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THE DIPOLE FIELD AND THE EFFECTIVE POTENTIAL IN QUANTUM FIELD THEORY

by

©

RANDY KOBES

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ABSTRACT

We start this work by considering two models that are fairly important in the present understanding of weak-electromagnetic interactions: the abelian Higg's model and electromagnetism. A covariant quantization leads to the presence of unphysical degrees of freedom in both. The usual method of eliminating these unphysical modes is to use an indefinite metric, whereby these modes are given negative or zero norm, and thus do not enter into the physical space. We inquire into whether or not these modes could be eliminated by some physical considerations if the quantization were to be performed on a positive metric Hilbert space. Restricted to that part of the Hilbert space where Poincaré transformations are implemented unitarily, we find that the models do contain the correct physical degrees of freedom if a certain gauge invariance is also imposed. However, if this gauge invariance is not postulated, it is found that the S operator for a coupling to an external c-number source is not unitary. In order to have a physically sensible theory, then, one is forced to impose gauge invariance, and the resulting models do contain the usual physical degrees of freedom.

We next consider quantum fluctuations in field theory, examining in particular the generating functional of one-particle-irreducible functions. We concentrate on the first two terms of the expansion of this functional in coordinate space: the effective potential $V(\phi)$ and a function $Z(\phi)$ that is related to the wave-function renormalization counterterm. We show that using the theory where the fields are shifted by a constant amount leads to a fairly efficient evaluation of these two functions. As well, the counterterms and hence the renormalization group equation coefficients can also be obtained in a relatively efficient

manner using the shifted theory. The amount of labour involved in these calculations is considerably less than in the more direct methods, and compares favourably with the path-integral approach, although one deals here directly with the Lagrange density. Finally, we present some conclusions, and point out some unresolved problems concerning the effective potential.

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NOTATIONS

We will use the following notations.

Lorentz indices, running from 0 to 3, will be denoted by Greek letters, while Latin indices run from 1 to 3, unless otherwise noted. Summation over repeated indices of either kind will always be implied.

A four vector B_μ is denoted by

$$B_\mu = (B_o, B_k) .$$

Indices are raised and lowered by the metric tensor $g_{\mu\nu}$, which has the form

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) .$$

A product of two four-vectors in this notation is then

$$B \cdot C = g_{\mu\nu} B^\mu C^\nu = B^o C_o + B^k C_k = B_o C_o - \vec{B} \cdot \vec{C} .$$

Derivatives are indicated by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} ,$$

although sometimes derivatives with respect to time are denoted by a dot:

$$\partial_t A = \dot{A} , \quad \partial_t^2 A = \ddot{A} , \ldots$$

The d'Alembertian symbol $\Box$ is given by

$$\Box = \partial^\mu \partial_\mu = \partial_t^2 - \vec{\nabla}^2 .$$

Poincaré transformations are generated by the operators $U(a,\Lambda)$, where a indicates a space-time translation and Λ the Lorentz transformation.

e.g. for a scalar field $\phi(x)$,

$$U(a,\Lambda)\phi(x)U^{-1}(a,\Lambda) = \phi(\Lambda x - a)$$

for a vector field $A_\mu(x)$,

$$U(a,\Lambda)A_\mu(x)U^{-1}(a,\Lambda) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x - a) .$$

The operators $U(a,\Lambda)$ are explicitly given by

$$U(a,\Lambda) = \exp(-ia_\mu P^\mu - i\epsilon_{\mu\nu} M^{\mu\nu})$$

where

$$\epsilon_{00} = \epsilon_{11} = 0 \,, \qquad \epsilon_{01} = -\epsilon_{10} = 1$$

$$P_\mu = \int d^3x \, T_{0\mu}$$

$$M_{\mu\nu} = \int d^3x [T_{0\mu} x_\nu - T_{0\nu} x_\mu]$$

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi^a} \partial_\nu \phi^a - g_{\mu\nu} \mathcal{L}$$

Finally, natural units, with $\hbar = c = 1$, are used throughout.

CHAPTER I

INTRODUCTION

Of all applications of quantum field theory, the interaction of matter with an electromagnetic field is the most tested and most successful. This is partly due to the fact that such processes are accessible at fairly low energies, and as such there is a strong classical analogue upon which to base the theory. An important concept to emerge from this theory is that of virtual particles, the exchange of which electromagnetic interactions proceed. For example, electron-positron scattering is accomplished by exchange of a virtual photon, while photon-photon scattering proceeds via an exchange of a virtual electron-positron pair.

On the other hand, the understanding of weak interactions took longer to develop. An early attempt by Fermi involved leptons and hadrons interacting at a single space-time point. However, the theory was not renormalizable: divergences that occur in higher order diagrams could not be removed with a finite number of counterterms. Faced with this, Gell-Mann and Feynman proposed that, in analogy with electromagnetism, weak interactions be mediated by exchange of a new type of particle. Because weak interactions are short range, this had to be a massive particle, and it turned out to be necessary for it to have spin one. If such a particle were neutral, the theory would indeed be renormalizable, but unfortunately charged particles were also necessary to account for certain reactions, and these theories were not renormalizable. Although the existence of such mediating particles in a sense unifies electromagnetic and weak interactions, it was many years before renormalizable theories of this type were finally produced. (For a more thorough review of these early attempts at explaining the weak interactions, see [M2] and [B1].)

An improbable beginning to the development of these theories occurred in 1954 when Yang and Mills discovered non-abelian gauge theories [Y1]. These again were built in analogy with electromagnetism, which is an abelian gauge theory. Consider a free charged scalar field described by the Lagrange density

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \quad . \tag{1.1}$$

This is invariant under the replacement

$$\phi \to e^{i\theta} \phi \tag{1.2}$$

if θ is a constant. If we demand (1.1) be invariant under the replacement (1.2) when θ is space-time dependent, it is necessary to modify (1.1) to

$$\mathcal{L} = [(\partial_\mu - ieA_\mu)\phi]^\dagger [(\partial^\mu - ieA^\mu)\phi] - m^2 \phi^\dagger \phi \quad , \tag{1.3}$$

with the four-vector A_μ transforming under (1.2) as

$$A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \theta \; . \tag{1.4}$$

Adding the invariant kinetic term for A_μ,

$$\mathcal{L}_A = -\frac{1}{4}(F_{\mu\nu})^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \quad , \tag{1.5}$$

it is seen that the resultant theory describes a charged scalar field minimally coupled to the electromagnetic field. The vector potential is, in this context, called the gauge field. The modifications required in order that (1.1) remain invariant when the non-abelian transformation (1.2) is space-time dependent were performed by Yang and Mills.

A seemingly even more remote contribution to this subject was made with the discovery of spontaneously broken symmetries [G4, G5, N4, A3]. Suppose we have a theory whose Lagrangian is invariant under a transformation $\phi(x) \to \phi'(x)$. By Noether's theorem, there will then exist a conserved current $J_\mu(x)$. Consider now a region of space-time R. The charge

$$Q(R) = \int d^3x \; J_o(x)\theta(\vec{x}) \;, \qquad \theta(\vec{x}) = 1 \text{ in } R \;, \tag{1.6}$$

generates the transformation of the field:

$$\phi'(x) = e^{iQ(R)} \phi(x) e^{-iQ(R)} \;, \qquad x \in R \;. \tag{1.7}$$

This symmetry is said to be spontaneously broken if

$$(\Omega, \phi(x)\Omega) \neq (\Omega, \phi'(x)\Omega) \;, \tag{1.8}$$

where Ω is the ground state. In such a case, the charge does not annihilate the vacuum.

An example of such a theory is the following. Consider two real scalar fields with Lagrange density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{24} g(\phi_1^2 + \phi_2^2 - a^2)^2 \;. \tag{1.9}$$

This is invariant under the transformation

$$\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta \;, \qquad \phi_2 \to -\phi_1 \sin\theta + \phi_2 \sin\theta \tag{1.10}$$

with θ a constant.

Now, to lowest order, the stable ground state of a system is determined by the minimum of the classical potential, the negative sum of all non-derivative terms in the Lagrange density. At this point, the classical energy density is minimized. In the model (1.9) with $a^2 < 0$, this minimum occurs at, to lowest order,

$$(\Omega, \phi_1 \Omega) = 0 = (\Omega, \phi_2 \Omega) \quad , \tag{1.11}$$

and the theory describes two massive interacting scalar fields. However, if $a^2 > 0$, the stable ground state, again to lowest order, occurs at

$$(\Omega, \phi_1 \Omega)^2 + (\Omega, \phi_2 \Omega)^2 = a^2 \quad . \tag{1.12}$$

The development of a non-zero vacuum expectation value of a field is a signal for the spontaneous breakdown of symmetry. We are free to choose, to satisfy (1.12),

$$(\Omega, \phi_1 \Omega) = a \quad , \qquad (\Omega, \phi_2 \Omega) = 0 \quad . \tag{1.13}$$

In order to have a particle interpretation, the fields must have a zero vacuum expectation value. We define such fields as

$$\psi = \phi_1 - a \quad , \qquad \phi = \phi_2 \quad , \tag{1.14}$$

with the resultant Lagrange density reading

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{24} g (\psi^2 + 2a\psi + \phi^2)^2 \quad . \tag{1.15}$$

The situation now is quite different. The ψ field has acquired a mass, while the ϕ field is massless. The presence of a massless scalar particle is a general result whenever a continuous symmetry is spontaneously broken, a result proved by what is known as Goldstone's theorem [G5, E2, K2, S3].

Since such theories predict massless Goldstone particles, it was doubted the concept of spontaneously broken symmetries would prove useful in relativistic quantum field theory, since no such particle exists in nature (at least, in that part of nature the theory wishes to describe). This was not to be the case. A curious result occurs when there are gauge fields present. Let us extend the model (1.9) to include an abelian gauge field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{24}g(\phi_1^2 + \phi_2^2 - a^2)^2 . \quad (1.16)$$

As before, if $a^2 > 0$, spontaneous symmetry breaking occurs. We again choose ϕ_1 to develop a nonzero vacuum expectation value, and define, to lowest order, the shifted fields

$$\psi = \phi_1 - a \quad , \quad \phi = \phi_2 \quad . \quad (1.17)$$

The Lagrange density becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi - eA_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\phi + eA_\mu\psi + eaA_\mu)^2 - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{24}g(\psi^2 + 2a\psi + \phi^2)^2 . \quad (1.18)$$

We see that a mass term $m = ea$ for the photon has appeared. The pleasant feature of this apparent round-about way of giving a mass to the photon is that the resulting theory, whether for charged or neutral gauge fields, is renormalizable [H5, H6, L2, L3, S4, T1, B2], as opposed to the theory with an explicit mass term present. This so-called Higg's mechanism [E1, H1, H2, H3, G7, K4, S6], together with non-abelian gauge fields, plays a major role in the presently accepted model of weak-electromagnetic interactions, the Weinberg-Salam model [W1, S1].

Of course, we still have to contend with the problem of the presence of the massless Goldstone field ϕ. In fact, a crucial feature of the Higg's model (1.18) that makes it so important is that the Goldstone field appears in an unphysical sector of the Hilbert space, and does not constitute a physical particle. This presence of unphysical degrees of freedom also occurs in electromagnetism, described by the Lagrange density (1.5). The vector potential A_μ possesses four degrees of freedom, while it is known that the photon has only two independent helicity states. One method of eliminating the two unphysical degrees of freedom is choosing the Coulomb gauge [B4]:

$$A_o = 0 \quad , \qquad \vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{1.19}$$

In this formalism, A_μ is no longer a local covariant four vector, although of course the undesirable non-local, non-covariant terms cancel in expressions involving measurable quantities. To keep A_μ a local covariant object, a "gauge-fixing term" is added to the Lagrange density (1.5). One such term is $-\frac{\alpha}{2}(\partial\cdot A)^2$, resulting in the field equations

$$\Box A_\mu + (\alpha - 1)\partial_\mu \partial\cdot A = 0 \ . \tag{1.20}$$

For $\alpha \neq 0$, we have $\Box\partial\cdot A = 0$, so that A_μ satisfies a massless dipole equation:

$$\Box^2 A_\mu = 0 \ . \tag{1.21}$$

How the two unphysical modes are eliminated from the quantized version of this equation will be dealt with in chapter IV.

Such a dipole term also arises in the Higg's model. Consider the field equations resulting from the Lagrange density (1.18) to lowest order, with the gauge-fixing term $-\frac{\alpha}{2}(\partial\cdot A)^2$ added:

$$\Box A_\mu + (\alpha-1)\partial_\mu \partial\cdot A + m^2 A_\mu - m\partial_\mu \phi = 0$$

$$\Box \phi - m\partial\cdot A = 0 \qquad (1.22)$$

$$\Box \psi + \mu^2 \psi = 0$$

where

$$m = ea$$

$$\mu^2 = \frac{1}{3} a^2 g \quad .$$

Since again $\Box\partial\cdot A = 0$, we have ϕ as a massless dipole field:

$$\Box^2 \phi = 0 \quad . \qquad (1.23)$$

The field ψ, the so-called Higg's field, is a genuine physical degree of freedom, while a massive vector field contains only three degrees of freedom. The reduction of the six degrees of freedom contained in the Higg's model (1.22) to four physical ones will be the subject of chapter III.

Of course, the elimination of unphysical modes in these models has been studied in the past. One successful approach used is the quantization of the fields on an indefinite metric space [N1, N2]. On these spaces, states of zero and negative norm are allowed, in contrast to a Hilbert space which contains only positive norm states. The unphysical degrees of freedom are then precisely those of negative and zero norm.

Such a formulation of these theories is completely consistent and is an elegant method of removing unphysical modes. However, unlike a Hilbert space, infinite dimensional indefinite metric spaces are not on a firm mathematical basis. A primary reason that such spaces are chosen, apart from the fact that they eliminate unphysical modes, is that hermiticity and unitarity, concepts employed in Hilbert spaces, are replaced by the similar concepts of pseudo-hermiticity and pseudo-unitarity (hermiticity and unitarity with respect to the indefinite metric, respectively). However, since in an indefinite metric space, only the positive norm states can be physical states, it might be wondered if it is possible to perform the quantization on a positive metric Hilbert space, and then establish some criterion that selects out what constitutes a physical space. In other words, can the unphysical degrees of freedom somehow be eliminated in a positive metric quantization? The situation might be compared with, given a Coulomb-type gauge quantization of Maxwell's equations, asking whether the presence of a non-local, non-covariant vector potential is essential in the quantization procedure. The indefinite metric approach shows that it is not. Here we are asking if it is necessary to use an indefinite metric to perform a covariant, local quantization, and still be able to eliminate in a consistent manner the unphysical modes. The answer to this question constitutes a major part of this work.

Of course, choosing the positive definite metric is not without some complications: what was pseudo-hermitean and pseudo-unitary on an indefinite metric space will not be hermitean or unitary on the corresponding Hilbert space. This immediately gives some restrictions a possible physical space must satisfy: restricted to it, operators corresponding to observables must be hermitean, the Poincaré group must be a symmetry group (i.e., implemented unitarily), and the S-operator must be unitary to conserve probabilities. Such a space, if it exists, would then contain any possible physical degrees of freedom of the model. By these considerations, we shall see how the concepts of pseudo-hermiticity and pseudo-unitarity translate from an indefinite metric space into a Hilbert space.

As a prelude to electromagnetism and the Higg's model, in chapter II we study the free massless dipole field. On an indefinite metric space, this field is pseudo-hermitean and Poincaré transformations are pseudo-unitarily implemented. No physical modes exist if the field is quantized using an indefinite metric. Using a positive definite metric, we find that the field is neither hermitean nor the Poincaré group implemented unitarily on the whole space. However, a subspace does exist where the Poincaré group is implemented unitarily. It is then possible to define an object containing only one of the two degrees of freedom that, restricted to this subspace, is a normal canonical scalar field. If this extra physical degree of freedom is still present in the two models we wish to study, it would be very paradoxical, as it does not occur in an indefinite metric approach, nor indeed is it observable in nature.

We first study in chapter III how this extra physical mode manifests itself in the quantization of the linearized Higg's model. In addition to the Higg's particle and the three physical modes of a massive vector field, we find that the space where the Poincaré group is unitarily implemented contains the extra additional degree of freedom of the dipole field. However, the presence of this additional mode violates a restricted gauge invariance. Since at the free field level there is no cogent reason to postulate gauge invariance, we couple the fields to external sources. This leads to a factorizeable S-operator; the part corresponding to the physical dipole degree of freedom is not unitary. We are thereby forced to adopt the principle of gauge invariance in order that the S-operator remain unitary and probabilities be conserved. The physical dipole degree of freedom is eliminated, leaving in the physical space four degrees of freedom.

We next consider the quantization of electromagnetism in chapter IV. Here we find there are actually two spaces where Poincaré transformations are implemented unitarily: one has one physical degree of freedom, the other two. However, only on the latter space are observables invariant under a restricted gauge transformation. As before, we couple the field to an external source, and again find that in order to have a unitary S-operator, we must choose the physical space to be the gauge invariant one. This space contains only two transverse photons.

In chapter V we explore a different aspect of the Higg's model, and of spontaneous breakdown of symmetry in general. In order for a symmetry to be spontaneously broken, at least one field in the theory must develop a nonzero vacuum expectation value. This value is fairly easy to calculate to lowest order, as it is just the minimum of the classical potential. However, how are quantum effects to be included? A function known as the effective potential allows one to calculate these effects in a systematic manner. Two methods are generally used to evaluate this function perturbatively: a direct infinite summation of graphs and a path-integral approach. In this chapter we describe a third method that is comparable in difficulty to the path-integral approach. In the course of setting up the formalism in which the effective potential is defined, we shall run across a function $Z(\phi)$ that is related to the wave-function renormalization of the fields. This function also has been computed by methods similar to those used for the effective potential. We will then describe a third method for its evaluation. Finally we show that the methods employed for computing the effective potential and $Z(\phi)$ lead to a fairly efficient evaluation of the counterterms of a theory, from which the coefficients of the renormalization group equations can be obtained.

A summary and conclusions are contained in chapter VI. As well, some potentially serious problems concerning the use of the effective potential in physically realistic theories are pointed out.

CHAPTER II

THE DIPOLE FIELD

In this chapter we will study the free massless dipole field. This field was first introduced in the hope that it could eliminate the ultraviolet divergences in quantum field theory [B3, T2]. However, Pais and Uhlenbeck pointed out that in these theories the energy is not hounded from below [P1]. This indicated an indefinite metric will be encountered upon quantization. More recently this field has turned up in gauge theories of gravitation [P2], in models where quark confinement is explicit [B5, K3, K5], and in the Schwinger model [C2], as well as in the Higg's model and electromagnetism.

As opposed to the massive dipole field [F4, L6, N1], the Fock space construction of the massless version is not so straightforward. The general solution in coordinate space is known [F1], and an indefinite metric quantization has been performed [K5]. This latter paper is interesting, in that it shows there are difficulties in giving a physical interpretation to a theory one would have thought elementary: a scalar field interacting quartically with a dipole field. This seems to show physically sensible theories containing dipole fields have to be judged case-by-case.

We start with the classical dipole field, but with an eye towards quantization we decompose the two degrees of freedom in a non-covariant manner. We then canonically quantize the model, and define a positive definite scalar product. The Poincaré group is found to be implemented unitarily on only a subspace of the full Hilbert space. It is then possible, restricted to this subspace, to define a hermitean scalar field that contains one of the two original degrees of freedom. Although this subspace does not carry a unitary representation of a certain restricted "gauge" transformation, the question of whether or not one can associate a physical particle with this degree of freedom has to be postponed until interactions are introduced.

A. The Classical Theory

In this section we introduce the model and study its main properties at the classical (c-number) level. We begin by giving the general wave packet solution of the field equations. Next we examine the transformation properties of the Fourier amplitudes under the Poincaré group and compute the corresponding Noether charges. Finally, we give a physical interpretation to this model at the classical level.

The fundamental variables we will use are two real scalar fields ϕ and χ, whose dynamics are specified through the Lagrange density

$$\mathcal{L} = \xi\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}\lambda^2\chi^2 - \partial_\mu\chi\partial^\mu\phi \tag{2.1}$$

where ξ is dimensionless and real and λ has dimensions of mass and is positive.

The corresponding field equations read

$$\begin{aligned} &\Box\chi = 0 \\ &\Box\phi = \lambda^2\chi \Longrightarrow \Box^2\phi = 0 \ . \end{aligned} \tag{2.2}$$

These are independent of ξ, indicating we could set ξ to zero in (2.1). However, the canonical commutation relations will contain ξ, and it turns out to be useful to keep it arbitrary.

Equations (2.2) with initial data $\phi(t_o,\cdot)$, $\dot{\phi}(t_o,\cdot)$, $\chi(t_o,\cdot)$, $\dot{\chi}(t_o,\cdot)$ pose a Cauchy problem which is equivalent to that of a dipole field ψ satisfying

$$\Box^2\psi = 0 \tag{2.3}$$

with initial data

$$\psi(t_o,\cdot) = \phi(t_o,\cdot) \qquad \dot{\psi}(t_o,\cdot) = \dot{\phi}(t_o,\cdot)$$

$$\Box\psi(t_o,\cdot) = \lambda^2\chi(t_o,\cdot) \qquad \Box\dot{\psi}(t_o,\cdot) = \lambda^2\dot{\chi}(t_o,\cdot) \quad . \tag{2.4}$$

This leads to the global identification $\phi(x) = \psi(x)$ and $\chi(x) = \frac{1}{\lambda^2}\Box\psi(x)$. In terms of the Fourier transform,

$$\psi(x) = \int \frac{d^2k}{(2\pi)^2}\,\tilde{\psi}(k)\,e^{-ik\cdot x} \quad ,$$

the field equation (2.3) reads

$$(k^2)^2\,\tilde{\psi}(k) = 0 \quad . \tag{2.5}$$

The general solution of (2.5) can be composed of four arbitrary distributions in $\mathcal{S}'(R^3)$, $\tilde{\phi}_o(\pm,k)$ and $\tilde{\phi}_1(\pm,k)$ as follows:

$$\tilde{\psi}(k) = [\tilde{\phi}_o(+,\vec{k})\theta(k_o) + \tilde{\phi}_o(-,\vec{k})\theta(-k_o)]|\vec{k}|\delta(k^2)$$

$$- i\lambda\frac{\partial}{\partial k^o}\{[\tilde{\phi}_1(+,\vec{k})\theta(k_o) + \tilde{\phi}_1(-,\vec{k})\theta(-k_o)]|\vec{k}|\delta(k^2)\} \quad . \tag{2.6}$$

We restrict $\tilde{\phi}_o$ and $\tilde{\phi}_1$ to lie in $L^2(R^3)$ so as $\tilde{\psi}(k)$ has a pointwise meaning almost everywhere. The choice $\frac{\partial}{\partial k^o}\delta(k^2)$ is arbitrary; what is needed is an unambigous definition of "$\delta'(k^2)$" as an element of $\mathcal{S}'(R^4)$. We could equally as well use $n^\mu\frac{\partial}{\partial k^\mu}\delta(k^2)$, with n^μ timelike, but $n_\mu = \delta_{o\mu}$ turns out to be particularly useful for canonical quantization.

The question now arises: given a particular $\tilde{\psi}(k)$, are the $\tilde{\phi}_i$ uniquely determined? As well, do the class of solutions specified by

(2.6) contain all the solutions of (2.5)? To answer this, we first perform the inverse Fourier transform of (2.6):

$$\phi_i(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|\vec{k}|} \{a_i(k)e^{-ik\cdot x} + \bar{a}_i(k)e^{ik\cdot x}\} \tag{2.7}$$

where

$$k_o = +|\vec{k}|$$

$$A(k) = \begin{pmatrix} a_o(k) \\ a_1(k) \end{pmatrix} = \frac{|\vec{k}|}{\sqrt{2\pi}} \begin{pmatrix} \tilde{\phi}_o(+,\vec{k}) \\ \tilde{\phi}_1(+,\vec{k}) \end{pmatrix}$$

$$\bar{A}(k) = (\bar{a}_o(k), \bar{a}_1(k)) = \frac{|\vec{k}|}{\sqrt{2\pi}} (\tilde{\phi}_o(-,-\vec{k}), \tilde{\phi}_1(-,-\vec{k})) .$$

In coordinate space, the solution (2.6) becomes

$$\psi(x) = \phi_o(x) + \lambda x_o \phi_1(x) , \tag{2.8}$$

where ϕ_o and ϕ_1 both satisfy the massless wave equation and belong to $L^2(R^3)$.

We can now express the Cauchy data for ψ in terms of that for ϕ_i as follows:

$$\psi(t,\vec{x}) = \phi_o(t,\vec{x}) + \lambda t \phi_1(t,\vec{x})$$

$$\dot{\psi}(t,\vec{x}) = \dot{\phi}_o(t,\vec{x}) + \lambda\phi_1(t,\vec{x}) + \lambda t\dot{\phi}_1(t,\vec{x})$$

$$\Box\psi(t,\vec{x}) = 2\lambda\dot{\phi}_1(t,\vec{x})$$

$$\Box\dot{\psi}(t,\vec{x}) = 2\lambda\nabla^2\phi_1(t,\vec{x}) \tag{2.9}$$

If $\psi(t,\vec{x})$ is an element of $\mathcal{S}(R^3)$ for fixed t, equations (2.9) can be inverted. The inversion reads

$$\phi_1(t,\vec{x}) = \frac{1}{2\lambda}\int \frac{d^3y}{4\pi|\vec{x}-\vec{y}|}\ \Box\dot{\psi}(t,\vec{y})$$

$$\dot{\phi}_1(t,\vec{x}) = \frac{1}{2\lambda}\ \Box\psi(t,\vec{x})$$

$$\phi_o(t,\vec{x}) = \psi(t,\vec{x}) - \lambda t\phi_1(t,\vec{x})$$

$$\dot{\phi}_o(t,\vec{x}) = \dot{\psi}(t,\vec{x}) - \lambda\phi_1(t,\vec{x}) - \lambda t\dot{\phi}_1(t,\vec{x})\ . \qquad (2.10)$$

This shows that ϕ_o and ϕ_1 are globally and uniquely determined by the Cauchy data for ψ, demonstrating the sufficient generality of the solution (2.8). Also notice that, because ψ is a local scalar field, both ϕ_o and ϕ_1 are non-local and are not scalar fields. They do provide, however, a useful separation of the two degrees of freedom of ψ in any given set of Minkowski coordinates.

