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TITLE OF THESIS STABILITY STUDIES ON NONLINEAR DISCRETE-DATA SYSTEMS THROUGH FUNCTIONAL ANALYSIS

UNIVERSITY ALBERTA

DEGREE FOR WHICH THESIS WAS PRESENTED Ph.D.

YEAR THIS DEGREE GRANTED 1969

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DATED September 17, 1969

THE UNIVERSITY OF ALBERTA

STABILITY STUDIES ON NONLINEAR DISCRETE-DATA SYSTEMS THROUGH FUNCTIONAL ANALYSIS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

FALL, 1969

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

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ABSTRACT

This thesis deals with the study of bounded-input bounded-output (BIBO) stability of certain classes of nonlinear discrete-data systems using the methods of Functional Analysis. Sufficient conditions for the BIBO stability of such systems are established employing the contraction mapping principle. The systems considered include those with the nonlinearity in the feedback path, as well as in the forward path. A system with a slope restricted nonlinearity is also studied. The method for finding the input and output bounds for stability in a special case of nonlinearity is discussed. Finally a method for generating the system solution by a process of successive approximations is given. It is shown that the solution comes out in the form of a discrete Volterra type series, whose convergence is assured in the region in which the BIBO stability conditions derived, are satisfied.

ACKNOWLEDGEMENTS

The author is deeply indebted to Dr. G.S. Christensen, the supervisor of this project, for his invaluable assistance and guidance throughout the course of this work. He would like to thank Dr. J.W. Macki, of the Department of Mathematics for constructive criticisms and helpful suggestions, and Professsor Y.J. Kingma, for initiating him into the study of sampled-data systems.

Further he wishes to acknowledge with gratitude the help and encouragement he received from Dr. G.B. Walker, the Head of this Department, throughout his stay in this University. Special thanks go to Dr. V. Gourishanker, who has been largely responsible for the author's decision to launch on this academic adventure, and whose friendly counsel has always been a source of great encouragement.

Acknowledgement is gratefully given to the National Research Council of Canada for providing assistance for the summer of 1967, and to the University of Alberta for awarding the Teaching Assistantship (1967-68), the Intersession Bursary (1968 Summer) and the Dissertation Fellowship (1968-69).

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LIST OF MAJOR SYMBOLS

r	$= r(t)$, continuous input.
c	$= c(t)$, continuous output.
e	$= e(t)$, continuous error.
r^*	$= r^*(t)$, sampled input.
c^*	$= c^*(t)$, sampled output.
e^*	$= e^*(t)$, sampled error.
T	Period of sampling.
$x(nT)$	Value of sampled variable x at the sampling instant nT , $n = 0, 1, \ldots, \infty$.
$\delta(t-nT)$	Unit impulse occurring at $t = nT$. $=1$ when $t = nT$, and 0 when $t \neq nT$.
x^*	$= x^*(t) = \sum_{n=o}^{\infty} x(nT)\delta(t - nT)$, using the δ notation.
	$= \{x_n\}_{n=o}^{\infty}$, where $x_n = x(nT)$, using the sequence notation.
s	Laplace Transform variable.
$x(s)$	Laplace Transform of the continuous function $x(t)$.
$x^*(s)$	Laplace Transform of the sampled function $x^*(t)$
z	Z-transform variable.
$x(z)$	Z-transform of $x^*(s)$.
$G(s)$	Transfer function of the linear plant G .

$G(z)$	Z-transfer function of the linear plant G .
$H(z)$	Overall Z-transfer function of the linearized system.
$h(kT)$	Value of the sampled impulse response of the linearized system at $t = kT$, $k = 0, 1, \ldots, \infty$.
$d(x_1,x_2)$	Distance between x_1 and x_2 .
F	Banach space.
$\lvert\cdot\rvert$	Absolute value sign.
$\lVert\cdot\rVert$	Norm sign.
R*	$= \lVert r^* \rVert$, norm of r^* .
C*	$= \lVert c^* \rVert$, norm of c^* .
B*	Convolution summation operator.
H*	$= \lVert B^* \rVert$, norm of the operator B^* .
A	Nonlinear operator.
λ	Lipchitz constant, $0 < \lambda < 1$.
a	radius of the sphere S .
$\in$	Belongs to.
$<$	Less than
$\leq$	Less than or equal to.
$>$	Greater then
$\geq$	Greater than or equal to
Sgn	Signum, $\text{Sgn } x = \frac{x}{\lvert x \rvert}$.
BIBO	Bounded-input bounded-output.
ε	Arbitrarily small real number > 0 .

CHAPTER I

INTRODUCTION

Discrete-data systems, or sampled-data systems as they are commonly called, have come to play a significant role in present day control technology. Although the use of sampled-data in control systems was recognized quite early in the history of feed-back control systems, interest in the analysis and design of sampled-data systems began perhaps in the early forties, when during the war, problems connected with radar tracking systems were encountered. The advent of the digital computers and their incorporation into control systems have given a fillip to the study of sampled-data systems, as the basic operation of such computers by necessity tends to reduce them to sampled-data systems.

An important area of investigation into the behavior of these systems is the determination of conditions for system stability. For this purpose sampled-data systems may be broadly divided into two categories: the linear sampled-data systems and the nonlinear sampled-data systems. For linear sampled-data systems the formulation of both necessary and sufficient conditions for system stability have been exhaustively investigated. However, many sampled-data systems of practical importance are nonlinear for which formulation of conditions for system stability is far from being complete, although it has focussed the attention of many recent investigators.

Much of the work done in this field deals with the direct application of Liapunov's second method extended to systems of difference equations[1*]. Although this is the most general of all known methods of stability analysis of nonlinear systems, it suffers from some arbitrariness in the choice of suitable functions.

A few years ago the Rumanian scientist V.M. Popov developed an entirely new approach to the classical problem of absolute stability by working in the frequency domain. Popov's results which originally applied to autonomous continuous systems with nonlinearities confined to specified gain sectors, have been extended to discrete-data systems by Tsypkin[2]. Jury and Lee[3-5] further extended the application of Popov's method to nonlinear sampled-data systems, and by placing constraints on the slope of the nonlinearity obtained less conservative results than did Tsypkin. All these stability criteria could only predict absolute stability of the null solution, which means that sampled inputs which tend to zero fast enough produce outputs which tend to zero.

Another very important kind of stability is the requirement that bounded inputs produce bounded outputs, usually referred to as bounded-input bounded-output (BIBO) stability. This thesis is primarily concerned with the stability of certain classes of nonlinear discrete-data systems in the BIBO sense. For linear systems uniform asymptotic stability of the autonomous system implies BIBO stability as a direct consequence. In general this is not true for nonlinear systems and many counterexamples have been constructed[6]. Also a nonlinear system can exhibit local BIBO stability even though it may not be BIBO stable in a global sense.

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* Numbers placed above the line of text refer to the references.

In a recent paper Iwens and Bergen[7] have shown that the condition given by Jury and Lee[3] for the absolute asymptotic stability of certain classes of autonomous nonlinear sampled-data systems establishes absolute BIBO stability as well. Apart from this reference, available technical literature provides little evidence of much work having been done in the area of BIBO stability of nonlinear discrete-data systems.

In this thesis the problem is approached in an altogether different way. The method of analysis is based on the fixed-point property of certain contraction mappings in Banach space. Although this concept is well established in Functional Analysis and cognate branches of modern mathematics, its application to the study of stability and convergence problems arising in nonlinear system analysis is relatively new. Observations on the use of this approach has been made by several authors in the past - Kalman and Bertram[1], Zames[8], and Sandberg[9], to name a few. On the application side papers by Desoer[10], Leon and Anderson[11] and Leibovic[12] may be mentioned. Recently the BIBO stability of a class of nonlinear continuous data systems has been successfully investigated by Christensen[13] using the same principle. However, as far as the BIBO stability of nonlinear discrete-data systems is concerned, this approach does not seem to have been attempted so far, though methods based on contraction mapping have been used in a different way to investigate asymptotic stability in the large of such systems[14,15].

The objective of this thesis is to extend the technique of contraction mapping to obtain sufficient conditions for the BIBO stability of different classes of nonlinear discrete-data systems. Some of the results have been presented by the author in a paper entitled "Contraction mapping applied to stability analysis of a class of nonlinear discrete-data systems", at the Second Asilomar Conference on Circuits and Systems, October 1968[16].

Chapter II considers the stability of linear discrete-data systems from a functional point of view. The mathematical concepts and functional nontations involved are introduced in this chapter.

Chapter III deals with the stability analysis of nonlinear discrete-data systems with a polynomial type of nonlinearity. The contraction mapping theorem is stated and proved, and its application for obtaining sufficient conditions for the BIBO stability of the systems considered is explained.

In Chapter IV the method of computation of input and output bounds in a special case of the nonlinearity, is discussed. Numerical examples illustrating the procedure are also given.

The analysis of systems with a slope restricted nonlinearity is treated in Chapter V. A criterion for the BIBO stability of such a system is stated and proved. Its application is illustrated with the aid of a numerical example and the results compared with those obtained by the criterion of Jury and Lee[3].

Chapter VI considers the generation of the actual system solution by a process of successive approximations, leading to a discrete Volterra type series, and shows how the stability conditions derived establishes the convergence of such a series.

In conclusion the advantages and disadvantages of the method of contraction mapping are discussed, and suggestions for future investigations in this field given.

CHAPTER II

STABILITY OF LINEAR DISCRETE-DATA SYSTEMS

2.1 Introduction.

The main aim of this chapter is to discuss the BIBO stability of linear discrete-data systems from a functional point of view, as this forms the basis for the development of the conditions for the stability of the nonlinear discrete-data systems considered later. This is essentially a time domain approach, and involves the representation of the system as a linear mapping in a function space and defining a norm for the linear operator which effects the transformation. Before introducing the functional notation, however, the more conventional time domain representation of the system will be first discussed, as also the Z-domain representation.

2.2 Conventional Time Domain and Z-Domain Representation of Linear Discrete-Data Systems[17-19].

A conventional linear single-loop discrete-data system is shown in Fig. 2.1.

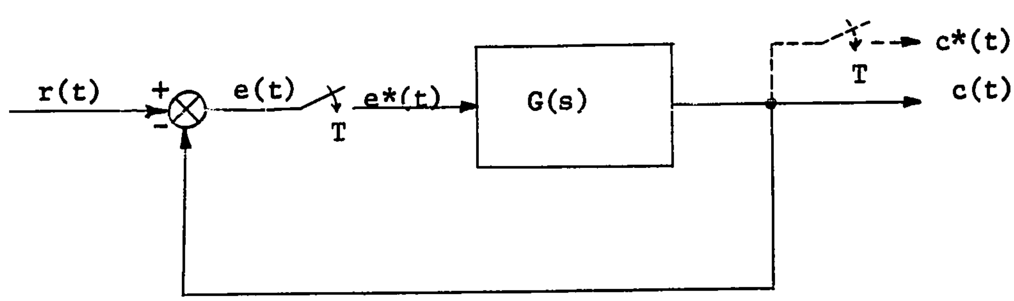

Fig. 2.1 A Linear Discrete-Data System.

G(s) is the transfer function of the linear plant assumed to be situated in the forward path. The sampler with uniform period T , and shown as a switch, is a device such that:

$$e^*(t) = e(t) \sum_{n=o}^{\infty} \delta(t-nT)$$

$$= \sum_{n=o}^{\infty} e(nT)\delta(t-nT) \quad . \qquad (2.1)$$

That is it produces a train of impulses, the area of which being the value of the sampled variable at the sampling instant nT . Note that the input to the sampler is the continuous error function e(t) assumed to be zero for $t < 0$, whereas the output is a discrete valued function of t , defined at the discrete instants $t = nT$, $n = 0,1,2,\ldots,\infty$.

If now equation (2.1) is Laplace transformed one gets:

$$e^*(s) = \sum_{n=o}^{\infty} e(nT)e^{-nTs} \qquad (2.2)$$

since the Laplace transform of the impulse $\delta(t-nT)$ is equal to e^{-nTs} .

Letting $e^{Ts} = z$, equation (2.2) may be written as:

$$e^*(s)\Big|_{s = \frac{1}{T}\log z} = e(z) = \sum_{n=o}^{\infty} e(nT)z^{-n} \quad . \qquad (2.3)$$

Equation (2.3) defines the Z-transform of the input function e(t) .

Consider now the forward path of the system shown in Fig. 2.1. The input to G(s) is e*(t) , a train of impulses with amplitudes e(nT) at $t = nT$, whereas the output is c(t) , a continuous function of time. Since our analysis is confined to the discrete function space, a fictitious sampler is introduced at the output synchronized with the input sampler for obtaining the sampled output, which is denoted by $c^*(t)$. A relation between $c^*(t)$ and $e^*(t)$ may be obtained as follows:

Suppose a unit impulse is applied to $G(s)$ at $t = 0$, the output response would be $g(t)$, the impulse response of G , and the output of the fictitious sampler would be:

$$g^*(t) = \sum_{n=o}^{\infty} g(nT)\delta(t-nT) \quad . \tag{2.4}$$

Hence if $e^*(t)$ is applied to G , the output sample $c(nT)$ at $t = nT$, would be the effects of all samples $e(nT)$, $e(nT-T)$, ..., $e(T)$, $e(0)$. That is:

$$c(nT) = e(0)g(nT) + e(T)g(nT-T) + \ldots + e(nT)g(0) \quad . \tag{2.5}$$

Equation (2.5) may also be written as:

$$\begin{aligned} c(nT) &= \sum_{k=o}^{n} e(kT)g(nT-kT) \\ &= \sum_{k=o}^{n} g(kT)e(nT-kT) \end{aligned} \tag{2.6}$$

$c^*(t)$ will be then given by:

$$c^*(t) = \sum_{n=o}^{\infty} \left[\sum_{k=o}^{n} e(kT)g(nT-kT) \right] \delta(t-nT) \tag{2.7a}$$

$$= \sum_{n=o}^{\infty} \left[\sum_{k=o}^{n} g(kT)e(nT-kT) \right] \delta(t-nT) \quad . \tag{2.7b}$$

Thus the output samples can be obtained from the input samples by a process of summation defined by equation (2.7a) or (2.7b). This operation is called the convolution summation.

The foregoing can be related to the Z-domain approach in the following way:

Taking the Laplace transform of equation (2.4) yields:

$$G^*(s) = \sum_{n=o}^{\infty} g(nT)e^{-nTs} \tag{2.8}$$

i.e.

$$G(z) = \sum_{n=o}^{\infty} g(nT)z^{-n} \quad . \tag{2.9}$$

Equation (2.9) defines the Z-transfer function of $G(s)$.

Now multiplying both sides of equation (2.5) by e^{-nTs} , and taking the summation for $n = 0$ to ∞ :

$$\sum_{n=o}^{\infty} c(nT)e^{-nTs} = \sum_{n=o}^{\infty} e(0)g(nT)e^{-nTs} + \sum_{n=o}^{\infty} e(T)g(nT-T)e^{-nTs} + \ldots + \sum_{n=o}^{\infty} e(nT)g(0)e^{-nTs} \tag{2.10}$$

or

$$\sum_{n=o}^{\infty} c(nT)e^{-nTs} = \sum_{n=o}^{\infty} g(nT)e^{-nTs} [e(0) + e(T)e^{-Ts} + + e(2T)e^{-2Ts} + \ldots] \tag{2.11}$$

from which,

$$\sum_{n=o}^{\infty} c(nT)e^{-nTs} = \sum_{n=o}^{\infty} g(nT)e^{-nTs} \sum_{n=o}^{\infty} e(nT)e^{-nTs} \tag{2.12}$$

or

$$c^*(s) = G^*(s)e^*(s) \quad . \tag{2.13}$$

Using the Z-transform notation, (2.13) may be written as:

$$c(z) = G(z)e(z) \quad . \tag{2.14}$$

Equation (2.14) is the Z-domain equivalent of equation (2.7a). Thus it is seen that convolution in the time domain is equivalent to multiplication in the Z-domain.

