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OSCILLATION CRITERIA FOR NONLINEAR DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

by

© Qingkai Kong

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Oscillation criteria for nonlinear differential and integro-differential equations submitted by Qingkai Kong in partial fulfilment of the requirements for the Degree of Master of Science.

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ABSTRACT

We obtain some Olech-Opial-Wazewski oscillation criteria for the second order differential equation

$$y''(t) + q(t)f(y(t)) = 0$$

and its prototype, the generalized Emden-Fowler equation

$$y''(t) + q(t)|y(t)|^{\alpha}\operatorname{sgn} y(t) = 0, \quad \alpha > 0.$$

Similar results are also obtained for the integro-differential equation

$$r(t)y'(t) = c - \int_a^t f(y(s))d\sigma(s)$$

and the difference equation

$$\Delta(c_{n-1}\Delta y_{n-1}) + b_n f(y_n) = 0 \ , \ n = 0,1,2,\ldots \ .$$

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CHAPTER 1.

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

We are mainly concerned in this thesis with the following second order nonlinear differential equation

$$y''(t) + q(t)f(y(t)) = 0 , \quad t \in R_+ = [0,\infty) \qquad (N)$$

and its prototype, the generalized Emden-Fowler equation

$$y''(t) + q(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) = 0 , \quad t \in R_+ = [0,\infty), \quad \alpha > 0 . \qquad (N_{\alpha})$$

In (N) we shall assume that $f \in C(R) \cup C^2[(-\infty,0) \cup (0,+\infty)]$ with

$$yf(y) > 0, \quad f'(y) > 0 \quad \text{for} \quad y \neq 0 . \qquad (1.1.1)$$

The function $q(t)$ is assumed to be continuous on R_+ but not necessarily to be nonnegative throughout this thesis.

Definition 1.1.1 A solution of (N) or (N_{α}) is said to be oscillatory if it has arbitrarily large zeros, i.e., for any $T \in [0,\infty)$ there exists $t \geq T$ such that $y(t) = 0$.

Equation (N) or (N_{α}) is said to be oscillatory if every continuable solution is oscillatory.

Remark 1.1.1. We wish to point out that not every solution of (N) or (N_α) is continuable throughout $[0,\infty)$ when $q(t)$ assumes negative values for arbitrarily large values of t. For results on continuability of solutions of the above equations we refer to Wong [39] and Butler [7]. There will always exist infinitely many continuble solutions under the above conditoins on q, f, α [7].

We shall first state a series of oscillation results for equation (N) and (N_α) based on the comparison theory and the integral averaging technique. And then we shall give some Olech-Opial-Wazewski oscillation criteria for (N) and (N_α).

We will extend our results to more general self-adjoint equations

$$(r(t)y'(t))' + q(t)f(y(t)) = 0 \qquad (\hat{N})$$

and

$$(r(t)y'(t))' + q(t)|y(t)|^\alpha \operatorname{sgn} y(t) = 0 , \qquad (\hat{N}_\alpha)$$

where $r(t) \gtrless 0$ is continuous on $[0,\infty)$.

We will also discuss the oscillatory behavior of the Stieltjes integro-differential equations

$$r(t)y'(t) = c - \int_a^t f(y(s))d\sigma(s) \qquad (\hat{S})$$

and

$$r(t)y'(t) = c - \int_a^t |y(s)|^\alpha \operatorname{sgn} y(s)d\sigma(s), \qquad (\hat{S}_\alpha)$$

where $c = (ry')(a)$, $r, \sigma: R_+ \to R$ are right continuous functions of locally bounded variation on R_+ such that $r(t) > 0$ and $r^{-1}(t) \in L(I)$ where $I = [a,t] \subset R_+$.

By a solution of $(\hat{S})$ or $(\hat{S}_\alpha)$ we will mean a function $y(t) \in LAC(R_+)$ (i.e., locally absolutely continuous on R_+) satisfying $(\hat{S})$ or $(\hat{S}_\alpha)$ for almost all points on R_+. $y'(t)$ denotes the right derivative of $y(t)$.

Remark 1.1.2. Equation $(\hat{S})$ and $(\hat{S}_\alpha)$ are of more general forms than equation $(\hat{N})$ and $(\hat{N}_\alpha)$. In fact, if we let $\sigma(t) \in C^1(R_+)$, and $\sigma'(t) = q(t)$, then equation $(\hat{S})$ and $(\hat{S}_\alpha)$ become equation $(\hat{N})$ and $(\hat{N}_\alpha)$, respectively.

When $r(t) \equiv 1$, we denote $(\hat{S})$ and $(\hat{S}_\alpha)$ by (S) and (S_α). As an application we will give some analogous oscillation criteria to the corresponding difference equations

$$\Delta(c_{n-1}\Delta y_{n-1}) + b_n f(y_n) = 0 , \quad n = 0,1,2,\ldots, \qquad (\hat{D})$$

and

$$\Delta(c_{n-1}\Delta y_{n-1}) + b_n |y_n|^\alpha \operatorname{sgn} y_n = 0 , \quad n = 0,1,2,\ldots, \qquad (\hat{D}'_\alpha)$$

where $c_n > 0$ $(n = -1,0,1,\ldots)$, $\{b_n\}$ $(n = 0,1,\ldots)$ is any given real sequence and f is assumed to satisfy all conditions for (N), $\Delta y_n = y_{n+1} - y_n$.

When $c_n = 1$ $(n = -1,0,1,\ldots)$, we denote $(\hat{D})$ and $(\hat{D}_\alpha)$ by (D) and (D_α), respectively.

<u>Definition 1.1.2</u> A solution of equation $(\hat{D})$ or $(\hat{D}_\alpha)$ is called oscillatory if for any integer $N \geq 0$ there is an $n \geq N$ such that $y_n y_{n+1} \leq 0$.

Equation $(\hat{D})$ or $(\hat{D}_\alpha)$ is called oscillatory if all continuable solutions of it are oscillatory.

Let $\{t_n\}$ be an increasing sequence of points such that $t_{-1} = a$ and $t_n \to \infty$ as $n \to \infty$. Define the step functions $\sigma(t)$ and $r(t)$ as follows:

$$\sigma(t_n) - \sigma(t_n - 0) = b_n \tag{1.1.2}$$

and

$$r(t) = c_{n-1}(t_n - t_{n-1}), \quad t \in [t_{n-1}, t_n). \tag{1.1.3}$$

Note that $\sigma(t)$ and $r(t)$ are right-continuous and of locally bounded variation on $[a,\infty)$. So according to [42] we can conclude that all solutions of $(\hat{S})$ are polygonal curves whose "vertices" are the points $(t_n, y(t_n))$ where $y_n = y(t_n)$ satisfy $(\hat{D})$. On the other hand, we can also prove that for any solution y_n of $(\hat{D})$ with $c_{-1}\Delta y_{-1} = c$ there is a solution of $(\hat{S})$ which is a polygonal curve whose "vertices" are the points (t_n, y_n). In fact, since all solutions of $(\hat{S})$ are polygonal curves, assume $y(t)$ is a solution of $(\hat{S})$ with $y(a) = y_{-1}$.

Thus $y'(a) = \dfrac{y(t_0) - y(a)}{t_0 - a}$, from which we have

$$c = r(a)y'(a) = c_{-1}(y(t_0) - y(a)).$$

Comparing with $c_{-1}\Delta y_{-1} = c$ we get $y(t_0) = y_0$. In general, if we assume $y(t_i) = y_i$, $i = -1,0,1,\ldots,n$, then from $(\hat{S})$ we have

$$(ry')(t_n) = (ry')(t_{n-1}) - \int_{t_{n-1}}^{t_n} f(y(s))d\sigma(s),$$

i.e.,

$$c_n(y(t_{n+1}) - y(t_n)) = c_{n-1}(y(t_n) - y(t_{n-1})) - b_n f(y(t_n)).$$

From $(\hat{D})$ we have

$$c_n(y_{n+1}-y_n) = c_{n-1}(y_n - y_{n-1}) - b_n f(y_n).$$

From the above discussion we obtain the following proposition.

Proposition 1.1 Equation $(\hat{D})$ is oscillatory if and only if equation $(\hat{S})$ with conditions (1.1.2) and (1.1.3) is oscillatory for any $c \in R$.

Theorem 4.5.1'. Let conditions (H) and (4.3.3) hold and $-\infty < \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i (i = 1,2,3,4) and $\varepsilon > 0$ satisfying (4.2.1), $(\bar{D}_1)$, $(\bar{D}_2)$ and $(\bar{D}_3)$. Then equation $(\hat{S}_\alpha)$ is oscillatory.

Theorem 4.5.2'. Let conditions (H) and (4.3.3) hold and $-\infty = \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i (i = 1,2,3,4) , $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (4.2.1) , $(\bar{D}_1')$, $(\bar{D}2)$ and $(\bar{D}3)$. Then equation $(\hat{S}_\alpha)$ is oscillatory.

Theorem 4.5.3'. Let conditions (H) and (4.3.3) hold and $\ell_1 < \ell_2$. Assume there are an $h_2 \in (-\infty,\ell_2)$ and a $k \in (\max\{m,0\},1)$ satisfying $(\bar{D}_1')$, and for any $h_3 \in (-\infty,\ell_2)$ there is an $\varepsilon > 0$ satisfying $(\bar{D}_2)$. Then equation $(\hat{S}_\alpha)$ is oscillatory.

4.6 Interpretation for difference equation $(\hat{D})$.

In this section we assume $\sum_{n=0}^{\infty} C_{n-1}^{-1} = +\infty$ in equation $(\hat{D})$. We omit the results for the case that $\sum_{n=0}^{\infty} C_{n-1}^{-1} < +\infty$ since it is easy to obtain it by following the idea described in sections 4.3 and 4.5. As in section 3.3. we choose $t_n = n$ (n = -1,0,1,...0). Then

$$\sigma(n) - \sigma(n-0) = b_n \text{ , } r(t) = C_{n-1} \quad t \in [n-1,n), \text{ and}$$

$$Q(t) = \int_0^t d\sigma(s) = \sum_{i=1}^{n} b_i \text{ for } t \in [n,n+1).$$

According to the definition

$$\ell_1 = \lim_{t\to\infty} \text{approx inf } Q(t) = \liminf_{n\to\infty} \sum_{i=1}^{n} b_i$$

and

$$\ell_2 = \lim_{t\to\infty} \text{approx sup } Q(t) = \limsup_{n\to\infty} \sum_{i=1}^{n} b_i .$$

For any $h_i \in (\ell_1,\ell_2)$ $(i = 1,2,3,4)$ satisfying (4.2.1) and for any $p \in N$ define

$$\sigma_{ip} = \{n\colon 1 \le n < p ,\ \sum_{j=1}^{n} b_j < h_i\}$$

$$s_{ip} = \{n\colon 1 \le n < p ,\ \sum_{j=1}^{n} b_j > h_i\}$$

and

$$\sigma_i(p) = \bigcup_{n\in\sigma_{ip}} [n-1,n),\quad s_i(p) = \bigcup_{n\in s_{ip}} [n-1,n)$$

where $i = 1,2,3,4$. By an easy calculation we see

$$\int_0^p r^{-1}(s)ds = \sum_{n=1}^{p} C_{n-1}^{-1}$$

and if $\alpha(p)$ is either $\sigma_i(p)$ or $s_i(p)$

$$\hat{\rho}(\alpha(p)) = \int_{\alpha(p)} r \text{[illegible]} / \int_0^p r^{-1}(s)ds$$

$$= \sum_{n\epsilon\alpha_p} C_{n-1}^{-1} / \sum_{n=1}^{p} C_{n-1}^{-1} .$$

Before stating some theorems we give the conditions corresponding to (D1) - (D3):

(E1) $\limsup_{p\to\infty} \left(\sum_{n\epsilon s_{2p}} C_{n-1}^{-1} - \varepsilon \sum_{n=1}^{p} C_{n-1}^{-1} \right) = +\infty ,$

(E1') $\limsup_{p\to\infty} \left(\sum_{n\epsilon s_{2p}} C_{n-1}^{-1} - \delta(M,n,k) \sum_{n=1}^{p} C_{n-1}^{-1} \right) = +\infty,$

(E2) $\limsup_{p\to\infty} \left(\sum_{n\epsilon\sigma_{3p}} C_{n-1}^{-1} - \varepsilon \sum_{n=1}^{p} C_{n-1}^{-1} \right) = +\infty,$

(E3) $\limsup_{p\to\infty} \left(\sum_{n\epsilon s_{4p}\cup\sigma_{1p}} C_{n-1}^{-1} - \varepsilon \sum_{n=1}^{p} C_{n-1}^{-1} \right) = +\infty.$

In particular, when $C_i \equiv 1$ $(i = -1,0,1,\ldots)$, we let $\mu(\alpha)$ be the cardinal measure of any finite set α. Then (E1')-(E3) become

(F1) $\limsup_{p\to\infty} (\mu(s_{2p})-\varepsilon p) = +\infty$,

(F1') $\limsup_{p\to\infty} (\mu(s_{2p})-\delta(M,m,k)p) = +\infty$,

(F2) $\limsup_{p\to\infty} (\mu(\sigma_{3p})-\varepsilon p) = +\infty$,

(F3) $\limsup_{p\to\infty} (\mu(s_{4p}\cup\sigma_{1p})-\varepsilon p) = +\infty$.

