq \{\mathscr{F}_t^X\}_{t \geq 0}$ is called the *augmented natural filtration of* X. Let $\xi(t,\omega) \circeq X_t(\omega)$ for each $\omega \in \Omega$ and $t \in \mathbf{R}^+$. Then, X is a *measurable process* if

$$\xi \in M\left(\mathbf{R}^+ \times \Omega, \mathscr{B}(\mathbf{R}^+) \otimes \mathscr{F}; E, \mathscr{B}(E)\right). \tag{2.5.12}$$

X is a $\mathscr{G}_t$-*progressive process* if

$$\xi|_{[0,t] \times \Omega} \in M\left([0,t] \times \Omega, \mathscr{B}([0,t]) \otimes \mathscr{G}_t; E, \mathscr{B}(E)\right), \ \forall t \in \mathbf{R}^+. \tag{2.5.13}$$

X is a **progressive process** if it is $\mathscr{F}_t^X$-progressive.

2.6 Conventions

In view of simplicity and brevity, the remainder of the paper **will always stick to** the following conventions **if not otherwise specified**:

- $\mathbf{I}$ is a non-empty index set.
- Subsets are non-empty.
- Measures are non-trivial.
- Subsets of topological spaces are equipped with their subspace topologies.
- Topological spaces are equipped with their Borel σ-algebras.
- Any Cartesian product of topological spaces is equipped with the product topology and, hence, is equipped with the Borel σ-algebra generated by the product topology.
- Linear spaces are over the scalar field $\mathbf{R}$.
- Linear operators are single-valued.
- $(\Omega, \mathscr{F}, \mathbb{P})$, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)\}_{n\in\mathbf{N}_0}$ and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i)\}_{i\in\mathbf{I}}$ denote complete probability spaces with expectation operators $\mathbb{E}$, $\{\mathbb{E}^n\}_{n\in\mathbf{N}_0}$ and $\{\mathbb{E}^i\}_{i\in\mathbf{I}}$, respectively.

2.7 Motivating examples

We mentioned in Introduction many inspiring settings for replication. Herein, we outline seven representative examples to further motivate the readers. In §2.7.1 - §2.7.6, we give a brief review of the pseudo-path topology of càdlàg functions, the $\mathcal{S}$-topology of càdlàg functions, the strong topology of Borel probability measures, strong dual of nuclear Frechét space, Banach spaces of finite p-variation or $1/p$-Hölder continuous paths, and Banach spaces of rough paths. These spaces forfeit the traditional assumptions of metric *completeness* (see [Munkres, 2000, §43, Definition, p.264]), compactness, separability and/or metrizability. Moreover, we review in §2.7.7 a version of Kolmogorov's Extension Theorem for standard Borel spaces, which is a good illustration of boosting results by space change.

2.7.1 Pseudo-path topology

In Meyer and Zheng [1984], the pseudo-path topology (also known as "Meyer-Zheng topology") was used to characterize tightness of càdlàg semimartingales with respect to the topology of convergence in measure. This is an example of a non-Polish metrizable Lusin space.

Example 2.28. Let $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ denote the space of all càdlàg members of $\mathbf{R}^{\mathbf{R}^+}$ equipped with the pseudo-path topology[15]. This topology is induced by the mapping ψ^{pp} associating each $x\in D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ to its λ'-almost everywhere unique *pseudo-path* $\psi^{\mathrm{pp}}(x)\in\mathcal{P}(K)$, where λ is the Lebesgue measure on $\mathbf{R}^+$, $K\mathrel{\mathring{=}}[0,\infty]\times[-\infty,\infty]$,

$$\lambda'(A)\mathrel{\mathring{=}}\int_A e^{-t}\lambda(dt),\ \forall A\in\mathscr{B}(\mathbf{R}^+),\tag{2.7.1}$$

and

$$\psi^{\mathrm{pp}}(x)(B)\mathrel{\mathring{=}}\lambda'\left(\left\{t\in\mathbf{R}^+:(t,x(t))\in B\right\}\right),\ \forall B\in\mathscr{B}(K).\tag{2.7.2}$$

[Meyer and Zheng, 1984, Theorem 2] showed that

$$\psi^{\mathrm{pp}}\in\mathbf{imb}\left(D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R});\mathcal{P}(K)\right)\tag{2.7.3}$$

and

$$\psi^{\mathrm{pp}}\left(D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})\right)\in\mathscr{B}\left(\mathcal{P}(K)\right).\tag{2.7.4}$$

K is a Polish space by Proposition A.12 (d). $\mathcal{P}(K)$ is a Polish space by Theorem A.44 (b). Hence, $\psi^{\mathrm{pp}}(D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R}))$ is a metrizable Lusin space by Proposition A.49 (a, d), so is its homeomorph $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$. However, [Meyer and

[15] Pseudo-path topology can be defined similarly on the family of all càdlàg members of $E^{\mathbf{R}^+}$ when E is a Polish space. In that case, K will be a metrizable compactification of $\mathbf{R}^+\times E$.

Zheng, 1984, p.355 - 356] pointed out that $\psi^{\mathrm{pp}}(D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R}))$ and $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ are not Polish spaces.

2.7.2 $\mathcal{S}$-topology

Jakubowski [1997b] defined the $\mathcal{S}$-topology by introducing the $\mathcal{S}$-convergence of càdlàg functions from $[0,T] \subset \mathbf{R}^+$ to $\mathbf{R}$. This sequential topology is related to the pseodo-path topology as the tightness conditions proposed by Stricker [1985] for the pseudo-path topology turns out to be superfluous (see Kurtz [1991]) but serves precisely for the $\mathcal{S}$-topology (see Jakubowski [1997b] and Jakubowski [2012]).

Example 2.29. We define the total variation of $x \in \mathbf{R}^{[0,T]}$ by

$$\|x\|_{1\text{-var},[0,T]} \mathrel{\hat{=}} |x(0)| + \sup_{0\le t_0<\ldots<t_n\le T, n\in\mathbf{N}} \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|, \tag{2.7.5}$$

and put

$$\mathbf{V} \mathrel{\hat{=}} \left\{x \in \mathbf{R}^{[0,T]} : x \text{ is càdlàg}, \|x\|_{1\text{-var},[0,T]} < \infty\right\}. \tag{2.7.6}$$

Càdlàg functions $\{x_n\}_{n\in\mathbf{N}} \subset \mathbf{R}^{[0,T]}$ $\mathcal{S}$-converge to càdlàg function $x_0 \in \mathbf{R}^{[0,T]}$ if for any $\epsilon \in (0,\infty)$, there exist $\{v_n^\epsilon\}_{n\in\mathbf{N}_0} \in \mathbf{V}$ such that

$$\sup_{n\in\mathbf{N}_0} \|x_n - v_n^\epsilon\|_{1\text{-var},[0,T]} < \epsilon \tag{2.7.7}$$

and

$$\lim_{n\to\infty} \int_{[0,T]} f(t)dv_n(t) = \int_{[0,T]} f(t)dv_0(t),\ \forall f \in C([0,T];\mathbf{R}). \tag{2.7.8}$$

Considering $\mathcal{S}$-convergence as convergence a priori, the induced sequential topology is the $\mathcal{S}$-topology on the space of all càdlàg members of $\mathbf{R}^{[0,T]}$. This $\mathcal{S}$-topological space is a topological coarsening of the Skorokhod $\mathscr{J}_1$-space $D([0,T];\mathbf{R})$ (see [Jakubowski, 2012, p.5]). However, it is neither necessarily a Tychonoff space, nor is it known to be a topological vector space (see Jakubowski [1997b] and [Jakubowski, 2012, p.5]). $\mathcal{S}*$-convergence, the convergence a posteriori induced by $\mathcal{S}$-convergence is different from $\mathcal{S}$-convergence (see [Jakubowski, 2012, p.3 - 4]). Moreover, $\mathcal{S}^*$-convergence turns out to be the topological convergence under some coarserning of the $\mathcal{S}$-topology, but their equality remains an open question (see [Jakubowski, 2012, p.4]).

2.7.3 Strong topology of Borel probability measures

The strong topological space of all Borel probability measures on a Polish space E was used in Dembo and Zeitouni [1998] for large deviation theory. This is an example of a non-metrizable and non-*separable* (see p.218) Tychonoff space.

Example 2.30. Let E be a Polish space and $\mathcal{P}_S$ be the space of all Borel probability measures on E equipped with the *strong topology*

$$\mathscr{O}\left[\mathcal{P}_S(E)\right] \circeq \mathscr{O}_{M_b(E;\mathbf{R})^*}\left[\mathcal{P}(E)\right]. \tag{2.7.9}$$

Then, $\mathcal{P}_S(E)$ is a topological refinement of $\mathcal{P}(E)$,

$$M_b(E;\mathbf{R})^* \subset C_b\left(\mathcal{P}_S(E);\mathbf{R}\right), \tag{2.7.10}$$

and $M_b(E;\mathbf{R})^*$ strongly separates points on $\mathcal{P}_S(E)$. Furthermore, from the fact

$$\{(\mathbf{1}_A)^* : A \in \mathscr{B}(E)\} \subset M_b(E;\mathbf{R})^* \tag{2.7.11}$$

it follows that $M_b(E;\mathbf{R})^*$ separates points $\mathcal{P}_S(E)$. Hence, $\mathcal{P}_S(E)$ is a Tychonoff space by Proposition A.25 (a, b). However, [Dembo and Zeitouni, 1998, p.263] argued that $\mathcal{P}_S(E)$ is neither metrizable nor separable.

2.7.4 Strong dual of nuclear Frechét space

[Jakubowski, 1986, §5.II] discussed tightness of probability measures on the Skorokhod $\mathscr{J}_1$-space $D([0,1];E)$ with E being the strong dual of a general nuclear Frechét space. This is an example of a possibly non-metrizable, Tychonoff topological vector space.

Example 2.31. Let E be the strong dual of some infinite-dimensional, unnormable, nuclear Frechét space. E is not metrizable by [Garling and Köthe, 2012, §29.1, (7), p.394], nor is $D([0,1];E)$ by [Jakubowski, 1986, Proposition

1.6 iii)]. According to [Jakubowski, 1986, Proposition 1.6 ii)], neither E nor $D([0,1];E)$ is necessarily separable. However, E is a nuclear space by [Schaefer and Wolff, 1999, §IV.9.6, Theorem, p.172] and the topology of E is induced by countably many *Hilbertian semi-norms* (see [Speed and Hida, 2012, Definition A.4]). Hence, E is a Tychonoff space by [Kallianpur and Xiong, 1995, Theorem 2.1.1], so is $D([0,1];E)$ by Proposition A.62 (e).

2.7.5 Spaces of finite-variation or Hölder continuous functions

The spaces of $\mathbf{R}^d$-valued continuous functions with finite p-variation or $\mathbf{R}^d$-valued $1/p$-Hölder continuuous functions are frequently used in stochastic differential equations driven by non-classical noises. They are examples of non-separable Banach spaces.

Example 2.32. Let $d,N\in\mathbf{N}$, $p\in[1,\infty)$ and $T\in(0,\infty)$. A path $x\in(\mathbf{R}^d)^{[0,T]}$ has finite p-variation or is $1/p$-Hölder continuous if the homogeneous p-variation norm of x defined by

$$\|x\|_{p\text{-var},[0,T]} \mathrel{\mathring{=}} |x(0)| + \sup_{0\le t_0<\ldots<t_n\le T, n\in\mathbf{N}} \left(\sum_{i=1}^{n} |x(t_i)-x(t_{i-1})|^p\right)^{1/p} \tag{2.7.12}$$

or the homogeneous $1/p$-Hölder norm of x defined by

$$\|x\|_{\frac{1}{p}-\text{Höl},[0,T]} \mathrel{\mathring{=}} \sup_{0\le s<t\le T} \frac{|x(t)-x(s)|}{|t-s|^{\frac{1}{p}}} \tag{2.7.13}$$

is finite respectively. The normed spaces

$$\left\{x\in C\left([0,T];\mathbf{R}^d\right) : \|x\|_{p-\text{var},[0,T]}<\infty\right\} \tag{2.7.14}$$

and

$$\left\{x\in\left(\mathbf{R}^d\right)^{[0,T]} : \|x\|_{\frac{1}{p}-\text{Höl},[0,T]}<\infty\right\} \tag{2.7.15}$$

are non-separable Banach spaces (see [Friz and Victoir, 2010, Theorem 5.25]).

2.7.6 Space of rough paths

The approach of rough path, initiated by the pioneering works of Lyons [1994] and Lyons [1998], is important to generalizing stochastic differential equations like

$$dY_t = \alpha(t, Y_t)dt + \sigma(t, Y_t)dX_t \tag{2.7.16}$$

to the case where the driven noise X is not necessarily a semimartingale. By this approach, the original driven noise X is *enhanced to a random rough path* $\mathfrak{X}$ (see [Friz and Victoir, 2010, §9.1]) and the Stratonovich solution of (2.7.16) is closely linked to the solution of

$$dY_t = \alpha(t, Y_t)dt + \sigma(t, Y_t)d\mathfrak{X}_t, \tag{2.7.17}$$

where (2.7.17) is considered as *rough differential equations driven by the realization paths of* $\mathfrak{X}$ *as a process* (see [Friz and Victoir, 2010, §10.3, §10.3, §17.1 and §17.2]). Friz and Victoir [2010] and Friz and Hairer [2014] considered the following spaces for the paths of $\mathfrak{X}$, which are also examples of non-separable Banach spaces.

Example 2.33. Let $d, N \in \mathbf{N}$, $p \in [1, \infty)$ and $T \in (0, \infty)$. A rough path is often considered as a mapping from $[0, T]$ to $G^N(\mathbf{R}^d)$, the *free nilpotent group of Step* N *over* $\mathbf{R}^d$ (see [Friz and Victoir, 2010, p.142-143]). As a Lie group, $G^N(\mathbf{R}^d)$ is equipped with the usual addition "+" of functions and the *Carnot-Caratheodory norm* $\|\cdot\|_{\text{cc}}$ (see [Friz and Victoir, 2010, Theorem 7.32]). Similar to $\mathbf{R}^d$-valued paths, a path $x \in G^N(\mathbf{R}^d)^{[0,T]}$ has finite p-variation or is $1/p$-Hölder continuous if the homogeneous p-variation cc-norm of x defined by

$$\|x\|_{\text{cc},p-\text{var},[0,T]} \mathrel{\mathring{=}} \sup_{0 \le t_0 < \ldots < t_n \le T, n \in \mathbf{N}} \left(\sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|_{\text{cc}}^p \right)^{1/p} \tag{2.7.18}$$

or the homogeneous $1/p$-Hölder cc-norm of x defined by

$$\|x\|_{\text{cc},\frac{1}{p}-\text{Höl},[0,T]} \mathrel{\mathring{=}} \sup_{0 \le s < t \le T} \frac{\|x(t) - x(s)\|_{\text{cc}}}{|t - s|^{\frac{1}{p}}} \tag{2.7.19}$$

is finite respectively. The random rough path $\mathfrak{X}$ in (2.7.17) may have paths in

$$\left\{x \in C\left([0,T]; G^N\left(\mathbf{R}^d\right)\right) : \|x\|_{\mathrm{cc},p-\mathrm{var},[0,T]} < \infty\right\} \tag{2.7.20}$$

or

$$\left\{x \in G^N\left(\mathbf{R}^d\right)^{[0,T]} : \|x\|_{\mathrm{cc},\frac{1}{p}-\mathrm{H\ddot{o}l},[0,T]} < \infty\right\}. \tag{2.7.21}$$

By [Friz and Victoir, 2010, Theorem 8.13], these normed spaces are non-separable Banach spaces.

2.7.7 Kolmogorov's Extension Theorem

The Kolmogorov's Extension Theorem is recognized as one of the cornerstones of modern probability theory and has been reproduced by many researchers in various settings. The *Kolmogorov's extension* (see [Aliprantis and Border, 2006, §15.6]) in its nature is a non-topological concept and depends purely on the relevant σ-algebras. Hence, existence of Kolmogorov's extension should be transferrable from a "nice" topological space to any "defective" topological space which are "indifferent" as measurable spaces.

A typical pair of "measurably indifferent" topological spaces is Polish and standard Borel spaces. Let $\{S_i\}_{i\in\mathbf{I}}$ be a family of standard Borel spaces,

$$(S,\mathscr{A}) \circeq \left(\prod_{i\in\mathbf{I}} S_i, \bigotimes_{i\in\mathbf{I}} \mathscr{B}(S_i)\right), \tag{2.7.22}$$

and

$$(S_{\mathbf{I}_0},\mathscr{A}_{\mathbf{I}_0}) \circeq \left(\prod_{i\in\mathbf{I}_0} S_i, \bigotimes_{i\in\mathbf{I}_0} \mathscr{B}(S_i)\right), \ \forall \mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I}). \tag{2.7.23}$$

For each $i \in \mathbf{I}$, Proposition A.52 (a, d) allows us to change the topology of S_i to a possibly different one $\mathscr{U}_i$ such that $(S_i,\mathscr{U}_i)$ is a Polish space and the Borel sets $\mathscr{B}(S_i) = \mathscr{B}(S_i,\mathscr{U}_i)$ remain unchanged. So, any $\mu_{\mathbf{I}_0} \in \mathfrak{P}(S_{\mathbf{I}_0},\mathscr{A}_{\mathbf{I}_0})$ can be viewed as a probability measure on $(S_{\mathbf{I}_0}, \bigotimes_{i\in\mathbf{I}_0}\mathscr{B}(S_i,\mathscr{U}_i))$ for each $\mathbf{I}_0 \subset \mathscr{P}_0(\mathbf{I})$, and any Kolmogorov's extension of $\{\mu_{\mathbf{I}_0}\}_{\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})}$ on $(S, \bigotimes_{i\in\mathbf{I}}\mathscr{B}(S_i,\mathscr{U}_i))$ would be a desired Kolmorogov's extension of them on $(S,\mathscr{A})$. Therefore, the well-known version of Kolmogorov's Extension Theorem for Polish spaces extends immediately to the standard Borel case.

Theorem 2.34 (**Kolmogorov's Extension Theorem**, [Kallenberg, 1997, Theorem 5.16])**.** *Let $\{S_i\}_{i\in\mathbf{I}}$ be standard Borel spaces, $(S,\mathscr{A})$ be as in (2.7.22), $\{(S_{\mathbf{I}_0},\mathscr{A}_{\mathbf{I}_0})\}_{\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})}$ be as in (2.7.23) and $\mu_{\mathbf{I}_0}\in\mathfrak{P}(S_{\mathbf{I}_0},\mathscr{A}_{\mathbf{I}_0})$ for each $\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})$. Suppose in addition that for each $\mathbf{I}_1,\mathbf{I}_2\in\mathscr{P}_0(\mathbf{I})$ with $\mathbf{I}_1\subset\mathbf{I}_2$, $\mu_{\mathbf{I}_1}$ is the push-forward measure of $\mu_{\mathbf{I}_2}$ by the projection from $S_{\mathbf{I}_2}$ to $S_{\mathbf{I}_1}$. Then, there exists a $\mu\in\mathfrak{P}(S,\mathscr{A})$ such that for each $\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})$, $\mu_{\mathbf{I}_0}$ is the push-forward measure of μ by the projection from S to $S_{\mathbf{I}_0}$.*

Chapter 3

Space Change in Replication

An ideal case where a topological space E needs no enhancement is when E is compact and metrizable. Otherwise, a problem on E might be greatly simplified if it is translated onto such a "perfect" space. Replication is a convenient scheme of space change and object transformation for this purpose.

The current chapter discusses the space change aspect of replication. §3.1 introduces the notion of *base* as our core platform to implement space change and other goals of replication. §3.2 and §3.3 discuss the existence and various properties of *baseable spaces* or *baseable subsets* with which one can construct the desired bases.

3.1 Base

The goal of space change in replication is to create a compact metric space $\widehat{E}$ related to the original space E. As illustrated by the following figure, the most natural way is to establish a metrizable *compactification* (see p.233) of E itself or, more generally, a Borel subset E_0 of E.

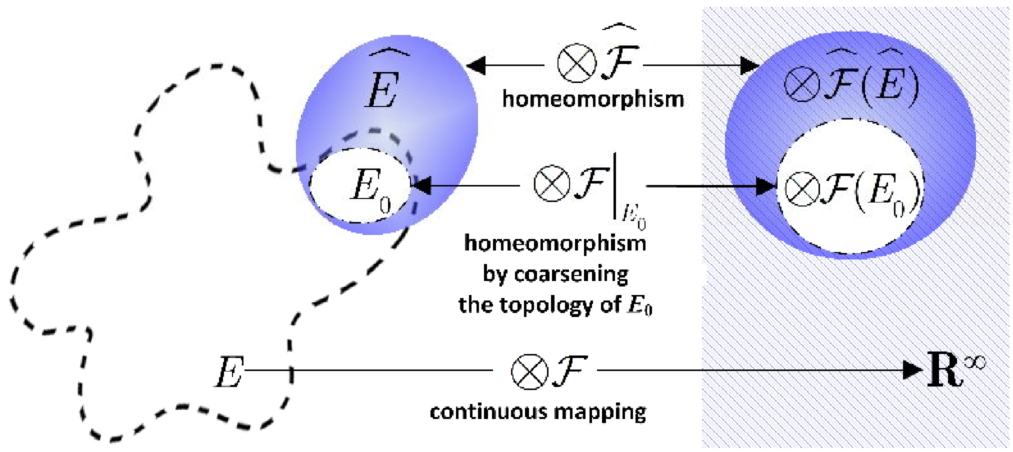

Figure 2: *Space change in replication*

3.1.1 Definition

Base, a foundational notion of replication concretizes the space change idea mentioned above.

Definition 3.1. Let E be a topological space. The quadruple $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is a **replication base over** E (a base over E or a base for short) if:

- E_0 is a non-empty Borel subset of E.
- $\mathcal{F} \subset C_b(E; \mathbf{R})$ is countable and contains the constant function 1.
- $\widehat{E}$ is a topological space containing E_0.
- $\widehat{\mathcal{F}} \subset \mathbf{R}^{\widehat{E}}$ is a countable collection, separates points on $\widehat{E}$ and satisfies[1]

$$\bigotimes \mathcal{F}|_{E_0} = \bigotimes \widehat{\mathcal{F}}|_{E_0} \tag{3.1.1}$$

and

$$\mathscr{O}(\widehat{E}) = \mathscr{O}_{\widehat{\mathcal{F}}}(\widehat{E}). \tag{3.1.2}$$

[1]The notations "$\mathcal{F}|_{E_0}$" and "$\bigotimes \mathcal{F}$" were defined in §2.1.2. "$\mathscr{O}(\cdot)$" denotes the family of all open subsets.

- $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$ is the closure of $\bigotimes \mathcal{F}(E_0)$ in $\mathbf{R}^\infty$.

Remark 3.2. In general, E need not be a subset of $\widehat{E}$.

The following lemma justifies that a base indeed brings about the compactification in Figure 2.

Lemma 3.3. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $A \subset E_0$. Then, the following statements are true:*

(a) $\widehat{\mathcal{F}} \subset C(\widehat{E}; \mathbf{R})$ is countable, contains the constant function 1 and strongly separates points on $\widehat{E}$. In particular,

$$\bigotimes \widehat{\mathcal{F}} \in \mathbf{imb}\left(\widehat{E}; \mathbf{R}^\infty\right). \tag{3.1.3}$$

(b) $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$ is a compactification of $\bigotimes \mathcal{F}(E_0)$ and $\widehat{E}$ is a compactification of $(E_0, \mathscr{O}_{\mathcal{F}}(E_0))$.

(c) $\widehat{E}$ is a Polish space and is completely metrized by $\rho_{\widehat{\mathcal{F}}}$[2].

(d) $\bigotimes \mathcal{F}|_A \in \mathbf{imb}(A, \mathscr{O}_{\widehat{E}}(A); \mathbf{R}^\infty)$. Moreover, $(A, \mathscr{O}_{\widehat{E}}(A))$ is a metrizable and separable topological coarsening of $(A, \mathscr{O}_E(A))$.

(e) $\bigotimes \mathcal{F} \in C(E; \mathbf{R}^\infty)$ is injective on A. Moreover, $\mathcal{F}$ separates points on the Hausdorff space $(A, \mathscr{O}_E(A))$.

Proof. (a) The members of $\widehat{\mathcal{F}}$ are continuous and $\widehat{\mathcal{F}}$ strongly separates points on $\widehat{E}$ by (3.1.2). $\widehat{\mathcal{F}}$ is countable and contains 1 by (3.1.1) and $1 \in \mathcal{F}$. Moreover, (3.1.3) follows by Lemma A.28 (a, c) (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$).

(b) $\mathbf{R}^\infty$ is a Polish space by Proposition A.11 (f). It follows by the Tychonoff Theorem (Proposition A.12 (b)) and Proposition A.12 (a) that

$$K_{\mathcal{F}} \mathrel{\mathring{=}} \prod_{f \in \mathcal{F}} [-\|f\|_\infty, \|f\|_\infty] \in \mathscr{K}(\mathbf{R}^\infty) \subset \mathscr{C}(\mathbf{R}^\infty). \tag{3.1.4}$$

$\bigotimes \widehat{\mathcal{F}}(\widehat{E})$ by definition is the closure of $\bigotimes \mathcal{F}(E_0)$ in $\mathbf{R}^\infty$. So,

$$\bigotimes \widehat{\mathcal{F}}(\widehat{E}) \in \mathscr{C}\left(K_{\mathcal{F}}, \mathscr{O}_{\mathbf{R}^\infty}(K_{\mathcal{F}})\right) \subset \mathscr{K}(\mathbf{R}^\infty) \subset \mathscr{C}(\mathbf{R}^\infty) \tag{3.1.5}$$

[2] $\rho_{\widehat{\mathcal{F}}}$ is defined by (2.2.22) with $\mathcal{D} = \widehat{\mathcal{F}}$.

and $\widehat{E}$ is compact by (3.1.4), (3.1.3) and Proposition A.12 (a, e).

Moreover, it follows by (a) and (3.1.1) that

$$\mathscr{O}_{\widehat{E}}(E_0) = \mathscr{O}_{\widehat{\mathcal{F}}}(E_0) = \mathscr{O}_{\mathcal{F}}(E_0). \tag{3.1.6}$$

$\bigotimes \mathcal{F}(E_0)$ by definition is dense in $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$, so E_0 is a dense subset of $\widehat{E}$ by (3.1.3). $\widehat{E}$ is a Hausdorff space by (3.1.2) and Proposition A.17 (c) (with $E = A = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$). Hence, $\widehat{E}$ is a compactification of $(E_0, \mathscr{O}_{\mathcal{F}}(E_0))$.

(c) $\rho_{\widehat{\mathcal{F}}}$ metrizes $\widehat{E}$ by (a) and Proposition A.17 (d) (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$). $\bigotimes \widehat{\mathcal{F}}$ is an *isometry* (see p.219) between $(\widehat{E}, \rho_{\widehat{\mathcal{F}}})$ and $(\bigotimes \widehat{\mathcal{F}}(\widehat{E}), d^\infty)$, where the metric[3]

$$d^\infty(x, y) \doteq \sum_{n=1}^{\infty} 2^{-n} \left(|\mathfrak{p}_n(x) - \mathfrak{p}_n(y)| \wedge 1\right), \ \forall x, y \in \mathbf{R}^\infty \tag{3.1.7}$$

completely metrizes $\mathbf{R}^\infty$ by Proposition A.7 (b) (with $(S_i, \mathfrak{r}_i) = \mathbf{R}$). $(\bigotimes \widehat{\mathcal{F}}(\widehat{E}), d^\infty)$ is complete by its compactness and Proposition A.12 (c). Thus, $(\widehat{E}, \rho_{\widehat{\mathcal{F}}})$ is a complete metric space by Proposition A.5 (a).

(d) The first statement of (d) follows by (3.1.3) and (3.1.1). $(A, \mathscr{O}_{\widehat{E}}(A))$ is metrizable and separable by (c) and Proposition A.11 (c). Moreover, one finds by (3.1.6) and $\mathcal{F} \subset C(E; \mathbf{R})$ that

$$\mathscr{O}_{\widehat{E}}(A) = \mathscr{O}_{\mathcal{F}}(A) \subset \mathscr{O}_E(A). \tag{3.1.8}$$

(e) The first statement of (e) follows by (d), $\mathcal{F} \subset C(E; \mathbf{R})$ and Fact 2.4 (b). The second part follows by Proposition A.17 (e) (with $\mathcal{D} = \mathcal{F}$). □

Corollary 3.4. *Let E be a topological space. If $\{(E_0, \mathcal{F}; \widehat{E}_i, \widehat{\mathcal{F}}_i)\}_{i=1,2}$ are bases over E, then $\widehat{E}_1$ and $\widehat{E}_2$ are isometric hence homeomorphic.*

Proof. Let $\mathcal{F} = \{f_n\}_{n \in \mathbf{N}}$. By (3.1.1) and Lemma 3.3 (a) (with $\widehat{E} = \widehat{E}_i$ and $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_i$), $\widehat{\mathcal{F}}_i \subset C(\widehat{E}_i; \mathbf{R})$ can be written as $\widehat{\mathcal{F}}_i = \{\widehat{f}_n^i\}_{n \in \mathbf{N}}$ for each $i \in \{1, 2\}$ such that $\widehat{f}_n^1|_{E_0} = f_n|_{E_0} = \widehat{f}_n^2|_{E_0}$ for all $n \in \mathbf{N}$. Then, $\widehat{f}_n^1 = \widehat{f}_n^2$ for all $n \in \mathbf{N}$ by the denseness of E_0 in $\widehat{E}$ and their continuities. Now, the corollary follows by Lemma 3.3 (c) (with $\widehat{E} = \widehat{E}_i$ and $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_i$) and Proposition A.5 (a) (with $E = \widehat{E}_1$ and $S = \widehat{E}_2$). □

[3] $\mathfrak{p}_n$ denotes the one-dimensional projection on $\mathbf{R}^\infty$ for $n \in \mathbf{N}$.

Remark 3.5. Compactification determined by *extending*[4] continuous functions was adopted by e.g. Ethier and Kurtz [1986], Bhatt and Karandikar [1993b], Blount and Kouritzin [2010] and Kouritzin [2016] to imbed stochastic processes into compact metric spaces. An ultimate case of such compactification is the well-known *Stone-Čech compactification* (see p.233) which exists for any compactifiable, or equivalently, Tychonoff space. While the Stone-Čech compactification is determined by the continuous extension of all bounded continuous function, the compactification $\widehat{E}$ of $(E_0, \mathscr{O}_{\mathcal{F}}(E_0))$ can be thought of as a "possibly smaller" compactification which might not extend all of $C_b(E_0, \mathscr{O}_{\mathcal{F}}(E_0); \mathbf{R})$.

Remark 3.6. As a cost of metrizability, $\widehat{E}$ *compactifies* (see p.233) E_0 with respect to a possibly *coarser* topology than its natural subspace topology induced from E. In fact, neither the original space E nor the subspace $(E_0, \mathscr{O}_E(E_0))$ is necessarily a Tychonoff space, hence need not have compactification.

Remark 3.7. *One-point compactifications* (see p.233) exist for locally compact Hausdorff spaces (see Proposition A.31). We do not presume E_0 to be a locally compact subspace of $\widehat{E}$, so $\widehat{E}$ is not necessarily a one-point compactification. Nonetheless, even Stone-Čech compactifications are sometimes one-point compactifcations. Corollary 4.8 to follow illustrates when the compactification establishing a base is of one-point compactification type.

The following theorem justifies the converse of Lemma 3.3 (e) and answers the question *when a base exists.*

Theorem 3.8. *Let E be a topological space, $E_0 \in \mathscr{B}(E)$ and $\mathcal{F} \ni 1$ be a countable subset of $C_b(E; \mathbf{R})$. Then, there exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E if and only if $\mathcal{F}$ separates points on E_0.*

Proof. Necessity follows by Lemma 3.3 (e) and we prove sufficiency. There exist a compactification $\widehat{E}$ of $(E_0, \mathscr{O}_{\mathcal{F}}(E_0))$ and an extension $\varphi \in \mathbf{imb}(\widehat{E}; \mathbf{R}^\infty)$ of $\bigotimes \mathcal{F}|_{E_0}$ by Lemma A.28 (a, b) (with $E = (E_0, \mathscr{O}_{\mathcal{F}}(E_0))$ and $\mathcal{D} = \mathcal{F}|_{E_0}$). Then, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is base over E with $\widehat{\mathcal{F}} \circeq \{\mathfrak{p}_n \circ \varphi\}_{n \in \mathbf{N}}$. □

[4]Recall that mapping g is a continuous extension of mapping f if g is continuous, the domain of g contains that of f and $g = f$ restricted to f's domain.

3.1.2 Properties

The following four results specify the basic finite-dimensional properties of bases.

Lemma 3.9. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$ and $A \subset E_0^d$. Then, the following statements are true:*

(a) $(E_0^d, \Pi^d(\mathcal{F}); \widehat{E}^d, \Pi^d(\widehat{\mathcal{F}}))$[5] *is a base over E^d.*

(b) $\Pi^d(\widehat{\mathcal{F}}) \subset C(\widehat{E}^d; \mathbf{R})$ *contains the constant function 1, separates points and strongly separates points on $\widehat{E}^d$. In particular,*

$$\bigotimes \Pi^d(\widehat{\mathcal{F}}) \in \mathbf{imb}\left(\widehat{E}^d; \mathbf{R}^\infty\right). \tag{3.1.9}$$

(c) $\widehat{E}^d$ *is a compactification of $(E_0^d, \mathscr{O}_{\mathcal{F}}(E_0)^d)$. Moreover, $\widehat{E}^d$ is a Polish space and is completely metrized by $\rho^d_{\widehat{\mathcal{F}}}$*[6]*.*

(d) $\bigotimes \mathcal{F}|_A \in \mathbf{imb}(A, \mathscr{O}_{\widehat{E}^d}(A); \mathbf{R}^\infty)$*. Moreover, $(A, \mathscr{O}_{\widehat{E}^d}(A))$ is a metrizable and separable coarsening of $(A, \mathscr{O}_{E^d}(A))$.*

(e) $\Pi^d(\mathcal{F} \backslash \{1\})$ *separates points on the Hausdorff space $(A, \mathscr{O}_{E^d}(A))$.*

Proof. (a) We verify the four properties of Definition 3.1 in four steps:

Step 1. We have by $E_0 \in \mathscr{B}(E)$ and Fact 2.3 (a) that

$$E_0^d = \bigcap_{i=1}^d \mathfrak{p}_i^{-1}(E_0) \in \mathscr{B}(E)^{\otimes d}, \ \forall d \in \mathbf{N}. \tag{3.1.10}$$

Step 2. We have by $1 \in \mathcal{F} \subset C_b(E; \mathbf{R})$, Fact B.16 (a) (with $\mathcal{D} = \mathcal{F}$) and Proposition A.21 (a) (with $\mathcal{D} = \mathcal{F}$) that $\Pi^d(\mathcal{F})$ is a countable collection and

$$1 \in \mathfrak{ca}\left(\Pi^d(\mathcal{F})\right) \subset C_b\left(E^d, \mathscr{O}_{\mathcal{F}}(E)^d; \mathbf{R}\right) \subset C_b(E^d; \mathbf{R}). \tag{3.1.11}$$

Step 3. We have by (3.1.1) that

$$\bigotimes \Pi^d(\mathcal{F})\Big|_{E_0^d} = \bigotimes \Pi^d(\widehat{\mathcal{F}})\Big|_{E_0^d}. \tag{3.1.12}$$

[5] The notation "$\Pi^d(\cdot)$" was defined in §2.2.3.

[6] $\rho^d_{\widehat{\mathcal{F}}}$ is defined by (2.2.23) with $\mathcal{D} = \widehat{\mathcal{F}}$.

We have by (3.1.2) and Proposition A.21 (a) (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$) that

$$\Pi^d(\widehat{\mathcal{F}}) \subset C(\widehat{E}^d; \mathbf{R}). \tag{3.1.13}$$

Then, $\Pi^d(\widehat{\mathcal{F}})$ separates points on $\widehat{E}^d$, strongly separates points on $\widehat{E}^d$ and satisfies[7]

$$\mathscr{O}_{\Pi^d(\widehat{\mathcal{F}})} = \mathscr{O}(\widehat{E}^d) = \mathscr{O}_{\widehat{\mathcal{F}}}(\widehat{E})^d \tag{3.1.14}$$

by Lemma 3.3 (a), Proposition A.21 (b) (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$), (3.1.13) and Proposition A.17 (b, e) (with $E = A = \widehat{E}^d$ and $\mathcal{D} = \Pi^d(\widehat{\mathcal{F}})$).

Step 4. E_0^d is dense in $\widehat{E}^d$ by the denseness of E_0 in $\widehat{E}$ and the definition of product topology. $\widehat{E}^d$ is a compact space by Lemma 3.3 (b) and the Tychonoff Theorem (see Proposition A.12 (b)). $\bigotimes \Pi^d(\widehat{\mathcal{F}}) \in C(\widehat{E}^d; \mathbf{R}^\infty)$ by (3.1.13) and Fact 2.4 (b).

$$\bigotimes \Pi^d(\widehat{\mathcal{F}})(\widehat{E}^d) \in \mathscr{K}(\mathbf{R}^\infty) \subset \mathscr{C}(\mathbf{R}^\infty) \tag{3.1.15}$$

by Proposition A.12 (a, e). So, $\bigotimes \Pi^d(\widehat{\mathcal{F}})(\widehat{E}^d)$ is the closure of $\bigotimes \Pi^d(\mathcal{F})(E_0^d)$ in $\mathbf{R}^\infty$ by (3.1.12) and [Munkres, 2000, Theorem 18.1 (a, b)].

(b) follows by (a) and Lemma 3.3 (a).

(c) The first statement follows by (a) and Lemma 3.3 (b). $\widehat{E}^d$ is a Polish space by (a) and Lemma 3.3 (c). Moreover, $\rho^d_{\widehat{\mathcal{F}}}$ metrizes $\widehat{E}^d$ by Lemma 3.3 (c) and Proposition A.7 (a) (with $\mathbf{I} = \{1, ..., d\}$ and $(E_i, \mathfrak{r}_i) = (\widehat{E}, \rho_{\widehat{\mathcal{F}}})$).

(d) follows by (a) and Lemma 3.3 (d).

(e) $\Pi^d(\mathcal{F}\backslash\{1\})$ separates points on E_0^d by Lemma 3.3 (e) and Proposition A.21 (b) (with $\mathcal{D} = \mathcal{F}\backslash\{1\}$). The rest of (e) follows by (a) and Lemma 3.3 (e). □

Corollary 3.10. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d \in \mathbf{N}$. Then*[8]*,*

$$C_c(\widehat{E}^d; \mathbf{R}) = C_0(\widehat{E}^d; \mathbf{R}) = C_b(\widehat{E}^d; \mathbf{R}) = C(\widehat{E}^d; \mathbf{R}) = \mathfrak{cl}\left[\mathfrak{ag}_{\mathbf{Q}}\left(\Pi^d(\widehat{\mathcal{F}})\right)\right]. \tag{3.1.16}$$

and

$$\mathfrak{cl}\left[\mathfrak{ag}_{\mathbf{Q}}\left(\Pi^d\left(\mathcal{F}|_{E_0}\right)\right)\right] = C(\widehat{E}^d; \mathbf{R})\Big|_{E_0^d}. \tag{3.1.17}$$

[7]In contrast, $\mathscr{O}(E^d)$ is not necessarily the same as $\mathscr{O}_{\mathcal{F}}(E)^d$.

[8]The notation "$\mathfrak{cl}(\cdot)$", "$\mathfrak{ag}_{\mathbf{Q}}(\cdot)$" and "$\mathfrak{ag}(\cdot)$" were defined in §2.2.3.

Proof. $\mathfrak{ag}(\Pi^d(\widehat{\mathcal{F}}))$ is uniformly dense in $C(\widehat{E}^d;\mathbf{R})$ by Lemma 3.9 (b, c) and the Stone-Weierstrass Theorem (see [Dudley, 2002, Theorem 2.4.11]). Thus, (3.1.16) follows by Lemma 3.9 (c), Fact B.43 (with $E = \widehat{E}$ and $k = 1$) and (2.2.15) (with $\mathcal{D} = \Pi^d(\widehat{\mathcal{F}})$). (3.1.17) follows by (3.1.16), the denseness of E_0^d in $\widehat{E}^d$ and properties of uniform convergence. □

Corollary 3.11. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$, $A \subset \widehat{E}^d$ and $\mathcal{D} \circeq \mathfrak{mc}[\Pi^d(\widehat{\mathcal{F}}\backslash\{1\})]$[9]. Then, the following statements are true:*

(a) $\mathcal{D}|_A^*$[10] separates points and strongly separates points on $\mathcal{P}(A, \mathscr{O}_{\widehat{E}^d}(A))$.

(b) $\mathcal{D}|_A \cup \{1\}$ (especially $\mathfrak{mc}[\Pi^d(\widehat{\mathcal{F}})]|_A$) is separating and convergence determining on $(A, \mathscr{O}_{\widehat{E}^d}(A))$.

(c) $\mathcal{M}^+(\widehat{E}^d)$ and $\mathcal{P}(\widehat{E}^d)$ are Polish spaces and, in particular, $\mathcal{P}(\widehat{E}^d)$ is compact.

Proof. $\widehat{E}$ is a Polish by Lemma 3.3 (c). So, (a) follows by Lemma A.35 (b) (with $E = (A, \mathscr{O}_{\widehat{E}^d}(A))$).

(b) follows by (a), Fact B.22 and the fact $(\mathcal{D} \cup \{1\}) \subset \mathfrak{mc}[\Pi^d(\widehat{\mathcal{F}})]$.

(c) follows by Lemma 3.9 (c) and Theorem A.44 (with $E = \widehat{E}^d$). □

Fact 3.12. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$ and $A \subset \widehat{E}^d$. Then, $\mathscr{B}_{\widehat{E}^d}(A) = \mathscr{B}(\widehat{E})^{\otimes d}|_A$[11].*

Proof. This fact is immediate by Lemma 3.3 (c) and Proposition B.46 (d) (with $\mathbf{I} = \{1, ..., d\}$ and $S_i = \widehat{E}$). □

Corollary 3.13. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d \in \mathbf{N}$. Then, the following statements are true:*

[9]The notation "$\mathfrak{mc}(\cdot)$" was defined in §2.2.3.

[10]The notation "$\mathcal{D}^*$" was defined in §2.3. Moreover, the terminologies "separating" and "convergence determining" were introduced in §2.3.

[11]The notations "$\mathscr{B}(\widehat{E})^{\otimes d}$" and "$\mathscr{B}(\widehat{E})^{\otimes d}\Big|_A$" were defined in §2.1.5.

(a) Any $A \subset E_0^d$ satisfies[12]

$$\begin{aligned}\mathscr{B}_{\left(E^d, \mathscr{O}_{\mathcal{F}}(E)^d\right)}(A) = \mathscr{B}(\widehat{E})^{\otimes d}\Big|_A = \mathscr{B}_{\widehat{E}^d}(A) = \mathscr{B}_{\Pi^d(\mathcal{F})}(A) = \mathscr{B}_{\mathcal{F}}(E)^{\otimes d}\big|_A \\ \subset \mathscr{B}(E)^{\otimes d}\big|_A \subset \mathscr{B}_{E^d}(A).\end{aligned} \tag{3.1.18}$$

In particular, E_0^d satisfies

$$\mathscr{B}_{\widehat{E}^d}(E_0^d) = \mathscr{B}_{\mathcal{F}}(E_0)^{\otimes d} \subset \mathscr{B}_E(E_0)^{\otimes d} \subset \mathscr{B}(E)^{\otimes d} \subset \mathscr{B}(E^d) \tag{3.1.19}$$

(b) $\mathcal{F}$ satisfies

$$\begin{aligned}\mathfrak{ca}\left(\Pi^d(\mathcal{F})\right) \subset M_b\left(E^d, \mathscr{B}_{\mathcal{F}}(E)^{\otimes d}; \mathbf{R}\right) \\ \subset M_b\left(E^d, \mathscr{B}(E)^{\otimes d}; \mathbf{R}\right) \subset M_b(E^d; \mathbf{R}).\end{aligned} \tag{3.1.20}$$

Proof. (a) We have that

$$\begin{aligned}&\mathscr{B}_{\left(E^d, \mathscr{O}_{\mathcal{F}}(E)^d\right)}(A) = \mathscr{B}_{(E_0^d, \mathscr{O}_{\mathcal{F}}(E_0)^d)}(A) \\ &= \mathscr{B}_{(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d)}(A) = \mathscr{B}_{\widehat{E}^d}(A) = \mathscr{B}(\widehat{E})^{\otimes d}\Big|_A \\ &= \mathscr{B}_{\widehat{E}}(E_0)^{\otimes d}\big|_A = \mathscr{B}_{\mathcal{F}}(E_0)^{\otimes d}\big|_A = \mathscr{B}_{\mathcal{F}}(E)^{\otimes d}\big|_A\end{aligned} \tag{3.1.21}$$

by (3.1.6), Fact 3.12 and the fact $A \subset E_0^d \subset (E^d \cap \widehat{E}^d)$. Then, the first line of (3.1.18) follows by (3.1.21), (3.1.12) and Lemma 3.9 (b). The second line of (3.1.18) follows by $\mathcal{F} \subset C(E; \mathbf{R})$ and Lemma B.46 (a). Now, (a) follows by (3.1.10).

(b) follows by $\mathcal{F} \subset C_b(E; \mathbf{R})$, Proposition A.21 (a) (with $\mathcal{D} = \mathcal{F}$) and (3.1.19). □

In general, E_0 is not necessarily a Borel subset of $\widehat{E}$, nor does $\widehat{E}$ endow E_0 with the same Borel sets as E. This measurability issue is avoided if E_0 is a standard Borel subset of E.

Lemma 3.14. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$ and $A \subset E_0^d$. Then, the following statements are true:*

[12] The notation "$\mathscr{B}_{\mathcal{F}}(E)$" was defined in §2.1.3.

(a) $A \in \mathscr{B}^{\mathbf{s}}(E^d)$[13] *if and only if*[14]

$$\mathscr{B}_{E^d}(A) = \mathscr{B}_{\widehat{E}^d}(A) \text{ and } A \in \mathscr{B}(\widehat{E}^d). \tag{3.1.22}$$

(b) If $A \in \mathscr{B}^{\mathbf{s}}(E^d)$*, then*

$$\mathscr{B}_{E^d}(A) = \mathscr{B}(E)^{\otimes d}\big|_A \subset \mathscr{B}(E)^{\otimes d} \subset \mathscr{B}(E^d). \tag{3.1.23}$$

(c) If $A = \bigcup_{n\in\mathbf{N}} A_n$ *and* $\{A_n\}_{n\in\mathbf{N}} \subset \mathscr{B}^{\mathbf{s}}(E^d)$*, then* $A \in \mathscr{B}^{\mathbf{s}}(E^d)$.

(d) If $A = \prod_{n=1}^d A_n$ *and* $\{A_n\}_{1\leq n\leq d} \subset \mathscr{B}^{\mathbf{s}}(E)$*, then*

$$\mathscr{B}_{\widehat{E}^d}(A) = \mathscr{B}(E)^{\otimes d}\big|_A \subset \left[\mathscr{B}(E)^{\otimes d} \cap \mathscr{B}(\widehat{E}^d)\right]. \tag{3.1.24}$$

Proof. (a - Necessity) Let f be the identity map on A, which is certainly injective. $f \in M(A, \mathscr{B}_{E^d}(A); \widehat{E}^d)$ by (3.1.18). $A \in \mathscr{B}^{\mathbf{s}}(E^d)$ and $A \subset E_0^d$ imply $A \in \mathscr{B}^{\mathbf{s}}(A, \mathscr{O}_{E^d}(A))$. $\widehat{E}^d$ is a Polish space by Lemma 3.9 (c). It then follows by Proposition A.57 (with $E = A$ and $S = \widehat{E}^d$) that $A \in \mathscr{B}^{\mathbf{s}}(\widehat{E}^d)$ and $\mathscr{B}_{E^d}(A) = \mathscr{B}_{\widehat{E}^d}(A)$. So, $A \in \mathscr{B}(\widehat{E}^d) = \mathscr{B}^{\mathbf{s}}(\widehat{E}^d)$ by Proposition A.56 (b) (with $E = \widehat{E}^d$).

(a - Sufficiency) follows by (3.1.22) and Fact A.48 (a).

(b) $A \in \mathscr{B}_{\widehat{E}^d}(E_0^d)$ by (3.1.22). Then, (3.1.23) follows by (3.1.19) and (3.1.18).

(c) We find by (a) (with $A = A_n$) that

$$\mathscr{B}_{E^d}(A_n) = \mathscr{B}_{\widehat{E}^d}(A_n) \subset \mathscr{B}(\widehat{E}^d),\ \forall n \in \mathbf{N}. \tag{3.1.25}$$

Then, A satisfies (3.1.22) by (3.1.25), Fact B.1 (with $E = A$, $\mathscr{U}_1 = \mathscr{B}_{E^d}(A)$ and $\mathscr{U}_2 = \mathscr{B}_{\widehat{E}^d}(A)$) and (3.1.18). Hence, $A \in \mathscr{B}^{\mathbf{s}}(E^d)$ by (a).

(d) $A \subset E_0^d$ implies $A_n = \mathfrak{p}_n(A) \subset E_0$ for all $1 \leq n \leq d$. We have that

$$\mathscr{B}_E(A_n) = \mathscr{B}_{\widehat{E}}(A_n) \subset \left[\mathscr{B}(E) \cap \mathscr{B}(\widehat{E})\right],\ \forall 1 \leq n \leq d \tag{3.1.26}$$

[13] $\mathscr{B}^{\mathbf{s}}(E^d)$ denotes the family of all standard Borel subsets of E^d.

[14] $\mathscr{B}_{E^d}(A) = \mathscr{B}_{\widehat{E}^d}(A)$ plus $A \in \mathscr{B}(\widehat{E}^d)$ is equivalent to $\mathscr{B}_{E^d}(A) = \mathscr{B}_{\widehat{E}^d}(A) \subset \mathscr{B}(\widehat{E}^d)$. Hereafter, we frequently use the latter notation.

by (a) (with $d=1$ and $A=A_n$) and the fact $E_0\in\mathscr{B}(E)$. It then follows by (3.1.26) and Corollary 3.13 (a) (with $d=1$ and $A=A_n$) that

$$A=\bigcap_{n=1}^{d}\mathfrak{p}_n^{-1}(A_n)\in\mathscr{B}(E)^{\otimes d}\cap\mathscr{B}(\widehat{E}^d) \tag{3.1.27}$$

and

$$\mathscr{B}_{\widehat{E}^d}(A)=\mathscr{B}(\widehat{E})^{\otimes d}\Big|_A=\bigotimes_{n=1}^{d}\mathscr{B}_{\widehat{E}}(A_n)=\bigotimes_{n=1}^{d}\mathscr{B}_E(A_n)=\mathscr{B}(E)^{\otimes d}\big|_A. \tag{3.1.28}$$

$\square$

Corollary 3.15. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E, $d\in\mathbf{N}$ and $A\subset E_0^d$. Then, the following statements are true:*

(a) If $A\in\mathscr{K}(E_0^d,\mathscr{O}_E(E_0)^d)$, then

$$(A,\mathscr{O}_{E^d}(A))=\left(A,\rho_{\mathcal{F}}^d\right)=\left(A,\mathscr{O}_{\widehat{E}^d}(A)\right) \tag{3.1.29}$$

and[15]

$$A\in\mathscr{K}^{\mathbf{m}}(E^d)\cap\mathscr{K}(\widehat{E}^d)\cap\mathscr{C}(\widehat{E}^d). \tag{3.1.30}$$

(b) If $A\in\mathscr{K}_\sigma(E_0^d,\mathscr{O}_E(E_0)^d)$, then

$$A\in\mathscr{B}^{\mathbf{s}}(E^d)\cap\mathscr{K}_\sigma^{\mathbf{m}}(E^d)\cap\mathscr{K}_\sigma(\widehat{E}^d)\cap\mathscr{B}(E)^{\otimes d}\cap\mathscr{B}(\widehat{E}^d). \tag{3.1.31}$$

Proof. (a) follows by Lemma 3.9 (b, c, e) and Fact B.51 (b) (with $E=E^d$, $\mathcal{D}=\Pi^d(\mathcal{F})$ and $K=A$).

(b) $A\in\mathscr{K}_\sigma^{\mathbf{m}}(E^d)\cap\mathscr{K}_\sigma(\widehat{E}^d)\cap\mathscr{B}(\widehat{E}^d)$ by (3.1.30). $A\in\mathscr{B}^{\mathbf{s}}(E^d)\cap\mathscr{B}(E)^{\otimes d}$ by (a) and Lemma 3.14 (b, c). $\square$

Note 3.16. Given a base $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ over E and $d\in\mathbf{N}$, Corollary 3.15 (b) shows that $\mathscr{K}_\sigma(E_0^d,\mathscr{O}_E(E_0)^d)$ lies in the domain of any $\mu\in\mathfrak{M}^+(E^d,\mathscr{B}(E)^{\otimes d})$.

Corollary 3.17. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E, $d\in\mathbf{N}$ and $\mu\in\mathfrak{M}^+(E^d,\mathscr{B}(E)^{\otimes d})$.*

[15] The notations "$\mathscr{C}(\cdot)$", "$\mathscr{K}(\cdot)$" and "$\mathscr{K}^{\mathbf{m}}(\cdot)$" as defined in §2.1.3 denote the families of closed, compact and metrizable compact subsets, respectively.

(a) If μ is supported on $A \subset E_0^d$[16] *and $A \in \mathscr{B}^{\mathrm{s}}(E^d)$, then $\mathfrak{be}(\mu)$*[17] *is a singleton.*

(b) If μ is tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$, then there exists a $\mu' = \mathfrak{be}(\mu)$ which is tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$.

Note 3.18. Please be noted that $\mu' \in \mathfrak{be}(\mu)$ (if any) is *not an expansion* of μ to a superspace. Rather than that, μ' is an *extension* of μ as set functions to the broader domain of all Borel sets. Hence, μ and μ' have the same total mass as mentioned in Note 2.11 and, in particular, any support of μ is also that of μ'.

Proof of Corollary 3.17. (a) One finds by Lemma 3.14 (b) that $A \in \mathscr{B}(E)^{\otimes d}$ and $\mathscr{B}_{E^d}(A) = \mathscr{B}(E)^{\otimes d}|_A$. Hence, (a) follows by Lemma B.48 (c) (with $\mathbf{I} = \{1, ..., d\}$, $S_i = E$, $S = E^d$ and $\mathscr{A} = \mathscr{B}(E)^{\otimes d}$).

(b) μ is supported on some $A \in \mathscr{K}_\sigma(E_0^d, \mathscr{O}_E(E_0)^d)$ by its tightness. $A \in \mathscr{B}^{\mathrm{s}}(E^d)$ by Corollary 3.15 (b). Hence, (b) follows by (a). □

The standard Borel property of $A \subset E_0^d$ also yields useful properties of the weak topological space $\mathcal{M}^+(A, \mathscr{O}_{E^d}(A))$.

Corollary 3.19. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$, $A \in \mathscr{B}^{\mathrm{s}}(E^d)$ with $A \subset E_0^d$ and $\mathcal{D} \mathrel{\mathring{=}} \mathfrak{mc}[\Pi^d(\mathcal{F} \backslash \{1\})]$. Then, the following statements are true:*

(a) $\mathcal{D}|_A^*$ separates points on $\mathcal{P}(A, \mathscr{O}_{E^d}(A))$. Moreover, $\mathcal{D}|_A^* \cup \{1\}$ (especially $\mathfrak{mc}[\Pi^d(\mathcal{F})]|_A$) is separating on $(A, \mathscr{O}_{E^d}(A))$.

(b) $\mathcal{M}^+(A, \mathscr{O}_{E^d}(A))$ and $\mathcal{P}(A, \mathscr{O}_{E^d}(A))$ are Tychonoff spaces.

Proof. (a) $\mathcal{D}|_A^*$ separates points on $\mathcal{P}(A, \mathscr{O}_{\widehat{E}^d}(A)) = \mathcal{P}(A, \mathscr{O}_{E^d}(A))$ by (3.1.1), Corollary 3.11 (a) and Lemma 3.14 (a). Now, (a) follows by Fact B.22 (a) and the fact $(\mathcal{D} \cup \{1\}) \subset \mathfrak{mc}[\Pi^d(\mathcal{F})]$.

(b) follows by (a) and Proposition A.34 (a, c) (with $E = (A, \mathscr{O}_{E^d}(A))$). □

[16] Support of measure was specified in §2.1.2.
[17] $\mathfrak{be}(\mu)$ as defined in §2.3 denotes the family of all Borel extensions of μ.

3.2 Baseable space

Theorem 3.8 equates the problem of establishing a base and that of finding a subset of E whose points can be separated by countably many bounded continuous functions. In fact, boundedness is not necessary.

Lemma 3.20. *Let E be a topological space, $E_0 \in \mathscr{B}(E)$ and $\mathcal{D} \subset C(E; \mathbf{R})$ be countable and separate points on E. Then, there exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E satisfying the following properties:*

(a) $\mathscr{O}_{\mathcal{D}}(E) \subset \mathscr{O}_{\mathcal{F}}(E)$.

(b) $(\mathcal{D} \cap C_b(E; \mathbf{R})) \subset \mathcal{F}$.

(c) $\mathcal{F}$ *can be taken to equal* $\mathcal{D} \cup \{1\}$ *whenever* $\mathcal{D} \subset C_b(E; \mathbf{R})$.

Proof. By the fact $\{(f \wedge n) \vee (-n)\}_{n \in \mathbf{N}, f \in \mathcal{D} \cup \{1\}} \subset C_b(E; \mathbf{R})$ and Lemma B.52 (with $\mathcal{G} = \mathcal{D} \cup \{1\}$ and $\mathcal{H} = C_b(E; \mathbf{R})$), there exists a countable $\mathcal{F} \subset C_b(E; \mathbf{R})$ which satisfies (a) - (c). $(E_0, \mathscr{O}_{\mathcal{D}}(E_0))$ is a Hausdorff space by Proposition A.17 (c) (with $A = E_0$), hence $(E_0, \mathscr{O}_{\mathcal{F}}(E_0))$ is also by (a) and Fact A.1. $\mathcal{F}$ separates points by Proposition A.17 (c) (with $A = E_0$ and $\mathcal{D} = \mathcal{F}$). Now, the result follows by Theorem 3.8. □

We introduce the words "*baseable*" and "*baseability*" to describe the above-mentioned *ability of inducing bases*.

Definition 3.21. Let E be a topological space and $A \subset E$ be non-empty.

- E is a **$\mathcal{D}$-baseable space** if $\mathcal{D} \subset C(E; \mathbf{R})$ has a countable subset separating points on E.
- A is a **$\mathcal{D}$-baseable subset of** E if $A \in \mathscr{B}(E)$, $\mathcal{D} \subset C(E; \mathbf{R})$ and $(A, \mathscr{O}_E(A))$ is a $\mathcal{D}|_A$-baseable space.
- E is a **baseable space** if E is a $C(E; \mathbf{R})$-baseable space.
- A is a **baseable subset of** E if A is a $C(E; \mathbf{R})$-baseable subset.

Remark 3.22. "being a baseable subset" equals "being a baseable subspace" plus Borel measurability. Moreover, the "$\mathcal{D}|_A$-baseable space" above is a proper statement since

$$\mathcal{D}|_A \subset C(E;\mathbf{R})|_A \subset C(A, \mathscr{O}_E(A);\mathbf{R}). \tag{3.2.1}$$

Baseable spaces or their analogues have appeared in many probabilistic literatures such as [Ethier and Kurtz, 1986, Chapter 3], Jakubowski [1986], Kurtz and Ocone [1988], Jakubowski [1997a], [Bogachev, 2007, Chapter 6], Blount and Kouritzin [2010], Kouritzin and Sun [2017] and Kouritzin [2016] etc. Herein, we first characterize and exhibit several concrete examples of baseable topological spaces. The next section will treat baseable subsets.

3.2.1 Characterization

The definition of baseability is not as explicit as usual topological notions because separating points is a *non-topological* property for function classes. Nonetheless, baseable spaces turn out to be a broad category with their own uniqueness.

Theorem 3.23. *Baseable spaces are precisely the topological refinements of metrizable and separable spaces.*

The theorem above addresses the accurate boarderline of baseable spaces and follows by two straightforward observations. First, we note that baseable spaces sit between Hausdorff spaces and separable metric spaces.

Fact 3.24. *The following statements are true:*

(a) If E is a baseable space, then E is a Hausdorff space.

(b) If E is a metrizable and separable space, then E is a $\mathcal{D}$-baseable space for some countable $\mathcal{D} \subset C_b(E;\mathbf{R})$ that strongly separates points on E.

Proof. (a) follows by Proposition A.17 (e) (with $A = E$).
(b) follows by Corollary A.30 (a, b). □

Corollary 3.25. *Metrizable Souslin spaces, metrizable Lusin spaces and metrizable standard Borel spaces are all baseable spaces.*

Proof. Souslin and Lusin spaces are separable by Proposition A.11 (d). Metrizable standard Borel spaces are Lusin spaces by Proposition A.53 (a, b). Now, the result follows by Fact 3.24 (b). □

Refining the topology of a metrizable and separable space E may cause a function class $\mathcal{D} \subset C(E; \mathbf{R})$ forfeit many *topology-dependent* properties like continuity or strongly separating points on E. However, we observe that refining topology does not affect the $\mathcal{D}$-baseability of E.

Fact 3.26. *If E is $\mathcal{D}$-baseable, then any topological refinement of E is also.*

Proof. Note that $\mathcal{D} \subset C(E; \mathbf{R}) \subset C(E, \mathscr{U}; \mathbf{R})$ if $\mathscr{U}$ refines $\mathscr{O}(E)$. □

Proof of Theorem 3.23. (Necessity) If E is a baseable space and $\mathcal{D} \subset C(E; \mathbf{R})$ is countable and separate points on E, then $(E, \mathscr{O}_{\mathcal{D}}(E))$ coarsens E and is a metrizable and separable space by Proposition A.17 (d).

(Sufficiency) follows by Fact 3.24 (b) and Fact 3.26. □

3.2.2 Examples of baseable spaces

The following figure illustrates the relationship of baseable spaces and other major categories of topological spaces.

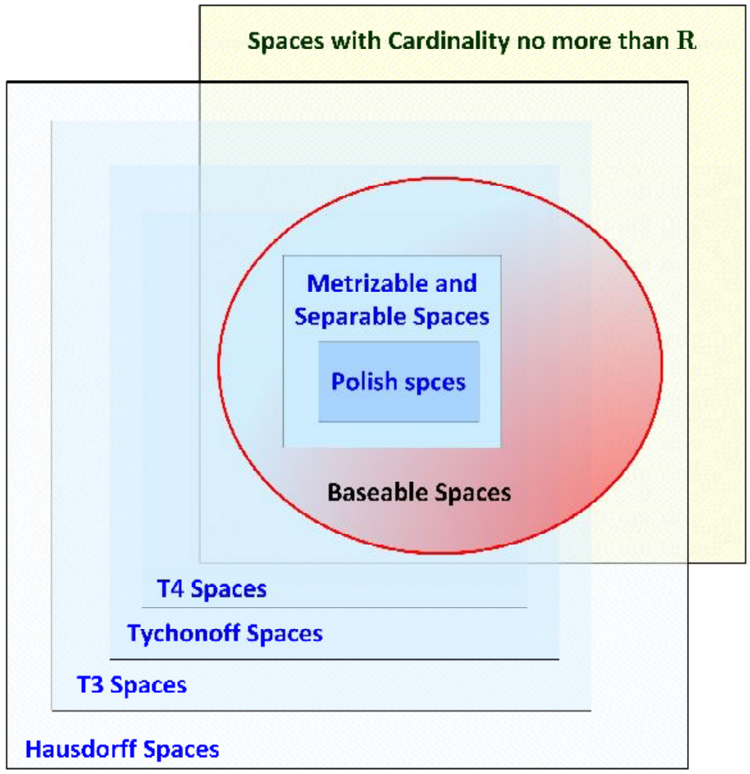

Figure 3: *Comparison of baseable spaces and other topological spaces*

On one hand, topological refinements of metrizable and separable spaces are varied, so baseable spaces range from Polish spaces to even non-*T3*[18] (see p.217) Hausdorff spaces as in the following examples.

Example 3.27.

(I) *A baseable, metrizable and non-Polish Lusin space*: Example 2.28 mentioned that the pseudo-path topological space $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ is a metrizable but non-Polish Lusin space. Lusin spaces are separable by Proposition A.11 (d). Hence, $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ is a baseable space.

(II) *Baseable and non-separable Banach spaces - 1*: Let l^∞ be the space of all bounded $\mathbf{R}$-valued sequences equipped with the supremum

[18]Herein, we use the terminologies "T3" and "T4" instead of "regular" and "normal" since the latter sometimes are used in a non-Hausdorff context.

norm, i.e.

$$l^\infty \mathrel{\hat{=}} \left\{ x \in \mathbf{R}^\infty : \|x\|_\infty = \sup_{n \in \mathbf{N}} |x_n| < \infty \right\}. \tag{3.2.2}$$

$(l^\infty, \mathscr{O}_{\mathbf{R}^\infty}(l^\infty))$ is metrizable and separable by Proposition A.11 (c, f). $(l^\infty, \|\cdot\|_\infty)$ is a Banach topological refinement of $(l^\infty, \mathscr{O}_{\mathbf{R}^\infty}(l^\infty))$ by [Munkres, 2000, Theorem 43.5 and Theorem 20.4], so $(l^\infty, \|\cdot\|_\infty)$ is a baseable space. However, $(l^\infty, \|\cdot\|_\infty)$ is non-separable by [Bachman and Narici, 2012, Example 6.6, p.83].

(III) *Baseable and non-separable Banach spaces - 2*: We now consider the non-separable Banach spaces mentioned in Example 2.32 and Example 2.33. [Friz and Victoir, 2010, Corollary 7.50] showed that $G^N(\mathbf{R}^d)$, the free nilpotent group of Step N over $\mathbf{R}^d$ is a separable Banach space equipped with the Carnot-Caratheodory norm $\|\cdot\|_{\mathrm{cc}}$. $C([0,T];\mathbf{R}^d)$ equipped with $\|\cdot\|_\infty$ and $C([0,T];G^N(\mathbf{R}^d))$ equipped with the supremum cc-norm

$$\|x\|_{\infty,\mathrm{cc}} \mathrel{\hat{=}} \sup_{t \in [0,T]} \|x(t)\|_{\mathrm{cc}} \tag{3.2.3}$$

are Polish spaces by [Srivastava, 1998, Theorem 2.4.3]. Then, the spaces in (2.7.14) and (2.7.15) equipped with $\|\cdot\|_\infty$, and those in (2.7.20) and (2.7.21) equipped with $\|\cdot\|_{\infty,\mathrm{cc}}$ are all metrizable and separable spaces by Proposition A.11 (c). It is known that the norms in (2.7.12) and (2.7.13) both induce finer topologies than $\|\cdot\|_\infty$, while the norms in (2.7.18) and (2.7.19) both induce finer topologies than $\|\cdot\|_{\infty,\mathrm{cc}}$ (see [Bayer and Friz, 2013, p.262 and Remark 3.6]). Thus, all the non-separable Banach spaces in Example 2.32 and Example 2.33 are baseable spaces.

(IV) *A baseable, non-second-countable* (see p.218)*, separable, Lindelöf* (see p.218) *and non-metrizable T4 space*: The *Sorgenfrey Line* $\mathbf{R}_l$ refers to the space of all real numbers equipped with the *lower limit* *topology* which is generated by the topological basis

$$\{[a,b) : a, b \in \mathbf{R}, a < b\}. \tag{3.2.4}$$

$\mathbf{R}_l$ as a topological refinement of $\mathbf{R}$ is a baseable space. $\mathbf{R}_l$ is separable, Lindelöf but not second-countable by [Munkres, 2000, §30, Example 3]. $\mathbf{R}_l$ is a *T4 space* (see p.217) by [Munkres, 2000, §31, Example 2]. Furthermore, $\mathbf{R}_l$ is non-metrizable by Proposition A.6 (c).

(V) *A baseable, non-Lindelöf, separable and non-T4 Tychonoff space*: The *Sorgenfrey Plane* $\mathbf{R}_l^2$ is a topological refinement of $\mathbf{R}^2$ and hence is a baseable space. Since $\mathbf{R}_l$ is a separable Tychonoff space, $\mathbf{R}_l^2$ is also by Proposition A.3 (c) and Proposition A.26 (c). However, $\mathbf{R}_l^2$ is neither a Lindelöf space nor a T4 space by [Munkres, 2000, §30, Example 4 and §31, Example 3].

(VI) *A baseable, non-separable and non-metrizable Tychonoff space*: When E is a Polish space, $\mathcal{P}(E)$ is also by Theorem A.44 (b). Example 2.30 explained that the strong topological space $\mathcal{P}_S(E)$ of all Borel probability measures on E is a non-metrizable, non-separable and Tychonoff refinement of $\mathcal{P}(E)$, so $\mathcal{P}_S(E)$ is a baseable space.

(VII) *A baseable, second-countable and non-T3 space*: Let $\mathbf{R}_K$ denote the space of all real numbers equipped with the *K-topology* which is generated by the countable topological basis

$$\{(a,b) : a, b \in \mathbf{Q}, a < b\} \cup \left\{(a,b) \backslash \{1/n\}_{n \in \mathbf{N}} : a, b \in \mathbf{Q}, a < b\right\}. \tag{3.2.5}$$

So, $\mathbf{R}_K$ is a second-countable topological refinement of $\mathbf{R}$ and hence is baseable. However, [Munkres, 2000, §31, Example 1] explained that $\mathbf{R}_K$ is not a T3 space, nor is it a Tychonoff space by Proposition A.26 (a).

(VIII) *A non-first-countable baseable space*: Example 2.8 (I) exhibited a simple countable set of continuous functions that separates points on the product topological space $(C([0,1];\mathbf{R}), \mathscr{O}(\mathbf{R})^{[0,1]})$. So, this space is a baseable space. However, the argument in [Munkres, 2000, §21, Example 2] explained that $C([0,1];\mathbf{R})$ is not even a first-countable space.

On the other hand, it is intuitive that a baseable space has no more points than the real line since its points are distinguishable by a countable function class.

Fact 3.28. *The cardinality of a baseable space is no greater than* $\aleph(\mathbf{R})$.

Proof. The cardinality of a metrizable and separable space never exceeds $\aleph(\mathbf{R}^\infty) = \aleph(\mathbf{R})$ by Corollary A.30 (a, c), nor can their topological refinements. $\square$

In general, Tychonoff spaces, metrizable spaces or separable spaces are not necessarily "small" enough to be baseable spaces.

Example 3.29. $\mathbf{R}^{[0,1]}$ equipped with the product topology is a Tychonoff space by Proposition A.26 (c) and is a separable space by [Munkres, 2000, §30, Exercise 16 (a)]. $(\mathbf{R}^{[0,1]}, \|\cdot\|_\infty)$ is a Banach space by [Munkres, 2000, Theorem 43.5]. However, $\mathbf{R}^{[0,1]}$ can not be baseable with any topology since its cardinality is strictly greater than $\aleph(\mathbf{R})$.

It is worth mentioning that some of the baseable spaces in Example 3.27 are also examples of non-Polish, non-separable or non-metrizable standard Borel spaces.

Example 3.30. Every metrizable Lusin space is a standard Borel space by Proposition A.53. In particular, the pseudo-path topological space $D^{\mathrm{pp}}(\mathbf{R}^+;\mathbf{R})$ is an example of a baseable, non-Polish, metrizable, separable, standard Borel space by Example 3.27 (I).

By Proposition A.52 (a, d), a topological space is standard Borel if its Borel σ-algebra can be generated by some Polish topology. The following examples are all of this type:

Example 3.31.

(I) The Sorgenfrey Line $\mathbf{R}_l$ is an example of a baseable, separable, non-metrizable, standard Borel space. According to Example 3.27 (IV), it suffices to show $\mathscr{B}(\mathbf{R}_l) \subset \mathscr{B}(\mathbf{R})$. By the definition of $\mathbf{R}_l$, [Munkres, 2000, Lemma 13.1] and Fact B.12, any $O \in \mathscr{O}(\mathbf{R}_l)$ satisfies

$$O = \bigcup_{i \in \mathbf{I}} [a_i, b_i) \in \mathscr{B}(\mathbf{R}) \tag{3.2.6}$$

for some $(\{a_i\}_{i\in\mathbf{I}} \cup \{b_i\}_{i\in\mathbf{I}}) \subset \mathbf{R}$, thus proving $\mathscr{B}(\mathbf{R}_l) \subset \mathscr{B}(\mathbf{R})$.

(II) A Polish space E . Hence, the strong topological space $\mathcal{P}_S(E)$ has the same Borel σ-algebra as the Polish space $\mathcal{P}(E)$ by Lemma A.47. According to Example 3.27 (VI), $\mathcal{P}_S(E)$ is a baseable, non-separable, non-metrizable, standard Borel space.

(III) The K-topological space $\mathbf{R}_K$ is an example of a baseable, second-countable, non-T3, standard Borel spaces. According to Example 3.27 (VII), it suffices to show $\mathscr{B}(\mathbf{R}_K) \subset \mathscr{B}(\mathbf{R})$. By the definition of $\mathbf{R}_K$ and [Munkres, 2000, Lemma 13.1], any $O \in \mathscr{O}(\mathbf{R}_K)$ satisfies

$$O = \left(\bigcup_{i \in \mathbf{I}} (a_i, b_i) \right) \backslash \{1/n\}_{n \in \mathbf{N}} \in \mathscr{B}(\mathbf{R}) \tag{3.2.7}$$

for some $(\{a_i\}_{i\in\mathbf{I}} \cup \{b_i\}_{i\in\mathbf{I}}) \subset \mathbf{Q}$, thus proving $\mathscr{B}(\mathbf{R}_K) \subset \mathscr{B}(\mathbf{R})$.

Remark 3.32. The baseable but non-separable Banach spaces in Example 3.27 (II, III) can not be standard Borel, since Lemma A.54 (b) shows metrizable standard Borel spaces must be separable. To a certain extent, the lack of standard-Borel property evidences the complexity of random rough paths and their distributions.

3.3 Baseable subset

When a topological space is not necessarily baseable, its baseable subsets are often used as "blocks" for building baseable subspaces.

3.3.1 General properties

The following three facts give initial descriptions of baseable subsets.

Fact 3.33. *Let E be a topological space and $A \subset E$. Consider the following statements:*

(a) A is a baseable subset of E.

(b) There exits a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E.

(c) A is a $C_b(E; \mathbf{R})$-baseable subset of E.

(d) $(A, \mathscr{O}_E(A))$ is a baseable space.

(e) There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \mathscr{O}_E(A))$.

Then, (a) - (c) are equivalent, so are (d) and (e). Moreover, (a) implies (d).

Proof. ((a) $\to$ (b)) follows by Lemma 3.20 (with $E_0 = A$). ((d) $\to$ (e)) follows by (a, b) (with $E = (A, \mathscr{O}_E(A))$). The other parts are immediate by definition. $\square$

Fact 3.34. *Let E be a topological space, $A \subset E$ and $\mathcal{D} \subset C(E; \mathbf{R})$. Consider the following statements:*

(a) $(A, \mathscr{O}_E(A))$ is a $\mathcal{D}|_A$-baseable space.

(b) A is a $\mathcal{D}$-baseable subset of E.

(c) A is a $\mathcal{D}_0$-baseable subset of E for some countable $\mathcal{D}_0 \subset \mathcal{D}$.

(d) A is a $\mathcal{D}'$-baseable subset of E for any $\mathcal{D}' \subset C(E; \mathbf{R})$ that includes $\mathcal{D}$.

(e) Any $B \in \mathscr{B}_E(A)$ is a $\mathcal{D}$-baseable subset of any topological refinement of E.

Then, (b) - (e) are equivalent and any of them implies (a). Moreover, if $A \in \mathscr{B}(E)$, then (a) - (e) are all equivalent.

Proof. We only prove ((b) $\to$ (e)) and the other parts are immediate by definition. To show this implication, it suffices to note that if $(E, \mathscr{U})$ is a topological refinement of E, then $\mathscr{B}_E(A) \subset \mathscr{B}_{(E,\mathscr{U})}(A) \subset \mathscr{B}(E, \mathscr{U})$ and $C(E; \mathbf{R}) \subset C(E, \mathscr{U}; \mathbf{R})$. □

Fact 3.35. *Let E be a topological space. Then, the following statements are equivalent:*

(a) E is a $\mathcal{D}$-baseable space (resp. baseable space).

(b) Any $A \subset E$ is a $\mathcal{D}|_A$-baseable subspace (resp. baseable subspace).

(c) All members of $\mathscr{B}(E)$ are $\mathcal{D}$-baseable subsets (resp. baseable subsets) of any topological refinement of E.

Proof. ((a) $\to$ (b)) follows by (3.2.1). ((b) $\to$ (c)) follows by Fact 3.34 (a, e) (with $A = E$ and $\mathcal{D} = \mathcal{D}$ or $C(E; \mathbf{R})$). ((c) $\to$ (a)) is automatic. □

The next three results look at the union and the product space of countably many baseable subsets.

Fact 3.36. *Let E be a topological space and A_n be a $\mathcal{D}_n$-baseable subset of E for each $n \in \mathbf{N}$. If $\{A_n\}_{n\in\mathbf{N}}$ is nested*[19] *(i.e. any A_{n_1} and A_{n_2} admit a common superset A_{n_3}), then $\bigcup_{n\in\mathbf{N}} A_n$ is a $\bigcup_{n\in\mathbf{N}} \mathcal{D}_n$-baseable subset of E.*

Proof. This result is immediate by Fact B.19. □

Proposition 3.37. *Let A_n be a $\mathcal{D}_n$-baseable subset of topological space S_n for each $n \in \mathbf{N}$. Then, the following statements are true:*

(a) $\prod_{n=1}^m A_n$ is a $\mathcal{D}^m$-baseable subset of $\prod_{n=1}^m S_n$ with $\mathcal{D}^m \doteq \{f \circ \mathfrak{p}_n : 1 \leq n \leq m, f \in \mathcal{D}_n\}$ for all $m \in \mathbf{N}$.

(b) $\prod_{n\in\mathbf{N}} A_n$ is a $\bigcup_{m\in\mathbf{N}} \mathcal{D}^m$-baseable subset of $\prod_{n\in\mathbf{N}} S_n$.

Proof. (a) $\prod_{n=1}^m A_n = \bigcap_{n=1}^m \mathfrak{p}_n^{-1}(A_n) \in \mathscr{B}(E)^{\otimes m} \subset \mathscr{B}(E^m)$ by Proposition B.46 (a). Let $\{f_{n,k}\}_{k\in\mathbf{N}} \subset \mathcal{D}_n$ separate points on A_n for each $n \in \mathbf{N}$ and

[19] A typical case of nested $\{A_n\}_{n\in\mathbf{N}}$ is when $A_n \subset A_{n+1}$ for all $n \in \mathbf{N}$.

$\mathcal{D}_0^m \stackrel{\circ}{=} \{f_{n,k} \circ \mathfrak{p}_n\}_{1 \leq n \leq m, k \in \mathbf{N}}$. Then, $\bigotimes \mathcal{D}_0^m(x) = \bigotimes \mathcal{D}_0^m(y)$[20] in $\mathbf{R}^{\mathcal{D}_0^m}$ implies $x = \bigotimes_{n=1}^m \mathfrak{p}_n(x) = \bigotimes_{n=1}^m \mathfrak{p}_n(y) = y$ in $\prod_{n=1}^m S_n$. Thus, $\mathcal{D}_0^m$ is a countable subset of $\mathcal{D}^m$ that separates points on $\prod_{n=1}^m A_n$.

(b) follows by an argument similar to (a). □

Corollary 3.38. *Let E be a topological space, $\{A_n\}_{n \in \mathbf{N}}$ be $\mathcal{D}$-baseable subsets of E and $d \in \mathbf{N}$. Then, $\prod_{n=1}^d A_n$ is a $\Pi^d(\mathcal{D})$-baseable subset of E^d.*

Proof. This corollary follows by Proposition 3.37 (a) (with $m = d$ and $\mathcal{D}_n = \mathcal{D}$), the definition of $\Pi^d(\mathcal{D})$ and Fact 3.34 (b, d). □

3.3.2 Selection of point-separating functions

When using $\mathcal{D}$-baseable subsets to construct a base, one can always include *arbitrary, up to countably many, bounded* members of $\mathcal{D}$ within the base. This is useful in many applications because one may include a desirable set of functions such as subdomains of operators, observation functions of nonlinear filters and test functions for measure-valued processes, etc.

Lemma 3.39. *Let E be a topological space, E_0 be a $\mathcal{D}$-baseable subset of E and $\mathcal{D}_0 \subset C_b(E; \mathbf{R})$ be countable. Then, the following statements are true:*

(a) There exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E with $\mathcal{D}_0 \subset \mathcal{F}$.

(b) If $\mathcal{D}$ is countable, then the $\mathcal{F}$ in (a) can be taken to contain $\mathcal{D} \cap C_b(E; \mathbf{R})$. If, in addition, $\mathcal{D} \subset C_b(E; \mathbf{R})$, then $\mathcal{F}$ can be taken to equal $\mathcal{D} \cup \{1\}$.

(c) If $\mathcal{D}_0 \subset \mathcal{D} \subset C_b(E; \mathbf{R})$, then the $\mathcal{F}$ in (a) can be taken within $\mathcal{D} \cup \{1\}$. If, in addition, $\mathcal{D}$ is closed under addition or multiplication, then $\mathcal{F}$ can be taken to have the same closedness.

Proof. Let $\mathcal{D}' \subset \mathcal{D}$ be countable and separate points on E_0. Then, (a) and (b) follow by Lemma 3.20 (with $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}_0$).

[20] Please be noted that the notation "$\bigotimes$" for functions has nothing to do with multiplication. The separability argument above does not require $1 \in \mathcal{D}_n$ for any $n \in \mathbf{N}$.

For (c), we define[21]

$$\mathcal{D}'' \circeq \begin{cases} \mathcal{D}_0 \cup \mathcal{D}' \cup \{1\}, & \text{in general,} \\ \mathfrak{ac}\left(\mathcal{D}_0 \cup \mathcal{D}' \cup \{1\}\right), & \text{if } \mathcal{D} \ni 1 \text{ is closed under addition,} \\ \mathfrak{mc}\left(\mathcal{D}_0 \cup \mathcal{D}' \cup \{1\}\right), & \text{if } \mathcal{D} \ni 1 \text{ is closed under multiplication,} \\ \mathfrak{ac}\left(\mathfrak{mc}\left(\mathcal{D}_0 \cup \mathcal{D}' \cup \{1\}\right)\right), & \text{if } \mathcal{D} \ni 1 \text{ is closed under both.} \end{cases} \tag{3.3.1}$$

In any case above, $\mathcal{D}''$ is a countable subset of $\mathcal{D} \cup \{1\}$, contains $\{1\} \cup \mathcal{D}_0$ and separates points on E_0. Now, (c) follows by (b) (with $\mathcal{D} = \mathcal{D}''$). ☐

A common situation of constructing bases is when $\mathcal{D} \subset C(E; \mathbf{R})$ (often uncountable) is known to separate points on E, and one desires if a subset $A \subset E$ is $\mathcal{D}$-baseable. In other words, one desires reducing this specific $\mathcal{D}$ to a countable subcollection separating points on A. One sufficient condition is the *hereditary Lindelöf property* (see p.218) of A.

Proposition 3.40. *Let E be a topological space, $A \in \mathscr{B}(E)$ and $\mathcal{D} \subset C(E; \mathbf{R})$ separate points on A. If $\{(x, x) : x \in A\}$ is a Lindelöf subspace of $E \times E$, especially if A is a Souslin or second-countable subspace of E, then A is a $\mathcal{D}$-baseable subset of E.*

Proof. If A is a Souslin or second-countable subspace of E, then $A \times A$ is a hereditary Lindelöf subspace of $E \times E$ by Proposition A.11 (d, f) and Proposition A.3 (b, c), which implies $\{(x, x) : x \in A\}$ is a Lindelöf subspace of $E \times E$. Now, the result follows by Proposition A.24 (a) (with $E = (A, \mathscr{O}_E(A))$ and $\mathcal{D} = \mathcal{D}|_A$). ☐

Remark 3.41. Separating points is usually weaker than strongly separating points. Compared to Proposition A.24 (b), the selection result above uses hereditary Lindelöf property, which is weaker than second-countability.

Remark 3.42. When E is a Tychonoff space, $C(E; \mathbf{R})$ separates points on E by Proposition A.25 (a, b). So, Proposition 3.40 (with $\mathcal{D} = C(E; \mathbf{R})$) slightly generalizes [Bogachev, 2007, Vol. II, Theorem 6.7.7 (ii)].

Moreover, the point-separating functions can always be selected from a uniformly dense collection.

[21] The notation "$\mathfrak{ac}(\cdot)$" was defined in §2.2.3.

Proposition 3.43. *Let E be a topological space and A be a $\mathcal{D}$-baseable subset. If $\mathcal{D}_0 \subset C_b(E;\mathbf{R})$ satisfies $\mathcal{D} \subset \mathfrak{cl}(\mathcal{D}_0)$, then E is a $\mathcal{D}_0$-baseable subset.*

Proof. Let $\{f_n\}_{n\in\mathbf{N}} \subset \mathcal{D}$ separate points on A. $\mathcal{D}$ and its superset $\mathfrak{cl}(\mathcal{D}_0)$ both lie in the normed space $(M_b(E;\mathbf{R}), \|\cdot\|_\infty)$, so there exist $\{f_{n,k}\}_{n,k\in\mathbf{N}} \subset \mathcal{D}_0$ such that $f_{n,k} \xrightarrow{u} f_n$ as $k \uparrow \infty$ for each $n \in \mathbf{N}$ by Fact A.9. Hence, $\{f_n\}_{n\in\mathbf{N}} \subset \mathfrak{cl}(\{f_{n,k}\}_{n,k\in\mathbf{N}})$ and $\{f_{n,k}\}_{n,k\in\mathbf{N}}$ separates points on A by Corollary A.19 (with $\mathcal{D} = \{f_n\}_{n\in\mathbf{N}}$ and $\mathcal{D}_0 = \{f_{n,k}\}_{n,k\in\mathbf{N}}$). □

3.3.3 Baseable standard Borel subsets

In general, standard Borel subsets and Borel subsets are not the same. However, they coincide for a baseable standard Borel subspace. It is worth noting that the following result generalizes its classical version established on metrizable spaces (see Proposition A.56 (b)).

Proposition 3.44. *Let E be a topological space and $A \in \mathscr{B}^{\mathrm{s}}(E)$. Then the following statements are true:*

(a) If $(A, \mathscr{O}_E(A))$ is a baseable space, then $\mathscr{B}^{\mathrm{s}}(A, \mathscr{O}_E(A)) = \mathscr{B}_E(A) \subset \mathscr{B}^{\mathrm{s}}(E)$.

(b) If E is a baseable space, then $\mathscr{B}^{\mathrm{s}}(E) \subset \mathscr{B}(E)$.

(c) If E is a baseable standard Borel space, then $\mathscr{B}(E) = \mathscr{B}^{\mathrm{s}}(E)$.

Proof. (a) $\mathscr{B}_E(A) \subset \mathscr{B}^{\mathrm{s}}(A, \mathscr{O}_E(A)) \subset \mathscr{B}^{\mathrm{s}}(E)$ by Proposition A.56 (a). Now, let $B \in \mathscr{B}^{\mathrm{s}}(A, \mathscr{O}_E(A))$. There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \mathscr{O}_E(A))$ by Fact 3.33 (d, e). Then, $B \in \mathscr{B}_E(A)$ by Lemma 3.14 (b) (with $d = 1$, $E = E_0 = (A, \mathscr{O}_E(A))$ and $A = B$).

(b) There exists a base $(E, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E by Fact 3.33 (d, e) (with $A = E$). Then, (b) follows by Lemma 3.14 (b) (with $d = 1$ and $E_0 = E$).

(c) follows immediately by (a) (with $A = E$). □

For $\mathcal{D}$-baseable standard Borel subsets, the function class $\mathcal{D}$ not only separates their points but also determines their subspace Borel σ-algebras.

Proposition 3.45. *Let E be a topological space and $A \in \mathscr{B}^{\mathrm{s}}(E)$. Then, A is a $\mathcal{D}$-baseable subset of E if and only if $A \in \mathscr{B}(E)$ and there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ such that $\mathscr{B}_E(A) = \mathscr{B}_{\mathcal{D}_0}(A) = \sigma(\mathcal{D}_0)|_A$*[22]*.*

Proof. The $\mathcal{D}$-baseability of A implies $A \in \mathscr{B}(E)$ and a countable $\mathcal{D}_0 \subset \mathcal{D} \subset C(E; \mathbf{R})$ that separates points on A. Then, the result follows by Proposition A.60 (b, c) (with $E = (A, \mathscr{O}_E(A))$ and $\mathcal{D} = \mathcal{D}_0|_A$). $\square$

Baseability facilitates countable union of standard Borel subsets.

Proposition 3.46. *Let E be a topological space, $\mathbf{I}$ be a countable set, $\{A_i\}_{i\in\mathbf{I}} \subset \mathscr{B}^{\mathrm{s}}(E)$ and $A = \bigcup_{i\in\mathbf{I}} A_i$. If $(A, \mathscr{O}_E(A))$ is a baseable space, then $A \in \mathscr{B}^{\mathrm{s}}(E)$.*

Proof. There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \mathscr{O}_E(A))$ by Fact 3.33 (d, e). Then, $A \in \mathscr{B}^{\mathrm{s}}(E)$ by Lemma 3.14 (c) (with $E = E_0 = (A, \mathscr{O}_E(A))$ and $d = 1$). $\square$

Baseabe standard Borel support often implies unique Borel extension.

Proposition 3.47. *Let E be a topological space, A be a baseable subset of E, $d \in \mathbf{N}$ and $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$. If μ is supported on $B \in \mathscr{B}^{\mathrm{s}}(E^d)$ and $B \subset A^d$, then $\mathfrak{be}(\mu)$ is a singleton.*

Proof. The baseability of A implies $A \in \mathscr{B}(E)$ and $A^d \in \mathscr{B}(E)^{\otimes d}$. One finds by Fact 2.1 (a) (with $\mathscr{U} = \mathscr{B}(E)^{\otimes d}$ and $A = A^d$) that $\mu|_{A^d} \in \mathfrak{M}^+(A^d, \mathscr{B}(E)^{\otimes d}|_{A^d})$. There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \mathscr{O}_E(A))$ by Fact 3.33 (d, e). $\mathfrak{be}(\mu|_{A^d})$ is a singleton by Corollary 3.17 (a) (with $E = E_0 = (A, \mathscr{O}_E(A))$, $A = B$ and $\mu = \mu|_{A^d}$). Now, $\mathfrak{be}(\mu)$ is a singleton by the fact $A^d \in \mathscr{B}(E)^{\otimes d}$ and Lemma B.48 (b) (with $\mathbf{I} = \{1, ..., d\}$, $S_i = E$, $S = E^d$, $\mathscr{A} = \mathscr{B}(E)^{\otimes d}$ and $A = A^d$). $\square$

The following three results relate the baseability of a standard Borel space E and that of $\mathcal{P}(E)$.

Proposition 3.48. *Let E be a topological space, $\mathcal{D} \subset C_b(E; \mathbf{R})$ and $A \in \mathscr{B}^{\mathrm{s}}(E)$. If $(A, \mathscr{O}_E(A))$ is a $\mathcal{D}|_A$-baseable space, then $\mathcal{P}(A, \mathscr{O}_E(A))$ is an $\mathfrak{mc}(\mathcal{D}|_A)^*$-baseable space.*

[22] The σ-algebra $\sigma(\mathcal{D}_0)$ was defined in §2.1.2. Recall that $\sigma(\mathcal{D}_0)|_A$ is generally smaller than $\mathscr{B}_{\mathcal{D}_0}(A)$.

Proof. There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \mathscr{O}_E(A))$ with $(\mathcal{F}\backslash\{1\}) \subset \mathcal{D}|_A$ by Lemma 3.39 (c) (with $E = E_0 = (A, \mathscr{O}_E(A))$ and $\mathcal{D} = \mathcal{D}|_A$). Then, $\mathfrak{mc}(\mathcal{F}\backslash\{1\})^*$ is a countable subset of $\mathfrak{mc}(\mathcal{D}|_A)^*$ by Fact B.15 and separates points on $\mathcal{P}(A, \mathscr{O}_E(A))$ by Corollary 3.19 (a) (with $E = (A, \mathscr{O}_E(A))$ and $d = 1$). ☐

Corollary 3.49. *Let E be a baseable standard Borel space. Then, $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are baseable spaces.*

Proof. This corollary follows by Fact 3.33 (a, c) (with $A = E$), Proposition 3.48 (with $\mathcal{D} = C_b(E; \mathbf{R})$ and $A = E$) and Fact B.22 (a) (with $\mathcal{D} = C_b(E; \mathbf{R})$). ☐

Proposition 3.50. *Let E be a first-countable space and $\{\{x\} : x \in E\} \subset \mathscr{B}(E)$*[23]*. If $\mathcal{P}(E)$ is a baseable space, then E is also.*

Proof. We suppose $\{g_n\}_{n\in\mathbf{N}} \subset C(\mathcal{P}(E); \mathbf{R})$ separates points on $\mathcal{P}(E)$ and define $f_n(x) \mathrel{\hat{=}} g_n(\delta_x)$[24] for all $x \in E$ and $n \in \mathbf{N}$. For distinct $x, y \in E$, $\delta_x \neq \delta_y$ by Proposition A.41 (a) and so $\bigotimes_{n\in\mathbf{N}} f_n(x) = \bigotimes_{n\in\mathbf{N}} g_n(\delta_x) \neq \bigotimes_{n\in\mathbf{N}} g_n(\delta_y) = \bigotimes_{n\in\mathbf{N}} f_n(y)$. Hence, $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{R}^E$ separates points on E. We show each $f_n \in C(E; \mathbf{R})$. If $x_k \to x$ as $k \uparrow \infty$ in E, then $\delta_{x_k} \Rightarrow \delta_x$ as $k \uparrow \infty$ in $\mathcal{P}(E)$ by Fact B.24. It follows by the continuity of g_n that $\lim_{k\to\infty} f_n(x_k) = \lim_{k\to\infty} g_n(\delta_{x_k}) = g_n(\delta_x) = f_n(x)$. Now, the continuity of f_n follows by the first-countability of E and [Munkres, 2000, Theorem 30.1 (b)]. ☐

3.3.4 Metrizable compact subsets

Metrizable compact subsets form an essential class of hereditary Lindelöf, standard Borel, baseable subsets for replication and weak convergence. The following proposition gives several equivalent forms of metrizable compact subsets.

Proposition 3.51. *Let E be a topological space, $K \in \mathscr{K}(E)$ and $\mathcal{D} \subset C(E; \mathbf{R})$. Consider the following statements:*

(a) K is a $\mathcal{D}$-baseable subset of E.

[23] That singletones are Borel sets is even milder than the Hausdorff property or the *T1 axiom* (see [Munkres, 2000, §17, p.99]).

[24] δ_x denotes the Dirac measure at x.

(b) K is a Souslin subspace of E.

(c) K is a Hausdorff subspace of E and $\{(x,x) : x \in K\}$ is a Lindelöf subspace of $E \times E$.

(d) $(K, \mathscr{O}_E(K))$ is a baseable space.

(e) K is a Hausdorff and second-countable subspace of E.

(f) $K \in \mathscr{K}^{\mathbf{m}}(E)$.

(g) $K \in \mathscr{B}^{\mathbf{s}}(E)$.

Then, (b) - (f) are equivalent and implied by (a). (f) implies (g). Morever, if $\mathcal{D}$ separates points on E, then (a) - (f) are all equivalent.

Proof. ((b) $\to$ (c)) follows by Proposition A.11 (d, f).

((c) $\to$ (d)) $(K, \mathscr{O}_E(K))$ is Tychonoff by Proposition A.12 (d) and Proposition A.26 (a). Then, (d) follows by Proposition A.25 (a, b) (with $E = (K, \mathscr{O}_E(K))$) and Proposition 3.40 (with $E = A = (K, \mathscr{O}_E(K))$ and $\mathcal{D} = C(K, \mathscr{O}_E(K); \mathbf{R})$).

((d) $\to$ (e, f)) $(K, \mathscr{O}_E(K))$ is a Hausdorff space by Fact 3.24 (a). Let $\mathcal{D} \subset C_b(K, \mathscr{O}_E(K); \mathbf{R})$ be countable and separate points on $(K, \mathscr{O}_E(K))$. $\mathcal{D}$ strongly separates points on $(K, \mathscr{O}_E(K))$ by Lemma A.20. Now, both (e) and (f) follow by Proposition A.17 (d).

((e) $\to$ (c)) follows by Proposition A.3 (b).

((f) $\to$ (b, g)) follows by Proposition A.12 (d), Proposition A.11 (a) and Fact A.48 (a).

((a) $\to$ (d)) follows by Fact 3.34 (b, d) (with $\mathcal{D}' = C(E; \mathbf{R})$) and Fact 3.33 (a, d) (with $A = K$).

Moreover, if $\mathcal{D}$ separates points on E, then $K \in \mathscr{B}(E)$ by Proposition A.17 (e) (with $A = E$) and Proposition A.12 (a), and (c) implies (a) by Proposition 3.40 (with $A = K$). $\square$

Baseable spaces, Lusin spaces and Souslin spaces are not necessarily metrizable, but all of them have metrizable compact subsets.

Corollary 3.52. *Let E be a topological space and $K \in \mathscr{K}(E)$. Then, the following statements are true:*

(a) If E is a baseable space, then K is a metrizable standard Borel subspace and is a baseable subset of E.

(b) If E is a Souslin or Lusin space, then K is a metrizable, baseable, standard Borel subspace of E. If, in addition, $C(E;\mathbf{R})$ separates points on E, then K is a baseable subset of E.

Proof. (a) follows by Fact 3.35 and Proposition 3.51 (a, f, g) (with $\mathcal{D} = C(E;\mathbf{R})$). Regarding (b), we note by Proposition A.12 (a), Proposition A.11 (a, b) and Proposition A.2 (c) that Lusin (resp. Souslin) spaces are Souslin (resp. Hausdorff) spaces and compact subsets of a Souslin space are closed, Souslin, Hausdorff subspaces. Then, (b) follows by Proposition 3.51 (a, b, d, f, g) (with $\mathcal{D} = C(E;\mathbf{R})$). □

The two results above indicated that many non-metrizable topological spaces like those in Example 3.27 have metrizable compact subsets. Meanwhile, we remind the readers of that having metrizable compact subsets is a strictly milder property than baseability.

Example 3.53. $([0,1]^{[0,1]}, \|\cdot\|_\infty)$ is a Banach space so it certainly has metrizable compact subsets. However, it is not baseable as in Example 3.29.

Here are several constructive properties of metrizable compact subsets.

Lemma 3.54. *Let E be a topological space, $m \in \mathbf{N}$ and $\{A_i\}_{1\leq i\leq m} \subset \mathscr{K}^{\mathbf{m}}(E)$. If $A = \bigcup_{i=1}^m A_i$ is a Hausdorff subspace of E, then $A \in \mathscr{K}^{\mathbf{m}}(E)$.*

Proof. $A \in \mathscr{K}(E)$ by Proposition A.12 (b). A is a Souslin subspace of E by Proposition A.11 (g). Now, the result follows by Proposition 3.51 (b, f). □

Lemma 3.55. *Let $\mathbf{I}$ be a countable index set, $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $S \doteq \prod_{i\in\mathbf{I}} S_i$. Then, the following statements are true:*

(a) If $A_i \in \mathscr{K}^{\mathbf{m}}(S_i)$ for all $i \in \mathbf{I}$, then $\prod_{i\in\mathbf{I}} A_i \in \mathscr{K}^{\mathbf{m}}(S)$.

(b) If $A \in \mathscr{K}^{\mathbf{m}}(S)$ and $\mathfrak{p}_i(A)$ is a Hausdorff subspace of S_i for some $i \in \mathbf{I}$, then $\mathfrak{p}_i(A) \in \mathscr{K}^{\mathbf{m}}(S_i)$.

Proof. (a) follows by Proposition A.12 (b) and Proposition A.8.

(b) $(A, \mathscr{O}_S(A))$ is hereditary Lindelöf by Proposition 3.51 (b, f) (with $E = S$), so $\{(x, x) : x \in A\}$ is a Lindelöf subspace of S. It follows that $\mathfrak{p}_i(A) \in \mathscr{K}(S_i)$ and

$$\{(y, y) : y \in \mathfrak{p}_i(A)\} = \{(\mathfrak{p}_i(x), \mathfrak{p}_i(x)) : x \in A\} \tag{3.3.2}$$

is a Lindelöf subspace of S_i by Fact 2.4 (a), Proposition A.12 (e) and Proposition A.3 (d). Thus, $\mathfrak{p}_i(A) \in \mathscr{K}^{\mathbf{m}}(S_i)$ by its Hausdorff proeprty and Proposition 3.51 (c, f). $\square$

When generalizing **m**-tightness to non-Borel measures, Definition 2.18 has to require a collection of compact sets lying in their domains. The next lemma shows that if E is a Hausdorff space, then this requirement is automatically satisfied by the members of $\mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$[25].

Lemma 3.56. *Let* $\mathbf{I}$ *be a countable index set,* $\{S_i\}_{i \in \mathbf{I}}$ *be topological spaces,* $(S, \mathscr{A})$ *be as in (2.7.22) and* $A \in \mathscr{K}^{\mathbf{m}}(S)$. *If* $B_i \in \mathscr{B}(S_i)$ *is a Hausdorff subspace of* S_i *and contains* $\mathfrak{p}_i(A)$ *for all* $i \in \mathbf{I}$, *then* $A \in \mathscr{A}$ *and* $\mathscr{B}_S(A) = \mathscr{A}|_A$.

Proof. $\mathfrak{p}_i(A) \in \mathscr{K}^{\mathbf{m}}(S_i)$ for all $i \in \mathbf{I}$ by Proposition A.2 (c) and Lemma 3.55 (b). As $B_i \in \mathscr{B}(S_i)$ for all $i \in \mathbf{I}$, we have that

$$A \subset F \circeq \prod_{i \in \mathbf{I}} \mathfrak{p}_i(A) \in \bigotimes_{i \in \mathbf{I}} \mathscr{B}_{S_i}(B_i) \subset \mathscr{A} \tag{3.3.3}$$

by Corollary A.13 (a) (with $S_i = B_i$). $F \in \mathscr{K}^{\mathbf{m}}(S)$ by Lemma 3.55 (a). F is a second-countable subspace of S by Proposition 3.51 (e, f).

$$\mathscr{B}_S(F) = \mathscr{A}|_F \subset \mathscr{A} \tag{3.3.4}$$

by Proposition B.46 (c) (with $S_i = \mathfrak{p}_i(A)$) and (3.3.3). This implies $\mathscr{B}_S(A) = \mathscr{A}|_A$ since $A \subset F$. Moreover, F is a Hausdorff subspace of S by Proposition A.2 (d). Hence, $A \in \mathscr{B}_S(F) \subset \mathscr{A}$ by Proposition A.12 (a) and (3.3.4). $\square$

We now show that **m**-tightness ensures unique and tight Borel extension.

[25] We remind the readers again that the Borel σ-algebra of the product topological space E^d is possibly different from the product σ-algebra $\mathscr{B}(E)^{\otimes d}$. Moreover, the notation $\mathfrak{M}^+(\cdot)$ means the family of all non-negative finite measures.

Proposition 3.57. *Let* $\mathbf{I}$ *be a countable index set,* $\{S_i\}_{i\in\mathbf{I}}$ *be topological spaces,* $(S,\mathscr{A})$ *be as in (2.7.22),* $\Gamma\subset\mathfrak{M}^+(S,\mathscr{A})$ *and* $A\subset S$*. Suppose in addition that* $\mathfrak{p}_i(A)\in\mathscr{B}(S_i)$ *is a Hausdorff subspace of* S_i *for all* $i\in\mathbf{I}$*. Then,* Γ *is* $\mathbf{m}$*-tight in* A *if and only if* $\{\mu'=\mathfrak{be}(\mu)\}_{\mu\in\Gamma}$[26] *is* $\mathbf{m}$*-tight in* A*.*

Proof. $\mathscr{K}^{\mathbf{m}}(A,\mathscr{O}_S(A))\subset\mathscr{A}$ by Lemma 3.56 (with $B_i=\mathfrak{p}_i(A)$). So, the $\mathbf{m}$-tightness of Γ and that of $\{\mu'\}_{\mu\in\Gamma}$ (if any) are equivalent. It now suffices to show the $\mathbf{m}$-tightness of Γ in A implies the existence of $\mu'=\mathfrak{be}(\mu)$ for each $\mu\in\Gamma$. Given such tightness, μ is supported on some $B\in\mathscr{K}^{\mathbf{m}}_{\sigma}(A,\mathscr{O}_S(A))$. $\mathscr{B}_S(B)=\mathscr{A}|_B$ by Lemma 3.56 (with $B_i=\mathfrak{p}_i(A)$). Then, the unique existence of μ' follows by Lemma B.48 (c) (with $A=B$). □

3.3.5 σ-metrizable compact subsets

σ-metrizable compact subsets inherit many nice properties from its metrizable compact components.

Proposition 3.58. *Let* E *be a topological space,* $\{K_n\}_{n\in\mathbf{N}}\subset\mathscr{K}(E)$*,* $A=\bigcup_{n\in\mathbf{N}}K_n$ *and* $\mathcal{D}\subset C(E;\mathbf{R})$*. Consider the following statements:*

(a) $(A,\mathscr{O}_E(A))$ *is a Souslin space.*

(b) $\{K_n\}_{n\in\mathbf{N}}\subset\mathscr{K}^{\mathbf{m}}(E)$ *(hence* $A\in\mathscr{K}^{\mathbf{m}}_{\sigma}(E)$*).*

(c) $(K_n,\mathscr{O}_E(K_n))$ *is a baseable space for all* $n\in\mathbf{N}$*.*

(d) $(A,\mathscr{O}_E(A))$ *is a baseable space.*

(e) A *is a* $\mathcal{D}$*-baseable subset of* E*.*

(f) $A\in\mathscr{B}^{\mathbf{s}}(E)$*.*

Then, (a) - (e) are successively stronger. (e) implies (f). Moreover, if $\mathcal{D}$ *separates points on* E*, then (a) - (e) are all equivalent.*

Proof. ((b) → (a)) Each $(K_n,\mathscr{O}_E(K_n))$ is Souslin by Proposition 3.51 (b, f). Hence, (a) follows by Proposition A.11 (g).

((c) → (b)) follows by Proposition 3.51 (d, f) (with $K=K_n$).

[26] The notation "$\mu'=\mathfrak{be}(\mu)$" defined in §2.3 means μ' is the unique Borel extension of μ.

((d) $\to$ (c)) follows by Fact 3.35 (a, b) (with $E = (A, \mathscr{O}_E(A))$ and $A = K_n$).

((e) $\to$ (d)) is automatic by definition.

((e) $\to$ (f)) We showed (e) implies (b) - (d). Then, (f) follows by (d), Proposition 3.51 (f, g) and Proposition 3.46.

