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Polynomial-Normal Extension of Black-Scholes Model

by

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Abstract

Black-Scholes Model is a widely used mathematical model for stock price behaviors, of which the return is assumed to be normally distributed. But this 'normally distributed' assumption is doubted and proved to be not true by realistic data. The main goal of this thesis is to explore polynomial-normal distribution, and use this distribution in the stock return, as a non-normal extension of the Black-Scholes Model. We will develop the properties of polynomial-normal distribution in the thesis, and also give the European call and put option price formulas under this model, and show how to use this model to estimate real stock returns.

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Contents

List of Figures

List of Tables

Chapter 1

Introductory

Stock price behavior has been an attracting and sophisticated problem for financial researcher for many years. After long developments, the opinions of researchers converge to geometric Brownian motion and log-normal distribution, which the stock prices are believed to follow. Famous scholars such as Jack L. Theynor, Paul Samuelson, A. James Boness, Sheen T. Kassouf, Edward O. Thorp did great contributions to the fundamental of this idea. In 1973, Fischer Black and Myron Scholes used these assumptions in the modeling of stock returns, and derived the formulas for risk-neutral prices of put and call options written on the underlying stock. This is Black-Scholes Model, the most famous and fundamental model in estimate stock prices, as it gives a nice framework for analyze stock price behavior, fair derivatives prices with relatively less complicated calculation.

However, deficiencies of Black-Scholes Model arise. F. Black wrote a special paper 'How to Use the Holes in Black-Scholes' (1993), devoted to the Black-Scholes formula and its holes, because there are both pure theoretical and practical evidences that stock price returns may follow other distributions different from normal. For instance, the Black-Scholes Model gives mis-pricing for some deep-in-the-money or some deep-out-of-money options. Another important example is the ‘volatility smile’. Empirical data shows that the implied volatility becomes higher when the strike price is relatively high or low, and this is a confliction with the assumption of constant volatility in Black-Scholes Model.

For better estimates of the stock prices, many researchers looked for extensions of Black-Scholes Model, or even turned to study other models. An example is Robert C. Merton, who derived Black-Scholes formula nearly at the same time which Black and Scholes, had extended Black-Scholes Model and relax the strong assumptions in the model. To make better estimation for extreme stock prices, scholars including Michael Sorensen, Svetlozar T. Rachev explored heavy-tailed distribution, which allows a higher probability for extreme stock prices than Black-Scholes Model. Another extension of Black-Scholes Model is the jump diffusions model, which largely based on Levy process and its generalizations. Some of the generalizations were tremendous and it’s difficult to expect closed form solutions on option pricing under such general approach.

One of the important drawbacks of Black-Scholes Model is that the skewness and kurtosis of the stock returns are constants (skewness is 0 and kurtosis is 3), which conflicts with the fact that skewness and kurtosis may vary in a considerable wild range. To make Black-Scholes Model become more accurate in estimate the skewness and kurtosis of the future stock price, Gram-Charlier Extension is introduced and has become popular in financial estimates, as this distribution allow different skewness and kurtosis. The paper 'Option Pricing under Extended Normal Distribution' in 2004 by Hosam Ki, etc. has introduced this model in detail.

Gram-Charlier distribution, is defined by its probability density function (pdf) p(x)h(x), where p is a 4^{th} order polynomial, and h is the pdf of a standard normal distribution. It is quite natural to extend p to be higher order polynomial. This extension is called Polynomial-Normal distribution.

While giving better estimates of the stock price by allowing moments parameters adjusted easily, polynomial-normal distribution is neglected by some researchers because the their computations are complicated. However, in this thesis we will extend many properties, from Black-Scholes Model to Polynomial-Normal Model (Let's call it Polynomial-Normal Model in latter part of the paper), and give the calculations of option prices, model parameters estimating and so on, in a not so complicated way.

The reminder of this thesis is written in basis of the three models we considered: Black-Scholes Model, Gram-Charlier Model, and Polynomial-Normal Model. We will give results in all these three models and do comparisons on them. Others chapters are recognized as follows. Chapter 2 will give the definitions of our three models, and discuss some tools for Polynomial-Normal distribution analyzing, and give a solution for positivity density functions. Chapter 3 will discuss the risk-neutral option prices under our three models, and give the Greeks, which helps us to analyze the changes of option prices. In Chapter 4 we will use examples of real data to show how to use these models and estimate parameters in the models.

Chapter 2

Preliminary Notions and Some Related Considerations

2.1 The Black-Scholes Model

Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a stochastic basis and $W = \{W_t, F_t\}_{t\geq 0}$ be a standard Wiener process (Brownian motion).

The Black-Scholes market model consists of two components. The first component is a bank account which is defined by $B_t = rB_t dt$, with $r \geq 0$ be the interest rate. And the second component is the stock price evolution. The stock price is modeled by

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad (2.1)$$

Here μ is the expected growing rate of the stock, and σ is its volatility. If the stock is calculated with respect to a unique risk neutral probability measure, the parameter μ equals the risk free rate r.

There is another representation for the stock price known as Geometric Brownian Motion: $S_T = S_0 Exp((\mu - \frac{\sigma^2}{2})T + \sigma W_T)$ (2.2)

We can also solve (2.1) by Ito's formula and get (2.2).

The return of the stock over the time interval $[0,T]$ is defined by

$R = Log(S_T / S_0)$, thus we have $R = (\mu - \frac{\sigma^2}{2})T + \sigma W_T$ (2.3)

When Black-Scholes Model was originally developed, the stock is supposed to pay no dividends. However, modifications of the model allow the stock to be paying continuous dividends, which is proportion to the stock price and time taking. In this case, we assume the continuous dividend rate to be q, and the expected return rate of the stock to be $\bar{\mu}$, where $\bar{\mu} = q + \mu$. (2.1) and (2.2) also hold and the return turns out to be $R = Log(S_T / S_0) + qT$ and thus

$$R = (\bar{\mu} - \frac{\sigma^2}{2})T + \sigma W_T \qquad (2.3^*)$$

As the model without dividend is the fundamental and originally designed Black-Scholes model, many considerations and results were given is a way that no dividends are considered. However, in the thesis, most of the analysis will be done in the way that non-zero continuous dividend rate q exists, because a full version of the model would make it more closed to reality and thus model the price better, and we will do some analysis of the dividend rate somewhere. In some considerations, where a zero dividend rate will simplify our analysis greatly, or other types of dividend is used instead of continuous dividend (like discrete dividend), we will mention how it should be considered.

2.2 Polynomial-Normal Model

Due to the connection of the Black-Scholes Model with Brownian motion, the parameters of the skewness and kurtosis of the return are fixed and become constants no matter whatever μ and σ are. There are many evidences in reality that the parameters of skewness and kurtosis are different from that in Black-Scholes Model, like in some large stock exchange market, smaller capitalized stocks indices would be more negative skewed than other stock indices, and the stock return would be more negative skewed in economic booming. (The paper 'Regularities in the Variation of Skewness of Asset Return' by Kling has introduced these results.)

To make better estimation of the stock prices, we need to generalize the Black-Scholes Model with the help of other distributions. Let's describe first a model which is the straightforward generalization of the Black-Scholes Model, known as Gram-Charlier Model.

Gram-Charlier Model can overcome the restriction of constant skewness and kurtosis which is implied in Black-Scholes Model. Under Gram-Charlier distribution, pdf of $Log(S_T / S_0)$ is

$$f(x) = h(\frac{x-m}{\tilde{\sigma}})\, p(\frac{x-m}{\tilde{\sigma}}) / \tilde{\sigma} \tag{2.4}$$

where $p(x) = 1 + \frac{\xi}{6} H_3(x) + \frac{\kappa - 3}{24} H_4(x)$. ξ denotes the parameter of skewness and κ denotes the parameter of kurtosis of $Log(S_T / S_0)$. H_i

is the i^th order Hermite polynomial. Hermite polynomial series is one kind of orthogonal polynomials arises from probability, and has lots of good properties. It will be discussed in section 2.4.

$\tilde{\sigma} = \sigma\sqrt{T}$ also holds in Gram-Charlier, but m is not. As the expected changing rate of logarithm of the price is μ, we have

$$\begin{aligned} e^{\mu T} &= E(e^{R}) \\ &= \int_{x=-\infty}^{\infty} e^{x} h(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) / \tilde{\sigma} dx \\ &= e^{\frac{\tilde{\sigma}^2}{2}+m}(1+\frac{\xi}{6}\tilde{\sigma}^3+\frac{\kappa-3}{24}\tilde{\sigma}^4) \end{aligned}$$

(This calculation is forward from propositions in section 2.4)

So $m = \mu T - \frac{\tilde{\sigma}^2}{2} - \ln(1+\frac{\xi}{6}\tilde{\sigma}^3+\frac{\kappa-3}{24}\tilde{\sigma}^4)$

Notice that if $\xi = 0$ and $\kappa = 3$, $m = \mu T - \frac{\tilde{\sigma}^2}{2}$ and it's the same with that in Black-Scholes Model. If under risk neutral probability,

$$m = (r-q)T - \frac{\tilde{\sigma}^2}{2} - \ln(1+\frac{\xi}{6}\tilde{\sigma}^3+\frac{\kappa-3}{24}\tilde{\sigma}^4) \quad (2.5)$$

Now we are going to give a more general theoretical framework for such approach using the Polynomial-Normal distribution.

Assume that the following represents the pdf of $Log(S_T / S_0)$:

$$f(x) = h(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) / \tilde{\sigma} \quad (2.4)$$

Here p is an N^th order polynomial: $p(x) = \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} b_n H_n(x)$. The n^th moment of the $Log(S_T / S_0)$ is determined by $b_1, b_2, ..., b_n$, and independent of $b_{n+1} ..., b_N$. And because the pdf should satisfy the following:

$$1 = \int_{x=-\infty}^{\infty} f(x)dx = \int_{y=-\infty}^{\infty} h(y)p(y)dy = b_0$$

We have $b_0 = 1$.

We can notice that this model includes the Gram-Charlier Model if we take $N = 4$ and $b_1 = b_2 = 0$.

Like Gram-Charlier Model, we can determine the value of m as follows:

$$\begin{aligned} e^{\mu T} &= E(e^R) \\ &= \int_{x=-\infty}^{\infty} e^x h(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) / \tilde{\sigma} dx \\ &= e^{\frac{\tilde{\sigma}^2}{2}+m} (\sum_{n=0}^{N} b_n \tilde{\sigma}^n) \end{aligned}$$

(This calculation is forward from propositions in Section 2.4)

So we have $m = \mu T - \dfrac{\tilde{\sigma}^2}{2} - \ln(p(\tilde{\sigma}))$. (2.5*)

Remark 2.1:

In Polynomial-Normal distribution, we also let $\tilde{\sigma} = \sigma\sqrt{T}$. But unlike Black-Scholes Model and Gram-Charlier Model, we no long have $\mathrm{var}(Log(S_T / S_0)) = \tilde{\sigma}^2 = \sigma^2 T$. Instead, we have

$$Exp(Log(S_T / S_0)) = \int_{x=-\infty}^{+\infty} xh(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) dx = m + \tilde{\sigma} b_1 \quad (2.6)$$

$$\begin{aligned} Var(Log(S_T / S_0)) &= \int_{x=-\infty}^{+\infty} x^2 h(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) dx - (m + \tilde{\sigma} b_1)^2 \\ &= 2\tilde{\sigma}^2 b_2 + 2m\tilde{\sigma} b_1 + \tilde{\sigma}^2 + m^2 - (m + \tilde{\sigma} b_1)^2 \quad (2.7) \\ &= \tilde{\sigma}^2 (2b_2 + 1 - b_1^2) \end{aligned}$$

Sometimes we just let $b_1 = b_2 = 0$ to simplify our computation, and use m and σ to adjust the mean and variance. For this reason, we will set $b_1 = b_2 = 0$ in the part of parameter estimation.

Remark 2.2:

A disadvantage of the Gram-Charlier distribution and Polynomial-Normal distribution is that not for all (ξ, κ) in Gram-Charlier distribution and not for all polynomials in Polynomial-Normal distribution would generate an acceptable density function, because the function would become negative somewhere, for some (ξ, κ) or some polynomials. But fortunately, the region of (ξ, κ) which guarantee positivity of the pdf has been found, and this method can also be generalized to Polynomial-Normal case. We will discuss the detail of it in section 2.6.

2.3 Hermite Polynomial Series

Hemite Polynomial Series is an important polynomial series in theoretical financial modeling analysis. This polynomial series is found very useful in the fields such as physics, combinatorics, probability theory, and also stock pricing estimating here. There are two types of Hermite polynomials, one is the 'physicists' type, while the other is called 'probabilists' type. We will only use the ‘probabilists’ type in this thesis, and we will call it Hermite polynomials in latter pages, with the name of ‘probabilists’ omitted.

Hermite polynomials can be seen as a kind of orthogonal polynomial series on probability space, which has a lot of nice properties. We usually need to decompose the polynomials in the integrals into a linear sum of Hermite polynomials to forward our calculation. In this section we introduce several basic properties of Hermite polynomials.