Now consider the closed-loop system of Fig. 2.1. To obtain the sampled response to any arbitrary input $r(t)$, one may proceed as follows: It is easy to see that,

$$e(z) = r(z) - c(z) \quad . \tag{2.15}$$

Substituting (2.15) into (2.14) and simplifying:

$$c(z) = \frac{G(z)}{1 + G(z)} \cdot r(z) \quad . \tag{2.16}$$

Putting $\frac{G(z)}{1 + G(z)} = H(z)$ one may write:

$$c(z) = H(z)r(z) \tag{2.17}$$

a relation which is analogous to equation (2.14). Here $H(z)$ is called the overall Z-transfer function of the closed-loop system. The corresponding time domain relationship between $c^*(t)$ and $r^*(t)$ will then be given by:

$$c^*(t) = \sum_{n=o}^{\infty} \left[\sum_{k=o}^{n} h(kT)r(nT-kT) \right] \delta(t-nT) \tag{2.18}$$

where $h(kT) = Z^{-1}H(z)$, the inverse Z-transform of $H(z)$. Note that

$h(kT)$ is the same as the sampled impulse response of the closed-loop system at $t = kT$. Given $H(z)$ one method of obtaining $h(kT)$ is to apply Cauchy's integral formula:

$$h(kT) = \frac{1}{2\pi j} \oint_{\Gamma} H(z) z^{k-1} dz$$

$$= \text{Sum of the residues of } H(z)z^{k-1} \text{ at the poles of } H(z)z^{k-1} . \qquad (2.19)$$

2.3 The Functional Representation[20,21].

The natural representation for any system is a correspondence between input and output elements. The classical time domain relationship between the sampled input and the sampled output given by equation (2.18) for the linear discrete-data system of Fig. 2.1, for example is one such representation. Here the emphasis is on the response of a particular system to a particular input. In order to obtain a qualitative understanding of the properties and behavior of systems in general, a characterization that yields a mathematical model compatible with a large class of systems and a large class of inputs is called for. This is the aim of the functional approach. This approach utilizes as its tools the concept of abstract spaces and operations on such spaces. The input and output functions are identified as belonging to an abstract space, endowed with a mathematical structure suited to the type of analysis desired, and the behavior of the system studied in this setting.

The space best suited for the purpose of our analysis is the Banach space. This space is chosen as it facilitates the application of the contraction mapping theorem, when the stability of the nonlinear discrete-data systems is considered. The following definitions will clarify what is meant by a Banach space.

Definition 1: A collection of elements together with a certain structure of relations between elements or of rules of manipulation and combination, the whole supporting a mathematical development is often called a space.

Definition 2: A space X is said to be a linear (or vector) space if addition and scalar multiplication are defined on X satisfying the commutative, associative and distributive laws. The scalar multiplication is related to some associated field of scalars, usually denoted by $\mathcal{F}$.

Definition 3: A space X is said to be a metric space, if we associate with any two points x_1, x_2 of X a distance function or metric such that:

(a) $d(x_1,x_2) > 0$, if $x_1 \neq x_2$.

(b) $d(x_1,x_2) = 0$, if and only if $x_1 = x_2$.

(c) $d(x_1,x_2) = d(x_2,x_1)$.

(d) $d(x_1,x_2) \leq d(x_1,x_3) + d(x_3,x_2)$ for every $x_3 \in X$.

Definition 4: A norm on a linear space X is a real valued function, whose value at x we denote by $\|x\|$, with the properties:

(a) $\|x\| \geq 0$.

(b) $\|x\| \neq 0$, if $x \neq 0$.

(c) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.

(d) $\|ax\| = |a|\|x\|$, for all $a \in \mathcal{F}$.

Definition 5: A linear space on which a norm is defined becomes a metric space if we define $d(x_1,x_2) = \|x_1 - x_2\|$. A linear space which is a metric space in this way is called a normed linear space or a normed vector space.

Definition 6: A sequence $\{x_n\}$ in a metric space is called a Cauchy sequence if $d(x_n,x_m) \to 0$ as m and $n \to \infty$.

Definition 7: If a metric space X has the property that every Cauchy sequence in X has a limit in X , X is said to be complete.

Definition 8: If a normed linear space is complete it is called a Banach space.

The input and output functions $r^*(t)$ and $c^*(t)$ are real valued functions of time defined only at the discrete instants $t = nT$, $n = 1,2,\ldots,\infty$, on the interval $-\infty < t < \infty$. They are thus sequences in a sequence space, say Y , which is a linear space. The definitions of addition and multiplication by scalars in this space are as follows:

Addition: If $x^* = \{x_n\}_{n=o}^{\infty}$ and $y^* = \{y_n\}_{n=o}^{\infty}$, then $z^* = x^* + y^*$, where $z^* = \{z_n\}_{n=o}^{\infty}$, $z_n = x_n + y_n$.

Multiplication: $ax^* = a\{x_n\}_{n=o}^{\infty} = \{ax_n\}_{n=o}^{\infty}$, $a \in \mathcal{F}$.

If furthermore, $r^*(t)$ and $c^*(t)$ are bounded, then they belong to the space of bounded sequences, say F , which is a subspace of Y . F is a Banach space if we define the norm:

$$\|x^*\| = \sup_{-\infty<t<\infty} |x^*| , \quad x^* \in F . \qquad (2.20)$$

Hereafter $r^*(t)$, $c^*(t)$ will be written as r^*, c^* and their norms as R^*, C^* if $r^*, c^* \in F$.

In the general situation when $r^*, c^* \in Y$, the relation between them may be looked upon as a transformation in the linear space Y, and hence can be represented by the functional equation:

$$c^* = B^*r^* \tag{2.21}$$

where B^* is an operator which maps the linear space Y into itself. This operator is said to be linear, if:

(a) $B^*(x_1^* + x_2^*) = B^*x_1^* + B^*x_2^*$, any $x_1^*, x_2^* \in Y$.

(b) $B^*(ax^*) = aB^*x^*$, any $x^* \in Y$, and any $a \in \mathcal{F}$.

B^* is then called a linear operator on Y.

For the discrete-data system shown in Fig. 2.1, the input-output relationship described by the equation (2.18) can be easily verified to be a linear operation in Y. Hence we can represent this relationship by the functional equation (2.21), where B^* is a linear operator defined by equation (2.18). B^* will be called the convolution summation operator.

Now suppose, $r^*, c^* \in F$, then B^* becomes a linear operator in a Banach space, and maps $F \to F$. How this concept is tied in with the stability of the linear system, and further properties of B^*, if the system is stable, are discussed in the next section.

2.4 Stability of Linear Discrete-Data Systems.

The following definition of stability will be adopted in this discussion: A discrete-data system is said to be stable in the BIBO sense if for any bounded input the corresponding sampled output is bounded, and for every real $\varepsilon > 0$ there exists a real $\delta(\varepsilon) > 0$ such that whenever two inputs r_1^*, r_2^* are less than $\delta(\varepsilon)$ apart, i.e. $\|r_1^* - r_2^*\| < \delta(\varepsilon)$, the corresponding outputs c_1^*, c_2^* are less than ε apart, i.e. $\|c_1^* - c_2^*\| < \varepsilon$.

Referring now to the linear mapping $c^* = B^*r^*$, the first part of the above definition implies that, if the system is stable, r^*, c^* belong to the space of bounded sequences; i.e. they belong to the Banach space F, and $B^* : F \to F$. The second part of the definition further demands that the mapping $B^* : F \to F$ is continuous. Since B^* is linear and continuous, B^* is bounded; i.e. a constant $M < \infty$ exists such that:

$$\|B^*r^*\| < M\|r^*\| \quad . \tag{2.22}$$

The smallest M for which (2.22) holds is called the norm of the linear operator B^*, denoted by $\|B^*\|$.

According to the above definition $\|B^*\|$ possesses the following properties:

$$\|B^*r^*\| \leq \|B^*\|\|r^*\| \tag{2.23}$$

for any arbitrary r^*, and there exists an r_ε^* for each $\varepsilon > 0$ such that:

$$\|B^*r_\varepsilon^*\| > (\|B^*\| - \varepsilon)\|r_\varepsilon^*\| \quad . \tag{2.24}$$

An alternate expression for the norm of B^* is:

$$\|B^*\| = \sup_{\|r^*\| \leq 1} \|B^* r^*\| \quad . \tag{2.25}$$

This can be shown to be true as follows:[22]

If $\|r^*\| \leq 1$, then:

$$\|B^* r^*\| \leq \|B^*\| \|r^*\| \leq \|B^*\|$$

and therefore,

$$\sup_{\|r^*\| \leq 1} \|B^* r^*\| \leq \|B^*\| \tag{2.26}$$

on the other hand there exists an r_ε^* for every $\varepsilon > 0$, such that:

$$\|B^* r_\varepsilon^*\| > (\|B^*\| - \varepsilon)\|r_\varepsilon^*\| \quad .$$

If we put

$$r_1^* = \frac{r_\varepsilon^*}{\|r_\varepsilon^*\|}$$

then

$$\begin{aligned} \|B^* r_1^*\| &= \frac{1}{\|r_\varepsilon^*\|} \|B^* r_\varepsilon^*\| \\ &> \frac{1}{\|r_\varepsilon^*\|} (\|B^*\| - \varepsilon)\|r_\varepsilon^*\| \\ &= \|B^*\| - \varepsilon \quad . \end{aligned}$$

Since $\|r_1^*\| = 1$, we have:

$$\sup_{\|r^*\| \leq 1} \|B^* r^*\| \geq \|B^* r_1^*\| \geq \|B^*\| - \varepsilon$$

and therefore:

$$\sup_{\|r^*\| \leq 1} \|B^*r^*\| \geq \|B^*\| \quad . \tag{2.27}$$

From (2.26) and (2.27) it immediately follows that:

$$\|B^*\| = \sup_{\|r^*\| \leq 1} \|B^*r^*\| \quad . \tag{2.28}$$

2.5 Computation of $\|B^*\|$ for a Given Linear Discrete-Data System.

For computational purpose it is necessary to evaluate an expression for the norm of B^* in terms of the system parameters for any given linear discrete-data system, known to be stable. To do this we go back to the explicit time domain representation of the linear discrete-data system given by equation (2.18), viz.:

$$c^* = \sum_{n=o}^{\infty} \left[\sum_{k=o}^{n} h(kT) r(nT-kT) \right] \delta(t-nT) \quad . \tag{2.29}$$

From (2.29):

$$\begin{aligned}
\|c^*\| &= \sup_{-\infty<t<\infty} \left| \sum_{n=o}^{\infty} \left[\sum_{k=o}^{n} h(kT) r(nT-kT) \right] \delta(t-nT) \right| \\
&= \sup_{o \leq n<\infty} \left| \sum_{k=o}^{n} h(kT) r(nT-kT) \right| \\
&\leq \sum_{k=o}^{\infty} |h(kT)| \sup_{o \leq n<\infty} |r(nT-kT)| \\
&= \sum_{k=o}^{\infty} |h(kT)| \sup_{-\infty<t<\infty} |r^*| \\
&= \sum_{k=o}^{\infty} |h(kT)| \, \|r^*\| \quad . \qquad (2.30)
\end{aligned}$$

Since

$$c^* = B^* r^*$$

$$\|B^*\| = \sup_{\|r^*\| \leq 1} \|B^* r^*\| = \sup_{\|r^*\| \leq 1} \|c^*\| \ . \tag{2.31}$$

From (2.30) letting $\|r^*\| = 1$,

$$\sup_{\|r^*\| \leq 1} \|c^*\| \leq \sum_{k=o}^{\infty} |h(kT)|$$

i.e.

$$\|B^*\| \leq \sum_{k=o}^{\infty} |h(kT)| \ . \tag{2.32}$$

A classical argument can now be used to show that:

$$\|B^*\| = \sum_{k=o}^{\infty} |h(kT)| \ .$$

To do this choose an element $c(nT)$ of the sequence c^* , n large.

$$c(nT) = \sum_{k=o}^{n} h(kT) r(nT-kT)$$

$$= \sum_{k=o}^{n} h(nT-kT) r(kT) \ .$$

For this n , choose $r(kT) = \text{Sgn}\ h(nT-kT) = \pm 1$. Then

$$c(nT) = \sum_{k=o}^{n} |h(nT-kT)| = \sum_{k=o}^{n} |h(kT)| \ .$$

Then

$$\|c^*\| = \|B^*r^*\| \geq \sum_{k=o}^{n} |h(kT)| \quad .$$

Therefore

$$\|B^*\| = \sup_{\|r^*\|\leq 1} \|B^*r^*\| \geq \sum_{k=o}^{n} |h(kT)| \ , \quad \text{for any } n \ .$$

i.e.

$$\|B^*\| \geq \sum_{k=o}^{\infty} |h(kT)| \quad . \tag{2.33}$$

From (2.32) and (2.33) it follows that:

$$\|B^*\| = \sum_{k=o}^{\infty} |h(kT)| \quad . \tag{2.34}$$

Hereafter $\|B^*\|$ will be denoted by H^*. Given $H(z)$ a method for finding an upper bound for H^* is discussed below.

It will be assumed that $H(z)$ is expressible as a ratio of two polynomials in z, and the number of zeros of $H(z)$ is at least one less than the number of poles. This is usually the case in most physical systems, and the impulse response of such a system will be zero at $t = 0$; in other words $h(kT) = 0$, when $k = 0$, so that

$$H^* = \sum_{k=1}^{\infty} |h(kT)| \quad .$$

Let

$$H(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{i=1}^{n} p_i z^{n-i}}{\sum_{i=o}^{n} q_i z^{n-i}} = \frac{K \sum_{i=1}^{n} p_i z^{n-i}}{\prod_{i=1}^{n} (z - z_i)} \tag{2.35}$$

where z_i are the n poles of $H(z)$. Here we are assuming that $H(z)$ has only simple poles. It may, however, be mentioned that the results can be extended to multiple poles also.

Applying equation (2.19), for $k > 0$:

$$h(kT) = \text{Sum of the residues of } H(z)z^{k-1} \text{ at the poles of } H(z) \; . \tag{2.36}$$

The residues for the rth pole will be:

$$\lim_{z \to z_r} \left[\frac{(z-z_r)P(z)}{Q(z)} \cdot z^{k-1} \right] = \frac{K \sum_{i=1}^{n} p_i z_r^{n-i}}{\prod_{\substack{i=1 \\ i \neq r}}^{n} (z_r - z_i)} \cdot z_r^{k-1}$$

$$= c_r z_r^{k-1} \tag{2.37}$$

where c_r is the coefficient of z_r^{k-1} and is a constant for all k.

Hence

$$H^* = \sum_{k=1}^{\infty} |h(kT)|$$

$$= \sum_{k=1}^{\infty} \left| \sum_{r=1}^{n} c_r z_r^{k-1} \right|$$

$$\leq \sum_{k=1}^{\infty} \sum_{r=1}^{n} |c_r| \, |z_r|^{k-1} \; .$$

Since we have assumed $H(z)$ is the overall Z-transfer function of a stable discrete-data system $|z_r| < 1$.

$$\therefore \qquad \sum_{k=1}^{\infty} \sum_{r=1}^{n} |c_r||z_r|^{k-1} = \sum_{r=1}^{n} \left[\sum_{k=1}^{\infty} |c_r||z_r|^{k-1} \right]$$

$$= \sum_{r=1}^{n} \frac{|c_r|}{1 - |z_r|}$$

Hence

$$H^* \leq \sum_{r=1}^{n} \frac{|c_r|}{1 - |z_r|} \quad . \tag{2.38}$$

The right hand side of inequality (2.38) gives an upper bound for H^* .