When $\mathcal{D}$ separates points on E, $A \in \mathscr{B}(E)$ by Proposition A.17 (e) (with $A = E$) and Proposition A.12 (a), and (a) implies (e) by Proposition 3.40. □

Remark 3.59. σ-metrizable compact subsets are baseable subspaces but not necessarily metrizable. So, "σ-metrizable compact" is generally weaker than "metrizable σ-compact".

Below are several constructive properties of σ-metrizable compact subsets.

Lemma 3.60. *Let E be a topological space and $\{A_n\}_{n\in\mathbf{N}} \subset \mathscr{K}_\sigma^{\mathbf{m}}(E)$. If $A = \bigcup_{n\in\mathbf{N}} A_n$ is a Hausdorff subspace of E, then there exist $\{K_q\}_{q\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ such that $A = \bigcup_{q\in\mathbf{N}} K_q$ and $K_q \subset K_{q+1}$ for all $q \in \mathbf{N}$.*

Proof. Let $A_n = \bigcup_{p\in\mathbf{N}} K_{p,n}$ with $\{K_{p,n}\}_{p\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ for each $n \in \mathbf{N}$. Define $K_q \circeq \bigcup_{p=1}^{q} \bigcup_{n=1}^{q} K_{p,n}$ for each $q \in \mathbf{N}$, so $K_q \subset K_{q+1}$ for all $q \in \mathbf{N}$ and $A = \bigcup_{q\in\mathbf{N}} K_q$. Furthermore, each $(K_q, \mathscr{O}_E(K_q))$ is a Hausdorff space by Proposition A.2 (c) and hence is metrizable by Lemma 3.54. □

Lemma 3.61. *Let $\mathbf{I}$ be a countable index set, $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $S \circeq \prod_{i\in\mathbf{I}} S_i$. Then, the following statements are true:*

(a) If $\mathbf{I}$ is finite and $A_i \in \mathscr{K}_\sigma^{\mathbf{m}}(S_i)$ for all $i \in \mathbf{I}$, then $\prod_{i\in\mathbf{I}} A_i \in \mathscr{K}_\sigma^{\mathbf{m}}(S)$.

(b) If $A \in \mathscr{K}_\sigma^{\mathbf{m}}(S)$ and $\mathfrak{p}_i(A)$ is a Hausdorff subspace of S_i for all $i \in \mathbf{I}$, then $\mathfrak{p}_i(A) \in \mathscr{K}_\sigma^{\mathbf{m}}(S_i)$ for all $i \in \mathbf{I}$.

Proof. (a) Without loss of generality, we suppose $\mathbf{I} = \{1, ..., d\}$ and let $A_i = \bigcup_{p\in\mathbf{N}} K_{p,i}$ with $\{K_{p,i}\}_{p\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(S_i)$ for each $1 \le i \le d$. We have that

$$\prod_{i=1}^{d} A_i \supset F_{p_1,...,p_d} \circeq \prod_{i\in\mathbf{I}} K_{p_i,i} \in \mathscr{K}^{\mathbf{m}}(S),\ \forall p_1, ..., p_d \in \mathbf{N} \tag{3.3.5}$$

by Lemma 3.55 (a). For any $x \in \prod_{i=1}^{d} A_i$, there exist $p_1, ..., p_d \in \mathbf{N}$ such that

$$\mathfrak{p}_i(x) \in K_{p_i,i},\ \forall 1 \le i \le d. \tag{3.3.6}$$

It then follows by (3.3.5) that

$$\prod_{i=1}^{d} A_i = \bigcup_{(p_1,\dots,p_d)\in\mathbf{N}^d} F_{p_1,\dots,p_d} \in \mathscr{K}_\sigma^{\mathbf{m}}(S). \tag{3.3.7}$$

(b) Let $A = \bigcup_{p\in\mathbf{N}} K_p$ with $\{K_p\}_{p\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(S)$. We have that $\mathfrak{p}_i(K_p) \in \mathscr{K}^{\mathbf{m}}(S_i)$ for all $p \in \mathbf{N}$ and $i \in \mathbf{I}$ by the fact $\mathfrak{p}_i(K_p) \subset \mathfrak{p}_i(A)$, the Hausdorff property of $\mathfrak{p}_i(A)$, Proposition A.2 (c) and Lemma 3.55 (b). Hence, $\mathfrak{p}_i(A) = \mathfrak{p}_i(\bigcup_{p\in\mathbf{N}} K_p) = \bigcup_{p\in\mathbf{N}} \mathfrak{p}_i(K_p) \in \mathscr{K}_\sigma^{\mathbf{m}}(S_i)$ for all $i \in \mathbf{I}$. $\square$

3.3.6 Baseability about Skorokhod $\mathscr{J}_1$-space

When E is a Tychonoff space, the associated Skorokhod $\mathscr{J}_1$-space $D(\mathbf{R}^+; E)$ is also Tychonoff (see Proposition A.62 (c)). The following proposition shows that baseability of E passes to $D(\mathbf{R}^+; E)$.

Proposition 3.62. *Let E be a Tychonoff space. Then, the following statements are true:*

(a) If E is a $\mathcal{D}$-baseable space with $\mathcal{D} \subset C_b(E; \mathbf{R})$, then $D(\mathbf{R}^+; E)$[27] is a $\{\alpha_{t,n}^f : f \in \mathcal{D}, t \in \mathbf{Q}^+\}$-baseable space with

$$\alpha_{t,n}^f(x) \circeq \frac{1}{n} \int_t^{t+1/n} f(x(s))ds \tag{3.3.8}$$

for each $f \in \mathcal{D}$, $t \in \mathbf{Q}^+$ and $n \in \mathbf{N}$.

(b) If E is a baseable space, then $D(\mathbf{R}^+; E)$ is also a baseable space and $J(x) \subset (0, \infty)$[28] is at most countable for all $x \in D(\mathbf{R}^+; E)$.

Proof. (a) Without loss of generality, we suppose $\mathcal{D}$ is countable. Then, (a) follows immediately by Proposition A.62 (b).

(b) There exists a countable $\mathcal{D} \subset C_b(E; \mathbf{R})$ separating points on E by Fact 3.33 (a, b) (with $A = E$) amd so $\varphi \circeq \bigotimes \mathcal{D}$ is injective. $\varphi \in C(E; \mathbf{R}^{\mathcal{D}})$ by Fact

[27] The Skorokhod $\mathscr{J}_1$-space $D(\mathbf{R}^+; E)$ was defined in §2.2.2.
[28] The notation "$J(x)$" was defined in §2.2.1.

2.4 (b) and $D(\mathbf{R}^+;E)$ is baseable by (a). $\mathbf{R}^{\mathcal{D}}$ is a Polish space by Proposition A.11 (f). Therefore,

$$J(x) = J\left[\varpi(\varphi)(x)\right], \ \forall x \in D(\mathbf{R}^+;E) \tag{3.3.9}$$

by Proposition A.62 (d) (with $S = E$, $E = \mathbf{R}^{\mathcal{D}}$ and $f = \varphi$) and both sets above are countable subsets of $(0,\infty)$ by [Ethier and Kurtz, 1986, §3.5, Lemma 5.1]. □

Metrizability of compact subsets of E also passes to $D(\mathbf{R}^+;E)$.

Proposition 3.63. *If E is a Tychonoff space with $\mathscr{K}(E) = \mathscr{K}^{\mathbf{m}}(E)$, then $\mathscr{K}(D(\mathbf{R}^+;E)) = \mathscr{K}^{\mathbf{m}}(D(\mathbf{R}^+;E))$.*

Proof. Let $K \in \mathscr{K}(D(\mathbf{R}^+;E))$. By Proposition A.67, there exist $\{A_n\}_{n\in\mathbf{N}} \subset \mathscr{K}(E) = \mathscr{K}^{\mathbf{m}}(E)$ such that $K \subset D(\mathbf{R}^+;A)$[29] with $A \circeq \bigcup_{n\in\mathbf{N}} A_n$. A is a baseable space by Proposition A.25 (a, c) and Proposition 3.58 (b, d) (with $K_n = A_n$ and $\mathcal{D} = C_b(E;\mathbf{R})$). $D(\mathbf{R}^+;A)$ is a baseable space by Proposition 3.62 (b) (with $E = A$). K is a baseable subspace of $D(\mathbf{R}^+;E)$ by Fact 3.35 and Corollary A.65. Hence, K is metrizable by Proposition 3.51 (d, f) (with $E = D(\mathbf{R}^+;E)$). □

The countability of fixed left-jump times of an E-valued càdlàg process is well-known when E is a metrizable and separable space. Herein, we extend this fact to baseable Tychonoff spaces.

Proposition 3.64. *Let E be a baseable Tychonoff space. Then, the following statements are true:*

(a) For any $\mu \in \mathfrak{M}^+(D(\mathbf{R}^+;E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$[30] (especially for any $\mu \in \mathcal{M}^+(D(\mathbf{R}^+;E))$), $J(\mu)$[31] is a well-defined countable subset of $(0,\infty)$.

(b) For any E-valued càdlàg process X, $J(X)$[32] is a well-defined countable subset of $(0,\infty)$.

[29] $D(\mathbf{R}^+;\bigcup_{n\in\mathbf{N}} A_n)$ is well-defined by Corollary A.65.

[30] Please be reminded that $\mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)}$ is generally smaller than $\mathscr{B}(D(\mathbf{R}^+;E))$.

[31] $J(\mu)$, the set of fixed left-jump times of μ was defined in (2.3.10).

[32] $J(X)$, the set of fixed left-jump times of X was defined in (2.5.8).

In particular, the conclusions above are true when E is a metrizable and separable space or a Polish space.

Proof. (a) Polish spaces, metrizable and separable spaces and baseable spaces are successively wider classes by Fact 3.24 (b) and Proposition A.11 (c). As E is a baseable space, there exists a countable $\mathcal{D} \subset C_b(E; \mathbf{R})$ separating points on E by Fact 3.33 (a, b) (with $A = E$) and so $\varphi \circeq \bigotimes \mathcal{D}$ is injective. We deduce $\varphi \in C(E; \mathbf{R}^{\mathcal{D}})$, (3.3.9) and

$$\begin{aligned} \varpi(\varphi) \in & M \Big(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)}; \\ & D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}), \mathscr{B}(\mathbf{R}^{\mathcal{D}})^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;\mathbf{R}^{\mathcal{D}})} \Big) \end{aligned} \tag{3.3.10}$$

from Fact 2.4 (b), Proposition A.62 (d) (with $S = E$, $E = \mathbf{R}^{\mathcal{D}}$ and $f = \varphi$) and Fact B.10 (b) (with $f = \varphi$).

$$\nu \circeq \mu \circ \varpi(\varphi)^{-1} \in \mathfrak{M}^+ \Big(D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}), \mathscr{B}(\mathbf{R}^{\mathcal{D}})^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;\mathbf{R}^{\mathcal{D}})} \Big) \tag{3.3.11}$$

by (3.3.10). $\mathbf{R}^{\mathcal{D}}$ as aforementioned is a Polish space, so (3.3.9) implies

$$\begin{aligned} & \mu \left(\left\{ x \in D(\mathbf{R}^+; E) : t \in J(x) \right\} \right) = \mu \left(\left\{ x \in D(\mathbf{R}^+; E) : t \in J\left[\varpi(\varphi)(x) \right] \right\} \right) \\ & = \nu \left(\left\{ y \in D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}) : t \in J(y) \right\} \right), \ \forall t \in \mathbf{R}^+, \end{aligned} \tag{3.3.12}$$

while the equalities in (3.3.12) as well as $J(\mu)$ and $J(\nu)$ are well-defined by Fact A.71. Hence, we have $J(\mu) = J(\nu)$ by (3.3.12) and this set is a countable subset of $(0, \infty)$ by [Ethier and Kurtz, 1986, §3.7, Lemma 7.7].

(b) follows immediately by Fact 2.26 (a), the definition of $J(X)$ and (a) (with $\mu = \mathrm{pd}(X)|_{D(\mathbf{R}^+;E)}$[33]). □

[33] $\mathrm{pd}(X)$ denotes the process distribution of X and was specified in §2.5.

Chapter 4

Replication of Function and Operator

The previous chapter discussed the space change through a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over topological space E and the derivative notions of baseable spaces and baseable subsets. Now, we start to discuss the replication of objects from E onto $\widehat{E}$ and the association of the original and replica objects. §4.1 of this chapter introduces replica of continuous function and §4.2 introduces replica of linear operator on $C_b(E; \mathbf{R})$. Under the regularity conditions proposed in §4.2.3, the replica operators constructed in §4.2.4 are strong generators of semigroups on $C(\widehat{E}; \mathbf{R})$ which play substantial roles in our companion papers Dong and Kouritzin [2017a,b,d].

4.1 Replica function

Given a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over topological space E, replicating a function $f \in M(E; \mathbf{R})$ onto $\widehat{E}$ basically means extending $f|_{E_0}$ onto $\widehat{E}$. A simplistic approach is preserving the values of f on E_0 and assigning constant value 0 on $\widehat{E} \backslash E_0$. This idea is also used in several other aspects of this work, so we make the following general notation for simplicity.

Notation 4.1. Let E, S_1 and S_2 be non-empty sets, A be an arbitrary subset of $S_1 \cap S_2$, $y_0 \in E$ and $f \in E^{S_1}$. By $\mathfrak{var}(f; S_2, A, y_0)$[1] we denote the mapping

$$\mathfrak{var}(f; S_2, A, y_0)(x) \stackrel{\circ}{=} \begin{cases} f(x), & \text{if } x \in A, \\ y_0, & \text{otherwise,} \end{cases} \quad \forall x \in S_2 \tag{4.1.1}$$

from S_2 to E.

Remark 4.2. $\mathfrak{var}(f; S_2, f^{-1}(\{y_0\}), y_0) = \mathfrak{var}(f; S_2, \varnothing, y_0)$ is the constant mapping that sends all $x \in S_2$ to y_0.

$\mathfrak{var}(f; \widehat{E}, E_0, 0)$ may not preserve the continuity or even the Borel measurability of f if E_0 is not a standard Borel set. Nonetheless, noticing that (3.1.1) links every member of $\mathcal{F}$ bijectively to a member of $\widehat{\mathcal{F}} \subset C(\widehat{E}; \mathbf{R})$, we can define the replica of suitable continuous function on E as a continuous function on $\widehat{E}$.

Definition 4.3. Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d, k \in \mathbf{N}$. The **replica of** $f \in C(E^d; \mathbf{R}^k)$ with respect to $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ (if any) refers to the continuous extension $\widehat{f}$ of $f|_{E_0^d}$ on $\widehat{E}^d$.

Remark 4.4. We mentioned in Remark 3.5 that the compactification inducing $\widehat{E}$ does not necessarily extend every member of $C_b(E_0, \mathscr{O}_{\mathcal{F}}(E_0); \mathbf{R})$ continuously onto $\widehat{E}$. So, a general $f \in C_b(E^d; \mathbf{R}^k)$ need not have a replica.

Notation 4.5. Let $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d, k \in \mathbf{N}$. Hereafter, we *will always* let $\overline{f}$ denote the function $\mathfrak{var}(f; \widehat{E}, E_0, 0)$ for $f \in (\mathbf{R}^k)^{E^d}$ and $\widehat{f}$ denote the replica of $f \in C(E^d; \mathbf{R}^k)$ if no confusion is caused.

Below are several basic properties of $\overline{f}$ and $\widehat{f}$.

Proposition 4.6. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d, k \in \mathbf{N}$. Then, the following statements are true:*

(a) If $f \in (\mathbf{R}^k)^{E^d}$ is bounded, then $\overline{f}$ is also bounded.

(b) If $f \in M(E^d; \mathbf{R}^k)$ and $f|_{E_0^d \setminus A} = 0$ for some $A \in \mathscr{B}^{\mathbf{s}}(E^d)$ with $A \subset E_0^d$, then $\overline{f} \in M(\widehat{E}^d; \mathbf{R}^k)$. In particular, this is true if $E_0^d \in \mathscr{B}^{\mathbf{s}}(E^d)$.

[1] "$\mathfrak{var}$" is "var" in fraktur font which stands for "variant".

(c) *The replica of* $f \in C(E^d;\mathbf{R}^k)$ *(if any) is unique.*

(d) *If* $f_1, f_2 \in C(E^d;\mathbf{R}^k)$ *have replicas, then* $a\widehat{f_1} + b\widehat{f_2}$ *(resp.* $\widehat{f_1}\widehat{f_2}$ *when* $k = 1$*) is the replica of* $af_1 + bf_2$ *for all* $a, b \in \mathbf{R}$ *(resp. of* $f_1 f_2$*).*

(e) $\widehat{\mathcal{F}} = \{\widehat{f} : f \in \mathcal{F}\}$, $\mathfrak{ag}(\widehat{\mathcal{F}}) = \{\widehat{f} : f \in \mathfrak{ag}(\mathcal{F})\}$ *and* $\mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})] = \{\widehat{f} : f \in \mathfrak{ag}(\Pi^d(\mathcal{F}))\}$.

(f) $f \in C(E^d;\mathbf{R}^k)$ *admits a replica if and only if*

$$\mathfrak{p}_i \circ f|_{E_0^d} \in \mathfrak{ca}\left[\Pi^d\left(\mathcal{F}|_{E_0}\right)\right],\ \forall 1 \le i \le k. \tag{4.1.2}$$

In particular, every $f \in \mathfrak{ca}[\Pi^d(\mathcal{F})]$ *admits a replica.*

Proof. (a) The definition of $\overline{f}$ implies $\|\overline{f}\|_\infty \le \|f\|_\infty$.

(b) $\overline{h}|_A = h|_A \in M(A, \mathscr{O}_{\widehat{E}^d}(A);\mathbf{R}^k)$ by Lemma 3.14 (a). So, $\overline{h} = \overline{h}\mathbf{1}_A \in M(\widehat{E}^d;\mathbf{R}^k)$[2] by Fact B.2 (with $E = \widehat{E}^d$, $\mathscr{U} = \mathscr{B}(\widehat{E}^d)$ and $f = \overline{h}$).

(c) follows by the denseness of E_0^d in $\widehat{E}^d$, the fact $f|_{E_0^d} = \widehat{f}|_{E_0^d}$ and the continuities of f and $\widehat{f}$.

(d) follows by the fact that $(af_1 + bf_2)|_{E_0^d} = (a\widehat{f_1} + b\widehat{f_2})|_{E_0^d}$ and $f_1 f_2|_{E_0^d} = \widehat{f_1}\widehat{f_2}|_{E_0^d}$, that $a\widehat{f_1} + b\widehat{f_2} \in C(\widehat{E}^d;\mathbf{R}^k)$ and that $\widehat{f_1}\widehat{f_2} \in C(\widehat{E}^d;\mathbf{R})$ when $k = 1$.

(e) Letting $\mathcal{F} = \{f_n\}_{n\in\mathbf{N}}$, we have $\widehat{\mathcal{F}} = \{\widehat{f}_n\}_{n\in\mathbf{N}}$ by (3.1.1) and Lemma 3.3 (a). For each $1 \le l \le d$ and $n_1, ..., n_l \in \mathbf{N}$, $f \mathrel{\hat{=}} \prod_{i=1}^l f_{n_i} \circ \mathfrak{p}_i$ and $g \mathrel{\hat{=}} \prod_{i=1}^l \widehat{f}_{n_i} \circ \mathfrak{p}_i$ satisfy

$$f|_{E_0^d} = \prod_{i=1}^l f_{n_i}|_{E_0} \circ \mathfrak{p}_i = \prod_{i=1}^l \widehat{f}_{n_i}|_{E_0} \circ \mathfrak{p}_i = g|_{E_0^d}. \tag{4.1.3}$$

So, $g = \widehat{f}$ by (3.1.13) and (4.1.3). Note that the members of $\mathfrak{ag}(\mathcal{F})$ (resp. $\mathfrak{ag}[\Pi^d(\mathcal{F})]$) correspond bijectively to those of $\mathfrak{ag}(\widehat{\mathcal{F}})$ (resp. $\mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})]$). Now, (e) follows by (d).

(f - Necessity) If $\widehat{f}$ exists, then $\{\mathfrak{p}_i \circ \widehat{f}\}_{1\le i\le k} \subset C(\widehat{E}^d;\mathbf{R})$ by Fact 2.4 (a). Hence, (4.1.2) follows by (3.1.17).

(f - Sufficiency) If (4.1.2) holds, then $\{\widehat{\mathfrak{p}_i \circ f}\}_{1\le i\le k}$ exists by Corollary 3.10.

[2] $\mathbf{1}_A$ denotes the indicator function of A.

Hence, $\bigotimes_{i=1}^{k}\widehat{\mathfrak{p}_i\circ f}=\widehat{f}$ by Fact 2.4 (b) and the fact

$$f|_{E_0^d}=\bigotimes_{i=1}^{k}\mathfrak{p}_i\circ f|_{E_0^d}=\bigotimes_{i=1}^{k}\widehat{\mathfrak{p}_i\circ f}|_{E_0^d}. \tag{4.1.4}$$

□

Note 4.7. For the sake of brevity, hereafter we may use the replica of $f\in\mathfrak{ca}(\Pi^d(\mathcal{F}))$ without referring to Proposition 4.6 (f) for its existence.

The following proposition shows a nice property of locally compact baseable spaces which recovers [Srivastava, 1998, Corollary 2.3.32]. This is also an example where $\overline{f}$ and $\widehat{f}$ coincide.

Proposition 4.8. *Let E be a locally compact space and $\mathcal{D}\subset C_0(E;\mathbf{R})$*[3]*. Consider the following statements:*

(a) E is a $\mathcal{D}$-baseable space.

(b) There exists a base $(E,\mathcal{F};\widehat{E},\widehat{\mathcal{F}}\}$ over E such that $\widehat{E}$ is a one-point compactification of E, $(\mathcal{F}\backslash\{1\})\subset\mathcal{D}\subset C_0(E;\mathbf{R})\subset\mathfrak{ca}(\mathcal{F})$ and $\mathcal{F}$ strongly separates points on E.

(c) E is a Polish space.

(d) E is a metrizable and separable space.

(e) E is a $C_0(E;\mathbf{R})$-baseable space.

Then, (a) - (e) are successively weaker. Moreover, (e) implies (a) when $\mathcal{D}$ is uniformly dense in $C_0(E;\mathbf{R})$.

Proof. ((a) → (b)) By (a), there exists a countable $\mathcal{F}\subset(\mathcal{D}\cup\{1\})\subset C_b(E;\mathbf{R})$ that separates points on E. E is a Hausdorff space by Fact 3.24 (a) and admits a one-point compactification $\widehat{E}$ by Proposition A.31. It follows by Lemma A.32 (with $\mathcal{D}=\mathcal{F}$) that $\mathcal{F}$ strongly separates points on E and

$$\widehat{\mathcal{F}}\mathrel{\hat{=}}\left\{\mathfrak{var}(f:\widehat{E},E,0):f\in\mathcal{F}\backslash\{1\}\right\}\cup\{1\}\subset C(\widehat{E};\mathbf{R}) \tag{4.1.5}$$

[3] $C_0(E;\mathbf{R})$, the family of all $\mathbf{R}$-valued continuous functions vanishing at infinity was defined in §2.2.3.

separates points and strongly separates points on $\widehat{E}$. Hence, $(E,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ by definition is a base over E. Moreover, we get $C_0(E;\mathbf{R})\subset \mathfrak{ca}(\mathcal{F})$ by Corollary 3.10 (with $d=1$ and $E_0=E$) and Fact B.43.

((b) $\to$ (c)) $\mathscr{O}(E)=\mathscr{O}_{\mathcal{F}}(E)$ by (b), so E is an open subspace of the Polish space $\widehat{E}$ by Proposition A.2 (a) and Lemma 3.3 (b, c). Hence, (c) follows by Proposition A.11 (b).

((c) $\to$ (d)) follows by Proposition A.11 (c).

((d) $\to$ (e)) $C_0(E;\mathbf{R})$ separates points on E by Proposition A.33 (a, d). Then, (e) follows by Proposition A.6 (c) and Proposition 3.40 (with $A=E$ and $\mathcal{D}=C_0(E;\mathbf{R})$).

Moreover, if $C_0(E;\mathbf{R})\subset \mathfrak{cl}(\mathcal{D})$, then (e) implies (a) by Proposition 3.43 (with $A=E$, $\mathcal{D}=C_0(E;\mathbf{R})$ and $\mathcal{D}_0=\mathcal{D}$). □

4.2 Replica operator

We now focus on replicating a linear operator $\mathcal{L}$ on the Banach space $(C_b(E;\mathbf{R}),\|\cdot\|_\infty)$ as a linear operator on $(C(\widehat{E};\mathbf{R}),\|\cdot\|_\infty)$. Most concepts about linear operators used below were reviewed in §2.2.5 and, as aforementioned in §2.6, we always consider single-valued operators.

4.2.1 Definition

Replicating $\mathcal{L}$ from $C_b(E;\mathbf{R})$ onto $C(\widehat{E};\mathbf{R})$ means constructing a linear operator on $C(\widehat{E};\mathbf{R})$ whose domain and range are formed by the replicas of the member of $\mathfrak{D}(\mathcal{L})$ and $\mathfrak{R}(\mathcal{L})$ respectively. Under the following conditions, the replica of $\mathcal{L}$ exists as a densely defined operator on $C(\widehat{E};\mathbf{R})$.

Proposition 4.9. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E and $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ such that*

$$\begin{aligned}&\mathfrak{mc}(\mathcal{F})\subset \mathfrak{D}(\mathcal{L}),\\&\mathfrak{R}\left(\mathcal{L}|_{\mathfrak{mc}(\mathcal{F})}\right)\subset \mathfrak{ca}(\mathcal{F}).\end{aligned}\tag{4.2.1}$$

Then, the following statements are true:

(a) There exists a unique linear operator $\widehat{\mathcal{L}}_0$ *on* $C(\widehat{E};\mathbf{R})$ *such that*

$$\mathfrak{D}(\widehat{\mathcal{L}}_0) = \mathfrak{ag}(\widehat{\mathcal{F}}) \tag{4.2.2}$$

and

$$\widehat{\mathcal{L}}_0\widehat{f} = \widehat{\mathcal{L}f},\ \forall \widehat{f} \in \mathfrak{ag}(\widehat{\mathcal{F}}). \tag{4.2.3}$$

(b) There exists a unique linear operator $\widehat{\mathcal{L}}_1$ *on* $C(\widehat{E};\mathbf{R})$ *such that*

$$\mathfrak{D}(\widehat{\mathcal{L}}_1) = \left\{\widehat{f} : (f,\mathcal{L}f) \in \mathcal{L} \cap (\mathfrak{ca}(\mathcal{F}) \times \mathfrak{ca}(\mathcal{F}))\right\} \tag{4.2.4}$$

and

$$\widehat{\mathcal{L}}_1\widehat{f} = \widehat{\mathcal{L}f},\ \forall \widehat{f} \in \mathfrak{D}(\widehat{\mathcal{L}}_1). \tag{4.2.5}$$

Proof. We have that

$$\mathfrak{ag}(\mathcal{F}) = \mathfrak{ac}\left(\{af : f \in \mathfrak{mc}(\mathcal{F}), a \in \mathbf{R}\}\right) \subset \mathfrak{D}(\mathcal{L}) \tag{4.2.6}$$

and

$$\begin{aligned}\mathfrak{R}\left(\mathcal{L}|_{\mathfrak{ag}(\mathcal{F})}\right) &= \{\mathcal{L}h : h \in \mathfrak{ac}\left(\{af : f \in \mathfrak{mc}(\mathcal{F}), a \in \mathbf{R}\}\right)\} \\ &= \mathfrak{ac}\left(\left\{ag : g \in \mathfrak{R}\left(\mathcal{L}|_{\mathfrak{mc}(\mathcal{F})}\right), a \in \mathbf{R}\right\}\right) \\ &= \mathfrak{ag}\left[\mathfrak{R}\left(\mathcal{L}|_{\mathfrak{mc}(\mathcal{F})}\right)\right] \subset \mathfrak{ca}(\mathcal{F})\end{aligned} \tag{4.2.7}$$

by (4.2.1), the linearity of $\mathcal{L}$ and linear space properties of $\mathfrak{D}(\mathcal{L})$ and $\mathfrak{ca}(\mathcal{F})$. Thus,

$$\widehat{\mathcal{L}}_0 \mathrel{\mathring{=}} \left\{(\widehat{f},\widehat{\mathcal{L}f}) : f \in \mathfrak{ag}(\mathcal{F})\right\} \tag{4.2.8}$$

and

$$\widehat{\mathcal{L}}_1 \mathrel{\mathring{=}} \left\{(\widehat{f},\widehat{g}) : (f,g) \in \mathcal{L} \cap (\mathfrak{ca}(\mathcal{F}) \times \mathfrak{ca}(\mathcal{F}))\right\} \tag{4.2.9}$$

are the desired linear operators by (4.2.6), (4.2.7) and Proposition 4.6 (d, f) (with $d = k = 1$). □

The $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_1$ above are defined as two (possibly) different replicas of $\mathcal{L}$.

Definition 4.10. Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E and $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$.

- $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is said to be **a base for** $\mathcal{L}$ if (4.2.1) holds.
- When $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is a base for $\mathcal{L}$, the operator $\widehat{\mathcal{L}}_0$ in (4.2.8) and the operator $\widehat{\mathcal{L}}_1$ in (4.2.9) are called the **core replica** and the **extended replica of** $\mathcal{L}$ respectively.

Hereafter, we use the following notations for brevity if no confusion is caused.

Notation 4.11. Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mathcal{L}$ be a linear operator on $C_b(E; \mathbf{R})$ and $\beta \in \mathbf{R}$.

- We define $\widetilde{f} \mathrel{\hat{=}} f|_{E_0^d}$ for each $d, k \in \mathbf{N}$ and $f \in M(E^d; \mathbf{R}^k)$. Moreover, $\widetilde{\mathcal{F}} \mathrel{\hat{=}} \mathcal{F}|_{E_0} = \{\widetilde{f} : f \in \mathcal{F}\}$.
- When $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is a base for $\mathcal{L}$, we define

$$\begin{aligned} &\mathcal{L}_0 \mathrel{\hat{=}} \mathcal{L}|_{\mathfrak{ag}(\mathcal{F})}, \\ &\mathcal{L}_1 \mathrel{\hat{=}} \mathcal{L} \cap (\mathfrak{ca}(\mathcal{F}) \times \mathfrak{ca}(\mathcal{F})), \\ &\widetilde{\mathcal{L}}_i \mathrel{\hat{=}} \left\{(\widetilde{f}, \widetilde{g}) : (f, g) \in \mathcal{L}_i\right\}, \ \forall i = 1, 2 \end{aligned} \tag{4.2.10}$$

  and $\widehat{\mathcal{L}}_0$ (resp. $\widehat{\mathcal{L}}_1$) always denote the core (resp. extended) replica of $\mathcal{L}$.
- We define the operator $\beta - \mathcal{L}$ by

$$(\beta - \mathcal{L})f \mathrel{\hat{=}} \beta f - \mathcal{L}f, \ \forall f \in \mathfrak{D}(\mathcal{L}). \tag{4.2.11}$$

  Similar notations apply to $\mathcal{L}_i$, $\widetilde{\mathcal{L}}_i$ and $\widehat{\mathcal{L}}_i$ for each $i = 0, 1$.

Below are several basic facts about the domain and range of the above-mentioned linear operators.

Proposition 4.12. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E; \mathbf{R})$ and $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E for $\mathcal{L}$. Then, the following statements are true:*

(a) The linear operators $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy $\mathcal{L}_0 = \mathcal{L}_1|_{\mathfrak{ag}(\mathcal{F})}$ and

$$\mathfrak{R}(\mathcal{L}_i) \subset \mathfrak{cl}(\mathfrak{D}(\mathcal{L}_i)) = \mathfrak{ca}(\mathcal{F}), \ \forall i = 0, 1. \tag{4.2.12}$$

(b) The linear operators $\widetilde{\mathcal{L}}_0$ *and* $\widetilde{\mathcal{L}}_1$ *satisfy* $\widetilde{\mathcal{L}}_0 = \widetilde{\mathcal{L}}_1|_{\mathfrak{ag}(\widetilde{\mathcal{F}})}$ *and*

$$\mathfrak{R}(\widetilde{\mathcal{L}}_i) \subset \mathfrak{cl}\left(\mathfrak{D}(\widetilde{\mathcal{L}}_i)\right) = \mathfrak{ca}(\widetilde{\mathcal{F}}), \ \forall i = 0, 1. \tag{4.2.13}$$

(c) The linear operators $\widehat{\mathcal{L}}_0$ *and* $\widehat{\mathcal{L}}_1$ *satisfy* $\widehat{\mathcal{L}}_0 = \widehat{\mathcal{L}}_1|_{\mathfrak{ag}(\widehat{\mathcal{F}})}$ *and*

$$\mathfrak{R}(\widehat{\mathcal{L}}_i) \subset \mathfrak{cl}\left(\mathfrak{D}(\widehat{\mathcal{L}}_i)\right) = \mathfrak{ca}(\widehat{\mathcal{F}}) = C(\widehat{E}; \mathbf{R}), \ \forall i = 0, 1. \tag{4.2.14}$$

(d) If $\mathcal{L}1 = 0$*, then* $\widehat{\mathcal{L}}_0 1 = 0$ *and* $\widehat{\mathcal{L}}_1 1 = 0$*.*

Proof. (a) The linearities of $\mathcal{L}_0$ and $\mathcal{L}_1$ follow by that of $\mathcal{L}$. It follows by (4.2.10) and (4.2.7) that

$$\mathfrak{R}(\mathcal{L}_0) = \mathfrak{R}\left(\mathcal{L}|_{\mathfrak{ag}(\mathcal{F})}\right) \subset \mathfrak{ca}(\mathcal{F}) = \mathfrak{cl}\left(\mathfrak{ag}(\mathcal{F})\right) = \mathfrak{cl}\left(\mathfrak{D}(\mathcal{L}_0)\right). \tag{4.2.15}$$

It follows by (4.2.10) and (4.2.15) that

$$\mathfrak{R}(\mathcal{L}_1) \subset \mathfrak{ca}(\mathcal{F}) = \mathfrak{cl}\left(\mathfrak{D}(\mathcal{L}_0)\right) \subset \mathfrak{cl}\left(\mathfrak{D}(\mathcal{L}_1)\right) \subset \mathfrak{ca}(\mathcal{F}). \tag{4.2.16}$$

Now, (a) follows by (4.2.15) and (4.2.16).

(b) The linearity of $\widetilde{\mathcal{L}}_0$ (resp. $\widetilde{\mathcal{L}}_1$) follows by that of $\mathcal{L}_0$ (resp. $\mathcal{L}_1$). It follows by (4.2.10), (4.2.7) and properties of uniform convergence that

$$\begin{aligned}\mathfrak{R}\left(\widetilde{\mathcal{L}}_0\right) &= \left\{\widetilde{\mathcal{L}f} : f \in \mathfrak{ag}(\mathcal{F})\right\} = \mathfrak{R}\left(\mathcal{L}|_{\mathfrak{ag}(\mathcal{F})}\right)\big|_{E_0} \\ &\subset \mathfrak{ca}(\mathcal{F})|_{E_0} \subset \mathfrak{ca}(\widetilde{\mathcal{F}}) = \mathfrak{cl}\left(\mathfrak{ag}(\widetilde{\mathcal{F}})\right) = \mathfrak{cl}\left(\mathfrak{D}(\widetilde{\mathcal{L}}_0)\right).\end{aligned} \tag{4.2.17}$$

It follows by (4.2.10) and (4.2.17) that

$$\begin{aligned}\mathfrak{R}(\widetilde{\mathcal{L}}_1) &= \{\widetilde{g} : g \in \mathfrak{R}(\mathcal{L}_1)\} \subset \mathfrak{ca}(\mathcal{F})|_{E_0} \\ &\subset \mathfrak{ca}(\widetilde{\mathcal{F}}) = \mathfrak{cl}\left(\mathfrak{D}(\widetilde{\mathcal{L}}_0)\right) \subset \mathfrak{cl}\left(\mathfrak{D}(\widetilde{\mathcal{L}}_1)\right) \subset \mathfrak{ca}(\widetilde{\mathcal{F}}).\end{aligned} \tag{4.2.18}$$

Now, (b) follows by (4.2.17) and (4.2.18).

(c) follows by Proposition 4.25 and Corollary 3.10 (with $d = 1$).

(d) $1 = \widehat{1} \in \widehat{\mathcal{F}} \subset \mathfrak{D}(\widehat{\mathcal{L}}_0)$ and $\widehat{\mathcal{L}}_0 1 = \widehat{\mathcal{L}1} = \widehat{0} = 0$ by the fact $1 \in \mathcal{F}$, (4.2.2), (4.2.3) and the denseness of E_0 in $\widehat{E}$. ☐

4.2.2 Markov-generator-type properties

The compact Polish space $\widehat{E}$ and the association of the orignal and replica functions allow the replica operators $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_1$ to inherit or refine many properties of the original operator $\mathcal{L}$. In particular, we desire replica operators with the following Markov-generator-type properties as in our companion papers Dong and Kouritzin [2017a,b,d].

Property.

P1 $\widehat{\mathcal{L}}_1 = \mathfrak{cl}(\widehat{\mathcal{L}}_0)$.

P2 $\widehat{\mathcal{L}}_0$ *satisfies the positive maximum principle.*

P3 $\widehat{\mathcal{L}}_1$ *satisfies the positive maximum principle.*

P4 $\widehat{\mathcal{L}}_0$ *is a strong generator on* $C(\widehat{E};\mathbf{R})$.

P5 $\widehat{\mathcal{L}}_0$ *is a Feller generator on* $C(\widehat{E};\mathbf{R})$.

The following proposition gives a sufficient condition for **P1** and explains why we call $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_1$ the core and extended replica of $\mathcal{L}$.

Lemma 4.13. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ and $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E for $\mathcal{L}$. Then, the following statements are true:*

(a) If ***P2*** *(resp.* ***P3****) holds, then $\widehat{\mathcal{L}}_0$ (resp. $\widehat{\mathcal{L}}_1$) is dissipative.*

(b) If ***P4*** *holds and $\widehat{\mathcal{L}}_1$ is dissipative (especially* ***P3*** *holds), then* ***P1*** *holds.*

Proof. $\widehat{E}$ is a compact Polish space by Lemma 3.3 (b, c). $C_0(\widehat{E};\mathbf{R}) = C(\widehat{E};\mathbf{R})$ by (3.1.16) (with $d = 1$). Then, the result follows by Proposition 4.12 (c) and [Ethier and Kurtz, 1986, §4.2, Lemma 2.1 and §1.4, Proposition 4.1]. □

Remark 4.14. $\widehat{\mathcal{L}}_1$ is a linear superspace of $\widehat{\mathcal{L}}_0$, so we call $\widehat{\mathcal{L}}_1$ the extended replica. "core replica" comes from the fact that $\mathfrak{D}(\widehat{\mathcal{L}}_0)$ is a *core* (see [Ethier and Kurtz, 1986, §1.3, p.17]) of $\widehat{\mathcal{L}}_1$ in the setting of Lemma 4.13 (b).

The following lemma specifies when **P2** - **P5** hold.

Lemma 4.15. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ and $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E for $\mathcal{L}$. Then, the following statements are true:*

(a) $\widehat{\mathcal{L}}_0$ (resp. $\widehat{\mathcal{L}}_1$) is dissipative if and only if $\widetilde{\mathcal{L}}_0$ (resp. $\widetilde{\mathcal{L}}_1$) is dissipative.

*(b) **P2** (resp. **P3**) holds if and only if for any $\epsilon\in(0,\infty)$ and $f\in\mathfrak{ag}(\mathcal{F})$ (resp. $f\in\mathfrak{D}(\mathcal{L}_1)$), there exists an $n_\epsilon^f\in\mathbf{N}$ such that*[4]

$$\sup_{x\in E_0}\left[\mathcal{L}f(x)-n_\epsilon^f\left(\left\|\widetilde{f^+}\right\|_\infty-f(x)\right)\right]\leq\epsilon. \tag{4.2.19}$$

*(c) **P4** holds if and only if: (1) $\widetilde{\mathcal{L}}_0$ is dissipative, and (2) There exists a $\beta\in(0,\infty)$ such that*[5]

$$\widetilde{\mathcal{F}}\subset\mathfrak{ca}\left(\left\{(\beta-\widetilde{\mathcal{L}})\widetilde{f}:f\in\mathfrak{mc}(\mathcal{F})\right\}\right). \tag{4.2.20}$$

*(d) **P5** holds if and only if (1) $\mathcal{L}1=0$, (2) There exists a $\beta\in(0,\infty)$ such that (4.2.20) holds, and (3) For any $\epsilon\in(0,\infty)$ and $f\in\mathfrak{ag}(\mathcal{F})$, there exists an $n_\epsilon^f\in\mathbf{N}$ such that (4.2.19) holds.*

Proof. (a) We have by Lemma B.72 (a) (with $d=k=1$) that

$$\|\widehat{f}\|_\infty=\|\widetilde{f}\|_\infty,\ \forall f\in\mathfrak{ca}(\mathcal{F}). \tag{4.2.21}$$

Letting $g\mathrel{\hat=}(\beta-\mathcal{L})f$, we have by (4.2.5) and Proposition 4.6 (d) that

$$\left\|\beta\tilde{f}-\widetilde{\mathcal{L}f}\right\|_\infty=\|\widetilde{g}\|_\infty=\|\widehat{g}\|_\infty=\left\|\beta\widehat{f}-\widehat{\mathcal{L}f}\right\|_\infty,\ \forall f\in\mathfrak{D}(\mathcal{L}_1),\beta\in(0,\infty). \tag{4.2.22}$$

Now, (a) follows by (4.2.21), (4.2.22) and the fact that $\widetilde{\mathcal{L}}_0\subset\widetilde{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_0\subset\widehat{\mathcal{L}}_1$.

(b - Sufficiency) We only prove the result for $\widehat{\mathcal{L}}_0$ since $\widehat{\mathcal{L}}_1$ follows by a similar argument. Let $\epsilon\in(0,\infty)$, $f\in\mathfrak{ag}(\mathcal{F})$ and $n_\epsilon^f\in\mathbf{N}$ satisfy (4.2.19). We have by Lemma B.72 (with $d=k=1$) that

$$\widehat{f^+}=\widehat{f}^+\text{ and }\|\widehat{f}^+\|_\infty=\|\widetilde{f^+}\|_\infty. \tag{4.2.23}$$

[4] f^+ was defined in §2.2.3 and $\widetilde{f^+}$ was defined in Notation 4.11.

[5] The operator $\lambda-\widetilde{\mathcal{L}}$ was defined in Notation 4.11.

Then, there exists an $x_0 \in \widehat{E}$ such that

$$\left\|\widetilde{f^+}\right\|_\infty = \left\|\widehat{f}^+\right\|_\infty = \hat{f}(x_0) \tag{4.2.24}$$

by (4.2.23), the compactness of $\widehat{E}$, the continuity of $\widehat{f}^+$ and [Munkres, 2000, Theorem 27.4]. $\widehat{E}$ is a metrizable space by Lemma 3.3 (c), so there exist $\{x_k\}_{k\in\mathbf{N}} \subset E_0$ such that

$$x_k \longrightarrow x_0 \text{ as } k \uparrow \infty \text{ in } \widehat{E} \tag{4.2.25}$$

by E_0's denseness in $\widehat{E}$ and Fact A.9 (with $E = \widehat{E}$ and $A = E_0$). It follows that

$$\begin{aligned}\widehat{\mathcal{L}f}(x_k) = \mathcal{L}f(x_k) &\le n_\epsilon^f\left(\left\|\widetilde{f^+}\right\|_\infty - f(x_k)\right) + \epsilon \\ &= n_\epsilon^f\left(\left\|\widehat{f}^+\right\|_\infty - \hat{f}(x_k)\right) + \epsilon \\ &= n_\epsilon^f\left(\widehat{f}(x_0) - \hat{f}(x_k)\right) + \epsilon,\ \forall k \in \mathbf{N}\end{aligned} \tag{4.2.26}$$

by (4.2.3), (4.2.19) and (4.2.24). Hence, we have that

$$\begin{aligned}\widehat{\mathcal{L}f}(x_0) &= \lim_{k\to\infty} \widehat{\mathcal{L}f}(x_k) \\ &\le \lim_{k\to\infty} n_\epsilon^f\left(\widehat{f}(x_0) - \hat{f}(x_k)\right) + \epsilon \\ &= n_\epsilon^f\left(\widehat{f}(x_0) - \lim_{k\to\infty}\hat{f}(x_k)\right) + \epsilon = \epsilon\end{aligned} \tag{4.2.27}$$

by the continuities of $\widehat{f}$ and $\widehat{\mathcal{L}f}$, (4.2.25), (4.2.26) and the independence of n_ϵ^f and $\{x_k\}_{k\in\mathbf{N}}$. Letting $\epsilon \downarrow 0$ in (4.2.27), we get $\widehat{\mathcal{L}}\widehat{f}(x_0) \le 0$.

(b - Necessity) Fix $\epsilon \in (0,\infty)$ and $f \in \mathfrak{ca}(\mathcal{F})$. Then, f satisfies (4.2.23) by Lemma B.72 (with $d = k = 1$). $\widehat{\mathcal{L}}_0$ is dissipative by **P2** and Lemma 4.13 (a). By (4.2.3), (4.2.21), the compactness of $\widehat{E}$, [Ethier and Kurtz, 1986, §4.5, Lemmas 5.3] and the development establishing [Ethier and Kurtz, 1986, §4.5, Theorem 5.4], there exist an $n_\epsilon^f \in \mathbf{N}$ and a *positive contraction* (see [Ethier and Kurtz, 1986, §1.1, p.6 and §4.2, p.165]) $\mathcal{S}_{1/n_\epsilon^f}$ on $C(\widehat{E};\mathbf{R})$ such that

$$\begin{aligned}\mathcal{L}f(x) = \widehat{\mathcal{L}f}(x) &\le n_\epsilon^f\left(\mathcal{S}_{1/n_\epsilon^f}\widehat{f}(x) - \widehat{f}(x)\right) + \epsilon \\ &\le n_\epsilon^f g(x) - \mathcal{S}_{1/n_\epsilon^f} g(x) + \epsilon \le n_\epsilon^f\left(\left\|\widetilde{f^+}\right\|_\infty - f(x)\right) + \epsilon\end{aligned} \tag{4.2.28}$$

for all $x \in E_0 \subset \widehat{E}$, where $g \mathrel{\hat{=}} \|\widehat{f}^+\|_\infty - f \geq 0$ satisfies $\mathcal{S}_{1/n_\epsilon^f} g(x) \geq 0$ by the positiveness of $\mathcal{S}_{1/n_\epsilon^f}$.

(c) It follows by (4.2.13), (4.2.20), the linearity of $\widetilde{\mathcal{L}}_0$ and (4.2.10) that

$$\begin{aligned}
\mathfrak{cl}\left[\mathfrak{D}(\widetilde{\mathcal{L}}_0)\right] &= \mathfrak{ca}(\widetilde{\mathcal{F}}) \\
&\subset \mathfrak{ca}\left[\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0|_{\mathfrak{mc}(\widetilde{\mathcal{F}})}\right)\right] \\
&\subset \mathfrak{ca}\left[\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0\right)\right] = \mathfrak{cl}\left[\mathfrak{ag}\left(\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0\right)\right)\right] \\
&= \mathfrak{cl}\left[\left\{\beta\widetilde{f} - \widetilde{\mathcal{L}f} : f \in \mathfrak{ag}\left(\mathfrak{D}(\mathcal{L}_0)\right)\right\}\right] \\
&= \mathfrak{cl}\left[\left\{\beta\tilde{f} - \widetilde{\mathcal{L}f} : f \in \mathfrak{D}(\mathcal{L}_0)\right\}\right] \\
&= \mathfrak{cl}\left[\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0\right)\right] \subset \mathfrak{ca}(\widetilde{\mathcal{F}}),
\end{aligned} \tag{4.2.29}$$

thus proving the equivalence between (4.2.20) and

$$\mathfrak{cl}\left[\mathfrak{D}(\widetilde{\mathcal{L}}_0)\right] = \mathfrak{cl}\left[\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0\right)\right]. \tag{4.2.30}$$

Next, we find by (4.2.10), (4.2.2), Proposition 4.6 (d, e) and (4.2.3) that

$$\mathfrak{D}(\widetilde{\mathcal{L}}_0) = \mathfrak{ag}(\widetilde{\mathcal{F}}) = \mathfrak{ag}(\widehat{\mathcal{F}})\Big|_{E_0} = \mathfrak{D}(\widehat{\mathcal{L}}_0)\Big|_{E_0} \tag{4.2.31}$$

and

$$\mathfrak{R}\left(\beta - \widetilde{\mathcal{L}}_0\right) = \mathfrak{R}\left(\beta - \widehat{\mathcal{L}}_0\right)\Big|_{E_0}. \tag{4.2.32}$$

Then, (4.2.30) is equivalent to

$$\mathfrak{cl}\left[\mathfrak{D}(\widehat{\mathcal{L}}_0)\right] = C(\widehat{E};\mathbf{R}) = \mathfrak{cl}\left[\mathfrak{R}\left(\beta - \widehat{\mathcal{L}}_0\right)\right] \tag{4.2.33}$$

by (4.2.31), (4.2.32), the denseness of E_0 in $\widehat{E}$, properties of uniform convergence and (4.2.14).

So far, we have shown the equivalence of (4.2.20), (4.2.30) and (4.2.33). Now, (c) follows by (a) and the Lumer-Phillips Theorem (see [Yosida, 1980, §IX.8]).

(d) follows by (b), Proposition 4.12 (d), [Ethier and Kurtz, 1986, §4.2, Theorem 2.2] and the equivalence between (4.2.20) and (4.2.33). □

4.2.3 Several regularity conditions about operator

Herein, we introduce several typical regularity conditions about the operator $\mathcal{L}$ under which: (1) One can construct bases in either of the following two forms, and (2) The associated replica operators satisfy one or more of (**P1**) - (**P5**).

Property.

P6 *There exists a base* $(E, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ *over* E *for* $\mathcal{L}$ *with* $\mathcal{D}_0 \subset \mathcal{F} = \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F})$.

P7 *There exists a base* $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ *over* E *for* $\mathcal{L}$ *such that* $A \subset E_0$, $E_0 \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ *and* $\mathcal{D}_0 \subset \mathcal{F} = \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F})$.

Remark 4.16. The $\mathcal{D}_0$ above, as appeared in Lemma 3.39, represents a desired set of bounded continuous functions from the domain of $\mathcal{L}$. For example, in the application of martingale problem, $\mathcal{D}_0$ can be a rich collection that approximates both the domain and range of $\mathcal{L}$ (see e.g. [Dong and Kouritzin, 2017a, §3.1]). In the application of filtering, $\mathcal{D}_0$ could be used to approximate the given sensor function (see e.g. Dong and Kouritzin [2017b]). With similar consideration, the A above stands for a desired subset of E to be contained in E_0. For example, in the application of martingale problems, the set A could be a support of the given initial distribution (see e.g. [Dong and Kouritzin, 2017a, §A.1]).

Our regularity conditions consist of four types. The first type is about the denseness of the domain and range of $\mathcal{L}$.

Condition (Denseness). The operator $\mathcal{L}$ satisfies:

D1 $\mathfrak{R}(\mathcal{L}) \subset \mathfrak{cl}(\mathfrak{D}(\mathcal{L}))$.

D2 $\mathfrak{D}(\mathcal{L}) \subset \mathfrak{cl}(\mathfrak{R}(\beta - \mathcal{L}))$ for some $\beta \in (0, \infty)$.

Remark 4.17. **D1** and **D2** are true for any strong generator $\mathcal{L}$ by the Lumer-Phillips Theorem.

The second type is about the point-separability of the domain of $\mathcal{L}$.

Condition (Separability). The operator $\mathcal{L}$ satisfies:

S1 $\mathfrak{D}(\mathcal{L})$ contains the constant function 1 and separates points on E.

S2 $\mathfrak{D}(\mathcal{L})$ contains the constant function 1 and E is a $\mathfrak{D}(\mathcal{L})$-baseable space.

S2 is stronger than **S1**. According to §3.2, they are mild requirements about the richness of $\mathfrak{D}(\mathcal{L})$. The next example further illustrates their generality.

Example 4.18.

(I) For martingale problems and nonlinear filternig problems, a common setting (see Szpirglas [1976], Bhatt et al. [1995] and Bhatt et al. [2000]) is that E is a metrizable Lusin (especially Polish) space and the domain $\mathfrak{D}(\mathcal{L})$ of $\mathcal{L}$ contains 1 and separates points on E. In this case, E is a second-countable space by Proposition A.11 (d) and Proposition A.6 (c). So, $\mathcal{L}$ satisfies **S2** by Proposition 3.40 (with $A = E$ and $\mathcal{D} = \mathfrak{D}(\mathcal{L})$).

(II) Another classical setting for martingale problems is that E is a locally compact separable metric space with one-point compactification $E \cup \{\Delta\}$, $\mathcal{L}$ is a linear operator on $C_0(E; \mathbf{R})$ and its domain $\mathfrak{D}(\mathcal{L})$ is uniformly dense in $C_0(E; \mathbf{R})$ (see [Ethier and Kurtz, 1986, Chapter 4] and Kurtz and Ocone [1988]). In this case, one can simply extend $\mathcal{L}$ to a linear operator $\mathcal{L}^*$ on $C_b(E; \mathbf{R})$ by defining $\mathfrak{D}(\mathcal{L}^*)$ as the linear span of $\mathfrak{D}(\mathcal{L}) \cup \{1\}$ and defining

$$\mathcal{L}^*(af + b) \circeq a\mathcal{L}f, \ \forall f \in \mathfrak{D}(\mathcal{L}), a, b \in \mathbf{R}. \tag{4.2.34}$$

By Proposition 4.8 (a, b, d, e) (with $\mathcal{D} = \mathfrak{D}(\mathcal{L})$), there exists a countable $\mathcal{F} \subset C_b(E; \mathbf{R})$ such that $\mathcal{F} \backslash \{1\} \subset \mathfrak{D}(\mathcal{L})$ strongly separates points on E and $C_0(E; \mathbf{R}) \subset \mathfrak{ca}(\mathcal{F})$. Thus, $\mathcal{L}^*$ satisfies **S2** by Proposition A.17 (a). Moreover, this $\mathcal{L}^*$ satisfies **D1**.

(III) Suppose that E is a (possibly non-metrizable) Tychonoff space and $\mathcal{L}$ is a strong generator on $C_b(E; \mathbf{R})$. Without loss of generality, one can consider $1 \in \mathfrak{D}(\mathcal{L})$. Otherwise, we extend $\mathcal{L}$ to $\mathcal{L}^*$ as in (II).

$\mathfrak{D}(\mathcal{L})$ is uniformly dense in $C_b(E;\mathbf{R})$ by the Lumer-Phillips Theorem, so **D1** holds. $C_b(E;\mathbf{R})$ separates points on E by Proposition A.25 (a, c), so **S1** holds by Corollary A.19.

The third type of our regularity conditions includes several analogues of dissipativeness and positive maximum principle.

Condition (Generator). The operator $\mathcal{L}$ satisfies:

G1 $\mathcal{L}$ is dissipative.

G2 For any $\epsilon, \beta \in (0,\infty)$ and $x \in E$, there exist $\{K_{q,\epsilon}^{x,\beta}\}_{q\in\mathbf{Q}\cap[0,1]} \subset \mathscr{K}^{\mathbf{m}}(E)$ (independent of f) such that each $f \in \mathfrak{D}(\mathcal{L})$ satisfies

$$\begin{aligned} &\beta\,|f(x)| - \left\| (\beta f - \mathcal{L}f)|_{K_{q,\epsilon}^{x,\beta}} \right\|_\infty \\ &\quad \leq (\beta\|f\|_\infty + \|\mathcal{L}f\|_\infty + 1)\,\epsilon, \ \forall f \in \mathfrak{D}(\mathcal{L}) \end{aligned} \tag{4.2.35}$$

for some $q = q_\epsilon^{f,x,\beta} \in \mathbf{Q} \cap [0,1]$.

G3 For any $\epsilon \in (0,\infty)$ and $f \in \mathfrak{D}(\mathcal{L})$, there exists an $n_\epsilon^f \in \mathbf{N}$ such that

$$\sup_{x\in E} \left[\mathcal{L}f(x) - n_\epsilon^f \left(\|f^+\|_\infty - f(x)\right)\right] \leq \epsilon. \tag{4.2.36}$$

G4 For any $\epsilon \in (0,\infty)$, $x \in E$ and $f \in \mathfrak{D}(\mathcal{L})$, there exist $\{K_{n,\epsilon}^x\}_{n\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ independent of f and an $n = n_\epsilon^f \in \mathbf{N}$ independent of x such that

$$\mathcal{L}f(x) - n_\epsilon^f \left(\left\| f^+|_{K_{n\epsilon}^x} \right\|_\infty - f(x) \right) \leq \epsilon. \tag{4.2.37}$$

Remark 4.19.

- **G2** can be thought of as local dissipativeness on metrizable compact subsets.
- When E is a locally compact metric space, positive maximum principle implies dissipativeness and is satisfied by Feller generators. **G3** extends this property to non-locally-compact spaces while **G4** localizes it on metrizable compact subsets.

The next example shows that **G2**, **G3** and **G4** are not unnatural.

Example 4.20. Let E, $\mathcal{L}$ and $\mathcal{L}^*$ be as in Example 4.18 (II) and $\epsilon \in (0,\infty)$.

(I) If $\mathcal{L}$ satisfies positive maximum principle, then $\mathcal{L}^*$ does also. Consequently, **G3** is satisfied by both $\mathcal{L}$ and $\mathcal{L}^*$ by an argument similar to the proof of Lemma 4.15 (b - "only if").

(II) When $\mathcal{L}$ is a Feller generator, the Feller semigroup $\{\mathcal{S}_t\}_{t\geq 0}$ generated by $\mathfrak{cl}(\mathcal{L})$ on $C_0(E;\mathbf{R})$ is often given by a *transition function* (see [Ethier and Kurtz, 1986, §4.1, p.156]) $\kappa : \mathbf{R}^+ \times E \times \mathscr{B}(E) \to [0,1]$. In the remainder of the example, we fix $x \in E$.

- We fix $\beta \in (0,\infty)$. E is a Polish space by Proposition 4.8 (c, d). So for each $q \in \mathbf{Q}$, $\kappa(q,x,\cdot)$ is tight in E by Ulam's Theorem (Theorem 2.21), so there exist $\{K_{q,\epsilon}^{x,\beta}\}_{q\in\mathbf{Q}[0,1]} \in \mathscr{K}(E)$ such that

$$\kappa\left(q,x,E\backslash K_{q,\epsilon}^{x,\beta}\right) \leq \epsilon,\ \forall q \in \mathbf{Q}\cap[0,1]. \tag{4.2.38}$$

One finds by [Ethier and Kurtz, 1986, §1.2, (2.1) and (2.6)], change of variable and Jensen's Inequality that

$$\begin{aligned}\beta\,|f(x)| &= \left|\int_0^\infty \beta e^{-\beta t}\mathcal{S}_t(\beta-\mathcal{L})f(x)dt\right| \\ &\leq \int_0^1 \left|\mathcal{S}_{-\frac{\ln u}{\beta}}(\beta-\mathcal{L})f(x)\right| du,\ \forall f\in\mathfrak{D}(\mathcal{L}).\end{aligned} \tag{4.2.39}$$

Then, there exist $\{t^{f,x,\beta}\}_{f\in\mathfrak{D}(\mathcal{L})} \subset (0,1)$ such that

$$\beta\,|f(x)| \leq \left|\mathcal{S}_{t^{f,x,\beta}}(\beta-\mathcal{L})f(x)\right|,\ \forall f\in\mathfrak{D}(\mathcal{L}). \tag{4.2.40}$$

by (4.2.39), Mean-Value Theorem and Jensen's Inequality. For each fixed $f\in\mathfrak{D}(\mathcal{L})$, there exists a $q_\epsilon^{f,x,\beta} \in \mathbf{Q}\cap[0,1]$ such that

$$\left\|\mathcal{S}_{q_\epsilon^{f,x,\beta}}(\beta-\mathcal{L})f - \mathcal{S}_{t^{f,x,\beta}}(\beta-\mathcal{L})f\right\| < \epsilon \tag{4.2.41}$$

by the strong continuity of $\{\mathcal{S}_t\}_{t\geq 0}$. From (4.2.40), (4.2.38) and (4.2.41) it follows that

$$\begin{aligned}\beta\,|f(x)| &\leq \int_E |(\beta-\mathcal{L})f(y)|\,\kappa\left(q_\epsilon^{f,x,\beta},x,dy\right)+\epsilon\\ &\leq \int_{K^{x,\beta}_{q_\epsilon^{f,x,\beta},\epsilon}} |(\beta f-\mathcal{L}f)(y)|\,\kappa\left(q_\epsilon^{f,x,\beta},x,dy\right)\\ &+(\|\beta f\|_\infty+\|\mathcal{L}f\|_\infty)\,\epsilon+\epsilon\\ &\leq \left\|(\beta f-\mathcal{L}f)|_{K^{x,\beta}_{q_\epsilon^{f,x,\beta},\epsilon}}\right\|_\infty+(\|\beta f\|_\infty+\|\mathcal{L}f\|_\infty+1)\,\epsilon.\end{aligned} \tag{4.2.42}$$

Thus, $\mathcal{L}$ satisfies **G2** as we select $\{K^{x,\beta}_{q,\epsilon}\}_{q\in\mathbf{Q}\cap[0,1]}$ without involving any f.

- Let x be fixed as above. For each $n\in\mathbf{N}$, the tightness of $\kappa(n^{-1},x,\cdot)$ implies a $K^x_{n,\epsilon}\in\mathscr{K}(E)$ satisfying

$$\kappa\left(n^{-1},x,E\backslash K^x_{n,\epsilon}\right)\leq\frac{\epsilon}{2n^2}. \tag{4.2.43}$$

Meanwhile, we fix $f\in\mathfrak{D}(\mathcal{L})$. $\mathfrak{cl}(\mathcal{L})$ is the infitesesimal generator of $\{\mathcal{S}_t\}_{t\geq 0}$, so there exists a sufficiently large $n_\epsilon^f\in\mathbf{N}$ such that

$$n_\epsilon^f\geq\|f\|_\infty \tag{4.2.44}$$

and

$$\sup_{z\in E}\left|\mathcal{L}f(z)-n_\epsilon^f\left[\int_E f(y)\kappa\left(1/n_\epsilon^f,z,dy\right)-f(z)\right]\right|\leq\frac{\epsilon}{2}. \tag{4.2.45}$$

The sequence of compact sets $\{K^x_{n,\epsilon}\}_{n\in\mathbf{N}}$ is determined by x and the transition function κ, which is independent of f. The convergence rate n_ϵ^f is an intrinsic parameter of f and is unrelated to x. From (4.2.43), (4.2.44) and (4.2.45) it follows

that

$$\begin{aligned}\mathcal{L}f(x) &\leq n_\epsilon^f \left[\int_{K^x_{n_\epsilon^f,\epsilon}} f^+(y)\kappa\left(1/n_\epsilon^f, x, dy\right) + \frac{\|f\|_\infty \epsilon}{2\left(n_\epsilon^f\right)^{-2}} - f(x)\right] + \frac{\epsilon}{2} \\ &\leq n_\epsilon^f \left(\left\| f^+|_{K^x_{n_\epsilon^f,\epsilon}} \right\|_\infty - f(x)\right) + \epsilon.\end{aligned} \tag{4.2.46}$$

Thus, $\mathcal{L}$ satisfies **G4**.

The fourth type is a common technical assumption. It is equivalent to assuming $\mathfrak{D}(\mathcal{L})$ is closed under multiplication since $\mathcal{L}$ and $\mathfrak{D}(\mathcal{L})$ are linear spaces.

Condition (DA). The domain of the operator $\mathcal{L}$ is a subalgebra of $C_b(E;\mathbf{R})$.

4.2.4 Existence of Markov-generator-type replica operator

Now, we give four constructions of Markov-generator-type replica operators under the aforementioned regularity conditions. The first two propositions assume **S2** and construct bases satisfying **P6**.

Lemma 4.21. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ and $\mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L})$ be countable. If **S2**, **D1**, **G3** and **DA** hold, then **P6**, **P2** and **P3** hold.*

Proof. We use induction to construct the $\mathcal{F}$ in **P6**. By **S2**, there exists a countable $\mathcal{D} \subset \mathfrak{D}(\mathcal{L})$ that separates on E. For $k = 0$,

$$\mathcal{F}_0 \mathrel{\hat{=}} (\mathcal{D}_0 \cup \mathcal{D} \cup \{1\}) \subset \mathfrak{D}(\mathcal{L}) \tag{4.2.47}$$

is countable, contains 1 and separates points on E. For $k \in \mathbf{N}$, we assume $\mathcal{F}_0 \subset \mathcal{F}_{k-1} \subset \mathfrak{D}(\mathcal{L})$ and find by **DA** that

$$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}) \subset \mathfrak{D}(\mathcal{L}). \tag{4.2.48}$$

Then, we define

$$\mathcal{F}_k \mathrel{\hat{=}} \bigcup_{f \in \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}), q \in \mathbf{N}} \left\{f, g_q^{f,k}\right\} \subset \mathfrak{D}(\mathcal{L}), \tag{4.2.49}$$

where each $g_q^{f,k} \in \mathfrak{D}(\mathcal{L})$ is chosen by **D1** to satisfy

$$\left\|g_q^{f,k} - \mathcal{L}f\right\|_\infty \leq 2^{-q}. \tag{4.2.50}$$

It follows immediately that

$$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}) \subset \mathcal{F}_k \tag{4.2.51}$$

and

$$\mathfrak{R}\left(\mathcal{L}|_{\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1})}\right) \subset \mathfrak{cl}\left(\mathcal{F}_k\right). \tag{4.2.52}$$

Based on the construction above[6],

$$\mathcal{F} \mathrel{\hat{=}} \bigcup_{k \in \mathbf{N}_0} \mathcal{F}_k \tag{4.2.53}$$

satisfies

$$\mathcal{D}_0 \cup \{1\} \cup \mathcal{D} = \mathcal{F}_0 \subset \mathcal{F} = \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}) \subset \mathfrak{D}(\mathcal{L}) \tag{4.2.54}$$

and (4.2.1). So, $\mathcal{F}$ separates points on E as $\mathcal{D}$ does. Now, **P6** follows by Lemma 3.39 (b) (with $E_0 = E$ and $\mathcal{D} = \mathcal{F}$) and (4.2.54). **P2** and **P3** follow by **G3** and Lemma 4.15 (b) (with $E_0 = E$, $\widetilde{\mathcal{L}}_0 = \mathcal{L}_0$ and $\widetilde{\mathcal{L}}_1 = \mathcal{L}_1$). ☐

Lemma 4.22. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ such that **S2**, **D1**, **D2** and **DA** hold, and $\mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L})$ be countable. Then, the following statements are true:*

*(a) If **G1** holds, then **P6**, **P1** and **P4** hold.*

*(b) If $\mathcal{L}1 = 0$ and **G3** holds, then **P6**, **P1**, **P3** and **P5** hold.*

Proof. (a) We follow the proof of Lemma 4.21 to establish the $\mathcal{F}$ in **P6** except

[6] $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$ denotes the non-negative integers.

for reconstructing

$$\mathcal{F}_k \mathrel{\hat{=}} \bigcup_{f \in \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}), q \in \mathbf{N}} \left\{f, g_q^{f,k}, h_q^{f,k}\right\} \subset \mathfrak{D}(\mathcal{L}). \tag{4.2.55}$$

As before, we choose each $g_q^{f,k} \in \mathfrak{D}(\mathcal{L})$ by **D1** to satisfy (4.2.50). Meanwhile, we find a constant $\beta \in (0, \infty)$ and choose each $h_q^{f,k} \in \mathfrak{D}(\mathcal{L})$ by **D2** to satisfy

$$\left\|(\beta - \mathcal{L})h_q^{f,k} - f\right\|_\infty < 2^{-q}. \tag{4.2.56}$$

Consequently, $\mathcal{F}_k$ defined in (4.2.55) satisfies not only (4.2.51) and (4.2.52) but also

$$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}) \subset \mathfrak{cl}\left[\mathfrak{R}\left(\beta - \mathcal{L}|_{\mathcal{F}_k}\right)\right]. \tag{4.2.57}$$

$\mathcal{F}$ defined in (4.2.53) not only satisfies (4.2.54) and (4.2.1) but also satisfies

$$\mathcal{F} \subset \mathfrak{cl}\left[\mathfrak{R}\left(\beta - \mathcal{L}|_{\mathcal{F}}\right)\right] \subset \mathfrak{ca}\left[\mathfrak{R}\left(\beta - \mathcal{L}|_{\mathfrak{mc}(\mathcal{F})}\right)\right]. \tag{4.2.58}$$

Now, **P6** follows by Lemma 3.39 (b) (with $E_0 = E$ and $\mathcal{D} = \mathcal{F}$) and (4.2.54). Both $\mathcal{L}_0$ and $\mathcal{L}_1$ are dissipative by **G1**, so **P4** and the dissipativeness of $\widehat{\mathcal{L}}_1$ follow by (4.2.58) and Lemma 4.15 (a, c) (with $E_0 = E$, $\widetilde{\mathcal{L}}_0 = \mathcal{L}_0$ and $\widetilde{\mathcal{L}}_1 = \mathcal{L}_1$). Moreover, **P1** follows by Lemma 4.13 (b).

(b) Let $\mathcal{F}$ be constructed as in (a). Then, **P6** follows by the same argument. **P5** and **P3** follow by **G3**, (4.2.58) and Lemma 4.15 (b, d) (with $E_0 = E$, $\widetilde{\mathcal{L}}_0 = \mathcal{L}_0$ and $\widetilde{\mathcal{L}}_1 = \mathcal{L}_1$). **P1** follows by Lemma 4.13 (b). □

The next proposition turns an arbitrary $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ and suitable metrizable compacts provided by **G4** into a base satisfying **P7**.

Lemma 4.23. *Let E be a topological space, $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ such that **S1**, **G4** and **DA** hold, and $\mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L})$ be countable. Then, the following statements are true:*

*(a) If **D1** holds, then **P7**, **P2** and **P3** hold.*

*(b) If $\mathcal{L}1 = 0$, and if **D1** and **D2** hold, then **P7**, **P1**, **P3** and **P5** hold.*

Proof. (a) We construct the $(E_0, \mathcal{F})$ in **P7** by induction. For $k = 0$, we define

$$A_0 \mathrel{\mathring{=}} A \in \mathscr{K}_\sigma^{\mathbf{m}}(E) \tag{4.2.59}$$

and

$$\mathcal{F}_0 \mathrel{\mathring{=}} (\mathcal{D}_0 \cup \{1\}) \subset \mathfrak{D}(\mathcal{L}). \tag{4.2.60}$$

For $k \in \mathbf{N}$, we assume $A_{k-1} \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ and $\mathcal{F}_0 \subset \mathcal{F}_{k-1} \subset \mathfrak{D}(\mathcal{L})$. $(A_{k-1}, \mathscr{O}_E(A_{k-1}))$ is a separable space by Proposition 3.58 (a, b) and Proposition A.11 (d), so it has a countable dense subset $\{x_j^{k-1}\}_{j\in\mathbf{N}}$. For each $i \in \mathbf{N}$ and $x \in A_{k-1}$, one finds by **G4** a sequence of metrizable compact sets $\{K_{n,2^{-i}}^x\}_{n\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ such that each $f \in \mathfrak{D}(\mathcal{L})$ satisfies

$$\mathcal{L}f(x) - n_{2^{-i}}^f \left(\left\| f^+|_{K^x_{n^f_{2^{-i}},2^{-i}}} \right\|_\infty - f(x) \right) \leq 2^{-i} \tag{4.2.61}$$

for some sufficiently large $n_{2^{-i}}^f \in \mathbf{N}$ independent of x. Then, we redefine

$$A_k \mathrel{\mathring{=}} A_{k-1} \cup \left(\bigcup_{i\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} \bigcup_{j\in\mathbf{N}} K_{n,2^{-i}}^{x_j^{k-1}} \right) \in \mathscr{K}_\sigma^{\mathbf{m}}(E). \tag{4.2.62}$$

By **S1** and Proposition 3.58 (b, e) (with $A = A_k$ and $\mathcal{D} = \mathfrak{D}(\mathcal{L})$), there exists a countable $\mathcal{J}_k \subset \mathfrak{D}(\mathcal{L})$ that separates points on A_k. **DA** implies

$$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1} \cup \mathcal{J}_k) \subset \mathfrak{D}(\mathcal{L}). \tag{4.2.63}$$

Then, we define

$$\mathcal{F}_k \mathrel{\mathring{=}} \bigcup_{f\in\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1}\cup\mathcal{J}_k), q\in\mathbf{N}} \left\{ f, g_q^{f,k} \right\} \subset \mathfrak{D}(\mathcal{L}), \tag{4.2.64}$$

where each $g_q^{f,k}$ is chosen by **D1** to satisfy (4.2.50).

By the construction above, $\{x_j^{k-1}\}_{j,k\in\mathbf{N}}$ is a countable dense subset of

$$E_0 \mathrel{\mathring{=}} \bigcup_{k\in\mathbf{N}_0} A_k \in \mathscr{K}_\sigma^{\mathbf{m}}(E) \tag{4.2.65}$$

under the subspace topology $\mathscr{O}_E(E_0)$. E is a Hausdorff space by **S1** and Proposition A.17 (e) (with $A = E$ and $\mathcal{D} = \mathfrak{D}(\mathcal{L})$), so $E_0 \in \mathscr{B}(E)$ by Proposition A.12 (a).

$\mathcal{F}$ defined by (4.2.53) satisfies (4.2.54) and (4.2.1). $\mathcal{F}$ contains $\mathcal{J}_k$ and separates points on A_k for all $k \in \mathbf{N}$. $\{A_k\}_{k\in\mathbf{N}}$ are nested[7] by (4.2.62), so $\mathcal{F}$ separates points on E_0 by Fact B.19. Moreover, fixing $i \in \mathbf{N}$ and $f \in \mathfrak{D}(\mathcal{L})$, we have by (4.2.61) that

$$\begin{aligned} &\mathcal{L}f(x_j^{k-1}) - n_{2^{-i}}^f \left(\left\| \widetilde{f^+} \right\|_\infty - f(x_j^{k-1}) \right) \\ &\leq \mathcal{L}f((x_j^{k-1}) - n_{2^{-i}}^f \left(\left\| f^+|_{K^{x_j^{k-1}}_{n^f_{2^{-i}},2^{-i}}} \right\|_\infty - f(x_j^{k-1}) \right) \leq 2^{-i} \end{aligned} \tag{4.2.66}$$

for all $j, k \in \mathbf{N}$ and a sufficiently large $n_{2^{-i}}^f \in \mathbf{N}$ *independent of any* x_j^{k-1}. Therefore, it follows that

$$\mathcal{L}f(x) - n_{2^{-i}}^f \left(\left\| \widetilde{f^+} \right\|_\infty - f(x) \right) \leq 2^{-i},\ \forall x \in E_0 \tag{4.2.67}$$

by the denseness of $\{x_j^{k-1}\}_{j,k\in\mathbf{N}}$ in $(E_0, \mathscr{O}_E(E_0))$ and the continuities of f and $\mathcal{L}f$.

Now, **P7** follows by Lemma 3.39 (b) (with $\mathcal{D} = \mathcal{F}$). **P2** and **P3** follow by (4.2.67) and Lemma 4.15 (b).

(b) We follow the proof of (a) to establish $(E_0, \mathcal{F})$ except for reconstructing

$$\mathcal{F}_k \triangleq \bigcup_{f \in \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}_{k-1} \cup \mathcal{J}_k), q \in \mathbf{N}} \left\{ f, g_q^{f,k}, h_q^{f,k} \right\} \subset \mathfrak{D}(\mathcal{L}). \tag{4.2.68}$$

Here, we choose each $g_q^{f,k} \in \mathfrak{D}(\mathcal{L})$ by **D1** to satisfy (4.2.50). We find a constant $\beta \in (0, \infty)$ and choose each $h_q^{f,k} \in \mathfrak{D}(\mathcal{L})$ by **D2** to satisfy (4.2.56). Consequently, $\mathcal{F}$ not only satisfies (4.2.54) and (4.2.1) but also satisfies (4.2.58).

Now, **P7** follows by the same argument of (a). (4.2.20) follows by (4.2.58) and properties of uniform convergence. Hence, **P5** follows by (4.2.67), Proposition 4.12 (d) and Lemma 4.15 (b, d). Moreover, **P1** follows by Lemma 4.13 (b). $\square$

The next proposition turns an arbitrary $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ and suitable metriz-

[7] The terminology "nested" was explained in Fact 3.36.

able compacts provided by **G2** into a base satisfying **P7**.

Lemma 4.24. *Let E be a topological space, $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ and $\mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L})$ be countable. If* ***S1***, ***D1***, ***D2***, ***G2*** *and* ***DA*** *hold, then* ***P7***, ***P1*** *and* ***P4*** *hold.*

Proof. We follow the proof of Lemma 4.23 (a) to establish the $(E_0, \mathcal{F})$ in **P7** except for reconstructing $E_0 = \bigcup_{k\in\mathbf{N}_0} A_k \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ as follows. For $k = 0$, we still define A_0 as in (4.2.59). For each $i \in \mathbf{N}$, $x \in A_{k-1}$ and $\beta \in \mathbf{Q}^+$[8], one finds by **G2** a sequence of metrizable compact sets $\{K_{q,2^{-i}}^{x,\beta}\}_{q\in\mathbf{Q}\cap[0,1]} \subset \mathscr{K}^{\mathbf{m}}(E)$ such that each $f \in \mathfrak{D}(\mathcal{L})$ satisfies

$$\beta\,|f(x)| - \left\| (\beta f - \mathcal{L}f)|_{K^{x,\beta}_{q^{f,x,\beta}_{2^{-i}},2^{-i}}} \right\|_\infty \le (\beta\|f\|_\infty + \|\mathcal{L}f\|_\infty + 1)\, 2^{-i} \tag{4.2.69}$$

for some $q_\epsilon^{f,x,\beta} \in \mathbf{Q} \cap [0,1]$

Then, we still take a countable dense subset $\{x_j^{k-1}\}_{j\in\mathbf{N}}$ of $(A_{k-1}, \mathscr{O}_E(A_{k-1}))$ and redefine A_k by

$$A_k \triangleq A_{k-1} \cup \left(\bigcup_{\beta\in\mathbf{Q}^+} \bigcup_{i\in\mathbf{N}} \bigcup_{q\in\mathbf{Q}\cap[0,1]} \bigcup_{j\in\mathbf{N}} K_{q,2^{-i}}^{x_j^{k-1},\beta} \right) \in \mathscr{K}_\sigma^{\mathbf{m}}(E). \tag{4.2.70}$$

By the reconstruction above, $(E_0, \mathcal{F})$ has almost the same properties as in the proof of Lemma 4.23 (b) except for (4.2.61). Fixing $f \in \mathfrak{D}(\mathcal{L})$, $\beta \in \mathbf{Q}^+$ and $i \in \mathbf{N}$, we alternatively have by (4.2.69) and (4.2.70) that

$$\begin{aligned} &\beta\left|f(x_j^{k-1})\right| - \left\|\beta\tilde{f} - \widetilde{\mathcal{L}f}\right\|_\infty \\ &\le \beta\left|f(x_j^{k-1})\right| - \left\| (\beta f - \mathcal{L}f)|_{K^{x_j^{k-1},\beta}_{q^{f,x_j^{k-1},\beta}_{2^{-i}},2^{-i}}} \right\|_\infty \\ &\le (\beta\|f\|_\infty + \|\mathcal{L}f\|_\infty + 1)\, 2^{-i} \end{aligned} \tag{4.2.71}$$

[8] $\mathbf{Q}^+$ denotes the non-negative rational numbers.

for all $j, k \in \mathbf{N}$ and some $q_{2^{-i}}^{f,x_j^{k-1},\beta} \in \mathbf{Q} \cap [0,1]$. Therefore, it follows that

$$\beta\|\tilde{f}\|_\infty - \|\beta\tilde{f} - \widetilde{\mathcal{L}f}\|_\infty \leq (\beta\|f\|_\infty + \|\mathcal{L}f\|_\infty + 1)\, 2^{-i} \tag{4.2.72}$$

by the denseness of $\{x_j^{k-1}\}_{j,k\in\mathbf{N}}$ in $(E_0, \mathscr{O}_E(E_0))$ and the continuities of f. Letting $i \uparrow \infty$ in (4.2.72), we obtain

$$\beta\|\tilde{f}\|_\infty \leq \|\beta\tilde{f} - \widetilde{\mathcal{L}f}\|_\infty,\ \forall f \in \mathfrak{D}(\mathcal{L}). \tag{4.2.73}$$

Next, we still fix $f \in \mathfrak{D}(\mathcal{L})$ and let $\beta \in (0,\infty)$. Choosing $\{\beta_m\}_{m\in\mathbf{N}} \subset \mathbf{Q}^+ \cap (0,\infty)$ with $\lim_{m\to\infty} \beta_m = \beta$, one finds that

$$\begin{aligned}
\beta\|\widetilde{f}\|_\infty &\leq \lim_{m\to\infty} \beta_m\|\widetilde{f}\|_\infty \\
&\leq \left\|\beta\widetilde{f} - \widetilde{\mathcal{L}f}\right\|_\infty + \lim_{m\to\infty}\left\|\beta\widetilde{f} - \beta_m\widetilde{f}\right\|_\infty \\
&\leq \left\|\beta\widetilde{f} - \widetilde{\mathcal{L}f}\right\|_\infty + \lim_{m\to\infty}\|\widetilde{f}\|_\infty\,|\beta_m - \beta| = \left\|\beta\widetilde{f} - \widetilde{\mathcal{L}f}\right\|_\infty
\end{aligned} \tag{4.2.74}$$

by (4.2.73) (with $\beta = \beta_m$) and Triangle Inequality, thus proving the dissipativeness of $\widetilde{\mathcal{L}}_0$ and $\widetilde{\mathcal{L}}_1$.

Now, **P7** follows by Lemma 3.39 (b) (with $\mathcal{D} = \mathcal{F}$). Herein, (4.2.58) holds as in the proof of Lemma 4.23 (a) and implies (4.2.20). Then, **P4** and the dissipativeness of $\widehat{\mathcal{L}}_1$ follow by Lemma 4.15 (a, c). Moreover, **P1** follows by Lemma 4.13 (b). □

Morever, the existence of replica operator needs less regularity of $\mathcal{L}$ if one does not require Markov-generator-type properties.

Proposition 4.25. *Let E be a topological space, $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$, $\mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L})$ be countable and $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$ such that* ***D1*** *and* ***DA*** *hold. Then, the following statements are true:*

(a) If ***S2*** *holds, then* ***P6*** *holds.*

(b) If ***S1*** *holds, then* ***P7*** *holds.*

Proof. The construction of the desired base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ for (a) is contained in the proof of Lemma 4.21. That for (b) is contained in the proof of Lemma 4.23 (a). □

Chapter 5

Weak Convergence and Replication of Measure

Weak convergence is a typical area of probability theory that benefits from replication. Using our results about baseable spaces and baseable subsets, §5.1 establishes mild conditions for the uniqueness, existence and consistency of weak limit points on the finite-dimensional Cartesian power E^d of a topological space E. §5.2.1 introduces (Borel) replica of possibly non-Borel measure on E^d. §5.2.2 discusses the association of weak convergence about the replica measures and that about the original ones, which will be a basic tool for our developments in **Theme 2** and **Theme 3**. By the aid of replica function and replica measure, we extend two fundamental theorems in probability theory to non-classical settings. In §5.3.1, we establish a version of the Radon-Riesz Representation Theorem on a non-locally-compact and even non-Tychonoff space. In 5.3.2, we establish the Skorokhod Representation Theorem under slightly milder conditions than Jakubowski [1997a].

5.1 Tightness and weak convergence

Given a general topological space E, tightness and $\mathbf{m}$-tightness unsurprisingly play a key part for establishing weak convergence on E^d. Existence of a tight subsequence implies existence of a weak limit point in most situations. For uniqueness, one needs slightly stronger tightness than just having a tight subsequence.

Definition 5.1. Let $(E, \mathscr{U})$ be a measurable space, S be a topological space and $\mathscr{A}$ be a σ-algebra on S.

- When $S \subset E$, $\Gamma \subset \mathfrak{M}^+(E, \mathscr{U})$ is said to be **sequentially tight** (resp. **m-tight**) **in** S if: (1) Γ is an infinite set, and (2) Any infinite subset of Γ admits a subsequence being tight (resp. **m**-tight) in S.

- $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$ is said to be **sequentially tight** (resp. **m-tight**) **in** $A \subset S$ if: (1) Γ is an infinite set and A is non-empty, and (2) Any infinite subset of Γ admits a subsequence being tight (resp. **m**-tight) in $(A, \mathscr{O}_S(A))$.

- $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$ is said to be **sequentially tight** (resp. **m-tight**) if Γ is sequentially tight (resp. **m**-tight) in S.

Note 5.2. Any type of sequential tightness in Definition 5.1 is defined for random variables by referring to the corresponding property of their distributions.

The following companion of Proposition 3.57 relates sequential **m**-tightness and unique existence of Borel extension.

Proposition 5.3. *Let* $\mathbf{I}$ *be a countable index set,* $\{S_i\}_{i\in\mathbf{I}}$ *be topological spaces,* $(S, \mathscr{A})$ *be as in (2.7.22),* $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$ *and* $A \subset S$*. Suppose in addition that* $\mathfrak{p}_i(A) \in \mathscr{B}(S_i)$ *is a Hausdorff subspace of* S_i *for all* $i \in \mathbf{I}$*. Then,* Γ *is sequentially* **m**-*tight in* A *if and only if there exists a* $\Gamma_0 \in \mathscr{P}_0(\Gamma)$[1] *such that* $\{\mu' = \mathfrak{be}(\mu)\}_{\mu\in\Gamma\backslash\Gamma_0}$ *is sequentially* **m**-*tight in* A.

Proof. For any $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma$, the **m**-tightness of $\{\mu_n\}_{n\in\mathbf{N}}\backslash\Gamma_0$ in A is equivalent to that of $\{\mu' : \mu \in \{\mu_n\}_{n\in\mathbf{N}}\backslash\Gamma_0\}$ (if any) by Proposition 3.57. Now, it suffices to show that the sequential **m**-tightness of Γ implies the existence of $\{\mu' = \mathfrak{be}(\mu)\}_{\mu\in\Gamma\backslash\Gamma_0}$ for some $\Gamma_0 \in \mathscr{P}_0(\Gamma)$. Suppose $\Gamma' \subset \Gamma$ is an infinite set and $\aleph(\mathfrak{be}(\mu)) \neq 1$ for each $\mu \in \Gamma'$. Given Γ's sequential **m**-tightness, there exists an **m**-tight subsequence $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma'$ and, by Proposition 3.57, $\mathfrak{be}(\mu_n)$ is a singleton for all $n \in \mathbf{N}$. Contradiction! □

We then give our conditions for Borel extensions of finite measures on the product measurable space $(E^d, \mathscr{B}(E)^{\otimes d})$ to have a unique weak limit point.

[1] $\mathscr{P}_0(\Gamma)$ is the family of all finite subsets of Γ.

Theorem 5.4. *Let E be a topological space, $d \in \mathbf{N}$, $\Gamma \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$, $\mathcal{D} \subset C_b(E;\mathbf{R})$ and $\mathcal{G} \triangleq \mathfrak{mc}[\Pi^d(\mathcal{D})]$. Suppose that:*

(i) Γ *is sequentially* $\mathbf{m}$*-tight.*

(ii) $\{\int_{E^d} f(x)\mu(dx)\}_{\mu\in\Gamma}$ *has at most one*[2] *limit point in* $\mathbf{R}$ *for all* $f \in \mathcal{G} \cup \{1\}$.

(iii) $\mathcal{D}$ *separates points on* E.

Then, the following statements are true:

(a) $\Gamma' \triangleq \{\mu' = \mathfrak{be}(\mu)\}_{\mu\in\Gamma\setminus\Gamma_0}$ *exists for some* $\Gamma_0 \in \mathscr{P}_0(\Gamma)$ *and is sequentially* $\mathbf{m}$*-tight.*

(b) Γ' *has at most one weak limit point in* $\mathcal{M}^+(E^d)$.

(c) If, in addition, $\{\mu(E^d)\}_{\mu\in\Gamma} \subset [a,b]$ *for some* $0 < a < b$*, then* Γ' *has a unique weak limit point* ν *in* $\mathcal{M}^+(E^d)$*. In particular,* ν *is an* $\mathbf{m}$*-tight measure with total mass*[3] *in* $[a,b]$ *and satisfies*

$$\text{w-}\lim_{n\to\infty} \mu_n' = \nu,\ \forall\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma\backslash\Gamma_0. \tag{5.1.1}$$

Note 5.5. The condition (iii) above is true for a wide subclass of Hausdorff spaces which need neither be Tychonoff nor be baseable.

Note 5.6. Any $\mathcal{D} \subset M_b(E;\mathbf{R})$ satisfies

$$\mathfrak{ca}\left[\Pi^d(\mathcal{D})\right] \subset M_b\left(E^d, \mathscr{B}(E)^{\otimes d}; \mathbf{R}\right) \tag{5.1.2}$$

and any $\mathcal{D} \subset C_b(E;\mathbf{R})$ satisfies

$$\mathfrak{ca}\left[\Pi^d(\mathcal{D})\right] \subset C_b\left(E^d, \mathscr{O}(E)^d; \mathbf{R}\right) \tag{5.1.3}$$

[2] $\{\int_{E^d} f(x)\mu(dx)\}_{\mu\in\Gamma}$ lies in $[-\|f\|_\infty, \|f\|_\infty]$ and has at least one limit point in $\mathbf{R}$ by the Bolzano-Weierstrass Theorem. So, it actually has a unique limit point.

[3] The notion of total mass was specified in §2.1.2. The notation "w-$\lim_{n\to\infty}\mu_n = \nu$" introduced in §2.3 means that ν is the weak limit of $\{\mu_n\}_{n\in\mathbf{N}}$. In other words, it means $\mu_n \Rightarrow \nu$ as $n \uparrow \infty$ and ν is the unique weak limit point of $\{\mu_n\}_{n\in\mathbf{N}}$.

by Proposition A.21 (a) and properties of uniform convergence. Hence, the integral $\int_{E^d} f(x)\mu(dx)$ is well-defined for all $f \in \mathfrak{ca}[\Pi^d(M_b(E;\mathbf{R}))]$ and $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$.

Before giving the proof, it is worth putting here a basic but useful Portmanteau-type lemma for compact sets.

Lemma 5.7. *Let E be a topological space and $C(E;\mathbf{R})$ separate points on E. Then, the following statements are true:*

(a) (2.3.4) implies $\mu(K) \geq \limsup_{n\to\infty} \mu_n(K)$ for all $K \in \mathscr{K}(E)$.

(b) If μ is a weak limit point of Γ in $\mathcal{M}^+(E)$, and if Γ is tight (resp. $\mathbf{m}$-tight) in $A \subset E$, then $\Gamma \cup \{\mu\}$ is tight (resp. $\mathbf{m}$-tight) in A.

Note 5.8. We remind the readers of that $C(E;\mathbf{R})$ separating points (resp. strongly separating points) on E is equivalent to $C_b(E;\mathbf{R})$ separating points (resp. strongly separating points) on E (see Corollary B.53).

Remark 5.9. The classical Portmanteau's Theorem asserts that the mass of a weakly convergent sequence in every *closed* set is confined without any escape. In the setting of Theorem 5.4 and Lemma 5.7, the mass may no longer be confined by a general closed subset of the possibly non-Tychonoff space E (see Theorem 2.17). Nonetheless, the lemma above confirms that the subclass $\mathscr{K}(E)$ of $\mathscr{C}(E)$ still maintains this property.

Proof of Lemma 5.7. (a) $(E, \mathscr{O}_{C(E;\mathbf{R})}(E))$ is a Tychonoff topological coarsening of E by Proposition A.25 (a, b). $K \in \mathscr{K}(E, \mathscr{O}_{C(E;\mathbf{R})}(E)) \subset \mathscr{C}(E, \mathscr{O}_{C(E;\mathbf{R})}(E))$ by Fact B.51 (b) (with $\mathcal{D} = C(E;\mathbf{R})$).

$$\mu_n \Longrightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+\left(E, \mathscr{O}_{C(E;\mathbf{R})}(E)\right) \tag{5.1.4}$$

by Fact B.26 (b) (with $\mathscr{U} = \mathscr{O}_{C(E;\mathbf{R})}(E;\mathbf{R})$). Now, (a) follows by Theorem 2.17 (a, b) (with $E = (E, \mathscr{O}_{C(E;\mathbf{R})}(E))$).

(b) Let $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma$ satisfy (2.3.4). By tightness (resp. $\mathbf{m}$-tightness) of Γ in A and (a), there exist $\{K_p\}_{p\in\mathbf{N}} \subset \mathscr{K}(A, \mathscr{O}_E(A))$ (resp. $\mathscr{K}^{\mathbf{m}}(A, \mathscr{O}_E(A))$) such that

$$\mu(E\backslash K_p) \leq \liminf_{n\to\infty} \mu_n(E\backslash K_p) \leq \sup_{\mu\in\Gamma} \mu(E\backslash K_p) \leq 2^{-p}, \ \forall p \in \mathbf{N}. \tag{5.1.5}$$

□

Proof of Theorem 5.4. (a) E is a Hausdorff space by Proposition A.17 (e) (with $A = E$). By Proposition 5.3 (with $\mathbf{I} = \{1, ..., d\}$, $S_i = E$ and $A = E^d$), there exists a $\Gamma_0 \subset \mathscr{P}_0(\Gamma)$ such that Γ' is well-defined and is sequentially $\mathbf{m}$-tight.

(b) Suppose $\{\mu'_{i,n} : i = 1, 2, n \in \mathbf{N}\} \subset \Gamma'$ satisfy

$$\mu'_{i,n} \Longrightarrow \mu_i \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d),\ \forall i = 1, 2. \tag{5.1.6}$$

The sequential $\mathbf{m}$-tightness of Γ' implies an $\mathbf{m}$-tight subsequence $\{\mu'_{i,n_k}\}_{k\in\mathbf{N}}$ for each $i = 1, 2$. $\mathcal{G}$ separates points on E^d by Proposition A.21 (b), so does $C(E^d; \mathbf{R})$ by (5.1.3). Then, $\{\mu'_{i,n_k}\}_{k\in\mathbf{N},i=1,2} \cup \{\mu_1\} \cup \{\mu_2\}$ is $\mathbf{m}$-tight by Lemma 5.7 (b) (with $E = A = E^d$ and $\Gamma = \{\mu'_{i,n_k}\}_{k\in\mathbf{N},i=1,2}$). It follows that[4]

$$\begin{aligned}\int_{E^d} f(x)\mu_1(dx) &= \lim_{n\to\infty} f^*(\mu'_{1,n}) \\ &= \lim_{n\to\infty} f^*(\mu'_{2,n}) = \int_{E^d} f(x)\mu_2(dx)\ \forall f \in \mathcal{G} \cup \{1\}\end{aligned} \tag{5.1.7}$$

by (5.1.6), (5.1.3) and the fact that $\{\int_{E^d} f(x)\mu_{i,n}(dx)\}_{n\in\mathbf{N},i=1,2}$ has at most one limit point in $\mathbf{R}$ for all $f \in \mathcal{G} \cup \{1\}$. Now, $\mu_1 = \mu_2$ by Lemma B.59 (a) (with $E = E^d$ and $\mathcal{D} = \mathcal{G}$).

(c) E^d is a Hausdorff space by Proposition A.2 (d). So, Γ' has a unique weak limit point ν in $\mathcal{M}^+(E^d)$ with $\nu(E^d) \in [a, b]$ by (b) and Lemma A.46 (with $E = E^d$ and $\Gamma = \Gamma'$). ν is $\mathbf{m}$-tight by Lemma 5.7 (b) (with $E = A = E^d$ and $\Gamma = \Gamma'$). Furthermore, (5.1.1) follows by the sequential $\mathbf{m}$-tightness of Γ', the fact

$$\mu'(E^d) = \mu(E^d) \in [a, b] \subset (0, \infty),\ \forall \mu \in \Gamma\backslash\Gamma_0 \tag{5.1.8}$$

and Corollary B.57 (with $E = A = E^d$, $\mu = \nu$, $\Gamma = \Gamma'$ and $\mu_n = \mu'_n$). □

For finite measures on infinite-dimensional Cartesian power of E, one can use the two results above to show unique existence of weak limit point for their finite-dimensional distributions. In the following theorem, we establish the Kolmogorov's Extension of these finite-dimensional weak limit points.

[4] The notation f^* was defined in (2.3.2).

Theorem 5.10. *Let E be a topological space, $\Gamma \subset \mathfrak{M}^+(E^{\mathbf{I}}, \mathscr{B}(E)^{\otimes \mathbf{I}})$ and $\mathcal{D} \subset C_b(E;\mathbf{R})$. Suppose that:*

(i) $\{\mu \circ \mathfrak{p}_{\mathbf{I}_0}^{-1}\}_{\mu \in \Gamma}$ *is sequentially* $\mathbf{m}$*-tight for all* $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$.

(ii) $\{\int_{E^{\mathbf{I}_0}} f(x)\mu \circ \mathfrak{p}_{\mathbf{I}_0}^{-1}(dx)\}_{\mu \in \Gamma}$ *has at most one limit point in* $\mathbf{R}$ *for all* $f \in \mathfrak{mc}[\Pi^{\mathbf{I}_0}(\mathcal{D})] \cup \{1\}$[5] *for all* $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$.

(iii) $\mathcal{D}$ *separates points on* E.

(iv) $\{\mu(E^{\mathbf{I}})\}_{\mu \in \Gamma} \subset [a,b]$ *for some* $0 < a < b$.

Then, there exist a unique $\mu^\infty \in \mathfrak{M}^+(E^{\mathbf{I}}, \mathscr{B}(E)^{\otimes \mathbf{I}})$ *such that:*

(a) The total mass of μ^∞ *lies in* $[a,b]$.

(b) $\mu^\infty \circ \mathfrak{p}_{\mathbf{I}_0}^{-1} \in \mathcal{M}^+(E^{\mathbf{I}_0})$ *is* $\mathbf{m}$*-tight for all* $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$.

(c) For each $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$*, there is some* $\Gamma^0_{\mathbf{I}_0} \in \mathscr{P}_0(\Gamma)$ *such that* $\mu^\infty \circ \mathfrak{p}_{\mathbf{I}_0}^{-1}$ *is the weak limit*[6] *of any subsequence of* $\{\mu'_{\mathbf{I}_0} = \mathfrak{be}(\mu \circ \mathfrak{p}_{\mathbf{I}_0}^{-1})\}_{\mu \in \Gamma \setminus \Gamma^0_{\mathbf{I}_0}}$ *in* $\mathcal{M}^+(E^{\mathbf{I}_0})$.

Remark 5.11. For each $\mu \in \Gamma \subset \mathfrak{M}^+(E^{\mathbf{I}}, \mathscr{B}(E)^{\otimes \mathbf{I}})$, the family of finite-dimensional distributions $\{\mu \circ \mathfrak{p}_{\mathbf{I}_0}^{-1}\}_{\mu \in \Gamma}$ naturally satisfies the *Kolmogorov's consistency* (see [Aliprantis and Border, 2006, §15.6, p.520]) as they are projected from the same μ. The goal of Theorem 5.10 is to show existence *and* consistency of weak limit points of $\{\mu' = \mu \circ \mathfrak{p}_{\mathbf{I}_0}^{-1}\}_{\mu \in \Gamma}$ on each finite dimension $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$. Then, these finite-dimensional weak limit points extend to a measure on $(E^{\mathbf{I}}, \mathscr{B}(E)^{\otimes \mathbf{I}})$.

Proof of Theorem 5.10. We fix $\mathbf{I}_1, \mathbf{I}_2 \in \mathscr{P}_0(\mathbf{I})$ with $\mathbf{I}_1 \subset \mathbf{I}_2$, let $\mathfrak{p}_{\mathbf{I}_j}$ denote the projection from $E^{\mathbf{I}}$ to $E^{\mathbf{I}_j}$ for each $j = 1, 2$, use $\widetilde{\mathfrak{p}}$ to specially denote the projection from $E^{\mathbf{I}_2}$ to $E^{\mathbf{I}_1}$ and observe $\mu \circ \mathfrak{p}_{\mathbf{I}_j}^{-1}(E^{\mathbf{I}_j}) = \mu(E^{\mathbf{I}}) \in [a,b]$ for all $\mu \in \Gamma$ and $j = 1, 2$.

[5] The notation "$\Pi^{\mathbf{I}_0}(\mathcal{D})$" was defined in §2.2.3. $\Pi^{\mathbf{I}_0}(\mathcal{D}) = \Pi^d(\mathcal{D})$ with $d \doteq \aleph(\mathbf{I}_0)$.

[6] The notion of weak limit was specified in §2.3.

By Theorem 5.4 (a, c) (with $d = \aleph(\mathbf{I}_j)$ and $\Gamma = \{\mu \circ \mathfrak{p}_{\mathbf{I}_j}^{-1}\}_{\mu\in\Gamma}$), there exist $\mu_{\mathbf{I}_j}^\infty \in \mathcal{M}^+(E^{\mathbf{I}_j})$ and $\Gamma_{\mathbf{I}_j}^0 \in \mathscr{P}_0(\Gamma)$ for each $j = 1, 2$ such that $\mu_{\mathbf{I}_j}^\infty$ is an $\mathbf{m}$-tight measure with total mass in $[a, b]$ and is the weak limit of any subsequence of

$$\Gamma'_{\mathbf{I}_j} \mathrel{\hat{=}} \left\{\mu'_{\mathbf{I}_j} = \mathfrak{be}(\mu \circ \mathfrak{p}_{\mathbf{I}_j}^{-1})\right\}_{\mu\in\Gamma\setminus\Gamma_{\mathbf{I}_j}^0} \subset \mathcal{M}^+(E^{\mathbf{I}_j}). \tag{5.1.9}$$

Suppose that $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma\setminus(\Gamma_{\mathbf{I}_1}^0 \cup \Gamma_{\mathbf{I}_2}^0)$ satisfies

$$\text{w-}\lim_{n\to\infty} \mu'_{n,\mathbf{I}_j} = \mu_{\mathbf{I}_j}^\infty,\ \forall j = 1, 2, \tag{5.1.10}$$

where $\mu'_{n,\mathbf{I}_j} = \mathfrak{be}(\mu_n \circ \mathfrak{p}_{\mathbf{I}_j}^{-1}) \in \Gamma'_{\mathbf{I}_j}$ for each $n \in \mathbf{N}$ and $j = 1, 2$. $\widetilde{\mathfrak{p}} \in C(E^{\mathbf{I}_2}; E^{\mathbf{I}_1})$ by Fact 2.4 (a). It follows that

$$\mu'_{n,\mathbf{I}_1} = \mu_n \circ \mathfrak{p}_{\mathbf{I}_2}^{-1} \circ \widetilde{\mathfrak{p}}^{-1} \Longrightarrow \mu_{\mathbf{I}_2}^\infty \circ \widetilde{\mathfrak{p}}^{-1} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^{\mathbf{I}_1}) \tag{5.1.11}$$

by (5.1.10) and the Continuous Mapping Theorem (Theorem B.25 (a)). So, (5.1.10) and (5.1.11) imply

$$\mu_{\mathbf{I}_1}^\infty = \mu_{\mathbf{I}_2}^\infty \circ \mathfrak{p}_{\mathbf{I}_1}^{-1}. \tag{5.1.12}$$

By the argument above, Γ uniquely determines $\{\mu_{\mathbf{I}_0}^\infty \in \mathcal{M}^+(E^{\mathbf{I}_0})\}_{\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})}$ such that: (1) $\{\mu_{\mathbf{I}_0}^\infty\}_{\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})}$ satisfies the Kolmogorov's consistency and admits a common total mass $c \in [a, b]$, and (2) each $\mu_{\mathbf{I}_0}^\infty$ is $\mathbf{m}$-tight and is the weak limit of any subsequence of $\{\mu'_{\mathbf{I}_0}\}_{\mu\in\Gamma\setminus\Gamma_{\mathbf{I}_0}^0}$ for some $\Gamma_{\mathbf{I}_0}^0 \in \mathscr{P}_0(\Gamma)$. $\{E^{\mathbf{I}_0}\}_{\mathbf{I}_0\in\mathscr{P}_0(\mathbf{I})}$ are all Hausdorff spaces by (5.1.3) (with $d = \aleph(\mathbf{I}_0)$) and Proposition A.17 (e) (with $E = A = E^{\mathbf{I}_0}$ and $\mathcal{D} = C(E^{\mathbf{I}_0}; \mathbf{R})$). Now, the unique existence of $\mu^\infty \in \mathfrak{M}^+(E^{\mathbf{I}}, \mathscr{B}(E)^{\otimes\mathbf{I}})$ satisfying $\mu^\infty(E^{\mathbf{I}}) = c$ and

$$\mu^\infty \circ \mathfrak{p}_{\mathbf{I}_0}^{-1} = \mu_{\mathbf{I}_0}^\infty,\ \forall \mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I}) \tag{5.1.13}$$

follows by a suitable version of the Kolmogorov's Extension Theorem (see [Aliprantis and Border, 2006, Corollary 15.28]). □

5.2 Replica measure

5.2.1 Definition

Given a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E and $d \in \mathbf{N}$, replicating a finite measure μ from the product measurable space $(E^d, \mathscr{B}(E)^{\otimes d})$ onto $\widehat{E}^d$ means expanding the concentrated measure $\mu|_{E_0^d}$ to a Borel replica measure on $\widehat{E}^d$.

Definition 5.12. Let E be a topological space and $d \in \mathbf{N}$. The **replica of** $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$ with respect to a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E is defined by[7]

$$\overline{\mu} \doteq \left(\mu|_{E_0}\right)\Big|^{\widehat{E}^d}. \tag{5.2.1}$$

The following fact justifies our definition of replica measure.

Fact 5.13. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d \in \mathbf{N}$. Then, any $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$ satisfies*

$$\begin{aligned} \mu|_{E_0^d} \in \mathfrak{M}^+\left(E_0^d, \mathscr{B}(E)^{\otimes d}\big|_{E_0^d}\right) &= \mathfrak{M}^+\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}\right) \\ \subset \mathfrak{M}^+\left(E_0^d, \mathscr{B}_{\mathcal{F}}(E_0)^{\otimes d}\right) &= \mathcal{M}^+\left(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d\right). \end{aligned} \tag{5.2.2}$$

Moreover,

$$\mathcal{M}^+\left(E_0^d, \mathscr{O}_E(E_0)^d\right) \subset \mathfrak{M}^+\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}\right). \tag{5.2.3}$$

Proof. The first line of (5.2.2) follows by (3.1.10) and Fact 2.1 (a) (with $E = E^d$, $\mathscr{U} = \mathscr{B}(E)^{\otimes d}$ and $A = E_0^d$). The second line of (5.2.2) and (5.2.3) follow by (3.1.19). □

Notation 5.14. If no confusion is caused, we *will always* let $\overline{\mu}$ denote the replica of μ with respect to the underlying base and do not make special mention.

Below are several basic properties of replica measure.

Proposition 5.15. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$ and $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$. Then, the following statements are true:*

[7] The concentrated measure "$\mu|_A$" and expanded measure "$\nu|^E$" were defined in §2.1.2.