Theorem 4.6.0. Let condition (H) hold and $\ell_2 = +\infty$. Assume

$$\limsup_{p\to\infty} \Big(\sum_{n\in s_{\ell p}} C_{n-1}^{-1}-\delta p\Big) = +\infty \qquad (4.6.1)$$

for all ℓ sufficiently large, where $\delta = \dfrac{M^2}{M^2-4m+4}$,

$$S_{\ell p} = \{n:\ 1 \le n < p\ ,\ \sum_{j=1}^{n} b_j > \ell\}.$$

Then equation $(\hat{D})$ is oscillatory.

Theorem 4.6.1. Let condition (H) hold and $-\infty < \ell_1 < \ell_2 \le +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$ and $\varepsilon > 0$ satisfying (4.2.1), (E1), (E2) and (E3). Then equation $(\hat{D})$ is oscillatory.

Theorem 4.6.2. Let condition (H) hold and $-\infty = \ell_1 < \ell_2 \le +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$, $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (2.1), (E1'), (E2) and (E3). Then equation $(\hat{D})$ is oscillatory.

Theorem 4.6.3. Let condition (H) hold and $\ell_1 < \ell_2$. Assume there are an $h_4 \in (-\infty, \ell_2)$ and a $k \in (\max\{m,0\},1)$ satisfying (E1'), and for any $h_3 \in (-\infty, \ell_2)$ there is an $\varepsilon > 0$ satisfying (E2). Then equation $(\hat{D})$ is oscillatory.

For equation (D) we can replace (E1) - (E3) by (F1) - (F3), and the above theorems also holds. We do not state them here.

Example 4.6.1. Consider the difference equation

$$\Delta^2 y_{n-1} + (-1)^n \left[\frac{n}{2}\right] |y_n|^\alpha \operatorname{sgn} y_n = 0, \quad n = 0,1,2,\ldots \tag{4.6.2}$$

where $\alpha > 0$, $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$. Here we have $c \equiv 1$, $n = -1,0,1,\ldots$, and $b_n = (-1)^n\left[\frac{n}{2}\right]$, $n = 0,1,2,\ldots$, i.e., $b_{2n} = n$, $b_{2n+1} = -n$. Hence $\sum_{i=1}^{2n} b_i = n-1$, $\sum_{i=1}^{2n+1} b_i = -1$, and

$$\ell_1 = \liminf_{n\to\infty} \sum_{i=1}^{n} b_i = \lim_{n\to\infty} \sum_{i=1}^{2n+1} b_i = -1$$

$$\ell_2 = \limsup_{n\to\infty} \sum_{i=1}^{n} b_i = \lim_{n\to\infty} \sum_{i=1}^{2n} b_i = +\infty .$$

Choose h_i $(i = 1,2,3,4)$ satisfying (4.2.1) and $0 < \varepsilon < \frac{1}{2}$. We find that $\mu(s_{2p}) \to \frac{1}{2}$, $\mu(\sigma_{3p}) \to \frac{1}{2}$, and $\mu(s_{4p} \cup \sigma_{1p}) \to 1$ as $p \to \infty$, therefore (F1), (F2) and (F3) hold. According to Theorem 4.6.1 equation (4.6.2) is oscillatory.

Note that for equation (4.6.2), if we choose $m = \frac{\alpha-1}{\alpha}$, $M = \left| \frac{\alpha-1}{\alpha} \right|$, then $\delta = \frac{M^2}{M^2-4m+4} = (\frac{\alpha-1}{\alpha+1})^2$, $\delta < \frac{1}{2}$ gives us that $\alpha \in (\frac{\sqrt{2}-1}{\sqrt{2}+1}, \frac{\sqrt{2}+1}{\sqrt{2}-1})$. Thus (4.6.1) hold only for $\alpha \in (\frac{\sqrt{2}-1}{\sqrt{2}+1}, \frac{\sqrt{2}+1}{\sqrt{2}-1})$. Therefore Theorem 4.6.0 fails when $\alpha \notin (\frac{\sqrt{2}-1}{\sqrt{2}+1}, \frac{\sqrt{2}+1}{\sqrt{2}-1})$.

Example 4.6.2. Consider the difference equation

$$\Delta^2 y_{n-1} + \cos \pi(n+\tau_n)|y_n|^\alpha \operatorname{sgn} y_n = 0 , \quad n = 0,1,2,\ldots \qquad (4.6.3)$$

where $\alpha > 0$, $\sum_{n=0}^{\infty} \tau_n^2 < \infty$. Here we have $c_n \equiv 1$, $n = -1,0,1,\ldots$, and

$$b_n = \cos \pi(n+\tau_n) = \cos(\pi n)\cos(\pi\tau_n)$$

$$= (-1)^n \cos(\pi\tau_n) , \quad n = 0,1,2,\ldots$$

Since $\tau_n \to 0$ as $n \to \infty$,

$$\cos(\pi\tau_n) = 1 - 2\sin^2\left(\frac{\pi}{2}\tau_n\right) = 1 - \frac{\pi^2}{2}\left(\tau_n^2 + o(\tau_n^2)\right) \quad \text{as} \quad n \to \infty.$$

Since $\sum_{n=0}^{\infty} \tau_n^2 < \infty$, we have

$$\sum_{n=0}^{\infty} \left(\tau_n^2 + |o(\tau_n^2)|\right) < \infty . \qquad (4.6.4)$$

Then $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (-1)^i \left[1 - \frac{\pi^2}{2}\left(\tau_n^2 + o(\tau_n^2)\right)\right]$ is uniformly bounded for all $n \in N$. That means $-\infty < \ell_1 \leq \ell_2 < +\infty$. From (4.6.4) we see there is a $N > 0$ such that

$$\sum_{i=2N}^{\infty} \left(\tau_i^2 + |o(\tau_i^2)|\right) < \frac{1}{2\pi^2} .$$

Let $\sum_{i=1}^{2N-1} b_i = M$. Then for any $k > N$

$$\sum_{i=1}^{2k-1} b_i = \sum_{i=1}^{2N-1} b_i + \sum_{i=2N}^{2k-1} b_i$$

$$= M + \sum_{i=2N}^{2k-1} (-1)^i \left[1 - \frac{\pi^2}{2}\left(\tau_i^2 + o(\tau_i^2)\right)\right]$$

$$\leq M + \frac{\pi^2}{2} \sum_{i=2N}^{2k-1} (\tau_i^2 + |o(\tau_i^2)|)$$

$$\leq M + \frac{\pi^2}{2} \sum_{i=2N}^{\infty} (\tau_i^2 + |o(\tau_i^2)|)$$

$$\leq M + \frac{1}{4} ,$$

and

$$\sum_{i=1}^{2k-1} b_i = \sum_{i=1}^{2N-1} b_i + \sum_{i=2N}^{2k} (-1)^i [1 - \frac{\pi^2}{2} (\tau_i^2 + o(\tau_i^2))]$$

$$\geq M + 1 - \frac{\pi^2}{2} \sum_{i=2N}^{2k} (\tau_i^2 + |o(\tau_i^2)|)$$

$$\geq M + 1 - \frac{\pi^2}{2} \sum_{i=2N}^{\infty} (\tau_i^2 + |o(\tau_i^2)|)$$

$$\geq M + \frac{3}{4} .$$

Therefore

$$\ell_1 = \liminf_{n\to\infty} \sum_{i=1}^{n} b_i \leq \liminf_{k\to\infty} \sum_{i=1}^{2k-1} b_i \leq M + \frac{1}{4} ,$$

$$\ell_2 = \limsup_{n\to\infty} \sum_{i=1}^{n} b_i \geq \limsup_{k\to\infty} \sum_{i=1}^{2k} b_i \geq M + \frac{3}{4} ,$$

Hence $\ell_1 < \ell_2$. Choose h_i ($i = 1,2,3,4$) satisfying (4.2.1) and

$0 > \varepsilon < \frac{1}{2}$, we find that $\mu(s_{2p}) \to \frac{1}{2}$, $\mu(\sigma_{3p}) \to \frac{1}{2}$ and $\mu(s_{4p} \cup \sigma_{1p}) \to 1$ as $p \to \infty$. Thus (F1), (F2) and (F3) hold. According to Theorem 4.6.1 equation (4.6.3) is oscillatory.

Example 4.6.3. Consider the difference equation

$$\Delta^2 y_{n-1} + b_n |y_n|^\alpha \operatorname{sgn} y_n = 0\ ,\ n = 0,1,2,\ldots, \qquad (4.6.5)$$

where $\alpha > 0$, $b_n = \begin{cases} -2 & , \ n = 2k-1 \\ 2^{1-\alpha} & , \ n = 2k \end{cases}$ $k \in N$. Here

$$\sum_{i=1}^{n} b_n \to -\infty\ , \text{ as } n \to \infty .$$

We can not conclude that equation (4.6.5) is oscillatory by Theorems 4.6.0-4.6.3. It is easy to check that equation (4.6.5) has a nonoscillatory solution $y_n = \begin{cases} 1 & , \ n = 2k-1 \\ 2 & , \ n = 2k \end{cases}$ $k \in N$. In fact,

$$\Delta y_{2n-1} = y_{2n} - y_{2n-1} = 1,$$

$$\Delta y_{2n} = y_{2n+1} - y_{2n} = -1,$$

$$\Delta^2 y_{2n-1} = \Delta(y_{2n} - y_{2n-1}) = \Delta y_{2n} - \Delta y_{2n-1} = -2\ ,$$

hence

$$\Delta^2 y_{2n-1} + b_{2n} y_{2n}^{\alpha} = 0 ,$$

and

$$\Delta^2 y_{2n} = \Delta(y_{2n+1} - y_{2n}) = \Delta y_{2n+1} - \Delta y_{2n} = 2 ,$$

hence

$$\Delta^2 y_{2n} + b_{2n+1} y_{2n+1}^{\alpha} = 0 .$$

This example shows that the conditions of Theorems 4.6.0-4.6.3 are sharp in a certain sense.

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1.2 Criteria for linear oscillation

Before stating some oscillation results for nonlinear equations we present some basic previous results on the second order linear differential equation

$$y''(t) + q(t)y(t) = 0 , \quad t \in R, \tag{L}$$

where $q(t) \in C[0,\infty)$. In the study of this equation from the point of view of disconjugacy on R_+, many criteria for oscillation have been found which involve the behavior of the integral of $q(t)$. Let

$$Q(t) = \int_0^t q(s)\, ds, \quad t \geq 0 . \tag{1.2.1}$$

Three of the most important such conditions which guarantee that all solutions of (L) oscillate are as follows:

(A1) $\lim_{t\to\infty} Q(t)$ +∞ , (Fite [17], Wintner [35])

(A2) $\lim_{t\to\infty} \frac{1}{t} \int_0^t Q(s)ds = +\infty$, (Wintner [35])

(A3) $-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_0^t Q(s)ds < \limsup_{t\to\infty} \frac{1}{t} \int_0^t Q(s)ds \leq +\infty$. (Hartman [18]).

Coles [11] and Willett [34] extended these criteria by considering weighted average of integrals of $q(t)$ of the form

$$Q_\varphi(t) = \int_0^t \varphi(s)Q(s)ds / \int_0^t \varphi(s)ds.$$

Thus Willett [34] showed that there is a class Φ_0 of nonnegative locally integrable, but not integrable on $[0,\infty)$, functions, which contains all such bounded functions, such that if for some $\varphi \in \Phi_0$, we have

(a2) $\lim_{t\to\infty} Q_\varphi(t) = +\infty$,

or

(a3) $-\infty < \liminf_{t\to\infty} Q_\varphi(t) < \limsup_{t\to\infty} Q_\varphi(t) \leq +\infty$.