The n^th^ order Hermite polynomial is defined as

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{\frac{-x^2}{2}}}{dx^n} \qquad (2.8)$$

The first few Hermite polynomials are as follows:

$H_0(x) = 1$

$H_1(x) = x$

$H_2(x) = x^2 - 1$

$H_3(x) = x^3 - 3x$

$H_4(x) = x^4 - 6x^2 + 3$

$H_5(x) = x^5 - 10x^3 + 15x$

$H_6(x) = x^6 - 15x^4 + 45x^2 - 15$

……

Proposition 2.1: $\dfrac{dH_n(x)}{dx} = nH_{n-1}(x)$ (2.9)

Proof: Use induction method, and assume the proposition holds for

$n = 0,1,2,\ldots,N$, by definition, we have

$$(H_n(x)e^{-\frac{x^2}{2}})' = -H_{n+1}(x)e^{-\frac{x^2}{2}} \qquad (2.10)$$

Expanding the derivative of the formula above and canceling the term $e^{-\frac{x^2}{2}}$, we get $-H_n'(x) + xH_n(x) = H_{n+1}(x)$ (2.11)

When $n = N+1$, we have the following

$$H_{N+1}'(x) = (-H_N'(x) + xH_N(x))'$$

$$= -NH_{N-1}'(x) + H_N(x) + NxH_{N-1}(x)$$

$$= N(-H_{N-1}'(x) + xH_{N-1}(x)) + H_N(x)$$

$$= (N+1)H_N(x)$$

So the proposition holds.

Proposition 2.2:

$$\int H_m(x)H_n(x)h(x)dx = \frac{(nH_m(x)H_{n-1}(x) - mH_{m-1}(x)H_n(x))h(x)}{m-n} + C \quad , \quad \text{if}$$

$m > 0$, $n > 0$ and $m \neq n$.

Proof:

$$\int H_m(x)H_n(x)h(x)dx$$

$$= -\int H_m(x)(H_{n-1}(x)h(x))'dx$$

$$= \int H_m'(x)H_{n-1}(x)h(x)dx - H_m(x)H_{n-1}(x)h(x)$$

$$= m\int H_{m-1}(x)H_{n-1}(x)h(x)dx - H_m(x)H_{n-1}(x)h(x)$$

$$= n\int H_{m-1}(x)H_{n-1}(x)h(x)dx - H_{m-1}(x)H_n(x)h(x)$$

When $m \neq n$, solve the linear equation and we have

$$\int H_m(x)H_n(x)h(x)dx = \frac{(nH_m(x)H_{n-1}(x) - mH_{m-1}(x)H_n(x))h(x)}{m-n} + C$$

So the proposition holds.

Proposition 2.3: $\int_{x=-\infty}^{\infty} H_m(x)H_n(x)h(x)dx = 0$ if $m \neq n$. And $\int_{x=-\infty}^{\infty} H_n^2(x)h(x)dx = n!$

Proof:

Suppose $m \geq n$, from proposition 2.2 we have

$$\int_{x=-\infty}^{\infty} H_m(x)H_n(x)h(x)dx$$

$$= -\int_{x=-\infty}^{\infty} H_m(x)\frac{d}{dx}\left(H_{n-1}(x)h(x)\right)dx$$

$$= \int_{x=-\infty}^{\infty} \frac{d}{dx}\left(H_{m-1}(x)\right)H_{n-1}(x)h(x)dx - H_m(x)H_{n-1}(x)h(x)\Big|_{-\infty}^{+\infty}$$

$$= m\int_{x=-\infty}^{\infty} H_{m-1}(x)H_{n-1}(x)h(x)dx$$

....

$$= \frac{m!}{(m-n)!}\int_{x=-\infty}^{\infty} H_{m-n}(x)h(x)dx$$

As $\int_{x=-\infty}^{\infty} h(x)dx = 1$, and $\int_{x=-\infty}^{\infty} H_{m-n}(x)h(x)dx = -H_{m-n-1}(x)h(x)\Big|_{-\infty}^{\infty} = 0$ when $m-n > 0$, the proposition holds.

If we define the inner product on the vector space of polynomials as $\langle p,q \rangle = \int_{x=-\infty}^{\infty} p(x)q(x)e^{-\frac{x^2}{2}}dx$, for any polynomial p and q. Proposition 2.2

shows that any Hermite polynomials are orthogonal to each other, over the inner product defined. Hermite polynomials can be seen as an orthogonal basis on the polynomial vector space, while $\left\{H_n(x)/\sqrt{n!}\right\}_n$ is the standardized basis. This nice property makes Hermite polynomial a perfect tool in analysis of Polynomial-Normal Model, and is the basis of many calculations. By expressing the polynomial used in density function as linear sum of Hermite polynomials, it's much more convenient to express everything in the form of the parameters of Hermite polynomials weights, than in the form of $\{a_n\}$, if we use the usual polynomial definition $p(x)=\sum_{n=0}^{N} a_n x^n$.

Proposition 2.4: $H_n(x+t)=\sum_{j=0}^{n}\binom{n}{j}t^{n-j}H_j(x)$ (2.12)

Proof: Use induction method. Assume that the proposition holds for $n=0,1,2,\ldots,N$. From proposition 2.1, we have

$$(H_{N+1}(x+t))'_t=(N+1)H_N(x+t)=(N+1)\sum_{j=0}^{N}\binom{N}{j}t^{N-j}H_j(x)$$

So by integration with t, we get

$$\begin{aligned}H_{N+1}(x+t)&=H_{N+1}(x)+\int_0^t (N+1)\sum_{j=0}^{N} s^{N-j}H_j(x)\binom{N}{j}ds\\&=H_{N+1}(x)+(N+1)\sum_{j=0}^{N}\frac{t^{N+1-j}}{N+1-j}H_j(x)\binom{N}{j}\\&=\sum_{j=0}^{N+1}\binom{N+1}{j}t^{N+1-j}H_j(x)\end{aligned}$$

Thus the proposition holds.

This proposition shows how we can express Hermite polynomials with a drift as linear sum of original Hermite polynomials. This proposition is used in the determination of m in the section 2.2 and 2.3, and is very useful in latter calculations, especially in option price formulas, as drifts such as $\tilde{\sigma}$ happens usually.

2.4 The Moments of $Log(S_T / S_0)$

Now we have a fine definition of the Polynomial-Normal distribution and a convenient tool of Hermite polynomials for its analysis. It's necessary to calculate the moments of the $Log(S_T / S_0)$ from the weights of Hermite polynomial parameters, as these moments would be obtained by stock data, and be plugged into the formulas derived in the following, thus determine what parameter we should use in the polynomial. The parameter estimating part would be in Chapter 4.

The n$^{\text{th}}$ moment (raw moment) of $Log(S_T / S_0)$ is denoted as μ_n. Let $(\tilde{\sigma} x + m)^n = \sum_{j=0}^{n} c_{n,j} H_j(x)$, we have the following:

$$\mu_n = \int_{x=-\infty}^{\infty} x^n f(x)dx$$
$$= \int_{y=-\infty}^{\infty} (\tilde{\sigma} y + m)^n h(y) p(y) dy \qquad (2.13)$$
$$= \sum_{j=0}^{n} j! c_{n,j} b_j$$

As $\tilde{\sigma} x + m = \tilde{\sigma} H_1(x) + mH_0(x)$, we have the mean $\mu_1 = \tilde{\sigma} b_1 + m$.

As $(\tilde{\sigma} x + m)^2 = \tilde{\sigma}^2 H_2(x) + 2m\tilde{\sigma} H_1(x) + (\tilde{\sigma}^2 + m^2)$, we have the variance $\mu_2 = 2\tilde{\sigma}^2 b_2 + 2m\tilde{\sigma} b_1 + \tilde{\sigma}^2 + m^2$

For high order moments, the following calculation can give $c_{n,j}$ in a relatively simple way.

Proposition 2.5: $c_{n,j} = \binom{n}{j} \tilde{\sigma}^j c_{n-j,0}$

Proof:

$$c_{n,j} = \frac{1}{j!} \int_{x=-\infty}^{\infty} (\tilde{\sigma} x + m)^n H_j(x) h(x) dx$$
$$= -\frac{1}{j!} \int_{x=-\infty}^{\infty} (\tilde{\sigma} x + m)^n (H_{j-1}(x) h(x))' dx$$
$$= \frac{n\tilde{\sigma}}{j!} \int_{x=-\infty}^{\infty} (\tilde{\sigma} x + m)^{n-1} H_{j-1}(x) h(x) dx$$

......

$$= \frac{n!\tilde{\sigma}^j}{j!(n-j)!} \int_{x=-\infty}^{\infty} (\tilde{\sigma} x + m)^{n-j} h(x) dx$$
$$= \binom{n}{j} \tilde{\sigma}^j c_{n-j,0}$$

So the proposition holds.

Proposition 2.6: $c_{n,0} = (-i\tilde{\sigma})^n H_n(\frac{mi}{\tilde{\sigma}})$, where $i = \sqrt{-1}$ stands for the imaginary units for complex numbers.

Proof:

From μ_1 and μ_2, we can see that $c_{1,0} = m$, and $c_{2,0} = m^2 + \tilde{\sigma}^2$.

$c_{n,0}$ is calculated by induction method as follows:

$$\begin{aligned}
c_{n,0} &= \int_{x=-\infty}^{\infty} (\tilde{\sigma}x + m)^n h(x)dx \\
&= \tilde{\sigma} \int_{x=-\infty}^{\infty} (\tilde{\sigma}x + m)^{n-1} xh(x)dx + mc_{n-1,0} \\
&= -\tilde{\sigma} \int_{x=-\infty}^{\infty} (\tilde{\sigma}x + m)^{n-1} h'(x)dx + mc_{n-1,0} \\
&= \tilde{\sigma}^2 (n-1) \int_{x=-\infty}^{\infty} (\tilde{\sigma}x + m)^{n-2} h(x)dx + mc_{n-1,0} \\
&= \tilde{\sigma}^2 (n-1) c_{n-2,0} + mc_{n-1,0}
\end{aligned}$$

It's obvious that the proposition holds when $n = 1, 2$. Assume that it holds for $n < N$, and we have the following:

$$\begin{aligned}
c_{N,0} &= \tilde{\sigma}^2 (N-1) c_{N-2,0} + mc_{N-1,0} \\
&= \tilde{\sigma}^2 (N-1)(-i\tilde{\sigma})^{N-2} H_{N-2}(\frac{mi}{\tilde{\sigma}}) + m(-i\tilde{\sigma})^{N-1} H_{N-1}(\frac{mi}{\tilde{\sigma}}) \\
&= (-i\tilde{\sigma})^N ((1-N) H_{N-2}(\frac{mi}{\tilde{\sigma}}) + \frac{mi}{\tilde{\sigma}} H_{N-1}(\frac{mi}{\tilde{\sigma}})) \\
&= (-i\tilde{\sigma})^N H_N(\frac{mi}{\tilde{\sigma}})
\end{aligned}$$

So the proposition holds.

Combining two propositions, we have

$$c_{n,j} = \binom{n}{j} \tilde{\sigma}^n (-i)^{n-j} H_{n-j}(\frac{mi}{\tilde{\sigma}}) \qquad (2.14)$$

2.5 Implementing Positivity into Density Functions

As mentioned in previous sections, a disadvantage of a Gram-Charlier or a Polynomial-Normal Distribution is that, the density function is not guaranteed to be globally positive for every skewness and kurtosis in Gram-Charlier Distribution, or for every polynomial in Polynomial-Normal Distribution. In other words, only when (ξ, κ) is addressed in a special region in R^2, the density function $f(x) = h(\frac{x-m}{\tilde{\sigma}})p(\frac{x-m}{\tilde{\sigma}})/\tilde{\sigma}$ is positive for every x. Polynomial-Normal Distribution is in similar case.

Barton and Dennis (1952) obtained the parameter conditions for positivity in this issue by numerical method. Later Jondeau and Rockinger (1999) found out the border of this region by theoretical method. We show in this section that the region of (ξ, κ) that ensure positivity for the Gram-Charlier Distribution, and also extend this method to Polynomial-Normal Distribution.

In Gram-Charlier Distribution, the density function of the $Log(S_T / S_0)$ is $f(x) = h(y)(1 + \frac{\xi}{6}H_3(y) + \frac{\kappa - 3}{24}H_4(y))/\tilde{\sigma}$, with $y = \frac{x-m}{\tilde{\sigma}}$.

We have to find out the region of (ξ, κ), such that $1 + \frac{\xi}{6}H_3(y) + \frac{\kappa - 3}{24}H_4(y) \geq 0$ for every y. For a fixed y, $1 + \frac{\xi}{6}H_3(y) + \frac{\kappa - 3}{24}H_4(y) = 0$ is a straight line in R^2 space of (ξ, κ). Holding (ξ, κ) fixed on this line, a small perturbation of y would give

$p(y)$ positive or negative value. Thus, we can determine the boundary of (ξ, κ) as a function of y, such that $p(y)$ remains zero for a small perturbation of y. So we have $p'(y) = \frac{\xi}{2} H_2(y) + \frac{\kappa - 3}{6} H_3(y) = 0$.

Solve the linear equations, we have

$$(\xi(y), \kappa(y)) = \left(\frac{-24H_3(y)}{4H_3^2(y) - 3H_4(y)H_2(y)}, 3 + \frac{-72H_2(y)}{4H_3^2(y) - 3H_4(y)H_2(y)}\right)$$

Figure 2.1 below shows the curve of $(\xi(y), \kappa(y))$ and the region with shadow denotes acceptable area that generates positive density function.