(a) (5.2.1) well defines $\overline{\mu} \in \mathcal{M}^+(\widehat{E}^d)$*. Moreover,*

$$\overline{\mu}(A) = \mu\left(A \cap E_0^d\right), \ \forall A \in \mathscr{B}(\widehat{E}^d) \tag{5.2.4}$$

(b) $\overline{\mu} \in \mathcal{P}(\widehat{E}^d)$ *if and only if* $\mu(E_0^d) = 1$*.*

(c) Any $\nu \in \mathfrak{be}(\mu)$ *satisfies* $\overline{\nu} = \overline{\mu}$*.*

(d) If $f \in M_b(E^d; \mathbf{R})$ *satisfies* $\overline{f} \in M_b(\widehat{E}^d; \mathbf{R})$[8]*, then*

$$\begin{aligned}\int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx) &= \int_{\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}\right)} f|_{E_0^d}(x)\mu|_{E_0^d}(dx) \\ &= \int_{\left(E_0^d, \mathscr{B}_{\widehat{E}^d}(E_0^d)\right)} f|_{E_0^d}(x)\mu|_{E_0^d}(dx) = \int_{\widehat{E}^d} \overline{f}(x)\overline{\mu}(dx).\end{aligned} \tag{5.2.5}$$

(e) If $f \in C(E^d; \mathbf{R})$ *has a replica* $\widehat{f}$*, then*

$$\begin{aligned}\int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx) &= \int_{\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}\right)} f|_{E_0^d}(x)\mu|_{E_0^d}(dx) \\ &= \int_{\left(E_0^d, \mathscr{B}_{\widehat{E}^d}(E_0^d)\right)} f|_{E_0^d}(x)\mu|_{E_0^d}(dx) = \int_{\widehat{E}^d} \widehat{f}(x)\overline{\mu}(dx).\end{aligned} \tag{5.2.6}$$

In particular, (5.2.6) is true for all $f \in \mathfrak{ca}[\Pi^d(\mathcal{F})]$*.*

Proof. (a) follows by (5.2.1), (5.2.2) and Fact 2.1 (b) (with $E = \widehat{E}^d$, $A = E_0^d$ and $\nu = \mu|_{E_0^d}$) and (b) is immediate by (5.2.4).

(c) Note that $\mu|_{E_0^d} = \nu|_{E_0^d}$ as members of $\mathfrak{M}^+(E_0^d, \mathscr{B}(E)^{\otimes d}|_{E_0^d})$.

(d) We have by (3.1.19) that

$$f|_{E_0^d} = \overline{f}|_{E_0^d} \in M_b\left(E_0^d, \mathscr{B}_{\widehat{E}^d}(E_0^d); \mathbf{R}\right) \subset M_b\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}; \mathbf{R}\right). \tag{5.2.7}$$

We then have by (5.2.7), (3.1.10) and Fact B.2 (with $E = E^d$, $\mathscr{U} = \mathscr{B}(E)^{\otimes d}$ and $A = E_0^d$) that

$$f\mathbf{1}_{E_0^d} \in M_b\left(E^d, \mathscr{B}(E)^{\otimes d}; \mathbf{R}\right). \tag{5.2.8}$$

So, (5.2.5) is well-defined and follows by (5.2.7) and (a).

[8] $\overline{f}$ was defined in Notation 4.5.

(e) We have by (3.1.19) that

$$\begin{aligned} f|_{E_0^d} = \widehat{f}|_{E_0^d} &\in C_b\left(E_0^d, \mathscr{O}_{\widehat{E}^d}(E_0^d); \mathbf{R}\right) \\ &\subset M_b\left(E_0^d, \mathscr{O}_{\widehat{E}^d}(E_0^d); \mathbf{R}\right) \subset M_b\left(E_0^d, \mathscr{B}_E(E_0)^{\otimes d}; \mathbf{R}\right). \end{aligned} \tag{5.2.9}$$

So, (5.2.6) is well-defined and follows by (5.2.9) and (a). $\square$

The next proposition gives a sufficient condition for a Borel measure on $\widehat{E}^d$ to be the replica of some Borel measure on E^d.

Proposition 5.16. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d \in \mathbf{N}$. If $\nu \in \mathcal{M}^+(\widehat{E}^d)$ is supported on $A \subset E_0^d$ and $A \in \mathscr{B}^{\mathbf{s}}(E^d)$, then*

$$\mu \mathrel{\hat{=}} (\nu|_A)|^{E^d} \in \mathcal{M}^+(E^d) \tag{5.2.10}$$

satisfies $\mu|_A = \nu|_A \in \mathcal{M}^+(A, \mathscr{O}_{E^d}(A))$ and $\nu = \overline{\mu}$.

Proof. We have by $A \in \mathscr{B}^{\mathbf{s}}(E^d)$ and Lemma 3.14 (a, b) that

$$\mathscr{B}_{E^d}(A) = \mathscr{B}_{\widehat{E}^d}(A) \subset \left[\mathscr{B}(E^d) \cap \mathscr{B}(\widehat{E}^d)\right] \tag{5.2.11}$$

It follows by Fact 2.1 (a) (with $\mu = \nu$ and $(E, \mathscr{U}) = (\widehat{E}^d, \mathscr{B}(\widehat{E}^d))$ and (5.2.11) that

$$\nu|_A \in \mathcal{M}^+\left(A, \mathscr{O}_{E^d}(A)\right) = \mathcal{M}^+\left(A, \mathscr{O}_{\widehat{E}^d}(A)\right). \tag{5.2.12}$$

Then, $\mu \in \mathcal{M}^+(E^d)$ by (5.2.12) and Fact 2.1 (b) (with $E = E^d$, $\mathscr{U} = \mathscr{B}(E^d)$ and $\nu = \nu|_A$). It follows by Fact 2.1 (c) (with $E = E^d$, $\mathscr{U} = \mathscr{B}(E^d)$ and $\nu = \nu|_A$) and (5.2.12) that

$$\mu|_A = \nu|_A \in \mathcal{M}^+\left(A, \mathscr{O}_{\widehat{E}^d}(A)\right). \tag{5.2.13}$$

It follows by the fact $\nu(\widehat{E}^d \backslash A) = 0$, (5.2.13) and Fact 2.1 (c) (with $E = \widehat{E}^d$, $\mathscr{U} = \mathscr{B}(\widehat{E}^d)$ and $\mu = \nu$) that

$$\nu = (\nu|_A)|^{\widehat{E}^d} = (\mu|_A)|^{\widehat{E}^d} = \overline{\mu}. \tag{5.2.14}$$

$\square$

5.2.2 Weak convergence of replica measures

We now consider the association of weak convergence of Borel extensions on E^d and that of replica measures on $\widehat{E}^d$. The next proposition discusses the direction from E^d to $\widehat{E}^d$.

Proposition 5.17. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$, $\mathcal{G} \doteq \mathfrak{mc}[\Pi^d(\mathcal{F}\backslash\{1\})]$ and $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\} \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$. Consider the following statements:*

(a) The replica measures $\{\overline{\mu}_n\}_{n\in\mathbf{N}}$ and $\overline{\mu}$ satisfy

$$\text{w-}\lim_{n\to\infty} \overline{\mu}_n = \overline{\mu} \text{ in } \mathcal{M}^+(\widehat{E}^d). \tag{5.2.15}$$

(b) The concentrated measures $\{\mu_n|_{E_0^d}\}_{n\in\mathbf{N}}$ and $\mu|_{E_0^d}$ satisfy

$$\text{w-}\lim_{n\to\infty} \mu_n|_{E_0^d} = \mu|_{E_0^d} \text{ in } \mathcal{M}^+\left(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d\right). \tag{5.2.16}$$

(c) The original measures $\{\mu_n\}_{n\in\mathbf{N}}$ and μ satisfy[9]

$$\lim_{n\to\infty} \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu_n(dx) = \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx), \ \forall f \in \mathcal{G} \cup \{1\}. \tag{5.2.17}$$

(d) E_0^d is a common support of $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\}$. Moreover,

$$\lim_{n\to\infty} \int_{E^d} f(x)\mu_n(dx) = \int_{E^d} f(x)\mu(dx), \ \forall f \in \mathcal{G} \cup \{1\}. \tag{5.2.18}$$

(e) E_0^d is a common support of $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\}$. Moreover, there exist $\{\mu_n' \in \mathfrak{be}(\mu_n)\}_{n\in\mathbf{N}}$ and $\mu' \in \mathfrak{be}(\mu)$ such that

$$\mu_n' \Longrightarrow \mu' \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d). \tag{5.2.19}$$

(f) E_0^d is a common support of $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\}$. Moreover, there exist $\{\mu_n' \in$

[9] The integrals in (5.2.17) are well-defined by Proposition 5.15 (d, e). Those in (5.2.18) are well-defined by Note 5.6 (with $\mathcal{D} = \mathcal{F}$).

$\mathfrak{be}(\mu_n)\}_{n\in\mathbf{N}}$ *and* $\mu' \in \mathfrak{be}(\mu)$ *such that*

$$\mu'_n|_{E_0^d} \Longrightarrow \mu'|_{E_0^d} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+\left(E_0^d, \mathscr{O}_E(E_0)^d\right). \tag{5.2.20}$$

Then, (a) - (c) are equivalent. (c) - (f) are successively stronger. Moreover, (e) and (f) are equivalent when E^d *is a Tychonoff space.*

Proof. ((a) $\to$ (b)) $\widehat{E}^d$ is a Tychonoff space by Lemma 3.9 (c) and Proposition A.26 (a). $(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d)$ is a metrizable and separable space by Lemma 3.9 (d) (with $A = E_0^d$), so is $\mathcal{M}^+(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d)$ by Corollary A.43. Now, (b) follows by (5.2.15), (5.2.1), Lemma B.55 (with $E = \widehat{E}^d$, $A = E_0^d$, $\nu_n = \mu_n|_{E_0^d}$ and $\nu = \mu|_{E_0^d}$) and the Hausdorff property of $\mathcal{M}^+(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d)$.

((b) $\to$ (c)) We have by Lemma 3.9 (b, d) (with $A = E_0^d$) that $\mathcal{G}|_{E_0^d} \subset C_b(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d; \mathbf{R})$. Hence, we have by (5.2.16) and (5.2.2) that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu_n(dx) \\
&= \lim_{n\to\infty} \int_{\left(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d\right)} f|_{E_0^d}(x)\mu_n|_{E_0^d}(dx) \\
&= \int_{\left(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d\right)} f|_{E_0^d}(x)\mu|_{E_0^d}(dx) \\
&= \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx), \ \forall f \in \mathcal{G} \cup \{1\}.
\end{aligned} \tag{5.2.21}$$

((c) $\to$ (a)) It follows by Proposition 5.15 (e) (with $\mu = \mu_n$ or μ) that

$$\begin{aligned}
\lim_{n\to\infty} \widehat{f}^*(\overline{\mu}_n) &= \lim_{n\to\infty} \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu_n(dx) \\
&= \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx) = \widehat{f}^*(\overline{\mu}), \ \forall f \in \mathcal{G} \cup \{1\}.
\end{aligned} \tag{5.2.22}$$

$\mathcal{M}^+(\widehat{E}^d)$ is a metrizable space by Corollary 3.11 (c). Now, (a) follows by (5.2.22), Corollary 3.11 (b) (with $A = \widehat{E}^d$) and the Hausdorff property of $\mathcal{M}^+(\widehat{E}^d)$.

((d) → (c)) Note that if E_0^d is a common support of $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\}$, then

$$\begin{aligned}
&\int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu_n(dx) - \int_{E^d} f(x)\mu_n(dx) \\
&= \int_{E^d} f(x)\mathbf{1}_{E_0^d}(x)\mu(dx) - \int_{E^d} f(x)\mu(dx) = 0,\ \forall n \in \mathbf{N}.
\end{aligned} \tag{5.2.23}$$

((e) → (d)) follows by $\mathcal{F} \subset C_b(E;\mathbf{R})$ and Fact B.54 (with $\mu = \mu_n$ or μ).

In both (e) and (f), E_0^d is a common support of $\{\mu_n'\}_{n\in\mathbf{N}} \cup \{\mu'\}$ and so $\mu_n' = (\mu_n'|_{E_0^d})|^{E^d}$ for all $n \in \mathbf{N}$ and $\mu' = (\mu'|_{E_0^d})|^{E^d}$ by Fact 2.1 (c) (with $E = E^d$, $\mathscr{U} = \mathscr{B}(E^d)$, $A = E_0^d$ and $\mu = \mu_n'$ or μ'). It then follows by Lemma B.55 (with $E = E^d$, $A = E_0^d$, $\mu_n = \mu_n'|_{E_0^d}$ and $\mu = \mu'|_{E_0^d}$) that (f) implies (e) in general, and (e) implies (f) when E^d is a Tychonoff space. □

The following corollary specializes Proposition 5.17 for probability measures.

Corollary 5.18. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$ and $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\mu\} \subset \mathfrak{P}(E^d, \mathscr{B}(E)^{\otimes d})$. Then, either of the following statements is equivalent to the statement of Proposition 5.17 (d):*

(a) The replica measures $\{\overline{\mu}_n\}_{n\in\mathbf{N}}$ and $\overline{\mu}$ satisfy

$$\overline{\mu}_n \Longrightarrow \overline{\mu} \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(\widehat{E}^d). \tag{5.2.24}$$

(b) The concentrated measures $\{\mu_n|_{E_0^d}\}_{n\in\mathbf{N}}$ and $\mu|_{E_0^d}$ satisfy

$$\mu_n|_{E_0^d} \Longrightarrow \mu|_{E_0^d} \text{ as } n \uparrow \infty \text{ in } \mathcal{P}\left(E_0^d, \mathscr{O}_{\widehat{E}}(E_0)^d\right). \tag{5.2.25}$$

Proof. This result follows by Proposition 5.15 (b) and Proposition 5.17. □

One can leverage proper tightness to transform a weak limit point of replica measures back into that of Borel extensions of the original measures.

Proposition 5.19. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$, $\mathcal{G} \mathrel{\hat{=}} \mathfrak{mc}[\Pi^d(\mathcal{F}\backslash\{1\})]$ and $\{\mu_n\}_{n\in\mathbf{N}} \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$. Suppose that:*

(i) $\{\mu_n\}_{n\in\mathbf{N}}$ is sequentially tight in E_0^d.

(ii) $\{\int_{E^d} f(x)\mu_n(dx)\}_{n\in\mathbf{N}}$ *is convergent in* $\mathbf{R}$ *for all* $f \in \mathcal{G}\cup\{1\}$.

(iii) $\{\mu_n(E^d)\}_{n\in\mathbf{N}} \subset [a,b]$ *for some* $0 < a < b$.

Then, there exist $\mu \in \mathcal{M}^+(E^d)$ *and* $N \in \mathbf{N}$ *such that:*

(a) μ *is* $\mathbf{m}$*-tight in* E_0^d *and* $\{\mu_n' = \mathfrak{be}(\mu_n)\}_{n>N}$ *exists.*

(b) $\{\mu_n'\}_{n>N}$ *satisfies*

$$\text{w-}\lim_{n\to\infty} \mu_n'|_{E_0^d} = \mu|_{E_0^d} \text{ in } \mathcal{M}^+\left(E_0^d, \mathscr{O}_E(E_0)^d\right) \tag{5.2.26}$$

and

$$\mu_n' \Longrightarrow \mu \text{ as } n\uparrow\infty \text{ in } \mathcal{M}^+(E^d). \tag{5.2.27}$$

Proof. $\{\mu_n\}_{n\in\mathbf{N}}$ is sequentially $\mathbf{m}$-tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$ by Corollary 3.15 (a). There exists an $N_1 \in \mathbf{N}$ such that $\{\mu_n\}_{n>N_1}$ are all supported on E_0^d by Fact B.29 (with $(E,\mathscr{U}) = (E^d, \mathscr{B}(E)^{\otimes d})$, $A = E_0^d$ and $\Gamma = \{\mu_n\}_{n\in\mathbf{N}}$). There exist $\nu \in \mathcal{M}^+(E_0^d, \mathscr{O}_E(E_0)^d)$ and $N_2 \in \mathbf{N}$ such that ν is $\mathbf{m}$-tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$, $\{\nu_n = \mathfrak{be}(\mu_n|_{E_0^d})\}_{n>N_2}$ exists and

$$\text{w-}\lim_{n\to\infty} \nu_n = \nu \text{ in } \mathcal{M}^+\left(E_0^d, \mathscr{O}_E(E_0)^d\right) \tag{5.2.28}$$

by Lemma 3.9 (e) and Theorem 5.4 (a, c) (with $E = (E_0, \mathscr{O}_E(E_0))$, $\Gamma = \{\mu_n|_{E_0^d}\}_{n\in\mathbf{N}}$ and $\mathcal{D} = \mathcal{F}\backslash\{1\}$).

$\mu \mathrel{\dot{=}} \nu|^{E^d}$ satisfies $\mu|_{E_0^d} = \nu$ by (3.1.10) and Fact 2.1 (c) (with $(E,\mathscr{U}) = (E^d, \mathscr{B}(E)^{\otimes d}$ and $A = E_0^d$).

$$\mu_n' \mathrel{\dot{=}} \nu_n|^{E^d} = \mathfrak{be}\left[\left(\mu_n|_{E_0^d}\right)\Big|^{E^d}\right] = \mathfrak{be}(\mu_n),\ \forall n > N \mathrel{\dot{=}} N_1 \vee N_2 \tag{5.2.29}$$

by (3.1.10), Fact 2.1 (c) (with $(E,\mathscr{U}) = (E^d, \mathscr{B}(E)^{\otimes d})$, $A = E_0^d$ and $\nu = \nu_n$) and Lemma B.48 (b) (with $\mathbf{I} = \{1,...,d\}$, $S_i = E$, $A = E_0^d$, $\mu = \mu_n$ and $\mathfrak{be}(\mu|_A) = \nu_n$).

Hence, (5.2.26) follows by (5.2.28). (5.2.27) follows by (5.2.26) and Lemma B.55 (with $E = E^d$, $A = E_0^d$, $\mu_n = \nu_n$ and $\mu = \nu$). □

Corollary 5.20. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $d \in \mathbf{N}$. If $\{\mu_n\}_{n\in\mathbf{N}} \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$ is sequentially tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$, and if*

$$\overline{\mu}_n \Longrightarrow \nu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(\widehat{E}^d), \tag{5.2.30}$$

then there exist $\mu \in \mathcal{M}^+(E^d)$ and $N \in \mathbf{N}$ satisfying Proposition 5.19 (a, b) and, in particular, $\nu = \overline{\mu}$.

Proof. There exists an $N_1 \in \mathbf{N}$ such that $\{\mu_n\}_{n>N_1}$ are all supported on E_0^d by Fact B.29 (with $(E, \mathscr{U}) = (E^d, \mathscr{B}(E)^{\otimes d})$, $A = E_0^d$ and $\Gamma = \{\mu_n\}_{n\in\mathbf{N}}$). It follows by (5.2.30) and Proposition 5.15 (e) (with $\mu = \mu_n$) that

$$\lim_{n\to\infty} \int_{E^d} f(x)\mu_n(dx) = \lim_{n\to\infty} \widehat{f}^*(\overline{\mu}_n) = \widehat{f}^*(\nu), \ \forall f \in \mathfrak{mc}\left[\Pi^d(\mathcal{F})\right]. \tag{5.2.31}$$

It follows by the fact $1 \in \Pi^d(\mathcal{F})$ and (5.2.31) that

$$\mu_n(E^d) \in \left(\frac{\nu(\widehat{E})}{2}, \frac{3\nu(\widehat{E})}{2}\right) \subset (0, \infty), \ \forall n > N_2 \tag{5.2.32}$$

for some $N_2 \in \mathbf{N} \cap (N_1, \infty)$. Now, we obtain the desired μ and N by (5.2.31), (5.2.32) and Proposition 5.19 (with $n = N_2 + n$, $a = \nu(\widehat{E})/2$ and $b = 3a$). $\{\mu_n\}_{n\in\mathbf{N}}$ satisfies (5.2.27), so we have[10]

$$\overline{\mu} = \text{w-}\lim_{n\to\infty} \overline{\mu}'_n = \text{w-}\lim_{n\to\infty} \overline{\mu}_n = \nu \tag{5.2.33}$$

by Proposition 5.15 (c) (with $\mu = \mu_n$ and $\nu = \mu'_n$) and Proposition 5.17 (a, e). □

5.3 Generalization of two fundamental results

5.3.1 Integral representation of linear functional

The celebrated Riesz-Radon Representation Theorem was established for *positive linear functionals on $C_0(E; \mathbf{R})$*[11] with E being a locally compact Hausdorff space. Herein, this result is extended to avoid the local compactness

[10] $\overline{\mu}'_n$ denote the replica of μ'_n.
[11] Positiveness of a functional on $C_0(E; \mathbf{R})$ means it maps non-negative functions into $\mathbf{R}^+$.

assumption which is violated by many infinite-dimensional spaces. Given reasonable regularity of the positive linear functional, we use the approach of replication and establish an analogue of the Riesz-Radon Representation Theorem on baseable spaces. As aforementioned in §3.2.2, baseable spaces need not be locally compact nor Tychonoff.

Theorem 5.21. *Let E be a $C_c(E;\mathbf{R})$-baseable space, φ be a linear functional on $C_c(E;\mathbf{R})$ and*

$$\mathcal{B} \mathrel{\hat{=}} \{g \in C_c(E;\mathbf{R}) : 0 < \|g\|_\infty \leq 1\}. \tag{5.3.1}$$

Then, the following statements are equivalent:

(a) There exists a positive linear functional Λ on $C_b(E;\mathbf{R})$ such that

$$\varphi(g) \leq \Lambda(g),\ \forall g \in C_c(E;\mathbf{R}) \tag{5.3.2}$$

and

$$\lambda_0 \mathrel{\hat{=}} \sup_{g\in\mathcal{B}} \varphi(g) = \Lambda(1) < \infty. \tag{5.3.3}$$

(b) There exists an $\mathbf{m}$-tight $\mu \in \mathcal{M}^+(E)$ such that

$$\varphi(g) = g^*(\mu),\ \forall g \in C_c(E;\mathbf{R}) \tag{5.3.4}$$

and

$$\mu(E) = \sup_{g\in\mathcal{B}} g^*(\mu). \tag{5.3.5}$$

Remark 5.22. In the theorem above, E is a Hausdorff space by Fact 3.24 (a). $C_c(E;\mathbf{R})$ is a possibly *non-unit*[12] subalgebra of $C_b(E;\mathbf{R})$ and is a function lattice[13] by Proposition B.44 (a). $C_c(E;\mathbf{R}) \neq \{0\}$ since $C_c(E;\mathbf{R})$ separates points on E, so

$$\mathcal{B} \neq \varnothing \tag{5.3.6}$$

and the supremum in (5.3.3) is well-defined.

[12] "non-unit" means excluding the constant function 1.
[13] The terminology "function lattice" was specified in §2.2.3.

Proof of Theorem 5.21. ((a) $\to$ (b)) We divide our proof into six steps.

Step 1: Extend φ to a positive linear functional on $C_b(E;\mathbf{R})$. $\varphi(g) \leq \Lambda(g) \leq 0$ for all non-positive $g \in C_c(E;\mathbf{R})$ by (5.3.2) and the positiveness of Λ, so φ is also a positive linear functional. Then, there exists a positive linear functional Φ on $C_b(E;\mathbf{R})$ satisfying

$$\varphi = \Phi|_{C_c(E;\mathbf{R})} \tag{5.3.7}$$

and

$$\Phi(g) \leq \Lambda(g),\ \forall g \in C_b(E;\mathbf{R}) \tag{5.3.8}$$

by a suitable version of the Hahn-Banach Theorem (see [Aliprantis and Border, 2006, Theorem 8.31]). In particular,

$$\lambda_0 = \Lambda(1) \geq \Phi(1) \geq \sup_{g \in C_b(E;\mathbf{R})\setminus\{0\}} \Phi\left(\frac{g}{\|g\|_\infty}\right) \geq \sup_{g\in\mathcal{B}} \Phi(g) = \lambda_0 \tag{5.3.9}$$

by (5.3.3), (5.3.8), the positiveness of Φ and the fact

$$g \leq \|g\|_\infty \leq 1,\ \forall g \in \mathcal{B}. \tag{5.3.10}$$

Step 2: Construct a suitable base. Letting

$$f_{g,a,b} \mathrel{\hat{=}} ag + b,\ \forall g \in C_c(E;\mathbf{R}), a, b \in \mathbf{R}, \tag{5.3.11}$$

we have that

$$\mathcal{D} \mathrel{\hat{=}} \mathfrak{ag}\left(C_c(E;\mathbf{R}) \cup \{1\}\right) = \left\{f_{g,a,b} : g \in C_c(E;\mathbf{R}), a, b \in \mathbf{R}\right\}. \tag{5.3.12}$$

E is a $\mathcal{D}$-baseable space by Fact 3.34 (d) (with $A = E$, $\mathcal{D} = C_c(E;\mathbf{R})$ and $\mathcal{D}' = \mathcal{D}$). There exist $\{g_p\}_{p\in\mathbf{N}} \subset \mathcal{B}$ satisfying

$$\Phi(1) = \lambda_0 = \lim_{p\to\infty} \varphi(g_p) = \lim_{p\to\infty} \Phi(g_p) \tag{5.3.13}$$

by (5.3.9), (5.3.3) and (5.3.7). We then find a base $(E, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E satisfying

$$\{g_p\}_{p\in\mathbf{N}} \subset (\mathcal{F} \cap \mathcal{B}) \subset (\mathcal{F}\setminus\{1\}) \subset C_c(E;\mathbf{R}) \subset \mathcal{D} \tag{5.3.14}$$

by Lemma 3.39 (c) (with $E_0 = E$ and $\mathcal{D}_0 = \{g_p\}_{p\in\mathbf{N}}$).

Step 3: Construct a replica positive linear functional on $C(\widehat{E};\mathbf{R})$. We have[14]

$$\widehat{f}_{g,a,b} = a\widehat{g} + b = a\overline{g} + b,\ \forall f_{g,a,b} \in \mathcal{D} \tag{5.3.15}$$

by Proposition 4.6 (d) (with $d = k = 1$ and $E_0 = E$), Lemma 3.3 (c) and Fact B.45 (with $E = \widehat{E}$, $A = E$ and $f = \widehat{g}$).

$$\mathfrak{ag}(\widehat{\mathcal{F}}) = \left\{\widehat{f}_{g,a,b} : f_{g,a,b} \in \mathfrak{ag}(\mathcal{F})\right\} \tag{5.3.16}$$

is a linear subspace of $C(\widehat{E};\mathbf{R})$ on which

$$\widehat{\Phi}(\widehat{f}_{g,a,b}) \doteq a\varphi(g) + b\lambda_0,\ \forall \widehat{f}_{g,a,b} \in \mathfrak{ag}(\widehat{\mathcal{F}}) \tag{5.3.17}$$

defines a positive linear functional. Moreover,

$$\widehat{\Phi}(\widehat{f}_{g,a,b}) = \Phi(f_{g,a,b}) \leq \lambda_0 \left\|f_{g,a,b}\right\|_\infty = \lambda_0 \left\|\widehat{f}_{g,a,b}\right\|_\infty,\ \forall f_{g,a,b} \in \mathfrak{ag}(\mathcal{F}) \tag{5.3.18}$$

by (5.3.7), the first equality of (5.3.13) and Lemma B.72 (a) (with $d = k = 1$, $E_0 = E$ and $f = f_{g,a,b}$). Hence, $\widehat{\Phi}$ extends linearly onto $C(\widehat{E};\mathbf{R})$ and satisfies

$$\lambda_0 = \Phi(1) = \widehat{\Phi}(1) = \sup_{h\in C(\widehat{E};\mathbf{R})} \frac{\widehat{\Phi}(h)}{\|h\|_\infty} < \infty \tag{5.3.19}$$

by (5.3.17), (5.3.18) (with $a = 0$ and $b = 1$) and the classical Hahn-Banach Theorem (see [Dudley, 2002, Theorem 6.1.4]).

$\mathfrak{ag}(\widehat{\mathcal{F}})$ is uniformly dense in $C(\widehat{E};\mathbf{R})$ by Corollary 3.10 (with $d = 1$ and $E_0 = E$). For each fixed $h \in C(\widehat{E};\mathbf{R})$, there exist $\{f_n\} \subset \mathfrak{ag}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \|h|_E - f_n\|_\infty = \lim_{n\to\infty} \|h|_E - \widehat{f}_n|_E\|_\infty \leq \lim_{n\to\infty} \|h - \widehat{f}_n\|_\infty = 0. \tag{5.3.20}$$

Φ and $\widehat{\Phi}$ are continuous functionals by (5.3.9), (5.3.19) and [Dudley, 2002, Theorem 6.1.2]. So, (5.3.18) and (5.3.20) imply

$$\widehat{\Phi}(h) = \lim_{n\to\infty} \widehat{\Phi}(\widehat{f}_n) = \lim_{n\to\infty} \Phi(f_n) = \Phi(h|_E). \tag{5.3.21}$$

[14] We noted in Notation 4.5 that $\overline{g} \doteq \mathfrak{var}(g;\widehat{E},E_0,0)$.

From the argument above and the positiveness of Φ it follows that

$$\widehat{\Phi}(h) = \Phi(h|_E),\ \forall h \in C(\widehat{E};\mathbf{R}) \tag{5.3.22}$$

and $\widehat{\Phi}(h) \geq 0$ for all non-negative $h \in C(\widehat{E};\mathbf{R})$. Thus, $\widehat{\Phi}$ is a positive functional.

Step 4: Establish integral representation of the replica functional. Since $\widehat{E}$ is a compact Polish space, we apply the classical Riesz Representation Theorem (see [Kallianpur and Xiong, 1995, Theorem 2.1.5]) to $\widehat{\Phi}$ and obtain a $\nu \in \mathcal{M}^+(\widehat{E})$ satisfying

$$\widehat{\Phi}(h) = h^*(\nu),\ \forall h \in C(\widehat{E};\mathbf{R}). \tag{5.3.23}$$

It follows by (5.3.22) and (5.3.23) that

$$\Phi(h|_E) = h^*(\nu),\ \forall h \in C(\widehat{E};\mathbf{R}). \tag{5.3.24}$$

Moreover, it follows by (5.3.24) and (5.3.13) that

$$\nu(\widehat{E}) = \Phi(1) = \lambda_0 = \lim_{p\to\infty} \Phi(g_p) = \lim_{p\to\infty} \widehat{\Phi}(\widehat{g}_p) = \lim_{p\to\infty} \widehat{g}_p^*(\nu). \tag{5.3.25}$$

Step 5: Establish the desired measure μ. We define

$$\mathcal{A} \mathrel{\mathring{=}} \left\{g \in C_c(E;\mathbf{R}) : f_{g,a,b} \in \mathfrak{ag}_{\mathbf{Q}}(\mathcal{F}) \text{ for some } a,b \in \mathbf{R}\right\}, \tag{5.3.26}$$

let $K_g \in \mathscr{K}(E)$ denote the closure of $E\backslash g^{-1}(\{0\})$ in E for each $g \in \mathcal{A}$, and have by Corollary 3.15 (a) (with $d = 1$ and $E_0 = E$) that

$$\{K_g\}_{g\in\mathcal{A}} \subset \mathscr{K}(\widehat{E}) \subset \mathscr{B}(\widehat{E}). \tag{5.3.27}$$

$\mathfrak{ag}_{\mathbf{Q}}(\mathcal{F})$ is a countable collection by Fact B.15, so $\mathcal{A}$ is also countable and

$$A \mathrel{\mathring{=}} \bigcup_{g\in\mathcal{A}} K_g \in \mathscr{K}_\sigma^{\mathbf{m}}(E) \cap \mathscr{B}(\widehat{E}) \tag{5.3.28}$$

by Corollary 3.15 (b) (with $d = 1$ and $E_0 = E$). We have $\{g_p\}_{p\in\mathbf{N}} \subset \mathcal{A}$ and

$$\widehat{g}_p = \overline{g}_p = \widehat{g}_p \mathbf{1}_{K_{g_p}} \leq 1,\ \forall p \in \mathbf{N} \tag{5.3.29}$$

by (5.3.14), (5.3.15) (with $g = g_p$, $a = 1$ and $b = 0$) and (5.3.10).

$$\begin{aligned}\nu(\widehat{E}) &\geq \nu(A) \geq \lim_{p\to\infty} \nu(K_{g_p}) \\ &\geq \lim_{p\to\infty} \left(\widehat{g}_p \mathbf{1}_{K_{g_p}}\right)^* (\nu) = \lim_{p\to\infty} \widehat{g}_p^*(\nu) = \nu(\widehat{E})\end{aligned} \tag{5.3.30}$$

by (5.3.27), (5.3.28), (5.3.29), (5.3.23) and (5.3.25). Hence, we have by (5.3.30) and Proposition 5.16 that ν is the replica of $\mu \doteq (\nu|_A)|^E$ and

$$\mu(E) = \mu(A) = \nu(A) = \nu(\widehat{E}) = \lambda_0. \tag{5.3.31}$$

Moreover, the $\mathbf{m}$-tightness of μ follows by (5.3.31) and the fact $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$.

Step 6: Redefine the integral representation of the replica functional as that of φ. First, we find that

$$\varphi(g) = \Phi(g) = \widehat{g}^*(\nu) = g^*(\mu), \ \forall g \in C_c(E;\mathbf{R}) \tag{5.3.32}$$

by (5.3.7), (5.3.15) (with $a = 1$ and $b = 0$), (5.3.24), the fact $\nu = \overline{\mu}$ and Proposition 5.15 (e) (with $d = 1$ and $E_0 = E$), thus proving (5.3.4). Secondly, we have

$$\mu(E) = \lambda_0 = \lim_{p\to\infty} \varphi(g_p) = \lim_{p\to\infty} g_p^*(\mu) \tag{5.3.33}$$

by (5.3.31), (5.3.13) and (5.3.32) (with $g = g_p$). Then, (5.3.5) follows by (5.3.33) and (5.3.10).

((b) $\to$ (a)) The functional defined by $\Lambda(g) \doteq g^*(\mu)$ for each $g \in C_b(E;\mathbf{R})$ satisfies $\Lambda|_{C_c(E;\mathbf{R})} = \varphi$ and is a positive linear functional. (5.3.3) follows by (5.3.5). □

Corollary 5.23. *Let E be a $C_c(E;\mathbf{R})$-baseable space, φ be a linear functional on $C_0(E;\mathbf{R})$ and $\mathcal{B}$ be as in (5.3.1). Then, the following statements are equivalent:*

(a) φ is continuous and there exists a positive linear functional Λ on $C_b(E;\mathbf{R})$ satisfying (5.3.2) and (5.3.3).

(b) There exists an $\mathbf{m}$-tight $\mu \in \mathcal{M}^+(E)$ satisfying (5.3.5) and

$$\varphi(g) = g^*(\mu), \ \forall g \in C_0(E;\mathbf{R}). \tag{5.3.34}$$

Proof. ((a) $\to$ (b)) There exists an $\mathbf{m}$-tight $\mu \in \mathcal{M}^+(E)$ satisfying (5.3.5) and (5.3.4) by Theorem 5.21. $C_c(E;\mathbf{R})$ is uniformly dense in $C_0(E;\mathbf{R})$ by Fact 3.24 (a) and Proposition B.44 (b). Hence, (5.3.34) follows by (5.3.4), the continuity of φ and the Dominated Convergence Theorem.

((b) $\to$ (a)) The functional defined by $\Lambda(g) \circeq g^*(\mu)$ for each $g \in C_b(E;\mathbf{R})$ satisfies $\Lambda|_{C_0(E;\mathbf{R})} = \varphi$, has linearity and is continuous by the Dominated Convergence Theorem. Moreover, (5.3.3) is immediate by (5.3.5). □

5.3.2 Almost sure representation of weak convergence

We now turn to generalizing the Skorokhod Representation Theorem in Jakubowski [1997a]. Commonly, the Skorokhod Representation Theorem is established on separable metric spaces. [Jakubowski, 1997a, Theorem 2] extended this result to sequences of tight probability measures on baseable spaces[15]. $\mathbf{m}$-tightness is equivalent to tightness in a baseable space E by Corollary 3.52. Hence, the conditions of the following theorem are strictly milder than those of Jakubowski [1997a].

Theorem 5.24. *Let E be a topological space, $C(E;\mathbf{R})$ separate points on E,*

$$\mu_n \Longrightarrow \mu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(E), \tag{5.3.35}$$

and $\{\mu_n\}_{n\in\mathbf{N}}$ be $\mathbf{m}$-tight. Then, there exist E-valued random variables $\{\xi_n\}_{n\in\mathbf{N}_0}$ defined on the same probability space such that μ_n is the distribution of ξ_n for all $n \in \mathbf{N}_0$ and $\{\xi_n\}_{n\in\mathbf{N}}$ converges to ξ_0 as $n \uparrow \infty$ almost surely.

Proof. $\{\mu_n\}_{n\in\mathbf{N}_0}$ is $\mathbf{m}$-tight by Lemma 5.7 (b) (with $\Gamma = \{\mu_n\}_{n\in\mathbf{N}}$). There exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ such that $\{\mu_n\}_{n\in\mathbf{N}_0}$ is tight in

$$E_0 \in \mathscr{K}_\sigma^{\mathbf{m}}(E) \cap \mathscr{B}^{\mathbf{s}}(E) \cap \mathscr{B}(E) \cap \mathscr{B}(\widehat{E}) \tag{5.3.36}$$

by Lemma B.74 (with $\Gamma_i = \{\mu_n\}_{n\in\mathbf{N}_0}$ and $\mathcal{D} = C(E;\mathbf{R})$) and Corollary 3.15 (b).

$$\inf_{n\in\mathbf{N}_0} \mu_n(E_0) = \inf_{n\in\mathbf{N}_0} \overline{\mu}_n(E_0) = 1 \tag{5.3.37}$$

[15] While Jakubowski [1997a] did not use the term "baseable", he did assume point-separability by countably many continuouos functions.

by the tightness of $\{\mu_n\}_{n\in\mathbf{N}_0}$ in E_0 and Proposition 5.15 (a). Furthermore,

$$\overline{\mu}_n \Longrightarrow \overline{\mu}_0 \text{ as } k\uparrow\infty \text{ in } \mathcal{P}(\widehat{E}) \tag{5.3.38}$$

by (5.3.35) and Proposition 5.17 (a, e) (with $d=1$, $\mu_n' = \mu_n$ and $\mu' = \mu_0$).

$\widehat{E}$ is a Polish space by Lemma 3.3 (c), so the classical Skorokhod Representation Theorem (see [Dudley, 2002, Theorem 11.7.2]) is applicable to $\{\overline{\mu}_n\}_{n\in\mathbf{N}_0}$, yielding random variables $\{\overline{\xi}_n\}_{n\in\mathbf{N}_0} \subset M(\Omega,\mathscr{F},\mathbb{P};\widehat{E})$ that satisfy $\mathbb{P}\circ\overline{\xi}_n^{-1} = \overline{\mu}_n$ for all $n\in\mathbf{N}_0$ and

$$\mathbb{P}\left(\overline{\xi}_n \longrightarrow \overline{\xi}_0 \text{ as } n\uparrow\infty\right) = 1. \tag{5.3.39}$$

Singletons are Borel sets in $\widehat{E}$ by Proposition A.2 (a, b). Hence, there exist

$$\xi_n \in M\left(\Omega,\mathscr{F};E_0,\mathscr{B}_{\widehat{E}}(E_0)\right),\ \forall n\in\mathbf{N}_0 \tag{5.3.40}$$

such that

$$\inf_{n\in\mathbf{N}_0}\mathbb{P}\left(\xi_n = \overline{\xi}_n\right) \geq \inf_{n\in\mathbf{N}_0}\mathbb{P}\left(\overline{\xi}_n\in E_0\right) = \inf_{n\in\mathbf{N}_0}\overline{\mu}_n(E_0) = 1 \tag{5.3.41}$$

by (5.3.37) and Fact B.3 (b) (with $(S,\mathscr{A}) = (\Omega,\mathscr{F})$, $(E,\mathscr{U}) = (\widehat{E},\mathscr{B}(\widehat{E}))$, $A = E_0$ and $f = \overline{\xi}_n$).

Now, we have by (5.3.39) and (5.3.41) that

$$\mathbb{P}\left(\xi_n \longrightarrow \xi_0 \text{ as } n\uparrow\infty\right) \geq \mathbb{P}\left(\overline{\xi}_n \longrightarrow \overline{\xi}_0 \text{ as } n\uparrow\infty\right) = 1. \tag{5.3.42}$$

(5.3.36) mentioned $E_0\in\mathscr{B}^{\mathbf{s}}(E)$. It then follows by Lemma 3.14 (a) (with $d=1$ and $A=E_0$) and (5.3.40) that

$$\xi_n \in M\left(\Omega,\mathscr{F};E_0,\mathscr{B}_E(E_0)\right) \subset M\left(\Omega,\mathscr{F};E\right),\ \forall n\in\mathbf{N}_0. \tag{5.3.43}$$

□

Remark 5.25. $([0,1]^{[0,1]}, \|\cdot\|_\infty)$ mentioned in Example 3.53 is non-baseable. Compact subsets of this normed space are automatically metrizable. This space is Tychonoff by Proposition A.26 (a) and so its points are separated by

$C([0,1]^{[0,1]}, \|\cdot\|_\infty; \mathbf{R})$ by Proposition A.25 (a, b). Theorem 5.24 applies in this case whereas [Jakubowski, 1997a, Theorem 2] does not.

Chapter 6

Replication of Stochastic Process

This chapter is devoted to the replication of E-valued stochastic process via a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E. §6.1 introduces and discusses the basic properties of replica process. §6.3 focuses on the special case of càdlàg replica. Compared to weak convergence about E-valued processes, it is generally easier to deal with weak convergece about their replica processes. Indeed, many properties like tightness and relative compactness are simpler to verify or even automatic on the compact Polish space $\widehat{E}$. Whereas, §6.2 associates the finite-dimensional convergence of general processes to that of their general replicas. §6.4 discusses tightness and weak convergence of càdlàg replicas as path-space-valued random variables. Finally, §6.5 considers when a family of processes can be contained in a large baseable set to perform the desired replication. If necessary, the readers are referred to §2.5 where we specify our terminologies and notations about stochastic processes.

6.1 Introduction to replica process

6.1.1 Definition

Given a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over topological space E, a replica of E-valued process X means a process that takes values in the compact Polish space $\widehat{E}$ and is analogous to X. Since X and its replicas may live in different spaces, they are generically associated by the mappings $\bigotimes \mathcal{F}$ and $\bigotimes \widehat{\mathcal{F}}$ rather than their own values.

Definition 6.1. Let E be a topological space and $(\Omega, \mathscr{F}, \mathbb{P}; X)$[1] be an E-valued process. With respect to a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E, an $\widehat{E}$-valued process $(\Omega, \mathscr{F}, \mathbb{P}; \widehat{X})$ is said to be a **replica of** X if

$$\mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t = \bigotimes \mathcal{F} \circ X_t\right) \geq \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right), \ \forall t \in \mathbf{R}^+. \tag{6.1.1}$$

Note 6.2. An E-valued process X may have multiple replicas, among which indistinguishability[2] is certainly an equivalence relation.

We make the following notations for simplicity.

Notation 6.3. Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and X be an E-valued process.

- $\mathfrak{rep}(X; E_0, \mathcal{F})$[3] denotes the family of all equivalence classes of X's replicas with respect to $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ under the equivalence relation of indistinguishability.

- $\mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$, $\mathfrak{rep}_{\mathrm{p}}(X; E_0, \mathcal{F})$ and $\mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ denote the measurable[4], progressive and càdlàg members of $\mathfrak{rep}(X; E_0, \mathcal{F})$, respectively.

- By $\widehat{X} = \mathfrak{rep}(X; E_0, \mathcal{F})$ we mean $\mathfrak{rep}(X; E_0, \mathcal{F})$ equals the singleton $\{\widehat{X}\}$. Similar notations apply to the above-mentioned subfamilies of $\mathfrak{rep}(X; E_0, \mathcal{F})$.

Remark 6.4. The notation "$\mathfrak{rep}(X; E_0, \mathcal{F})$" merely specifies the first two components $(E_0, \mathcal{F})$ of the base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ since Theorem 3.8 showed that this base is totally determined by $(E_0, \mathcal{F})$.

Note 6.5.

- $\mathbf{R}$, $\mathbf{R}^\infty$, $\widehat{E}$ and $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$ (as subspace of $\mathbf{R}^\infty$) are Polish spaces by Proposition A.11 (f), Lemma 3.3 (c) and (3.1.3).

[1] "$(\Omega, \mathscr{F}, \mathbb{P}; X)$" as defined in §2.4 means an E-valued random variable or process X defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We imposed in §2.6 that the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete.

[2] The terminology "indistinguishability" was explained in §2.5.

[3] "$\mathfrak{rep}$" is "rep" in fraktur font which stands for "replica".

[4] The notions of measurable process and progressive process were reviewed in §2.5.

- $D(\mathbf{R}^+;\mathbf{R})$, $D(\mathbf{R}^+;\mathbf{R}^\infty)$, $D(\mathbf{R}^+;\widehat{E})$ and $D(\mathbf{R}^+;\bigotimes\widehat{\mathcal{F}}(\widehat{E}))$ are well-defined Polish spaces by Proposition A.72 (d).
- $\mathscr{B}(\widehat{E}^d)=\mathscr{B}(\widehat{E})^{\otimes d}$ for all $d\in\mathbf{N}$ by Fact 3.12, so finite-dimensional distributions of any $\widehat{E}$-valued process (especially any replica process) are all Borel probability measures[5]. This is also true for any Polish-space-valued process.

The following proposition justifies the general existence of replica proecsses.

Proposition 6.6. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E and $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued process. Then, the following statements are true:*

(a) $\mathfrak{rep}(X;E_0,\mathcal{F})$ is non-empty.

(b) If X is a measurable process, then $\mathfrak{rep}_{\mathrm{m}}(X;E_0,\mathcal{F})$ is non-empty.

Proof. (a) We fix $x_0\in E_0$, let φ be the identity mapping on $\mathbf{R}^\infty$ and define[6]

$$\varphi_{x_0}\mathrel{\hat{=}}\mathfrak{var}\left(\varphi;\mathbf{R}^\infty,\bigotimes\widehat{\mathcal{F}}(\widehat{E}),\bigotimes\mathcal{F}(x_0)\right). \tag{6.1.2}$$

$\bigotimes\widehat{\mathcal{F}}(\widehat{E})\in\mathscr{B}(\mathbf{R}^\infty)$ by (3.1.5). $\mathbf{R}^\infty$ is a Polish space, so $\{\bigotimes\mathcal{F}(x_0)\}\in\mathscr{B}(\mathbf{R}^\infty)$ by Proposition A.2 (a, b). We then have that

$$\varphi_{x_0}(y)=y,\ \forall y\in\bigotimes\widehat{\mathcal{F}}(\widehat{E}) \tag{6.1.3}$$

and

$$\varphi_{x_0}\in M\left(\mathbf{R}^\infty;\bigotimes\widehat{\mathcal{F}}(\widehat{E})\right) \tag{6.1.4}$$

by Fact B.3 (b) (with $(S,\mathscr{A})=(E,\mathscr{U})=(\mathbf{R}^\infty,\mathscr{B}(\mathbf{R}^\infty))$, $A=\bigotimes\widehat{\mathcal{F}}(\widehat{E})$, $f=\varphi$ and $y_0=\bigotimes\mathcal{F}(x_0)$). It follows that

$$\left(\bigotimes\widehat{\mathcal{F}}\right)^{-1}\circ\varphi_{x_0}\circ\bigotimes\mathcal{F}\in M(E;\widehat{E}) \tag{6.1.5}$$

[5] As aforementioned in §2.5, this is not necessarily true for general processes.
[6] "$\mathfrak{var}(\cdot)$" was introduced in Notation 4.1.

by (3.1.3), (6.1.4) and Lemma 3.3 (e). Hence,

$$\widehat{X} \doteq \varpi \left[\left(\bigotimes \widehat{\mathcal{F}} \right)^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F} \right] \circ X \tag{6.1.6}$$

well defines an $\widehat{E}$-valued process $(\Omega, \mathscr{F}, \mathbb{P}; \widehat{X})$ by Fact B.32 (a) (with $S = \widehat{E}$ and $f = (\bigotimes \widehat{\mathcal{F}})^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F}$). It follows by (6.1.3) and (6.1.6) that

$$\begin{aligned}
&\mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E}) \right) \\
&= \mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ \left(\bigotimes \widehat{\mathcal{F}} \right)^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F} \circ X_t = \bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E}) \right) \\
&\leq \mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t = \bigotimes \mathcal{F} \circ X_t \right), \ \forall t \in \mathbf{R}^+,
\end{aligned} \tag{6.1.7}$$

thus proving $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$ by (6.1.1).

(b) Let $\widehat{X}$ be as above and define $\xi(t,\omega) \doteq X_t(\omega)$ and $\widehat{\xi}(t,\omega) \doteq \widehat{X}_t(\omega)$ for each $(t,\omega) \in \mathbf{R}^+ \times \Omega$. If X is a measurable process, then

$$\widehat{\xi} = \left(\bigotimes \widehat{\mathcal{F}} \right)^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F} \circ \xi \in M\left(\mathbf{R}^+ \times \Omega, \mathscr{B}(\mathbf{R}^+) \otimes \mathscr{F}; \widehat{E}, \mathscr{B}(\widehat{E}) \right) \tag{6.1.8}$$

by (6.1.5), thus proving $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$. $\square$

6.1.2 Association with the original process

In the next two propositions, the original and replica processes have further association than just (6.1.1) due to suitable properties of the base.

Proposition 6.7. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mathbf{T} \subset \mathbf{R}^+$, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process satisfying*

$$\inf_{t \in \mathbf{T}} \mathbb{P}(X_t \in E_0) = 1 \tag{6.1.9}$$

and $\widehat{X}, \widehat{X}^1, \widehat{X}^2 \in \mathfrak{rep}(X; E_0, \mathcal{F})$. Then, the following statements are true:

(a) $\widehat{X}$ satisfies

$$\inf_{t \in \mathbf{T}} \mathbb{P}\left(X_t = \widehat{X}_t \in E_0 \right) = 1. \tag{6.1.10}$$

Moreover, $\widehat{X}^1$ and $\widehat{X}^2$ satisfy

$$\inf_{t\in\mathbf{T}} \mathbb{P}\left(\widehat{X}_t^1 = \widehat{X}_t^2 \in E_0\right) = 1. \tag{6.1.11}$$

(b) $\mathbb{P}\circ\widehat{X}_{\mathbf{T}_0}^{-1}$ is the replica measure of $\mathbb{P}\circ X_{\mathbf{T}_0}^{-1}$ for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$[7].

Proof. (a) (6.1.10) follows by (6.1.1) and Lemma B.75 (with $Y = \widehat{X}$). (6.1.11) is immediate by (6.1.10) (with $\widehat{X} = \widehat{X}^1$ or $\widehat{X}^2$).

(b) Note 6.5 argued that $\mathbb{P}\circ\widehat{X}_{\mathbf{T}_0}^{-1} \in \mathcal{P}(\widehat{E}^{\mathbf{T}_0})$. Then, we have by (a) that

$$\mathbb{P}\left(\widehat{X}_{\mathbf{T}_0} \in A\right) = \mathbb{P}\left(X_{\mathbf{T}_0} \in A \cap E_0^{\mathbf{T}_0}\right),\ \forall A \in \mathscr{B}(\widehat{E}^{\mathbf{T}_0}) \tag{6.1.12}$$

and

$$\mathbb{P}\left(X_{\mathbf{T}_0} = \widehat{X}_{\mathbf{T}_0} \in E_0^{\mathbf{T}_0}\right) = 1. \tag{6.1.13}$$

□

Proposition 6.8. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mathbf{T} \subset \mathbf{R}^+$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process satisfying*

$$\inf_{t\in\mathbf{T}} \mathbb{P}\left(\bigotimes \mathcal{F}\circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1. \tag{6.1.14}$$

Then, the following statements are true:

(a) Any $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$ satisfies

$$\mathbb{P}\left(f\circ X_{\mathbf{T}_0} = \widehat{f}\circ\widehat{X}_{\mathbf{T}_0}\right) = 1,\ \forall f \in \mathfrak{ca}\left[\Pi^{\mathbf{T}_0}(\mathcal{F})\right], \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T}) \tag{6.1.15}$$

and

$$\mathbb{E}\left[f\circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f}\circ\widehat{X}_{\mathbf{T}_0}\right],\ \forall f \in \mathfrak{ca}\left[\Pi^{\mathbf{T}_0}(\mathcal{F})\right], \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T}). \tag{6.1.16}$$

(b) If $\mathbf{T} \subset \mathbf{R}^+$ is dense, then $\mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ is at most a singleton.

[7] $\widehat{X}_{\mathbf{T}_0}$, the section of $\widehat{X}$ for $\mathbf{T}_0$ was defined in §2.5.

Proof. (a) Any $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$ satisfies

$$\begin{aligned} &\inf_{t\in\mathbf{T}} \mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t = \bigotimes \mathcal{F} \circ X_t\right) \\ &\geq \inf_{t\in\mathbf{T}} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1 \end{aligned} \tag{6.1.17}$$

by (6.1.1) and (6.1.14). Now, (6.1.15) follows by (6.1.17) and properties of uniform convergence. (6.1.16) is immediate by (6.1.15).

(b) $\mathbf{T}$ must have a countable subset $\mathbf{T}_0$ being dense in $\mathbf{R}^+$. $\varpi(\bigotimes \widehat{\mathcal{F}})$[8] is injective on $D(\mathbf{R}^+; \widehat{E})$ by Lemma 3.3 (a) and Fact B.20 (with $E = A = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$). Given any $\widehat{X}^1, \widehat{X}^2 \in \mathfrak{rep}_c(X; E_0, \mathcal{F})$, $\{\varpi(\bigotimes \widehat{\mathcal{F}}) \circ \widehat{X}\}_{i=1,2}$ are càdlàg processes by (3.1.3) and Fact B.34 (a) (with $E = \widehat{E}$, $S = \mathbf{R}^\infty$ and $f = \bigotimes \widehat{\mathcal{F}}$), and

$$\begin{aligned} \mathbb{P}\left(\widehat{X}^1 = \widehat{X}^2\right) &= \mathbb{P}\left(\varpi(\bigotimes \widehat{\mathcal{F}}) \circ \widehat{X}^1 = \varpi(\bigotimes \widehat{\mathcal{F}}) \circ \widehat{X}^2\right) \\ &\geq \mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ \widehat{X}^1_t = \bigotimes \widehat{\mathcal{F}} \circ \widehat{X}^2_t, \forall t \in \mathbf{T}_0\right) = 1 \end{aligned} \tag{6.1.18}$$

by (a) (with $\widehat{X} = \widehat{X}^i$), Proposition B.33 (g) and the injectiveness of $\varpi(\bigotimes \widehat{\mathcal{F}})$. $\square$

The following consequence of (3.1.1) is apparent but indispensable.

Fact 6.9. *(6.1.9) is stronger than (6.1.14).*

6.1.3 Application to replicating measure-valued processes

Lemma 6.10. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\varphi \circeq \bigotimes \mathfrak{mc}(\mathcal{F})^*$[9], $\widehat{\varphi} \circeq \bigotimes \mathfrak{mc}(\widehat{\mathcal{F}})^*$, $S_0 \in \mathscr{B}(\mathbf{R}^\infty)$ be contained in $\widehat{\varphi}[\mathcal{M}^+(\widehat{E})]$, $y_0 \in S_0$, ϕ be the identity mapping on $\mathbf{R}^\infty$ and $X \in M(\Omega, \mathscr{F}; \mathcal{M}^+(E))$[10] satisfy*

$$\mathbb{P}\left(\varphi \circ X \in S_0\right) = 1. \tag{6.1.19}$$

Then, the following statements are true:

[8]The notations "$\varpi(f)$" and "$\varpi(\bigotimes \widehat{\mathcal{F}})$" were defined in §2.2.1.
[9]The notation "$\mathfrak{mc}(\mathcal{F})^*$" was specified in §2.3.
[10]$X \in M(\Omega, \mathscr{F}; \mathcal{M}^+(E))$ means X is a non-negative finite Borel random measure on E. That X is $\mathbb{P}$-almost surely supported on E means .

(a) $\Psi \circeq \widehat{\varphi}^{-1} \circ \mathfrak{var}(\phi; \mathbf{R}^\infty, S_0, y_0)$ *satisfies*

$$\Psi \in M\left[\mathbf{R}^\infty; \widehat{\varphi}^{-1}(S_0), \mathscr{O}_{\mathcal{M}^+(\widehat{E})}\left(\widehat{\varphi}^{-1}(S_0)\right)\right]. \tag{6.1.20}$$

(b) $Y \circeq \Psi \circ \varphi \circ X \in M(\Omega, \mathscr{F}; \mathcal{M}^+(\widehat{E}))$ *satisfies*[11]

$$\mathbb{P}\left(f^* \circ X = \widehat{f}^* \circ Y\right) = 1, \ \forall f \in \mathfrak{ca}(\mathcal{F}). \tag{6.1.21}$$

(c) If

$$\{\omega \in \Omega : X(\omega)(E \backslash E_0) > 0\} \in \mathscr{N}(\mathbb{P}), \tag{6.1.22}$$

then $Y(\omega)$ *equals the replica (measure) of* $X(\omega)$ *for* $\mathbb{P}$ *almost all* $\omega \in \Omega$.

(d) If $A \in \mathscr{B}^{\mathbf{s}}(E)$ *satisfies* $A \subset E_0$ *and*

$$\{\omega \in \Omega : X(\omega)(E \backslash A) > 0\} \in \mathscr{N}(\mathbb{P}), \tag{6.1.23}$$

then[12] $(hf)^* \circ X$ *and* $(\overline{h\mathbf{1}_A}\widehat{f})^* \circ Y$ *belong to* $M(\Omega, \mathscr{F}; \mathbf{R}^k)$ *and satisfy*

$$\mathbb{P}\left((hf)^* \circ X = (\overline{h\mathbf{1}_A}\widehat{f})^* \circ Y\right) = 1 \tag{6.1.24}$$

for all $f \in \mathfrak{ca}(\mathcal{F})$, $h \in M_b(E; \mathbf{R}^k)$ *and* $k \in \mathbf{N}$.

Remark 6.11. Every $f \in C_b(E; \mathbf{R})$ satisfies $f^* \in C_b(\mathcal{M}^+(E); \mathbf{R})$ by the definition of weak topology and so $f^* \circ X \in M(\Omega, \mathscr{F}; \mathbf{R})$. For $f \in M_b(E; \mathbf{R})$, however, f^* does not necessarily belong to $M_b(\mathcal{M}^+(E); \mathbf{R})$ in general, nor is $f^* \circ X$ always a random variable.

Proof of Lemma 6.10. (a) $\mathfrak{mc}(\mathcal{F})^*$ (resp. $\mathfrak{mc}(\widehat{\mathcal{F}})^*$) is a countable subset of $C_b(\mathcal{M}^+(E); \mathbf{R})$ (resp. $C_b(\mathcal{M}^+(\widehat{E}); \mathbf{R})$) by Fact B.15 (with $E = E$ or $\widehat{E}$, $\mathcal{D} = \mathcal{F}$ or $\widehat{\mathcal{F}}$ and $d = k = 1$), Definition 3.1 and Lemma 3.3 (a). Then, we have by Fact 2.4 (b) that

$$\varphi \in C\left(\mathcal{M}^+(E); \mathbf{R}^\infty\right). \tag{6.1.25}$$

[11] The notation "f^*" was specified in §2.3.

[12] The k-dimensional integration function $(hf)^*$ was defined in §2.3. $\overline{h\mathbf{1}_A}$ denotes the function $\mathfrak{var}(h1_A; E, A, 0)$.

At the same time, we have that

$$\widehat{\varphi} \in \mathbf{imb}\left(\mathcal{M}^+(\widehat{E}); \mathbf{R}^\infty\right) \tag{6.1.26}$$

by Corollary 3.11 (b) (with $d = 1$ and $A = \widehat{E}$) and Lemma B.7 (b) (with $E = \mathcal{M}^+(\widehat{E})$, $S = \mathbf{R}^\infty$ and $\mathcal{D} = \mathfrak{mc}(\widehat{\mathcal{F}})^*$). Furthermore,

$$\mathfrak{var}\left(\phi; \mathbf{R}^\infty, S_0, y_0\right) \in M\left(\mathbf{R}^\infty; S_0\right) \tag{6.1.27}$$

by the fact $S_0 \in \mathscr{B}(\mathbf{R}^\infty)$ and Fact B.3 (b) (with $(S, \mathscr{A}) = (E, \mathscr{U}) = (\mathbf{R}^\infty, \mathscr{B}(\mathbf{R}^\infty))$, $A = S_0$ and $f = \phi$). Now, (a) follows by (6.1.26) and (6.1.27).

(b) $Y \in M(\Omega, \mathscr{F}; \mathcal{M}^+(\widehat{E}))$ by (6.1.25) and (a). Ψ equals $\widehat{\varphi}^{-1}$ restricted to S_0, hence

$$\mathbb{P}\left(\varphi \circ X_t = \widehat{\varphi} \circ Y \in S_0\right) \geq \mathbb{P}\left(\varphi \circ X_t \in S_0\right) = 1 \tag{6.1.28}$$

by (6.1.14), which implies

$$\mathbb{P}\left(g^* \circ X = \widehat{g}^* \circ Y, \forall g \in \mathfrak{ag}(\mathcal{F})\right) = 1 \tag{6.1.29}$$

by linearity of integral. Fixing $f \in \mathfrak{ca}(\mathcal{F})$ and $g \in \mathfrak{ag}(\mathcal{F})$, we find that

$$\begin{aligned} \left| f^* \circ X - \widehat{f}^* \circ Y \right| (\omega) &\leq \|f - g\|_\infty + \|\widehat{f} - \widehat{g}\|_\infty + \left| g^* \circ X - \widehat{g}^* \circ Y \right| (\omega) \\ &\leq 2\|f - g\|_\infty + \left| g^* \circ X - \widehat{g}^* \circ Y \right| (\omega), \ \forall \omega \in \Omega \end{aligned} \tag{6.1.30}$$

by Triangle Inequality, (6.1.29), Proposition 4.6 (d) (with $a = 1$ and $b = -1$) and Lemma B.72 (a) (with $f = f - g$). Now, (b) follows by (6.1.29), (6.1.30) and (2.2.15) (with $\mathcal{D} = \widehat{\mathcal{F}}$).

(c) We let ν^ω denote the replica of $X(\omega)$ for each fix $\omega \in \Omega$ and find that

$$\begin{aligned} &\mathbb{P}\left(\left\{\omega \in \Omega : \varphi \circ X(\omega) = \widehat{\varphi}^*(\nu^\omega) \in \widehat{\varphi}^*\left[\mathcal{M}^+(\widehat{E})\right]\right\}\right) \\ &= \mathbb{P}\left(\left\{\omega \in \Omega : X(\omega)(E \backslash E_0) = 0\right\}\right) = 1 \end{aligned} \tag{6.1.31}$$

by the countability of $\mathfrak{mc}(\mathcal{F})$, Proposition 5.15 (a, e) (with $d = 1$, $\mu = X(\omega)$ and $\overline{\mu} = \nu^\omega$) and (6.1.22). Since S_0 is contained in the closure of $\widehat{\varphi}^*[\mathcal{M}^+(\widehat{E})]$, it follows by (6.1.28) and (6.1.31) that

$$\mathbb{P}\left(\{\omega \in \Omega : \widehat{\varphi} \circ Y(\omega) = \widehat{\varphi}(\nu^\omega)\}\right) = 1. \tag{6.1.32}$$

Hence, (c) follows by (6.1.26) and (6.1.32).

(d) We fix $f \in \mathfrak{ca}(\mathcal{F})$ and $h \in M_b(E; \mathbf{R}^k)$, get $\{\overline{h\mathbf{1}_A}, \overline{hf\mathbf{1}_A}\} \subset M_b(\widehat{E}; \mathbf{R}^k)$ from Proposition 4.6 (b) (with $d = 1$ and $f = h\mathbf{1}_A$ or $hf\mathbf{1}_A$), and find

$$\begin{aligned}&\left\{\omega \in \Omega : (hf)^* \circ X(\omega) = (hf\mathbf{1}_A)^* \circ X(\omega) = (\overline{hf\mathbf{1}_A})^*(\nu^\omega) = (\overline{h\mathbf{1}_A}\widehat{f})^*(\nu^\omega)\right\}\\ &\supset \{\omega \in \Omega : X(\omega)(E \backslash A) = 0\}\end{aligned} \tag{6.1.33}$$

by Proposition 5.15 (d) (with $d = 1$), the fact $\overline{hf\mathbf{1}_A}|_{E_0} = \overline{h\mathbf{1}_A}\widehat{f}|_{E_0}$ and the definition of ν^ω. Thus, (6.1.24) follows by (6.1.33) and (c). Moreover, we have $(\overline{h\mathbf{1}_A}\widehat{f})^* \circ Y \in M(\Omega, \mathscr{F}; \mathbf{R}^k)$ by the fact $\overline{h\mathbf{1}_A}\widehat{f} \in M_b(\widehat{E}; \mathbf{R}^k)$, Lemma 3.3 (c) and Proposition B.71 (b) (with $E = \widehat{E}$, $f = \overline{h\mathbf{1}_A}\widehat{f}$ and $\xi = Y$). Hence, $(hf)^* \circ X \in M(\Omega, \mathscr{F}; \mathbf{R}^k)$ by Lemma B.31 (a) (with $E = S = \mathbf{R}^k$, $\mathscr{U} = \mathscr{B}(\mathbf{R}^k)$, $X = (hf)^* \circ X$ and $Z = (\overline{h\mathbf{1}_A}\widehat{f})^* \circ Y$). □

Corollary 6.12. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an $\mathcal{M}^+(E)$-valued process satisfying*

$$\{\omega \in \Omega : X_t(\omega)(E \backslash E_0) > 0\} \in \mathscr{N}(\mathbb{P}),\ \forall t \in \mathbf{R}^+. \tag{6.1.34}$$

Then, there exists an $\mathcal{M}^+(\widehat{E})$-valued $\mathscr{F}_t^X$-adapted process $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ satisfying the following properties:

(a) $\varpi(\widehat{f^*}) \circ Y$ is a modification of $\varpi(f^*) \circ X$ for all $f \in \mathfrak{ca}(\mathcal{F})$.

(b) If X satisfies

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(X_t \in \mathcal{P}(E)\right) = 1, \tag{6.1.35}$$

then Y can be a $\mathcal{P}(\widehat{E})$-valued process.

(c) For each $t \in \mathbf{R}^+$, there exists an $\Omega_t \in \mathscr{N}(\mathbb{P})$ such that $Y_t(\omega)$ equals the replica (measure) of $X_t(\omega)$ for all $\omega \in \Omega \backslash \Omega_t$.

(d) If $A \in \mathscr{B}^{\mathrm{s}}(E_0)$ satisfies $A \subset E_0$ and

$$\{\omega \in \Omega : X_t(\omega)(E \backslash A) > 0\} \in \mathscr{N}(\mathbb{P}),\ \forall t \in \mathbf{R}^+, \tag{6.1.36}$$

then $\varpi((\overline{h\mathbf{1}_A}\widehat{f})^*) \circ Y$ is a modification of $\varpi((hf)^*) \circ X$ for all $f \in \mathfrak{ca}(\mathcal{F})$, $h \in M_b(E; \mathbf{R}^k)$ and $k \in \mathbf{N}$.

Remark 6.13. An implication of the statements "$\varpi(\widehat{f}^*) \circ Y$ is a modification of $\varpi(f^*) \circ X$" in (a) and "$\varpi((\overline{h\mathbf{1}_A}\widehat{f})^*) \circ Y$ is a modification of $\varpi((hf)^*) \circ X$" in (d) is that $\varpi(f^*) \circ X$, $\varpi(\widehat{f}^*) \circ Y$, $\varpi((\overline{h\mathbf{1}_A}\widehat{f})^*) \circ Y$ and $\varpi((hf)^*) \circ X$ are indeed processes. For (a), we know $f \in \mathfrak{ca}(\mathcal{F}) \subset C_b(E; \mathbf{R})$ and $\widehat{f} \in C(\widehat{E}; \mathbf{R})$, so $\varpi(f^*) \circ X$ and $\varpi(\widehat{f}^*) \circ Y$ are processes by Proposition B.71 (a). For (d), we shall justify $\varpi((\overline{h\mathbf{1}_A}\widehat{f})^*) \circ Y$ and $\varpi((hf)^*) \circ X$ are processes by Lemma 6.10 (d) (with $X = X_t$ and $Y = Y_t$).

Proof. We set φ, $\widehat{\varphi}$ y_0 and Ψ as in Lemma 6.10. In general, we let $S_0 = \widehat{\varphi}[\mathcal{M}^+(\widehat{E})]$. If (6.1.35) holds, we let $S_0 = \widehat{\varphi}[\mathcal{P}(\widehat{E})]$. Recall that $\widehat{\varphi}$ satisfies (6.1.26). $\mathcal{M}^+(\widehat{E})$, $\mathcal{P}(\widehat{E})$ and $\mathbf{R}^\infty$ are Polish spaces by Corollary 3.11 (c) (with $d = 1$) and Note 6.5. Consequently, $S_0 \in \mathscr{B}(\mathbf{R}^\infty)$ in both cases by Proposition A.57 (with $E = A = \mathcal{M}^+(\widehat{E})$ or $\mathcal{P}(\widehat{E})$, $f = \widehat{\varphi}$ and $S = \mathbf{R}^\infty$) and Proposition A.56 (b) (with $E = \mathbf{R}^\infty$). Hence, $Y \circeq \varpi(\Psi \circ \varphi) \circ X$ is the desired process by (6.1.25), Lemma 6.10 (with $X = \mathfrak{p}_t \circ X$ and $Y = \mathfrak{p}_t \circ Y$) and Fact B.32 (a) (with $E = \mathcal{M}^+(E)$, $f = \Psi \circ \varphi$ and $\mathscr{G}_t = \mathscr{F}_t^X$). $\square$

6.2 Finite-dimensional convergence about replica process

6.2.1 Definition

We start this section with the precise definition of and two important notions for establishing finite-dimensional convergence. Given a general space E, the finite-dimensional convergence of E-valued processes is about the Borel extensions of their possibly non-Borel finite-dimensional distributions.

Definition 6.14. Let E be a topological space and $(\Omega, \mathscr{F}, \mathbb{P}; X)$[13], $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ and $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ and be E-valued processes.

- $\{X^n\}_{n\in\mathbf{N}}$ **converges finite-dimensionally to** X **along** $\mathbf{T}$ if: (1) $\mathbf{T} \subset \mathbf{R}^+$ is non-empty, (2) For each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, there exist $N_{\mathbf{T}_0} \in \mathbf{N}$, $\{\mu_n \in \mathfrak{be}(\mathbb{P}^n \circ (X^n_{\mathbf{T}_0})^{-1})\}_{n>N_{\mathbf{T}_0}}$ and $\mu \in \mathfrak{be}(\mathbb{P} \circ X^{-1}_{\mathbf{T}_0})$, and (3) $\mu_n \Rightarrow \mu$ as $n \uparrow \infty$ in $\mathcal{P}(E^{\mathbf{T}_0})$.

- X is a **finite-dimensional limit point of** $\{X^i\}_{i\in\mathbf{I}}$ **along** $\mathbf{T}$ if there exists a subsequence of $\{X^i\}_{i\in\mathbf{I}}$ converging finite-dimensionally to X along $\mathbf{T}$.

- Two finite-dimensional limit points of $\{X^i\}_{i\in\mathbf{I}}$ along $\mathbf{T}$ are equivalent if their finite-dimensional distributions for any $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ are identical.

- X is a **finite-dimensional limit of** $\{X^n\}_{n\in\mathbf{N}}$ **along** $\mathbf{T}$ if $\{X^n\}_{n\in\mathbf{N}}$ converges finite-dimensionally to X along $\mathbf{T}$ and any finite-dimensional limit point of $\{X^n\}_{n\in\mathbf{N}}$ is equivalent to X.

- $\{X^i\}_{i\in\mathbf{I}}$ is **finite-dimensionally convergent along T under** $\mathcal{D}$ ($(\mathbf{T}, \mathcal{D})$-FDC[14] for short) if: (1) $\mathbf{I}$ is infinite, (2) $\mathbf{T} \subset \mathbf{R}^+$ and $\mathcal{D} \subset M_b(E; \mathbf{R})$ are non-empty, and (3) $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0}]\}_{i\in\mathbf{I}}$[15] has a unique[16] limit point in $\mathbf{R}$ for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

- $\{X^i\}_{i\in\mathbf{I}}$ is **asymptotically stationary along T under** $\mathcal{D}$ ($(\mathbf{T}, \mathcal{D})$-AS for short) if: (1) $\mathbf{I}$ is infinite, (2) $\mathbf{T} \subset \mathbf{R}^+$ and $\mathcal{D} \subset M_b(E; \mathbf{R})$ are non-empty, and (3) The unique limit point of $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0} - f \circ X^i_{\mathbf{T}_0+c}]\}_{i\in\mathbf{I}}$[17] in $\mathbf{R}$ is 0 for all $c \in (0, \infty)$, $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

Note 6.15. Given E-valued process X, the expectation $\mathbb{E}[f \circ X_{\mathbf{T}_0}]$ is well-defined for any $f \in \mathfrak{mc}[\Pi^d(M_b(E; \mathbf{R}))]$ by Fact 2.24 (d) and Note 5.6 (with $\mu = \mathbb{P} \circ X^{-1}_{\mathbf{T}_0}$).

[13] $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i)\}_{i\in\mathbf{I}}$ and $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)\}_{n\in\mathbf{N}_0}$ were assumed in §2.6 to be complete probability spaces. Completeness of measure space was specified in 2.1.2.

[14] Hereafter, "$(\mathbf{T}, \mathcal{D})$-FDC" and "$(\mathbf{T}, \mathcal{D})$-AS" also stand for "$(\mathbf{T}, \mathcal{D})$-finite-dimensional convergence" and "$(\mathbf{T}, \mathcal{D})$-asymptotical stationarity".

[15] $\mathbb{E}^i$ denotes the expectation operator of $(\Omega^i, \mathscr{F}^i, \mathbb{P}^i)$.

[16] In the definitions of $(\mathbf{T}, \mathcal{D})$-FDC and $(\mathbf{T}, \mathcal{D})$-AS, $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0}]\}_{i\in\mathbf{I}}$ and $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0} - f \circ X^i_{\mathbf{T}_0+c}]\}_{i\in\mathbf{I}}$ both lie in $[-2\|f\|_\infty, 2\|f\|_\infty]$. Each of them has at least one limit point in $\mathbf{R}$ by the Bolzano-Weierstrass Theorem, so it is enough to assume "at most one limit point".

[17] The notation "$\mathbf{T}_0 + c$" was defined in (2.5.10).

Note 6.16. Let X and Y be E-valued processes and $\mathbf{T} \subset \mathbf{R}^+$. We define $X \sim Y$ if $X_{\mathbf{T}_0}$ and $Y_{\mathbf{T}_0}$ have the same distribution for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$. This "$\sim$", which Definition 6.14 uses to define equivalence of finite-dimensional limit points, is indeed an equivalence relation among E-valued stochastic processes.

We make the following notations for simplicity.

Notation 6.17. Let X, $\{X^i\}_{i\in\mathbf{I}}$ and $\{X^n\}_{n\in\mathbf{N}}$ be E-valued processes.

- $\{X^n\}_{n\in\mathbf{N}}$ converging finite-dimensionally to X along $\mathbf{T}$ is denoted by

$$X^n \xrightarrow{\mathrm{D}(\mathbf{T})} X \text{ as } n \uparrow \infty. \tag{6.2.1}$$

- By $X = \mathfrak{fl}_{\mathbf{T}}(\{X^n\}_{n\in\mathbf{N}})$[18] we mean X is the finite-dimensional limit of $\{X^n\}_{n\in\mathbf{N}}$ along $\mathbf{T}$.
- By $\mathfrak{flp}_{\mathbf{T}}(\{X^i\}_{i\in\mathbf{I}})$ we denote the family of all equivalence classes (see Note 6.16) of finite-dimensional limit points of $\{X^i\}_{i\in\mathbf{I}}$ along $\mathbf{T}$.
- By $X = \mathfrak{flp}_{\mathbf{T}}(\{X^i\}_{i\in\mathbf{I}})$ we mean X is the unique member of $\mathfrak{flp}_{\mathbf{T}}(\{X^i\}_{i\in\mathbf{I}})$.

Remark 6.18. In general, $X = \mathfrak{fl}_{\mathbf{T}}(\{X^n\}_{n\in\mathbf{N}})$ is stronger than (6.2.1) because: (1) Each of the finite-dimensional distributions of $\{X^n\}_{n\in\mathbf{N}}$ may have multiple Borel extensions. (2) $\mathcal{P}(E^{\mathbf{T}_0})$ is not necessarily a Hausdorff space and a weakly convergent sequence may have multiple limits.

The following fact is straightforward.

Fact 6.19. *Let E be a topological space, $\mathbf{T} \subset \mathbf{R}^+$ and $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be E-valued processes. If (6.2.1) holds, then*[19]

$$\lim_{n\to\infty} \mathbb{E}^n \left[f \circ X^n_{\mathbf{T}_0}\right] = \mathbb{E}\left[f \circ X_{\mathbf{T}_0}\right] \tag{6.2.2}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E;\mathbf{R}))]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$. As a consequence, $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, C_b(E;\mathbf{R}))$-FDC.

[18] "$\mathfrak{fl}$" and "$\mathfrak{flp}$" are "fl" and "flp" in fraktur font which stand for "finite-dimensional limit" and "finite-dimensional limit point" respectively.

[19] $\mathbb{E}^n$ denotes the expectation operator of $(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)$.

Proof. Fixing $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E;\mathbf{R}))]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, it follows by Fact B.54 (with $d = \aleph(\mathbf{T}_0)$ and $X = X^n_{\mathbf{T}_0}$ or $X_{\mathbf{T}_0}$), (5.1.3) (with $d = \aleph(\mathbf{T}_0)$ and $\mathcal{D} = C_b(E;\mathbf{R})$) and (6.2.1) that

$$\lim_{n\to\infty} \mathbb{E}^n\left[f \circ X^n_{\mathbf{T}_0}\right] = \lim_{n\to\infty} f^*(\mu_n) = f^*(\mu) = \mathbb{E}\left[f \circ X_{\mathbf{T}_0}\right] \tag{6.2.3}$$

for some $\{\mu_n \in \mathfrak{be}(\mathbb{P} \circ (X^n_{\mathbf{T}_0})^{-1}\}_{n > N_{\mathbf{T}_0}}$ with $N_{\mathbf{T}_0} \in \mathbf{N}$ and $\mu \in \mathfrak{be}(\mathbb{P} \circ X^{-1}_{\mathbf{T}_0})$. □

6.2.2 Transformation of finite-dimensional convergence

The following theorem is our main tool for transforming a finite-dimensional limit point of replica processes back into that of original processes.

Theorem 6.20. *Let E be a topological space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $x_0 \in E_0$, $\mathbf{T} \subset \mathbf{R}^+$[20], and*

$$X_t \mathrel{\mathring{=}} \begin{cases} \mathfrak{var}\left(Y_t; \Omega, Y_t^{-1}(E_0), x_0\right), & \text{if } t \in \mathbf{T}, \\ \mathfrak{var}\left(Y_t; \Omega, Y_t^{-1}(\{x_0\}), x_0\right), & \text{if } t \in \mathbf{R}^+ \backslash \mathbf{T}, \end{cases} \tag{6.2.4}$$

where $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ is an $\widehat{E}$-valued process. Suppose that:

(i) $\{X^n_t\}_{n\in\mathbf{N}}$ *is sequentially tight*[21] *in E_0 for all $t \in \mathbf{T}$.*

(ii) $\{X^n\}_{n\in\mathbf{N}}$ *and Y satisfy*

$$\lim_{n\to\infty} \mathbb{E}^n\left[f \circ X^n_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f} \circ Y_{\mathbf{T}_0}\right] \tag{6.2.5}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

Then, the following statements are true:

(a) $X \mathrel{\mathring{=}} \{X_t\}_{t\geq 0}$ is an E-valued process with paths in $E_0^{\mathbf{R}^+}$[22] *and for each*

[20] Remark 4.2 mentioned that $\mathfrak{var}(Y_t; \Omega, Y_t^{-1}(\{x_0\}), x_0)$ is the constant mapping that sends every $\omega \in \Omega$ to x_0. We do not use x_0 to denote this mapping for clarity.

[21] Sequential tightness and sequential **m**-tightness of measures and random variabels was specified in Definition 5.1 and Note 5.2 respectively.

[22] "with paths in $E_0^{\mathbf{R}^+}$" means all paths of the process lying in $E_0^{\mathbf{R}^+}$. Of course, an E-valued process with paths in $E_0^{\mathbf{R}^+}$ is equivalent to an $(E_0, \mathscr{O}_E(E_0))$-valued process.

$\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, *there exist*[23]

$$\mu'_{\mathbf{T}_0,n} = \mathfrak{be}\left(\mathbb{P} \circ \left(X^n_{\mathbf{T}_0}\right)^{-1}\right) \in \mathcal{P}(E^{\mathbf{T}_0}),\ \forall n > N_{\mathbf{T}_0} \tag{6.2.6}$$

for some $N_{\mathbf{T}_0} \in \mathbf{N}$ *and*

$$\mu_{\mathbf{T}_0} = \mathfrak{be}\left(\mathbb{P} \circ X^{-1}_{\mathbf{T}_0}\right) \in \mathcal{P}(E^{\mathbf{T}_0}) \tag{6.2.7}$$

such that

$$\text{w-}\lim_{n\to\infty} \mu'_{\mathbf{T}_0,n}\big|_{E_0^{\mathbf{T}_0}} = \mu_{\mathbf{T}_0}\big|_{E_0^{\mathbf{T}_0}} \ \ \text{in } \mathcal{P}\left(E_0^{\mathbf{T}_0}, \mathscr{O}_E(E_0)^{\mathbf{T}_0}\right). \tag{6.2.8}$$

Moreover, $X_{\mathbf{T}_0}$ *is* $\mathbf{m}$*-tight in* $E_0^{\mathbf{T}_0}$ *for all* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$,

$$\inf_{t\in\mathbf{T}} \mathbb{P}\left(X_t = Y_t \in E_0\right) = 1 \tag{6.2.9}$$

and (6.2.1) holds.

(b) If $\mathbf{T} = \mathbf{R}^+$ *and* $\{X^n\}_{n\in\mathbf{N}}$ *is* $(\mathbf{R}^+, \mathcal{F}\backslash\{1\})$*-AS, then* X *and* Y *are both stationary processes*[24].

(c) If $C_b(E;\mathbf{R})$ *separates points on* E[25], *then* $X = \mathfrak{fl}_{\mathbf{T}}(\{X_n\}_{n\in\mathbf{N}})$.

Remark 6.21. The condition (i) above will ensure the following two facts for $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ with finite exception of $n \in \mathbf{N}$: (1) $\mathbb{P}^n \circ (X^n_{\mathbf{T}_0})^{-1}$ admits a unique Borel extension, and (2) $\mathbb{P}^n \circ (\widehat{X}^n_{\mathbf{T}_0})^{-1}$ is the replica measure of $\mathbb{P}^n \circ (X^n_{\mathbf{T}_0})^{-1}$. Hence, transforming finite-dimensional convergence from the replicas $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ to the original processes $\{X^n\}_{n\in\mathbf{N}}$ comes down to transforming weak convergence from replica measures to Borel extensions of original measures.

Proof of Theorem 6.20. (a) We divide the proof of (a) into four steps.

Step 1: Constructing $\mu_{\mathbf{T}_0}$ *for each* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$. We know from (i) and Fact B.76 (with $\mathbf{I} = \mathbf{N}$) that $\{\mu_{\mathbf{T}_0,n} \stackrel{\circ}{=} \mathbb{P}^n \circ (X^n_{\mathbf{T}_0})^{-1}\}_{n\in\mathbf{N}}$ is sequentially tight

[23] The notation "$\mu'_{\mathbf{T}_0,n} = \mathfrak{be}(\mathbb{P} \circ (X^n_{\mathbf{T}_0})^{-1})$" as defined in §2.1.3 means $\mu'_{\mathbf{T}_0,n}$ is the unique Borel extension of $\mathbb{P} \circ (X^n_{\mathbf{T}_0})^{-1}$.

[24] The notion of stationary process was specified in §2.5

[25] Note 5.8 argued that $C_b(E;\mathbf{R})$ separating points on E is equivalent to $C(E;\mathbf{R})$ separating points on E.

in $E_0^{\mathbf{T}_0}$ and there exists an $N_{\mathbf{T}_0} \in \mathbf{N}$ such that

$$\inf_{n>N_{\mathbf{T}_0}, t\in\mathbf{T}_0} \mathbb{P}^n\left(X_t^n \in E_0\right) = 1 \tag{6.2.10}$$

and the $\{\mu'_{\mathbf{T}_0,n} = \mathfrak{be}(\mu'_{\mathbf{T}_0,n})\}_{n>N_{\mathbf{T}_0}}$ in (6.2.6) exist. We then have

$$\inf_{n>N_{\mathbf{T}_0}, t\in\mathbf{T}_0} \mathbb{P}^n\left(\bigotimes \mathcal{F}\circ X_t^n \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1 \tag{6.2.11}$$

by (6.2.10) and Fact 6.9 (with $X = X^n$). From (6.2.11), the condition (ii) above and Lemma B.78 (c, e) it follows that

$$\widehat{X}^n \xrightarrow{\mathrm{D}(\mathbf{T})} Y \text{ as } n\uparrow\infty. \tag{6.2.12}$$

From (6.2.12) and Proposition 6.7 (b) (with $X = X^n$ and $\mathbf{T} = \mathbf{T}_0$) it follows that[26]

$$\overline{\mu}_{\mathbf{T}_0,n} = \mathbb{P}^n\circ\left(\widehat{X}^n_{\mathbf{T}_0}\right)^{-1} \Longrightarrow \nu_{\mathbf{T}_0} \mathrel{\hat{=}} \mathbb{P}\circ Y_{\mathbf{T}_0}^{-1} \text{ as } n\uparrow\infty \text{ in } \mathcal{P}(\widehat{E}^{\mathbf{T}_0}). \tag{6.2.13}$$

Now, by Corollary 5.20 (with $d \mathrel{\hat{=}} \aleph(\mathbf{T}_0)$, $\mu_n = \mu_{\mathbf{T}_0,n+N_{\mathbf{T}_0}}$, $\mu'_n = \mu'_{\mathbf{T}_0,n+N_{\mathbf{T}_0}}$ and $\nu = \nu_{\mathbf{T}_0}$) and Fact B.23 (with $E = E^{\mathbf{T}_0}$ or $(E_0^{\mathbf{T}_0}, \mathscr{O}_E(E_0)^{\mathbf{T}_0})$), there exists a $\mu_{\mathbf{T}_0} \in \mathcal{P}(E^{\mathbf{T}_0})$ such that[27]

$$\text{w-}\lim_{n\to\infty} \mu'_{\mathbf{T}_0,n}\big|_{E_0^d} = \mu_{\mathbf{T}_0}\big|_{E_0^d} \text{ in } \mathcal{P}\left(E_0^{\mathbf{T}_0}, \mathscr{O}_E(E_0)^{\mathbf{T}_0}\right), \tag{6.2.14}$$

$\mu_{\mathbf{T}_0}$ is tight in E_0^d[28], $\nu_{\mathbf{T}_0} = \overline{\mu}_{\mathbf{T}_0}$ and

$$\mu'_{\mathbf{T}_0,n} \Longrightarrow \mu_{\mathbf{T}_0} \text{ as } n\uparrow\infty \text{ in } \mathcal{P}\left(E^{\mathbf{T}_0}\right). \tag{6.2.15}$$

Step 2: Verify $X = \{X_t\}_{t\geq 0}$ defined by (6.2.4) is a process and satisfies (6.2.9). For each $t \in \mathbf{T}$, we let $\mu_{\{t\}} \in \mathcal{P}(E)$ be the measure constructed in Step 1 with $\mathbf{T}_0 = \{t\}$. By our argument above, each $\mu_{\{t\}}$ is tight in $(E_0, \mathscr{O}_E(E_0))$ and so is supported on some $S_t \in \mathscr{K}_\sigma(E_0, \mathscr{O}_E(E_0))$. $S_t \in \mathscr{B}(\widehat{E})$ and $\mathscr{B}_E(S_t) =$

[26] $\overline{\mu}_{\mathbf{T}_0,n}$ as specified in Notation 5.14 denotes the replica measure of $\mu_{\mathbf{T}_0,n}$.

[27] The notation "w-lim" was introduced in §2.3 and means weak limit of a sequence of non-negative finite Borel measures.

[28] The tightness of the limit measure $\mu_{\mathbf{T}_0}$ in E_0^d is given by Corollary 5.20.

$\mathscr{B}_{\widehat{E}}(S_t)$ by Corollary 3.15 (b) and Lemma 3.14 (a). Let $\nu_{\{t\}} = \mathbb{P} \circ Y_t^{-1} = \overline{\mu}_{\{t\}}$ be defined as in (6.2.13) with $\mathbf{T}_0 = \{t\}$. It follows by Proposition 5.15 (a) that

$$\mathbb{P}\left(Y_t \in S_t\right) = \nu_{\{t\}}(S_t) = \mu_{\{t\}}(S_t) = 1, \ \forall t \in \mathbf{T}. \tag{6.2.16}$$

Hence,

$$X_t \in M\left(\Omega, \mathscr{F}; E_0, \mathscr{B}_E(E_0)\right), \ \forall t \in \mathbf{T} \tag{6.2.17}$$

satisfy (6.2.9) by Lemma B.31 (b, c) (with $(E, \mathscr{U}) = (\widehat{E}, \mathscr{B}(\widehat{E}))$, $S_0 = S_t$, $(S, \mathscr{U}') = (E_0, \mathscr{B}_E(E_0))$, $X = Y_t$ and $Y = X_t$). Furthermore, we have

$$\{x_0\} \in \mathscr{B}(\widehat{E}) \cap \mathscr{B}\left(E_0, \mathscr{O}_E(E_0)\right) \cap \mathscr{B}(E), \tag{6.2.18}$$

by Lemma 3.3 (c, e), the fact $E_0 \in \mathscr{B}(E)$ and Proposition A.2 (a), which implies

$$X_t \in M\left(\Omega, \mathscr{F}; E_0, \mathscr{O}_E(E_0)\right), \ \forall t \in \mathbf{R}^+ \backslash \mathbf{T}. \tag{6.2.19}$$

Now, X is an $(E_0, \mathscr{O}_E(E_0))$-valued process by Fact 2.24 (b).

Step 3: Verify the $\mathbf{m}$*-tightness of* $X_{\mathbf{T}_0}$ *in* $E_0^{\mathbf{T}_0}$ *and (6.2.7) for each* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$. Letting $\{S_t\}_{t \in \mathbf{T}}$ be as in Step 2, we have that

$$S_{\mathbf{T}_0} \circeq \prod_{t \in \mathbf{T}_0} S_t \in \mathscr{K}_\sigma\left(E_0^{\mathbf{T}_0}, \mathscr{O}_E(E_0)^{\mathbf{T}_0}\right) \tag{6.2.20}$$

by Corollary A.15 (b) (with $\mathbf{I} = \mathbf{T}_0$ and $S_i = (E_0, \mathscr{O}_E(E_0))$). It follows that

$$S_{\mathbf{T}_0} \in \mathscr{B}(\widehat{E}^{\mathbf{T}_0}) \cap \mathscr{B}(E)^{\otimes \mathbf{T}_0} \cap \mathscr{K}_\sigma^{\mathbf{m}}(E^{\mathbf{T}_0}) \tag{6.2.21}$$

and

$$\mathscr{B}_{E^{\mathbf{T}_0}}(S_{\mathbf{T}_0}) = \mathscr{B}_{\widehat{E}^{\mathbf{T}_0}}(S_{\mathbf{T}_0}) \tag{6.2.22}$$

by Corollary 3.15 (b) and Lemma 3.14 (a). Now,

$$\nu_{\mathbf{T}_0}\left(S_{\mathbf{T}_0}\right) = \mathbb{P}\left(Y_{\mathbf{T}_0} \in S_{\mathbf{T}_0}\right) = 1 \tag{6.2.23}$$

by (6.2.21) and (6.2.16). Moreover,

$$\nu_{\mathbf{T}_0}\left(A \cap S_{\mathbf{T}_0}\right) = \mu_{\mathbf{T}_0}\left(A \cap S_{\mathbf{T}_0}\right) = \mu_{\mathbf{T}_0}(A), \ \forall A \in \mathscr{B}(E^{\mathbf{T}_0}) \tag{6.2.24}$$

by (6.2.21), (6.2.22), the fact $\nu_{\mathbf{T}_0} = \overline{\mu}_{\mathbf{T}_0}$, (6.2.23) and Proposition 5.15 (a) (with $\mu = \mu_{\mathbf{T}_0}$). It follows that

$$\begin{aligned}\mathbb{P}\left(X_{\mathbf{T}_0} \in A\right) &= \mathbb{P}\left(Y_{\mathbf{T}_0} \in A\right) \\ &= \nu_{\mathbf{T}_0}\left(A \cap S_{\mathbf{T}_0}\right) = \mu_{\mathbf{T}_0}(A),\ \forall A \in \mathscr{B}(E)^{\bigotimes \mathbf{T}_0}\end{aligned} \tag{6.2.25}$$

by (6.2.9), (6.2.23) and (6.2.24). Thus, $X_{\mathbf{T}_0}$ is $\mathbf{m}$-tight in $E_0^{\mathbf{T}_0}$ by (6.2.20), (6.2.21), (6.2.23) and (6.2.25). (6.2.7) follows by (6.2.25) and Proposition 3.57 (with $\mathbf{I} = \mathbf{T}_0$, $S_i = E$, $A = E_0^{\mathbf{T}_0}$ and $\Gamma = \{\mathbb{P} \circ X_{\mathbf{T}_0}^{-1}\}$).

Step 4: Verify (6.2.8) and (6.2.1). One obtains (6.2.8) from (6.2.14) established in Step 1 and (6.2.7) established in Step 3. (6.2.1) follows from (6.2.15) and (6.2.7).

(b) We have by (a) (with $\mathbf{T} = \mathbf{R}^+$) that

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(X_t = Y_t \in E_0\right) = 1 \tag{6.2.26}$$

and

$$X^n \xrightarrow{\mathrm{D}(\mathbf{R}^+)} X \text{ as } n \uparrow \infty. \tag{6.2.27}$$

It then follows that

$$\begin{aligned}\mathbb{E}\left[\widehat{f} \circ Y_{\mathbf{T}_0} - \widehat{f} \circ Y_{\mathbf{T}_0+c}\right] &= \mathbb{E}\left[f \circ X_{\mathbf{T}_0} - f \circ X_{\mathbf{T}_0+c}\right] \\ &= \lim_{n \to \infty} \mathbb{E}^n\left[f \circ X^n_{\mathbf{T}_0} - f \circ X^n_{\mathbf{T}_0+c}\right] = 0\end{aligned} \tag{6.2.28}$$

for all $c > 0$, $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ and $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ by (6.2.27), Fact B.35 (b) (with $\mathbf{T} = \mathbf{R}^+$), Fact 6.19 (with $\mathbf{T} = \mathbf{R}^+$), (6.2.26) and Lemma B.77 (a) (with $\mathbf{T} = \mathbf{R}^+$). Hence, the stationarity of Y follows by Corollary 3.11 (a)[29] (with $d = \aleph(\mathbf{T}_0)$ amd $A = \widehat{E}^d$). $\mathbb{P} \circ X_0^{-1}$ is $\mathbf{m}$-tight in E_0 by (a), so there exists an $A \in \mathscr{K}_\sigma(E_0, \mathscr{O}_E(E_0))$ such that

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(X_t = Y_t \in A\right) = 1, \tag{6.2.29}$$

by (6.2.26) and the stationarity of Y. Now, X is stationary by Lemma B.77

[29] $\mathbb{P} \circ \widehat{X}_{\mathbf{T}_0}^{-1}$ and each $\mathbb{P}^n \circ (\widehat{X}^n_{\mathbf{T}_0})^{-1}$ are Borel probability measures as mentioned in Note 6.5.

(e).

(c) We fix $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ and let each $\mu_{\mathbf{T}_0,n}$, $\mu'_{\mathbf{T}_0,n}$ and $\mu_{\mathbf{T}_0}$ be as in (a). It follows by (6.2.15), (5.1.3) (with $\mathcal{D} = C_b(E;\mathbf{R})$) and Fact B.54 (with $d = \aleph(\mathbf{T}_0)$, $\mu = \mu_{\mathbf{T}_0,n}$ and $\nu_1 = \mu'_{\mathbf{T}_0,n}$) that

$$\lim_{n\to\infty} \int_{E^{\mathbf{T}_0}} f(x)\mu_{\mathbf{T}_0,n}(dx) = \lim_{n\to\infty} f^*\left(\mu'_{\mathbf{T}_0,n}\right) = f^*(\mu_{\mathbf{T}_0}) \tag{6.2.30}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E;\mathbf{R}))]$. Hence, (6.2.15) implies

$$\text{w-}\lim_{n\to\infty} \mu'_{\mathbf{T}_0,n} = \mu_{\mathbf{T}_0} \tag{6.2.31}$$

by Theorem 5.4 (a, b) (with $d = \aleph(\mathbf{T}_0)$, $\Gamma = \{\mu_{\mathbf{T}_0,n}\}_{n\in\mathbf{N}}$ and $\mathcal{D} = C_b(E;\mathbf{R})$). Now, (c) follows by (6.2.6), (6.2.31) and Fact B.36 (with $\mathbf{I} = \mathbf{N}$). □

Remark 6.22. As mentioned in Note 6.5, Polish-space-valued processes (especially replica processes) have Borel finite-dimensional distributions and their finite-dimensional convergence refers exactly to the weak convergence of their finite-dimensional distributions. Moreover, $\mathcal{P}(\widehat{E})$ is a Polish space by Corollary 3.11 (c). Hence, (6.2.12) is equivalent to $Y = \mathfrak{fl}_{\mathbf{T}}(\{\widehat{X}^n\}_{n\in\mathbf{N}})$.

The next corollary leverages Theorem 6.20 to establish finite-dimensional convergence to a given limit process.

Corollary 6.23. *Let E be a topological space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $\mathbf{T} \subset \mathbf{R}^+$. Suppose that:*

(i) $\{X_t^n\}_{n\in\mathbf{N}}$ *is sequentially tight in E_0 for all $t \in \mathbf{T}$.*

(ii) (6.2.2) holds for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(iii) E-valued process $(\Omega, \mathscr{F}, \mathbb{P}; X)$ satisfies (6.1.14) (especially (6.1.9)).

Then, the following statements are true:

(a) (6.2.6), (6.2.7), (6.2.8) and (6.2.1) hold for some $\{N_{\mathbf{T}_0}\}_{\mathbf{T}_0\in\mathscr{P}_0(\mathbf{T})} \subset \mathbf{N}$. Moreover, $X_{\mathbf{T}_0}$ is $\mathbf{m}$-tight for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(b) If $\mathbf{T} = \mathbf{R}^+$ and $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{R}^+, \mathcal{F}\backslash\{1\})$-AS, then X is stationary.

(c) If $C_b(E;\mathbf{R})$ separates points on E, then $X = \mathfrak{fl}_{\mathbf{T}}(\{X_n\}_{n\in\mathbf{N}})$.

Proof. Letting $Y = \widehat{X}$, we obtain (6.2.5) for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ by the conditions (ii) and (iii) above, Fact 6.9 and Proposition 6.8 (a). Now, the result follows by Theorem 6.20 immediately. □

6.3 Càdlàg replica

Càdlàg replicas have features that can ease establishing relative compactness, tightness, well-posedness of martingale problems and convergence properties of nonlinear filters etc. As Polish-space-valued càdlàg processes, they have the following nice measurability.

Fact 6.24. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E and $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued process. Then, $\mathfrak{rep}_{\mathrm{c}}(X;E_0,\mathcal{F}) \subset M(\Omega,\mathscr{F};D(\mathbf{R}^+;\widehat{E}))$[30] and $\mathfrak{rep}_{\mathrm{c}}(X;E_0,\mathcal{F}) \subset \mathfrak{rep}_{\mathrm{p}}(X;E_0,\mathcal{F}) \subset \mathfrak{rep}_{\mathrm{m}}(X;E_0,\mathcal{F})$[31].*

Proof. The first statement follows by Fact A.77 (b) (with $E = \widehat{E}$ and $X = \widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X;E_0,\mathcal{F})$) and Fact A.76 (a) (with $E = \widehat{E}$). The second statement follows by Proposition B.33 (a, c). □

Due to the topological difference of E and $\widehat{E}$, non-càdlàg E-valued processes can have càdlàg replicas if they are almost càdlàg on the functions in $\mathcal{F}$.

Definition 6.25. Let E be a topological space and $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued process. X is said to be **weakly càdlàg along $\mathbf{T}$ under $\mathcal{D}$** ($(\mathbf{T},\mathcal{D})$-càdlàg for short) if: (1) $\mathbf{T} \subset \mathbf{R}^+$ and $\mathcal{D} \subset M(E;\mathbf{R})$ are non-empty, and (2) There exist $\mathbf{R}$-valued càdlàg processes $\{(\Omega,\mathscr{F},\mathbb{P};\zeta^f)\}_{f\in\mathcal{D}}$ such that

$$\inf_{f\in\mathcal{D},t\in\mathbf{T}} \mathbb{P}\left(f\circ X_t = \zeta_t^f\right) = 1. \tag{6.3.1}$$

[30] $M(\Omega,\mathscr{F};D(\mathbf{R}^+;\widehat{E}))$ denotes the family of all $D(\mathbf{R}^+;\widehat{E})$-valued random variables defined on measurable space $(\Omega,\mathscr{F})$.

[31] $\mathfrak{rep}_{\mathrm{m}}(X;E_0,\mathcal{F})$, $\mathfrak{rep}_{\mathrm{p}}(X;E_0,\mathcal{F})$ and $\mathfrak{rep}_{\mathrm{c}}(X;E_0,\mathcal{F})$ were introduced in Notation 6.3 and stand for all equivalence classes of measurable, progressive and càdlàg replicas of X with respect to $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$.

Note 6.26. X is $(\mathbf{R}^+, \mathcal{D})$-càdlàg if $\{\varpi(f) \circ X\}_{f \in \mathcal{D}}$ are all càdlàg processes, especially if X is càdlàg and $\mathcal{D} \subset C(E; \mathbf{R})$ by Fact B.34 (a) (with $S = \mathbf{R}$). Apparently, $(\mathbf{R}^+, \mathcal{D})$-càdlàg property is transitive between modifications.

Remark 6.27. A special case of an $(\mathbf{R}^+, \mathcal{D})$-càdlàg X is when $\varpi(f) \circ X$ is càdlàg for all $f \in \mathcal{D}$. This can happen to a non-càdlàg X even if $E = \mathbf{R}$.

Here are three typical sufficient conditions for unique existence of càdlàg replica.

Proposition 6.28. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process and $\mathbf{T} \subset \mathbf{R}^+$ be dense. Then, the following statements are true:*

(a) If X is $(\mathbf{R}^+, \mathcal{F})$-càdlàg and (6.1.14) holds, then $\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$[32] *exists and $\varpi(\bigotimes \widehat{\mathcal{F}}) \circ \widehat{X}$ (resp. $\varpi(\widehat{f}) \circ \widehat{X}$) is the unique càdlàg modification*[33] *of $\varpi(\bigotimes \mathcal{F}) \circ X$ (resp. $\varpi(f) \circ X$ for each $f \in \mathcal{F}$) up to indistinguishability.*

(b) If $\{\varpi(f) \circ X\}_{f \in \mathcal{F}}$ are all càdlàg and (6.1.14) holds, then $\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$ exists and[34]

$$\mathbb{P}\left(\varpi(f) \circ X = \varpi(\widehat{f}) \circ \widehat{X}, \forall f \in \mathfrak{ca}(\mathcal{F})\right) = 1. \tag{6.3.2}$$

(c) If X is càdlàg and satisfies (6.1.9), then $\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$ exists and

$$\mathbb{P}\left(\varpi(f) \circ X = \varpi(\widehat{f}) \circ \widehat{X}, \forall f \in C(E; \mathbf{R}) \text{ having a replica } \widehat{f}\right) = 1. \tag{6.3.3}$$

Remark 6.29. The *functional* indistinguishability of X and $\widehat{X}$ in (6.3.2) is a valuable property of càdlàg replica. Corollary 3.10 showed $C(\widehat{E}; \mathbf{R}) = \mathfrak{ca}(\widehat{\mathcal{F}})$, so (6.3.2) allows many properties to be transferred between X and $\widehat{X}$.

Our construction of $\widehat{X}$ is based on the following technical lemma.

[32] We specified in Notation 6.3 that "$\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$" means $\widehat{X}$ is the unique càdlàg replica of X up to indistinguishability. "Unique up to indistinguishability" means any two processes with the relevant property is indistinguishable.

[33] The terminology "modification" was explained in §2.5.

[34] Please be reminded that $\widehat{f}$ denotes the continuous replica of f.

Lemma 6.30. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process satisfying (6.1.14) for some dense $\mathbf{T} \subset \mathbf{R}^+$ and $\mathbf{T} \subset \mathbf{S} \subset \mathbf{R}^+$. Then, the following statements are equivalent:*

(a) X is $(\mathbf{S}, \mathcal{F})$-càdlàg.

(b) There exists an $\mathbf{R}^\infty$-valued càdlàg process $(\Omega, \mathscr{F}, \mathbb{P}; \zeta)$ such that

$$\inf_{t \in \mathbf{S}} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \zeta_t\right) = 1. \tag{6.3.4}$$

(c) There exists an $\widehat{X} \in M(\Omega, \mathscr{F}; D(\mathbf{R}^+; \widehat{E}))$ such that

$$\inf_{t \in \mathbf{S}} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t\right) = 1. \tag{6.3.5}$$

Proof. ((a) $\to$ (b)) is immediate by Fact B.37 (with $\mathcal{D} = \mathcal{F}$ and $\mathbf{T} = \mathbf{S}$).

((b) $\to$ (c)) Let $\mathbf{T}_0 \subset \mathbf{T}$ be countable and dense in $\mathbf{R}^+$. $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$ is a closed subspace of $\mathbf{R}^\infty$ by (3.1.5).

$$\begin{aligned} &\mathbb{P}\left[\zeta \in D\left(\mathbf{R}^+; \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right)\right] \\ &\geq \mathbb{P}\left(\zeta_t = \bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E}), \forall t \in \mathbf{T}_0\right) = 1 \end{aligned} \tag{6.3.6}$$

by (6.3.4), $\mathbf{T}_0 \subset \mathbf{S}$, (6.1.14), the càdlàg property of ζ and the closedness of $\bigotimes \widehat{\mathcal{F}}(\widehat{E})$. Then, there exists a

$$\zeta' \in M\left[\Omega, \mathscr{F}; D\left(\mathbf{R}^+; \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right)\right] \tag{6.3.7}$$

satisfying

$$\mathbb{P}\left(\zeta = \zeta'\right) = 1 \tag{6.3.8}$$

by (6.3.6), Proposition A.72 (b) (with $E = \mathbf{R}^\infty$) and Lemma B.70 (b) (with $E = \mathbf{R}^\infty$, $E_0 = \bigotimes \widehat{\mathcal{F}}(\widehat{E})$, $S_0 = D(\mathbf{R}^+; \bigotimes \widehat{\mathcal{F}}(\widehat{E}))$ and $X = \zeta$).