CHAPTER III

STABILITY OF NONLINEAR DISCRETE-DATA SYSTEMS

3.1 Introduction.

In this chapter we will develop sufficient conditions for the BIBO stability of a specific class of discrete-data systems with a single continuous nonlinearity. Two such systems will be considered: in system A, the nonlinearity will be assumed to be situated in the feedback path, and in system B it will be assumed to be in the forward path. Since the method of analysis of both systems is almost similar, the analysis of system A will be done in detail, while that of system B will be done more briefly. In either case the boundedness of the output is studied only at the discrete instants of sampling. The general method of approach is to first express the discrete output of the systems as a mapping of a Banach space into itself, and obtain sufficient conditions for the existence of a fixed point within a bounded region in the space making use of the contraction mapping theorem of Functional Analysis. The theorem will be stated and proved as the analysis of system A is developed.

3.2 System A: Description and Assumptions.

The configuration of system A is shown in Fig. 3.1. It is a single-loop negative feedback system with a single nonlinear element N in the feedback path. The sampler is situated in the forward path and has a constant period of T. It is followed by a linear plant G, which may be made of continuous controlling devices, continuous and/or discrete compensating networks as well as a zero order hold. r, c, and e denote the input, output and error signals respectively; they are real valued continuous functions of time t . r*, c* and e* denote the corresponding discrete sampled functions. r is assumed to be bounded.

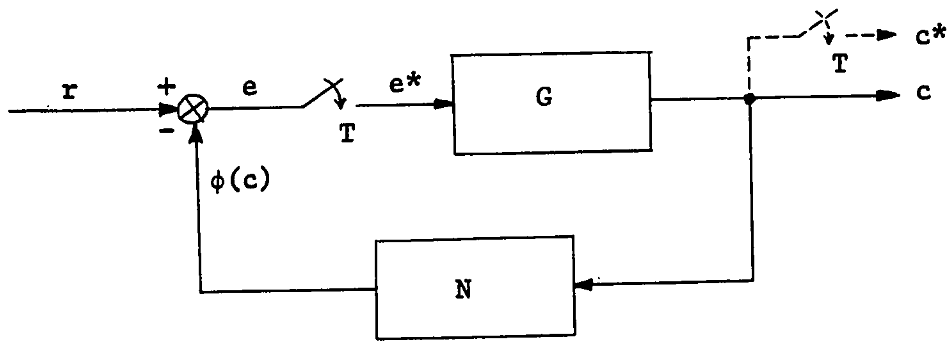

Fig. 3.1 Nonlinear Discrete-Data System A

The nonlinear gain function $\phi(c)$ is assumed to be continuous and further satisfies the condition $\phi(0) = 0$, which permits its representation in the form:

$$\phi(c) = K_1 c + \sum_{i=2}^{N} K_i c^i \tag{3.1}$$

where K_1, K_i are constants, and N a finite integer. Thus $\phi(c)$ may be looked upon as consisting of a linear part proportional to c , and a nonlinear partcontaining all the higher order terms in c . If the higher order terms were not present the system reduces to a linear system with linear gain K_1 in the feedback path; the system will then be said to have been linearized. The BIBO stability conditions to be derived for the nonlinear system will be based on the assumption that the linearized system is BIBO stable.

3.3 Functional Representation of System A.

Let G(s) be the transfer function of the linear plant G , and G(z) the corresponding Z-transfer function. Consider the linearized system first. It is shown in Fig. 3.2

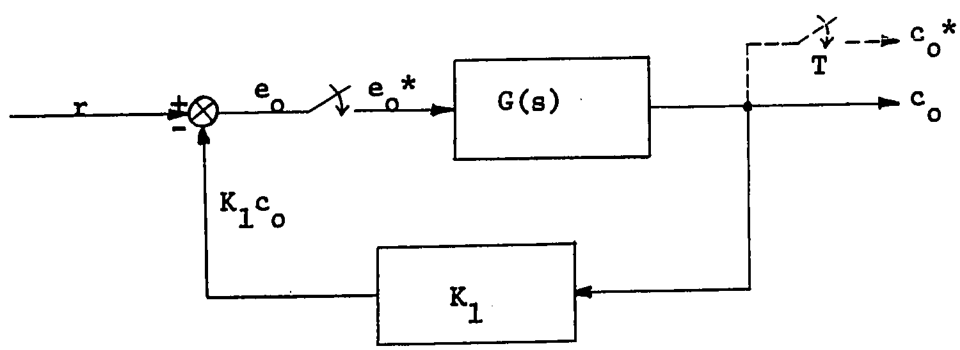

Fig. 3.2 Linearized System Corresponding to System A

Let c_o be the continuous output of the linearized system for the input r , and c_o* be the corresponding sampled output. Also let H(z) be the overall Z-transfer function of the linearized system. Then it can be easily deduced from Z-transform theory that:

$$H(z) = \frac{G(z)}{1 + K_1G(z)} \tag{3.2}$$

and

$$c_o(z) = H(z)r(z) \quad . \tag{3.3}$$

Inverse transforming equation (3.3) into the time domain and using functional notation:

$$c_o^* = B^*r^* \tag{3.4}$$

where B^* is the convolution summation operator. Since we have assumed that the linearized system is stable, c_o^* is bounded whenever r^* is bounded, and B^* is a bounded operator; i.e. $\|B^*\|$ is finite. $\|B^*\|$ can be determined using the formula:

$$\|B^*\| = H^* = \sum_{k=o}^{\infty} |h(kT)| \tag{3.5}$$

where

$$h(kT) = Z^{-1}H(z) \quad , \tag{3.6}$$

$H(z)$ being defined by equation (3.2).

Also let

$$F = \left\{ x^* \begin{cases} = 0 & \text{for} \quad t < 0 \\ = \sum_{n=o}^{\infty} x(nT)\delta(t-nT) & \text{for} \quad t \geq 0 \end{cases} \middle| \ \sup_{-\infty<t<\infty} |x^*| < \infty \right\} \tag{3.7}$$

This is a Banach space (essentially ℓ_∞) if:

$$\|x^*\| = \sup_{-\infty<t<\infty} |x^*| \quad . \tag{3.8}$$

In this setting, for our stable linearized system, whenever $r^* \in F$, $c_o^* \in F$; and $B^* : F \to F$. In other words B^* is a mapping of the Banach space F into itself.

Now consider the nonlinear system A. Referring back to Fig. 3.1, it is clear that the error signal e is given by the relation:

$$e = r - \phi(c) = r - K_1 c - \sum_{i=2}^{N} K_i c^i \quad . \tag{3.9}$$

The input to G is the sampled error signal e^* , and

$$e^* = r^* - \phi^*(c) = r^* - K_1 c^* - \sum_{i=2}^{N} K_i c^{*i} \quad . \tag{3.10}$$

Laplace transforming equation (3.10):

$$e^*(s) = r^*(s) - K_1 c^*(s) - \left(\sum_{i=2}^{N} K_i c^{*i} \right)(s) \quad . \tag{3.11}$$

Rewriting equation (3.11) in Z-transform notation:

$$e(z) = r(z) - K_1 c(z) - \left(\sum_{i=2}^{N} K_i c^{*i} \right)(z) \quad . \tag{3.12}$$

Multiplying both sides of equation (3.12) by $G(z)$, and remembering that:

$$e(z)G(z) = c(z) \quad \text{and} \quad \frac{G(z)}{1 + K_1 G(z)} = H(z)$$

the following expression for $c(z)$ may be derived:

$$c(z) = H(z)r(z) - H(z) \cdot \left(\sum_{i=2}^{N} K_i c^{*i} \right)(z) \quad . \tag{3.13}$$

Transforming equation (3.13) into the time domain and using the operator notation:

$$c^* = B^* r^* - B^* \sum_{i=2}^{N} K_i c^{*i} \tag{3.14}$$

with r^* fixed, equation (3.14) defines a relation of the form

$$c^* = Ac^* \tag{3.15}$$

where A is a nonlinear operator, such that

$$Ac^* = B^*r^* - B^* \sum_{i=2}^{N} K_i c^{*i} \quad . \tag{3.16}$$

3.4 Conditions for the BIBO Stability of System A.

Whether the system represented by equation (3.14) is BIBO stable is equivalent to asking under what conditions does it have a solution c^* belonging to F, if $r^* \in F$; or when does A have a fixed point in F for each $r^* \in F$. To answer this question, we resort to a contraction mapping theorem[23], which is stated and proved next. The statement of the theorem is slightly modified to suit our purpose, but the proof is essentially the same as that given in reference 23.

THEOREM: Let F be a Banach space, and let A be a mapping of F into F. Let S be a sphere contained in F of centre x_o, and radius a, such that:

(i) $\|Ax_1 - Ax_2\| \leq \lambda \|x_1 - x_2\|$ (3.17)

for every $x_1, x_2 \in S$, and some $\lambda : 0 < \lambda < 1$

(ii) $\|Ax_o - x_o\| < (1 - \lambda)a$ (3.18)

then there is a unique fixed point x in S, that is a point for which $Ax = x$.

Condition (i) will be called the contraction condition, and condition (ii) the fixed point condition.

PROOF: Define the sequence $\{x_n\}$ which is given recursively by:

$$x_n = Ax_{n-1} , \quad n = 1,2,\ldots \tag{3.19}$$

(and in which x_o is the vector of our hypothesis). By hypothesis x_o and x_1 lie in the sphere S . If:

$$\|x_j - x_o\| < (1 - \lambda^j)a , \quad j = 1,2,\ldots,n \tag{3.20}$$

(so that $x_1, x_2, \ldots, x_n$ lie in S), then:

$$\begin{aligned} \|x_{n+1} - x_o\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_o\| \\ &\leq \|Ax_n - Ax_{n-1}\| + (1 - \lambda^n)a \quad . \end{aligned} \tag{3.21}$$

But

$$\begin{aligned} \|Ax_n - Ax_{n-1}\| &\leq \lambda\|x_n - x_{n-1}\| \\ &= \lambda\|Ax_{n-1} - Ax_{n-2}\| \\ &\leq \lambda^2\|x_{n-1} - x_{n-2}\| \\ &\quad \cdots\cdots\cdots \\ &\leq \lambda^n\|x_1 - x_o\| \\ &\leq \lambda^n(1 - \lambda)a \quad . \end{aligned} \tag{3.22}$$

It follows that:

$$\begin{aligned} \|x_{n+1} - x_o\| &\leq \lambda^n(1 - \lambda)a + (1 - \lambda^n)a \\ &= (1 - \lambda^{n+1})a \end{aligned} \tag{3.23}$$

so that x_n lies in S for every n . We see also that because

$$\|x_{n+1} - x_n\| \leq \lambda^n \|x_1 - x_o\| \tag{3.24}$$

we have

$$\begin{aligned} \|x_{n+r} - x_n\| &\leq \sum_{j=n}^{n+r-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=n}^{\infty} \|x_{j+1} - x_j\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_1 - x_o\| \ . \end{aligned} \tag{3.25}$$

For $r \geq 1$, the sequence $\{x_n\}$ is a Cauchy sequence, $x_n \to x$ as $n \to \infty$, and x belongs to S .

Since Ax is continuous on S (by contraction condition, which is a Lipchitz condition) we now let $n \to \infty$ in $x_n = Ax_{n-1}$ and get

$$x = Ax \ . \tag{3.26}$$

There cannot be a second point y in S for which $y = Ay$ for if there were we would have:

$$\|x - y\| = \|Ax - Ay\| \leq \lambda\|x - y\| < \|x - y\| \tag{3.27}$$

which is impossible. The proof is now complete.

In order to apply the above theorem to our nonlinear mapping A , we must first make sure that A maps the Banach space F into F . This means that in equation (3.16) defining the mapping A , Ac* should belong to F , if r* and c* belong to F . That it is indeed so can be shown as follows:

Let

$$Ac^* = B^*r^* - B^* \sum_{i=2}^{N} K_i c^{*i}$$

$$= B^*r^* - B^*Tc^* \qquad (3.28)$$

where

$$T : c^* \rightarrow \sum_{i=2}^{N} K_i c^{*i} \quad .$$

It has already been shown that $B^* : F \rightarrow F$, if $r^* \in F$. Also $T : F \rightarrow F$, if $c^* \in F$, since

$$\|Tc^*\| = \| \sum_{i=2}^{N} K_i c^{*i} \| \leq \sum_{i=2}^{N} |K_i| \|c^*\|^i < \infty \qquad (3.29)$$

Therefore it follows that $A : F \rightarrow F$, if $r^*, c^* \in F$.

Next we have to choose a sphere S in F , with centre x_o and radius 'a' such that the contraction and fixed point conditions are satisfied in S . To do this we arbitrarily pick x_o as the output c_o^* of the linearized system for an input $r^* \in F$; that is we let $x_o = c_o^* = B^*r^*$, and proceed to determine the conditions to be satisfied by r^* and 'a' , in order that the contraction and fixed point conditions are satisfied in S . The choice of c_o^* as the centre of the sphere S is justified on the basis that the nonlinear system A may be looked upon as a perturbed version of the linearized system, and hence if it has a fixed point for a given $r^* \in F$, that fixed point is likely to be in a neighbourhood 'a' of the solution of the linearized system for the same input $r^* \in F$.