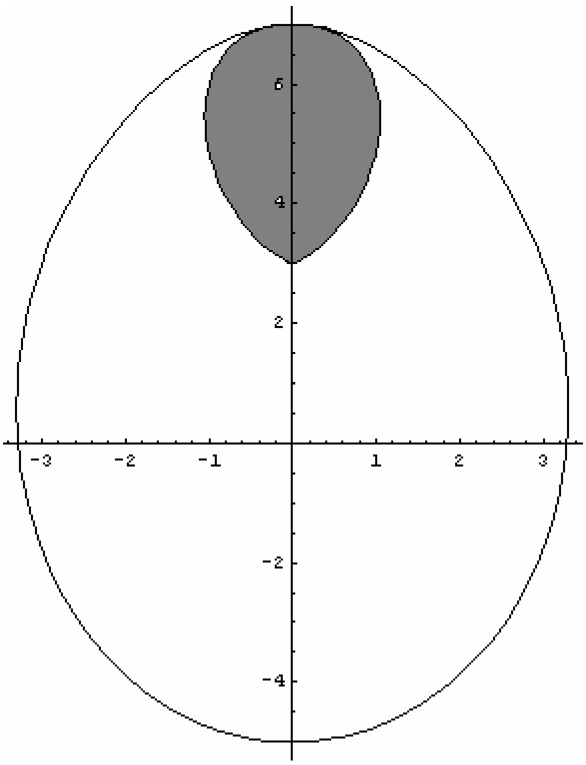

Figure 2.1

For general Polynomial-Normal Distributions where $p(x) = \sum_{n=0}^{N} b_n H_n(x)$, suppose array $(b_1, b_2, ..., b_N)$ is a point on the border of the region where positivity is satisfied, then $p(x) = \sum_{n=0}^{N} b_n H_n(x) = 0$ for some x, and $p(x)$ have the minimum at x. So x is a critical point of the polynomial and $p'(x) = 0$. Thus the set of the points on the border is a subset of $A = \{(b_1, b_2, ..., b_N), p(x) = p'(x) = 0 \textit{ for some } x\}$. A can be seen as a union of a series of moving $n-2$ dimensional manifolds. Each of these manifolds will be tangent to the border of the region we required. As the manifold moves, the border of the region forms and it becomes a curved surfaces (or $N-1$ dimensional manifold) in R^N space.

As the set of A would generate a complicated manifold in R^N space, which divide the space into many parts, we need to decide which part is the region we wanted. Consider $(b_1, b_2, ..., b_N)$ such that $b_{2k+1} = 0$, $b_{2k} = \varepsilon$ for every nature number k, where ε is sufficiently small positive number. Then $\min(p(x)) \geq 1 + \varepsilon \sum_{N \geq 2n > 0} \min(H_{2n}(x)) > 0$ if ε is small enough. So the region required should include this point while other parts without this point would have negative $p(x)$ somewhere.

Figure 2.2 shows the border of the region required when $p(x) = 1 + b_3 H_3(x) + b_4 H_4(x) + b_6 H_6(x)$. The three axis denote b_3, b_4, b_6 respectively, and the plot range is $b_3 \in [-0.2, 0.2]$, $b_4 \in [-0.02, 0.2]$, $b_6 \in [-0.0015, 0.018]$.

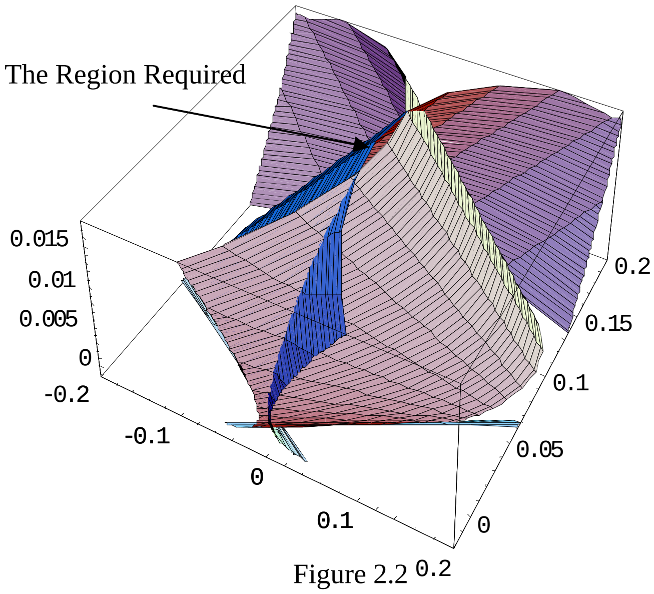

Figure 2.2

Chapter 3

Option Pricing Formulas

There are many types of options. Classified by the assets written on, there are options written on stocks, written on indices, written on commodities and so on. Classified by exercise maturity, the European type options only authorize the owner to exercise it exactly at maturity, and the American type allows the owner to exercise it at any time before maturity. Classify by the right to buy or sell, the call option gives the owner the right to buy underlying assets at some fixed prices (also called strike price) while the put option gives the right to sell it. Classified by payoff, there is usual option in which the strike price is a fixed amount, and there is Asian option in which average price of the stock is used instead, and in Russian option the maximum of minimum of the stock price over a time period is used. And the so-called exotic options are also used in many ways. We can generalize this concept to a contract that will pay the owner an amount which is depend on the price behavior over a time period, of the asset that the option written on.

The most simple and often used options are call options and put options, with European and American type. European call option gives its holder the right but not the obligation to buy the asset that the option written on at strike price at maturity, while European put option gives the right to sell it. American option gives this right at any time before maturity to buy of sell the stock at strike price.

It has become a problem how to price options fairly. Because the payoff of these options are uncertain and usually depend on the behavior of the prices of the stocks, the risk-neutral price became a popular pricing method in many researching. The risk-neutral price gives the price of options as the average payoff under some probability measure, less the expected interest and dividend earned over time. Despite that risk-free assets are usually considered more valuable than risky assets like options, it is still important to find the risk-neutral price as it is the foundation of the whole pricing model.

In this chapter, we will give the famous Black-Scholes formula, and extend this formula to Gram-Charlier Model and Polynomial-Normal Model. To estimate how option price would change when the formula parameters change, we will use Greeks to analyze the change rate of the option price.

3.1 The Black-Scholes Formula

In Black-Scholes Model, the European call and put price is determined by Black-Scholes formula. This formula is first articulate by Fischer Black and Myron Scholes in the paper 'the Pricing of Options and Corporate Liabilities'

(1973). This famous formula is given as follows:

$$C = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2) \qquad (3.1)$$

$$P = -S_0 e^{-qT}\Phi(-d_1) + Ke^{-rT}\Phi(-d_2) \qquad (3.2)$$

Where $d_1 = \dfrac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

C and P denote the price European call and put option, with strike price K. r and q denote the risk free interest rate and continuous dividend rate respectively. And T is time to maturity while σ is the volatility of the stock.

Black-Scholes formula can be simply derived by taking the discounted expected value of the payoff of the option at maturity. So actually it’s the risk-neutral option price under log-normal distribution.

Remark 3.1:

The original Black-Scholes Formula is $C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$, where d_1 and d_2 is like what is defined above without the dividend term. This formula is the non-dividend version of Black-Scholes formula. We can see that our modified Black-Scholes formula have the stock price part discounted, because if the option is exercise, the stock purchased is valued less as it pays dividend before time T .

3.2 The Gram-Charlier Model: Formulas for Calls and Puts

When Gram-Charlier distribution is used, the risk-neutral option prices can be determined as follows, which is similar to Black-Scholes Model.

$$
\begin{aligned}
C &= e^{-rT} E(S_T - K)^+ \\
&= e^{-rT} \int_{x=\ln(K/S_0)}^{+\infty} (S_0 e^x - K) h(\frac{x-m}{\tilde{\sigma}}) p(\frac{x-m}{\tilde{\sigma}}) / \tilde{\sigma} dx \\
&= S_0 e^{-qT} \Phi(D_1) - K e^{-rT} \Phi(D_2) \\
&\quad + \frac{\xi}{6} S_0 e^{-qT} h(D_1)(H_2(D_1) - 3\tilde{\sigma} H_1(D_1) + 3\tilde{\sigma}^2) / p(\tilde{\sigma}) \\
&\quad + \frac{\kappa-3}{24} S_0 e^{-qT} h(D_1)(-H_3(D_1) + 4\tilde{\sigma} H_2(D_1) - 6\tilde{\sigma}^2 H_1(D_1) + 4\tilde{\sigma}^3) / p(\tilde{\sigma}) \\
&\quad + K e^{-rT} h(D_2)(\frac{\xi}{6} H_2(D_2) + \frac{\kappa-3}{24} H_3(D_2))
\end{aligned}
$$

where $D_1 = \dfrac{\ln(S_0/K) + (r-q)T + \tilde{\sigma}^2/2 - \ln(1 + \frac{\xi}{6}\tilde{\sigma}^3 + \frac{\kappa-3}{24}\tilde{\sigma}^4)}{\tilde{\sigma}}$, and $D_2 = D_1 - \tilde{\sigma}$, $\tilde{\sigma} = \sigma\sqrt{T}$.

Remark 3.1: D_1 and D_2 are different to d_1 and d_2 we used in Black-Scholes Model, but they are the same when $p(x) = 1$, or identically $\xi = 0$ and $\kappa = 3$. Also we can notice that when $\xi = 0$ and $\kappa = 3$, the two latter terms will be cancel out, and the Gram-Charlier call price would equal to Black-Scholes call price. In latter text, we will use d_1, d_2 in Black-Scholes Model and D_1, D_2 in Gram-Charlier and Polynomial-Normal Model.

Using similar calculation, we get the Gram-Charlier put price as

$$
\begin{aligned}
P &= e^{-rT}E(K-S_T)^+ \\
&= e^{-rT}\int_{x=\ln(K/S_0)}^{+\infty}(K-S_0e^x)h(\frac{x-m}{\tilde{\sigma}})p(\frac{x-m}{\tilde{\sigma}})/\tilde{\sigma}dx \\
&= -S_0e^{-qT}\Phi(-D_1)+Ke^{-rT}\Phi(-D_2) \\
&\quad +\frac{\xi}{6}S_0e^{-qT}h(D_1)(H_2(D_1)-3\tilde{\sigma}H_1(D_1)+3\tilde{\sigma}^2)/p(\tilde{\sigma}) \\
&+\frac{\kappa-3}{24}S_0e^{-qT}h(D_1)(-H_3(D_1)+4\tilde{\sigma}H_2(D_1)-6\tilde{\sigma}^2H_1(D_1)+4\tilde{\sigma}^3)/p(\tilde{\sigma}) \\
&+Ke^{-rT}h(D_2)(\frac{\xi}{6}H_2(D_2)+\frac{\kappa-3}{24}H_3(D_2))
\end{aligned}
$$

We can also use the put-call parity to get this formula.

3.3 The Polynomial-Normal Model: General Formulas for Calls and Puts

In Polynomial-Normal Model, risk-neutral option price is determined as follows:

$$
\begin{aligned}
C &= e^{-rT}E(S_T-K)^+ \\
&= e^{-rT}\int_{x=\ln(K/S_0)}^{+\infty}(S_0e^x-K)h(\frac{x-m}{\tilde{\sigma}})p(\frac{x-m}{\tilde{\sigma}})/\tilde{\sigma}dx \\
&= S_0e^{-qT}\Phi(D_1)-Ke^{-rT}\Phi(D_2)+Ke^{-rT}h(D_2)(\sum_{n=0}^{N-1}b_{n+1}H_n(D_2)) \\
&\quad +S_0e^{-qT}h(D_1)\sum_{n=1}^{N}b_n(\sum_{j=0}^{n-1}(-1)^j\begin{pmatrix}n\\j+1\end{pmatrix}\tilde{\sigma}^{n-j}H_j(D_1))/p(\tilde{\sigma})
\end{aligned}
$$

(The properties of Hermite Polynomials is used here)

(3.3)

$$
\begin{aligned}
P &= e^{-rT} E(K - S_T)^{+} \\
&= e^{-rT} \int_{x=-\infty}^{\ln(K/S_0)} (K - S_0 e^{x}) h\left(\frac{x-m}{\tilde{\sigma}}\right) p\left(\frac{x-m}{\tilde{\sigma}}\right) / \tilde{\sigma} dx \\
&= -S_0 e^{-qT} \Phi(-D_1) + K e^{-rT} \Phi(-D_2) + K e^{-rT} h(D_2) \left(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)\right) \\
&\quad + S_0 e^{-qT} h(D_1) \sum_{n=1}^{N} b_n \left(\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \tilde{\sigma}^{n-j} H_j(D_1)\right) / p(\tilde{\sigma})
\end{aligned}
$$

(The properties of Hermite Polynomials is used here)

(3.4)

Where $D_1 = \dfrac{\ln(S_0 / K) + (r-q)T + \tilde{\sigma}^2 / 2 - \ln(p(\tilde{\sigma}))}{\tilde{\sigma}}$ and $D_2 = D_1 - \tilde{\sigma}$

We can also notice that the option formulas for Polynomial-Normal Model have the term $S_0 e^{-qT} \Phi(D_1) - K e^{-rT} \Phi(D_2)$ for the calls, and the term $-S_0 e^{-qT} \Phi(-D_1) + K e^{-rT} \Phi(-D_2)$ for the puts. These terms are the option prices under Black-Scholes Model, if we ignore the difference between D_1, D_2 and d_1, d_2. In other words, option prices under Polynomial-Normal Model are the ones under Black-Scholes Model plus some other terms. We will call these terms moment premium in latter analysis.

Moment Premium:

$$
\begin{aligned}
MP = {} & S_0 e^{-qT} h(D_1) \sum_{n=1}^{N} b_n \left(\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \tilde{\sigma}^{n-j} H_j(D_1)\right) / p(\tilde{\sigma}) \\
& + K e^{-rT} h(D_2) \left(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)\right)
\end{aligned}
$$

3.4 Greeks for Polynomial-Normal Model

Greeks is used to measure the sensitivity of option prices, in response to the changes of underlying parameters. The name of 'Greek' is obtained by the reason that people usually use Greek letters to represent these sensitivity rates.