Next we examine the transformation properties of the objects ϕ_o and ϕ_1 under Poincaré transformations of the scalar field ψ. We will express these properties in terms of the Fourier amplitudes, and with a view to the quantized version, we do not use the "reality condition" $\bar{a} = a^{*t}$.

The Fourier amplitudes can be expressed in terms of the Cauchy data as follows:

$$a_o(\vec{k}) = (e^{-ik\cdot x},\psi(x))_{x_o=0} - \frac{i\lambda}{2|\vec{k}|}\{a_1(\vec{k}) + \bar{a}_1(-\vec{k})\}$$

$$a_1(\vec{k}) = \frac{i}{2\lambda|\vec{k}|}(e^{-ik\cdot x},\Box\psi(x))_{x_o}$$

$$\bar{a}_o(\vec{k}) = -(e^{ikx},\psi(x))_{x_o=0} + \frac{i\lambda}{2|\vec{k}|}\{a_1(-\vec{k}) + \bar{a}_1(\vec{k})\}$$

$$\bar{a}_1(\vec{k}) = \frac{i}{2\lambda|\vec{k}|} (e^{ik\cdot x}, \Box\psi(x))_{x_o} \tag{2.11}$$

where

$$(f,g)_{x_o} = \frac{i}{(2\pi)^{3/2}} \int d^3x\, f^*(x_o,\vec{x})(\vec{\partial}_o - \overleftarrow{\partial}_o) g(x_o,\vec{x})$$

is the usual Klein-Gordon scalar product.

The Poincaré group forms an invariance group on the set of solutions of (2.3). Under the action of space-time transactions,

$$\psi(x) \to \hat{\psi}(x) = \psi(x-a)$$

the Fourier amplitudes transform as

$$\hat{A}(k) = e^{ik\cdot a} \begin{pmatrix} 1 & -\lambda a_o \\ 0 & 1 \end{pmatrix} A(k)$$

$$\hat{\bar{A}}(k) = e^{-ik\cdot a}\, \bar{A}(k) \begin{pmatrix} 1 & 0 \\ -\lambda a_o & 1 \end{pmatrix} \tag{2.12}$$

Under action of Lorentz transformations, we find

$$\psi(x) \to \hat{\psi}(x) = \psi(\Lambda x)$$

$$\hat{A}(k) = \vec{D}(\Lambda,k) A(\Lambda k)$$

$$\hat{\bar{A}}(k) = \bar{A}(\Lambda k)\, \overleftarrow{D}^{\dagger}(\Lambda,k) \tag{2.13}$$

where

$$\vec{D}(\Lambda,\vec{k}) = \begin{pmatrix} 1 & -i\lambda|\vec{k}|(\Lambda)_o^{\ell} \frac{\vec{\partial}}{\partial k^{\ell}} \left(\frac{1}{|\vec{k}|}\right) \\ 0 & \frac{(\Lambda k)_o}{k_o} \end{pmatrix}$$

We see that only under pure space rotations and translations do ϕ_o and ϕ_1 transform as scalar fields.

Because of the invariance of the theory under the Poincaré group, conservation laws arise. Space-time translational invariance yields the symmetric, conserved canonical tensor $K_{\mu\nu}$.

$$K_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^{\mu}\phi} \partial_{\nu}\phi + \frac{\partial \mathcal{L}}{\partial \partial^{\mu}\phi} \partial_{\nu}\chi - g_{\mu\nu} \mathcal{L} = \xi U_{\mu\nu} + V_{\mu\nu} \tag{2.14}$$

where

$$U_{\mu\nu} = \partial_{\mu}\chi\partial_{\nu}\chi - \frac{1}{2} g_{\mu\nu}\partial_{\sigma}\chi\partial^{\sigma}\chi$$

$$V_{\mu\nu} = -\partial_{\mu}\chi\partial_{\nu}\phi - \partial_{\nu}\chi\partial_{\nu}\phi + g_{\mu\nu}\{\frac{1}{2} \lambda^2\chi^2 + \partial_{\sigma}\chi\partial^{\sigma}\phi\} \;.$$

Both $U_{\mu\nu}$ and $V_{\mu\nu}$ are symmetric and conserved, but since only U_{oo} is positive definite, only $U_{\mu\nu}$ can be considered as a viable candidate for an energy-momentum tensor (at this classical level). In terms of the Fourier amplitudes, the Hamiltonian and 3-momentum read

$$H = \int d^3x \, K_{oo} = \frac{1}{\lambda^2} \int d^3k |\vec{k}| \bar{A}(k) \begin{pmatrix} 0 & i\lambda \\ -i\lambda & 2\xi|\vec{k}| + \frac{2\lambda^2}{|\vec{k}|} \end{pmatrix} A(k) \tag{2.15}$$

$$P_{\ell} = \int d^3x \, K_{o\ell} = \frac{1}{\lambda^2} \int d^3k \, k_{\ell} \, \bar{A}(k) \begin{pmatrix} 0 & i\lambda \\ -i\lambda & 2\xi|\vec{k}| + \frac{\lambda^2}{|\vec{k}|} \end{pmatrix} A(k) \;. \tag{2.16}$$

The invariance of the model under Lorentz transformations leads to the conserved density $K_{\mu\nu}x_{\sigma} - K_{\mu\sigma}x_{\nu}$, from which follows the Noether charge:

$$M_{\mu\nu} = \int d^3x \, \{ K_{o\mu}x_{\nu} - K_{o\nu}x_{\mu} \} \tag{2.17}$$

In terms of the Fourier amplitudes, the boosts read

$$M_{o\ell} = -\frac{i}{\lambda^2}\int d^3k \left\{ \partial_\ell \bar{A}(k) \begin{pmatrix} 0 & i\lambda|\vec{k}| \\ 0 & \xi\vec{k}^2+\lambda^2 \end{pmatrix} A(k) \right.$$

$$\left. - \bar{A}(k) \begin{pmatrix} 0 & 0 \\ -i\lambda|\vec{k}| & \xi\vec{k}^2+\lambda^2 \end{pmatrix} \partial_\ell A(k) \right\} , \qquad (2.18)$$

while the pure spatial rotations become

$$M_{m\ell} = -\frac{i}{\lambda^2}\int d^3k \; k_m \left\{ \partial_\ell \bar{A}(k) \begin{pmatrix} 0 & i\lambda \\ 0 & -\beta/2 \end{pmatrix} A(k) \right.$$

$$\left. - \bar{A}(k) \begin{pmatrix} 0 & 0 \\ -i\lambda & -\beta/2 \end{pmatrix} \partial_\ell A(k) \right\} - (m \longleftrightarrow \ell) , \qquad (2.19)$$

where

$$\beta(\vec{k}) = -2\xi|\vec{k}| - \frac{\lambda^2}{|\vec{k}|} .$$

In addition to these invariances, the model is also invariant under the following local gauge transformation:

$$\chi \to \chi \quad , \qquad \phi \to \phi + \Lambda \qquad (2.20)$$

with

$$\Box \Lambda = 0 .$$

The conserved current corresponding to this invariance is

$$J_\mu(\Lambda) = \chi\partial_\mu\Lambda - \Lambda\partial_\mu\chi \qquad (2.21)$$

If we define

$$\Lambda(\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{2|\vec{k}|} \left\{ c(k)e^{-ik\cdot x} + \bar{c}(k)e^{ik\cdot x} \right\} ,$$

the Fourier amplitudes transform as follows:

$$a_o(k) \to a_o(k) + c(k) \qquad \bar{a}_o(k) \to \bar{a}_o(k) + \bar{c}(k)$$

$$a_1(k) \to a_1(k) \qquad a_1(k) \to a_1(k) \quad . \qquad (2.22)$$

The corresponding charge is

$$\begin{aligned} Q_\Lambda &= \int d^3x\, J_o(x,\Lambda) \\ &= \frac{1}{\lambda} \int d^3k \{\bar{c}(k) a_1(k) + \bar{a}_1(k) c(k)\} \quad . \end{aligned} \qquad (2.23)$$

By insisting that the field χ is real and that observables be real valued gauge invariant functionals of the fields, we can arrive at a physical interpretation of the model at the classical level. To any given solution (χ_o, ϕ_o) we associate the gauge equivalence class

$$<\chi_o> = \{(\phi,\chi) | \chi = \chi_o\} \quad . \qquad (2.24)$$

These classes represent the physical states of the model, elements within each class differing only by their nonobservable ϕ degree of freedom. Observables, which are real valued functionals of the field χ, would then be gauge invariant. The energy momentum tensor would then be the gauge invariant $U_{\mu\nu}$: it is symmetric, conserved, and U_{oo} is positive definite.

However, as we shall see in the next section, this particular interpretation of the classical model cannot be carried over to the quantized version.

B. The Quantized Version

In this section we develop an explicit Fock construction of the Hilbert space $\mathcal{H}$ carrying the free fields (ϕ,χ) as operator valued distributions obeying the field equations (2.2) and canonical equal time commutation relations (CETCRs) as initial conditions. The construction is carried out in three steps. First we derive some general properties of the fields based solely on their equations of motion and the CETCRs. We next construct a linear space L carrying a representation of these fields and investigate the spectrum of the generators of space-time translations on this space. Finally we define a class of positive definite nondegenerate scalar products $(\, , \,)_f$ on L. The vectors in L which have finite f-norm $||\phi||^2 = (\phi,\phi)_f < \infty$ form the Hilbert space $\mathcal{H}_f \subset$ L. We then investigate the unitarity of the operator representation of the Poincaré group on the spaces $\mathcal{H}_f$. Although time tranlsations and boosts are nonunitary on all of the spaces $\mathcal{H}_f$, we find there does exist exactly one space $\mathcal{H}$ in the family $\{\mathcal{H}_f\}$ which contains a nontrivial subspace upon which the whole Poincaré group acts unitarily. This distinguished subspace would then be a candidate for a reasonable physical subspace. We then investigate the possible physical interpretations for this quantized model.

We begin our program by imposing canonical equal time commutation relations among the fields ϕ and χ.

$$[\pi_\chi(t,\vec{x}),\chi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y})$$

$$[\pi_\phi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y}) \qquad (2.25)$$

where

$$\pi_\chi = \xi\dot{\chi} - \dot{\phi} \quad , \quad \pi_\phi = -\dot{\chi} \; .$$

All others are assumed to vanish. In terms of the dipole field ψ the only nonvanishing CETCRs corresponding to (2.25) are

$$[\dot{\psi}(t,\vec{x}),\psi(t,\vec{y})] = i\xi\delta(\vec{x}-\vec{y})$$

$$[\Box\psi(t,\vec{x}),\dot{\psi}(t,\vec{y})] = -i\lambda^2\delta(\vec{x}-\vec{y})$$

$$[\Box\dot{\psi}(t,\vec{x}),\psi(t,\vec{y})] = i\lambda^2\delta(\vec{x}-\vec{y}) \ , \qquad (2.26)$$

or, in terms of the quantities ϕ_o and ϕ_1

$$[\phi_o(t,\vec{x}),\phi_1(t,\vec{y})] = -\frac{i\lambda}{2}\,\frac{1}{4\pi|\vec{x}-\vec{y}|}$$

$$[\dot{\phi}_o(t,\vec{x}),\dot{\phi}_1(t,\vec{y})] = \frac{i\lambda}{2}\,\delta(\vec{x}-\vec{y})$$

$$[\dot{\phi}_o(t,\vec{x}),\phi_o(t,\vec{y})] = i\xi\delta(\vec{x}-\vec{y}) - \frac{i\lambda^2}{2}\,\frac{1}{4\pi|\vec{x}-\vec{y}|} \ . \qquad (2.27)$$

This reveals the nonlocal nature of ϕ_o and ϕ_1.

Let us pause here for a moment. One might wonder whether or not one could use the "standard" Cauchy initial data ψ, $\dot{\psi}$, $\ddot{\psi}$, $\dddot{\psi}$ in imposing CETCRs for just the pure dipole field ψ, without introducing the pair (ϕ,χ). Such a procedure is possible (see [K5]), but requires an extension of the usual canonical quantization rules to include Lagrange densities with higher order derivatives of the form

$$\mathcal{L} = (\partial_\mu\partial^\mu\psi)(\partial_\nu\partial^\nu\psi) \ . \qquad (2.28)$$

Such modifications are possible, [illegible] lead to the same physical results as the use of the Lagrange density [illegible] we employ.

By using (2.11) we [illegible] the nonvanishing commutation relations for the Fourier amplitu[illegible]

$$[A(k),\bar{A}(q)]_{ij} = [A_i(k),\bar{A}_j(q)] = C_{ij}(k)\delta(\vec{k}-\vec{q}) \qquad (2.29)$$

where

$$C(k) = \begin{pmatrix} \beta(k) & i\lambda \\ -i\lambda & 0 \end{pmatrix}$$

$$\beta(k) = -\frac{1}{|\vec{k}|}[2\xi|\vec{k}|^2 + \lambda^2] \quad .$$

The following combination of Fourier amplitudes leads to a diagonal commutator matrix:

$$B(k) = NMA(k) \qquad (2.30)$$

where

$$N = \sqrt{2|\vec{k}|}\begin{pmatrix} \sqrt{\varepsilon_+} & 0 \\ 0 & \sqrt{|\varepsilon_-|} \end{pmatrix}$$

$$M = \begin{pmatrix} \dfrac{\varepsilon_+}{\sqrt{\varepsilon_+^2 + \lambda^2}} & \dfrac{i\lambda}{\sqrt{\varepsilon_+^2 + \lambda^2}} \\ \dfrac{\varepsilon_-}{\sqrt{\varepsilon_-^2 + \lambda^2}} & \dfrac{i\lambda}{\sqrt{\varepsilon_-^2 + \lambda^2}} \end{pmatrix}$$

$$\varepsilon_\pm = \frac{1}{2}\{ \beta \pm \sqrt{\beta^2 + 4\lambda^2} \} \quad .$$

The result is

$$[B(k),\bar{B}(q)] = 2|\vec{k}|\eta\delta(\vec{k}-\vec{q}) \qquad (2.31)$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

The commutators for arbitrary times are, for the nonlocal objects ϕ_o and ϕ_1:

$$[\phi_o(x+y),\phi_o(y)] = \frac{i\xi}{2\pi}\,\varepsilon(x_o)\delta(x^2) + \frac{i\lambda^2}{8\pi}\,\{\varepsilon(x_o)\theta(x^2) + \frac{x_o}{|\vec{x}|}\,\theta(-x^2)\}$$

$$[\phi_o(x+y),\phi_1(y)] = \frac{i\lambda^2}{8\pi}\,\frac{\theta(-x^2)}{|\vec{x}|}$$

$$[\phi_1(x+y),\phi_1(y)] = 0\ , \tag{2.32}$$

while for the fields ϕ and χ:

$$[\phi(x+y),\phi(y)] = \frac{i\xi}{2\pi}\,\varepsilon(x_o)\delta(x^2) + \frac{i\lambda^2}{8\pi}\,\varepsilon(x_o)\theta(x^2)$$

$$[\phi(x+y),\chi(y)] = \frac{i}{2\pi}\,\varepsilon(x_o)\delta(x^2)$$

$$[\chi(x+y),\chi(y)] = 0\ . \tag{2.33}$$

Notice that the nonlocal, noncovariant terms present in (2.32) have disappeared in the expressions (2.33). The commutators (2.33) coincide with those quoted by Ferarri [F1], obtained using Poincaré invariance arguments.

In terms of the Fourier amplitudes B(k), the Hamiltonian and 3-momentum (2.16) become

$$H = \int \frac{d^2k}{2|\vec{k}|}\,\bar{B}(k)\,h'\,B(k) \tag{2.34}$$

where

$$h' = \frac{1}{|\vec{k}|\gamma}\begin{pmatrix} \lambda^2 + |\vec{k}|^2\gamma & \lambda^2 \\ \lambda^2 & \lambda^2 - |\vec{k}|^2\gamma \end{pmatrix}$$

$$\gamma = \frac{1}{|\vec{k}|}\sqrt{\beta^2 + 4\lambda^2}$$

and

$$P_m = \int \frac{d^3k}{2|\vec{k}|}\,k_m\,\bar{B}(k)\,\eta\,B(k)\ . \tag{2.35}$$

We now define a linear space L and an operator representation of the fields by the following postulate of cyclicity:

L contains exactly one element Ω (the vacuum) which is Poincaré invariant and obeys $B(k)\Omega = 0 \; \forall \; |\vec{k}|$. The linearity independent vectors $\bar{B}_{i_1}(k_1)\ldots\bar{B}_{i_n}(k_n)\Omega$, with $n = 0,1,\ldots\infty$ form a basis of L (of course, this assumes a meaning only after a scalar product on L is defined).

The assumption of a cyclic vacuum makes $\bar{B}$ a creation and B a destruction operator.

Having done this, we are now in a position to calculate the various two-point functions of the theory. As might be expected, however, we will encounter some infrared difficulties. Consider, for example, the following expression:

$$I = (\Omega, \int d^4x \, \phi_o(x+y) h(x) \phi_o(y)\Omega) \qquad (2.36)$$

where h(x) is a real valued test function from a subset of $\mathcal{S}(R^4)$ to be specified. Introducing the Fourier transform,

$$h(x) = \int \frac{d^4k}{(2\pi)^2} \tilde{h}(k) \, e^{-ik\cdot x} \quad ,$$

we arrive at

$$I = \frac{1}{8\pi} \int \frac{d^3k}{|\vec{k}|} \, \beta(k) \, \frac{\tilde{h}(k)}{|\vec{k}|} \quad . \qquad (2.37)$$

This integral exists only if $\tilde{h}(k)/|\vec{k}|$ is finite at $|\vec{k}| = 0$. For an arbitrary h(x) in $\mathcal{S}(R^4)$, a regularization of (2.37) must be performed (see [G3]). We choose the following scheme:

$$I_{reg} = \frac{1}{8\pi} \int_{|\vec{k}|<\mu} d^3k \, \frac{\beta(k)}{|\vec{k}|^2} \{\tilde{h}(k) - \tilde{h}(0)\} + \frac{1}{8\pi} \int_{|\vec{k}|>\mu} d^3k \, \frac{\beta(k)}{|\vec{k}|^2} \tilde{h}(k) \quad . \qquad (2.38)$$

With such a regularization scheme, we obtain the following results:

$$(\Omega,\phi_o(x+y)\phi_o(y)\Omega)_{reg} = \frac{\xi}{4\pi^2}\left\{\mathcal{P}(\frac{1}{x^2}) + i\pi\varepsilon(x_o)\delta(x^2)\right\} + \frac{i\lambda^2}{16\pi^2}\left\{\pi\varepsilon(x_o)\theta(x^2) + \frac{-x_o}{|\vec{x}|}\theta(-x^2) + 2\pi i(1-c-\ln\mu) + \frac{x_o}{|\vec{x}|}\ln\left|\frac{|x_o-|\vec{x}||}{|x_o+|\vec{x}||}\right| - 2\ln|x^2|\right\}$$

$$(\Omega,\phi_o(x+y)\phi_1(y)\Omega) = \frac{\lambda}{16\pi^2}\frac{1}{|\vec{x}|}\left\{\ln\left|\frac{|x_o+|\vec{x}||}{|x_o-|\vec{x}||}\right| + i\pi\theta(-x^2)\right\}$$

$$(\Omega,\phi_1(x+y)\phi_1(y)\Omega) = 0 \qquad (2.39)$$

where c is Euler's constant.

Notice that, as well as being noncovariant, these objects also display nonlocality. However, these undesirable terms drop out in the expressions for the fields ϕ and χ.

$$(\Omega,\phi(x+y)\phi(y)\Omega)_{reg} = \frac{\xi}{4\pi^2}\left\{\mathcal{P}(\frac{1}{x^2}) + i\pi\varepsilon(x_o)\delta(x^2)\right\} + \frac{\lambda^2}{16\pi^2}\left\{\ln|x^2| + i\pi\varepsilon(x_o)\theta(x^2)\right\}$$

$$(\Omega,\phi(x+y)\chi(y)\Omega)_{reg} = \frac{1}{4\pi^2}\left\{\mathcal{P}(\frac{1}{x^2}) + i\pi\varepsilon(x_o)\delta(x^2)\right\}$$

$$(\Omega,\chi(x+y)\chi(y)\Omega) = 0 \quad . \qquad (2.40)$$

We have dropped the constant terms. The corresponding time ordered functions are, again neglecting constants,

$$(\Omega,T\{\phi(x+y)\phi(y)\}\Omega)_{reg} = \frac{\xi}{4\pi^2}\left\{\mathcal{P}(\frac{1}{x^2}) + i\pi\delta(x^2)\right\} + \frac{\lambda^2}{16\pi^2}\left\{\ln|x^2| + i\pi\theta(x^2)\right\}.$$

$$(\Omega,T\{\phi(x+y)\chi(y)\}\Omega)_{reg} = \frac{1}{4\pi^2}\left\{\mathcal{P}(\frac{1}{x^2}) + i\pi\delta(x^2)\right\}$$

$$(\Omega,T\{\chi(x+y)\chi(y)\}\Omega) = 0 \quad . \qquad (2.41)$$

These results ((2.40) and (2.41)) are again the same as those found in [F1].

We now turn our attention to the spectrum of the space-time translation generators. From (2.35), it is seen that B ($\bar{B}$)-functions as a lowering (raising) operator with respect to P_m:

$$[P_m, B(k)] = -k_m B(k)$$
$$[P_m, \bar{B}(k)] = k_m \bar{B}(k) \quad (2.42)$$

However, they do not do so for the Hamiltonian (2.34):

$$[H, B(k)] = -\eta h' B(k)$$
$$[H, \bar{B}(k)] = \bar{B}(k)\, h' \eta \quad (2.43)$$

To find the linear combinations that do function as lowering and raising operators for H, we must solve the equation

$$[H, \upsilon^\dagger B(k)] = -\upsilon^\dagger \eta\, h' B(k) = -\mu\, \upsilon^\dagger B(k)$$
$$[H, \bar{B}(k)\, \upsilon] = \bar{B}(k)\, h' \eta\, \upsilon = \mu \bar{B}(k)\, \upsilon \quad (2.44)$$

The solution is given by

$$\mu = + |\vec{k}| \; , \qquad \upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad (2.45)$$

there being only one such combination stemming from the fact that the matrices $\eta h'$ and $h'\eta$ do not have a complete set of eigenvectors and are not diagonizable by a similarity transformation.

Contrary to what one might expect, the operators $\upsilon^\dagger B$ do not act as annihilation operators for the quanta created by $\bar{B}\upsilon$, as they commute:

$$[\upsilon^\dagger B(k), \bar{B}(q)\upsilon] = 0 \quad (2.46)$$

Instead, the $\upsilon^\dagger B(k)$ act as annihilation operators for the quanta created by $\bar{B}(k)v$ (where $v = \frac{1}{\sqrt{2}}\binom{1}{-1}$), since

$$[\upsilon^{+}B(k),\bar{B}(q)v] = 2|\vec{k}|\delta(\vec{k}-\vec{q}) . \quad (2.47)$$

Thus, the only eigenstates of the four-momentum P_μ are those states created by $\bar{B}(k)\upsilon$, and satisfy

$$P_\mu[\bar{B}_{i_1}(k_1)\upsilon]\dots[\bar{B}_{i_n}(k_n)\upsilon]\Omega = (\sum_{i=1}^{n} k_{\mu_i})[\bar{B}_{i_1}(k_1)\upsilon]\dots[\bar{B}_{i_n}(k_n)\upsilon]\Omega. \quad (2.48)$$

We shall call the subspace constructed by application of the operator $\bar{B}(k)\upsilon$ $\mathcal{H}'$.

Before going on to define a scalar product, let us see how such a theory would be handled using an indefinite metric. On would first define an indefinite sesquilinear form $<\ ,\ >$ through $\bar{B}(k) = B^{+}(k)$ and $<\Omega,\Omega> = 1$. As mentioned in the Introduction, the attractive feature of this form is that the Poincaré group turns out to be "unitary" relative to it (actually pseudo-unitary since the form is indefinite). Only the $\bar{B}(k)\upsilon = B^{+}(k)\upsilon$ quanta would then be candidates for physical particles, as they create the only eigenstates of P_μ. However, since these states have zero norm (see (2.46)), this leads to the conclusion that the theory lacks a Poincaré invariant probability interpretation. The philosophy we adopt here is that, having noticed only a part of the whole space has a chance of being physical, why cannot one adopt any form Poincaré invariant on a subspace of the whole space? In particular, we wish to explore the consequences of choosing the positive definite scalar product.

To proceed, we now introduce a family of positive definite scalar products $(\ ,\ )_f$ on L leading to a family of Hilbert spaces $\{\mathcal{H}_f\} \subset L$ of finite norm states. To do so we make the following postulates:

$$(\Omega,\Omega) = 1$$

$$\bar{B}(k) = f(k)B^{\dagger}(k) \qquad , \tag{2.49}$$

where f(k) is some positive real-valued function. Here, $B^{\dagger}$ means the hermitean adjoint of B with respect to $(\cdot,\cdot)_f$, and defines this scalar product implicitly. The commutator between B and $B^{\dagger}$ now reads

$$[B(k),B^{\dagger}(q)] = \frac{2|\vec{k}|}{f(k)}\,\delta(\vec{k}-\vec{q}) \quad . \tag{2.50}$$

The positive definiteness of the scalar product follows from the positivity of this commutator matrix.