Consider now the contraction condition. Let c_1^*, c_2^* be any two points belonging to S. Then the contraction condition may be stated as:

$$\|Ac_1^* - Ac_2^*\| \leq \lambda \|c_1^* - c_2^*\| \ , \quad \lambda : 0 < \lambda < 1 \ . \tag{3.30}$$

From equation (3.16):

$$\begin{aligned}
&\|Ac_1^* - Ac_2^*\| \\
&\quad = \|B^* \sum_{i=2}^{N} K_i c_2^{*i} - B^* \sum_{i=2}^{N} K_i c_1^{*i}\| \\
&\quad \leq \|B^*\| \sum_{i=2}^{N} \|K_i (c_2^{*i} - c_1^{*i})\| \\
&\quad = H^* \sum_{i=2}^{N} \|K_i (c_2^* - c_1^*) \sum_{k=o}^{i-1} c_1^{*\,i-1-k} c_2^{*\,k}\| \\
&\quad \leq H^* \|c_2^* - c_1^*\| \sum_{i=2}^{N} |K_i| \sum_{k=o}^{i-1} \|c_1^*\|^{i-1-k} \|c_2^*\|^{k} \ .
\end{aligned} \tag{3.31}$$

Since $c_1^*, c_2^* \in S$:

$$\begin{aligned}
\|c_1^*|, \|c_2^*| &\leq \|c_o^*\| + a \\
&= \|B^* r^*\| + a \\
&\leq \|B^*\| \|r^*\| + a \\
&= H^* R^* + a
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\therefore \quad &\|Ac_1^* - Ac_2^*\| \\
&\leq H^* \|c_2^* - c_1^*\| \sum_{i=2}^{N} |K_i| \sum_{k=o}^{i-1} (H^*R^*+a)^{i-1-k} (H^*R^*+a)^{k} \\
&\leq H^* \|c_2^* - c_1^*\| \sum_{i=2}^{N} |K_i| \ i (H^*R^*+a)^{i-1} \ .
\end{aligned} \tag{3.33}$$

Hence (3.30) will be satisfied if:

$$H^*\| c_2^* - c_1^*\| \sum_{i=2}^{N} |K_i| i(H^*R^*+a)^{i-1} \leq \lambda \| c_1^* - c_2^*\| \ . \qquad (3.34)$$

Since $\| c_2^* - c_1^*\| = \| c_1^* - c_2^*\|$, (3.34) is equivalent to:

$$H^* \sum_{i=2}^{N} |K_i| \ i(H^*R^*+a)^{i-1} \leq \lambda \ . \qquad (3.35)$$

Now consider the fixed point condition. With c_o^* as the centre of the sphere S , this may be stated as follows:

$$\| Ac_o^* - c_o^*\| \leq (1 - \lambda)a \ . \qquad (3.36)$$

Since

$$c_o^* = B^*r^* \qquad (3.37)$$

$$\| c_o^*\| \leq H^*R^* \ . \qquad (3.38)$$

Also it follows from equation (3.16) that:

$$Ac_o^* = c_o^* - B^* \sum_{i=2}^{N} K_i c_o^{*i} \ . \qquad (3.39)$$

Transposing c_o^* to the left hand side and taking norms on both sides:

$$\| Ac_o^* - c_o^*\| = \| -B^* \sum_{i=2}^{N} K_i c_o^{*i}\|$$

$$\leq \| B^*\| \sum_{i=2}^{N} |K_i| \| c_o^*\|^i$$

$$\leq H^* \sum_{i=2}^{N} |K_i| (H^*R^*)^i \ . \qquad (3.40)$$

Hence (3.36) will be satisfied if:

$$H^* \sum_{i=2}^{N} |K_i| \ (H^*R^*)^i < (1 - \lambda)a \quad . \tag{3.41}$$

Recapitulating, given an aribtrary input $r^* \in F$, with $\|r^*\| = R^*$, inequalities (3.35) and (3.41) give the sufficient conditions for the nonlinear mapping expressed by equation (3.16) to have a fixed point c^* in a neighbourhood 'a' of the solution of the stable linearized system for the same input r^* . Moreover, since c^* belongs to the sphere S , an upper bound for c^* is given by:

$$\|c^*\| \leq H^*R^* + a \quad . \tag{3.42}$$

We are now in a position to state the following theorem which embodies sufficient conditions for the existence of a region in which the system A exhibits bounded-input bounded-output stability:

<u>THEOREM 1</u>: The system A, satisfying the conditions set forth in section 3.2, and driven by an arbitrary input r , with $\|r^*\| = R^*$, possesses a bounded sampled output c^* in a neighbourhood 'a' of the solution of the stable linearized system for the same input r , if a $\lambda : 0 < \lambda < 1$, exists such that the following inequalities are satisfied:

$$H^* \sum_{i=2}^{N} |K_i| \, i(H^*R^*+a)^{i-1} \leq \lambda \tag{3.43}$$

$$H^* \sum_{i=2}^{N} |K_i| (H^*R^*)^i < (1 - \lambda)a \quad . \tag{3.44}$$

An upper bound for c^* is then given by:

$$\|c^*\| \leq H^*R^* + a \quad . \tag{3.45}$$

3.5 Conditions for the BIBO Stability of System B.

System B differs from system A in that the nonlinearity is situated in the forward path instead of in the feedback path. The configuration of system B is shown in Fig. 3.3, and that of the corresponding linearized system in Fig. 3.4.

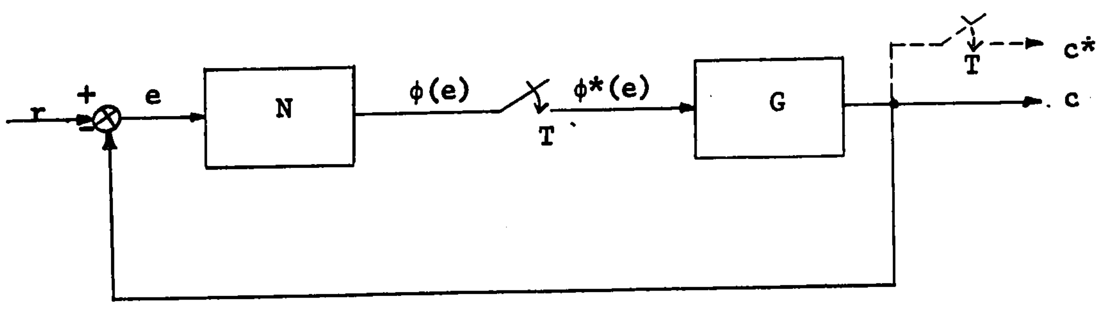

Fig. 3.3 Nonlinear Discrete-Data System B

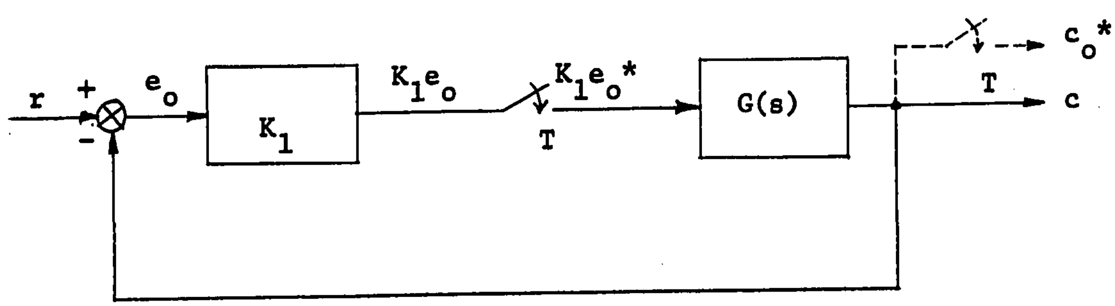

Fig. 3.4 Linearized System Corresponding to System B

The equation for the gain is:

$$\phi(e) = K_1 e + \sum_{i=2}^{N} K_i e^i \tag{3.46}$$

where

$$e = r - c \quad . \tag{3.47}$$

The linearized system is assumed to be stable as before. The overall Z-transfer function of the linearized system is:

$$H(z) = \frac{K_1 G(z)}{1 + K_1 G(z)} \quad . \tag{3.48}$$

The equation for the nonlinear system can be derived as follows:

$$\begin{aligned} c(z) &= G(z)\phi(e)(z) \\ &= G(z)\ [K_1 e(z) + (\sum_{i=2}^{N} K_i e^{*i})(z)] \\ &= K_1 G(z)[r(z) - c(z)] + G(z)\ (\sum_{i=2}^{N} K_i e^{*i})(z) \quad . \end{aligned} \tag{3.49}$$

Transposing $-K_1 G(z)c(z)$ to the left hand side, and dividing both sides by $1 + K_1 G(z)$:

$$\begin{aligned} c(z) &= \frac{K_1 G(z) r(z)}{1 + K_1 G(z)} + \frac{1}{K_1} \cdot \frac{K_1 G(z)}{1 + K_1 G(z)}\ (\sum_{i=2}^{N} K_i e^{*i})(z) \\ &= H(z) r(z) + \frac{1}{K_1} H(z)(\sum_{i=2}^{N} K_i e^{*i})(z) \quad . \end{aligned} \tag{3.50}$$

Inverse transforming (3.50) and using operator notation:

$$c^* = B^* r^* + \frac{1}{K_1} B^* \sum_{i=2}^{N} K_i e^{*i} \tag{3.51}$$

where

$$e^* = r^* - c^* \quad . \tag{3.52}$$

With r^* fixed, equation (3.51) defines a relation of the form:

$$c^* = Ac^* \tag{3.53}$$

where A is a nonlinear operator, such that:

$$Ac^* = B^*r^* + \frac{1}{K_1} B^* \sum_{i=2}^{N} K_i e^{*i} \quad . \tag{3.54}$$

It is easy to show as before that A maps a Banach space F into F if $r^*, c^* \in F$.

To investigate the existence of a fixed point for the mapping A in a sphere S with centre $c_o^* = B^*r^*$, and radius 'a' consider first the contraction condition. Let c_1^*, c_2^* be any two points in the sphere S . Let e_1^*, e_2^* be the sampled error functions corresponding to c_1^*, c_2^* .

From equation (3.54):

$$\begin{aligned} &\|Ac_1^* - Ac_2^*\| \\ &= \frac{1}{K_1} \left\| B^* \sum_{i=2}^{N} K_i e_1^{*i} - B^* \sum_{i=2}^{N} k_i e_2^{*i} \right\| \\ &\le \frac{1}{K_1} \|B^*\| \sum_{i=2}^{N} \|K_i (e_1^{*i} - e_2^{*i})\| \\ &= \frac{1}{K_1} H^* \sum_{i=2}^{N} \|K_i (e_1^* - e_2^*) \sum_{k=o}^{i-1} e_1^{*i-1-k} e_2^{*k}\| \\ &\le \frac{1}{K_1} H^* \|e_1^* - e_2^*\| \sum_{i=2}^{N} |K_i| \sum_{k=o}^{i-1} \|e_1^*\|^{i-1-k} \|e_2^*\|^k \quad . \end{aligned} \tag{3.55}$$

Note: Since the linearized system is assumed to be stable, K_1 has got to be positive; hence, the absolute value sign is not put on K_1 .

Now for a given r^* it follows from equation (3.52) that:

$$\|e_1^* - e_2^*\| = \|c_1^* - c_2^*\| \tag{3.56}$$

$$\|e_1^*\| \leq R^* + \|c_1^*\| \tag{3.57}$$

$$\|e_2^*\| \leq R^* + \|c_2^*\| \ . \tag{3.58}$$

Since $c_1^*, c_2^* \in S$, we also have:

$$\|c_1^*\|, \|c_2^*\| \leq H^*R^* + a \ . \tag{3.59}$$

Therefore

$$\|e_1^*\| \leq R^* + H^*R^* + a \tag{3.60}$$

$$\|e_2^*\| \leq R^* + H^*R^* + a \ . \tag{3.61}$$

Making use of (3.56), (3.60) and (3.61) we may write:

$$\|Ac_1^* - Ac_2^*\|$$

$$\leq \frac{1}{K_1} H^*\|c_1^* - c_2^*\| \sum_{i=2}^{N} |K_i| \sum_{k=o}^{i-1} (R^*+H^*R^*+a)^{i-1-k}(R^*+H^*R^*+a)^k$$

$$= \frac{1}{K_1} H^*\|c_1^* - c_2^*\| \sum_{i=2}^{N} |K_i| \, i(R^*+H^*R^*+a)^{i-1} \ . \tag{3.62}$$

In order to satisfy the contraction condition we require:

$$\|Ac_1^* - Ac_2^*\| \leq \lambda\|c_1^* - c_2^*\| \ , \quad \lambda : 0 < \lambda < 1 \ . \tag{3.63}$$

By virtue of (3.62), (3.63) will be satisfied if:

$$\frac{1}{K_1} H^*\|c_1^* - c_2^*\| \sum_{i=2}^{N} |K_i| i(R^*+H^*R^*+a)^{i-1} \leq \lambda\|c_1^* - c_2^*\| \tag{3.64}$$

or if:

$$\frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| i(R^*+H^*R^*+a)^{i-1} \leq \lambda \quad . \tag{3.65}$$

Now consider the fixed point condition, which requires that:

$$\|Ac_o^* - c_o^*\| < (1 - \lambda)a \quad . \tag{3.66}$$

If e_o^* is the sampled error corresponding to c_o^*, it follows from equation (3.54) that:

$$Ac_o^* = c_o^* + \frac{1}{K_1} B^* \sum_{i=2}^{N} K_i e_o^{*i} \quad . \tag{3.67}$$

Transposing c_o^* to the left hand side and taking norms on both sides:

$$\|Ac_o^* - c_o^*\| = \|\frac{1}{K_1} B^* \sum_{i=2}^{N} K_i e_o^{*i}\|$$

$$\leq \frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| \|e_o^*\|^i \quad . \tag{3.68}$$

But

$$\|e_o^*\| \leq R^* + \|c_o^*\|$$

$$\leq R^* + H^*R^* \quad . \tag{3.69}$$

Therefore

$$\|Ac_o^* - c_o^*\| \leq \frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| (R^* + H^*R^*)^i \quad . \tag{3.70}$$

Hence the fixed point condition will be satisfied if:

$$\frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| (R^* + H^*R^*)^i < (1 - \lambda)a \quad . \tag{3.71}$$

Thus inequalities (3.65) and (3.71) are sufficient for the mapping A to have a fixed point c^* inside the sphere S . Since c^* lies inside S , an upper bound for c^* is given by:

$$\|c^*\| \leq H^*R^* + a \quad . \tag{3.72}$$

We may now state the following theorem concerning the bounded-input bounded-output stability of system B.

<u>THEOREM 2</u>: The system B, conforming to the description and assumptions set forth at the beginning of this section, and driven by an arbitrary input r , with $\|r^*\| = R^*$, possesses a bounded sampled output c^* in a neighbourhood 'a' of the solution of the stable linearized system for the same input r , if a $\lambda : 0 < \lambda < 1$, exists such that the following inequalities are satisfied:

$$\frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| i (R^* + H^*R^* + a)^{i-1} \leq \lambda \tag{3.73}$$

$$\frac{1}{K_1} H^* \sum_{i=2}^{N} |K_i| (R^* + H^*R^*)^{i} < (1 - \lambda) a \quad . \tag{3.74}$$

An upper bound for c^* is then given by:

$$\|c^*\| \leq H^*R^* + a \quad . \tag{3.75}$$

CHAPTER IV

COMPUTATION OF INPUT AND OUTPUT BOUNDS

4.1 Introduction.

In the last chapter we formulated the sufficient conditions for the existence of a bounded output for two different types of non-linear discrete-data systems. In practice we are interested in determining the maximum value of R^* for which the sytems exhibit a bounded output. To do this we have to choose the optimum values of 'a' and λ which give the maximum value of R^* satisfying the contraction and fixed point conditions. This has to be done only by trial and error in the general case of the nonlinearity. However, if the nonlinear part of the gain function consists of a single n^{th} degree term a direct approach is available to determine the maximum value of R^* to ensure a bounded output, and the corresponding upper bound for the output. This method will be described in the following sections, together with numerical examples to illustrate the computational procedure.

4.2 Input and Output Bounds for System A with a Single n^{th} Degree Term in the Nonlinearity.

The above system differs from the general system in that the nonlinear gain function is given by:

$$\phi(c) = K_1 c + K_n c^n , \quad n \geq 2 . \tag{4.1}$$

The equation for c^* will then be:

$$c^* = B^*r^* - B^*K_n c^{*n} \quad . \tag{4.2}$$

The contraction condition reduces to:

$$nH^*|K_n|(H^*R^* + a)^{n-1} \le \lambda \tag{4.3}$$

and the fixed point condition to:

$$H^*|K_n|(H^*R^*)^n < (1 - \lambda)a \quad . \tag{4.4}$$

Note that $\lambda : 0 < \lambda < 1$, and $a > 0$. Letting

$$a = bH^*R^* \tag{4.5}$$

inequality (4.3) may be written as:

$$nH^*|K_n|(H^*R^* + bH^*R^*)^{n-1} \le \lambda$$

or

$$nH^*|K_n|(H^*R^*)^{n-1}(1 + b)^{n-1} \le \lambda$$

or

$$|K_n|H^*(H^*R^*)^{n-1} \le \frac{\lambda}{n(1 + b)^{n-1}} \quad . \tag{4.6}$$

Similarly inequality (4.4) may be written as:

$$H^*|K_n|(H^*R^*)^n < (1 - \lambda)bH^*R^*$$

or

$$|K_n|H^*(H^*R^*)^{n-1} < (1 - \lambda)b \quad . \tag{4.7}$$

Let

$$|K_n|H^*(H^*(H^*R^*)^{n-1} = x \quad . \tag{4.8}$$

Then (4.6) implies

$$x \le \frac{\lambda}{n(1 + b)^{n-1}} \tag{4.9}$$

(4.7) implies

$$x < (1 - \lambda)b \quad . \tag{4.10}$$

To find the maximum value of x, satisfying (4.9) and (4.10) note that for a given n, and $b > 0$, the right hand side of (4.9) is an increasing function of λ, whereas the right hand side of (4.10) is a decreasing function of λ. Hence for a given n and b, x will be a maximum for that value of λ at which the two curves:

$$y = \frac{\lambda}{n(1 + b)^{n-1}} \quad \text{and} \quad y = (1 - \lambda)b \quad \text{intersect} .$$

Similarly for a given n and λ, $0 < \lambda < 1$, the right hand side of (4.9) is a decreasing function of b, whereas the right hand side of (4.10) is an increasing function of b. Hence for a given n and λ, x will be a maximum for that value of b at which the two curves:

$$y = \frac{\lambda}{n(1 + b)^{n-1}} \quad \text{and} \quad y = (1 - \lambda)b \quad \text{intersect} .$$

Therefore when λ and b are both varied for a given n, the maximum value of x will lie on the curve of intersection between the two surfaces:

$$y = \frac{\lambda}{n(1 + b)^{n-1}} \quad \text{and} \quad y = (1 - \lambda)b \quad ,$$

and will correspond to the maximum height of that curve, from the λ-b plane.