Then all solutions of (L) are oscillatory.

It is obvious that (a2) and (a3) are extensions of the criteria (A2) and (A3), which together correspond to (a2) and (a3) with $\varphi \equiv 1$.

Kamenev gave another condition for oscillation of (L), i.e.,

(A4) $\limsup_{t\to\infty} \frac{1}{t^n} \int_0^t (t-s)^n q(s)ds = +\infty$, $n > 1$.

We can easily check that (A1) implies (A4) with $n = 2$. We should note that (A4) with $n = 1$, i.e.,

(A5) $\limsup_{t\to\infty} \frac{1}{t} \int_0^t Q(s)ds = +\infty$

alone is not sufficient for the oscillation of (L), see Hartman [18].

In a different direction, Olech, Opial and Wazewski gave another criterion for oscillation of (L). To state the results we need the following definition.

Definition 1.2 Assume $h = h(t)$ is a real-valued function defined on $[a,\infty)$, $a \geq 0$, and $-\infty \leq \ell$, $L \leq +\infty$, $\mu(s)$ denote the Labesgue measure. Then we write

$$\lim_{t\to\infty} \text{approx sup } h(t) = L$$

in case

$$\mu\{t: h(t) > L_1\} = +\infty \quad \text{for all } L_1 < L \text{ and}$$

$$\mu(t: h(t) \geq L_2\} < +\infty \quad \text{for all } L_2 > L.$$

Similarly,

$$\lim_{t\to\infty} \text{approx inf } h(t) = \ell$$

in case

$$\mu\{t: h(t) < \ell_1\} < +\infty \quad \text{for all } \ell_1 < \ell \text{ and}$$

$$\mu(t: h(t) < \ell_2\} = +\infty \quad \text{for all } \ell_2 > \ell.$$

Finally, $\lim_{t\to\infty} \text{approx } h(t) = \ell$, $-\infty \leq \ell \leq +\infty$ in case

$$\lim_{t\to\infty} \text{approx sup } h(t) = \lim_{t\to\infty} \text{approx inf } h(t) = \ell.$$

According to the definition we see that

$$\liminf_{t\to\infty} h(t) \leq \lim_{t\to\infty} \text{approx} \inf h(t) \leq \lim_{t\to\infty} \text{approx} \sup h(t) \leq \limsup_{t\to\infty} h(t),$$

and if $\lim_{t\to\infty} h(t) = \ell$, then $\lim_{t\to\infty} \text{approx}\, h(t) = \ell$.

Theorem 1.2 [31]. Equation (L) is oscillatory in case

$$\lim_{t\to\infty} \text{approx}\ Q(t) = +\infty \tag{1.2.2}$$

or in case

$$\lim_{t\to\infty} \text{approx} \inf Q(t) < \lim_{t\to\infty} \text{approx} \sup Q(t). \tag{1.2.3}$$

In particular, if

$$\lim_{t\to\infty} \text{approx} \sup Q(t) = +\infty ,$$

then equation (L) is oscillatory.

CHAPTER 2.

COMPARISON THEOREMS

2.1 Introduction.

In 1836 Sturm proved a famous result usually known as the Sturm-Comparison Theorem for the linear equation of the form

$$(p(t)y'(t))' - q(t)y(t) = 0.$$

In that thesis Sturm considered the equations

$$(p_1y')' - q_1y = 0 \tag{2.1.1}$$

and

$$(p_2y')' - q_2y = 0 \tag{2.1.2}$$

on a finite interval and showed that if $0 < p_2 \leq p_1$, $q_2 \leq q_1$, equality not holding everwhere on the interval, then between any two zeros of some solution of (2.1.1) there is at least one zero of any solution of (2.1.2).

This theorem gives us a way to get the oscillatory property of a linear equation by comparing it with another linear equation which is known to be oscillatory. Several extensions of the Theorem have been made for the linear oscillation. In a similar way we can obtain some interesting oscillation criteria for nonlinear equations by comparing them with related linear equations.

In the following we present some results mainly from [28].

2.2 For equation (N)

Consider the relation between equation (N) and the linear equation

$$y''(t) + q(t)y(t) = 0 \ , \ t \in [0,\infty). \qquad (2.2.1)$$

<u>Definition 2.2</u> Equation (2.2.1) is called strongly oscillatory if the equation

$$y''(t) + cq(t)y(t) = 0 \ , \qquad (2.2.2)$$

is oscillatory for every positive constant c.

Equation (2.2.1) is called weakly oscillatory if equation (2.2.2) is oscillatory for some positive constant c.

<u>Theorem 2.2.1</u> (Wong [38]) Suppose that f(y) satisfies (1.1.1) and there is a positive constant c such that $f'(y) \geq c > 0$ for all $y \in (-\infty,+\infty)$. If, for this fixed c, (2.2.2) is oscillatory, then equation (N) is also oscillatory.

<u>Theorem 2.2.2</u> (Erbe [14]) Suppose that the function f(y) satisfies

$$f'(y) \geq \frac{f(y)}{y} > 0 \quad \text{for} \quad y \neq 0, \qquad (2.2.3)$$

(Note that condition (2.2.3) implies (1.1.1)) and that $q(t)$ satisfies

$$\liminf_{t\to\infty} \int_T^t q(s)ds > 0$$

for all large T. If (2.2.1) is strongly oscillatory, then (N) is oscillatory.

Let us point out that an application of Theorem 2.2.1 gives the following oscillation criterion for equation (N).

Corollary 2.2 Let $f(y)$ satisfy (1.1.1) and let there be a constant $c > 0$ such that $f'(y) \geq c$ for all $y \in (-\infty,+\infty)$. Suppose condition (A5) holds and there exists a constant $b > 0$ such that

$$\int_{t_0}^t q(s)\, ds \geq -e^{-bt}, \quad t \geq t_0 \tag{2.2.4}$$

for some $t_0 \geq 0$. Then equation (N) is oscillatory.

Remark 2.2 This result generalizes a theorem of Onose [32, Theorem 3, p. 71], where condition (2.2.4) is replaced by the stronger assumption

$$\liminf_{t\to\infty} Q(t) \geq -L > -\infty,$$

and $f(y)$ satisfies an additional nonlinearity condition.

2.3 For equation (N_α)

Because of the limited applications of the theorems in section 2.2, we present some further criteria for the oscillation of equation (N_α).

Suppose $\lim_{t\to\infty} Q(t)$ exists and is finite first, where $Q(t)$ is defined by (1.2.1). Denote $P(t) = \int_t^\infty q(s)ds$ and

$p(t) = \max\{1, -\int_0^t P(s)ds\}$.

<u>Theorem 2.3.1</u> Suppose equation

$$(p(t)z')' + q(t)z = 0$$

is strongly oscillatory. Then (N_α) is oscillatory for $\alpha > 1$.

<u>Corollary 2.3.1</u> Suppose equation (2.2.1) is strongly oscillatory, and

$$\liminf_{t\to\infty} \int_0^t P(s)ds > -\infty . \tag{2.3.1}$$

Then (N_α) is oscillatory for $\alpha > 1$.

<u>Corollary 2.3.2</u> Suppose (2.3.1) holds and, assume that there exists a

positive nondecreasing function $\varphi \in C^1[0,\infty)$ satisfying

$$\int^{\infty} \frac{\varphi'^2}{\varphi} < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_0^t \varphi(s)q(s)ds = +\infty .$$

Then (N_α) is oscillatory for $\alpha > 1$.

Now we assume $Q(t)$ is bounded below instead of the existence of $\lim_{t\to\infty} Q(t)$.

<u>Theorem 2.3.2</u> Suppose

$$\liminf_{t\to\infty} Q(t) > -\infty .$$

If equation

$$(tz'(t))' + cq(t)z(t) = 0 \qquad (2.3.2)$$

is weakly oscillatory, then (N_α) is oscillatory for $\alpha > 0$.

<u>Remark 2.3</u> Many well-known linear oscillation criteria are stated only for equation (2.2.1). To apply theorem 2.3.2 we can transform (2.3.2) by the familiar Liouville transformation $t = e^s$ and $z(s) = y(t)$ to

$$z''(s) + ce^s q(e^s)z(s) = 0, \quad s \geq 0. \qquad (2.3.2)$$

Clearly the oscillatory behavior of (2.3.2) and (2.3.3) remains the same.

<u>Corollary 2.3.3</u> Let $tq(t)$ be either bounded above or below and satisfy

$$-\infty < \liminf_{t\to\infty} Q(t) < \limsup_{t\to\infty} Q(t) \leq +\infty.$$

Then (N_α) is oscillatory.

Before stating the next theorem we introduce some new concepts. Consider two equations

$$(p(t)z')' + q(t)z = 0 \qquad (2.3.4)$$

$$(p(t)z')' + (q(t)+\eta(t))z = 0 \qquad (2.3.5)$$

where $p(t) > 0$, $q(t)$ and $\eta(t)$ are continuous.

<u>Definition 2.3</u> Suppose equation (2.3.4) is (strongly) oscillatory, and for any $\eta(t)$ satisfying $\lim_{t\to\infty} \int_0^t \eta(s)ds$ exists and is finite, equation (2.3.5) is also (strongly) oscillatory. Then equation (2.3.4) is (strongly) S1-oscillatory.

Suppose equation (2.3.4) is (strongly) oscillatory, and for any $\eta(t)$ satisfying $\int_0^t \eta(s)ds$ is bounded, equation (2.3.5) is also (strongly) oscillatory. Then equation (2.3.4) is (stongly) S2-oscillatory.

Here we mention two conditions which are relevant to the above concepts. We say that function $R(t)$ satisfies condition (*) or (**) if

(*) There exists an increasing sequence $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$ such that

$$\liminf_{n\to\infty} \lambda_n \int_{s_n} \frac{dt}{t} > 0 \, ,$$

where s_n is the set $\{t \in [0,\infty): R(t) \geq \lambda_n\}$; or

(**) There exist two numbers $\lambda_1 < \lambda_2$ such that

$$\int_0^\infty \frac{1}{t} [R(t) - \lambda_2]_+^2 \, dt = \int_0^\infty \frac{1}{t} [R(t) - \lambda_1]_-^2 \, dt = \infty,$$

where $[g(t)]_+ = \max\{g(t),0\}$, $[g(t)]_- = \max\{-g(t),0\}$ for any function $g(t)$.

<u>Proposition 2.3</u> Suppose equation (2.3.4) is (strongly) oscillatory.

i) If $p(t) = t$ and $Q(t)$ satisfies (*) or (**), then (2.3.4) is (strongly) oscillatory;

ii) if $p(t) = t$ and $Q(t)$ satisfies (*) or $p(t) = t$ and $Q(t)$

satisfies (**) for some λ_1 and all $\lambda_2 > \lambda_1$, then (2.3.4) is strongly S2-oscillatory.

<u>Theorem 2.3.3</u> Suppose $\varphi \in C^2[0,\infty)$ satisfying

$$\varphi'^2(t) \geq \varphi(t)\varphi''(t)$$

for all large t, and

$$\int_0^t \frac{q(s)}{\varphi(s)}\,ds \geq -L > -\infty \ .$$

If the linear equation $(tz')' + q(t)z = 0$ is strongly S1-oscillatory, then

$$y''(t) + \varphi(t)q(t)|y(t)|^\alpha \operatorname{sgn} y(t) = 0 \ , \ \alpha > 1 \ , \ t \in [0,\infty)$$

is oscillatory.

2.4 Further results for (N_α), $0 < \alpha < 1$.

<u>Theorem 2.4.1</u> Suppose there exists a function $\varphi \in C^2[0,\infty)$ satisfying $\varphi'(t) \geq 0$, $\varphi''(t) \leq 0$, $t \geq 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s \varphi^\alpha(\tau)q(\tau)d\tau ds > -\infty \ . \qquad (2.4.1)$$

If the equation

$$(tz'(t))' + \varphi^\alpha(t)q(t)z(t) = 0 \ , \quad t \in [0,\infty) \qquad (2.4.2)$$

is strongly S2-oscillatory, then (N_α) with $0 < \alpha < 1$ is oscillatory.

<u>Theorem 2.4.2</u> Suppose there exists a function $\varphi \in C^2[0,\infty)$ satisfying $\varphi'(t) \geq 0$, $\varphi''(t) \leq 0$, $t \geq 0$, and

$$\lim_{t\to\infty} t\varphi'(t)/\varphi(t) \text{ exists and is finite} \tag{2.4.3}$$

such that $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s \varphi^\alpha(\tau)q(\tau)d\tau ds$ exists and is finite. If (2.4.2) is strongly S1-oscillatory, then (N_α) with $0 < \alpha < 1$ is oscillatory.