Bearing in mind that the linear combination $\bar{B}(k)\,u$ serves as the only raising operator for the Hamiltonian, we now form the following objects:

$$G(k) = \begin{pmatrix} g_o(k) \\ g_1(k) \end{pmatrix} = \frac{f(k)}{\sqrt{\gamma}}\,S\,B(k)$$

$$G^{\dagger}(k) = (g_o^{\dagger}(k),g_1^{\dagger}(k)) = \frac{f(k)}{\sqrt{\gamma}}\,B^{\dagger}(k)\,S^{\dagger} \tag{2.51}$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad .$$

The commutator matrix in terms of G is given by

$$[G(k),G^{\dagger}(q)] = 2|\vec{k}|\,\delta(\vec{k}-\vec{q})\,\frac{f(k)}{\gamma(k)} \quad . \tag{2.52}$$

We now investigate the action of the Poincaré group on this space to see if a subspace exists where it acts unitarily. We begin with space-time translations $\psi(x) \to \hat{\psi}(x) = \psi(x-a)$:

$$\hat{G}(k) = e^{ik\cdot a} \begin{pmatrix} 1 & \frac{2i\lambda^2 a_o}{|\vec{k}|\gamma} \\ 0 & 1 \end{pmatrix} G(k)$$

$$\hat{G}^\dagger(k) = e^{-ik\cdot a}\, G^\dagger(k) \begin{pmatrix} 1 & \frac{-2i\lambda^2 a_o}{|\vec{k}|\gamma} \\ 0 & 1 \end{pmatrix} . \tag{2.53}$$

The generators of space-time translations, (2.34) and (2.35), now become

$$H = \int \frac{d^3k}{2|\vec{k}|}\, G^\dagger(k) \begin{pmatrix} |\vec{k}| & \frac{2\lambda^2}{|\vec{k}|\gamma} \\ 0 & |\vec{k}| \end{pmatrix} G(k) \tag{2.54}$$

$$P_m = \int \frac{d^3k}{2|\vec{k}|}\, k_m\, G^\dagger(k)\, G(k) . \tag{2.55}$$

For homogeneous Lorentz transformations, we find for $\psi(x) \to \psi(x) = \psi(\Lambda x)$,

$$\hat{g}_o(k) = \frac{f(k)}{f(\Lambda k)} \frac{\gamma(\Lambda k)}{\gamma(k)}\, g_o(\Lambda k) + \frac{f(k)}{f(\lambda k)} \frac{1}{\gamma(k)} \left\{ \frac{\beta(\Lambda k)}{(\Lambda k)_o} - 2\lambda^2 |k| \Lambda_o^{\ell} \frac{\partial}{\partial k^\ell} \frac{1}{|\vec{k}|(\Lambda k)_o} - \frac{\beta(k)}{|\vec{k}|} \right\} \left[\frac{g_1(\Lambda k)}{\sqrt{\gamma(\Lambda k)}} \right]$$

$$\hat{g}_1(k) = \frac{f(k)}{f(\Lambda k)}\, g_1(\Lambda k)$$

$$\hat{g}_o^\dagger(k) = g_o^\dagger(\Lambda k)$$

$$\hat{g}_1^\dagger(k) = \frac{\gamma(\Lambda k)}{\gamma(k)}\, g_1^\dagger(\Lambda k) + \left\{ \frac{\beta(\Lambda k)}{(\Lambda k)_o} - 2\lambda^2 |\vec{k}| \Lambda_o^{\ell} \frac{\partial}{\partial k^\ell} \frac{1}{|\vec{k}|(\Lambda k)_o} - \frac{\beta(k)}{|\vec{k}|} \right\} \left[\frac{g_o^\dagger(\Lambda k)}{\sqrt{\gamma(\Lambda k)}} \right] \tag{2.56}$$

Now, from (2.53), it follows that space-time translations are implemented unitarily only on a subspace $\mathcal{H}'$ defined by

$$g_1\big|_{\mathcal{H}'} = 0 \quad . \tag{2.57}$$

Restricted to this space, the Hamiltonian (2.54) is hermitean and positive definite. By (2.53) and (2.56), the condition (2.57) is Lorentz invariant. Furthermore, if we choose $f(k) = \gamma(k)$, we find that the homogeneous Lorentz transformations given in (2.56) will also be implemented unitarily on the same space. Thus, for this choice of f(k), the whole Poincaré group is implemented unitarily on the subspace $\mathcal{H}'$ defined in (2.57). Indeed, this subspace is the same as constructed using the only raising operator of the Hamiltonian (2.34), $\bar{B}(k)\cup$ (see (2.48)).

In fact, this distinct subspace $\mathcal{H}'$ fulfills all the criteria for a set of physically realizable states, a physical subspace:

- $\mathcal{H}'$ is a Hilbert space.
- The Poincaré group is implemented unitarily on it.
- The spectrum of the space-time translation generators is given by the forward light cone and their eigenstates form an improper basis of $\mathcal{H}'$.

Before deciding if $\mathcal{H}'$ us a reasonable physical subspace or not, we have to consider whether the states in $\mathcal{H}'$ can be identified by performing local measurements. To this end we now examine the following local selfadjoint quantity

$$\Gamma(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|\vec{k}|} \left\{ g_o(k)\, e^{-ik\cdot x} + g_o^\dagger\, e^{ik\cdot x} \right\} . \tag{2.58}$$

$\Gamma(x)$ leaves $\mathcal{H}'$ invariant and, from the commutation relations (2.52) with the special choice $f(k) = \gamma(k)$,

$$[G(k), G^{\dagger}(q)] = 2|\vec{k}|\delta(\vec{k}-\vec{q}) \quad , \tag{2.59}$$

will satisfy

$$[\Gamma(x+y), \Gamma(y)] = -\frac{i}{2\pi}\varepsilon(x_o)\delta(x^2) \quad . \tag{2.60}$$

Although $\Gamma(x)$ is not a scalar field, its matrix elements between vectors from $\mathcal{H}'$ are c-number scalar fields and satisfy (for Ψ and Φ in $\mathcal{H}'$):

$$(U(a,\Lambda)\Psi, \Gamma(x)U(a,\Lambda)\Phi) = (\Psi, \Gamma(\Lambda^{-1}x-a)\Phi) \quad . \tag{2.61}$$

Thus, $\Gamma(x)\big|_{\mathcal{H}'}$ is a selfadjoint canonical scalar field.

Now, suppose we have an observeable θ whose restriction to $\mathcal{H}'$ is selfadjoint on $\mathcal{H}'$ (i.e., $\theta^{+}\big|_{\mathcal{H}'} = \theta\big|_{\mathcal{H}'}$). The states in $\mathcal{H}'$ can then be identified by performing local, covariant measurements of the observables represented by the selfadjoint elements of the polynomial algebra of $\Gamma(x$[illegible]. All of these considerations suggest that $\mathcal{H}'$ does indeed form a reasonable physical subspace.

So far in this section we have ignored the role of local gauge transformations (2.20):

$$\chi \to \chi$$

$$\phi \to \phi + \Lambda \tag{2.62}$$

with $\Box\Lambda = 0$. We will now consider them. The operator implementing these transformations is generated (up to a c-number) by the Q_Λ of (2.23):

$$Q_\Lambda = \frac{1}{\lambda}\int d^3k \, \{\bar{c}(k)a_1(\vec{k}) + \bar{a}_1(k)c(k)\}$$

$$= -2i\int \frac{d^3k}{2|\vec{k}|} \, \{c(k)\gamma^{-1}g_1(k) - g_o^{+}(k)c(k)\} \quad . \tag{2.63}$$

One can then derive

$$e^{iQ_\Lambda} G(k)\, e^{-iQ_\Lambda} = G(k) + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} c(k)\,,$$

$$e^{iQ_\Lambda} G^{\dagger}(k)\, e^{-iQ_\Lambda} = G^{\dagger}(k) + 2\,(0,\ 1)\, \bar{c}(k)\, \gamma^{-1} \quad . \qquad (2.64)$$

Although these transformations leave $\mathcal{H}'$ invariant, they do not leave invariant the improper basis of $\mathcal{H}'$, namely the vectors $g_o^{\dagger}(k_1) \ldots g_o^{\dagger}(k_n)\Omega$, since the vacuum is mapped onto a coherent state of g_o quanta:

$$e^{iQ_\Lambda}\,\Omega = \exp\left\{-2\int \frac{d^2k}{2|\vec{k}|}\, g_o^{\dagger}(\vec{k})\, c(\vec{k})\right\} \Omega \quad . \qquad (2.65)$$

Furthermore, these transformations are not implemented unitarily on $\mathcal{H}'$ since Q_Λ is not selfadjoint there. This fact has important implications for the maximal physical interpretation of the model, since if $\mathcal{H}'$ is to be the space of physical states, then local gauge transformations do not constitute a symmetry on it. If one adopts the principle that observables must be gauge invariant as well as being selfadjoint, then this model is left without any nontrivial observables, the reason being that observables would then have to commute with Q_Λ, and this excludes the polynomial algebra of $\Gamma(x)$ from the set of observables.

The considerations undertaken so far do not provide us with a cogent reason for postulating or not postulating gauge invariance of observables. At the free field level we thus cannot exclude the g_o quanta from being a potential observable degree of freedom. The decision on this point has to come from an analysis of actual measurement processes performed on the g_o quanta, which would involve interaction with some other measurable degree of freedom. We therefore have to extend the present free dipole model to an interacting field theory.

The crucial question then is whether the interacting theory too possesses a positive norm subspace with a unitary implementation of the Poincare group and a unitary S operator. In the next two chapters we will examine two simple (but solvable) examples of an interacting dipole field: the linearized Higg's model and the Maxwell field, both with external c-number sources. We find that a physical interpretation for the g_o quanta is not allowed, primarily due to the requirement that the S operator be unitary and to the related question of gauge invariance of observables. We suspect that this result is typical for any interacting dipole model and that these quanta are unobservable in principle.

One final comment is in order concerning global gauge transformations (2.62) with Λ a constant. The corresponding Noether current $J_\mu = -\partial_\mu \chi$ does not lead to an implementable symmetry since

$$[J_o(t,\vec{x}),\phi(t,\vec{y})] = [\pi_\phi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y}) \qquad (2.66)$$

indicates the symmetry is spontaneously broken. Both the g_o and g_1 degrees of freedom contribute to this commutator and hence neither alone plays the role of a "Goldstone mode".

CHAPTER III

THE LINEARIZED ABELIAN HIGG'S MODEL

In this chapter we will study the linearized Abelian Higg's model coupled to external c-number sources. This will provide us with our first example of an interacting dipole field theory. Beginning with the model as outlined in the Introduction, we define a mapping that allows us to carry over the results of the previous chapter with essentially no modifications. We find that the subspace of the whole Hilbert space upon which the Poincaré group acts unitarily factorizes into three spaces: one corresponding to the massive scalar Higg's particle, one to the massive vector boson particle, and the final one to the one of the two dipole degrees of freedom that was potentially physical found in the last chapter. The S operator is found to factorize similarly; however, that part corresponding to the one dipole degree of freedom is not unitary. In order to conserve probabilities, we must therefore mask the appearance of the dipole mode as a physical one. This is done by postulating a restricted gauge invariance of observables; the resulting physical subspace then contains only the massive scalar Higg's particle and the massive vector boson. This shows the important role gauge invariance plays in making such interacting dipole theories physically sensible.

We begin our study by outlining briefly the model. As mentioned in the Introduction, it is an extension of a model of two interacting scalar fields:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 - \frac{1}{24} g (\phi_1^2 + \phi_2^2 - a^2)^2 \qquad (3.1)$$

This is invariant under the global gauge transformations

$$\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta$$

$$\phi_2 \to -\phi_1 \sin\theta + \phi_2 \cos\theta \tag{3.2}$$

with θ a constant. If one wishes to extend the transformations (3.2) to local space-time dependent ones, one must add a gauge field A_μ and modify the Lagrange density (3.1) to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2$$

$$- \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{24} g(\phi_1^2 + \phi_2^2 - a^2)^2 \tag{3.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is then invariant under the local gauge transformations

$$\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta$$

$$\phi_2 \to -\phi_1 \sin\theta + \phi_2 \cos\theta$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\theta \tag{3.4}$$

where θ is now space-time dependent.

For $a^2 > 0$, spontaneous symmetry breakdown occurs. Choosing ϕ_1 to develop a nonzero vacuum expectation value, we define, to lowest order, the shifted fields

$$\psi = \phi_1 - a \quad , \quad \phi = \phi_2 \tag{3.5}$$

The Lagrange density becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\mu^2\psi^2 + \frac{1}{2}(\partial_\mu\phi + mA_\mu)^2 - \frac{1}{4}(F_{\mu\nu})^2 - S_\mu A^\mu + \frac{1}{m}\partial\cdot S\phi + \tau\psi \tag{3.6}$$

where

$$m = ea$$

$$\mu^2 = \frac{1}{3} a^2 g$$

$$S_\mu = e\phi\partial_\mu\psi - e\psi\partial_\mu\phi - meA_\mu\psi - \frac{1}{2} e^2 A_\mu \phi^2 - \frac{1}{2} e^2 A_\mu \psi^2$$

$$\tau = e\partial_\mu\phi A^\mu + meA_\mu A^\mu + \frac{1}{2} e^2 A_\mu A^\mu \psi - \frac{1}{6} ag\psi^2 - \frac{1}{6} ag\phi^2 - \frac{1}{12} g\psi\phi^2 - \frac{1}{24} g\psi^3 .$$

Let us now mimic the effect of the interaction terms by replacing them with c-number external sources. The field equations resulting from (3.6) read

$$\Box A_\mu - \partial_\mu \partial\cdot A + m^2 A_\mu + m\partial_\mu\phi = S_\mu$$

$$\Box\phi + m\partial\cdot A = \frac{1}{m} \partial\cdot S$$

$$\Box\psi + \mu^2\psi = \tau \qquad (3.7)$$

where it is understood that now S_μ and τ are merely external c-number sources, not containing nonlinear terms in the fields. We see that the Higgs field ψ decouples, but the fields A_μ and ϕ do not. However, if we form the combination

$$B_\mu = A_\mu + \frac{1}{m} \partial_\mu\phi \qquad (3.8)$$

we find that ϕ does in fact decouple:

$$\Box B_\mu - \partial_\mu\partial\cdot B + m^2 B_\mu = S_\mu$$

$$\Box\psi + \mu^2\psi = \tau . \qquad (3.9)$$

The "transformation" (3.8) looks like a gauge transformation, and makes us suspicious that the degree of freedom associated with ϕ is somehow a gauge degree of freedom and will not appear as a physical one. It appears from equations (3.9) that the only physical degrees of freedom are those associated with a massive vector field B_μ and a massive scalar ψ. These conclusions, while correct, must nevertheless be verified in the full quantum version.

In fact, there does exist a choice of gauge, the unitarity gauge, where only physical fields appear and the system is indeed described by (3.9). The Higg's model in this gauge has one main advantage and one major drawback. Since only physical degrees of freedom appear, the S matrix is manifestly unitary. However, it is not immediately apparent that the theory is renormalizable; indeed, it has been called quasi-renormalizable in this gauge because of some "miraculous" cancellation of divergences that would have destroyed renormalizability (see [W2, F5] for a more complete discussion of the unitarity gauge). Because of the lack of a proof of renormalizability in the unitarity gauge, people were led to search for another choice of gauge where such a proof could be made. From experience with electrodynamics, it seemed natural to try adding to the Lagrange density a gauge fixing term $-\frac{\alpha}{2}(\partial\cdot A)^2$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\mu^2\psi^2 + \frac{1}{2}(\partial_\mu\phi + mA_\mu)^2 - \frac{1}{4}(F_{\mu\nu})^2 - \frac{\alpha}{2}(\partial\cdot A)^2$$
$$- S_\mu A^\mu + \frac{1}{m}\partial\cdot S\phi + \tau\psi \qquad (3.10)$$

The field equations are then modified to

$$\Box A_\mu + (\alpha-1)\partial_\mu\partial\cdot A + m^2A_\mu + m\partial_\mu\phi = S_\mu$$

$$\Box\phi + m\partial\cdot A = \frac{1}{m}\,\partial\cdot S$$

$$\Box\psi + \mu^2\psi = \tau \qquad (3.11)$$

These are still invariant under the following transformations:

$$A_\mu \to A_\mu - \frac{1}{m}\,\partial_\mu\Lambda$$

$$\phi \to \phi + \Lambda$$

$$\psi \to \psi \qquad (3.12)$$

with $\Box\Lambda = 0$. However, because in the absence of sources $\Box\phi \neq 0$, the unitarity gauge with this particular gauge fixing term cannot be chosen. To see how the degrees of freedom associated with ϕ and one of those of A_μ do in fact decouple from the physical sector will occupy the remainder of this chapter.

(As an aside, let us note that the gauge fixing term $-\frac{\alpha}{2}(\partial\cdot A)^2$ is still not the most convenient choice for a discussion of renormalizability; the first successful proof of renormalizability was done using a slightly different such term (see [H6, F5]). However, the discussion of what constitutes a physical subspace will be similar for this modified choice, so we will continue to use $-\frac{\alpha}{2}(\partial\cdot A)^2$, since this is what concerns us at present).

To begin our discussion we define a mapping from the set of solutions $\{A_\mu, \phi, \psi\}$ of (3.11) onto a set $\{C_\mu, \phi, \chi, \psi\}$ by the following:

$$A_\mu = C_\mu - \frac{1}{m}(\partial_\mu\phi - n\partial_\mu\chi)$$

$$\phi = \phi$$

$$\psi = \psi \qquad (3.13)$$

where η is a real dimensionless parameter. The set $\{C_\mu,\phi,\chi,\psi\}$ satisfy

$$\Box C_\mu - \partial_\mu \partial\cdot C + m^2 C_\mu = S_\mu$$

$$\Box\phi - \frac{m^2\eta}{\alpha}\chi = \frac{1}{m}\partial\cdot S$$

$$\Box\chi = 0$$

$$\Box\psi + \mu^2\psi = \tau \qquad (3.14)$$

By using

$$\alpha\partial\cdot A = -m\eta\chi \qquad (3.15)$$

one can show that the mapping (3.13) is invertible. With this mapping we have decoupled the massive spin-one field C_μ from the fields ϕ and χ (note that $m^2\partial\cdot C = \partial\cdot S$, so, say, C_o is not an independent degree of freedom).

The gauge transformations (3.12) in terms of C_μ,ϕ,χ and ψ now read

$$C_\mu \to C_\mu$$

$$\phi \to \phi + \Lambda$$

$$\chi \to \chi$$

$$\psi \to \psi. \qquad (3.16)$$

These are exactly the type of transformations considered in the previous chapter (see (2.62)). The operator that implements these transformations is thus generated by

$$Q_\Lambda = -2i\int\frac{d^3k}{2|\vec{k}|}\{\bar{c}(k)\gamma^{-1}g_1(k) - g_o^\dagger(k)c(k)\} \qquad (3.17)$$

where

$$\Lambda(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2|\vec{k}|}\{c(k)e^{-ik\cdot x} + \bar{c}(k)e^{ik\cdot x}\},$$

and satisfies

$$e^{iQ_\Lambda} C_\mu e^{-iQ_\Lambda} = C_\mu ,$$

$$e^{iQ_\Lambda} \phi e^{-iQ_\Lambda} = \phi + \Lambda$$

$$e^{iQ_\Lambda} \chi e^{-iQ_\Lambda} = \chi$$

$$e^{iQ_\Lambda} \psi e^{-iQ_\Lambda} = \psi \qquad (3.18)$$

These transformations will play an important role in deciding on the physical interpretation of the model.

We now turn our attention to the quantized version. From the Lagrange density (3.10), we find the canonical variables to be

$$A_\mu ; \pi_\mu = F_{\mu o} - \alpha g_{o\mu} \partial \cdot A$$

$$\phi ; \pi = mA_o + \dot{\phi}$$

$$\psi ; \pi_\psi = \dot{\psi} \qquad (3.19)$$

We then impose CETCRs among these:

$$[\pi_\mu(t,\vec{x}), A_\nu(t,\vec{y})] = ig_{\mu\nu}\delta(\vec{x}-\vec{y})$$

$$[\pi(t,\vec{x}), \phi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y})$$

$$[\pi_\psi(t,\vec{x}), \psi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y}) , \qquad (3.20)$$

with all others vanishing. These in turn induce the following ETCRs for the new variables $\{C_\mu, \phi, \chi, \psi\}$:

$$[\pi_\phi(t,\vec{x}), \phi(t,\vec{y})] = \frac{i}{\eta}\delta(\vec{x}-\vec{y})$$

$$[\pi_\chi(t,\vec{x}), \phi(t,y)] = i(1 - \frac{\xi}{\eta})\delta(\vec{x}-\vec{y})$$

$$[\pi_\chi(t,\vec{x}),\chi(t,\vec{y})] = \frac{i}{\eta}\,\delta(\vec{x}-\vec{y})$$

$$[\pi_C^k(t,\vec{x}),C_\ell(t,\vec{y})] = -i\delta^k{}_\ell\,\delta(\vec{x}-\vec{y})$$

$$[\pi_\psi(t,\vec{x}),\psi(t,\vec{y})] = -i\delta(\vec{x}-\vec{y}) \qquad (3.21)$$

where the conjugate momenta used correspond to the canonical variables of chapter II (see (2.25)):

$$C^k \; ; \; \pi_C{}^\ell = \partial^\ell C^0 - \dot{C}^\ell$$

$$\phi \; ; \; \pi_\phi = -\dot{\chi}$$

$$\chi \; ; \; \pi_\chi = \xi\chi - \dot{\phi}$$

$$\psi \; ; \; \pi_\psi = \dot{\psi} \qquad (3.22)$$

The remaining ETCRs vanish. Since the fields are coupled to c-number external sources, these relations hold unchanged for the asymptotic fields. In particular, the CETCRs for the pure dipole field (see (2.25)) are recovered for χ and ϕ with the choice $\eta = -1 = \xi$. The value of λ^2 introduced in chapter II (see (2.2)) can then be read off from the asymptotic form of equations (3.14):

$$\lambda^2 = -m^2/\alpha \qquad . \qquad (3.23)$$

We can thus take over bodily the results of chapter II in the case $\alpha < 0$, since then $\lambda^2 > 0$ (this will be supposed in the following).

It is now fairly straightforward to calculate the various commutators and two-point functions for the theory. The massive vector field C_μ satisfies (see, for example, [I3])

$$[C_\mu(x+y),C_\nu(y)] = -\int \frac{d^4k}{(2\pi)^3}\, e^{-ik\cdot x}\,\varepsilon(k_o)\,\delta(k^2-m^2)\left\{g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right\}. \qquad (3.24)$$

With this and the results of the previous chapter (see (2.33)), we then find

$$[A_\mu(x+y), A_\nu(y)] = -\int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x} \Big\{ \varepsilon(k_o)\delta(k^2-m^2)[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}]$$

$$- \frac{k_\mu k_\nu}{m^2} [\varepsilon(k_o)\delta(k^2) + \frac{m^2}{\alpha} \varepsilon(k_o)\delta'(k^2)]\Big\}$$

$$[A_\mu(x+y), \phi(y)] = -\frac{i}{m} \int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x} k_\mu \Big\{\varepsilon(k_o)\delta(k^2) + \frac{m^2}{\alpha} \varepsilon(k_o)\delta'(k^2)\Big\}$$

$$[\phi(x+y), \phi(y)] = -\int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x} \Big\{\varepsilon(k_o)\delta(k^2) + \frac{m^2}{\alpha} \varepsilon(k_o)\delta'(k^2)\Big\}$$

$$[\psi(x+y), \psi(y)] = \int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x} \varepsilon(k_o)\delta(k^2-\mu^2) , \tag{3.25}$$

with all other commutators vanishing.

For the propagators, using

$$(\Omega, T\{C_\mu(x+y)C_\nu(y)\}\Omega) = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \Big\{\frac{g_{\mu\nu} - k_\mu k_\nu/m^2}{k^2 - m^2 + i\varepsilon}\Big\} , \tag{3.26}$$

one then calculates with the help of (2.41)

$$(\Omega, T\{A_\mu(x+y)A_\nu(y)\}\Omega)_{reg} = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}$$

$$\times \Big\{(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+i\varepsilon}) \frac{1}{k^2-m^2+i\varepsilon} + \frac{1}{\alpha} \frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\Big\}$$

$$(\Omega, T\{A_\mu(x+y)\phi(y)\}\Omega)_{reg} = \frac{m}{\alpha} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{k_\mu}{(k^2+i\varepsilon)^2}$$

$$(\Omega, T\{\phi(x+y)\phi(y)\}\Omega)_{reg} = i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \Big\{\frac{1}{k^2+i\varepsilon} - \frac{m^2}{\alpha} \frac{1}{(k^2+i\varepsilon)^2}\Big\}$$

$$(\Omega, T\{\psi(x+y)\psi(y)\}\Omega) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{1}{k^2-\mu^2+i\varepsilon} \tag{3.27}$$

with all others vanishing. The bad high-momentum behaviour of the massive vector field C_μ is evident in the term $k_\mu k_\nu/m^2$ of the propagator (3.26); it is this term which spoils the renormalizability of a theory of weak interactions containing such a field. However, this term does not appear in the propagator (3.27) for A_μ, and indeed a theory giving a mass to the photon in this manner is renormalizable.

We can also see now how Goldstone's theorem is verified. Recall the asymptotic equation of motion (3.11) for A_μ:

$$\Box A_\mu + (\alpha-1)\partial_\mu \partial\cdot A = K_\mu \tag{3.28}$$

where

$$K_\mu = -m^2(A_\mu + \frac{1}{m}\partial_\mu\phi) = -m^2(C_\mu - \frac{1}{m}\partial_\mu\chi) .$$

The current K_μ generates global gauge transformations (3.16). Now form the quantity

$$(\Omega, \int d^3x[K_0(x+y),\phi(y)]\Omega) = im \tag{3.29}$$

The fact that this quantity does not vanish is a signal that these transformations are spontaneously broken. According to Goldstone's theorem, then, the Fourier transform of

$$(\Omega,[K_\mu(x+y),\phi(y)]\Omega) = \frac{im}{2\pi}\partial_\mu\varepsilon(x_0)\delta(x^2) \tag{3.30}$$

should contain a $\delta(p^2)$ singularity, as indeed it does. Similar results are obtained if one replaces ϕ in (3.29) by any product of fields A_μ and ϕ.