It follows by by (3.1.3) and Proposition A.62 (d) (with $S = \bigotimes \widehat{\mathcal{F}}(\widehat{E})$, $E = \widehat{E}$ and $f = (\bigotimes \widehat{\mathcal{F}})^{-1}$) that

$$\varpi\left[\left(\bigotimes \widehat{\mathcal{F}}\right)^{-1}\right] \in C\left[D\left(\mathbf{R}^+; \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right); D(\mathbf{R}^+; \widehat{E})\right]. \tag{6.3.9}$$

It follows by (6.3.7) and (6.3.9) that

$$\widehat{X} \doteq \varpi \left[\left(\bigotimes \widehat{\mathcal{F}}\right)^{-1}\right] \circ \zeta' \in M\left(\Omega, \mathscr{F}; D(\mathbf{R}^+; \widehat{E})\right). \tag{6.3.10}$$

It follows by (6.3.8), (6.3.10) and (3.1.3) that

$$\mathbb{P}\left(\zeta = \varpi\left(\bigotimes \widehat{\mathcal{F}}\right) \circ \varpi\left[\left(\bigotimes \widehat{\mathcal{F}}\right)^{-1}\right] \circ \zeta' = \varpi\left(\bigotimes \widehat{\mathcal{F}}\right) \circ \widehat{X}\right) = 1. \tag{6.3.11}$$

Now, (6.3.5) follows by (6.3.4) and (6.3.11).

((c) → (a)) is automatic. □

Proof of Proposition 6.28. (a) By Lemma 6.30 (with $\mathbf{S} = \mathbf{R}^+$), there exists an $\widehat{X} \in M(\Omega, \mathscr{F}; D(\mathbf{R}^+; \widehat{E}))$ such that

$$\begin{aligned} &\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t\right) \\ &= 1 \geq \sup_{t \in \mathbf{R}^+} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right). \end{aligned} \tag{6.3.12}$$

$\varpi(\bigotimes \widehat{\mathcal{F}}) \circ \widehat{X}$ (resp. $\varpi(\widehat{f}) \circ \widehat{X}$) is a càdlàg modification of $\varpi(\bigotimes \mathcal{F}) \circ X$ (resp. $\varpi(f) \circ X$ for each $f \in \mathcal{F}$) by (3.1.3), the fact $\widehat{\mathcal{F}} \subset C(\widehat{E}; \mathbf{R})$, (6.3.12) and Fact B.34 (a) (with $E = \widehat{E}$, $X = \widehat{X}$ and $f = \widehat{f}$ or $\bigotimes \widehat{\mathcal{F}}$). Now, (a) follows by Proposition 6.8 (b) and Proposition B.33 (h).

(b) Given any $\widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$, one finds that

$$\begin{aligned} &\left\{\omega \in \Omega : \varpi(f) \circ X(\omega) = \varpi(\widehat{f}) \circ \widehat{X}(\omega), \forall f \in \mathfrak{ca}(\mathcal{F})\right\} \\ &= \left\{\omega \in \Omega : \varpi\left(\bigotimes \mathcal{F}\right) \circ X(\omega) = \varpi\left(\bigotimes \widehat{\mathcal{F}}\right) \circ \widehat{X}(\omega)\right\} \end{aligned} \tag{6.3.13}$$

by properties of uniform convergence. Then, (b) follows by (6.3.13) and (a).

(c) Let $\mathbf{T}_0 \subset \mathbf{T}$ be countable and dense in $\mathbf{R}^+$. Given a càdlàg X and any $\widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$, $\varpi(f) \circ X$ and $\varpi(\widehat{f}) \circ \widehat{X}$ are càdlàg process for all $f \in C(E; \mathbf{R})$ by Fact B.34 (a).

$$\begin{aligned} &\left\{\omega \in \Omega : \varpi(f) \circ X(\omega) = \varpi(\widehat{f}) \circ \widehat{X}(\omega), \forall f \in C(E; \mathbf{R}) \text{ having a replica } \widehat{f}\right\} \\ &\supset \left\{\omega \in \Omega : X_t(\omega) = \widehat{X}_t(\omega) \in E_0, \forall t \in \mathbf{T}_0\right\} \end{aligned} \tag{6.3.14}$$

by the fact $f|_{E_0} = \widehat{f}|_{E_0}$ and Proposition B.33 (g). Now, (c) follows by Fact 6.9, (b), Proposition 6.7 (a) and (6.3.14). $\square$

Remark 6.31. The càdlàg property of $\varpi(\bigotimes \mathcal{F}) \circ X(\omega)$ in $(\mathbf{R}^\infty)^{\mathbf{R}^+}$ does not guarantee that of $X(\omega)$ in $E^{\mathbf{R}^+}$ since $\bigotimes \mathcal{F}$ is not necessarily an imbedding on E.

If E_0 is large enough for X to almost surely live in $E_0^{\mathbf{R}^+}$, then one can modify merely a $\mathbb{P}$-negligible amount of paths of X and obtain an indistinguishable replica of X.

Proposition 6.32. *Let E be a topological space, $E_0 \in \mathscr{B}(E)$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process satisfying*

$$\mathbb{P}\left(X \in S_0 \subset E_0^{\mathbf{R}^+}\right) = 1. \tag{6.3.15}$$

Then, there exists an $\widehat{X} \in S_0^\Omega$ satisfying the following properties:

(a) $\widehat{X}$ is an $(E_0, \mathscr{O}_E(E_0))$-valued process and $\mathbb{P}(X = \widehat{X} \in S_0) = 1$.

(b) $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$ for any base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E.

(c) If every element of S_0 is a càdlàg member of $(E_0^{\mathbf{R}^+}, \mathscr{O}_E(E_0)^{\mathbf{R}^+})$[35], then $\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$ for any base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E.

Proof. We fix $y_0 \in S_0$ and define $\widehat{X} \triangleq \mathfrak{var}(X; \Omega, X^{-1}(S_0), y_0)$[36]. Then, (a) follows by Lemma B.31 (b, c) (with $(E, \mathscr{U}) = (E^{\mathbf{R}^+}, \mathscr{B}(E)^{\otimes \mathbf{R}^+})$, $S = S_0$, $\mathscr{U}' = \mathscr{U}|_{S_0}$ and $Y = \widehat{X}$). Given any base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$, $(E_0, \mathscr{O}_{\widehat{E}}(E_0))$ is coarser than $(E_0, \mathscr{O}_E(E_0))$ by Lemma 3.3 (d) and so $\widehat{X}$ is an $\widehat{E}$-valued process. We have by (3.1.1) that

$$\begin{aligned}
&\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t\right) \\
&\geq \mathbb{P}\left(X_t = \widehat{X}_t \in E_0, \forall t \in \mathbf{R}^+\right) \geq \mathbb{P}\left(X = \widehat{X} \in S_0\right) \\
&= 1 \geq \sup_{t \in \mathbf{R}^+} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right),
\end{aligned} \tag{6.3.16}$$

[35] $(E_0, \mathscr{O}_E(E_0))$ need not be a Tychonoff space, so we avoid the notation $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$.

[36] $\mathfrak{var}(\cdot)$ was introduced in Notation 4.1.

thus proving (b). The càdlàg property of $\widehat{X}(\omega) : \mathbf{R}^+ \to (E_0, \mathscr{O}_E(E_0))$ implies $\widehat{X}(\omega) \in D(\mathbf{R}^+; \widehat{E})$ for all $\omega \in \Omega$[37] by Fact B.14 (b) (with $E = (E_0, \mathscr{O}_E(E_0))$, $S = (E_0, \mathscr{O}_{\widehat{E}}(E_0))$ and f being the identity mapping on E_0). Hence, (c) follows by (b), (6.3.16) and Proposition 6.8 (b) (with $\mathbf{T} = \mathbf{R}^+$). □

Remark 6.33. In general, many paths of a càdlàg replica could live outside $E_0^{\mathbf{R}^+}$. It is not necessarily an E-valued process, nor is it (pathwisely) indistinguishable from X. Specifically, if X is càdlàg and satisfies

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}(X_t \in E_0) = 1, \tag{6.3.17}$$

then $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ satisfies

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(X_t = \widehat{X}_t \in E_0\right) = 1 \tag{6.3.18}$$

by Proposition 6.28 (c) (with $\mathbf{T} = \mathbf{R}^+$) and Proposition 6.7 (a) (with $\mathbf{T} = \mathbf{R}^+$). However, this does not necessarily imply $\mathbb{P}(X \in E_0^{\mathbf{R}^+}) = 1$ nor $\mathbb{P}(X = \widehat{X} \in E_0^{\mathbf{R}^+}) = 1$ since E_0 might not be a closed subspace of E or $\widehat{E}$.

Moreover, we transform $\mathcal{M}^+(E)$-valued weakly càdlàg[38] processes into $\mathcal{P}(\widehat{E})$-valued càdlàg processes by similar construction techniques for càdlàg replica, which furthers Corollary 6.12. An analogue for the measurable process case is given in Lemma B.92 in §B.4.

Lemma 6.34. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an $\mathcal{M}^+(E)$-valued $(\mathbf{R}^+, \mathfrak{mc}(\mathcal{F})^*)$-càdlàg process satisfying (6.1.34) and (6.1.35). Then, there exists an $\mathscr{F}_t^X$-adapted $D(\mathbf{R}^+; \mathcal{P}(\widehat{E}))$-valued random variable $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ satisfying Corollary 6.12 (a, c, d).*

Remark 6.35. In the proof below, we let φ, $\widehat{\varphi}$ y_0 and Ψ be as in Lemma 6.10 and set $S_0 = \widehat{\varphi}[\mathcal{P}(\widehat{E})]$. Recall that $\widehat{\varphi}$ satisfies (6.1.26). $\mathcal{P}(\widehat{E})$ is a compact Polish space Corollary 3.11 (c) (with $d = 1$). Hence, $S_0 \in \mathscr{C}(\mathbf{R}^\infty)$ is a Polish subspace of $\mathbf{R}^\infty$ by Proposition A.12 (a, e) and Proposition A.11 (b, f). $D(\mathbf{R}^+; \mathcal{P}(\widehat{E}))$ and $D(\mathbf{R}^+; S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$ are Polish spaces by Proposition A.72 (d) (with $E = \mathcal{P}(\widehat{E})$ or $(S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$). Therefore, $D(\mathbf{R}^+; \mathcal{P}(\widehat{E}))$-valued

[37]This statement is stronger than being an $\widehat{E}$-valued càdlàg process.
[38]The notion of weakly càdlàg was introduced in Definition 6.25.

and $D(\mathbf{R}^+; S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$-valued random variables are càdlàg processes by Fact A.76 (a), for which $\mathscr{F}_t^X$-adaptedness is a proper concept.

Proof of Lemma 6.34. The proof of Lemma (6.10) mentioned that $\mathfrak{mc}(\mathcal{F})$ is countable and φ satisfies (6.1.25), so $\varphi \circ X$ has an $\mathbf{R}^\infty$-valued càdlàg modification ζ by Fact B.37 (with $E = \mathcal{M}^+(E)$, $\mathcal{D} = \mathfrak{mc}(\mathcal{F})^*$ and $\mathbf{T} = \mathbf{R}^+$) and ζ is $\mathscr{F}_t^X$-adapted by Proposition B.33 (e).

Similar to (6.1.31), we have that

$$\begin{aligned} &\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\zeta_t = \varphi \circ X_t \in S_0\right) \\ &= \inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\{\omega \in \Omega : \varphi \circ X_t(\omega) = \widehat{\varphi}^*(\nu^\omega) \in S_0\}\right) \\ &= \mathbb{P}\left(\{\omega \in \Omega : X(\omega)(E) = X(\omega)(E_0) = 1\}\right) = 1 \end{aligned} \tag{6.3.19}$$

by the countability of $\mathfrak{mc}(\mathcal{F})$, Proposition 5.15 (a, b, e) (with $d = 1$, $\mu = X_t(\omega)$ and $\overline{\mu} = \nu^\omega$), (6.1.34) and (6.1.35). It follows that

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\zeta \in D\left(\mathbf{R}^+; S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0)\right)\right) = 1 \tag{6.3.20}$$

by (6.3.19), the closedness of S_0 and the càdlàg property of ζ. Then, there exists a

$$\zeta' \in M\left[\Omega, \mathscr{F}; D\left(\mathbf{R}^+; S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0)\right)\right] \tag{6.3.21}$$

satisfying (6.3.8) by (6.3.20), Proposition A.72 (b) (with $E = \mathbf{R}^\infty$) and Lemma B.70 (b) (with $E = E_0 = \mathbf{R}^\infty$, $E_0 = S_0$, $S_0 = D(\mathbf{R}^+; S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$ and $X = \zeta$). As ζ is $\mathscr{F}_t^X$-adapted, ζ' is $\mathscr{F}_t^X$-adapted by (6.3.8) and Proposition B.33 (e). Furthermore,

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\zeta'_t = \varphi \circ X_t \in S_0\right) = 1. \tag{6.3.22}$$

by (6.3.8) and (6.3.19).

The proof of Lemma (6.10) mentioned that $\widehat{\varphi}$ satisfies (6.1.26) and Ψ equals $\widehat{\varphi}^{-1}$ restricted to S_0. Hence, we have that: (1)

$$Y \mathrel{\hat{=}} \varpi(\Psi) \circ \zeta' = \varpi(\widehat{\varphi}^{-1}) \circ \zeta' \in M\left[\Omega, \mathscr{F}; D\left(\mathbf{R}^+; \mathcal{P}(\widehat{E})\right)\right] \tag{6.3.23}$$

by (6.3.21) and Proposition A.62 (d) (with $S = (S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$, $E = \mathcal{P}(\widehat{E})$ and $f = \widehat{\varphi}^{-1}$), (2) Y is $\mathscr{F}_t^X$-adapted by (6.1.26), the $\mathscr{F}_t^X$-adaptedness of ζ' and Fact B.32 (a) (with $E = (S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$, $S = \mathcal{P}(\widehat{E})$, $f = \widehat{\varphi}^{-1}$ and $X = \zeta'$), and (3)

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\varphi \circ X_t = \zeta'_t = \widehat{\varphi} \circ Y_t\right) = 1 \tag{6.3.24}$$

by (6.3.23) and (6.3.22).

Now, the result follows by Corollary 6.12 (a) - (d), (6.3.24) and (6.1.26). □

6.4 Weak convergence about càdlàg replica

6.4.1 Several regularity conditions about processes

Before discussing weak convergence of càdlàg replicas on their path space, we give a series of regularity conditions about stochastic processes for our later use.

Definition 6.36. Let E be a topological space and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i \in \mathbf{I}}$ be E-valued processes.

- When $(E, \mathfrak{r})$ is a metric space, $\{X^i\}_{i \in \mathbf{I}}$ satisfies **Mild Pointwise Containment Condition for T** (**T**-MPCC for short) if $\mathbf{T} \subset \mathbf{R}^+$ is non-empty and for any $\epsilon \in (0, \infty)$ and $t \in \mathbf{T}$, there exists a *totally bounded* (see p. 222) set $A_{\epsilon,t} \in \mathscr{B}(E)$ satisfying[39]

$$\inf_{i \in \mathbf{I}} \mathbb{P}^i\left(X_t^i \in A_{\epsilon,t}^{\epsilon}\right) \geq 1 - \epsilon. \tag{6.4.1}$$

- $\{X^i\}_{i \in \mathbf{I}}$ satisfies **T-Pointwise m-Tightness Condition** or **T-Pointwise Sequential m-Tightness Condition in** $A \subset E$ (**T**-PMTC or **T**-PSMTC in A for short) if $\mathbf{T} \subset \mathbf{R}^+$ is non-empty and $\{X_t^i\}_{i \in \mathbf{I}}$ is **m**-tight or sequentially **m**-tight in A for all $t \in \mathbf{T}$, respectively. Moreover, we say $\{X^i\}_{i \in \mathbf{I}}$ satisfies **T**-PMTC (resp. **T**-PSMTC) if it satisfies **T**-PMTC (resp. **T**-PSMTC) in E.

[39] The notation "$A_{\epsilon,t}^{\epsilon}$" was defind in §2.1.3.

- $\{X^i\}_{i\in\mathbf{I}}$ satisfies **Metrizable Compact Containment Condition in** A (MCCC in A for short) if for each $\epsilon, T \in (0,\infty)$, there exists a $K_{\epsilon,T} \in \mathscr{K}^{\mathbf{m}}(E)$ such that $K_{\epsilon,T} \subset A$,

$$\bigcap_{t\in[0,T]} (X_t^i)^{-1}(K_{\epsilon,T}) \in \mathscr{F}^i, \ \forall i \in \mathbf{I} \tag{6.4.2}$$

  and

$$\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X_t^i \in K_{\epsilon,T}, \forall t \in [0,T]\right) \geq 1-\epsilon. \tag{6.4.3}$$

  Moreover, by $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC we mean it satisfies MCCC in E.

- When $\{X^i\}_{i\in\mathbf{I}}$ are measurable processes, $\{X^i\}_{i\in\mathbf{I}}$ is said to satisfy **Long-time-average m-Tightness Condition in** A **for** $\{T_k\}_{k\in\mathbf{N}}$ (T_k-LMTC in A for short) if $T_k \uparrow \infty$[40] and

$$\left\{\frac{1}{T_k}\int_0^{T_k} \mathbb{P}^i \circ (X_\tau^i)^{-1} d\tau\right\}_{k\in\mathbf{N}, i\in\mathbf{I}} \subset \mathcal{P}(E) \tag{6.4.4}$$

  is $\mathbf{m}$-tight in A. Moreover, by $\{X^i\}_{i\in\mathbf{I}}$ satisfies T_k-LMTC we mean it satisfies T_k-LMTC in E.

- $\{X_i\}_{i\in\mathbf{I}}$ satisfies **Modulus of Continuity Condition for** $\mathfrak{r}$ ($\mathfrak{r}$-MCC for short) if: (1) $\mathfrak{r}$ is a pseudometric on E, and (2) For any $\epsilon, T \in (0,\infty)$, there exists a $\delta_{\epsilon,T} \in (0,\infty)$ such that[41]

$$\left\{\omega \in \Omega : w'_{\mathfrak{r},\delta_{\epsilon,T},T} \circ X^i(\omega) \geq \epsilon\right\} \in \mathscr{F}^i, \ \forall i \in \mathbf{I} \tag{6.4.5}$$

  and

$$\sup_{i\in\mathbf{I}} \mathbb{P}^i\left(w'_{\mathfrak{r},\delta_{\epsilon,T},T} \circ X^i \geq \epsilon\right) \leq \epsilon. \tag{6.4.6}$$

- $\{X_i\}_{i\in\mathbf{I}}$ satisfies **Modulus of Continuity Condition** (MCC for short) if there exist a family of pseudometrics $\mathcal{R}$ that induces $\mathscr{O}(E)$[42] and $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathfrak{r}$-MCC for all $\mathfrak{r} \in \mathcal{R}$.

[40] "$T_k \uparrow \infty$" as usual denotes a non-decreasing sequence $\{T_k\}_{k\in\mathbf{N}} \subset \mathbf{R}$ that converges to ∞.

[41] "$w'_{|\cdot|,\delta,T}$" is defined by (2.2.3) with $E = \mathbf{R}$ and $\mathfrak{r} = |\cdot|$.

[42] The meaning of $\mathcal{R}$ inducing $\mathscr{O}(E)$ was explained in §2.1.3.

- $\{X_i\}_{i\in\mathbf{I}}$ satisfies **Functional Modulus of Continuity Condition for** $\mathcal{D}$ ($\mathcal{D}$-FMCC for short) if: (1) $\varpi(f)\circ X^i$ admits a càdlàg modification $\zeta^{f,i}$ for each $f\in\mathcal{D}\subset M(E;\mathbf{R})$ and $i\in\mathbf{I}$, and (2) $\{\zeta^{f,i}\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC[43] for all $f\in\mathfrak{ae}(\mathcal{D})$.

- $\{X^i\}_{i\in\mathbf{I}}$ satisfies **Weak Modulus of Continuity Condition** (WMCC for short) if: (1) There exists a $\mathcal{D}\subset C(E;\mathbf{R})$ separating points on E, and (2) $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{D}$-FMCC.

Note 6.37. An E-valued process X is said to satisfy any of the properties above except $\mathbf{T}$-PSMTC (in A)[44] if the singleton $\{X\}$ does.

Note 6.38. If $\{X^i\}_{i\in\mathbf{I}}$ and $\{Y^i\}_{i\in\mathbf{I}}$ are two bijectively indistinguishable families of E-valued processes (i.e. X^i and Y^i are indistinguishable for all $i\in\mathbf{I}$), then each of the conditions above is transitive between between $\{X^i\}_{i\in\mathbf{I}}$ and $\{Y^i\}_{i\in\mathbf{I}}$. Moreover, $\mathcal{D}$-FMCC and WMCC are transitive between $\{X^i\}_{i\in\mathbf{I}}$ and $\{Y^i\}_{i\in\mathbf{I}}$ if Y^i is a modification of X^i for all $i\in\mathbf{I}$.

Remark 6.39.

- Assuming total boundedness in lieu of compactness for each $A_{\epsilon,t}$, $\mathbf{R}^+$-MPCC weakens the Pointwise Containment Property in [Ethier and Kurtz, 1986, §3.7, Theorem 3.7.2] and [Kouritzin, 2016, §5].

- MCCC is a variant of the famous *Compact Containment Condition* (see [Jakubowski, 1986, §4, (4.8)] and [Ethier and Kurtz, 1986, §3.7, (7.9)]) with respect to $\mathbf{m}$-tightness, which become indifferent if E has metrizable compact sets. $\mathbf{R}^+$-PMTC is a similar variant of the Pointwise Tight Condition in [Kouritzin, 2016, §5] and [Ethier and Kurtz, 1986, §3.7, (7.7)].

- T_k-LMTC often appears in constructing stationary distributions (see Kunita [1971] and Bhatt et al. [2000]). The measures in (6.4.4) are well-defined by properties of measurable process and Fubini's Theorem.

[43] $|\cdot|$-MCC means MCC for Euclidean metric $|\cdot|$. The notation "$\mathfrak{ae}(\mathcal{D})$" was defined in §2.2.3.

[44] Note that sequential $\mathbf{m}$-tightness is for infinite collections of measures or random variables.

Remark 6.40.

- MCC was used in Jakubowski [1986] and Kouritzin [2016] (in its finite-time-horizon form) for general Tychonoff spaces. As long as E is Hausdorff, the assumption of pseudometrics $\mathcal{R}$ inducing $\mathscr{O}(E)$ in MCC implies E is Tychonoff (see Proposition A.25 (a, d)).
- WMCC is a special case of $\mathcal{D}$-FMCC. Both of them are generically milder than MCC as $\mathcal{D}$ need not strongly separate points on E.

Note 6.41. If $\{X^i\}_{i\in\mathbf{I}}$ satisfy $\mathcal{D}$-FMCC, then they are apparently $(\mathbf{R}^+,\mathcal{D})$-càdlàg processes.

Remark 6.42. In many standard texts like Billingsley [1968] and Ethier and Kurtz [1986], $\mathfrak{r}$-MCC and MCCC are two fundamental criteria for establishing tightness or relative compactness in Skorokhod $\mathscr{J}_1$-spaces. The common setting of E for $\mathfrak{r}$-MCC and MCCC is that $(E,\mathfrak{r})$ is a separable metric space. Herein, we specify the measurability conditions (6.4.2) and (6.4.5) in the definitions of MCCC and MCC respectively since they are not necessarily true for a general E. Given a càdlàg X^i, Lemma A.79 justifies (6.4.2) under very mild conditions about E and $K_{\epsilon,T}$, and Lemma A.80 justifies (6.4.5) for the following four cases:

(1) $(E,\mathfrak{r})$ is a metric space and X^i is a $D(\mathbf{R}^+;E)$-valued random variable.

(2) $(E,\mathfrak{r})$ is a separable metric space.

(3) $\mathfrak{r}=\rho_{\{f\}}$[45] with $f\in C(E;\mathbf{R})$.

(4) $\mathfrak{r}=\rho_{\mathcal{D}}$ with $\mathcal{D}\subset C(E;\mathbf{R})$ being a countable point-separating collection (hence E is baseable).

Consequently, $\mathcal{D}$-FMCC never incurs measurability issue like (6.4.5) by case (2) above (with $E=\mathbf{R}$), nor does $\rho_{\{f\}}$-MCC (resp. $\rho_{\mathcal{D}}$-MCC) for càdlàg processes and $\mathcal{D}$ consistent with case (3) (resp. case (4)) above.

Besides the measurability conditions (6.4.2) and (6.4.5), §A.7 of Appendix A also provides several results about the relationship among $\mathfrak{r}$-MCC, MCC, $\mathcal{D}$-FMCC and WMCC. The above-mentioned containment or tightness conditions will be further discussed in §6.5.

[45] The pseudometric $\rho_{\{f\}}$ is defined by (2.2.22) with $\mathcal{D}=\{f\}$.

6.4.2 Tightness of càdlàg replicas

Given E-valued processes $\{X^i\}_{i\in\mathbf{I}}$, we first consider tightness of their càdlàg replicas $\{\widehat{X}^i \in \mathfrak{rep}_c(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}}$ in the path space $D(\mathbf{R}^+; \widehat{E})$.

Remark 6.43. Càdlàg replicas are always $D(\mathbf{R}^+; \widehat{E})$-valued random variables (see Fact 6.24) and their tightness in $D(\mathbf{R}^+; \widehat{E})$ has the usual meaning (compared to our general interpretation in §2.4).

Note 6.44. Thanks to the compactness of $\widehat{E}$ (see Lemma 3.3 (b)), the stringent MCCC becomes an automatic condition for any family of $\widehat{E}$-valued processes.

Given MCCC, tightness of $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+; \widehat{E})$ can be reduced to that $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{F}$-FMCC.

Proposition 6.45. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes satisfying*

$$\inf_{t\in\mathbf{T}, i\in\mathbf{I}} \mathbb{P}^i\left(\bigotimes \mathcal{F} \circ X_t^i \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1 \tag{6.4.7}$$

for some dense $\mathbf{T} \subset \mathbf{R}^+$. Then, the following statements are true:

(a) If $\{X_i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{F}$-FMCC, then $\{\widehat{X}^i = \mathfrak{rep}_c(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}}$ satisfies $\widehat{\mathcal{F}}$-FMCC, satisfies $\rho_{\widehat{\mathcal{F}}}$-MCC and is tight in $D(\mathbf{R}^+; \widehat{E})$.

(b) The converse of (a) is true when $\mathbf{T} = \mathbf{R}^+$ or $\{X_i\}_{i\in\mathbf{I}}$ are all càdlàg.

(c) If $\mathbf{I}$ is an infinite set and any subsequence of $\{X_i\}_{i\in\mathbf{I}}$ has a sub-subsequence satisfying $\mathcal{F}$-FMCC, then $\{\widehat{X}^i = \mathfrak{rep}_c(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}\setminus\mathbf{I}_0}$ is sequentially tight in $D(\mathbf{R}^+; \widehat{E})$ for some $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$.

Proof. (a) Suppose $\zeta^{f,i}$ is a càdlàg modification of $\varpi(f) \circ X^i$ for each $f \in \mathfrak{ae}(\mathcal{F})$ and $i \in \mathbf{I}$ and $\{\zeta^{f,i}\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC for all $f \in \mathfrak{ae}(\mathcal{F})$. It follows by (6.4.7) and Proposition 6.28 (a) (with $X = X^i$) that $\{\widehat{X}^i = \mathfrak{rep}_c(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}}$ exists and satisfies

$$\inf_{f\in\mathfrak{ae}(\mathcal{F}), i\in\mathbf{I}} \mathbb{P}^i\left(\varpi(\widehat{f}) \circ \widehat{X}^i = \zeta^{f,i}\right) = 1, \tag{6.4.8}$$

thus proving $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $\widehat{\mathcal{F}}$-FMCC. $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $\rho_{\widehat{\mathcal{F}}}$-MCC by Lemma 3.3 (a) and Proposition A.84 (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}$). Now, (a) follows by Note 6.44, Lemma 3.3 (c) and Theorem A.88 (with $(E, \mathfrak{r}) = (\widehat{E}, \rho_{\widehat{\mathcal{F}}})$ and $X^i = \widehat{X}^i$).

(b) Given tightness of $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+; \widehat{E})$, $\{\varpi(\widehat{f}) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+; \mathbf{R})$ for all $\widehat{f} \in \mathfrak{ae}(\widehat{\mathcal{F}})$ by Proposition A.62 (d) (with $E = \widehat{E}$ and $S = \mathbf{R}$) and Fact B.60 (a) (with $E = A = D(\mathbf{R}^+; \widehat{E})$, $S = D(\mathbf{R}^+; \mathbf{R})$, $f = \varpi(\widehat{f})$ and $\Gamma = \{\mathbb{P}^i \circ (\widehat{X}^i)^{-1}\}_{i\in\mathbf{I}}$). Then, $\{\varpi(\widehat{f}) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC for all $\widehat{f} \in \mathfrak{ae}(\widehat{\mathcal{F}})$ by Theorem A.88 (with $(E, \mathfrak{r}) = (\mathbf{R}, |\cdot|)$ and $X^i = \varpi(\widehat{f}) \circ \widehat{X}^i$).

We have that

$$\begin{aligned}&\inf_{t\in\mathbf{T}, i\in\mathbf{I}, f\in\mathfrak{ae}(\mathcal{F})} \mathbb{P}^i\left(f \circ X_t = \widehat{f} \circ \widehat{X}^i_t\right)\\ &\geq \inf_{t\in\mathbf{T}, i\in\mathbf{I}} \mathbb{P}^i\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1\end{aligned} \tag{6.4.9}$$

by (6.4.7) and (6.1.1) (with $X = X^i$ and $\widehat{X} = \widehat{X}^i$), so $\varpi(\widehat{f}) \circ \widehat{X}^i$ is a càdlàg modification of $\varpi(f) \circ X^i$ for all $i \in \mathbf{I}$ and $f \in \mathfrak{ae}(\mathcal{F})$. If $\mathbf{T} \neq \mathbf{R}^+$ and $\{X^i\}_{i\in\mathbf{I}}$ are all càdlàg, then $\varpi(\widehat{f}) \circ \widehat{X}^i$ is indistinguishable from $\varpi(f) \circ X^i$ for all $i \in \mathbf{I}$ and $f \in \mathfrak{ae}(\mathcal{F})$ by Proposition 6.28 (c). In either case, $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{F}$-FMCC.

(c) follows immediately by (a) and a subsequential argument. □

Next, we consider tightness of the indistinguishable càdlàg replicas constructed by Proposition 6.32 in $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$ or $D(\mathbf{R}^+; E)$.

Proposition 6.46. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Suppose that:*

(i) $(E_0, \mathscr{O}_E(E_0))$ is a Tychonoff space.

(ii) $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC in E_0.

Then, there exists an $S_0 \subset E^{\mathbf{R}^+}$ such that:

(a) $\mathbb{D}_0 \mathrel{\hat{=}} D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$ satisfies[46]

$$\mathscr{B}(\mathbb{D}_0)|_{S_0} = \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{S_0} \subset \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{\mathbb{D}_0} \subset \mathscr{B}(\mathbb{D}_0). \tag{6.4.10}$$

[46] $\mathscr{B}(\mathbb{D}_0)$ is generated by the Skorokhod $\mathscr{J}_1$-topology $\mathscr{J}(E_0, \mathscr{O}_E(E_0))$.

(b) $\{\widehat{X}^i = \mathfrak{rep}_{\mathrm{c}}(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}}$ *satisfies*

$$\widehat{X}^i \in M\left[\Omega^i, \mathscr{F}^i; S_0, \mathscr{O}_{\mathbb{D}_0}(S_0)\right], \ \forall i \in \mathbf{I} \tag{6.4.11}$$

and

$$\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X^i = \widehat{X}^i \in S_0\right) = 1. \tag{6.4.12}$$

(c) $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ *is* $\mathbf{m}$*-tight in* S_0 *as* $\mathbb{D}_0$*-valued random variables if and only if* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{F}$*-FMCC.*

Remark 6.47. The càdlàg replicas $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ above are $\mathbb{D}_0$-valued random variables and their tightness in $S_0 \subset \mathbb{D}_0$ has the usual meaning. We noted in §2.3 that $\mathbb{D}_0$ and S_0 always satisfy[47]

$$\mathscr{B}(\mathbb{D}_0)|_{S_0} \supset \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{S_0} \tag{6.4.13}$$

but not the equality in (6.4.10). So, $\mathbb{D}_0$-valued random variable (like $\widehat{X}^i$ in (6.4.11)) is generally a stronger concept than $(E_0, \mathscr{O}_E(E))$-valued càdlàg process.

In fact, the developments of Proposition 6.46 do not require a base. We establish the following more general result without imposing the boundedness of the point-separating functions.

Theorem 6.48. *Let E be a topological space, $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes, $E_0 \in \mathscr{B}(E)$ and $\mathcal{D} \subset C(E; \mathbf{R})$. Suppose that:*

(i) $\mathcal{D}$ *separates points on* E_0.

(ii) $(E_0, \mathscr{O}_E(E_0))$ *is a Tychonoff space.*

(iii) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies MCCC in* E_0.

Then, there exist $S_0 \subset E^{\mathbf{R}^+}$ and $\{\widehat{X}^i \in S_0^{\Omega^i}\}_{i\in\mathbf{I}}$ such that:

(a) $\mathbb{D}_0 \triangleq D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$ *satisfies (6.4.10).*

(b) $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ *satisfies (6.4.11) and (6.4.12).*

[47] $\mathscr{B}(\mathbb{D}_0)$ is generated by the Skorokhod $\mathscr{J}_1$-topology of $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$.

(c) $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ *is* $\mathbf{m}$*-tight in* S_0 *as* $\mathbb{D}_0$*-valued random variables if and only if* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC.*

Proof. We divide the proof into five steps. We equip E_0 with the subspace topology $\mathscr{O}_E(E_0)$ throughout the proof, which we make implicit for convenience.

Step 1: Construct S_0. By the condition (iii) above,

$$\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X_t^i \in A_{p,q}, \forall t \in [0,q]\right) \geq 1 - 2^{-p-q},\ \forall p,q \in \mathbf{N}. \tag{6.4.14}$$

holds for some $\{A_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E_0)$. It follows that

$$K_{p,q} \stackrel{\circ}{=} \bigcup_{i=1}^{q} A_{p,i} \in \mathscr{K}^{\mathbf{m}}(E_0) \subset \mathscr{C}(E_0),\ \forall p,q \in \mathbf{N} \tag{6.4.15}$$

by the Hausdorff property of E_0, Proposition A.2 (c), Lemma 3.54 and Proposition A.12 (a). From (6.4.14) and (6.4.15) we obtain that

$$K_{p,q} \subset K_{p,q+1},\ \forall p,q \in \mathbf{N} \tag{6.4.16}$$

and

$$\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X_t^i \in K_{p,q}, \forall t \in [0,q]\right) \geq 1 - 2^{-p-q},\ \forall p,q \in \mathbf{N}. \tag{6.4.17}$$

Letting

$$V_p \stackrel{\circ}{=} \bigcap_{q\in\mathbf{N}} \left\{x \in \mathbb{D}_0 : x|_{[0,q)} \in K_{p,q}^{[0,q)}\right\},\ \forall p \in \mathbf{N}, \tag{6.4.18}$$

one finds by the fact $E_0 \in \mathscr{B}(E)$, Lemma B.50 (b) (with $E = E_0$, $A = K_{p,q}$ and $T = q$) and and Proposition A.68 (b) (with $E = E_0$) that

$$V_p \in \mathscr{B}_E(E_0)^{\otimes\mathbf{R}^+}\Big|_{\mathbb{D}_0} \subset \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{\mathbb{D}_0} \subset \mathscr{B}(\mathbb{D}_0),\ \forall p \in \mathbf{N}, \tag{6.4.19}$$

which immediately implies

$$S_0 \stackrel{\circ}{=} \bigcup_{p\in\mathbf{N}} V_p \in \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{S_0} \subset \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{\mathbb{D}_0} \subset \mathscr{B}(\mathbb{D}_0). \tag{6.4.20}$$

Step 2: Verify (a). Each of $\{K_{p,q}\}_{p,q\in\mathbf{N}}$ is a $\mathcal{D}|_{E_0}$-baseable subset of E_0 by (6.4.15) and Proposition 3.51 (a, f) (with $E = E_0$, $K = K_{p,q}$ and $\mathcal{D} = \mathcal{D}|_{E_0}$).

So, $\mathcal{D}$ has a countable subset that separates points and strongly separates points on each of $\{K_{p,q}\}_{p,q\in\mathbf{N}}$ by Lemma A.20. For simplicity, we assume $\mathcal{D}$ is countable in Step 2 - 4 of the proof.

Letting $\Psi \mathrel{\dot=} \varpi[\mathfrak{ae}(\mathcal{D})]$, we have

$$\Psi|_{V_p} \in \mathbf{imb}\left(V_p, \mathscr{O}_{\mathbb{D}_0}(V_p); D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right), \ \forall p \in \mathbf{N} \tag{6.4.21}$$

and

$$\mathscr{B}(\mathbb{D}_0)|_{V_p} = \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{V_p}, \ \forall p \in \mathbf{N} \tag{6.4.22}$$

by Lemma B.65 (with $E = E_0$, $V = V_p$, $p = q$ and $A_p = K_{p,q}$). One then finds by (6.4.19), (6.4.22) and Fact B.1 (with $E = S_0$, $n = p$, $A_n = V_p$, $\mathscr{U}_1 = \mathscr{B}(\mathbb{D}_0)|_{S_0}$ and $\mathscr{U}_2 = \mathscr{B}(E)^{\otimes\mathbf{R}^+}|_{S_0}$) that

$$\mathscr{B}(\mathbb{D}_0)|_{S_0} \subset \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{S_0}. \tag{6.4.23}$$

Now, (a) follows by (6.4.20), (6.4.13) and (6.4.23).

Step 3: Construct $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ and verify (b). It follows by (6.4.16) and (6.4.17) that

$$\begin{aligned}\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X^i \in V_p\right) &\geq 1 - \sup_{i\in\mathbf{I}} \sum_{q\in\mathbf{N}} \left[1 - \mathbb{P}^i\left(X_t^i \in K_{p,q}, \forall t \in [0,q]\right)\right] \\ &\geq 1 - 2^{-p}, \ \forall p \in \mathbf{N}.\end{aligned} \tag{6.4.24}$$

Then, (6.4.20) and (6.4.24) imply

$$\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X^i \in S_0 \subset E_0^{\mathbf{R}^+}\right) = 1. \tag{6.4.25}$$

By Proposition 6.32 (a) (with $X = X^i$), there exist

$$\widehat{X}^i \in M\left(\Omega^i, \mathscr{F}^i; S_0, \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{S_0}\right), \ \forall i \in \mathbf{I} \tag{6.4.26}$$

satisfying (6.4.12). Now, (b) follows by (6.4.26) and (a).

Step 4: Verify sufficiency of (c). We have that[48]

$$\inf_{f\in\mathfrak{ae}(\mathcal{D}),i\in\mathbf{I}} \mathbb{P}^i\left(\varpi(f)\circ X^i = \varpi(f)\circ \widehat{X}^i\right) = 1 \tag{6.4.27}$$

and

$$\varpi(f)\circ \widehat{X}^i \in M\left(\Omega^i,\mathscr{F}^i;D(\mathbf{R}^+;\mathbf{R})\right),\ \forall f\in\mathfrak{ae}(\mathcal{D}), i\in\mathbf{I} \tag{6.4.28}$$

by (6.4.12) and Proposition A.62 (d) (with $S=E_0$ and $E=\mathbf{R}$). Fixing $f\in\mathfrak{ae}(\mathcal{D})$, $\varpi(f)\circ\widehat{X}^i$ is the unique càdlàg modification of $\varpi(f)\circ X^i$ up to indistinguishability for all $i\in\mathbf{I}$ by (6.4.27) and Proposition B.33 (f, h). $\{\varpi(f)\circ\widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC by (6.4.27) and $\{X^i\}_{i\in\mathbf{I}}$ satisfying $\mathcal{D}$-FMCC. $\{\varpi(f)\circ\widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies MCCC by the boundedness of f. Hence, $\{\varpi(f)\circ\widehat{X}^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})$ by Theorem A.88 (with $(E,\mathfrak{r})=(\mathbf{R},|\cdot|)$ and $X^i=\varpi(f)\circ\widehat{X}^i$).

Letting Ψ be as above, $\{\Psi\circ\widehat{X}^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ by tightness of $\{\varpi(f)\circ\widehat{X}^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+;\mathbf{R})$ and Proposition B.63 (a) (with $E=E_0$, $\mathcal{D}=\mathfrak{ae}(\mathcal{D})$ and $\mu^i=\mathbb{P}^i\circ(\widehat{X}^i)^{-1}\in\mathcal{P}(\mathbb{D}_0)$). (We assumed $\mathcal{D}$ is countable in this step, so $\mathfrak{ae}(\mathcal{D})$ is countable by Fact B.15.) $\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}$ is a Polish space by Proposition A.11 (f).

$$\varphi \mathrel{\mathring{=}} \bigotimes \mathfrak{ae}(\mathcal{D}) \in C\left(E;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}\right) \tag{6.4.29}$$

by Fact 2.4 (b). Letting $\{K_{p,q}\}_{p,q\in\mathbf{N}}$ be as in (6.4.15), we have that

$$\bigotimes \mathfrak{ae}(\mathcal{D})(K_{p,q}) \in \mathscr{K}\left(\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}\right) \subset \mathscr{C}\left(\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}\right) \tag{6.4.30}$$

by (6.4.29) and Proposition A.12 (a, e). It follows by Lemma B.66 (with $E=E_0$, $V=V_p$, $p=q$ and $A_q=K_{p,q}$) that

$$\Psi(V_p)\in\mathscr{C}\left(D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right),\ \forall p\in\mathbf{N}. \tag{6.4.31}$$

Hence, tightness of $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ in S_0 follows by Lemma B.67 (with $E=\mathbb{D}_0$, $S=D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$, $A_p=V_p$, $E_0=S_0$ and $f=\Psi$).

$A\mathrel{\mathring{=}}\bigcup_{p,q\in\mathbf{N}}K_{p,q}\in\mathscr{K}_\sigma^{\mathbf{m}}(E_0)$ is a $\mathcal{D}|_{E_0}$-baseable subset of E_0 by Proposition 3.58 (b, e) (with $E=E_0$, and $\mathcal{D}=\mathcal{D}|_{E_0}$). S_0 is a baseable subspace of $\mathbb{D}_0$ by

[48] Herein, the replica process $\widehat{X}^i$ is an E_0-valued process and so $f\circ\widehat{X}^i_t$ is well-defined.

Proposition 3.62 (b) (with $E = E_0$) and Fact 3.35. Thus, $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ is $\mathbf{m}$-tight in S_0 by Corollary 3.15.

Step 5: Verify necessity of (c). In this step, we do not require $\mathcal{D}$ to be countable and suppose $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ is $\mathbf{m}$-tight in S_0. For each fixed $f \in \mathfrak{ae}(\mathcal{D})$, $\{\varpi(f) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})$ by Proposition A.62 (d) (with $E = E_0$ and $S = \mathbf{R}$) and Fact B.60 (with $E = \mathbb{D}_0$, $S = D(\mathbf{R}^+;\mathbf{R})$, $f = \varpi(f)$, $\mu^i = \mathbb{P}^i \circ (\widehat{X}^i)^{-1} \in \mathcal{P}(\mathbb{D}_0)$ and $\Gamma = \{\mu^i\}_{i\in\mathbf{I}}$). $\{\varpi(f) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC by Theorem A.88 (with $(E,\mathfrak{r}) = (\mathbf{R}, |\cdot|)$ and $X^i = \varpi(f) \circ \widehat{X}^i$). Hence, $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{D}$-FMCC by (6.4.27) and Proposition B.33 (f). □

Proof of Proposition 6.46. This result follows immediately by Theorem 6.48 (with $\mathcal{D} = \mathcal{F}$) and Proposition 6.32 (with $X = X^i$). □

6.4.3 Weak convergence of càdlàg replicas on path space

The following proposition connects the weak convergence of càdlàg replicas on path space and their finite-dimensional convergence.

Proposition 6.49. *Let E be a topological space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\widehat{X}^n \in \mathfrak{rep}_c(X^n; E_0, \mathcal{F})$ for each $n \in \mathbf{N}$ and $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ be a $D(\mathbf{R}^+; \widehat{E})$-valued random variable. Then, the following statements are true:*

(a) If $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ and Y satisfy[49]

$$\widehat{X}^n \Longrightarrow Y \text{ as } n \uparrow \infty \text{ on } D(\mathbf{R}^+; \widehat{E}), \tag{6.4.32}$$

then

$$\widehat{X}^n \xrightarrow{\mathrm{D}(\mathbf{R}^+ \setminus J(Y))} Y \text{ as } n \uparrow \infty. \tag{6.4.33}$$

(b) If (6.2.12) holds for some dense $\mathbf{T} \subset \mathbf{R}^+$, and if $\{X^n\}_{n\in\mathbf{N}}$ satisfies

$$\inf_{t\in\mathbf{T}, n\in\mathbf{N}} \mathbb{P}^n \left(\bigotimes \mathcal{F} \circ X_t^n \in \bigotimes \widehat{\mathcal{F}}(\widehat{E}) \right) = 1 \tag{6.4.34}$$

and $\mathcal{F}$-FMCC, then (6.4.32) holds.

[49] The meaning of (6.4.32) follows our interpretation in §2.4. Moreover, the notation "$J(Y)$" is defined by (2.5.8) with $X = Y$.

(c) If (6.4.32) holds, $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T},\mathcal{F}\backslash\{1\})$-AS[50] *and (6.4.34) holds for some conull*[51] *$\mathbf{T}\subset\mathbf{R}^+$, then Y is an $\widehat{E}$-valued stationary process.*

Note 6.50. If Y is an $\widehat{E}$-valued càdlàg process (especially a càdlàg replica), then $J(Y)\subset(0,\infty)$ is countable by Lemma 3.3 (c) and Proposition 3.64. In other words, $\mathbf{R}^+\backslash J(Y)$ is a cocountable[52] (hence non-empty and dense) subset of $\mathbf{R}^+$.

Proof of Proposition 6.49. (a) follows by Lemma 3.3 (c) and Theorem A.87 (a) (with $E=\widehat{E}$, $X^n=\widehat{X}^n$ and $X=Y$).

(b) $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ is tight in the Polish space $D(\mathbf{R}^+;\widehat{E})$ by Proposition 6.45 (a) (with $\mathbf{I}=\mathbf{N}$). It is relatively compact in $D(\mathbf{R}^+;\widehat{E})$[53] by the Prokhorov's Theorem (Theorem 2.22 (b)). Now, (b) follows by Theorem A.87 (b) (with $E=\widehat{E}$, $X^n=\widehat{X}^n$ and $X=Y$).

(c) $\mathbf{T}\backslash J(Y)$ is a conull set, so

$$\mathbf{S}_{Y,\mathbf{T}_0}\mathrel{\mathring{=}}\bigcap_{t\in\mathbf{T}_0}\{c\in(0,\infty):t+c\in\mathbf{T}\backslash J(Y)\}\tag{6.4.35}$$

is a conull hence dense subset of $\mathbf{R}^+$. Fixing $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{T}\backslash J(Y))$ and $f\in\mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$, we have

$$\mathbb{E}\left[\widehat{f}\circ Y_{\mathbf{T}_0}\right]=\mathbb{E}\left[\widehat{f}\circ Y_{\mathbf{T}_0+c}\right]\tag{6.4.36}$$

for all $c\in\mathbf{S}_{Y,\mathbf{T}_0}$ by (a) and Lemma B.78 (b, e) (with $\mathbf{T}=\mathbf{T}\backslash J(Y)$ and $\mathbf{S}_{\mathbf{T}_0}=\mathbf{S}_{Y,\mathbf{T}_0}$). $\{Y_{t+c}\}_{c\geq 0}$ is a càdlàg process for all $t\in\mathbf{T}_0$ since Y is càdlàg. $\zeta\mathrel{\mathring{=}}\{\varpi(\widehat{f})\circ Y_{\mathbf{T}_0+c}\}_{c\geq 0}$ is also a càdlàg process by Fact B.34 (a, b) (with $\mathbf{I}=\mathbf{T}_0$, $i=t$, $X^i=Y_{t+c}$, $X=\{Y_{\mathbf{T}_0+c}\}_{t\geq 0}$ and $f=\widehat{f}$). Then, (6.4.36) extends to all $c\in(0,\infty)$ by the denseness of $\mathbf{S}_{\mathbf{T}_0,Y}$ in $\mathbf{R}^+$, the càdlàg property of ζ and the Dominated Convergence Theorem. Now, (c) follows by Corollary 3.11 (a) (with $d=\aleph(\mathbf{T}_0)$ and $A=\widehat{E}^d$) and Fact A.76 (a) (with $E=\widehat{E}$). □

[50] The notion of $(\mathbf{T},\mathcal{F}\backslash\{1\})$-AS was introduced in Definition 6.14.

[51] Conull set was specified in §2.1.5. Conull subset of $\mathbf{R}$ is in the Lebesgue sense.

[52] The notion of cocountable set was defined in §2.1.1.

[53] Relative compactness of $D(\mathbf{R}^+;\widehat{E})$-valued random variables $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ follows our interpretation in §2.4.

The next proposition connects weak convergence of càdlàg replicas on $D(\mathbf{R}^+;\widehat{E})$ and that on the restricted path space $D(\mathbf{R}^+;E_0,\mathscr{O}_E(E_0))$ (if well-defined).

Proposition 6.51. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E, $(\Omega,\mathcal{F},\mathbb{P};X)$ and $\{(\Omega^n,\mathscr{F}^n,\mathbb{P}^n;X^n)\}_{n\in\mathbf{N}}$ be E-valued càdlàg processes, $\widehat{X}\in\mathfrak{rep}_{\mathrm{c}}(X;E_0,\mathcal{F})$ and $\widehat{X}^n\in\mathfrak{rep}_{\mathrm{c}}(X^n;E_0,\mathcal{F})$ for each $n\in\mathbf{N}$. In addition, suppose $(E_0,\mathscr{O}_E(E_0))$ is a Tychonoff space. Then, the following statements are true:*

(a) If $\widehat{X}$ and $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ satisfy[54]

$$\widehat{X}^n \Longrightarrow \widehat{X} \text{ as } n\uparrow\infty \text{ on } D\left(\mathbf{R}^+;E_0,\mathscr{O}_E(E_0)\right), \tag{6.4.37}$$

then they satisfy

$$\widehat{X}^n \Longrightarrow \widehat{X} \text{ as } n\uparrow\infty \text{ on } D(\mathbf{R}^+;\widehat{E}). \tag{6.4.38}$$

(b) If there exists an $S_0\subset E^{\mathbf{R}^+}$ satisfying (6.3.15) and

$$\inf_{n\in\mathbf{N}}\mathbb{P}^n\left(X^n\in S_0\subset E_0^{\mathbf{R}^+}\right)=1, \tag{6.4.39}$$

and if $\mathcal{F}$ strongly separates points on E_0, then (6.4.38) implies (6.4.37).

Proof. (a) For ease of notation, we let $(\Omega^0,\mathscr{F}^0,\mathbb{P}^0;X^0)\doteq(\Omega,\mathscr{F},\mathbb{P};X)$, $\mathbb{D}_0\doteq D(\mathbf{R}^+;E_0,\mathscr{O}_E(E_0))$ and $\widehat{\mathbb{D}}\doteq D(\mathbf{R}^+;\widehat{E})$. $(E_0,\mathscr{O}_E(E_0))$ is a topological refinement of $(E_0,\mathscr{O}_{\widehat{E}}(E_0))$ by Lemma 3.3 (d). $D(\mathbf{R}^+;E_0,\mathscr{O}_{\widehat{E}}(E_0))$ is a subspace of $\widehat{\mathbb{D}}$ by Corollary A.65 (with $E=\widehat{E}$ and $A=E_0$). It then follows by (6.4.37) and Proposition A.62 (e) (with $E=(E_0,\mathscr{O}_{\widehat{E}}(E_0))$ and $S=(E_0,\mathscr{O}_E(E_0))$) that $\mathbb{D}_0\subset\widehat{\mathbb{D}}$, $\widehat{\mathbb{D}}_0\doteq(\mathbb{D}_0,\mathscr{O}_{\widehat{\mathbb{D}}}(\mathbb{D}_0))$ is a topological coarsening of $\mathbb{D}_0$ and

$$\widehat{X}^n\in M\left(\Omega^n,\mathscr{F}^n;\mathbb{D}_0\right)\subset M\left(\Omega^n,\mathscr{F}^n;\widehat{\mathbb{D}}_0\right),\ \forall n\in\mathbf{N}_0. \tag{6.4.40}$$

Let μ_n, $\widehat{\nu}_n$ and ν_n denote the distribution of $\widehat{X}^n$ as $\mathbb{D}_0$-valued, $\widehat{\mathbb{D}}$-valued and $\widehat{\mathbb{D}}_0$-valued random variables for each $n\in\mathbf{N}_0$, respectively. It follows by

[54]As specified in §2.4, (6.4.37) abbreviates the statement that $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ and $\widehat{X}$ are $\mathbb{D}_0$-valued random variables and the distributions of $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ converge weakly to that of $\widehat{X}$ in $\mathcal{P}(\mathbb{D}_0)$.

(6.4.37) and Fact B.26 (with $E=\mathbb{D}_0$, $\mathscr{U}=\mathscr{O}(\widehat{\mathbb{D}}_0)$ and $\mu=\mu_0$) that

$$\nu_n \Longrightarrow \nu_0 \text{ as } n\uparrow\infty \text{ in } \mathcal{P}(\widehat{\mathbb{D}}_0). \tag{6.4.41}$$

It follows by (6.4.41) and Lemma B.55 (with $E=\widehat{\mathbb{D}}$, $A=\widehat{\mathbb{D}}_0$, $\mu_n=\nu_n$ and $\mu=\nu_0$) that

$$\widehat{\nu}_n=\nu_n\big|^{\widehat{\mathbb{D}}} \Longrightarrow \nu_0\big|^{\widehat{\mathbb{D}}}=\widehat{\nu}_0 \text{ as } n\uparrow\infty \text{ in } \mathcal{P}(\widehat{\mathbb{D}}), \tag{6.4.42}$$

which proves (6.4.38).

(b) (6.3.15) and (6.4.39) imply

$$\inf_{n\in\mathbf{N}_0}\mathbb{P}^n\left(X^n\in\mathbb{D}_0\right)\geq\inf_{n\in\mathbf{N}_0}\mathbb{P}^n\left(X^n\in S_0\cap\mathbb{D}_0\right)=1. \tag{6.4.43}$$

We have $\mathscr{O}_E(E_0)=\mathscr{O}_{\widehat{E}}(E_0)$ and $\mathbb{D}_0=\widehat{\mathbb{D}}_0$ by $\mathcal{F}$ strongly separating points on E_0 and Lemma 3.3 (b). According to Proposition 6.32 (c) (with $S=\mathbb{D}_0$ and $X=X^n$ or X), one can take

$$\begin{aligned}\widehat{X}^n &= \mathfrak{rep}_{\mathrm{c}}(X^n;E_0,\mathcal{F})\\ &\in M\left(\Omega^n,\mathscr{F}^n;\mathbb{D}_0\right)=M\left(\Omega^n,\mathscr{F}^n;\widehat{\mathbb{D}}_0\right)\subset M\left(\Omega^n,\mathscr{F}^n;\widehat{\mathbb{D}}\right),\ \forall n\in\mathbf{N}_0\end{aligned} \tag{6.4.44}$$

and each μ_n, $\widehat{\nu}_n$ and ν_n in (a) are all well-defined with

$$\mu_n=\nu_n\in\mathcal{P}(\widehat{\mathbb{D}}_0)=\mathcal{P}(\mathbb{D}_0),\ \forall n\in\mathbf{N}_0. \tag{6.4.45}$$

(6.4.38) implies (6.4.42). As $D(\mathbf{R}^+;\widehat{E})$ is a Polish space, (6.4.42) implies (6.4.41) by Lemma B.55 (with $E=\widehat{\mathbb{D}}$, $A=\widehat{\mathbb{D}}_0$, $\mu_n=\nu_n$ and $\mu=\nu_0$). Now, (6.4.37) follows by (6.4.44), (6.4.41) and (6.4.45). □

6.5 Containment in large baseable subsets

Given E-valued processes $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ and a base $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ over E, most developments of §6.1, §6.2 and §6.4 need E_0 to have containment properties like (6.4.7),

$$\inf_{t\in\mathbf{T},i\in\mathbf{I}}\mathbb{P}^i\left(X_t^i\in E_0\right)=1 \tag{6.5.1}$$

or (6.4.25) for $\{X^i\}_{i\in\mathbf{I}}$. In practice, one usually constructs a basesable set E_0 satisfying the non-functional[55] conditions (6.5.1) or (6.4.25) first, and then select proper functions to establish the base $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$. From Fact 6.9 we immediately observe that:

Fact 6.52. *(6.4.7), (6.5.1) and (6.4.25) are successively stronger for any index set* $\mathbf{I}$ *and* $\mathbf{T}\subset\mathbf{R}^+$.

The simplest case is when E itself is a baseable space. Then, one easily obtains a base $(E,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ by Lemma 3.39 and the containment properties in Fact 6.52 are automatic. When E is non-baseable, one can use $\mathbf{T}$-MPCC, MCCC, $\mathbf{T}$-PMTC or T_k-LMTC introduced in §6.4.1 to construct the desired E_0 in (6.4.25) or (6.5.1).

When $\{X^i\}_{i\in\mathbf{I}}$ are all càdlàg, the following proposition uses $\mathbf{T}$-MPCC and $\mathfrak{r}$-MCC to construct an E_0 satisfying (6.4.25).

Proposition 6.53. *Let* $(E,\mathfrak{r})$ *be a metric space,* $\mathcal{D}\subset C(E;\mathbf{R})$ *separate points on* E*,* $\mathbf{T}$ *be a countable dense subset of* $\mathbf{R}^+$ *and* $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ *be* E*-valued càdlàg processes satisfying* $\mathbf{T}$*-MPCC and* $\mathfrak{r}$*-MCC. Then, there exist* $\{A_{p,q}\}_{p,q\in\mathbf{N}}\subset\mathscr{C}(E)$ *satisfying the following properties:*

(a) $\{A_{p,q}\}_{p,q\in\mathbf{N}}$ *are totally bounded and satisfy*

$$A_{p,q}\subset A_{p,q+1},\ \forall p,q\in\mathbf{N} \tag{6.5.2}$$

and

$$\inf_{i\in\mathbf{I}}\mathbb{P}^i\left(X_t^i\in A_{p,q},\forall t\in[0,q]\right)\geq 1-2^{-p-q},\ \forall p,q\in\mathbf{N}. \tag{6.5.3}$$

(b) $E_0\doteq\bigcup_{p,q\in\mathbf{N}}A_{p,q}$ *is a second-countable subspace and is a* $\mathcal{D}$*-baseable subset of* E.

(c) E_0 *and* $S_0\doteq\bigcup_{p\in\mathbf{N}}V_p$ *satisfy (6.4.25), where*

$$V_p\doteq\bigcap_{q\in\mathbf{N}}\left\{x\in\mathbb{D}_0: x|_{[0,q)}\in A_{p,q}^{[0,q)}\right\},\ \forall p\in\mathbf{N}. \tag{6.5.4}$$

[55](6.4.7) by contrast is a functional condition depending on the choice of $\mathcal{F}$.

(d) If $(E, \mathfrak{r})$ is complete, then $\{A_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{K}(E)$ and $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC in E_0.

As noted in §3.3.4, metrizable compact subsets provided by MCCC, **T**-PMTC and T_k-LMTC are nice baseable "blocks" for building E_0. In metric spaces, totally bounded subsets provided by **T**-MPCC form another category of such blocks.

Fact 6.54. *Let $(E, \mathfrak{r})$ be a metric space, $\mathcal{D} \subset C(E; \mathbf{R})$ separate points on E and $\{A_n\}_{n\in\mathbf{N}}$ be totally bounded Borel subsets of E. Then, $\bigcup_{n\in\mathbf{N}} A_n$ is a second-countable subspace and, in particular, is a $\mathcal{D}$-baseable subset of E.*

Proof. $A \doteq \bigcup_{n\in\mathbf{N}} A_n$ is a separable subspace of E by Proposition A.10 (a) and Proposition A.3 (e). Now, the result follows by Proposition A.6 (c) and Proposition 3.40. □

Proof of Proposition 6.53. (a) An inspection of the proof of [Kouritzin, 2016, Theorem 17] shows that **T**-MPCC is enough for their developments. So, one follows Kouritzin [2016] to construct totally bounded $\{A_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{C}(E)$ satisfying (a).

(b) follows by (a) and Fact 6.54.

(c) One finds by (a) that

$$\begin{aligned}\inf_{i\in\mathbf{I}} \mathbb{P}^i\left(X^i \in V_p\right) &\geq 1 - \sup_{i\in\mathbf{I}} \sum_{q\in\mathbf{N}} \left[1 - \mathbb{P}^i\left(X^i_t \in A_{p,q}, \forall t \in [0,q]\right)\right] \\ &\geq 1 - 2^{-p}, \ \forall p \in \mathbf{N}.\end{aligned} \tag{6.5.5}$$

(d) Each $(A_{p,q}, \mathfrak{r})$ is complete by the fact $A_{p,q} \subset \mathscr{C}(E)$ and Proposition A.5 (c). Then, (d) follows by Proposition A.16. □

The next proposition uses MCCC to construct an E_0 satisfying (6.4.25).

Proposition 6.55. *Let E be a topological space, $\mathcal{D} \subset C(E; \mathbf{R})$ separate points on E and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes satisfying MCCC in $A \subset E$. Then, there exist $\{K_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ satisfying the following properties:*

(a) (6.4.16) and (6.4.17) hold.

(b) $E_0 \mathrel{\hat{=}} \bigcup_{p,q\in\mathbf{N}} K_{p,q} \subset A$ *is a* $\mathcal{D}$*-baseable subset of* E*. Moreover,* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies MCCC in* E_0*.*

(c) E_0 *and* $S_0 \mathrel{\hat{=}} \bigcup_{p\in\mathbf{N}} V_p$ *satisfy (6.4.25), where* $\{V_p\}_{p\in\mathbf{N}}$ *are defined as in (6.4.18).*

Proof. (a) We pick $\{A_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E)$ satisfying $A_{p,q} \subset A$ for all $p,q \in \mathbf{N}$ and (6.4.14). E is a Hausdorff space by Proposition A.17 (e) (with $A = E$). Then,

$$K_{p,q} \mathrel{\hat{=}} \bigcup_{i=1}^{q} K_{p,i} \in \mathscr{K}^{\mathbf{m}}(E) \subset \mathscr{C}(E) \subset \mathscr{B}(E),\ \forall p,q \in \mathbf{N} \tag{6.5.6}$$

by Proposition A.2 (c), Lemma 3.54 and Proposition A.12 (a). Now, (6.4.16) and (6.4.17) follow by (6.4.14) and (6.5.6).

(b) follows by (6.5.6), (a) and Proposition 3.58 (b, e).

(c) E_0 and S satisfy (6.4.24) by (a), which implies (6.4.25) immediately. □

In the following fact, one gets an E_0 satisfying (6.5.1) by $\mathbf{T}$-PMTC for a countable $\mathbf{T} \subset \mathbf{R}^+$.

Fact 6.56. *Let* E *be a topological space,* $A \subset E$ *and* $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ *be* E*-valued processes. Then, the following statements are true:*

(a) If $\mathbf{I}$ *is an infinite set and* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PMTC in* A*, then* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PSMTC in* A*.*

(b) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PMTC in* A *if and only if* $\{X^i_{\mathbf{T}_0}\}_{i\in\mathbf{I}}$ *is* $\mathbf{m}$*-tight in* $A^{\mathbf{T}_0}$ *for all* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$*.*

(c) If $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PSMTC in* A*, then* $\{X^i_{\mathbf{T}_0}\}_{i\in\mathbf{I}}$ *is sequentially* $\mathbf{m}$*-tight in* $A^{\mathbf{T}_0}$ *for all* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$*.*

(d) If $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PMTC in* A *for a countable* $\mathbf{T} \subset \mathbf{R}^+$ *and* $\mathcal{D} \subset C(E;\mathbf{R})$ *separates points on* E*, then there exists a* $\mathcal{D}$*-baseable subset* $E_0 \in \mathscr{K}^{\mathbf{m}}_\sigma(E)$ *such that* $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{T}$*-PMTC in* $E_0 \subset A$*.*

(e) When $(E, \mathfrak{r})$ *is a metric space,* $\{X^i\}_{i\in\mathbf{I}}$ *satisfying* $\mathbf{T}$*-PMTC implies* $\{X^i\}_{i\in\mathbf{I}}$ *satisfying* $\mathbf{T}$*-MPCC and the converse is true if* $(E, \mathfrak{r})$ *is complete.*

(f) If $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC in A, then $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathbf{R}^+$-PMTC in A.

Proof. (a) and (f) are automatic by definition. (b) and (c) follow by Lemma B.61 (with $\mathbf{I}=\mathbf{T}_0$, $S_i=E$, $A_i=A$ and $\Gamma=\{\mathbb{P}^i\circ(X^i_{\mathbf{T}_0})^{-1}\}_{i\in\mathbf{I}}$). (d) follows by Lemma B.74 (with $\mathbf{I}=\mathbf{T}$, $i=t$ and $\Gamma_i=\{\mathbb{P}^i\circ(X^i_t)^{-1}\}_{i\in\mathbf{I}}$).

(e) The first part is immediate by Proposition A.16. Then, we suppose $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathbf{T}$-MPCC and $(E,\mathfrak{r})$ is complete. For any $\epsilon\in(0,\infty)$ and $t\in\mathbf{T}$, there exists a totally bounded $B_{\epsilon,t}\subset E$ such that (6.4.1) holds for the closure $A_{\epsilon,t}$ of $B_{\epsilon,t}$. $A_{\epsilon,t}$ is totally bounded by Proposition A.10 (c). $(A_{\epsilon,t},\mathfrak{r})$ is complete by Proposition A.5 (c). Hence, $A_{\epsilon,t}\in\mathscr{K}(E)$ by Proposition A.16. □

Given countably many processes satisfying T_k-LMTC, the next proposition constructs an E_0 satisfying (6.5.1) for a conull $\mathbf{T}\subset\mathbf{R}^+$.

Proposition 6.57. *Let E be a topological space, $\mathcal{D}\subset C(E;\mathbf{R})$ separate points on E and $\{(\Omega^n,\mathscr{F}^n,\mathbb{P}^n;X^n)\}_{n\in\mathbf{N}}$ be E-valued measurable processes satisfying T_k-LMTC in $A\subset E$. Then, there exists a $\mathcal{D}$-baseable subset $E_0\in\mathscr{K}^{\mathbf{m}}_\sigma(E)$ with $A\supset E_0$ and a conull $\mathbf{T}\subset\mathbf{R}^+$ such that*

$$\inf_{t\in\mathbf{T},n\in\mathbf{N}}\mathbb{P}^n\left(X^n_t\in E_0\right)=1 \tag{6.5.7}$$

and $\{X^n\}_{n\in\mathbf{N}}$ satisfies T_k-LMTC in E_0.

Proof. We take $\{K_p\}_{p\in\mathbf{N}}\subset\mathscr{K}^{\mathbf{m}}(A,\mathscr{O}_E(A))$ satisfying

$$\inf_{k,n\in\mathbf{N}}\frac{1}{T_k}\int_0^{T_k}\mathbb{P}^n\left(X^n_\tau\in K_p\right)d\tau\geq 1-2^{-p},\ \forall p\in\mathbf{N} \tag{6.5.8}$$

and let $E_0\mathrel{\hat{=}}\bigcup_{p\in\mathbf{N}}K_p\in\mathscr{K}^{\mathbf{m}}_\sigma(A,\mathscr{O}_E(A))$. It follows that

$$\sup_{k,n\in\mathbf{N}}\int_0^{T_k}\mathbb{P}^n\left(X^n_\tau\notin E_0\right)d\tau=0 \tag{6.5.9}$$

by (6.5.8) and continuity of measure. Hence, (6.5.7) holds for the conull set

$$\mathbf{T}\mathrel{\hat{=}}\mathbf{R}^+\backslash\bigcup_{k,n\in\mathbf{N}}\left\{t\in[0,T_k]:\mathbb{P}^n\left(X^n_t\notin E_0\right)>0\right\}. \tag{6.5.10}$$

Now, the result follows by (6.5.8) and Proposition 3.58 (b, e) (with $A = E_0$). ☐

The relationship among MCCC, **T**-MPCC, **T**-PMTC and T_k-LMTC is illustrated in Figure 4 below, where green arrows means definite implication, blue arrow means conditional implication and red crossed arrow means false converse.

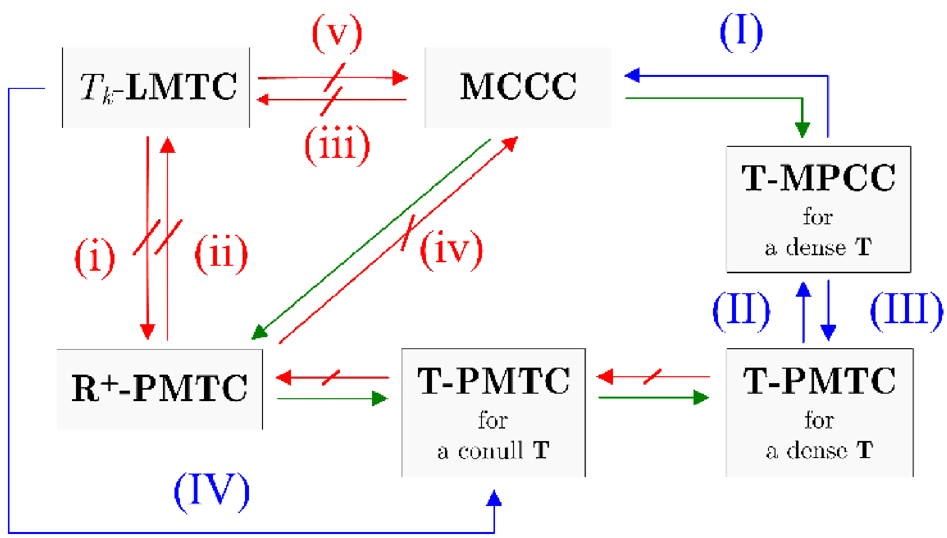

Figure 4: *The relationship among tightness/containment conditions*

Remark 6.58. All the unlabelled arrows in Figure 4 are immediate. Below is some explanation for the labelled ones:

- (I) was justified in Proposition 6.53 (a, d) for càdlàg processes living on a complete (but not necessarily separable) metric space $(E, \mathfrak{r})$ and satisfying $\mathfrak{r}$-MCC. This is a generalization of [Kouritzin, 2016, Theorem 17] on infinite time horizon since **T**-MPCC with a dense **T** is weaker than the Pointwise Containment Property in [Kouritzin, 2016, §5].

- By Fact 6.56 (e), (II) is true on arbitrary metric spaces and (III) is true on complete metric spaces.

- (IV) was justified in Proposition 6.57 for a countable collection of measurable processes.

- (i) is not true because T_k-LMTC will not be affected by changing the distributions of $\{X_t^i\}_{i\in\mathbf{I}}$ to a non-tight family for each $t \in \mathbf{Q}^+$.
- (ii) and (iii) are disproved by the constant process $\{t\}_{t\geq 0}$.
- (iv) and (v) are disproved by Example 6.59 below, where we construct a non-stationary càdlàg process that satisfies T_k-LMTC and $\mathbf{R}^+$-PMTC but violates MCCC. (iv) was also disproved by [Kouritzin, 2016, Example 2].

Example 6.59. Let μ be the uniform distribution on $(0, 1)$ and

$$\eta_t(\omega) \mathrel{\mathring{=}} \begin{cases} 1-\omega+t, & \text{if } t \in [0,\omega), \\ \frac{1}{2}, & \text{if } t \in [\omega,\infty), \end{cases} \quad \forall \omega \in (0,1), t \in \mathbf{R}^+. \tag{6.5.11}$$

$\eta = \{\eta_t\}_{t\geq 0}$ satisfies $\mathbf{R}^+$-PMTC since $(0, 1)$ is σ-compact. However, η violates MCCC because for any $a, b \in (0, 1)$,

$$\begin{aligned} &\mu\left(\eta_t \in [a,b], \forall t \in [0,1]\right) \\ &\leq 1-\mu\left(\{\omega \in (0,1) : 0 \leq \omega - t < 1-b, \exists t \in [0,\omega)\}\right) = 0. \end{aligned} \tag{6.5.12}$$

For each $\tau > 0$ and $\epsilon \in (0, 1/2)$,

$$\{\eta_\tau \in [\epsilon, 1-\epsilon]\} = ((\tau \wedge 1) \vee (\epsilon+\tau), 1 \wedge (1+\tau-\epsilon)) \cup (0, \tau \wedge 1). \tag{6.5.13}$$

Letting $T > 1/\epsilon$, one finds by (6.5.13) that

$$\frac{1}{T}\int_0^T \mu\left(\eta_\tau \in [\epsilon, 1-\epsilon]\right) d\tau \geq \frac{1}{T}\int_1^T 1 d\tau \geq 1-\epsilon. \tag{6.5.14}$$

Hence, η satisfies T_k-LMTC for any $T_k \uparrow \infty$. Moreover, η is non-stationary since η_0 and $\eta_{1/2}$ have distinct expectations.

Chapter 7

Application to Finite-Dimensional Convergence

The previous four chapters elaborate **Theme 1** of this work. With the help of replication, we have developed in §6.2 several tool results for **Theme 2**, the finite-dimensional convergence of possibly non-càdlàg processes. Now, we are going to answer the target questions **Q1**, **Q2** and **Q3** of **Theme 2** in the following three sections. §7.1, establishing finite-dimensional convergence to processes with general paths, answers **Q2**. §7.2, establishing finite-dimensional convergence of weakly càdlàg processes to weakly càdlàg or progressive limit processes, provides answers to both **Q2** and **Q3**. In §7.3, we answer **Q1** by establishing finite-dimensional convergence to long-time typical behaviors of a given measurable process.

7.1 Convergence of process with general paths

Let $\{X^i\}_{i\in\mathbf{I}}$ be E-valued processes and $\mathbf{S} \subset \mathbf{R}^+$. We give in this section a set of sufficient conditions for the unique existence of $X \in \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$[1]. The nature of establishing an $X \in \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ with general paths is establishing a Kolmogorov's extension of weak limit points of the finite-dimensional distributions of $\{X^i\}_{i\in\mathbf{I}}$ for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$. Hence, our goal can be achieved by directly applying Theorem 5.10 established in §5.1.

[1]The readers are referred to §6.2 for definitions and notations about finite-dimensional convergence, finite-dimensional limit point and finite-dimensional limit.

Theorem 7.1. *Let E be a topological space, $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes, $\mathcal{D} \subset C_b(E;\mathbf{R})$ separate points on E*[2] *and $\mathbf{S} \subset \mathbf{R}^+$. Then, the following statements are true:*

(a) If $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathbf{S}$-PSMTC[3] *in $A \subset E$, then any $X \in \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ satisfies $\mathbf{S}$-PMTC in A.*

(b) If $\{X^i\}_{i\in\mathbf{I}}$ is $(\mathbf{S},\mathcal{D})$-FDC[4] *and satisfies $\mathbf{S}$-PSMTC, then there exists an $X = \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ satisfying $\mathbf{S}$-PMTC and $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{i_n}\}_{n\in\mathbf{N}})$ for any $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$.*

(c) If $\{X^i\}_{i\in\mathbf{I}}$ is $(\mathbf{R}^+,\mathcal{D})$-FDC, is $(\mathbf{R}^+,\mathcal{D})$-AS and satisfies $\mathbf{R}^+$-PSMTC, then there exists a stationary $X = \mathfrak{flp}_{\mathbf{R}^+}(\{X^i\}_{i\in\mathbf{I}})$ satisfying $\mathbf{R}^+$-PMTC and $X = \mathfrak{fl}_{\mathbf{R}^+}(\{X^{i_n}\}_{n\in\mathbf{N}})$ for any $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$.

Remark 7.2. We pointed out in Fact 6.56 (b) (with $\mathbf{T} = \mathbf{S}$) that X satisfying $\mathbf{S}$-PMTC in A is equivalent to $X_{\mathbf{T}_0}$ being $\mathbf{m}$-tight in $A^{\mathbf{T}_0}$ for every $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$. In particular, X satisfying $\mathbf{R}^+$-PMTC is equivalent to all finite-dimensional distributions of X being $\mathbf{m}$-tight.

Proof of Theorem 7.1. (a) Without of loss of generality, we suppose X is defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and

$$X^{i_n} \xrightarrow{\mathrm{D}(\mathbf{S})} X \text{ as } n \uparrow \infty. \tag{7.1.1}$$

It follows by (7.1.1) and Fact 2.24 (d) that

$$\mathbb{P}^{i_n} \circ (X_t^{i_n})^{-1} \Longrightarrow \mathbb{P} \circ X_t^{-1} \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(E),\ \forall t \in \mathbf{S}. \tag{7.1.2}$$

$\{X_t^{i_n}\}_{n\in\mathbf{N}}$ is sequentially $\mathbf{m}$-tight in A for all $t \in \mathbf{S}$ as $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathbf{S}$-PSMTC in A. Hence, (a) follows by (7.1.2) and Lemma 5.7 (with $\Gamma = \{\mathbb{P}^{i_n} \circ (X_t^{i_n})^{-1}\}_{n\in\mathbf{N}}$ and $\mu = \mathbb{P} \circ X_t^{-1}$).

[2] As mentioned in Note 5.5, the assumption of $\mathcal{D} \subset C_b(E;\mathbf{R})$ separating points on E below does not require E to be a Tychonoff or baseable space.

[3] $\mathbf{S}$-PMTC (in A) and $\mathbf{S}$-PSMTC (in A) were introduced in §6.4.1. As specified in Note 6.37, that X satisfies $\mathbf{S}$-PMTC in A means the singleton $\{X\}$ satisfies $\mathbf{S}$-PMTC in A.

[4] The notion of $(\mathbf{S},\mathcal{D})$-FDC was introduced in §6.2.

(b) For each $i \in \mathbf{I}$, we let μ_i be the restriction of $\mathrm{pd}(X)$ to $\mathscr{B}(E)^{\otimes \mathbf{R}^+}$[5]. For each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$, the probability measures

$$\mu_i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \mathbb{P}^i \circ (X^i_{\mathbf{T}_0})^{-1} \in \mathfrak{P}\left(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0}\right), \ \forall i \in \mathbf{I} \tag{7.1.3}$$

form a sequentially $\mathbf{m}$-tight family by Fact 6.56 (c) (with $A = E$). For each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$ and $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$, the integrals

$$\int_{E^{\mathbf{T}_0}} f(x)\mu_i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) = \mathbb{E}^i\left[f \circ X^i_{\mathbf{T}_0}\right], \ \forall i \in \mathbf{I} \tag{7.1.4}$$

admit at most one limit point in $\mathbf{R}$ since $\{X^i\}_{i\in\mathbf{I}}$ is $(\mathbf{S}, \mathcal{D})$-FDC. Hence, it follows by Theorem 5.10 (with $\Gamma = \{\mu_i\}_{i\in\mathbf{I}}$, $\mathbf{I} = \mathbf{S}$, $\mathbf{I}_0 = \mathbf{T}_0$ and $a = b = 1$) that there exists a unique $\mu \in \mathfrak{P}(E^{\mathbf{S}}, \mathscr{B}(E)^{\otimes \mathbf{S}})$ and some $\{\mathbf{I}_{\mathbf{T}_0} \in \mathscr{P}_0(\mathbf{I})\}_{\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})}$ such that $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathcal{P}(E^{\mathbf{T}_0})$ is the weak limit of *any subsequence of* and, hence, is the unique weak limit point of $\{\mu_{\mathbf{T}_0,i} = \mathfrak{be}(\mu_i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1})\}_{i\in\mathbf{I}\setminus\mathbf{I}_{\mathbf{T}_0}}$ for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$.

We fix $t_0 \in \mathbf{S}$ and define

$$X_t \mathrel{\hat{=}} \begin{cases} \mathfrak{p}_t, & \text{if } t \in \mathbf{S}, \\ \mathfrak{p}_{t_0}, & \text{if } t \in \mathbf{R}^+ \backslash \mathbf{S}, \end{cases} \quad \forall t \in \mathbf{R}^+. \tag{7.1.5}$$

By Fact 2.3 (a) and Fact 2.24 (b), $X \mathrel{\hat{=}} \{X_t\}_{t\geq 0}$ well-defines an E-valued process on the probability space $(E^{\mathbf{S}}, \mathscr{B}(E)^{\otimes \mathbf{S}}, \mu)$ and certainly satisfies

$$\mu \circ X_{\mathbf{T}_0}^{-1} = \mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}, \ \forall \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}). \tag{7.1.6}$$

Now, (b) follows by (a) and Fact B.36 (with $\mathbf{T} = \mathbf{S}$).

(c) One obtains by (b) (with $\mathbf{S} = \mathbf{R}^+$) an $X = \mathfrak{flp}_{\mathbf{R}^+}(\{X^i\}_{i\in\mathbf{I}})$ satisfying all conclusions of (c) except for stationarity. Without loss of generality, we suppose X is defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and

$$X^{i_n} \xrightarrow{\mathrm{D}(\mathbf{R}^+)} X \text{ as } n \uparrow \infty. \tag{7.1.7}$$

[5]Restriction of measure to sub-σ-algebra and X's process distribution $\mathrm{pd}(X)$ were specified in §2.1.2 and §2.5 respectively.

Fixing $c \in (0, \infty)$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$, it follows by (7.1.7), Fact B.35 (b) (with $n = i_n$) and Fact 6.19 (with $n = i_n$) that

$$\mathbb{E}\left[f \circ X^n_{\mathbf{T}_0}\right] - \mathbb{E}\left[f \circ X^n_{\mathbf{T}_0+c}\right] = \lim_{n\to\infty} \mathbb{E}^n\left[f \circ X^n_{\mathbf{T}_0} - f \circ X^n_{\mathbf{T}_0+c}\right] = 0 \quad (7.1.8)$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$. Hence, $\mathbb{P} \circ X^{-1}_{\mathbf{T}_0} = \mathbb{P} \circ X^{-1}_{\mathbf{T}_0+c}$ by their $\mathbf{m}$-tightness and Lemma B.59 (b) (with $d = \aleph(\mathbf{T}_0)$). □

Theorem 7.1 can be used to identify a given E-valued limit process as the unique finite-dimensional limit point.

Corollary 7.3. *Let E be a topological space, $\mathbf{S} \subset \mathbf{R}^+$ and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be E-valued processes. Suppose that:*

(i) $\mathcal{D} \subset C_b(E; \mathbf{R})$ *separates points on E.*

(ii) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{S}$*-PSMTC.*

(iii) X *satisfies* $\mathbf{S}$*-PMTC*[6]*.*

(iv) $\mathbb{E}[f \circ X_{\mathbf{T}_0}]$ *is the unique limit point of* $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0}]\}_{i\in\mathbf{I}}$ *in* $\mathbf{R}$ *for all* $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$*.*

Then, the following statements are true:

(a) $X = \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ *and* $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{i_n}\}_{n\in\mathbf{N}})$ *for any* $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$*.*

(b) If $\mathbf{S} = \mathbf{R}^+$ *and* $\{X^i\}_{i\in\mathbf{I}}$ *is* $(\mathbf{R}^+, \mathcal{D})$*-AS, then* $X = \mathfrak{flp}_{\mathbf{R}^+}(\{X^i\}_{i\in\mathbf{I}})$ *is a sationary process and* $X = \mathfrak{fl}_{\mathbf{R}^+}(\{X^{i_n}\}_{n\in\mathbf{N}})$ *for any* $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$*.*

Proof. (a) $\{X^i\}_{i\in\mathbf{I}}$ is $(\mathbf{S}, \mathcal{D})$-FDC by the condition (iv) above. By Theorem 7.1 (b), there exists a $Z = \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ such that Z satisfies $\mathbf{S}$-PMTC and $Z = \mathfrak{fl}_{\mathbf{S}}(\{X^{i_n}\}_{n\in\mathbf{N}})$ for any $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$. We suppose Z is defined on $(\Omega, \mathscr{F}, \mathbb{P})$ for simplicity, fix $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$ and show $\mathbb{P} \circ X^{-1}_{\mathbf{T}_0} = \mathbb{P} \circ Z^{-1}_{\mathbf{T}_0}$ for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$. Since

$$X^{i_n} \xrightarrow{\mathrm{D}(\mathbf{S})} Z \text{ as } n \uparrow \infty, \quad (7.1.9)$$

[6] X satisfying $\mathbf{S}$-PMTC means the singleton $\{X\}$ satisfies $\mathbf{S}$-PMTC.

we have by (iv) and Fact 6.19 (with $X = Z$) that

$$\mathbb{E}\left[f \circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[f \circ Z_{\mathbf{T}_0}\right] \tag{7.1.10}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$. $X_{\mathbf{T}_0}$ and $Z_{\mathbf{T}_0}$ are $\mathbf{m}$-tight by the condition (iii) above and Remark 7.2 (with $X = X$ or Z). Hence, $\mathbb{P} \circ X_{\mathbf{T}_0}^{-1} = \mathbb{P} \circ Z_{\mathbf{T}_0}^{-1}$ by Lemma B.59 (b) (with $d = \aleph(\mathbf{T}_0)$).

(b) follows by (a) (with $\mathbf{S} = \mathbf{R}^+$) and Theorem 7.1 (c). □

Remark 7.4. A variant of Theorem 7.1 will be given in Proposition 7.15 that relies heavily on our results herein.

7.2 Convergence of weakly càdlàg processes

This section is concerned with finite-dimensional convergence of E-valued processes satisfying $\mathcal{D}$-FMCC[7]. Such processes as mentioned in Note 6.41 are $(\mathbf{R}^+, \mathcal{D})$-càdlàg[8] processes.

Given $\{X^n\}_{n\in\mathbf{N}}$ satisfying $\mathcal{D}$-DMCC, part (a) of the next theorem establishes an $(\mathbf{S}, \mathcal{D})$-càdlàg $X \in \mathfrak{flp}_{\mathbf{S}}(\{X^n\}_{n\in\mathbf{N}})$ and gives an alternative answer to **Q2** in Introduction. Part (b) further imposes the standard Borel property and establishes a progressive member of $\mathfrak{flp}_{\mathbf{S}}(\{X^n\}_{n\in\mathbf{N}})$. In lieu of a standard Borel assumption, part (c) assumes the $(\mathbf{T}, \mathcal{D})$-AS of $\{X^n\}_{n\in\mathbf{N}}$ for a conull $\mathbf{T} \supset \mathbf{S}$ and establishes a stationary and progressive member of $\mathfrak{flp}_{\mathbf{S}}(\{X^n\}_{n\in\mathbf{N}})$. These two parts provide answers to **Q3** in Introduction.

Theorem 7.5. *Let E be a topological space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes and $\mathbf{S} \subset \mathbf{T} \subset \mathbf{R}^+$ with $\mathbf{S}$ being dense. Suppose that:*

(i) $C_b(E; \mathbf{R})$ *separates points on* E.

(ii) $\mathcal{D} \subset C_b(E; \mathbf{R})$ *is countable and* E_0 *is a* $\mathcal{D}$*-baseable subset of* E.

(iii) $\{X^n\}_{n\in\mathbf{N}}$ *satisfies (6.5.7).*

(iv) $\{X^n\}_{n\in\mathbf{N}}$ *satisfies* $\mathbf{S}$*-PSMTC in* E_0 *and* $\mathcal{D}$*-FMCC.*

[7] The notion of $\mathcal{D}$-FMCC was introduced in Definition 6.36.
[8] The notion of $(\mathbf{S}, \mathcal{F})$-càdlàg process was introduced in Definition 6.25.

(v) $\mathfrak{flp}_{\mathbf{S}}(\{\varpi(\varphi)\circ X^n\}_{n\in\mathbf{N}})$ *has at least one càdlàg member with* $\varphi \mathrel{\mathring{=}} \bigotimes \mathcal{D}$.

Then, there exist a stochastic basis[9] $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t\geq 0}, \mathbb{P})$, *some* $\{n_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$ *and* $X, X' \in (E_0^{\mathbf{R}^+})^\Omega$ *such that the following statements are true:*

(a) $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$ *is an E-valued* $(\mathbf{S}, \mathcal{D})$*-càdlàg process and satisfies* $\mathbf{S}$*-PMTC in* E_0.

(b) If $E_0 \in \mathscr{B}^{\mathbf{s}}(E)$, *then* $X' = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$ *is an E-valued,* $(\mathbf{S}, \mathcal{D})$*-càdlàg,* $\mathscr{G}_t$*-progressive*[10] *process and satisfies* $\mathbf{S}$*-PMTC in* E_0. *In particular,* X' *has an* $(\mathbf{S}, \mathcal{D})$*-càdlàg progressive modification with paths in* $E_0^{\mathbf{R}^+}$[11].

(c) If $\mathbf{T}$ *is conull and* $\{X^{n_k}\}_{k\in\mathbf{N}}$ *is* $(\mathbf{T}, \mathcal{D})$*-AS, then* $X' = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$ *is an E-valued, stationary,* $(\mathbf{R}^+, \mathcal{D})$*-càdlàg process and satisfies* $\mathbf{R}^+$*-PMTC in* E_0. *In particular,* X' *has an* $(\mathbf{R}^+, \mathcal{D})$*-càdlàg progressive modification with paths in* $E_0^{\mathbf{R}^+}$.

Remark 7.6. If E_0 is a $\mathcal{D}$-baseable subset for a general $\mathcal{D} \subset C(E; \mathbf{R})$, then E_0 is $\mathcal{D}_0$-baseable for some countable $\mathcal{D}_0 \subset \mathcal{D}$ (see Fact 3.34 (c)) and $\mathcal{D}_0$-FMCC is a weaker assumption than $\mathcal{D}$-FMCC. Hence, it is no less general to make $\mathcal{D}$ a countable collection in the theorem above.

Remark 7.7. Any compact subset contained in a baseable set E_0 is metrizable by Corollary 3.52. So, the $\mathbf{m}$-tightness within $\mathbf{S}$-PSMTC in E_0 is reduced to ordinary tightness.

Remark 7.8. The proof of Theorem 7.5 relies on Theorem 6.20 in which the limit processs X was initiated as a collection of E_0-valued mappings $\{X_t\}_{t\geq 0}$. Of course, this is equivalent to initiating the limit process as an $E_0^{\mathbf{R}^+}$-valued mapping (like the X and X' in Theorem 7.5). Moreover, both Theorem 6.20 and Theorem 7.5 consider the limit processes as E-valued processes with paths in $E_0^{\mathbf{R}^+}$ for the desired finite-dimensional convergence.

We use the next lemma to establish progressiveness in Theorem 7.5 (b, c).

[9] The notion of stochastic basis was reviewed in §2.5.

[10] The notion of $\mathscr{G}_t$-progressive processes was specified in §2.5.

[11] Given $E_0 \subset E$, an E-valued process with paths in $E_0^{\mathbf{R}^+}$ is equivalent to an $(E_0, \mathscr{O}_E(E_0))$-valued process.

Lemma 7.9. *Let E be a topological space, $x_0 \in E_0 \subset E$, $\mathbf{T} \subset \mathbf{R}^+$ and $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ be an E-valued process satisfying*

$$\inf_{t \in \mathbf{T}} \mathbb{P}(Y_t \in E_0) = 1. \tag{7.2.1}$$

Then, the mapping

$$X \mathrel{\mathring{=}} \bigotimes_{t \in \mathbf{R}^+} \mathfrak{var}\left(Y_t; \Omega, Y_t^{-1}(E_0), x_0\right) \in \left(E_0^{\mathbf{R}^+}\right)^{\Omega} \tag{7.2.2}$$

satisfies the following statements:

(a) $X_t \mathrel{\mathring{=}} \mathfrak{p}_t \circ X \in M(\Omega, \mathscr{F}; E_0, \mathscr{B}_E(E_0))$ *for all* $t \in \mathbf{T}$ *and (6.2.9) holds.*

(b) If $\mathbf{T} = \mathbf{R}^+$*, then* X *is an* E*-valued process with paths in* $E_0^{\mathbf{R}^+}$*, satisfies (6.2.26) and is a modification of* Y*.*

(c) If $E_0 \in \mathscr{B}(E)$*, then* X *is an* $(E_0, \mathscr{O}_E(E_0))$*-valued* $\mathscr{F}_t^Y$*-adapted process. If, in addition,* Y *is a measurable or progressive process, then* X *is measurable or* $\mathscr{F}_t^Y$*-progressive respectively.*

(d) If $E_0 \in \mathscr{B}(E) \cap \mathscr{B}^{\mathrm{s}}(E)$*, and if* Y *is a measurable process, then* X *is an* $(E_0, \mathscr{O}_E(E_0))$*-valued, measurable,* $\mathscr{F}_t^Y$*-adapted process and admits an* $\mathscr{F}_t^Y$*-progressive modification.*

Proof. (a) follows by Lemma B.31 (b, c) (with $\mathscr{U} = \mathscr{B}(E)$, $S = S_0 = E_0$, $\mathscr{U}' = \mathscr{B}_E(E_0)$, $X = Y_t$, $Y = X_t$ and $y_0 = x_0$).

(b) follows by (a) (with $\mathbf{T} = \mathbf{R}^+$) and Fact 2.24 (b) (with $E = (E_0, \mathscr{O}_E(E_0))$).

(c) Let φ denote the identity mapping on E. We find that

$$\varphi' \mathrel{\mathring{=}} \mathfrak{var}(\varphi; E, E_0, x_0) \in M\left(E; E_0, \mathscr{O}_E(E_0)\right) \tag{7.2.3}$$

by $E_0 \in \mathscr{B}(E)$ and Fact B.3 (b) (with $(S, \mathscr{A}) = (E, \mathscr{B}(E))$, $(E, \mathscr{U}) = (E_0, \mathscr{B}_E(E_0))$, $A = E_0$, $f = \varphi$ and $y_0 = x_0$). Then, (c) follows by (7.2.3), the fact $X = \varpi(\varphi') \circ Y$ and Fact B.32 (a) (with $S = (E_0, \mathscr{O}_E(E_0))$, $f = \varphi'$, $X = Y$ and $\mathscr{G}_t = \mathscr{F}_t^Y$).

(d) X is an $(E_0, \mathscr{O}_E(E_0))$-valued, measurable, $\mathscr{F}_t^Y$-adapted process by (c). Let φ_0 denote the identity mapping on E_0. By $E_0 \in \mathscr{B}^{\mathrm{s}}(E)$ and Proposition A.52 (a, d), there exists a topology $\mathscr{U}$ on E_0 such that $(E_0, \mathscr{U})$ is a

Polish space and $\varphi_0 \in \mathbf{biso}(E_0, \mathscr{O}_E(E_0); E_0, \mathscr{U})$. Then, $(E_0, \mathscr{O}_E(E_0))$-valued measurable (resp. $\mathscr{F}_t^Y$-progressive) processes are equivalent to $(E_0, \mathscr{U})$-valued measurable (resp. $\mathscr{F}_t^Y$-progressive) processes by Fact B.32 (a) (with E (or S) being $(E_0, \mathscr{O}_E(E_0))$, S (or E) being $(E_0, \mathscr{U})$, $f = \varphi_0$ and $\mathscr{G}_t = \mathscr{F}_t^Y$). We owe to [Ondrejat and Seidler, 2013, Theorem 0.1] the proof of that every Polish-space-valued, measurable, $\mathscr{F}_t^Y$-adapted process (like X) admits an $\mathscr{F}_t^Y$-progressive modification. Thus (d) follows immediately. □

Corollary 7.10. *Let E be a topological space, $E_0 \in \mathscr{B}(E) \cap \mathscr{B}^{\mathbf{s}}(E)$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process satisfying (6.3.17). Then, X has a progressive modification with paths in $E_0^{\mathbf{R}^+}$.*

Proof. This corollary follows by Lemma 7.9 (b, d) (with $Y = X$ and $X = Y$) and Proposition B.33 (e). □

Corollary 7.11. *Let E be a topological space, $\mathcal{D} \subset C_b(E; \mathbf{R})$, E_0 be a $\mathcal{D}$-baseable standard Borel subset of E and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued $(\mathbf{R}^+, \mathcal{D})$-càdlàg process satisfying (6.3.17). Then, X has a progressive modification with paths in $E_0^{\mathbf{R}^+}$.*

Proof. There exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E with $\mathcal{F} \subset (\mathcal{D} \cup \{1\})$ by Lemma 3.39 (c). $\widehat{X} = \mathfrak{rep}_c(X; E_0, \mathcal{F})$ exists by the fact $(\mathcal{F} \backslash \{1\}) \subset \mathcal{D}$ and Proposition 6.28 (a). It follows by Fact 6.24 and Proposition 6.7 (a) (with $\mathbf{T} = \mathbf{R}^+$) that X and $\widehat{X}$ satisfy (6.3.18) and $\widehat{X}$ is a progressive process. $\mathscr{F}^X = \mathscr{F}^{\widehat{X}}$[12] by Lemma B.77 (e) (with $A = E_0$ and $Y = \widehat{X}$), so $\widehat{X}$ is $\mathscr{F}_t^X$-progressive. Furthermore, we have

$$\mathscr{B}_E(E_0) = \mathscr{B}_{\widehat{E}}(E_0) \subset \mathscr{B}(\widehat{E}) \tag{7.2.4}$$

by Lemma 3.14 (a) (with $d = 1$ and $A = E_0$). $\widehat{X}$ has an $\mathscr{F}_t^X$-progressive modification Z with paths in $E_0^{\mathbf{R}^+}$ by (6.3.18), (7.2.4) and Lemma 7.9 (b, c) (with $E = \widehat{E}$ and $Y = \widehat{X}$). Z is an $(E_0, \mathscr{O}_E(E_0))$-valued process by (7.2.4). Z is a modification of X by (6.3.18). So, Z is progressive by Proposition B.33 (e). □

Remark 7.12. A special case of Corollary 7.10 and Corollary 7.11 is when $E = E_0$ is a $\mathcal{D}$-baseable standard Borel space and (6.3.17) becomes automatic.

[12] The notation "$\mathscr{F}^X$" as defined in §2.5 means the augmented natural filtration of X.

The following proposition shows that the condition (v) of Theorem 7.5 is realizable.

Proposition 7.13. *Let E be a topological space, $\mathcal{D} \subset C_b(E;\mathbf{R})$ be countable, E_0 be a $\mathcal{D}$-baseable subset of E and $\mathbf{I}$ be an infinite index set. If E-valued processes $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ satisfy (6.4.7) and $\mathcal{D}$-FMCC, then there exists a countable $\mathbf{J} \subset (0,\infty)$ such that $\mathfrak{flp}_{\mathbf{R}^+\backslash\mathbf{J}}(\{\varpi(\mathcal{D}) \circ X^i\}_{i\in\mathbf{I}})$ has at least one càdlàg member.*

Proof. There exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E with $\mathcal{F} = \mathcal{D} \cup \{1\}$ by Lemma 3.39 (b), so $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{F}$-FMCC. It follows by Proposition 6.45 (a) and Note 6.5 that $\{\widehat{X}^i = \mathfrak{rep}_{\mathrm{c}}(X^i; E_0, \mathcal{F})\}_{i\in\mathbf{I}}$ is tight in the Polish space $D(\mathbf{R}^+; \widehat{E})$. $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ admits at least one weak limit point Y on $D(\mathbf{R}^+; \widehat{E})$ by the Prokhorov's Theorem (Theorem 2.22 (b)). $\mathbf{J} \mathrel{\hat{=}} J(Y) \subset (0,\infty)$ is countable by Note 6.50. Now, the result follows by Proposition 6.49 (a) and Lemma B.81. □

We now prove the main theorem of this section.

Proof of Theorem 7.5. The proof is divided into six steps.

Step 1: Establish a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ and càdlàg replicas $\{\widehat{X}^n\}_{n\in\mathbf{N}}$. There exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E with $\mathcal{F} = \mathcal{D}\cup\{1\}$ by the condition (ii) above and Lemma 3.39 (b). (6.4.34) holds by the condition (iii) above and Fact 6.52 (with $\mathbf{I} = \mathbf{N}$). $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{F}$-FMCC by the condition (iv) above and the fact $\mathcal{F}\backslash\{1\} = \mathcal{D}$, so they are $(\mathbf{R}^+, \mathcal{F})$-càdlàg. It then follows by Proposition 6.45 (a) (with $\mathbf{I} = \mathbf{N}$), Proposition 6.28 (a) (with $X = X^n$) and Note 6.5 that $\{\widehat{X}^n = \mathfrak{rep}_{\mathrm{c}}(X^n; E_0, \mathcal{F})\}_{n\in\mathbf{N}}$ is tight in the Polish space $D(\mathbf{R}^+; \widehat{E})$ and satisfies

$$\inf_{n\in\mathbf{N}} \mathbb{P}^n \left(\varphi \circ X_t^n = \widehat{\varphi} \circ \widehat{X}_t^n\right) = 1 \tag{7.2.5}$$

with $\widehat{\varphi} \mathrel{\hat{=}} \bigotimes \widehat{\mathcal{F}}\backslash\{1\}$.

Step 2: Establish $\{n_k\}_{k\in\mathbf{N}}$ and a $D(\mathbf{R}^+; \widehat{E})$-valued random variable Y such that

$$\widehat{X}^{n_k} \xrightarrow{\mathrm{D}(\mathbf{S})} Y \text{ as } k \uparrow \infty. \tag{7.2.6}$$

By the condition (v) above, the tightness of $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ in $D(\mathbf{R}^+; \widehat{E})$ and Prokhorov's Theorem (Theorem 2.22 (b)), there exist $\{n_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$, a $D(\mathbf{R}^+; \widehat{E})$-valued random variable Y (see Remark 7.14 below for an explicit construction) and an $\mathbf{R}^{\mathcal{D}}$-valued càdlàg process Z such that

$$\widehat{X}^{n_k} \Longrightarrow Y \text{ as } k \uparrow \infty \text{ on } D(\mathbf{R}^+; \widehat{E}) \tag{7.2.7}$$

and

$$\varpi(\varphi) \circ X^{n_k} \xrightarrow{\mathrm{D}(\mathbf{S})} Z \text{ as } k \uparrow \infty. \tag{7.2.8}$$

Without loss of generality, we suppose Y and Z are both defined on $(\Omega, \mathscr{F}, \mathbb{P})$.

Since $\mathcal{D}$ is countable, $\mathbf{R}^{\mathcal{D}}$ and $D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}})$ are Polish spaces as mentioned in Note 6.5. So, Z can be considered as a $D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}})$-valued random variable by Fact A.77 (b) (with $E = \mathbf{R}^{\mathcal{D}}$). $\widehat{\mathcal{F}}\backslash\{1\}$ separates points on $\widehat{E}$ by Lemma 3.3 (a). $\widehat{\mathcal{F}}\backslash\{1\}$ strongly separates points on $\widehat{E}$ by Lemma 3.3 (a) and Lemma A.20 (with $E = \widehat{E}$ and $\mathcal{D} = \widehat{\mathcal{F}}\backslash\{1\}$). So,

$$\widehat{\varphi} \in \mathbf{imb}(\widehat{E}; \mathbf{R}^{\mathcal{D}}) \tag{7.2.9}$$

by the fact $\mathcal{D} = \mathcal{F}\backslash\{1\}$ and Lemma B.7 (b) (with $E = \widehat{E}$, $S = \mathbf{R}$ and $\mathcal{D} = \widehat{\mathcal{F}}\backslash\{1\}$).

$$\varpi(\widehat{\varphi}) \in C\left(D(\mathbf{R}^+; \widehat{E}); D(\mathbf{R}^+; \mathbf{R})\right) \tag{7.2.10}$$

by Proposition A.62 (d) (with $S = \widehat{E}$, $E = \mathbf{R}^{\mathcal{D}}$ and $f = \widehat{\varphi}$).

It follows by (7.2.7), (7.2.10) and Continuous Mapping Theorem (Theorem B.25 (a)) that

$$\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k} \Longrightarrow \varpi(\widehat{\varphi}) \circ Y \text{ as } k \uparrow \infty \text{ on } D\left(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}\right). \tag{7.2.11}$$

It follows by (7.2.5) and (7.2.8) that

$$\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k} \xrightarrow{\mathrm{D}(\mathbf{S})} Z \text{ as } k \uparrow \infty. \tag{7.2.12}$$

$\{\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k}\}_{k\in\mathbf{N}}$ is tight in $D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}})$ by (7.2.10) and Fact B.60 (with $E = D(\mathbf{R}^+; \widehat{E})$, $S = D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}$ and $f = \varpi(\widehat{\varphi})$). Hence, (7.2.15) implies

$$\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k} \Longrightarrow Z \text{ as } k \uparrow \infty \text{ on } D\left(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}\right) \tag{7.2.13}$$

by the Prokhorov's Theorem (Theorem 2.22 (b)), the denseness of $\mathbf{S}$ and Theorem A.87 (b) (with $E = \mathbf{R}^{\mathcal{D}}$, $X^n = \varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k}$, $X = Z$ and $\mathbf{T} = \mathbf{S}$). $\mathcal{P}(D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}}))$ is a Polish space by Theorem A.44 (b) (with $E = D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}})$), so

$$\mathbb{P} \circ Z^{-1} = \mathbb{P} \circ (\varpi(\widehat{\varphi}) \circ Y)^{-1} \in \mathcal{P}\left(D(\mathbf{R}^+; \mathbf{R}^{\mathcal{D}})\right) \tag{7.2.14}$$

by (7.2.11) and (7.2.13). Y and Z as $\mathbf{R}^{\mathcal{D}}$-valued processes have the same finite-dimensional distributions by (7.2.14) and Fact A.76 (b) (with $E = \mathbf{R}^{\mathcal{D}}$), so (7.2.15) implies

$$\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k} \xrightarrow{\mathrm{D}(\mathbf{S})} \varpi(\widehat{\varphi}) \circ Y \text{ as } k \uparrow \infty. \tag{7.2.15}$$

We fix $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$ and put $d = \aleph(\mathbf{T}_0)$. $\{\widehat{X}^{n_k}\}_{k \in \mathbf{N}}$, $\{\varpi(\widehat{\varphi}) \circ \widehat{X}^{n_k}\}_{k \in \mathbf{N}}$, Y and $\varpi(\widehat{\varphi}) \circ Y$ all have Borel finite-dimensional distributions as mentioned in Note 6.5, so (7.2.15) implies

$$\left(\bigotimes_{t \in \mathbf{T}_0} \widehat{\varphi} \circ \mathfrak{p}_t\right) \circ \widehat{X}^{n_k} \Longrightarrow \left(\bigotimes_{t \in \mathbf{T}_0} \widehat{\varphi} \circ \mathfrak{p}_t\right) \circ Y \text{ as } k \uparrow \infty \text{ on } \mathbf{R}^d. \tag{7.2.16}$$

One finds that

$$\Psi \mathrel{\mathring{=}} \bigotimes_{t \in \mathbf{T}_0} \widehat{\varphi}^{-1} \circ \mathfrak{p}_t \in C\left[\widehat{\varphi}(\widehat{E})^d, \mathscr{O}_{\mathbf{R}^d}\left(\widehat{\varphi}(\widehat{E})^d\right); \widehat{E}^d\right] \tag{7.2.17}$$

by (7.2.9) and Fact 2.4 (a, b). Hence, it follows by (7.2.17), (7.2.16) and Continuous Mapping Theorem (Theorem B.25 (a)) that

$$\begin{aligned} \widehat{X}^{n_k}_{\mathbf{T}_0} &= \Psi \circ \left(\bigotimes_{t \in \mathbf{T}_0} \widehat{\varphi} \circ \mathfrak{p}_t\right) \circ \widehat{X}^{n_k} \\ &\Longrightarrow \Psi \circ \left(\bigotimes_{t \in \mathbf{T}_0} \widehat{\varphi} \circ \mathfrak{p}_t\right) \circ Y = Y \text{ as } k \uparrow \infty \text{ on } \widehat{E}^{\mathbf{T}_0}. \end{aligned} \tag{7.2.18}$$

Step 3: Construct $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t \geq 0}, \mathbb{P})$*, X and X'.* We fix an arbitrary $x_0 \in E_0$, set $\{\mathscr{G}_t\}_{t \geq 0} \mathrel{\mathring{=}} \mathscr{F}^Y$, define X by (6.2.4) with $\mathbf{T} = \mathbf{S}$ and define X' by (7.2.2) with X replaced by X'.