Suppose instead of (2.4.3) φ satisfies the stronger condition

$$\lim_{t\to\infty} t\varphi'(t)/\varphi(t) = 0$$

and $q(t)$ satisfies (2.4.1). If (2.4.2) is strongly S1-oscillatory, then (N_α) with $0 < \alpha < 1$ is oscillatory.

<u>Theorem 2.4.3</u> Suppose that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t Q(s)ds > -\infty.$$

If the linear equation

$$(tz'(t))' + q(t)z(t) = 0$$

is weakly oscillatory, then (N_α) with $0 < \alpha < 1$ is oscillatory.

<u>Remark 2.4.</u> Theorem 2.4.3 includes a result of Butler in [9], which state the condition (A3) in section 1.2 implies (N_α) with $0 < \alpha < 1$ is oscillatory.

CHAPTER 3

INTEGRAL AVERAGING TECHNIQUE FOR OSCILLATION

In this section we will first state a series of criteria for nonlinear oscillation, especially for equation (N_α), which are the analogues of the integral averaging technique for linear oscillation. The proofs can be found in [9,19,33,37,40].

Wong gave a new proof for all of the results in [41]. Using his idea we will extend all of the results to the Stieltjes integro-differential equation (S_α) and the differential equation (D_α).

3.1. For equation (N_α)

We list below the major results relating to the generalization of oscillation criteria (A1)-(A5) in chapter 1, and the conditions

(A3') $\liminf_{t\to\infty} \frac{1}{t}\int_0^t Q(s)ds < \limsup_{t\to\infty} \frac{1}{t}\int_0^t Q(s)ds$

(A6) $\liminf_{t\to\infty} Q(t) > -\infty$

to equation (N_α):

(I) Waltman [33]: (A1) => (N_α) oscillatory for $\alpha > 0$,

(II) Kamenev [19]: (A5) => (N_α) oscillatory for $0 < \alpha < 1$,

(III) Butler [9]: (A2) => (N_α) oscillatory for $\alpha > 0$,

(IV) Butler [9]: (A3) => (N_α) oscillatory for $\alpha > 1$,

(V) Butler [9]: (A3') => (N_α) oscillatory for $0 < \alpha < 1$,

(VI) Wong [40]: (A4)(A6) => (N_α) oscillatory for $\alpha > 0$,

(VII) Wong [37]: (A5)(A6) => (N_α) oscillatory for $\alpha > 1$.

With the exception that (III) => (I), since condition (A1) follows from (A2), all other theorems are independent of one another. The most sophisticated results are three theorems of Butler (III) (IV) and (V) which were in fact proved for the more general equation (N) with some rather complicated assumptions which are satisfied by (N_α), $\alpha > 0$. However as far as applying general averaging conditions of the type (a2) or (a3) to (N) is concerned, we have had only limited success.

Wong gave a new simple method to prove all of the results (I)-(VII). His main idea is depicted below:

Assume (N_α) has a nonoscillatory solution $y(t)$, which can be assumed to be positive on $[t_0,\infty)$ for some t_0. Define $z(t) = y^{1-\alpha}(t)$ when $\alpha \neq 1$. It is easy to verify from (N_α) that $z(t)$ satisfies the following equation

$$z'' = (\alpha-1)q + \alpha(\alpha-1)^{-1}z^{-1}z'^2 \tag{3.1.1}$$

on $[t_0,\infty)$. Denote $m = \alpha-1$ and $n = \alpha(\alpha-1)^{-1}$. Integrating both sides of (3.1.1) twice and dividing them by t, we have

$$-\frac{n}{t}\int_{t_0}^{t}\int_{t_0}^{s} z^{-1}z'^2 + \frac{1}{t}z(t) = u(t_0) + \frac{1}{t}z(t_0) + \frac{m}{t}\int_{t_0}^{t} Q \ , \tag{3.1.2}$$

where $u(t) = z'(t) - mQ(t)$. (3.1.2) would in each case lead to a desired contradiction.

3.2 For Stieltjes integro-differential equation (S_α)

Theorem 3.2 If we replace $Q(t)$ by $\int_0^t d\sigma(s)$ in conditions (A1)-(A6), then all of the results (I)-(VII) hold for equation (S_α), where c is any constant.

Proof. Assume (S_α) has a non-oscillatory solution $y(t)$, which can be assued to be positive on $[t_0,\infty)$ for some t_0. Then $y(t)$ satisfies

$$y'(t) = c - \int_{t_0}^{t} y^\alpha(s)d\sigma(s),$$

where $c = y'(t_0)$. Define $z(t) = y^{1-\alpha}(t)$ when $\alpha \neq 1$. Then $y(t) = z^{\frac{1}{1-\alpha}}(t)$, $y'(t) = \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} z'$. Substituting them into equation (3.2.1) we have

$$z'(t) = c(1-\alpha)z^{-\frac{\alpha}{1-\alpha}}(t) - (1-\alpha)z^{-\frac{\alpha}{1-\alpha}}(t) \int_{t_0}^{t} z^{\frac{\alpha}{1-\alpha}}(s)d\sigma(s)$$

$$= c(1-\alpha)z^{-\frac{\alpha}{1-\alpha}}(t)-(1-\alpha)\int_{t_0}^{t} d\sigma(s)-(1-\alpha)\int_{t_0}^{t} [\int_{t_0}^{s} z^{\frac{\alpha}{1-\alpha}}(\tau)d\sigma(\tau)]d(z^{-\frac{\alpha}{1-\alpha}}(s))$$

$$= c(1-\alpha)z^{-\frac{\alpha}{1-\alpha}}(t) -(1-\alpha)\int_{t_0}^{t} d\sigma(s) -(1-\alpha)\int_{t_0}^{t} (c- \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} z')d(z^{-\frac{\alpha}{1-\alpha}})$$

$$= c(1-\alpha)^{-} z^{\frac{\alpha}{1-\alpha}} (t_0) -(1-\alpha)\int_{t_0}^{t} d\sigma(s) - \alpha(1-\alpha)^{-1}\int_{t_0}^{t} z^{-1}z'^{2}ds$$

$$= z'(t_0) - (1-\alpha)\int_{t_0}^{t} d\sigma(s) + \alpha(\alpha-1)^{-1}\int_{t_0}^{t} z^{-1}z'^{2}ds.$$

Let $m = \alpha-1$, $n = \alpha(\alpha-1)^{-1}$, and $u(t) = z'(t) - m\int_0^t d\sigma(s)$. Then

$$u(t) = u(t_0) + n\int_{t_0}^{t} z^{-1}z'^{2}ds. \tag{3.2.2}$$

Integrating (3.2.2) and dividing it by t we have

$$-\frac{n}{t}\int_{t_0}^{t}\int_{t_0}^{s} z^{-1}z'^{2} + \frac{1}{t} z(t) = u(t_0) + \frac{1}{t} y(t_0) + \frac{m}{t}\int_{t_0}^{t} (\int_0^s d\sigma(\tau)). \tag{3.2.3}$$

If we replace $Q(t)$ by $\int_0^t d\sigma(s)$, then (3.1.2) becomes (3.2.3) immediately. Since all of the results (I)-(VII) contradict (3.1.2), we can conclude that after changing $Q(t)$ to $\int_0^t d\sigma(s)$, every condition in (I) - (VII) also guarantees the oscillation of equation (S_α).

3.3 For difference equation (D_α)

As indicated in section 1.1, under the assumptions (1.1.2) and (1.1.3) equation (S_α) becomes equation (D_α). Now we are going to derive some oscillatory results on equation (D_α) from Theorem 3.2.1. To simplify the discussion we assume $t_n = n$ $(n = -1, 0, 1, \ldots)$. Then $\sigma(n) - \sigma(n-0) = b_n$, and $r(t) = c_{n-1}$, $t \in [n-1, n)$. It is easy to see

$$Q(t) = \int_0^t d\sigma(s) = \sum_{i=1}^{n} b_i \quad \text{for} \quad t \in [n, n+1).$$

Therefore conditions (A1)-(A6) in sections 1.2 and 3.1 become conditions (B1)-(B6) respectively:

(B1) $\sum_{i=1}^{\infty} b_i = +\infty$,

(B2) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} b_i = +\infty$,

(B3) $-\infty < \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i < \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i \le +\infty$, (B3') $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i < \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i$,

(B4) $\limsup_{n\to\infty} \frac{1}{n^k} \sum_{i=1}^{n} (n-i)^k b_i = +\infty,\ k > 1$,

(B5) $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i = +\infty$,

(B6) $\liminf_{n\to\infty} \sum_{i=1}^{n} b_i > -\infty$.

According to theorem 3.2 and the discussion in section 1.1 we obtain

<u>Theorem 3.3</u>

(B1) => (D_α) oscillatory for $\alpha > 0$,

(B5) => (D_α) oscillatory for $0 < \alpha < 1$,

(B2) => (D_α) oscillatory for $\alpha > 0$,

(B3) => (D_α) oscillatory for $\alpha > 1$,

(B3') => (D_α) oscillatory for $0 < \alpha < 1$,

(B4)(B6) => (D_α) oscillatory for $\alpha > 1$.

CHAPTER 4

OLECH-OPIAL-WAZEWSKI OSCILLATION CRITERIA

4.1 Introduction

In section 1.1 we have stated the Olech-Opial-Wazewski criterion (Theorem 1.1) for linear oscillation which uses the special limits given by Definition 1.1. Now we shall extend the result to nonlinear oscillation. More precisely, we will obtain conditions under which all continuable solutions of (N) or (N_α) are oscillatory by considering the asymptotic density of the sets $\{t: Q(t) > h\}$ and $\{t: Q(t) < k\}$ for various values of h,k.

In [10] Theorem 1.1 was shown to have one kind of nonlinear analogues expressed in terms of the nonlinearity as well as the asymptotic density of the set where $Q(t)$ is sufficiently positive. And it was shown that this was sharp in a certain sense.
In this paper we shall show that an extension of (1.2.3) is also valid for (N) and (N_α). This involves a more detailed analysis than in [10]. As an example, we are then able to show that (N_α) is oscillatory for all $\alpha > 0$ if $q(t) = t^{\beta-1}\varphi(t^\beta)$, $\beta > 0$, where $\varphi(u)$ is periodic with period $\omega > 0$ and $\int_0^\omega \varphi(s)ds \geq 0$. An example is also given to indicate that the conditions are again, in some sense sharp. We refer to [10,28,36,38,39] for additional discussions of the oscillation/nonoscillation problem for (N) and (N_α). The results which we obtain also are somewhat related to results of Kwong and Zettl (see [28] for further discussion of these results). Moreover, we hope

that our results make a little more precise the general idea that oscillation for (N) and (N_α) is caused by Q(t) becoming sufficiently positive on a "sufficiently large" set or by its oscillatory behavior on two "sufficiently large sets".

In section 4.2 we present the main results with proofs and in section 4.3 we give some extensions to the more general self-adjoint equation

$$(r(t)y'(t))' + q(t)f(y(t)) = 0 \qquad (\hat{N})$$

and

$$(r(t)y'(t))' + q(t)|(y(t)|^\alpha \operatorname{sgn} y(t) = 0 \qquad (\hat{N}_\alpha)$$

where $r(t) > 0$.

In section 4.4 we will show some examples and applications, in sections 4.5 and 4.6 we will give the interpretation of the results for equations ($\hat{S}$), ($\hat{S}_\alpha$) and ($\hat{D}$), ($\hat{D}_\alpha$).

Before giving the main results in the next section, we introduce some notation. For any set $S \subset [0,+\infty)$, we define $S(t) = S \cap [0,t]$ and the density function by

$$\rho(S(t)) \equiv \frac{\mu(S(t))}{t} \qquad (4.1.1)$$

where μ denotes Lebesgue measure. We denote by g(y) the expression

$$g(y) \equiv \frac{f''(y)f(y)}{(f'(y))^2}, \quad y \neq 0 \tag{4.1.2}$$

and note that $g(y)$ is continuous for $y \neq 0$. If $f(y) = |y|^\alpha \operatorname{sgn} y$, $\alpha > 0$, then $g(y) = \frac{\alpha-1}{\alpha}$. We shall assume the following condition holds:

(H) there exist numbers $M > 0$ and $m < 1$ such that

$$g(y) \leq m < 1$$

and for all $y \neq 0$

$$|g(y)| \leq M$$

For later comparison purposes we state the following result which is Theorem 2.1 of [10].