In Black-Scholes formula, call and put prices depend on 6 parameters, and they also depend on the polynomial parameters in Gram-Charlier Model and Polynomial-Normal Model. After the option is written, its price on derivative market may change as the value of these arguments change. By differentiating the price respect to these arguments, we can estimate the change in price of the option in response to change in one value of these arguments. This is useful when one wishes to estimate the gain and loss of the options, to hedge the option for some particular parameters, and so on.

The following table gives the definitions of the 6 Greek measures that is used mostly.

Name	Equation	Description
Δ (Delta)	$\frac{\partial C}{\partial S_0}, \frac{\partial P}{\partial S_0}$	Increase in option price to the increase in stock price
Γ (Gamma)	$\frac{\partial^2 C}{\partial {S_0}^2}, \frac{\partial^2 P}{\partial {S_0}^2}$	Increase in Delta to the increase in stock price
(Vega)	$0.01\frac{\partial C}{\partial \sigma}$	Increase in option price to the increase in volatility. Times 0.01, because

	$0.01\frac{\partial P}{\partial \sigma}$	volatility is measured in the percentage form. (There is no Greek letter for it)
θ (Theta)	$\frac{-1}{365}\frac{\partial C}{\partial T}$ $\frac{-1}{365}\frac{\partial P}{\partial T}$	Increase in option price to the increase in time to maturity. Divided by 365, because the time is measured in days. Negative sign is taken because time to maturity decreases as time approaches.
ρ (Rho)	$0.01\frac{\partial C}{\partial r}$ $0.01\frac{\partial P}{\partial r}$	Increase in option price to the increase in interest rate. Times 0.01, because the interest rate is measured in percentage form.
ψ (Psi)	$0.01\frac{\partial C}{\partial q}$ $0.01\frac{\partial P}{\partial q}$	Increase in option price to the increase in dividend yield. Times 0.01, because the dividend rate is measured in percentage form.

Table 3.1: definitions of the 6 Greek Measures that usually used

In this part, we will give the Greeks formula for Polynomial-Normal Model, and additionally we will pay attention to how the put-call parity affects the relations of Greeks for calls and puts. For some Greeks, we can derive the formulas from the pdf of the return, and it's a good way to see what mathematical meanings these Greeks have.

Greeks for Black-Scholes Model and Gram-Charlier Model can be derived as special cases from the formulas of Polynomial-Normal Model, so we will not use other formulas for Greeks of these two models here.

Maybe it's easy to think that the Greeks under Polynomial-Normal Model is that under Black-Scholes model plus the derivatives of moment premium respect to the corresponding variables. However, this is not true in the Greeks of Vega and Theta because D_1 and D_2 in Polynomial-Normal Model is different from d_1 and d_2 in Black-Scholes Model, so are their derivatives. For other four Greeks, we can use this consideration, because $D_{1,2} = d_{1,2} + p(\tilde{\sigma}) / \tilde{\sigma}$, which means they only differ in derivatives respect to σ and T.

Delta:

Delta measures the change in option price in response to the change in stock price. Delta is often used in Delta hedging, which let the investor have extremely low risk if the stock price doesn't change too much. Delta hedging is the most important and fundamental Greek letter hedging, and is often combined with other hedging.

If risk-neutral option price is used, we have the following.

$$\begin{aligned}
\Delta_{call} &= \frac{\partial}{\partial S_0} e^{-rT} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^x - K) f(x) dx \\
&= e^{-rT} \int_{\ln(K/S_0)}^{+\infty} e^x f(x) dx + e^{-rT} (S_0 e^x - K) f(x) \Big|_{x = \ln(S_0/K)} \\
&= e^{-rT} \int_{\ln(K/S_0)}^{+\infty} e^x f(x) dx
\end{aligned}$$

By differentiating the put-call parity equation $C - P = S_0 e^{-qT} - K e^{-rT}$, we have $\Delta_{call} - \Delta_{put} = e^{-qT}$.

In Polynomial-Normal Model,

$$\Delta_{call} = e^{-qT} \Phi(D_1) + e^{-qT} h(D_1) \sum_{n=1}^{N} b_n \left(\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} \tilde{\sigma}^{n-j} H_j(D_1) \right) / p(\tilde{\sigma})$$

$$\Delta_{put} = -e^{-qT}\Phi(-D_1) + e^{-qT}h(D_1)\sum_{n=1}^{N} b_n(\sum_{j=0}^{n-1}(-1)^j \binom{n}{j+1}\tilde{\sigma}^{n-j}H_j(D_1)) / p(\tilde{\sigma})$$

In Black-Scholes Model, $\Delta_{call} = e^{-qT}\Phi(d_1)$, $\Delta_{put} = -e^{-qT}\Phi(-d_1)$

From the formula $\Delta_{call} = e^{-rT}\int_{\ln(K/S_0)}^{+\infty} e^x f(x)dx$, and $\Delta_{put} = e^{-rT}\int_{-\infty}^{\ln(K/S_0)} (-e^x) f(x)dx$, it's obvious that $\Delta_{call} > 0$ and $\Delta_{put} < 0$. We can see that Delta tends to 0 or $\pm e^{-qT}$ when S_0 is extremely high or low, such like some deep-in-the-money options or deep-out-of-money options. For options with a short time to maturity, Delta remains close to $\pm e^{-qT}$ or 0 most of the time, and varies rapidly near strike price.

Gamma:

Gamma is the second derivatives of option price in respect to stock price. Gamma hedging is also used as a complementary of Delta hedging, allowing the investors to make their positions further less risky with change of stock prices. By differentiating the put-call parity equation twice, we have $\Gamma_{call} = \Gamma_{put}$. We will not identify Γ_{call} and Γ_{put} in latter analysis.

$$\begin{aligned}\Gamma &= \frac{\partial}{\partial S_0}(e^{-rT}\int_{\ln(K/S_0)}^{+\infty} e^x f(x)dx) \\ &= \frac{e^{-rT}K}{S_0^{\ 2}} f(\ln(\frac{K}{S_0})) > 0\end{aligned}$$

Because Γ is always positive, the option price is a convex function in respect to the stock price, and Delta is an increasing function in respect to the stock price.

In Polynomial-Normal Model, we have the following formula by differentiating Delta to S_0:

$$\Gamma = \frac{e^{-qT} h(D_1)}{S_0 \tilde{\sigma}} + \frac{e^{-qT}}{S_0 \tilde{\sigma}} h(D_1) \sum_{n=1}^{N} b_n (\sum_{j=1}^{n} (-1)^j \binom{n}{j} \tilde{\sigma}^{n+1-j} H_j(D_1)) / p(\tilde{\sigma})$$

In Black-Scholes Model, we have $\Gamma = \frac{e^{-qT} h(d_1)}{S_0 \sigma \sqrt{T}}$.

Vega:

Vega is used to see how changes in volatility affect option prices. In traditional Black-Scholes Models, the volatility is assumed to be constant. However, as reality data conflicts with this simple assumption, the changing of volatility must be taken into consideration.

Differentiating put-call parity equation $C - P = S_0 e^{-qT} - Ke^{-rT}$ by σ, we have $Vega_{call} = Vega_{put}$. We will not identify $Vega_{call}$ or $Vega_{put}$ in latter analysis.

As the density function f mainly depends σ, T, r and q, these Greek letter would be related to derivatives of f in respect to these parameters.

In Polynomial-Normal Model,

$$\begin{aligned} Vega = 0.01 \Bigg\{ & S_0 e^{-qT} h(D_1) \frac{\partial D_1}{\partial \sigma} - Ke^{-rT} h(D_2) \frac{\partial D_2}{\partial \sigma} \\ & + S_0 e^{-qT} h(D_1) \sum_{n=1}^{N} b_n (\sum_{j=1}^{n} (-1)^j \binom{n}{j} \tilde{\sigma}^{n+1-j} H_j(D_1)) \frac{\partial D_1}{\partial \sigma} / p(\tilde{\sigma}) \\ & + S_0 e^{-qT} h(D_1) \sum_{n=1}^{N} b_n (\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} H_j(D_1)) \sqrt{T} \\ & \frac{(n-j)\tilde{\sigma}^{n-1-j} p(\tilde{\sigma}) - \tilde{\sigma}^{n-j} p'(\tilde{\sigma})}{p^2(\tilde{\sigma})} - Ke^{-rT} h(D_2) \sum_{n=1}^{N} b_n H_n(D_2) \frac{\partial D_2}{\partial \sigma} \Bigg\} \end{aligned}$$

where $\frac{\partial D_1}{\partial \sigma} = -\frac{\ln(S_0/K)+(r-q)T-\ln(p(\sigma\sqrt{T}))}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2} - \frac{p'(\sigma\sqrt{T})\sqrt{T}}{p(\sigma\sqrt{T})\sigma\sqrt{T}}$

And $\frac{\partial D_2}{\partial \sigma} = \frac{\partial D_1}{\partial \sigma} - \sqrt{T}$

In Black-Scholes Model,

$Vega = 0.01(-d_2 S_0 e^{-qT} h(d_1) + d_1 K e^{-rT} h(d_2))/\sigma$

Theta:

Theta is used to measure how option prices changes as time approach, if other parameters stay the same. This measures the decay effect of option prices. Theta hedging is usually done by borrowing or investing in the bank account.

For Polynomial-Normal Model, we have

$$
\begin{aligned}
\theta_{call} &= \frac{-1}{365}\frac{\partial C}{\partial T} \\
&= \frac{-1}{365}\Bigg\{ -qS_0 e^{-qT}\Phi(D_1) + S_0 e^{-qT} h(D_1)\frac{\partial D_1}{\partial T} + rKe^{-rT}\Phi(D_2) - Ke^{-rT}h(D_2)\frac{\partial D_2}{\partial T} \\
&\quad -qS_0 e^{-qT} h(D_1)\sum_{n=1}^{N} b_n \Big(\sum_{j=0}^{n-1}(-1)^j \begin{pmatrix} n \\ j+1 \end{pmatrix} \tilde{\sigma}^{n-j} H_j(D_1)\Big)/p(\tilde{\sigma}) \\
&\quad -S_0 e^{-qT} h(D_1)\sum_{n=1}^{N} b_n \Big(\sum_{j=0}^{n-1}(-1)^j \begin{pmatrix} n \\ j+1 \end{pmatrix} \tilde{\sigma}^{n-j} H_{j+1}(D_1)\Big)\frac{\partial D_1}{\partial T}/p(\tilde{\sigma}) \\
&\quad +S_0 e^{-qT} h(D_1) \\
&\quad \sum_{n=1}^{N} b_n \Big(\sum_{j=0}^{n-1}(-1)^j \begin{pmatrix} n \\ j+1 \end{pmatrix} H_j(D_1)\big((n-j)\tilde{\sigma}^{n-j} p(\tilde{\sigma}) - \tilde{\sigma}^{n+1-j} p'(\tilde{\sigma})\big)\Big)/(2Tp^2(\tilde{\sigma})) \\
&\quad -rKe^{-rT} h(D_2)\Big(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)\Big) - Ke^{-rT} h(D_2)\Big(\sum_{n=1}^{N} b_n H_n(D_2)\Big)\frac{\partial D_2}{\partial T} \Bigg\}
\end{aligned}
$$

$$\theta_{put} = \frac{-1}{365}\frac{\partial P}{\partial T}$$
$$= \frac{-1}{365}\Bigg\{ qS_0 e^{-qT}\Phi(-D_1) + S_0 e^{-qT} h(D_1)\frac{\partial D_1}{\partial T} - rKe^{-rT}\Phi(-D_2)$$
$$-Ke^{-rT}h(D_2)\frac{\partial D_2}{\partial T} - qS_0 e^{-qT} h(D_1)\sum_{n=1}^{N} b_n (\sum_{j=0}^{n-1}(-1)^j \binom{n}{j+1}\tilde{\sigma}^{n-j} H_j(D_1)) / p(\tilde{\sigma})$$
$$-S_0 e^{-qT} h(D_1)\sum_{n=1}^{N} b_n (\sum_{j=0}^{n-1}(-1)^j \binom{n}{j+1}\tilde{\sigma}^{n-j} H_{j+1}(D_1))\frac{\partial D_1}{\partial T} / p(\tilde{\sigma})$$

$$+S_0 e^{-qT} h(D_1)\sum_{n=1}^{N} b_n (\sum_{j=0}^{n-1}(-1)^j \binom{n}{j+1} H_j(D_1)((n-j)\tilde{\sigma}^{n-j} p(\tilde{\sigma}) - \tilde{\sigma}^{n+1-j} p'(\tilde{\sigma})))$$
$$/(2Tp^2(\tilde{\sigma}))$$
$$-rKe^{-rT}h(D_2)(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)) - Ke^{-rT}h(D_2)(\sum_{n=1}^{N} b_n H_n(D_2))\frac{\partial D_2}{\partial T}\Bigg\}$$

where $\frac{\partial D_1}{\partial T} = -\frac{\ln(S_0/K)}{2\sigma T^{\frac{3}{2}}} + \frac{r-q+\sigma^2/2}{2\sigma\sqrt{T}} + (\frac{\ln(p(\tilde{\sigma}))}{\tilde{\sigma}^2} - \frac{p'(\tilde{\sigma})}{\tilde{\sigma}p(\tilde{\sigma})})\frac{\sigma}{2\sqrt{T}}$, and

$\frac{\partial D_2}{\partial T} = \frac{\partial D_1}{\partial T} - \frac{\sigma}{2\sqrt{T}}$.