To obtain y_{max} proceed as follows:

At the point of intersection of the two surfaces:

$$\frac{\lambda}{n(1 + b)^{n-1}} = (1 - \lambda)b$$

from which:

$$\lambda = \frac{nb(1 + b)^{n-1}}{nb(1 + b)^{n-1} + 1} \quad . \tag{4.11}$$

Note that from (4.11) it is obvious that the condition $0 < \lambda < 1$ is satisfied for a given n and $b > 0$. Also

$$y = (1 - \lambda)b$$

$$= \frac{b}{nb(1 + b)^{n-1} + 1} \quad . \tag{4.12}$$

The condition for y to be maximum are:

$$\frac{dy}{db} = 0 \ , \tag{4.13}$$

and

$$\frac{d^2y}{db^2} < 0 \quad , \tag{4.14}$$

for the value of b for which (4.13) holds.

Differentiating (4.12) with respect to b , and equating to zero we get:

$$1 - n(n - 1)b^2(1 + b)^{n-2} = 0 \quad . \tag{4.15}$$

The positive real root of equation (4.15) gives the value of b for which y is a maximum. It may be observed that the left hand side of equation (4.15) is a decreasing function of b , for $b > 0$, and since it is equal to 1 when $b = 0$, the equation possesses one and only one positive real root. Moreover, it can be easily verified that for this root the second derivative of y with respect to b is less than zero. Substituting this root in equation (4.12) we get the maximum value of y .

Since y_{max} represents the maximum value of x , satisfying the inequalities (4.9) and (4.10), and x is given by equation (4.8) we may conclude that the maximum value of R^* satisfying both the contraction and fixed point conditions is given by the inequality:

$$|K_n|H^*(H^*R^*)^{n-1} < \frac{b_o}{nb_o(1 + b_o)^{n-1} + 1} \tag{4.16}$$

where b_o represents the positive real root of equation (4.15). From (4.16) it follows that the best possible bound on R^* for the system to exhibit a bounded output is given by the following inequality:

$$R^* < \frac{1}{H^*}\left[\frac{b_o}{|K_n|H^*\{nb_o(1 + b_o)^{n-1} + 1\}}\right]^{\frac{1}{n-1}} \tag{4.17}$$

The corresponding bound on c^* is then given by:

$$\begin{aligned} C^* &\le H^*R^* + a \\ &= H^*R^*(1 + b) \\ &< (1 + b_o)\left[\frac{b_o}{|K_n|H^*\{nb_o(1 + b_o)^{n-1} + 1\}}\right]^{\frac{1}{n-1}} \end{aligned} \tag{4.18}$$

<u>Numerical Example 1</u>:

As a numerical example consider the nonlinear discrete-data system shown in Fig. 4.1.

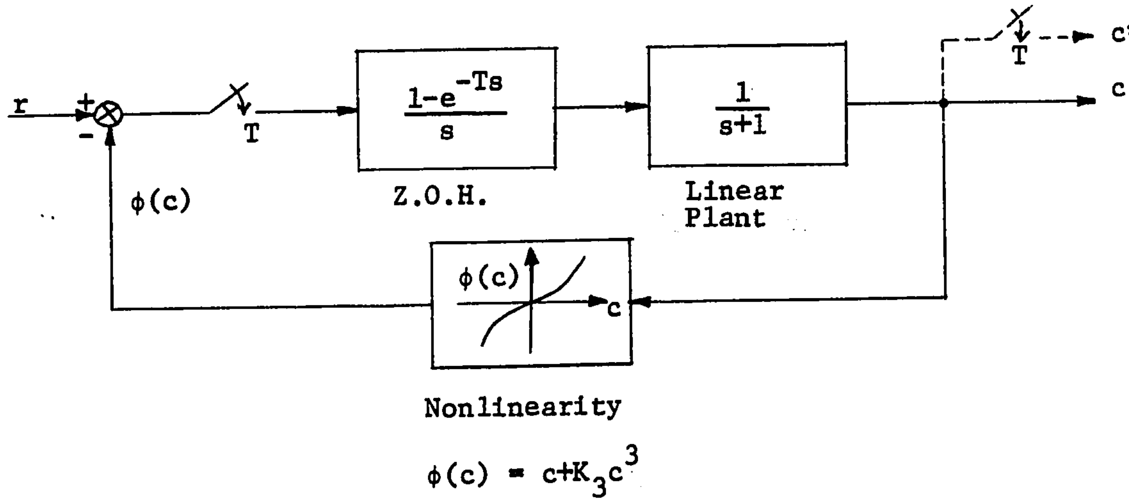

Fig. 4.1 Discrete-Data System with a Cubic Nonlinearity in the Feedback Path.

It has a cubic type of nonlinearity represented by the gain function:

$$\phi(c) = c + K_3c^3 \quad . \tag{4.19}$$

Referring to equation (4.1) this corresponds to the case when $K_1 = 1$, and $n = 3$. The forward path of the system contains a sampler with period $T = 1$ second, followed by a linear plant consisting of a zero order hold and a controller with transfer function $\frac{1}{s+1}$, so that the Z-transfer function of the linear plant may be written as:

$$\begin{aligned} G(z) &= Z\left[\frac{1 - e^{-Ts}}{s} \cdot \frac{1}{s+1}\right] \\ &= \frac{1 - e^{-T}}{z - e^{-T}} \\ &= \frac{0.632}{z - 0.368} \end{aligned} \tag{4.20}$$

and

$$H(z) = \frac{G(z)}{1 + G(z)}$$

$$= \frac{0.632}{z + 0.264} \quad . \tag{4.21}$$

Inverse transforming $H(z)$:

$$h(kT) = \sum \text{ Residues of } H(z)z^{k-1} \text{ at the poles of } H(z)z^{k-1} \quad . \tag{4.22}$$

Inserting numerical values and evaluating the residues:

$$h(kT) \begin{cases} = 0, & \text{for } k = 0 \\ = 0.632\ (-0.264)^{k-1} & \text{for } k > 0 \end{cases} \tag{4.23}$$

Hence

$$H^* = \sum_{k=o}^{\infty} |h(kT)|$$

$$= \sum_{k=1}^{\infty} |0.632\ (-0.264)^{k-1}|$$

$$= 0.632 \sum_{k=1}^{\infty} 0.264^{k-1}$$

$$= 0.86 \quad . \tag{4.24}$$

Now for $n = 3$, equation (4.15) reduces to:

$$1 - 6b^2(1 + b) = 0 \quad . \tag{4.25}$$

(4.25) is a cubic equation, solving which we find that it has a positive real root $= 0.35$. Hence $b_o = 0.35$. Now letting $H^* = 0.86$, $n = 3$ and $b_o = 0.35$ in inequality (4.17) the maximum permissible sampled input to ensure the existence of a bounded sampled output will be given by:

$$R^* < \frac{1}{0.86}\left[\frac{0.35}{|K_3| 0.86\{3 \times 0.35 \times 1.35^2 + 1\}}\right]^{\frac{1}{2}}$$

$$= 0.435\ |K_3|^{-\frac{1}{2}} \quad . \tag{4.26}$$

To obtain the corresponding bound on the sampled output use inequality (4.18) and obtain:

$$C^* < 0.795\ |K_3|^{-\frac{1}{2}} \quad . \tag{4.27}$$

Note that the optimum value of λ in this case is 0.66 from equation (4.11) by letting $n = 3$, and $b_o = 0.35$.

Inequality (4.26) can also be used to find the permissible maximum value of $|K_3|$ for a specified bound on R^*. For example in the case of a unit step input $R^* = 1$, and $|K_3|_{max}$ works out to 0.188, and the corresponding bound on C^* becomes 1.83 from (4.27).

An example of the same system, where the solution obtained by solving the difference equation of the system by means of a recurrence relation for the case $K_3 = 0.1$, and a unit step input, may be found in reference 24. In that example, c^*_{max} occurs at the end of the first sampling period, and is equal to 0.632. Using the contraction technique outlined above the bounds for R^* and C^* work out to 1.37 and 2.5 when $K_3 = 0.1$, which are consistent with the results obtained in reference 24.

4.3 Input and Output Bounds for System B with a Single n^{th} Degree Term in the Nonlinearity.

For the above system the nonlinear gain function is given by:

$$\phi(e) = K_1 e + K_n e^n , \quad n \geq 2 , \quad K_1 > 0 \tag{4.28}$$

and the equation for c^* is:

$$c^* = B^* r^* + \frac{1}{K_1} B^* K_n e^{*n} . \tag{4.29}$$

The contraction condition reduces to:

$$\frac{1}{K_1} nH^* |K_n| (R^* + H^* R^* + a)^{n-1} \leq \lambda \tag{4.30}$$

and the fixed point condition to:

$$\frac{1}{K_1} H^* |K_n| (R^* + H^* R^*)^n < (1 - \lambda) a . \tag{4.31}$$

Now we let

$$a = b(R^* + H^* R^*) .$$

Then inequality (4.30) may be written as:

$$\frac{1}{K_1} nH^* |K_n| (R^* + H^* R^*)^{n-1} (1 + b)^{n-1} \leq \lambda$$

or

$$\frac{|K_n|}{K_1} H^* (R^* + H^* R^*)^{n-1} \leq \frac{\lambda}{n(1 + b)^{n-1}} . \tag{4.32}$$

Similarly inequality (4.31) may be written as:

$$\frac{|K_n|}{K_1} H^* (R^* + H^* R^*)^{n-1} < (1 - \lambda) b . \tag{4.33}$$

Letting

$$\frac{|K_n|}{K_1} H^*(R^* + H^*R^*)^{n-1} = x \tag{4.34}$$

(4.32) implies

$$x \leq \frac{\lambda}{n(1 + b)^{n-1}} \tag{4.35}$$

(4.33) implies

$$x < (1 - \lambda)b \quad . \tag{4.36}$$

Note that the inequalities (4.35) and (4.36) are the same as inequalities (4.9) and (4.10) for system A considered in section 4.2, though x in this case represents a different quantity. The maximum value of x for which the inequalities (4.35) and (4.36) are satisfied can therefore be found as before, and the condition for BIBO stability of the system reduces to:

$$\frac{|K_n|}{K_1} H^*(R^* + H^*R^*)^{n-1} < \frac{b_o}{nb_o(1 + b_o)^{n-1} + 1} \tag{4.37}$$

where b_o is the real positive root of equation (4.15).

From (4.37) it can be easily deduced that the best bound for R^* for the existence of a bounded sampled output is given by:

$$R^* < \frac{1}{1 + H^*}\left[\frac{K_1 b_o}{|K|_n H^*\{nb_o(1 + b_o)^{n-1} + 1\}}\right]^{\frac{1}{n-1}} \quad . \tag{4.38}$$

The corresponding bound on C^* is given by:

$$C^* \leq H^*R^* + a$$

$$= H^*R^* + b(R^* + H^*R^*)$$

$$\leq H^* + b(1 + H^*)\ R^*$$

$$< \left[\frac{H^*}{1 + H^*} + b_o\right]\left[\frac{K_1\ b_o}{|K_n|H^*\{nb_o(1 + b_o)^{n-1} + 1\}}\right]^{\frac{1}{n-1}} . \qquad (4.39)$$

Numerical Example 2:

As a numerical example we will consider the nonlinear

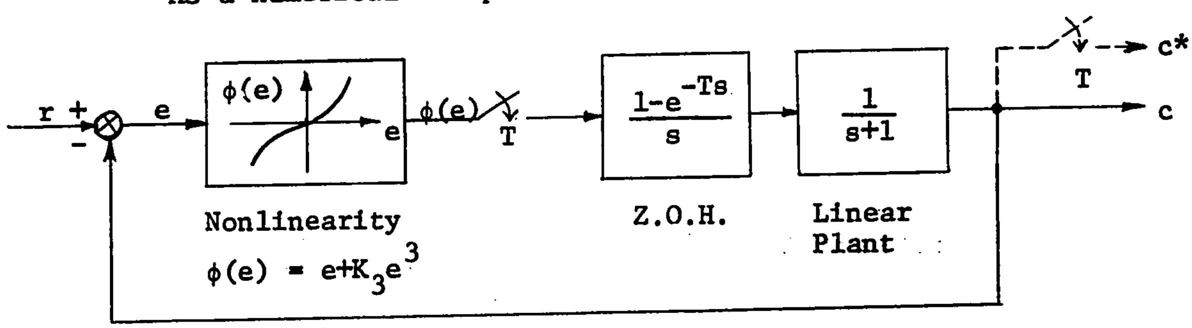

Fig. 4.2 Discrete-Data System with a Cubic Nonlinearity in the Forward Path.

discrete-data system shown in Fig. 4.2. It will be observed that the linear part of the system is the same as that of the system considered in example 1; the nonlinearity is however situated in the forward path instead of in the feedback path. The nonlinear gain function is given by:

$$\phi(e) = e + K_3e^3 . \qquad (4.40)$$

The period T of the sampler is again assumed to be 1 second. Since the linearized system corresponding to the above system is the same as that of the one considered in example 1, $H^* = 0.86$, for this system as well. Also $n = 3$, $K_1 = 1$, and $b_o = 0.35$. Substituting these values in inequality (4.38) the maximum permissible sampled input to ensure the existence of a bounded sampled output will be given by:

$$R^* < \frac{1}{1.86}\left[\frac{0.35}{|K_3|0.86\ \{3 \times 0.35 \times 1.35^2 + 1\}}\right]^{\frac{1}{2}}$$

$$< \quad 0.2\ |K_3|^{-\frac{1}{2}} \quad . \qquad (4.41)$$

The corresponding bound on the sampled output can be obtained using inequality (4.39). It is given by:

$$C^* < [\frac{0.86}{1.86} + 0.35]\left[\frac{0.35}{|K_3|0.86\ \{3 \times 0.35 \times 1.35^2 + 1\}}\right]^{\frac{1}{2}}$$

$$< \quad 0.478\ |K_3|^{-\frac{1}{2}} \quad . \qquad (4.42)$$

As before, inequality (4.41) can be used to find the permissible maximum value of $|K_3|$ for a specified bound on R^* . Thus in the case of a unit step input $|K_3|_{max}$ works out to 0.04 , and the corresponding bound on C^* becomes 2.39 from (4.42).

CHAPTER V

STABILITY OF A DISCRETE-DATA SYSTEM WITH A SLOPE RESTRICTED NONLINEARITY

5.1 Introduction.

In this chapter we will present and prove a criterion for the bounded-input bounded-output stability of a class of discrete-data systems with a slope restricted nonlinearity. The proof is based on the principle of contraction mapping. A numerical example is worked out to illustrate the application of the criterion, and to discuss its implication in relation to some existing results.

