Directional Splitting On Grid With Local Refinement For Parabolic Problems

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Applied Mathematics

Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

A cell centered finite difference scheme for 2D parabolic problems on grids with a local refinement is presented. Peaceman and Rachford directional splitting is used in the discretization of time.

For cell centered grids, grid points are always at the center of each cell and the grid is generated first by discretizing the domain with rectangle cells; for region having local property and being required to be investigated, further local refinement is performed by dividing original coarse cells into several similar rectangles(fine cells).

Scheme is generated by approximating the flux and integrals in the balance equation which is the result of integrating the partial differential equation over the cell and using the Green's flux theorem. Special attention is paid on the flux approximation of the irregular cells in the area of intersection between coarse and fine cells; and the scheme is designed to be symmetric and later proved to be positive. This implies that it is unconditionally stable.

By applying classical results in directional splitting, the scheme is proven to be of second order convergence in time. Numerical experiments indicate that it should also be of second order in space; however, by Bramble Hilbert lemma, we can only prove $\frac{3}{2}$ convergence rate given a certain regularity condition of the exact solution. A numerical experiment which demonstrates the benefit of the proposed scheme is also conducted and presented.

It should be noted that we can use the Schur compliment technique to construct the solution of the linear systems induced by the scheme. What is more, the algorithm can be implemented in parallel machines very efficiently.

Preface

This is a work of Zecheng Zhang under the supervision of Prof. Peter Minev. The idea of the work is originally came up by Peter. He proposed the scheme using Bramble Hilbert lemma. My work includes proving the positivity of the operators, stability and convergence of the scheme. I also conducted all numerical experiments. To prove the theorem, Peter gives me the direction usually by pointing out the papers or books which may be useful for proving, I then give the proof.

Acknowledgements

The author would like to thank Prof. Yaushu Wong and Prof. Peter Minev for their patient, careful and inspiring supervision. In addition, I would like to thank Yaushu and Peter for their great help on my application to graduate school. I really really appreciate any help Prof. Yaushu Wong and Prof. Peter Minev have given me since I decided to apply a degree of University of Alberta.

Also, I would like to thank all my course instructors in University of Alberta and Hong Kong Baptist University. I learned a lot from the courses you have taught me. In particular, I want to mention Dr. Tieyong Zeng, Prof. Xiaonan Wu for 8 wonderful courses you have taught me; and Dr. Yutian Li, Prof. Hermann Brunner for your supervision of my honor project; and Prof. Volker Runde and Prof. Thomas Hillen for your strict requirement and great content of your course, I learned a lot from your courses. In addition, I would like to thank Prof. Rolf Jeltsch and Prof. Tao Tang. Your consistent warm help and suggestion benefit me a lot.

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Chapter 1

Introduction

Local refinement is an important idea in scientific computing since it greatly reduces the requirement on computer facilities and improves the efficiency. Meanwhile, there are many real life problems which have local properties and needs to be investigated locally.

In this paper, we consider solving parabolic problems with local refinement on a cell centered grid including hanging nodes. The essential idea is that we show that it is stable when combined with a directional splitting scheme. The advantage of this approach is that the resulting system is tri-diagonal dominated (the matrix is tri-diagonal except a small part) and can be solved by Schur-complement technique efficiently. The algorithm is parallelizable; and hence efficiency is further improved.

Cell-centered approximation was developed by Samarskii in [9]. The coefficients of a cell centered scheme are some functions of harmonic mean of coefficients of a partial differential equation. This is an important feature and the advantage of cell centered scheme since it allows us to solve problems with discontinuous coefficients [21]. Due to the physics (conservation law) inherent in the cell centered finite difference, petroleum engineers are also interested in cell centered scheme [16].

Russell and Wheeler [16] proved the equivalence of a cell centered scheme and the mixed finite element method for elliptic problems in the case of Raviart-Thomas elements of lowest order [18] using special quadrature rule. It follows by several works regarding this area. Readers can refer to Baranger, Maitre and Oudin in [17]. Chen and Yu later in [19] studied Brezzi-Douglas-Marini elements and Douglas-Duran-Fortin elements and showed that the mixed finite element with a certain quadrature rule is also equivalent to a cell centered finite difference scheme.

Previous mentioned works all focused on uniform mesh. Weiser and Wheeler in [15] did convergence analysis for the cell centered finite difference scheme for elliptic and parabolic problems on non-uniform grids. It is shown that the cell centered finite difference scheme has second order convergence in approximating both solution and the first derivative of the PDE given sufficient smooth data.

Ewing, Lazarov and Vassilevski later introduced several algorithms on cell centered grid with a local refinement for elliptic [3] problems. The derivation is based on approximating balance equation. Two approximations are actually included. First one is the approximation to the flux which is the first order derivative; second one is approximating the integration by a mid-point rule. With exception of cells on intersection area in the grid, the derivation of other cells results in standard five point scheme given constant coefficients. The author paid special attention to irregular cells and proposed three algorithms which gave different convergence rates.

They later extended their work for parabolic problems on Cartesian grids [4] and on triangular grids [5] using the same derivation. The proposed schemes were implicit of backward Euler type. It was unconditionally stable with first order convergence in time.

Alternating directional splitting method was developed in 1950s' by Peaceman, Rachford and Douglas [13][12][11][10]. In [12], Douglas and Rachford (D-R) proposed a first order in time unconditionally stable scheme dealing with 2D parabolic and elliptic problems. The scheme can be easily extended to 3D and is similar with a backward Euler scheme with an additional term. Peaceman and Rachford (P-R) in [13] devised a similar unconditionally stable splitting scheme; compared to D-R splitting, P-R scheme is second order in time but loses stability when applied to 3D problems. Douglas later in [10] derived a new scheme which is equivalent to P-R scheme in 2D and can be extended to 3D problems. This scheme has excellent stability and accuracy. Compared to the conventional Crank-Nicolson method for solving parabolic problems, splitting method results in a tri-diagonal system which can be easily solved by tri-diagonal solver [14] meanwhile preserves stability and accuracy.

Gornak, Iliev and Minev later introduced an algorithm [1] involving direction splitting for parabolic problems with a local refinement. The authors considered the point centered finite difference scheme. The scheme proposed is unconditionally stable; but since the authors did not provide the methodology of generating scheme, the refinement is restricted to one to two (each coarse cell is subdivided into 4 finer cells) and is hard to be generalized. The directional splitting results in a sequence of linear systems; they are tri-diagonal for rows or columns without refinement and tri-diagonal dominated for other rows. The tri-diagonal dominated systems are then easily solved by the Schur compliment technique using direct solver.

The authors also stressed in their paper that the algorithm can be implemented in parallel computers. This greatly improves the efficiency. The idea is demonstrated

by Minev and Guermond in [25]. In our work, we will adopt their strategy and hence the method proposed in this paper will be very efficient.

The paper is organized as follow. In section 2, basic notations and grids will be introduced. In section 3, we derive the finite difference scheme. Time discretization will be performed with Peaceman and Rachford directional splitting. Integrating the semi-discretized equation over the cell and applying Green's flux theorem will give us the balance equation. The essential idea is to derive finite dimensional operators which approximate first order derivative(flux). It is crucial that operators devised are positive and symmetric. This results in the stability of the scheme.

We focus on the stability in section 4. Positivity of operators proposed will be proven on the first place. The method is based on Samarskii in [8]. We do inner product and use discrete summation by parts formula . Then the unconditional stability in a norm defined in (4.5) can be proven using classical results in [2].

Section 5 presents the error analysis. This is done in the framework of Samarskii,Lazarov and Makarov in [7]. We first focus on deriving the local truncation error of the scheme using Bramble-Hilbert Lemma; by applying the stability results in section 4, it follows the convergence estimates. It should be noted that we loss one half order of convergence for cells in the interface region between coarse and fine cells; but numerical experiments later demonstrate that the scheme is of second order convergence both in space and time.

Numerical experiments will be presented in section 6. It should be noted that resulting linear system for rows and columns with refinement is tri-diagonal except a small part which corresponds to intersection region involving irregular nodes.

This reminds us of using the Schur compliment technique. Detailed implementation of the scheme by this technique in parallel computers is demonstrated in section 7.

Chapter 2

Formulation and Notation

2.1 Notation and Problem Formulation

For any $u \in \mathbb{R}^n$, $\|u\|$ denotes standard L_2 norm. $H^m(\Omega)$ is the Sobolev space, that is,

$$H^m(\Omega) = \{D^\alpha u \in L^2(\Omega), \text{where } \alpha \text{ is the multi-index such that } |\alpha| \leq m\}$$

and it is equipped with norm $\|.\|_{H^m}$ and semi-norm $|.|_m$.