In Black-Scholes Model, we have

$$\theta_{call} = \frac{-1}{365}\frac{\partial C}{\partial T}$$
$$= \frac{-1}{365}\left[-qS_0 e^{-qT}\Phi(d_1) + rKe^{-rT}\Phi(d_2) + S_0 e^{-qT} h(d_1)\frac{\partial d_1}{\partial T} - Ke^{-rT}h(d_2)\frac{\partial d_2}{\partial T}\right]$$

$$\theta_{put} = \frac{-1}{365}\frac{\partial P}{\partial T}$$
$$= \frac{-1}{365}\left[qS_0 e^{-qT}\Phi(-d_1) - rKe^{-rT}\Phi(-d_2) + S_0 e^{-qT} h(d_1)\frac{\partial d_1}{\partial T} - Ke^{-rT}h(d_2)\frac{\partial d_2}{\partial T}\right]$$

Where $\frac{\partial d_1}{\partial T} = (-\ln(S_0/K) + (r-q+\frac{\sigma^2}{2})T)/(2\sigma T^{\frac{3}{2}})$, and $\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{\sigma}{2\sqrt{T}}$

Rho:

Rho is used to measure how changes in interest rate would affect option prices.

Differentiating the put-call parity equation, we have $\rho_{call} - \rho_{put} = 0.01TKe^{-rT}$.

For Polynomial-Normal Model, we have

$$\rho_{call} = 0.01\left\{ S_0 e^{-qT} h(D_1)\frac{\sqrt{T}}{\sigma} + TKe^{-rT}\Phi(D_2) - Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma} \right.$$
$$+ S_0 e^{-qT} h(D_1)\frac{\sqrt{T}}{\sigma}\sum_{n=1}^{N} b_n (\sum_{j=1}^{n} (-1)^j \binom{n}{j} \tilde{\sigma}^{n+1-j} H_j(D_1)) / p(\tilde{\sigma})$$
$$\left. - Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma}(\sum_{n=1}^{N} b_n H_n(D_2)) - TKe^{-rT}h(D_2)(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)) \right\}$$

$$\rho_{put} = 0.01\left\{ S_0 e^{-qT} h(D_1)\frac{\sqrt{T}}{\sigma} - TKe^{-rT}\Phi(-D_2) - Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma} \right.$$
$$+ S_0 e^{-qT} h(D_1)\frac{\sqrt{T}}{\sigma}\sum_{n=1}^{N} b_n (\sum_{j=1}^{n} (-1)^j \binom{n}{j} \tilde{\sigma}^{n+1-j} H_j(D_1)) / p(\tilde{\sigma})$$
$$\left. - Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma}(\sum_{n=1}^{N} b_n H_n(D_2)) - TKe^{-rT}h(D_2)(\sum_{n=0}^{N-1} b_{n+1} H_n(D_2)) \right\}$$

In Black-Scholes Model, Rho is given as follows:

$$\rho_{call} = 0.01\left[S_0 e^{-qT} h(d_1)\frac{\sqrt{T}}{\sigma} + TKe^{-rT}\Phi(d_2) - Ke^{-rT}h(d_2)\frac{\sqrt{T}}{\sigma} \right]$$

$$\rho_{put} = 0.01\left[S_0 e^{-qT} h(d_1)\frac{\sqrt{T}}{\sigma} - TKe^{-rT}\Phi(-d_2) - Ke^{-rT}h(d_2)\frac{\sqrt{T}}{\sigma} \right]$$

Rho for calls is always positive and for puts is always negative in these three models, no matter what polynomial is used. This is basically because the strike price is discount greater when a higher interest rate is used. The following proposition is a simple proof for this consideration.

Proposition 3.1: $\rho_{call} > 0$ and $\rho_{put} < 0$ in Polynomial Model.

Proof:

Consider another interest rate $r + \Delta r$ is used instead of r, and $\Delta r > 0$.

Denote the original and new call options prices as C_1 and C_2, the original and new put options prices as P_1 and P_2.

$$
\begin{aligned}
C_2 &= e^{-(r+\Delta r)T} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^x - K) f(x - \Delta rT) dx \\
&= e^{-rT} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^{x-\Delta rT} - K e^{-\Delta rT}) f(x - \Delta rT) dx \\
&= e^{-rT} \int_{\ln(K/S_0)-\Delta rT}^{+\infty} (S_0 e^x - K e^{-\Delta rT}) f(x) dx \\
&> e^{-rT} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^x - K) f(x) dx = C_1
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= e^{-(r+\Delta r)T} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) f(x - \Delta rT) dx \\
&= e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K e^{-\Delta rT} - S_0 e^{x-\Delta rT}) f(x - \Delta rT) dx \\
&= e^{-rT} \int_{-\infty}^{\ln(K/S_0)-\Delta rT} (K e^{-\Delta rT} - S_0 e^x) f(x) dx \\
&< e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) f(x) dx = P_1
\end{aligned}
$$

So the call option price should be an increasing function of r, while the put option price is a decreasing function of r. Thus $\rho_{call} > 0$ and $\rho_{put} < 0$. The proposition holds.

Psi:

Psi is used to measure how the changes of dividend rate q would affect the option prices.

Differentiating the put-call parity equation, we have $\psi_{call} - \psi_{put} = -0.01TS_0e^{-qT}$.

In Polynomial-Normal Model, we have

$$\begin{aligned}\psi_{call} = 0.01\Bigg\{ & -TS_0e^{-qT}\Phi(D_1) - S_0e^{-qT}h(D_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma} \\ & -TS_0e^{-qT}h(D_1)\sum_{n=1}^{N}b_n(\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}\tilde{\sigma}^{n-j}H_j(D_1))/p(\tilde{\sigma}) \\ & -S_0e^{-qT}h(D_1)\sum_{n=1}^{N}b_n(\sum_{j=1}^{n}(-1)^j\binom{n}{j}\tilde{\sigma}^{n+1-j}H_j(D_1))/p(\tilde{\sigma})\frac{\sqrt{T}}{\sigma} \\ & +Ke^{-rT}h(D_2)(\sum_{n=1}^{N}b_nH_n(D_2))\frac{\sqrt{T}}{\sigma}\Bigg\}\end{aligned}$$

$$\begin{aligned}\psi_{call} = 0.01\Bigg\{ & TS_0e^{-qT}\Phi(-D_1) - S_0e^{-qT}h(D_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}h(D_2)\frac{\sqrt{T}}{\sigma} \\ & -TS_0e^{-qT}h(D_1)\sum_{n=1}^{N}b_n(\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}\tilde{\sigma}^{n-j}H_j(D_1))/p(\tilde{\sigma}) \\ & -S_0e^{-qT}h(D_1)\sum_{n=1}^{N}b_n(\sum_{j=1}^{n}(-1)^j\binom{n}{j}\tilde{\sigma}^{n+1-j}H_j(D_1))/p(\tilde{\sigma})\frac{\sqrt{T}}{\sigma} \\ & +Ke^{-rT}h(D_2)(\sum_{n=1}^{N}b_nH_n(D_2))\frac{\sqrt{T}}{\sigma}\Bigg\}\end{aligned}$$

In Black-Scholes Model, we have the following:

$$\psi_{call} = 0.01\left[-TS_0e^{-qT}\Phi(d_1) - S_0e^{-qT}h(d_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}h(d_2)\frac{\sqrt{T}}{\sigma}\right]$$

$$\psi_{put} = 0.01\left[TS_0e^{-qT}\Phi(-d_1) - S_0e^{-qT}h(d_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}h(d_2)\frac{\sqrt{T}}{\sigma}\right]$$

As dividend rate here is playing a role of reversing interest rate, we can prove $\psi_{call} < 0$ and $\psi_{put} > 0$ similarly.

Proposition 3.2: $\psi_{call} < 0$ and $\psi_{put} > 0$ in Polynomial-Normal Model.

Proof:

Consider another dividend rate $q + \Delta q$ is used instead of q, and $\Delta q > 0$.

Denote the original and new call options prices as C_1 and C_2, the original and new put options prices as P_1 and P_2.

$$
\begin{aligned}
C_2 &= e^{-rT} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^x - K) f(x + \Delta qT) dx \\
&= e^{-rT} \int_{\ln(K/S_0)+\Delta qT}^{+\infty} (S_0 e^{x-\Delta qT} - K) f(x) dx \\
&< e^{-rT} \int_{\ln(K/S_0)}^{+\infty} (S_0 e^x - K) f(x) dx = C_1
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) f(x + \Delta qT) dx \\
&= e^{-rT} \int_{-\infty}^{\ln(K/S_0)+\Delta qT} (K - S_0 e^{x-\Delta qT}) f(x) dx \\
&> e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) f(x) dx = P_1
\end{aligned}
$$

So the call option price should be a decreasing function of q, while the put option price is an increasing function of q. Thus $\psi_{call} < 0$ and $\psi_{put} > 0$. The proposition holds.

Chapter 4

Parameters Estimating and Examples

In this section, we will use realistic stock data to see how our models actually work in real data analysis, and estimate the efficiency of our models. Stock data of a few large companies and some index will be used in our empirical work. We will use monthly return and prices changing to evaluate the return distribution function.

For Polynomial-Normal Models, We will assume that $b_1 = b_2 = 0$ in this section, so the mean and variance of the return would be m and $\tilde{\sigma}$ respectively and our estimation would be simplified greatly. We also assume that the moment parameters up to order three stay the same with different time periods, for the simplification of the model. However this is not true in the real world, like the skewness of the annual return is likely to be different from that of monthly return. We have derived the formula for moments in section 2.5. The equation 2.13 and equation 2.14 show us how to get the moments from polynomial parameters. Basically we will estimate the moments from our data and solve the linear equations to get the polynomial parameters. Another part of our example is to use the estimated parameters to calculate the option prices, from the formulas we derived in Chapter 3, to see which model would give the most precise option prices.

4.1 Empirical Performance: Royal Bank of Canada

We are using the data of Royal Bank of Canada in our first example. Royal Bank of Canada is the largest financial organization in Canada, with the most measured deposits, revenues and market capitalization. It also has a large proportion of business in the United States and some other Caribbean countries. Its stock is traded on Toronto Stock Exchange (TSE).

The data source is Canada Financial Market Research Center (CFMRC). The data was lastly revised at 2008 when we cited it into this thesis. The time period of our data is between the beginning of 2001 and the end of 2006. We use monthly return to estimate the changing of the stock prices, so the time unit considered is $\Delta t = 1/12$, and the sample size is $n = 72$. Assume that $Log(S_{t+\Delta t} / S_t)$ is independent identically distributed (IDD) random variable. The average annual dividend rate is $\hat{q} = 0.07153$. And the average annual risk free rate period is calculated from the Geometric mean of long-term government bond yields between 2001 and 2006, which gives us $\hat{r} = 0.069425$. Here we will use Black-Scholes Model, Gram-Charlier Model and a Polynomial-Normal Model up to degree 6 to analyze our data.

In our models, we can calculate from the data that the sample mean and standard deviation of $Log(S_{t+1}/S_t)$ are the following:

$$\hat{m} = mean(Log(S_{i+1}/S_i))$$
$$= \sum_{i=1}^{n} Log(S_{i+1}/S_i)/n$$
$$= 0.0108449$$

$$\hat{\sigma} = \sqrt{\mathrm{var}(Log(S_{i+1}/S_i))}$$
$$= \sqrt{\sum_{i=1}^{n} (Log(S_{i+1}/S_i) - \hat{m})^2 / n}$$
$$= 0.0437284$$

The third sample moment $\hat{\mu}_3$ should be

$$\hat{\mu}_3 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^3 / n$$
$$= 7.3531 \times 10^{-5}$$

The fourth sample moment $\hat{\mu}_4$ should be

$$\hat{\mu}_4 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^4 / n$$
$$= 1.5039 \times 10^{-5}$$

The fifth sample moment $\hat{\mu}_5$ should be

$$\hat{\mu}_5 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^5 / n$$
$$= 1.0250 \times 10^{-6}$$

The sixth sample moment $\hat{\mu}_6$ should be

$$\hat{\mu}_6 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^6 / n$$
$$= 2.0199 \times 10^{-7}$$

Using equation 2.13 and 2.14, we can solve the polynomial parameters as following: $b_3 = 0.020019$, $b_4 = 0.025875$, $b_5 = 0.0040980$, $b_6 = -0.0019260$.

The pdf under three models can be plot as following:

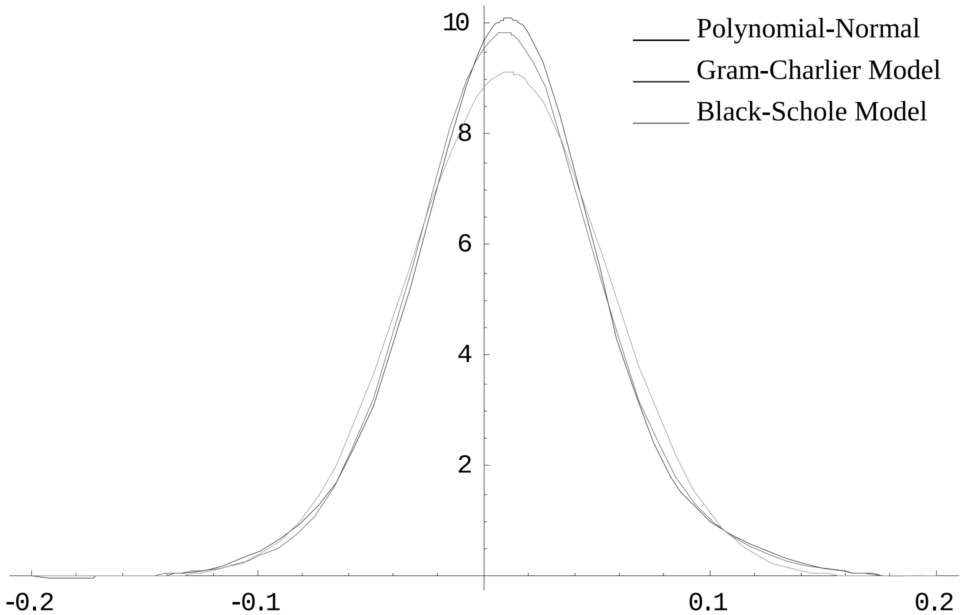

Figure 4.1: phf of rbc stock returns distribution

The call and put option prices under the three models can be calculated from equation 3.3 and 3.4. The following table shows us the option prices under $T = 1$, $S_0 = K = 50$.