Step 4: Verify (a). It follows by the conditions (i, iv) above, (7.2.6), Lemma B.78 (c, e) (with $n = n_k$) and Theorem 6.20 (a, c) (with $n = n_k$ and $\mathbf{T} = \mathbf{S}$)

that the X defined in Step 3 is an E-valued process with paths in $E_0^{\mathbf{R}^+}$ and satisfies: (1) $\mathbf{S}$-PMTC in E_0, (2)

$$\inf_{t\in\mathbf{S}} \mathbb{P}\left(X_t = Y_t \in E_0\right) = 1, \tag{7.2.19}$$

and (3) $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$. Hence, (a) follows by the fact $\mathcal{D} \subset \mathcal{F}$, (7.2.19) and Lemma B.77 (b) (with $\mathbf{T} = \mathbf{S}$).

Step 5: Verify (b). Y is càdlàg hence $\mathscr{G}_t$-progressive by Proposition B.33 (a). Given $E_0 \in \mathscr{B}^{\mathbf{s}}(E)$, (7.2.4) holds by Lemma 3.14 (a) (with $d = 1$ and $A = E_0$). Hence, the X' defined in Step 3 is an $(E_0, \mathscr{O}_E(E_0))$-valued $\mathscr{G}_t$-progressive process satisfying

$$\inf_{t\in\mathbf{S}} \mathbb{P}\left(X_t = Y_t = X'_t \in E_0\right) = 1 \tag{7.2.20}$$

by (7.2.19), (7.2.4) and Lemma 7.9 (a, c) (with $E = \widehat{E}$ and $\mathbf{T} = \mathbf{S}$ and $X = X'$). Then, $X' = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$ by (a) and (7.2.20). X' is $(\mathbf{S}, \mathcal{D})$-càdlàg by the fact $\mathcal{D} \subset \mathcal{F}$, (7.2.20) and Lemma B.77 (b) (with $\mathbf{T} = \mathbf{S}$ and $X = X'$). X is a measurable process by Proposition B.33 (c). Now, (b) follows by the fact $E_0 \in \mathscr{B}(E) \cap \mathscr{B}^{\mathbf{s}}(E)$, Corollary 7.10 (with $X = X'$) and Note 6.26.

Step 6: Verify (c). $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{F}\backslash\{1\})$-AS since $\mathcal{D} = \mathcal{F}\backslash\{1\}$. Then, Y is a stationary process by (7.2.7) and Proposition 6.49 (c) (with $n = n_k$). We know from (a) that X satisfies $\mathbf{S}$-PMTC in E_0 and (7.2.19). So, there exists an $A \in \mathscr{K}_\sigma^{\mathbf{m}}(E_0, \mathscr{O}_E(E_0))$ such that

$$\inf_{t\in\mathbf{R}^+} \mathbb{P}\left(Y_t \in A \subset E_0\right) = 1. \tag{7.2.21}$$

A is a $\mathcal{D}$-baseable standard Borel subset of E and satisfies

$$\mathscr{B}_E(A) = \mathscr{B}_{\widehat{E}}(A) \subset \mathscr{B}(\widehat{E}) \tag{7.2.22}$$

by Corollary 3.15 (b) (with $d = 1$), Lemma 3.14 (a) (with $d = 1$), the fact $\mathcal{F}\backslash\{1\} = \mathcal{D}$ and Fact 3.34 (a, b). Hence, the X' defined in Step 3 is an $(E_0, \mathscr{O}_{\widehat{E}}(E_0))$-valued process satisfying both (7.2.20) and

$$\inf_{t\in\mathbf{R}^+} \mathbb{P}\left(X'_t = Y_t \in A \subset E_0\right) = 1 \tag{7.2.23}$$

by (7.2.21), (7.2.22), Lemma 7.9 (b) (with $E = \widehat{E}$, $E_0 = A$ and $X = X'$) and (7.2.19). X' is an $(E_0, \mathscr{O}_E(E_0))$-valued process and satisfies $\mathbf{R}^+$-PMTC in E_0 by (7.2.23), (7.2.22), Lemma B.31 (b, c) (with $E = \widehat{E}$, $\mathscr{U} = \mathscr{B}(\widehat{E})$, $S_0 = A$, $S = E_0$, $\mathscr{U}' = \mathscr{B}_E(E_0)$, $X = Y_t$ and $Y = X'_t$) and Fact 2.24 (b). Thus, $X' = \mathfrak{fl}_{\mathbf{S}}(\{X^{n_k}\}_{k\in\mathbf{N}})$ by (a) and (7.2.20). X' is stationary by (7.2.23) and Lemma B.77 (e) (with $X = X'$). X' is $(\mathbf{R}^+, \mathcal{D})$-càdlàg by the fact $\mathcal{D} \subset \mathcal{F}$, (7.2.23) and Lemma B.77 (b) (with $\mathbf{T} = \mathbf{R}^+$ and $X = X'$). Finally, (c) follows by Corollary 7.11 (with $E_0 = A$ and $X = X'$) and Note 6.26. □

Remark 7.14. Let $\{\widehat{X}^{n_k}\}_{k\in\mathbf{N}}$ be as in the proof of Theorem 7.5. One can realize (7.2.7) by letting $\Omega = D(\mathbf{R}^+; \widehat{E})$, $\mathbb{P}$ be the weak limit of the distributions of $\{\widehat{X}^{n_k}\}_{k\in\mathbf{N}}$ in $\mathcal{P}(D(\mathbf{R}^+; \widehat{E}))$, $\mathscr{F}$ be the completion[13] of $\mathscr{B}(\Omega)$ with respect to $\mathbb{P}$ and Y be the identity mapping on $D(\mathbf{R}^+; \widehat{E})$. This process $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ is often called the coordinate process or canonical process on $D(\mathbf{R}^+; \widehat{E})$.

With the help of Lemma 7.9, we give a slightly weaker form of Theorem 7.1 (b) that can be used to show uniqueness in the settings of Theorem 7.5.

Proposition 7.15. *Let E be a topological space, $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes and $\mathbf{S} \subset \mathbf{T} \subset \mathbf{R}^+$. Suppose that:*

(i) $C_b(E; \mathbf{R})$ *separates points on* E.

(ii) $\mathcal{D} \subset C_b(E; \mathbf{R})$ *separates points on* $E_0 \in \mathscr{B}(E)$.

(iii) *(6.5.1) holds.*

(iv) $\{X^i\}_{i\in\mathbf{I}}$ *is* $(\mathbf{S}, \mathcal{D})$*-FDC and satisfies* $\mathbf{S}$*-PSMTC in* E_0.

Then, there exists an $X = \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ with paths in $E_0^{\mathbf{R}^+}$ and satisfying $\mathbf{S}$-PMTC in E_0. Moreover, $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{i_n}\}_{n\in\mathbf{N}})$ for any $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$.

Proof. We let $\widetilde{f} \doteq f|_{E_0^d}$[14] for each $f \in C_b(E^d; \mathbf{R})$ and $d \in \mathbf{N}$, put $\widetilde{\mathcal{D}} \doteq \mathcal{D}|_{E_0} = \{\widetilde{f} : f \in \mathcal{D}\}$, fix $x_0 \in E_0$ and define $\{Z_t^i\}_{t\geq 0} \subset E_0^{\Omega_i}$ for each $i \in \mathbf{I}$ by (6.2.4) with X_t, Y_t, Ω replaced by Z_t^i, X_t^i, Ω^i respectively.

[13]Completion of measure space was specified in §2.1.2.
[14]Similar notations were used in Notation 4.11.

It follows by (6.5.1) and Lemma 7.9 (a) (with $(\Omega, \mathscr{F}, \mathbb{P}; Y) = (\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)$ and $X_t = Z_t^i$) that

$$Z_t^i \in M\left(\Omega^i, \mathscr{F}^i; E_0, \mathscr{O}_E(E_0)\right), \ \forall t \in \mathbf{T}, i \in \mathbf{I} \tag{7.2.24}$$

and

$$\inf_{t \in \mathbf{T}, i \in \mathbf{I}} \mathbb{P}^i(X_t^i = Z_t^i \in E_0) = 1. \tag{7.2.25}$$

E is a Hausdorff space by Proposition A.17 (e) (with $A = E$ and $\mathcal{D} = C_b(E; \mathbf{R})$). So, $(E_0, \mathscr{O}_E(E_0))$ is a Hausdorff subspace and $\{x_0\} \in \mathscr{B}(E)$ by Proposition A.2 (a, c) and the fact $E_0 \in \mathscr{B}(E)$. This immediately implies

$$Z_t^i \in M\left(\Omega^i, \mathscr{F}^i; E_0, \mathscr{O}_E(E_0)\right), \ \forall t \in \mathbf{R}^+ \backslash \mathbf{T}, i \in \mathbf{I}. \tag{7.2.26}$$

Hence, $Z^i \triangleq \{Z_t^i\}_{t \geq 0}$ is an $(E_0, \mathscr{O}_E(E_0))$-valued process for all $i \in \mathbf{I}$ by Fact 2.24 (b) (with $E = (E_0, \mathscr{O}_E(E_0))$).

$\{Z^i\}_{i \in \mathbf{I}}$ satisfies $\mathbf{S}$-PSMTC by (7.2.25) and the condition (iv) above. At the same time, we observe by (7.2.25) that

$$\mathbb{E}^i\left[\widetilde{f} \circ Z^i_{\mathbf{T}_0}\right] = \mathbb{E}^i\left[f \circ X^i_{\mathbf{T}_0}\right] \tag{7.2.27}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E; \mathbf{R}))]$, $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ and $i \in \mathbf{I}$, so $\{Z^i\}_{i \in \mathbf{I}}$ is $(\mathbf{S}, \widetilde{\mathcal{D}})$-FDC.

Now, we apply Theorem 7.1 (b) (with $E = (E, \mathscr{O}_E(E_0))$, $\mathcal{D} = \widetilde{\mathcal{D}}$, $X^i = Z^i$ and $X = Z$) and obtain an $(E_0, \mathscr{O}_E(E_0))$-valued process $(\Omega, \mathscr{F}, \mathbb{P}; Z)$ satisfying: (1) $\mathbf{S}$-PMTC (in E_0), and (2) $Z = \mathfrak{fl}_{\mathbf{S}}(\{Z^{i_n}\}_{n \in \mathbf{N}})$ for any $\{i_n\}_{n \in \mathbf{N}} \subset \mathbf{I}$.

Considering Z as an E-valued process with paths in $E_0^{\mathbf{R}^+}$, it follows by Z's property (2) above, Fact 6.19 (with $X^n = Z^{i_n}$ and $X = Z$) and (7.2.27) that $\mathbb{E}[f \circ Z_{\mathbf{T}_0}]$ is the unique limit point of $\{\mathbb{E}^i[f \circ X_{\mathbf{T}_0}]\}_{i \in \mathbf{I}}$ for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E; \mathbf{R}))]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$. Now, the result follows by Z's property (1) and Corollary 7.3 (a) (with $X = Z$ and $\mathcal{D} = C_b(E; \mathbf{R})$). □

Remark 7.16. The tiny difference between Theorem 7.1 (b) and Proposition 7.15 lies in that Proposition 7.15 does not require $\mathcal{D}$ to separate points on the entire space E as in Theorem 7.1 (b).

Corollary 7.17. *Let E be a topological space, $\mathbf{S} \subset \mathbf{T} \subset \mathbf{R}^+$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes. Suppose that:*

(i) $C_b(E;\mathbf{R})$ *separates points on E.*

(ii) $\mathcal{D} \subset C_b(E;\mathbf{R})$ *separates points on $E_0 \in \mathscr{B}(E)$.*

(iii) *(6.5.1) holds.*

(iv) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathbf{S}$*-PSMTC in E_0.*

(v) *X satisfies* $\mathbf{S}$*-PMTC.*

(vi) $\mathbb{E}[f \circ X_{\mathbf{T}_0}]$ *is the unique limit point of* $\{\mathbb{E}^i[f \circ X^i_{\mathbf{T}_0}]\}_{i\in\mathbf{I}}$ *in* $\mathbf{R}$ *for all* $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$.

Then, $X = \mathfrak{flp}_{\mathbf{S}}(\{X^i\}_{i\in\mathbf{I}})$ *and* $X = \mathfrak{fl}_{\mathbf{S}}(\{X^{i_n}\}_{n\in\mathbf{N}})$ *for any* $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$.

Proof. This corollary follows immediately by Proposition 7.15 and a similar argument to the proof of Corollary 7.3 (a). □

7.3 Stationary long-time typical behavior

We now come to **Q1** in Introduction that motivates our interest in finite-dimensional convergence of stochastic processes. In order to utilize our results in §7.1 and §7.2, we introduce the randomly advanced processes of a given measurable process X whose finite-dimensional distributions are the long-time-averaged distributions in (1.5).

Definition 7.18. Let E be a topological space and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process.

- For each $T \in (0,\infty)$, by $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{P}^T; X^T) = \mathfrak{rap}_T(X)$[15] ($X^T = \mathfrak{rap}_T(X)$ for short) we denote that $\widetilde{\Omega} \doteq \mathbf{R}^+ \times \Omega$, $\widetilde{\mathscr{F}} \doteq \mathscr{B}(\mathbf{R}^+) \otimes \mathscr{F}$,

$$\mathbb{P}^T(A) \doteq \frac{1}{T}\int_0^T \int_\Omega \mathbf{1}_A(\tau,\omega)\mathbb{P}(d\omega)d\tau,\ \forall A \in \widetilde{\mathscr{F}}, \tag{7.3.1}$$

[15] "$\mathfrak{rap}$" is "rap" in fraktur font which stands for randomly advanced process.

and

$$X^T(\tau,\omega)(t) \doteq X_{\tau+t}(\omega),\ \forall t \in \mathbf{R}^+, (\tau,\omega) \in \widetilde{\Omega}. \tag{7.3.2}$$

$X^T \in (E^{\mathbf{R}^+})^{\widetilde{\Omega}}$ defined by (7.3.2) is called the **T-randomly advanced process of X**.

- A **long-time typical behavior of X along $\mathbf{T}$** refers to a member of $\mathfrak{flp}_{\mathbf{T}}(\{X^{T_k}\}_{k\in\mathbf{N}})$ with $T_k \uparrow \infty$, $X^{T_k} = \mathfrak{rap}_{T_k}(X)$ for each $k \in \mathbf{N}$ and $\mathbf{R}^+\backslash\mathbf{T}$ being a countable subset of $(0,\infty)$[16].

Remark 7.19. As its name implies, the T-randomly advanced process of X is defined by advancing X to start at a random time $(\tau,\omega) \mapsto \tau$ defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$.

Below is a justification of our definition of randomly advanced process.

Proposition 7.20. *Let E be a topological space, $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued measurable process and $T \in (0,\infty)$. Then, $(\widetilde{\Omega},\widetilde{\mathscr{F}},\mathbb{P}^T;X^T) = \mathfrak{rap}_T(X)$ is an E-valued measurable process.*

Proof. (7.3.1) well defines $\mathbb{P}^T \in \mathfrak{P}(\widetilde{\Omega},\widetilde{\mathscr{F}})$ by Fubini's Theorem. Let $\xi(t,\omega) \doteq X_t(\omega)$, $\xi^T(t,(\tau,\omega)) \doteq X_{\tau+t}(\omega)$ and $\varphi(t,(\tau,\omega)) \doteq (\tau+t,\omega)$ for each $t \in \mathbf{R}^+$ and $(\tau,\omega) \in \widetilde{\Omega}$. It is well-known that

$$\varphi \in M\left(\mathbf{R}^+ \times \widetilde{\Omega}, \mathscr{B}(\mathbf{R}^+) \otimes \widetilde{\mathscr{F}}; \widetilde{\Omega}, \widetilde{\mathscr{F}}\right). \tag{7.3.3}$$

X being a measurable process implies $\xi \in M(\widetilde{\Omega},\widetilde{\mathscr{F}};E)$ and

$$\xi^T = \xi \circ \varphi \in M\left(\mathbf{R}^+ \times \widetilde{\Omega}, \mathscr{B}(\mathbf{R}^+) \otimes \widetilde{\mathscr{F}}; E\right), \tag{7.3.4}$$

thus proving X^T is a measurable process. □

We present several further properties of randomly advanced process in §B.1 of Appendix B. Now, we give our answer to **Q1**.

Theorem 7.21. *Let E be a topological space, $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued measurable process satisfying T_k-LMTC in $A \subset E$[17] and $\mathcal{D} \subset C_b(E;\mathbf{R})$ separate points on E. Then, the following statements are true:*

[16] This means $\mathbf{T}$ is cocountable.

[17] The terminology "X satisfying T_k-LMTC in A" was specified in Definition 6.36 and Note 6.37.

(a) If $\{\frac{1}{T_k}\int_0^{T_k}\mathbb{E}[f\circ X_{\mathbf{T}_0+\tau}]\}_{k\in\mathbf{N}}$ *is convergent in* $\mathbf{R}$ *for all* $f\in\mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ *and* $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$ *with* $0\in\mathbf{T}_0$*, then* X *has a stationary long-time typical behavior along* $\mathbf{R}^+$*.*

(b) If $\mathcal{D}$ *is countable and* $\{X^{T_k}\}_{k\in\mathbf{N}}$ *satisfies* $\mathcal{D}$*-FMCC, then there exist a cocountable* $\mathbf{S}\subset\mathbf{R}^+$*, an* $E_0\in\mathscr{K}_\sigma^{\mathbf{m}}(E)$ *such that* $\{\frac{1}{T_k}\int_0^{T_k}\mathbb{P}\circ X_\tau^{-1}d\tau\}_{k\in\mathbf{N}}$ *is* $\mathbf{m}$*-tight in* $E_0\subset A$*, and a stationary long-time typical behavior of* X *along* $\mathbf{S}$ *which is an* E*-valued,* $(\mathbf{R}^+,\mathcal{D})$*-càdlàg, progressive process with paths in* $E_0^{\mathbf{R}^+}$*.*

Proof. (a) Let $X^{T_k}=\mathfrak{rap}_{T_k}(X)$ for each $k\in\mathbf{N}$. By Proposition 6.57 (with $\{X^n\}_{n\in\mathbf{N}}=\{X\}$), there exists a $\mathcal{D}$-baseable subset $E_0\subset\mathscr{K}_\sigma^{\mathbf{m}}(E)$ such that $E_0\subset A$ and (6.1.9) holds for some conull $\mathbf{T}\subset\mathbf{R}^+$ and $\{X_0^{T_k}\}_{k\in\mathbf{N}}$ is $\mathbf{m}$-tight in E_0. By Lemma B.42 (b, c, d) (with $A=E_0$), $\{X^{T_k}\}_{k\in\mathbf{N}}$ satisfies $\mathbf{R}^+$-PSMTC in E_0, is $(\mathbf{R}^+,\mathcal{D})$-AS and is $(\mathbf{R}^+,\mathcal{D})$-FDC. Now, (a) follows by Theorem 7.1 (c) (with $\mathbf{I}=\{T_k\}_{k\in\mathbf{I}}$).

(b) Let E_0 be as above. As (6.1.9) holds for the conull set $\mathbf{T}$, we have that

$$\inf_{t\in\mathbf{R}^+,k\in\mathbf{N}}\mathbb{P}^{T_k}\left(X_t^{T_k}\in E_0\right)=1. \tag{7.3.5}$$

by Lemma B.80 (a) (with $T=T_k$). Now, (b) follows by (7.3.5), Proposition 7.13 (with $i=T_k$) and Theorem 7.5 (c) (with $n=T_k$, $\mathbf{T}=\mathbf{R}^+$ and $\mathbf{S}=\mathbf{R}^+\backslash\mathbf{J}$). □

Chapter 8

Application to Weak Convergence on Path Space

The current chapter addresses the target problems **Q5** and **Q6** of **Theme 3** using the replication tools developed in §6.4. Throughout this chapter, we consider càdlàg processes taking values in a (at least) Tychonoff space E, whose common path space is the Skorokhod $\mathscr{J}_1$-space $D(\mathbf{R}^+; E)$. If necessary, the readers can look back at §2.2.2, §2.4 and §2.5 for our terminologies and notations about the Skorokhod $\mathscr{J}_1$-space and càdlàg process. Also, §A.6 of Appendix A together with §B.2 of Appendix B provide a short, almost self-contained review of Skorokhod $\mathscr{J}_1$-space.

The results of this chapter are sketched in Figure 5 below.

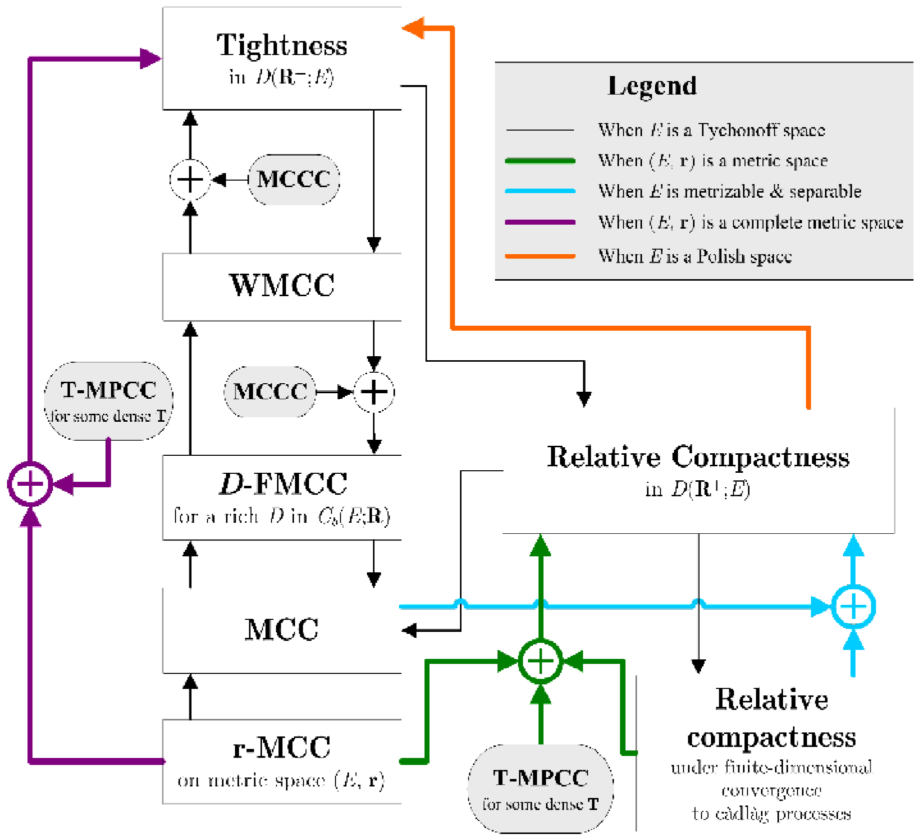

Figure 5: *Tightness and relative compactness in* $D(\mathbf{R}^+;E)$

Given MCCC, §8.1 establishes the equivalence among tightness in $D(\mathbf{R}^+;E)$ and the MCC-type conditions introduced in §6.4.2, which answers **Q5**. §8.2 looks into the relationship between weak convergnce on $D(\mathbf{R}^+;E)$ and finite-dimensional convergence. Based upon the developments of §8.2, §8.3 establishes several results connecting finite-dimensional convergence and relative compactness in $D(\mathbf{R}^+;E)$, which answers **Q6**.

Prior to the formal discussion, we recall several basic but essential facts for this chapter. Let E be a Tychonoff space, $\mu \in \mathcal{M}^+(D(\mathbf{R}^+;E))$, $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued càdlàg process. Then:

- $D(\mathbf{R}^+;E)$ is a Tychonoff space as mentioned in §3.3.6.
- $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$[1] lies in $\mathfrak{M}^+(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0})$ (see Corollary A.69).

[1]Herein, $\mathfrak{p}_{\mathbf{T}_0}$ denotes the projection on $E^{\mathbf{R}^+}$ for $\mathbf{T}_0 \subset \mathbf{R}^+$ restricted to $D(\mathbf{R}^+;E)$.

- The set $J(\mu)$[2] of fixed left-jump times of μ and the set $J(X)$[3] of fixed left-jump times of X are well-defined when a countable subset of $M(E;\mathbf{R})$ separates points on E. $\mathbf{R}^+\backslash J(\mu)$ and $\mathbf{R}^+\backslash J(X)$ are cocountable (hence non-empty and dense) subsets of $\mathbf{R}^+$ when E is baseable (see Proposition 3.64).

- $D(\mathbf{R}^+;E)$-valued random variables are E-valued càdlàg processes (see Fact A.76 (a)) but the converse is not necessarily true.

- When X is a $D(\mathbf{R}^+;E)$-valued random variable, $\mathbb{P}\circ X^{-1}$ is the restriction of $\mathrm{pd}(X)|_{D(\mathbf{R}^+;E)}$ to $\sigma(\mathscr{J}(E))$[4] and $(\mathbb{P}\circ X^{-1})\circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ is the finite-dimensional distribution of X for $\mathbf{T}_0$ as an E-valued process (see Fact A.76 (b, c)).

Hereafter, we may not always make special reference for the facts above.

8.1 Tightness

Our treatment of tightness in Skorokhod $\mathscr{J}_1$-space continues Kouritzin [2016] in the infinite time horizon setting. Tightness in $D(\mathbf{R}^+;E)$ is stronger than the Compact Containment Condition in Jakubowski [1986] and Ethier and Kurtz [1986] (MCCC if E has metrizable compact subsets).

Fact 8.1. *Let E be a Tychonoff space and $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ be a tight family of $D(\mathbf{R}^+;E)$-valued random variables. Then, $\{X^i\}_{i\in\mathbf{I}}$ satisfies the Compact Containment Condition in [Jakubowski, 1986, §4, (4.8)]. In particular, $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC when $\mathscr{K}(E)=\mathscr{K}^{\mathbf{m}}(E)$.*

Proof. This fact is immediate by Proposition 3.63. ☐

The following theorem is a version of [Kouritzin, 2016, Theorem 20] on infinite time horizon. This result plus Fact 8.1 answer **Q5** in Introduction.

Theorem 8.2. *Let E be a Tychonoff space and $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Consider the following statements:*

[2] $J(\mu)$ was defined in (2.3.10).

[3] $J(X)$ was defined in (2.5.8).

[4] $\mathscr{J}(E)$ denotes the Skorokhod $\mathscr{J}_1$-topology of $D(\mathbf{R}^+;E)$. Restriction of measure to sub-σ-algebra and X's process distribution $\mathrm{pd}(X)$ were specified in §2.1.2 and §2.5 respectively.

(a) X^i *is indistinguishable from some* $\widehat{X}^i \in M(\Omega^i, \mathscr{F}^i; D(\mathbf{R}^+; E))$ *for all* $i \in \mathbf{I}$*, and* $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ *is* $\mathbf{m}$*-tight in* $D(\mathbf{R}^+; E)$*.*

(b) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC and the closure of* $\mathcal{D} \subset C(E; \mathbf{R})$ *under the topology of compact convergence (see [Munkres, 2000, §46, Definition, p.283]) contains* $C_b(E; \mathbf{R})$*.*

(c) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies MCC.*

(d) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies WMCC.*

Then, (a) implies (b), (c) implies (d), and (a) - (d) are all equivalent when $\{X^i\}_{i\in\mathbf{I}}$ *satisfies MCCC.*

Note 8.3. Part (a) of the theorem above addresses a stronger statement than tightness of $\{X^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+; E)$. Moreover, tightness of the $D(\mathbf{R}^+; E)$-valued random variables $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ above is in the usual sense[5].

Remark 8.4. The condition in Theorem 8.2 (b) was used in [Kurtz, 1975, p.628-629] to show tightness in $D(\mathbf{R}^+; E)$ with E being a locally compact Polish space. For general Polish spaces, it appeared in [Ethier and Kurtz, 1986, §3.9].

Remark 8.5. When $(E, \mathfrak{r})$ is a metric space, the standard combination of $\mathfrak{r}$-MCC plus MCCC was used as a sufficient condition for relative compactness in $D(\mathbf{R}^+; E)$ by [Ethier and Kurtz, 1986, §3.7, Theorem 7.6]. Its necessity was treated in [Ethier and Kurtz, 1986, §3.7, Theorem 7.2, Remark 7.3] with E being a Polish space. Theorem 8.2 refines these two results as well as [Ethier and Kurtz, 1986, §3.9, Theorem 9.1] and a few other analogues in the following four aspects:

- We establish tightness which is often stronger than relative compactness.
- The E herein need not be metrizable or separable.

[5] Random variables do not always live on topological spaces. The E-valued processes $\{X^i\}_{i\in\mathbf{I}}$ in Theorem 8.2 are $(E^{\mathbf{R}^+}, \mathscr{B}(E)^{\otimes\mathbf{R}^+})$-valued but not necessarily $D(\mathbf{R}^+; E)$-valued random variables, where $D(\mathbf{R}^+; E)$ is a subset of the measurable space $(E^{\mathbf{R}^+}, \mathscr{B}(E)^{\otimes\mathbf{R}^+})$ and has the Skorokhod $\mathscr{J}_1$-topology. Then, their tightness in $D(\mathbf{R}^+; E)$ follows our generalized definition of tightness introduced in §2.4.

- We allow unbounded functions in $\mathcal{D}$, which can be handy when working with algebras of polynomials for random measures as in [Dawson, 1993, §2.1].

- $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ satisfying $|\cdot|$-MCC is milder than $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ being relatively compact if f is not necessarily bounded. So, WMCC is weaker than the analogous condition in [Jakubowski, 1986, Theorem 4.6, (4.9)] which was shown very useful for establishing tightness of measure-valued processes in [Dawson, 1993, §3.7] and [Perkins, 2002, §II.4].

Proof of Theorem 8.2. ((a) $\to$ (b)) For each fixed $f \in \mathcal{D} \mathrel{\hat{=}} C(E;\mathbf{R})$, $\{\varpi(f) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})$ by Proposition A.62 (d) and Fact B.60 (with $E = D(\mathbf{R}^+;E)$, $S = D(\mathbf{R}^+;\mathbf{R})$, $f = \varpi(f)$ and $\Gamma = \{\mathbb{P}^i \circ (\widehat{X}^i)^{-1}\}_{i\in\mathbf{I}}$). $\{\varpi(f) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC by Theorem A.88 (a) (with $E = \mathbf{R}$). Now, (b) follows by the bijective indistinguishability of $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ and $\{\varpi(f) \circ \widehat{X}^i\}_{i\in\mathbf{I}}$ [6].

((c) $\to$ (d)) is proved in Fact A.85.

((b) $\to$ (c) given MCCC) For each $g \in C_b(E;\mathbf{R})$, $\epsilon \in (0, 1/2)$ and $T \in (0,\infty)$, there exist $K_{\epsilon,T} \in \mathscr{K}^{\mathbf{m}}(E)$ and $f_{g,\epsilon,T} \in \mathcal{D}$ such that

$$\begin{aligned}\left\| f_{g,\epsilon,T}|_{K_{\epsilon,T}} - g|_{K_{\epsilon,T}} \right\|_\infty &\le \epsilon < 1 - \epsilon \\ &\le \inf_{i\in\mathbf{I}} \mathbb{P}^i \left(X_t^i \in K_{\epsilon,T}, \forall t \in [0,T] \right),\end{aligned} \tag{8.1.1}$$

which implies that

$$\begin{aligned}&\sup_{i\in\mathbf{I}} \mathbb{P}^i \left(\sup_{t\in[0,T]} \left| f_{g,\epsilon,T} \circ X_t^i - g \circ X_t^i \right| > \epsilon \right) \\ &\le 1 - \inf_{i\in\mathbf{I}} \mathbb{P}^i \left(X_t^i \in K_{\epsilon,T}, \forall t \in [0,T] \right) < \epsilon.\end{aligned} \tag{8.1.2}$$

Then, $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCC by (8.1.2), Proposition A.25 (a, c) and Proposition A.82 (a, b) (with $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_2 = C_b(E;\mathbf{R})$).

((d) $\to$ (a) given MCCC) follows by Theorem 6.48 (with $E_0 = E$). $\square$

When $(E, \mathfrak{r})$ is a complete (but not necessarily separable) metric space and $\mathfrak{r}$-MCC is given, we have shown in Proposition 6.53 the equivalence be-

[6]Note 6.38 mentioned the terminology "bijective indistinguishability" and the transitivity of $|\cdot|$-MCC between two bijectively indistinguishable families of processes.

tween MCCC and $\mathbf{T}$-MPCC with a dense $\mathbf{T}$. This gives us one more tightness criterion.

Proposition 8.6. *Let $(E, \mathfrak{r})$ be a metric space, $\mathbf{T}$ be a dense subset of $\mathbf{R}^+$ and E-valued càdlàg processes $\{X^i\}_{i\in\mathbf{I}}$ satisfy $\mathfrak{r}$-MCC and $\mathbf{T}$-MPCC. Then, the following statements are true:*

(a) There exists an $E_0 \in \mathscr{B}(E)$ such that E_0 is a separable subspace of E and X^i is indistinguishable from a $D(\mathbf{R}^+; E)$-valued random variable $\widehat{X}^i$ with paths in $E_0^{\mathbf{R}^+}$ for all $i \in \mathbf{I}$.

(b) If $(E, \mathfrak{r})$ is complete, then the $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ in (a) is tight in $D(\mathbf{R}^+; E)$.

Remark 8.7.

- Compared to [Ethier and Kurtz, 1986, §3.7, Theorem 7.2], part (b) above applies to non-separable spaces, looses compact containment to totally bounded containment and improves relative compactness into tightness in $D(\mathbf{R}^+; E)$.
- Compared to [Ethier and Kurtz, 1986, §3.7, Lemma 7.5], part (a) above replaces MCCC by $\mathfrak{r}$-MCC plus $\mathbf{T}$-MPCC with a dense $\mathbf{T}$. $\mathbf{T}$-MPCC is weaker than MCCC for any $\mathbf{T} \subset \mathbf{R}^+$ on metric spaces. In practice, $\mathfrak{r}$-MCC is usually no more difficult than MCCC to verify.

Proof of Proposition 8.6. (a) $C(E; \mathbf{R})$ separates points on E by Proposition A.26 (a) and Proposition A.25 (a, c). By Proposition 6.53 (a, b, c) (with $\mathcal{D} = C(E; \mathbf{R})$) and Proposition 6.32 (a) (with $X = X^i$ and $S_0 \circeq D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$), there exists an $E_0 \in \mathscr{B}(E)$ such that $(E_0, \mathscr{O}_E(E_0))$ is a separable subspace and X^i is indistinguishable from an E-valued process $\widehat{X}^i$ with paths in $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$[7] for all $i \in \mathbf{I}$. These $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ are $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$-valued variables by Fact A.77 (a) (with $E = (E_0, \mathscr{O}_E(E_0))$). They are $D(\mathbf{R}^+; E)$-valued random variables by Corollary A.65 (with $A = E_0$).

(b) When $(E, \mathfrak{r})$ is complete, the $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ above satisfies MCCC in E_0 by Proposition 6.53 (d). Then, (b) follows by Corollary A.86 (a) and Theorem 8.2 (a, c). □

[7] "with paths in $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$" means all paths of the process lying in $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$.

8.2 Weak convergence and finite-dimensional convergence

We discuss in this section the relationship of the following properties of E-valued càdlàg processes $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$.

Property.

P8 $\mathbf{S} \subset \mathbf{R}^+$ *is dense and* $\{X^n\}_{n\in\mathbf{N}_0}$ *satisfies*[8]

$$\lim_{n\to\infty} \mathbb{E}^n\left[f \circ X^n_{\mathbf{T}_0}\right] = \mathbb{E}^0\left[f \circ X^0_{\mathbf{T}_0}\right] \tag{8.2.1}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$.

P9 ***P8*** *holds with* $\mathcal{D}$ *separating points on* E.

P10 ***P8*** *holds with* $\mathcal{D}$ *strongly separating points on* E.

P11 $\mathbf{S} \subset \mathbf{R}^+$ *is dense and* $\{X^n\}_{n\in\mathbf{N}_0}$ *satisfies*

$$X^n \xrightarrow{\mathrm{D}(\mathbf{S})} X^0 \text{ as } n \uparrow \infty. \tag{8.2.2}$$

P12 *There exist* $\{\widehat{X}^n \in M(\Omega^n, \mathscr{F}^n; D(\mathbf{R}^+; E))\}_{n\in\mathbf{N}_0}$ *such that* X^n *and* $\widehat{X}^n$ *are indistinguishable for all* $n \in \mathbf{N}_0$ *and*[9]

$$\widehat{X}^n \Longrightarrow \widehat{X}^0 \text{ as } n \uparrow \infty \text{ on } D(\mathbf{R}^+; E). \tag{8.2.3}$$

Remark 8.8. $\mathcal{P}(D(\mathbf{R}^+; E))$ is a Tychonoff space by Corollary A.36, so (8.2.3) is equivalent to that $\widehat{X}^0$ is the weak limit of $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ on $D(\mathbf{R}^+; E)$.

Below are several immediate observations about **P8** - **P11**:

Fact 8.9. *Let* E *be a topological space and* $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$ *be* E*-valued processes. Then,* ***P11*** *implies* ***P8*** *with* $\mathcal{D} = C_b(E; \mathbf{R})$*. Moreover, if* E *is a Hausdorff (Tychonoff) space, then* ***P10*** *(resp.* ***P11****) implies* ***P9*** *(resp.* ***P10****).*

[8] Note 6.15 argued that the expectations in (6.2.2) are well-defined.

[9] Weak convergence, weak limit point and weak limit of random variables were interpreted in §2.4.

Proof. This fact follows by Fact 6.19 (with $X = X^0$ and $\mathbf{T} = \mathbf{S}$), Proposition A.17 (a) (with $A = E$) and Proposition A.25 (a, c). □

When E is a metrizable and separable space, weak convergence on $D(\mathbf{R}^+; E)$ is well-known to imply finite-dimensional convergence along all time points with no fixed left-jumps (see Theorem A.87 (a)). Recall that every metrizable and separable space is baseable (see Fact 3.24 (b)), so the result below generalizes Theorem A.87 (a).

Theorem 8.10. *Let E be a Tychonoff space and $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$ be E-valued càdlàg processes. Then, the following statements are true:*

(a) If $M(E;\mathbf{R})$ has a countable subset separating points on E[10]*, and if $\mathbf{S} \circeq \mathbf{R}^+\backslash J(X^0) \neq \varnothing$, then **P12** implies (8.2.2).*

*(b) If E is a baseable space, then **P12** implies **P11** with $\mathbf{S} \circeq \mathbf{R}^+\backslash J(X^0)$.*

We prove the more general result below, and Theorem 8.10 then follows.

Lemma 8.11. *Let E be a Tychonoff space and $\{\mu_n\}_{n\in\mathbf{N}_0} \subset \mathcal{M}^+(D(\mathbf{R}^+; E))$ satisfy*

$$\mu_n \Longrightarrow \mu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(D(\mathbf{R}^+; E)). \tag{8.2.4}$$

If $M(E;\mathbf{R})$ has a countable subset separating points on E[11] *and $\mathbf{R}^+\backslash J(\mu_0)$ is non-empty, especially if E is a baseable space, then there exist $\{\nu_{\mathbf{T}_0,n} \in \mathfrak{be}(\mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1})\}_{n\in\mathbf{N}_0}$ such that*

$$\nu_{\mathbf{T}_0,n} \Longrightarrow \nu_{\mathbf{T}_0,0} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^{\mathbf{T}_0}) \tag{8.2.5}$$

and[12]

$$\lim_{n\to\infty} \int_{E^{\mathbf{T}_0}} f(x)\mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) = \int_{E^{\mathbf{T}_0}} f(x)\mu_0 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) \tag{8.2.6}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E;\mathbf{R}))]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+\backslash J(\mu_0))$.

[10] This condition ensures $J(X^0)$ is well-defined.

[11] This condition ensures $J(\mu_0)$ is well-defined.

[12] $\mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ and $\mu_0 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ are members of $\mathfrak{M}^+(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\mathbf{T}_0})$, so the integrals in (8.2.6) are well-defined by Note 5.6.

Proof. The introductory part of this chapter noted that $\mathbf{R}^+\backslash J(\mu_0) \neq \varnothing$ when E is baseable. Let $\mathbb{D} \mathrel{\hat{=}} D(\mathbf{R}^+; E)$ and $(\mathbb{D}, \mathscr{S}_n, \nu_n)$ be the completion of $(\mathbb{D}, \mathscr{B}(\mathbb{D}), \mu_n)$ for each $n \in \mathbf{N}_0$. It follows by Lemma B.62 (with $(\mathscr{S}, \mu, \nu) = (\mathscr{S}_n, \mu_n, \nu_n)$) that

$$\nu_{\mathbf{T}_0,n} \mathrel{\hat{=}} \nu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathfrak{be}\left(\mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}\right), \ \forall n \in \mathbf{N}_0. \tag{8.2.7}$$

It follows by (8.2.4) and Fact B.27 (with $E = \mathbb{D}$ and $\mathscr{U}_n = \mathscr{S}_n$) that

$$\nu_n \Longrightarrow \nu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(\mathbb{D}). \tag{8.2.8}$$

The *set of discontinuity points* (see [Munkres, 2000, p.104]) of $\mathfrak{p}_{\mathbf{T}_0}$ has zero measure under μ_0 (hence under ν_0) by the definition of $J(\mu_0)$ and Proposition A.68 (c). Now, (8.2.5) follows by (8.2.8) and the Continuous Mapping Theorem (Theorem B.25 (b) with $E = \mathbb{D}$, $S = E^{\mathbf{T}_0}$, $\mu_n = \nu_n$, $\mu = \nu_0$ and $f = \mathfrak{p}_{\mathbf{T}_0}$). (8.2.6) follows by (8.2.5) and Fact B.54 (with $d = \aleph(\mathbf{T}_0)$, $\mu = \mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ and $\nu_1 = \nu_{\mathbf{T}_0,n}$). □

Proof of Theorem 8.10. (a) follows by Lemma 8.11 (with $\mu_n = \mathbb{P}^n \circ (\widehat{X}^n)^{-1}$) and the indistinguishability of X^n and $\widehat{X}^n$. When E is baseable, $\mathbf{S}$ is a dense subset of $\mathbf{R}^+$. Thus (b) follows by (a). □

The remainder of this section is about the converse of Theorem 8.10. First of all, we establish a result about the converse of Lemma 8.11.

Lemma 8.12. *Let E be a baseable Tychonoff space, $\{\mu_n\}_{n\in\mathbf{N}_0} \subset \mathcal{M}^+(D(\mathbf{R}^+; E))$ and $\mathcal{D} \subset C_b(E; \mathbf{R})$. Suppose that:*

(i) *$\mathbf{S}$ is a dense subset of $\mathbf{R}^+$ and (8.2.6) holds for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup 1$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$.*

(ii) *There exists an $S_0 \in \mathscr{B}(D(\mathbf{R}^+; E))$ such that μ_0 is supported on S_0 and*

$$\mathscr{B}_{D(\mathbf{R}^+;E)}(S_0) = \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_{S_0}. \tag{8.2.9}$$

(iii) *There exist $\{V_p\}_{p\in\mathbf{N}} \subset \mathscr{C}(D(\mathbf{R}^+; E))$ such that $V_p \subset S_0$ for all $p \in \mathbf{N}$ and*

$$\liminf_{n\to\infty} \mu_n\left(D(\mathbf{R}^+; E)\backslash V_p\right) \leq 2^{-p}, \ \forall p \in \mathbf{N}. \tag{8.2.10}$$

(iv) $\{\mu_n\}_{n\in\mathbf{N}}$ *is relatively compact.*

Then, the following statements are true:

(a) If $\mathcal{D}$ *strongly separates points on* E*, then (8.2.4) holds.*

(b) If $\mathcal{D}$ *separates points on* E*,* $\mu_0 \circ \mathfrak{p}_t^{-1}$ *is tight and* $\{\mu_n \circ \mathfrak{p}_t^{-1}\}_{n\in\mathbf{N}}$ *is sequentially tight for all* t *in a conull* $\mathbf{T} \subset \mathbf{R}^+$*, then (8.2.4) holds.*

Proof. Let $\gamma^1 \circeq \mu_0$ and $\mathbb{D} \circeq D(\mathbf{R}^+; E)$. By the condition (iv) above and Fact B.6, it suffices to show $\gamma^1 = \gamma^2$ for any weak limit point γ^2 of $\{\mu_n\}_{n\in\mathbf{N}}$ in $\mathcal{M}^+(\mathbb{D})$. By passing to a subsequence if necessary, we suppose

$$\mu_n \Longrightarrow \gamma^2 \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(\mathbb{D}) \tag{8.2.11}$$

and let $\mathbf{S}^1 \circeq \mathbf{S}$ and $\mathbf{S}^2 \circeq J(\gamma^2)$. The rest of the proof is divided into three steps.

Step 1: Verify

$$\lim_{n\to\infty} \int_{E^{\mathbf{T}_0}} f(x)\mu_n \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) = \int_{E^{\mathbf{T}_0}} f(x)\gamma^i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) \tag{8.2.12}$$

for each $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup 1$*,* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}^i)$ *and* $i = 1, 2$. For $i = 1$, (8.2.12) is given by the condition (i) above. For $i = 2$, (8.2.12) follows by (8.2.11) and Lemma 8.11 (with $\mu_0 = \gamma^2$).

Step 2: Verify

$$\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \text{ in } \mathfrak{M}^+\left(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0}\right), \ \forall \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+). \tag{8.2.13}$$

Under the conditions of (a), (8.2.13) follows immediately by (2.3.9), Step 1 and Lemma B.69 (a).

It takes a bit more work to show (8.2.13) for (b). E is baseable, so $\mathbf{S}^2$ is cocountable. $\mathbf{T}$ is conull by the hypothesis of (b), so $\mathbf{S}^2 \cap \mathbf{T}$ is a conull hence dense subset of $\mathbf{R}^+$. Fixing $t \in \mathbf{S}^2 \cap \mathbf{T}$, we find from Step 1 that

$$\lim_{n\to\infty} \int_E f(x)\mu_n \circ \mathfrak{p}_t^{-1}(dx) = \int_E f(x)\gamma^2 \circ \mathfrak{p}_t^{-1}(dx) \tag{8.2.14}$$

for each $f \in \mathfrak{mc}(\mathcal{D}) \cup 1$. Letting $f = 1$ in (8.2.14), we find that $\{\mu_n \circ \mathfrak{p}_t^{-1}(E)\}_{n\in\mathbf{N}}$ must be contained in a compact sub-interval of $(0, \infty)$. $\gamma^1 \circ \mathfrak{p}_t$ is $\mathbf{m}$-tight and $\{\mu_n \circ \mathfrak{p}_t\}_{n\in\mathbf{N}}$ is sequentially $\mathbf{m}$-tight by the hypothesis of (b), the baseability of E and Corollary 3.52 (a). It then follows by (8.2.14) and Theorem 5.4 (c) (with $d = 1$ and $\Gamma = \{\mu_n\}_{n\in\mathbf{N}}$) that

$$\text{w-}\lim_{n\to\infty} \mu_n \circ \mathfrak{p}_t^{-1} = \gamma^2 \circ \mathfrak{p}_t^{-1} \tag{8.2.15}$$

and $\gamma^2 \circ \mathfrak{p}_t^{-1}$ is $\mathbf{m}$-tight.

For each $\mathbf{T}_0 \in \mathscr{P}(\mathbf{S}^2 \cap \mathbf{T})$, both $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ and $\gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ are $\mathbf{m}$-tight by Lemma B.61 (a) (with $\mathbf{I} = \mathbf{T}_0$, $S_i = A_i = E$, $A = E^{\mathbf{T}_0}$ and $\Gamma = \{\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}\}$ or $\{\gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}\}$). Thus, (8.2.13) follows by (2.3.9) and Lemma B.69 (b) (with $\mathbf{S} = \mathbf{S}^2 \cap \mathbf{T}$).

Step 3: Verify $\gamma^1 = \gamma^2$ in $\mathcal{M}^+(\mathbb{D})$. As $\mathbb{D}$ is a Tychonoff space, we have that

$$\gamma^2(\mathbb{D}\backslash V_p) \leq \liminf_{n\to\infty} \mu_n(\mathbb{D}\backslash V_p) \leq 2^{-p},\ \forall p \in \mathbf{N} \tag{8.2.16}$$

by $\{V_p\}_{p\in\mathbf{N}} \subset \mathscr{C}(\mathbb{D})$, (8.2.10) and the Portmanteau's Theorem (Theorem 2.17 (a, c) with $E = \mathbb{D}$). As $S_0 \in \mathscr{B}(\mathbb{D})$ contains every V_p, we have by (8.2.16) and (ii) that

$$\gamma^1(\mathbb{D}\backslash S_0) = \gamma^2(\mathbb{D}\backslash S_0) = 0. \tag{8.2.17}$$

It follows by Step 2, the definition of $\mathscr{B}(E)^{\otimes\mathbf{R}^+}$ and (8.2.17) that

$$\gamma^1(A \cap S_0) = \gamma^2(A \cap S_0),\ \forall A \in \mathscr{B}(E)^{\otimes\mathbf{R}^+}\Big|_{\mathbb{D}}. \tag{8.2.18}$$

It follows by (8.2.18) and (8.2.9) that

$$\gamma^1|_{S_0} = \gamma^2|_{S_0} \text{ in } \mathcal{M}^+\left(S_0, \mathscr{O}_{\mathbb{D}}(S_0)\right). \tag{8.2.19}$$

It then follows that

$$\gamma^1 = \gamma^1|_{S_0}\Big|^{\mathbb{D}} = \gamma^2|_{S_0}\Big|^{\mathbb{D}} = \gamma^2 \text{ in } \mathcal{M}^+(\mathbb{D}) \tag{8.2.20}$$

by the fact , (8.2.19) and Fact 2.1 (c) (with $E = \mathbb{D}$, $\mathscr{U} = \mathscr{B}(E)$, $A = S_0$ and $\mu = \gamma^1$ or γ^2). $\square$

The following proposition treats a typical case of Lemma 8.12 where each μ_n is the distribution of $D(\mathbf{R}^+;E)$-valued random variable X^n and the $\{V_p\}_{p\in\mathbf{N}}$ in condition (iii) are compact sets provided by tightness.

Proposition 8.13. *Let E be a baseable Tychonoff space, $\mathcal{D}\subset C_b(E;\mathbf{R})$, $\mathbf{S}\subset\mathbf{R}^+$ and $X^n\in M(\Omega^n,\mathscr{F}^n,\mathbb{P}^n;D(\mathbf{R}^+;E))$ for each $n\in\mathbf{N}_0$. Suppose that:*

(i) There is an $S_0\in\mathscr{B}(D(\mathbf{R}^+;E))$ satisfying $\mathbb{P}^0(X^0\in S_0)=1$ and (8.2.9).

(ii) $\{X^n\}_{n\in\mathbf{N}}$ is tight in S_0.

(iii) $\mathbf{T}\subset\mathbf{R}^+$ is conull and X^0 satisfies $\mathbf{T}$-PMTC.

Then, ***P9*** *implies* ***P12****.*

Remark 8.14. In Proposition 8.13, tightness in S_0 is different from tightness in $D(\mathbf{R}^+;E)$ since $S_0=D(\mathbf{R}^+;E)$ does not necessarily satisfy (8.2.9).

Proof of Proposition 8.13. Let $\mu_n=\mathbb{P}^n\circ(X^n)^{-1}\in\mathcal{P}(D(\mathbf{R}^+;E))$ for each $n\in\mathbf{N}_0$. **P9** implies $\mathbf{S}$ is dense in $\mathbf{R}^+$ and (8.2.6) holds for $f\in\mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]\cup\{1\}$ and $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{S})$. The condition (i) above implies $\mu_0(S_0)=1$. By the condition (ii) above, there exist $\{V_p\}_{p\in\mathbf{N}}\subset\mathscr{K}(D(\mathbf{R}^+;E))$ such that $V_p\subset S_0$ for all $p\in\mathbf{N}$ and $\inf_{n\in\mathbf{N}}\mu_n(V_p)\geq 1-2^{-p}$. As $D(\mathbf{R}^+;E)$ is a Tychonoff space, $\{V_p\}_{p\in\mathbf{N}}\subset\mathscr{C}(D(\mathbf{R}^+;E))$ by Proposition A.12 (a) and the tight sequence $\{\mu_n\}_{n\in\mathbf{N}}$ is relatively compact by the Prokhorov's Theorem (Theorem 2.22 (b)). As E is a baseable space, the tight sequence $\{X^n\}_{n\in\mathbf{N}}$ satisfies MCCC by Fact 8.1 and Corollary 3.15. $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathbf{R}^+$-PMTC by Fact 6.56 (f) (with $\mathbf{I}=\mathbf{N}$, $i=n$ and $A=E$), so $\{\mu_n\circ\mathfrak{p}_t^{-1}\}_{n\in\mathbf{N}}$ is $\mathbf{m}$-tight for all $t\in\mathbf{R}^+$. Moreover, $\mu_0\circ\mathfrak{p}_t^{-1}$ is $\mathbf{m}$-tight for all $t\in\mathbf{T}$ by the condition (iii) above. So far, we have justified all conditions of Lemma 8.12 (b) for $\{\mu_n\}_{n\in\mathbf{N}_0}$, thus (8.2.4) and **P12** hold. $\square$

The following proposition uses our tightness criteria established in §8.1 to realize the "tightness in S_0" desired by Proposition 8.13.

Proposition 8.15. *Let E be a Tychonoff space, $\{(\Omega^n,\mathscr{F}^n,\mathbb{P}^n;X^n)\}_{n\in\mathbf{N}_0}$ be E-valued càdlàg processes, $\mathcal{D}\subset C_b(E;\mathbf{R})$ and $\mathbf{S}\subset\mathbf{R}^+$. Suppose that $\{X^n\}_{n\in\mathbf{N}_0}$ satisfies MCCC and $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}$-FMCC. Then,* ***P9*** *implies* ***P12****.*

Proof. The proof is divided into three steps.

Step 1: Construct a suitable base. By Proposition 6.55 (with $\mathbf{I} = \mathbf{N}_0$ and $i = n$), there exists a $\mathcal{D}$-baseable subset E_0 of E such that $\{X^n\}_{n\in\mathbf{N}_0}$ satisfies MCCC in E_0. By Proposition 3.39 (c), there exists a base $(E_0, \mathcal{F}; \widehat{E}; \widehat{\mathcal{F}})$ with $\mathcal{F} \subset (\mathcal{D} \cup \{1\})$.

Step 2: Construct $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$. E_0 is a Tychonoff subspace of E by Proposition A.26 (b). $\mathbb{D}_0 \circeq D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$ is a Tychonoff subspace of $D(\mathbf{R}^+; E)$ by Corollary A.65 (with $A = E_0$). $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{F}$-FMCCC since $(\mathcal{F}\backslash\{1\}) \subset \mathcal{D}$. By Proposition 6.46 (with $E_0 = E$ and $\mathbf{I} = \mathbf{N}_0$), there exists an

$$S_0 \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{\mathbb{D}_0} \subset \mathscr{B}(\mathbb{D}_0) \tag{8.2.21}$$

and

$$\begin{aligned}\widehat{X}^n = \mathfrak{rep}_c(X^n; E_0, \mathcal{F}) &\in M\left(\Omega^n, \mathscr{F}^n; S_0, \mathscr{O}_{\mathbb{D}_0}(S_0)\right) \\ &\subset M\left(\Omega^n, \mathscr{F}^n; \mathbb{D}_0\right) \subset M\left(\Omega^n, \mathscr{F}^n; D(\mathbf{R}^+; E)\right), \ \forall n \in \mathbf{N}_0\end{aligned} \tag{8.2.22}$$

such that (8.2.9) holds,

$$\inf_{n\in\mathbf{N}_0} \mathbb{P}^n\left(X^n = \widehat{X}^n \in S_0\right) = 1 \tag{8.2.23}$$

and $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$ is $\mathbf{m}$-tight in S_0[13].

Step 3: Show (8.2.3). It follows by (8.2.23), Fact 6.52 (with $\mathbf{I} = \mathbf{N}_0$ and $i = n$), **P9**, the fact $(\mathcal{F}\backslash\{1\}) \subset \mathcal{D}$ and Lemma B.78 (d) (with $X = X^0$ and $Y = \widehat{X}^0$) that

$$\lim_{n\to\infty} \mathbb{E}^n\left[f \circ \widehat{X}^n_{\mathbf{T}_0}\right] = \mathbb{E}^0\left[f \circ \widehat{X}^0_{\mathbf{T}_0}\right] \tag{8.2.24}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$. X^0 satisfies MCCC, so it satisifes $\mathbf{R}^+$-PMTC by Fact 6.56 (f) (with $A = E$ and $\mathbf{I} = \{0\}$). It then follows that

$$\widehat{X}^n \Longrightarrow \widehat{X}^0 \text{ as } n \uparrow \infty \text{ on } \mathbb{D}_0 \tag{8.2.25}$$

by (8.2.22) and Proposition 8.13 (with $E = E_0$, $\mathcal{D} = \mathcal{F}|_{E_0}\backslash\{1\}$, $X^n = \widehat{X}^n$ and $\mathbf{T} = \mathbf{R}^+$). Now, (8.2.3) follows by (8.2.22), (8.2.25) and Lemma B.55 (with

[13] It is the subsequence $\{X^n\}_{n\in\mathbf{N}}$ that satisfies $\mathcal{F}$-FMCCC, so Proposition 6.46 (c) is just applied to $\{X^n\}_{n\in\mathbf{N}}$ with X^0 removed.

$E = D(\mathbf{R}^+; E)$, $A = \mathbb{D}_0$, $\mu_n = \mathbb{P}^n \circ (\widehat{X}^n)^{-1} \in \mathcal{P}(\mathbb{D}_0)$ and $\mu = \mathbb{P}^0 \circ (\widehat{X}^0)^{-1} \in \mathcal{P}(\mathbb{D}_0)$). □

Another typical case of Lemma 8.12 is when E has a metrizable and separable subspace E_0, and the S_0 in condition (ii) and the $\{V_p\}_{p\in\mathbf{N}}$ in condition (iii) are all taken to be $D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$. Then, the assumption of relative compactness can be loosen to $\mathcal{D}$-FMCC.

Theorem 8.16. *Let E be a Tychonoff space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$ be E-valued càdlàg processes, $\mathcal{D} \subset C_b(E; \mathbf{R})$ and $\mathbf{S} \subset \mathbf{R}^+$. Suppose that:*

(i) $\mathcal{D}$ is countable and strongly separates points on $E_0 \in \mathscr{B}(E)$.

(ii) $\{X_n\}_{n\in\mathbf{N}_0}$ satisfies

$$\inf_{n\in\mathbf{N}_0} \mathbb{P}^n \left(X^n \in E_0^{\mathbf{R}^+} \right) = 1. \tag{8.2.26}$$

(iii) $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}$-FMCC.

Then, ***P8*** *implies* ***P12****.*

Remark 8.17. The condition (i) above implies E_0 is a second-countable subspace of E by Proposition A.17 (d) (with $A = E_0$). Given such E_0, the condition (i, iii) above is weaker than relative compactness by Theorem 8.21 (a) to follow, Corollary A.83 and Proposition A.24 (b).

Proof of Theorem 8.16. The proof is divided into four steps.

Step 1: Construct a suitable base by E_0 and $\mathcal{D}$. E_0 is a Tychonoff subspace and is a $\mathcal{D}$-baseable subset of E by Proposition A.26 (b) and Proposition A.17 (a) (with $A = E_0$). As $\mathcal{D}$ is countable, there exists a base $(E_0, \mathcal{F}; \widehat{E}; \widehat{\mathcal{F}})$ with $\mathcal{F} = \mathcal{D} \cup \{1\}$ strongly separating points on E_0 by Proposition 3.39 (b).

Step 2: Construct $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$. Let $\mathbb{D} \doteq D(\mathbf{R}^+; E)$, $S_0 \doteq D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$ and $\widehat{\mathbb{D}} \doteq D(\mathbf{R}^+; \widehat{E})$. $(E_0, \mathscr{O}_E(E_0))$ is metrizable and separable by Corollary A.30 (a, b) (with $E = E_0$ and $\mathcal{D} = \mathcal{F}|_{E_0}$). S_0 is a subspace of $\mathbb{D}$ by Corollary A.65 (with $A = E_0$). S_0 satisfies (8.2.9) by Proposition A.72 (b) (with

$E = (E_0, \mathscr{O}_E(E_0)))$. Then, there exist

$$\begin{aligned}\widehat{X}^n &= \mathfrak{rep}_c(X^n; E_0, \mathcal{F}) \\ &\in M(\Omega^n, \mathscr{F}^n; S_0) \cap M(\Omega^n, \mathscr{F}^n; \mathbb{D}) \cap M\left(\Omega^n, \mathscr{F}^n; \widehat{\mathbb{D}}\right), \ \forall n \in \mathbf{N}_0\end{aligned} \tag{8.2.27}$$

satisfying (8.2.23) by (8.2.26), Proposition 6.32 (with $X = X^n$) and Fact 6.24.

Step 3: Show $\widehat{X}^0$ is the weak limit of $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ on $\widehat{\mathbb{D}}$. In this step, we consider $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$ as $\widehat{\mathbb{D}}$-valued random variables. As mentioned in Note 6.5, $\widehat{E}$ is a compact Polish space, so $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$ automatically satisfies MCCC by Note 6.44. $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{F}$-FMCC since $\mathcal{F}\backslash\{1\} = \mathcal{D}$, so $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ satisfies $\widehat{\mathcal{F}}$-FMCC by Proposition 6.45 (a). (8.2.24) holds for all $\widehat{f} \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\widehat{\mathcal{F}}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$ by (8.2.26), Fact 6.52 (with $\mathbf{I} = \mathbf{N}_0$ and $i = n$), **P8**, the fact $\mathcal{F}\backslash\{1\} = \mathcal{D}$ and Lemma B.78 (d) (with $X = X^0$ and $Y = \widehat{X}^0$). $\widehat{\mathcal{F}}\backslash\{1\}$ is a subset of $C_b(\widehat{E}; \mathbf{R})$ by Corollary 3.10 (a) and separates points on $\widehat{E}$ by definition of base. Now, the conclusion of Step 3 follows by Proposition 8.15 (with $E = \widehat{E}$, $X^n = \widehat{X}^n$ and $\mathcal{D} = \widehat{\mathcal{F}}\backslash\{1\}$).

Step 4: Show (8.2.3). $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$ as S_0-valued random variables satisfies (8.2.25) with $\mathbb{D}_0 = S_0$ by Step 3 and Proposition 6.51 (b). $\{\widehat{X}^n\}_{n\in\mathbf{N}_0}$ as $\mathbb{D}$-valued random variables satisfies (8.2.3) by (8.2.27), (8.2.25) and Lemma B.55 (with $E = \mathbb{D}$, $A = S_0$, $\mu_n = \mathbb{P}^n \circ (\widehat{X}^n)^{-1} \in \mathcal{P}(S_0)$ and $\mu = \mathbb{P}^0 \circ (\widehat{X}^0)^{-1} \in \mathcal{P}(S_0)$). □

If E itself is a metrizable and separable space, then the E_0 in Theorem 8.16 can be taken to equal E.

Corollary 8.18. *Let E be a metrizable and separable space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$ be E-valued càdlàg processes and $\mathbf{S} \subset \mathbf{R}^+$. Then, the following statements are successively weaker:*

*(a) $\{X^n\}_{n\in\mathbf{N}}$ satisfies MCC and **P11**.*

*(b) $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}$-FMCC and **P10** for some $\mathcal{D} \subset C_b(E; \mathbf{R})$.*

*(c) $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}$-FMCC and **P10** for some countable $\mathcal{D} \subset C_b(E; \mathbf{R})$.*

*(d) $\{X^n\}_{n\in\mathbf{N}}$ satisfies **P12**.*

Proof. ((a) → (b)) follows by Fact 8.9 and Corollary A.86 (b). ((b) → (c)) follows by Proposition A.6 (c) and Proposition A.24 (b). ((c) → (d)) follows by Theorem 8.16 (with $E_0 = E$). □

Remark 8.19. Compared to Theorem A.87 (b), Corollary 8.18 (a, d) reduces relative compactness to MCC which is a weaker condition by Theorem 8.21 (a) to follow.

When E is a non-separable metric space, one can obtain the E_0 in Theorem 8.16 by $\mathfrak{r}$-MCC and $\mathbf{T}$-MPCC.

Proposition 8.20. *Let $(E, \mathfrak{r})$ be a metric space, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}_0}$ be E-valued càdlàg processes and $\mathbf{S} \subset \mathbf{R}^+$. Then, the following statements are successively weaker:*

*(a) $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathfrak{r}$-MCC and $\mathbf{T}$-MPCC with a dense $\mathbf{T} \subset \mathbf{R}^+$. X satisfies $\mathfrak{r}$-MCC. Moreover, **P11** holds.*

*(b) $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathfrak{r}$-MCC, $\mathcal{D}$-FMCC with $\mathcal{D} \subset C_b(E;\mathbf{R})$ and $\mathbf{T}_1$-MPCC with a dense $\mathbf{T}_1 \subset \mathbf{R}^+$. X^0 satisfies $\mathfrak{r}$-MCC and $\mathbf{T}_2$-MPCC with a dense $\mathbf{T}_2 \subset \mathbf{R}^+$. Moreover, **P10** holds.*

*(c) $\{X^n\}_{n\in\mathbf{N}_0}$ satisfies **P12**.*

Proof. ((a) → (b)) By Corollary A.86 (a), $\{X^n\}_{n\in\mathbf{N}}$ satisfies MCC. By Fact 8.9 and Corollary A.83 (a, b), there exists a $\mathcal{D} \subset C_b(E;\mathbf{R})$ such that $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}$-FMCC and **P10** holds. By Proposition 6.53 (a), there exist $\{A_{p,q}\}_{p,q\in\mathbf{N}} \subset \mathscr{C}(E)$ such that each $A_{p,q}$ is a totally bounded set and

$$\inf_{n\in\mathbf{N}} \mathbb{P}^n\left(X_t^n \in A_{p,q}\right) \geq 1 - 2^{-p},\ \forall t \in [0, q],\ p, q \in \mathbf{N}. \tag{8.2.28}$$

Hence, $\{X^n\}_{n\in\mathbf{N}}$ satisfies $\mathbf{R}^+$-MPCC. For each $t \in \mathbf{S}$ and $n \in \mathbf{N}_0$, the (one-dimensional) distribution $\mathbb{P}^n \circ (X_t^n)^{-1} \in \mathcal{P}(E)$ by Fact 2.24 (d) and so **P11** implies

$$X_t^n \Longrightarrow X_t^0 \text{ as } n \uparrow \infty \text{ on } E. \tag{8.2.29}$$

As E is a Tychonoff space, it follows by (8.2.28), (8.2.29), the closedness of each $A_{p,q}$ and the Portmanteau's Theorem (Theorem 2.17 (a, b)) that

$$\begin{aligned}\mathbb{P}^0\left(X_t^0 \in A_{p,q}\right) &\geq \inf_{n\in\mathbf{N}} \mathbb{P}^n\left(X_t^n \in A_{p,q}\right) \\ &\geq 1-2^{-p},\ \forall t \in \mathbf{S}\cap[0,q], p,q \in \mathbf{N},\end{aligned} \tag{8.2.30}$$

thus proving X^0 satisfies $\mathbf{S}$-MPCC. Now, (b) follows by letting $\mathbf{T}_1 = \mathbf{T}$ and $\mathbf{T}_2 = \mathbf{S}$.

((b) $\to$ (c)) The union of two second-countable subspaces of E is still second-countable by Proposition A.6 (c) and Proposition A.3 (b, e). So, we apply Proposition 6.53 (a - c) to $\{X^n\}_{n\in\mathbf{N}}$ and the singleton $\{X^0\}$ respectively and find a second-countable subspace E_0 of E satisfying (8.2.26). There exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ strongly separating points on E_0 by **P10** and Proposition A.24 (b). $\{X^n\}_{n\in\mathbf{N}_0}$ certainly satisfies the weaker property **P8** than **P10**. Now, (c) follows by Theorem 8.16 (with $\mathcal{D} = \mathcal{D}_0$). □

8.3 Relative compactness and finite-dimensional convergence

When $(E, \mathfrak{r})$ is a separable metric space, relative compactness in $D(\mathbf{R}^+; E)$ implies $\mathfrak{r}$-MCC (see e.g. [Ethier and Kurtz, 1986, §3.7, Theorem 7.2]). For a general Tychonoff space E, we now justify the sufficiency of relative compactness in $D(\mathbf{R}^+; E)$ for MCC. If E is also baseable, we leverage Theorem 8.10 and establish the sufficiency of relative compactness in $D(\mathbf{R}^+; E)$ for "relative compactness" under finite-dimensional convergence.

Theorem 8.21. *Let E be a Tychonoff space, $\mathbf{I}$ be an infinite index set and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be a relatively compact*[14] *family of $D(\mathbf{R}^+; E)$-valued random variables. Then, the following statements are true:*

(a) $\{X^i\}_{i\in\mathbf{I}}$ satisfies $C(E;\mathbf{R})$-FMCC and MCC.

(b) If E is baseable, then any infinite subset of $\{X^i\}_{i\in\mathbf{I}}$ has a subsequence that converges finite-dimensionally to some $D(\mathbf{R}^+; E)$-valued random

[14] Relative compactness of random variables was interpreted in §2.4.

variable X along $\mathbf{R}^+\backslash J(X)$.

Proof. By the relative compactness of $\{X^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+;E)$, any infinite subset of $\mathbf{I}$ has a subsequence $\{i_n\}_{n\in\mathbf{N}}\subset\mathbf{I}$ such that

$$\mathbb{P}^{i_n}\circ(X^{i_n})^{-1}\Longrightarrow\mu \text{ as } n\uparrow\infty \text{ in } \mathcal{P}\left(D(\mathbf{R}^+;E)\right) \tag{8.3.1}$$

for some $\mu\in\mathcal{P}(D(\mathbf{R}^+;E))$. Let $\Omega\doteq D(\mathbf{R}^+;E)$, $\mathscr{F}\doteq\mathscr{B}(\Omega)$, $\mathbb{P}\doteq\mu$ and X be the identity mapping on Ω. Then, $(\Omega,\mathscr{F},\mathbb{P};X)$[15] is a $D(\mathbf{R}^+;E)$-valued random variable and satisfies

$$X^{i_n}\Longrightarrow X \text{ as } n\uparrow\infty \text{ on } D(\mathbf{R}^+;E). \tag{8.3.2}$$

For each $f\in C(E;\mathbf{R})$, it follows that

$$\varpi(f)\circ X^{i_n}\Longrightarrow\varpi(f)\circ X \text{ as } n\uparrow\infty \text{ on } D(\mathbf{R}^+;\mathbf{R}) \tag{8.3.3}$$

by (8.3.2), Proposition A.62 (d) (with $S=\mathbf{R}$) and the Continuous Mapping Theorem (Theorem B.25 with $E=D(\mathbf{R}^+;E)$, $S=D(\mathbf{R}^+;\mathbf{R})$ and $f=\varpi(f)$). The argument above proves the relative compactness of $\{\varpi(f)\circ X^i\}_{i\in\mathbf{I}}$ in $D(\mathbf{R}^+;\mathbf{R})$. $D(\mathbf{R}^+;\mathbf{R})$, as mentioned in Note 6.5, is a Polish space, so $\{\varpi(f)\circ X^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})$ by the Prokhorov's Theorem (Theorem 2.22 (a)) and satisfies $|\cdot|$-MCC by Theorem A.88 (with $(E,\mathfrak{r})=(\mathbf{R},|\cdot|)$. $C(E;\mathbf{R})$ strongly separates points on E by Proposition A.25 (a, b). Now, (a) follows by Fact A.81 (b) (with $\mathcal{D}=C(E;\mathbf{R})$) and Corollary A.83 (a, d) (with $\mathcal{D}=C(E;\mathbf{R})$). If E is also baseable, then we have by Theorem 8.10 (b) (with $n=i_n$) that

$$X^{i_n}\xrightarrow{\mathrm{D}(\mathbf{R}^+\backslash J(X))}X \text{ as } n\uparrow\infty, \tag{8.3.4}$$

thus proving (b). □

We then consider the converse of Theorem 8.21.

Theorem 8.22. *Let E be a Tychonoff space, $\mathbf{I}$ be an infinite index set and $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Suppose that for each infinite $\mathcal{I}^*\subset\mathbf{I}$, there exists a subsequence $\mathcal{I}\doteq\{i_n\}_{n\in\mathbf{N}}\subset\mathcal{I}^*$, an $E_{0,\mathcal{I}}\in\mathscr{B}(E)$,*

[15]The X herein is known as the coordinate process on $D(\mathbf{R}^+;E)$. We did a similar construction in 7.14.

a $\mathcal{D}_{\mathcal{I}} \subset C_b(E;\mathbf{R})$*, an* $\mathbf{S}_{\mathcal{I}} \subset \mathbf{R}^+$ *and an* E*-valued càdlàg process* $(\Omega, \mathscr{F}, \mathbb{P}; X^{\mathcal{I}})$ *such that:*

(i) $\mathcal{D}_{\mathcal{I}}$ *is countable and strongly separates points on* $E_{0,\mathcal{I}}$.

(ii) $\{X^{i_n}\}_{n\in\mathbf{N}}$ *and* $X^{\mathcal{I}}$ *satisfy*

$$\inf_{n\in\mathbf{N}} \mathbb{P}^{i_n}\left(X^{i_n} \in E_{0,\mathcal{I}}^{\mathbf{R}^+}\right) = \mathbb{P}\left(X^{\mathcal{I}} \in E_{0,\mathcal{I}}^{\mathbf{R}^+}\right) = 1. \tag{8.3.5}$$

(iii) $\{X^{i_n}\}_{n\in\mathbf{N}}$ *satisfies* $\mathcal{D}_{\mathcal{I}}$*-FMCC.*

(iv) $\mathbf{S}_{\mathcal{I}}$ *is dense in* $\mathbf{R}^+$ *and*

$$\lim_{n\to\infty} \mathbb{E}^{i_n}\left[f \circ X^{i_n}_{\mathbf{T}_0}\right] = \mathbb{E}\left[f \circ X^{\mathcal{I}}_{\mathbf{T}_0}\right] \tag{8.3.6}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D}_{\mathcal{I}})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}_{\mathcal{I}})$.

Then, there exist an $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$ *and* $D(\mathbf{R}^+;E)$*-valued random variables* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *such that* $\widehat{X}^i$ *is indistinguishable from* X^i *for all* $i \in \mathbf{I}\backslash\mathbf{I}_0$ *and* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *is relatively compact in* $D(\mathbf{R}^+;E)$.

Proof. Let $\mathbb{D} \doteq D(\mathbf{R}^+;E)$ and $\mathbb{D}_0 \doteq D(\mathbf{R}^+;E_0,\mathscr{O}_E(E_0))$. For any infinite $\mathcal{I}^* \subset \mathbf{I}$, we take $\mathcal{I} = \{i_n\}_{n\in\mathbf{N}} \subset \mathcal{I}^*$, $E_{0,\mathcal{I}} \in \mathscr{B}(E)$, $\mathcal{D}_{\mathcal{I}} \subset C_b(E;\mathbf{R})$, $\mathbf{S}_{\mathcal{I}} \subset \mathbf{R}^+$ and E-valued càdlàg process $(\Omega, \mathscr{F}, \mathbb{P}; X^{\mathcal{I}})$ satisfy the conditions (i) - (iv) above. It follows by (8.3.5) and the càdlàg properties of $\{X^i\}_{i\in\mathbf{I}}$ and $X^{\mathcal{I}}$ that

$$\inf_{n\in\mathbf{N}} \mathbb{P}^{i_n}\left(X^{i_n} \in \mathbb{D}_0\right) = \mathbb{P}\left(X^{\mathcal{I}} \in \mathbb{D}_0\right) = 1. \tag{8.3.7}$$

By Proposition 6.32 (a) (with $S_0 = \mathbb{D}_0$ and $X = X^{i_n}$ or $X^{\mathcal{I}}$), there exist

$$\left(\{\widehat{X}^{i_n}\}_{n\in\mathbf{N}} \cup \{\widehat{X}^{\mathcal{I}}\}\right) \subset M\left(\Omega^{i_n}, \mathscr{F}^{i_n}; \mathbb{D}_0\right) \subset M\left(\Omega^{i_n}, \mathscr{F}^{i_n}; \mathbb{D}\right) \tag{8.3.8}$$

satisfying

$$\inf_{n\in\mathbf{N}} \mathbb{P}^{i_n}\left(X^{i_n} = \widehat{X}^{i_n} \in \mathbb{D}_0\right) = \mathbb{P}\left(X^{\mathcal{I}} = \widehat{X}^{\mathcal{I}} \in \mathbb{D}_0\right) = 1. \tag{8.3.9}$$

It follows by (8.3.9) and the condition (iv) above that

$$\lim_{n\to\infty} \mathbb{E}^{i_n}\left[f \circ \widehat{X}^{i_n}_{\mathbf{T}_0}\right] = \mathbb{E}\left[f \circ \widehat{X}^{\mathcal{I}}_{\mathbf{T}_0}\right] \tag{8.3.10}$$

for each $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D}_{\mathcal{I}})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}_{\mathcal{I}})$. It then follows by Theorem 8.16 (with $X^n = \widehat{X}^{i_n}$, $X^0 = \widehat{X}^{\mathcal{I}}$, $E_0 = E_{0,\mathcal{I}}$, $\mathcal{D} = \mathcal{D}_{\mathcal{I}}$ and $\mathbf{S} = \mathbf{S}_{\mathcal{I}}$) that

$$\widehat{X}^{i_n} \Longrightarrow \widehat{X}^{\mathcal{I}} \text{ as } n \uparrow \infty \text{ on } \mathbb{D}. \tag{8.3.11}$$

From the argument above we draw two conclusions: (1) There would be at most finite members of $\{X^i\}_{i\in\mathbf{I}}$ which may not admit an indistinguishable $\mathbb{D}$-valued copy. Let $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$ be the indices of these exceptions. For each $i \in \mathbf{I}\backslash\mathbf{I}_0$, different $\{i_n\}_{n\in\mathbf{N}} \subset \mathbf{I}$ that contains i may induce different $\mathbb{D}$-valued copies of X^i. However, such copy can be thought of as a unique one up to indistinguishability, which we denote by $\widehat{X}^i$. (2) For any infinite $\mathcal{I}^* \subset (\mathbf{I}\backslash\mathbf{I}_0)$, there exist a subsequence $\{i_n\}_{n\in\mathbf{N}} \subset \mathcal{I}^*$ and a $\mathbb{D}$-valued random variable $\widehat{X}^{\mathcal{I}}$ such that (8.3.11) holds. In other words, $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ is relatively compact in $\mathbb{D}$. □

The following special cases of Theorem 8.21 correspond to the settings of Corollary 8.18 and Proposition 8.20 respectively.

Corollary 8.23. *Let E be a metrizable and separable space, $\mathbf{I}$ be an infinite index set and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Then, the following statements are successively weaker:*

(a) Any infinite subset of $\{X^i\}_{i\in\mathbf{I}}$ has a subsequence that satisfies MCC and converges finite-dimensionally to some E-valued càdlàg process along a dense subset of $\mathbf{R}^+$.

(b) For any infinite $\mathcal{I}^* \subset \mathbf{I}$, there exist a subsequence $\mathcal{I} \circeq \{i_n\}_{n\in\mathbf{N}} \subset \mathcal{I}^*$, a $\mathcal{D}_{\mathcal{I}} \subset C_b(E;\mathbf{R})$, a dense $\mathbf{S}_{\mathcal{I}} \subset \mathbf{R}^+$ and an E-valued càdlàg process $(\Omega, \mathscr{F}, \mathbb{P}; X^{\mathcal{I}})$ such that: (i) $\mathcal{D}$ is countable and strongly separates points on E, (ii) $\{X^{i_n}\}_{n\in\mathbf{N}}$ satisfies $\mathcal{D}_{\mathcal{I}}$-FMCC, and (iii) (8.3.6) holds for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D}_{\mathcal{I}})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}_{\mathcal{I}})$.

(c) There exist an $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$ *and* $D(\mathbf{R}^+; E)$*-valued random variables* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *such that* $\widehat{X}^i$ *is indistinguishable from* X^i *for all* $i \in \mathbf{I}\backslash\mathbf{I}_0$ *and* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *is relatively compact in* $D(\mathbf{R}^+; E)$*.*

Proof. ((a) → (b)) follows by Fact 8.9 (with $X^n = X^{i_n}$ and $X = X^{\mathcal{I}}$) and Corollary A.86 (b). ((b) → (c)) follows by Theorem 8.22 (with $E_0 = E$). □

Proposition 8.24. *Let* $(E, \mathfrak{r})$ *be a metric space,* $\mathbf{I}$ *be an infinite index set and* $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ *be E-valued càdlàg processes. Then, the following statements are successively weaker:*

(a) For any infinite $\mathcal{I}^* \subset \mathbf{I}$*, there exist a subsequence* $\mathcal{I} \circeq \{i_n\}_{n\in\mathbf{N}} \subset \mathcal{I}^*$*, an E-valued càdlàg process* $(\Omega, \mathscr{F}, \mathbb{P}; X^{\mathcal{I}})$ *and dense subsets* $\mathbf{T}_{\mathcal{I}}$ *and* $\mathbf{S}_{\mathcal{I}}$ *of* $\mathbf{R}^+$ *such that: (i)* $\{X^{i_n}\}_{n\in\mathbf{I}}$ *satisfies* $\mathfrak{r}$*-MCC and* $\mathbf{T}_{\mathcal{I}}$*-MPCC, (ii)* $X^{\mathcal{I}}$ *satisfies* $\mathfrak{r}$*-MCC, and (iii)*

$$X^{i_n} \xrightarrow{\mathrm{D}(\mathbf{S}_{\mathcal{I}})} X^{\mathcal{I}} \text{ as } n \uparrow \infty. \tag{8.3.12}$$

(b) For any infinite $\mathcal{I}^* \subset \mathbf{I}$*, there exist a sub-subsequence* $\mathcal{I} \circeq \{i_n\}_{n\in\mathbf{N}} \subset \mathcal{I}^*$*, a* $\mathcal{D}_{\mathcal{I}} \subset C_b(E; \mathbf{R})$*, an E-valued càdlàg process* $(\Omega, \mathscr{F}, \mathbb{P}; X^{\mathcal{I}})$ *and dense subsets* $\mathbf{T}^1_{\mathcal{I}}$*,* $\mathbf{T}^2_{\mathcal{I}}$*,* $\mathbf{S}_{\mathcal{I}}$ *of* $\mathbf{R}^+$ *such that: (i)* $\mathcal{D}$ *strongly separates points on E, (ii)* $\{X^{i_n}\}_{n\in\mathbf{N}}$ *satisfies* $\mathfrak{r}$*-MCC,* $\mathcal{D}_{\mathcal{I}}$*-FMCC and* $\mathbf{T}^1_{\mathcal{I}}$*-MPCC, (iii)* $X^{\mathcal{I}}$ *satisfies* $\mathfrak{r}$*-MCC and* $\mathbf{T}^2_{\mathcal{I}}$*-MPCC, and (iv) (8.3.6) holds for all* $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D}_{\mathcal{I}})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}_{\mathcal{I}})$*.*

(c) There exist an $\mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})$ *and* $D(\mathbf{R}^+; E)$*-valued random variables* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *such that* $\widehat{X}^i$ *is indistinguishable from* X^i *for all* $i \in \mathbf{I}\backslash\mathbf{I}_0$ *and* $\{\widehat{X}^i\}_{i\in\mathbf{I}\backslash\mathbf{I}_0}$ *is relatively compact in* $D(\mathbf{R}^+; E)$*.*

Proof. This result follows by: (1) Applying Proposition 8.15 (with $X^n = X^{i_n}$, $X^0 = X^{\mathcal{I}}$, $\mathbf{T}_1 = \mathbf{T}^1_{\mathcal{I}}$, $\mathbf{T}_2 = \mathbf{T}^2_{\mathcal{I}}$, $\mathbf{S} = \mathbf{S}_{\mathcal{I}}$ and $\mathcal{D} = \mathcal{D}_{\mathcal{I}}$) to a suitable sub-subsequence $\{i_n\}_{n\in\mathbf{N}}$ of each infinite $\mathcal{I}^* \subset \mathbf{I}$, and (2) Applying our argument about the finite index set $\mathbf{I}_0$ of exceptions in the proof of Theorem 8.22. A formal proof is omitted to avoid redundancy. □

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Appendix A

Background

This appendix presents a series of background results used in Chapters 3 - 8. We limit our discussions to the most necessary material.

§A.1, §A.2 and §A.3 are about the point-set topology. More details are found in [Munkres, 2000, Chapter 1 - 7], [Bogachev, 2007, Vol. II, Chapter 6], [Ethier and Kurtz, 1986, §3.4] and Blount and Kouritzin [2010]. §A.4 deals with weak topology of Borel measures in the spirit of [Ethier and Kurtz, 1986, §3.1 - 3.4], [Kallianpur and Xiong, 1995, Chapter 1] and Blount and Kouritzin [2010]. §A.5 discusses standard Borel property of topological spaces and their subsets, where we refer the readers to [Srivastava, 1998, §3.3] and [Bogachev, 2007, Vol. II, Chapter 6] for further materials. §A.6 gives a short review of Skorokhod $\mathscr{J}_1$-spaces. Excellent treatments of this topic are available in [Ethier and Kurtz, 1986, §3.5 - 3.10], Jakubowski [1986], Blount and Kouritzin [2010] and Kouritzin [2016]. §A.7 reviews some basic properties of càdlàg processes. Moreover, a collection of miscellaneous results about these topics are presented in §B.2 of Appendix B.

This appendix complies with all our notations, terminologies and conventions introduced before. Several general technicalities used herein are provided in §B.1 of Appendix B.

A.1 Point-set topology

A.1.1 Separability

Separability of a topological space E refers to the separation of points by open sets. E is a Hausdorff space if for any distinct $x, y \in E$, there exist disjoint $O_x, O_y \in \mathscr{O}(E)$[1] such that $x \in O_x$ and $y \in O_y$. From this definition one immediately observes that:

Fact A.1. *Any topological refinement[2] of a Hausdorff space is also Hausdorff.*

E is a T3 space if E is a Hausdorff space and for any $x \in E$ and $F \in \mathscr{C}(E)$ excluding x, there exist disjoint $O_x, O_F \in \mathscr{O}(E)$ such that $x \in O_x$ and $F \subset O_F$. E is a T4 space if E is a Hausdorff space and for any disjoint $F_1, F_2 \in \mathscr{C}(E)$, there exist disjoint $O_1, O_2 \in \mathscr{O}(E)$ such that $F_1 \subset O_1$ and $F_2 \subset O_2$. Below are several basic properties of Hausdorff, T3 and T4 spaces.

Proposition A.2. *The following statements are true:*

(a) Any finite subset of a Hausdorff space is closed.

(b) The families of T4, T3 and Hausdorff spaces are successively larger.

(c) Subspaces of a T3 or Hausdorff space are T3 or Hausdorff spaces respectively. Moreover, closed subsets of a T4 space are T4 subspaces.

(d) Any product space of T3 or Hausdorff spaces is a T3 or Hausdorff space respectively.

Proof. (a) was proved in [Munkres, 2000, Theorem 17.8]. (b) is immediate by (a). (c) was justified in [Munkres, 2000, Theorem 31.2 and §32, Exercise 1]. (d) was proved in [Munkres, 2000, Theorem 31.2]. □

A.1.2 Countability

Intuitively, countability of a topological space E describes the "number of open sets". E is first-countable if for each $x \in E$, there exists a countable

[1] $\mathscr{O}(E)$ and $\mathscr{C}(E)$ denotes the family of all open and closed subsets of E respectively.
[2] The terminology "topological refinement" was introduced in §2.1.3.

$\mathcal{O}_x \subset \mathscr{O}(E)$ such that $O \ni x$ for all $O \in \mathcal{O}_x$ and any $U \in \mathscr{O}(E)$ containing x is the superset of some member of $\mathcal{O}_x$. E is separable if E has a countable dense subset. E is a Lindelöf space if any $\{O_i\}_{i\in\mathbf{I}} \subset \mathscr{O}(E)$ satisfying $E = \bigcup_{i\in\mathbf{I}} O_i$ admits a countable subset $\{O_{i_n}\}_{n\in\mathbf{N}}$ satisfying $E = \bigcup_{n\in\mathbf{N}} O_{i_n}$. E is a hereditary Lindelöf space if any subspace of E is a Lindelöf space. E is a second-countable space if it admits a countable topological basis. Below are some basic facts about the countability notions above.

Proposition A.3. *The following statements are true:*

(a) Subspaces of a first-countable, second-countable or heredetary Lindelöf space are first-countable, second-countable or heredetary Lindelöf spaces, respectively. Moreover, closed subsets of a Lindelöf space are Lindelöf subspaces.

(b) Every second-countable space is first-countable, separable and heriditary Lindelöf simultaneously.

(c) The product space of countably many first-countable, second-countable or separable spaces is first-countable, second-countable or separable, respectively.

(d) The image of a separable, Lindelöf or heriditary Lindelöf space under a continuous mapping is separable, Lindelöf or heriditary Lindelöf, respectively.

(e) The union of countably many separable, Lindelöf or hereditary Lindelöf subspaces is a separable, Lindelöf or hereditary Lindelöf subspace, respectively.

Proof. (a) was proved in [Munkres, 2000, Theorem 30.2 and §30, Exercise 9].

(b) Second-countable spaces are first-countable by definition. They are separable and Lindelöf by [Munkres, 2000, Theorem 30.3]. Moreover, they are hereditary Lindelöf spaces by (a).

(c) was justified in [Munkres, 2000, Theorem 30.2 and §30, Exercise 10].

(d) Let E and S be topological spaces and $f \in C(E; S)$ be surjective. If E is separable or Lindelöf, then S is also by [Munkres, 2000, §30, Exercise 11].

Now let E be hereditary Lindelöf, $A \subset S$ and $B \doteq f^{-1}(A)$. $(B, \mathscr{O}_E(B))$ is Lindelöf and $f|_B \in C(B, \mathscr{O}_E(B); A, \mathscr{O}_S(A))$ is surjective. Hence, $(A, \mathscr{O}_S(A))$ is also Lindelöf.

(e) Let $E = \bigcup_{n\in\mathbf{N}} A_n$ be a topological space. If $\{A_n\}_{n\in\mathbf{N}}$ are all separable subspaces, then there exist $\{x_{n,i}\}_{n,i\in\mathbf{N}} \subset E$ such that $\{x_{n,i}\}_{i\in\mathbf{N}}$ is dense in A_n for all $n \in \mathbf{N}$. Hence, $\{x_{n,i}\}_{n,i\in\mathbf{N}}$ is a countable dense subset of E. If $\{A_n\}_{n\in\mathbf{N}}$ are all Lindelöf subspaces of E and $E = \bigcup_{i\in\mathbf{I}} O_i$ with $\{O_i\}_{i\in\mathbf{I}} \subset \mathscr{O}(E)$, then there exist $\{O_{i_{n,k}}\}_{k\in\mathbf{N}} \subset \{O_i\}_{i\in\mathbf{I}}$ satisfying $A_n \subset \bigcup_{k\in\mathbf{N}} O_{i_{n,k}}$ for each $n \in \mathbf{N}$. Hence, $E = \bigcup_{n,k\in\mathbf{N}} O_{i_{n,k}}$. So far we proved the conclusions about separable and Lindelöf subspaces. That about hereditary Lindelöf subspaces follows immediately. □

A.1.3 Metrizability

The metric space is the most well-known type of topological spaces and we merely recall a few necessary facts. Let $(E, \mathfrak{r})$ and $(S, \mathfrak{d})$ be metric spaces. $f \in S^E$ is an *isometry between* $(E, \mathfrak{r})$ *and* $(S, \mathfrak{d})$ if f is a surjective and $\mathfrak{r}(x, y) = \mathfrak{d}(f(x), f(y))$ for all $x, y \in E$. $(E, \mathfrak{r})$ and $(S, \mathfrak{d})$ are *isometric* if there exists an isometry between them. $(S, \mathfrak{d})$ is a completion of $(E, \mathfrak{r})$ if $(S, \mathfrak{d})$ is complete and E is isometric to a dense subspace of S.

Note A.4. Without loss of generality, a metric space $(E, \mathfrak{r})$ can always be treated as a dense subset of its completion $(S, \mathfrak{d})$ (if any) such that $\mathfrak{d}$ restricted to $E \times E$ equals $\mathfrak{r}$.

Below are several facts about the completeness of metric spaces.

Proposition A.5. *Let* $(E, \mathfrak{r})$ *and* $(S, \mathfrak{d})$ *be metric spaces. Then, the following statements are true:*

(a) If $(E, \mathfrak{r})$ *and* $(S, \mathfrak{d})$ *are isometric, then they are homeomorphic. In particular,* $(E, \mathfrak{r})$ *is complete precisely when* $(S, \mathfrak{d})$ *is complete.*

(b) There exists a unique completion of $(E, \mathfrak{r})$ *up to isometry.*

(c) If $(E, \mathfrak{r})$ *is complete, then the closure of* $A \subset E$ *equipped with (the restricted metric of)* $\mathfrak{r}$ *is the completion of* A.

Proof. (a) Any isometry between E and S maps their open balls bijectively and equates Cauchy property or convergence of their sequences.

(b) was proved in [Munkres, 2000, Theorem 43.7].

(c) If $(E, \mathfrak{r})$ is complete, then every Cauchy sequence in A's closure $\overline{A}$ converges to a member of $\overline{A}$. $\square$

If E's topology is the same as the topology of $(E, \mathfrak{r})$ for some metric $\mathfrak{r}$, then E is said to be *metrizable*, $\mathfrak{r}$ is said to *metrize* E and $(E, \mathfrak{r})$ is called a *metrization* of E. Below are several elementary results about metrizable spaces.

Proposition A.6. *The following statements are true:*

(a) Metrizable spaces are T4 (hence T3 and Hausdorff) spaces.

(b) Subspaces of a metrizable space are metrizable.

(c) Subspaces of a metrizable and separable space are metrizable and second-countable (hence separable and hereditary Lindelöf).

(d) Homeomprhs of metrizable spaces are metrizable.

Proof. (a) follows by [Munkres, 2000, Theorem 32.2] and Proposition A.2 (a). (b) was justified in [Munkres, 2000, §21, Exercise 1]. (c) follows by [Munkres, 2000, §30, Exercise 5], (b) and Proposition A.3 (a, b). Regarding (d), we note that if $(E, \mathfrak{r})$ is a metric space and $f \in \mathbf{hom}(E; S)$, then S is metrized by the metric $\mathfrak{d}$ defined by $\mathfrak{d}(x, y) \doteq \mathfrak{r}(f^{-1}(x), f^{-1}(y))$ for each $x, y \in S$. $\square$

The next two results summarize metrizability of countable Cartesian products.

Proposition A.7. *Let $\{(S_i, \mathfrak{r}_i)\}_{i \in \mathbf{I}}$ be metric spaces and $S \doteq \prod_{i \in \mathbf{I}} S_i$. Then, the following statements are true:*

(a) When $\mathbf{I} = \{1, ..., d\}$, S is metrized by[3]

$$\mathfrak{r}^d(x, y) \doteq \max_{1 \leq i \leq d} \mathfrak{r}_i \left(\mathfrak{p}_i(x), \mathfrak{p}_i(y)\right), \ \forall x, y \in S. \tag{A.1.1}$$

If $\{(S_i, \mathfrak{r}_i)\}_{1 \leq i \leq d}$ are all complete, then $(S, \mathfrak{r}^d)$ is also.

[3] The notation "$\mathfrak{p}_i$" as defined in §2.1.1 denotes one-dimensional projection on S for $i \in \mathbf{I}$.

(b) When $\mathbf{I} = \mathbf{N}$, *S is metrized by*

$$\mathfrak{r}_1^\infty(x,y) \doteq \sup_{i\in\mathbf{N}} i^{-1}\left[\mathfrak{r}_i\left(\mathfrak{p}_i(x),\mathfrak{p}_i(y)\right)\wedge 1\right],\ \forall x,y\in S, \tag{A.1.2}$$

or alternatively by

$$\mathfrak{r}_2^\infty(x,y) \doteq \sum_{i=1}^{\infty} 2^{-i+1}\left[\mathfrak{r}_i\left(\mathfrak{p}_i(x),\mathfrak{p}_i(y)\right)\wedge 1\right],\ \forall x,y\in S. \tag{A.1.3}$$

If $\{(S_i,\mathfrak{r}_i)\}_{i\in\mathbf{N}}$ *are all complete, then* $(S,\mathfrak{r}_1^\infty)$ *and* $(S,\mathfrak{r}_2^\infty)$ *are also.*

Proof. (a) was proved in [Munkres, 2000, §21, Exercise 3 and Theorem 43.5].

(b) We prove $\mathfrak{r}_2^\infty$ metrizes S and the case for $\mathfrak{r}_1^\infty$ follows by a similar argument. Let $x\in S$, $\epsilon\in(0,1)$, $\{i_1,...,i_d\}\subset\mathbf{N}$,

$$B_i^{x,\epsilon} \doteq \begin{cases}\{y\in S_i : \mathfrak{r}_i\left(\mathfrak{p}_i(x),y\right)<\epsilon\}, & \text{if } i\in\{i_1,...,i_d\},\\ S_i, & \text{otherwise}\end{cases} \tag{A.1.4}$$

and $N \doteq \max\{i_1,...,i_d\}$. From the fact

$$\left\{y\in S : \mathfrak{r}_2^\infty(x,y) < \frac{d\epsilon}{2^{N-1}}\right\} \subset \prod_{i\in\mathbf{N}} B_i^{x,\epsilon} \subset \left\{y\in S : \mathfrak{r}_2^\infty(x,y) < d\epsilon + 2^{-d}\right\} \tag{A.1.5}$$

and the openness of the left side of (A.1.5) it follows that $\mathfrak{r}_2^\infty$ metrizes S. When $\{(S_i,\mathfrak{r}_i)\}_{i\in\mathbf{N}}$ are all complete, the completeness of $(S,\mathfrak{r}_1^\infty)$ and $(S,\mathfrak{r}_2^\infty)$ follows by the argument establishing [Munkres, 2000, Theorem 43.4]. □

Proposition A.8. *Let* $\mathbf{I}$ *be a countable index set,* $\{S_i\}_{i\in\mathbf{I}}$ *be topological spaces and* $S \doteq \prod_{i\in\mathbf{I}} S_i$. *Then, A is a metrizable subspace of S if and only if* $\mathfrak{p}_i(A)$ *is a metrizable subspace of* S_i *for all* $i\in\mathbf{I}$.

Proof. Necessity follows by the argument establishing [Aliprantis and Border, 2006, Theorem 3.36]. Regarding sufficiency, we get the metrizability of $\prod_{i\in\mathbf{I}}\mathfrak{p}_i(A)$ from Proposition A.7 (with $S_i = \mathfrak{p}_i(A)$) and then get that of $A\subset\prod_{i\in\mathbf{I}}\mathfrak{p}_i(A)$ by Proposition A.6 (b). □

The following property of first-countable and metrizable spaces is indispensable.

Fact A.9. *Let E be a topological space, $x \in E$ and $A \subset E$. Then, the following statements are true:*

(a) If E is metrizable, then E is first-countable.

(b) If there exist $\{x_n\}_{n\in\mathbf{N}} \subset A$ converging to x in E, then x is a limit point[4] *of A. The converse is true when E is first-countable space.*

Proof. This fact was proved by [Munkres, 2000, p.131 and Theorem 30.1]. □

A subset A of metric space $(E, \mathfrak{r})$ is said to be totally bounded if for any $\epsilon \in (0, \infty)$, there exists an $A_\epsilon \in \mathscr{P}_0(E)$[5] such that $E = \bigcup_{x\in A_\epsilon}\{y \in E : \mathfrak{r}(x,y) < \epsilon\}$. We used in §6.5 the following properties of totally bounded sets.

Proposition A.10. *Let $(E, \mathfrak{r})$ be a metric space. Then, the following statements are true:*

(a) If $A \subset E$ is totally bounded, $(A, \mathscr{O}_E(A))$ is a second-countable space.

(b) The union of finitely many totally bounded subsets of E is totally bounded.

(c) If $A \subset E$ is totally bounded, then its closure is also.

Proof. (a) For each $p \in \mathbf{N}$, we pick an $A_p \in \mathscr{P}_0(E)$ such that $A \subset \bigcup_{x\in A_p}\{y \in E : \mathfrak{r}(x,y) < 2^{-p}\}$. Then, $\bigcup_{p\in\mathbf{N}} A_p$ is a countable dense subset of E. Now, (a) follows by Proposition A.6 (c).

(b) is immediate by the definition of totally bounded sets.

(c) For any $\epsilon > 0$, there exists an $A_\epsilon \in \mathscr{P}_0(A)$ such that $A \subset \bigcup_{x\in A_\epsilon}\{y \in E : \mathfrak{r}(x,y) < \epsilon/2\}$. Then, A's closure is contained in $\bigcup_{x\in A_\epsilon}\{y \in E : \mathfrak{r}(x,y) < \epsilon\}$. □

A.1.4 Polish, Lusin and Souslin spaces

Polish, Lusin and Souslin spaces are topological variations of complete separable metric spaces. Homeomorphs of complete separable metric spaces are called Polish spaces. E is a Lusin space (resp. Souslin space) if E is a

[4] The notion of "limit point" was mentioned in p.26.

[5] $\mathscr{P}_0(E)$ denotes the family of all finite subsets of E.

Hausdorff space and there exists a bijective (resp. surjective) $f \in C(S;E)$ with S being a Polish space. The following are several basic facts about these spaces above and §A.5 will discuss the measurability aspect of Polish and Lusin spaces further.

Proposition A.11. *The following statements are true:*

(a) Every Polish (resp. Lusin) space is a Lusin (resp. Souslin) space.

(b) Open and closed subsets of a Polish, Lusin or Souslin space are Polish, Lusin or Souslin subspaces, respectively.

(c) Subspaces of a Polish space are metrizable and second-countable.

(d) Subspaces of a Polish, Lusin or Souslin space are separable and heriditary Lindelöf.

(e) A metric space is separable if and only if its completion is a Polish space.

(f) The product space of countably many Polish, Lusin or Souslin spaces is a Polish, Lusin or Souslin space, respectively. In particular, $\mathbf{R}^\infty$ *and its subspace* $\mathbf{N}^\infty$ *are Polish spaces.*

(g) The intersection or union of countably many Souslin subspaces is a Souslin subspace.

Proof. (a) Note that the identity mapping is a continuous bijection between any topological space and itself.

(b) Let E be a topological space, $O \in \mathscr{O}(E)$ and $F \in \mathscr{C}(E)$. If E is a Polish space, then O and F are Polish subspaces of E by [Bogachev, 2007, Vol. II, Example 6.1.11]. If E is a Lusin (resp. Souslin) space, then there exist a Polish space S and a bijective (resp. surjective) $f \in C(S;E)$. $f^{-1}(O) \in \mathscr{O}(S)$ and $f^{-1}(F) \in \mathscr{C}(S)$ are Polish subspaces of S. $f|_{f^{-1}(O)}$ and $f|_{f^{-1}(F)}$ are bijective (resp. surjective) continuous mappings from $f^{-1}(O)$ and $f^{-1}(F)$ to O and F respectively. Thus, O and F are Lusin (resp. Souslin) subspaces of E.

(c) follows by Proposition A.6 (c, d).

(d) follows by (c) and Proposition A.3 (b, d).

(e) Sufficiency follows by Proposition A.3 (a). For necessity, we note that any countable dense subset of a metric space is also dense in its completion.

(f) If $\{S_n\}_{n\in\mathbf{N}}$ are all Polish spaces, then $\prod_{n\in\mathbf{N}} S_n$ is a Polish space by [Srivastava, 1998, §2.2, (v), p.52]. If $\{S'_n\}_{n\in\mathbf{N}}$ are all Lusin (resp. Souslin) spaces and $\{f_n \in C(S_n; S'_n)\}_{n\in\mathbf{N}}$ are bijective (resp. surjective), then $\bigotimes_{n\in\mathbf{N}} f_n \circ \mathfrak{p}_n \in C(\prod_{n\in\mathbf{N}} S_n; \prod_{n\in\mathbf{N}} S'_n)$[6] is bijective by Fact 2.4 (a, b). Hence, $\prod_{n\in\mathbf{N}} S'_n$ is a Lusin (resp. Souslin) space.

(g) was proved in [Bogachev, 2007, Vol. II, Theorem 6.6.6]. □

A.1.5 Compactness

Let E be a topological space and $A \subset E$ be non-empty. E is compact if any $\{O_i\}_{i\in\mathbf{I}} \subset \mathscr{O}(E)$ satisfying $E = \bigcup_{i\in\mathbf{I}} O_i$ admits a finite subset $\{O_{i_j}\}_{1\leq j\leq n}$ satisfying $E = \bigcup_{j=1}^n O_{i_j}$. E is sequentially compact (resp. limit point compact) if any infinite subset of E has a convergent subsequence (resp. a limit point). A is a compact, sequentially compact or limit point compact subset of E if $(A, \mathscr{O}_E(A))$ is compact, sequentially compact or limit point compact, respectively. A is a *precompact*[7] subset of E if A's closure is compact. E is locally compact if for any $x \in E$, there exist $K_x \in \mathscr{K}(E)$[8] and $O_x \in \mathscr{O}(E)$ such that $x \in O_x \subset K_x$. Compact spaces have the convenient properties below.

Proposition A.12. *The following statements are true:*

(a) Closed subsets of a compact space are compact. Moreover, compact subsets of a Hausdorff space are closed and hence Borel subsets.

(b) The union of finitely many compact subsets is compact. Moreover, any product space of compact spaces is a compact space.

(c) Compact metric spaces are complete.

(d) Hausdorff (resp. metrizable) compact spaces are T4 (resp. Polish) spaces.