We consider the following initial and boundary value problem,

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial W_1}{\partial x_1} + \frac{\partial W_2}{\partial x_2} = f, \\ &u(x,0) = g(x), x \in \Omega \subset \mathbb{R}^2, \\ &u(x,t) = h(x,t), x \in \Gamma \text{ and } t \in [0,T], \end{aligned} \tag{2.1}$$

where $W_1 = -a_1(x)\frac{\partial u}{\partial x_1} = A_1 u$ and $W_2 = -a_2(x)\frac{\partial u}{\partial x_2} = A_2 u$. Solution $u \in H^\alpha$, where α will be determined later for the purpose of convergence; $a_1(x)$ and $a_2(x)$ are the coefficients and are independent of time; we require $0 < b_1 \leq a_1(x) \leq b_2$

and $0 < b_1 \leq a_2(x) \leq b_2$.

u is denoted as exact solution to (2.1); and $\bar{u}$ and $\tilde{u}$ will be denoted as semi-discrete (discreitzation in time) and full discrete solution respectively. Let us consider the following grid cell denoted by $e(x)$ as shown in figure (2.1) in page 7

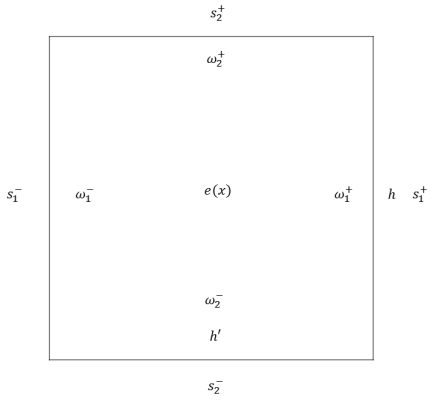

Figure 2.1: One Grid cell

For cell $e(x)$ in figure (2.1), we have following notations,

s_1^+ and $s_1^-$are east and west boundary

s_2^+ and $s_2^-$are the north and south boundary

w_l^+ donates the total approximate flux crossing the $s_l^+, l = 1, 2$

w_l^- donates the total approximate flux crossing the $s_l^-, l = 1, 2$

2.2 The Grids

Let us assume the Ω is a rectangle in $\mathbb{R}^2$. We cover Ω with uniform rectangle cells defined above and the grid point is at the center of each cell. We denote all grid cells by ω. Now choose a region in ω (depends on the solution of (2.1)) and

subdivide each cell inside the region into $m - by - m$ fine cells as shown in the figure (2.2) in page 8. Now, ω can be divided into three parts

1. ω_1, regular cells;

2. ω_2, irregular cells in coarse part;

3. ω_3, irregular cells in fine part;

The 'regular' mentioned here means the neighboring cells of $e(x)$ are of the same size with the $e(x)$.

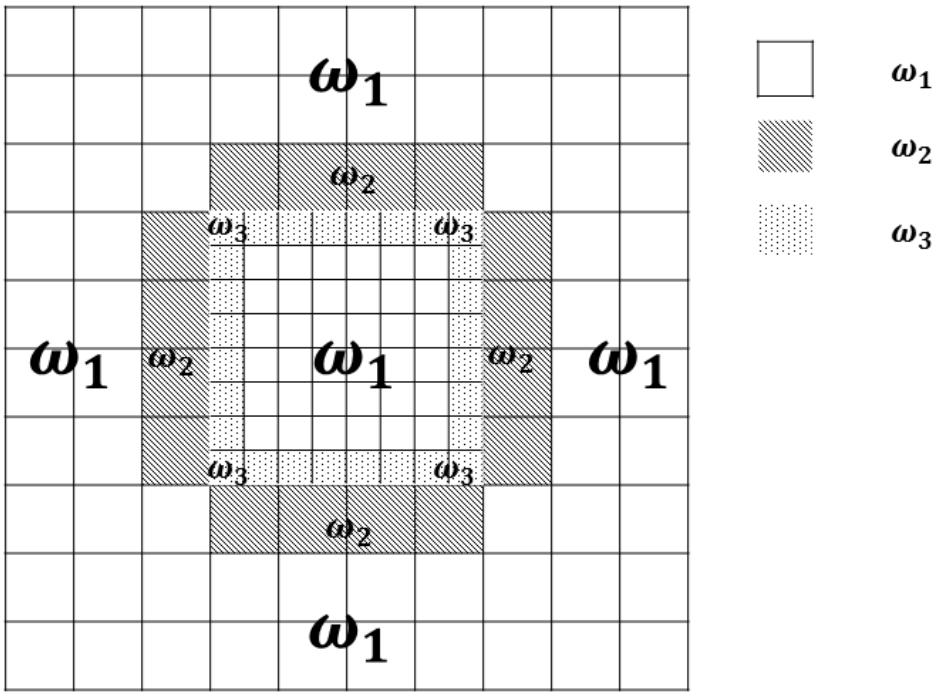

Figure 2.2: Grids with local refinement

If the height and width of coarse cells are denoted as h_c and h'_c, the height and width of fine cells can be denoted as $h_f = \frac{h_c}{m}$ and $h'_f = \frac{h'_c}{m}$. h and h' are also used as a general notation of height and width of a cell. In the remaining part the thesis, $m = 2$ for the purpose of easy demonstration. The analysis for $m > 2$ can be conducted in similar fashion.

Chapter 3

Approximation

We derive the finite difference scheme in this section. Directional splitting is used first in the discretization of time. By integrating the semi-discrete equation over cell $e(x)$ and applying the Gauss-flux law, we can get the balance equation. The idea is to approximate the flux $\frac{\partial u}{\partial x_l}$ through s_l , for $l = 1, 2$. The discussion is based on the location of cells. It is easy to see that the approximation to regular cells leads to standard five points scheme given constant $a_1(x)$ and $a_2(x)$; hence we will focus on the approximation of irregular cells. For simplicity, we consider only $1 - to - 2$ refinement on rectangle girds; the flux approximation of general $1 - to - n$ refinement on arbitrary rectangle grid is given in the remark (3.2.2).

3.1 Semi Discretized Scheme and the Balance Equation

Let us start with the semi-discrete scheme. Peaceman Rachford directional splitting for (2.1) will be applied in the discretization in time, this gives us,

$$\begin{aligned}
&\frac{\bar{u}^{n+1/2}-\bar{u}^{n}}{\tau/2}+A_1\bar{u}^{n+1/2}+A_2\bar{u}^{n}=f^{n+1/2},\\
&\frac{\bar{u}^{n+1}-\bar{u}^{n+1/2}}{\tau/2}+A_1\bar{u}^{n+1/2}+A_2\bar{u}^{n+1}=f^{n+1/2}.
\end{aligned} \tag{3.1}$$

Without loss of generality, our analysis is done only for the first equation in (3.1), analysis for the other equation can be done in the same way. Consider cell $e_{i,j}$ and integrate over the cell with respect to both directions. By the Green flux theorem, it follows that

$$\begin{aligned}
&\int_{e(x)}\frac{\bar{u}^{n+1/2}-\bar{u}^{n}}{\tau/2}dx-\Big(\int_{s_1^+}a_1(s)\frac{\partial\bar{u}^{n+1/2}}{\partial x_1}ds-\int_{s_1^-}a_1(s)\frac{\partial\bar{u}^{n+1/2}}{\partial x_1}ds\Big)\\
&-\Big(\int_{s_2^+}a_2(s)\frac{\partial\bar{u}^{n}}{\partial x_2}ds-\int_{s_2^-}a_2(s)\frac{\partial\bar{u}^{n}}{\partial x_2}ds\Big)\\
=&\int_{e(x)}\frac{\bar{u}^{n+1/2}-u^{n}}{\tau/2}dx+\Big(\int_{s_1^+}W_1^{n+1/2}ds-\int_{s_1^-}W_1^{n+1/2}ds\Big)\\
&+\Big(\int_{s_2^+}W_2^{n}ds-\int_{s_2^-}W_2^{n}ds\Big)\\
=&\int_{e(x)}f dx=\Phi.
\end{aligned} \tag{3.2}$$

The equation (3.2) is called the balance equation and our scheme is genereated by approximating the integration and total flux in balance equation. The first integral in (3.2) can be approximated by mid-point rule, which is

$$\int_{e_{i,j}(x)}\frac{\bar{u}^{n+1/2}-\bar{u}^{n}}{\tau/2}dx=hh'\frac{\tilde{u}_{i,j}^{n+1/2}-\tilde{u}_{i,j}^{n}}{\tau/2}.$$

Approximate $\int_{s_l^+} W_l ds$ and $\int_{s_l^-} W_l ds$ by w_l^+ and w_l^- which are evaluated by the full-discrete solution $\tilde{u}$, it follows that

$$hh' \frac{\tilde{u}^{n+1/2} - \tilde{u}^n}{\tau/2} + w_1^{+,n+1/2} - w_1^{-,n+1/2} + w_2^{+,n} - w_2^{-,n} = \Phi. \tag{3.3}$$

3.2 Approximation of the Flux

We now focus on approximating the total flux. Without loss of generality, we consider x_1 direction only, that is, we will derive w_1^+ and w_1^-; x_2 direction approximation can be derived in the same fashion. Our discussion depends on the location of the cell $e_{i,j}$.