	Black-Scholes	Gram-Charlier	Polynomial-Normal
Call Price	4.68767	4.09483	4.38713
Put Price	1.63133	1.03849	1.33079

Table 4.1: option prices on RBC stocks under the three models

Remark 5.1:

For Gram-Charlier and Polynomial-Normal Model, the pdf doesn't satisfy global positivity actually. The pdf of Gram-Charlier Model falls below 0 around ± 1, and rise to above 0 again at around ± 2. The pdf of 6-degree Polynomial-Normal Model falls below 0 at about -4 and 6, and never rise above 0 again (because the 6th order polynomial parameter is negative). But as the main part of the pdf is positive, this won't be big problem for our modeling.

Remark 5.2:

Because $\hat{\sigma}$ is the sample standard deviation of return in month, so the yearly volatility should be $\sigma = \sqrt{12}\hat{\sigma} = 0.1599$. These two volatilities should be identified with each other.

Remark 5.3:

There are two stock splits for stocks of RBC in the period considered. The first one was in Oct 5, 2000, and the second one was in March 3, 2006. Both stock splits are 2-1 stock splits. We use $Log(2S_{t+1} / S_t)$ as a substitution of $Log(S_{t+1} / S_t)$ in these two months, as a compensation to the influence of the stock split.

The data of stock prices of RBC is showed as following:

Date	closing price	return	dividend	Date	closing price	return	dividend
2000-1-31	59.05	-0.061575	1	2004-1-30	63.19	0.029935	1
2000-2-29	62.25	0.054191	0	2004-2-27	63.51	0.005064	0
2000-3-31	68	0.092369	0	2004-3-31	62.7	-0.012754	0
2000-4-28	69.9	0.035882	1	2004-4-30	60.96	-0.019458	1
2000-5-31	77.85	0.113734	0	2004-5-31	59.02	-0.031824	0
2000-6-30	75.75	-0.026975	0	2004-6-30	58.96	-0.001017	0

2000-7-31	79.3	0.054786	1	2004-7-30	61.5	0.0519	1
2000-8-31	86.15	0.086381	0	2004-8-31	59.42	-0.033821	0
2000-9-29	44.7	n/a	1	2004-9-30	59.63	0.003534	0
2000-10-31	48.3	0.09396	1	2004-10-29	63.4	0.071944	1
2000-11-30	45.65	-0.054865	0	2004-11-30	62.48	-0.014511	0
2000-12-29	50.85	0.11391	0	2004-12-31	64.18	0.027209	0
2001-1-31	48.2	-0.045624	1	2005-1-31	63	-0.009816	1
2001-2-28	46.85	-0.028008	0	2005-2-28	72.55	0.151587	0
2001-3-30	47.18	0.007044	0	2005-3-31	73.56	0.013921	0
2001-4-30	43.25	-0.076304	1	2005-4-29	74.93	0.026101	1
2001-5-31	48.94	0.131561	0	2005-5-31	75.22	0.00387	0
2001-6-29	48.57	-0.00756	0	2005-6-30	76	0.01037	0
2001-7-31	50.96	0.056619	1	2005-7-29	77.39	0.026316	1
2001-8-31	49.6	-0.026688	0	2005-8-31	80.75	0.043416	0
2001-9-28	48.15	-0.029234	0	2005-9-30	84.96	0.052136	0
2001-10-31	46.8	-0.020561	1	2005-10-31	83.54	-0.009181	1
2001-11-30	49.24	0.052137	0	2005-11-30	89.27	0.06859	0
2001-12-31	51.83	0.0526	0	2005-12-30	90.65	0.015459	0
2002-1-31	50	-0.028362	1	2006-1-31	89.22	-0.008715	1
2002-2-28	50.71	0.0142	0	2006-2-28	94.87	0.063327	0
2002-3-28	53.2	0.049103	0	2006-3-31	49.3	0.039317	1
2002-4-30	54.97	0.040414	1	2006-4-28	47.8	-0.023124	1
2002-5-31	58.6	0.066036	0	2006-5-31	45.24	-0.053556	0
2002-6-28	52.5	-0.104096	0	2006-6-30	45.51	0.005968	0
2002-7-31	53.45	0.025333	1	2006-7-31	46.06	0.019996	1
2002-8-30	56	0.047708	0	2006-8-31	48.92	0.062093	0
2002-9-30	52.7	-0.058929	0	2006-9-29	49.62	0.014309	0
2002-10-31	54.41	0.040038	1	2006-10-31	49.89	0.013503	1
2002-11-29	58.55	0.076089	0	2006-11-30	53.24	0.067148	0
2002-12-31	57.85	-0.011956	0	2006-12-29	55.51	0.042637	0
2003-1-31	55.3	-0.037165	1	2007-1-31	54.52	-0.017835	0
2003-2-28	58.1	0.050633	0	2007-2-28	54.32	-0.003668	0
2003-3-31	57.14	-0.016523	0	2007-3-30	57.61	0.060567	0
2003-4-30	59.8	0.054078	1	2007-4-30	57.78	0.010936	1
2003-5-30	58.89	-0.015217	0	2007-5-31	58.33	0.009519	0
2003-6-30	57.38	-0.025641	0	2007-6-29	56.55	-0.030516	0
2003-7-31	58.9	0.033984	1	2007-7-31	54.25	-0.032538	1
2003-8-29	59.59	0.011715	0	2007-8-31	54.38	0.002396	0
2003-9-30	59.45	-0.002349	0	2007-9-28	55.1	0.01324	0
2003-10-31	63.48	0.075526	1	2007-10-31	56.04	0.026134	1
2003-11-28	61.7	-0.02804	0	2007-11-30	53.12	-0.052106	0
2003-12-31	61.8	0.001621	0	2007-12-31	50.69	-0.045745	0

Table 4.2: RBC stock prices history

4.2 Empirical Performance: Wal-Mart Store Inc.

We choose Wal-Mart Store Inc. for our second real data analysis. Wal-Mart Store Inc. is an American company that runs a chain of department stores, which spread on many countries in the world. In the corporation ranking of Fortune Global 500 in 2008, Wal-Mart Store Inc. is the largest public corporation by revenue. Its stock is traded on New York Stock Exchange (NYSE).

We take the data for Wal-Mart Store Inc. from finance.yahoo.com. The data period is from 2000 to 2007. As we are using monthly return in our analysis, Similar to the stock analysis for RBC, we set $\Delta T = 1/12$, and the sample size $n = 96$. Continuous dividend rate is calculated from the geometric mean of the dividend discounted prices by non-discounted prices. Thus we have $\hat{q} = 0.0112484$. The risk free rate is calculated as the geometric mean of the yields of 10 Years Treasury Notes (^TNX), thus we have $\hat{r} = 0.04560$. We what we did in the RBC analysis, we assume the monthly return of stocks for Wal-Mart Store Inc. follows a IID Polynomial-Normal distribution and perform our analysis as follow:

$$\begin{aligned}\hat{m} &= mean(Log(S_{i+1}/S_i)) \\ &= \sum_{i=1}^{n} Log(S_{i+1}/S_i)/n \\ &= -0.0039009\end{aligned}$$

$$\hat{\sigma} = \sqrt{\mathrm{var}(Log(S_{i+1}/S_i))}$$
$$= \sqrt{\sum_{i=1}^{n}(Log(S_{i+1}/S_i) - \hat{m})^2/n}$$
$$= 0.063560$$

The third sample moment $\hat{\mu}_3$ should be

$$\hat{\mu}_3 = \sum_{i=1}^{n}(Log(S_{i+1}/S_i)^3/n$$
$$= -1.5536 \times 10^{-4}$$

The fourth sample moment $\hat{\mu}_4$ should be

$$\hat{\mu}_4 = \sum_{i=1}^{n}(Log(S_{i+1}/S_i)^4/n$$
$$= 6.9716 \times 10^{-5}$$

The fifth sample moment $\hat{\mu}_5$ should be

$$\hat{\mu}_5 = \sum_{i=1}^{n}(Log(S_{i+1}/S_i)^5/n$$
$$= -7.5684 \times 10^{-6}$$

The sixth sample moment $\hat{\mu}_6$ should be

$$\hat{\mu}_6 = \sum_{i=1}^{n}(Log(S_{i+1}/S_i)^6/n$$
$$= 2.30945 \times 10^{-6}$$

Using equation 2.13 and 2.14, we can solve the polynomial parameters as following: $b_3 = -0.070076$, $b_4 = 0.047632$, $b_5 = -0.014970$, $b_6 = 0.00055747$.

The pdf under three models can be plot as following:

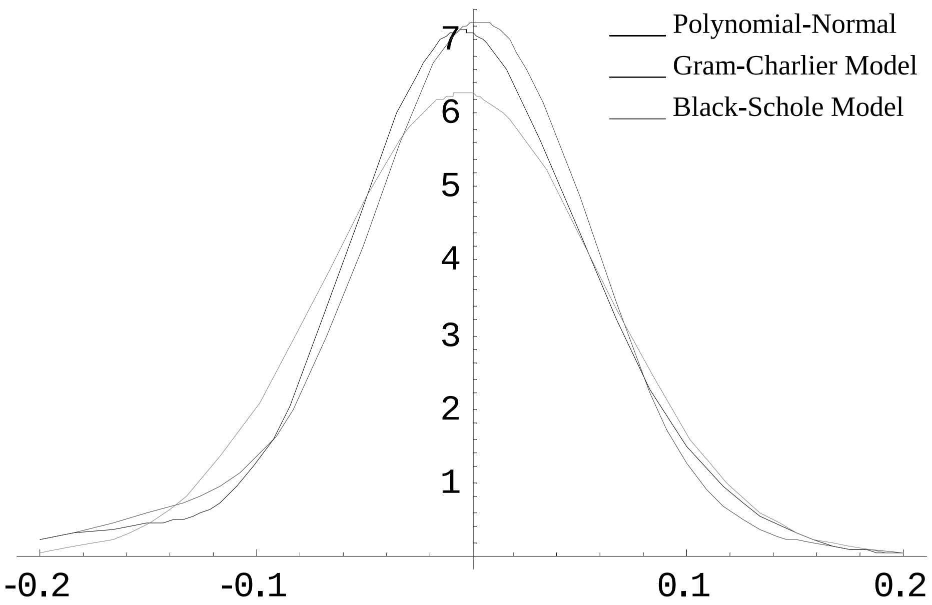

Figure 4.2: pdf of Wal-Mart Inc. stock returns distribution

If we let $T=1$, $S_0 = K = 50$, the call and put options prices under the three models can be calculated by equation 3.3 and 3.4 as follows:

	Black-Scholes	Gram-Charlier	Polynomial-Normal
Call Price	5.46240	6.06516	6.18909
Put Price	3.28045	3.88320	4.00714

Table 4.3: option prices of Wal-Mart Inc. stocks under the three models

Remark 5.4

Like in the RBC data analysis, under the Gram-Charlier and Polynomial-Normal model, the pdf of the return didn't satisfy global positivity actually. The pdf goes negative at a tiny interval around ± 0.1 in both Gram-Charlier and Polynomial-Normal Models, and additionally it falls negative in Polynomial-Normal Model in a relatively large interval at about (0.2,1.5). But this would not cause too much problem as the main part of the pdf stays positive.