We now concentrate on constructing the Hilbert space of the model. In order to do this, we must consider the asymptotic fields. These are the solutions to the homogeneous version of equation (3.14):

$$\Box C_\mu^{ex} - \partial_\mu \partial\cdot C^{ex} + m^2 C_\mu^{ex} = 0$$

$$\Box \phi^{ex} - \frac{m^2\eta}{\alpha}\chi^{ex} = 0$$

$$\Box \chi^{ex} = 0$$

$$\Box \psi^{ex} + \mu^2 \psi^{ex} = 0 \qquad (3.31)$$

where the superscript "ex" indicates either in or out fields. The Hilbert space of the full model is then obtained by performing the Fock construction for the fields C_μ^{ex}, ϕ^{ex}, $\chi^{ex} = \chi$ and ψ^{ex}. Since C_μ^{ex} and ψ^{ex} decouple from ϕ^{ex} and χ, the complete Hilbert space $\mathcal{H}$ is the symmetric tensor product of the space V carrying C_μ^{ex}, the space H carrying ψ^{ex}, and the space D (containing both g_0 and g_1 quanta) constructed in chapter II for the fields ϕ^{ex} and χ:

$$\mathcal{H} = V \otimes H \otimes D \quad . \qquad (3.32)$$

The interpolating fields C_μ, ϕ, χ and ψ are represented as operators on $\mathcal{H}$ which satisfy (3.14). The solution to these equations is given by

$$C_\mu(x) = C_\mu^{ex}(x) + \int d^4y\, D_{\mu\nu}^{ex}(x-y,m^2) S^\nu(y)$$

$$\phi(x) = \phi^{ex}(x) + \frac{1}{m}\int d^4y\, \Delta^{ex}(x-y,0)\,\partial\cdot S(y)$$

$$\chi(x) = \chi^{ex}(x)$$

$$\psi(x) = \psi^{ex}(x) + \int d^4y\, \Delta^{ex}(x-y,\mu^2)\tau(y) \qquad (3.33)$$

where

$$\Delta^{ex}(x,m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{(m^2-k^2)_{ex}}$$

$$D^{ex}_{\mu\nu}(x,m^2) = (g_{\mu\nu} + \frac{1}{m^2}\partial_\mu\partial_\nu)\Delta^{ex}(x,m^2)$$

and, as before, the subscript "ex" indicates the iε-prescription corresponding to the boundary conditions yielding advanced or retarded propagators.

Now, recalling the results of the previous chapter, it is apparent that the Poincaré group is not implemented unitarily on the entire Hilbert space $\mathcal{H}$. However, on a subspace $\mathcal{H}$ defined by (see (2.57))

$$g_1\Big|_{\mathcal{H}} = 0 , \qquad (3.34)$$

it is implemented unitarily. Since we have

$$\partial\cdot A = -\frac{mn}{\alpha}\chi$$

$$= \frac{2mn}{\alpha}\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2|\vec{k}|}\left\{g_1(k)\gamma^{-1}e^{-ik\cdot x} + g_o^\dagger(k)e^{ikx}\right\} , \qquad (3.35)$$

the condition (3.34) is identical with the weak Lorentz condition

$$\partial\cdot A^{(+)}\Big|_{\mathcal{H}} = 0 , \qquad (3.36)$$

where $\partial\cdot A^{(+)}$ denotes the positive frequency part of $\partial\cdot A$. Note that, because $\Box\chi = 0 = \Box\partial\cdot A$ even when sources are present, the condition (3.36) retains its meaning in the presence of interactions.

Having found the subspace where the Poincaré group acts unitarily, we now address the crucial question of whether or not the S operator too is unitary on it. The S operator is defined as that operator which connects in-fields to out-fields:

$$S\,\Sigma^{in}\,S^{-1} = \Sigma^{out} . \qquad (3.37)$$

where Σ stands for any of the fields C_μ, ϕ, χ or ψ. As did the Hilbert space $\mathcal{H}$, the full S operator factorizes into a part acting on the massive vector boson space V, one onto the Higg's space H, and the final one onto the dipole space D:

$$S = S_V \otimes S_H \otimes S_D \tag{3.38}$$

with

$$S_V C_\mu^{in} S_V^{-1} = C_\mu^{out}$$

$$S_H \psi^{in} S_H^{-1} = \psi^{out}$$

$$S_D \phi^{in} S_D^{-1} = \phi^{out}$$

$$S_D \chi S_D^{-1} = \chi$$

The operators S_V and S_H are unitary on the Hilbert space $\mathcal{H}$, and present no problems. The same cannot be said for S_D. Up to an arbitrary c-number phase factor, it is given by

$$S_D = \exp\left\{-\frac{2i}{m}\sqrt{2\pi}\int \frac{d^3k}{2|\vec{k}|}\left[ik^\mu \tilde{S}_\mu(k)\gamma^{-1} g_1^{in}(k) - ik^\mu \tilde{S}_\mu(k) g_o^{+in}(k)\right]\right\} , \tag{3.39}$$

where $\tilde{S}_\mu(k)$ is the Fourier transform of the current $S_\mu(x)$. Recalling the effect of the local gauge transformations (3.18), it is seen that S_D actually implements these transformations, as (ϕ^{out},χ) and (ϕ^{in},χ) are connected by a gauge transformation.

This shows that scattering and creation of the scalar Goldstone degrees of freedom takes place, as one expects from the fact that ϕ is not a free field. This is a rather simple interaction because, even though $g_o^{+in} = g_o^{+out}$, the out vacuum $\Omega_D^{out} = S_D \Omega_D^{in}$ is a coherent state relative to the P_μ^{in} basis.

It is also now apparent that S_D is neither unitary on $\mathcal{H}$ nor on $\tilde{\mathcal{H}}$. If states of $\mathcal{H}_D$ were to be observable, then the fact that S_D is not unitary on $\mathcal{H}_o$ would violate the principle of probability conservation. In order to have a physically sensible theory, we must therefore make the g_o quanta unobservable. Now, a look at (3.17) indicates that restricted gauge transformations neither will be implemented unitarily on $\mathcal{H}$, and therefore will not constitute a symmetry there. Observability of the g_o quanta can thus be prevented by adopting the postulate that observables must be gauge invariant (i.e., invariant under (3.16)). The subspace of physical states $\mathcal{H}_P$ would then have to be chosen as that gauge invariant one containing only the Higg's scalar and the massive spin-one degrees of freedom:

$$\mathcal{H}_P = \mathcal{H}_H \otimes \mathcal{H}_V . \tag{3.40}$$

Observables would be described by operators of the type

$$\theta\Big|_{\mathcal{H}_P} = \theta_H \otimes \theta_V \tag{3.41}$$

with θ_H and θ_V each being selfadjoint. Thus, the only observable degrees of freedom are those associated with the Higg's and with the massive vector boson quanta. The resulting physical subspace $\mathcal{H}_P$ carries a unitary representation of the Poincaré group and is the maximal one where the S operator is unitary, guaranteed by the postulate that observables must be gauge invariant on this space.

This, of course, is the same interpretation as that obtained using the indefinite metric approach. As seen in chapter II, there are no physical observables associated with the two dipole degrees of freedom when the indefinite form is used, and this immediately reduces the original six degrees of freedom of the Higg's model to four physical ones. However, as we have seen in the linearized Higg's model with vanishing sources, no reason that this form must be used can be found in the interaction free case. Using the positive definite scalar product, it is only the demand that, in order to have a physically sensible theory, the S operator must be unitary that leaves the model with only four physical degrees of freedom. In the next chapter we will study another model containing a dipole field - the Maxwell field in a non-Feynman type gauge. Again we shall find that differences in physical interpretation arise (in the absence of interaction) when the positive definite scalar product is used as opposed to the indefinite sesquilinear form. As well, we shall also again see the important role gauge invariance plays in making the theory physically sensible.

CHAPTER IV

THE MAXWELL FIELD

In this chapter we will study the quantization of the electromagnetic field in a non-Feynman gauge. When coupled to an external source it will provide us with another example of an interacting dipole field. Again we shall see that quantization on a positive metric leads to a different physical interpretation at the free field level than that found using an indefinite metric, and that unitarity of the S operator and the related question of gauge invariance in the interacting theory reconciles these differences.

Quantizing Maxwell's equation has proven to be one of the most difficult problems in the history of quantization. Work by Strocchi has shown just how deep the problems associated with quantizing electromagnetism lie (see [S7, S8], and also [W5, M3]). Assuming only that a Poincaré invariant vacuum exists, the field strength $F_{\mu\nu}$ is covariant, and the two-point function in the forward light cone is analytic, he has shown that Maxwell's equations

$$\partial^{\mu}F_{\mu\nu} = 0 = \epsilon^{\mu\nu\sigma\delta}\partial_{\nu}F_{\sigma\delta} \tag{4.1}$$

do not admit a nontrivial solution of the form

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{4.2}$$

if either the vector potential A_{μ} is covariant or is local. No assumptions on the positivity of the metric are necessary in this proof.

One evidently has two options available in avoiding these conclusions. The first one, and the one taken historically, is to give up the covariance and locality of the vector potential. This essentially leads to a Coulomb-like gauge formalism (see [B4], as well as [G1, M4, M5] for alternate formulations). In such theories only the two transverse degrees of freedom are quantized, the other two remaining unphysical ones being eliminated in some suitable manner. It is this splitting that renders the vector potential nonlocal and noncovariant, although of course the field strength is still local and covariant.

Since only the components of the field strength are observable, and not the potential itself, there appears no problem with this approach. Nonlocal and noncovariant terms present in intermediate steps drop out in expressions for observables. However, the concept of the vector potential seems unavoidable in formulating electromagnetic interactions, as such interactions cannot be written down in a local manner using just the field strengths. This observation along with the advent of axiomatic field theory [W5, S5] and the powerful results that follow from, among others, the assumptions of locality and covariance, led some to wonder if a local covariant formulation of the Maxwell field was possible. It must be emphasized that there is nothing wrong with a Coulomb gauge quantization; the question asked here is a more theoretical one; is there something intrinsic in the quantization of Maxwell's equations that makes the use of a noncovariant, nonlocal vector potential necessary?

The answer to this question is that it is not necessary. A second way out of Strocchi's conclusions is to modify Maxwell's equations themselves, but then recover them in a weak form on certain physical states. This results in the well-known Gupta [G6]-Bleuler [B6] formalism, where the vector potential satisfies the massless Klein-Gordon equation (which is the Feynman gauge). This was later extended to an arbitrary gauge by Lautrup [L1] and Nakanishi [N2]. (Incidentally, the original seemingly theoretical motives were not without practical benefits; many calculations are actually more easily performed using this type of quantization).

A feature of these theories is that an indefinite metric is used, and a way of disallowing states of zero and negative norm from the physical sector must be found. In the Gupta-Bleuler method the following is used. Physical states Ψ are said to satisfy a weak Lorentz condition

$$\partial \cdot A^{(+)} \Psi = 0 \ . \tag{4.3}$$

Norms of such states are positive semi-definite. These states are then grouped into equivalence classes: two states are said to be equivalent if they differ only by a state of zero norm. The physical space is then defined to be the quotient space

$$\mathcal{H}_P = \mathcal{H}'/\mathcal{H}'' \tag{4.4}$$

where $\Psi \in \mathcal{H}'$ and $\mathcal{H}''$ is the space of zero norm vectors. By means of (4.3), Maxwell's equations supplemented by the Lorentz condition are recovered in a weak form on $\mathcal{H}_P$. In this way the two unphysical degrees of freedom are eliminated, leaving in $\mathcal{H}_P$ two physical transverse photons. (For a more detailed discussion of these and other types of gauges, see [S9].)

Of course, using either a Coulomb gauge formalism or an indefinite metric formulation leads to the same physical results. It is largely a matter of taste which method to employ. The Gupta-Bleuler formalism is a completely consistent and elegant approach to electromagnetism. However, we now ask the same question as we did for the Higg's model: is it possible to quantize the electromagnetic field in a covariant local manner on a positive metric Hilbert space and still eliminate the two unphysical degrees of freedom? As before, using the positive definite scalar product, we will have to contend with the fact that A_μ is not hermitean nor is the Poincaré group implemented unitarily on the whole space. We will find:

(1) The representation of the Poincaré group carried by A_μ is irreducible but not indecomposable. Demanding space-time translations be implemented unitarily then reproduces an equation like (4.3) and removes one degree of freedom.

(2) If one then imposes a restricted gauge invariance, then the only space where the whole Poincaré group acts unitarily is a quotient space similar to that in (4.4). The remaining unphysical degree of freedom is eliminated, and we are left with two physical transverse photons.

This is not the whole story, though. If one does not postulate gauge invariance, one finds a second space, containing only one physical degree of freedom, where Poincaré transformations are implemented unitarily. This interpretation, as opposed to the other, however, leads to a nonunitary S operator for a coupling to a conserved external current. In order to conserve probabilities, one is forced to choose the physical space to be the gauge invariant one containing the two transverse photons. In a sense one derives gauge invariance of observables in this framework; one does not have to postulate it.

A. Quantizing the Electromagnetic Field

We begin our study by finding a suitable decomposition of the four degrees of freedom of the Maxwell field. As mentioned earlier, the usual procedure used in a covariant quantization is to modify Maxwell's equations. One such modification is the addition of a gauge fixing term $-\frac{\alpha}{2}(\partial\cdot A)^2$ to the Lagrange density:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{\alpha}{2}(\partial\cdot A)^2 \tag{4.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\alpha \neq 0$. The restricted gauge invariance alluded to previously is that of the following:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \tag{4.6}$$

with $\Box\Lambda = 0$.

The field equations resulting from (4.5) read

$$\Box A_\mu + (\alpha-1)\partial_\mu \partial\cdot A = 0 \tag{4.7}$$

The choice $\alpha = 1$ is the Feynman gauge, used by Gupta and Bleuler. In Fourier space, these equations become

$$[k^2 g_{\mu\nu} + (\alpha-1)k_\mu k_\nu]\tilde{A}^\nu(k) = 0 \tag{4.8}$$

where $\tilde{A}_\mu(k)$ is the Fourier transform of $A_\mu(x)$. For a nontrivial solution, $\tilde{A}_\mu(k)$ must have support on the cone $k^2 = 0$. However, in order to solve (4.8) for $\alpha \neq 1$, $\tilde{A}_\mu(k)$ must involve a $\delta'(k^2)$ as well as a $\delta(k^2)$ singularity since $A_\mu(x)$ in general satisfies a dipole equation:

$$\Box^2 A_\mu(x) = 0 . \tag{4.9}$$

Guided by the results of chapter II, we write the solution to (4.7) as follows:

$$A_\mu(x) = A_\mu^t(x) + \partial_\mu \phi_3(x) + (1-\alpha)\partial_\mu[x_o \phi_4(x)] + 2\alpha g_{o\mu}\phi_4(x) \qquad (4.10)$$

where

$$A_\mu^t(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|\vec{k}|} \Big\{ [\varepsilon_\mu(k) a_+(k) + \varepsilon_\mu^*(k) a_-(k)] e^{-ik\cdot x}$$

$$+ [\varepsilon_\mu^*(k)\bar{a}_+(k) + \varepsilon_\mu(k)\bar{a}_-(k)] e^{ik\cdot x} \Big\}$$

$$\phi_3(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|\vec{k}|} \Big\{ a_3(k) e^{-ik\cdot x} + \bar{a}_3(k) e^{ik\cdot x} \Big\}$$

$$\phi_4(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2} \Big\{ a_4(k) e^{-ik\cdot x} + \bar{a}_4(k) e^{ik\cdot x} \Big\}$$

$$k^2 = 0 \; ; \qquad k_o = + |\vec{k}|$$

The objects $\varepsilon_\mu(k)$ describe states of circular polarization, and satisfy

$$\varepsilon^* \cdot \varepsilon = -1 \quad , \quad \varepsilon \cdot \varepsilon = 0 \quad , \quad \varepsilon \cdot k = 0 \quad . \qquad (4.11)$$

In addition, for canonical quantization it is found convenient to work with the class of $\varepsilon_\mu(k)$ satisfying

$$\varepsilon_o(k) = 0 \qquad (4.12)$$

Although the separation of the degrees of freedom in (4.10) is non-covariant, A_μ itself is still a four-vector. This is the same situation as in the free dipole field (see (2.81)), where ψ was a proper scalar field but the objects of the decomposition ϕ_o and ϕ_1 were not.

With the Klein-Gordon product,

$$(f,g)_{x_o} = \frac{i}{(2\pi)^{3/2}} \int d^3x \; f^*(x_o,\vec{x}) [\vec{\partial}_o - \overleftarrow{\partial}_o] g(x_o,\vec{x}) \quad , \qquad (4.13)$$

we can project out the Fourier amplitudes from $A_\mu(x)$ as follows:

$$a_+(k) = -(e^{-ik\cdot x},\ \varepsilon^*\cdot A(x))_{x_o}$$

$$a_-(k) = -(e^{-ik\cdot x},\ \varepsilon\cdot A(x))_{x_o}$$

$$a_3(k) = -\frac{i}{2}(\alpha+3)a_4(k) - \frac{i}{2}(\alpha-1)\bar{a}_4(k) + \frac{i}{|\vec{k}|}(e^{-ik\cdot x},\ A_o(x))_{x_o=0}$$

$$a_4(k) = \frac{i}{2|\vec{k}|^2}(e^{-ik\cdot x},\ \partial\cdot A(x))_{x_o} \quad . \qquad (4.14)$$

The corresponding expressions for the barred objects are

$$\bar{a}_+(k) = (e^{ik\cdot x},\ \varepsilon\cdot A(x))_{x_o}$$

$$\bar{a}_-(k) = (e^{ik\cdot x},\ \varepsilon^*\cdot A(x))_{x_o}$$

$$\bar{a}_3(k) = \frac{i}{2}(\alpha+3)\bar{a}_4(k) + \frac{i}{2}(\alpha-1)a_4(k) + \frac{i}{|\vec{k}|}(e^{ik\cdot x},\ A_o(x))_{x_o=0}$$

$$\bar{a}_4(k) = \frac{i}{2|\vec{k}|^2}(e^{ik\cdot x},\ \partial\cdot A(x))_{x_o} \quad . \qquad (4.15)$$

By imposing canonical equal-time commutation relations,

$$[A_\mu(t,\vec{x}),A_\nu(t,\vec{y})] = 0 = [\pi_\mu(t,\vec{x}),\pi_\nu(t,\vec{y})]$$

$$[A_\mu(t,\vec{x}),\pi_\nu(t,\vec{y})] = ig_{\mu\nu}\delta(\vec{x}-\vec{y}) \qquad (4.16)$$

where

$$\pi_\mu = F_{\mu o} - \alpha g_{o\mu}\partial\cdot A \quad ,$$

we can find the commutation relations among the Fourier amplitudes using (4.14) and (4.15). Defining A(k) by

$$A(k) \stackrel{c}{=} \begin{pmatrix} a_+(k) \\ a_-(k) \\ a_3(k) \\ a_4(k) \end{pmatrix} \quad ,$$

these relations turn out to be

$$[A_i(k),\bar{A}_j(q)] = \frac{1}{\alpha|\vec{k}|} C_{ij}\,\delta(\vec{k}-\vec{q}) \tag{4.17}$$

where

$$C = \begin{pmatrix} 2\alpha|\vec{k}|^2 & 0 & 0 & 0 \\ 0 & 2\alpha|\vec{k}|^2 & 0 & 0 \\ 0 & 0 & \alpha+1 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

If we now perform the transformation

$$B(k) = \rho(k)\gamma MA(k) \quad , \qquad \bar{B}(k) = \rho(k)\gamma\bar{A}(k)M^{\dagger} \quad , \tag{4.18}$$

where

$$B(k) = \begin{pmatrix} a_+(k) \\ a_-(k) \\ b_3(k) \\ b_4(k) \end{pmatrix}$$

$$\rho(k) = \sqrt{2\alpha}\ |\vec{k}| \qquad \alpha > 0$$

$$= i\sqrt{2|\alpha|}\ |\vec{k}| \qquad \alpha < 0$$

$$\gamma = [(\alpha+1)^2 + 4]^{-1/4}$$

$$M = \begin{pmatrix} (\rho\gamma)^{-1} & 0 & 0 & 0 \\ 0 & (\rho\gamma)^{-1} & 0 & 0 \\ 0 & 0 & 1 & i\varepsilon_- \\ 0 & 0 & -1 & -i\varepsilon_+ \end{pmatrix}$$

$$\varepsilon_\pm = \frac{1}{2}\left\{\alpha + 1 \pm \sqrt{(\alpha+1)^2 + 4}\right\} ,$$

the commutation relations become diagonal:

$$[B(k),\bar{B}(q)] = 2|\vec{k}|\eta\delta(\vec{k}-\vec{q}) \tag{4.19}$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

At this point we pause and calculate the commutation relations for arbitrary space-time separations. One finds, for the objects of the decomposition of A_μ,

$$[A^t_\mu(x+y),A^t_\nu(y)] = \frac{i}{2\pi} g_{\mu\nu}\varepsilon(x_o)\delta(x^2) - \frac{i}{4\pi}\partial_\mu\partial_\nu[\varepsilon(x_o)\theta(x^2)]$$

$$- \frac{i}{4\pi} x_o\partial_\mu\partial_\nu\left[\frac{\theta(-x^2)}{|\vec{x}|}\right]$$

$$[\phi_3(x+y),\phi_3(y)] = -\frac{i}{8\pi}\frac{\alpha+1}{\alpha}\left\{\varepsilon(x_o)\theta(x^2) + \frac{x_o}{|\vec{x}|}\theta(-x^2)\right\}$$

$$[\phi_3(x+y),\phi_4(y)] = -\frac{i}{8\pi}\frac{1}{\alpha}\frac{\theta(-x^2)}{|\vec{x}|}$$

$$[\phi_4(x+y),\phi_4(y)] = 0 \tag{4.20}$$

with all others vanishing. Notice that, as well as being noncovariant, these expressions also display nonlocality. However, the commutator for A_μ itself is local and covariant:

$$[A_\mu(x+y),A_\nu(y)] = \frac{i}{2\pi} g_{\mu\nu}\varepsilon(x_o)\delta(x^2) + \frac{i}{8\pi}\frac{1-\alpha}{\alpha}\partial_\mu\partial_\nu[\varepsilon(x_o)\theta(x^2)]$$

$$= \int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x}\left\{g_{\mu\nu}\varepsilon(k_o)\delta(k^2) + \frac{1-\alpha}{\alpha}k_\mu k_\nu\varepsilon(k_o)\delta'(k^2)\right\} . \tag{4.21}$$

The fact that the noncovariant and nonlocal terms in (4.20) have cancelled in this last expression just reflects the noncovariant and nonlocal decomposition of the local covariant field A_μ.

We now direct our attention to the behaviour of the Fourier amplitudes under Poincaré transformations. For space-time translations,

$$A_\mu(x) \to \hat{A}_\mu(x) = A_\mu(x-a) \ ,$$

we obtain

$$\hat{a}_+(k) = e^{ik\cdot a}\, a_+(k)$$

$$\hat{a}_-(k) = e^{ik\cdot a}\, a_-(k)$$

$$\hat{a}_3(k) = e^{ik\cdot a}\,\{a_3(k) + a_o(\alpha-1)a_4(k)\}$$

$$\hat{a}_4(k) = e^{ik\cdot a}\, a_4(k) \ . \qquad (4.22)$$

For Lorentz transformations,

$$A_\mu(x) \to \hat{A}_\mu(x) = \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1}x) \ ,$$

we have

$$\hat{a}_+(k) = a_+(\Lambda^{-1}k) - (\alpha+1)(\Lambda^{-1}k)_o\,\varepsilon_\mu^*(k)\Lambda^{\mu o}a_4(\Lambda^{-1}k)$$

$$\hat{a}_-(k) = a_-(\Lambda^{-1}k) - (\alpha+1)(\Lambda^{-1}k)_o\,\varepsilon_\mu^*(k)\Lambda^{\mu o}a_4(\Lambda^{-1}k)$$

$$\hat{a}_3(k) = a_3(\Lambda^{-1}k) + \frac{i}{|\vec{k}|}\,\Lambda_o^{\ \nu}\{\varepsilon_\nu(\Lambda^{-1}k)a_+(\Lambda^{-1}k) + \varepsilon_\nu^*(\Lambda^{-1}k)a_-(\Lambda^{-1}k)\}$$

$$+\, i\{(\alpha-1)(\Lambda^{-1})_o^{\ \ell}\frac{\partial}{\partial k^\ell} - (\alpha+1)(\Lambda^{-1})_o^{\ \ell}\frac{k_\ell}{|\vec{k}|^2}\}[(\Lambda^{-1}k)_o\,a_4(\Lambda^{-1}k)]$$

$$\hat{a}_4(k) = \left(\frac{(\Lambda^{-1}k)_o}{k_o}\right)^2 a_4(\Lambda^{-1}k) \ . \qquad (4.23)$$

The corresponding expressions for the transformed barred objects are obtained by complex conjugating the c-numbers and barring the q-numbers in (4.22) and (4.23). We note that the noncovariance of these amplitudes is reflected in time-translations and in boosts.

For the generators of space-time translations, one obtains for the the Hamiltonian

$$H = \int \frac{d^3k}{2|\vec{k}|} \, \vec{k} \, \bar{B}(k) h'' B(k) \quad , \tag{4.24}$$

where

$$h'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon+1 & \varepsilon \\ 0 & 0 & \varepsilon & \varepsilon-1 \end{pmatrix}$$

$$\varepsilon = \frac{1-\alpha}{\sqrt{(\alpha+1)^2 + 4}} \quad ,$$

and for the 3-momentum,

$$P_m = \int \frac{d^3k}{2|\vec{k}|} \, k_m \bar{B}(k) \eta B(k) \quad , \tag{4.25}$$

where η is given in (4.19). Notice that, except for $\alpha = 1$, the Hamiltonian (4.24) is not diagonalizable by a transformation that leaves the diagonal form of the commutation relations (4.19) intact.