- Case 1, $e_{i,j} \in \omega_1$, i.e., $e(x)$ is a regular cell. Let us approximate the flux w_1^+ and w_1^- by

$$a_1(x) \frac{\partial u_{i,j}}{\partial x_1}\bigg|_{s_1^+} = a_1(x) \frac{\tilde{u}_{i+1,j} - \tilde{u}_{i,j}}{h'},$$
$$a_1(x) \frac{\partial u_{i,j}}{\partial x_1}\bigg|_{s_1^-} = a_1(x) \frac{\tilde{u}_{i,j} - \tilde{u}_{i-1,j}}{h'}.$$

  It follows that

$$\begin{aligned} & hh' \frac{\tilde{u}_{i,j}^{n+1/2} - \tilde{u}_{i,j}^n}{\tau/2} \\ & - \Big\{ \frac{\tilde{u}_{i+1,j}^{n+1/2} - \tilde{u}_{i,j}^{n+1/2}}{h'} \int_{s_1^+} a_1(s) ds - \frac{\tilde{u}_{i,j}^{n+1/2} - \tilde{u}_{i-1,j}^{n+1/2}}{h'} \int_{s_1^-} a_1(s) ds \Big\} \\ & - \Big\{ \frac{\tilde{u}_{i,j+1}^{n} - \tilde{u}_{i,j}^{n}}{h} \int_{s_2^+} a_2(s) ds - \frac{\tilde{u}_{i,j}^{n} - \tilde{u}_{i,j-1}^{n}}{h} \int_{s_2^-} a_2(s) ds \Big\} \\ & = \Phi. \end{aligned} \tag{3.4}$$

Let the harmonic means of the coefficients be defined as

$$k_{i,j}^{+} = \frac{1}{h}\int_{s_1^{+}(e_{i,j})} a_1(s)ds \text{ and } k_{i,j}^{-} = \frac{1}{h}\int_{s_1^{-}(e_{i,j})} a_1(s)ds,$$
$$g_{i,j}^{+} = \frac{1}{h'}\int_{s_2^{+}(e_{i,j})} a_2(s)ds \text{ and } g_{i,j}^{-} = \frac{1}{h'}\int_{s_2^{-}(e_{i,j})} a_2(s)ds,$$

where h' and h are length of s_2 and s_1; then (3.4) can be simplified to

$$\begin{aligned} & hh'\frac{\tilde{u}_{i,j}^{n+1/2} - \tilde{u}_{i,j}^{n}}{\tau/2} \\ & - \Big\{\frac{\tilde{u}_{i+1,j}^{n+1/2} - \tilde{u}_{i,j}^{n+1/2}}{h'} hk_{i,j}^{+} - \frac{\tilde{u}_{i,j}^{n+1/2} - \tilde{u}_{i-1,j}^{n+1/2}}{h'} hk_{i,j}^{-}\Big\} \\ & - \Big\{\frac{\tilde{u}_{i,j+1}^{n} - \tilde{u}_{i,j}^{n}}{h} h'g_{i,j}^{+} - \frac{\tilde{u}_{i,j}^{n} - \tilde{u}_{i,j-1}^{n}}{h} h'g_{i,j}^{-}\Big\}. \end{aligned} \tag{3.5}$$

If $a_1(x) = a_2(x) = 1$ and the square grids is used, the scheme is exactly the standard 5 points stencil approximation,

$$\begin{aligned} & h^2\frac{\tilde{u}^{n+1/2} - \tilde{u}^{n}}{\tau/2} - \big(\tilde{u}_{i+1,j}^{n+1/2} - 2\tilde{u}_{i,j}^{n+1/2} + \tilde{u}_{i-1,j}^{n+1/2}\big) - \big(\tilde{u}_{i,j+1}^{n} - 2\tilde{u}_{i,j}^{n} + \tilde{u}_{i,j-1}^{n}\big) \\ = & \int_{e(x)} f dx \equiv \Phi. \end{aligned}$$

To deal with the boundary nodes; let us introduce the ghost point g_1. Given a homogeneous boundary condition, we have

$$\frac{g_1 + \tilde{u}_{0,j}}{2} = 0.$$

This implies that $g_1 = -\tilde{u}_{0,j}$; hence we have

$$\begin{aligned} & -\int_{s_1^{+}} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds + \int_{s_1^{-}} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds \\ \approx & w_1^{+} - w_1^{-} = -\frac{\tilde{u}_{1,j} - \tilde{u}_{0,j}}{h'} \cdot hk_{0,j}^{+} + \frac{\tilde{u}_{0,j} + \tilde{u}_{0,j}}{h'} \cdot hk_{0,j}^{-}. \end{aligned} \tag{3.6}$$

- Case 2, $e_{i,j} \in \omega_2$, where $e_{i,j}$ is the irregular cell in coarse part.

  We number part of cells in the intersection region as shown in the figure (3.1) in page 14. Let us consider cell 0. The approximation to flux through s_1^+ is

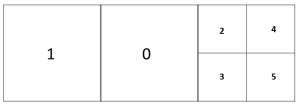

Figure 3.1: Intersection region in x_1 direction

given by

$$u_x\Big|_{s_1^+} = \frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}},$$

where $\bar{h'} = \frac{h_f^{'} + h_c^{'}}{2}$. The approximation to the flux crossing s_1^- is defined as follow,

$$u_x\Big|_{s_1^-} = \frac{\tilde{u}_0 - \tilde{u}_1}{h_c^{'}}.$$

It then follows that,

$$\begin{aligned} & -\int_{s_1^+} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds + \int_{s_1^-} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds \\ \approx & w_1^+ - w_2^- \\ = & -\frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}}\int_{s_1^+} a_1(s)ds + \frac{\tilde{u}_0 - \tilde{u}_1}{h_c^{'}}\int_{s_1^-} a_1(s)ds \\ = & -\frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}} \cdot h_c k_0^+ + \frac{\tilde{u}_0 - \tilde{u}_1}{h_c^{'}} \cdot h_c k_0^-. \end{aligned} \tag{3.7}$$

The approximation in x_2 direction follows directly from case 1.

- Case 3, $e_{i,j} \in \omega_3$, where $e_{i,j}$ is the irregular cell in fine part. Let us consider the singular cell 2 as shown in figure (3.1). The approximation to the flux

through s_1^- is given by

$$u_x\Big|_{s_1^-} = \frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}}.$$

The approximation to the flux crossing s_1^+ is given as follow,

$$u_x\Big|_{s_1^+} = \frac{\tilde{u}_4 - \tilde{u}_2}{h_f'}.$$

Hence we have,

$$\begin{aligned}
&-\int_{s_1^+} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds + \int_{s_1^-} a_1(s)\frac{\partial u^{n+1/2}}{\partial x_1}ds \\
\approx& w_1^+ - w_1^- \\
=& -\frac{\tilde{u}_4 - \tilde{u}_2}{h_f'}\int_{s_1^+} a_1(s)ds + \frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}}\int_{s_1^-} a_1(s)ds \\
=& -\frac{\tilde{u}_4 - \tilde{u}_2}{h_f'}\cdot h_f k_2^+ + \frac{\frac{1}{2}(\tilde{u}_2 + \tilde{u}_3) - \tilde{u}_0}{\bar{h'}}\cdot h_f k_2^-.
\end{aligned} \tag{3.8}$$

The approximation in x_2 direction follows directly from case 1.

Remark 3.2.1. *For the purpose of proving stability in section 4, we make the following assumption,*

$$k_0^+ = \frac{1}{h_c}\int_{s_1^+(e_0)} a_1(s)ds = \frac{2}{h_c}\int_{s_1^-(e_3)} a_1(s)ds = \frac{2}{h_c}\int_{s_1^-(e_2)} a_1(s)ds. \tag{3.9}$$

That is, $k_0^+ \approx k_3^- \approx k_2^-$.

Remark 3.2.2. *Here we derive the scheme of arbitrary* $1-to-n$ *refinement on a general rectangle grid. We only give the approximation to the flux through the intersection edge of the cell, other steps are same with* $1-to-2$ *approximation.*

Let us consider cell 0 in figure (3.2) in page 16. It follows that

$$u_x\Big|_{s_1^+} = \frac{\frac{1}{n}\sum_{i=1}^{n}\tilde{u}_i - \tilde{u}_0}{\bar{h'}}, \tag{3.10}$$

where $\bar{h'} = \frac{h_f^{'}+h_c^{'}}{2}$. Consider cell 0, then we have

$$\int_{s_1^+} a_1(s)\frac{\partial u}{\partial x_1}ds \cong \frac{\frac{1}{n}\sum_{i=1}^{n}\tilde{u}_i - \tilde{u}_0}{\bar{h'}}\int_{s_1^+} a_1(s)ds. \tag{3.11}$$

If $a_1(x) = 1$, we then have,

$$\int_{s_1^+} a_1(s)\frac{\partial u}{\partial x_1}ds \approx \frac{h_c}{\bar{h'}}\big(\frac{1}{n}\sum_{i=1}^{n}\tilde{u}_i - \tilde{u}_0\big).$$