[6] $\mathfrak{p}_n$ herein denotes the projection on $\mathbf{R}^\infty$ for $n \in \mathbf{N}$.

[7] Having a compact closure is commonly known as relative compactness. Herein, we use the alternative terminology "precompactness" to distinguish this notion from the relative compactness about finite Borel measures.

[8] $\mathscr{K}(E)$ denotes the family of all compact subsets of E.

(e) Compact spaces are Lindelöf spaces. Moreover, the image of a compact space under any continuous mapping is a compact space.

(f) Compactness implies limit point compactness. Moreover, compactness, sequential compactness and limit point compactness are equivalent in metrizable spaces.

Proof. (a) was proved in [Munkres, 2000, Theorem 26.2 and Theorem 26.3]. The first statement of (b) was proved in [Munkres, 2000, §26, Exercise 3]. The second statement of (b) is the well-known Tychonoff Theorem (see [Munkres, 2000, Theorem 37.3]). (c) was justified in [Munkres, 2000, Theorem 45.1 and §30, Exercise 4]. (d) follows by [Munkres, 2000, Theorem 32.3] and (c). The first statement of (e) is automatic. The second statement of (e) was proved in [Munkres, 2000, Theorem 26.5]. (f) was proved in [Munkres, 2000, Theorem 28.1 and Theorem 28.2]. □

Corollary A.13. *Let $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $(S,\mathscr{A})$ be as in (2.7.22). Then, the following statements are true:*

(a) If $A_i \in \mathscr{K}(S_i)$ for all $i \in \mathbf{I}$, then $\prod_{i\in\mathbf{I}} A_i \in \mathscr{K}(S)$. If, in addition, $\mathbf{I}$ is countable and $\{S_i\}_{i\in\mathbf{I}}$ are all Hausdorff spaces, then $A_i \in \mathscr{B}(S_i)$ for all $i \in \mathbf{I}$ and $\prod_{i\in\mathbf{I}} A_i \in \mathscr{A}$.

(b) If $A \in \mathscr{K}(S)$, then $\mathfrak{p}_i(A) \in \mathscr{K}(S_i)$ for all $i \in \mathbf{I}$. If, in addition, $\{S_i\}_{i\in\mathbf{I}}$ are all Hausdorff spaces, then $\mathfrak{p}_i(A) \in \mathscr{B}(S_i)$ for all $i \in \mathbf{I}$.

Proof. (a) The first part follows by Proposition A.12 (b). The second part follows by Proposition A.12 (a), Fact 2.3 (a) and the fact $\prod_{i\in\mathbf{I}} A_i = \bigcap_{i\in\mathbf{I}} \mathfrak{p}_i^{-1}(A_i)$.

(b) The first part follows by Fact 2.4 (a) and Proposition A.12 (e). The second part follows by Proposition A.12 (a). □

Corollary A.14. *Let $\mathbf{I}$ be a countable index set, $\{S_i\}_{i\in\mathbf{I}}$ be Hausdorff spaces, $(S,\mathscr{A})$ be as in (2.7.22), $\Gamma \subset \mathfrak{M}^+(S,\mathscr{A})$ and $\mu' \in \mathfrak{bc}(\mu)$ for each $\mu \in \Gamma$. Then, $\{\mu'\}_{\mu\in\Gamma}$ is tight or sequentially tight[9] if and only if Γ is.*

[9] The notion of sequential tightness was introduced in §5.1.

Proof. We only prove the tightness of $\{\mu'\}_{\mu\in\Gamma}$ implies that of Γ and the rest is immediate. Given the tightness of $\{\mu'\}_{\mu\in\Gamma}$, there exist $\{K_p\}_{p\in\mathbf{N}} \subset \mathscr{K}(S)$ such that $\sup_{\mu\in\Gamma}\mu'(S\backslash K_p) \geq 1-2^{-p}$ for all $p\in\mathbf{N}$. It follows by Corollary A.13 (a) that $F_p \circeq \prod_{i\in\mathbf{I}}\mathfrak{p}_i(K_p) \in \mathscr{A}\cap\mathscr{K}(S)$ for all $\forall p\in\mathbf{N}$. Now, we have that

$$\sup_{\mu\in\Gamma}\mu(S\backslash F_p) = \sup_{\mu\in\Gamma}\mu'(S\backslash F_p) \leq \sup_{\mu\in\Gamma}\mu'(S\backslash K_p) \geq 1-2^{-p},\ \forall p\in\mathbf{N}. \tag{A.1.6}$$

□

Corollary A.15. *Let $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $(S,\mathscr{A})$ be as in (2.7.22). Then, the following statements are true:*

(a) If $\mathbf{I}$ is countable, $A_i\in\mathscr{K}_\sigma(S_i)$ for all $i\in\mathbf{I}$ and $\{S_i\}_{i\in\mathbf{I}}$ are all Hausdorff spaces, then $A_i\in\mathscr{B}(S_i)$ for all $i\in\mathbf{I}$ and $\prod_{i\in\mathbf{I}}A_i\in\mathscr{A}$.

(b) If $\mathbf{I}$ is finite and $A_i\in\mathscr{K}_\sigma(S_i)$ for all $i\in\mathbf{I}$, then $\prod_{i\in\mathbf{I}}A_i\in\mathscr{K}_\sigma(S)$.

(c) If $A\in\mathscr{K}_\sigma(S)$, then $\mathfrak{p}_i(A)\in\mathscr{K}_\sigma(S_i)$ for all $i\in\mathbf{I}$. If, in addition, $\{S_i\}_{i\in\mathbf{I}}$ are all Hausdorff spaces, then $\mathfrak{p}_i(A)\in\mathscr{B}(S_i)$ for all $i\in\mathbf{I}$.

Proof. (a) follows by Proposition A.12 (a), Fact 2.3 (a) and the fact $\prod_{i\in\mathbf{I}}A_i = \bigcap_{i\in\mathbf{I}}\mathfrak{p}_i^{-1}(A_i)$.

(b) follows by a similar argument to the proof of Lemma 3.61 (b).

(c) Let $A=\bigcup_{p\in\mathbf{N}}K_p$ with $\{K_p\}_{p\in\mathbf{N}}\subset\mathscr{K}(S)$. We have that

$$\mathfrak{p}_i(A) = \mathfrak{p}_i\left(\bigcup_{p\in\mathbf{N}}K_p\right) = \bigcup_{p\in\mathbf{N}}\mathfrak{p}_i(K_p)\in\mathscr{K}_\sigma(S_i),\ \forall i\in\mathbf{I} \tag{A.1.7}$$

by Corollary A.13 (b). The second part of (c) follows by Proposition A.12 (a). □

We used in §6.5 the following connection of total boundedness and compactness.

Proposition A.16. *Compactness of a metric space is equivalent to total boundedness plus completeness.*

Proof. This result was proved in [Munkres, 2000, Theorem 45.1]. □

A.2 Point-separation properties of functions

§2.2.4 introduced three functional separabilities of points: separating points, strongly separating points and determining point convergence. The following proposition specifies the relationship among these three separabilities.

Proposition A.17. *Let E be a topological space, $A \subset E$ be non-empty, $\mathcal{D} \subset \mathbf{R}^E$ and $d \in \mathbf{N}$. Then, the following statements are true:*

(a) If $\{\{x\} : x \in A\} \subset \mathscr{C}(E)$, especially if A is a Hausdorff subspace of A, then $\mathcal{D}$ strongly separating points on A implies $\mathcal{D}$ separating points on A.

(b) If $\mathcal{D}$ strongly separates points on A, then $\mathcal{D}$ determines point convergence on A. The converse is true when $(A, \mathscr{O}_E(A))$ is a Hausdorff space.

(c) $\mathcal{D}$ separates points on A if and only if $(A, \mathscr{O}_{\mathcal{D}}(A))$[10] is a Hausdorff space.

(d) $\mathscr{O}_{\mathcal{D}}(A)$ is induced by pseudometrics $\{\rho_{\{f\}}\}_{f \in \mathcal{D}}$[11]. If $\mathcal{D}$ is countable, then $(A, \mathscr{O}_{\mathcal{D}}(A))$ is a second-countable space pseudometrized by $\rho_{\mathcal{D}}$. If, in addition, $\mathcal{D}$ separates points on A, then $\mathscr{O}_{\mathcal{D}}(A)$ is metrized by $\rho_{\mathcal{D}}$.

(e) If $\mathcal{D}|_A \subset C(A, \mathscr{O}_E(A); \mathbf{R})$ separates points (resp. strongly separates points) on A, then $(A, \mathscr{O}_E(A))$ is a Hausdorff space (resp. $\mathscr{O}_E(A) = \mathscr{O}_{\mathcal{D}}(A)$).

Proof. (a) The Hausdorff property of $(A, \mathscr{O}_E(A))$ (if any) implies $\{\{x\} : x \in A\} \subset \mathscr{C}(A, \mathscr{O}_E(A))$ by Proposition A.2 (a). We then prove (a) by contradiction. If $\mathcal{D}$ fails to separate points on A, then there exist distinct $x, y \in A$ such that $\bigotimes \mathcal{D}(x) = \bigotimes \mathcal{D}(y)$. Since $\{y\}$ is a closed set and $\mathcal{D}$ strongly separates points on A, there exist $\mathcal{D}_x \in \mathscr{P}_0(\mathcal{D})$ and $\epsilon \in (0, \infty)$ such that $y \in \{z \in A : \max_{f \in \mathcal{D}_x} |f(x) - f(z)|\} < \epsilon \subset A \backslash \{y\}$. Contradiction!

(b) was proved in [Blount and Kouritzin, 2010, Lemma 4].

[10] The notation $\mathscr{O}_{\mathcal{D}}(A)$ was introduced in §2.1.3.

[11] The pseudometrics $\rho_{\mathcal{D}}$ and $\rho_{\mathcal{D}}^d$ were defined in §2.2.4 and $\rho_{\{f\}}$ refers to $\rho_{\mathcal{D}}$ with $\mathcal{D} = \{f\}$. The meaning of $\{\rho_{\{f\}} : f \in \mathcal{D}\}$ inducing $\mathscr{O}_{\mathcal{D}}(A)$ was explained in §2.1.3.

(c - Sufficiency) follows by (a) (with $A = (E, \mathscr{O}_{\mathcal{D}}(E))$).

(c - Necessity) Let $x_1, x_2 \in A$ be distinct. Since $\mathcal{D}$ separates points on A, there exists an $f \in \mathcal{D}$ such that $\epsilon_0 \doteq |f(x) - f(y)| > 0$. Then, we define $O_i \doteq \{z \in A : |f(x) - f(z)| < \frac{\epsilon_0}{3}\} \in \mathscr{O}_{\mathcal{D}}(A)$ for each $i = 1, 2$ and observe that $x_1 \in \mathscr{O}_1$, $x_2 \in \mathscr{O}_2$ and $\mathscr{O}_1 \cap \mathscr{O}_2 = \varnothing$.

(d) The first statement follows by the fact that

$$\left\{y \in A : \max_{f \in \mathcal{D}_0} |f(x) - f(y)| < 2^{-n}\right\} = \bigcap_{f \in \mathcal{D}_0} \left\{y \in A : \rho_{\{f\}}(x, y) < 2^{-n}\right\} \tag{A.2.1}$$

for all $\mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D})$, $x \in A$ and $n \in \mathbf{N}$. If $\mathcal{D} = \{f_j\}_{j \in \mathbf{N}}$ is countable, then

$$\left\{\left\{y \in A : \max_{1 \le j \le m} |f_j(x) - f_j(y)| < 2^{-n}\right\} : x \in A, m, n \in \mathbf{N}\right\} \tag{A.2.2}$$

defines a countable topological basis of $\mathscr{O}_{\mathcal{D}}(A)$. From the fact

$$\begin{aligned} &\left\{y \in A : \rho_{\mathcal{D}}(x, y) < 2^{-m-n+1}\right\} \\ &\subset \left\{y \in A : \max_{1 \le j \le m} |f_j(x) - f_j(y)| < 2^{-n}\right\} \\ &\subset \left\{y \in A : \rho_{\mathcal{D}}(x, y) < m2^{-n} + 2^{-m}\right\}, \ \forall x \in A, m, n \in \mathbf{N} \end{aligned} \tag{A.2.3}$$

it follows that $\rho_{\mathcal{D}}$ induces $\mathscr{O}_{\mathcal{D}}(E)$. If, in addition, $\mathcal{D}$ separates points on A, then $\rho_{\mathcal{D}}(x, y) = 0$ implies $\bigotimes \mathcal{D}(x) = \bigotimes \mathcal{D}(y)$ and so $x = y$, thus proving $\rho_{\mathcal{D}}$ is a metric.

(e) follows by the observation $\mathscr{O}_{\mathcal{D}}(A) \subset \mathscr{O}_E(A)$, (c) and Fact A.1. $\square$

Corollary A.18. *Let E be a topological space, $\mathcal{D} \subset \mathbf{R}^E$ be countable and $d \in \mathbf{N}$. Then, $(E^d, \mathscr{O}_{\mathcal{D}}(E)^d)$ is a second-countable space pseudometrized by $\rho_{\mathcal{D}}^d$. If, in addition, $\mathcal{D}$ separates points on E, then $\mathscr{O}_{\mathcal{D}}(E)^d$ is metrized by $\rho_{\mathcal{D}}^d$.*

Proof. This corollary follows by Proposition A.17 (d) and the argument establishing Proposition A.7 (a). $\square$

Corollary A.19. *Let E be a topological space and the members of $\mathcal{D}_0 \subset \mathbf{R}^E$ and $\mathcal{D} \subset \mathbf{R}^E$ are bounded. If $\mathcal{D} \subset \mathfrak{cl}(\mathcal{D}_0)$*[12]*, then $\mathscr{O}_{\mathcal{D}}(E) \subset \mathscr{O}_{\mathcal{D}_0}(E)$. In*

[12] Recall that "$\mathfrak{cl}(\cdot)$" refers to closure under supremum norm.

particular, if $\mathcal{D}$ separates points or strongly separates points on E, then $\mathcal{D}_0$ does also.

Proof. $\mathcal{D} \subset C(E, \mathscr{O}_{\mathcal{D}_0}(E); \mathbf{R})$ by [Munkres, 2000, Theorem 43.6] and so $\mathscr{O}_{\mathcal{D}}(E) \subset \mathscr{O}_{\mathcal{D}_0}(E)$. If $\mathcal{D}$ separates points on E, then $(E, \mathscr{O}_{\mathcal{D}}(E))$ is a Hausdorff space by Proposition A.17 (c) and $(E, \mathscr{O}_{\mathcal{D}_0}(E))$ is also by Fact A.1. So, $\mathcal{D}_0$ separates points on E by Proposition A.17 (c) (with $A = E$ and $\mathcal{D} = \mathcal{D}_0$). That $\mathcal{D}$ strongly separates points on E implies $\mathscr{O}(E) \subset \mathscr{O}_{\mathcal{D}}(E) \subset \mathscr{O}_{\mathcal{D}_0}(E)$, so $\mathcal{D}_0$ strongly separates points on E too. ☐

The next lemma shows that strongly separating points and separating points coincide for continuous functions on compact spaces.

Lemma A.20. *Let E be a compact space and $\mathcal{D} \subset C(E; \mathbf{R})$. Then, E is a Hausdorff space and $\mathcal{D}$ strongly separates points on E if and only if $\mathcal{D}$ separates points on E.*

Proof. (Necessity) follows by Proposition A.17 (a).

(Sufficiency) E is a Hausdorff space by Proposition A.17 (e) (with $A = E$). So, we need only show $\mathcal{D}$ determines point convergence on E by Proposition A.17 (b). Suppose $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in \mathcal{D}$. $\{x_n\}_{n\in\mathbf{N}}$ has at least one convergent subsequence by the compactness of E and Proposition A.12 (f). If $x_{n_k} \to y \in E$ as $k \uparrow \infty$, then $f(y) = \lim_{k\to\infty} f(x_{n_k}) = f(x)$ for all $f \in \mathcal{D}$. This implies $x = y$ and $x_{n_k} \to x$ as $k \uparrow \infty$ as $\mathcal{D}$ separates points on E. ☐

Below are two useful properties of the function class $\Pi^d(\mathcal{D})$ introduced in §2.2.3.

Proposition A.21. *Let E be a topological space and $d \in \mathbf{N}$. Then, the following statements are true:*

(a) Any $\mathcal{D} \subset \mathbf{R}^E$ satisfies[13]

$$\mathfrak{ag}\left[\Pi^d(\mathcal{D})\right] \subset \left[C\left(E^d, \mathscr{O}_{\mathcal{D}}(E)^d; \mathbf{R}\right) \cap M\left(E^d, \sigma(\mathcal{D})^{\otimes d}; \mathbf{R}\right)\right]. \quad \text{(A.2.4)}$$

[13]The notations "$\mathfrak{ag}(\cdot)$" and "$\mathfrak{ca}(\cdot)$" were defined in §2.2.3.

If, in addition, the members of $\mathcal{D}$ are bounded, then,

$$\mathfrak{ca}\left[\Pi^d(\mathcal{D})\right] \subset \left[C_b\left(E^d, \mathscr{O}_{\mathcal{D}}(E)^d; \mathbf{R}\right) \cap M_b\left(E^d, \sigma(\mathcal{D})^{\otimes d}; \mathbf{R}\right)\right]. \quad \text{(A.2.5)}$$

(b) If $\mathcal{D}$ separates points (resp. determines point convergence) on E, then $\Pi^d(\mathcal{D})$[14] *separates points (resp. determines point convergence) on E^d.*

Remark A.22. Please be reminded that the σ-algebra $\sigma(\mathcal{D})$ induced by $\mathcal{D}$ is possibly smaller than the Borel σ-algebra $\mathscr{B}_{\mathcal{D}}(E)$ induced by $\mathcal{D}$ (see Fact B.5).

Remark A.23. $\Pi^d(\mathcal{D})$ does not only include functions like $\prod_{k=1}^d f_k \circ \mathfrak{p}_k$ with $\{f_k\}_{1\leq k\leq d} \subset \mathcal{D}$. Hence, we do not need $1 \in \mathcal{D}$ in Propoisition A.21 (b).

Proof of Proposition A.21. (a) For each $g \in \Pi^d(\mathcal{D})$, there exist $k \in \mathbf{N}$ and $f_1, ..., f_k \in \mathcal{D}$ such that $g = \varphi \circ (\bigotimes_{i=1}^k f_i \circ \mathfrak{p}_i)$ with $\varphi \mathrel{\hat{=}} \prod_{i=1}^k \mathfrak{p}_i \in \mathbf{R}^{\mathbf{R}^k}$.

$$\{\mathfrak{p}_i\}_{1\leq i\leq k} \subset \left[M\left(E^d, \sigma(\mathcal{D})^{\otimes d}; E, \sigma(\mathcal{D})\right) \cap C\left(E^d, \mathscr{O}_{\mathcal{D}}(E)^d; E, \mathscr{O}_{\mathcal{D}}(E)\right)\right] \quad \text{(A.2.6)}$$

by Fact 2.3 (a) and Fact 2.4 (a).

$$\bigotimes_{i=1}^k f_i \circ \mathfrak{p}_i \in \left[C\left(E^d, \mathscr{O}_{\mathcal{D}}(E)^d; \mathbf{R}^k\right) \cap M\left(E^d, \sigma(\mathcal{D})^{\otimes d}; \mathbf{R}^k, \mathscr{B}(\mathbf{R})^{\otimes k}\right)\right] \quad \text{(A.2.7)}$$

by Fact 2.3 (b) and Fact 2.4 (b). $\mathscr{B}(\mathbf{R}^k) = \mathscr{B}(\mathbf{R})^{\otimes k}$ by Proposition B.46 (d) (with $S_i = \mathbf{R}$), so

$$\varphi \in C(\mathbf{R}^k; \mathbf{R}) \subset M\left(\mathbf{R}^k, \mathscr{B}(\mathbf{R})^{\otimes k}; \mathbf{R}\right). \quad \text{(A.2.8)}$$

Now, (A.2.4) follows by (A.2.7) and (A.2.8). (A.2.5) follows by (A.2.4), Fact B.16 (b) and the fact that uniform convergence of functions preserves boundedness, continuity and measurability.

(b) If $x \neq y$ in E and $\mathcal{D}$ separates points on E, then there exist $1 \leq i \leq d$ and $f \in \mathcal{D}$ such that $\mathfrak{p}_i(x) \neq \mathfrak{p}_i(y)$ and $f \circ \mathfrak{p}_i(x) \neq f \circ \mathfrak{p}_i(y)$. So, $\Pi^d(\mathcal{D}) \ni f \circ \mathfrak{p}_i$ separates points on E^d. $\bigotimes \Pi^d(\mathcal{D})(x_n) \to \bigotimes \Pi^d(\mathcal{D})(x)$ as $n \uparrow \infty$ in $\mathbf{R}^{\Pi^d(\mathcal{D})}$ implies $\bigotimes \mathcal{D} \circ \mathfrak{p}_i(x_n) \to \bigotimes \mathcal{D} \circ \mathfrak{p}_i(x)$ as $n \uparrow \infty$ in $\mathbf{R}^{\mathcal{D}}$ for all $1 \leq i \leq d$ by Fact B.11. It follows that $\mathfrak{p}_i(x_n) \to \mathfrak{p}_i(x)$ as $n \uparrow \infty$ in E for all $1 \leq i \leq d$ since $\mathcal{D}$

[14]The definition of $\Pi^d(\mathcal{D})$ refers to (2.2.14).

determines point convergence on E. Hence, $x_n \to x$ as $n \uparrow \infty$ in E^d by Fact B.11. □

The following proposition describes two typical cases where one can select countably many (strongly) point-separating functions.

Proposition A.24. *Let E be a topological space and $\mathcal{D} \subset C(E;\mathbf{R})$. Then, the following statements are true:*

(a) If $\{(x,x) : x \in E\}$ is a Lindelöf subspace of $E \times E$ and $\mathcal{D}$ separates points on E, then there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ that separates points on E.

(b) If E is a second-countable space and $\mathcal{D}$ strongly separates points on E, then there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ that strongly separates points on E.

Proof. (a) follows by the argument establishing [Bogachev, 2007, Vol. II, Proposition 6.5.4]. (b) was proved in [Blount and Kouritzin, 2010, Lemma 2]. □

A.3 Tychonoff space and compactification

E is a Tychonoff (or T3$\frac{1}{2}$) space if E is a Hausdorff space and for any $x \in E$ and $F \in \mathscr{C}(E)$ that excludes x, there exists an $f_{x,F} \in C(E;[0,1])$ such that $f_{x,F}(x) = 0$ and (the image) $f_{x,F}(F) = \{1\}$. Besides the functional separability of points and closed sets above, Tychonoff spaces are also defined as the spaces whose topology is induced by some family of $\mathbf{R}$-valued functions, or alternatively by some family of pseudometrics.

Proposition A.25. *Let E be a topological space. Then, the following statements are equivalent:*

(a) E is a Tychonoff space.

(b) $C(E;\mathbf{R})$ separates points and strongly separates points on E.

(c) $C_b(E;\mathbf{R})$ separates points and strongly separates points on E.

(d) E is a Hausdorff space and $\mathscr{O}(E)$ is induced by a family of pseudometrics.

(e) E is a Hausdorff space and $\mathscr{O}(E) = \mathscr{O}_{\mathcal{D}}(E)$ for some $\mathcal{D} \subset \mathbf{R}^E$.

Proof. ((a) $\to$ (b)) For each non-empty $O \in \mathscr{O}(E)$, there exists an $f \subset C(E;\mathbf{R})$ such that $\{y \in E : f(y) < \epsilon\} \subset O$ for all $\epsilon \in (0,1)$ by the definition of Tychonoff spaces. This implies $\mathscr{O}_{C(E;\mathbf{R})}(E)$ is finer than $\mathscr{O}(E)$ and $C(E;\mathbf{R})$ strongly separates points on E. The Hausdorff property of $(E, \mathscr{O}_{C(E;\mathbf{R})}(E))$ follows by that of E and Fact A.1. Hence, $C(E;\mathbf{R})$ separates points on E by Proposition A.17 (c) (with $\mathcal{D} = C(E;\mathbf{R})$).

((b) $\to$ (c)) Observing $\{(f \wedge n) \vee (-n)\}_{n \in \mathbf{N}, f \in C(E;\mathbf{R})} \subset C_b(E;\mathbf{R})$, one finds $\mathscr{O}(E) = \mathscr{O}_{C_b(E;\mathbf{R})}(E)$ by Lemma B.52 (with $\mathscr{O}(E) = \mathscr{O}_{C(E;\mathbf{R})}(E)$, $\mathcal{G} = C(E;\mathbf{R})$ and $\mathcal{H} = C_b(E;\mathbf{R})$).

((c) $\to$ (a)) Let $x \in E$ and $F \in \mathscr{C}(E)$ such that $x \notin F$. Since $C_b(E;\mathbf{R})$ strongly separates points on E, there exist $\mathcal{D}_x \in \mathscr{P}_0(C_b(E;\mathbf{R}))$ and $\epsilon \in (0,\infty)$ such that

$$x \in \left\{ y \in E : \max_{f \in \mathcal{D}_x} |g_f(y)| < \epsilon \right\} \subset E \backslash F, \tag{A.3.1}$$

where $g_f(y) \mathrel{\hat=} f(y) - f(x)$ for each $y \in E$ and $f \in \mathcal{D}_x$. One finds by (A.3.1) that

$$h(y) \mathrel{\hat=} 1 \wedge \epsilon^{-1} \max_{f \in \mathcal{D}_x} |g_f(y)| \begin{cases} = 1, & \text{if } y \in F, \\ = 0, & \text{if } y = x, \\ \in [0,1], & \text{otherwise,} \end{cases} \tag{A.3.2}$$

proving that E is a Tychonoff space.

((b) $\to$ (d)) follows by Proposition A.17 (d, e) (with $A = E$ and $\mathcal{D} = C(E;\mathbf{R})$).

((d) $\to$ (e)) Given pseudometrics $\{\mathfrak{r}_i\}_{i \in \mathbf{I}}$ inducing $\mathscr{O}(E)$, [Kouritzin, 2016, p.5687] showed $\mathscr{O}(E) = \mathscr{O}_{\mathcal{D}}(E)$ with $\mathcal{D} \mathrel{\hat=} \{f_{\mathbf{I}_0,\mathbf{y}}\}_{\mathbf{y} \in E^{\mathbf{I}_0}, \mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I})}$ and

$$f_{\mathbf{I}_0,\mathbf{y}}(x) \mathrel{\hat=} \prod_{i \in \mathbf{I}_0} \frac{1 - \mathfrak{r}_i(x, y_i)}{2} \vee 0, \ \forall x \in E, \mathbf{y} \in E^{\mathbf{I}_0}, \mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I}). \tag{A.3.3}$$

((e) $\to$ (b)) is immediate by Proposition A.17 (c) (with $\mathscr{O}_{\mathcal{D}}(E) = \mathscr{O}(E)$). $\square$

Below are a few more properties of Tychonoff spaces.

Proposition A.26. *The following statements are true:*

(a) T4 spaces, especially metrizable spaces are Tychonoff space. Moreover, Tychonoff spaces are T3 spaces.

(b) Subspaces of a Tychonoff space are Tychonoff spaces.

(c) Any product space of Tychonoff spaces is a Tychonoff space.

Proof. (a) The first statement follows by the Urysohn's Lemma (see [Munkres, 2000, Theorem 33.1]) and Proposition A.6 (a). The definition of a Tychonoff space E implies $(E, \mathscr{O}_{C_b(E;\mathbf{R})}(E))$ is T3. Then, E is T3 by Proposition A.25 (a, c).

(b) and (c) were proved in [Munkres, 2000, Theorem 33.2]. $\square$

Tychonoff space has close link to compactification. S is called a compactification of E (or S compactifies E) if S is a compact Hausdorff space and E is a dense *subspace* of S. S is the Stone-Čech compactification of E if S compactifies E and $\bigotimes C_b(E;\mathbf{R})$ extends to a member of $\mathbf{imb}(S;\mathbf{R}^{C_b(E;\mathbf{R})})$. S is the one-point compactification of E if S compactifies E and $S\backslash E$ is a singleton. The Tychonoff property, general compactifiability and Stone-Čech compactifiability are equivalent.

Proposition A.27. *Let E be a topological space. Then, the following statements are equivalent:*

(a) E has a compactification.

(b) E is a Tychonoff space.

(c) E has a unique Stone-Čech compactification up to homeomorphism[15].

Stone-Čech compactification is a special case of the following result.

Lemma A.28. *Let E be a topological space and $\mathcal{D} \subset \mathbf{R}^E$ be a collection of bounded functions. Then, the following statements are equivalent:*

[15] "Unique up to homeomorphism" means any two spaces with the relevant property are homeomorphic.

(a) $\mathcal{D} \subset C_b(E; \mathbf{R})$ *separates points and strongly separates points on* E.

(b) E *admits a unique compactification* S *up to homeomorphism such that* $\bigotimes \mathcal{D}$ *extends to a homeomorphism between* S *and the closure of* $\bigotimes \mathcal{D}(E)$ *in* $\mathbf{R}^{\mathcal{D}}$.

(c) $\bigotimes \mathcal{D} \in \mathbf{imb}(E; \mathbf{R}^{\mathcal{D}})$.

Proof. ((a) → (b)) is adapted from [Kouritzin, 2016, Theorem 6 (1 - 3)]. ((b) → (c)) is immediate as E is a subspace of S. ((c) → (a)) follows by Lemma B.7 (b) (with $S = \mathbf{R}$). □

Remark A.29. If the $\mathcal{D}$ above is countable, then the induced compactification has a homeomorph in $\mathbf{R}^\infty$ and hence is metrizable. This is the foundation of the replication bases.

The next result is about compactification of metrizable and separable space.

Corollary A.30. *Let* E *be a topological space. Then, the following statements are equivalent:*

(a) E *is metrizable and separable.*

(b) There exists a countable subset of $\mathcal{D} \subset C_b(E; \mathbf{R})$ *that separates points and strongly separates points on* E.

(c) E *has a compactification that is homeomorphic to a compact subset of* $\mathbf{R}^\infty$.

(d) E *admits a metrizable compactification.*

(e) E *is a dense subspace of some Polish space.*

Proof. ((a) → (b)) follows by Proposition A.26 (a), Proposition A.25 (a, c), Proposition A.6 (c), Proposition A.24 (b) and Proposition A.17 (a). ((b) → (c)) follows by Lemma A.28. ((c) → (d)) follows by Proposition A.6 (b, d). ((d) → (e)) follows by Proposition A.12 (d). ((e) → (a)) follows by Proposition A.11 (c). □

Proof of Proposition A.27. ((a) $\to$ (b)) follows by Proposition A.12 (d) and Proposition A.26 (a, b). ((b) $\to$ (c)) follows by Proposition A.25 (a, c) and Lemma A.28 (a, b) (with $\mathcal{D} = C_b(E;\mathbf{R})$). ((c) $\to$ (a)) is automatic. $\square$

Locally compact Hausdorff spaces are well-known to have a unique one-point compactifiable up to Homeomorphism.

Proposition A.31. *Let E be a locally compact space. Then, the following statements are equivalent:*

(a) E is a Hausdorff space.

(b) E has a unique one-point compactification up to homeomorphism.

(c) E is a Tychonoff space.

Proof. ((a) $\to$ (b)) was proved in [Munkres, 2000, Theorem 29.1]. ((b) $\to$ (c)) follows by A.27 (a, b). ((c) $\to$ (a)) is automatic. $\square$

The next lemma is an analogue of Lemma A.20 for locally compact spaces.

Lemma A.32. *Let E be a locally compact space and $\mathcal{D} \subset C_0(E;\mathbf{R})$. Then, the following statements are equivalent:*

(a) $\mathcal{D}$ separates points on E.

(b) $\mathcal{D}^\Delta \doteq \{f^\Delta\}_{f\in\mathcal{D}} \cup \{1\}$ is a subset of $C(E^\Delta;\mathbf{R})$ which separates points and strongly separates points on E^Δ, where E^Δ is a one-point compactification of E and $f^\Delta \doteq \mathfrak{var}(f;E^\Delta,E,0)$[16] for each $x \in E^\Delta$ and $f \in \mathcal{D}$.

(c) E is a Hausdorff space and $\mathcal{D}$ strongly separates points on E.

Proof. ((a) $\to$ (b)) E is a Hausdorff space by Proposition A.17 (e) (with $A = E$), so E admits a one-point compactification $E^\Delta = E \cup \{\Delta\}$ by Proposition A.31. $\mathscr{K}(E) \subset \mathscr{C}(E^\Delta)$ by the Hausdorff property of E^Δ and Proposition A.12 (a). f^Δ is a continuous extension of each $f \in \mathcal{D} \subset C_0(E;\mathbf{R})$ by Lemma B.17 (with $S = E^\Delta$). $\mathcal{D}^\Delta$ separates points on E^Δ since $1 \in \mathcal{D}^\Delta$, $\{f^\Delta : f \in \mathcal{D}\}$

[16] "$\mathfrak{var}(\cdot)$" was defined in Notation 4.1.

separates points on E and $f^{\Delta}(\Delta) = 0$ for all $f \in \mathcal{D}$. Thus, $\mathcal{D}^{\Delta}$ strongly separates points on E^{Δ} by Lemma A.20.

((b) $\to$ (c)) follows by Proposition A.2 (c) and the fact $\mathscr{O}(E) = \mathscr{O}_{E^{\Delta}}(E) = \mathscr{O}_{\mathcal{D}^{\Delta}}(E) = \mathscr{O}_{\mathcal{D}}(E)$. ((c) $\to$ (a)) follows by Proposition A.17 (a). □

The local compactness of E leads to the following point-separability of $C_0(E;\mathbf{R})$.

Proposition A.33. *Let E be a locally compact space. Then, the following statements are equivalent:*

(a) E is a Hausdorff space.

(b) $C_c(E;\mathbf{R})$ separates points on E.

(c) $C_c(E;\mathbf{R})$ separates points and strongly separates points on E.

(d) $C_0(E;\mathbf{R})$ separates points and strongly separates points on E.

(e) E is a Tychonoff space.

Proof. ((a) $\to$ (b)) Let $x, y \in E$ be distinct. By the local compactness and Hausdorff property of E, there exist $O_x, O_y \in \mathscr{O}(E)$ and $K_x \in \mathscr{K}(E)$ such that $x \in O_x \subset K_x$, $y \in O_y$ and $O_x \cap O_y = \varnothing$. $K_x \backslash O_y \in \mathscr{K}(E) \subset \mathscr{C}(E)$ and $\{y\} \in \mathscr{C}(E)$ by the Hausdorff property of E, Proposition A.12 (a) and Proposition A.2 (a). So, O_x's closure F lies in $K_x \backslash O_y$, $E \backslash \{y\}$ is an open superset of F and $F \in \mathscr{K}(E)$ by Proposition A.12 (a). Now, by a version of the Urysohn's Lemma for locally compact Hausdorff spaces (see [Kantorovitz, 2003, Theorem 3.1]), there exist an $f \in C_c(E;\mathbf{R})$ such that $f(F) = 1$ and the closure of $E \backslash f^{-1}(\{0\})$ lies in $E \backslash \{y\}$. Hence, $f(x) = 1 \neq 0 = f(y)$.

((b) $\to$ (c)) follows by Fact B.43 and Lemma A.32 (a, c) (with $\mathcal{D} = C_c(E;\mathbf{R})$). ((c) $\to$ (d)) is immediate by Fact B.43. ((d) $\to$ (e)) follows by Proposition A.25 (a, b). ((e) $\to$ (a)) is automatic. □

A.4 Weak topology of non-negative finite Borel measures

Recall that the weak topology of $\mathcal{M}^+(E)$ is induced by $C_b(E;\mathbf{R})^*$[17]. Hence, the Tychonoff properties of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ is reduced to their Hausdorff properties.

Proposition A.34. *Let E be a topological space. Then, the following statements are equivalent:*

(a) $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are Tychonoff spaces.

(b) $\mathcal{P}(E)$ is a Hausdorff space.

(c) $C_b(E;\mathbf{R})^*$ separates points on $\mathcal{P}(E)$.

Proof. ((a) $\to$ (b)) is automatic. ((b) $\to$ (c)) follows by Proposition A.17 (c) (with $E = \mathcal{M}^+(E)$, $A = \mathcal{P}(E)$ and $\mathcal{D} = C_b(E;\mathbf{R})^*$). Given (c), $C_b(E;\mathbf{R})^*$ separates points on $\mathcal{M}^+(E)$ by Fact B.22 (a) (with $\mathcal{D} = C_b(E;\mathbf{R})$). Then, (a) follows by (2.3.1), Proposition A.17 (c) (with $E = \mathcal{M}^+(E)$ and $\mathcal{D} = C_b(E;\mathbf{R})^*$), Proposition A.25 (a, e) and Proposition A.26 (b). □

E's functional separabilities of points is related to those of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$.

Lemma A.35. *Let E be a topological space, $\mathcal{D} \subset M_b(E;\mathbf{R})$, $d \in \mathbf{N}$ and $\mathcal{G} \overset{\circ}{=} \mathfrak{mc}(\Pi^d(\mathcal{D}))$[18]. Then, the following statements are true:*

(a) If $\mathcal{D} \subset C_b(E;\mathbf{R})$ strongly separates points on E, then $\mathcal{G}^*$ separates points on $\mathcal{P}(E^d)$ and $\mathcal{G} \cup \{1\}$ is separating on E^d.

(b) If $\mathcal{D}$ is countable and strongly separates points on E, then $\mathcal{G}^*$ separates points and strongly separates points on $\mathcal{P}(E^d)$, and $\mathcal{G} \cup \{1\}$ is separating and convergence determining on E^d.

[17] The notation "$C_b(E;\mathbf{R})^*$" was specified in §2.3.

[18] As mentioned in Remark A.23, we need not impose $1 \in \mathcal{D}$ in Lemma A.35 by the definition of $\Pi^d(\mathcal{D})$.

Proof. (a) $\mathcal{G}$ determines point convergence on E^d by Proposition A.17 (b) (with $A = E$) and Proposition A.21 (b) (with $\mathcal{D} = C_b(E;\mathbf{R})$). $\mathcal{G}$ strongly separates points on E^d by Proposition A.17 (a) (with $A = E = E^d$ and $\mathcal{D} = \mathcal{G}$). $\mathscr{O}_{\mathcal{G}}(E) = \mathscr{O}(E)$ by (5.1.3) and Proposition A.17 (e) (with $A = E$ and $\mathcal{D} = \mathcal{G}$). Now, (a) follows by [Blount and Kouritzin, 2010, Theorem 11 (a)] and Fact B.22 (a) (with $E = E^d$ and $\mathcal{D} = \mathcal{G} \cup \{1\}$).

(b) $\mathcal{G}$ strongly separates points on E^d by a similar argument to above. $\mathcal{G}$ is a countable subset of $M_b(E^d;\mathbf{R})$ by (5.1.2) and Fact B.15. Now, (b) follows by [Blount and Kouritzin, 2010, Theorem 6 (b) and Theorem 11 (c)] and Fact B.22 (b) (with $E = E^d$ and $\mathcal{D} = \mathcal{G} \cup \{1\}$). □

We now investigate the connection between the Tychonoff property of E and those of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$. On one hand, we give a generalization of [Kallianpur and Xiong, 1995, Theorem 2.1.4] without the restriction to Radon measures.

Corollary A.36. *Let E be a Tychonoff space and $d \in \mathbf{N}$. Then, $\mathcal{M}^+(E^d)$ and $\mathcal{P}(E^d)$ are Tychonoff spaces and $\mathfrak{mc}[\Pi^d(C_b(E;\mathbf{R}))]$ is separating on E^d.*

Proof. $\mathfrak{mc}[\Pi^d(C_b(E;\mathbf{R}))]^*$ by definition is a subset of $C_b(\mathcal{M}^+(E^d);\mathbf{R})$ and it separates points on $\mathcal{M}^+(E^d)$ by Proposition A.25 (a, c) and Lemma A.35 (a) (with $\mathcal{D} = C_b(E;\mathbf{R})$). Hence, the result follows by Proposition A.34 (a, c) (with $E = \mathcal{M}^+(E^d)$) and Proposition A.26 (b). □

On the other hand, we give an explicit example showing that the converse of Corollary A.36 is not true.

Example A.37. Example 3.27 (VII) and Example 3.31 (III) mentioned that $\mathbf{R}_K$ is a non-T3 (hence non-Tychonoff) topological refinement of $\mathbf{R}$ with $\mathscr{B}(\mathbf{R}_K) = \mathscr{B}(\mathbf{R})$. $\mathcal{P}(\mathbf{R})$ is a Tychonoff space by Corollary A.36. $\mathcal{P}(\mathbf{R}_K)$ is a Hausdorff topological refinement of $\mathcal{P}(\mathbf{R})$ by Fact B.26 (a) and Fact A.1. Thus, $\mathcal{P}(\mathbf{R}_K)$ and $\mathcal{M}^+(\mathbf{R}_K)$ are Tychonoff spaces by Proposition A.34 (a, b).

The two examples below illustrate that the Hausdorff property of E and those of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ do not imply each other.

Example A.38. Let $E = \{1,2,3,4\}$, $A \doteq \{1,2\}$, $B \doteq \{3,4\}$ and equip E with the topology $\mathscr{O}(E) \doteq \{\varnothing, A, B, E\}$. Then, $\mathscr{B}(E) = \mathscr{C}(E) = \mathscr{O}(E)$

and singletons in E are neither closed nor Borel. So, E is non-Hausdorff by Proposition A.2 (a). Letting[19] $\mu_A \circeq (\delta_1 + \delta_2)/2$ and $\mu_B = (\delta_3 + \delta_4)/2$, we observe that $C_b(E; \mathbf{R}) = \{a\mathbf{1}_A + b\mathbf{1}_B : a, b \in \mathbf{R}\}$ and $\mathcal{P}(E) = \{\mu_a \circeq a\mu_A + (1-a)\,\mu_B : a \in [0,1]\}$. $C_b(E; \mathbf{R})^*$ separates points on $\mathcal{P}(E)$ since for any $a_1, a_2 \in [0,1]$,

$$\int_E \mathbf{1}_A(x)\mu_{a_1}(dx) = a_1 = a_2 = \int_E \mathbf{1}_A(dx)\mu_{a_2}(dx) \tag{A.4.1}$$

implies $\mu_{a_1} = \mu_{a_2}$. Thus, $\mathcal{P}(E)$ and $\mathcal{M}^+(E)$ are Hausdorff by Proposition A.34.

Example A.39. Due to the limit of space, we refer the readers to [Munkres, 2000, §33, Exercise 11] for the non-trivial construction of a topological space E satisfying: (1) E is a T3 (*hence Haudorff*) but non-Tychonoff space, and (2) there exist $a \neq b$ in E such that $f^*(\delta_a) = f(a) = f(b) = f^*(\delta_b)$[20] for all $f \in C(E; \mathbf{R})$. δ_a and δ_b are distinct measures by the Hausdorff property of E and Proposition A.41 (a), but they can not be separated by $C_b(E; \mathbf{R})^*$. So, neither $\mathcal{P}(E)$ nor $\mathcal{M}^+(E)$ is Hausdorff by Proposition A.34.

Remark A.40. The difference of the two examples above is at the Borel measurability of singletons and the distinctiveness of Dirac measures at distinct points.

As long as the extreme non-Borel singletons are avoided, the Hausdorff property of (the usually more complicated space) $\mathcal{P}(E)$ implies that of E.

Proposition A.41. *Let E be a topological space satisfying $\{\{x\} : x \in E\} \subset \mathscr{B}(E)$ and $\mathcal{D} \subset M_b(E; \mathbf{R})$. Then, the following statements are true:*

(a) $\delta_x \neq \delta_y$ for any distinct $x, y \in E$.

(b) If $\mathcal{D}^*$ separates points on $\mathcal{P}(E)$, then $\mathcal{D}$ separates points on E.

(c) If $\mathcal{P}(E)$ is a Hausdorff space, then $C_b(E; \mathbf{R})$ separates points on E and E is a Hausdorff space.

[19] δ_x denotes the Dirac measure at x.
[20] The notation "f^*" was specified in §2.3.

Proof. (a) $x \neq y$ in E implies $\delta_x(\{x\}) = 1 \neq 0 = \delta_y(\{x\})$.

(b) One finds by (a) an $f_{x,y} \in \mathcal{D}$ with $f_{x,y}(x) = f^*_{x,y}(\delta_x) \neq f^*_{x,y}(\delta_y) = f_{x,y}(y)$.

(c) follows by Proposition A.34 (b, c), (b) (with $\mathcal{D} = C_b(E;\mathbf{R})$) and Proposition A.17 (e) (with $A = E$ and $\mathcal{D} = C_b(E;\mathbf{R})$). $\square$

Below are two corollaries of our previous developments.

Corollary A.42. *Let E be a metrizable and separable space. Then, there exists a countable $\mathcal{D} \subset C_b(E;\mathbf{R})$ satisfying the following:*

(a) $\mathcal{D} \ni 1$ is closed under addition and multiplication.

(b) $\mathcal{D}$ separates points and strongly separates points on E.

(c) $\mathcal{D}$ is separating and convergence determining on E.

Proof. There exists a countable $\mathcal{D}_0 \subset C_b(E;\mathbf{R})$ that separates points and strongly separates points on E by Corollary A.30 (a, b). $\mathcal{D} \stackrel{\circ}{=} \mathfrak{mc}(\mathcal{D}_0 \cup \{1\})$[21] is countable by Fact B.15 and satisfies (a, b). Now, (c) follows by Lemma A.35 (b). $\square$

Corollary A.43. *The following statements are equivalent:*

(a) E is a Tychonoff space, and $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are metrizable and separable spaces.

(b) E is a metrizable and separable space.

Proof. ((a) $\to$ (b)) By Proposition A.24 (b) (with $E = \mathcal{M}^+(E)$ and $\mathcal{D} = C_b(E;\mathbf{R})^*$) and Proposition A.17 (a, b), there exists a countable $\mathcal{D} \subset C_b(E;\mathbf{R})$ that is separating and convergence determining on $\mathcal{M}^+(E)$. $\mathcal{D}$ separates points and strongly separates points on E by Proposition A.41 (b), Lemma B.58 (a, b) and Proposition A.2 (b). Now, (b) follows by Corollary A.30 (a, b).

((b) $\to$ (a)) follows by Proposition A.26 (a), Corollary A.42 (c), Proposition A.17 (b) (with $E = \mathcal{M}^+(E)$ and $\mathcal{D} = \mathcal{D}^*$), Corollary A.30 (a, b) (with $E = \mathcal{M}^+(E)$ and $\mathcal{D} = \mathcal{D}^*$) and Proposition A.6 (c). $\square$

The properties of $\mathcal{M}^+$ and $\mathcal{P}(E)$ below are vital.

[21] The notation "$\mathfrak{mc}(\cdot)$" was defined in §2.2.3.

Theorem A.44. *The following statements are true:*

(a) If E is a compact Hausdorff space, then $\mathcal{P}(E)$ is also.

(b) If E is a Polish space, then $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are also.

Proof. (a) follows by Proposition A.12 (d), Corollary A.36 (with $d = 1$), the Prokhorov's Theorem (Theorem 2.22 (b) with $\Gamma = \mathcal{P}(E)$) and Fact B.23.

(b) The case for $\mathcal{P}(E)$ was proved in [Bogachev, 2007, Vol. II, Theorem 8.9.5]. The case for $\mathcal{M}^+(E)$ refers to [Ethier and Kurtz, 1986, §9.5, Problem 6]. □

As noted in p.26, the sequential concepts "weak limit point[22]" and "relative compactness" in $\mathcal{M}^+(E)$ may cause ambiguity in general, but one can get rid of that when E is a metriable space.

Proposition A.45. *Let E be a topological space and $\Gamma \subset \mathcal{M}^+(E)$. Then, the following statements are true:*

(a) If E is metrizable, then $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are metrizable by the same metric.

(b) If ν is a weak limit point of Γ, then ν is a limit point of Γ with respect to weak topology. The converse is true when E is metrizable.

(c) If E is metrizable, then the relative compactness of Γ is equivalent to its precompactness with respect to weak topology.

Proof. (a) The construction of the Lévy-Prokhorov metric on $\mathcal{P}(E)$ can be found in [Ethier and Kurtz, 1986, §3.1]. A natural extension of this metric to $\mathcal{M}^+(E)$ was given in [Ethier and Kurtz, 1986, §9.5, Problem 6].

(b) follows by (a) and Fact A.9 (b) (with $E = \mathcal{M}^+(E)$ and $A = \Gamma$).

(c) Let $(\mathcal{M}^+(E), \mathfrak{r})$ be a metrization[23] of $\mathcal{M}^+(E)$, $\overline{\Gamma}$ be Γ's closure in $\mathcal{M}^+(E)$ and let $\{\mu_n\}_{n\in\mathbf{N}} \subset \overline{\Gamma}$. $\overline{\Gamma}$'s compactness implies Γ's relative compactness by (b) and Proposition A.12 (f). Conversely, we suppose Γ is relatively

[22] Weak limit point and relative compactness of finite Borel measures were reviewed in §2.3.

[23] The notion of metrization was specified in §A.1.3.

compact and pick a $\nu_n \in \Gamma$ satisfying $\mathfrak{r}(\mu_n, \nu_n) < 2^{-n}$ for each $n \in \mathbf{N}$. Any weak limit point of Γ lies in $\overline{\Gamma}$ by (b) and the closedness of $\overline{\Gamma}$. Then, there must exist $\nu \in \overline{\Gamma}$ and $\{n_k\} \subset \mathbf{N}$ such that

$$\lim_{k\to\infty} \mathfrak{r}(\mu_{n_k}, \nu) \leq \lim_{k\to\infty} \mathfrak{r}(\mu_{n_k}, \nu_{n_k}) + \lim_{k\to\infty} \mathfrak{r}(\nu_{n_k}, \nu) = 0. \tag{A.4.2}$$

This proves the sequential compactness of $\overline{\Gamma}$, hence (c) follows by Proposition A.12 (f). □

The next lemma extends Theorem 2.22 (b) to the non-probabilistic case.

Lemma A.46. *Let E be a Hausdorff space, $\Gamma \subset \mathcal{M}^+(E)$ be sequentially tight and $0 < a < b$ satisfy $\{\mu(E)\}_{\mu\in\Gamma} \subset [a, b]$. Then, Γ is relatively compact and the total mass*[24] *of any weak limit point of Γ lies in $[a, b]$.*

Proof. Let $\mu' \stackrel{\circ}{=} \mu/\mu(E) \in \mathcal{P}(E)$ for each $\mu \in \Gamma$. By the sequential tightness of Γ, there exist a tight subsequence $\{\mu_n\}_{n\in\mathbf{N}}$ and $\{K_p\}_{p\in\mathbf{N}} \subset \mathscr{K}(E)$ such that $\sup_{n\in\mathbf{N}} \mu_n(E\backslash K_p) \leq 2^{-p}a$ for all $p \in \mathbf{N}$ and

$$\sup_{n\in\mathbf{N}} \mu'_n(E\backslash K_p) \leq \frac{\sup_{n\in\mathbf{N}} \mu_n(E\backslash K_p)}{\inf_{n\in\mathbf{N}} \mu_n(E)} \leq 2^{-p},\ \forall p \in \mathbf{N}. \tag{A.4.3}$$

Thus, $\{\mu'_n\}_{n\in\mathbf{N}}$ is tight by (A.4.3) and is relatively compact by the Prokhorov's Theorem (Theorem 2.22 (b)). Then, there exist $\nu \in \mathcal{M}^+(E)$, $\{n_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$ and $c \in [a, b]$ such that $\mu'_{n_k} \Rightarrow \nu$ as $k \uparrow \infty$ in $\mathcal{M}^+(E)$ and $\lim_{k\to\infty} \mu_{n_k}(E) = c$. Hence, $\mu_{n_k} \Rightarrow c\nu$ as $k \uparrow \infty$ in $\mathcal{M}^+(E)$ by Fact B.21 (b). □

Morever, given a *perfectly normal* (see e.g. [Munkres, 2000, §33, Exercise 6]) space E, we equate the Borel sets of $\mathcal{M}^+(E)$ generated by its *strong*[25] and weak topologies, which generalizes [Bolthausen and Schmock, 1989, Lemma 2.1].

Lemma A.47. *Let E be a perfectly normal (especially metrizable or Polish) space. Then, $\mathscr{B}_{M_b(E;\mathbf{R})^*}(\mathcal{M}^+(E)) = \mathscr{B}(\mathcal{M}^+(E))$.*

[24] The notion of total mass was specified in §2.1.2.

[25] Strong topology of Borel probability measures was reviewed in Example 2.30, Example 3.27 and Example 3.31. It can be defined for non-negative finite Borel measures in almost the same way.

Proof. Let $O \in \mathscr{O}(E)$. By the definition of perfectly normal space, there exist $\{F_n\}_{n\in\mathbf{N}} \subset \mathscr{C}(E)$ such that $F_n \subset F_{n+1}$ for all $n \in \mathbf{N}$ and $O = \bigcup_{n\in\mathbf{N}} F_n$. E is a T4 space by [Munkres, 2000, §32, Exerise 6 and §33, Exercise 6 (b)]. Then, there exist $\{f_n\} \subset C_b(E;\mathbf{R})$ such that $\mathbf{1}_{F_n} \leq f_n \leq \mathbf{1}_O$ for all $n \in \mathbf{N}$ by [Munkres, 2000, §33, Exercise 5]. Consequently, $f_n^* \to \mathbf{1}_O^*$ as $n \uparrow \infty$ by the Dominated Convergence Theorem. The O above is arbitrary, so

$$\mathscr{O}(E) \subset \left\{B \in \mathscr{B}(E) : \mathbf{1}_B^* \in M_b\left(\mathcal{M}^+(E)\right)\right\}. \tag{A.4.4}$$

The right-hand side above is a Dynkin system (see e.g. [Bogachev, 2007, Vol. I, Definition 1.9.2]). It then follows that

$$\mathbf{1}_B^* \in M_b\left(\mathcal{M}^+(E)\right),\ \forall B \in \mathscr{B}(E) \tag{A.4.5}$$

by a suitable Monotone Class Theorem (see [Bogachev, 2007, Vol. I, Theorem 1.9.3 (ii)]), thus proving

$$\mathscr{B}_{M_b(E;\mathbf{R})^*}\left(\mathcal{M}^+(E)\right) \subset \mathscr{B}\left(\mathcal{M}^+(E)\right). \tag{A.4.6}$$

The converse containment of (A.4.6) is immediate. Moreover, Polish and metrizable spaces are perfectly normal by Proposition A.6 (d) and [Munkres, 2000, §33, Exercise 6 (a)]. □

A.5 Standard Borel space

We review in this section a few fundamental properties of standard Borel spaces and standard Borel subsets.

Fact A.48. *The following statements are true:*

(a) Borel isomorphs of standard Borel spaces are still standard Borel spaces. In particular, Polish spaces, their Borel isomorphs and their Borel subspaces[26] *are standard Borel spaces.*

(b) The cardinality of a standard Borel space can never exceed $\aleph(\mathbf{R})$.

[26] The notion of Borel subspace was introduced in Definition 2.2.

Proof. (a) is automatic by definition. Regarding (b), we note that subsets of Polish spaces can be injectively mapped into $\mathbf{R}^\infty$ by Proposition A.11 (c) and Corollary A.30 (a, c). So, the cardinalities of their Borel isomorphs will not exceed $\aleph(\mathbf{R}^\infty) = \aleph(\mathbf{R})$. □

Standard Borel spaces are Borel isomorphic to Borel subsets of Polish spaces. The latter turns out to be precisely the metrizable Lusin spaces.

Proposition A.49. *Let E be a metrizable space. Then, the following statements are equivalent:*

(a) E is a Lusin space.

(b) E has a Polish topological refinement $(E, \mathscr{U})$ with $\mathscr{B}(E) = \sigma(\mathscr{U})$.

(c) E is separable and for any metrization $(E, \mathfrak{r})$ of E, E is a Borel subset of the completion of $(E, \mathfrak{r})$.

(d) E is a Borel subspace of some Polish space.

(e) There exist an $S \in \mathscr{C}(\mathbf{N}^\infty)$ and a bijective $f \in C(S; E)$.

The key to prove the equivalence above is the preservation of Borel sets under bijective Borel measurable mappings. Here is a standard result about this.

Lemma A.50. *Let E be a Lusin space, S be a Polish space, $f \in M(S; E)$ and*[27]

$$\mathscr{U}_f \doteq \left\{ O \subset E : f^{-1}(O) \in \mathscr{O}(S) \right\}. \tag{A.5.1}$$

Then, the following statements are true:

(a) If f is continuous and bijective, then $f \in \mathbf{hom}(S; E, \mathscr{U}_f)$ and $(E, \mathscr{U}_f)$ is a Polish topological refinement of E.

(b) If E is metrizable (especially a Polish space) and f is injective, then $f(B) \in \mathscr{B}(E)$ for all $B \in \mathscr{B}(S)$.

[27] $\mathscr{U}_f$ is known as the "push-forward topology of f". In any case, $f \in C(S; E, \mathscr{U}_f)$.

(c) If E is metrizable (especially a Polish space) and f is bijective, then $f \in \mathbf{biso}(S; E)$[28]*.*

Proof. (a) $f \in C(S; E)$ implies $\mathscr{O}(E) \subset \mathscr{U}_f$, so (a) follows by Lemma B.8 (a).

(b) Polish and Lusin spaces are separable by Proposition A.11 (d). E is a dense subspace of some Polish space S' by Corollary A.30 (a, e), so $f \in M(S; S')$ is injective. Now, (b) follows by [Srivastava, 1998, Theorem 4.5.4].

(c) follows immediately by (b). □

Corollary A.51. *Lusin spaces (resp. Souslin spaces) are precisely the Hausdorff topological coarsenings*[29] *of Polish spaces (resp. Lusin spaces).*

Proof. (Necessity) The case of Lusin spaces follows directly by Lemma A.50 (a). The case of Souslin spaces follows by a similar argument.

(Sufficiency) Note that the identity mapping on a topological space is a continuous bijection from any of its topological refinement(s) to itself. □

Proof of Proposition A.49. ((a) → (b)) As E is a Lusin space, there exist a Polish space S and a bijective $f \in C(S; E)$. Let $\mathscr{U}_f$ be as in (A.5.1) and $\mathscr{U} \circeq \mathscr{U}_f$. Then, $(E, \mathscr{U})$ is a Polish topological refinement of E and $f \in \mathbf{biso}(S; E)$ by Lemma A.50 (a, c). $\mathscr{B}(E) = \sigma(\mathscr{U})$ by Lemma B.8 (b).

((b) → (c)) Let f be the identity mapping on E. $f \in C(E, \mathscr{U}; E)$ since $\mathscr{O}(E) \subset \mathscr{U}$. So, E is separable by Proposition A.11 (c) and Proposition A.3 (d). The completion S' of $(E, \mathfrak{r})$ is a Polish space by Proposition A.11 (e). $f \in \mathbf{biso}((E, \mathscr{U}); E)$ by $\mathscr{B}(E) = \sigma(\mathscr{U})$. Since $(E, \mathscr{U})$ and S' are Polish spaces, $E = f(E) \in \mathscr{B}(\overline{E})$ by Lemma A.50 (b) (with $S = (E, \mathscr{U})$ and $E = S'$).

((c) → (d)) follows by Proposition A.11 (e).

((d) → (e)) We refer this non-trivial result to [Bogachev, 2007, Vol. II, Lemma 6.8.4 and Corollary 6.8.5].

((e) → (a)) $(S, \mathscr{O}_{\mathbf{N}^\infty}(S))$ is a Polish space by Proposition A.11 (b, f), so E is a Lusin space. □

From above we observe that a general (resp. metrizable) Lusin space is a topological coarsening of some Polish space that does not necessarily preserve (resp. does preserve) the Borel σ-algebra. By contrast, the next proposition

[28] The notation "**biso**" was defined in §2.2.2.

[29] The terminology "topological coarsening" was specified in §2.1.3.

shows that a general standard Borel space is a topological *variant* (not necessarily a coarsening or refinement) of some Polish space that preserves the Borel σ-algbera.

Proposition A.52. *Let E be a topological space. Then, the following statements are equivalent:*

(a) E is a standard Borel space.

(b) E is Borel isomorphic to some metrizable Lusin space.

(c) There exists a topology $\mathscr{U}_1$ on E such that $(E, \mathscr{U}_1)$ is a metrizable Lusin space and $\mathscr{B}(E) = \sigma(\mathscr{U}_1)$.

(d) There exists a topology $\mathscr{U}_2$ on E such that $(E, \mathscr{U}_2)$ is a Polish space and $\mathscr{B}(E) = \sigma(\mathscr{U}_2)$.

(e) E is Borel isomorphic to some Polish space.

Proof. ((a) $\to$ (b)) follows by Proposition A.49 (a, d).

((b) $\to$ (c)) Let S be a metrizable Lusin space, $f \in \mathbf{biso}(S; E)$, $\mathscr{U}_f$ be as in (A.5.1) and $\mathscr{U}_1 \overset{\circ}{=} \mathscr{U}_f$. It follows by Lemma B.8 that $f \in \mathbf{hom}(S; (E, \mathscr{U}_1))$ and $\mathscr{B}(E) = \sigma(\mathscr{U}_1)$, thus proving $(E, \mathscr{U}_1)$ is a metrizable Lusin space.

((c) $\to$ (d)) It follows by Proposition A.49 (a, b) (with $E = (E, \mathscr{U}_1)$) that $(E, \mathscr{U}_1)$ has a Polish topological refinement $(E, \mathscr{U}_2)$ with $\mathscr{B}(E) = \sigma(\mathscr{U}_1) = \sigma(\mathscr{U}_2)$.

((d) $\to$ (e)) The identity mapping on E is a Borel isomorphism between E and Polish space $(E, \mathscr{U}_2)$.

((e) $\to$ (a)) is immediate by Fact A.48 (a). $\square$

Given metrizability, standard Borel property and Lusin property become indifferent. Such spaces coarsen the topology but preserve the Borel σ-algebras of certain Polish spaces.

Proposition A.53. *Let E be a metrizable space. Then, the following statements are equivalent:*

(a) E is a standard Borel space.

(b) E is a Lusin space.

(c) E admits a Polish topological refinement $(E, \mathscr{U})$ satisfying $\mathscr{B}(E) = \sigma(\mathscr{U})$.

Our proof is based on the following interesting result which illustrates that a Borel measurable mapping may preserve some topological properties.

Lemma A.54. *Let E be a standard Borel space and S be a metrizable space. Then, the following statements are true:*

(a) If there is a bijective $f \in M(E; S)$, then S is separable.

(b) If E is metrizable, then E is separable (hence second-countable).

Proof. (a) There exist a Polish space S' and an $f' \in \mathbf{biso}(S'; E)$ by Proposition A.52 (a, e). $f \circ f' \in M(S'; S)$ is injective. Now, (a) follows by [Srivastava, 1998, Proposition 4.3.8].

(b) Let S' and f' be as above. S' is a standard Borel space by Fact A.48 (a). Then, $E = f'(S')$ is separable and second-countable by (a) (with $E = S'$, $S = E$ and $f = f'$) and Proposition A.6 (c). □

Proof of Proposition A.53. ((a) → (b)) E is separable by Lemma A.54 (b). E is a dense subspace of some Polish space S' by Corollary A.30 (a, e). There exist a Polish space S and an $f \in \mathbf{biso}(S; E)$ by Proposition A.52 (a, e). $E = f(S) \in \mathscr{B}(S')$ follows by Lemma A.50 (b) (with $E = S'$ and $S = B = E$). Thus, E is a Lusin space by Proposition A.49 (a, d).

((b) → (c)) Let f be the identity mapping on E. E admits a Polish refinement $(E, \mathscr{U})$ by Proposition A.49 (a, b), so $f \in M((E, \mathscr{U}); E)$. It then follows by Lemma A.50 (c) (with $S = (E, \mathscr{U})$) that $f \in \mathbf{biso}((E, \mathscr{U}); E)$ and $\mathscr{B}(E) = \sigma(\mathscr{U})$.

((c) → (a)) is immediate by Fact A.48 (a). □

Corollary A.55. *$\mathcal{P}(E)$ is a metrizable standard Borel space whenever E is.*

Proof. E admits a Polish refinement $(E, \mathscr{U})$ with $\sigma(\mathscr{U}) = \mathscr{B}(E)$ by Proposition A.53 (a, c). $\mathcal{P}(E, \mathscr{U})$ is a Polish refinement of $\mathcal{P}(E)$ by Fact B.26 (a) and Theorem A.44 (b). $\mathcal{P}(E)$ is a metrizable Lusin space by Corollary A.43 and Corollary A.51. Thus, $\mathcal{P}(E)$ is standard Borel by Proposition A.53 (a, b). □

The next proposition compares standard Borel substes and Borel subsets which are likely to be different in general topological spaces.

Proposition A.56. *Let E be a topological space. Then, the following statements are true:*

(a) If $A \in \mathscr{B}^{\mathrm{s}}(E)$, then $\mathscr{B}_E(A) \subset \mathscr{B}^{\mathrm{s}}(E)$. In particular, if E is a standard Borel space, then $\mathscr{B}(E) \subset \mathscr{B}^{\mathrm{s}}(E)$.

(b) If E is a metrizable standard Borel space, especially if E is a Polish space, then $\mathscr{B}(E) = \mathscr{B}^{\mathrm{s}}(E)$.

Proof. (a) There exist Polish space S and $f \in \mathbf{biso}(A; S)$ by Proposition A.52 (a, e) (with $E = (A, \mathscr{O}_E(A))$). Any $B \in \mathscr{B}_E(A)$ satisifes $f(B) \in \mathscr{B}(S)$ and $f|_B \in \mathbf{biso}(B; f(B))$, so $B \in \mathscr{B}^{\mathrm{s}}(E)$.

(b) Let $A \in \mathscr{B}^{\mathrm{s}}(E)$. There exist a Polish space S and an $f \in \mathbf{biso}(S; A)$ by Proposition A.52 (a, e). $A = f(S) \in \mathscr{B}(E)$ by Lemma A.50 (b) (with $B = S$). Now, (b) follows by (a). □

Recall that if E is compact and S is Hausdorff, then any bijective $f \in C(E; S)$ belongs to $\mathbf{hom}(E; S)$ and S is also compact (see [Munkres, 2000, Theorem 26.6]). One of Kuratowski's theorems provides a similar identification for bijective Borel measurable mappings from standard Borel spaces to metrizable spaces. Herein, we give a short proof for integrity.

Proposition A.57. *Let E be a topological space, $A \in \mathscr{B}^{\mathrm{s}}(E)$, S be a metrizable space and $f \in M(E; S)$ be injective. Then, $f(A) \in \mathscr{B}^{\mathrm{s}}(S)$ and $f|_A \in$ $\mathbf{biso}(A; f(A))$.*

Proof. There exist a Polish space S' and an $f' \in \mathbf{biso}(S'; A)$. $f(A)$ is a metrizable subspace of S by Proposition A.6 (b). Observing that $f|_A \circ f' \in M(S'; S)$ is injective, we have that $f(A) = f|_A \circ f'(S')$ is a separable subspace of S by Fact A.48 (a) and Lemma A.54 (b) (with $E = S'$ and $f = f|_A \circ f'$).

$f(A)$ is a dense subspace of some Polish space S'' by Corollary A.30 (a, e). Observing that $f|_A \circ f' \in M(S'; S'')$ is injective, one finds by Lemma A.50 (b, c) (with $E = S''$ and $S = B = S'$) that $f(A) = f|_A \circ f'(S')$ is a Borel subspace of Polish space S'' and $f|_A \circ f' \in \mathbf{biso}(S'; f(A))$. Hence, $f(A) \in \mathscr{B}^{\mathrm{s}}(S)$ by Fact A.48 (a) and $f|_A = (f|_A \circ f') \circ (f')^{-1} \in \mathbf{biso}(A; f(A))$. □

Remark A.58. The E in Lemma A.50 (b, c) is standard Borel by Proposition A.53 (a, b) and the Polish space S in Lemma A.50 satisfies $\mathscr{B}(S) = \mathscr{B}^{\mathbf{s}}(S)$ by Proposition A.56 (b). Hence, Proposition A.57 truly generalizes Lemma A.50 (b, c).

The next result treats standard Borel property of countable Cartesian product.

Proposition A.59. *Let $\mathbf{I}$ be a countable index set, $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces, $(S, \mathscr{A})$ be as in (2.7.22) and $A \in \mathscr{B}^{\mathbf{s}}(S)$. Then, the following statements are true:*

(a) If $\{\{x\} : x \in S_i\} \subset \mathscr{B}(S_i)$ for all $i \in \mathbf{I}$, then $\mathfrak{p}_i(A) \in \mathscr{B}^{\mathbf{s}}(S_i)$ for all $i \in \mathbf{I}$.

(b) If $\{S_i\}_{i\in\mathbf{I}}$ are standard Borel spaces, and if $\mathscr{B}(S) = \mathscr{A}$ (especially $\{S_i\}_{i\in\mathbf{I}}$ are all metrizable), then S is a standard Borel space.

Proof. (a) We fix $i \in \mathbf{I}$ and $x \in A$. For each $B \in \mathscr{B}(S_i)$ and $j \neq i$, we define $B_i^x \circeq B$, $x_j \circeq \mathfrak{p}_j(x)$, $B_j^x \circeq \{x_j\}$ and $B^x \circeq \prod_{j\in\mathbf{I}} B_j^x$. Since $\mathbf{I}$ is countable and $\{x_j\} \in \mathscr{B}(S_j)$ for all $j \in \mathbf{I}$, we have that

$$B^x = \mathfrak{p}_i^{-1}(B) \cap \bigcap_{j\in\mathbf{I}\setminus\{i\}} \mathfrak{p}_j^{-1}(\{x_j\}) \in \mathscr{A},\ \forall B \in \mathscr{B}(S_i). \tag{A.5.2}$$

Observing that

$$\begin{aligned}\mathscr{O}_S(S_i^x) &= \left\{ S_i^x \cap \left[\bigcap_{j\in\mathbf{I}_0} \mathfrak{p}_j^{-1}(O_j) \right] : O_j \in \mathscr{O}(S_j), \mathbf{I}_0 \in \mathscr{P}_0(\mathbf{I}) \right\} \\ &= \{O^x : O \in \mathscr{O}(S_i)\},\end{aligned} \tag{A.5.3}$$

we have by (A.5.2) that

$$\mathscr{B}_S(S_i^x) = \sigma\left(\{O^x : O \in \mathscr{O}(S_i)\}\right) \subset \mathscr{A}. \tag{A.5.4}$$

It follows that

$$\mathscr{B}_S(A \cap S_i^x) = \mathscr{A}|_{A\cap S_i^x} \subset \mathscr{B}_S(A) \subset \mathscr{B}^{\mathbf{s}}(S) \tag{A.5.5}$$

by Proposition B.46 (a) and Proposition A.56 (a). One further finds

$$\mathfrak{p}_i|_{A\cap S_i^x} \in \mathbf{biso}\left[A \cap S_i^x, \mathscr{B}_S(A\cap S_i^x); \mathfrak{p}_i(A), \mathscr{B}_{S_i}\left[\mathfrak{p}_i(A)\right]\right] \tag{A.5.6}$$

by (A.5.5) and Fact 2.3 (a). Hence, $\mathfrak{p}_i(A) \in \mathscr{B}^{\mathbf{s}}(S_i)$ by (A.5.5), (A.5.6) and Fact A.48 (a).

(b) If $\{S_i\}_{i\in\mathbf{I}}$ are all metrizable, then $\mathscr{B}(S) = \mathscr{A}$ holds by Lemma A.54 (b) and Proposition B.46 (d). For each $i \in \mathbf{I}$, there exist a Polish space S_i' and an $f_i \in \mathbf{biso}(S_i'; S_i)$ by Proposition A.52 (a, e). $\prod_{i\in\mathbf{I}} S_i'$ is a Polish space by Proposition A.11 (f). $\mathscr{B}(\prod_{i\in\mathbf{I}} S_i') = \bigotimes_{i\in\mathbf{I}} \mathscr{B}(S_i')$ by Proposition B.46 (d). It then follows that $\bigotimes_{i\in\mathbf{I}} f_i \circ \mathfrak{p}_i \in \mathbf{biso}(\prod_{i\in\mathbf{I}} S_i'; S)$ by $\mathscr{B}(S) = \mathscr{A}$ and Fact 2.3 (b), thus proving $S \in \mathscr{B}^{\mathbf{s}}(S)$. $\square$

The next proposition is about the functional separabilities of points and probability measures on standard Borel spaces.

Proposition A.60. *Let E be a standard Borel space. Then, the following statements are true:*

(a) There exists a countable susbet of $M_b(E;\mathbf{R})$ that separates points on E.

(b) If $\mathcal{D} \subset \mathbf{R}^E$ satisfies $\mathscr{B}_{\mathcal{D}}(E) = \mathscr{B}(E)$, then $\mathcal{D}$ separates points on E.

(c) If $\mathcal{D} \subset M(E;\mathbf{R})$ is countable and separates points on E, then $\sigma(\mathcal{D}) = \mathscr{B}_{\mathcal{D}}(E) = \mathscr{B}(E)$.

(d) If $\mathcal{D} \subset M_b(E;\mathbf{R})$ is countable, is closed under multiplication and separates points on E, then $\mathcal{D}^*$ separates points on $\mathcal{P}(E)$.

Proof. (a) There exist a Polish space S and an $f \in \mathbf{biso}(E; S)$ by Proposition A.52 (a, e). There exist $\{g_n\}_{n\in\mathbf{N}} \subset C_b(S;\mathbf{R})$ separating points on S by Corollary A.30 (a, b). Hence, $\{g_n \circ f\}_{n\in\mathbf{N}} \subset M_b(E;\mathbf{R})$ separates points on E.

(b) Let S and f be as in (a). It follows by [Bogachev, 2007, Vol. II, Example 6.5.2] that $\mathscr{B}(S)$ has a countable subcollection $\mathscr{U}$ such that for any distinct $x, y \in A$, there exists an $B_{x,y} \in \mathscr{U}$ containing $f^{-1}(x)$ but excluding $f^{-1}(y)$. It follows by the fact $f \in \mathbf{biso}(S; E)$ that $\{x, y\}\backslash f(B_{x,y}) = \{y\}$ and

$f(B_{x,y}) \in \{f(B) : B \in \mathscr{U}\} \subset \mathscr{B}(E) = \mathscr{B}_{\mathcal{D}}(E)$. Hence, $\mathcal{D}$ separate points on A by [Bogachev, 2007, Vol. II, Lemma 6.5.3].

(c) $\bigotimes \mathcal{D} \in M(E; \mathbf{R}^\infty)$ by Fact 2.3 (b). $\bigotimes \mathcal{D} \in \mathbf{biso}(E; \bigotimes \mathcal{D}(E))$ and $\bigotimes \mathcal{D}(E) \in \mathscr{B}(\mathbf{R}^\infty)$ by the injectiveness of $\bigotimes \mathcal{D}$ and Proposition A.57 (with $A = E$ and $S = \mathbf{R}^\infty$). $\mathscr{B}(\mathbf{R}^\infty) = \mathscr{B}(\mathbf{R})^{\otimes \mathbf{N}}$ by Proposition B.46 (d). It then follows that

$$\begin{aligned}\mathscr{B}(E) &= \sigma\left[\left\{\left(\bigotimes \mathcal{D}\right)^{-1}(B) : B \in \mathscr{B}(\mathbf{R}^\infty) = \mathscr{B}(\mathbf{R})^{\otimes \mathbf{N}}\right\}\right] \\ &= \sigma\left(\left\{f^{-1}(B) : B \in \mathscr{B}(\mathbf{R}), f \in \mathcal{D}\right\}\right) = \sigma(\mathcal{D}).\end{aligned} \tag{A.5.7}$$

Now, (c) follows by (A.5.7) and Fact B.5 (with $S = A = E$ and $E = \mathbf{R}$).

(d) follows by (c) and Lemma A.35 (a) (with $E = (E, \mathscr{O}_{\mathcal{D}}(E))$). □

We end this section with a lemma about existence of conditional distribution. Preciously, such existence is known on Polish spaces. Now, we extend it to a fairly mild setting.

Lemma A.61. *Let E be a topological space, $X \in M(\Omega, \mathscr{F}; E)$, $A \in \mathscr{B}^{\mathrm{s}}(E)$ and $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. If $\mathbb{P}(X \in A) = 1$, then:*

(a) $\mathbb{P}(\mathbb{E}[\mathbf{1}_{\{X \in A\}} | \mathscr{G}] = 1) = 1$.

(b) The conditional distribution $\mathbb{P}_{X|\mathscr{G}}$ of X given $\mathscr{G}$ (see e.g. [Dudley, 2002, §10.2, p.342]) exists as a member of $M(\Omega, \mathscr{F}; \mathcal{P}(E))$. In particular,

$$\mathbb{P}\left(\left\{\omega \in \Omega : \mathbb{P}_{X|\mathscr{G}}(\omega)(A) = 1\right\}\right) = 1. \tag{A.5.8}$$

Proof. (a) follows by the fact

$$\mathbb{E}\left[1 - \mathbb{E}\left[\mathbf{1}_{\{X \in A\}} | \mathscr{G}\right]\right] = 1 - \mathbb{P}\left(X \in A\right) = 0 = \mathbb{P}\left(\mathbb{E}\left[\mathbf{1}_{\{X \in A\}} | \mathscr{G}\right] > 1\right). \tag{A.5.9}$$

(b) There exist a Polish space S and an $f \in \mathbf{biso}(S; A, \mathscr{O}_E(A))$ since $A \in \mathscr{B}^{\mathrm{s}}(E)$. We fix an arbitrary $y_0 \in A$ and find

$$Y \mathrel{\hat=} f^{-1} \circ \mathfrak{var}\left(X; \Omega, X^{-1}(A), y_0\right) \in M(\Omega, \mathscr{F}; S) \tag{A.5.10}$$

by Lemma B.31 (c) (with $S_0 = S = A$, $\mathscr{U} = \mathscr{B}(E)$ and $\mathscr{U}' = \mathscr{B}_E(A)$). We let $\mathbb{P}_{Y|\mathscr{G}}$ denote the conditional distribution of Y given $\mathscr{G}$, which is well-known to exist on the Polish space S (see [Dudley, 2002, Theorem 10.2.2]). Now, we define

$$\mathbb{P}_{X|\mathscr{G}}(\omega)(B) \triangleq \begin{cases} \mathbb{P}_{Y|\mathscr{G}}(\omega) \circ f^{-1}(B \cap A), & \text{if } \omega \in X^{-1}(A), \\ \mathbb{P}(X \in B), & \text{otherwise}, \end{cases} \quad \forall \omega \in \Omega, B \in \mathscr{B}(E), \tag{A.5.11}$$

and check that: (1) $\mathbb{P}_{X|\mathscr{G}}$ satisfies the definition of X's conditional distribution given $\mathscr{G}$, and (2) (A.5.8) holds and $\mathbb{P}_{X|\mathscr{G}} \in M(\Omega, \mathscr{F}; \mathcal{P}(E))$.

Regarding claim (1), we observe that

$$\mathbb{P}_{X|\mathscr{G}}(\omega) = \mathbb{P}_{Y|\mathscr{G}}(\omega) \circ f^{-1}\big|^{E}, \ \forall \omega \in X^{-1}(A) \tag{A.5.12}$$

and

$$\mathbb{P}_{Y|\mathscr{G}}(\omega) \circ f^{-1} \in \mathcal{P}\left(A, \mathscr{O}_E(A)\right), \ \forall \omega \in X^{-1}(A) \tag{A.5.13}$$

from (A.5.11) and properties of $\mathbb{P}_{Y|\mathscr{G}}$[30]. Thus,

$$\mathbb{P}_{X|\mathscr{G}}(\omega) \in \mathcal{P}(E), \ \forall \omega \in \Omega \tag{A.5.14}$$

by (A.5.12), (A.5.13), Fact 2.1 (b) (with $\mathscr{U} = \mathscr{B}(E)$, $\mu = \mathbb{P}_{X|\mathscr{G}}(\cdot, \omega)$ and $\nu = \mathbb{P}_{Y|\mathscr{G}}(\cdot, \omega) \circ f^{-1}$) and (A.5.11). Next, we fix $B \in \mathscr{B}(E)$ and find that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \mathbb{P}_{X|\mathscr{G}}(\omega)(B) = \mathbb{E}\left[\mathbf{1}_{\{Y \in f^{-1}(B \cap A)\}} \middle| \mathscr{G}\right](\omega)\right\}\right) = 1 \tag{A.5.15}$$

by (A.5.11) and properties of $\mathbb{P}_{Y|\mathscr{G}}$. We find that

$$\mathbb{P}\left(\mathbb{E}\left[\mathbf{1}_{\{Y \in f^{-1}(B \cap A)\}} \middle| \mathscr{G}\right] = \mathbb{E}\left[\mathbf{1}_{\{X \in B\}} \middle| \mathscr{G}\right]\right) = 1 \tag{A.5.16}$$

by A.5.10 and (a). It follows by (A.5.15) and (A.5.16) that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \mathbb{P}_{X|\mathscr{G}}(\omega)(B) = \mathbb{E}\left[\mathbf{1}_{\{X \in B\}} \middle| \mathscr{G}\right]\right\}\right) = 1. \tag{A.5.17}$$

Therefore, $\mathbb{P}_{X|\mathscr{G}}(\cdot)(B) \in M(\Omega, \mathscr{G}; [0,1])$ by the fact $\mathbb{E}[\mathbf{1}_{\{X \in B\}} | \mathscr{G}] \in M(\Omega, \mathscr{G}; [0,1])$

[30] $\mathbb{P}_{Y|\mathscr{G}}$ is sometimes defined to take values almost surely in $\mathcal{P}(S)$. In this case, one can use the argument of [Dudley, 2002, §10.2, p.341, Notes] to remove the null set of exception.

and Lemma B.31 (a) (with $E = S = [0,1]$, $\mathscr{U} = \mathscr{B}(E)$, $X = \mathbb{P}_{X|\mathscr{G}}(\cdot)(B)$ and $Z = \mathbb{E}[\mathbf{1}_{\{X\in B\}}|\mathscr{G}]$). So far, our claim (1) above is justified.

Regarding claim (2), we fix $f \in C_b(E;\mathbf{R})$ and $\mu \in \mathcal{P}(E)$. Since $\mathbb{P}_{X|\mathscr{G}}$ is the desired conditional distribution, one finds by [Dudley, 2002, Theorem 10.2.5] that

$$\mathbb{P}\left(\left\{\omega \in \Omega : f^* \circ \mathbb{P}_{X|\mathscr{G}}(\omega) = \mathbb{E}\left[f \circ X|\mathscr{G}\right](\omega)\right\}\right) = 1 \tag{A.5.18}$$

and

$$\mathbb{P}\left(\left\{\omega \in \Omega : \mathbb{P}_{X|\mathscr{G}}(\omega)(A) = \mathbb{E}\left[\mathbf{1}_{\{X\in A\}}\middle|\mathscr{G}\right](\omega)\right\}\right) = 1. \tag{A.5.19}$$

Since $\mathbb{E}\left[f \circ X|\mathscr{G}\right] \in M(\Omega,\mathscr{F};\mathbf{R})$,

$$\{\omega \in \Omega : |\mathbb{E}\left[f \circ X|\mathscr{G}\right](\omega) - f^*(\mu)| < \epsilon\} \in \mathscr{F}. \tag{A.5.20}$$

It then follows that

$$\left\{\omega \in \Omega : \left|f^* \circ \mathbb{P}_{X|\mathscr{G}}(\omega) - f^*(\mu)\right| < \epsilon\right\} \in \mathscr{F} \tag{A.5.21}$$

by (A.5.18), (A.5.20) and the $\mathbb{P}$-completeness of $\mathscr{F}$. Thus, claim (2) follows by (A.5.19), (a), (A.5.14), (A.5.21) and the definition of $\mathscr{O}(\mathcal{P}(E))$. □

A.6 Skorokhod $\mathscr{J}_1$-space

This section contains necessary materials about Skorokhod $\mathscr{J}_1$-space for the developments in Chapter 6 - Chapter 8. We start with several most essential properties of $D(\mathbf{R}^+;E)$.

Proposition A.62. *Let E and S be Tychonoff spaces and $\mathcal{D} \subset C(E;\mathbf{R})$. Then, the following statements are true:*

(a) If $\mathcal{D}$ strongly separates points on E (especially $\mathcal{D} = C(E;\mathbf{R})$), then $\{\varpi(f) : f \in \mathfrak{ae}(\mathcal{D})\} \subset D(\mathbf{R}^+;\mathbf{R})^{D(\mathbf{R}^+;E)}$[31] satisfies[32]

$$\varpi[\mathfrak{ae}(\mathcal{D})] \in \mathbf{imb}\left(D(\mathbf{R}^+;E); D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right) \tag{A.6.1}$$

[31] The notations "$\varpi(f)$" and "$\varpi(\mathcal{D})$" were defined in §2.2.1. The notation "$\mathfrak{ae}(\cdot)$" was defined in §2.2.3.

[32] $\mathscr{J}(E)$ denotes the Skorokhod $\mathscr{J}_1$-topology of $D(\mathbf{R}^+;E)$.

and

$$\mathscr{J}(E) = \mathscr{O}_{\{\varpi(f): f\in\mathfrak{ae}(\mathcal{D})\}}\left(D(\mathbf{R}^+; E)\right). \tag{A.6.2}$$

(b) If $\mathcal{D} \subset C_b(E; \mathbf{R})$ separates points on E, then $\{\alpha^f_{t,n} : f \in \mathcal{D}, t \in \mathbf{Q}^+, n \in \mathbf{N}\}$[33] is a subset of $C(D(\mathbf{R}^+; E); \mathbf{R})$ separating points on $D(\mathbf{R}^+; E)$ with each $\alpha^f_{t,n}$ defined as in (3.3.8).

(c) $D(\mathbf{R}^+; E)$ is a Tychonoff space.

(d) If $f \in \mathcal{D} \subset C(S; E)$, then

$$\varpi(f) \in C\left(D(\mathbf{R}^+; S); D(\mathbf{R}^+; E)\right) \tag{A.6.3}$$

and

$$\varpi(\mathcal{D}) \subset C\left(D(\mathbf{R}^+; S); D(\mathbf{R}^+; E)^{\mathcal{D}}\right). \tag{A.6.4}$$

(e) If E is a topological coarsening of S, then $D(\mathbf{R}^+; S) \subset D(\mathbf{R}^+; E)$ and $\mathscr{J}(S) \supset \mathscr{O}_{D(\mathbf{R}^+;E)}(D(\mathbf{R}^+; S))$.

Note A.63. For any index set $\mathbf{I}$, $\mathbf{R}^{\mathbf{I}}$ is a Tychonoff space by Proposition A.26 (c). So, $D(\mathbf{R}^+; \mathbf{R})^{\mathbf{I}}$ and $D(\mathbf{R}^+; \mathbf{R}^{\mathbf{I}})$ are well-defined Tychonoff spaces by Proposition A.62 (c) and Proposition A.26 (c).

Proof of Proposition A.62. (a) If $\mathcal{D} = C(E; \mathbf{R})$, then $\mathcal{D}$ strongly separates points on E by Proposition A.25 (a, b). $\varpi(f) \in D(\mathbf{R}^+; \mathbf{R})^{D(\mathbf{R}^+;E)}$ for all $f \in C(E; \mathbf{R})$ and $\varpi[\mathfrak{ae}(\mathcal{D})] \in D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ by Fact B.14 (b). [Blount and Kouritzin, 2010, Theorem 5] clarified that (A.6.1) follows from the proof of [Jakubowski, 1986, Theorem 1.7 and Theorems 4.3 (ii)] for E being a metric space. An inspection of the details justifies this conclusion for the general Tychonoff space case. $\mathcal{D}$ separates points on E by the Hausdorff property of E and Proposition A.17 (a), so $\varpi[\mathfrak{ae}(\mathcal{D})]$ is injective by Fact B.20 (with $A = E$ and $\mathcal{D} = \mathfrak{ae}(\mathcal{D})$). Then, (A.6.2) follows by (A.6.1) and Lemma B.7 (b) (with $E = D(\mathbf{R}^+; E)$, $S = D(\mathbf{R}^+; \mathbf{R})$ and $\mathcal{D} = \{\varpi(f)\}_{f\in\mathfrak{ae}(\mathcal{D})}$).

(b) The integral in (3.3.8) is well-defined by Fact B.14 (a) and each $\alpha^f_{t,n}$ is continuous by the argument establishing [Ethier and Kurtz, 1986, §3.7, Proposition 7.1]. If $x, y \in D(\mathbf{R}^+; E)$ satisfies $\alpha^f_{t,n}(x) = \alpha^f_{t,n}(y)$ for all $f \in \mathcal{D}$,

[33] $\mathbf{Q}^+$ denotes non-negative rational numbers.

$t \in \mathbf{Q}^+$ and $n \in \mathbf{N}$, then $f(x(t)) = \lim_{n\to\infty} \alpha^f_{t,n}(x) = \lim_{n\to\infty} \alpha^f_{t,n}(y) = f(y(t))$ for all $f \in \mathcal{D}$ and $t \in \mathbf{Q}^+$. This implies $x(t) = y(t)$ for all $t \in \mathbf{Q}^+$ as $\mathcal{D}$ separates points on E. Hence, $x = y$ by their right-continuities.

(c) follows by (a) and Proposition A.25 (a, b).

(d) $\varpi(C(E;\mathbf{R})) \in \mathbf{imb}(D(\mathbf{R}^+;E); D(\mathbf{R}^+;\mathbf{R})^{C(E;\mathbf{R})})$ by (a). For each $g \in C(E;\mathbf{R})$, it follows by $g \circ f \in C(S;\mathbf{R})$ and (a) that $\varpi(g) \circ \varpi(f) = \varpi(g \circ f) \in C(D(\mathbf{R}^+;S); D(\mathbf{R}^+;\mathbf{R}))$. Hence, we have by Fact 2.4 (b) that

$$\begin{aligned}\varpi(f) &= \varpi\left(C(E;\mathbf{R})\right)^{-1} \circ \left[\bigotimes_{g\in C(E;\mathbf{R})} \left(\varpi(g) \circ \varpi(f)\right)\right] \\ &\in C\left(D(\mathbf{R}^+;S); D(\mathbf{R}^+;E)\right).\end{aligned} \tag{A.6.5}$$

Moreover, (A.6.4) follows by (A.6.5) and Fact 2.4 (b).

(e) follows by the fact $C(E;\mathbf{R}) \subset C(S;\mathbf{R})$ and (a). □

Note A.64. Proposition A.62 (a) confirms that $\mathscr{J}(E)$ is uniquely determined by $\mathscr{O}(E)$ and does not depend on the choice of the pseudometrics in its definition.

Corollary A.65. *Let E be a Tychonoff space. Then, $D(\mathbf{R}^+;A,\mathscr{O}_E(A))$ is a topological subspace of $D(\mathbf{R}^+;E)$ for any non-empty $A \subset E$.*

Proof. $(A,\mathscr{O}_E(A)) = (A,\mathscr{O}_{C(E;\mathbf{R})}(A))$ is a Tychonoff space by Proposition A.25 (a, b). We then have by Proposition A.62 (a) that

$$\begin{aligned}&\mathscr{O}_{D(\mathbf{R}^+;E)}\left[D(\mathbf{R}^+;A,\mathscr{O}_E(A))\right] \\ &= \mathscr{O}_{\{\varpi(f|_A):f\in C(E;\mathbf{R})\}}\left[D(\mathbf{R}^+;A,\mathscr{O}_E(A))\right] = \mathscr{J}\left(A,\mathscr{O}_E(A)\right).\end{aligned} \tag{A.6.6}$$

□

Corollary A.66. *The one-dimensional projections $\mathcal{J} = \{\mathfrak{p}_i\}_{i\in\mathbf{I}}$ on $\mathbf{R}^{\mathbf{I}}$ satisfy*

$$\varpi\left[\mathfrak{ae}(\mathcal{J})\right] \in \mathbf{imb}\left(D(\mathbf{R}^+;\mathbf{R}^{\mathbf{I}}); D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{J})}\right) \tag{A.6.7}$$

and

$$\varpi\left[\bigotimes \mathfrak{ae}(\mathcal{J})\right] \in C\left(D(\mathbf{R}^+;\mathbf{R}^{\mathbf{I}}); D\left(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{J})}\right)\right). \tag{A.6.8}$$

Proof. $\mathfrak{ae}(\mathcal{J})$ is a subset of $C(\mathbf{R}^{\mathbf{I}};\mathbf{R})$ and strongly separates points on $\mathbf{R}^{\mathbf{I}}$ since $\mathscr{O}(\mathbf{R}^{\mathbf{I}}) = \mathscr{O}_{\mathcal{J}}(\mathbf{R}^{\mathbf{I}})$. $\bigotimes \mathfrak{ae}(\mathcal{J}) \in C(\mathbf{R}^{\mathbf{I}};\mathbf{R}^{\mathfrak{ae}(\mathcal{J})})$ by Fact 2.4 (b). Hence, (A.6.7) follows by Proposition A.62 (a) (with $E = \mathbf{R}^{\mathbf{I}}$ and $\mathcal{D} = \{p_i\}_{i\in\mathbf{N}}$), and (A.6.8) follows by Proposition A.62 (d) (with $S = \mathbf{R}^{\mathbf{I}}$, $E = \mathbf{R}^{\mathfrak{ae}(\mathcal{J})}$ and $f = \bigotimes \mathfrak{ae}(\mathcal{J})$). □

Below is a well-known property of compact subsets of $D(\mathbf{R}^+;E)$.

Proposition A.67. *Let E be a Tychonoff space and $K \in \mathscr{K}(D(\mathbf{R}^+;E))$. Then, there exist $\{K_n\}_{n\in\mathbf{N}} \subset \mathscr{K}(E)$ such that*

$$K \subset \bigcap_{n\in\mathbf{N}} \left\{x \in D(\mathbf{R}^+;E) : x(t) \in K_n, \forall t \in [0,n)\right\}. \tag{A.6.9}$$

Proof. This result follows by the argument of [Ethier and Kurtz, 1986, §3.6, Theorem 3.6.3 and Remark 3.6.4]. □

The next proposition discusses finite-dimensional projections on $D(\mathbf{R}^+;E)$.

Proposition A.68. *Let E be a Tychonoff space and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$. Then, the following statements are true:*

(a) $\mathfrak{p}_{\mathbf{T}_0} \in M(D(\mathbf{R}^+;E);E^{\mathbf{T}_0},\mathscr{B}(E)^{\otimes\mathbf{T}_0})$[34].

(b) (2.3.8) and (2.3.9) hold.

(c) $\mathfrak{p}_{\mathbf{T}_0}$ *is continuous at* $x \in D(\mathbf{R}^+;E)$ *whenever* $\mathbf{T}_0 \subset \mathbf{R}^+\backslash J(x)$.

Proof. (a) We fix $f \in C_b(E;\mathbf{R})$ and $t \in \mathbf{R}^+$, let $\alpha^f_{t,n}$ be as in (3.3.8) for each $n \in \mathbf{N}$, observe $\lim_{n\to\infty} \alpha^f_{t,n} = f\circ\mathfrak{p}_t$ so $f\circ\mathfrak{p}_t \in M(D(\mathbf{R}^+;E);\mathbf{R})$.

$$\bigotimes_{f\in C_b(E;\mathbf{R})} (f\circ\mathfrak{p}_t) \in M\left(D(\mathbf{R}^+;E);\mathbf{R}\right)^{C_b(E;\mathbf{R})} \tag{A.6.10}$$

by Fact 2.3 (b). We have by Proposition A.25 (a, c) and Lemma A.28 (a, c) (with $\mathcal{D} = C_b(E;\mathbf{R})$) that

$$\bigotimes C_b(E;\mathbf{R}) \in \mathbf{imb}\left(E;\mathbf{R}^{C_b(E;\mathbf{R})}\right). \tag{A.6.11}$$

[34]Herein, $\mathfrak{p}_{\mathbf{T}_0}$ denotes the projection on $E^{\mathbf{R}^+}$ for $\mathbf{T}_0 \subset \mathbf{R}^+$ restricted to $D(\mathbf{R}^+;E)$.

Now, (a) follows by (A.6.10), the fact

$$\mathfrak{p}_t = \left(\bigotimes C_b(E;\mathbf{R})\right)^{-1} \circ \left[\bigotimes_{f\in C_b(E;\mathbf{R})} (f\circ \mathfrak{p}_t)\right],\ \forall t\in \mathbf{R}^+, \tag{A.6.12}$$

(A.6.11) and Fact 2.3 (b).

(b) follows by (a) and the definition of $\mathscr{B}(E)^{\otimes \mathbf{R}^+}$.

(c) Let $\mathfrak{p}_t$ and $\mathfrak{p}'_t$ denote the one-dimensional projections for $t\geq 0$ on $D(\mathbf{R}^+;E)$ and $D(\mathbf{R}^+;\mathbf{R})$ repectively. Fixing $t\in \mathbf{T}_0$ and $f\in C(E;\mathbf{R})$, we have by Proposition A.62 (d) (with $S=E$ and $E=\mathbf{R}$) and [Munkres, 2000, Theorem 18.1] that $\varpi(f)$ is *continuous at* x[35] and so $t\notin J[\varpi(f)(x)]$. Then, $\mathfrak{p}'_t$ is continuous at $\varpi(f)(x)$ by [Ethier and Kurtz, 1986, §3.6, Proposition 6.5 (a)] (with $E=\mathbf{R}$ and $t_n=t$) and so $f\circ\mathfrak{p}_t=\mathfrak{p}'_t\circ\varpi(f)$ is continuous at x. Now, the continuity of $\mathfrak{p}_t$ at x follows by (A.6.12), Fact 2.4 (c) and (A.6.11). The continuity of $\mathfrak{p}_{\mathbf{T}_0}$ at x follows by that of $\mathfrak{p}_t$ for each $t\in\mathbf{T}_0$ and Fact 2.4 (c). □

Corollary A.69. *Let E be a Tychonoff space. Then, $\mu\circ\mathfrak{p}_{\mathbf{T}_0}^{-1}$ is a member of $\mathfrak{M}^+(E^{\mathbf{T}_0},\mathscr{B}(E)^{\otimes\mathbf{T}_0})$ for all $\mu\in\mathfrak{M}^+(D(\mathbf{R}^+;E),\mathscr{B}(E)^{\otimes\mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$ (especially $\mu\in\mathcal{M}^+(D(\mathbf{R}^+;E))$) and $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{R}^+)$.*

Proof. This result is immediate by Proposition A.68 (a, b). □

We now look at measurability of the modulus of continuity $w'_{\mathfrak{r},\delta,T}$ on $D(\mathbf{R}^+;E)$.

Proposition A.70. *Let E be a Tychonoff space and $\delta,T\in(0,\infty)$. Then, the following statements are true:*

(a) $w'_{\mathfrak{r},\delta,T}\in M(D(\mathbf{R}^+;E);\mathbf{R})$[36] if E allows a metrization $(E,\mathfrak{r})$.

(b) $w'_{\rho_{\{f\}},\delta,T}\in M(D(\mathbf{R}^+;E);\mathbf{R})$ for all $f\in C(E;\mathbf{R})$.

Proof. (a) was proved in [Ethier and Kurtz, 1986, §3.6, Lemma 6.2 (c)].

(b) $\{\rho_{\{f\}}\}_{f\in C(E;\mathbf{R})}$ induces $\mathscr{O}(E)$ by Proposition A.25 (a, b) and Proposition A.17 (d, e) (with $A=E$ and $\mathcal{D}=C(E;\mathbf{R})$). Hence, $\mathscr{O}(D(\mathbf{R}^+;E))$ by

[35] The notion of continuity at a point was specified at [Munkres, 2000, §18, p.104].

[36] The notation "$w'_{\mathfrak{r},\delta,T}$" was defined in §2.2.1. "$w'_{\rho_{\{f\}},\delta,T}$" (resp. "$w'_{|\cdot|,\delta,T}$") is defined by (2.2.3) with $\mathfrak{r}=\rho_{\{f\}}$ (resp. $E=\mathbf{R}$ and $\mathfrak{r}=|\cdot|$).

definition is induced by pseudometrics $\{\varrho^{\rho_{\{f\}}}\}_{f\in C(E;\mathbf{R})}$[37]. Now, (b) follows by applying the argument establishing (a) to each $\rho_{\{f\}}$. □

The next fact clarifies the measurability issue about (2.3.10).

Fact A.71. *Let E be a Tychonoff space. If $M(E;\mathbf{R})$ has a countable subset separating points on E, then*

$$\big\{x \in D(\mathbf{R}^+;E) : t \in J(x)\big\} \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)},\ \forall t \in \mathbf{R}^+ \tag{A.6.13}$$

and $J(\mu)$[38] is well-defined for all $\mu \in \mathfrak{M}^+(D(\mathbf{R}^+;E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$ (especially $\mu \in \mathcal{M}^+(D(\mathbf{R}^+;E))$).

Proof. This fact follows by Lemma B.50 (a) (with $V = D(\mathbf{R}^+;E)$) and Proposition A.68 (b). □

The next proposition discusses the metrizability of $D(\mathbf{R}^+;E)$.

Proposition A.72. *Let E be a metrizable space. Then, the following statements are true:*

(a) If $(E,\mathfrak{r})$ is a metrization of E, then $D(\mathbf{R}^+;E)$ is metrized by $\varrho^{\mathfrak{r}}$.

(b) If E is separable (espeically a Polish space), then $D(\mathbf{R}^+;E)$ is also separable and $\mathscr{B}(D(\mathbf{R}^+;E)) = \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)}$.

(c) If a metric $\mathfrak{r}$ completely metrizes E, then $\varrho^{\mathfrak{r}}$ completely metrizes $D(\mathbf{R}^+;E)$.

(d) If E is a Polish space, then $D(\mathbf{R}^+;E)$ is also.

Proof. (a) was shown in [Ethier and Kurtz, 1986, §3.5, p.117 - 118]. (b) was proved in [Ethier and Kurtz, 1986, §3.5, Theorem 5.6 and §3.7, Proposition 7.1]. (c) was proved in [Ethier and Kurtz, 1986, §3.5, Theorem 5.6]. (d) follows by (c). □

The following three results relate convergence in $D(\mathbf{R}^+;E)$ and that in the Skorokhod $\mathscr{J}_1$-space $D([0,u];E)$[39] with finite time horizon. These results

[37] The pseudometric $\varrho^{\rho_{\{f\}}}$ is defined by (2.2.8) with $\mathfrak{r} = \rho_{\{f\}}$.

[38] $J(\mu)$, the set of fixed left-jump times of μ was defined in (2.3.10).

[39] As aforementioned in §2.2.2, $D([0,u];E)$ denotes the Skorokhod $\mathscr{J}_1$-space of all càdlàg mappings from $[0,u]$ to E.

are contained explicitly or implicitly in standard references like Jacod and Shiryaev [2003] and Jakubowski [1986]. Herein, we clarify these fundamental facts for readers' convenience.

Proposition A.73. *Let E be a metrizable space, $\{y_k\}_{k\in\mathbf{N}_0} \subset D(\mathbf{R}^+;E)$*[40]*, and*

$$y_k^u \circeq \mathfrak{var}\left(y_k;[0,u+1],[0,u],y_k(u)\right),\ \forall k\in\mathbf{N}_0, u\in(0,\infty). \tag{A.6.14}$$

Then for each $u\in\mathbf{R}^+\backslash J(y_0)$,

$$y_k \longrightarrow y_0 \text{ as } k\uparrow\infty \text{ in } D(\mathbf{R}^+;E) \tag{A.6.15}$$

implies

$$y_k^u \longrightarrow y_0^u \text{ as } k\uparrow\infty \text{ in } D([0,u+1];E). \tag{A.6.16}$$

Proof. Let $(E,\mathfrak{r})$ be a metrization of E. $\varrho^{\mathfrak{r}}_{[0,u+1]}$[41] by definition metrizes the Skorokhod $\mathscr{J}_1$-topology of $D([0,u+1];E)$. So, we prove (A.6.16) by establishing

$$\lim_{k\to\infty}\varrho^{\mathfrak{r}}_{[0,u+1]}(y_k^u,y_0^u) \leq \lim_{k\to\infty}|||\lambda_k^u|||\vee\mathfrak{r}_{[0,u+1]}\left(y_k^u\circ\lambda_k^u,y_0^u\right)=0. \tag{A.6.17}$$

For each $\lambda\in\mathbf{TC}(\mathbf{R}^+)$[42] with $|||\lambda|||<1$,

$$\lambda^u(t) \circeq \begin{cases}\lambda(t), & \text{if } t\in[0,u],\\ \lambda(u)+(u+1-\lambda(u))(t-u), & \text{if } t\in[u,u+1]\end{cases} \tag{A.6.18}$$

defines a member of $\mathbf{TC}([0,u+1])$. By [Ethier and Kurtz, 1986, §3.5, Proposition 5.3 (b)], there exist $\{\lambda_k\}_{k\in\mathbf{N}}\subset\mathbf{TC}(\mathbf{R}^+)$ such that

$$\lim_{k\to\infty}\mathfrak{r}_{[0,u]}\left(y_k\circ\lambda_k,y_0\right)=\lim_{k\to\infty}|||\lambda_k|||=0. \tag{A.6.19}$$

We fix $\epsilon\in(0,1)$. By y_0's *left-continuity at u*, there exists a $\delta_{\epsilon,u}\in(0,\infty)$ such

[40] $\mathbf{N}_0$ denotes the non-negative integers. Our notation $y_k^u \circeq \mathfrak{var}(x;[0,u+1],[0,u],x(u))$ represents the function $y_k^u(t)\circeq y_k(t)$ for all $t\in[0,u]$ and $y_k^u(t)\circeq y_k(u)$ for all $t\in(u,u+1]$.