5.2 System Description.

The configuration of the system considered is shown in Fig. 5.1(a).

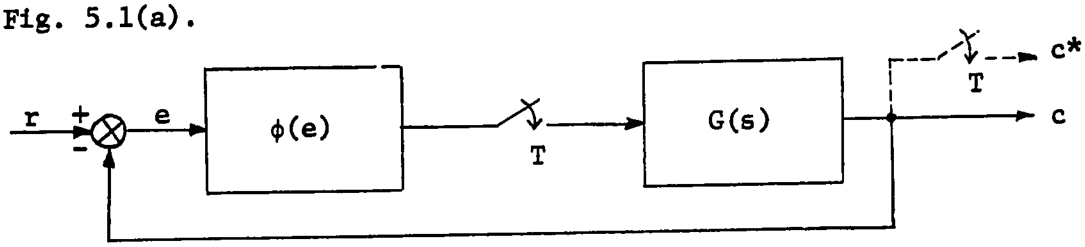

Fig. 5.1(a)

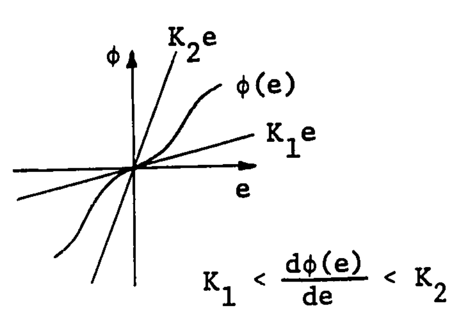

Fig. 5.1(b)

Fig. 5.1 Class N Nonlinear Discrete-Data System
(a) System Configuration (b) Gain Function.

The nonlinear gain function $\phi(e)$ (shown in Fig. 5.1(b)) is assumed to be continuous and differentiable, and further satisfies the condtions:

$$\phi(0) = 0 \tag{5.1}$$

$$0 < K_1 < \frac{d\phi(e)}{de} < K_2 \quad . \tag{5.2}$$

Note that (5.1) and (5.2) imply that $\phi(e)$ is confined to the gain sector $[K_1, K_2]$ i.e. $0 < K_1 < \frac{\phi(e)}{e} < K_2$. The rest of the system comprises a sampler with period T , followed by a linear plant with transfer function $G(s)$. Negative unity feedback is assumed. The system so described will be designated as belonging to the class N .

5.3 <u>Stability Criterion for N</u> .

Let $\phi(e)$ be replaced by a linear gain K , and the resulting linear system designated by L . Define

$$H^* = \sum_{k=o}^{\infty} |h(kT)| \quad ,$$

where $h(kT)$ is the closed-loop impulse response sequence of L . Then the stability criterion for N may be formulated as follows:

<u>THEOREM 1:</u> A nonlinear discrete-data system belonging to the class N is BIBO stable, if a positive K exists for which L is stable, and $K - \frac{K}{H^*} < K_1$ and $K_2 < K + \frac{K}{H^*}$.

<u>PROOF:</u> Choose a K for which L is stable; this implies that, if the input - output relation of L is represented by the linear functional equation:

$$c^* = B^*r^* \tag{5.3}$$

where B^* is the convolution summation operator, then:

$$\|B^*\| = H^* < \infty \quad . \tag{5.4}$$

Note that in (5.3) r^*, c^* denote the sampled input and output respectively, and since L is stable r^*, c^* belong to a Banach space F , equipped with the uniform norm $\|x^*\| = \sup_{x^* \in F} |x^*|$, and (5.4) defines the norm of B^* , induced by the norm defined in F .

Now consider the nonlinear system N . From Z-transform theory:

$$\begin{aligned} c(z) &= \phi(e)(z)G(z) \\ &= Ke(z)G(z) + \phi(e)(z)G(z) - Ke(z)G(z) \\ &= K[r(z) - c(z)]G(z) + \phi(e)(z)G(z) - Ke(z)G(z) \quad . \end{aligned} \tag{5.5}$$

Transposing $- Kc(z)G(z)$ to the left hand side, dividing both sides by $1 + KG(z)$, and putting $\frac{KG(z)}{1 + KG(z)} = H(z)$, (5.5) reduces to:

$$c(z) = H(z)r(z) + H(z) \left[\frac{\phi(e)(z)}{K} - e(z)\right] \quad . \tag{5.6}$$

Inverse transforming (5.6) and using the operator notation:

$$c^* = B^*r^* + B^*\left[\frac{\phi(e^*)}{K} - e^*\right] \quad . \tag{5.7}$$

Since for a given r^* ,

$$e^* = r^* - c^* \tag{5.8}$$

(5.7) defines a relation of the form:

$$c^* = Ac^* \tag{5.9}$$

where A is a nonlinear operator, such that:

$$Ac^* = B^*r^* + B^*\left[\frac{\phi(e^*)}{K} - e^*\right] \quad . \tag{5.10}$$

Now if $r^*, c^* \in F$, A maps $F \to F$. This can be shown as follows:

$$\|Ac^*\| = \left\|B^*r^* + B^*e^* \left[\frac{1}{K} \cdot \frac{\phi(e^*)}{e^*} - 1\right]\right\|$$

$$\leq \|B^*\|\|r^*\| + \|B^*\|\|e^*\|\left[\frac{1}{K}\left\|\frac{\phi(e^*)}{e^*}\right\| + 1\right] \quad . \tag{5.11}$$

Note that in (5.11):

(1) $\|B^*\| = H^* < \infty$, since L is stable.

(2) $\|r^*\| < \infty$, as $r^* \in F$.

(3) $\|e^*\| \leq \|r^*\| + \|c^*\|$, from (5.8)

$< \infty$, as $r^*, c^* \in F$.

(4) $\left\|\frac{\phi(e^*)}{e^*}\right\| < K_2 < \infty$, which is a consequence of the hypotheses (5.1) and (5.2).

Thus all quantities on the right hand side of (5.11) are bounded, and hence $\|Ac^*\| < \infty$, which means $A : F \to F$.

We can therefore apply the contraction mapping theorem to determine the conditions under which the mapping A has a fixed point in F . To do this we choose the output c_o^* of the linear system L to a bounded input r , as a first approximation for the solution of the nonlinear system N for the same input r , and show that a sphere S with a finite radius 'a' , centred on c_o^* exists in which the fixed point condition is always satisfied. Next we apply the contraction condition to the points inside this sphere, and obtain the requirement for the existence of a fixed point in S .

Consider first the fixed point condition which states:

$$\|Ac_o^* - c_o^*\| < (1 - \lambda)a \ , \quad \lambda : 0 < \lambda < 1 \ . \tag{5.12}$$

Since L is stable c_o^* is bounded; so also is e_o^* the corresponding sampled error. Also:

$$c_o^* = B^*r^* \ . \tag{5.13}$$

Hence from equation (5.10):

$$Ac_o^* - c_o^* = B^*\left[\frac{\phi(e_o^*)}{K} - e_o^*\right]$$

$$= B^*e_o^*\left[\frac{\phi(e_o^*)}{Ke_o^*} - 1\right]. \tag{5.14}$$

Therefore

$$\|Ac_o^* - c_o^*\| \leq \|B^*\|\|e_o^*\|\left[\frac{1}{K}\left\|\frac{\phi(e_o^*)}{e_o^*}\right\| + 1\right]. \tag{5.15}$$

Note that in (5.15):

(1) $\|B^*\| = H^* < \infty$, since L is stable.

(2) $\|e_o^*\| < \infty$, since e_o^* is bounded.

(3) $\left\|\frac{\phi(e_o^*)}{e_o^*}\right\| < K_2 < \infty$, which is a consequence of the hypotheses (5.1) and (5.2)

Thus all quantities on the right hand side of the inequality (5.15) are bounded, and hence $\|Ac_o^* - c_o^*\|$ is bounded. Therefore a finite 'a' exists, for $\lambda : 0 < \lambda < 1$, satisfying the fixed point condition given by (5.12).

Now consider the contraction condition which states:

$$\|Ac_1^* - Ac_2^*\| \leq \lambda \|c_1^* - c_2^*\| \tag{5.16}$$

for every $c_1^*, c_2^* \in S$, and some $\lambda : 0 < \lambda < 1$.

Choose some $c_1^*, c_2^* \in S$. Let e_1^*, e_2^* be the sampled error functions corresponding to c_1^*, c_2^*. For a given r^* it follows from (5.8) that:

$$\|c_1^* - c_2^*\| = \|e_1^* - e_2^*\| \quad . \tag{5.17}$$

Also from (5.10):

$$\begin{aligned}
&\|Ac_1^* - Ac_2^*\| \\
&= \left\| B^*\left[\frac{\phi(e_1^*)}{K} - e_1^*\right] - B^*\left[\frac{\phi(e_2^*)}{K} - e_2^*\right] \right\| \\
&\leq \|B^*\| \left\| \frac{\phi(e_1^*) - \phi(e_2^*)}{K} - (e_1^* - e_2^*) \right\| \\
&= H^*\left\| (e_1^* - e_2^*)\left[\frac{\phi(e_1^*) - \phi(e_2^*)}{K\ (e_1^* - e_2^*)} - 1\right] \right\| \\
&\leq H^*\|e_1^* - e_2^*\| \left\| \frac{\phi(e_1^*) - \phi(e_2^*)}{K\ (e_1^* - e_2^*)} - 1 \right\| \quad .
\end{aligned} \tag{5.18}$$

From (5.17) and (5.18) it follows that the contraction condition will be satisfied if:

$$H^*\|e_1^* - e_2^*\| \left\| \frac{\phi(e_1^*) - \phi(e_2^*)}{K\ (e_1^* - e_2^*)} - 1 \right\| \leq \lambda \|e_1^* - e_2^*\| \qquad (5.19)$$

i.e.
$$\left\| \frac{\phi(e_1^*) - \phi(e_2^*)}{K\ (e_1^* - e_2^*)} - 1 \right\| \leq \frac{\lambda}{H^*} \qquad (5.20)$$

i.e.
$$-\frac{\lambda}{H^*} \leq \frac{\phi(e_1^*) - \phi(e_2^*)}{K\ (e_1^* - e_2^*)} - 1 \leq \frac{\lambda}{H^*} \qquad (5.21)$$

i.e.
$$K - \frac{\lambda K}{H^*} \leq \frac{\phi(e_1^*) - \phi(e_2^*)}{e_1^* - e_2^*} \leq K + \frac{\lambda K}{H^*} \quad . \qquad (5.22)$$

Now by the mean value theorem:

$$\frac{\phi(e_1^*) - \phi(e_2^*)}{e_1^* - e_2^*} = \left.\frac{d\phi(e)}{de}\right|_{e=e_m^*,\ e_m^* \in S} \quad . \qquad (5.23)$$

Hence (5.22) will be satisfied if:

$$K - \frac{\lambda K}{H^*} \leq \frac{d\phi(e)}{de} \leq K + \frac{\lambda K}{H^*} \quad . \qquad (5.24)$$

Since by hypothesis $K_1 < \frac{d\phi(e)}{de} < K_2$, and $\lambda : 0 < \lambda < 1$, it follows from (5.24) that the contraction condition will be satisfied if:

$$K - \frac{K}{H^*} < K_1 \quad \text{and} \quad K_2 < K + \frac{K}{H^*} \quad . \qquad (5.25)$$

Hence condition (5.25) alone is sufficient for the mapping A to have a fixed point in F for every bounded input r . In other words system N is BIBO stable if (5.25) is satisfied. This completes the proof of the theorem.

5.4 Numerical Example 1.

To illustrate the application of the criterion, consider the nonlinear integral-control sampled-data system show in Fig. 5.2.

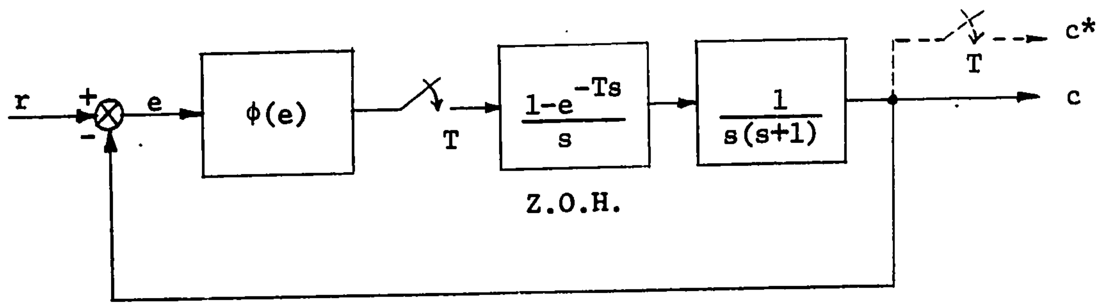

Fig. 5.2 Nonlinear Integral-Control Sampled-Data System

Let $\phi(e)$ be replaced by some linear gain K, for which the resulting linear system L is stable. The problem is to pick the optimum value of K which gives the widest possible bounds for the slope of the nonlinearity for the BIBO stability of the system. To do this the behavior of H^* when K is varied is to be studied. The procedure is as follows:

The Laplace transform of the total linear part is given by:

$$G(s) = \frac{1 - e^{-Ts}}{s^2(s + 1)} \quad . \tag{5.26}$$

The corresponding Z-transform is:

$$G(z) = \frac{T}{z - 1} - \frac{1 - e^{-T}}{z - e^{-T}} \tag{5.27}$$

from which it can be determined:

$$H(z) = \frac{KG(z)}{1 + KG(z)}$$

$$= \frac{K[(T+e^{-T}-1)z + 1-e^{-T}-Te^{-T}]}{z^2 + [(T+e^{-T}-1)K-1-e^{-T}]z + K(1-e^{-T}-Te^{-T})+e^{-T}} \quad . \tag{5.28}$$

Let z_1, z_2 be roots of the characteristic equation:

$$z^2 + [(T+e^{-T}-1)K-1-e^{-T}]z + K(1-e^{-T}-Te^{-T})+e^{-T} = 0 \quad . \tag{5.29}$$

Since L is stable K should be such that $|z_1| < 1$, and $|z_2| < 1$. In order to find the range of values of K, for which $|z_1|, |z_2|$ are less than 1, we may use the following criterion applicable to a quadratic polynomial $p(z)$, with real coefficients and the coefficient of z^2 unity[25]:

The necessary and sufficient conditions for $p(z)$ to have no zeros outside the unit circle are:

$$|p(0)| < 1 \qquad p(1) > 0 \qquad p(-1) > 0 \quad . \tag{5.30}$$

The three conditions of (5.30) applied to the polynomial on the left hand side of equation (5.29) lead to the relations:

$$|K(1-e^{-T}-Te^{-T})+e^{-T}| < 1 \tag{5.31}$$

$$T(1-e^{-T})K > 0 \tag{5.32}$$

$$K < \frac{2(1+e^{-T})}{T(1+e^{-T})-2(1-e^{-T})} \quad . \tag{5.33}$$

From (5.32) it is obvious that K has to be positive. From (5.31) it follows that:

$$-1 < K(1-e^{-T}-Te^{-T})+e^{-T} < 1 \tag{5.34}$$

or

$$-\frac{1+e^{-T}}{1-e^{-T}-Te^{-T}} < K < \frac{1-e^{-T}}{1-e^{-T}-Te^{-T}} \quad . \tag{5.35}$$