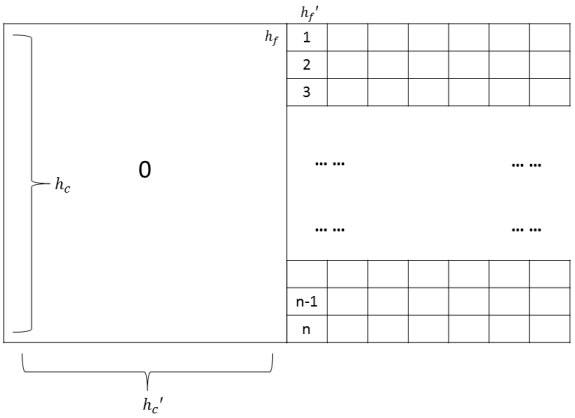

Figure 3.2: General Refinements

3.3 Formulating the Scheme as Operator Equations

In order to conduct stability and convergence analysis later in a more convenient way, we will formulate the problem as a system of operator equations. Now let us define,

$$\begin{aligned}\bar{A}_1 &= -\big(\int_{s_1^+} a_1(s)\frac{\partial .}{\partial x_1}ds - \int_{s_1^-} a_1(s)\frac{\partial .}{\partial x_1}ds\big),\\ \bar{A}_2 &= -\big(\int_{s_2^+} a_2(s)\frac{\partial .}{\partial x_2}ds - \int_{s_2^-} a_2(s)\frac{\partial .}{\partial x_2}ds\big).\end{aligned} \tag{3.12}$$

which are two operators defined on $H^\alpha(e(x))$, where α will be defined for the purpose of the accuracy later in section 5. Then we can write the balance equation with the first integration being approximated by a mid-point rule in operator form,

$$\begin{aligned}M\frac{\hat{u}^{n+1/2}-\hat{u}^n}{\tau/2} + \bar{A}_1\hat{u}^{n+1/2} + \bar{A}_2\hat{u}^n &= \Phi,\\ M\frac{\hat{u}^{n+1}-\hat{u}^{n+1/2}}{\tau/2} + \bar{A}_1\hat{u}^{n+1/2} + \bar{A}_2\hat{u}^{n+1} &= \Phi,\end{aligned} \tag{3.13}$$

where $\hat{u}$ is the exact solution to the above equations. If we apply (3.4),(3.7) and (3.8), the above differential problems (3.13) can be approximated by

$$\begin{aligned}M\frac{\tilde{u}^{n+1/2}-\tilde{u}^n}{\tau/2} + \tilde{A}_1\tilde{u}^{n+1/2} + \tilde{A}_2\tilde{u}^n &= \Phi,\\ M\frac{\tilde{u}^{n+1}-\tilde{u}^{n+1/2}}{\tau/2} + \tilde{A}_1\tilde{u}^{n+1/2} + \tilde{A}_2\tilde{u}^{n+1} &= \Phi,\end{aligned} \tag{3.14}$$

where $\tilde{u} \in \mathbb{R}^N$, for some N is the finite dimension finite difference solution; M is a diagonal matrix consisting of the volumes of each cell; $\tilde{A}_1$ and $\tilde{A}_2$ are finite dimensional operators constructed by the combination of (3.4),(3.7) and (3.8)

which defines the scheme for regular cells, irregular cells in coarse part and irregular cells in fine part respectively. It is easy to that see M, $\tilde{A}_1$ and $\tilde{A}_2$ are symmetric.

Chapter 4

Stability Analysis

In order to prove the unconditional stability of the scheme, we need to prove the positiveness of $\tilde{A}_1$ and $\tilde{A}_2$ first. It is shown that the scheme is unconditionally stable. The remainder of this section will be organized as follow. In 4.1, we will prove the positiveness of $\tilde{A}_1$ and $\tilde{A}_2$. It should be noted that positiveness of $\tilde{A}_1$ and $\tilde{A}_2$ are easy to see if they are defined for rows or columns involving regular cells only; hence we only prove positiveness of rows involving irregular cells. Same as what we did before, we study $\tilde{A}_1$ only. The discussion for $\tilde{A}_2$ is similar with that of $\tilde{A}_1$. In 4.2, we are going to study the stability of the scheme. The scheme is unconditional stable with respect to a special normed defined later. For simplicity, throughout the section, we assume one to two refinement on square cells, that is, $h_c^{'} = h_c = 2h_f = 2h_f^{'}$.

4.1 Positiveness of Devised Operators

Let $u, v \in \mathbb{R}^{N+1}$. We define an inner product

$$[u, v] = \sum_{i=0}^{N} u_i v_i,$$

and the following two operators,

$$(u, v] = \sum_{i=1}^{N} u_i v_i \text{ and } [u, v) = \sum_{i=0}^{N-1} u_i v_i.$$

It is easy to derive summation by parts formula,

$$\begin{aligned}(u, v_x) &= \sum_{i=1}^{N-1} u_i v_{i+1} - \sum_{i=1}^{N-1} u_i v_i \\ &= \sum_{i=1}^{N} u_{i-1} v_i - \sum_{i=1}^{N} u_i v_i - u_0 v_1 + u_N u_N \\ &= -(u_{\overline{x}}, v] - u_0 v_1 + u_N v_N,\end{aligned}$$

where $u_{\bar{x},i} = u_i - u_{i-1}$ and $u_{x,i} = u_{i+1} - u_i$. Let us consider rows with local refinement. The cells are numbered from n_0 to n_{13} as shown in figure (4.1).

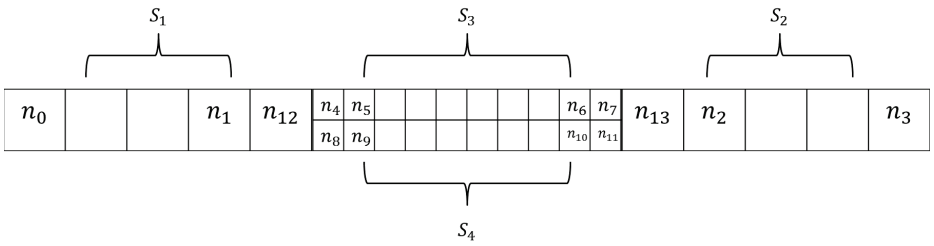

Figure 4.1: A row with refinement

Let $x, y \in \mathbb{R}^m$, where m is the dimension of the solution $\tilde{u}$ of the rows with local

refinement. We define the following operators,

$$
\begin{aligned}
(x,y)_1 &= \sum_{i=n_0+1}^{n_1} x_i y_i; \text{ and } (x,y]_1 = \sum_{i=n_0+1}^{n_{12}} x_i y_i,\\
(x,y)_2 &= \sum_{i=n_2}^{n_3-1} x_i y_i; \text{ and } (x,y]_2 = \sum_{i=n_2}^{n_3} x_i y_i,\\
(x,y)_3 &= \sum_{i=n_5}^{n_6} x_i y_i; \text{ and } (x,y]_3 = \sum_{i=n_5}^{n_7} x_i y_i,\\
(x,y)_4 &= \sum_{i=n_9}^{n_{10}} x_i y_i; \text{ and } (x,y]_4 = \sum_{i=n_9}^{n_{11}} x_i y_i.
\end{aligned}
$$

Then, we have,

$$
\begin{aligned}
[\tilde{A}_1 u, u] = &+ \big((w^+ - w^-), u\big)_1 + \big((w^+ - w^-), u\big)_2\\
&+ \big((w^+ - w^-), u\big)_3 + \big((w^+ - w^-), u\big)_4\\
&- \Big((u_{n_5} - u_{n_4})u_{n_4}k^+_{n_4} + \frac{u_{n_{12}} - \frac{1}{2}(u_{n_4} + u_{n_8})}{\overline{h}} \cdot h_f k^-_{n_4} \cdot u_{n_4}\Big)\\
&- \Big((u_{n_6} - u_{n_7})u_{n_7}k^-_{n_7} + \frac{u_{n_{13}} - \frac{1}{2}(u_{n_7} + u_{n_{11}})}{\overline{h}} \cdot h_f k^+_{n_7} \cdot u_{n_7}\Big)\\
&- \Big((u_{n_9} - u_{n_8})u_{n_8}k^+_{n_8} + \frac{u_{n_{12}} - \frac{1}{2}(u_{n_4} + u_{n_8})}{\overline{h}} \cdot h_f k^-_{n_8} \cdot u_{n_8}\Big)\\
&- \Big((u_{n_{10}} - u_{n_{11}})u_{n_{11}}k^-_{n_{11}} + \frac{u_{n_{13}} - \frac{1}{2}(u_{n_7} + u_{n_{11}})}{\overline{h}} \cdot h_f k^+_{n_{11}} \cdot u_{n_{11}}\Big)\\
&- \Big((u_{n_1} - u_{n_{12}})u_{n_{12}}k^-_{n_{12}} + \frac{\frac{1}{2}(u_{n_4} + u_{n_8}) - u_{n_{12}}}{\overline{h}} \cdot h_c k^+_{n_{12}} \cdot u_{n_{12}}\Big)\\
&- \Big((u_{n_2} - u_{n_{13}})u_{n_{13}}k^+_{n_{13}} + \frac{\frac{1}{2}(u_{n_7} + u_{n_{11}}) - u_{n_{13}}}{\overline{h}} \cdot h_c k^-_{n_{13}} \cdot u_{n_{13}}\Big)\\
&- (u_{n_0+1} - u_{n_0})u_{n_0}k^+_{n_0} + 2u^2_{n_0}k^-_{n_0} - (u_{n_3-1} - u_{n_3})k^-_{n_3}u_{n_3} + 2u^2_{n_3}k^+_{n_3}.
\end{aligned}
$$

Using summation by parts formula, it follows that,

$$\begin{aligned}\big((w^+ - w^-), u\big)_1 &= \sum_{i=n_0+1}^{n_1} (w^-_{i+1} - w^-_i)u_i \\ &= \sum_{i=n_0+1}^{n_{12}} w^-_i(u_{i-1} - u_i) - w^-_{n_0+1}u_{n_0} + w^-_{n_{12}}u_{n_{12}} \\ &= (w^-, \frac{w^-}{k^-}]_1 - w^-_{n_0+1}u_{n_0} + w^-_{n_{12}}u_{n_{12}}.\end{aligned}$$