[41] "$\mathfrak{r}_{[0,u+1]}$" is defined by (2.2.4) with $[a,b]=[0,u+1]$ and "$\varrho^{\mathfrak{r}}_{[0,u+1]}$" is defined by (2.2.7) with $[a,b]=[0,u+1]$

[42] The notations "$\mathbf{TC}(\mathbf{R}^+)$", "$\mathbf{TC}([0,u+1])$" and "$|||\cdot|||$" were defined in §2.2.2.

that

$$\mathfrak{r}\left(y_0(t), y_0(u)\right) \leq \epsilon,\ \forall t \in \left(u - \delta_{u,\epsilon}, u\right). \tag{A.6.20}$$

By (A.6.19), there exists an $N_{u,\epsilon} \in \mathbf{N}$ such that

$$|||\lambda_k||| < 1 \wedge \left(-\log\left|1 - \frac{\delta_{u,\epsilon} \wedge (e^\epsilon - 1)}{u}\right|\right),\ \forall k > N_{u,\epsilon} \tag{A.6.21}$$

and

$$\mathfrak{r}_{[0,u]}\left(y_k \circ \lambda_k, y_0\right) < \epsilon,\ \forall k > N_{u,\epsilon}. \tag{A.6.22}$$

Fixing $k > N_{u,\epsilon}$, we have by (A.6.21) that

$$\begin{aligned}\sup_{t \in (u,u+1]} |\lambda_k^u(t) \wedge u - u| &\leq |\lambda_k(u) - u| \\ &\leq \left(1 - e^{-|||\lambda_k|||}\right) u < \delta_{u,\epsilon} \wedge (e^\epsilon - 1)\end{aligned} \tag{A.6.23}$$

and

$$|||\lambda_k^u||| \leq \log\left(u + 1 - \lambda_k(u)\right) < \epsilon. \tag{A.6.24}$$

If $t \in [0, u]$, we have by (A.6.22) that

$$\mathfrak{r}\left(y_k^u \circ \lambda_k^u(t), y_0^u(t)\right) = \mathfrak{r}\left(y_k \circ \lambda_k(t), y_0(t)\right) < \epsilon. \tag{A.6.25}$$

If $t \in (u, u+1]$, we have by (A.6.22), (A.6.23) and (A.6.20) that

$$\begin{aligned}&\mathfrak{r}\left(y_k^u \circ \lambda_k^u(t), y_0^u(t)\right) = \mathfrak{r}\left[y_k\left(\lambda_k^u(t) \wedge u\right), y_0(u)\right] \\ &\leq \mathfrak{r}\left[y_k\left(\lambda_k^u(t) \wedge u\right), y_0\left(\lambda_k^u(t) \wedge u\right)\right] + \mathfrak{r}\left(y_0\left(\lambda_k^u(t) \wedge u\right), y_0(u)\right) \leq 2\epsilon.\end{aligned} \tag{A.6.26}$$

Now, the desired (A.6.17) follows by (A.6.24), (A.6.25) and (A.6.26). □

Lemma A.74. *Let E be a Tychonoff space, $\mathcal{D} \subset C(E; \mathbf{R})$ strongly separate points on E, $\Psi \mathrel{\hat{=}} \varpi[\mathfrak{ae}(\mathcal{D})]$, $\{y_k\}_{k \in \mathbf{N}_0} \subset D(\mathbf{R}^+; E)$ and $\{y_k^u\}_{k \in \mathbf{N}_0}$ be as in (A.6.14). Then for each $u \in \mathbf{R}^+ \backslash J(y_0)$,*

$$\Psi(y_k) \longrightarrow \Psi(y_0) \ \textit{as}\ k \uparrow \infty\ \textit{in}\ D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})} \tag{A.6.27}$$

implies (A.6.16). In particular, (A.6.15) implies (A.6.16).

Proof. (A.6.15) implies (A.6.27) by Proposition A.62 (a). We now suppose

(A.6.27) holds and fix $f \in \mathfrak{ae}(\mathcal{D})$. It follows by Fact B.11 and Proposition A.62 (d) (with $S = E$ and $E = \mathbf{R}$) that $\varpi(f)(y_k) \to \varpi(f)(y_0)$ as $k \uparrow \infty$ in $D(\mathbf{R}^+;\mathbf{R})$ and $u \in \mathbf{R}^+ \backslash J[\varpi(f)(y_0)]$. It follows by Proposition A.73 (with $E = \mathbf{R}$ and $y_k = \varpi(f)(y_k)$) that[43]

$$\varpi_{u+1}(f)(y_k^u) \longrightarrow \varpi_{u+1}(f)(y_0^u) \text{ as } k \uparrow \infty \text{ in } D([0,u+1];\mathbf{R}). \tag{A.6.28}$$

It follows by (A.6.28) and Fact B.11 that

$$\Psi_{u+1}(y_k^u) \longrightarrow \Psi_{u+1}(y_0^u) \text{ as } k \uparrow \infty \text{ in } D([0,u+1];\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}, \tag{A.6.29}$$

where

$$\Psi_{u+1} \mathrel{\hat{=}} \varpi_{u+1}\left(\mathfrak{ae}(\mathcal{D})\right) \in \mathbf{imb}\left(D\left([0,u+1];E\right); D([0,u+1];\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right) \tag{A.6.30}$$

by [Kouritzin, 2016, Theorem 22] (with $S = E$, $[a,b] = [0,u+1]$, $\mathcal{H}^1 = \mathcal{D}$ and $\mathcal{H} = \mathfrak{ae}(\mathcal{D})$). Hence, (A.6.16) follows by (A.6.29) and (A.6.30). □

Lemma A.75. *Let E be a Tychonoff space, $\{y_k\}_{k\in\mathbf{N}_0} \subset D(\mathbf{R}^+;E)$ and $\{y_k^u\}_{k\in\mathbf{N}_0}$ be as in (A.6.14). If $\mathbf{R}^+ \backslash J(y_0)$ is dense in $\mathbf{R}^+$, and if (A.6.16) holds for all $u \in \mathbf{R}^+ \backslash J(y_0)$, then (A.6.15) holds.*

Proof. We show (A.6.15) by verifying that every subsequence of $\{y_k\}_{k\in\mathbf{N}}$ has a sub-subsequence $\{y_{k_p}\}_{p\in\mathbf{N}}$ converging to y_0 as $p \uparrow \infty$ in $D(\mathbf{R}^+;E)$. Without loss of generality, we let the subsequence be $\{y_k\}_{k\in\mathbf{N}}$ itself. Letting $f \in C(E;\mathbf{R})$, $p \in \mathbf{N}$ and $\mathfrak{r}$ temporarily denoting the Euclidean metric, we find that

$$\lim_{k\to\infty} \varrho^{\mathfrak{r}}_{[0,u+1]}\left[\varpi_{u+1}(f)(y_k^u), \varpi_{u+1}(f)(y_0^u)\right] = 0,\ \forall u \in \mathbf{R}^+ \backslash J(y_0) \tag{A.6.31}$$

by (A.6.16) and [Jakubowski, 1986, Theorem 1.7]. $\mathbf{R}^+ \backslash J(y_0)$ is dense by our hypothesis, so there exist $u_p \in (p,\infty) \backslash J(y_0)$, $k_p \in \mathbf{N}$ and $\lambda_p \in \mathbf{TC}([0,u_p+1])$ such that

$$|||\lambda_p||| \vee \mathfrak{r}_{[0,u_p+1]}\left[\varpi_{u_p+1}(f)\left(y_{k_p}^{u_p} \circ \lambda_p\right), \varpi_{u_p+1}(f)(y_0^{u_p})\right] < 2^{-p}. \tag{A.6.32}$$

[43] The notation "$\varpi_{u+1}(\cdot)$" was defined in §2.2.1.

We define

$$\lambda'_p(t) \circeq \begin{cases} \lambda_p(t), & \text{if } t \in [0, u+1], \\ t, & \text{if } t \in (u+1, \infty) \end{cases} \tag{A.6.33}$$

for each $p \in \mathbf{N}$ and find by (A.6.32) that

$$|||\lambda'_p||| \leq |||\lambda_p||| < 2^{-p} \tag{A.6.34}$$

and

$$\begin{aligned} &\sup_{u \in (0, u_p]} \mathfrak{r}_{[0,u]} \left[\varpi(f)\left(y_{k_p} \circ \lambda'_p\right), \varpi(f)(y_0)\right] \\ &\leq \mathfrak{r}_{[0,u_p]} \left[\varpi(f)\left(y_{k_p} \circ \lambda'_p\right), \varpi(f)(y_0)\right] \\ &\leq \mathfrak{r}_{[0,u_p]} \left[\varpi_{u_p+1}(f)\left(y_{k_p}^{u_p} \circ \lambda_p\right), \varpi_{u_p+1}(f)(y_0^{u_p})\right] < 2^{-p}. \end{aligned} \tag{A.6.35}$$

It follows by (A.6.34), (A.6.35) and the fact $u_p \in (p, \infty)$ that

$$\begin{aligned} &\varrho^{\mathfrak{r}} \left[\varpi(f)(y_{k_p}), \varpi(f)(y_0)\right] \\ &\leq |||\lambda'_p||| \vee \int_0^\infty e^{-u} \mathfrak{r}_{[0,u]} \left[\varpi(f)(y_{k_p}) \circ \lambda'_p, \varpi(f)(y_0)\right] du \\ &\leq 2^{-p} \int_0^{u_p} e^{-u} du + \int_{u_p}^\infty e^{-u} du \leq 2^{-p} + e^{-p}. \end{aligned} \tag{A.6.36}$$

$\varrho^{\mathfrak{r}}$ metrizes the Skorokhod $\mathscr{J}_1$-topology of $D([0, u+1]; \mathbf{R})$ by Proposition A.72 (a) (with $E = \mathbf{R}$). From (A.6.36) it follows that

$$\varpi(f)(y_{k_p}) \longrightarrow \varpi(f)(y_0) \text{ as } p \uparrow \infty \text{ in } D(\mathbf{R}^+; \mathbf{R}). \tag{A.6.37}$$

The $f \in C(E; \mathbf{R})$ above is arbitrary, so we have by (A.6.37) and Fact B.11 that

$$\begin{aligned} &\varpi\left(\bigotimes C(E; \mathbf{R})\right)(y_{k_p}) \\ &\longrightarrow \varpi\left(\bigotimes C(E; \mathbf{R})\right)(y_0) \text{ as } p \uparrow \infty \text{ in } D(\mathbf{R}^+; \mathbf{R})^{C(E;\mathbf{R})}. \end{aligned} \tag{A.6.38}$$

Now, $y_{k_p} \to y_0$ as $p \uparrow \infty$ in $D(\mathbf{R}^+; E)$ by (A.6.38) and Proposition A.62 (a). $\square$

A.7 Càdlàg process

The following are two well-known facts about the relationship between E-valued càdlàg processes and $D(\mathbf{R}^+;E)$-valued random variables.

Fact A.76. *Let E be a Tychonoff space and $\mu \in \mathcal{M}^+(D(\mathbf{R}^+;E))$ be the distribution of $D(\mathbf{R}^+;E)$-valued random variable X. Then, the following statements are true:*

(a) X is an E-valued càdlàg process

(b) μ equals the restriction of $\mathrm{pd}(X)|_{D(\mathbf{R}^+;E)}$ to $\sigma(\mathscr{J}(E))$[44]*.*

(c) The finite-dimensional distribution of X for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ is $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$.

Proof. This fact follows by Proposition A.68 (b), Fact 2.26 (a) and Corollary A.69. □

Fact A.77. *Let E be a metrizable and separable space and $(\Omega, \mathscr{F}, \mathbb{P}; X)$*[45] *be an E-valued process. Then, the following statements are true:*

(a) If all paths of X lie in $D(\mathbf{R}^+;E)$, then $X \in M(\Omega, \mathscr{F}; D(\mathbf{R}^+;E))$.

(b) If X is càdlàg, then there exists a $Y \in M(\Omega, \mathscr{F}; D(\mathbf{R}^+;E))$ that is indistinguishable from X.

In particular, the above statements are true when E is a Polish space.

Proof. This fact follows by Proposition A.11 (c), Proposition A.72 (b) and Lemma B.70 (b) (with $E_0 = E$ and $S_0 \circeq D(\mathbf{R}^+;E)$). □

The set of fixed left-jump times of an E-valued càdlàg process is well-defined for fairly general E.

[44] Restriction of measure to sub-σ-algebra and X's process distribution $\mathrm{pd}(X)$ were specified in §2.1.2 and §2.5 respectively.

[45] Please be reminded that $(\Omega, \mathscr{F}, \mathbb{P})$ and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i)\}_{i \in \mathbf{I}}$ are complete probability spaces as arranged in §2.6. Completeness of measure space was specified in 2.1.2.

Fact A.78. *Let E be a topological space and X be an E-valued càdlàg process. If there exists a countable subset of $M(E;\mathbf{R})$ that separates points on E, then $J(X)$[46] is well-defined.*

Proof. This result follows by Lemma B.50 (a). $\square$

The next lemma solves the measurability issue in (6.4.2) under mild conditions.

Lemma A.79. *Let E be a Hausdorff space, V be the family of all càdlàg members of $E^{\mathbf{R}^+}$[47], $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued càdlàg process and $T\in(0,\infty)$. Then, $\bigcap_{t\in[0,T]}X_t^{-1}(A)\in\mathscr{F}$ for all $A\in\mathscr{C}(E)$, especially for all $A\in\mathscr{K}(E)$ when E is a Hausdorff space.*

Proof. When E is Hausdorff, $\mathscr{K}(E)\subset\mathscr{C}(E)$ by Proposition A.12 (a). Let $F\circeq\bigcap_{t\in[0,T)}X_t^{-1}(A)$. There exists $B\in\mathscr{B}(E)^{\otimes\mathbf{R}^+}$ such that $B\cap V=\{x\in V: x|_{[0,T)}\in A^{[0,T)}\}$ by Lemma (B.50) (b). $X^{-1}(V)$ and $F\backslash X^{-1}(V)$ both belong to $\mathscr{F}$ by the fact $\mathbb{P}(X\in V)=1$ and the completeness of $(\Omega,\mathscr{F},\mathbb{P})$. $X_T^{-1}(A)\in\mathscr{F}$ by the fact $A\in\mathscr{C}(E)\subset\mathscr{B}(E)$ and act 2.24 (a). Hence,

$$\bigcap_{t\in[0,T]}X_t^{-1}(A)=X_T^{-1}(A)\cap\left[\left(F\backslash X^{-1}(V)\right)\cup\left(X^{-1}(B)\cap X^{-1}(V)\right)\right]\in\mathscr{F}. \tag{A.7.1}$$

$\square$

The next lemma treats the measurability issue in (6.4.5).

Lemma A.80. *Let E be a topological space, $(\Omega,\mathscr{F},\mathbb{P};X)$ be an E-valued càdlàg process, $\mathfrak{r}$ be a pseudometric on E and $\delta,T\in(0,\infty)$. Then, $w'_{\mathfrak{r},\delta,T}\circ X\in M(\Omega,\mathscr{F};\mathbf{R})$ in each of the following settings:*

(a) $(E,\mathfrak{r})$ is a metric space and $X\in M(\Omega,\mathscr{F};D(\mathbf{R}^+;E))$.

(b) $(E,\mathfrak{r})$ is a separable metric space.

(c) $\mathfrak{r}=\rho_{\{f\}}$ with $f\in C(E;\mathbf{R})$.

[46] $J(X)$, the set of fixed left-jump times of X was defined in (2.5.8).
[47] E need not be a Tychonoff space, so we avoid the notation $D(\mathbf{R}^+;E)$ for clarity.

(d) $\mathfrak{r} = \rho_{\mathcal{D}}$ *with* $\mathcal{D} \subset C(E;\mathbf{R})$ *being countable and separating points on* E[48].

Proof. (a) is immediate by Proposition A.70 (a).

(b) X is indistinguishabe from some $Y \in M(\Omega, \mathscr{F}; D(\mathbf{R}^+; E))$ by Fact A.77 (b). $w'_{\mathfrak{r},\delta,T} \circ Y \in M(\Omega, \mathscr{F}; \mathbf{R})$ by (a). Now, (b) follows by the fact $\mathbb{P}(w'_{\mathfrak{r},\delta,T} \circ X = w'_{\mathfrak{r},\delta,T} \circ Y) = 1$ and Lemma B.31 (a) (with $(E, \mathscr{U}) = S = \mathbf{R}$, $X = w'_{\mathfrak{r},\delta,T} \circ X$ and $Z = w'_{\mathfrak{r},\delta,T} \circ Y$).

(c) $\varpi(f) \circ X$ is a càdlàg process by Fact B.34 (a) (with $S = \mathbf{R}$). It follows by (b) (with $X = \varpi(f) \circ X$ and $(E, \mathfrak{r}) = (\mathbf{R}, |\cdot|)$) that

$$w'_{\rho_{\{f\}},\delta,T} \circ X = w'_{|\cdot|,\delta,T} \circ (\varpi(f) \circ X) \in M\left(\Omega, \mathscr{F}; \mathbf{R}\right). \tag{A.7.2}$$

(d) $(E, \rho_{\mathcal{D}})$ is a separable metric space and is a topological coarsening of E by $\mathcal{D} \subset C(E;\mathbf{R})$ and Proposition A.17 (d, e) (with $A = E$). So, X is an $(E, \rho_{\mathcal{D}})$-valued càdlàg process by Fact B.34 (c). Now, (d) follows by (b) (with $\mathfrak{r} = \rho_{\mathcal{D}}$). □

The following five results discuss the relationship among $\mathfrak{r}$-MCC[49], MCC, $\mathcal{D}$-FMCC and WMCC for càdlàg processes.

Fact A.81. *Let* E *be a topological space,* $\mathcal{D} \subset M(E;\mathbf{R})$ *and* $\{X^i\}_{i\in\mathbf{I}}$ *be* E*-valued processes such that* $\{\varpi(f) \circ X^i\}_{f\in\mathcal{D},i\in\mathbf{I}}$ *are all càdlàg. Then, the following statements are true:*

(a) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\rho_{\{f\}}$*-MCC for all* $f \in \mathcal{D}$ *if and only if* $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ *satisfies* $|\cdot|$*-MCC for all* $f \in \mathcal{D}$.

(b) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC if and only if* $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ *satisfies* $|\cdot|$*-MCC for all* $f \in \mathfrak{ae}(\mathcal{D})$.

In particular, the two statements above are true when $\mathcal{D} \subset C(E;\mathbf{R})$ *and* $\{X^i\}_{i\in\mathbf{I}}$ *are all càdlàg.*

[48] The E in (d) is a $\mathcal{D}$-baseable space.

[49] $\mathfrak{r}$-MCC, MCC, $\mathcal{D}$-FMCC and WMCC were introduced in Definition 6.36.

Proof. If X^i is càdlàg and $f \in C(E;\mathbf{R})$, then $\varpi(f)\circ X^i$ is càdlàg by Fact B.34 (a) (with $X = X^i$ and $S = \mathbf{R}$). If $\varpi(f)\circ X^i$ is càdlàg, then it is indistinguishable from any of its càdlàg modification by Proposition B.33 (h). Now, the result follows by the fact that

$$w'_{\rho_{\{f\}},\delta,T}\circ X^i = w'_{|\cdot|,\delta,T}\circ\varpi(f)\circ X^i,\ \forall f \in M(E;\mathbf{R}), i \in \mathbf{I}, \delta, T \in (0,\infty). \quad \text{(A.7.3)}$$

□

The following proposition is a version of [Kouritzin, 2016, Proposition 14] on infinite time horizon.

Proposition A.82. *Let E be a Hausdorff space and $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;X^i)\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Then, the following statements are equivalent:*

(a) $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCC.

(b) There exist a $\mathcal{D}_1 \subset C(E;\mathbf{R})$ and a $\mathcal{D}_2 \subset C_b(E;\mathbf{R})$ such that: (1) $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{D}_1$-FMCC, (2) $\mathcal{D}_2 = \mathfrak{ac}(\mathcal{D}_2)$[50] strongly separates points on E, and (3) for any $g \in \mathcal{D}_2$ and $\epsilon, T > 0$, there exists an $f_{g,\epsilon,T} \in \mathcal{D}_1$ satisfying

$$\sup_{i\in\mathbf{I}}\mathbb{P}^i\left(\sup_{t\in[0,T]}\left|f_{g,\epsilon,T}\circ X^i_t - g\circ X^i_t\right| \geq \epsilon\right) \leq \epsilon. \quad \text{(A.7.4)}$$

(c) There exist $\mathcal{D} \subset C_b(E;\mathbf{R})$ and $\{(\Omega^i,\mathscr{F}^i,\mathbb{P}^i;\zeta^{i,f,\epsilon,T})\}_{i\in\mathbf{I},f\in\mathcal{D},\epsilon,T>0}$ such that: (1) $\mathcal{D} = \mathfrak{ac}(\mathcal{D})$ strongly separates points on E, and (2) for each $f \in \mathfrak{ac}(\mathcal{D})$ and $\epsilon, T > 0$, $\mathbf{R}$-valued processes $\{\zeta^{i,f,\epsilon,T}\}_{i\in\mathbf{I}}$ satisfy $|\cdot|$-MCC[51] and

$$\sup_{i\in\mathbf{I}}\mathbb{P}^i\left(\sup_{t\in[0,T]}\left|f\circ X^i_t - \zeta^{i,f,\epsilon,T}_t\right| \geq \epsilon\right) \leq \epsilon. \quad \text{(A.7.5)}$$

(d) There exists an $\mathcal{D} \subset C(E;\mathbf{R})$ such that: (1) $\mathcal{D}$ strongly separates points on E, and (2) $\{\varpi(\bigotimes \mathcal{D}_0)\circ X^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC and MCCC[52] for

[50] $\mathcal{D}_2 = \mathfrak{ac}(\mathcal{D}_2)$ means $\mathcal{D}_2$ is closed under multiplication.
[51] $|\cdot|$-MCC means MCC for the Euclidean norm metric $|\cdot|$.
[52] The notion of MCCC was specified in Definition 6.36.

all $\mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D})$.

(e) There exists a $\mathcal{D} \subset C(E;\mathbf{R})$ *such that: (1)* $\mathcal{D}$ *strongly separates points on* E*, and (2)* $\{\varpi(g) \circ X^i\}_{i\in\mathbf{I}}$ *satisfies* $|\cdot|$*-MCC and MCCC for all* $g \in \mathfrak{ac}(\{af : f \in \mathcal{D}, a \in \mathbf{R}\})$[53].

Proof. The proof is almost the same as that of [Kouritzin, 2016, Proposition 14] and we omit it herein to avoid redundancy. □

Corollary A.83. *Let* E *be a Hausdorff space and* $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ *be* E*-valued càdlàg processes. Then, the following statements are equivalent:*

(a) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies MCC.*

(b) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC for some* $\mathcal{D} = \mathfrak{ac}(\mathcal{D}) \subset C_b(E;\mathbf{R})$ *and* $\mathcal{D}$ *strongly separates points on* E.

(c) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC for some* $\mathcal{D} = \mathfrak{ac}(\mathcal{D}) \subset C(E;\mathbf{R})$ *and* $\mathcal{D}$ *strongly separates points on* E.

(d) $\{X^i\}_{i\in\mathbf{I}}$ *satisfies* $\mathcal{D}$*-FMCC for some* $\mathcal{D} \subset C(E;\mathbf{R})$ *and* $\mathcal{D}$ *strongly separates points on* E.

Proof. ((a) → (b)) If Proposition A.82 (c) holds, by observing that

$$\begin{aligned} &\left|g \circ X_u^i - g \circ X_v^i\right| \\ &\leq \left|g \circ X_u^i - \zeta_u^{i,f,\epsilon,T}\right| + \left|g \circ X_v^i - \zeta_v^{i,f,\epsilon,T}\right| + \left|\zeta_u^{i,f,\epsilon,T} - \zeta_v^{i,f,\epsilon,T}\right| \\ &\leq 2 \sup_{t\in[0,T]} \left|g \circ X_t^i - \zeta_t^{i,f,\epsilon,T}\right| + \left|\zeta_u^{i,f,\epsilon,T} - \zeta_v^{i,f,\epsilon,T}\right| \end{aligned} \tag{A.7.6}$$

for all $g \in \mathcal{D}$, $u, v \in \mathbf{R}^+$, $\epsilon, T \in (0,\infty)$ and $i \in \mathbf{I}$ we conclude $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ satisfies $|\cdot|$-MCC for all $f \in \mathcal{D}$. Thus, (b) follows by Proposition A.82 (a, c).

((b) → (c)) and ((c) → (d)) are automatic.

((d) → (a)) $\{\rho_{\{f\}}\}_{f\in\mathcal{D}}$ generates the Hausdorff topology of E by Proposition A.17 (d, e) (with $A = E$). Then, (a) follows by Fact A.81. □

[53] The notation "$\mathfrak{ac}(\cdot)$" was defined in §2.2.3.

Proposition A.84. *Let E be a topological space, $\{X^i\}_{i\in\mathbf{I}}$ be E-valued processes and $\mathcal{D} \subset M(E;\mathbf{R})$ be countable and separate points on E. If $\{\varpi(f) \circ X^i\}_{f\in\mathcal{D},i\in\mathbf{I}}$ are all càdlàg, especially if $\{X^i\}_{i\in\mathbf{I}}$ are all càdlàg and $\mathcal{D} \subset C(E;\mathbf{R})$, then $\{X^i\}_{i\in\mathbf{I}}$ satisfying $\rho_{\mathcal{D}}$-MCC is equivalent to $\{\varpi(f) \circ X^i\}_{i\in\mathbf{I}}$ satisfying $|\cdot|$-MCC for all $f \in \mathcal{D}$.*

Proof. Necessity is straightforward by the definition of $\rho_{\mathcal{D}}$. Sufficiency was proved in the argument establishing [Kouritzin, 2016, Theorem 15]. For brevity, we do not duplicate this small technicality herein. □

Fact A.85. *Let E be a Hausdorff space. If E-valued càdlàg processes $\{X^i\}_{i\in\mathbf{I}}$ satisfy MCC, then $\{X^i\}_{i\in\mathbf{I}}$ satisfies WMCC.*

Proof. This fact is immediate by Corollary A.83 (a, c) and Proposition A.17 (a) (with $A = E$). □

Corollary A.86. *Let E be a metrizable space and $\{X^i\}_{i\in\mathbf{I}}$ be E-valued càdlàg processes. Then, the following statements are true:*

(a) *If $(E, \mathfrak{r})$ is a metrization of E and $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathfrak{r}$-MCC, then $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCC.*

(b) *If E is separable and $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCC, then $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{D}$-FMCC for some countable $\mathcal{D} = \mathfrak{ac}(\mathcal{D}) \subset C_b(E;\mathbf{R})$ and $\mathcal{D}$ strongly separates points on E. Moreover, $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathfrak{r}$-MCC for some metrization $(E, \mathfrak{r})$ of E.*

Proof. (a) is automatic by definition. Regarding (b), we have by Corollary A.83 (a, b) that $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{G}$-FMCC for some $\mathcal{G} = \mathfrak{ac}(\mathcal{G}) \subset C_b(E;\mathbf{R})$ and $\mathcal{G}$ strongly separates points on E. There exists a countable $\mathcal{G}_0 \subset \mathcal{G}$ such that $\mathcal{G}_0$ strongly separates points on E by Proposition A.6 (c) and Proposition A.24 (b). $\mathcal{D} \circeq \mathfrak{ac}(\mathcal{G}_0)$ is a countable subset of $\mathcal{G}$ by Fact B.15 (with $\mathcal{D} = \mathcal{G}_0$), so $\{X^i\}_{i\in\mathbf{I}}$ satisfies $\mathcal{D}$-measurable or $\mathscr{G}_t$-progressive process, then $\varpi(f) \circ X$ is an S-valued process with the corresponding measurability.FMCC. $\mathcal{D}$ separates points on E and $\mathfrak{r} \circeq \rho_{\mathcal{D}}$ metrizes E by Proposition A.6 (a) and Proposition A.17 (a, d, e) (with $A = E$). Now, (b) follows by Fact A.81 (b) and Proposition A.84. □

For readers' convenience, we quote from Ethier and Kurtz [1986] the following two classical result about Skorokhod-$\mathscr{J}_1$-spaces-valued random variables.

Theorem A.87 ([Ethier and Kurtz, 1986, §3.7, Theorem 7.8])**.** *Let E be a metrizable and separable space and $\{X^n\}_{n\in\mathbf{N}} \cup \{X\}$ be $D(\mathbf{R}^+; E)$-valued random variables. Then, the following statements are true:*

(a) (1.6)[54] *implies*

$$X^n \xrightarrow{\mathrm{D}(\mathbf{R}^+\backslash J(X))} X \text{ as } n \uparrow \infty. \tag{A.7.7}$$

(b) If $\{X^n\}_{n\in\mathbf{N}}$ is relatively compact in $D(\mathbf{R}^+; E)$ and (6.2.1) holds for some dense $\mathbf{T} \subset \mathbf{R}^+$, then (1.6) holds.

Theorem A.88. *Let $(E, \mathfrak{r})$ be a complete separable metric space and $\{X^i\}_{i\in\mathbf{I}}$ be $D(\mathbf{R}^+; E)$-valued random variables. Then, $\{X^i\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+; E)$*[55] *if and only if $\{X^i\}_{i\in\mathbf{I}}$ satisfies MCCC and $\mathfrak{r}$-MCC.*

Proof. As E is a Polish space, tightness is equivalent to relative compactness in $D(\mathbf{R}^+; E)$ by Proposition A.72 (d) and the Prokhorov's Theorem (Theorem 2.22 (a)). Hence, the result is equivalent to [Ethier and Kurtz, 1986, §3.7, Theorem 7.2 and Remark 7.3]. $\square$

[54] Weak convergence and relative compactness of random variables were specified in §2.4.

[55] Tightness of random variables was specified in §2.4.

Appendix B

Miscellaneous

This chapter consists of auxiliary results related to all aspects of this work. In view of clarity and self-containment, we collect in §B.1 a set of basic and general technicalities. §B.2 supplements a few results that involves the topics or developments of Appendix A. §B.3 houses several auxiliary lemmas about replication which is used in Chapter 3 - Chapter 8 and the companion papers Dong and Kouritzin [2017a,b,c,d]. All notations, terminologies and conventions introduced before apply to this appendix.

B.1 General technicalities

Fact B.1. *Let E be the union of non-empty sets $\{A_n\}_{n\in\mathbf{N}}$. If σ-algebras $\mathscr{U}_1$ and $\mathscr{U}_2$ on E satisfy $A_n \in \mathscr{U}_2$ and $\mathscr{U}_1|_{A_n} = \mathscr{U}_2|_{A_n}$ for all $n \in \mathbf{N}$, then $\mathscr{U}_1 \subset \mathscr{U}_2$.*

Proof. We observe that

$$\begin{aligned}\mathscr{U}_1 &= \left\{\bigcup_{n\in\mathbf{N}} B \cap A_n : B \in \mathscr{U}_1\right\} \subset \left\{\bigcup_{n\in\mathbf{N}} B_n : B_n \in \mathscr{U}_1|_{A_n}, \forall n \in \mathbf{N}\right\} \\ &= \left\{\bigcup_{n\in\mathbf{N}} B_n : B_n \in \mathscr{U}_2|_{A_n}, \forall n \in \mathbf{N}\right\} \subset \sigma\left(\bigcup_{n\in\mathbf{N}} \mathscr{U}_2|_{A_n}\right) \subset \mathscr{U}_2.\end{aligned} \tag{B.1.1}$$

□

Fact B.2. *Let $(E, \mathscr{U})$ be a measurable space, $A \in \mathscr{U}$ and $k \in \mathbf{N}$. If $f \in (\mathbf{R}^k)^E$*

satisfies $f|_A \in M(A, \mathscr{U}|_A; \mathbf{R}^k)$, *then* $f\mathbf{1}_A \in M(E, \mathscr{U}; \mathbf{R}^k)$[1].

Proof. We observe for each $B \in \mathscr{B}(\mathbf{R}^k)$ that

$$(f\mathbf{1}_A)^{-1}(B) = \begin{cases} (f|_A)^{-1}(B) \in \mathscr{U}|_A \subset \mathscr{U}, & \text{if } 0 \notin B, \\ (f|_A)^{-1}(B) \cup (E\backslash A) \in \mathscr{U}, & \text{otherwise.} \end{cases} \tag{B.1.2}$$

□

Fact B.3. *Let* E *and* S *be non-empty sets,* $y_0 \in A \subset E$*,* $f \in E^S$ *and* $g \doteq \mathfrak{var}(f; S, f^{-1}(A), y_0)$*. Then, the following statements are true:*

(a) $\{x \in S : f(x) = g(x)\} \supset f^{-1}(A)$.

(b) If $(E, \mathscr{U})$ *and* $(S, \mathscr{A})$ *are measurable spaces,* $f \in M(S, \mathscr{A}; E, \mathscr{U})$*,* $A \in \mathscr{U}$ *and* $\{y_0\} \in \mathscr{U}$*, then* $g \in M(S, \mathscr{A}; A, \mathscr{U}|_A) \subset M(S, \mathscr{A}; E, \mathscr{U})$.

Proof. (a) is immediate by the definition of $\mathfrak{var}(f; S, f^{-1}(A), y_0)$.

(b) We find for each $B \in \mathscr{U}$ that

$$g^{-1}(B) = \begin{cases} f^{-1}(B \cap A) \in \mathscr{A}, & \text{if } y_0 \notin B, \\ f^{-1}\left[(B \cap A)\backslash\{y_0\}\right] \cup (S\backslash f^{-1}(A)) \in \mathscr{A}, & \text{if } y_0 \in B. \end{cases} \tag{B.1.3}$$

□

Fact B.4. *Let* E *and* S *be topological space,* $\mu \in \mathcal{M}^+(E)$ *and* A *denote the set of discontinuity points*[2] *of* $f \in S^E$*. If* $(E, \mathscr{U}, \nu)$ *is the completion*[3] *of* $(E, \mathscr{B}(E), \mu)$ *and* $A \in \mathscr{N}(\mu)$*, then* $f \in M(E, \mathscr{U}; S)$.

Proof. Fixing $O \in \mathscr{O}(S)$, we have that

$$(f^{-1}(O)\backslash A) = (f|_{E\backslash A})^{-1}(O) \in \mathscr{O}_E(E\backslash A) \subset \mathscr{B}_E(E\backslash A) \subset \mathscr{U} \tag{B.1.4}$$

and

$$f^{-1}(O) \cap A \in \mathscr{N}(\mu) \subset \mathscr{U} \tag{B.1.5}$$

by the continuity of $f|_{E\backslash A}$ and the fact $A \in \mathscr{N}(\mu) \subset \mathscr{U}$, so $f^{-1}(O) \in \mathscr{U}$. □

[1] $\mathbf{1}_A$ denotes the indicator function of A.

[2] The notion of set of discontinuity points was mentioned in the proof of Lemma 8.11.

[3] The notation "$\mathscr{N}(\mu)$" and completion of measure space were specified in §2.1.2.

The fact below confirms that $\sigma(C(E;\mathbf{R}))$, the Baire σ-algebra on E is generally smaller than $\mathscr{B}(E)$.

Fact B.5. *Let E be a topological space, S be a non-empty set and $A \subset S$. Then, $\sigma(\mathcal{D})|_A \subset \mathscr{B}_{\mathcal{D}}(A)$ for any $\mathcal{D} \subset E^S$ and the equality holds if E is a second-countable space and $\mathcal{D}$ is countable.*

Proof. In any case, we have that

$$\begin{aligned}\mathscr{B}_{\mathcal{D}}(A) &\supset \sigma\left(\bigcup_{f\in\mathcal{D}} \left\{f^{-1}(O)\cap A : O \in \mathscr{O}(E)\right\}\right)\\ &= \sigma\left[\bigcup_{f\in\mathcal{D}} \sigma\left(\left\{f^{-1}(O)\cap A : O \in \mathscr{O}(E)\right\}\right)\right]\\ &= \sigma\left(\left\{f^{-1}(B)\cap A : B \in \sigma\left(\mathscr{O}(E)\right) = \mathscr{B}(E), f\in\mathcal{D}\right\}\right) = \sigma(\mathcal{D})\big|_A.\end{aligned} \tag{B.1.6}$$

If $\{O_n\}_{n\in\mathbf{N}}$ is a countable topological basis of E and $\mathcal{D}$ is countable, then every member of $\mathscr{O}_{\mathcal{D}}(A)$ is a union of members of the countable topological basis

$$\left\{\bigcap_{f\in\mathcal{D}_0} f^{-1}(O_n)\cap A : n\in\mathbf{N}, \mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D})\right\} \tag{B.1.7}$$

by [Munkres, 2000, Lemma 13.1]. As a result,

$$\begin{aligned}\mathscr{O}_{\mathcal{D}}(A) &\subset \sigma\left(\left\{\bigcap_{f\in\mathcal{D}_0} f^{-1}(O_n)\cap A : n\in\mathbf{N}, \mathcal{D}_0 \in \mathscr{P}_0(\mathcal{D})\right\}\right)\\ &\subset \sigma\left(\left\{f^{-1}(B)\cap A : B\in\mathscr{B}(E), f\in\mathcal{D}\right\}\right) = \sigma(\mathcal{D})\big|_A.\end{aligned} \tag{B.1.8}$$

□

Fact B.6. *Let E be a topological space and $\{x_n\}_{n\in\mathbf{N}} \subset E$. If every convergent subsequence of $\{x_n\}_{n\in\mathbf{N}}$ must converge to x as $n\uparrow\infty$, and if any infinite subset of $\{x_n\}_{n\in\mathbf{N}}$ has a convergent subsequence, then $x_n \to x$ as $n\uparrow\infty$ in E.*

Proof. Suppose $\{x_n\}_{n\in\mathbf{N}}$ does not converge to x as $n\uparrow\infty$. Then, there exist an $O_x \in \mathscr{O}(E)$ containing x and $\{x_{n_k}\}_{k\in\mathbf{N}} \subset E\backslash O_x$ with $n_k\uparrow\infty$. However, $\{x_{n_k}\}_{k\in\mathbf{N}}$ has a convergent subsequence which must converge to x and stay in O_x with finite exception. Contradiction! □

Lemma B.7. *Let E and S be topological spaces, $\mathcal{D} \subset S^E$ and equip $V \mathrel{\mathring{=}} \bigotimes \mathcal{D}(E)$ with the subspace topology $\mathscr{O}_{S^{\mathcal{D}}}(V)$. Then, the following statements are true:*

(a) $(\bigotimes \mathcal{D})^{-1} \in C(V; E)$ *if and only if* $\mathscr{O}(E) \subset \mathscr{O}_{\mathcal{D}}(E)$ *and* $\bigotimes \mathcal{D}$ *is injective.*

(b) $\bigotimes \mathcal{D} \in \mathbf{hom}(E; V)$ *if and only if* $\mathscr{O}(E) = \mathscr{O}_{\mathcal{D}}(E)$ *and* $\bigotimes \mathcal{D}$ *is injective.*

Proof. (a) We find by Fact 2.4 (b) that $\bigotimes \mathcal{D} \in \mathbf{imb}(E, \mathscr{O}_{\mathcal{D}}(E); V)$ if and only if $\bigotimes \mathcal{D}$ is injective. Given the injectiveness of $\bigotimes \mathcal{D}$, $(\bigotimes \mathcal{D})^{-1} \in C(V; E)$ precisely when $\mathscr{O}(E)$ is coarser than $\mathscr{O}_{\mathcal{D}}(E)$.

(b) is immediate by (a). □

Lemma B.8. *Let E and S be topological spaces, $f \in E^S$ and $\mathscr{U}_f$ be defined as in (A.5.1). Then, the following statements are true:*

(a) If f is bijective, then $f \in \mathbf{hom}(S; (E, \mathscr{U}_f))$.

(b) If $f \in \mathbf{biso}(S; E)$, *then* $\mathscr{B}(E) = \sigma(\mathscr{U}_f)$.

Proof. (a) $\mathscr{U}_f$ is well-known to be a topology. $f \in C(S; E, \mathscr{U}_f)$ is immediate by (A.5.1). The bijectiveness of f implies that

$$\mathscr{U}_f = \{f(B) : B \in \mathscr{O}(S)\}, \tag{B.1.9}$$

thus proving $f^{-1} \in C(E, \mathscr{U}_f; S)$.

(b) $f \in \mathbf{biso}(S; E)$ satisfies (B.1.9) and further satisfies

$$\mathscr{B}(E) = \{f(B) : B \in \mathscr{B}(S)\} = \sigma\left(\{f(B) : B \in \mathscr{O}(S)\}\right) = \sigma(\mathscr{U}_f). \tag{B.1.10}$$

□

Remark B.9. The lemma above shows that a Borel isomorphism can always be turned into a homeomorphism by changing the generating topology of the underlying Borel σ-algebra.

Fact B.10. *Let $\mathbf{I}$, E and S be non-empty sets and $f \in S^E$. Then, the following statements are true:*

(a) If f is injective or surjective, then $\varpi_{\mathbf{I}}(f)$ is also.

(b) If $(E, \mathscr{U})$ and $(S, \mathscr{A})$ are measurable spaces and $f \in M(E, \mathscr{U}; S, \mathscr{A})$, then $\varpi_{\mathbf{I}}(f) \in M(E^{\mathbf{I}}, \mathscr{U}^{\otimes \mathbf{I}}; S^{\mathbf{I}}, \mathscr{A}^{\otimes \mathbf{I}})$.

(c) If E and S are topological spaces and $f \in C(E; S)$, then $\varpi_{\mathbf{I}}(f) \in C(E^{\mathbf{I}}; S^{\mathbf{I}})$.

Proof. This result is immediate by Fact 2.3 and Fact 2.4. □

Fact B.11. *Let $\{S_i\}_{i \in \mathbf{I}}$ be topological spaces. Then, $x_k \to x$ as $k \uparrow \infty$ in $\prod_{i \in \mathbf{I}} S_i$ if and only if $\mathfrak{p}_i(x_k) \to \mathfrak{p}_i(x)$ as $k \uparrow \infty$ in S_i for all $i \in \mathbf{I}$.*

Proof. This fact was justified in [Munkres, 2000, §19, Exercise 6]. □

Fact B.12. *Let $\mathbf{I}$ be an arbitrary index set and $\{a_i, b_i\} \subset \mathbf{R}$ satisfy $a_i < b_i$ for all $i \in \mathbf{I}$. Then, $\bigcup_{i \in \mathbf{I}} [a_i, b_i) \in \mathscr{B}(\mathbf{R})$.*

Proof. For each $\{i_1, i_2\} \subset \mathbf{I}$, we define $i_1 \sim i_2$ if there exist some $\{c, d\} \subset \mathbf{R}$ with $c < d$ and $\mathbf{I}_0 \subset \mathbf{I}$ such that

$$[c, d) = \bigcup_{i \in \mathbf{I}_0 \cup \{i_1, i_2\}} [a_i, b_i). \tag{B.1.11}$$

It is not difficult to see "$\sim$" defines an equivalence relation on $\mathbf{I}$. Let $\{\mathbf{I}_j : j \in \mathbf{J}\}$ be the "$\sim$" equivalence classes of the members of $\mathbf{I}$. Then for each $j \in \mathbf{J}$, there exist $\{c_j, d_j\} \subset \mathbf{R}$ such that $c_j < d_j$ and

$$\bigcup_{i \in \mathbf{I}_j} [a_i, b_i) = [c_j, d_j). \tag{B.1.12}$$

$\{[c_j, d_j) : j \in \mathbf{J}\}$ are pairwisely disjoint by the definition of "$\sim$", so $\mathbf{J}$ is countable by [Munkres, 2000, §30, Exercise 13]. Hence,

$$\bigcup_{i \in \mathbf{I}} [a_i, b_i) = \bigcup_{j \in \mathbf{J}} [c_j, d_j) \in \mathscr{B}(\mathbf{R}). \tag{B.1.13}$$

□

Fact B.13. *Let the $\star$-convergence in $\mathbf{R}$ be defined in Example 2.15 and $A \subset \mathbf{R}$ be $\star$-closed if and only if $\{x_n\}_{n \in \mathbf{N}} \subset A$ $\star$-converging to $x \in \mathbf{R}$ implies $x \in A$.*

Then, $A \subset \mathbf{R}$ is $\star$-closed if and only it is closed with respect to the Euclidean topology.

Proof. $\star$-convergence is stronger than the Euclidean one, so every Euclidean closed A is immediately $\star$-closed. Conversely, suppose $A \neq \varnothing$ is $\star$-closed and x is an Euclidean limit point of A. We show $x \in A$ by constructing $\{x_n\}_{n\in\mathbf{N}} \subset A$ that $\star$-converges to x. For $n = 1$, let $x_1 \in A$ be arbitrary. For $n > 1$, we let $\epsilon_n \stackrel{\circ}{=} |x_{n-1} - x| \wedge 10^{-n}$ so $\{x_k\}_{1\leq k<n} \subset \mathbf{R}\backslash(x - \epsilon_n, x + \epsilon_n)$. Since x is an Euclidean limit point of A, $A \cap (x - \epsilon_n, x + \epsilon_n) \neq \varnothing$ and we pick an arbitrary x_n from this set. By induction, we obtain distinct points $\{x_n\}_{n\in\mathbf{N}} \subset A$ such that $|x_n - x| \leq 10^{-n} < 2^{-n}$ for all $n \in \mathbf{N}$. Hence, $\{x_n\}_{n\in\mathbf{N}}$ $\star$-converges to x. □

Fact B.14. *Let E, S and $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $f \in C(E; S)$. Then, the following statements are true:*

(a) If $x \in E^{\mathbf{R}^+}$ is right-continuous, then $x \in M(\mathbf{R}^+; E)$.

(b) If $x \in E^{\mathbf{R}^+}$ is càdlàg and $f \in C(E; S)$, then $\varpi(f)(x) \in S^{\mathbf{R}^+}$ is also càdlàg.

(c) $\bigotimes_{i\in\mathbf{I}} f_i : E \to S^i$ is càdlàg if and only if $f_i : E \to S_i$ is càdlàg for all $i \in \mathbf{I}$.

Proof. (a) Note that

$$x_n \stackrel{\circ}{=} \sum_{i=1}^{n2^n} x\left(\frac{i}{2^n}\right) \mathbf{1}_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)} + x(n)\mathbf{1}_{[n,\infty)} \in M(\mathbf{R}^+; E),\ \forall n \in \mathbf{N} \qquad \text{(B.1.14)}$$

and $x_n \to x$ as $n \uparrow \infty$ in $E^{\mathbf{R}^+}$. Then, $x \in M(\mathbf{R}^+; E)$ as pointwise convergence preserves measurability.

(b) and (c) are immediate by the definitions of $\varpi(f)$, $\bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i)$ and product topology. □

Fact B.15. *Let E be a non-empty set, $d, k \in \mathbf{N}$ and $\mathcal{D} \subset (\mathbf{R}^k)^E$ be a countable collection. Then, $\mathfrak{ae}(\mathcal{D})$ and $\mathfrak{ac}(\mathcal{D})$ are countable collections. When $k = 1$, $\mathfrak{mc}(\mathcal{D})$ and $\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D})$ are also countable collections.*

Proof. Let $\mathcal{D} = \{f_n\}_{n\in\mathbf{N}}$. For each $m \in \mathbf{N}$, we observe that $\sum_{i=1}^m f_{n_i} \mapsto (n_1, ..., n_m)$ defines an injective mapping from $\mathcal{D}_m \doteq \{\sum_{i=1}^m f_{n_i} : n_1, ..., n_m \in \mathbf{N}\}$ to the countable set $\mathbf{N}^m$, so $\mathcal{D}_m$ is countable. As a result, $\mathfrak{ae}(\mathcal{D}) = \mathcal{D}_2$ and $\mathfrak{ac}(\mathcal{D}) = \bigcup_{m\in\mathbf{N}} \mathcal{D}_m$ are both countable.

Next, we let $k = 1$ and observe that $\prod_{i=1}^m f_{n_i} \mapsto (n_1, ..., n_m)$ defines an injective mapping from $\mathcal{D}'_m \doteq \{\prod_{i=1}^m f_{n_i} : n_1, ..., n_m \in \mathbf{N}\}$ to the countable set $\mathbf{N}^m$, so $\mathcal{D}'_m$ is countable. As a result, $\mathfrak{mc}(\mathcal{D}) = \bigcup_{m\in\mathbf{N}} \mathcal{D}'_m$ is also countable.

Furthermore, we ndex $\mathfrak{mc}(\mathcal{D})$ by $\mathbf{N}$ as $\{g_j\}_{j\in\mathbf{N}}$ and observe that $ag_j \longmapsto (j, a)$ defines an injective mapping from $\mathcal{D}_{\mathbf{Q}} \doteq \{ag_j : j \in \mathbf{N}, a \in \mathbf{Q}\}$ to the countable set $\mathbf{N} \times \mathbf{Q}$, so $\mathcal{D}_{\mathbf{Q}}$ is countable. As a result, $\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D}) = \mathfrak{ac}(\mathcal{D}_{\mathbf{Q}})$ is also countable by our conclusion about additive closure[4]. □

Fact B.16. *Let E be a non-empty set, $d \in \mathbf{N}$ and $\mathcal{D} \subset \mathbf{R}^E$. Then, the following statements are true:*

(a) $\Pi^d(\mathcal{D})$ is a countable collection whenever $\mathcal{D}$ is. Moreover,

$$\begin{aligned} \Pi^d(\mathfrak{ac}(\mathcal{D})) &\subset \mathfrak{ac}\left(\Pi^d(\mathcal{D})\right), \\ \Pi^d(\mathfrak{mc}(\mathcal{D})) &= \mathfrak{mc}\left(\Pi^d(\mathcal{D})\right), \\ \Pi^d\left(\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D})\right) &\subset \mathfrak{ag}_{\mathbf{Q}}\left(\Pi^d(\mathcal{D})\right), \\ \Pi^d(\mathfrak{ag}(\mathcal{D})) &\subset \mathfrak{ag}\left(\Pi^d(\mathcal{D})\right). \end{aligned} \tag{B.1.15}$$

(b) If the members of $\mathcal{D}$ are bounded, then those of $\Pi^d(\mathcal{D})$ are also. Moreover,

$$\begin{aligned} \Pi^d(\mathfrak{cl}(\mathcal{D})) &\subset \mathfrak{cl}\left(\Pi^d(\mathcal{D})\right), \\ \Pi^d(\mathfrak{ca}(\mathcal{D})) &\subset \mathfrak{ca}\left(\Pi^d(\mathcal{D})\right). \end{aligned} \tag{B.1.16}$$

Proof. (a) If $\mathcal{D} = \{f_n\}_{n\in\mathbf{N}}$ is countable, we observe for each $k \in \{1, ..., d\}$ that $\prod_{i=1}^k f_{n_i} \circ \mathfrak{p}_i \mapsto (n_1, ..., n_k)$ defines an injective mapping from $\mathcal{D}_k \doteq \{\prod_{i=1}^k f_{n_i} \circ \mathfrak{p}_i : n_i \in \mathbf{N}\} \subset \mathbf{R}^{E^d}$ to the countable set $\mathbf{N}^k$ and so $\mathcal{D}_k$ is countable. As a result, $\Pi^d(\mathcal{D}) = \bigcup_{k=1}^d \mathcal{D}_k$ is also a countable set.

Letting $k \in \{1, ..., d\}$, $n_1, ..., n_k \in \mathbf{N}$ and $\{f_{i,j}\}_{1\le j\le n_i, 1\le i\le k} \subset \mathcal{D}$, we observe

[4]Recall that "$\mathfrak{ac}(\cdot)$" stands for additive closure.

that

$$\prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} f_{i,j}\right)\circ \mathfrak{p}_i = \sum_{j_1=1}^{n_1}\ldots\sum_{j_k=1}^{n_k}\left(\prod_{i=1}^{k} f_{i,j_i}\circ \mathfrak{p}_i\right) \in \mathfrak{ac}\left(\Pi^d(\mathcal{D})\right), \tag{B.1.17}$$

and

$$\prod_{i=1}^{k}\left(\prod_{j=1}^{n_i} f_{i,j}\right)\circ \mathfrak{p}_i = \prod_{1\le j\le n_i, 1\le i\le k} f_{i,j}\circ \mathfrak{p}_i \in \mathfrak{mc}\left(\Pi^d(\mathcal{D})\right). \tag{B.1.18}$$

Letting $N \in \mathbf{N}$, $k_1, ..., k_N \in \{1, ..., d\}$ and $\{f_{i,j}\}_{1\le i\le k_i, 1\le j\le N} \subset \mathcal{D}$, we observe that

$$\prod_{j=1}^{N}\left(\prod_{i=1}^{k_j} f_{i,j}\circ \mathfrak{p}_i\right) = \prod_{i=1}^{k^*}\left(\prod_{j\in J_i} f_{i,j}\right)\circ \mathfrak{p}_i \in \Pi^d\left(\mathfrak{mc}(\mathcal{D})\right), \tag{B.1.19}$$

where $k^* \circeq \max\{k_1, ..., k_N\}$ and $J_i \circeq \{j \in \{1, ..., n\} : k_j \ge i\}$ for each $1 \le i \le k^*$. Then, the first two lines of (B.1.15) follow by (B.1.17), (B.1.18) and (B.1.19).

Using the second line of (B.1.15), we have that

$$\begin{aligned}\Pi^d\left(\{af : f \in \mathfrak{mc}(\mathcal{D}), a \in \mathbf{Q}\}\right) &= \left\{af : f \in \Pi^d\left(\mathfrak{mc}(\mathcal{D})\right), a \in \mathbf{Q}\right\}\\ &= \left\{af : f \in \mathfrak{mc}\left(\Pi^d(\mathcal{D})\right), a \in \mathbf{Q}\right\}.\end{aligned} \tag{B.1.20}$$

Using (B.1.20) and the first two lines of (B.1.15), we have that

$$\begin{aligned}\Pi^d\left(\mathfrak{ag}_{\mathbf{Q}}(\mathcal{D})\right) &= \Pi^d\left[\mathfrak{ac}\left(\{af : f \in \mathfrak{mc}(\mathcal{D}), a \in \mathbf{Q}\}\right)\right]\\ &\subset \mathfrak{ac}\left[\Pi^d\left(\{af : f \in \mathfrak{mc}(\mathcal{D}), a \in \mathbf{Q}\}\right)\right]\\ &= \mathfrak{ac}\left(\left\{af : f \in \mathfrak{mc}\left(\Pi^d(\mathcal{D})\right), a \in \mathbf{Q}\right\}\right) = \mathfrak{ag}_{\mathbf{Q}}\left(\Pi^d(\mathcal{D})\right),\end{aligned} \tag{B.1.21}$$

which proves the third line of (B.1.15). The fourth line of (B.1.15) follows by a similar argument with $\mathbf{Q}$ replaced by $\mathbf{R}$.

(b) We fix $1 \le k \le d$. If all members of $\mathcal{D}$ are bounded, then

$$\left\|\prod_{i=1}^{k} f_i\circ \mathfrak{p}_i\right\|_\infty \le \prod_{i=1}^{k}\|f_i\circ \mathfrak{p}_i\|_\infty = \prod_{i=1}^{k}\|f_i\|_\infty < \infty \tag{B.1.22}$$

for all $f_1, ..., f_k \in \mathcal{D}$. Next, we suppose $\{f_1, ..., f_k\} \subset \mathfrak{cl}(\mathcal{D})$. By Fact A.9 (with $E = (\mathfrak{cl}(\mathcal{D}), \|\cdot\|_\infty)$ and $A = \mathcal{D}$), there exist $\{f_{i,n}\}_{1\leq i\leq k, n\in\mathbf{N}} \subset \mathcal{D}$ such that $f_{i,n} \overset{u}{\to} f$ as $n \uparrow \infty$ for all $1 \leq i \leq k$. We let $c \mathrel{\hat{=}} (\sup_{1\leq i\leq d} \|f_i\|_\infty)^{k-1}$ and find that

$$\lim_{n\to\infty} \left\| \prod_{i=1}^{k} f_i \circ \mathfrak{p}_i - \prod_{i=1}^{k} f_{i,n} \circ \mathfrak{p}_i \right\| \leq c \lim_{n\to\infty} \sum_{i=1}^{k} \|f_i - f_{i,n}\|_\infty = 0, \tag{B.1.23}$$

thus proving the first line of (B.1.16). The second line of (B.1.16) is immediate by the first line (with $\mathcal{D} = \mathfrak{ag}(\mathcal{D})$). □

Lemma B.17. *Let E be an open subspace of S and $f \in C(E; \mathbf{R}^k)$. If for any $\epsilon \in (0, \infty)$, there exists an $A_\epsilon \subset E$ such that $A_\epsilon \in \mathscr{C}(S)$ and $\|f|_{A_\epsilon}\|_\infty < \epsilon$, then $g \mathrel{\hat{=}} \mathfrak{var}(f; S, E, 0)$[5] is a continuous extension of f on S.*

Proof. We need only prove the case of $k = 1$ and the general result follows by Fact 2.4 (b). Let $\epsilon \in \mathbf{R}\backslash\{0\}$ and $A_\epsilon \subset E$ satisfy $A_\epsilon \in \mathscr{C}(S)$ and $\|f|_{A_\epsilon}\|_\infty < |\epsilon|$. From the facts

$$g^{-1}\left[(-\infty, \epsilon)\right] \backslash A_\epsilon = \begin{cases} S\backslash A_\epsilon \in \mathscr{O}(S), & \text{if } \epsilon > 0, \\ \varnothing, & \text{if } \epsilon < 0 \end{cases} \tag{B.1.24}$$

and $E \in \mathscr{O}(S)$ it follows that

$$g^{-1}\left[(-\infty, \epsilon)\right] = \begin{cases} f^{-1}\left[(-\infty, \epsilon)\right] \cup (S\backslash A_\epsilon) \in \mathscr{O}(S), & \text{if } \epsilon > 0, \\ f^{-1}\left[(-\infty, \epsilon)\right] \in \mathscr{O}(E) \subset \mathscr{O}(S), & \text{if } \epsilon < 0, \end{cases} \tag{B.1.25}$$

thus proving the continuity of g. □

Fact B.18. *Let E be a non-empty set and $\{f_n\}_{n\in\mathbf{N}_0} \subset \mathbf{R}^E$. Then, the following statements are true:*

(a) If $|f_n - f_0| < \epsilon$, then $\left|f_n^+ - f_0^+\right| < 3\epsilon$.

(b) If $f_n \overset{u}{\to} f_0$ as $n \uparrow \infty$, then $f_n^+ \overset{u}{\to} f_0^+$ as $n \uparrow \infty$.

[5] "$\mathfrak{var}(\cdot)$" was defined in Notation 4.1.

Proof. (a) For each $x \in E$, from the fact

$$f_n^+(x) \begin{cases} = f_n(x), & \text{if } f_0(x) \geq \epsilon, \\ \in \left[0, \frac{3}{2}\epsilon\right), & \text{if } f_0(x) < \epsilon \end{cases} \tag{B.1.26}$$

it follows that

$$\left|f_n^+(x) - f_0^+(x)\right| \begin{cases} = |f_n(x) - f_0(x)| < 2^{-n}, & \text{if } f(x) \geq \epsilon, \\ \leq |f_n^+(x)| + |f^+(x)| < \frac{5}{2}\epsilon, & \text{if } f(x) < \epsilon. \end{cases} \tag{B.1.27}$$

(b) follows immediately by (a). □

Fact B.19. *Let $\{A_n\}_{n\in\mathbf{N}}$ be nested*[6] *non-empty subsets of E and $\mathcal{D}_n \subset \mathbf{R}^E$ separate points on A_n for each $n \in \mathbf{N}$. Then, $\bigcup_{n\in\mathbf{N}} \mathcal{D}_n$ separates points on $\bigcup_{n\in\mathbf{N}} A_n$.*

Proof. For any distinct $x_1, x_2 \in \bigcup_{n\in\mathbf{N}} A_n$, there exist $n_1, n_2, N \in \mathbf{N}$ such that $x_i \in A_{n_i} \subset A_N$ for each $i = 1, 2$. Then, $\bigotimes \mathcal{D}(x_1) \neq \bigotimes \mathcal{D}(x_2)$ since $\bigcup_{n\in\mathbf{N}} \mathcal{D}_n$ contains $\mathcal{D}_N$ and separates points on A_N. □

Fact B.20. *Let A be a non-empty subset of E and $\mathcal{D} \subset \mathbf{R}^E$ separate points on $A \subset E$. Then, $\varpi(\bigotimes \mathcal{D})$ and $\varpi(\mathcal{D})$ are both injective restricted to $A^{\mathbf{R}^+}$.*

Proof. The injectiveness of $\varpi(\bigotimes \mathcal{D})$ on $A^{\mathbf{R}^+}$ is immediate by Fact B.10 (a) (with $E = A$, $\mathbf{I} = \mathbf{R}^+$, $S = \mathbf{R}^{\mathcal{D}}$ and $f = \bigotimes \mathcal{D}|_A$). Furthermore, we note that $\varpi(\mathcal{D})(x) = \varpi(\mathcal{D})(y)$ in $(A^{\mathbf{R}^+})^{\mathcal{D}}$ implies $\bigotimes \mathcal{D}[x(t)] = \bigotimes \mathcal{D}[y(t)]$ for all $t \in \mathbf{R}^+$. This indicates $x(t) = y(t)$ for all $t \in \mathbf{R}^+$, i.e. $x = y$. □

Fact B.21. *Let E be a topological space. Then, the following statements are true:*

(a) $\mu_1 = \mu_2$ in $\mathcal{M}^+(E)$ if and only if $\mu_1/\mu_1(E) = \mu_2/\mu_2(E)$ in $\mathcal{P}(E)$ and $\mu_1(E) = \mu_2(E)$.

(b) (2.3.4) holds if and only if $\lim_{n\to\infty} \mu_n(E) = \mu(E)$ and

$$\frac{\mu_n}{\mu_n(E)} \Longrightarrow \frac{\mu}{\mu(E)} \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(E). \tag{B.1.28}$$

[6] We explained the meaning of "nested" in Fact 3.36.

Proof. (a) Note that $\mu_1(A) = \mu(E)[\mu/\mu(E)](A)$ for all $A \in \mathscr{B}(E)$ and $\mu \in \mathcal{M}^+(E)$.

(b - Necessity) $1 \in C_b(E;\mathbf{R})$ implies

$$\lim_{n\to\infty} \mu_n(E) = \lim_{n\to\infty} \int_E 1\mu_n(dx) = \int_E 1\mu(dx) = \mu(E). \tag{B.1.29}$$

Then,

$$\lim_{n\to\infty} f^*\left(\frac{\mu_n}{\mu_n(E)}\right) = \lim_{n\to\infty} \frac{f^*(\mu_n)}{\mu_n(E)} = \frac{f^*(\mu)}{\mu(E)} = f^*\left(\frac{\mu}{\mu(E)}\right) \tag{B.1.30}$$

follows immediately for all $f \in C_b(E;\mathbf{R})$[7].

(b- Sufficiency) We observe for each $f \in C_b(E;\mathbf{R})$ that

$$\lim_{n\to\infty} f^*(\mu_n) = \lim_{n\to\infty} \mu_n(E) f^*\left(\frac{\mu_n}{\mu_n(E)}\right) = \mu(E) f^*\left(\frac{\mu}{\mu(E)}\right) = f^*(\mu). \tag{B.1.31}$$

□

Fact B.22. *Let E be a topological space and $1 \in \mathcal{D} \subset M_b(E;\mathbf{R})$. Then, the following statements are true:*

(a) $\mathcal{D}$ is separating on E if and only if $\mathcal{D}^*$ separates points on $\mathcal{P}(E)$.

(b) $\mathcal{D}$ is convergence determining on E if and only if $\mathcal{D}^*$ determines point convergence on $\mathcal{P}(E)$.

Proof. This result is immediate by Fact B.21. □

Fact B.23. *Let E be a topological space. Then, $\mathcal{P}(E) \in \mathscr{C}[\mathcal{M}^+(E)]$.*

Proof. Let μ be a limit point[8] of $\mathcal{P}(E)$ in $\mathcal{M}^+(E)$. For any $p \in \mathbf{N}$, there exists a $\mu_p \in \mathcal{P}(E)$ such that $|\mu(E) - \mu_p(E)| = |\mu(E) - 1| < 2^{-p}$. Hence, $\mu(E) = 1$ as $p \uparrow 0$. □

Fact B.24. *If $x_n \to x$ as $n \uparrow \infty$ in topological space E, then $\delta_{x_n} \Rightarrow \delta_x$ as $n \uparrow \infty$ in $\mathcal{P}(E)$.*

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[7] The notation "f^*" was specified in §2.3.

[8] $\mathcal{M}^+(E)$ as aforementioned is not necessarily first-countable. So, μ being a limit point of $\mathcal{P}(E)$ does not necessarily imply a subsequence of $\mathcal{P}(E)$ converging weakly to μ.

Proof. This fact follows by the integral convergence test

$$\lim_{n\to\infty} f^*(\delta_{x_n}) = \lim_{n\to\infty} f(x_n) = f(x) = f^*(\delta_x),\ \forall f \in C_b(E;\mathbf{R}). \tag{B.1.32}$$

□

The generalized Portmanteau's Theorem helps to establish the Continuous Mapping Theorem on fairly general topological spaces.

Theorem B.25 (**Continuous Mapping Theorem**). *Let E and S be topological spaces. Then, the following statements are true:*

(a) If $f \in C(E;S)$, then (2.3.4) implies

$$\mu_n \circ f^{-1} \Longrightarrow \mu \circ f^{-1} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(S). \tag{B.1.33}$$

(b) If E is a Tychonoff space and the set of discontinuity points of $f \in M(E;S)$ has zero measure under μ, then (2.3.4) implies (B.1.33).

Proof. (a) follows by the fact that

$$\begin{aligned} \lim_{n\to\infty} g^*\left(\mu_n \circ f^{-1}\right) &= \lim_{n\to\infty} (g\circ f)^*(\mu_n) \\ &= (g\circ f)^*(\mu) = g^*\left(\mu\circ f^{-1}\right),\ \forall g \in C_b(S;\mathbf{R}). \end{aligned} \tag{B.1.34}$$

(b) Let $O \in \mathscr{O}(S)$ and $A \subset E$ be the set of discontinuity points of f. If $x \in f^{-1}(O)\backslash A$, then there exists an $O_x \in \mathscr{O}(E)$ such that $x \in O_x \subset f^{-1}(O)$, i.e. x is an interior point of $f^{-1}(O)$. So, $f^{-1}(O)\backslash A \in \mathscr{O}(E)$. Then, it follows by the Tychonoff property of E and Theorem 2.17 (a, c) that

$$\begin{aligned} \mu \circ f^{-1}(O) &= \mu\left(f^{-1}(O)\backslash A\right) \\ &\le \liminf_{n\to\infty} \mu_n\left(f^{-1}(O\backslash A)\right) \le \liminf_{n\to\infty} \mu_n \circ f^{-1}(O). \end{aligned} \tag{B.1.35}$$

Now, (b) follows by (B.1.35) and Theorem 2.17 (a, c). □

Fact B.26. *Let E be a topological space, $(E,\mathscr{U})$ be a topological coarsening of E and $S \doteq \mathcal{M}^+(E,\mathscr{U})$. Then, the following statements are true:*

(a) $(\mathcal{M}^+(E), \mathscr{O}_S[\mathcal{M}^+(E)])$ *and* $(\mathcal{P}(E), \mathscr{O}_S[\mathcal{P}(E)])$ *are topological coarsenings of* $\mathcal{M}^+(E)$ *and* $\mathcal{P}(E)$ *respectively.*

(b) If $\mu_n \Rightarrow \mu$ *as* $n \uparrow \infty$ *in* $\mathcal{M}^+(E)$*, then* $\mu_n \Rightarrow \mu$ *as* $n \uparrow \infty$ *in* S*.*

Proof. (a) $\mathscr{U} \subset \mathscr{O}(E)$ implies $\mathscr{B}(E, \mathscr{U}) \subset \mathscr{B}(E)$, so every $\mu \in \mathcal{M}^+(E)$ is naturally a member of S. $\mathscr{U} \subset \mathscr{O}(E)$ implies $C_b(E, \mathscr{U}; \mathbf{R}) \subset C_b(E; \mathbf{R})$. Then, (a) follows by the fact that

$$\begin{aligned} \mathscr{O}\left[\mathcal{M}^+(E)\right] &= \mathscr{O}_{C_b(E;\mathbf{R})^*}\left[\mathcal{M}^+(E)\right] \\ &\supset \mathscr{O}_{C_b(E,\mathscr{U};\mathbf{R})^*}\left[\mathcal{M}^+(E)\right] = \mathscr{O}_S\left[\mathcal{M}^+(E)\right]. \end{aligned} \tag{B.1.36}$$

(b) is immediate by (a). □

Fact B.27. *Let* E *be a topological space,* $\mu_n \Rightarrow \mu_0$ *as* $n \uparrow \infty$ *in* $\mathcal{M}^+(E)$ *and* $(E, \mathscr{U}_n, \nu_n)$ *be the completion of* $(E, \mathscr{B}(E), \mu_n)$ *for each* $n \in \mathbf{N}_0$*. Then,* $\nu_n \Rightarrow \nu_0$ *as* $n \uparrow \infty$ *in* $\mathcal{M}^+(E)$*.*

Proof. $\{\nu_n\}_{n\in\mathbf{N}_0} \subset \mathcal{M}^+(E)$ since $\mathscr{U}_n \supset \mathscr{B}(E)$ for all $n \in \mathbf{N}_0$. Then, the result follows by the apparent fact that $f^*(\nu_n) = f^*(\mu_n)$ for all $n \in \mathbf{N}_0$ and $f \in C_b(E; \mathbf{R})$. □

Fact B.28. *Let* E *be a topological space and* μ *be the unique weak limit point of* $\{\mu_n\}_{n\in\mathbf{N}}$ *in* $\mathcal{M}^+(E)$*. If* $\{\mu\}_{n\in\mathbf{N}}$ *is relatively compact, then (2.3.5) holds.*

Proof. The fact follows by Fact B.6 (with $(E, x_n, x) = (\mathcal{M}^+(E), \mu_n, \mu)$). □

Fact B.29. *Let* E *be a topological space and* $\mathscr{U}$ *be a* σ*-algebra on* E*. If* $\Gamma \subset \mathfrak{M}^+(E, \mathscr{U})$ *is sequentially tight in* $A \subset E$*, then there exists a* $\Gamma_0 \in \mathscr{P}_0(\Gamma)$ *such that* A *is a common support of all members of* $\Gamma \backslash \Gamma_0$*.*

Proof. Suppose none of $\{\mu_n\}_{n\in\mathbf{N}} \subset \Gamma$ is supported on A. The sequential tightness of Γ implies a subsequence $\{\mu_{n_k}\}_{k\in\mathbf{N}}$ being tight in A. In other words, $\{\mu_{n_k}\}_{k\in\mathbf{N}}$ are all supported on some $B \in \mathscr{K}_\sigma(E)$ with $B \subset A$. Contradiction! □

Fact B.30. *Let* $\mathscr{U}$ *and* $\mathscr{A}$ *be* σ*-algebras on topological spaces* E *and* S *respectively. If* $\Gamma \subset \mathfrak{M}^+(E, \mathscr{U})$ *is tight in* $A \subset E$[9]*, and if* $f \in M(E, \mathscr{U}; S, \mathscr{A})$

[9]Please be reminded that we generalized the definition of tightness and **m**-tightness to possibly non-Borel measures in Definition 2.18.

satisfies $f(K) \in \mathscr{K}(S) \cap \mathscr{A}$ *for all* $K \in \mathscr{K}(E) \cap \mathscr{U}$*, then* $\{\mu \circ f^{-1}\}_{\mu \in \Gamma}$ *is tight (resp.* $\mathbf{m}$*-tight) in* $f(A)$*. This implication is also true if tightness,* $\mathscr{K}(S)$ *and* $\mathscr{K}(S)$ *are replaced by* $\mathbf{m}$*-tightness,* $\mathscr{K}^{\mathbf{m}}(E)$ *and* $\mathscr{K}^{\mathbf{m}}(S)$*, respectively.*

Proof. It suffices to note that $f(K) \subset f(A)$ and $K \subset f^{-1}(f(K))$. $\square$

Lemma B.31. *Let* $(E, \mathscr{U})$ *be a measurable space,* $S_0 \subset S \subset E$*,* $y_0 \in S$*,* X *be a mapping from* $(\Omega, \mathscr{F}, \mathbb{P})$ *to* E *and* $Y \circeq \mathfrak{var}(X; \Omega, X^{-1}(S), y_0)$*. Then, the following statements are true:*

(a) If $\mathbb{P}(X = Z) = 1$ *for some* $Z \in M(\Omega, \mathscr{F}; S, \mathscr{U}|_S)$*, then* $X \in M(\Omega, \mathscr{F}; E, \mathscr{U})$*.*

(b) If $X \in M(\Omega, \mathscr{F}; E, \mathscr{U})$ *satisfies* $\mathbb{P}(X \in S_0) = 1$*, then* $\mathbb{P}(X = Y \in S) = 1$*.*

(c) If, in addition to the condition of (b), $(S, \mathscr{U}')$ *is a measurable space satisfying* $\mathscr{U}'|_{S_0} = \mathscr{U}|_{S_0}$*, then* $Y \in M(\Omega, \mathscr{F}; S, \mathscr{U}')$*.*

Proof. (a) Let $\Omega_0 \circeq \{\omega \in \Omega : X(\omega) = Z(\omega)\}$. It follows by $\mathbb{P}(X = Z) = 1$ and the completeness of $(\Omega, \mathscr{F}, \mathbb{P})$[10] that $\Omega_0 \in \mathscr{F}$ and $X^{-1}(A) \backslash \Omega_0 \in \mathscr{N}(\mathbb{P}) \subset \mathscr{F}$ for all $A \in \mathscr{U}$. Hence, we have that

$$X^{-1}(A) = \left[Z^{-1}(A \cap S) \cap \Omega_0\right] \cup \left(X^{-1}(A) \backslash \Omega_0\right) \in \mathscr{F}, \ \forall A \in \mathscr{U}. \tag{B.1.37}$$

(b) We find $\mathbb{P}(X = Y \in S) \geq \mathbb{P}(X \in S_0) = 1$ by Fact B.3 (a) (with $(S, \mathscr{A}) = (\Omega, \mathscr{F})$ and $A = S$) and the completeness of $(\Omega, \mathscr{F}, \mathbb{P})$.

(c) We fix $A \in \mathscr{U}'$ and find $A \cap S_0 \in \mathscr{U}|_{S_0}$ by $\mathscr{U}'|_{S_0} = \mathscr{U}|_{S_0}$. It follows by $\mathbb{P}(X \in S_0) = 1$ and the completeness of $(\Omega, \mathscr{F}, \mathbb{P})$ that $X^{-1}(S_0) \in \mathscr{F}$ and $Y^{-1}(A \cap S) \backslash X^{-1}(S_0) \in \mathscr{N}(\mathbb{P}) \subset \mathscr{F}$. Hence, we have that

$$Y^{-1}(A \cap S) = \left[X^{-1}(A) \cap X^{-1}(S_0)\right] \cup \left[Y^{-1}(A \cap S) \backslash X^{-1}(S_0)\right] \in \mathscr{F}. \tag{B.1.38}$$

$\square$

Fact B.32. *Let* E *and* S *be topological spaces,* $\{X^i\}_{i \in \mathbf{I}}$ *and* X *be* E*-valued processes defined on stochastic basis* $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t \geq 0}, \mathbb{P})$ *and* $f \in M(E; S)$*. Then, the following statements are true:*

[10] Completeness of measure space was specified in 2.1.2.

(a) If X is a general, $\mathscr{G}_t$-adapted, measurable or $\mathscr{G}_t$-progressive process, then $\varpi(f) \circ X$ is an S-valued process with the corresponding measurability.

(b) If $\{X^i\}_{i\in\mathbf{I}}$ are general, $\mathscr{G}_t$-adapted, measurable or $\mathscr{G}_t$-progressive processes, then $\{\bigotimes_{i\in\mathbf{I}} X_t^i\}_{t\geq 0}$ is an $E^{\mathbf{I}}$-valued process with the corresponding measurability.

Proof. This result follows straightforwardly by Fact B.10 (b), Fact 2.3 (b), Fact 2.24 (b) and the definitions of measurable processes, $\mathscr{G}_t$-progressive processes and product topology. □

Proposition B.33. *Let E be a topological space and X and Y be E-valued processes defined on stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t\geq 0}, \mathbb{P})$[11]. Then, the following statements are true:*

(a) If X has càdlàg paths (resp. is a càdlàg process), then it is (resp. is indistinguishable from) an E-valued progressive process.

(b) If X is progressive and $\mathscr{G}_t$-adapted, then it is $\mathscr{G}_t$-progressive.

(c) If X is $\mathscr{G}_t$-progressive, then it is $\mathscr{G}_t$-adapted and measurable.

(d) If X is measurable, then $X(\omega) \in M(\mathbf{R}^+; E)$ for all $\omega \in \Omega$.

(e) If X and Y are modifications of each other, then $\mathscr{F}^X = \mathscr{F}^Y$[12].

(f) If X and Y are indistinguishable, then they are modifications of each other. If, in addition, X is a measurable, $\mathscr{G}_t$-progressive, progressive or càdlàg process, then Y is also.

(g) If $\inf_{t\in\mathbf{T}} \mathbb{P}(X_t = Y_t) = 1$ for some dense $\mathbf{T} \subset \mathbf{R}^+$, and if X and Y are càdlàg, then X and Y are indistinguishable.

(h) If X is càdlàg, then it is indistinguishable from any of its càdlàg modifications and such modification is at most unique up to indistinguishability.

[11] The notion of stochastic basis was specified in §2.5.
[12] The filtrations $\mathscr{F}^X$ and $\mathscr{F}^Y$ were defined in §2.5.

Proof. The well-known facts above are treated in standard texts like [Ethier and Kurtz, 1986, Chapter 2], [Protter, 1990, Chapter 1] and Nikeghbali [2006] for E being a Euclidean or metric space. An inspection into their proofs shows that there is no problem to make E a general topological space. □

Fact B.34. *Let E and S be topological spaces, $\{X^i\}_{i\in\mathbf{I}}$ and X be E-valued càdlàg processes defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and $f \in C(E; S)$. Then, the following statements are true:*

(a) $\varpi(f) \circ X$ is an S-valued càdlàg process.

(b) If $\mathbf{I}$ is countable, then $\{\bigotimes_{i\in\mathbf{I}} X_t^i\}_{t\geq 0}$ is an $E^{\mathbf{I}}$-valued càdlàg process.

(c) If S is a topological coarsening of E, then X is an S-valued càdlàg process.

Proof. $Y \doteq \varpi(f) \circ X$ and $Z \doteq \{\bigotimes_{i\in\mathbf{I}} X_t^i\}_{t\geq 0}$ are processes by Fact B.32. We note by Fact B.14 (b, c) that

$$\{\omega \in \Omega : Y(\omega) \text{ is càdlàg}\} \supset \{\omega \in \Omega : X(\omega) \text{ is càdlàg}\} \tag{B.1.39}$$

and

$$\{\omega \in \Omega : Z(\omega) \text{ is càdlàg}\} \supset \bigcap_{i\in\mathbf{I}} \left\{\omega \in \Omega : X^i(\omega) \text{ is càdlàg}\right\}. \tag{B.1.40}$$

Now, (a) follows by (B.1.39). (b) follows by (B.1.40) and the countability of $\mathbf{I}$. Moreover, (c) is immediate by (a) (with f being the identity mapping on E). □

Fact B.35. *Let E be a topological space, $\mathbf{T} \subset \mathbf{R}^+$ and $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be E-valued processes. Then, the following statements are true:*

(a) If $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{D})$-FDC, then $\{\mathbb{E}^n[f \circ X^n_{\mathbf{T}_0}]\}_{n\in\mathbf{N}}$ is a convergent sequence in $\mathbf{R}$ for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(b) If $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{D})$-AS, then

$$\lim_{n\to\infty} \mathbb{E}^n \left[f \circ X^n_{\mathbf{T}_0} - f \circ X^n_{\mathbf{T}_0+c}\right] = 0 \tag{B.1.41}$$

for all $c \in (0, \infty)$, $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})]$ *and* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

Proof. This result is immediate by the Bolzano-Weierstrass Theorem and Fact B.6 (with $E = \mathbf{R}$ and $x_n = \mathbb{E}^n[f \circ X^n_{\mathbf{T}_0}]$ or $\mathbb{E}^n[f \circ X^n_{\mathbf{T}_0} - f \circ X^n_{\mathbf{T}_0+c}]$). □

Fact B.36. *Let E be a topological space, $\mathbf{T} \subset \mathbf{R}^+$ and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes. If for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, there exists some $\mathbf{I}_{\mathbf{T}_0} \in \mathscr{P}_0(\mathbf{I})$ such that $\{\mu_{\mathbf{T}_0,i} = \mathfrak{be}(\mathbb{P} \circ (X^i_{\mathbf{T}_0})^{-1})\}_{i\in\mathbf{I}\setminus\mathbf{I}_{\mathbf{T}_0}}$ has at most one weak limit point, then $\mathfrak{flp}_{\mathbf{T}}(\{X^i\}_{i\in\mathbf{I}})$[13] is at most a singleton.*

Proof. Without loss of generality, we suppose $(\Omega, \mathscr{F}, \mathbb{P}; Y^j) \in \mathfrak{flp}_{\mathbf{T}}(\{X^i\}_{i\in\mathbf{I}})$ for each $j = 1, 2$. Fixing $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, there exist $\{\nu_j \in \mathfrak{be}(\mathbb{P} \circ (Y^j_{\mathbf{T}_0})^{-1})\}_{j=1,2}$ such that ν_1 and ν_2 are both weak limit points of $\{\mu_{\mathbf{T}_0,i}\}_{i\in\mathbf{I}\setminus\mathbf{I}_{\mathbf{T}_0}}$. So, we must have $\nu_1 = \nu_2$ and hence $\mathbb{P} \circ (Y^1_{\mathbf{T}_0})^{-1} = \mathbb{P} \circ (Y^2_{\mathbf{T}_0})^{-1}$. □

Fact B.37. *Let E be a topological space, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process and $\mathbf{T} \subset \mathbf{R}^+$. If there exists an $\mathbf{R}^{\mathcal{D}}$-valued càdlàg process $(\Omega, \mathscr{F}, \mathbb{P}; \zeta)$ such that*

$$\inf_{t\in\mathbf{T}} \mathbb{P}\left(\bigotimes \mathcal{D} \circ X_t = \zeta_t\right) = 1, \tag{B.1.42}$$

then X is $(\mathbf{T}, \mathcal{D})$-càdlàg. The converse is true when $\mathcal{D}$ is a countable collection.

Proof. $\{\zeta^f \doteq \varpi(\mathfrak{p}_f) \circ \zeta\}_{f\in\mathcal{D}}$[14] are $\mathbf{R}$-valued càdlàg processes satisfying

$$\begin{aligned} &\inf_{t\in\mathbf{T}} \mathbb{P}\left(\mathfrak{p}_f \circ \zeta_t = \zeta^f_t = f \circ X_t = \mathfrak{p}_f \circ \bigotimes \mathcal{D} \circ X_t, \forall f \in \mathcal{D}\right) \\ &= \inf_{t\in\mathbf{T}} \mathbb{P}\left(\zeta_t = \bigotimes \mathcal{D} \circ X_t\right) = 1 \end{aligned} \tag{B.1.43}$$

by Fact 2.4 (a) and Fact B.34 (a) (with $E = \mathbf{R}^{\mathcal{D}}$ and $f = \mathfrak{p}_f$).

Conversely, we suppose $\mathcal{D}$ is countable and $\mathbf{R}$-valued càdlàg processes $\{\zeta^f\}_{f\in\mathcal{D}}$ satisfy (6.3.1). Letting $\zeta_t \doteq \bigotimes_{f\in\mathcal{D}} \zeta^f_t$ for each $t \in \mathbf{R}^+$, we find that $\{\zeta_t\}_{t\geq 0}$ is an $\mathbf{R}^\infty$-valued càdlàg process satisfying (B.1.43) by the countability of $\mathcal{D}$, Fact B.34 (b) (with $\mathbf{I} = \mathcal{D}$, $i = f$ and $X^i = \zeta^f$) and (6.3.1). □

Fact B.38. *Let E be a topological space, $A \subset E$ and $\mathbf{T} \subset \mathbf{R}^+$ be countable. If E-valued processes $\{X^i_t\}_{i\in\mathbf{I}}$ is sequentially tight in A for all $t \in \mathbf{T}$ (resp. satisfies $\mathbf{T}$-PSMTC in A[15]), then there exist $\{i_n\}_{n\in\mathbf{I}} \subset \mathbf{I}$ such that $\{X^{i_n}_t\}_{n\in\mathbf{N}}$*

[13] The notation "$\mathfrak{flp}_{\mathbf{T}}(\{X^n\}_{n\in\mathbf{N}})$" was introduced in §6.2 and stands for the family of all equivalence classes of finite-dimensional limit points of $\{X^n\}_{n\in\mathbf{N}}$ along $\mathbf{T}$.

[14] Recall that $\mathfrak{p}^f$ denotes the projection on $\mathbf{R}^{\mathcal{D}}$ for $f \in \mathcal{D}$.

[15] The notions of $\mathbf{T}$-PMTC and $\mathbf{T}$-PSMTC were introduced in Definition 6.36.

is tight in A for all $t \in \mathbf{T}$ (resp. satisfies $\mathbf{T}$-PMTC in A).

Proof. This result follows immediately by a triangular array argument. ☐

Lemma B.39. *Let E be a topological space, $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued measurable processes, $T_k \uparrow \infty$, $\{A_p\}_{p\in\mathbf{N}} \subset \mathscr{B}(E)$ and $(\widetilde{\Omega}^i, \widetilde{\mathscr{F}}^i, \mathbb{P}^{i,T_k}; X^{i,T_k}) = \mathfrak{rap}_{T_k}(X^i)$[16] for each $i \in \mathbf{I}$ and $k \in \mathbf{N}$. If*

$$\inf_{i\in\mathbf{I},k\in\mathbf{N}} \mathbb{P}^{i,T_k}\left(X_0^{i,T_k} \in A_p\right) \geq 1 - 2^{-p},\ \forall p \in \mathbf{N}, \tag{B.1.44}$$

then for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ with $d \mathrel{\hat=} \aleph(\mathbf{T}_0)$, there exist $\{N_{\mathbf{T}_0,p}\}_{p\in\mathbf{N}} \subset \mathbf{N}$ such that

$$\inf_{i\in\mathbf{I},k>N_{\mathbf{T}_0,p}} \mathbb{P}^{i,T_k}\left(X_{\mathbf{T}_0}^{i,T_k} \in A_p^d\right) \geq 1 - (d+1)2^{-p},\ \forall p \in \mathbf{N}. \tag{B.1.45}$$

Proof. Let $\mathbf{T}_0 = \{t_1, ..., t_d)$, $t \mathrel{\hat=} \sum_{i=1}^d t_i$, $N_{\mathbf{T}_0,0} \mathrel{\hat=} 0$ and $T_{n_{\mathbf{T}_0,0}} \mathrel{\hat=} 0$. Define $\{N_{\mathbf{T}_0,p}\}_{p\in\mathbf{N}}$ inductively by

$$N_{\mathbf{T}_0,p} \mathrel{\hat=} \min\left\{k \in \mathbf{N} : T_k > (2^{p+1}td) \vee T_{N_{\mathbf{T}_0,p-1}}\right\},\ \forall p \in \mathbf{N}. \tag{B.1.46}$$

For each $p \in \mathbf{N}$, it follows by (B.1.44) and (B.1.46) that

$$\begin{aligned}
&\inf_{i\in\mathbf{I},k>N_{\mathbf{T}_0,p}} \mathbb{P}^{i,T_k}\left(X_{\mathbf{T}_0}^{i,T_k} \in A_p^d\right)\\
&\geq 1 - \sup_{i\in\mathbf{I},k>N_{\mathbf{T}_0,p}} \sum_{i=1}^d \mathbb{P}^{i,T_k}\left(X_{t_i}^{i,T_k} \notin A_p\right)\\
&\geq 1 - d \sup_{i\in\mathbf{I},k>N_{\mathbf{T}_0,p}} \mathbb{P}^{i,T_k}\left(X_0^{i,T_k} \notin A_p\right)\\
&\quad - \sup_{i\in\mathbf{I},k>N_{\mathbf{T}_0,p}} \frac{d}{T_k}\int_{[0,t]\cup[T_k,T_k+t]} \mathbb{P}^i\left(X_\tau^i \notin A_p\right)d\tau\\
&\geq 1 - d2^{-p} - \frac{2td}{2^{p+1}td} = 1-(d+1)2^{-p}.
\end{aligned} \tag{B.1.47}$$

☐

Lemma B.40. *Let E be a topological space, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued càdlàg process, $\epsilon, \delta, T, c \in (0,\infty)$ and $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{P}^T; X^T) = \mathfrak{rap}_T(X)$. Then, the*

[16]Randomly advanced process and related notations were introduced in §7.3.

following statements are true:

(a) $\xi^\tau \doteq \{X_{\tau+t}\}_{t\geq 0}$ *well defines an* E*-valued càdlàg process for all* $\tau \in \mathbf{R}^+$.

(b) X^T *is an* E*-valued càdlàg process.*

(c) If $(E, \mathfrak{r})$ *is a separable metric space, then*

$$\frac{1}{T}\int_0^T \mathbb{P}\left(w'_{\mathfrak{r},\delta,c} \circ \xi^\tau \geq \epsilon\right) d\tau = \mathbb{P}^T\left(w'_{\mathfrak{r},\delta,c} \circ X^T \geq \epsilon\right). \tag{B.1.48}$$

Proof. $\{\xi^\tau\}_{\tau\in\mathbf{R}^+}$ are E-valued processes by Fact 2.24 (b). Letting $\Omega_0 \doteq \{\omega \in \Omega : X(\omega)$ is càdlàg$\}$, we find that

$$\{\omega \in \Omega : \xi^\tau(\omega) \text{ is càdlàg}\} \supset \Omega_0,\ \forall \tau \in \mathbf{R}^+ \tag{B.1.49}$$

and

$$\left\{(\tau,\omega) \in \widetilde{\Omega} : X^T(\tau,\omega) \text{ is càdlàg}\right\} \supset \mathbf{R}^+ \times \Omega_0. \tag{B.1.50}$$

Then, (a, b) follows by (B.1.49), (B.1.50) and the fact $\mathbb{P}^T(\mathbf{R}^+ \times \Omega_0) = \mathbb{P}(\Omega_0) = 1$.

(c) It follows by (a), (b) and Lemma A.80 (b) (with $X = \xi^\tau$ or X^T) that $w'_{\mathfrak{r},\delta,c} \circ \xi^\tau \in M(\Omega, \mathscr{F}; \mathbf{R})$ for all $\tau \in \mathbf{R}^+$ and $w'_{\mathfrak{r},\delta,c} \circ X^T \in M(\widetilde{\Omega}, \widetilde{\mathscr{F}}; \mathbf{R})$. Hence, both sides of (B.1.48) are all well-defined and (B.1.48) is true since $\xi^\tau(\omega) = X^T(\tau,\omega)$ for all $(\tau,\omega) \in \widetilde{\Omega}$. □

Fact B.41. *Let* $(E, \mathfrak{r})$ *be a metric space,* $(\Omega, \mathscr{F}, \mathbb{P}; X)$ *be an* E*-valued càdlàg process,* $T_k \uparrow \infty$ *and* $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{P}^{T_k}; X^{T_k}) = \mathfrak{rap}_{T_k}(X)$ *for each* $k \in \mathbf{N}$. *If for any* $\epsilon, T \in (0,\infty)$, *there exists a* $\delta_{\epsilon,T} \in (0,\infty)$ *satisfying*

$$\sup_{k\in\mathbf{N}} \frac{1}{T_k}\int_0^{T_k} \mathbb{P}\left(w'_{\mathfrak{r},\delta_{\epsilon,T},T} \circ \xi^\tau \geq \epsilon\right) d\tau \leq \epsilon \tag{B.1.51}$$

with $\xi^\tau \doteq \{X_{\tau+t}\}_{t\geq 0}$, *then* $\{X^{T_k}\}_{k\in\mathbf{N}}$ *is a family of* E*-valued càdlàg processes satisfying* $\mathfrak{r}$*-MCC.*

Proof. This fact follows by Lemma B.40 (c) (with $T = T_k$ and $c = T$). □

Lemma B.42. *Let E be a topological space, $A \subset E$, $\mathcal{D} \subset M_b(E;\mathbf{R})$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process, $T_k \uparrow \infty$ and $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{P}^{T_k}; X^{T_k}) = \mathfrak{rap}_{T_k}(X)$ for each $k \in \mathbf{N}$. Then, the following statements are true:*

(a) X satisfies T_k-LMTC in A[17] *if and only if $\{X_0^{T_k}\}_{n\in\mathbf{N}}$ is $\mathbf{m}$-tight in A.*

(b) If $\{X_0^{T_k}\}_{n\in\mathbf{N}}$ is $\mathbf{m}$-tight in A, then $\{X^{T_k}\}_{k\in\mathbf{N}}$ satisfies $\mathbf{R}^+$-PSMTC in A.

(c) $\{X^{T_k}\}_{k\in\mathbf{N}}$ is $(\mathbf{R}^+, M_b(E;\mathbf{R}))$-AS.

(d) If $\{X^{T_k}\}_{k\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{D})$-FDC, then it is $(\mathbf{T}+c, \mathcal{D})$-FDC for all $c \in (0,\infty)$.

Proof. (a) is automatic by the definition of $\{X^{T_k}\}_{k\in\mathbf{N}}$.

(b) follows by (a) and Lemma B.39 (with $\{X^i\}_{i\in\mathbf{I}} = \{X\}$).

(c) and (d) follow immediately by the fact that

$$\begin{aligned} &\lim_{k\to\infty} \left| \mathbb{E}^{T_k}\left[f \circ X_{\mathbf{T}_0}^{T_k} - f \circ X_{\mathbf{T}_0+c}^{T_k} \right] \right| \\ &\leq \lim_{k\to\infty} \frac{1}{T_k} \int_0^c \mathbb{E}\left[|f \circ X_{\mathbf{T}_0} - f \circ X_{\mathbf{T}_0+\tau+T_k}| \right] d\tau \leq \lim_{k\to\infty} \frac{2c\|f\|_\infty}{T_k} = 0 \end{aligned} \tag{B.1.52}$$

for all $c \in (0,\infty)$, $f \in \Pi^{\mathbf{T}_0}(M_b(E;\mathbf{R}))$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$, where $\mathbb{E}^{T_k}$ denotes the expectation operator of $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{P}^{T_k})$ for each $k \in \mathbf{N}$. □

B.2 Supplementary results for Appendix A

Fact B.43. *Let E be a topological space and $k \in \mathbf{N}$. Then, $C_c(E;\mathbf{R}^k) \subset C_0(E;\mathbf{R}^k) \subset C_b(E;\mathbf{R}^k)$ and they are indifferent if E is compact.*

Proof. This result follows by [Munkres, 2000, Theorem 27.4]. □

Proposition B.44. *Let E be a Hausdorff space. Then, the following statements are true:*

(a) $C_c(E;\mathbf{R})$ is a subalgebra of $C_b(E;\mathbf{R})$ and is a function lattice.

(b) $C_c(E;\mathbf{R}) \subset C_0(E;\mathbf{R}) \subset \mathfrak{cl}(C_c(E;\mathbf{R}))$.

[17] The terminology "X satisfying T_k-LMTC in A" was specified in Definition 6.36 and Note 6.37.

Proof. (a) $C_c(E;\mathbf{R}) \subset C_b(E;\mathbf{R})$ by Fact B.43. $0 \in C_c(E;\mathbf{R})$ since $\varnothing$ is compact. Fixing $f, g \in C_c(E;\mathbf{R})$ and $a \in \mathbf{R}\backslash\{0\}$, we observe that

$$(f+g)^{-1}(\mathbf{R}\backslash\{0\}) \subset f^{-1}(\mathbf{R}\backslash\{0\}) \cup g^{-1}(\mathbf{R}\backslash\{0\}), \tag{B.2.1}$$

that

$$(af)^{-1}(\mathbf{R}\backslash\{0\}) = f^{-1}(\mathbf{R}\backslash\{0\}), \tag{B.2.2}$$

and that

$$(fg)^{-1}(\mathbf{R}\backslash\{0\}) = f^{-1}(\mathbf{R}\backslash\{0\}) \cap g^{-1}(\mathbf{R}\backslash\{0\}). \tag{B.2.3}$$

Let K_f denote the compact closure of $f^{-1}(\mathbf{R}\backslash\{0\})$ for each $f \in C_c(E;\mathbf{R})$. We have $K_{f+g} \subset (K_f \cup K_g)$, $K_{af} = K_f$ and $K_{fg} \subset (K_f \cap K_g)$ by (B.2.1), (B.2.2) and (B.2.3), respectively. It then follows that

$$\{K_f \cup K_g, K_{f+g}, K_{af}, K_{fg}\} \subset \mathscr{K}(E) \tag{B.2.4}$$

by the fact $\{K_f, K_g\} \in \mathscr{K}(E)$, the Hausdorff property of E and Proposition A.12 (a, b). Hence, $C_c(E;\mathbf{R})$ is an algebra.

From the fact

$$|f|^{-1}(\{0\}) = f^{-1}(\{0\}),\ \forall f \in C(E;\mathbf{R}) \tag{B.2.5}$$

it follows that

$$|f| \in C_c(E;\mathbf{R}),\ \forall f \in C_c(E;\mathbf{R}), \tag{B.2.6}$$

that

$$f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \in C_c(E;\mathbf{R}),\ \forall f, g \in C_c(E;\mathbf{R}), \tag{B.2.7}$$

and that

$$f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \in C_c(E;\mathbf{R}),\ \forall f, g \in C_c(E;\mathbf{R}), \tag{B.2.8}$$

thus proving $C_c(E;\mathbf{R})$ is a function lattice.

(b) The first inclusion is automatic and we prove the second one. We fix $f \in C_0(E;\mathbf{R})$, $p \in \mathbf{N}$ and a $K_p \in \mathscr{K}(E)$ such that $\|f|_{E\backslash K_p}\|_\infty < 2^{-p}$.

The case where $K_p = E$ is trivial. Otherwise, we define $A \doteq f^{-1}((2^{-p}, \infty))$, $B \doteq f^{-1}[(-\infty, -2^{-p})]$ and

$$f_p \doteq \left(f^+(x) - 2^{-p}\right)^+ - \left(f^-(x) - 2^{-p}\right)^+. \tag{B.2.9}$$

A and B are disjoint subsets of K_p. The fact

$$f_p(x) = \begin{cases} f(x) - 2^{-p} > 0, & \text{if } x \in A, \\ f(x) + 2^{-p} < 0, & \text{if } x \in B, \\ 0, & \text{if } x \in E\backslash(A \cup B) \end{cases} \tag{B.2.10}$$

implies $A \cup B = E\backslash f^{-1}(\{0\})$. Letting F be the closure of $A \cup B$ in E, we have by Proposition A.12 (a) that $K_p \in \mathscr{C}(E)$, $F \subset K_p$ and $F \in \mathscr{K}(E)$, thus proving $f_p \in C_c(E; \mathbf{R})$. Furthermore, from the fact

$$f_p(x) - f(x) = \begin{cases} -2^{-p}, & \text{if } x \in A, \\ 2^{-p}, & \text{if } x \in B, \\ -f(x) \in (-2^{-p}, 2^{-p},), & \text{if } x \in E\backslash(A \cup B) \end{cases} \tag{B.2.11}$$

it follows that $\|f_p - f\|_\infty \leq 2^{-p}$. □

Fact B.45. *Let E be a topological space, $f \in C(E; \mathbf{R})$ and A be a dense subset of E with $E\backslash A \neq \varnothing$. If E is a first-countable space and $(A\backslash B) \subset f^{-1}(\{0\})$ for some $B \in \mathscr{C}(E)$ with $B \subset A$, then $f|_{E\backslash A} = 0$. In particular, this is true when E is a metrizable space and $f \in C_c(A, \mathscr{O}_E(A))$.*

Proof. For each $x \in E\backslash A$, the first-countability of E implies a sequence $\{x_n\}_{n\in\mathbf{N}} \subset A$ converging to x as $n \uparrow \infty$. As $x \in E\backslash B$ and $E\backslash B \in \mathscr{O}(E)$, there exists an $N \in \mathbf{N}$ such that $x_n \in A\backslash K$ and $f(x_n) = 0$ for all $n > N$. Hence, the continuity of f implies $f(x) = \lim_{n\to\infty} f(x_n) = 0$.

If E is a metrizable space, then it is first-countable by Fact A.9. If, in addition, $f \in C_c(A, \mathscr{O}_E(A))$, then we let B be the closure of $A\backslash f^{-1}(\{0\})$ and $B \in \mathscr{C}(E)$ by Proposition A.6 (a) and Proposition A.12 (a). □

Proposition B.46. *Let $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces and $(S, \mathscr{A})$ be defined as in (2.7.22). Then, the following statements are true:*

(a) $\mathscr{B}(S) \supset \mathscr{A}$.

(b) If $\mathbf{I}$ *is countable and* S *is hereditary Lindelöf, then* $\mathscr{B}(S) = \mathscr{A}$.

(c) If $\mathbf{I}$ *is countable and* $\{S_i\}_{i\in\mathbf{I}}$ *are all second-countable, then* $\mathscr{B}(S) = \mathscr{A}$.

(d) If $\mathbf{I}$ *is countable and* $\{S_i\}_{i\in\mathbf{I}}$ *are all metrizable and separable spaces (especially Polish spaces), then* $\mathscr{B}(S) = \mathscr{A}$.

Note B.47. As arranged in §2.6, the Cartesian product $S \doteq \prod_{i\in\mathbf{I}} S_i$ above is equipped with the product topology $\mathscr{O}(S) \doteq \bigotimes_{i\in\mathbf{I}} \mathscr{O}(S_i)$ and its Borel σ-algebra is $\mathscr{B}(S) \doteq \sigma[\mathscr{O}(S)]$.

Proof of Proposition B.46. (a) follows by the argument establishing [Bogachev, 2007, Vol. II, Lemma 6.4.1].

(b) and (c) were proved in [Bogachev, 2007, Vol. II, Lemma 6.4.2 (ii)].

(d) follows by (c), Proposition A.6 (c) and Proposition A.11 (c). □

Lemma B.48. *Let* $\{S_i\}_{i\in\mathbf{I}}$ *be topological spaces,* $(S, \mathscr{A})$ *be as in (2.7.22),* $A \in \mathscr{A}$, $\{A_n\}_{n\in\mathbf{N}} \subset \mathscr{A}$ *and* $\mu \in \mathfrak{M}^+(S, \mathscr{A})$. *Then, the following statements are true:*

(a) If $S = \bigcup_{n\in\mathbf{N}} A_n$ *and* $\mathscr{A}|_{A_n} = \mathscr{B}_S(A_n)$ *for all* $n \in \mathbf{N}$, *then* $\mathscr{B}(S) = \mathscr{A}$.

(b) If μ *is supported on* A *and* $\mathfrak{be}(\mu|_A)$[18] *is a singleton, then* $\mathfrak{be}(\mu)$ *is a singleton.*

(c) If μ *is supported on* A *and* $\mathscr{B}_S(A) = \mathscr{A}|_A$, *then* $\mathfrak{be}(\mu)$ *is a singleton.*

Proof. (a) follows by Proposition B.46 (a) and Fact B.1 (with $E = S$, $\mathscr{U}_1 = \mathscr{B}(S)$ and $\mathscr{U}_2 = \mathscr{A}$).

(b) Let $\nu = \mathfrak{be}(\mu|_A)$[19]. $\mu_1 \doteq \nu|^S \in \mathcal{M}^+(S)$ by Fact 2.1 (b) (with $E = S$ and $\mathscr{U} = \mathscr{B}(E)$). Since $A \in \mathscr{A}$, we have that

$$B \cap A \in \mathscr{A}|_A \subset \mathscr{B}_S(A),\ \forall B \in \mathscr{A}. \tag{B.2.12}$$

[18] "$\mu|_A$" and $\nu|^E$ denote the concentration of μ on A and the expansion of ν onto E. "$\mathfrak{be}(\mu|_A)$" denotes the Borel extension(s) of μ.

[19] "$\nu = \mathfrak{be}(\mu|_A)$" means ν is the unique Borel extension of $\mu|_A$.

Note that ν and $\mu|_A$ are identical restricted to $\mathscr{A}|_A$. It then follows by (B.2.12) that

$$\mu|_A(B \cap A) = \nu(B \cap A) = \mu_1(B),\ \forall B \in \mathscr{A}. \tag{B.2.13}$$

It follows by the fact $\mu(A) = 1$, the fact $A \in \mathscr{A}$ and (B.2.13) that

$$\mu(B) = \mu(B \cap A) = \mu|_A(B \cap A) = \mu_1(B),\ \forall B \in \mathscr{A}, \tag{B.2.14}$$

thus proving $\mu_1 \in \mathfrak{be}(\mu)$. If $\mu_2 \in \mathfrak{be}(\mu)$, then we have that

$$\mu_2|_A = \mathfrak{be}(\mu|_A) = \nu \in \mathscr{M}^+ \left(A, \mathscr{O}_S(A)\right). \tag{B.2.15}$$

It follows that

$$\mu_2 = (\mu_2|_A)|^S = \nu|^S = \mu_1 \tag{B.2.16}$$

by (B.2.15) and Fact 2.1 (a, c) (with $E = S$, $\mathscr{U} = \mathscr{B}(E)$ and $\mu = \mu_2$).

(c) $\mathscr{B}_S(A) = \mathscr{A}|_A$ immediately implies $\mu|_A = \mathfrak{be}(\mu|_A)$. Then, (c) follows by (b). □

Lemma B.49. *Let E and S be measurable spaces and $f, g \in M(S; E)$. If there exists a countable subset of $M(E; \mathbf{R})$ separating points on E, then $\{x \in S : f(x) = g(x)\}$ is a measurable subset of S. In particular, this is true when E is baseable.*

Proof. Let $\{h_n\}_{n\in\mathbf{N}} \subset M(E; \mathbf{R})$ separate points on E. $\mathscr{B}(\mathbf{R}^2) = \mathscr{B}(\mathbf{R})^{\otimes 2}$ by Proposition B.46 (d) (with $S_i = \mathbf{R}$).

$$\varphi_n \doteq (h_n \circ f, h_n \circ g) \in M\left(S; \mathbf{R}^2, \mathscr{B}(\mathbf{R}^2)\right),\ \forall n \in \mathbf{N} \tag{B.2.17}$$

by Fact 2.3 (b). The diagonal line $\Delta \doteq \{(y, y) : y \in \mathbf{R}\} \in \mathscr{C}(\mathbf{R}^2) \subset \mathscr{B}(\mathbf{R}^2)$ by [Munkres, 2000, §17, Exercise 13]. Hence,

$$\begin{aligned} &\{x \in S : f(x) = g(x)\} \\ &= \{x \in S : h_n \circ f(x) = h_n \circ g(x),\ \forall n \in \mathbf{N}\} = \bigcap_{n\in\mathbf{N}} \varphi_n^{-1}(\Delta) \end{aligned} \tag{B.2.18}$$

is a measurable subset of S. □

Lemma B.50. *Let E be a topological space, V be the family of all càdlàg members of $E^{\mathbf{R}^+}$*[20]*, $t \in \mathbf{R}^+$ and $T \in (0, \infty)$. Then, the following statements are true:*

(a) If $M(E; \mathbf{R})$ has a countable subset separating points on E, then $\{x \in V : t \in J(x)\} \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V$.

(b) $\{x \in V : x|_{[0,T)} \in A^{[0,T)}\} \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V$ for all $A \in \mathscr{C}(E)$, especially for all $A \in \mathscr{K}(E)$ when E is a Hausdorff space.

Proof. (a) We fix $t \in \mathbf{R}^+$, let $\mathfrak{p}_{t-}$ denote the mapping associating each $x \in V$ to its left limit at t and find by Fact 2.3 (a) that

$$\mathfrak{p}_{t-}^{-1}(A) = \bigcap_{p \in \mathbf{Q}^+ \cap [0,t)} \bigcup_{q \in \mathbf{Q}^+ \cap (p,t)} \mathfrak{p}_q^{-1}(A) \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V, \ \forall A \in \mathscr{O}(E), \tag{B.2.19}$$

so $\mathfrak{p}_{t-} \in M(V, \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V; E)$. It then follows by Lemma B.49 (with $S = V$, $f = \mathfrak{p}_{t-}$ and $g = \mathfrak{p}_t$) that

$$\{x \in V : t \in J(x)\} = \{x \in V : \mathfrak{p}_{t-}(x) = \mathfrak{p}_t(x)\} \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V. \tag{B.2.20}$$

(b) When E is Hausdorff, $\mathscr{K}(E) \subset \mathscr{C}(E)$ by Proposition A.12 (a). It follows by the closedness of A and the right-continuity of each x that

$$\left\{x \in V : x|_{[0,T)} \in A^{[0,T)}\right\} = \bigcap_{t \in \mathbf{Q} \cap [0,T)} V \cap \mathfrak{p}_t^{-1}(A). \tag{B.2.21}$$

It follows by the fact $A \in \mathscr{C}(E) \subset \mathscr{B}(E)$ and Fact 2.3 (a) that

$$V \cap \mathfrak{p}_t^{-1}(A) \in \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_V, \ \forall t \in \mathbf{R}^+. \tag{B.2.22}$$

Now, (b) follows by (B.2.21), (B.2.22) and the countability of $\mathbf{Q} \cap [0,T)$. □

Fact B.51. *Let E is a topological space and $K \in \mathscr{K}(E)$. Then, the following statements are true:*

(a) $K \in \mathscr{K}(E, \mathscr{U})$ for any topological coarsening $(E, \mathscr{U})$ of E.