B. Constructing the Hilbert Space

As a first step in constructing the Hilbert space, we postulate the existence of a Poincaré invariant vacuum with the following properties

$$B(k)\Omega = 0 \quad \forall \; |\vec{k}| \; ,$$

$$U(a,\Lambda)\Omega = \Omega \; . \tag{4.26}$$

The existence of such a vacuum allows us to calculate the various two-point functions of the theory. As in the free dipole case, however, matters become fairly delicate. Consider, for example, the following expression:

$$I = (\Omega, \int d^4x \; \phi_3(x+y)\, h(x)\, \phi_3(y)\Omega) \quad , \tag{4.27}$$

where h(x) is again a real-valued test function from a subset of $\mathcal{S}(\mathbf{R}^4)$. Introducing the Fourier transform,

$$h(x) = \int \frac{d^4k}{(2\pi)^2}\, e^{-ik\cdot x}\, \tilde{h}(k) \quad ,$$

one arrives at

$$I = \frac{1}{2\pi}\,\frac{\alpha+1}{\alpha} \int \frac{d^3k}{4|\vec{k}|^3}\; \tilde{h}(k) \quad . \tag{4.28}$$

This integral exists only if $\tilde{h}(k)/|\vec{k}|$ is finite at $|\vec{k}| = 0$. If one allows any h(x) belonging to $\mathcal{S}(\mathbf{R}^4)$, a regularization of (4.27) must be made (see [G3]). We choose the following scheme:

$$I_{reg} = \frac{1}{2\pi}\,\frac{\alpha+1}{\alpha} \left\{ \int_{|\vec{k}|<\mu} \frac{d^3k}{4|\vec{k}|^3}\,[\tilde{h}(k)-\tilde{h}(0)] + \int_{|\vec{k}|>\mu} \frac{d^3k}{4|\vec{k}|^3}\,\tilde{h}(k) \right\} \quad . \tag{4.29}$$

With such a regularization, one again finds all noncovariant terms that appear in intermediate calculations cancel, leaving us with the following covariant expressions:

$$\begin{aligned}(\Omega, A_\mu(x+y)A_\nu(y)\Omega)_{reg} &= \frac{1}{4\pi^2}\, g_{\mu\nu}\{\mathcal{P}(\frac{1}{x^2}) + i\pi\varepsilon(x_o)\delta(x^2)\} \\ &+ \frac{1}{16\pi^2}\,\frac{1-\alpha}{\alpha}\,\partial_\mu\partial_\nu\{\ln|x^2| + i\pi\varepsilon(x_o)\theta(x^2)\}\end{aligned} \tag{4.30}$$

$$\begin{aligned}(\Omega, T\{A_\mu(x+y)A_\nu(y)\}\Omega)_{reg} &= \frac{1}{4\pi^2}\, g_{\mu\nu}\{\mathcal{P}(\frac{1}{x^2}) + i\pi\delta(x^2)\} \; + \\ &+ \frac{1}{16\pi^2}\,\frac{1-\alpha}{\alpha}\,\partial_\mu\partial_\nu\{\ln|x^2| + i\pi\theta(x^2)\} \\ &= -i\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\left\{\frac{g_{\mu\nu}}{k^2+i\varepsilon} + \frac{1-\alpha}{\alpha}\,\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right\} \quad .\end{aligned} \tag{4.31}$$

We now return to the problem of the construction of the Hilbert space. To this end, we first construct a linear space that is spanned by the vectors

$$\Phi^{(n)}(k_1,\ldots k_n) = \bar{B}_{i_1}(k_1)\ldots\bar{B}_{i_n}(k_n)\,\Omega \qquad (4.32)$$

with $n = 0,1,\ldots\infty$. The vacuum is the unique vector in this space satisfying (4.26). Now, recall that for the pure dipole field the scalar product was implicitly defined by $\bar{B}(k) = f(k)B^{+}(k)\eta$ (see (2.49)), where f(k) was a scalar factor later adjusted so that the Poincaré group acted unitarily on a subspace of the whole space. We can, in fact, follow a different but equivalent path. A scalar product (,) is introduced on the linear space by the following:

$$(\Omega,\Omega) = 1$$
$$\bar{B}(k) = B^{+}(k)\eta \quad . \qquad (4.33)$$

Here, $B^{+}(k)$ is the hermitean adjoint of B(k) with respect to (,), which defines this scalar product implicitly, so that

$$(\Phi^{(n)}, B\Psi^{(n)}) = (B^{+}\Phi^{(n)},\Psi^{(n)}) \quad . \qquad (4.34)$$

With this identification, the positivity of the commutation relations

$$[B(k),B^{+}(q)] = 2|\vec{k}|\,\delta(\vec{k}-\vec{q})\;1 \qquad (4.35)$$

ensures that the norm of a state $\Phi^{(n)}$,

$$||\Phi^{(n)}||^2 = (\Phi^{(n)},\Phi^{(n)}) \qquad (4.36)$$

is positive definite. Those states with finite norm then form the Hilbert space of the model.

The generators of space-time translations, (4.24) and (4.25), now become

$$H = \int \frac{d^3k}{2|\vec{k}|} \, |\vec{k}| B^\dagger(k) h' B(k) \quad , \qquad (4.37)$$

where

$$h' = \eta h'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta+1 & \beta \\ 0 & 0 & -\beta & -\beta+1 \end{pmatrix} ,$$

and

$$P_m = \int \frac{d^3k}{2|\vec{k}|} k_m \, B^\dagger(k) B(k) \quad . \qquad (4.38)$$

For $\alpha \neq 1$, the matrix h' possesses an incomplete set of eigenvectors and, in fact, only $\{a_+^\dagger, a_-^\dagger, b_3^\dagger - b_4^\dagger\}$ and $\{a_+, a_-, b_3+b_4\}$ function as raising and lowering operators, respectively, for H (i.e., the two sets satisfy

$$[H, a_R^\dagger(k)] = +|\vec{k}| a_R^\dagger \quad , \qquad [H, a_L(k)] = -|\vec{k}| a_L \qquad (4.39)$$

respectively). With this in mind, we define

$$G(k) = \begin{pmatrix} a_+(k) \\ a_-(k) \\ g_3(k) \\ g_4(k) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & \text{[illegible]} & 1 & -1 \\ \text{[illegible]} & \text{[illegible]} & 1 & 1 \end{pmatrix} B(k)$$

$$G^{\dagger}(k) = (a_{+}^{\dagger}(k),\ a_{-}^{\dagger}(k),\ g_{3}^{\dagger}(k),\ g_{4}^{\dagger}(k))$$

$$= B^{\dagger}(k)\ \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad . \tag{4.40}$$

We then have in terms of G(k):

$$[G(k), G^{\dagger}(q)] = 2|\vec{k}|\,\delta(\vec{k}-\vec{q})\,1 \tag{4.41}$$

$$H = \int \frac{d^3k}{2|\vec{k}|}\ |\vec{k}|\,G^{\dagger}(k)\,h\,G(k) \tag{4.42}$$

where

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ,$$

and

$$P_m = \int \frac{d^3k}{2|\vec{k}|}\ k_m\, G^{\dagger}(k)\,G(k) \quad . \tag{4.43}$$

Now, notice in the expressions (4.41) to (4.43) we could replace G(k) by f(k)G(k) and $G^{\dagger}(k)$ by $f^{-1}(k)G^{\dagger}(k)$, where f(k) is some real scalar multiple of the unit matrix, without altering the form or physical content of these expressions. The only effect this replacement would have would be to change the transformation properties of G(k) and $G^{\dagger}(k)$ under Poincaré transformations. We can now use this freedom to allow for the existence of a part of the whole space where the Poincaré group is implemented unitarily. It turns out that the necessary choice is $f(k) = |\vec{k}|^{-1}$. With this choice we obtain, for space-time translations

$$A_\mu(x) \to \hat{A}_\mu(x) = A_\mu(x-a) \quad ,$$

the following:

$$\hat{a}_+(k) = e^{ik\cdot a}\, a_+(k)$$

$$\hat{a}_-(k) = e^{ik\cdot a}\, a_-(k)$$

$$\hat{g}_3(k) = e^{ik\cdot a}\, \{ g_3(k) - 2i\beta a_o g_4(k) \}$$

$$\hat{g}_4(k) = e^{ik\cdot a}\, g_4(k) \tag{4.44}$$

$$\hat{a}_+^+(k) = e^{-ik\cdot a}\, a_+^+(k)$$

$$\hat{a}_-^+(k) = e^{-ik\cdot a}\, a_-^+(k)$$

$$\hat{g}_3^+(k) = e^{-ik\cdot a}\, g_3^+(k)$$

$$\hat{g}_4^+(k) = e^{-ik\cdot a}\, \{ g_4^+(k) + 2i\beta a_o g_3^+(k) \} \quad , \tag{4.45}$$

where β is given in (4.42). Under the action of Lorentz transformations,

$$\hat{A}_\mu(x) \to A_\mu(x) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x) \quad ,$$

we have

$$\hat{a}_+(k) = a_+(\Lambda^{-1}k) - i(\alpha+1)\, \frac{\gamma}{\sqrt{\alpha}}\, (\Lambda^{-1}k)_o\, \varepsilon_\mu^*(k)\, \Lambda^{\mu o} g_4(\Lambda^{-1}k)$$

$$\hat{a}_-(k) = a_-(\Lambda^{-1}k) - i(\alpha+1)\, \frac{\gamma}{\sqrt{\alpha}}\, (\Lambda^{-1}k)_o\, \varepsilon_\mu^*(k)\, \Lambda^{\mu o} g_4(\Lambda^{-1}k)$$

$$\begin{aligned} \hat{g}_3(k) = {} & g_3(\Lambda^{-1}k) + 2i\sqrt{\alpha}\, \frac{\gamma}{|\vec{k}|}\, \Lambda_o{}^\nu \{ \varepsilon_\nu(\Lambda^{-1}k) a_+(\Lambda^{-1}k) + \varepsilon_\nu^*(\Lambda^{-1}k) a_-(\Lambda^{-1}k) \} \\ & + \gamma \left\{ \frac{\alpha+1}{(\Lambda^{-1}k)_o} \left[1 - \left(\frac{(\Lambda^{-1}k)_o}{k_o} \right)^2 \right] - 2(\alpha-1)(\Lambda^{-1})_o{}^\ell \frac{\partial}{\partial k^\ell} \right. \\ & \left. - 2(\alpha+1)(\Lambda^{-1})_o{}^\ell \frac{k_\ell}{|\vec{k}|^2} \right\} \left[(\Lambda^{-1}k)_o\, g_4(\Lambda^{-1}k) \right] \end{aligned}$$

$$\hat{g}_4(k) = \left(\frac{(\Lambda^{-1}k)_o}{k_o}\right)^2 g_4(\Lambda^{-1}k) \quad . \tag{4.46}$$

$$\hat{a}_+^+(k) = a_+^+(\Lambda^{-1}k) + \frac{i(\alpha+1)}{\sqrt{\alpha}} \frac{\gamma}{(\Lambda^{-1}k)_o} \varepsilon_\mu(k)\Lambda^{\mu o} g_3^+(\Lambda^{-1}k)$$

$$\hat{a}_-^+(k) = a_-^+(\Lambda^{-1}k) + \frac{i(\alpha+1)}{\sqrt{\alpha}} \frac{\gamma}{(\Lambda^{-1}k)_o} \varepsilon_\mu^*(k)\Lambda^{\mu o} g_3^+(\Lambda^{-1}k)$$

$$\hat{g}_3^+(k) = g_3^+(\Lambda^{-1}k)$$

$$\hat{g}_4^+(k) = \left(\frac{k_o}{(\Lambda^{-1}k)_o}\right)^2 g_4^+(\Lambda^{-1}k) - 2i\sqrt{\alpha}\,\gamma|\vec{k}|\Lambda_o^{\ \nu}\{\varepsilon_\nu^*(\Lambda^{-1}k)a_+^+(\Lambda^{-1}k)$$

$$+\varepsilon_\nu(\Lambda^{-1}k)a_-^+(\Lambda^{-1}k)\} + \gamma\{(\alpha+1)(\Lambda^{-1}k)_o\left[\left(\frac{(\Lambda^{-1}k)_o}{k_o}\right)^2 - 1\right]$$

$$-2(\alpha-1)|\vec{k}|^2(\Lambda^{-1})_o^{\ \ell}\frac{\partial}{\partial k^\ell} + 2(\alpha+1)(\Lambda^{-1})_o^{\ \ell}\, k_\ell\}\left(\frac{g_3^+(\Lambda^{-1}k)}{(\Lambda^{-1}k)_o}\right) \tag{4.47}$$

where γ is given in (4.18).

We are now in a position to discuss possible physical interpretations of the model. Since the Poincaré group does not act unitarily on the whole Hilbert space $\mathcal{H}$, we first search for a space where it does.

A look at the Hamiltonian (4.42) reveals that, except for $\alpha = 1$, time translations are not implemented unitarily on $\mathcal{H}$. However, on a subspace $\mathcal{H}'$ defined by

$$g_4\Big|_{\mathcal{H}'} = 0 \quad , \tag{4.48}$$

the Hamiltonian is hermitean, as well as being positive definite and independent of α. By (4.44) and (4.46), this condition is Lorentz invariant. Since we have

$$\partial\cdot A = 2\dot{\xi}_4 = \frac{1}{(2\pi)^{3/2}}\,\frac{2\gamma}{\sqrt{\alpha}}\int\frac{d^3k}{2|\vec{k}|}\,\{|\vec{k}|^2 g_4(k)e^{-ik\cdot x} + g_3^{+}(k)\,e^{ik\cdot x}\} \quad (4.49)$$

condition (4.48) is equivalent to the weak Lorentz condition

$$\partial\cdot A^{(+)}\Big|_{\mathcal{H}'} = 0 \quad . \quad (4.50)$$

The elimination of the $g_4(k)$ quanta then allows two possible interpretations of the model to be made.

(1) We could define a physical subspace $\mathcal{H}_P$ by

$$a_{\pm}(k)\Big|_{\mathcal{H}_P} = 0 \quad . \quad (4.51)$$

In this space, which contains only the $g_3(k)$ quanta, the Poincaré group is implemented unitarily. The vector potential in this scheme describes one physical degree of freedom.

(2) Alternatively, we could group states created by $a_{+}^{+}(k)$ and $a_{-}^{+}(k)$ into equivalence classes: two states in $\mathcal{H}'$ are said to belong to the same equivalence class if they differ only by the number of $g_3(k)$ quanta they contain. The physical space would then be taken as the quotient space

$$\mathcal{H}_P = \mathcal{H}'/\mathcal{H}'' \quad (4.52)$$

where $\mathcal{H}''$ is the Poincaré invariant subspace of $g_3(k)$ quanta. Again on $\mathcal{H}_P$, the Poincaré group is unitarily implemented. In this interpretation, the vector potential describes the usual two transverse degrees of freedom.

Of course, in the latter case one can also construct a physical space describing only one of the two transverse photons. However, in theories involving some unphysical degrees of freedom, one expects that the maximal possible physical interpretation should be given.

For this reason we would strongly favour the second of the two interpretations. From a pragmatic point of view, however, either of the two is viable at this time.

C. A Physical Interpretation

Up to now, we have ignored the existence of the invariance of the theory under the restricted gauge transformation (4.6),

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{4.53}$$

with $\Box\Lambda = 0$. We now consider it. Under such a transformation, the objects of the decomposition change as follows:

$$A^t_\mu \to A^t_\mu \quad , \qquad \phi_3 \to \phi_3 + \Lambda \quad , \qquad \phi_4 \to \phi_4 \quad . \tag{4.54}$$

If we define

$$\Lambda(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|\vec{k}|} \left[c(k)e^{-ik\cdot x} + \bar{c}(k)e^{ik\cdot x} \right] \quad , \tag{4.55}$$

the Fourier amplitudes transform as

$$\begin{aligned}
a_+(k) &\to a_+(k) & \bar{a}_+(k) &\to \bar{a}_+(k) \\
a_-(k) &\to a_-(k) & \bar{a}_-(k) &\to \bar{a}_-(k) \\
a_3(k) &\to a_3(k) + c(k) & \bar{a}_3(k) &\to \bar{a}_3(k) + \bar{c}(k) \\
a_4(k) &\to a_4(k) & \bar{a}_4(k) &\to \bar{a}_4(x) \quad .
\end{aligned} \tag{4.56}$$

The generator Q_Λ of this transformation, which satisfies

$$[Q_\Lambda, A_\mu(x)] = -i\partial_\mu \Lambda(x) \quad , \tag{4.57}$$

is given by

$$Q_\Lambda = -\alpha \int d^3k \{\bar{c}(k)a_4(k) + c(k)\bar{a}_4(k)\}$$

$$= -2i\gamma\sqrt{\alpha}\int \frac{d^3k}{2|\vec{k}|}\{|\vec{k}|^2\bar{c}(k)g(k) - c(k)g_3^\dagger(k)\} \quad . \qquad (4.58)$$

We thus see that if we take the first of the two interpretations of the model to be the physical one, restricted gauge transformations would not constitute a symmetry because Q_Λ is not selfadjoint on the physical subspace. Only the second interpretation allows such transformations to be a symmetry of the theory.

It is tempting to exclude the existence of the $g_3(k)$ quanta by postulating gauge invariance of observables, as then the theory would be given the maximal physical interpretation. As with the Higg's model, however, there is no cogent reason at the free field level to make such a postulate, and thus at this time we cannot rule out the first interpretation. The question now is whether or not each interpretation allows for a physically sensible set of observables, and how these observables are to be measured. We must, therefore, as we had to do with the Higg's model, consider an interacting theory.

Of course, the simplest nontrivial interaction we can consider is coupling the vector field to a conserved c-number current $J_\mu(x)$, and this is the one we study. The field equations now read

$$\Box A_\mu + (\alpha-1)\partial_\mu \partial\cdot A = J_\mu \quad . \qquad (4.59)$$

The solution is given by

$$A_\mu(x) = A_\mu^{in}(x) + \int d^4y\, G_{\mu\nu}^{ret}(x-y)J^\nu(y)$$

$$= A_\mu^{out}(x) + \int d^4y\, G_{\mu\nu}^{adv}(x-y)J^\nu(y) \qquad (4.60)$$

where

$$G^{ret}_{\mu\nu}(x) = i\theta(x_o)D_{\mu\nu}(x)$$

$$G^{adv}_{\mu\nu}(x) = -i\theta(-x_o)D_{\mu\nu}(x)$$

$$D_{\mu\nu}(x) = [A_\mu(x+y), A_\nu(y)]$$
$$= \frac{i}{2\pi} g_{\mu\nu}\varepsilon(x_o)\delta(x^2) + \frac{i}{8\pi}\frac{1-\alpha}{\alpha}\partial_\mu\partial_\nu[\varepsilon(x_o)\theta(x^2)]$$

and $A^{in}_\mu(x)$ or $A^{out}_\mu(x)$ satisfy the asymptotic form of equations (4.59) with appropriate boundary conditions.

From the solutions (4.60) we can find the S operator, up to an arbitrary c-number phase factor. We have

$$A^{out}_\mu(x) = S^{-1}A^{in}_\mu(x)S$$
$$= A^{in}_\mu(x) - i\int d^4y\,[A^{in}_\mu(x), A^{in}_\nu(y)]J^\nu(y) \quad , \qquad (4.61)$$

which gives

$$S = \exp\{-i\int d^4y\, A^{in}(y)\cdot J(y)\} \quad . \qquad (4.62)$$

This furthermore factorizes:

$$S = S_t \otimes S_g \quad , \qquad (4.63)$$

where

$$S_t = \exp\{-i\int d^4y\, A^t_{in}(y)\cdot J(y)\} \qquad (4.64)$$

carries the two transverse degrees of freedom, and

$$S_g = \exp\{-2i\alpha\int d^4y\, \phi^{in}_4(y)J_o(y)\} \qquad (4.65)$$

carries the $g_3(k)$ degree of freedom.

Now, since $A^t_{\mu,in}$ is hermitean but

$$\phi^{in}_4(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2}\{a_4(k)e^{-ik\cdot x} + \bar{a}_4(k)e^{ik\cdot x}\}$$
$$= \frac{1}{(2\pi)^{3/2}}\frac{i\gamma}{\sqrt{\alpha}}\int\frac{d^3k}{2|\vec{k}|}\{|\vec{k}|g_4(k)e^{-ik\cdot x} - \frac{1}{|\vec{k}|}g^\dagger_3(k)e^{ik\cdot x}\}, \qquad (4.66)$$

we see that only on the physical space defined in the second of the two interpretations is the S operator unitary. In order to conserve probabilities, and therefore have a physically sensible interacting theory, we are forced to choose the second of the two interpretations. As we have seen, this may be enacted by postulating that observables must be invariant under restricted gauge transformations. The physical space $\mathcal{H}_P$ is then the quotient space $\mathcal{H}'/\mathcal{H}''$, where we recall $g_4\big|_{\mathcal{H}'} = 0$ and $\mathcal{H}''$ is the Poincare invariant space carrying the g_3 quanta. On $\mathcal{H}_P$, Poincare transformations are unitarily implemented, restricted gauge transformations are a symmetry, and the S operator for a coupling to a conserved external current is unitary, as well as independent of α. The only physical degrees of freedom are then the two transverse photons.

Finally, let us consider how observables are described in this theory. A potential observable is constructed using an operator θ that, on $\mathcal{H}'$, factorizes into a physical (transverse) part and a part corresponding to the g_3 quanta:

$$\theta\big|_{\mathcal{H}'} = \theta_t \otimes \theta_g \qquad (4.67)$$

Restricted to the physical space $\mathcal{H}_P$, θ must assume the form

$$\theta\big|_{\mathcal{H}_P} = \theta_t \otimes 1 \qquad (4.68)$$

with θ_t being selfadjoint. Since the generator Q_Λ of restricted gauge transformations is the unit operator on $\mathcal{H}_P$ (see (4.58)), any θ of the form (4.68) will commute with it, verifying that these transformations are indeed a symmetry of the theory. As a last point, we can check that for vectors $\Psi, \Phi \in \mathcal{H}_P$,

$$(\Psi, \Box A_\mu + (\alpha-1)\partial_\mu \partial\cdot A\Phi) = (\Psi, \Box A^t_\mu \Phi) = 0 \ . \tag{4.69}$$

Thus, we have recovered Maxwell's equations supplemented by the Lorentz condition in a weak form on $\mathcal{H}_P$.

We have thus seen that it is possible to quantize the electromagnetic field in a covariant manner on a positive metric Hilbert space. What is unusual in this treatment is the existence of two different physical spaces where the Poincaré group acts unitarily, and the corresponding two different physical interpretations at the free field level. Only when an interaction is present is it possible to give a physical argument favouring one of the two. For probability conservation, the S operator must be unitary. On only one of the two physical spaces does this happen, and this is the only one also where a restricted gauge invariance holds. As with the Higg's model, then, postulating gauge invariance of observables guarantees that the resulting subspace describes a physically sensible interacting theory.

Although certainly more involved than an indefinite metric quantization, and therefore not so amiable to practical calculations, it is satisfying to see that the positive metric quantization leads to the maximal physical interpretation of these theories. Just as using an indefinite metric showed that a covariant local formulation of these theories was possible (as opposed to a Coulomb-type gauge formalism), the use of the positive definite scalar product has shown how the concepts of negative and zero norm states, pseudo-unitarity and pseudo-hermiticity translate from an indefinite metric space into a Hilbert space. It is an indication of the power of these theories that their physical content is the same no matter which formalism is chosen.

At this point the classical potential (the negative sum of all non-derivative terms in the Lagrange density (5.4)) is minimized ($b^2 = 0$ is a maximum). However, quantum effects would be expected to alter this relation. How are they to be taken into account? A function known as the effective potential allows just such an investigation to be undertaken systematically, and we do see perturbatively how the relation (5.5) is modified.

We begin by setting up the general formalism through which the effective potential is defined. Two methods that are generally used to calculate it perturbatively are then outlined: a direct infinite summation of graphs and a path integral approach. We describe a third method that, although recognized before but not widely used, actually involves about the same amount of labour a the path-integral approach. The method is illustrated both for a loop expansion of the potential and in the 1/N approximation. In the course of defining the effective potential in this formalism, we shall encounter a function $Z(\tilde{\phi})$ that is related to the wave function renormalization of the field. This function also has been evaluated by two methods similar to those used to find the effective potential. We next describe a third method that can be used to find this function. Finally we point out that the methods used here, independent of contact with the effective potential and $Z(\tilde{\phi})$, lead to a relatively efficient evaluation of the counterterms of a theory, and hence to a determination of the renormalization group equation coefficients.