Similarly, we have

$$\big((w^+ - w^-), u\big)_2 = (w^-, \frac{w^-}{k^-}]_2 - w^-_{n_2}u_{n_{13}} + w^-_{n_3}u_{n_3}, \tag{4.1}$$

$$\big((w^+ - w^-), u\big)_3 = (w^-, \frac{w^-}{k^-}]_3 - w^-_{n_5}u_{n_4} + w^-_{n_7}u_{n_7}, \tag{4.2}$$

$$\big((w^+ - w^-), u\big)_4 = (w^-, \frac{w^-}{k^-}]_4 - w^-_{n_9}u_{n_8} + w^-_{n_{11}}u_{n_{11}}. \tag{4.3}$$

Substitute into the equation and recall remark(3.2.1), above equations lead to canceling,

$$\begin{aligned}[\tilde{A}_1 u, u] = &- \tilde{y}_1 h_f k^+_4 u_{n_4} - \tilde{y}_1 h_f k^+_{11} u_{n_8} + 2\tilde{y}_1 h_c k^-_{12} u_{n_{12}} \\ &- \tilde{y}_2 h_f k^+_7 u_{n_7} - \tilde{y}_2 h_f k^+_{11} u_{n_{11}} + 2\tilde{y}_2 h_c k^-_{13} u_{n_{13}} \\ &+ (w^-, \frac{w^-}{k^-}]_1 + (w^-, \frac{w^-}{k^-}]_2 + (w^-, \frac{w^-}{k^-}]_3 + (w^-, \frac{w^-}{k^-}]_4 \\ &+ 2u^2_{n_0} k^-_{n_0} + 2u^2_{n_3} k^-_{n_3},\end{aligned}$$

where $\tilde{y}_1 = -\frac{\frac{1}{2}(u_{n_4}+u_{n_8})-u_{n_{12}}}{\bar{h}}$ and $\tilde{y}_2 = -\frac{\frac{1}{2}(u_{n_7}+u_{n_{11}})-u_{n_{13}}}{\bar{h}}$; it follows immediately

that,

$$[\tilde{A}_1 u, u] = (w^-, \frac{w^-}{k^-}]_1 + (w^-, \frac{w^-}{k^-}]_2 + (w^-, \frac{w^-}{k^-}]_3 + (w^-, \frac{w^-}{k^-}]_4 + 2k_4^- \tilde{y}_1^2 \bar{h} h_f + 2k_7^- \tilde{y}_2^2 \bar{h} h_f + 2u_{n_0}^2 k_{n_0}^- + 2u_{n_3}^2 k_{n_3}^- \geq 0.$$

Above result uses the assumption in remark (3.2.1), which is $k_{n_4}^- = k_{n_8}^- = k_{n_{12}}^+$. Since $k_i > 0$ for all i, it follows the positiveness of $\tilde{A}_1$ and $\tilde{A}_2$.

4.2 Stability of Scheme

Recall the operator equations (3.14), without loss of generality, we study the first equation only. Hence, we have

$$M(\tilde{u}^{n+1/2} - \tilde{u}^n) + \frac{\tau}{2}\tilde{A}_1 \tilde{u}^{n+1/2} + \frac{\tau}{2}\tilde{A}_2 \tilde{u}^n = \frac{\tau}{2}\Phi.$$

This implies

$$(M + \frac{\tau}{2}\tilde{A}_1)\tilde{u}^{n+1/2} = (M - \frac{\tau}{2}\tilde{A}_2)\tilde{u}^n + \frac{\tau}{2}\Phi.$$

Since M is positive and symmetric, there exists a positive and symmetric matrix $M^{\frac{1}{2}}$ such that $M = M^{\frac{1}{2}} M^{\frac{1}{2}}$. Denoting $M^{-\frac{1}{2}}$, the inverse of $M^{\frac{1}{2}}$, it follows that,

$$(M^{\frac{1}{2}} + \frac{\tau}{2}\tilde{A}_1 M^{-\frac{1}{2}}) M^{\frac{1}{2}} \tilde{u}^{n+1/2} = (M^{\frac{1}{2}} - \frac{\tau}{2}\tilde{A}_2 M^{-\frac{1}{2}}) M^{\frac{1}{2}} \tilde{u}^n + \frac{\tau}{2}\Phi.$$

Multiply both sides of the equation by $M^{-\frac{1}{2}}$ to the left , we have

$$(I + \frac{\tau}{2} M^{-\frac{1}{2}} \tilde{A}_1 M^{-\frac{1}{2}}) y^{n+1/2} = (I - \frac{\tau}{2} M^{-\frac{1}{2}} \tilde{A}_2 M^{-\frac{1}{2}}) y^n + \frac{\tau}{2} M^{-\frac{1}{2}} \Phi,$$

where $y = M^{\frac{1}{2}}\tilde{u}$. Let $\hat{A}_1 = M^{-\frac{1}{2}}\tilde{A}_1 M^{-\frac{1}{2}}$, $\hat{A}_2 = M^{-\frac{1}{2}}\tilde{A}_2 M^{-\frac{1}{2}}$ and $F = \frac{\tau}{2}M^{-\frac{1}{2}}\Phi$; then we have

$$y^{n+1/2} + \frac{\tau}{2}\hat{A}_1 y^{n+1/2} = y^n - \frac{\tau}{2}\hat{A}_2 y^n + F.$$

Thus,

$$y^{n+1/2} - y^n + \frac{\tau}{2}\hat{A}_1 y^{n+1/2} + \frac{\tau}{2}\hat{A}_2 y^n = F.$$

Applying the same method to the second equation, we then have,

$$y^{n+1} - y^{n+1/2} + \frac{\tau}{2}\hat{A}_1 y^{n+1/2} + \frac{\tau}{2}\hat{A}_2 y^{n+1} = F.$$

Since $\hat{A}_1$ and $\hat{A}_2$ are positive, we establish the stability

$$\begin{aligned} \|(E + \frac{\tau}{2}\hat{A}_2)y^{n+1}\| &= \|(M^{\frac{1}{2}} + \frac{\tau}{2}M^{-\frac{1}{2}}\tilde{A}_2)\tilde{u}^{n+1}\| \\ &\leq \|(M^{\frac{1}{2}} + \frac{\tau}{2}M^{-\frac{1}{2}}\tilde{A}_2)\tilde{u}^0\| + 2\sum_{k=0}^{n}\tau\|F^k/(\frac{\tau}{2})\| \\ &\leq \|(M^{\frac{1}{2}} + \frac{\tau}{2}M^{-\frac{1}{2}}\tilde{A}_2)\tilde{u}^0\| + 2\|M^{-\frac{1}{2}}\|\sum_{k=0}^{n}\tau\|\Phi^k\|, \end{aligned} \tag{4.4}$$

where E is the identity matrix. The above estimate is presented in [2] page 125. For $\forall w \in \mathbb{R}^n$, if we define

$$\|(M^{\frac{1}{2}} + \frac{\tau}{2}M^{-\frac{1}{2}}\tilde{A}_2)w\| = \|w\|_O. \tag{4.5}$$

We can simplify the result in the following theorem,

Theorem 4.2.1. *The scheme proposed is unconditionally stable and we have the following estimate,*

$$\|\tilde{u}^{n+1}\|_O \leq \|\tilde{u}^0\|_O + C\sum_{k=0}^{n}\tau\|\Phi^k\|,$$

where C is a constant.

Chapter 5

Convergence Analysis

The convergence analysis will be done in the framework of the book [7] by Samarskii. The analysis will be conducted in following steps. We first derive the difference between the total flux and approximated total flux by Bramble Hilbert lemma [6]; that is, we want to estimate

$$\int_{s_l^{+,-}} W_l ds - w_l^{+,-} \text{ for } l = 1, 2.$$

In the operator form, we want to build the following relation,

$$\bar{A}_l u - \tilde{A}_l u = \mathcal{O}(h^\alpha), \tag{5.1}$$

for $l = 1, 2$ and some α to be determined later. With the help of the above relation, it follows the estimation of the truncation error,

$$\bar{\psi}_1 = \frac{e^{n+1/2} - e^n}{\tau/2} + M^{-1}\tilde{A}_1 e^{n+1/2} + M^{-1}\tilde{A}_2 e^n, \tag{5.2}$$

which is obtained by substituting the error $e = \tilde{u} - u$ into equation (3.14) multiplied by M^{-1}. It should be noted that the classical result in directional splitting [2] is used when estimating the consistency error. Finally, substitute e into (3.14), we can get the $\mathcal{O}(\tau^2 + h^\alpha)$ consistency error; where α will be determined later. The convergence analysis can be easily done with the help of the stability result in section 4. Without loss of generality, the analysis is conducted only on the first equation; and for simplicity, we assume $a_1(x) = a_2(x) = 1$ and $1 - to - 2$ refinement on square grids.

5.1 Truncation Error Estimate

Let $\eta_{1,2}^{+,-}$ be functionals of u at some time level and defined as below,

$$\eta_l^+ \equiv \int_{s_l^+} W_l ds - w_l^+ \text{ and } \eta_l^- \equiv \int_{s_l^-} W_l ds - w_l^- \text{ where } l = 1, 2 \ .$$

It should be noted that $\eta_{1,2}^{+,-}$ are the difference between total flux and approximated total flux which is evaluated by the values of u at the grid points. Without loss of generality, we estimate η_1^+ only; the estimate for the other functionals can be done in the same fashion. Convergence analysis here is done in the framework of [7]. $\eta_l(x)$ can be taken as a linear functional bounded in H^m, where $m \geq \frac{3}{2}$. Since the total flux $\int_{s_l^+} W_1 ds$ is approximated in different ways because of the non-uniform grids; linear functionals vanish for polynomials of different orders. By Bramble Hilbert Lemma [6], we have the following estimate $|\eta_l| = \mathcal{O}(h^\alpha)$. If we use the operators (3.12),(3.4),(3.7) and (3.8) defined in section 3, that will be,

$$\bar{A}_l u - \tilde{A}_l u = \mathcal{O}(h^\alpha), \tag{5.3}$$

where u is exact solution. Above relation will be the target and let us now start the estimation with respect to different locations of the cells.