<u>Theorem 4.1.1</u>: Let condition (H) hold and assume there exists a set $S \subset [0,+\infty)$ such that

(i) $\limsup_{t\to\infty} t(\rho(S(t)) - \delta) = +\infty$

and

(ii) $\lim_{\substack{t\to\infty \\ t\in S}} Q(t) = +\infty$,

where $Q(t) = \int_0^t q(s)ds$ and $\delta = \frac{M^2}{M^2-4m+4}$. Then equation (N) is oscillatory.

An equivalent version of Theorem 4.1.1 may be reformulated as follows.

<u>Theorem 4.1.2:</u> Let condition (H) hold and assume

(i)' $\lim\limits_{t\to\infty} \text{approx} \sup Q(t) = +\infty$

and

(ii)' $\limsup\limits_{t\to\infty} t(\rho(S_\ell(t) - \delta) = +\infty$

for all ℓ sufficiently large, where

$$S_\ell(t) = \{t \in [0,\infty): Q(t) > \ell\}.$$

Then equation (N) is oscillatory.

To verify that the condition (i) and (ii) of Theorem 4.1.1 are equivalent to (i)' and (ii)' of Theorem 4.1.2, suppose that (i)' and (ii)' hold. Choose $t_1 \geq 1$ with $\mu(S_1) \geq 1$, where $S_1 = \{t \in [0,t_1]: Q(t) \geq 1\}$. Then choose $t_2 \geq 2$ such that $\mu(S_2) \geq 1$ where $S_2 = \{t \in [t_1,t_2]: Q(t) \geq 2\}$, and in general choose $t_n \geq n$ such that $\mu(S_n) \geq 1$ where $S_n = \{t \in [t_{n-1},t_n]: Q(t) \geq n\}$. Clearly $S_n \neq \emptyset$, $n = 1,2,\ldots$ and if we set $S = \bigcup_{n=1}^{\infty} S_n$ then it follows that

$$\lim_{\substack{t\to\infty \\ t\in S}} Q(t) = +\infty .$$

For $\ell = n$ there is a sequence $t_n^k \to \infty$ as $k \to \infty$ such that

$\lim_{k\to\infty} t_n^k(\rho(S_n(t_n^k))-\delta) = +\infty$, $n = 1,2,\ldots$. For $n = 1,2,\ldots$ we let $\tau_n = t_n^{k_n} \to \infty$ as $n \to \infty$ and $\lim \tau_n(\rho(S_n(\tau_n))-\delta) = +\infty$. From the definition of S we see that $\rho(S(\tau_n)) \geq \rho(S_n(\tau_n))$ and hence $\tau_n(\rho(S(\tau_n))-\delta) \geq \tau_n(\rho(S_n(\tau_n))-\delta) \to \infty$ as $n \to \infty$ and so $\limsup_{t\to\infty} t(\rho(S(t))-\delta) = +\infty$. Suppose that (i) and (ii) hold. Then for any ℓ sufficiently large there is a $T > 0$ such that when $t \in S \cap [T,+\infty)$ $Q(t) > \ell$. Hence $S_\ell \supset S \cap [T,+\infty)$ and so $\lim\text{ approx}_{t\to\infty} \sup Q(t) = +\infty$ and $\limsup_{t\to\infty} t(\rho(S_\ell(t))-\delta) = +\infty$. That is, (i)' and (ii)' hold.

4.2. Main results.

In the sequel we will use the following notation. For the function $q(t)$ given in (N) and ($\hat{N}$), let $Q(t) = \int_0^t q(s)ds$ and

$$\lim_{t\to\infty}\text{ approx inf } Q(t) = \ell_1 ,$$

$$\lim_{t\to\infty}\text{ approx sup } Q(t) = \ell_2 .$$

For any h_i $(i = 1,2,3,4) \in (\ell_1,\ell_2)$ satisfying

$$\ell_1 < h_1 < h_2 < h_3 < h_4 < \ell_2 \qquad (4.2.1)$$

define

$$\sigma_i = \{t \in [0,\infty): Q(t) < h_i\}$$

$$S_i = \{t \in [0,\infty): Q(t) > h_i\}$$

and

$$S_i(t) = S_i \cap [0,t],\ \sigma_i(t) = \sigma_i \cap [0,t] .$$

It is obvious that $S_1(t) \supset S_2(t) \supset S_3(t) \supset S_4(t)$ and $\sigma_1(t) \subset \sigma_2(t) \subset \sigma_3(t) \subset \sigma_4(t)$.

For convenience we introduce some conditions.

(C1) $\limsup_{t\to\infty} t(\rho(S_2(t))-\varepsilon) = +\infty$,

(C1') $\limsup_{t\to\infty} t(\rho(S_2(t))-\delta(M,m,k)) = +\infty$,

(C2) $\limsup_{t\to\infty} t(\rho(\sigma_3(t))-\varepsilon) = +\infty$,

(C3) $\limsup_{t\to\infty} t(\rho((S_4\cup\sigma_1)(t))-\varepsilon) = +\infty$,

where $\varepsilon > 0$ is a constant, $\delta(M,m,k) = \dfrac{M^2}{M^2-4k(m-k)}$ for some constant k.

THEOREM 4.2.1. Let condition (H) hold and $-\infty < \ell_1 < \ell_2 \le +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$ and $\varepsilon > 0$ satisfying (4.2.1), (C1), (C2), and (C3). Then equation (N) is oscillatory.

THEOREM 4.2.2. Let condition (H) hold and $-\infty = \ell_1 < \ell_2 \le +\infty$. Assume there are constants h_i ($i = 1,2,3,4$), $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (4.2.1), (C1'), (C2) and (C3). Then equation (N) is oscillatory.

Theorem 4.2.3. Let condition (H) hold and $\ell_1 < \ell_2$. Assume there are $h_2 \in (-\infty,\ell_2)$ and $k \in (\max\{m,0\},1)$ satisfying (C1'), and for any $h_3 \in (-\infty,\ell_2)$ there is an $\varepsilon > 0$ satisfying (C2). Then equation (N) is oscillatory.

Remark 4.2. The conditions of Theorem 4.2.3 imply that $\ell_1 = -\infty$. But Theorem 4.2.3 gives different conditions for oscillation of equation (N) from those of Theorem 4.2.2. The conditions (C1), (C1)', (C2), (C3) imply, of course, that the limit of the asymptotic density of the sets appearing in these conditions cannot be zero. Thus, the measure of these sets cannot be of the order of t^{η}, $0 < \eta < 1$.

Proof of Theorem 4.2.1: Suppose equation (N) is not oscillatory and let $y = y(t)$ be a nontrivial nonoscillatory solution on $[0,\infty)$ which we may suppose satisfies $y(t) > 0$ on $[t_0,\infty)$, $t_0 \geq 0$. Define the Riccati variable

$$z(t) = -y'(t)/f(y(t)), \qquad (4.2.2)$$

and we obtain the Riccati equation

$$z'(t) = q(t) + f'(y(t))z^2(t), \quad t \geq t_0$$

which we may integrate to obtain

$$z(t) = z(t_0) + Q(t) + \int_{t_0}^{t} f'(y(s))z^2(s)ds - Q(t_0)$$

(4.2.3)

$$= z(t_0) + Q(t) + w(t) - Q(t_0), \quad t \geq t_0$$

where

$$w(t) = \int_{t_0}^{t} f'(y(s))z^2(s)ds \ , \quad t \geq t_0 \ . \qquad (4.2.4)$$

Now $\ell_1 < \ell_2$ implies that $q(t) \not\equiv 0$ on any half-line $[t_1,\infty)$ and so $z(t) \not\equiv 0$ on any half-line $[t_1,\infty)$. It follows that $w(t) > 0$ for t sufficiently large. Without loss of generality we may assume that $w(t) > 0$ for $t \geq t_0$. Then we can set

$$\lim_{t\to\infty} w(t) = \alpha$$

where $0 < \alpha \leq +\infty$.

i) If $\alpha = +\infty$, define functions $p(t)$ and $u(t)$ as follows:

$$p(t) = z(t)/w(t) \ , \quad u(t) = \frac{1}{f'(y(t))w(t)} \ .$$

We see that $u(t) > 0$ for $t \geq t_0$. From (2.2) and (2.4) we have

$$y'(t) = -f(y(t))z(t)$$

and

$$w'(t) = f'(y(t))z^2(t) \ . \qquad (4.2.5)$$

Thus we get

$$u'(t) = -\frac{f''(y(t))y'(t)}{[f'(y(t))]^2 w(t)} - \frac{w'(t)}{f'(y(t))w^2(t)}$$

$$= \frac{f''(y(t))f(y(t))z(t)}{[f'(y(t))]^2 w(t)} - \frac{z^2(t)}{w^2(t)}$$

$$= g(y(t))p(t) - p^2(t) , \quad t \geq t_0 . \qquad (4.2.6)$$

Choose $h_0 < \ell_1$ and let

$$S_0 = \{t \in [0,\infty): Q(t) > h_0\}$$

Then $\mu([t_0,\infty)\backslash S_0) < +\infty$, i.e., for any $\delta \in (0,1)$

$$\lim_{t\to\infty} t(\rho(S_0(t))-\delta) = +\infty. \qquad (4.2.7)$$

According to (4.2.3) we know for $t \in S_0$

$$(p(t)-1)w(t) > z(t_0) + h_0 - Q(t_0) \equiv K_0$$

i.e.,

$$p(t) > 1 + \frac{K_0}{w(t)} .$$

Since $\lim_{t\to\infty} w(t) = \infty$, for any $k \in (\max\{m,0\},1)$, there is a $T \geq t_0$ such that $p(t) > k$ for $t \in S_0 \cap [T,\infty)$. Hence from (4.2.6), when

$t \in S_0 \cap [T,\infty)$,

$$u'(t) \leq p(t)(m-p(t)) \leq k(m-k) < 0.$$

When $t \in [T,\infty)\backslash S_0$, also from (4.2.6) we see

$$u'(t) \leq \frac{1}{4} g^2(y(t)) \leq \frac{1}{4} M^2 .$$

Denote $S_0(t) = S_0 \cap [T,t]$, $(S_0(t))^c = [T,t]\backslash S_0(t)$, $t \geq T$. Then $\mu(S_0(t)) \to \infty$ as $t \to \infty$ and $\mu((S_0(t))^c) \leq A < +\infty$. From

$$u(t) = u(T) + \int_{S_0(t)} u'(s)ds + \int_{(S_0(t))^c} u'(s)ds , \quad t \geq T$$

and $u(t) > 0$, $t \geq T$ we know there is a $c > 0$ such that when $t \geq T$

$$-\infty < -c < \int_{S_0(t)} u'(s)ds + \int_{(S_0(t))^c} u'(s)ds$$

$$\leq k(m-k)\mu(S_0(t)) + \frac{M^2}{4} \mu((S_0(t))^c) \to -\infty , \text{ as } t \to -\infty,$$

which is a contradiction.

ii) If $0 < \alpha < \infty$, define functions $p(t)$ and $u(t)$ as follows:

$$p(t) = z(t)/w(t), \quad u(t) = \frac{1}{f'(y(t))w^{\lambda}(t)}, \quad \lambda > 1 . \tag{4.2.8}$$

From (4.2.5) we get

$$u'(t) = g(y(t)) \frac{z(t)}{w^{\lambda}(t)} - \lambda \frac{z^2(t)}{w^{\lambda+1}(t)}$$

$$= w^{-\lambda+1}(t)(g(y(t))p(t) - \lambda p^2(t)). \tag{4.2.9}$$

For any $\lambda > 1$ and $y > 0$ let $P = \{p: g(y)p - \lambda p^2 > -1\}$. Since $g(y)$ is bounded and for any $p \neq 0$

$$\lim_{\lambda \to \infty} (g(y)p - \lambda p^2) = -\infty ,$$

we can choose λ sufficiently large so that the length of the interval P is sufficiently small. Let $\lambda = \lambda_0$ be so large that the length of the interval P is less than $\frac{1}{2\alpha} \min_{0 \leq i \leq 4} \{h_{i+1} - h_i\}$, where $h_0 = \ell_1$, $h_5 = \ell_2$. Set $r(t_0) = z(t_0) - Q(t_0)$ and for any $n > 0$ let

$$L = \{\ell: 1 + \frac{r(t_0)+\ell}{\alpha} < p < 1 + \frac{r(t_0)+\ell}{(n-1)\alpha/n}, \; p \in P\}.$$

Then for $\ell \in L$, $\frac{n-1}{n} \alpha < \frac{r(t_0)+\ell}{p-1} < \alpha$, where $p \in P$. Therefore we can choose $n = n_0$ so large that the length of the interval L is less than $\min_{0 \le i \le 4} \{h_{i+1} - h_i\}$ when $\lambda \geq \lambda_0$ and $n \geq n_0$. For these values of λ and n we can conclude that

$$[h_2, +\infty) \cap L = \emptyset, \text{ or } [-\infty, h_3] \cap L = \emptyset, \text{ or } ((-\infty, h_1) \cup [h_4, +\infty)) \cap L = \emptyset.$$

We only consider the case for $[h_2, +\infty) \cap L = \emptyset$. The discussion for the other cases is similar.