The stock data of Wal-Mart Store Inc. is listed as following:

Date	Close Price	Dividend	Date	Close Price	Dividend
2007-12-3	47.53	0.22	2003-12-1	53.05	0.09
2007-11-1	47.9	0	2003-11-3	55.64	0
2007-10-1	45.21	0	2003-10-1	58.95	0.09
2007-9-4	43.65	0	2003-9-2	55.85	0
2007-8-1	43.63	0.22	2003-8-1	59.17	0
2007-7-2	45.95	0	2003-7-1	55.91	0
2007-6-1	48.11	0	2003-6-2	53.67	0.09
2007-5-1	47.6	0.22	2003-5-1	52.61	0
2007-4-2	47.92	0	2003-4-1	56.32	0
2007-3-1	46.95	0.22	2003-3-3	52.03	0.09
2007-2-1	48.31	0	2003-2-3	48.06	0
2007-1-3	47.69	0	2003-1-2	47.8	0
2006-12-1	46.18	0.168	2002-12-2	50.51	0.075
2006-11-1	46.1	0	2002-11-1	53.9	0
2006-10-2	49.28	0	2002-10-1	53.55	0
2006-9-1	49.32	0	2002-9-3	49.24	0.075
2006-8-1	44.72	0.168	2002-8-1	53.48	0
2006-7-3	44.5	0	2002-7-1	49.18	0
2006-6-1	48.17	0	2002-6-3	55.01	0.075
2006-5-1	48.45	0.168	2002-5-1	54.1	0
2006-4-3	45.03	0	2002-4-1	55.86	0
2006-3-1	47.24	0.168	2002-3-1	61.3	0.075
2006-2-1	45.36	0	2002-2-1	62.01	0
2006-1-3	46.11	0	2002-1-2	59.98	0
2005-12-1	46.8	0.15	2001-12-3	57.55	0.07
2005-11-1	48.56	0	2001-11-1	55.15	0
2005-10-3	47.31	0	2001-10-1	51.4	0
2005-9-1	43.82	0	2001-9-4	49.5	0.07
2005-8-1	44.96	0.15	2001-8-1	48.05	0
2005-7-1	49.35	0	2001-7-2	55.9	0
2005-6-1	48.2	0	2001-6-1	48.8	0.07
2005-5-2	47.23	0.15	2001-5-1	51.75	0
2005-4-1	47.14	0	2001-4-2	51.74	0
2005-3-1	50.11	0.15	2001-3-1	50.5	0.07
2005-2-1	51.61	0	2001-2-1	50.09	0
2005-1-3	52.4	0	2001-1-2	56.8	0
2004-12-1	52.82	0.13	2000-12-1	53.13	0.06
2004-11-1	52.06	0	2000-11-1	52.19	0

2004-10-1	53.92	0	2000-10-2	45.38	0
2004-9-1	53.2	0	2000-9-1	48.13	0.06
2004-8-2	52.67	0.13	2000-8-1	47.63	0
2004-7-1	53.01	0	2000-7-3	55.25	0
2004-6-1	52.5	0	2000-6-1	57.63	0.06
2004-5-3	55.73	0.13	2000-5-1	57.63	0
2004-4-1	57	0	2000-4-3	55.38	0
2004-3-1	59.69	0.13	2000-3-1	56.5	0.06
2004-2-2	59.56	0	2000-2-1	48.75	0
2004-1-2	53.85	0	2000-1-3	54.75	0

Table 4.4: Wal-Mart Inc. stock price history

4.3 Empirical Performance: S&P 500 Index

Our last empirical analysis will be about S&P 500 Index. Established at 1957, S&P 500 index served as a weight index for 500 large publicly traded stocks in the United States. Most of the stocks in S&P 500 are those with the largest capitalization in American stock market. Compared to Dow Jones Industries Index, S&P 500 included more companies, has more diversified risk, and reflects the stock market behaviors better. S&P 500 index uses the market capitalization as the weight of stocks.

Our data for S&P 500 is from finance.yahoo.com. The period we considered is from the beginning of 1979 to the end of 2008. As our analysis is based on monthly return of the index, we set $\Delta t = 1/12$, and the sample size $n = 360$. As the index would not pay any dividend, so we set $\hat{q} = 0$. The risk free rate we use here is calculated from the geometric mean of the yield of 10-year treasury notes, thus we have $\hat{r} = 0.071288$. Instead of using the 6th degree

Polynomial-Normal Model in the previous examples, we are using the 8th degree model this time, to see what result will generates from a much longer time period and a more precise model.

The mean and variance is calculated as follows:

$$\hat{m} = mean(Log(S_{i+1}/S_i))$$
$$= \sum_{i=1}^{n} Log(S_{i+1}/S_i)/n$$
$$= 0.0062236$$

$$\hat{\sigma} = \sqrt{\text{var}(Log(S_{i+1}/S_i))}$$
$$= \sqrt{\sum_{i=1}^{n} (Log(S_{i+1}/S_i) - \hat{m})^2 / n}$$
$$= 0.044232$$

The third sample moment $\hat{\mu}_3$ should be

$$\hat{\mu}_3 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^3 / n$$
$$= -4.8528 \times 10^{-5}$$

The fourth sample moment $\hat{\mu}_4$ should be

$$\hat{\mu}_4 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^4 / n$$
$$= 2.3701 \times 10^{-5}$$

The fifth sample moment $\hat{\mu}_5$ should be

$$\hat{\mu}_5 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^5 / n$$
$$= -3.2163 \times 10^{-6}$$

The sixth sample moment $\hat{\mu}_6$ should be

$$\hat{\mu}_6 = \sum_{i=1}^{n} (Log(S_{i+1}/S_i))^6 / n$$
$$= 8.2898 \times 10^{-7}$$

The seventh sample moment $\hat{\mu}_7$ should be

$$\hat{\mu}_7 = \sum_{i=1}^{n} (Log(S_{i+1} / S_i)^7 / n$$
$$= -1.7451 \times 10^{-7}$$

The eighth sample moment $\hat{\mu}_8$ should be

$$\hat{\mu}_8 = \sum_{i=1}^{n} (Log(S_{i+1} / S_i)^8 / n$$
$$= 4.2021 \times 10^{-8}$$

Using equation 2.13 and 2.14, we can solve the polynomial parameters as following: $b_3 = -0.164276$, $b_4 = 0.15114$, $b_5 = -0.11362$, $b_6 = 0.082215$, $b_7 = -0.050499$, $b_8 = 0.024831$.

The pdf under three models can be plot as following:

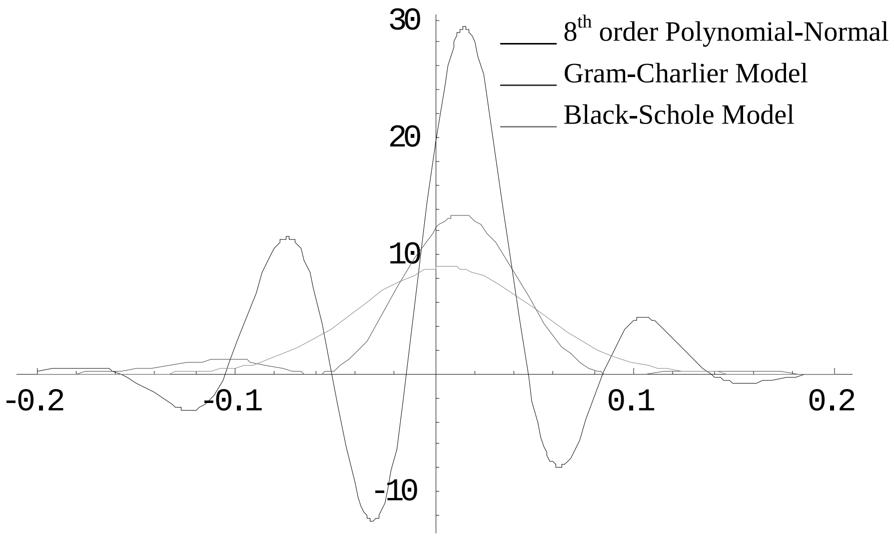

Figure 4.3: pdf of S&P 500 index return, 30 years model

If we let $T = 1$, $S_0 = K = 1000$, the call and put options prices under the three models can be calculated by equation 3.3 and 3.4 as follows:

	Black-Scholes	Gram-Charlier	Polynomial-Normal
Call Price	84.6193	-31.4912	0
Put Price	40.0379	-76.0671	-44.576

Table 4.5: option prices of S&P 500 index, 30 years data, under the three models

Remark 4.5:

We can notice that there's a relatively greater negative part in the pdf of the Polynomial-Normal Model. This is a sign that even the Gram-Charlier distribution function cannot model this distribution well, as the pdf under two models looks so different. The negative option price under Gram-Charlier Model is another evidence of the flawed modeling. But this may be because of the reason that the monthly return are not following the same distribution and independent with each other, because the time period taken into consideration is so long. We would see the return pdf with only 15 years taken into account would looks much better.

The 8^{th} order Polynomial-Normal model shows us that the actual pdf of the return may be not reaching the peak around its mean value, which Black-Scholes Model assumes to be true, and it may has multiple peaks. Another thing caught our concern is that such a 'negative somewhere' pdf function would possibly lead to a negative call price or put price, if the risk neutral price is used.

The 8^{th} order Polynomial-Normal Model with 15 years data:

With only 15^{th} years data accounted, it's much likely that the return would be the same distributed, as the distribution wouldn't change so much in a relatively short period of time.

With 15th years taken into account, the sample size would be 180. The mean and standard deviation of the return and moments of different orders is list as follows:

$\hat{m} = 0.0036714$, $\hat{\sigma} = 0.043900$, $\hat{\mu}_3 = -6.1967 \times 10^{-5}$, $\hat{\mu}_4 = 1.7258 \times 10^{-5}$, $\hat{\mu}_5 = -1.9040 \times 10^{-6}$, $\hat{\mu}_6 = 3.5403 \times 10^{-7}$, $\hat{\mu}_7 = -5.5917 \times 10^{-8}$, $\hat{\mu}_8 = 9.9394 \times 10^{-9}$.

The polynomial parameter is listed as follows:

$b_3 = -0.16399$, $b_4 = 0.080576$, $b_5 = -0.031988$, $b_6 = 0.016405$, $b_7 = -0.0049211$, $b_8 = 0.00016531$.

Thus we have the return pdf under the three models as following:

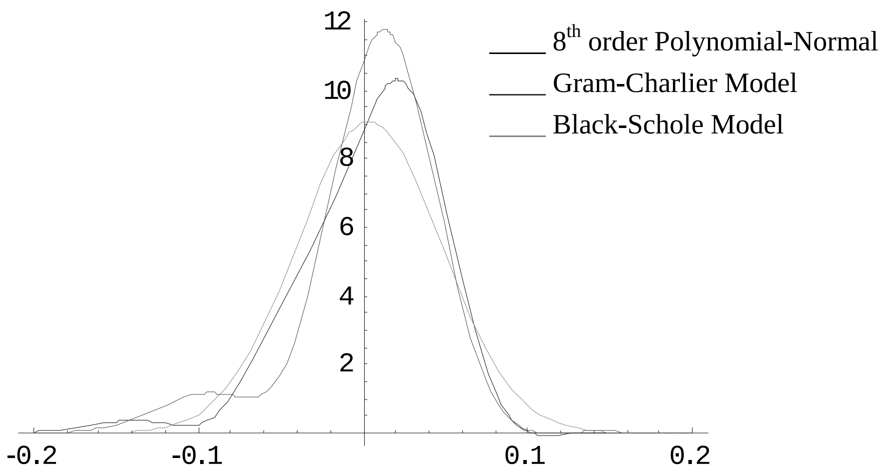

Figure 4.4: pdf of S&P 500 Index return, 15 years model

Similarly, if we let $T = 1$, $S_0 = K = 1000$, the call and put options prices under the three models can be calculated as follows:

	Black-Scholes	Gram-Charlier	Polynomial-Normal
Call Price	84.1857	2.1353	120.6050
Put Price	39.6089	-42.4407	72.0292

Table 4.6: option prices of S&P 500 index, 15 years data, under the three models

Now we use the historical option prices to compare the results generated by these three models.

The historical optional prices can be calculated from the implied volatility of the specific option and the Black-Scholes framework. However, we don't know how much the implied volatility is before the option written, because the implied volatility is calculated from the market price of the option such that the Black-Scholes option prices equal to the market prices. We use the CBOT Volatility Index value at the time of the option written as the volatility in our option formulas, and use the three prices we got in our three models and compare with the market price of the option. The option prices between 2005 and 2007 are considered. Time to maturity is selected as one month, and strike prices are equal to index value. (It means we only consider at-the-money options.) We used the 10-year treasury notes rates as interest rates. The standard error of the three prices is listed as follows.

Black-Scholes Model:

$$SE_{BS} = \sqrt{\frac{1}{2n}\sum_t \left[(C_{BS}(t) - C_{mk}(t))^2 + (P_{BS}(t) - P_{mk}(t))^2 \right]} = 4.3602$$

Gram-Charlier Model:

$$SE_{GC} = \sqrt{\frac{1}{2n}\sum_t \left[(C_{GC}(t) - C_{mk}(t))^2 + (P_{GC}(t) - P_{mk}(t))^2 \right]} = 25.0353$$

Polynomial-Normal Model:

$$SE_{PN} = \sqrt{\frac{1}{2n}\sum_t \left[(C_{PN}(t) - C_{mk}(t))^2 + (P_{PN}(t) - P_{mk}(t))^2 \right]} = 20.4595$$

Here, C_{BS}, P_{BS}, C_{GC}, P_{GC}, C_{PN}, P_{PN}, C_{mk}, P_{mk} stand for the call and put prices under Black-Scholes Model, Gram-Charlier Model, Polynomial-Normal Model, and the market prices respectively. We can see that the Black-Scholes formula gives the closest option prices to the market prices. The Polynomial-Normal formula is worse than Black-Scholes formula, but better than Gram-Charlier formula.

In spite of the result that Black Scholes Model is providing the most closing price estimate for the option price, we still think that Gram Charlier and Polynomial Normal Model would be good extensions of the well known Black Scholes Model. Why these model extensions are not giving better results? That's because the market option prices C_{mk} and P_{mk} are calculated in Black Scholes Model from implied volatility, which an average of that of options with different strike prices. In fact, the market prices of different strike prices options at different time would have different implied volatility, and this lack of information impairs the accuracy of the estimation greatly. Another reason would be from the calculation of polynomial parameters. As we can see in the section 2.5, which is for find the region of polynomial parameters that makes the density function positive globally, the range of b_6 is about $[0, 0.02]$. Any number of b_6 out of this range would make the pdf negative at some point. In fact, the accepted range of b_i is very narrow, and it becomes narrower for higher order parameters. However, the results of these parameters in our real data example fall out of the range and none of the pdf is a valid one that satisfies positivity. And this is why we even have negative option prices some time.