- For regular cells $e_{i,j}(x)$, we have

$$\eta_1^+(x) = -\int_{x_{i,j-1/2}}^{x_{i,j+1/2}} \frac{\partial u}{\partial x_1}(x_{1,i} + \frac{h}{2}, s)ds + u_{i+1,j} - u_{i,j},$$

  where subscript 1 of $x_{1,i}$ denotes x_1 direction and η_1^+ is a linear functional bounded for $u \in H^{\alpha+1}(e_{i,j} \cup e_{i+1,j})$, $\alpha \geq \frac{1}{2}$. It is easy to verify that this functional vanishes for all polynomials of second degree. By Bramble Hilbert Lemma, we get

$$|\eta_1^+| \leq Ch^\alpha |u|_{\alpha+1, e_{i,j} \cup e_{i+1,j}}, \tag{5.4}$$

  and $\frac{1}{2} \leq \alpha \leq 2$.

- For irregular cells; let us consider the cell 0 in figure (3.1) in page 14. We have

$$\eta_1^+(x) = -\int_{x_{i,j-1/2}}^{x_{i,j+1/2}} \frac{\partial u}{\partial x_1}(x_{1,i} + \frac{h}{2}, s)ds + \frac{h_c}{\bar{h}}\big(\frac{1}{2}(u_2 + u_3) - u_0\big). \tag{5.5}$$

  η_1^+ is a linear functional bounded for $u \in H^{\alpha+1}(e_0 \cup e_1 \cup e_2)$, $\alpha \geq \frac{1}{2}$. This functional vanishes for all polynomials of degree one. The Bramble Hilbert lemma then implies that,

$$|\eta_1^+| \leq Ch^\alpha |u|_{\alpha+1, e_0 \cup e_1 \cup e_2}, \tag{5.6}$$

  and $\frac{1}{2} \leq \alpha \leq 1$.

Introducing $e = \tilde{u} - u$, denoting the error between full-discrete solution and exact solution; multiplying the scheme (3.14) by M^{-1} and substituting the error e into the modified scheme, we can get the error equation as follow,

$$\begin{aligned}\bar{\psi}_1 &= \frac{e^{n+1/2} - e^n}{\tau/2} + M^{-1}\tilde{A}_1 e^{n+1/2} + M^{-1}\tilde{A}_2 e^n \\ &= M^{-1}\Phi - \frac{u^{n+1/2} - u^n}{\tau/2} - M^{-1}\tilde{A}_1 u^{n+1/2} - M^{-1}\tilde{A}_2 u^n.\end{aligned} \tag{5.7}$$

Using the result in page 126 of [2], it follows that

$$\bar{\psi}_1 = -\frac{M^{-1}\tilde{A}_1}{2}(u^{n+1} + u^n) - \frac{M^{-1}\tilde{A}_2}{2}(u^{n+1} + u^n) - \frac{u^{n+1} - u^n}{\tau} + M^{-1}\Phi + \mathcal{O}(\tau^2). \tag{5.8}$$

If relation (5.3) is used, that is, $\bar{A}_l u = \tilde{A}_l u + \mathcal{O}(h^\alpha)$, we can get,

$$\begin{aligned}\bar{\psi}_1 = &-\frac{M^{-1}\bar{A}_1}{2}(u^{n+1} + u^n) - \frac{M^{-1}\bar{A}_2}{2}(u^{n+1} + u^n) - \frac{u^{n+1} - u^n}{\tau} + M^{-1}\Phi \\ &+ \mathcal{O}(M^{-1}h^\alpha + \tau^2) = \mathcal{O}(M^{-1}h^\alpha + \tau^2).\end{aligned} \tag{5.9}$$

The above cancellation is due to the Taylor's expansion of u at the time level $n + \frac{1}{2}$. Finally, substituting the error e into the scheme (3.14), it follows the truncation error ψ of the scheme,

$$\begin{aligned}\psi =&\frac{M}{\tau/2}(e^{n+1/2} - e^n) + \tilde{A}_1 e^{n+1/2} + \tilde{A}_2 e^n \\ =&M\left(\frac{(\tilde{u} - u)^{n+1/2} - (\tilde{u} - u)^n}{\tau/2} + M^{-1}\tilde{A}_1(\tilde{u} - u)^{n+1/2} + M^{-1}\tilde{A}_2(\tilde{u} - u)^n\right) \\ =&\mathcal{O}(\tau^2 + h^\alpha).\end{aligned}$$

Hence the consistency error of the proposed scheme is $\mathcal{O}(\tau^2 + h^\alpha)$, where $\frac{1}{2} \leq \alpha \leq 1$.

5.2 Error Estimate

Recall stability result (4.4) in section 4, it follows that,

$$\|e^{n+1}\|_O \leq \|e^0\|_O + C\sum_{k=0}^{n}\tau\|\psi_1^k\|.$$

Since $e^0 = 0$, we then can get the error estimate in time,

$$\sum_{k=0}^{n}\tau\|\psi^k\| = \mathcal{O}(\tau^2).$$

For the error in space, we have

$$\begin{aligned}\sum_{x\in\omega}\psi^2(x) &\leq C\Big(\sum_{x\in\omega,irregular}h^{2\alpha}|u|^2_{\alpha+1} + \sum_{x\in\omega,regular}h^{2\beta}|u|^2_{\beta+1}\Big)\\ &\leq C(h^{2\alpha}|u|^2_{\alpha+1,\Omega_h} + h^{2\beta}|u|^2_{\beta+1,\Omega/\Omega_h}),\end{aligned} \tag{5.10}$$

where Ω_h is intersection region and of width d and $\frac{1}{2}\leq\alpha\leq 1$ and $\frac{1}{2}\leq\beta\leq 2$.

Hence we have,

$$\begin{aligned}\|\eta_1\| = \big(\sum_{x\in\omega}\eta_1^2\big)^{\frac{1}{2}} &= Ch^{\alpha}\big(\sum_{|m|=\alpha+1}|D^m u|^2_{0,\Omega_h}\big)^{\frac{1}{2}}\\ &\leq Ch^{\alpha+\frac{1}{2}}\big(\sum_{|m|=\alpha+1}\|D^m u\|^2_{\alpha,\Omega}\big)^{\frac{1}{2}}\\ &\leq Ch^{\alpha+\frac{1}{2}}\|u\|_{2\alpha+1,\Omega},\end{aligned} \tag{5.11}$$

for $\frac{1}{2}\leq\alpha\leq 1$ and first inequality is due to Il's inequality [7]. The result is summarized in the following theorem.

Theorem 5.2.1. *For problem (2.1) with solution u and constant coefficients $a_1(x)$ and $a_2(x)$, if $u\in H^{2\alpha+1}$, $\frac{1}{2}\leq\alpha\leq 1$, the convergence rate of proposed scheme in*

space is $\mathcal{O}(h^{\alpha+\frac{1}{2}})$*, that is*

$$\|\tilde{u} - u\|_O \leq Ch^{\alpha+\frac{1}{2}}\|u\|_{2\alpha+1,\Omega} \ , \frac{1}{2} \leq \alpha \leq 1,$$

where C *is a constant independent of* h *and* $u(x)$*.*

Chapter 6

Numerical Results

Two numerical experiments will be presented. In the first experiment, we consider a problem with smooth solution. It should be noted that second order convergence in space for both irregular cells (cells in the interface area) and the whole grid cells can be observed.

The second experiment will be the comparison of the regular 2D Peaceman Rachford directional splitting scheme [13] on uniform cell centered grid and the proposed scheme solving a problem which has large source term f. This problem generates singularity in its solution and demonstrates the advantage of the proposed scheme. Programs are run on Intel(R) Core(TM) i7-4510U CPU @ 2.00GHz workstation with PYTHON.

6.1 Continuous problem

We consider solving problem (2.1) with homogeneous boundary and initial condition on $\Omega = [0,1] \times [0,1]$ and $t \in [0, 2\pi]$. We assume $a_1(x) = a_2(x) = 1$ and

$$f = 2\pi^2 \sin(t) \sin(\pi x) \sin(\pi y) + \cos(t) \sin(\pi x) \sin(\pi y).$$

The exact solution is,

$$u(x,t) = \sin(t) \sin(\pi x) \sin(\pi y).$$

One to two refinement is considered and without loss of generality, refinement area is chosen to be $[\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$. Let $e^j \in \mathbb{R}^n$ be error at time level j, we define norm $\|.\|_{l^2(\Omega)}$ as follow,

$$\|e^j\|_{l^2(\Omega)} = \big(\sum_{i=1}^{n} h^2 |e_i^j|^2\big)^{\frac{1}{2}}.$$

For simplicity, this norm is also denoted as $\|.\|_{h_{(1)}}$. Let $[e^1, e^2, ..., e^m] \in \mathbb{R}^{m \times n}$, where m is the total time levels. We define norm $\|.\|_{l^2(0,T;l^2(\Omega))}$ as follow,

$$\|.\|_{l^2(0,T;l^2(\Omega))} = \big(\sum_{j=1}^{m} \tau \|e^j\|_{l^2(\Omega)}^2\big)^{\frac{1}{2}},$$

where τ is the time step; for simplicity, this norm is also denoted as $\|.\|_{h_{(2)}}$. Numerical results for error in $\|.\|_{l^2(\Omega)}$ of irregular cells with respect to different space steps are presented in the table (6.1). The data have been plotted in figure (6.1). The time level of the left hand side graph in the figure (6.1) is chosen at $t = \pi$. Table (6.2) presents error in $\|.\|_{l^2(\Omega)}$ of all grid cells with respect to different space steps. The data have been plotted in figure (6.2). The time level of graph in the figure (6.2) on the left is chosen at $t = \pi$.