From (4.2.3) and (4.2.8) we get

$$p(t) = 1 + \frac{1}{w(t)} (r(t_0) + Q(t)).$$

It is easy to show that $p(t) \notin P$ for $t \in S_2$ and $t \geq T$, where T is sufficiently large. For otherwise, there is a $t \in S_2$, $t \geq T$, such that $p(t) \in P$. Then

$$1 + \frac{r(t_0)+\ell}{\alpha} < p(t) < 1 + \frac{r(t_0)+\ell}{(n-1)\alpha/n}$$

for some $\ell \in [h_2, +\infty)$, if T is large enough. Therefore $\ell \in L$, which contradicts the assumption.

From (4.2.9) we know for $t \in S_2 \cap [T, \infty)$

$$u'(t) \leq -w^{-\lambda+1}(t).$$

When $t \in [T, \infty) \setminus S_2$, also from (4.2.9) we have

$$u'(t) \le \frac{g^2(y(t))}{4\lambda} w^{-\lambda+1}(t) \le \frac{M^2}{4\lambda} w^{-\lambda+1}(t).$$

Let T be so large that for $t \ge T$

$$w^{-\lambda+1}(t) \le 2\alpha^{-\lambda+1}.$$

Denote $\hat{S}_2(t) = \hat{S}_2 \cap [T,t]$, $(\hat{S}_2(t))^c = [T,t]\setminus\hat{S}_2(t)$, $t \ge T$. Then from

$$u(t) = u(T) + \int_{\hat{S}_2(t)} u'(s)ds + \int_{(\hat{S}_2(t))^c} u'(s)ds$$

and $u(t) > 0$, $t \ge T$, we have

$$-\infty < -c < \int_{\hat{S}_2(t)} u'(s)ds + \int_{(\hat{S}_2(t))^c} u'(s)ds$$

$$\le -\alpha^{-\lambda+1}\mu(\hat{S}_2(t)) + \frac{M^2}{4\lambda}\cdot 2\alpha^{-\lambda+1}\mu((\hat{S}_2(t))^c)$$

$$\le -\alpha^{-\lambda+1}[\rho(S_2(t))t - \rho(S_2(T))T] + \frac{M^2}{2\lambda}\alpha^{-\lambda+1}[1-\rho(S_2(t))]t$$

for some $c > 0$ and $t \ge T$, so that

$$t[(\frac{M^2}{2\lambda} + 1)\alpha^{-\lambda+1}\rho(S_2(t)) - \frac{M^2}{2\lambda}\alpha^{-\lambda+1}] \le c'$$

for some c' and $t \geq T$. Therefore

$$t\left(\rho(S_2(t)) - \frac{M^2}{M^2+2\lambda}\right) \leq \frac{2\lambda c'}{(M^2+2\lambda)\alpha^{-\lambda+1}} < \infty .$$

Letting $\lambda = \lambda_0$ be so large that $\frac{M^2}{M^2+2\lambda} \leq \varepsilon$ for $\lambda \geq \lambda_0$, we obtain

$$t(\rho(S_2(t))-\varepsilon) < \frac{2\lambda c'}{(M^2+2\lambda)\alpha^{-\lambda+1}} < \infty ,$$

which contradicts the assumption (C1). The proof is complete.

Proof of Theorem 4.2.2. The only part of the proof which is different from that of Theorem 4.2.1 is for the case that $\alpha = +\infty$.

In this case we know from (4.2.3) that for $t \in S_2$

$$(p(t)-1)w(t) > r(t_0) + h_2 .$$

i.e.,

$$p(t) > 1 + (r(t_0)+h_2)/w(t).$$

Since $\lim_{t\to\infty} w(t) = \infty$, for any $k \in (\max\{m,0\},1)$, there is a $T \geq t_0$ such that $p(t) > k$ for $t \in S_2 \cap [T,\infty)$. Hence from (4.2.6), when $t \in S_2 \cap [T,\infty)$,

$$u'(t) \leq p(t)(m-p(t)) \leq k(m-k) < 0.$$

When $t \in [T,\infty)\backslash S_2$, also from (4.2.6) we see

$$u'(t) \le \frac{1}{4} g^2(y(t)) \le \frac{1}{4} M^2 .$$

Denote $\hat{S}_2(t) = S_2 \cap [T,t]$, $(\hat{S}_2(t))^c = [T,t]\backslash S_2(t)$, $t \ge T$. Then from

$$u(t) = u(T) + \int_{\hat{S}_2(t)} u'(s)ds + \int_{(\hat{S}_2(t))^c} u'(s)ds \quad , \quad t \ge T$$

and $u(t) > 0$, $t \ge T$ we obtain that

$$-\infty < -c < \int_{\hat{S}_2(t)} u'(s)ds + \int_{(\hat{S}_2(t))^c} u'(s)ds$$

$$\le k(m-k)[\rho(S_2(t))t - \rho(\hat{S}_2(T))T] + \frac{M^2}{4} [1-\rho(S_2(t))]t$$

for some $c > 0$, and $t \ge T$, from which we have

$$t\{[\frac{M^2}{4} - k(m-k)]\rho(S_2(t)) - \frac{M^2}{4}\} \le c'$$

for some $c' > 0$ and $t \ge T$. Therefore

$$t\left(\rho(S_2(t)) - \frac{M^2}{M^2-4k(m-k)}\right) \leq \frac{4c'}{M^2-4k(m-k)} < \infty ,$$

which contradicts the assumption (Cl').

Proof of Theorem 4.2.3. The only part of the proof which is different from that of Theorem 4.2.2 is for the case that $0 < \alpha < +\infty$.

In this case we define functions $p(t)$ and $u(t)$ just the same as in part i). From (4.2.3) we can see for any $h_3 \in (-\infty, \ell_2)$, when $t \in \sigma_3$, $t \geq t_0$

$$p(t) < 1 + (r(t_0)+h_3)/w(t).$$

So for any $a < -M$ we can choose h_3 so small that

$$p(t) < 1 + (r(t_0)+h_3)/w(t) < a$$

for $t \in \sigma_3$ and $t \geq t_0$. From (4.2.6) when $t \in \sigma_3 \cap [t_0,\infty)$,

$$u'(t) \leq -Mp(t) - p^2(t) \leq -Ma - a^2 .$$

When $t \in [t_0,\infty)\backslash\sigma_3$, also from (4.2.6) we have

$$u'(t) \leq \frac{1}{4} g^2(y(t)) \leq \frac{M^2}{4} .$$

Denote $\hat{\sigma}_3(t) = \sigma_3 \cap [t_0,t]$, $(\hat{\sigma}_3(t))^c = [t_0,t]\backslash\sigma_3(t)$, $t \geq t_0$. Then from

$$u(t) = u(t_0) + \int_{\hat{\sigma}_3(t)} u'(s)ds + \int_{(\hat{\sigma}_3(t))^c} u'(s)ds \ , \ t \geq t_0$$

and $u(t) > 0$, $t \geq t_0$ we obtain that

$$-\infty < -c < \int_{\hat{\sigma}_3(t)} u'(s)ds + \int_{(\hat{\sigma}_3(t))^c} u'(s)ds$$

$$\leq -a(M+a)[\rho(\sigma_3(t))t - \rho(\sigma_3(t_0))t_0] + \frac{M^2}{4}[1-\rho(\sigma_3(t))]t$$

for some $c > 0$ and $t \geq t_0$, from which

$$t\{[\frac{M^2}{4} + a(M+a)]\rho(\sigma_3(t)) - \frac{M^2}{4}\} \leq c'$$

for some c' and $t \geq t_0$. Therefore

$$t(\rho(\sigma_3(t)) - \frac{M^2}{M^2+4a(M+a)}) \leq \frac{4c'}{M^2+4a(M+a)} .$$

Let a be sufficiently large and negative, $a < -M$, so that

$$\frac{M^2}{M^2+4a(M+a)} < \varepsilon ,$$

then we have

$$t(\rho(\sigma_3(t))-\varepsilon) \leq \frac{4c'}{M^2+4a(M+a)} < +\infty ,$$

which contradicts the assumption (C2). This completes the proof.

4.3. Extensions of the previous results

Concerning equations $(\hat{N})$ or $(\hat{N}_\alpha)$ similar results can be obtained. Before stating those results we introduce some conditions.

For any h_i $(i = 1,2,3,4) \in (\ell_1,\ell_2)$ satisfying (4.2.1) let

(D1) $\limsup_{t\to\infty} (\int_0^t r^{-1}(s)ds)(\hat{\rho}(S_2(t))-\varepsilon) = +\infty$,

(D1') $\limsup_{t\to\infty} (\int_0^t r^{-1}(s)ds)(\hat{\rho}(S_2(t))-\delta(M,m,k)) = +\infty$,

(D2) $\limsup_{t\to\infty} (\int_0^t r^{-1}(s)ds)(\hat{\rho}(\sigma_3(t))-\varepsilon) = +\infty$,

(D3) $\limsup_{t\to\infty} (\int_0^t r^{-1}(s)ds)(\hat{\rho}(S_4 \cup \sigma_1)(t))-\varepsilon) = +\infty$, where $S_i(t)$, $\sigma_i(t)$ $(i = 1,2,3,4)$, $\delta(M,m,k)$ and ε are the same as in (C1)-(C3), and

$$\hat{\rho}(\alpha(t)) = \frac{\int_{\alpha(t)}^{t} r^{-1}(s)ds}{\int_0^t r^{-1}(s)ds}.$$

For the case that $\int_0^\infty r^{-1}(t)dt = +\infty$ we have the following results.

<u>THEOREM 4.3.1.</u> Let condition (H) hold and $-\infty < \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$ and $\varepsilon > 0$ satisfying (4.2.1), (D1), (D2) and (D3) and there exists an $h_0 < \ell_1$ such that $\limsup_{t\to\infty} \hat{\rho}(S_0(t)) = 1$. Then equation $(\hat{N})$ is oscillatory.

<u>THEOREM 4.3.2.</u> Let condition (H) hold and $-\infty = \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$, $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (4.2.1), (D1'), (D2) and (D3). Then equation $(\hat{N})$ is oscillatory.

<u>THEOREM 4.3.3.</u> Let condition (H) hold and $\ell_1 < \ell_2$. Assume there are an $h_2 \in (-\infty,\ell_2)$ and a $k \in (\max\{m,0\},1)$ satisfying (D1'), and for any $h_3 \in (-\infty,\ell_2)$ there is an $\varepsilon > 0$ satisfying (D2). Then equation $(\hat{N})$ is oscillatory.

The proofs of theorems 4-6 are similar to those of theorem

4.2.1-3. As an example we give a sketch of the proof of Theorem 4.3.1.