Other than data-relevant biases in our model, non-data biases are also important when considering the accuracy of our models. From the view of company management, events such as stock split, stock issued or repurchase would be a source of distortion in the stock return. This leads to bias in our model estimation, which can be adjusted using company accounting information. Other non-data biases that would occur include the systematic risk of the whole stock market, the upturn or downturn of the national economy, etc. Actually, our models are good for measuring and estimating the non-systematic risk, while the systematic risk estimation would better off using some models relevant to the whole markets behavior.

Further improvement for our new models would be for the approximation of polynomial parameters. We should make it more accurate and provide better approximation of the stock prices.

The following table shows the data used for S&P 500 Index analysis.

Date	Closing Price	Date	Closing Price	Date	Closing Price
2008-12-1	903.25	1998-11-2	1163.63	1988-11-1	273.7
2008-11-3	896.24	1998-10-1	1098.67	1988-10-3	278.97
2008-10-1	968.75	1998-9-1	1017.01	1988-9-1	271.91
2008-9-2	1164.74	1998-8-3	957.28	1988-8-1	261.52
2008-8-1	1282.83	1998-7-1	1120.67	1988-7-1	272.02
2008-7-1	1267.38	1998-6-1	1133.84	1988-6-1	273.5
2008-6-2	1280	1998-5-1	1090.82	1988-5-2	262.16
2008-5-1	1400.38	1998-4-1	1111.75	1988-4-4	261.33

2008-4-1	1385.59	1998-3-2	1101.75	1988-3-1	258.89
2008-3-3	1322.7	1998-2-2	1049.34	1988-2-1	267.82
2008-2-1	1330.63	1998-1-2	980.28	1988-1-4	257.07
2008-1-2	1378.55	1997-12-1	970.43	1987-12-1	247.08
2007-12-3	1468.36	1997-11-3	955.4	1987-11-2	230.3
2007-11-1	1481.14	1997-10-1	914.62	1987-10-1	251.79
2007-10-1	1549.38	1997-9-2	947.28	1987-9-1	321.83
2007-9-4	1526.75	1997-8-1	899.47	1987-8-3	329.8
2007-8-1	1473.99	1997-7-1	954.31	1987-7-1	318.66
2007-7-2	1455.27	1997-6-2	885.14	1987-6-1	304
2007-6-1	1503.35	1997-5-1	848.28	1987-5-1	290.1
2007-5-1	1530.62	1997-4-1	801.34	1987-4-1	288.36
2007-4-2	1482.37	1997-3-3	757.12	1987-3-2	291.7
2007-3-1	1420.86	1997-2-3	790.82	1987-2-2	284.2
2007-2-1	1406.82	1997-1-2	786.16	1987-1-2	274.08
2007-1-3	1438.24	1996-12-2	740.74	1986-12-1	242.17
2006-12-1	1418.3	1996-11-1	757.02	1986-11-3	249.22
2006-11-1	1400.63	1996-10-1	705.27	1986-10-1	243.98
2006-10-2	1377.94	1996-9-3	687.33	1986-9-2	231.32
2006-9-1	1335.85	1996-8-1	651.99	1986-8-1	252.93
2006-8-1	1303.82	1996-7-1	639.95	1986-7-1	236.12
2006-7-3	1276.66	1996-6-3	670.63	1986-6-2	250.84
2006-6-1	1270.2	1996-5-1	669.12	1986-5-1	247.35
2006-5-1	1270.09	1996-4-1	654.17	1986-4-1	235.52
2006-4-3	1310.61	1996-3-1	645.5	1986-3-3	238.9
2006-3-1	1294.87	1996-2-1	640.43	1986-2-3	226.92
2006-2-1	1280.66	1996-1-2	636.02	1986-1-2	211.78
2006-1-3	1280.08	1995-12-1	615.93	1985-12-2	211.28
2005-12-1	1248.29	1995-11-1	605.37	1985-11-1	202.17
2005-11-1	1249.48	1995-10-2	581.5	1985-10-1	189.82
2005-10-3	1207.01	1995-9-1	584.41	1985-9-3	182.08
2005-9-1	1228.81	1995-8-1	561.88	1985-8-1	188.63
2005-8-1	1220.33	1995-7-3	562.06	1985-7-1	190.92
2005-7-1	1234.18	1995-6-1	544.75	1985-6-3	191.85
2005-6-1	1191.33	1995-5-1	533.4	1985-5-1	189.55
2005-5-2	1191.5	1995-4-3	514.71	1985-4-1	179.83
2005-4-1	1156.85	1995-3-1	500.71	1985-3-1	180.66
2005-3-1	1180.59	1995-2-1	487.39	1985-2-1	181.18
2005-2-1	1203.6	1995-1-3	470.42	1985-1-2	179.63
2005-1-3	1181.27	1994-12-1	459.27	1984-12-3	167.24
2004-12-1	1211.92	1994-11-1	453.69	1984-11-1	163.58

2004-11-1	1173.82	1994-10-3	472.35	1984-10-1	166.09
2004-10-1	1130.2	1994-9-1	462.71	1984-9-4	166.1
2004-9-1	1114.58	1994-8-1	475.49	1984-8-1	166.68
2004-8-2	1104.24	1994-7-1	458.26	1984-7-2	150.66
2004-7-1	1101.72	1994-6-1	444.27	1984-6-1	153.18
2004-6-1	1140.84	1994-5-2	456.5	1984-5-1	150.55
2004-5-3	1120.68	1994-4-4	450.91	1984-4-2	160.05
2004-4-1	1107.3	1994-3-1	445.77	1984-3-1	159.18
2004-3-1	1126.21	1994-2-1	467.14	1984-2-1	157.06
2004-2-2	1144.94	1994-1-3	481.61	1984-1-3	163.41
2004-1-2	1131.13	1993-12-1	466.45	1983-12-1	164.93
2003-12-1	1111.92	1993-11-1	461.79	1983-11-1	166.4
2003-11-3	1058.2	1993-10-1	467.83	1983-10-3	163.55
2003-10-1	1050.71	1993-9-1	458.93	1983-9-1	166.07
2003-9-2	995.97	1993-8-2	463.56	1983-8-1	164.4
2003-8-1	1008.01	1993-7-1	448.13	1983-7-1	162.56
2003-7-1	990.31	1993-6-1	450.53	1983-6-1	167.64
2003-6-2	974.5	1993-5-3	450.19	1983-5-2	162.39
2003-5-1	963.59	1993-4-1	440.19	1983-4-4	164.43
2003-4-1	916.92	1993-3-1	451.67	1983-3-1	152.96
2003-3-3	848.18	1993-2-1	443.38	1983-2-1	148.06
2003-2-3	841.15	1993-1-4	438.78	1983-1-3	145.3
2003-1-2	855.7	1992-12-1	435.71	1982-12-1	140.64
2002-12-2	879.82	1992-11-2	431.35	1982-11-1	138.53
2002-11-1	936.31	1992-10-1	418.68	1982-10-1	133.72
2002-10-1	885.76	1992-9-1	417.8	1982-9-1	120.42
2002-9-3	815.28	1992-8-3	414.03	1982-8-2	119.51
2002-8-1	916.07	1992-7-1	424.21	1982-7-1	107.09
2002-7-1	911.62	1992-6-1	408.14	1982-6-1	109.61
2002-6-3	989.82	1992-5-1	415.35	1982-5-3	111.88
2002-5-1	1067.14	1992-4-1	414.95	1982-4-1	116.44
2002-4-1	1076.92	1992-3-2	403.69	1982-3-1	111.96
2002-3-1	1147.39	1992-2-3	412.7	1982-2-1	113.11
2002-2-1	1106.73	1992-1-2	408.78	1982-1-4	120.4
2002-1-2	1130.2	1991-12-2	417.09	1981-12-1	122.55
2001-12-3	1148.08	1991-11-1	375.22	1981-11-2	126.35
2001-11-1	1139.45	1991-10-1	392.45	1981-10-1	121.89
2001-10-1	1059.78	1991-9-3	387.86	1981-9-1	116.18
2001-9-4	1040.94	1991-8-1	395.43	1981-8-3	122.79
2001-8-1	1133.58	1991-7-1	387.81	1981-7-1	130.92
2001-7-2	1211.23	1991-6-3	371.16	1981-6-1	131.21

2001-6-1	1224.38	1991-5-1	389.83	1981-5-1	132.59
2001-5-1	1255.82	1991-4-1	375.34	1981-4-1	132.81
2001-4-2	1249.46	1991-3-1	375.22	1981-3-2	136
2001-3-1	1160.33	1991-2-1	367.07	1981-2-2	131.27
2001-2-1	1239.94	1991-1-2	343.93	1981-1-2	129.55
2001-1-2	1366.01	1990-12-3	330.22	1980-12-1	135.76
2000-12-1	1320.28	1990-11-1	322.22	1980-11-3	140.52
2000-11-1	1314.95	1990-10-1	304	1980-10-1	127.47
2000-10-2	1429.4	1990-9-4	306.05	1980-9-2	125.46
2000-9-1	1436.51	1990-8-1	322.56	1980-8-1	122.38
2000-8-1	1517.68	1990-7-2	356.15	1980-7-1	121.67
2000-7-3	1430.83	1990-6-1	358.02	1980-6-2	114.24
2000-6-1	1454.6	1990-5-1	361.23	1980-5-1	111.24
2000-5-1	1420.6	1990-4-2	330.8	1980-4-1	106.29
2000-4-3	1452.43	1990-3-1	339.94	1980-3-3	102.09
2000-3-1	1498.58	1990-2-1	331.89	1980-2-1	113.66
2000-2-1	1366.42	1990-1-2	329.08	1980-1-2	114.16
2000-1-3	1394.46	1989-12-1	353.4	1979-12-3	107.94
1999-12-1	1469.25	1989-11-1	345.99	1979-11-1	106.16
1999-11-1	1388.91	1989-10-2	340.36	1979-10-1	101.82
1999-10-1	1362.93	1989-9-1	349.15	1979-9-4	109.32
1999-9-1	1282.71	1989-8-1	351.45	1979-8-1	109.32
1999-8-2	1320.41	1989-7-3	346.08	1979-7-2	103.81
1999-7-1	1328.72	1989-6-1	317.98	1979-6-1	102.91
1999-6-1	1372.71	1989-5-1	320.52	1979-5-1	99.08
1999-5-3	1301.84	1989-4-3	309.64	1979-4-2	101.76
1999-4-1	1335.18	1989-3-1	294.87	1979-3-1	101.59
1999-3-1	1286.37	1989-2-1	288.86	1979-2-1	96.28
1999-2-1	1238.33	1989-1-3	297.47	1979-1-2	99.93
1999-1-4	1279.64	1988-12-1	277.72	1978-12-3	96.11
1998-12-1	1229.23				

Table 4.7: S&P 500 Index history

Date	Closing price of S&P 500	Implied volatility of call option	Implied volatility of put option	Closing value of CBOT volatility index	10-year treasury note

2007-12-3	1468.36	0.23514	0.23089	0.225	0.0403
2007-11-1	1481.14	0.20716	0.24245	0.2287	0.0397
2007-10-1	1549.38	0.17369	0.17238	0.1853	0.0447
2007-9-4	1526.75	0.20848	0.22611	0.18	0.0458
2007-8-1	1473.99	0.24274	0.21858	0.2338	0.0454
2007-7-2	1455.27	0.13712	0.14425	0.2352	0.0477
2007-6-1	1503.35	0.11317	0.115	0.1623	0.0503
2007-5-1	1530.62	0.11562	0.11982	0.1305	0.0489
2007-4-2	1482.37	0.12445	0.12542	0.1422	0.0463
2007-3-1	1420.86	0.1723	0.14417	0.1464	0.0465
2007-2-1	1406.82	0.09108	0.09677	0.1542	0.0455
2007-1-3	1438.24	0.10676	0.12415	0.1042	0.0483
2006-12-1	1418.3	0.10185	0.11285	0.1156	0.0471
2006-11-1	1400.63	0.10346	0.10988	0.1091	0.0446
2006-10-2	1377.94	0.12776	0.11455	0.111	0.0461
2006-9-1	1335.85	0.11475	0.11198	0.1198	0.0463
2006-8-1	1303.82	0.13487	0.13541	0.1231	0.0473
2006-7-3	1276.66	0.11726	0.1212	0.1495	0.0499
2006-6-1	1270.2	0.12894	0.12596	0.1308	0.0514
2006-5-1	1270.09	0.10896	0.10917	0.1644	0.0511
2006-4-3	1310.61	0.12259	0.10704	0.1159	0.0507
2006-3-1	1294.87	0.11286	0.10963	0.1139	0.0485
2006-2-1	1280.66	0.11202	0.12587	0.1234	0.0455
2006-1-3	1280.08	0.10437	0.10812	0.1295	0.0453
2005-12-1	1248.29	0.10925	0.10168	0.1207	0.0439
2005-11-1	1249.48	0.12048	0.12255	0.1206	0.045
2005-10-3	1207.01	0.11782	0.11427	0.1532	0.0456
2005-9-1	1228.81	0.12862	0.11509	0.1192	0.0433
2005-8-1	1220.33	0.10746	0.10748	0.126	0.0402
2005-7-1	1234.18	0.11992	0.10856	0.1157	0.0429
2005-6-1	1191.33	0.11155	0.11202	0.1204	0.0394
2005-5-2	1191.5	0.13503	0.1376	0.1329	0.0401
2005-4-1	1156.85	0.1278	0.1299	0.1531	0.042
2005-3-1	1180.59	0.11664	0.11013	0.1402	0.045
2005-2-1	1203.6	0.10239	0.10916	0.1208	0.0436
2005-1-3	1181.27	0.12562	0.12843	0.1282	0.0413

Table 4.8: implied volatility, volatility index, yields of 10-year treasury notes

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