	$n = 96$	$n = 48$	$n = 32$	$n = 24$	$n = 16$
$t = \frac{\pi}{2}$	6.34263e-06	3.51007e-05	9.42574e-05	1.90305e-04	5.29469e-04
$t = \pi$	6.06405e-07	1.82685e-06	3.99325e-06	7.52024e-06	2.09874e-05
$t = \frac{3\pi}{2}$	6.34338e-06	3.51029e-05	9.42623e-05	1.90314e-04	5.29496e-05
$t = 2\pi$	6.06405e-07	1.82685e-06	3.99325e-06	7.52024e-06	2.09874e-05

Table 6.1: Error changes(Irregular) w.r.t. different h

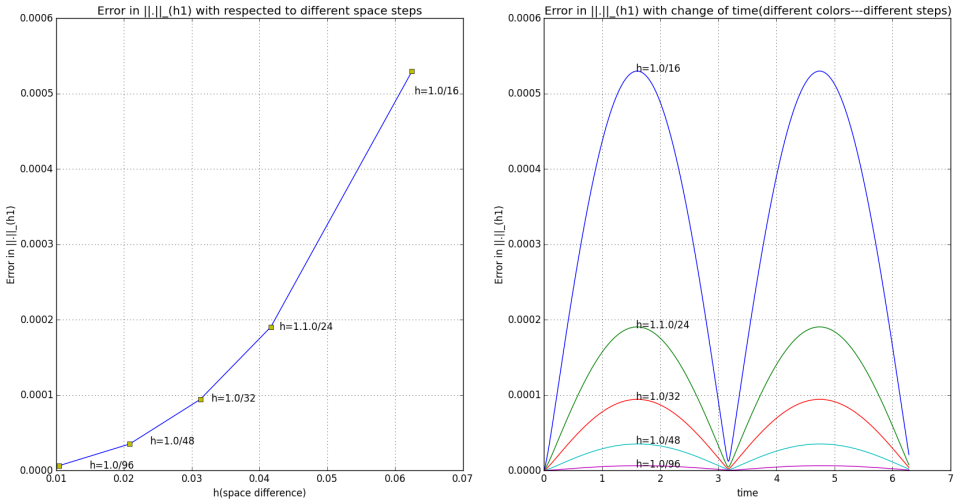

Figure 6.1: Error changes with respect to space steps (Intersection Cells)

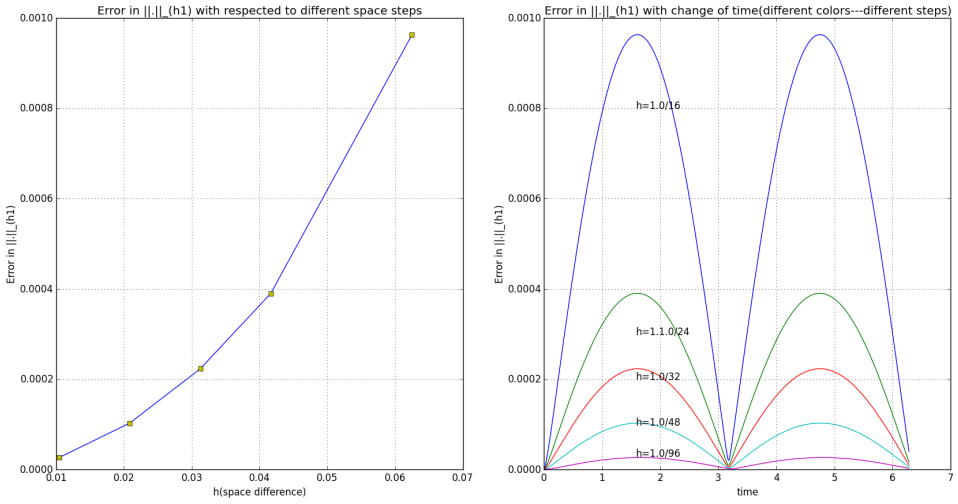

Figure 6.2: Error changes with respect to space steps (ALL cells)

	$n = 96$	$n = 48$	$n = 32$	$n = 24$	$n = 16$
$t = \frac{\pi}{2}$	2.64850e-05	1.02814e-04	2.23069e-04	3.90110e-04	9.62619e-04
$t = \pi$	2.58987e-06	5.61133e-06	1.01595e-05	1.65519e-05	4.00996e-05
$t = \frac{3\pi}{2}$	2.64850e-05	1.02814e-04	2.23069e-04	3.90110e-04	9.62619e-04
$t = 2\pi$	2.58987e-06	5.61133e-06	1.01595e-05	1.65519e-05	4.00996e-05

Table 6.2: Error changes(ALL cells) w.r.t. different h

Table (6.3) lists error in $\|.\|_{l^2(0,T;l^2(\Omega))}$ norm with respect to different time steps τ. It is clear to see the second order convergence. Figure (6.3) presents the error

	$m = 800$	$m = 400$	$m = 200$	$m = 100$	$m = 75$	$m = 50$
Err.	8.50153e-05	2.75331e-04	1.075980e-03	4.300317e-03	7.655054e-03	1.726870e-02

Table 6.3: Error changes in $\|.\|_{l^2(0,T;l^2(\Omega))}$ norm w.r.t. different τ

changing with respect to different time steps. The error in $\|.\|_{l^2(\Omega)}$ is periodic, which can be observed from (6.2). Hence for the graph on the left in figure (6.3) , we choose the time level whose error is the largest. For the graph on the right, the error is measured in $\|.\|_{l^2(0,T;l^2(\Omega))}$.

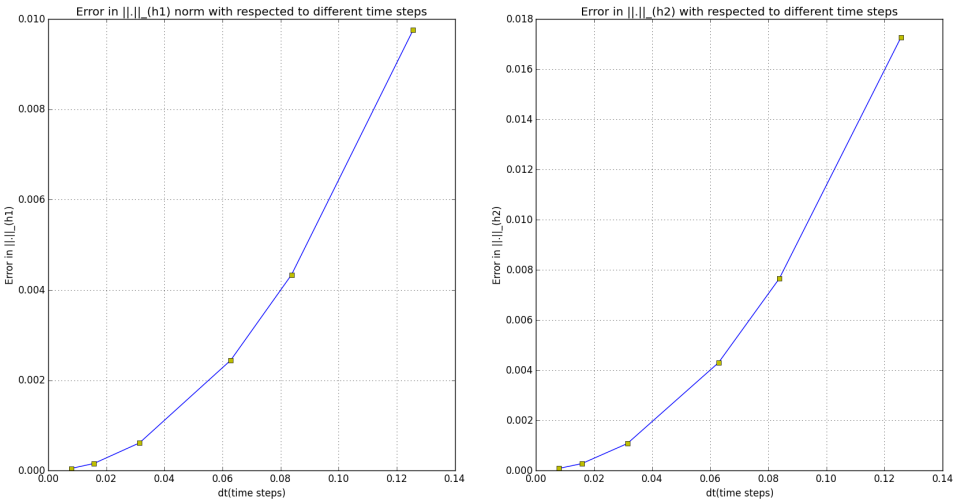

Figure 6.3: Error changes with respect to time steps(ALL cells)

6.2 Comparison of Proposed Scheme with General Splitting Scheme

We now consider solving problem (2.1) with a homogeneous boundary and initial condition on $\Omega = [0,1] \times [0,1]$ and $t \in [0, 0.5]$. We assume $a_1(x) = a_2(x) = 1$ and

$$f = \begin{cases} D & x \in \delta, \\ 0 & \text{otherwise}, \end{cases} \tag{6.1}$$

where $\delta = [0.4615, 0.5128] \times [0.4615, 0.5128]$ and $D = 10000$. δ is chosen to be a square so that it is covered exactly by four coarse cells and later, when one-to-three refinement is introduced, it is covered by 36 fine cells. There is no explicit solution of the problem. A high resolution grid which is three times finer than the fine grid of the proposed scheme is used to generate a relative high accuracy solution. One-to-three refinement is used on cells with large source term. We compare numerical results of P-R directional splitting method without local refinement and scheme proposed. In figure (6.4), the left hand side graph presents the solution on two refined cells; meanwhile the right one presents solution of the whole row.

Execution time for the proposed scheme is $0.9220s$; for high accuracy scheme is $135.7460s$; for scheme without refinement and whose grid size is equal to the grid size of the coarse cells of the proposed scheme is $0.6100s$ and for a scheme without refinement and whose grid size is same with the grid size of the fine cells of the proposed scheme is $11.9010s$. From the figure (6.4) and execution time, we can see that the scheme proposed reaches high accuracy meanwhile it saves time.