A. General Formalism

We start with an introduction of a formalism that is found convenient to use in theories with spontaneous breakdown of symmetry [T2, C3, C4, I2]. A major problem in field theory is to calculate the Green functions, and for this the concept of generating functionals is useful. For notational simplicity, we work with a single real scalar field with Lagrange density $\mathcal{L}(\phi)$. We now consider the effect of adding to this Lagrange density an external c-number source J(x) in the following manner:

$$\mathcal{L}(\phi) \rightarrow \mathcal{L}(\phi) + J(x)\phi(x) \quad . \tag{5.6}$$

The complete generating functional H(J) is defined as

$$\begin{aligned} H(J) &= \langle 0^+ | 0^- \rangle_J \\ &= \langle 0 | T\{\exp i \int d^4x \, \phi(x) J(x)\} | 0 \rangle \quad . \end{aligned} \tag{5.7}$$

By expanding the exponential in (5.7), it is seen that H(J) generates the n-point Green functions $G^{(n)}(x_1, \ldots x_n) = \langle 0 | T\{\phi(x_1) \ldots \phi(x_n)\} | 0 \rangle$, and hence can be written as

$$H(J) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, J(x_1) \ldots J(x_n) G^{(n)}(x_1, \ldots x_n) \quad . \tag{5.8}$$

Differentiating H(J) n times with respect to $J(x_1), \ldots, J(x_n)$, and then setting $J = 0$, yields $G^{(n)}(x_1 \ldots x_n)$. These Green functions contain both connected and disconnected parts (see Fig. 1(a)). It would be convenient to be able to work with only the connected graphs, however, as the disconnected graphs can easily be reconstructed from them. The connected generating functional $W(J) = -i \ln H(J)$ can in fact be shown to generate only connected graphs:

$$\begin{aligned} W(J) &:= -i \ln H(J) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, J(x_1) \ldots J(x_n) G_c^{(n)}(x_1 \ldots x_n) \quad . \end{aligned} \tag{5.9}$$

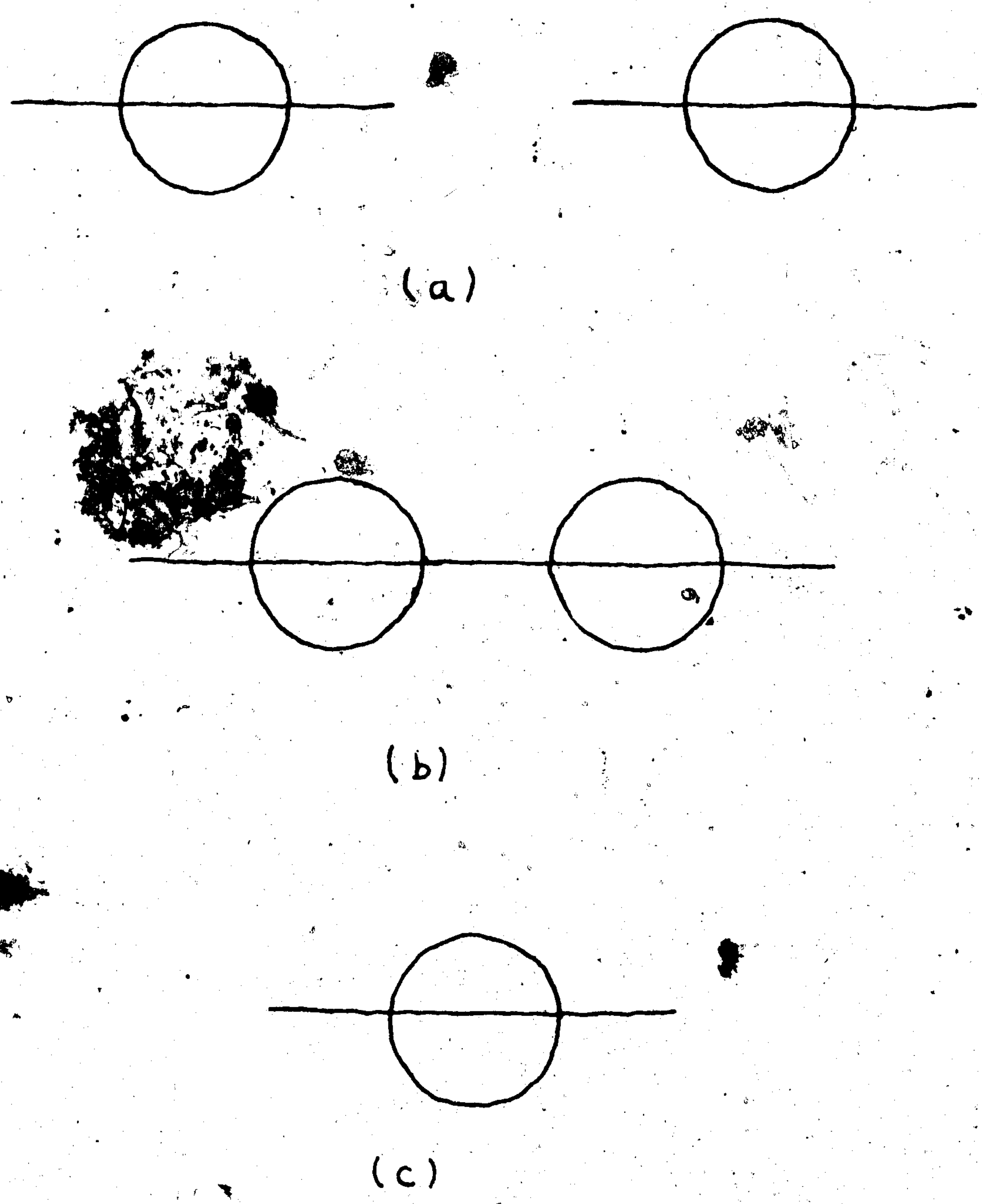


Fig.1 (a) Example of a disconnected four-point function in a ϕ^4 theory.
(b) Example of a connected, but not 1PI, two-point function in a ϕ^4 theory.
(c) Example of a 1PI two-point function in a ϕ^4 theory.

Here, $G_c^{(n)}(x_1 \ldots x_n)$ is the n-point Green function composed only of connected graphs. There is a further simplification possible, however. Suppose we have a graph that can be made disconnected by cutting a single internal line (see Fig. 1(b)). The complete graph would then factorize into a product of one-particle-irreducible (1PI) graphs (those that can't be made disconnected by cutting a single internal line - see Fig. 1(c)). For this reason we would like a generating functional that generates only 1PI graphs, since the complete Green functions of the theory can be reconstructed by knowledge of them. Such a functional can indeed be found. We first define the classical field $\tilde{\phi}(x)$:

$$\tilde{\phi}(x) = \frac{\delta W(J)}{\delta J(x)} = \frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle} \tag{5.10}$$

The effective action $\Gamma(\tilde{\phi})$ is then defined by a functional Legendre transformation:

$$\Gamma(\tilde{\phi}) = W(J) - \int d^4x\, J(x)\tilde{\phi}(x) \quad . \tag{5.11}$$

A functional Taylor expansion of $\Gamma(\tilde{\phi})$ leads to

$$\Gamma(\tilde{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, \tilde{\phi}(x_1) \ldots \tilde{\phi}(x_n)\, \Gamma^{(n)}(x_1 \ldots x_n) \ , \tag{5.12}$$

where the coefficients $\Gamma^{(n)}(x_1 \ldots x_n)$ are in fact the 1PI n-point Green functions.

We can also expand $\Gamma(\tilde{\phi})$ in an alternate [illegible] instead of an expansion in powers of $\tilde{\phi}(x)$, we can expand in powers of momentum, about the point where all external momenta vanish. Such an expansion starts like

$$\Gamma(\tilde{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x [\tilde{\phi}(x)]^n\, \tilde{\Gamma}^{(n)}(0) + \text{momentum dependent terms}, \tag{5.13}$$

where $\tilde{\Gamma}^{(n)}(p_i)$ is the Fourier transform of $\Gamma^{(n)}(x_i)$. In position space, the expansion looks like

$$\Gamma(\tilde{\phi}) = \int d^4x\, [-V(\tilde{\phi}) + \frac{1}{2} Z(\tilde{\phi})(\partial_\mu \tilde{\phi})^2 + \text{higher order derivatives of } \tilde{\phi}(x)] \tag{5.14}$$

where $V(\tilde{\phi})$ is called the effective potential. Comparing (5.13) and (5.14), we see that

$$V(\tilde{\phi}) = -\sum_{n=0}^{\infty} \frac{1}{n!} [\tilde{\phi}]^n \tilde{\Gamma}^{(n)}(0) \quad . \tag{5.15}$$

Taylor expanding V about $\tilde{\phi} = 0$, we see that the n^{th} derivative of V with respect to $\tilde{\phi}$, evaluated at $\tilde{\phi} = 0$, is the sum of all 1PI graphs with n vanishing external momenta. In the tree approximation (i.e., neglecting all closed loop graphs), V is just the classical potential, the negative sum of all nonderivative terms in the Lagrange density.

We now inquire as to how spontaneous symmetry breaking can be recognized in this formalism. It is in fact the ease with which this can be done that makes the formalism so useful. From (5.11), we see that

$$\frac{\delta\Gamma(\tilde{\phi})}{\delta\tilde{\phi}(x)} = -J(x) \quad . \tag{5.16}$$

When the source J(x) vanishes, the classical field will no longer be space-time dependent (if we assume the vacuum to be Poincaré invariant; see (5.10)). Therefore, for vanishing source (5.16) simplifies to

$$\frac{dV(\tilde{\phi})}{d\tilde{\phi}} = 0 \quad . \tag{5.17}$$

The presence of a solution $\tilde{\phi} \neq 0$ to this equation indicates that the field has developed a nonzero vacuum expectation value, and hence spontaneous symmetry breakdown has occurred. To be stable against small perturbations, this solution must be a minimum of V. Thus, if we can succeed in evaluating the effective potential, we have a way of exploring the effects of quantum fluctuations on the classical potential.

Of course, in most realistic theories the exact effective potential cannot be found, and so it is important to have some suitable approximation scheme for it. Before dealing with this, though, let us anticipate a little and question how we are to remove the infinities that will inevitably occur in the calculations. We consider for definiteness a massive ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} g \phi^4 + \text{counterterms} . \qquad (5.18)$$

The usual renormalization conditions for this theory are that the propagator has a pole of unit residue at $p^2=m^2$, the physical mass:

$$\Gamma^{(2)}(p^2=m^2) = 0$$

$$\left. \frac{\partial \Gamma^{(2)}}{\partial p^2} \right|_{p^2=m^2} = -i \qquad (5.19)$$

and that the four-point function be equal to the negative physical coupling constant at the symmetric point S(m):

$$\left. \Gamma^{(4)}(p_i) \right|_{S(m)} = -ig \qquad (5.20)$$

where S(m) is defined by

$$p_i^2 = m^2 \qquad i = 1,2,3,4$$

$$(p_i+p_j)^2 = \frac{4}{3} m^2 \qquad i \neq j \quad .$$

Once these conditions are met by adjustment of the counterterms, all further Green functions will be finite. Here we have renormalized using the physical mass and physical coupling constant, and the "physical"

Green functions so calculated will be directly related to the S matrix.

However, sometimes it is not so convenient to choose the physical mass m as the renormalization point. We could, in fact, choose an arbitrary point M:

$$\Gamma^{(2)}(p^2=M^2) = 0$$

$$\left.\frac{\partial\Gamma^{(2)}}{\partial p^2}\right|_{p^2=M^2} = -i$$

$$\Gamma^{(4)}(p_i)\Big|_{S(M)} = -ig_M \quad . \tag{5.21}$$

The renormalizability of the theory then implies that a change in the subtraction point M can be compensated by a change in the value of the coupling constant and a rescaling of the fields:

$$\Gamma^{(n)}(p_i, M_1^2, g_{M_1}) = Z^{n/2}(M_2^2, M_1^2, g_{M_1})\,\Gamma^{(n)}(p_i, M_2^2, g_{M_2}) \quad . \tag{5.22}$$

This is a finite renormalization now. In this way we can, from any convenient choice of M, recover the physical Green functions. It often happens that the choice M = 0 is particularly convenient:

$$\Gamma^{(2)}(p^2=0) = im^2$$

$$\left.\frac{\partial\Gamma^{(2)}}{\partial p^2}\right|_{p^2=0} = -i$$

$$\Gamma^{(4)}(p_i=0) = -ig \quad . \tag{5.23}$$

The quantities m^2 and g^2 are no longer necessarily the physical ones, but are related to them by a finite renormalization. (Since we shall not be concerned with this finite renormalization, we shall not distinguish the physical parameters from m^2 and g, but one should always keep that in mind.)

The reason the choice $M = 0$ is so convenient here is that the first and third conditions in (5.23) can be readily transferred into conditions on the effective potential. Since the n^{th} derivative of V with respect to $\tilde{\phi}$, evaluated at $\tilde{\phi} = 0$, is the sum of all 1PI graphs with n vanishing external momenta, the two conditions become

$$\left.\frac{d^2V}{d\tilde{\phi}^2}\right|_{\tilde{\phi}=0} = m^2$$

$$\left.\frac{d^4V}{d\tilde{\phi}^4}\right|_{\tilde{\phi}=0} = g \qquad (5.24)$$

The second condition in (5.23), which determines the wave-function renormalization counterterm, is given by

$$Z(0) = 1 \qquad (5.25)$$

where [illegible] defined in (5.14).

B. Evaluating the Effective Potential

In this section we will outline briefly two common methods of obtaining a perturbative approximation to the effective potential, and then give a more detailed account of a third method. This latter method, while known for some time, does not seem to have been fully exploited. When used in conjunction with dimensional regularization, it is a fairly efficient procedure for doing such calculations. We illustrate the method both in the loop expansion and in the 1/N approximation.

1. The Loop Expansion

The first perturbative approximation beyond the tree level of the effective potential was done by S. Coleman and E. Weinberg [C3].

They directly applied the definition (5.15):

$$V(\tilde{\phi}) = - \sum_{n=0}^{\infty} [\tilde{\phi}]^n \tilde{\Gamma}^{(n)}(0) \tag{5.26}$$

where $\tilde{\Gamma}^{(n)}(0)$ is the n-point IPI Green function in momentum space with zero external momenta. Of course, the best we can hope for is some suitable approximation scheme for these functions. One particularly useful scheme in this context is the loop expansion – a function is expanded according to the number of closed loops in a graph. The reason this is useful is the following. Suppose we introduce into the Lagrange density an overall multiplicative factor a:

$$\mathcal{L}(\phi,a) = a^{-1} \mathcal{L}(\phi) \ . \tag{5.27}$$

The power of a associated with a graph can then be shown to be L-1, where L is the number of loops in the graph (see [C3]). The loop expansion is thus equivalent to a power series expansion in a. Such an expansion is not good for the reason that a is small; indeed, a is equal to one. (However, since the set of graphs with up to n loops contains the set of all graphs of n^{th} order or less in the coupling constants, the expansion is at least as reliable as one in the coupling constants.) The reason it is useful here is that the factor a multiplies the whole Lagrange density, and as such the expansion is unaffected by any shift of a field. Thus, the expansion preserves an important feature of the effective potential: it allows us to survey all possible vacuum states at once, before the theory decides which one to pick.

Without going into the [illegible] (see [C3]), let us see how this expansion is applied to a ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} g\phi^4 \quad . \tag{5.28}$$

One first expands each n-point 1PI momentum space Green function with vanishing external momenta according to the number of loops (see Fig. 2(a)). According to (5.26), [illegible] then the sum of all such n-point functions, the sum again ordered according to the number of loops, with an appropriate factor of ϕ (see Fig. 2(b)). Each order of the loop expansion thus contains an infinite summation of graphs, although at the one-loop level this is a fairly manageable sum. However, combinatorial factors quickly make going beyond the one-loop level a formidable task. For this reason a more efficient evaluation of the effective potential was sought.

The path-integral approach proved to be just such an evaluation. Again without going into details, let us sketch how the effective potential may be evaluated in this framework (see [J1, I2]). One starts with the path integral expression for the generating functional H(J), equation (5.7),

$$H(J) = e^{iW(J)} = \langle 0|T\{\exp i\int d^4x\, J(x)\phi(x)\}|0\rangle$$

$$= H^{-1}(0)\int \mathcal{D}\phi\, \exp\{i\int d^4x[\mathcal{L}(\phi) + J(x)\phi(x)]\} \quad . \tag{5.29}$$

Let us again consider the ϕ^4 theory in (5.28). A standard method of handling expressions like that in (5.29) is the method of steepest descent. One expands the exponent in the numerator of the right-hand-side in the second equation of (5.29) about the point $\phi_0(J)$ at which it is stationary. This stationary point is the solution of the classical equation

$$(\Box + m^2)\phi_0 + \frac{1}{6} g\phi_0^3 = J(x) \quad . \tag{5.30}$$

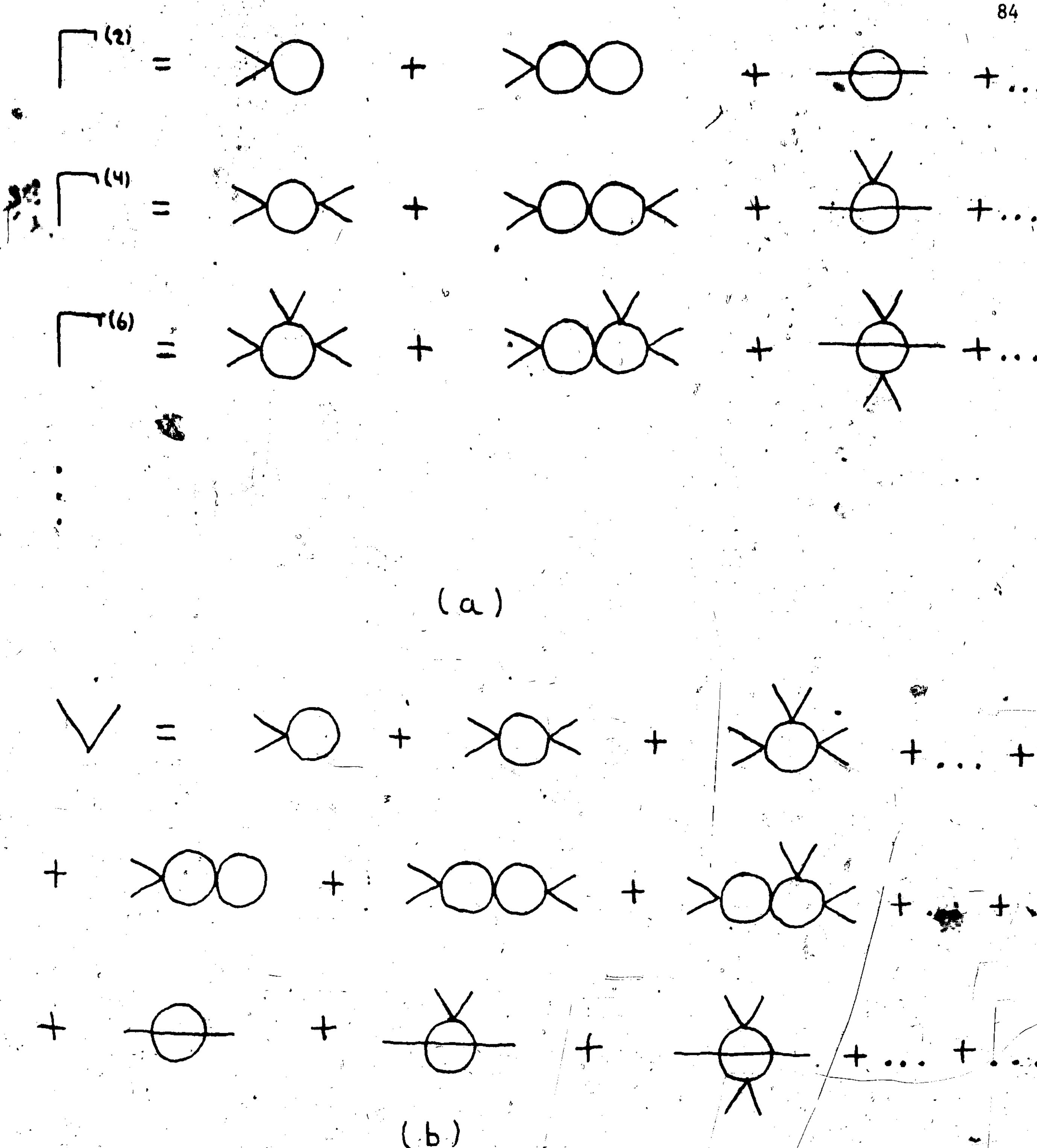


Fig. 2 (a) Loop expansion of 1PI functions with zero external momenta in a ϕ^4 theory.
(b) The corresponding loop expansion of the effective potential.

If we demand J(x) vanish at infinity, then the solution of (5.30) is unique if it is also required to vanish at infinity. The expansion about this point can be shown to be equivalent to the loop expansion just discussed. Performing the Legendre transformation (5.11), one obtains the corresponding expansion for $\Gamma(\phi)$, and from this the loop expansion of $V(\tilde{\phi})$ is found. This approach turns out to be very economical: at each stage of the loop expansion, only a finite number of graphs must be evaluated. For the ϕ^4 theory, at the one-loop level only one graph has to be calculated (see Fig. 3(a)); at the two-loop level, only two graphs (see Fig. 3(b)). As opposed to an infinite summation of graphs at each level, there is a substantial reduction of labour involved in this approach. (Note the similarities in the graphs of Figs. (2) and (3)).

We intend now to describe a third method of evaluating the effective potential. Like the functional approach, at each level of the loop expansion only a finite number of graphs must be calculated (although this number is more than in the path-integral method). The method was first found by Weinberg ([W3], see also [L4]); however, with the advent of dimensional regularization its efficiency in evaluating the effective potential doesn't seem to have been fully recognized. Although it involves more algebra than the path-integral approach, essentially no new integrals emerge, and one deals directly with the Lagrange density. (In effect, the graphs involved are obtained by attaching a single external zero momentum leg to the graphs in the path-integral approach, Fig. 3.)

Recall the definition of the effective potential (5.26): given a Lagrange density $\mathcal{L}(\phi)$, $V(\tilde{\phi})$ is defined by

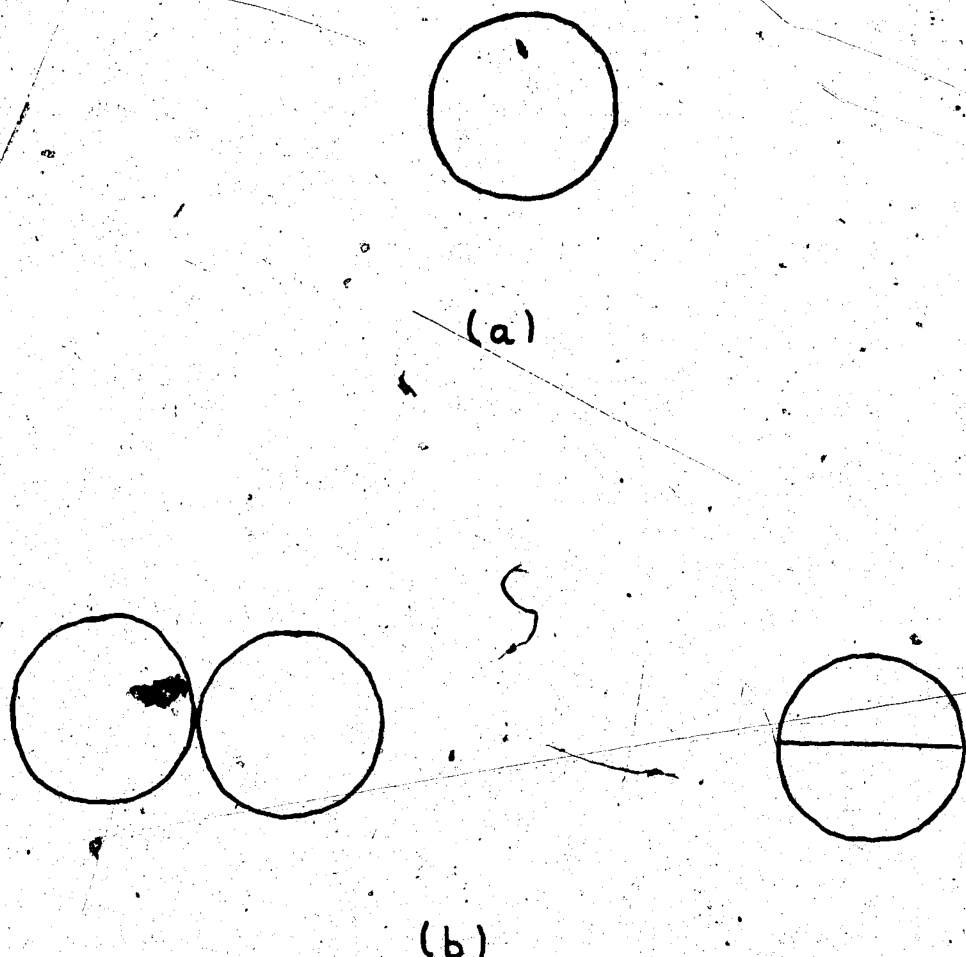


Fig. 3 (a) The graph involved at the one-loop level for $V(\tilde{\phi})$ in the path-integral approach.

(b) The graphs involved at the two-loop level for $V(\tilde{\phi})$ in the path-integral approach.

$$V(\tilde{\phi}) = -\sum_{n=0}^{\infty} [\tilde{\phi}]^n \tilde{\Gamma}^{(n)}(0) \quad , \tag{5.31}$$

where $\tilde{\Gamma}^{(n)}(0)$ is the n-point 1PI Green function with zero external momentum in momentum space, corresponding to $\mathcal{L}(\phi)$. Suppose instead we had used a Lagrange density $\mathcal{L}(\phi-a)$, where a is some constant. There would then be different n-point 1PI Green functions $\tilde{\Gamma}^{(n)'}(0)$, and this would lead to the expansion

$$V(\tilde{\phi}) = -\sum_{n=0}^{\infty} [\tilde{\phi}-a]^n \tilde{\Gamma}^{(n)'}(0) \quad . \tag{5.32}$$

Differentiating this relation once with respect to $\tilde{\phi}$ and evaluating at $\tilde{\phi} = a$ gives

$$\left.\frac{dV(\tilde{\phi})}{d\tilde{\phi}}\right|_{\tilde{\phi}=a} = -\tilde{\Gamma}^{(1)'}(0) \quad . \tag{5.33}$$

Here, $\tilde{\Gamma}^{(1)'}(0)$ is the one-point 1PI Green function, calculated using $\mathcal{L}(\phi-a)$, and represents a particle decaying into the vacuum. Thus, if we make a loop expansion of the one-point function in the theory described by $\mathcal{L}(\phi-a)$, we have in effect a loop expansion of the first derivative of the effective potential. An integration then yields the effective potential itself. We thus have a picture of what setting the first derivative of the effective potential to zero means: in the theory with Lagrange density $\mathcal{L}(\phi-a)$, all particle-to-vacuum transitions must vanish, this allowing a particle interpretation of the theory.

As an example of a calculation using this method, we will evaluate the effective potential to two loops of an n-component massless ϕ^4 theory. This result has already been obtained by Jackiw using a path-integral approach [J1]. The Lagrange density, including the relevant counterterm, is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{24} g(\phi^a \phi^a)^2 - \frac{1}{24} C(\phi^a \phi^a)^2 \tag{5.34}$$

where $a = 1,2,\ldots n$. To the level we will be working, the wave-function renormalization does not enter. We now perform the shift $\phi_1 \to \phi_1 + a$, and obtain

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{6} g a^3 \phi_1 - \frac{1}{2}\mu^2 \phi_1^2 - \frac{1}{2} m^2 \phi_i^2 \\ & - \frac{1}{6} g a \phi_1 (\phi_1^2 + \phi_i^2) - \frac{1}{24} g(\phi_1^2 + \phi_i^2)^2 \\ & - \frac{1}{6} C a^3 \phi_1 - \frac{1}{4} a^2 C \phi_1^2 - \frac{1}{12} a^2 C \phi_i^2 \\ & - \frac{1}{6} a C \phi_1 (\phi_1^2 + \phi_i^2) - \frac{1}{24} C(\phi_1^2 + \phi_i^2)^2 \end{aligned} \tag{5.35}$$

where

$$i = 2,3,\ldots n \quad , \quad \mu^2 = \frac{1}{2} a^2 g \quad , \quad m^2 = \frac{1}{6} a^2 g \ .$$

We now proceed to evaluate the IPI one-point function. Let T denote the complete function, and let us expand it as

$$T = T_o + T_1 + T_2 + \ldots \tag{5.36}$$

where T_o denotes the no-loop (tree) contribution, T_1 the one loop term, etc. A similar expansion is made for the counterterm C, except that for the renormalization conditions we will use, $C_o = 0$.