Since (5.32) has to be satisfied (5.31) reduces to the requirement:

$$0 < K < \frac{1-e^{-T}}{1-e^{-T}-Te^{-T}} \quad . \tag{5.36}$$

Hence for stability K should be such that (5.33) and (5.36) are satisfied. By giving various values for T it can be seen that for small values of T, (5.36) gives the upper bound for K, whereas for larger values of T, (5.33) gives the upper bound for K. We will confine our discussion to the case when T is small, i.e. of the order of 1 or 2 seconds. In this case the bound on K for stability will be given by inequality (5.36). For this range of values of K, the roots z_1, z_2 of the characteristic equation (5.29) may be real and unequal, real and equal or complex conjugate. By equating the discriminant of (5.29) to zero, and solving for K we can determine the value of K for which $z_1 = z_2$. The value of K thus obtained turns out to be:

$$K = \frac{1-e^{-T}}{(1+T-e^{-T})+2\sqrt{T(1-e^{-T})}} \quad . \tag{5.37}$$

It can be easily verified that for values of K in the range 0 to that given by equation (5.37) z_1, z_2 are real, unequal and positive, whereas for values of K greater than that given by equation (5.37) upto the limiting value for stability given by (5.36) z_1, z_2 are complex conjugate. Thus we can say z_1, z_2 are:

(1) Real, unequal and positive if:

$$0 < K < \frac{1-e^{-T}}{(1+T-e^{-T})+2\sqrt{T(1-e^{-T})}} \quad . \tag{5.38}$$

(2) Real and equal if:

$$K = \frac{1-e^{-T}}{(1+T-e^{-T})+2\sqrt{T(1-e^{-T})}} \quad . \tag{5.39}$$

(3) Complex conjugate if:

$$\frac{1-e^{-T}}{(1+T-e^{-T})+2\sqrt{T(1-e^{-T})}} < K < \frac{1-e^{-T}}{1-e^{-T}-Te^{-T}} \quad . \tag{5.40}$$

Consider case (1). Let $z_1 > z_2$. Inverse transforming $H(z)$ using the residue theorem:

$$h(kT) = \begin{cases} 0\,, & \text{for } k = 0\,. \\ \dfrac{K}{z_1-z_2}[(T+e^{-T}-1)(z_1{}^k-z_2{}^k)+(1-e^{-T}-Te^{-T})(z_1{}^{k-1}-z_2{}^{k-1})]\,. \\ \quad \text{for } k > 0\,. \end{cases} \tag{5.41}$$

Hence

$$H^* = \sum_{k=o}^{\infty} |h(kT)| = \sum_{k=1}^{\infty} h(kT) \quad . \tag{5.42}$$

Note that the absolute value sign can be dispensed with in summing the series, as z_1, z_2 are real and positive, and $z_1 > z_2$, which ensures that each term of the series is positive for all $k = 1, 2, \ldots, \infty$. Moreover, each of the geometric series can be separately summed and are convergent because of the fact z_1, z_2 are less than 1 . Performing the summation we get:

$$\sum_{k=1}^{\infty} h(kT)$$

$$= \frac{K}{z_1-z_2}\left[\frac{(T+e^{-T}-1)z_1}{1-z_1} - \frac{(T+e^{-T}-1)z_2}{1-z_2} + \frac{1-Te^{-T}-e^{-T}}{1-z_1} - \frac{1-Te^{-T}-e^{-T}}{1-z_2}\right]$$

$$= \frac{K}{z_1-z_2}\left[\frac{(T+e^{-T}-1)(z_1-z_2)}{(1-z_1)(1-z_2)} + \frac{(1-e^{-T}-Te^{-T})(z_1-z_2)}{(1-z_1)(1-z_2)}\right]$$

$$= \frac{KT(1-e^{-T})}{(1-z_1-z_2+z_1z_2)} \quad . \tag{5.43}$$

Since z_1, z_2 are the roots of the equation (5.29) we have:

$$-(z_1+z_2) = K(T+e^{-T}-1)-e^{-T}-1 \tag{5.44}$$

and

$$z_1z_2 = K(1-e^{-T}-Te^{-T})+e^{-T} \quad . \tag{5.45}$$

Substituting (5.44) and (5.45) in (5.43):

$$H^* = \frac{KT(1-e^{-T})}{1+K(T+e^{-T}-1)-e^{-T}-1+K(1-e^{-T}-Te^{-T})+e^{-T}}$$

$$= \frac{KT(1-e^{-T})}{KT(1-e^{-T})}$$

$$= 1 \quad . \tag{5.46}$$

Now consider case (2). Here $z_1 = z_2 = z_o$ (say). Application of the reside theorem in this special case yields:

$$h(kT) = \begin{cases} 0\ , & \text{for } k = 0\ . \\ K[(T+e^{-T}-1)kz_o^{k-1}+(1-e^{-T}-Te^{-T})(k-1)z_o^{k-2}] & \text{for } k > 0\ . \end{cases} \quad (5.47)$$

$$\therefore\ H^* = \sum_{k=o}^{\infty} |h(kT)|$$

$$= K(T+e^{-T}-1) \sum_{k=1}^{\infty} kz_o^{k-1} + K(1-e^{-T}-Te^{-T}) \sum_{k=1}^{\infty} (k-1)z_o^{k-2}\ . \quad (5.48)$$

Note that

$$\sum_{k=1}^{\infty} kz_o^{k-1} \quad \text{and} \quad \sum_{k=1}^{\infty} (k-1)z_o^{k-2}$$

are the first derivatives of the geometric series

$$\sum_{k=1}^{\infty} z_o^{k} \quad \text{and} \quad \sum_{k=1}^{\infty} z_o^{k-1}$$

respectively, and hence their sums are

$$\frac{d}{dz}\left[\frac{z}{1-z}\right]\Bigg|_{z=z_o} \quad \text{and} \quad \frac{d}{dz}\left[\frac{1}{1-z}\right]\Bigg|_{z=z_o}$$

respectively. Performing the differentiation we get:

$$\sum_{k=1}^{\infty} kz_o^{k-1} = \frac{1}{(1-z_o)^2} \quad (5.49)$$

and

$$\sum_{k=1}^{\infty} (k-1)z_o^{k-2} = \frac{1}{(1-z_o)^2}\ . \quad (5.50)$$

Substituting (5.49) and (5.50) in equation (5.48):

$$H^* = \frac{K(T+e^{-T}-1)+K(1-e^{-T}-Te^{-T})}{(1-z_o)^2}$$

$$= \frac{KT(1-e^{-T})}{1-2z_o+z_o^2} \quad . \tag{5.51}$$

Since $z_o = z_1 = z_2$ we have:

$$-2z_o = -(z_1+z_2) = K(T+e^{-T}-1)-e^{-T}-1 \tag{5.52}$$

and

$$z_o^2 = z_1 z_2 = K(1-e^{-T}-Te^{-T})+e^{-T} \quad . \tag{5.53}$$

Substituting (5.52) and (5.53) in (5.51) we get as before:

$$H^* = 1 \quad . \tag{5.54}$$

As for case (3), $h(kT)$ can be found as in case (1); but it is difficult to get an expression for H^* in a closed form. However, an estimate for H^* can be found using equation (2.38). One may also conjecture that H^* increases from 1 to ∞ as the complex roots move from z_o to the circumference of the unit circle.

The above analysis shows that as long as K is such that z_1, z_2 are real, $H^* = 1$, and the permissible bounds for $\frac{d\phi(e)}{de}$ are given by:

$$0 < \frac{d\phi(e)}{de} < 2K \quad . \tag{5.55}$$

The best bounds are got when $z_1 = z_2 = z_o$; i.e. when

$$K = \frac{1-e^{-T}}{(1+T-e^{-T})+2\sqrt{T(1-e^{-T})}} \quad . \tag{5.56}$$

These bounds for various values of T are shown in Table 1.

TABLE 1

BEST BOUNDS FOR $\frac{d\phi(e)}{de}$ FOR BIBO STABILITY

USING CONTRACTION MAPPING

Sampling period T (in seconds)	1	2	3	4	5
Bounds for $\frac{d\phi(e)}{de}$: $0 < \frac{d\phi(e)}{de} <$	0.42	0.31	0.26	0.22	0.19

5.5 Comparison with some Existing Results:

In this section we will compare the results obtained above with those obtained by the application of a frequency domain criterion, originally proposed by Jury and Lee[3] for the absolute asymptotic stability of a class of nonlinear discrete-data systems, but later shown to be sufficient for the absolute BIBO stability as well by Iwens and Bergen[7]. The type of system considered by them has the same configuration as that represented in Fig. 5.1(a), but the constraints on the nonlinear gain function $\phi(e)$ are:

$$\phi(0) = 0 \tag{5.57}$$

$$0 < \frac{\phi(e)}{e} < K \tag{5.58}$$

$$\left|\frac{d\phi(e)}{de}\right| < K' \; . \tag{5.59}$$

It may be mentioned that (5.59) implies that:

$$K \leq K' \quad . \tag{5.60}$$

The criterion states that a sufficient condition for the absolute asymptotic as well as BIBO stability of the type of system considered is that the following relationship is satisfied on the unit circle for some non-negative q :

$$\text{Re } G(z)[1+q(z-1)] + \frac{1}{K} - \frac{K'q}{2} \left|(z-1)G(z)\right|^2 \geq 0 \quad . \tag{5.61}$$

In the above G(z) is the Z-transfer function of the linear plant in the forward path of the system.

Jury and Lee in their paper[3] consider the same system that has been discussed in section 5.4 above, as an example to illustrate the application of their criterion. By drawing the loci of G(z) , (z-1)G(z) and $(z-1)G(z) - \frac{K'}{2}\left|(z-1)G(z)\right|^2$ for $z = \exp(j\omega T)$ and several values of K' , they deduce that the stability criterion for this particular example can be reduced to the following inequality:

$$-(\frac{T}{2} + 1) + \frac{1}{K} + q(T - \frac{K'}{2} T^2) \geq 0 \quad . \tag{5.62}$$

Then they assert that if K and K' are restricted to:

$$K \leq K' < \frac{2}{T} \tag{5.63}$$

there always exists a non-negative q for which (5.62) is satisfied. Thus the condition for stability (both asymptotic as well as BIBO) of the system reduces to the requirement that the absolute value of the slope of the nonlinearity is less than $\frac{2}{T}$. Table 2 gives the bounds on the slope of the nonlinearity for stability for various values of T. For comparison, the corresponding bounds on the slope obtained by the contraction mapping technique are also given in the same table.

TABLE 2

BEST BOUNDS FOR $\frac{d\phi(e)}{de}$ FOR BIBO STABILITY

(JURY'S METHOD VS CONTRACTION MAPPING)

Sampling period T (in seconds)		1	2	3	4	5
Stability theorem of Jury and Lee	$\left\|\frac{d\phi(e)}{de}\right\| <$	2	1	0.67	0.50	0.40
Stability via contraction mapping	$0 < \frac{d\phi(e)}{de} <$	0.42	0.31	0.26	0.22	0.19

It must be conceded that the results obtained via contraction mapping are quite conservative compared with those of Jury and Lee. In fact, Jury and Lee state in a foot note to their paper[3], that Professor Y.Z. Tsypkin has indicated that condition (5.59) may be replaced by the more general condition:

$$-\infty \leq \frac{d\phi(e)}{de} < K' \qquad (5.63)$$

which gives still better bounds for K' for stability. However, as far as the contraction mapping method is concerned, the important point to be noted is that within the bounds given by the method, the actual solution of the system can be generated by a process of successive approximations, starting from the solution of the linear system as a first approximation, and using equation (5.10) recursively; and the convergence of the process is assured. Jury's method does not possess this advantage.

CHAPTER VI

GENERATION OF SYSTEM SOLUTION

6.1 Introduction.

In this chapter we will discuss the generation of the solution of a nonlinear discrete-data system, satisfying the contraction and fixed point conditions, by a process of successive approximations. It will be shown that the process results in a discrete Volterra type series, whose convergence is assured in the region in which the contraction and fixed point conditions hold.

6.2 Derivation of the Solution.

To illustrate the derivation of the solution consider the discrete-data system with a single cubic type of nonlinearity in the feedback path. The general configuration of the system is shown in Fig. 6.1.

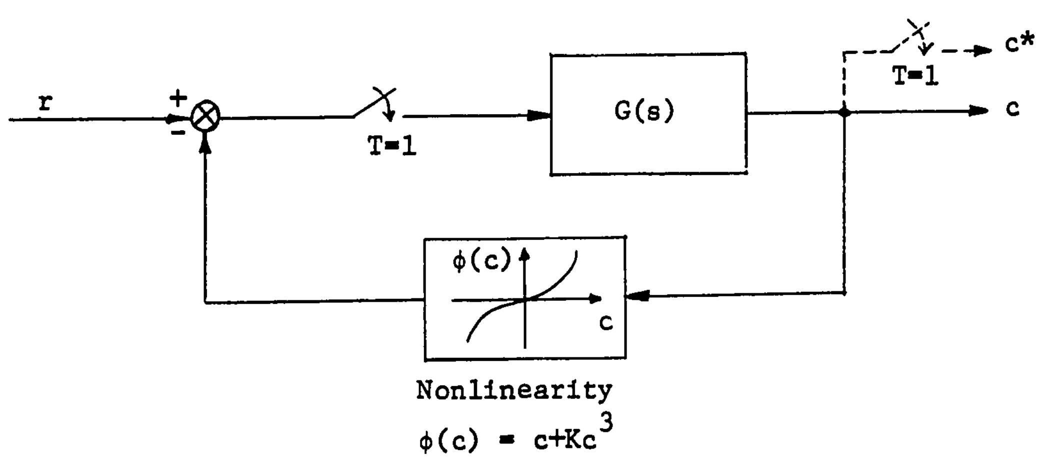

Fig. 6.1 Discrete-Data System with a Cubic Nonlinearity in the Feedback Path

It will be readily recognized that the system discussed in example 1, chapter 4.2 belongs to this type. For convenience the sampling time will be normalized, so that for the rest of the derivation it is understood that $T = 1$. The functional equation for c^* for the system is:

$$c^* = B^*r^* - B^*Kc^{*3} \quad . \tag{6.1}$$

For numerical evaluation, it is more convenient to rewrite the above equation for the output at the end of the n^{th} sampling period thus:

$$c(n) = Br(n) - BK[c(n)]^3 \tag{6.2}$$

where B represents the convolution summation operator for that sampling instant. B is defined by the relation:

$$Bx(n) = \sum_{k=o}^{n} h(n-k)x(k) \tag{6.3}$$

where $h(k)$ represents the impulse response of the linearized system at the end of the k^{th} sampling period. If equation (6.2) is solved for $c(n)$ we can determine the sampled outputs at the various sampling instants, by substituting $n = 1, 2, \ldots,$ etc. in the solution.

In deriving the solution we will be making the assumption that all initial conditions are zero, and the linearized system corresponding to the given system is stable. This means that for the linear case, i.e. when $K = 0$, the solution is:

$$c(n) = \sum_{k=o}^{n} h(n-k)r(k) \quad . \tag{6.4}$$

Since $r(n)$ is such that the contraction and fixed point conditions are satisfied, we can generate the solution of the system by a process of successive approximations, starting from the solution of the linearized system as a first approximation, and using equation (6.2) recursively. Let the output of the linearized system at the end of the n^{th} sampling period be denoted by $c_o(n)$. Then:

$$c_o(n) = \sum_{k=o}^{n} h(n-k)r(k) \tag{6.5}$$

will be our first approximation for the solution of (6.2). Substituting (6.5) on the right hand side of (6.2), we obtain the second approximation for $c(n)$, viz. $c_1(n)$.

$$c_1(n) = \sum_{k=o}^{n} h(n-k)r(k) - K \sum_{k=o}^{n} h(n-k)\left[\sum_{k_1=o}^{k} h(k-k_1)r(k_1)\right]^3 . \tag{6.6}$$

The last term in (6.6) can be written in the following way:

$$K \sum_{k=o}^{n} h(n-k) \left[\sum_{k_1=o}^{k} h(k-k_1)r(k_1)\right]^3$$

$$= K \sum_{k=o}^{n} h(n-k)\left[\sum_{k_1=o}^{k} h(k-k_1)r(k_1) \sum_{k_2=o}^{k} h(k-k_2)r(k_2) \sum_{k_3=o}^{k} h(k-k_3)r(k_3)\right]$$

$$= K \sum_{k=o}^{k} \sum_{k_1=o}^{k} \sum_{k_2=o}^{k} \sum_{k_3=o}^{k} h(n-k)h(k-k_1)h(k-k_2)h(k-k_3)r(k_1)r(k_2)r(k_3)$$

$$= K \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} \sum_{k_3=o}^{n} \left[\sum_{k = \max k_1,k_2,k_3}^{n} h(n-k)h(k-k_1)h(k-k_2)h(k-k_3)\right] \times$$

$$r(k_1)r(k_2)r(k_3) \quad . \tag{6.7}$$

The last expression in (6.7) is obtained by rearranging the order of summation.