[20] E need not be a Tychonoff space, so we avoid the notation $D(\mathbf{R}^+; E)$ for clarity.

(b) If $\mathcal{D} \subset C(E;\mathbf{R})$ *separate points on* K*, then* $\mathscr{O}_E(K) = \mathscr{O}_{\mathcal{D}}(K)$ *and* $K \in \mathscr{K}(E,\mathscr{O}_{\mathcal{D}}(E)) \subset \mathscr{C}(E,\mathscr{O}_{\mathcal{D}}(E))$.

Proof. (a) is immediate by the definition of compactness.

(b) $\mathscr{O}_E(K) = \mathscr{O}_{\mathcal{D}}(K)$ is a Hausdorff topology by Lemma A.20 (with $E = K$ and $\mathcal{D} = \mathcal{D}|_K$) and Proposition A.17 (c) (with $A = E$). Now, (b) follows by Proposition A.12 (a). □

Lemma B.52. *Let* E *be a non-empty set,* $\mathcal{G} \subset \mathbf{R}^E$ *and* $\mathcal{H} \subset \mathbf{R}^E$*. Suppose that for any* $g \in \mathcal{G}$ *and* $n \in \mathbf{N}$*, there exists a bounded function* $f_{g,n} \in \mathcal{H}$ *such that*

$$A_{g,n} \stackrel{\circ}{=} \{x \in E : |g(x)| < n\} = \{x \in E : |f_{g,n}(x)| < n\} \tag{B.2.23}$$

and

$$g\mathbf{1}_{A_{g,n}} = f_{g,n}\mathbf{1}_{A_{g,n}}. \tag{B.2.24}$$

Then, there exists a subset $\mathcal{F} \subset \mathcal{H}$ *such that:*

(a) The members of $\mathcal{F}$ *are all bounded and include all the bounded members of* $\mathcal{G}$*. In particular,* $\mathcal{F} = \mathcal{G}$ *when the members of* $\mathcal{G}$ *are all bounded.*

(b) $\mathcal{F}$ *is countable if* $\mathcal{G}$ *is.*

(c) $\mathscr{O}_{\mathcal{G}}(E) \subset \mathscr{O}_{\mathcal{F}}(E)$*. Moreover, if* $\mathcal{G}$ *separates points on* E*, or if* E *is a topological space and* $\mathcal{G}$ *strongly separates points on* E*, then* $\mathcal{F}$ *has the same property.*

Proof. (a, b) Let $\mathcal{G}_0$ be the family of all bounded members of $\mathcal{G}$. For each $g \in \mathcal{G}\backslash\mathcal{G}_0$ (if any), we take a bounded $f_{g,n} \in \mathcal{H}$ satisfying (B.2.24) for all $n \in \mathbf{N}$. For each $g \in \mathcal{G}_0$ (if any), we let

$$n_g \stackrel{\circ}{=} \min\{n \in \mathbf{N} : n > \|g\|_\infty\}, \tag{B.2.25}$$

and find an $f_{g,n_g} \in \mathcal{H}$ satisfying

$$A_{g,n_g} = E \text{ and } f_{g,n} = f_{g,n}\mathbf{1}_{A_{g,n}} = g\mathbf{1}_{A_{g,n}} = g. \tag{B.2.26}$$

Now, we define $\mathcal{F} \stackrel{\circ}{=} \{f_{g,n} : g \in \mathcal{G}, n \in \mathbf{N}\}$.

(c) It follows by (B.2.23) and (B.2.24) that

$$\begin{aligned}\{x \in E : g(x) < a\} &= \bigcup_{n>a} \{x \in E : g(x) < a, |g(x)| < n\} \\ &= \bigcup_{n>a} \{x \in E : f_{g,n}(x) < a, |f_{g,n}(x)| < n\} \\ &= \bigcup_{n>a} \{x \in E : -n < f_{g,n}(x) < a\} \in \mathscr{O}_{\mathcal{F}}(E), \ \forall a \in \mathbf{R}, g \in \mathcal{G},\end{aligned} \tag{B.2.27}$$

thus proving $\mathscr{O}_{\mathcal{G}}(E) \subset \mathscr{O}_{\mathcal{F}}(E)$.

The Hausdorff property of $\mathscr{O}_{\mathcal{G}}(E)$ implies that of $\mathscr{O}_{\mathcal{F}}(E)$ by Fact A.1. So, $\mathcal{G}$ separating points on E implies $\mathcal{F}$ separating points on E by Proposition A.17 (c).

If $\mathcal{G}$ strongly separates points on topological space E, then $\mathscr{O}(E) \subset \mathscr{O}_{\mathcal{G}}(E) \subset \mathscr{O}_{\mathcal{F}}(E)$ and so $\mathcal{F}$ strongly separates points on E. □

Corollary B.53. *Let E be a topological space. Then, $C(E;\mathbf{R})$ separates points (resp. strongly separates points) on E if and only if $C_b(E;\mathbf{R})$ does.*

Proof. Sufficiency is immediate. Necessity follows by Lemma B.52 (with $\mathcal{G} = C(E;\mathbf{R})$ and $\mathcal{H} = C_b(E;\mathbf{R})$). □

Fact B.54. *Let E be a topological space, $\mu \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$, $\nu_1 \in \mathfrak{be}(\mu)$, $X \in M(\Omega, \mathscr{F}, \mathbb{P}; E^d, \mathscr{B}(E)^{\otimes d})$ and $\nu_2 \in \mathfrak{be}(\mathbb{P} \circ X^{-1})$. Then, $\int_{E^d} f(x)\mu(dx) = f^*(\nu_1)$ and $\mathbb{E}[f \circ X] = f^*(\nu_2)$ for all $f \in \mathfrak{ca}[\Pi^d(M_b(E;\mathbf{R}))]$.*

Proof. This result follows by Proposition A.21 (a) (with $\mathcal{D} = M_b(E;\mathbf{R})$) and the fact that ν_1 (resp. ν_2) and μ (resp. $\mathbb{P} \circ X^{-1}$) are the same measures on $(E^d, \mathscr{B}(E)^{\otimes d})$. □

Lemma B.55. *Let E be a topological space and $A \subset E$. Then,*

$$\mu_n \Longrightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(A, \mathscr{O}_E(A)) \tag{B.2.28}$$

implies

$$\mu_n|^E \Longrightarrow \mu|^E \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E). \tag{B.2.29}$$

The converse is true when E is a Tychonoff space.

Proof. It follows by (B.2.28) and $C_b(E;\mathbf{R})|_A \subset C_b(A, \mathscr{O}_E(A);\mathbf{R})$ that

$$\begin{aligned}\lim_{n\to\infty}\int_E f(x)\mu_n|^E(dx) &= \lim_{n\to\infty}\int_A f|_A(x)\mu_n(dx) = \int_A f|_A(x)\mu(dx)\\ &= \int_E f(x)\mu|^E(dx),\ \forall f\in C_b(E;\mathbf{R}),\end{aligned} \tag{B.2.30}$$

thus proving (B.2.29). Conversely, if E is a Tychonoff space, then (B.2.29) implies

$$\limsup_{n\to\infty}\mu_n(F\cap A) = \limsup_{n\to\infty}\mu_n|^E(F) \le \mu|^E(F) = \mu(F\cap A),\ \forall F\in\mathscr{C}(E) \tag{B.2.31}$$

by Theorem 2.17 (a, b) (with $\mu_n = \mu_n|^E$ and $\mu = \mu|^E$). $(A, \mathscr{O}_E(A))$ is a Hausdorff subspace by Proposition A.26 (b). Now, (B.2.28) follows by (B.2.31) and Theorem 2.17 (a, b). □

Fact B.56. *Let E be a topological space and $A \subset E$. If $\Gamma \subset \mathcal{M}^+(A, \mathscr{O}_E(A))$ is relatively compact, then $\{\mu|^E\}_{\mu\in\Gamma}$ is relatively compact in $\mathcal{M}^+(E)$.*

Proof. This result is immediate by Lemma B.55. □

Corollary B.57. *Let E be a topological space, $A \in \mathscr{B}(E)$ be a Hausdorff subspace of E and $\{\mu_n\}_{n\in\mathbf{N}}$ be sequentially tight in A and satisfy $\{\mu_n(E)\}_{n\in\mathbf{N}} \subset [a,b]$ for some $0 < a < b$. If μ is the unique weak limit point of $\{\mu_n\}_{n\in\mathbf{N}}$ in $\mathcal{M}^+(E)$, then (2.3.5) holds.*

Proof. $\{\mu_n\}_{n\in\mathbf{N}}$ are all supported on A with finite exception by Fact B.29 (with $\mathscr{U} = \mathscr{B}(E)$ and $\Gamma = \{\mu_n\}_{n\in\mathbf{N}}$) and $\{\mu_n|_A\}_{n\in\mathbf{N}}$ is relatively compact in $\mathcal{M}^+(A, \mathscr{O}_E(A))$ by Lemma A.46 (with $E = (A, \mathscr{O}_E(A))$ and $\Gamma = \{\mu_n|_A\}_{n\in\mathbf{N}}$). Then, $\{\mu_n\}_{n\in\mathbf{N}}$ is relatively compact in $\mathcal{M}^+(E)$ by Fact 2.1 (c) (with $\mathscr{U} = \mathscr{B}(E)$ and $\nu = \mu|_A$) and Fact B.56. Now, the corollary follows by Fact B.28. □

Lemma B.58. *Let E be a Hausdorff space. Then, the following statements are equivalent:*

(a) E is a Tychonoff space.

(b) $\delta_{x_n} \Rightarrow \delta_x$ as $n\uparrow\infty$ in $\mathcal{P}(E)$ implies $x_n \to x$ as $n\uparrow\infty$ in E.

(c) Convergence determining implies determining point convergence on E.

Proof. ((a) $\to$ (b)) $\delta_{x_n} \Rightarrow \delta_x$ as $n \uparrow \infty$ in $\mathcal{P}(E)$ implies

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f^*(\delta_{x_n}) = f(\delta_x) = f(x) \tag{B.2.32}$$

for all $f \in C_b(E;\mathbf{R})$. $C_b(E;\mathbf{R})$ determines point convergence on E by Proposition A.25 (a, c) and Proposition A.17 (b). Hence, (B.2.32) implies $x_n \to x$ as $n \uparrow \infty$.

((b) $\to$ (c)) If $\mathcal{D} \subset M_b(E;\mathbf{R})$ satisfies $\bigotimes \mathcal{D}(x_n) \to \bigotimes \mathcal{D}(x)$ as $n \uparrow \infty$, then (B.2.32) holds for all $f \in \mathcal{D}$. This implies $\delta_{x_n} \Rightarrow \delta_x$ as $n \uparrow \infty$ since $\mathcal{D}$ is convergence determining on E. Now, we have $x_n \to x$ as $n \uparrow \infty$ by (b).

((c) $\to$ (a)) $C_b(E;\mathbf{R})$ determines point convergence on E by (c). It strongly separates points and separates points on E by Proposition A.17 (a, b). Now, (a) follows by Proposition A.25 (a, c). $\square$

Lemma B.59. *Let E be a topological space, $\mathcal{D} \subset C_b(E;\mathbf{R})$ separate points on E and $d \in \mathbf{N}$. Then, the following statements are true:*

(a) If each of $\mu_1, \mu_2 \in \mathcal{M}^+(E)$ is tight and $\mathcal{D}$ is closed under multiplication, then $f^*(\mu_1) = f^*(\mu_2)$ for all $f \in \mathcal{D} \cup \{1\}$ implies $\mu_1 = \mu_2$.

(b) If each of $\mu_1, \mu_2 \in \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$ is $\mathbf{m}$-tight, then $f^*(\mu_1) = f^*(\mu_2)$ for all $f \in \mathfrak{mc}[\Pi^d(\mathcal{D})] \cup \{1\}$ implies $\mu_1 = \mu_2$.

Proof. (a) Let $a \mathrel{\hat{=}} \mu_1(E) = \mu_2(E)$ and $\nu_i \mathrel{\hat{=}} \mu_i/a$ for each $i \in \{1,2\}$. Each of ν_1 and ν_2 is a tight member of $\mathcal{P}(E)$ and they satisfy $f^*(\nu_1) = f^*(\nu_2)$ for all $f \in \mathcal{D}$. Then, $\nu_1 = \nu_2$ by [Blount and Kouritzin, 2010, Theorem 11 (d)] and so $\mu_1 = \mu_2$.

(b) E is a Hausdorff space by Proposition A.17 (e) (with $A = E$). For each $j = 1, 2$, there exists an $\mathbf{m}$-tight $\mu'_j = \mathfrak{be}(\mu_j)$ by Proposition 3.57 (with $\mathbf{I} = \{1, ..., d\}$, $S_i = E$, $A = E^d$ and $\Gamma = \{\mu_j\}$). $\mathfrak{mc}[\Pi^d(\mathcal{D})]$ separates points on E^d by Proposition A.21 (b). Now, (b) follows by (a) (with $E = E^d$, $\mathcal{D} = \mathfrak{mc}[\Pi^d(\mathcal{D})]$ and $\mu_j = \mu'_j$). $\square$

Fact B.60. *Let E be a topological space and S be a Hausdorff space. If $\Gamma \subset \mathcal{M}^+(E)$ is tight in $A \subset E$ and $f \in C(E;S)$, then $\{\mu \circ f^{-1} : \mu \in \Gamma\}$ is tight in $f(A)$.*

Proof. $f(K) \in \mathscr{K}(S) \subset \mathscr{B}(S)$ for all $K \in \mathscr{K}(E)$ by Proposition A.12 (a, e). Now, the result follows by Fact B.30 (with $\mathscr{U} = \mathscr{B}(E)$ and $\mathscr{A} = \mathscr{B}(S)$). $\square$

Lemma B.61. *Let $\mathbf{I}$ be a countable index set, $\{S_i\}_{i\in\mathbf{I}}$ be topological spaces, $(S, \mathscr{A})$ be as in (2.7.22), $\Gamma \subset \mathfrak{M}^+(S, \mathscr{A})$, $A_i \subset S_i$ for each $i \in \mathbf{I}$ and $A \mathrel{\mathring{=}} \prod_{i\in\mathbf{I}} A_i$. Then, the following statements are true:*

(a) If $\{\mu \circ \mathfrak{p}_i^{-1}\}_{\mu\in\Gamma}$ is tight (resp. $\mathbf{m}$-tight) in A_i for all $i \in \mathbf{I}$, then Γ is tight (resp. $\mathbf{m}$-tight) in A. The converse is true when $(A_i, \mathscr{O}_{S_i}(A_i))$ is a Hausdorff subspace of S_i and $A_i \in \mathscr{B}(S_i)$.

(b) If $\{\mu \circ \mathfrak{p}_i^{-1}\}_{\mu\in\Gamma}$ is sequentially tight (resp. $\mathbf{m}$-tight) in A_i for all $i \in \mathbf{I}$, then Γ is sequentially tight (resp. $\mathbf{m}$-tight) in A. The converse is true when $(A_i, \mathscr{O}_{S_i}(A_i))$ is a Hausdorff subspace of S_i and $A_i \in \mathscr{B}(S_i)$.

Proof. (a) Without loss of generality, we suppose $\mathbf{I} = \mathbf{N}$. Each A_i is equipped with the subspace topology $\mathscr{O}_{S_i}(A_i)$ throughout the proof. If $\{\mu \circ \mathfrak{p}_i^{-1}\}_{\mu\in\Gamma}$ is tight in A_i for all $i \in \mathbf{I}$, then there exist

$$A_i \supset K_{p,i} \in \mathscr{K}(S_i) \cap \mathscr{B}(S_i),\ \forall i, p \in \mathbf{N} \tag{B.2.33}$$

such that

$$\sup_{\mu\in\Gamma} \mu \circ \mathfrak{p}_i^{-1}\left(S_i \backslash \mathfrak{p}_i(K_p)\right) \leq 2^{-p-i},\ \forall i, p \in \mathbf{N}. \tag{B.2.34}$$

It follows that

$$A \supset \prod_{i\in\mathbf{N}} K_{p,i} \in \mathscr{K}(S) \cap \mathscr{A},\ \forall p \in \mathbf{N} \tag{B.2.35}$$

by Proposition A.12 (b), Fact 2.4 (a) and the fact $\prod_{i\in\mathbf{I}} K_{p,i} = \bigcap_{i\in\mathbf{I}} \mathfrak{p}_i^{-1}(K_{p,i})$. Now, we conclude the tightness of Γ in A by observing that

$$\sup_{\mu\in\Gamma} \mu\left(S \backslash \prod_{i\in\mathbf{N}} K_{p,i}\right) \leq \sum_{i=1}^{\infty} \sup_{\mu\in\Gamma} \mu\left(S_i \backslash K_{p,i}\right) \leq 2^{-p},\ \forall p \in \mathbf{N}. \tag{B.2.36}$$

If $\{\mu \circ \mathfrak{p}_i^{-1}\}_{\mu\in\Gamma}$ is $\mathbf{m}$-tight in A_i for all $i \in \mathbf{I}$, then we retake each $K_{p,i}$ above from $\mathscr{K}^{\mathbf{m}}(S_i) \cap \mathscr{B}(S_i)$, find $\prod_{i\in\mathbf{I}} K_{p,i} \in \mathscr{K}^{\mathbf{m}}(S)$ by Lemma 3.55 (a) (with $A_i = K_{p,i}$) and verify the $\mathbf{m}$-tightness of Γ by a similar argument.

Next, we suppose $A_i \in \mathscr{B}(S_i)$ is a Hausdorff subspace for all $i \in \mathbf{I}$ and justify the converse statement. We have that

$$\mathfrak{p}_i(K) \in \mathscr{K}(A_i) \subset \mathscr{B}(A_i) \subset \mathscr{B}(S_i),\ \forall K \in \mathscr{K}(S), i \in \mathbf{I} \tag{B.2.37}$$

and

$$\mathfrak{p}_i(K) \in \mathscr{K}^{\mathbf{m}}(A_i) \subset \mathscr{B}(A_i) \subset \mathscr{B}(S_i),\ \forall K \in \mathscr{K}^{\mathbf{m}}(S), i \in \mathbf{I} \tag{B.2.38}$$

by Corollary A.13 (b) (with $A = K$ and $S_i = A_i$), Lemma 3.55 (b) (with $A = K$ and $S_i = A_i$) and the fact $A_i \in \mathscr{B}(S_i)$. If Γ is tight (resp. $\mathbf{m}$-tight) in A, then for each $i \in \mathbf{I}$, the tightness (resp. $\mathbf{m}$-tightness) of $\{\mu \circ \mathfrak{p}_i^{-1}\}_{\mu \in \Gamma}$ in A_i follows by (B.2.37), (B.2.38), the fact $A_i = \mathfrak{p}(A)$, Fact 2.3 (a) and Fact B.30 (with $(E, \mathscr{U}) = (S, \mathscr{A})$, $(S, \mathscr{A}) = (S_i, \mathscr{B}(S_i))$ and $f = \mathfrak{p}_i$).

(b) follows immediately by (a) and a triangular array argument. □

Lemma B.62. *Let E be a Tychonoff space and $(D(\mathbf{R}^+; E), \mathscr{S}, \nu)$ be the completion of $(D(\mathbf{R}^+; E), \sigma(\mathscr{J}(E)), \mu)$*[21]*. If $M(E; \mathbf{R})$ has a countable subset separating points on E, especially if E is baseable, then $\nu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathcal{M}^+(E^{\mathbf{T}_0})$ is a Borel extension of $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathfrak{M}^+(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0})$ for all non-empty $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+ \backslash J(\mu))$*[22].

Proof. $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ is a member of $\mathfrak{M}^+(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0})$ by Proposition A.68 (a). $\mathfrak{p}_{\mathbf{T}_0} \in M[D(\mathbf{R}^+; E), \mathscr{S}; E^{\mathbf{T}_0}, \mathscr{B}(E^{\mathbf{T}_0})]$ by Proposition A.68 (c), the definition of $J(\mu)$ and Fact B.4 (with $E = D(\mathbf{R}^+; E)$, $\mathscr{U} = \mathscr{S}$, $S = E^{\mathbf{T}_0}$ and $f = \mathfrak{p}_{\mathbf{T}_0}$). Hence, $\nu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathfrak{be}(\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1})$ as ν is an extension[23] of μ to $\mathscr{S}$. □

Lemma B.63. *Let E be a Tychonoff space, $\mathcal{D} \subset C(E; \mathbf{R})$ be countable and $\{\mu_i\}_{i \in \mathbf{I}} \subset \mathcal{M}^+(D(\mathbf{R}^+; E))$. Then, the following statements are true:*

(a) If $\{\mu_i \circ \varpi(f)^{-1}\}_{i \in \mathbf{I}}$ is tight in $D(\mathbf{R}^+; \mathbf{R})$ for all $f \in \mathcal{D}$, then $\{\mu_i \circ \varpi(\mathcal{D})^{-1}\}_{i \in \mathbf{I}}$ is tight in $D(\mathbf{R}^+; \mathbf{R})^{\mathcal{D}}$.

[21] $\mathscr{B}(D(\mathbf{R}^+; E)) \stackrel{\circ}{=} \sigma[\mathscr{J}(E)]$, so the measure space notation "$(D(\mathbf{R}^+; E), \mathscr{B}(D(\mathbf{R}^+; E)), \mu)$" implies $\mu \in \mathcal{M}^+(D(\mathbf{R}^+; E))$.

[22] The $J(\mu)$ herein is well-defined by Proposition A.68 (b) and Fact A.71.

[23] Extension of measure was specified in §2.1.2.

(b) If $\mathcal{D}$ strongly separates points on E and $\{\mu_i \circ \varpi(f)^{-1}\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})$ for all $f \in \mathfrak{ae}(\mathcal{D})$, then $\{\mu_i \circ \varpi[\bigotimes \mathfrak{ae}(\mathcal{D})]^{-1}\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$.

Note B.64. When $\mathcal{D}$ is countable, $\mathfrak{ae}(\mathcal{D})$ is also by Fact B.15. Then, $\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}$, $D(\mathbf{R}^+;\mathbf{R})$, $D(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$ and $D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ are all Polish spaces by Proposition A.11 (f) and Proposition A.72 (d).

Proof of Lemma B.63. (a) follows by the fact

$$\left(\mu_i \circ \varpi(\mathcal{D})^{-1}\right) \circ \mathfrak{p}_f^{-1} = \mu_i \circ \varpi(f)^{-1} \in \mathcal{M}^+\left(D(\mathbf{R}^+;\mathbf{R})\right),\ \forall f \in \mathcal{D} \quad \text{(B.2.39)}$$

and Lemma B.61 (a) (with $\mathbf{I} = \mathcal{D}$, $S_i = A_i = D(\mathbf{R}^+;\mathbf{R})$ and $\Gamma = \{\mu_i \circ \varpi(\mathcal{D})^{-1}\}_{i\in\mathbf{I}}$).

(b) $\{\mu_i \circ \varpi(\mathfrak{ae}(\mathcal{D}))^{-1}\}_{i\in\mathbf{I}}$ is tight in $D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ by (a) (with $\mathcal{D} = \mathfrak{ae}(\mathcal{D})$). Let $\phi \mathrel{\hat{=}} \varpi[\bigotimes \mathfrak{ae}(\mathcal{J})]$, $\mathcal{J} = \{\mathfrak{p}_f\}_{f\in\mathcal{D}}$ be the one-dimensional projections on $\mathbf{R}^{\mathcal{D}}$ and $\nu_i \mathrel{\hat{=}} \mu_i \circ \phi^{-1}$ for each $i \in \mathbf{I}$. From Corollary A.66 (with $\mathbf{I} = \mathcal{D}$) it follows that

$$\phi \circ \varpi\left(\mathfrak{ae}(\mathcal{J})\right)^{-1} \in C\left[D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}; D\left(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}\right)\right]. \quad \text{(B.2.40)}$$

Observing that

$$\nu_i = \left[\mu_i \circ \varpi\left(\mathfrak{ae}(\mathcal{D})\right)^{-1}\right] \circ \left[\phi \circ \varpi\left(\mathfrak{ae}(\mathcal{J})\right)^{-1}\right]^{-1},\ \forall i \in \mathbf{I}, \quad \text{(B.2.41)}$$

we conclude the desired tightness of $\{\nu_i\}_{i\in\mathbf{I}}$ by (B.2.40), (B.2.41) and Fact B.60 (with $E = D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$, $S = D(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$ and $f = \phi \circ \varpi\left(\mathfrak{ae}(\mathcal{J})\right)^{-1}$). $\square$

Lemma B.65. *Let E be a Tychonoff space, $\mathcal{D} \subset C(E;\mathbf{R})$ be countable, $\Psi \mathrel{\hat{=}} \varpi[\mathfrak{ae}(\mathcal{D})]$, $V \subset D(\mathbf{R}^+;E)$ and $\{A_p\}_{p\in\mathbf{N}} \subset \mathscr{B}(E)$. If $A_p \subset A_{p+1}$, $\mathcal{D}$ strongly separates points on A_p and $x|_{[0,p)} \in A_p^{[0,p)}$ for all $x \in V$ and $p \in \mathbf{N}$, then*

$$\Psi|_V \in \mathbf{imb}\left(V, \mathscr{O}_{D(\mathbf{R}^+;E)}(V); D(\mathbf{R}^+\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right) \quad \text{(B.2.42)}$$

and

$$\mathscr{B}_{D(\mathbf{R}^+;E)}(V) = \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_V. \quad \text{(B.2.43)}$$

Proof. Step 1: Show $\Psi|_V$ is injective. $E_0 \doteq \bigcup_{p\in\mathbf{N}} A_p \in \mathscr{B}(E)$ satisfies

$$V \subset D\left(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0)\right). \tag{B.2.44}$$

$\mathfrak{ae}(\mathcal{D})$ separates points on E_0 by the Hausdorff property of E, Proposition A.17 (a) and Fact B.19 (with $n = p$ and $\mathcal{D}_n = \mathfrak{ae}(\mathcal{D})$). So, $\Psi|_V$ is injective by (B.2.44) and Fact B.20 (with $A = E_0$ and $\mathcal{D} = \mathfrak{ae}(\mathcal{D})$).

Step 2: Show the continuity of $\Psi|_V$. We have by Fact B.15 and Proposition A.17 (d) (with $A = E_0$) that $(E_0, \rho_{\mathcal{D}})$ is a separable metric space, $\mathfrak{ae}(\mathcal{D})$ is a countable subset of $C(E_0, \rho_{\mathcal{D}}; \mathbf{R})$ and $\mathfrak{ae}(\mathcal{D})$ strongly separates points on $(E_0, \rho_{\mathcal{D}})$. It follows that

$$\Psi|_{D(\mathbf{R}^+;E_0,\rho_{\mathcal{D}})} \in \mathbf{imb}\left(D\left(\mathbf{R}^+; E_0, \rho_{\mathcal{D}}\right); D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right) \tag{B.2.45}$$

by Proposition A.62 (a) (with $\mathcal{D} = \mathfrak{ae}(\mathcal{D})$). $(E_0, \rho_{\mathcal{D}}) = (E_0, \mathscr{O}_{\mathcal{D}}(E_0))$ is a topological coarsening of $(E_0, \mathscr{O}_E(E_0))$ since $\mathcal{D} \subset C(E; \mathbf{R}^+)$. So,

$$\Psi|_{D(\mathbf{R}^+;E_0,\mathscr{O}_E(E_0))} \in C\left(D\left(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0)\right); D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right) \tag{B.2.46}$$

by (B.2.45) and Proposition A.62 (e) (with $E = (E_0, \rho_{\mathcal{D}})$ and $S = (E_0, \mathscr{O}_E(E_0))$). Hence, we have by (B.2.44) and (B.2.46) that

$$\Psi|_V \in C\left(V, \mathscr{O}_{D(\mathbf{R}^+;E)}(V); D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right). \tag{B.2.47}$$

Step 3: Show the continuity of $(\Psi|_V)^{-1}$. We aforementioned that $D(\mathbf{R}^+; \mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ is a Polish space, so its subspace $\Psi(V)$ is metrizable by Proposition A.6 (b). According to [Munkres, 2000, Theorem 21.3], showing the continuity of $(\Psi|_V)^{-1}$ is equivalent to showing that (A.6.27) implies (A.6.15) for all $\{y_k\}_{k\in\mathbf{N}_0} \subset V$.

We suppose (A.6.27) holds, fix $u \in \mathbf{R}^+\backslash J(y_0)$ and define $p_u \doteq \min\{p \in \mathbf{N} : p > u + 1\}$. Observing that

$$\left\{y_k|_{[0,u+1]}\right\}_{k\in\mathbf{N}} \cup \left\{y|_{[0,u+1]}\right\} \subset D\left([0, u + 1]; A_{p_u}, \mathscr{O}_E(A_{p_u})\right), \tag{B.2.48}$$

we have

$$y_k^u \longrightarrow y_0^u \text{ as } k \uparrow \infty \text{ in } D\left([0,u+1]; A_{p_u}, \mathscr{O}_E(A_{p_u})\right) \tag{B.2.49}$$

by Lemma A.74 (with $E = (A_{p_u}, \mathscr{O}_E(A_{p_u}))$ and $\mathcal{D} = \mathcal{D}|_{A_{p_u}}$). This implies

$$y_k^u \longrightarrow y_0^u \text{ as } k \uparrow \infty \text{ in } D\left([0,u+1]; E_0, \mathscr{O}_E(E_0)\right) \tag{B.2.50}$$

by Corollary A.65 (with $E = (E_0, \mathscr{O}_E(E_0))$ and $A = A_{p_u}$). We aforementioned that the countable collection $\mathcal{D} \subset C(E;\mathbf{R})$ separates points on E_0, so $(E_0, \mathscr{O}_E(E_0))$ is a baseable space. Hence, it follows by (B.2.50) and Lemma A.75 (with $E = (E_0, \mathscr{O}_E(E_0))$) that

$$y_k \longrightarrow y_0 \text{ as } k \uparrow \infty \text{ in } D\left(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0)\right). \tag{B.2.51}$$

Now, the desired (A.6.15) follows by Corollary A.65 (with $A = E_0$).

Step 4: Show (B.2.43). The three steps above established (B.2.42).

$$\varpi(f) \in M\left(E^{\mathbf{R}^+}, \mathscr{B}(E)^{\otimes \mathbf{R}^+}; \mathbf{R}^{\mathbf{R}^+}, \mathscr{B}(\mathbf{R})^{\otimes \mathbf{R}^+}\right), \ \forall f \in \mathfrak{ae}(\mathcal{D}) \tag{B.2.52}$$

by $\mathcal{D} \subset C(E;\mathbf{R})$ and Fact B.10 (b) (with $\mathbf{I} = \mathbf{R}^+$). $D(\mathbf{R}^+;\mathbf{R})$ is a Polish space, so

$$\left[\mathscr{B}(\mathbf{R})^{\otimes \mathbf{R}^+}\right]^{\otimes \mathfrak{ae}(\mathcal{D})} = \sigma\left[\mathscr{J}(\mathbf{R})\right]^{\otimes \mathfrak{ae}(\mathcal{D})} = \sigma\left[\mathscr{J}(\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right] \tag{B.2.53}$$

by Proposition A.72 (b) (with $E = \mathbf{R}$) and Proposition (B.46) (d) (with $S_i = D(\mathbf{R}^+;\mathbf{R})$).

$$\Psi|_V \in M\left(V, \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_V; \Psi(V), \sigma\left[\mathscr{J}(\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right]\big|_{\Psi(V)}\right) \tag{B.2.54}$$

by (B.2.52), Fact 2.3 (b) and (B.2.53).

$$\begin{aligned}\mathscr{B}_{D(\mathbf{R}^+;E)}(V) &= \sigma\left(\left\{(\Psi|_V)^{-1}(O) : O \in \mathscr{J}(\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right\}\right) \\ &= \left\{(\Psi|_V)^{-1}(B) : B \in \sigma\left(\mathscr{J}(\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}\right)\right\} \subset \mathscr{B}(E)^{\otimes \mathbf{R}^+}\Big|_V\end{aligned} \tag{B.2.55}$$

by (B.2.42) and (B.2.54). Now, (B.2.43) follows by Proposition A.68 (b). □

Lemma B.66. *Let E be a Tychonoff space, $\mathcal{D} \subset C(E;\mathbf{R})$ be countable, $\Psi \mathrel{\mathring{=}} \varpi[\mathfrak{ae}(\mathcal{D})]$, $\varphi \mathrel{\mathring{=}} \bigotimes \mathfrak{ae}(\mathcal{D})$, $\{A_p\}_{p\in\mathbf{N}} \subset \mathscr{B}(E)$ and*

$$V \mathrel{\mathring{=}} \bigcap_{p\in\mathbf{N}} \left\{x \in D(\mathbf{R}^+;E) : x|_{[0,p)} \in A_p^{[0,p)}\right\}. \tag{B.2.56}$$

If $A_p \subset A_{p+1}$, $\mathcal{D}$ strongly separates points on A_p and $\varphi(A_p) \in \mathscr{C}(\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$ for all $p \in \mathbf{N}$, then $\Psi(V) \in \mathscr{C}(D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})})$.

Proof. $D(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$ is a Polish space, so $\Psi(V)$ as a subspace is metrizable by Proposition A.6 (b). Hence, showing the closeness of $\Psi(V)$ is reduced by Fact A.9 (with $E = D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$ and $A = \Psi(V)$) to showing that

$$\Psi(y_k) \longrightarrow z \text{ as } k \uparrow \infty \text{ in } D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})} \tag{B.2.57}$$

imply $z \in \Psi(V)$ for any $\{y_k\}_{k\in\mathbf{N}} \subset V$.

Let $\{\mathfrak{p}_f\}_{f\in\mathcal{D}}$ be the one-dimensional projections on $D(\mathbf{R}^+;\mathbf{R})^{\mathfrak{ae}(\mathcal{D})}$.

$$z'(t) \mathrel{\mathring{=}} \bigotimes_{f\in\mathfrak{ae}(\mathcal{D})} \mathfrak{p}_f(z)(t),\ \forall t \in \mathbf{R}^+ \tag{B.2.58}$$

defines a member of $D(\mathbf{R}^+;\mathbf{R}^{\mathfrak{ae}(\mathcal{D})})$ by Fact B.14 (c).

$$\mathbf{T} \mathrel{\mathring{=}} \bigcap_{f\in\mathfrak{ae}(\mathcal{D})} \mathbf{R}^+\backslash J(\mathfrak{p}_f(z)) \tag{B.2.59}$$

is cocountable by Proposition 3.62 (b) (with $E = \mathbf{R}$ and $x = \mathfrak{p}_f(z)$).

$$\varphi \circ y_k(t) \longrightarrow z'(t) \text{ as } k \uparrow \infty \text{ in } \mathbf{R}^{\mathfrak{ae}(\mathcal{D})},\ \forall t \in \mathbf{T} \tag{B.2.60}$$

by (B.2.57), Fact B.11 and Proposition A.68 (c). It then follows that

$$z'(t) \in \varphi(A_p),\ \forall t \in [0,p), p \in \mathbf{N} \tag{B.2.61}$$

by (B.2.60), the closedness of each $\varphi(A_p)$ in $\mathbf{R}^{\mathfrak{ae}(\mathcal{D})}$, the denseness of $\mathbf{T}$ in $\mathbf{R}^+$ and the right-continuity of z'.

$$\varphi|_{A_p} \in \mathbf{imb}\left(A_p, \mathscr{O}_E(A_p); \mathbf{R}^{\mathfrak{ae}(\mathcal{D})}\right),\ \forall p \in \mathbf{N} \tag{B.2.62}$$

by Lemma A.28 (a, c). So, $(\varphi|_{A_p})^{-1} \circ z'|_{[0,p)}$ is a càdlàg mapping from $[0,p)$ to $(A_p, \mathscr{O}_E(A_p))$ for all $p \in \mathbf{N}$ and, hence,

$$y(t) \circeq (\varphi|_{A_p})^{-1}(z'(t)),\ \forall t \in [0,p), p \in \mathbf{N} \tag{B.2.63}$$

well defines a member of V. Now, one observes $\Psi(y) = z$ from (B.2.58), (B.2.63) and the definitions of Ψ and φ. ☐

The next lemma is adapted from Kouritzin [2016] and restated befittingly.

Lemma B.67. *Let E and S be topological spaces, $\{A_p\}_{p\in\mathbf{N}} \subset \mathscr{B}(E)$ and $f \in S^E$ satisfy $\{f(A_p)\}_{p\in\mathbf{N}} \subset \mathscr{C}(S)$ and*

$$f|_{A_p} \in \mathbf{hom}\left[A_p, \mathscr{O}_E(A_p); f(A_p), \mathscr{O}_S\left(f(A_p)\right)\right],\ \forall p \in \mathbf{N}, \tag{B.2.64}$$

$E_0 \circeq \bigcup_{p\in\mathbf{N}} A_p$ satisfy $f \in M(E_0, \mathscr{O}_E(E_0); S)$ and $\{\mu_i\}_{i\in\mathbf{I}} \subset \mathcal{P}(E)$ satisfy

$$\inf_{i\in\mathbf{I}} \mu_i(A_p) \geq 1 - 2^{-p},\ \forall p \in \mathbf{N}. \tag{B.2.65}$$

Then, tightness of $\{\mu_i \circ f^{-1}\}_{i\in\mathbf{I}}$ implies that of $\{\mu_i\}_{i\in\mathbf{I}}$ in E_0.

Proof. We refer the proof to [Kouritzin, 2016, Lemma 24]. ☐

Lemma B.68. *Let E be a Tychonoff space, $\mathbf{T} \subset \mathbf{R}^+$ be dense, $d \in \mathbf{N}$ and $f \in C_b(E^d; \mathbf{R})$. Then, the following statements are true:*

(a) For each $\mathbf{T}_0 = \{t_1, ..., t_d\} \in \mathscr{P}_0(\mathbf{R}^+)$, there exists a $\mathbf{T}_p = \{t_{p,1}, ..., t_{p,d}\} \in \mathscr{P}_0(\mathbf{T})$ for each $p \in \mathbf{N}$ such that[24]

$$\lim_{n\to\infty} \int_{E^{\mathbf{T}_0}} f(x)\mu \circ \mathfrak{p}_{\mathbf{T}_p}^{-1}(dx) = \int_{E^{\mathbf{T}_0}} f(x)\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx). \tag{B.2.66}$$

for all $\mu \in \mathfrak{M}^+(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$.

(b) If $\gamma^1, \gamma^2 \in \mathfrak{M}^+(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$ satisfy (8.2.12) for all $\mathbf{T}_0 = \{t_1, ..., t_d\} \in \mathscr{P}_0(\mathbf{T})$, then they satisfy

$$\int_{E^{\mathbf{T}_0}} f(x)\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) = \int_{E^{\mathbf{T}_0}} f(x)\gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) \tag{B.2.67}$$

[24] Recall that Corollary A.69 verified $\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} \in \mathfrak{M}^+(E^{\mathbf{T}_0}, \mathscr{B}(E)^{\otimes \mathbf{T}_0})$ for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ and $\mu \in \mathfrak{M}^+(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes \mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$.

for all $\mathbf{T}_0 = \{t_1, ..., t_d\} \in \mathscr{P}_0(\mathbf{R}^+)$.

Proof. (a) For ease of notation, we define $\phi_{\mathbf{S}} \circeq f \circ \mathfrak{p}_{\mathbf{S}}$ for each $\mathbf{S} \in \mathscr{P}_0(\mathbf{R}^+)$. The denseness of $\mathbf{T}$ in $\mathbf{R}^+$ allows us to take $\mathbf{T}_p = \{t_{p,1}, ..., t_{p,d}\} \in \mathscr{P}_0(\mathbf{T})$ for each $p \in \mathbf{N}$ such that

$$\lim_{p\to\infty} \sup_{1\leq i\leq d} |t_i - t_{p,i}| = 0. \tag{B.2.68}$$

It follows that

$$\lim_{p\to\infty} \left|\phi_{\mathbf{T}_0}(x) - \phi_{\mathbf{T}_p}(x)\right| = 0,\ \forall x \in D(\mathbf{R}^+; E) \tag{B.2.69}$$

by (B.2.68), the right-continuity of $x \in S$, Fact B.11 and the continuity of f. The boundeness of f implies

$$\sup_{p\in\mathbf{N}} \left\|\phi_{\mathbf{T}_p}\right\|_\infty \leq \|f\|_\infty < \infty. \tag{B.2.70}$$

Now, we have by (B.2.69), (B.2.70) and the Dominated Convergence Theorem that

$$\begin{aligned}\lim_{n\to\infty} \int_{E^{\mathbf{T}_0}} f(x)\mu \circ \mathfrak{p}_{\mathbf{T}_p}^{-1}(dx) &= \lim_{n\to\infty} \int_S \phi_{\mathbf{T}_p}(y)\mu(dy)\\ &= \int_S \phi_{\mathbf{T}_0}(y)\mu(dy) = \int_{E^{\mathbf{T}_0}} f(x)\mu \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx).\end{aligned} \tag{B.2.71}$$

(b) follows immediately by (a) (with $\mu = \gamma^1$ or γ^2). □

Lemma B.69. *Let E be a Tychonoff space, $\mathbf{S}^1$, $\mathbf{S}^2$ and $\mathbf{S}$ be dense subsets of $\mathbf{R}^+$, $\mathcal{D} \subset C_b(E; \mathbf{R})$ and $\{\mu_n\}_{n\in\mathbf{N}} \cup \{\gamma^1, \gamma^2\} \subset \mathfrak{M}^+(D(\mathbf{R}^+; E), \mathscr{B}(E)^{\otimes\mathbf{R}^+}|_{D(\mathbf{R}^+;E)})$ satisfy (8.2.12) for each $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$, $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S}^i)$ and $i = 1, 2$. Then, the following statements are true:*

(a) If $\mathcal{D}$ strongly separates points on E, then (8.2.13) holds.

(b) If $\mathcal{D}$ separates points on E, and if each of $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ and $\gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ is $\mathbf{m}$-tight for all $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$, then (8.2.13) holds.

Proof. For ease of notation, we define $\phi_{\mathbf{T}} \circeq f \circ \mathfrak{p}_{\mathbf{T}}$ for each $\mathbf{T} \in \mathscr{P}_0(\mathbf{R}^+)$ and $f \in M_b(E^{\mathbf{T}}; \mathbf{R})$. The proof is divided into four steps.

Step 1: We show (B.2.67) *for each* $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$ *and* $\mathbf{T}_0 = \{t_1, ..., t_d\} \in \mathscr{P}_0(\mathbf{S}^1)$. Note 5.6 argued that $f \in C_b(E^d; \mathbf{R})$. By Lemma B.68 (a) (with $\mathbf{T} = \mathbf{S}^2$), there exists a $\mathbf{T}_p = \{t_{p,1}, ..., t_{p,d}\} \in \mathscr{P}_0(\mathbf{S}^2)$ for each $p \in \mathbf{N}$ such that

$$\begin{aligned} &\lim_{p\to\infty} \int_{D(\mathbf{R}^+;E)} \left(\phi_{\mathbf{T}_p} - \phi_{\mathbf{T}_0}\right)(x)\gamma^2(dx) \\ &= \lim_{p\to\infty} \int_{D(\mathbf{R}^+;E)} \left(\phi_{\mathbf{T}_p} - \phi_{\mathbf{T}_0}\right)(x)\mu_n(dx) = 0,\ \forall n \in \mathbf{N}. \end{aligned} \tag{B.2.72}$$

From *(8.2.12)* we get

$$\lim_{n\to\infty} \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\mu_n(dx) = \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\gamma^1(dx) \tag{B.2.73}$$

and

$$\lim_{n\to\infty} \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_p}(x)\mu_n(dx) = \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_p}(x)\gamma^2(dx),\ \forall p \in \mathbf{N}. \tag{B.2.74}$$

Let $\epsilon \in (0, \infty)$ be arbitrary and $n_0 \doteq 1$. By (B.2.73) and (B.2.74), we inductively choose an $n_p \in \mathbf{N} \cap (n_{p-1}, \infty)$ for each $p \in \mathbf{N}$ such that

$$\begin{aligned} &\left|\int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\mu_{n_p}(dx) - \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\gamma^1(dx)\right| \vee \\ &\left|\int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_p}(x)\mu_{n_p}(dx) - \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_p}(x)\gamma^2(dx)\right| < \frac{\epsilon}{2}. \end{aligned} \tag{B.2.75}$$

From Triangle Inequality and (B.2.75) it follows that

$$\begin{aligned} &\left|\int_{E^{\mathbf{T}_0}} f(x)\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) - \int_{E^{\mathbf{T}_0}} f(x)\gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx)\right| \\ &= \left|\int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\gamma^1(dx) - \int_{D(\mathbf{R}^+;E)} \phi_{\mathbf{T}_0}(x)\gamma^2(dx)\right| \\ &\leq \left|\int_{D(\mathbf{R}^+;E)} \left(\phi_{\mathbf{T}_0} - \phi_{\mathbf{T}_p}\right)(x)\gamma^2(dx)\right| \\ &+ \left|\int_{D(\mathbf{R}^+;E)} \left(\phi_{\mathbf{T}_p} - \phi_{\mathbf{T}_0}\right)(x)\mu_{n_p}(dx)\right| + \epsilon,\ \forall p \in \mathbf{N}. \end{aligned} \tag{B.2.76}$$

Now, (B.2.67) follows by (B.2.72), letting $p \uparrow \infty$ in (B.2.76) and then letting

$\epsilon \downarrow 0$.

Step 2: We show (B.2.67) for each $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$ *and* $\mathbf{T}_0 = \{t_1, ..., t_d\} \in \mathscr{P}_0(\mathbf{R}^+)$. This step follows by Step 1, the denseness of $\mathbf{S}^1$ in $\mathbf{R}^+$ and Lemma B.68 (b) (with $\mathbf{T} = \mathbf{S}^1$).

Step 3: Verify $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ *for each* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ *in (a).* For each $i = 1, 2$, we let $(D(\mathbf{R}^+; E), \mathscr{A}_{n_0}, \nu^i)$ be the completion of $(D(\mathbf{R}^+; E), \sigma[\mathscr{J}(E)], \gamma^i)$ and find by Lemma B.62 (with $\mu = \gamma^i$ and $\nu = \nu^i$) that $\nu^i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ is a Borel extension of $\gamma^i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$. It follows that

$$\int_{E^{\mathbf{T}_0}} f(x) \nu^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) = \int_{E^{\mathbf{T}_0}} f(x) \nu^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}(dx) \tag{B.2.77}$$

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$ by Step 2 and Fact B.54 (with $d = \aleph(\mathbf{T}_0)$, $\mu = \gamma^i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ and $\nu_1 = \nu^i \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$). Then, $\nu^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \nu^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ by Lemma A.35 (a) (with $d = \aleph(\mathbf{T}_0)$) and Fact B.22 (a) (with $E = E^{\mathbf{T}_0}$ and $\mathcal{D} = \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{D})] \cup \{1\}$), which of course implies $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$.

Step 4: Verify $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ *for each* $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ *in (b).* When $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{S})$, $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$ by Step 2 and Lemma B.59 (b) (with $d = \aleph(\mathbf{T}_0)$) and so (B.2.67) holds for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E; \mathbf{R}))]$. For general $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$, the key equality (B.2.67) holds for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(C_b(E; \mathbf{R}))]$ by (5.1.3) (with $\mathcal{D} = C_b(E; \mathbf{R})$ and $d = \aleph(\mathbf{T}_0)$), the denseness of $\mathbf{S}$ in $\mathbf{R}^+$ and Lemma B.68 (b) (with $\mathbf{T} = \mathbf{S}$). $C_b(E; \mathbf{R})$ strongly separates points on Proposition A.25 (a, c). Then, one follows the argument of Step 3 (with $\mathcal{D} = C_b(E; \mathbf{R})$) to show $\gamma^1 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1} = \gamma^2 \circ \mathfrak{p}_{\mathbf{T}_0}^{-1}$. □

Lemma B.70. *Let* E *be a topological space,* $(E_0, \mathscr{O}_E(E_0))$ *be a Tychonoff subspace of* E, $y_0 \in S_0 \subset \mathbb{D}_0 \doteq D(\mathbf{R}^+; E_0, \mathscr{O}_E(E_0))$, $\mathscr{U} \doteq \mathscr{B}(E)^{\otimes \mathbf{R}^+}$ *and* X *be a mapping from* $(\Omega, \mathscr{F}, \mathbb{P})$ *to* $E^{\mathbf{R}^+}$. *Then, the following statements are true:*

(a) If $\mathbb{P}(X = Z) = 1$ *for some* $Z \in M(\Omega, \mathscr{F}; \mathbb{D}_0)$, *then* X *is an* E*-valued càdlàg process.*

(b) If X *is an* E*-valued process,* $\mathbb{P}(X \in S_0) = 1$ *and* S_0 *satisfies* $\mathscr{B}(\mathbb{D}_0)|_{S_0} = \mathscr{U}|_{S_0}$, *then*

$$Y \doteq \mathfrak{var}\left(X; \Omega, X^{-1}(S_0), y_0\right) \in M\left(\Omega, \mathscr{F}; S_0, \mathscr{O}_{\mathbb{D}_0}(S_0)\right) \tag{B.2.78}$$

and $\mathbb{P}(X = Y \in S_0) = 1$.

Proof. (a) follows by Proposition A.68 (b) (with $E = (E_0, \mathscr{O}_E(E_0))$) and Lemma B.31 (a) (with $E = E^{\mathbf{R}^+}$ and $S = \mathbb{D}_0$).

(b) follows by Lemma B.31 (b, c) (with $(E, S, \mathscr{U}') = (E^{\mathbf{R}^+}, S_0, \mathscr{B}_{\mathbb{D}_0}(S_0))$). □

Proposition B.71. *Let E be a topological space, $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis, $k \in \mathbf{N}$, $\xi \in M(\Omega, \mathscr{F}; \mathcal{M}^+(E))$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an $\mathcal{M}^+(E)$-valued process. In addition, suppose either of the following hypotheses is true:*

(i) $f \in C_b(E; \mathbf{R}^k)$.

(ii) *E is a perfectly normal*[25] *(especially metrizable or Polish) space and* $f \in M_b(E; \mathbf{R}^k)$.

Then, $f^* \circ \xi \in M(\Omega, \mathscr{F}; \mathbf{R}^k)$ and $\varpi(f^*) \circ X$ is an $\mathbf{R}^k$-valued process. If, in addition, X is a $\mathscr{G}_t$-adapted, measurable or $\mathscr{G}_t$-progressive process, then $\varpi(f^*) \circ X$ also has the corresponding measurability.

Proof. Under the hypothesis (i), $f^* \in C_b(\mathcal{M}^+(E); \mathbf{R}^k)$ by the definition of weak topology and Fact 2.4 (b). Under the hypothesis (ii), $f^* \in M_b(\mathcal{M}^+(E); \mathbf{R}^k)$ by Lemma A.47 and Fact 2.4 (b). Now, the result follows by Fact B.32 (a) (with $E = \mathcal{M}^+(E)$, $S = \mathbf{R}^k$ and $f = f^*$). □

B.3 Auxiliary results about replication

Lemma B.72. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be base over E and $d, k \in \mathbf{N}$. If $f \in C(E^d; \mathbf{R}^k)$ has a replica $\widehat{f}$, then the following statements are true:*

(a) $\|\widehat{f}\|_\infty = \|f|_{E_0^d}\|_\infty \leq \|f\|_\infty$.

(b) $\widehat{f^+} = \widehat{f}^+$ *and* $\widehat{f^-} = \widehat{f}^-$.

[25] The notion of perfectly normal space was mentioned in §A.47.

Proof. (a) follows by the fact $f|_{E_0^d} = \widehat{f}|_{E_0^d}$, the deneseness of E_0 in $\widehat{E}$ and the continuities of f and $\widehat{f}$.

(b) follows by the fact $\widetilde{f^+} = \widehat{f}^+|_{E_0^d}$, the fact $\widetilde{f^-} = \widehat{f}^-|_{E_0^d}$ and the continuities of $\widehat{f}^+$ and $\widehat{f}^-$. □

Lemma B.73. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be base over E, $d \in \mathbf{N}$ and $\{\mu_n\}_{n\in\mathbf{N}} \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$. Then, (5.2.30) implies*

$$\nu\left(\widehat{E}^d \backslash K\right) \leq \liminf_{n\to\infty} \mu_n\left(E^d \backslash K\right), \ \forall K \in \mathscr{K}\left(E_0^d, \mathscr{O}_E(E_0)^d\right). \tag{B.3.1}$$

Proof. We have that

$$K \in \mathscr{K}(\widehat{E}^d) \subset \mathscr{C}(\widehat{E}^d) \subset \mathscr{B}(\widehat{E}^d) \tag{B.3.2}$$

by Corollary 3.15. It then follows that[26]

$$\nu\left(\widehat{E}^d \backslash K\right) \leq \liminf_{n\to\infty} \overline{\mu}_n\left(\widehat{E}^d \backslash K\right) = \liminf_{n\to\infty} \mu_n\left(E_0^d \backslash K\right) \tag{B.3.3}$$

by (5.2.30), (B.3.2), Theorem 2.17 (a, b) (with $E = \widehat{E}^d$, $F = K$, $\mu_n = \overline{\mu}_n$ and $\mu = \nu$) and Proposition 5.15 (a). □

Lemma B.74. *Let E be a topological space, $\mathcal{D} \subset C(E; \mathbf{R})$ separate points on E, $d \in \mathbf{N}$, $\mathbf{I}$ be a countable index set and $\Gamma_i \subset \mathfrak{M}^+(E^d, \mathscr{B}(E)^{\otimes d})$ be $\mathbf{m}$-tight for each $i \in \mathbf{I}$. Then, there exists a base $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over E such that $E_0 \in \mathscr{K}_\sigma^{\mathbf{m}}(E)$ and Γ_i is tight in $(E_0^d, \mathscr{O}_E(E_0)^d)$ for all $i \in \mathbf{I}$. In particular, $\mathcal{F}$ can be taken within $\mathcal{D} \cup \{1\}$ when $\mathcal{D} \subset C_b(E; \mathbf{R})$.*

Proof. Without loss of generality, we let $\mathbf{I} = \mathbf{N}$. By the $\mathbf{m}$-tightness of each Γ_i, there exist $\{K_{p,i}\}_{p,i\in\mathbf{N}} \subset \mathscr{K}^{\mathbf{m}}(E^d)$ satisfying

$$\sup_{\mu\in\Gamma_i} \mu(E^d \backslash K_{p,i}) \geq 1 - 2^{-p}, \ \forall p, i \in \mathbf{N}. \tag{B.3.4}$$

E is a Hausdorff space by Proposition A.17 (e) (with $A = E$).

$$\{K_{p,i,j} \circeq \mathfrak{p}_j(K_{p,i}) : 1 \leq j \leq d, p, i \in \mathbf{N}\} \subset \mathscr{K}^{\mathbf{m}}(E) \subset \mathscr{B}(E) \tag{B.3.5}$$

[26] $\overline{\mu}_n$ denotes the replica measure of μ_n which was introduced in §5.2.

by Proposition A.2 (c) and Lemma 3.55 (b) (with $A = K_{p,i}$). So,

$$E_0 \doteq \bigcup_{i\in\mathbf{N}} \bigcup_{p\in\mathbf{N}} \bigcup_{j=1}^{d} K_{p,i,j} \in \mathscr{K}_\sigma^{\mathbf{m}}(E). \tag{B.3.6}$$

We have by Corollary A.13 (a) and (B.3.4) that

$$\prod_{j=1}^{d} K_{p,i,j} \subset \mathscr{K}\left(E_0^d, \mathscr{O}_E(E_0)^d\right), \ \forall p, i \in \mathbf{N} \tag{B.3.7}$$

and

$$\mu(E_0^d) \geq \mu\left(\prod_{j=1}^{d} K_{p,i,j}\right) \geq \mu(K_{p,i}) \geq 1 - 2^p, \ \forall \mu \in \Gamma_i, p, i \in \mathbf{N}. \tag{B.3.8}$$

thus proving the tightness of each Γ_i in $(E_0^d, \mathscr{O}_E(E_0)^d)$. E_0 is a $\mathcal{D}$-baseable subset of E by (B.3.6) and Proposition 3.58 (b, e) (with $A = E_0$). Now, the result follows by Lemma 3.39 (a, c) (with $\mathcal{D}_0 = \varnothing$). □

Lemma B.75. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be base over E, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process, $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ be an $\widehat{E}$-valued process and $\mathbf{T} \subset \mathbf{R}^+$. If X satisfies (6.1.9), and if*

$$\mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \bigotimes \widehat{\mathcal{F}} \circ Y_t\right) \geq \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right), \ \forall t \in \mathbf{T}, \tag{B.3.9}$$

then X and Y satisfy (6.2.9).

Proof. It follows by (B.3.9), (3.1.1) and (3.1.3) that

$$\begin{aligned} 1 &= \mathbb{P}\left(X_t \in E_0\right) \\ &= \mathbb{P}\left(\bigotimes \widehat{\mathcal{F}} \circ Y_t = \bigotimes \mathcal{F} \circ X_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E}), X_t \in E_0\right) \\ &\leq \mathbb{P}\left(Y_t = \left(\bigotimes \widehat{\mathcal{F}}\right)^{-1} \circ \bigotimes \mathcal{F} \circ X_t = X_t \in E_0\right), \ \forall t \in \mathbf{T}. \end{aligned} \tag{B.3.10}$$

□

Fact B.76. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes satisfying $\mathbf{T}$-PSMTC in E_0. Then,* *the following statements are true for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$:*

(a) $\{X^i_{\mathbf{T}_0}\}_{i\in\mathbf{I}}$ *is sequentially* $\mathbf{m}$*-tight in* $E_0^{\mathbf{T}_0}$ *and* $\mathbb{P}^i(X^i_{\mathbf{T}_0} \in E_0^{\mathbf{T}_0}) = 1$.

(b) $\{\mu^i = \mathfrak{be}(\mathbb{P}^i \circ X^{-1}_{\mathbf{T}_0})\}_{i\in\mathbf{I}\setminus\mathbf{I}_{\mathbf{T}_0}}$ *exists for some* $\mathbf{I}_{\mathbf{T}_0} \in \mathscr{P}_0(\mathbf{I})$.

Proof. Let $\Gamma \triangleq \{\mathbb{P}^i \circ X^{-1}_{\mathbf{T}_0}\}_{i\in\mathbf{I}}$ and $A \triangleq E_0^{\mathbf{T}_0}$. Then, (a) follows by Lemma B.61 (b) (with $\mathbf{I} = \mathbf{T}_0$, $S_i = E$ and $A_i = E_0$) and Fact B.29 (with $E = E^{\mathbf{T}_0}$ and $\mathscr{U} = \mathscr{B}(E)^{\otimes\mathbf{T}_0}$). (b) follows by Lemma 3.3 (e) (with $A = E_0$) and Proposition 5.3 (with $\mathbf{I} = \mathbf{T}_0$ and $S_i = E$). $\square$

Lemma B.77. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mathbf{T} \subset \mathbf{R}^+$, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be E-valued process and $(\Omega, \mathcal{F}, \mathbb{P}; Y)$ be an $\widehat{E}$-valued process. Then, the following statements are true:*

(a) *If*

$$\inf_{f\in\mathcal{F},t\in\mathbf{T}} \mathbb{P}\left(f \circ X_t = \widehat{f} \circ Y_t\right) = 1, \tag{B.3.11}$$

then

$$\mathbb{P}\left(f \circ X_{\mathbf{T}_0} = \widehat{f} \circ Y_{\mathbf{T}_0}\right) = 1,\ \forall f \in \mathfrak{ca}\left[\Pi^{\mathbf{T}_0}(\mathcal{F})\right], \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T}) \tag{B.3.12}$$

and

$$\mathbb{E}\left[f \circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f} \circ Y_{\mathbf{T}_0}\right],\ \forall f \in \mathfrak{ca}\left[\Pi^{\mathbf{T}_0}(\mathcal{F})\right], \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T}). \tag{B.3.13}$$

Moreover, (6.2.9) implies (B.3.11).

(b) *If (B.3.11) holds (especially (6.2.9) holds) and Y is càdlàg, then X is $(\mathbf{T}, \mathcal{F})$-càdlàg.*

(c) *If $\mathbf{T} = \mathbf{R}^+$, (B.3.13) holds (especially (B.3.11) or (6.2.9) holds) and X is stationary, then Y is also stationary.*

(d) *If (B.3.13) holds (especially (B.3.11) or (6.2.9) holds), $\mathbf{T}$ is conull, X is stationary and Y is càdlàg, then Y is also stationary.*

(e) *If $A \in \mathscr{B}^{\mathbf{s}}(E)$ (especially $A \in \mathscr{K}_\sigma(E)$) satisfies $A \subset E_0$ and (6.2.29) holds, then $\mathscr{F}^X = \mathscr{F}^Y$. If, in addition, Y is stationary, then X is also stationary.*

(f) If (6.2.9) holds, and if $f \in M_b(E^{\mathbf{T}_0};\mathbf{R})$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ satisfy $\overline{f} \in M_b(\widehat{E}^{\mathbf{T}_0};\mathbf{R})$ (especially if $E_0^{\mathbf{T}_0} \in \mathscr{B}^{\mathbf{s}}(E^{\mathbf{T}_0})$), then

$$\mathbb{P}\left(f \circ X_{\mathbf{T}_0} = \overline{f} \circ Y_{\mathbf{T}_0}\right) = 1 \tag{B.3.14}$$

and

$$\mathbb{E}\left[f \circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[\overline{f} \circ Y_{\mathbf{T}_0}\right]. \tag{B.3.15}$$

Proof. (a) (B.3.11) implies (B.3.12) by properties of uniform convergence. (B.3.13) is immediate by (B.3.12). Moreover, (6.2.9) implies (B.3.11) by (3.1.1).

(b) $\{\varpi(\widehat{f}) \circ Y\}_{f \in \mathcal{F}}$ are all càdlàg processes by Fact B.34 (a) (with $E = \widehat{E}$, $S = \mathbf{R}$ and $X = Y$). Then, (b) follows by (a).

(c) Fixing $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$, one finds by (a) (with $\mathbf{T} = \mathbf{R}^+$) and the stationarity of X that

$$\mathbb{E}\left[\widehat{f} \circ Y_{\mathbf{T}_0} - \widehat{f} \circ Y_{\mathbf{T}_0+c}\right] = \mathbb{E}\left[f \circ X_{\mathbf{T}_0} - f \circ X_{\mathbf{T}_0+c}\right] = 0 \tag{B.3.16}$$

for all $c \in (0,\infty)$. Then, (c) follows by Corollary 3.11 (a) (with $d = \aleph(\mathbf{T}_0)$ and $A = \widehat{E}^d$).

(d) Fixing $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$, one finds by (a) and the stationarity of X that (B.3.16) holds for all c in the conull set

$$\mathbf{S}_{\mathbf{T}_0} \doteq \bigcap_{t \in \mathbf{T}_0} \{c \in (0,\infty) : t + c \in \mathbf{T}\}. \tag{B.3.17}$$

Then, (d) follows by a similar argument to the proof of Proposition 6.49 (c).

(e) Any $A \in \mathscr{K}_\sigma(E)$ satisfying $A \subset E_0$ belongs to $\mathscr{B}^{\mathbf{s}}(E)$ by Corollary 3.15 (b) (with $d = 1$). For each fixed $t \in \mathbf{R}^+$, we let $\Omega_t^0 \doteq \{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \in A\}$ and find by (6.2.29), the $\mathbb{P}$-completeness of $\mathscr{F}$[27] and Lemma 3.14 (a) (with $d = 1$) that $\Omega \backslash \Omega_t^0 \in \mathscr{N}(\mathbb{P}) \subset \mathscr{F}$,

$$\begin{aligned} X_t^{-1}(B) \cap \Omega_t^0 &= X^{-1}(B \cap A) \cap \Omega_t^0 \\ &\in \left\{Y_t^{-1}(V) \cap \Omega_t^0 : V \in \mathscr{B}(\widehat{E})\right\},\ \forall B \in \mathscr{B}(E) \end{aligned} \tag{B.3.18}$$

[27] Completeness of measure space was specified in 2.1.2. Completeness of filtration was specified in §2.5.

and

$$\begin{aligned} Y_t^{-1}(V) \cap \Omega_t^0 &= Y^{-1}(V \cap A) \cap \Omega_t^0 \\ &\in \left\{ X_t^{-1}(B) \cap \Omega_t^0 : B \in \mathscr{B}(E) \right\}, \ \forall V \in \mathscr{B}(\widehat{E}). \end{aligned} \tag{B.3.19}$$

Thus, $\mathscr{F}^X = \mathscr{F}^Y$ by their $\mathbb{P}$-completeness. When Y is stationary, we fix $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ and find by Lemma 3.14 (d) (with $d = \aleph(\mathbf{T}_0)$) and (6.2.29) that

$$\begin{aligned} \mathbb{P}\left(X_{\mathbf{T}_0} \in B\right) &= \mathbb{P}\left(Y_{\mathbf{T}_0} \in B \cap A^{\mathbf{T}_0}\right) \\ &= \mathbb{P}\left(Y_{\mathbf{T}_0+c} \in B \cap A^{\mathbf{T}_0}\right) = \mathbb{P}\left(X_{\mathbf{T}_0+c} \in B\right) \end{aligned} \tag{B.3.20}$$

for all $B \in \mathscr{B}(E)^{\otimes \mathbf{T}_0}$ and $c \in (0, \infty)$, which gives the stationarity of X.

(f) follows by the definition of $\overline{f}$ and Proposition 4.6 (b) (with $d = \aleph(\mathbf{T}_0)$). □

Lemma B.78. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mathbf{T} \subset \mathbf{R}^+$, $\mathcal{G}_{\mathbf{T}_0} \mathrel{\mathring{=}} \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ for each $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes satisfying (6.4.34), $\widehat{X}^n \in \mathfrak{rep}(X^n; E_0, \mathcal{F})$ for each $n \in \mathbf{N}$, $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ be an $\widehat{E}$-valued process, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued process satisfying (6.1.14) and $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$. Then, the following statements are true:*

(a) $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{F}\backslash\{1\})$-FDC if and only if $\{\widehat{X}^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \widehat{\mathcal{F}}\backslash\{1\})$-FDC.

(b) If $\{X^n\}_{n\in\mathbf{N}}$ is $(\mathbf{T}, \mathcal{F}\backslash\{1\})$-AS, then

$$\lim_{n\to\infty} \mathbb{E}^n \left[\widehat{f} \circ \widehat{X}^n_{\mathbf{T}_0} - \widehat{f} \circ \widehat{X}^n_{\mathbf{T}_0+c}\right] = 0 \tag{B.3.21}$$

for all $f \in \mathcal{G}_{\mathbf{T}_0}$, $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ and c (if any) in the set $\mathbf{S}_{\mathbf{T}_0}$ defined in (B.3.17).

(c) (6.2.5) is equivalent to

$$\lim_{n\to\infty} \mathbb{E}^n \left[\widehat{f}\left(\widehat{X}^n_{\mathbf{T}_0}\right)\right] = \mathbb{E}\left[\widehat{f}\left(Y_{\mathbf{T}_0}\right)\right] \tag{B.3.22}$$

for all $f \in \mathcal{G}_{\mathbf{T}_0}$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(d) If (B.3.12) holds (especially (6.2.9) holds), then (6.2.5) is equivalent to (B.3.22) for all $f \in \mathcal{G}_{\mathbf{T}_0}$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(e) (6.2.12) holds if and only if (B.3.22) holds for all $f \in \mathcal{G}_{\mathbf{T}_0}$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(f) (6.2.2) holds for all $f \in \mathcal{G}_{\mathbf{T}_0}$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$ if and only if

$$\widehat{X}^n \xrightarrow{\mathrm{D}(\mathbf{T})} \widehat{X} \text{ as } n \uparrow \infty. \tag{B.3.23}$$

(g) (6.2.1) implies (B.3.23).

In particular, the above conclusions are when if $\{X^n\}_{n\in\mathbf{N}}$ (resp. X) satisfies the stronger condition[28] (6.5.7) (resp. (6.1.9)) than (6.4.34) (resp. (6.1.14)).

Proof. (a) - (c) follow by Fact B.35 (b) and Proposition 6.8 (a).

(d) follows by Proposition 6.8 (a) (with $X = X^n$) and Lemma B.77 (a).

(e) follows by (3.1.16) and Corollary 3.11 (a) (with $(d, A) = (\aleph(\mathbf{T}_0), \widehat{E}^d)$).

(f) follows by Proposition 6.8 (a) and (d, e) (with $Y = \widehat{X}$).

(g) follows by $\mathcal{F} \subset C_b(E; \mathbf{R})$, Fact 6.19 and (f). □

Lemma B.79. *Let E be a topological space, $\{(\Omega^i, \mathscr{F}^i, \mathbb{P}^i; X^i)\}_{i\in\mathbf{I}}$ be E-valued processes, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be base over E and $\widehat{X}^i \in \mathfrak{rep}_c(X^i; E_0, \mathcal{F})$ for each $i \in \mathbf{I}$. If (6.4.7) holds for some conull $\mathbf{T} \subset \mathbf{R}^+$, and if $\{X^i\}_{i\in\mathbf{I}}$ is $(\mathbf{T}, \mathcal{F}\backslash\{1\})$-FDC, then $\{\widehat{X}^i\}_{i\in\mathbf{I}}$ has at most one weak limit point on $D(\mathbf{R}^+; \widehat{E})$[29].*

Proof. Suppose $\widehat{X}^{i_{j,n}} \Rightarrow Y^j$ as $n \uparrow \infty$ on $D(\mathbf{R}^+; \widehat{E})$ for each $j = 1, 2$. Without loss of generality, we suppose Y^1 and Y^2 are both defined on $(\Omega, \mathscr{F}, \mathbb{P})$. It follows by Proposition 6.49 (a) (with $n = i_{j,n}$ and $Y = Y^j$) and Lemma B.78 (a, e) (with $n = i_{j,n}$) that

$$\begin{aligned} \mathbb{E}\left[\widehat{f} \circ Y^1_{\mathbf{T}_0}\right] &= \lim_{n\to\infty} \mathbb{E}^{i_{1,n}}\left[\widehat{f} \circ \widehat{X}^{i_{1,n}}_{\mathbf{T}_0}\right] \\ &= \lim_{n\to\infty} \mathbb{E}^{i_{2,n}}\left[\widehat{f} \circ \widehat{X}^{i_{2,n}}_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f} \circ Y^2_{\mathbf{T}_0}\right] \end{aligned} \tag{B.3.24}$$

[28] We compared these conditions in Fact 6.9 and Fact 6.52.

[29] Càdlàg replicas are $D(\mathbf{R}^+; \widehat{E})$-valued random variables. Weak limit point of random variables was interpreted in §2.4.

for all $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})]$ and $\mathbf{T}_0 \in \mathscr{P}_0[\mathbf{T}\backslash(J(Y^1) \cup J(Y^2))]$. $\mathbf{T}\backslash(J(Y^1) \cup J(Y^2))$ is conull by Note 6.50. It then follows that

$$\mathbb{E}\left[\widehat{f} \circ Y^1_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f} \circ Y^2_{\mathbf{T}_0}\right], \ \forall f \in \mathfrak{mc}\left[\Pi^{\mathbf{T}_0}(\mathcal{F}\backslash\{1\})\right], \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+) \quad \text{(B.3.25)}$$

by the continuity of $\widehat{f}$, the càdlàg property of Y^j and the Dominated Convergence Theorem. It follows that

$$\mathbb{P} \circ (Y^1_{\mathbf{T}_0})^{-1} = \mathbb{P} \circ (Y^2_{\mathbf{T}_0})^{-1}, \ \forall \mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+) \quad \text{(B.3.26)}$$

by (B.3.25) and Corollary 3.11 (a) (with $d = \aleph(\mathbf{T}_0)$ and $A = \widehat{E}^d$). Now, the result follows by (B.3.26), Lemma 3.3 (c) and Proposition A.72 (b) (with $E = \widehat{E}$). □

Lemma B.80. *Let E be a topological space, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $T \in (0, \infty)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P}^T; X^T) = \mathfrak{rap}_T(X)$. Then, the following statements are true:*

(a) If (6.1.14) or (6.1.9) holds for some conull $\mathbf{T} \subset \mathbf{R}^+$, then X^T satisfies

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}^T\left(\bigotimes \mathcal{F} \circ X^T_t \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1 \quad \text{(B.3.27)}$$

or

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}^T\left(X^T_t \in E_0\right) = 1. \quad \text{(B.3.28)}$$

respectively.

(b) If (6.3.15) holds for $S_0 \subset E_0^{\mathbf{R}^+}$, then

$$\mathbb{P}^T\left(X^T \in S_0 \subset E_0^{\mathbf{R}^+}\right) = 1. \quad \text{(B.3.29)}$$

(c) If $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$, then $\widehat{X}^T = \mathfrak{rap}_T(X) \in \mathfrak{rep}_{\mathrm{m}}(X^T; E_0, \mathcal{F})$.

(d) If $\widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$, then $\widehat{X}^T = \mathfrak{rap}_T(X) \in \mathfrak{rep}_{\mathrm{c}}(X^T; E_0, \mathcal{F})$.

Proof. (a) One finds by the conullity of $\mathbf{T}$ that

$$\begin{aligned}&\inf_{t\in\mathbf{R}^+}\mathbb{P}^T\left(\bigotimes\mathcal{F}\circ X_t^T\in\bigotimes\widehat{\mathcal{F}}(\widehat{E})\right)\\&\geq\frac{1}{T}\int_{[t,T+t]\cap\mathbf{T}}\mathbb{P}\left(\bigotimes\mathcal{F}\circ X_\tau\in\bigotimes\widehat{\mathcal{F}}(\widehat{E})\right)d\tau=1\end{aligned}\tag{B.3.30}$$

and

$$\inf_{t\in\mathbf{R}^+}\mathbb{P}^T\left(X_t^T\in E_0\right)\geq\frac{1}{T}\int_{[t,T+t]\cap\mathbf{T}}\mathbb{P}\left(X_\tau\in E_0\right)d\tau=1.\tag{B.3.31}$$

(b) follows by the fact that $(X^T)^{-1}(S_0)\supset\mathbf{R}^+\times X^{-1}(S_0)$.

(c) follows by Proposition 7.20 (with $X=\widehat{X}$), (6.1.1) and the fact that

$$\begin{aligned}&\frac{1}{T}\int_0^T\mathbb{P}\left(\bigotimes\mathcal{F}\circ X_{\tau+t}=\bigotimes\widehat{\mathcal{F}}\circ\widehat{X}_{\tau+t}\right)d\tau\\&\geq\frac{1}{T}\int_0^T\mathbb{P}\left(\bigotimes\mathcal{F}\circ X_{\tau+t}\in\bigotimes\widehat{\mathcal{F}}(\widehat{E})\right)d\tau,\ \forall t\in\mathbf{R}^+.\end{aligned}\tag{B.3.32}$$

(d) follows by (c), Fact 6.24 and Lemma B.40 (b) (with $X=\widehat{X}$). ☐

Lemma B.81. *Let E be a topological space, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E, $\mathbf{T}\subset\mathbf{R}^+$, $\{(\Omega^n,\mathscr{F}^n,\mathbb{P}^n;X^n)\}_{n\in\mathbf{N}}$ be E-valued processes satisfying (6.4.34) (especially (6.5.7)), $\widehat{X}^n\in\mathfrak{rep}(X^n;E_0,\mathcal{F})$ for each $n\in\mathbf{N}$ and $(\Omega,\mathscr{F},\mathbb{P};Y)$ be an $\widehat{E}$-valued process. Then, (6.2.12) implies $\varpi(\bigotimes\widehat{\mathcal{F}})\circ Y=\mathfrak{fl}_{\mathbf{T}}(\{\varpi(\bigotimes\mathcal{F})\circ X^n\}_{n\in\mathbf{N}})$.*

Proof. We prove the result with (6.4.34) which is weaker than (6.5.7) by Fact 6.52 (with $\mathbf{I}=\mathbf{N}$). We define $Z\doteq\bigotimes\widehat{\mathcal{F}}\circ Y$, $Z^n\doteq\bigotimes\widehat{\mathcal{F}}\circ\widehat{X}^n$ and $\zeta^n\doteq\bigotimes\mathcal{F}\circ X^n$ for each $n\in\mathbf{N}$. One finds that

$$\inf_{t\in\mathbf{T},n\in\mathbf{N}}\mathbb{P}^n\left(\zeta_t^n=Z_t^n\right)=1\tag{B.3.33}$$

by Proposition 6.8 (a). We fix $\mathbf{T}_0\in\mathscr{P}_0(\mathbf{T})$ and put $d\doteq\aleph(\mathbf{T}_0)$. As mentioned in Note 6.5, $\{\widehat{X}\}_{n\in\mathbf{N}}$, $\{\zeta^n\}_{n\in\mathbf{N}}$, $\{Z^n\}_{n\in\mathbf{N}}$, Y and Z all have Borel finite-dimensional distributions, so (6.2.12) implies

$$\widehat{X}^n_{\mathbf{T}_0}\Longrightarrow Y_{\mathbf{T}_0}\text{ as }k\uparrow\infty\text{ on }\mathbf{R}^d.\tag{B.3.34}$$

One finds that

$$\varphi \mathrel{\mathring{=}} \bigotimes_{t\in\mathbf{T}_0}\left(\bigotimes \widehat{\mathcal{F}}\right)\circ \mathfrak{p}_t \in C\left[\widehat{E}^d;(\mathbf{R}^\infty)^d\right] \tag{B.3.35}$$

by (3.1.3) and Fact 2.4 (a, b). Hence, it follows by (B.3.35), (B.3.34) and Continuous Mapping Theorem (Theorem B.25 (a)) that

$$Z^n_{\mathbf{T}_0} = \varphi\circ \widehat{X}^n_{\mathbf{T}_0} \Longrightarrow \varphi\circ Y_{\mathbf{T}_0} = Z_{\mathbf{T}_0} \text{ as } k\uparrow\infty \text{ on } (\mathbf{R}^\infty)^d. \tag{B.3.36}$$

$(\mathbf{R}^\infty)^d$ and $\mathcal{P}((\mathbf{R}^\infty)^d)$ are Polish spaces by Proposition A.11 (f) and Theorem A.44 (b) (with $E=(\mathbf{R}^\infty)^d$). Now, the result follows by (B.3.36), (B.3.33) and Fact B.36 (with $E=\mathbf{R}^\infty$ and $X^i=\zeta^n$). □

B.4 Auxiliary results for companion papers

Fact B.82. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E for $\mathcal{L}$*[30]*, $i\in\{0,1\}$ and $\mu\in\mathcal{M}^+(E)$ satisfy $\mu(E\backslash E_0)=0$. Then, the following statements are true:*

(a) $\mu(\mathcal{L}f)=\overline{\mu}(\widehat{\mathcal{L}}_i\widehat{f})$[31] *for each $\widehat{f}\in\mathfrak{D}(\widehat{\mathcal{L}}_i)$*[32].

(b) If $E_0\in\mathscr{B}^{\mathbf{s}}(E)$, then $\mu(\mathcal{L}f)=\overline{\mu}(\overline{\mathcal{L}f})$[33] *for all $f\in\mathfrak{D}(\mathcal{L})$.*

Proof. (a) follows by Proposition 4.25 and Proposition 5.15 (a, e) (with $d=1$). (b) follows by Proposition 4.6 (a, b) (with $d=k=1$) and Proposition 5.15 (a, d) (with $d=1$). □

Fact B.83. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$, $(E_0,\mathcal{F};\widehat{E},\widehat{\mathcal{F}})$ be a base over E with $E_0\in\mathscr{B}^{\mathbf{s}}(E)$ and $\mathcal{F}\subset\mathfrak{D}(\mathcal{L})$, and the operator $\overline{\mathcal{L}}$ be defined by $\overline{\mathcal{L}}\widehat{f}\mathrel{\mathring{=}}\overline{\mathcal{L}f}$ for each $f\in\mathfrak{D}(\mathcal{L})\cap\mathfrak{ca}(\mathcal{F})$. Then, $\overline{\mathcal{L}}$ is a linear operator on $M_b(\widehat{E};\mathbf{R})$.*

Proof. $\mathcal{F}\subset\mathfrak{D}(\mathcal{L})$ implies $\mathfrak{D}(\mathcal{L})\cap\mathfrak{ca}(\mathcal{F})\neq\varnothing$. f's replica $\widehat{f}$ exists and $\overline{\mathcal{L}}\widehat{f}\in M_b(\widehat{E};\mathbf{R})$ for all $f\in\mathfrak{D}(\mathcal{L})\cap\mathfrak{ca}(\mathcal{F})$ by Proposition 4.6 (a, b, f) (with $d=k=1$).

[30] The notions of base for $\mathcal{L}$ and the replica operators $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_1$ of $\mathcal{L}$ were introduced in §4.2.
[31] Recall that $\overline{\mu}$ denotes the replica of μ.
[32] $\mathfrak{D}(\widehat{\mathcal{L}}_i)$ denotes the domain of operator $\widehat{\mathcal{L}}_i$.
[33] $\overline{\mathcal{L}f}$ was defined in Notation 4.1 and Notation 4.5.

Then, the result follows by the linearity of $\mathcal{L}$ and Proposition 4.6 (d) (with $d = k = 1$). $\square$

Lemma B.84. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E; \mathbf{R})$, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process, $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ be an $\widehat{E}$-valued measurable process, $\mathbf{T} \subset \mathbf{R}^+$ be conull and $i \in \{0, 1\}$. Then, the following statements are true:*

(a) If (B.3.11) (especially (6.2.9)) holds and $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ is a base for $\mathcal{L}$, then

$$\begin{aligned} &\mathbb{E}\left[f \circ X_t g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\widehat{f} \circ Y_t \widehat{g} \circ Y_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\left(\int_s^t \mathcal{L}f \circ X_u du\right) g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\left(\int_s^t \widehat{\mathcal{L}}_i \widehat{f} \circ Y_u du\right) \widehat{g} \circ Y_{\mathbf{T}_0}\right] = 0 \end{aligned} \tag{B.4.1}$$

for all $s < t$ in $\mathbf{T}$, $f \in \mathfrak{D}(\mathcal{L}_i)$[34], $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(b) If (6.2.9) holds, $E_0 \in \mathscr{B}^{\mathbf{s}}(E)$ and $\mathcal{F} \subset \mathfrak{D}(\mathcal{L})$[35], then

$$\begin{aligned} &\mathbb{E}\left[f \circ X_t g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\widehat{f} \circ Y_t \widehat{g} \circ Y_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\left(\int_s^t \mathcal{L}f \circ X_u du\right) g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\left(\int_s^t \overline{\mathcal{L}f} \circ Y_u du\right) \widehat{g} \circ Y_{\mathbf{T}_0}\right] \end{aligned} \tag{B.4.2}$$

for all $s < t$ in $\mathbf{T}$, $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

Remark B.85. One always has $\mathcal{L}f \in C_b(E; \mathbf{R})$ and $\widehat{\mathcal{L}}_i \widehat{f} \in C_b(\widehat{E}; \mathbf{R})$. If $E_0 \in \mathscr{B}^{\mathbf{s}}(E)$, then $\overline{\mathcal{L}f} \in M_b(\widehat{E}; \mathbf{R})$ by Fact B.83. Hence, the integrals $\int_0^t \mathcal{L}f \circ X_u du$ and $\int_0^t \widehat{\mathcal{L}}_i \widehat{f} \circ Y_u du$ in (B.4.1) and $\int_0^t \overline{\mathcal{L}f} \circ Y_u du$ in (B.4.2) are well-defined for measurable processes X and Y by Proposition B.33 (d).

Proof of Lemma B.84. We fix $t, u \in \mathbf{T}$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$. Under the hypothesis of (a), we have

$$\begin{aligned} &\mathbb{E}\left[f \circ X_t g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\widehat{f} \circ Y_t \widehat{g} \circ Y_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\mathcal{L}f \circ X_u g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\widehat{\mathcal{L}f} \circ Y_u \widehat{g} \circ Y_{\mathbf{T}_0}\right] = 0 \end{aligned} \tag{B.4.3}$$

[34] The operators $\mathcal{L}_0$ and $\mathcal{L}_1$ were defined in Notation 4.11.

[35] As mentioned in the proof of Fact B.83, $\mathcal{F} \subset \mathfrak{D}(\mathcal{L})$ ensures $\mathfrak{D}(\mathcal{L}) \cap \mathfrak{ca}(\mathcal{F}) \neq \varnothing$ and $\widehat{f}$, $\mathcal{L}f$ and $\overline{\mathcal{L}f}$ are well-defined for each $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$.

for all $f \in \mathfrak{D}(\mathcal{L}_i)$ by Proposition 4.25 and Lemma B.77 (a). Under the hypothesis of (b), we have

$$\begin{aligned} &\mathbb{E}\left[f \circ X_t g \circ X_{\mathbf{T}_0}\right] - \mathbb{E}\left[\widehat{f} \circ Y_t \widehat{g} \circ Y_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\mathcal{L} f \circ X_u g \circ X_{\mathbf{T}_0})\right] - \mathbb{E}\left[\overline{\mathcal{L} f} \circ Y_u \widehat{g} \circ Y_{\mathbf{T}_0}\right] = 0 \end{aligned} \tag{B.4.4}$$

for all $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$ by Lemma B.77 (a, f). Now, the result follows by Fubini's Theorem and the conullity of $\mathbf{T}$. □

Lemma B.86. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E for $\mathcal{L}$, $i \in \{0,1\}$, $\mathbf{T} \subset \mathbf{R}^+$ and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process satisfying (6.1.14). Then, the following statements are true:*

(a) If $\mathbf{T}$ is conull, then there exists an $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$[36] satisfying[37]

$$\begin{aligned} &\mathbb{E}\left[\left(f \circ X_t - f \circ X_s - \int_s^t \mathcal{L} f \circ X_u du\right) g \circ X_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\left(\widehat{f} \circ \widehat{X}_t - \widehat{f} \circ \widehat{X}_s - \int_s^t \widehat{\mathcal{L}}_i \widehat{f} \circ \widehat{X}_u du\right) \widehat{g} \circ \widehat{X}_{\mathbf{T}_0}\right] \end{aligned} \tag{B.4.5}$$

for all $s < t$ in $\mathbf{T}$, $f \in \mathfrak{D}(\mathcal{L}_i)$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(b) If X is $(\mathbf{R}^+, \mathcal{F})$-càdlàg and $\mathbf{T}$ is dense, then there exists an $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ satisfying

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}\left(\bigotimes \mathcal{F} \circ X_t = \bigotimes \widehat{\mathcal{F}} \circ \widehat{X}_t\right) = 1 \tag{B.4.6}$$

and (B.4.5) for all $s < t$ in $\mathbf{R}^+$, $f \in \mathfrak{D}(\mathcal{L}_i)$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$.

In particular, if X satisfies (6.1.9) instead of (6.1.14), then the conclusions above are true and (6.1.10) holds.

[36] $\mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ and $\mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ were introduced in Notation 6.3 and stand for all equivalence classes of measurable and càdlàg replicas of X with respect to $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ respectively.

[37] $\int_0^t \widehat{\mathcal{L}}_i \widehat{f} \circ \widehat{X}_u du$ in (B.4.5) is well-defined for $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ as explained in Remark B.85. It is well-defined for $\widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ by Fact 6.24.

Proof. (a) follows by Proposition 6.6 (b), Proposition 6.8 (a) and Lemma B.84 (a) (with $Y = \widehat{X}$).

(b) There exists an $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ satisfying (B.4.6) by Proposition 6.28 (a). Then, (b) follows Lemma B.84 (a) (with $\mathbf{T} = \mathbf{R}^+$ and $Y = \widehat{X}$).

Moreover, if (6.1.9) holds, then (6.1.14) holds by Fact 6.9 and (6.1.10) holds by Proposition 6.7 (a). □

Lemma B.87. *Let E be a topological space, $\mathcal{L}$ be a linear operator on $C_b(E;\mathbf{R})$, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E with $E_0 \in \mathscr{B}^{\mathrm{s}}(E)$ and $\mathcal{F} \subset \mathfrak{D}(\mathcal{L})$, $\mathbf{T} \subset \mathbf{R}^+$ be conull and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued measurable process satisfying (6.1.9). Then, the following statements are true:*

(a) There exists an $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ satisfying (6.1.10) and[38]

$$\begin{aligned} &\mathbb{E}\left[\left(f\circ X_t - f\circ X_s - \int_s^t \mathcal{L}f\circ X_u du\right) g\circ X_{\mathbf{T}_0}\right] \\ &= \mathbb{E}\left[\left(\widehat{f}\circ \widehat{X}_t - \widehat{f}\circ \widehat{X}_s - \int_s^t \overline{\mathcal{L}f}\circ \widehat{X}_u du\right) \widehat{g}\circ \widehat{X}_{\mathbf{T}_0}\right] \end{aligned} \tag{B.4.7}$$

for all $s < t$ in $\mathbf{T}$, $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{T})$.

(b) If X is $(\mathbf{R}^+, \mathcal{F})$-càdlàg, then there exists an $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ satisfying (6.1.10), (B.4.6) and (B.4.7) for all $s < t$ in $\mathbf{R}^+$, $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$.

Proof. (a) follows by Proposition 6.6 (b), Proposition 6.7 (a) and Lemma B.84 (b) (with $Y = \widehat{X}$).

(b) The conull set $\mathbf{T}$ is certainly dense in $\mathbf{R}^+$. Then, there exists an $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ satisfying (B.4.6) and (6.1.10) by Fact 6.9, Proposition 6.28 (a) and Proposition 6.7 (a). Fixing $f \in \mathfrak{ca}(\mathcal{F}) \cap \mathfrak{D}(\mathcal{L})$, we have

$$\mathbb{E}\left[f\circ X_t g\circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[\widehat{f}\circ \widehat{X}_t \widehat{g}\circ \widehat{X}_{\mathbf{T}_0}\right] \tag{B.4.8}$$

for all $t \in \mathbf{R}^+$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ by (B.4.6) and Lemma B.77

[38] $\int_0^t \overline{\mathcal{L}f}\circ \widehat{X}_u du$ in (B.4.7) is well-defined for $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ as explained in Remark B.85. It is well-defined for $\widehat{X} \in \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ by Fact 6.24.

(a) (with $\mathbf{T} = \mathbf{R}^+$ and $Y = \widehat{X}$). We have

$$\mathbb{E}\left[\left(\int_s^t \mathcal{L}f \circ X_u du\right) g \circ X_{\mathbf{T}_0}\right] = \mathbb{E}\left[\left(\int_s^t \overline{\mathcal{L}f} \circ \widehat{X}_u du\right) \widehat{g} \circ \widehat{X}_{\mathbf{T}_0}\right] \tag{B.4.9}$$

for all $s < t$ in $\mathbf{R}^+$, $g \in \mathfrak{ca}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0(\mathbf{R}^+)$ by Lemma B.84 (b) (with $Y = \widehat{X}$), the càdlàg properties of $\widehat{X}$, the continuity of $\widehat{g}$, the absolute continuity of Lebesgue integral, the boundedness of $\overline{\mathcal{L}f}$ and $\widehat{g}$, the denseness of $\mathbf{T}$ in $\mathbf{R}^+$ and Dominated Convergence Theorem. □

Fact B.88. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $\mu \in \mathcal{P}(E)$ be supported on E_0, $\{(\Omega^n, \mathscr{F}^n, \mathbb{P}^n; X^n)\}_{n\in\mathbf{N}}$ be E-valued processes satisfying*

$$\inf_{n\in\mathbf{N}} \mathbb{P}^n \left(\bigotimes \mathcal{F} \circ X_0^n \in \bigotimes \widehat{\mathcal{F}}(\widehat{E})\right) = 1 \tag{B.4.10}$$

and $\widehat{X}^n \in \mathfrak{rep}(X^n; E_0, \mathcal{F})$ for each $n \in \mathbf{N}$. If $\lim_{n\to\infty} \mathbb{E}^n [f \circ X_0^n] = f^*(\mu)$ for all $f \in \mathfrak{mc}(\mathcal{F}\backslash\{1\})$, then $\mathbb{P} \circ (\widehat{X}_0^n)^{-1} \Rightarrow \overline{\mu}$ as $n \uparrow \infty$ in $\mathcal{P}(\widehat{E})$. In particular, this is true when

$$\inf_{n\in\mathbf{N}} \mathbb{P}^n (X_0^n \in E_0) = 1. \tag{B.4.11}$$

Proof. (B.4.11) implies (B.4.10) by Fact 6.52 (with $\mathbf{T} = \{0\}$). It follows by the fact $\mu(E_0) = 1$, Proposition 6.8 (a) (with $X = X^n$ and $\mathbf{T} = \{0\}$) and Proposition 5.15 (a, b, e) (with $d = 1$) that $\overline{\mu} \in \mathcal{P}(\widehat{E})$ and

$$\lim_{n\to\infty} \mathbb{E}^n \left[\widehat{f} \circ \widehat{X}_0^n\right] = \widehat{f}^*(\overline{\mu}),\ \forall \widehat{f} \in \mathfrak{mc}(\widehat{\mathcal{F}}\backslash\{1\}). \tag{B.4.12}$$

Now, the result follows by Corollary 3.11 (a) (with $d = 1$ and $A = \widehat{E}$). □

Lemma B.89. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $d \in \mathbf{N}$, $X \in M_b(\Omega, \mathscr{F}; \mathbf{R})$[39] and $Y \in M(\Omega, \mathscr{F}; \widehat{E}^d)$. If $\mathbb{E}[X \widehat{f} \circ Y] = 0$ for all $\widehat{f} \in \mathfrak{mc}[\Pi^d(\widehat{\mathcal{F}})]$, then $\mathbb{E}[X|Y] = 0$.*

Proof. We fix $h \in M_b(\widehat{E}^d; \mathbf{R})$ and $\epsilon > 0$. $\mathbb{E}[X\widehat{f} \circ Y] = 0$ for all $\widehat{f} \in \mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})]$ by linearity of expectation. $\widehat{E}^d$ is a compact Polish space by Lemma 3.9 (c), so $\mathbb{P}\circ Y^{-1}$ is automatically a regular Borel measure. $\mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})]$ is uniformly dense in $C_b(\widehat{E}^d; \mathbf{R})$ by Corollary 3.10. Then, there exists a $\widehat{f}_{h,\epsilon} \in \mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})]$ and a

[39] Recall that $X \in M_b(\Omega, \mathscr{F}; \mathbf{R})$ means X is a bounded $\mathbf{R}$-valued random variable defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

$K_{h,\epsilon} \in \mathscr{K}(\widehat{E}^d)$ such that $\widehat{f}_{h,\epsilon}|_{K_{h,\epsilon}} = h|_{K_{h,\epsilon}}$, $\|\widehat{f}_{h,\epsilon}\|_\infty \leq \|h\|_\infty$ and $\mathbb{P}(Y \in K_{h,\epsilon}) \geq 1 - \epsilon$ by Lusin's Theorem ([Dudley, 2002, Theorem 7.5.2]), Tietz Extension Theorem ([Munkres, 2000, Theorem 35.1 (b)]) and the denseness of $\mathfrak{ag}[\Pi^d(\widehat{\mathcal{F}})]$ in $C_b(\widehat{E}^d; \mathbf{R})$. Using Jensen's Inequality, we have that

$$|\mathbb{E}[Xh \circ Y]| \leq \mathbb{E}\left[\mathbf{1}_{\{Y \notin K_{h,\epsilon}\}} \left|X(h - \widehat{f}_{h,\epsilon}) \circ Y\right|\right] \leq 2\|h\|_\infty \|X\|_\infty \epsilon. \quad \text{(B.4.13)}$$

Letting $\epsilon \downarrow 0$, we have $\mathbb{E}[Xh \circ Y] = 0$ by and the boundedness of h and X. Now, the result follows by [Dudley, 2002, Theorem 4.2.8]. □

Lemma B.90. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and X, Y and Z be E-valued, $\widehat{E}$-valued and $\mathbf{R}$-valued processes defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$, respectively. In addition, suppose that $Z_t - Z_s \in M_b(\Omega, \mathscr{F}; \mathbf{R})$ for all $s < t$ in $\mathbf{R}^+$. Then, the following statements are true:*

(a) If Z is $\mathscr{F}_t^Y$-adapted and $\mathbb{E}[(Z_t - Z_s)\widehat{f} \circ Y_{\mathbf{T}_0}] = 0$ for all $t > s$ in $\mathbf{R}^+$, $\widehat{f} \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\widehat{\mathcal{F}})]$ and $\mathbf{T}_0 \in \mathscr{P}_0([0, s])$, then Z is an $\mathscr{F}_t^Y$-martingale.

(b) If Z is $\mathscr{F}_t^X$-adapted, $\mathbb{E}[(Z_t - Z_s)f \circ X_{\mathbf{T}_0}] = 0$ for all $t > s$ in $\mathbf{R}^+$, $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0([0, s])$, and

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}(X_t \in A) = 1 \quad \text{(B.4.14)}$$

for some $A \in \mathscr{B}^{\mathbf{s}}(E)$ with $A \subset E_0$, then Z is an $\mathscr{F}_t^X$-martingale.

Proof. (a) Fixing $t > s$ in $\mathbf{R}^+$, $f \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\mathcal{F})]$ and $\mathbf{T}_0 \in \mathscr{P}_0([0, s])$, we have $\mathbb{E}[Z_t - Z_s | Y_{\mathbf{T}_0}] = 0$ by Lemma B.89 (with $X = Z_t - Z_s$, $Y = Y_{\mathbf{T}_0}$ and $d = \aleph(\mathbf{T}_0)$). Letting $\mathbf{T}_0 \in \mathscr{P}_0([0, s])$ be arbitrary, we have $\mathbb{E}[Z_t - Z_s | \mathscr{F}_s^Y] = 0$.

(b) There exists an $\widehat{X} \in \mathfrak{rep}(X; E_0, \mathcal{F})$ satisfying (6.3.18) by Proposition 6.6 (a) and Proposition 6.7 (a) (with $\mathbf{T} = \mathbf{R}^+$). It follows by (6.3.18), (B.4.14), the fact $A \subset E_0$ and Lemma B.77 (a, e) (with $Y = \widehat{X}$) that $\mathscr{F}^X = \mathscr{F}^{\widehat{X}}$ and $\mathbb{E}[(Z_t - Z_s)\widehat{f} \circ \widehat{X}_{\mathbf{T}_0}] = \mathbb{E}[(Z_t - Z_s)f \circ X_{\mathbf{T}_0}] = 0$ for all $t > s$ in $\mathbf{R}^+$, $\widehat{f} \in \mathfrak{mc}[\Pi^{\mathbf{T}_0}(\widehat{\mathcal{F}})]$ and $\mathbf{T}_0 \in \mathscr{P}_0([0, s])$. Now, (b) follows by (a) (with $Y = \widehat{X}$). □

Lemma B.91. *Let $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E, $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t \geq 0}, \mathbb{P})$ be a stochastic basis and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an E-valued, $\mathscr{G}_t$-adapted, measurable process. Then, the following statements are true:*

(a) If (6.1.14) holds with $\mathbf{T} = \mathbf{R}^+$*, then there exists an* $\widehat{X} = \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ *that is* $\mathscr{G}_t$*-adapted and satisfies (B.4.6) and*

$$\mathbb{E}\left[f \circ X_t | \mathscr{G}_s\right] = \mathbb{E}\left[\widehat{f} \circ \widehat{X}_t | \mathscr{G}_s\right] \tag{B.4.15}$$

for all $s \leq t$ *in* $\mathbf{R}^+$ *and* $f \in \mathfrak{ca}(\mathcal{F})$*. In particular, if (6.1.9) holds, then (6.3.18) holds.*

(b) If X *satisfies (B.4.14) for some* $A \in \mathscr{B}^{\mathbf{s}}(E)$ *with* $A \subset E_0$*, then then there exists an* $\widehat{X} = \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ *that is* $\mathscr{G}_t$*-adapted and satisfies (B.4.6), (6.3.18), (B.4.15) and*[40]

$$\mathbb{P}\left(h \circ X_t = \overline{h\mathbf{1}_A} \circ \widehat{X}_t\right) = 1 \tag{B.4.16}$$

for all $s \leq t$ *in* $\mathbf{R}^+$*,* $f \in \mathfrak{ca}(\mathcal{F})$*,* $h \in M_b(E; \mathbf{R}^k)$ *and* $k \in \mathbf{N}$*. In particular,*

(c) If X *is* $(\mathbf{R}^+, \mathcal{F})$*-càdlàg and satisfies (6.1.14) for some dense* $\mathbf{T} \subset \mathbf{R}^+$*, then there exists an* $\widehat{X} = \mathfrak{rep}_{\mathrm{c}}(X; E_0, \mathcal{F})$ *that is* $\mathscr{G}_t$*-adapted and satisfies (B.4.6) and (B.4.15) for all* $s \leq t$ *in* $\mathbf{R}^+$ *and* $f \in \mathfrak{ca}(\mathcal{F})$*. In particular, if* X *satisfies (6.1.9), then (6.1.10) holds.*

Proof. (a) There exists an $\widehat{X} \in \mathfrak{rep}_{\mathrm{m}}(X; E_0, \mathcal{F})$ satisfying (B.4.6) by Proposition 6.6 (b) and Proposition 6.8 (a). For each fixed $t \in \mathbf{R}^+$, (B.4.6) implies $\mathbb{P}(\widehat{X}_t = Z) = 1$ with $Z \doteq (\bigotimes \widehat{\mathcal{F}})^{-1} \circ \bigotimes \mathcal{F} \circ X_t$. $Z \in M(\Omega, \mathscr{G}_t; \widehat{E})$ by the fact $X_t \in M(\Omega, \mathscr{G}_t; E)$ and Lemma 3.3 (a, e). $\widehat{X}_t \in M(\Omega, \mathscr{G}_t; \widehat{E})$ by Lemma B.31 (a) (with $E = S = \widehat{E}$, $\mathscr{F} = \mathscr{G}_t$, $\mathscr{U} = \mathscr{B}(\widehat{E})$ and $X = \widehat{X}_t$). Hence, $\widehat{X}$ is $\mathscr{G}_t$-adapted. (B.4.15) follows by (B.4.6) and Lemma B.77 (a) (with $Y = \widehat{X}$, $\mathbf{T}_0 = \{t\}$ and $\mathbf{T} = \mathbf{R}^+$). Now, (a) follows by Proposition 6.7 (a) (with $\mathbf{T} = \mathbf{R}^+$).

(b) Let $\widehat{X}$ be as in (a) and fix $h \in M_b(E; \mathbf{R}^k)$. $\overline{h\mathbf{1}_A} \in M_b(\widehat{E}; \mathbf{R}^k)$ by Proposition 4.6 (b) (with $f = h\mathbf{1}_A$). It then follows by (B.4.14) and Lemma B.77 (f) (with $Y = \widehat{X}$, $f = h\mathbf{1}_A$, $\mathbf{T}_0 = \{t\}$ and $\mathbf{T} = \mathbf{R}^+$) that

$$\mathbb{P}\left(h \circ X_t = h\mathbf{1}_A \circ X_t = \overline{h\mathbf{1}_A} \circ \widehat{X}_t\right) = 1. \tag{B.4.17}$$

[40] $\overline{h\mathbf{1}_A}$ denotes the function $\mathfrak{var}(h1_A; E, A, 0)$.

(c) One establishes $\widehat{X}$ by Proposition 6.28 (a), then (c) follows by a similar argument to (a). □

Lemma B.92. *Let E be a topological space, $(E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ be a base over E and $(\Omega, \mathscr{F}, \mathbb{P}; X)$ be an $\mathcal{M}^+(E)$-valued process satisfying (6.1.34). In addition, suppose that $\varpi(f^*) \circ X$ has a measurable modification ζ^f for all $f \in \mathfrak{mc}(\mathcal{F})$. Then, there exists an $\mathcal{M}^+(\widehat{E})$-valued $\mathscr{F}_t^X$-progressive process $(\Omega, \mathscr{F}, \mathbb{P}; Y)$ satisfying Corollary 6.12 (a) - (d).*

Proof. We set φ, $\widehat{\varphi}$, y_0 and Ψ as in Lemma 6.10, define $S_0 \in \mathscr{B}(\mathbf{R}^\infty)$ as in the proof of Corollary 6.12 and let $Y = \varpi(\Psi \circ \varphi) \circ X$. Then, from the proof of Corollary 6.12 we know that Y is an $\mathcal{M}^+(\widehat{E})$-valued $\mathscr{F}_t^X$-adapted process satisfying Corollary 6.12 (a) - (d). Then, $\varpi(\widehat{\varphi}) \circ Y$ is an S_0-valued process by (6.1.20).

ζ^f is a modification of $\varpi(\widehat{f}^*) \circ Y$ by Corollary 6.12 (a). $\mathfrak{mc}(\mathcal{F})$ is a countable collection as mentioned in the proof of Lemma 6.10, hence $\zeta \mathrel{\mathring{=}} \{\bigotimes_{f \in \mathfrak{mc}(\mathcal{F})} \zeta_t^f\}_{t \geq 0}$ is a measurable modification of $\varpi(\widehat{\varphi}) \circ Y$ by Fact B.32 (b) (with $i = f$ and $X^i = \zeta^f$) and so

$$\inf_{t \in \mathbf{R}^+} \mathbb{P}(\zeta_t \in S_0) = 1. \tag{B.4.18}$$

$S_0 \in \mathscr{B}(\mathbf{R}^\infty) \cap \mathscr{B}^{\mathrm{s}}(\mathbf{R}^\infty)$ by the fact $S_0 \in \mathscr{B}(\mathbf{R}^\infty)$, Note 6.5 and Proposition A.56 (b) (with $E = \mathbf{R}^\infty$). Therefore, the ζ above admits an S_0-valued progressive modification ζ' by (B.4.18) and Corollary 7.10 (with $E = \mathbf{R}^\infty$, $E_0 = S_0$ and $X = \zeta$). Recall that $\widehat{\varphi}$ satisfies (6.1.26). Thus, $\varpi(\widehat{\varphi}^{-1}) \circ \zeta'$ is an $\mathscr{F}_t^X$-progressive modification of Y by Fact B.32 (a) (with $E = (S_0, \mathscr{O}_{\mathbf{R}^\infty}(S_0))$, $S = \mathcal{M}^+(\widehat{E})$, $f = \widehat{\varphi}^{-1}$, $X = \zeta'$ and $\mathscr{G}_t = \mathscr{F}_t^X$). Now, we retake Y as $\varpi(\widehat{\varphi}^{-1}) \circ \zeta'$ and observe that: (1) Corollary 6.12 (a, b, d) are preservable among and still hold, (2) If (6.1.35) holds, then the S_0 above is $\widehat{\varphi}[\mathcal{P}(\widehat{E})]$ and so $\varpi(\widehat{\varphi}^{-1}) \circ \zeta'$ will be a desired $\mathcal{P}(\widehat{E})$-valued process. □