The tree approximation to T is pictured in Fig. 4(a), and is given by

$$T_o = \frac{1}{6} g a^3 \ . \tag{5.37}$$

At the one-loop level we will encounter ultraviolet divergences, and thus a regularization scheme is required. One particularly useful scheme is dimensional regularization [H7, H8, B7, A4]: an integral in four dimensions is analytically continued to d dimensions, and the

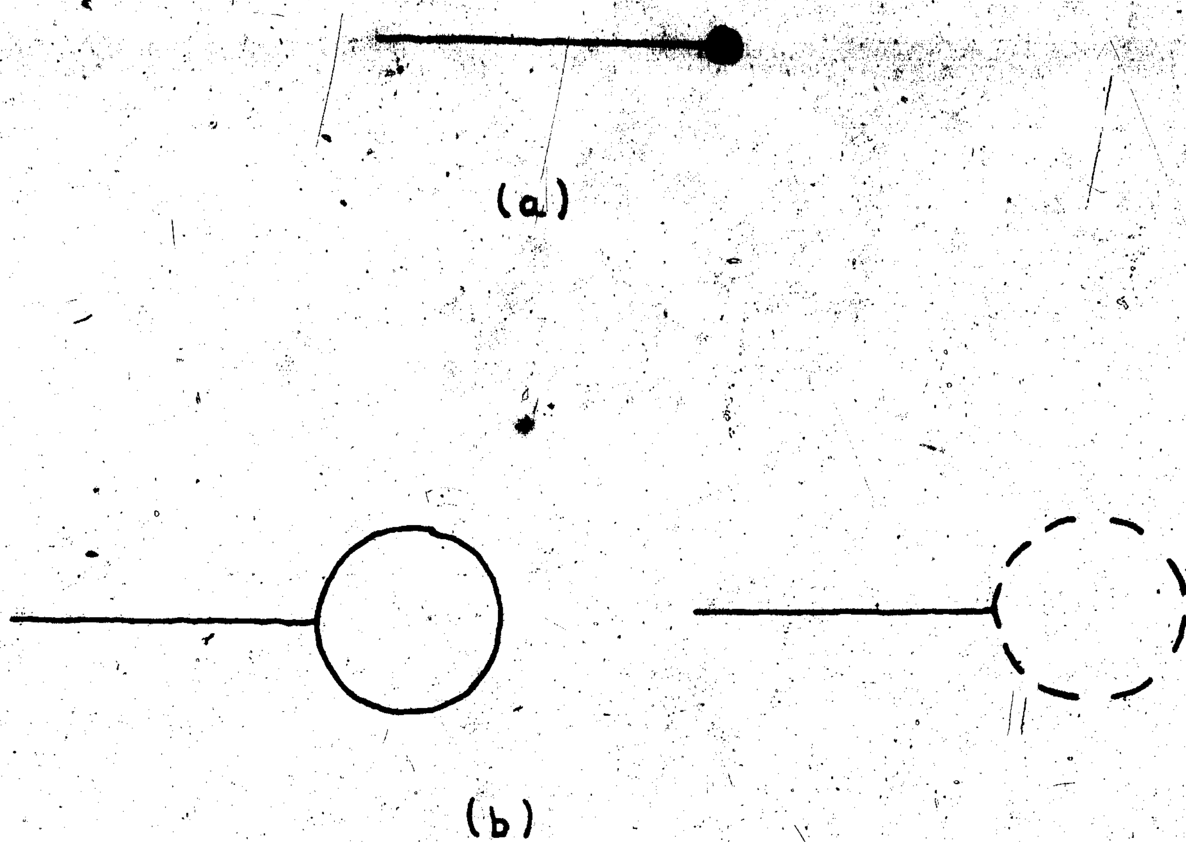


Fig. 4 (a) The zero-loop approximation to T.
(b) The one-loop contribution to T. A solid line denotes ϕ_1, a dashed line ϕ_i.

poles at $d = 4$, if they exist, are isolated. It is then up to the renormalization program to remove these poles in an unambiguous manner.

There are two one-loop graphs, which are shown in Fig. 4(b). These give a contribution

$$\frac{iag}{2}\int\frac{d^dk}{(2\pi)^4}\,\frac{1}{k^2-\mu^2} = \frac{a^3g^2}{2^6\pi^2}\left(-\frac{2}{\varepsilon}+\gamma-1+\ln\mu^2\right) \tag{5.38}$$

for the one on the left, and

$$\frac{iag}{2}\,\frac{n-1}{3}\int\frac{d^dk}{(2\pi)^4}\,\frac{1}{k^2-m^2} = \frac{a^3g^2}{2^6\pi^2}\,\frac{n-1}{9}\left(-\frac{2}{\varepsilon}+\gamma-1+\ln m^2\right) \tag{5.39}$$

for the one on the right. Here we have used the formulas [H7]

$$\int\frac{d^dk}{(k^2-Q)^m} = i(-1)^m\,\pi^{d/2}\,\frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)}\,Q^{\frac{d}{2}-m}$$

$$\Gamma(1-\frac{d}{2}) = -\frac{2}{\varepsilon}+\gamma-1+O(\varepsilon) \quad ; \quad \varepsilon = 4-d$$

$$(x)^{-\varepsilon} = 1-\varepsilon\ln x+\ldots \tag{5.40}$$

where γ is Euler's constant.

We now need a renormalization scheme. Observe ~~first~~ that differentiating the one-point function, found to some order, m times with respect to a is equivalent to finding the ϕ_1 (m+1)-point function in the shifted theory (5.35) with zero external momenta, to the same order. If we then set a to zero, this is equivalent to finding the (m+1)-point function with zero external momenta in the original, unshifted theory (5.34). Now, recall the renormalization conditions (5.24) derived on the effective potential for the massive scalar ϕ^4 theory:

$$\left.\frac{d^2V}{d\tilde{\phi}^2}\right|_{\tilde{\phi}=0} = m^2 \qquad \left.\frac{d^4V}{d\tilde{\phi}^4}\right|_{\tilde{\phi}=0} = g \tag{5.41}$$

These state that the value of the two-point function, in the unshifted theory, with zero external momentum is the mass m^2 and that for the four-point function with zero external momentum is the coupling constant g. These two functions, however, are exactly those found by differentiating the one-point function (in the shifted theory) with respect to a once and three times, respectively, and then setting a to zero (thereby recovering those in the unshifted theory). Since we are here working with a massless theory, we adopt as one of the renormalization conditions

$$\left.\frac{d^2V}{d\tilde{\phi}^2}\right|_{\tilde{\phi}=0} = 0 \implies \left.\frac{dT}{da}\right|_{a=0} = 0 \ . \tag{5.42}$$

However, because of the absence of a mass, we cannot adopt the other condition on V in (5.41) without modification, because $\tilde{\phi}=0$ is a singular point for this function. We can, however, modify it by choosing to renormalize at some nonsingular point $\tilde{\phi}=M$:

$$\left.\frac{d^4V}{d\tilde{\phi}^4}\right|_{\tilde{\phi}=M} = g \implies \left.\frac{d^3T}{da^3}\right|_{a=M} = g \ . \tag{5.43}$$

The point M is completely arbitrary; a change in M can be compensated by a change in g and a rescaling of the fields, and will not affect the functional forms of V or T. This freedom is in fact the basis of a set of renormalization group equations for this theory (see [C3]). The conditions (5.42) and (5.43) will now serve as our renormalization prescription.

To the one loop level we have, from (5.37), (5.38) and (5.39) and including the counterterm C_1:

$$T = T_o + T_1 = \frac{a^3 g}{6} + \frac{a^3 g^2}{2^6 \pi^2}\left(1+\frac{n-1}{9}\right)\left(-\frac{2}{\varepsilon}+\gamma-1\right) + \frac{a^3 g^2}{2^6\pi^2}\left(\ell n\ \mu^2 + \frac{n-1}{9}\ \ell n\ m^2\right) + \frac{a^3 C_1}{6} \ . \tag{5.44}$$

Imposing the renormalization conditions (5.42) and (5.43) leads to

$$T = \frac{a^3 g}{6} + \frac{a^3 g^2}{2^6 \pi^2}\left(1 + \frac{n-1}{9}\right)\left[\ln\left(\frac{a^2}{M^2}\right) - \frac{11}{3}\right] \tag{5.45}$$

with the one-loop counterterm C_1 given by

$$C_1 = -\frac{3g^2}{2^5 \pi^2}\left(1+\frac{n-1}{9}\right)\left(-\frac{2}{\epsilon}+\gamma-1\right) - \frac{3g^2}{2^5\pi^2}\left[\ln\left(\frac{gM^2}{2}\right)+\frac{n-1}{9}\ln\left(\frac{gM^2}{6}\right)\right]$$
$$-\frac{11g^2}{2^5\pi^2}\left(1+\frac{n-1}{9}\right) \quad . \tag{5.46}$$

[redacted] that if we had used a different set of renormalization conditi[redacted], or had regula[redacted]grals by introducing a cut-off, say, then a mass counterterm would also have been required [C3, J1].

We can check that the result (5.45) is indeed independent of M. Suppose we decide to choose a different renormalization point $\tilde{M}$. We can then define a new coupling constant $\tilde{g}$:

$$g = \tilde{g} + \frac{3\tilde{g}^2}{2^5\pi^2}\left(1 + \frac{n-1}{9}\right)\ln\left(\frac{M^2}{\tilde{M}^2}\right) \quad . \tag{5.47}$$

The expression for T becomes

$$T = \frac{a^3\tilde{g}}{6} + \frac{a^3\tilde{g}^2}{2^6\pi^2}\left(1 + \frac{n-1}{9}\right)\left(\ln\frac{a^2}{\tilde{M}^2} - \frac{11}{3}\right) + O(\tilde{g}^3) \tag{5.48}$$

which is the same functional form as in (5.45), and thus contains the same physical information to the order we are working.

We now come to the two loop terms, pictured in Fig. 5. For these it is found convenient to define the following integrals:

$$I(x,y) := -\frac{(ag)g}{4(2\pi)^8}\int \frac{d^dk\, d^dp}{(k^2-x)^2\,(p^2-y)} \tag{5.49}$$

$$J(x,y) := -\frac{(ag)g}{6(2\pi)^8}\int \frac{d^dk\, d^dp}{(p^2-x)(k^2-x)[(p+k)^2-y]} \tag{5.50}$$

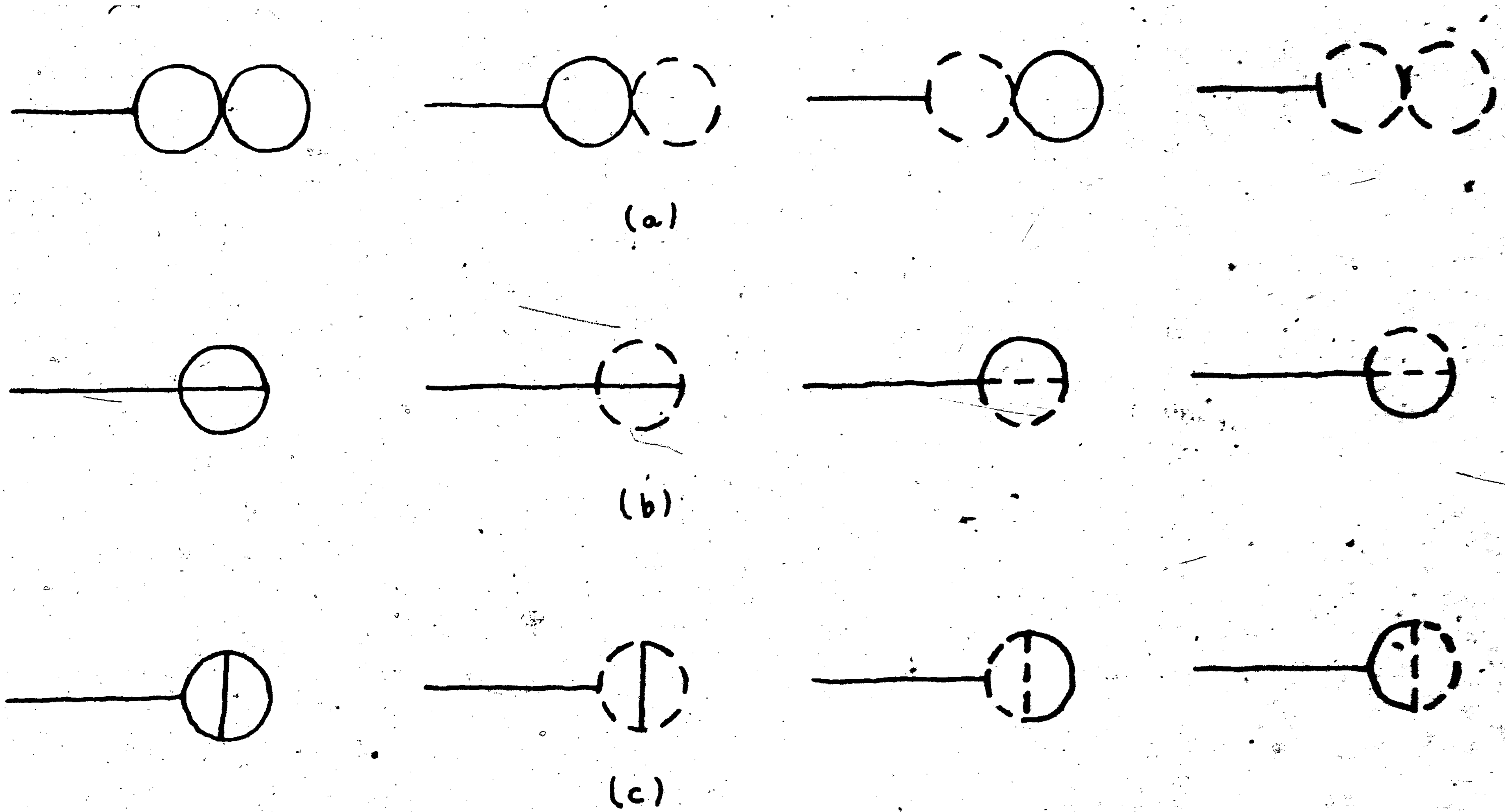


Fig. 5. The two-loop contribution to T.

$$K(x,y,z) = -\frac{(ag)^3}{4(2\pi)^8} \int \frac{d^dk\, d^dp}{(p^2-x)^2(k^2-y)[(p+k)^2-z]} . \qquad (5.51)$$

The two-loop terms then become the following. Fig. 5(a) gives, from left to right,

$$I(\mu^2,\mu^2) + \frac{n-1}{3} I(\mu^2,m^2) + \frac{n-1}{9} I(m^2,\mu^2) + \frac{n-1}{3}\left(1 + \frac{n-2}{3}\right) I(m^2,m^2)$$

$$= -\frac{a^2g^3}{2^{11}\pi^4}\left(\frac{2}{\varepsilon}-\gamma\right)\left(-\frac{2}{\varepsilon}+\gamma-1\right)\{(\mu^2)^{-\varepsilon} + \frac{2(n-1)}{9}(\mu^2)^{-\varepsilon/2}(m^2)^{-\varepsilon/2}$$

$$+ \frac{n-1}{27}[2 + (n-1)](m^2)^{-\varepsilon}\} . \qquad (5.52)$$

Fig. 5(b) yields,

$$J(\mu^2,\mu^2) + (1+1+1)\frac{n-1}{9} J(m^2,\mu^2) =$$

$$= -\frac{a^3g^3}{2^{10}\pi^4}\left(-\frac{1}{\varepsilon}+\gamma-1\right)\left(\frac{2}{\varepsilon}+1\right)\{(\mu^2)^{-\varepsilon} + \frac{n-1}{9}[\frac{2}{3}(m^2)^{-\varepsilon} + (\mu^2)^{-\varepsilon}\} \qquad (5.53)$$

Fig. 5(c) gives the contribution

$$K(\mu^2,\mu^2,\mu^2) + (1+1)\frac{n-1}{27}K(m^2,m^2,\mu^2) + \frac{n-1}{9}K(\mu^2,m^2,m^2)$$

$$= -\frac{a^3g^3}{2^{10}\pi^4}\left(-\frac{1}{\varepsilon}+\gamma\right)\left(\frac{2}{\varepsilon}+1\right)\{(\mu^2)^{-\varepsilon} + \frac{n-1}{9}[\frac{2}{3}(m^2)^{-\varepsilon} + (\mu^2)^{-\varepsilon}]\} . \qquad (5.54)$$

For the calculation of the latter nontrivial integrals, see the Appendix. In addition to these terms, we have the contribution from the one-loop counterterm, shown in Fig. 6. Fig. 6(a) gives rise to

$$\frac{iaC_1}{2}\int \frac{d^dk}{(2\pi)^4}\,\frac{1}{k^2-\mu^2} = \frac{a^3gC_1}{2^6\pi^2}\left(-\frac{2}{\varepsilon}+\gamma-1\right)(\mu^2)^{-\varepsilon/2} \qquad (5.55)$$

and

$$\frac{iaC_1}{2}\,\frac{n-1}{3}\int \frac{d^dk}{(2\pi)^4}\,\frac{1}{k^2-m^2} = \frac{a^3gC_1}{2^6\pi^2}\,\frac{n-1}{9}\left(-\frac{2}{\varepsilon}+\gamma-1\right)(m^2)^{-\varepsilon/2} . \qquad (5.56)$$

Fig. 6(b) contributes

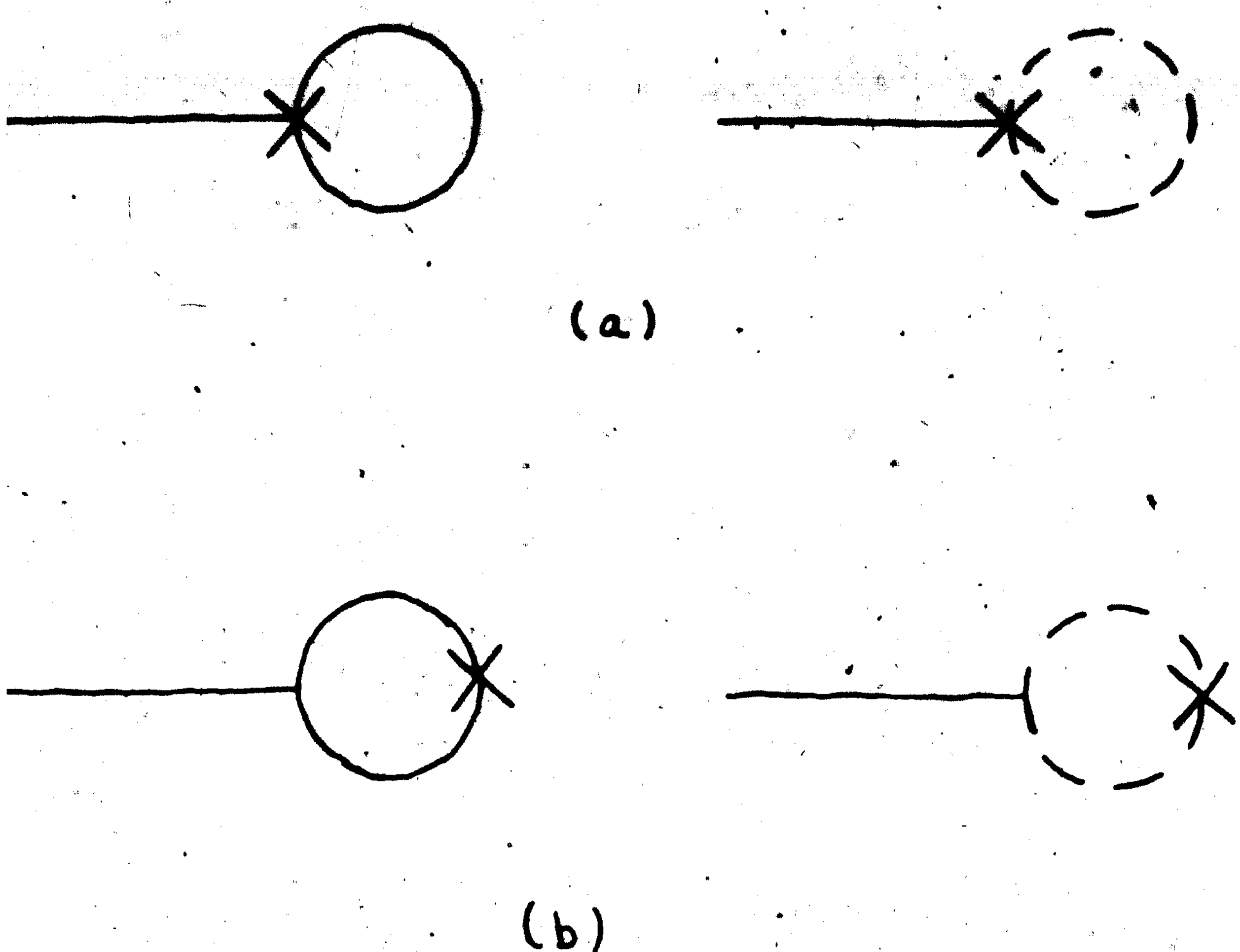


Fig. 6. The two-loop contribution from the one-loop counterterm.

$$\frac{iag}{2}\left(\frac{a^2C_1}{2}\right)\int\frac{d^dk}{(2\pi)^4}\frac{1}{(k^2-\mu^2)^2}=\frac{a^3gC_1}{2^6\pi^2}\left(-\frac{2}{\epsilon}+\gamma\right)(\mu^2)^{-\epsilon/2} \qquad (5.57)$$

and

$$\frac{iag}{2}\left(\frac{a^2C_1}{6}\right)\int\frac{d^dk}{(2\pi)^4}\frac{1}{(k^2-m^2)^2}=\frac{a^3gC_1}{2^6\pi^2}\frac{n-1}{9}\left(-\frac{2}{\epsilon}+\gamma\right)(m^2)^{-\epsilon/2} \qquad (5.58)$$

where C_1 is given in (5.44).

In some of the terms (5.52) to (5.58), there occur expressions like $(1/\epsilon)\ \ln m^2$ and $(1/\epsilon)\ \ln\mu^2$, which, were they to survive, could not be eliminated by a local counterterm. Fortunately, when the sum of (5.52) to (5.58) is evaluated, such expressions cancel, leaving us with the two-loop-plus-counterterm contribution

$$\begin{aligned}T_2 = \frac{a^3g^3}{2^{11}\pi^4}\Big\{&\left[3+\frac{2(n-1)}{9}\right]\ln^2\mu^2-\left[23+\frac{23(n-1)}{9}+6\ \ln\left(\frac{gM^2}{2}\right)\right.\\&\left.+\frac{6(n-1)}{9}\ln\left(\frac{gM^2}{6}\right)\right]\ln\mu^2+\frac{2(n-1)}{9}\ln\mu^2\ln m^2+\left[\frac{2(n-1)}{9}+\frac{(n-1)^2}{27}\right]\ln^2 m^2\\&-\left[\frac{65(n-1)}{27}+\frac{19(n-1)^2}{81}+\frac{6(n-1)}{9}\ln\left(\frac{gM^2}{2}\right)+\frac{6(n-1)^2}{81}\ln\left(\frac{gM^2}{6}\right)\right]\ln m^2\Big\}+\frac{a^3C_2}{6},\end{aligned} \qquad (5.59)$$

plus terms that will be absorbed in the counterterm C_2. The counterterm is then determined by the renormalization conditions (5.42) and (5.43):

$$\left.\frac{dT_2}{da}\right|_{a=0}=0=\left.\frac{d^3T_2}{da^3}\right|_{a=M} \qquad (5.60)$$

Adding the zero- and one-loop contributions from (5.45), we find for the renormalized one-point function to two loops the expression

$$\begin{aligned}T=\frac{a^3g}{6}+\frac{a^3g^2}{2^6\pi^2}\left(1+\frac{n-1}{9}\right)\left[\ln\left(\frac{a^2}{M^2}\right)-\frac{11}{3}\right]+\frac{a^3g^3}{2^{11}\pi^4}\Big\{&3\left(1+\frac{n-1}{9}\right)^2\left[\ln^2\left(\frac{a^2}{M^2}\right)-8\right]\\-19\left(1+\frac{n-1}{9}\right)^2\left[\ln\left(\frac{a^2}{M^2}\right)-\frac{11}{3}\right]-4\left(1+\frac{5(n-1)}{27}\right)&\left[\ln\left(\frac{a^2}{M^2}\right)-\frac{11}{3}\right]\Big\}.\end{aligned} \qquad (5.61)$$

This agrees with the result for the first derivative of the effective potential as obtained by Jackiw [J1].

2. The 1/N Expansion

The 1/N expansion is an expansion of the various quantities of physical interest in a theory in terms of the number of fields, N, present. Sometimes a coupling constant or loop expansion may be deceptive: by including only a finite number of terms to any given order, some of the nonlinear structure of the complete theory is lost. The 1/N expansion, although still perturbative, is an attempt to retain more terms of the series at each level and thereby to incorporate more of the nonlinear structure [C5, S2, R1, K6, A1, C6].

We again consider a theory of N real scalar fields ϕ^a, a=1,2,..N, with an O(N)-symmetric quartic coupling but with a mass term allowed:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)^2 - \frac{1}{2}m^2\phi^a\phi^a - \frac{1}{4!N}g(\phi^a\phi^a)^2 . \tag{5.62}$$

We wish to find for this theory, if possible, an expansion that is in powers of 1/N. We expect that the approximation will become better as N grows larger.