Let us now define:

$$h_3 = \sum_{\substack{k = \max \\ k_1, k_2, k_3}}^{n} h(n-k)h(k-k_1)h(k-k_2)h(k-k_3) \quad . \tag{6.8}$$

Then (6.7) becomes:

$$K \sum_{k=o}^{n} h(n-k) \left[\sum_{k_1=o}^{k} h(k-k_1)r(k_1) \right]^3$$

$$= K \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} \sum_{k_3=o}^{n} h_3 r(k_1)r(k_2)r(k_3) \tag{6.9}$$

and substituting (6.9) into (6.6) we obtain:

$$c_1(n) = \sum_{k=o}^{n} h(n-k)r(k) - K \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} \sum_{k_3=o}^{n} h_3 r(k_1)r(k_2)r(k_3) \quad . \tag{6.10}$$

If we desire higher order approximations we must then substitute equation (6.10) into the right hand side of (6.2). We will then obtain the solution of equation (6.2) in the form of a discrete Volterra type series as follows:

$$c(n) = \sum_{k=o}^{n} h(n-k)r(k) - K \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} \sum_{k_3=o}^{n} h_3 r(k_1)r(k_2)r(k_3)$$

$$+ \ldots + \ldots \quad . \tag{6.11}$$

The above series is analogous to that obtained by Barrett[26] for a continuous system with a single cubic nonlinearity. The system discussed by him is described by the following differential equation:

$$L(\frac{d}{dt})y(t) + \varepsilon y^3(t) = x(t) \tag{6.12}$$

where ε is a constant and

$$L(\frac{d}{dt}) = L(p) = p^m + a_1 p^{m-1} + \ldots + a_m \tag{6.13}$$

$a_1, a_2, \ldots, a_m$ being constants. $x(t)$ is the continuous input and $y(t)$ the output. Letting $h(t)$ be the impulse response of the linear part of the differential equation (6.12), and assuming that for $t < 0$, $x(t) = y(t) = 0$, (6.12) can be converted to a nonlinear integral equation:

$$y(t) + \int_0^t \varepsilon h(t-u) y^3(u) du = \int_0^t h(t-u) x(u) du \quad . \tag{6.14}$$

If we denote the convolution operation by the symbol β , the above equation can be written as:

$$y(t) + \beta \varepsilon y^3(t) = \beta x(t) \tag{6.15}$$

or alternately:

$$y(t) = \beta x(t) - \beta \varepsilon y^3(t) \quad . \tag{6.16}$$

Comparing equation (6.16) with equation (6.1) or (6.2) it will be readily seen that they are analogous to each other. Proceeding by the method of successive approximations, Barrett gets the Volterra series solution for the equation (6.14) as:

$$y(t) = \int_0^t h(t-u)x(u)du - \int_0^t \int_0^t \int_0^t \varepsilon h_3 x(u_1)x(u_2)x(u_3)du_1du_2du_3 + \dots , \qquad (6.17)$$

where

$$h_3 = \int_0^t h(t-u)h(u-u_1)h(u-u_2)h(u-u_3)du \quad . \qquad (6.18)$$

NOTE: In the solution given by Barrett the limits of integration have been shown as from $-\infty$ to ∞; however, since $x(u) = 0$ for $u < 0$, and $h(t-u) = 0$ for $u > t$, it is enough if the integration is performed over the finite range 0 to t.

Comparing (6.17) and (6.11) the analogy between the continuous Volterra series solution and the discrete Volterra series solution will be evident. Moreover, the contraction mapping principle shows that the series represented by the equation (6.11) exists, is unique and is convergent in the region in which the contraction and fixed point conditions hold.

The validity of the Volterra series representation of the solution of a discrete-data system can be justified on the same basis as has been done for the continuous-data systems by Christensen[27]. The following extract from reference 27 will be relevant to our discussion:

"When can it be inferred that a Volterra series exists and is convergent for a given system if the contraction mapping principle shows that system to exhibit bounded input bounded-output stability? The answer to this question is that the given system in addition to having a unique solution must also be analytic.

To be more specific, consider the usual form of the Volterra series:

$$x_1(t) = \int_{-\infty}^{\infty} h_1(u)y(t-u)du + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u_1,u_2)y(t-u_1) y(t-u_2)du_1du_2 + \ldots + \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_i(u_1,\ldots,u_i)y(t-u_1)\ldots y(t-u_i)du_1\ldots du_i \tag{6.19}$$

which gives an output $x_1(t)$ for a given input $y(t)$ to the system in question. Brilliant[28] showed that if:

$$\|h_i\| = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} |h_i(u_1,\ldots,u_i)|du_1\ldots du_i < \infty \tag{6.20}$$

for all i such that (6.19) can be cast into the form:

$$X_1 \leq \sum_{i=o}^{\infty} \|h_i\| Y^i \tag{6.21}$$

where the right side of (6.21) is a convergent series then the given system is an analytic system."

For a discrete-data system, the discrete Volterra series analogous to (6.19) would be:

$$c(n) = \sum_{k=o}^{n} h_1(k)r(n-k) + \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} h_2(k_1,k_2)r(n-k_1)r(n-k_2)$$

$$+ \ldots + \sum_{k_1=o}^{n} \cdots \sum_{k_i=o}^{n} h_i(k_1,\ldots,k_i)r(n-k_1)\ldots r(n-k_i)+\ldots \quad . \tag{6.22}$$

Note that in (6.22) the integrals have been replaced by summations. Moreover, since we are making the assumption $r(n) = 0$ for $n < 0$, and $r(n-k) = 0$ for $k > n$, the summation need be performed only over the finite range 0 to n.

(6.22) can also be written as:

$$c(n) = \sum_{k=o}^{n} h_1(k)r(n-k) + \sum_{k_1=o}^{n} \sum_{k_2=o}^{n} h_2(n-k_1,n-k_2)r(k_1)r(k_2) +$$

$$\ldots + \sum_{k_1=o}^{n} \cdots \sum_{k_i=o}^{n} h_i(n-k_1,\ldots,n-k_i)r(k_1)\ldots r(k_i) + \ldots \tag{6.23}$$

Note that (6.23) corresponds to the series (6.11) generated by the process of iteration.

By analogy with the continuous case, the condition for analyticity of the discrete-data system may be stated as follows:

If

$$\|h_i\| = \sum_{k_1=o}^{\infty} \cdots \sum_{k_i=o}^{\infty} |h_i(k_1,\ldots,k_i)| < \infty \tag{6.24}$$

for all i such that (6.22) can be cast into the form:

$$\|c(n)\| \le \sum_{i=o}^{\infty} \|h_i\| \, \|r(n)\|^i \tag{6.25}$$

where the right hand side of (6.25) is a convergent series then the given discrete-data system is an analytic system.

Consider now the series generated in (6.11). It is readily recognized that this series can be cast into the form (6.25). Thus the system considered is analytic. As for the uniqueness of the solution, the contraction mapping principle shows that the solution generated is unique in the region in which the contraction and fixed point conditions hold. Hence it can be stated that a convergent Volterra series solution exists for the system considered.

CHAPTER VII

CONCLUSION

7.1. Summary.

In this thesis we have treated the bounded-input bounded-output stability of certain classes of nonlinear discrete-data systems using the methods of functional analysis, in particular, the contraction mapping principle. After discussing the stability of linear discrete-data systems from a functional point of view, sufficient conditions for the BIBO stability of discrete-data systems with a single polynomial type of nonlinearity have been developed using the contraction mapping technique. The results have been stated in the form of two theorems applicable to two different types of systems with nonlinearities located in different parts of the systems. A method of computing the bounds on the system inputs and outputs for stability in a special case of the nonlinearity has been presented, together with numerical examples to illustrate the technique. Another type of system investigated is the one with a slope restricted nonlinearity. A sufficient criterion for the BIBO stability of such a system has been formulated, and its application also illustrated using a system which has been treated by Jury and Lee by a frequency domain method. Finally a method of generating the solution of a nonlinear discrete-data system, satisfying the contraction and fixed point conditions, by a process of successive approximations has been given. It has been shown that the process results in a discrete Volterra type series, whose convergence is assured in the region in which the contraction and fixed point conditions hold.

7.2 Comments and Suggestions.

Since the contraction condition is a strong requirement, the stability bounds obtainable by this method are bound to be on the conservative side. The frequency domain method is superior in this respect, as is observable in the case of the example discussed under slope restricted nonlinearity. But the most important advantage of the contraction mapping method lies in the fact that once the stability region is found, it enables one to generate the system solution by a process of successive approximations, starting from an initial estimate within the region, and the convergence of the process is assured. The frequency domain method does not possess this advantage. Moreover, the frequency domain criteria stipulate that the nonlinearity must be entirely contained in a certain sector; they thus usually refer only to global stability. The contraction mapping method can, however, be applied to study local stability, particularly when the nonlinearity is explicitly given in a functional form, such as a polynomial.

Although the contraction mapping technique presented in this thesis deals exclusively with a single-input single-output system, it is believed that the method can be generalised and made applicable to multiple-input multiple-output systems. It is also believed that by choosing suitable norms, it should be possible to investigate asymptotic stability, as well as stability in between sampling period. These are some of the avenues open for research in this area.

In conclusion the author wishes to state that the main contribution of this thesis in the field of Control Engineering is the exploitation of some of the basic principles of Functional Analysis for the investigation of the BIBO stability of a class of nonlinear discrete-data systems. The scope of the thesis is perforce confined to theoretical exposition, though the author realises that the value of the investigations could have been enhanced by supplementing it with some computer simulation results; this could perhaps form part of a separate project for future investigations.

Some attempt has been made to bring out the practical importance of BIBO stability, by choosing standard practical models such as the integral control servo to illustrate the application of theory. Since, for practical purposes, the mere assurance of BIBO stability is not enough, an attempt has also been made to obtain an estimate for the input and output bounds in the examples discussed in Chapter IV. In the present state of the development of the nonlinear theory, it is indeed difficult to obtain exact bounds; it is hoped future work in this area will help to resolve this difficulty.

Finally the author wishes to remark that while the practising engineer is apt to have little patience with theoretical work, it is still a rewarding field as even the purest of pure mathematics has the habit of eventually finding its way into application.

REFERENCES

1. R.E. Kalman and J.E. Bertram, "Control system analysis and design via the second method of Liapunov, II, Discrete systems," Transactions ASME, Journal of Basic Engineering, Vol. 82, pp 397-400, June 1960.

2. Ya. Z. Tsypkin, "Fundamentals of the theory of nonlinear pulse control systems:, Automatic and Remote Control (Theory), Proceedings of the Second Congress of the IFAC, 1963, pp 172-180, Butterworth, 1964.

3. E.I. Jury and B.W. Lee, "On the stability of certain class of nonlinear sampled-data systems", IEEE Transactions Automatic Control, Vol. AC-9, pp 51-61, January 1964.

4. E.I. Jury and B.W. Lee, "On the absolute stability of nonlinear sampled-data systems", IEEE Transactions, Automatic Control, Vol. AC-9, pp 551-554, October 1964.

5. E.I. Jury and B.W. Lee, "A stability theory for multinonlinear control systems", Proc. 1966 IFAC Conf., pp 28A.1-28A.11.

6. G.W. Johnson, "On Liapunov stability vs. bounded-input bounded-output," IEEE Transactions, Automatic Control, Vol. AC-9, pp 178, 179, April 1964.

7. R.P. Iwens and A.R. Bergen, "Frequency criteria for bounded-input bounded-output stability of nonlinear sampled-data systems", IEEE Transactions, Automatic Control, Vol. AC-12 No. 1, pp 46-53, February 1967.

8. G. Zames, "Functional analysis applied to nonlinear feedback sytems", IEEE Transactions, Circuit Theory, Vol. CT-10, No. 3, pp 392-404, September 1963.

9. I.W. Sandberg, "A note on the application of the contraction mapping fixed point theorem to a class of nonlinear functional equations", SIAM Review, 7, No. 2, pp 199-204, April 1965.

10. C.A. Desoer, "Nonlinear distortion in feedback amplifiers," IRE Transactions, Circuit Theory, Vo. CT-9, No. 1, pp 2-6, March 1962.

11. B.J. Leon and D.R. Anderson, "The stability of pumped nonlinear reactance circuits", IEEE Transactions, Circuit Theory, Bol. CT-10, pp 468-476, December 1963.

12. K.N. Leibovic, "The principle of contraction mapping in nonlinear and adaptive control systems", IEEE Transactions, Automatic Control, Vol. AC-9, No. 4, pp 393-398, October 1964.

13. G.S. Christensen, "Aspects of nonlinear system stability", Ph. D. Thesis, University of British Columbia, Vancouver, B.C., Canada, October 1966.

14. Z. Bubnicki, "On the stability condition of nonlinear sampled-data systems", IEEE Transactions, Automatic Control, Vol. AC-9, pp 280-281, July 1964.

15. M.S. Epel'man, "A criterion for the absolute stability of nonlinear sampled-data automatic systems", Avtomatika i Telemekhanica, Vol. 27, No. 10, pp 74-77, October 1966.

16. R.S. Rao and G.S. Christensen, "Contraction mapping applied to stability analysis of a class of nonlinear discrete-data systems", Conference record of Second Asilomar Conference on Circuits and Systems, pp 56-60, October 1968.

17. E. I. Jury, "Sampled-data control systems", John Wiley and Sons, New York, 1958.

18. E.I. Jury, "Theory and application of the Z-transform method", John Wiley and Sons, New York, 1964.

19. B. C. Kuo, "Analysis and synthesis of sampled-data control systems", Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

20. A.E. Taylor, "Introduction to Functional Analysis", John Wiley and Sons, New York, 1967.

21. W. A. Porter, "Modern foundations of systems engineering", Macmillan Co., New York, 1966.

22. L.A. Lusternik and V.J. Sobolev, "Elements of functional analysis", Translated from Russian, Hindustan Publishing Corpn. (India), Delhi, 1961.

23. T.L. Saaty and J. Bram, "Nonlinear mathematics", McGraw-Hill Book Co., New York, 1964.

24. E.I. Jury and M.A. Pai, "Convolution Z-transform method applied to certain nonlinear discrete systems", IRE Transactions, Automatic Control, Vol. AC-7, pp 57-64, January 1962.

25. J.G. Truxal, "Control system synthesis", McGraw-Hill Book Co., New York, 1955 (pp 523).

26. J.F. Barrett, "The use of Volterra series to find region of stability of a nonlinear differential equation", International Journal of Control, 1965, pp 209-216.

27. G.S. Christensen, "On the convergence of Volterra series", IEEE Transactions, Automatic Control, Vol. AC-13, No. 6, December, 1968, pp 736, 737.

28. M.B. Brilliant, "Theory of the analysis of nonlinear systems", MIT research Lab. of Electronics, Report 345, 1959.