<u>Proof of Theorem 4.3.1.</u> We employ the Riccati substitution $z(t) = -r(t)y'(t)/f(y(t))$, where $y(t)$ is a nonoscillatory positive solution of $(\hat{N})$. Instead of (4.2.3) we get

$$z(t) = z(t_0) + Q(t) + w(t) - Q(t_0) \ , \quad t \geq t_0 \qquad (4.3.1)$$

where

$$w(t) = \int_{t_0}^{t} r^{-1}(s)f'(y(s))z^2(s)ds.$$

Then we can let

$$\lim_{t\to\infty} w(t) = \alpha \ , \ 0 < \alpha \leq +\infty \ .$$

i) If $\alpha = +\infty$, defining

$$p(t) = z(t)/w(t) \ , \ u(t) = \frac{1}{f'(g(t))w(t)} \ ,$$

we get

$$u'(t) = r^{-1}(t)[g(y(t))p(t) - p^2(t)], \quad t \geq t_0 \ .$$

Since $\limsup_{t\to\infty} \hat{\rho}(S_0(t)) = 1$, and $\int_0^t r^{-1}(s)ds = \infty$, for any $\delta \in (0,1)$

$$\limsup_{t\to\infty} \left(\int_0^t r^{-1}(s)ds\right)(\hat{\rho}(S_0(t))-\delta) = +\infty . \qquad (4.3.2)$$

According to (4.3.1) we know for $t \in S_0$

$$p(t) > 1 + (z(t_0)-Q(t_0)+h_0)/w(t).$$

Thus for any $k \in (\max\{m,0\},1)$ there is a $T \geq t_0$ such that $p(t) > k$ for $t \in S_0 \cap [T,\infty)$. It follows that $u'(t) \leq k(m-k)r^{-1}(t)$ for $t \in S_0 \cap [T,\infty)$ and $u'(t) \leq \frac{1}{4} M^2 r^{-1}(t)$ for $t \in [T,\infty)\backslash S_0$. From

$$u(t) = u(T) + \int_{S_0(t)} u'(s)ds + \int_{(S_0(t))^c} u'(s)ds, \ t \geq T$$

and $u(t) > 0$, $t \geq T$ we can obtain finally that

$$\int_0^t r^{-1}(s)ds \ [\hat{\rho}(S_0(t))- \frac{M^2}{M^2-4k(m-k)}] \leq \frac{4c'}{M^2-4k(m-k)} < \infty$$

which contradicts (4.3.2).

ii) If $0 < \alpha < \infty$, defining

$$p(t) = z(t)/w(t), \ u(t) = \frac{1}{f'(y(t))w^{\lambda}(t)} , \quad \lambda > 1$$

we get

$$u'(t) = r^{-1}(t)w^{-\lambda+1}(t)[g(y(t))p(t)-p^2(t)].$$

The remainder of the proof proceeds as in part ii) of Theorem 4.2.1.

Choose λ, n so large that

$$[h_2,+\infty) \cap L = \emptyset \text{ , or } (-\infty,h_3] \cap L = \emptyset \text{ , or } ((-\infty,h_1] \cup [h_4,+\infty)) \cap L = \emptyset.$$

For the case that $[h_2,+\infty) \cap L = \emptyset$ we have $u'(t) \leq -r^{-1}(t)w^{-\lambda+1}(t)$ for $t \in S_2 \cap [T,\infty)$, and $u'(t) \leq \frac{M^2}{4\lambda} r^{-1}(t)w^{-\lambda+1}(t)$ for $t \in [T,\infty)\backslash S_2$. Let T be so large that for $t \geq T$ $w^{-\lambda+1}(t) \leq 2\alpha^{-\lambda+1}$. Then from

$$u(t) = u(T) + \int_{\hat{S}_0(t)} u'(s)ds + \int_{(\hat{S}_0(t))^c} u'(s)ds \text{ , } t \geq T$$

and $u(t) > 0$, $t \geq T$ we can obtain finally that

$$\int_0^t r^{-1}(s)ds[\hat{\rho}(S_2(t)) - \frac{M^2}{M^2+2\lambda}] \leq \frac{2\lambda c'}{(M^2+2\lambda)\alpha^{1-\lambda}}$$

Letting λ be so large that $\frac{M^2}{M^2+2\lambda} \leq \varepsilon$, we obtain

$$\int_0^t r^{-1}(s)ds(\hat{\rho}(S_2(t))-\varepsilon) \leq \frac{2\lambda c'}{(M^2+2\lambda)\alpha^{1-\lambda}} < \infty \text{ ,}$$

which contradicts assumption (D1).

For the case that

$$\int_0^\infty r^{-1}(s)\, ds < +\infty \tag{4.3.3}$$

using some kind of variable transformation we can get some corresponding results for equation $(\hat{N}_\alpha)$.

In that case we introduce the following transformation

$$s(t) = \left(\int_t^\infty r^{-1}(\tau)d\tau\right)^{-1}, \quad z(s) = sy(t). \tag{4.3.4}$$

Then we obtain the equation in new variables

$$\frac{d^2z}{ds^2} + s^{-(\alpha+3)}r(t(s))q(t(s))|z(s)|^\alpha \operatorname{sgn} z(s) = 0 , \qquad (\hat{N}'_\alpha)$$

where $t(s)$ is determined by (4.3.4). It is easy to see that $s \to +\infty$ as $t \to +\infty$, and the oscillation of equation $(\hat{N}'_\alpha)$ is equivalent to that of equation $(\hat{N}_\alpha)$. Now we only need to consider the equation $(\hat{N}'_\alpha)$ in order to investigate the oscillatory property of equation $(\hat{N}_\alpha)$. Replacing t and $Q(t)$ by s and $\int_0^s \tau^{-(\alpha+3)}r(t(\tau))q(t(\tau))d\tau$ in Chapter 1 and 2, where $t(s)$ is the inverse function of $s(t)$, and applying Theorems 4.1.2, 4.2.1-4.2.3 to equation $(\hat{N}'_\alpha)$, we find that all of the theorems are of the corresponding forms for equation $(\hat{N}_\alpha)$

while $\int_0^\infty r^{-1}(s)ds < +\infty$. We state them below.

For ℓ_i $(i = 1,2)$, h_i $(i = 1,2,3,4)$ satisfying (4.2.1) and s satisfying (4.3.4) define

$$\bar{Q}(s) = \int_0^s \tau^{-(\alpha+3)} r(t(\tau))q(t(\tau))d\tau,$$

$$\bar{\sigma}_i = \{s \in [0,\infty): \bar{Q}(s) < h_i\}.$$

$$\bar{S}_i = \{s \in [0,\infty): \bar{Q}(s) > h_i\} , \tag{4.3.5}$$

and $\bar{\sigma}_i(s) = \bar{\sigma}_i \cap [0,s]$, $\bar{S}_i(s) = \bar{S}_i \cap [0,s]$. Let

$$\bar{\rho}(\sigma_i(t_s)) = \rho(\bar{\sigma}_i(s)),\ \bar{\rho}(\bar{S}_i(t_s)) = \rho(\bar{S}_i(s)).$$

We introduce the following conditions:

$$(\bar{D}1)\quad \limsup_{t\to\infty}\left(\int_t^\infty r^{-1}(\tau)d\tau\right)^{-1}(\bar{\rho}(\bar{S}_2(t_s))-\varepsilon) = +\infty ,$$

$$(\bar{D}1')\quad \limsup_{t\to\infty}\left(\int_t^\infty r^{-1}(\tau)d\tau\right)^{-1}(\bar{\rho}(\bar{S}_2(t_s))-\delta(M,m,k)) = +\infty,$$

$$(\bar{D}2)\quad \limsup_{t\to\infty}\left(\int_t^\infty r^{-1}(\tau)d\tau\right)^{-1}(\bar{\rho}(\bar{\sigma}_3(t_s))-\varepsilon) = +\infty ,$$

$(\bar{D}3)$ $\limsup_{t\to\infty}(\int_t^\infty r^{-1}(\tau)d\tau)^{-1}(\bar{\rho}(\bar{S}_4\cup\bar{\sigma}_1)(t_s)-\varepsilon) = +\infty.$

Theorem 4.3.0'. Let conditions (H) and (4.3.3) hold and $\ell_2 = +\infty$. Assume

$$\limsup_{n\to\infty}(\int_t^\infty r^{-1}(\tau)d\tau)^{-1}(\bar{\rho}(\bar{S}_\ell(t_s)-\delta) = +\infty$$

for every ℓ sufficiently large, where $\delta = \frac{M^2}{M^2-4m+4}$. Then equation $(\hat{N}_\alpha)$ is oscillatory.

Theorem 4.3.1'. Let conditions (H) and (4.3.3) hold and $-\infty < \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$ and $\varepsilon > 0$ satisfying (4.2.1), $(\bar{D}1)$, $(\bar{D}2)$ and $(\bar{D}3)$. Then equation $(\hat{N}_\alpha)$ is oscillatory.

Theorem 4.3.2' Let conditions (H) and (4.3.3) hold and $-\infty = \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$, $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (4.2.1), $(\bar{D}1')$, $(\bar{D}2)$ and $(\bar{D}3)$. Then equation $(\hat{N}_\alpha)$ is oscillatory.

Theorem 4.3.3'. Let conditions (H) and (4.3.3) hold and $\ell_1 < \ell_2$. Assume there are an $h_2 \in (-\infty,\ell_2)$ and a $k \in (\max\{m,0),1)$ satisfying

$(\bar{D}1')$, and for any $h_3 \in (-\infty, \ell_2)$ there is an $\varepsilon > 0$ satisfying $(\bar{D}2)$. Then equation $(\hat{N}_\alpha)$ is oscillatory.

4.4. Examples and applications

As an example we consider the oscillatory behavior of the equation

$$y''(t) + \varphi(\tau(t))\tau'(t)f(y(t)) = 0. \tag{4.4.1}$$

THEOREM 4.4.1. Let condition (H) hold. Assume that $\varphi(u)$ is continuous and periodic with period $\omega > 0$, and $\int_0^\omega \varphi(s)ds \geq 0$, $\tau(t)$ is differentiable on R^+ and $\lim_{t\to\infty} \tau(t) = +\infty$. Then equation (4.4.1) is oscillatory.

Proof. Here $q(t) = \varphi(\tau(t))\tau'(t)$. Thus

$$Q(t) = \int_0^t q(s)ds = \int_0^t \varphi(\tau(s))\tau'(s)ds = \int_{\tau(0)}^{\tau(t)} \varphi(s)ds .$$

If $\int_0^\omega \varphi(s)ds > 0$, then $\lim_{t\to\infty} Q(t) = +\infty$ since $\lim_{t\to\infty} \tau(t) = +\infty$. It follows that $\lim_{t\to\infty}$ approx $Q(t) = +\infty$. According to Corollary 2.2 in [10] equation (4.4.1) is oscillatory.

If $\int_0^{\omega} \varphi(s)ds = 0$, then $\int_0^t \varphi(s)ds$ is periodic with period ω.

Let $\ell_1 = \min_{t\in[0,\omega]} \{\int_0^t \varphi(s)ds\}$, $\ell_2 = \max_{t\in[0,\omega]} \{\int_0^t \varphi(s)ds\}$, $\varepsilon \in (0, \frac{1}{3})$.

Then

$$\ell_1 \leq Q(t) \leq \ell_2 \text{ for all } t \in R^+ .$$

Denote

$$S_\ell = \{t \in [0,\infty): Q(t) > \ell\} \text{ and } S_\ell(t) = S_\ell \cap [0,t].$$

Since

$$\lim_{t\to\infty} \rho(S_{\ell_1}(t)) = 1, \quad \lim_{t\to\infty} \rho(S_{\ell_2}(t)) = 0 ,$$

and $\limsup_{t\to\infty} \rho(S_\ell(t))$ is continuous and decreasing in ℓ, there exist h_2 and $h_4 \in (\ell_1,\ell_2)$, $h_2 < h_4$, such that

$$\limsup_{t\to\infty} \rho(S_4(t)) = \varepsilon,$$

and

$$\limsup_{t\to\infty} \rho(S_2(t)) = 2\varepsilon.$$

Similarly, there exist h_1, $h_3 \in (\ell_1,\ell_2)$, $h_1 < h_3$, such that

$$\limsup_{t\to\infty} \rho(\sigma_1(t)) = \varepsilon, \tag{4.4.2}$$

and

$$\limsup_{t\to\infty} \rho(\sigma_3(t)) = 2\varepsilon.$$

We can show that $h_1 \in (\ell_1, h_2)$. For otherwise, $h_1 \geq h_2$. Then $\rho(S_2 \cup \sigma_1)(t)) \equiv 1$ for $t \in R^+$. Hence $\rho(\sigma_1(t)) \geq 1 - \rho(S_2(t))$ for $t \in R^+$. Thus

$$\limsup_{t\to\infty} \rho(\sigma_1(t)) \geq 1 - \limsup_{t\to\infty} \rho(S_2(t)) = 1 - 2\varepsilon > \varepsilon ,$$

which contradicts (4.4.2).

In the same way we can show that $h_4 \in (h_3, \ell_2)$.

Now we want $h_2 < h_3$. If $h_2 \geq h_3$, then choose $h_3' \in (h_2, h_4)$ and substitute h_3' for h_3. For these h_i $(i = 1,2,3,4)$ conditions (C1), (C2) and (C3) are satisfied. Therefore, by Theorem 4.2.1 equation (4.4.1) is oscillatory.