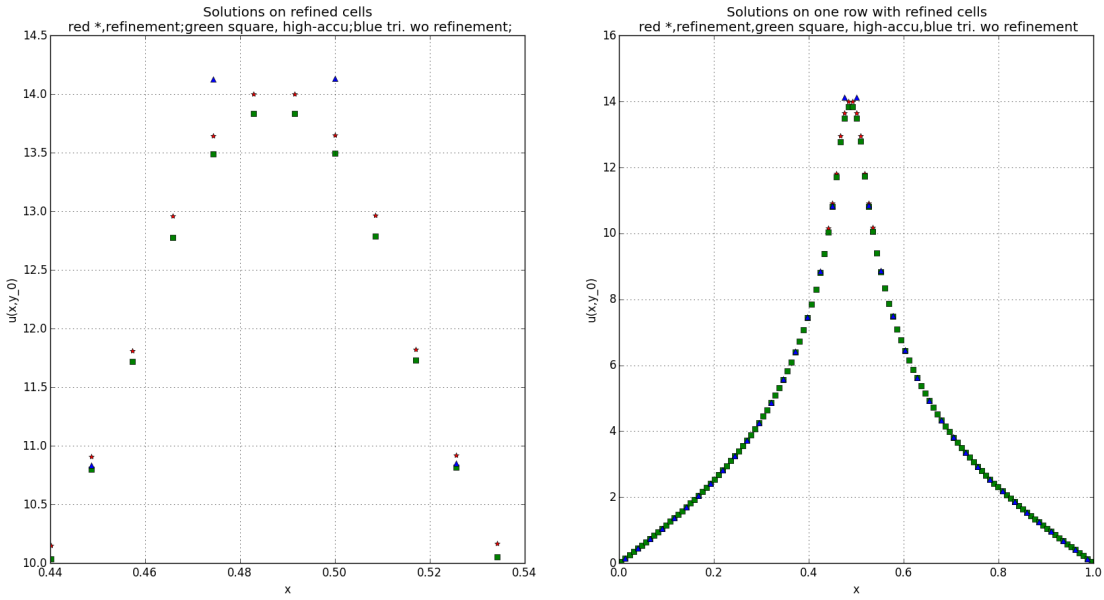

Figure 6.4: Comparasion of scheme proposed and scheme without refinement

Chapter 7

Implementation of Scheme in Parallel Computers

In this section, we report the motivation of our scheme. In fact, the scheme proposed can be implemented in parallel computer very fast and efficiently. Due to the directional splitting, we will solve a sequence of linear systems (first in x direction and then in y direction). For rows or columns without refinement, the computation can be done using a direct solver and the algorithm can be implemented in parallel machines. Here, we demonstrate the parallel computation idea inherent in our scheme when solving linear systems of rows or columns with local refinement. We will consider using the Schur compliment technique; with the help of the this technique, the problem of solving a huge linear system by iterative method will be simplified to solving several small problems by using a direct solver. This simplifies the computation; what is more, the algorithm is parallelizible.

As shown in figure (7.1), the solution is divided into internal nodes $u_i \in \mathbb{R}^{n_i}$ which consists of all regular nodes and interface nodes $u_e \in \mathbb{R}^{n_e}$. Here n_i and n_e are number of internal nodes and interface nodes respectively.

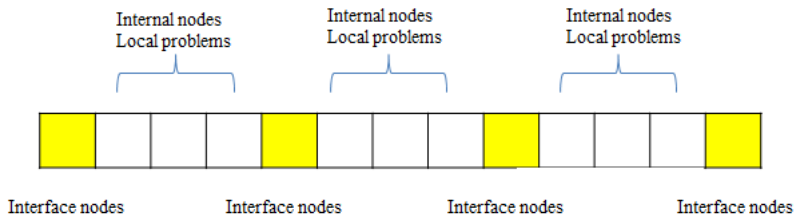

Figure 7.1: The split of the solution. Yellow cells denote interface nodes; white cells denote internal nodes

For our scheme, the splitting can be roughly shown in figure (7.2).

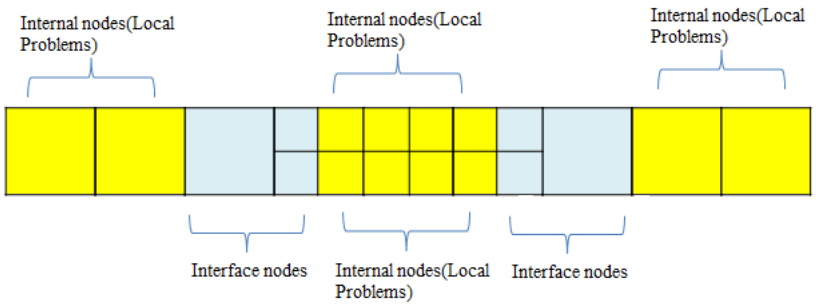

Figure 7.2: The split of the solution. Yellow cells denote internal nodes; blue cells denote interface nodes

To implement the Schur compliment algorithm, we rewrite the system in following way,

$$\begin{pmatrix} A_{ii} & A_{ie} \\ A_{ei} & A_{ee} \end{pmatrix} \begin{pmatrix} u_i \\ u_e \end{pmatrix} = \begin{pmatrix} f_i \\ f_e \end{pmatrix},$$

where A_{ii} is tridiagonal and A_{ee} is penta-diagonal of size 6 for our scheme. The solution can be constructed first by solving $Su_e = f_e - A_{ei}A_{ii}^{-1}f_i$, where $S = A_{ee} - A_{ei}A_{ii}^{-1}A_{ie} \in \mathbb{R}^{n_e \times n_e}$ is called the Schur compliment. It is followed by solving $A_{ii}u_i = f_i - A_{ie}u_e$. The idea is implemented with two processors and there is data communication between two processors. One processor is in charge of

calculating interface nodes u_e and the Schur compliment which can be determined at the very beginning of the computation; the other processor is in charge of solving local problems(as shown in figure 7.2). To be more specific, we first solve $A_{ii}x_i = f_i$ in local processor $\mathcal{L}$; hence we can assemble $A_{ei}x_i = A_{ei}A_{ii}^{-1}f_i$. This data (vector) of length n_e will then be transfered to processor $\mathcal{S}$ where the Schur compliment S is constructed. By assembling $f_e - A_{ei}A_{ii}^{-1}f_i$, we can solve $Su_e = f_e - A_{ei}A_{ii}^{-1}f_i$ and get intersection solution u_e. Later $A_{ie}u_e$ of length n_e will be transfered back to processor $\mathcal{L}$ where u_i can then be solved.

It should be noted that each local processor $\mathcal{L}$ deals with solving a linear system by a direct solver and has similar amount of computation workload(if optimal parallelization is reached). This means that processors do not have to wait for too long and hence improves the efficiency. In addition, the amount of data transfered between processors is equal to the number of intersection nodes which is a small portion of the total number of nodes. This speeds up the computation. Data communication for x direction is demonstrated in figure (7.3)

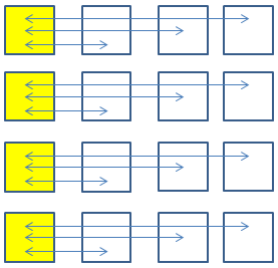

Figure 7.3: Communication of data between processors. Yellow squares are processors in charge of the Schur compliment for x direction

To sum up, the algorithm can be implemented in parallel machines. All processors can be efficiently used by solving a linear system using a direct solver and the amount of data communication is small. Compared to solving a huge system on

one processor by a iterative method, this method is fast and efficient.

Chapter 8

Conclusion

8.1 Conclusion

We derive a cell centered finite difference scheme on grids with local refinement for 2D parabolic problems. The scheme is stably combined with a directional splitting which gives the unconditional stability and results in a tri-diagonal dominated linear system. We prove the stability and conducted the convergence analysis using Bramble Hilbert Lemma. Due to the limitation of the Bramble Hilbert lemma resulted from non-uniform mesh, the optimal order of convergence in space cannot be proven; but the numerical experiments presented demonstrate the second order convergence both in space and time; they also shows the advantages of the proposed scheme. It should also be noted that the implementation algorithm of the proposed scheme is parallelizable. This improves the efficiency.

8.2 Further Topics

There are several further topics regarding this work. The analysis conducted here assume $a(x)$ be constant and can only show $\frac{3}{2}$ order of convergence. This makes

the proof less powerful. Several authors have showed the equivalence of mixed methods and cell centered methods; hence the analysis of the scheme proposed here may be conducted in the framework of mixed finite elements method. The discontinuous Galerkin method can be also considered as a generalization of the method mentioned here. In [22], Chen and Cockburn have proven that when non-conformal semimatching mesh are used and degree of approximation is chosen suitably on mesh with refinement, the optimal order of convergence can be reached for both flux and scalar variable. Hence, if we can build the equivalence between HDG mentioned by Cockburn and cell-centered finite difference proposed here, the optimal convergence can be proven directly by applying HDG results. In fact, there have been some works regarding this area. Readers can refer to Kanschat [23].

We can also extend our work by introducing a local refinement in time. There have been some works which study the local refinement in time; readers can refer to Lazarov [24],etc. 3D problems are also of great interest; but for stability, Douglas and Rachford splitting should be used [10].

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