Example 4.4.1. Equation

$$y''(t) + t^{\beta-1}\varphi(t^\beta)|y(t)|^\alpha \operatorname{sgn} y(t) = 0 \quad (\alpha > 0 , \beta > 0), \tag{4.4.3}$$

where $\varphi(u)$ is continuous and periodic with period ω and $\int_0^\omega \varphi(s)\, ds \geq 0$, satisfies all conditions of Theorem 4.4.1. Therefore (4.4.3) is oscillatory.

Example 4.4.2. Consider equation

$$y''(t) + \frac{\sin t}{(2+\sin t)^{\alpha}} |y(t)|^{\alpha} \operatorname{sgn} y(t) = 0 \quad (\alpha > 0) . \qquad (4.4.4)$$

We can let $\varphi(u) = \frac{\sin u}{(2+\sin u)^{\alpha}}$ and $\tau(t) = t$. And it is easy to see that $\int_0^{2\pi} \frac{\sin u}{(2+\sin u)^{\alpha}} ds < 0$. So we can not conclude that equation (4.4.4) is oscillatory by Theorem 4.4.1. In fact, equation (4.4.4) has a nonoscillatory solution $y(t) = 2 + \sin t$. This example shows that the conditions of Theorem 4.4.1 are sharp.

The result in Theorem 4.4.1 can be extended to more general forms.

Corollary 4.4.1. Consider equation

$$y''(t) + [\psi(t) + \varphi(\tau(t))\tau'(t)]f(y(t)) = 0. \qquad (4.4.5)$$

In addition to the conditions of Theorem 4.4.1, assume that $\psi(t)$ is continuous on R^+ and $\lim_{t\to\infty} \int_0^t \psi(s)ds = k$, $-\infty < k \leq +\infty$. Then equation (4.4.5) is oscillatory.

Corollary 4.4.2. Consider equation

$$(r(t)y'(t))' + [\psi(t) + \varphi(\tau(t))\tau'(t)]f(y(t)) = 0. \quad (4.4.6)$$

In addition to the conditions of Corollary 4.4.1, assume that

$$\int_0^\infty r^{-1}(s)ds = \infty.$$

Then equation (4.4.6) is oscillatory.

<u>Proof of Corollary 4.4.1</u>. Here $q(t) = \psi(t) + \varphi(\tau(t))\tau'(t)$. Thus

$$Q(t) = \int_0^t q(s)ds = \int_0^t \psi(s)ds + \int_{\tau(0)}^{\tau(t)} \varphi(s)ds.$$

If $\int_0^\omega \varphi(s)ds > 0$, then $\lim_{t\to\infty} Q(t) = +\infty$ since $\lim_{t\to\infty} \tau(t) = +\infty$ and $\int_0^\infty \psi(s)ds = k$. It follows that $\lim_{n\to\infty}$ approx $Q(t) = +\infty$. According to Corollary 2.2 in [10] equation (4.4.5) is oscillatory.

If $\int_0^\omega \varphi(s)ds = 0$, then $\int_0^t \varphi(s)ds$ is periodic with period ω. Let $\ell_1 = \min_{t\in[0,\omega]} \{\int_0^t \varphi(s)ds\}$, $\ell_2 = \max_{t\in[0,\omega]} \{\int_0^t \varphi(s)ds\}$, $\varepsilon \in (0, \frac{1}{3})$. Then there exists constants ℓ_1', ℓ_2', such that $\ell_1' < \ell_1$, $\ell_2' > \ell_2$ and

$$\ell_1' \le Q(t) \le \ell_2' \text{ for sufficiently large } t.$$

Denote

$$S_\ell = \{t \in [0,\infty): Q(t) > \ell\} \text{ and } S_\ell(t) = S_\ell(t) = S_\ell \cap [0,t].$$

Then

$$\lim_{t\to\infty} \rho(S_{\ell_1}(t)) = 1, \quad \lim_{t\to\infty} \rho(S_{\ell_2}(t)) = 0.$$

The rest of the proof is similar to that of Theorem 4.4.1, we omit it.

Proof of Corollary 4.4.2. The proof is straightforward by the same discussion as in Corollary 4.4.1 and using Theorem 4.3.1.

4.5 Extension to equation $(\hat{S})$

At first, we give some extensions of the previous results to equation $(\hat{S})$ under the assumption that $\int_0^\infty r^{-1}(s)ds = +\infty$. Let $Q(t) = \int_0^t d\sigma(s)$, $t \geq 0$, ℓ_i $(i = 1,2)$ and h_i $(i = 1,2,3,4)$ are given in section 4.2, (D1)-(D3) are given in section 4.3.

Theorem 4.5.0 Let condition (H) hold and $\ell_2 = +\infty$. Assume

$$\limsup_{t\to\infty}\left(\int_0^t r^{-1}(s)ds\right)(\hat{\rho}(S_\ell(t))-\delta) = +\infty$$

for every sufficently large ℓ, where $\delta = \frac{M^2}{M^2-4m+4}$. Then equation $(\hat{S})$ is oscillatory.

Theorem 4.5.1. Let condition (H) hold and $-\infty < \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$ and $\varepsilon > 0$ satisfying (4.2.1), (D1), (D2) and (D3), and there exists an $h_0 < \ell_1$ such that $\limsup_{t\to\infty} \hat{\rho}(S_0(t)) = 1$. Then equation $(\hat{S})$ is oscillatory.

Theorem 4.5.2. Let condition (H) hold and $-\infty = \ell_1 < \ell_2 \leq +\infty$. Assume there are constants h_i $(i = 1,2,3,4)$, $\varepsilon > 0$ and $k \in (\max\{m,0\},1)$ satisfying (4.2.1), (D1'), (D2) and (D3). Then equation $(\hat{S})$ is oscillatory.

Theorem 4.5.3. Let condition (H) hold and $\ell_1 < \ell_2$. Assume there are an $h_2 \in (-\infty,\ell_2)$ and a $k \in (\max\{m,0\},1)$ satisfying (D1'), and for any $h_3 \in (-\infty,\ell_2)$ there is an $\varepsilon > 0$ satisfying (D2). Then equation $(\hat{S})$ is oscillatory.

To prove the above theorems we introduce the following lemma, the direct proof of which can be found in [42].

Lemma 4.5.1. Let $f(t)$ and $g(t)$ be continuous functions on $[a,b]$ and $\sigma(t)$ be of bounded variation on $[a,b]$. Then

$$\int_a^b f(s)g(s)d\sigma(s) = f(b)\int_a^b g(s)d\sigma(s) - \int_a^b [\int_a^u g(s)d\sigma(s)]df(u).$$

Proofs of Theorems 4.5.0-4.5.3. Suppose equation $(\hat{S})$ is not oscillatory and let $y = y(t)$ be a nontrivial nonsocillatory solution on $[0,\infty)$ which we may suppose satisfies $y(t) > 0$ on $[t_0,\infty)$, $t_0 \geq 0$.

Equation (S) is equivalent to the equation

$$r(t)y'(t) = c' - \int_{t_0}^{t} f(y(s))d\sigma(s)$$

where $c' = (py')(t_0)$. Let $z(t) = -r(t)y'(t)/f(y(t))$. Then

$$z(t) = -\frac{c'}{f(y(t))} + \frac{1}{f(y(t))}\int_{t_0}^{t} f(y(s))d\sigma(s)$$

$$= -\frac{c'}{f(y(t))} + \int_{t_0}^{t} d\sigma(s) + \int_{t_0}^{t} (c'-r(s)y'(s))d(f^{-1}(y(s))$$

$$= z(t_0) - Q(t_0) + Q(t) + \int_{t_0}^{t} r(s)y'^2(s)f^{-2}(y(s))f'(y(s))ds$$

$$= z(t_0) - Q(t_0) + Q(t) + \int_{t_0}^{t} r^{-1}(s)f'(y(s))z^2(s)ds$$

$$= z(t_0) - Q(t_0) + Q(t) + w(t),$$

where

$$w(t) = \int_{t_0}^{t} r^{-1}(s)f'(y(s))z^2(s)ds.$$

This is just the same as (4.3.1) and a corresponding result to (4.7) in [10].

The rest of the proofs of the theorems are similar to those of Theorem 4.2 in [10] and Theorems 4.3.1-4.3.3 in this paper. We omit them here.

Now we give some oscillation results for equation $(\hat{S}_\alpha)$ relating to the case that $\int_0^\infty r^{-1}(s)ds < +\infty$. Here we also use the transformation (4.3.3).

Lemma 4.5.2. Under the variable transformation (4.3.3) equation $(\hat{S}_\alpha)$ becomes equation

$$z'(s) = d - \int_{s(a)}^{s} \tau^{-(\alpha+1)}|z(\tau)|^\alpha \operatorname{sgn} z(\tau)d\sigma(t(\tau)) \qquad (\hat{S}'_\alpha)$$

where $d = z'(s(a))$, $t(s)$ is the inverse functin of $s(t)$.

Proof. Suppose $y(t)$ is a solution of equation $(\hat{S}_\alpha)$. According to (4.3.3) we have

$$z'(s) = y(t) + sy'(t)\frac{dt}{ds} = y(t) + sy'(t)\frac{r(t)}{s^2}$$

$$= y(t) + \frac{1}{s} y'(t)r(t)$$

$$= \frac{z(s)}{s} + \frac{1}{s}\left[c - \int_a^t |y(\ell)|^\alpha \operatorname{sgn} y(\ell)\,d\sigma(\ell)\right]$$

$$= \frac{z(s)}{s} + \frac{c}{s} - \frac{1}{s}\int_{s(a)}^{s} \tau^{-\alpha}|z(\tau)|^\alpha \operatorname{sgn} z(\tau)\,d\sigma(t(\tau)).$$

Using Lemma 4.5.1 we get

$$z'(s) = \frac{z(s)}{s} + \frac{c}{s} - \int_{s(a)}^{s} \tau^{-(\alpha+1)}|z(\tau)|^\alpha \operatorname{sgn} z(\tau)\,d\sigma(t(\tau))$$

$$- \int_{s(a)}^{s} \left(\int_{s(a)}^{\ell} \tau^{-\alpha}|z(\tau)|^\alpha \operatorname{sgn} z(\tau)\,d\sigma(t(\tau))\right)d\left(\frac{1}{\ell}\right)$$

$$= \frac{z(s)}{s} + \frac{d}{s} - \int_{s(a)}^{s} \tau^{-(\alpha+1)}|z(\tau)|^\alpha \operatorname{sgn} z(\tau)\,d\sigma(t(\tau))$$

$$- \int_{s(a)}^{s} [c - r(\ell(t))y'(\ell(t))]\,d\left(\frac{1}{\ell}\right) \qquad (4.5.1)$$

Since $r(t)y'(t) = z'(s)s - z(s)$,

$$\int_{s(a)}^{s} r(\ell(t))y'(\ell(t))\,d\left(\frac{1}{\ell}\right) = -\int_{s(a)}^{s} \frac{z'(\ell)\ell - z(\ell)}{\ell^2}\,d\ell$$

$$= -\int_{s(a)}^{s} \left(\frac{z(\ell)}{\ell}\right)'d\ell = -\frac{z(s)}{s} + \frac{z(s(a))}{s(a)}. \qquad (4.5.2)$$

Combining (4.5.1) and (4.5.2) we obtain

$$z'(s) = \frac{z(s(a))}{s(a)} + \frac{c}{s(a)} - \int_{s(a)}^{s} \tau^{-(\alpha+1)} z^{\alpha}(\tau) \operatorname{sgn} z(\tau) d\sigma(t(\tau))$$

$$= z'(s(a)) - \int_{s(a)}^{s} \tau^{-(\alpha+1)} z^{\alpha}(\tau) \operatorname{sgn} z(\tau)\, d\sigma(t(\tau)).$$

The proof is complete.

Equation $(\hat{S}'_\alpha)$ is of the form of equation (S_α), which is a special case of equation $(\hat{S}_\alpha)$. According to Theorems 4.5.0 - 4.5.3, if we replace $\bar{Q}(R)$ by $\int_0^s \tau^{-(\alpha+1)} d\sigma(t(\tau))$ in (4.3.5), we obtain

<u>Theorem 4.5.0'</u>. Let conditions (H) and (4.3.3) hold and $\ell_2 = +\infty$. Assume

$$\limsup_{t\to\infty} \left(\int_t^\infty r^{-1}(\tau)d\tau\right)^{-1} (\bar{\rho}(\bar{S}_\ell(t_s)) - \delta) = +\infty$$

for every ℓ sufficiently large, where $\delta = \frac{M^2}{M^2-4m+4}$. Then equation $(\hat{S}_\alpha)$ is oscillatory.