However, from the Lagrange density (5.62), it is fairly difficult to deduce which diagrams are important in the large-N limit. Consider, for example, scattering of an a-type meson from one of type b (see Fig. 7). An explicit factor of 1/N comes at each vertex, while a factor of N comes from an internal summation. Therefore, Figs. 7(a), 7(b) and 7(c) are all proportional to 1/N, while Fig. 7(d) is proportional to $1/N^2$. This indicates how complicated our task is going to be: Figs. 7(b) and 7(d) have the same topological structure, but enter at a different order in 1/N. On the other hand, Figs. 7(a), 7(b) and 7(c)

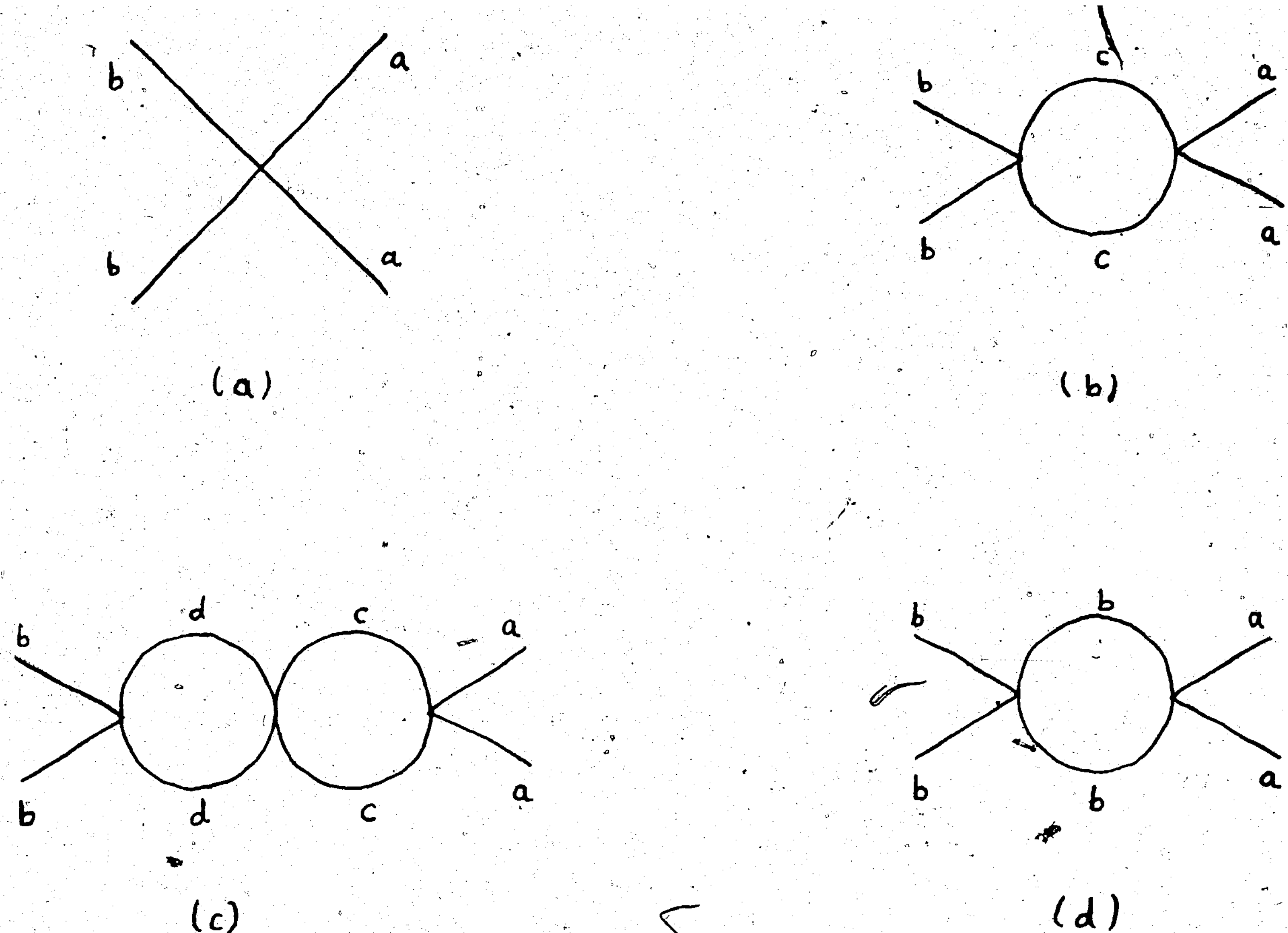


Fig. 7. Some diagrams of meson-meson scattering. A summation over internal indices in the loops is implied.

enter at the same order in 1/N, but their topological structures are quite different.

Matters simplify considerably by employing the following trick [C5]. Add to the Lagrange density (5.66) the term

$$\frac{3N}{2g}\left(\chi - \frac{g}{6N}\phi^a\phi^a - m^2\right)^2 \quad . \tag{5.63}$$

This term has no effect on the dynamics of the system, as the Euler-Lagrange equation for χ involves no time derivatives, and hence is an equation of constraint. However, the Feynman rules for this theory are quite different. For the new Lagrange density, one finds

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{3N}{2g}\chi^2 - \frac{1}{2}\chi\phi^a\phi^a - \frac{3N}{g}m^2\chi \quad , \tag{5.64}$$

As will be seen, finding the graphs that are important in the large-N limit is decidedly easier with the theory in this form.

As an example, we will calculate the first two terms in the 1/N expansion of the effective potential. This result has been obtained by Root using a path-integral formalism [R1]. As before, we begin by performing the shifts

$$\begin{aligned} \phi_1 &\to \sigma + \sqrt{N}\,a \\ \chi &\to \chi + b \\ \phi_i &\to \phi_i \quad ; \quad i = 2,\ldots N \; . \end{aligned} \tag{5.65}$$

The Lagrange density becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{1}{2}b\phi_i\phi_i + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}b\sigma^2 + \frac{3N}{2g}\chi^2 - \frac{1}{2}\chi(\sigma^2+\phi_i\phi_i)$$
$$- \sqrt{N}\,a\sigma\chi - \sqrt{N}\,ab\sigma + \frac{3N}{g}\left(b - m^2 - \frac{1}{6}a^2 g\right)\chi \quad , \tag{5.66}$$

plus counterterms. Because the quadratic terms are not diagonal in the fields, we list the propagators:

$$D_{ij} = \frac{i\delta_{ij}}{k^2 - b} \quad , \quad D_{\chi\chi} = \frac{ig}{3N} \frac{k^2 - b}{k^2 - b - \frac{1}{3} a^2 g}$$

$$D_{\sigma\sigma} = \frac{i}{k^2 - b - \frac{1}{3} a^2 g} \quad , \quad D_{\chi\sigma} = \frac{iag}{3\sqrt{N}} \frac{1}{k^2 - b - \frac{1}{3} a^2 g} \quad . \tag{5.67}$$

We now calculate the one-point function for χ, again denoted by T, and demand that it vanish (this is equivalent to setting the first derivative of the effective potential with respect to χ to zero). To order N, we have the two graphs in Fig. 8 contributing. Fig. 8(a) contributes

$$\frac{3N}{g} \left(\frac{1}{6} a^2 g + m^2 - b\right) \quad , \tag{5.68}$$

while Fig. 8(b) gives

$$\frac{N-1}{2} \int \frac{d^d k}{(2\pi)^4} \frac{i}{k^2 - b} \quad . \tag{5.69}$$

Setting the sum of (5.68) and (5.69) to zero results in the "gap" equation:

$$b = \frac{1}{6} a^2 g + m^2 + \frac{1}{6} g \int \frac{d^d k}{(2\pi)^4} \frac{i}{k^2 - b} \quad . \tag{5.70}$$

As Coleman, Jackiw and Politzer show [C5], this equation may be renormalized by introducing the renormalized quantities g_R and m_R^2 by

$$m^2 = m_R^2 - \frac{1}{6} g \int \frac{d^d k}{(2\pi)^4} \frac{i}{k^2 - M^2}$$

$$\frac{1}{g} = \frac{1}{g_R} + \frac{1}{6} g \int \frac{d^d k}{(2\pi)^4} \frac{i}{k^2 - M^2} \tag{5.71}$$

where M is arbitrary. The gap equation (5.70) then becomes

$$b = m_R^2 + \frac{1}{6} \alpha_R a^2 + \frac{\alpha_R}{96\pi^2} b \, \ln\left(\frac{b}{M^2}\right) \tag{5.72}$$

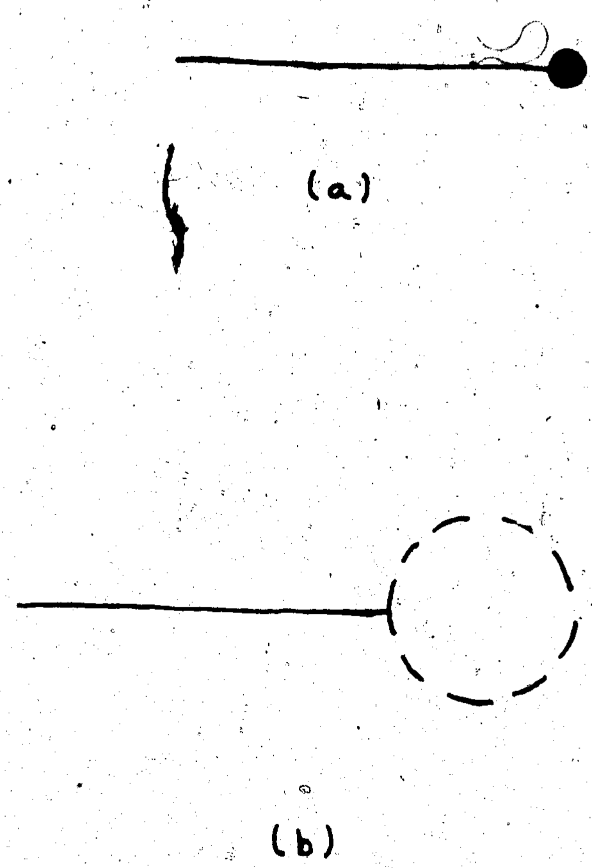


Fig. 8. The order-N contribution to T. A solid line denotes χ, a dashed one ϕ_i.

where $\alpha_R = g_R/(1 + g_R/96\pi^2)$.

The order-one contribution to T is now calculated. To begin with, we have a contribution from equation (5.69) that is of order one:

$$-\frac{1}{2}\int \frac{d^dk}{(2\pi)^4}\ \frac{i}{k^2-b} \quad . \tag{5.73}$$

We also have a term pictured in Fig. 9(a), where the σ propagator running around the closed loop must be modified to take into account the relevant factors of N. These modifications are calculated in two steps. The correction to the χ propagator, shown in Fig. 9(c), is

$$\frac{ig}{3N} + \frac{ig}{3N}\left[\frac{3N}{ig}B(k^2,b)\right]\left(\frac{ig}{3N}\right) + \ldots = \frac{ig}{3N}\left\{1 - \frac{3N}{ig}B(k^2,b)\left(\frac{ig}{3N}\right)\right\}^{-1}$$

$$= \frac{ig}{3N}\ \frac{1}{1-B(k^2,b)} \quad . \tag{5.74}$$

where we have defined

$$B(k^2,b) = \frac{g}{6}\int \frac{d^dp}{(2\pi)^4}\ \frac{i}{[(p+k)^2-b](p^2-b)} \quad . \tag{5.75}$$

We can now calculate the modified σ propagator, shown in Fig. 9(b). This is

$$\frac{i}{k^2-b} + \frac{i}{k^2-b}\left[(-i\sqrt{N}a)^2\ \left(\frac{ig}{3N}\right)\ \frac{1}{1-B}\right]\frac{i}{k^2-b} + \ldots$$

$$= \frac{i}{k^2-b}\left\{1 - (-i\sqrt{N}a)^2\ \left(\frac{ig}{3N}\right)\ \frac{1}{1-B}\ \frac{i}{k^2-b}\right\}^{-1}$$

$$= i\left\{k^2 - b - \frac{1}{3}a^2g\ \frac{1}{1-B}\right\}^{-1} \quad . \tag{5.76}$$

Thus, Fig. 9(a) contributes

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^4}\ \frac{i}{k^2-b-\frac{1}{3}a^2g[1-B]^{-1}} \quad . \tag{5.77}$$

Adding (5.73) and (5.77) results in

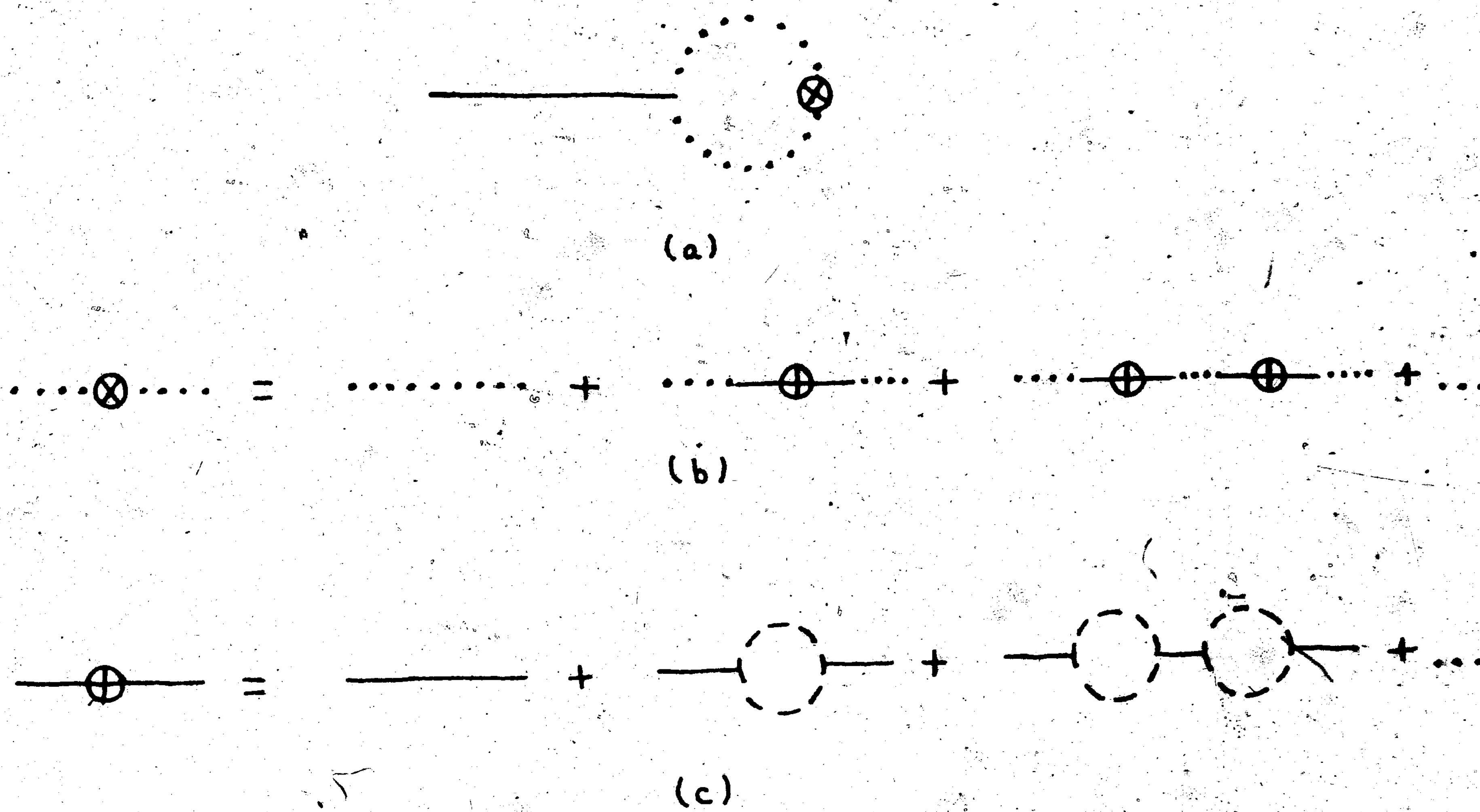


Fig. 9. An order-one contribution to T. A solid line denotes χ, a dashed one ϕ_i, a dotted one σ.

$$\frac{a^2g}{6}\int\frac{d^dk}{(2\pi)^4}\;\frac{i}{(k^2-b)\{(k^2-b)[1-B]-\frac{1}{3}a^2g\}}\;. \tag{5.78}$$

We now come to the final graph, Fig. 10(a). In this graph the χ propagator must include the corrections shown in Fig. 10(b). These corrections modify the χ propagator to

$$\frac{ig}{3N}\,\frac{k^2-b}{k^2-b-\frac{1}{3}a^2g}+\frac{ig}{3N}\,\frac{k^2-b}{k^2-b-\frac{1}{3}a^2g}\left[\frac{3N}{ig}\,B(k^2,b)\right]\frac{ig}{3N}\,\frac{k^2-b}{k^2-b-\frac{1}{3}a^2g}+\ldots$$

$$=\frac{ig}{3N}\,\frac{k^2-b}{k^2-b-\frac{1}{3}a^2g}\left\{1-\frac{(k^2-b)B(k^2,b)}{k^2-b-\frac{1}{3}a^2g}\right\}^{-1}$$

$$=\frac{ig}{3N}\,\frac{k^2-b}{(k^2-b)(1-B)-\frac{1}{3}a^2g}\;. \tag{5.79}$$

With these modifications, Fig. 10(a) contributes

$$\frac{ig}{6}\int\frac{d^dk}{(2\pi)^4}\int\frac{d^dp}{(2\pi)^4}\;\frac{i(k^2-b)}{(k^2-b)(1-B)-\frac{1}{3}a^2g}\;\frac{i}{[(p+k)^2-b]^2(p^2-b)}$$

$$=\frac{1}{2}\int\frac{d^dk}{(2\pi)^4}\;\frac{i(k^2-b)}{(k^2-b)(1-B)-\frac{1}{3}a^2g}\;\frac{\partial B(k^2,b)}{\partial b}\;. \tag{5.80}$$

Adding the two contributions (5.78) and (5.80), including the order-N terms (5.68) and (5.67), and then setting the result to zero results in the gap equation

$$b=m^2+\frac{1}{6}a^2g+\frac{1}{6}g\int\frac{d^dk}{(2\pi)^4}\,\frac{i}{k^2-b}+\frac{a^2g}{3}\,\frac{g}{6N}\int\frac{d^dk}{(2\pi)^4}\,\frac{i}{(k^2-b)\{(k^2-b)(1-B)-\frac{1}{3}a^2g\}}$$

$$+\frac{g}{6N}\int\frac{d^dk}{(2\pi)^4}\;\frac{i(k^2-b)}{(k^2-b)(1-B)-\frac{1}{3}a^2g}\;\frac{\partial B(k^2,b)}{\partial b} \tag{5.81}$$

where $B=B(k^2,b)$ is given in (5.75). This equation can be thought of as one which eliminates b in terms of a. Inserting the value of b thus found in the expression for the one-point function T and integrating

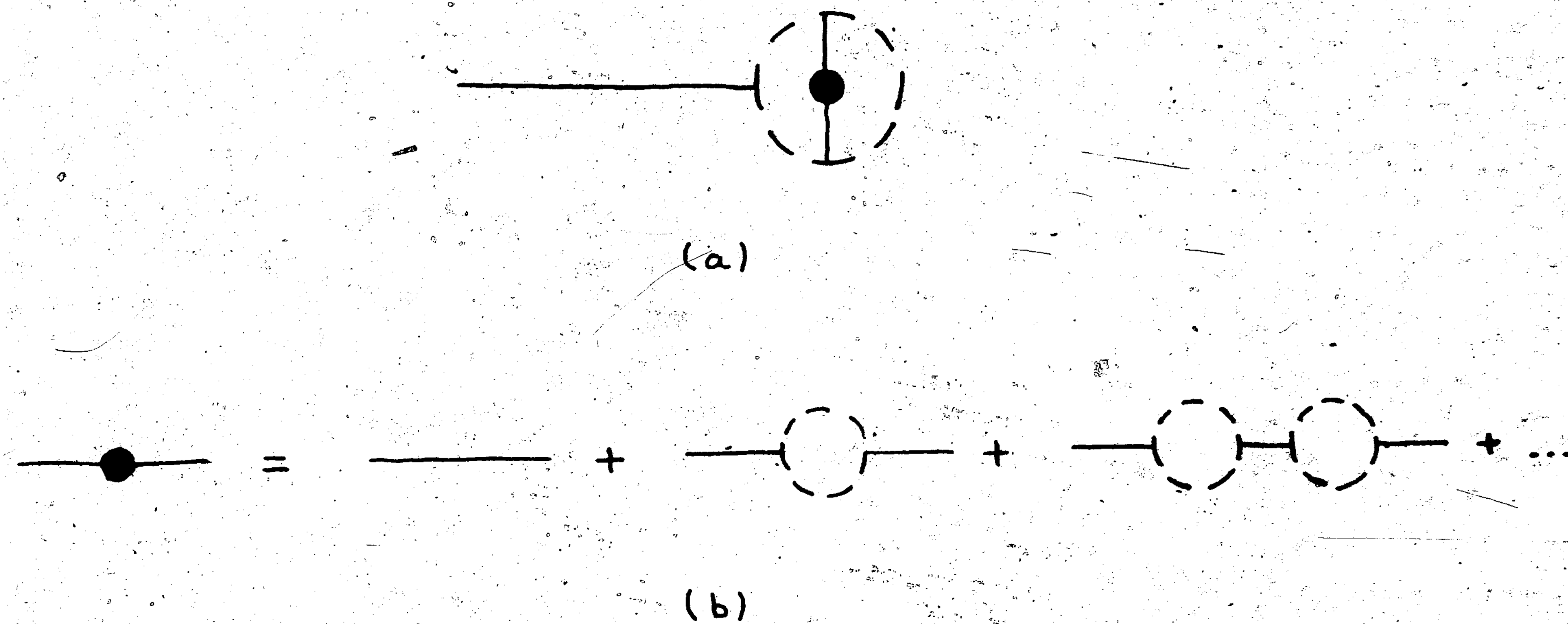


Fig. 10. An order-one contribution to T. A solid line denotes χ, a dashed one ϕ_i.

with respect to a would then give the effective potential as a function of a alone. The ground state would correspond to that value of a which minimizes the effective potential.

The equation (5.79) agrees with that found by Root [R1]. Of course, renormalization is required. This is a fairly involved task, and will not be done here (see [R1]). The main point we wish to bring out is that, because an infinite subset of graphs are included, more of the nonlinear structure of the theory is expected to survive in each order of this perturbation scheme. Indeed, for this model the 1/N expansion gives a substantially different physical interpretation than a coupling constant or loop expansion does. There, for a range of parameters ($m^2 < 0$, $g > 0$), spontaneous breakdown of the symmetry occurs. Using the 1/N approximation, it . . found [A1]:

(1) for a certain range of parameters, the effective potential is everywhere complex, and no consistent theory is possible,

(2) for the remaining choice of parameters, the ground state occurs in the symmetric state $\phi^2 = 0$.

Of course, the question of convergence of either series is still open: is the 1/N expansion with $N = 5$ a better approximation than a coupling constant expansion with $g = 0.2$? For small N we expect the 1/N approximation to fail, and the symmetry will be spontaneously broken. For large N, however, where the approximation should be good, the symmetry will be restored. It remains to be seen how the critical value of N, where the transition between the two phases occurs, can be calculated.

C. The $Z(\tilde{\phi})$ Function

In this section we will be interested in finding an approximation for the function $Z(\tilde{\phi})$ found in the expansion (5.14) of the effective action $\Gamma(\tilde{\phi})$:

$$\Gamma(\tilde{\phi}) = \int d^4x[-V(\tilde{\phi}) + \frac{1}{2} Z(\tilde{\phi})(\partial_\mu\tilde{\phi})^2 + \ldots] \quad . \qquad (5.82)$$

Two methods again have generally been used for finding $Z(\tilde{\phi})$ perturbatively. One involves an infinite summation of graphs at each order of the loop expansion like that for the effective potential, except the external momenta are now nonzero (see [C3]). The other method relies on the steepest descent approximation to the generating functional similar to that used to find the effective potential; however, this time the expansion is not about a constant $\tilde{\phi}(x)$ but rather some x-dependent one (see [I2]). Like for the effective potential calculations, this latter method avoids an infinite summation of diagrams at each order of the loop expansion, simplifying the combinatorial factors considerably, but some care must be taken in choosing the correct space-time dependence of $\tilde{\phi}(x)$. We will now describe a method that, although involves more graphs at each level of the loop expansion than in the path-integral approach (but still a finite number), has the property that only a constant $\tilde{\phi}(x)$ need be considered.

Suppose we have a real scalar field $\phi(x)$ with Lagrange density $\mathcal{L}(\phi)$. The effective action $\Gamma(\tilde{\phi})$ for this theory can be expanded as follows (see (5.12) and (5.14)):

$$\Gamma(\tilde{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \; \tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n)\Gamma^{(n)}(x_1\ldots x_n)$$

$$= \int d^4x[-V(\tilde{\phi}) + \frac{1}{2} Z(\tilde{\phi})(\partial_\mu\tilde{\phi})^2 + \ldots] \qquad (5.83)$$

where $\Gamma^{(n)}(x_1 \ldots x_n)$ is the n-point IPI function. Rather than expanding about $\tilde{\phi}=0$, let us see the effect of expanding $\Gamma(\tilde{\phi})$ about some constant $\tilde{\phi}=a$. In momentum space, such an expansion has the form

$$\Gamma(\tilde{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^4p_1 \ldots d^4p_n}{(2\pi)^{2n}} (2\pi)^4 \delta(p_1+\ldots+p_n)[\tilde{\phi}(p_1)-a]\ldots[\tilde{\phi}(p_n)-a]\Gamma'^{(n)}(p_1\ldots p_n)$$

$$= -\int d^4x V(\tilde{\phi}) - \frac{1}{2} Z(\tilde{\phi}) \int d^4p_1 d^4p_2 \, \delta(p_1+p_2) p_1 \cdot p_2 [\tilde{\phi}(p_1)-a][\tilde{\phi}(p_2)-a] + \ldots$$

$$= -\int d^4x V(\tilde{\phi}) + \frac{1}{2} Z(\tilde{\phi}) \int d^4p_1 d^4p_2 \, \delta(p_1+p_2) p_1^2 [\tilde{\phi}(p_1)-a][\tilde{\phi}(p_2)-a] + \ldots \tag{5.84}$$

where $\tilde{\phi}(k)$ is the Fourier transform of $\tilde{\phi}(x)$ and $\Gamma'^{(n)}(p_1\ldots p_n)$ is the n-point IPI function in momentum space associated with the Lagrange density $\mathcal{L}(\phi-a)$. Comparing the first and last equations in (5.84), we see that

$$\frac{\partial}{\partial p^2} \Gamma'^{(2)}(p) \Big|_{p^2=0} = Z(\tilde{\phi}) \Big|_{\tilde{\phi}=a} \tag{5.85}$$

Thus, to obtain $Z(\tilde{\phi})$ we first compute the two-point function in the theory described by $\mathcal{L}(\phi-a)$, then differentiate it with respect to p^2, and finally set $p^2=0$.

As an example of this procedure, let us evaluate $Z(\tilde{\phi})$ to two loops for a single component massive ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!} g\phi^4 + \frac{A}{2}(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4 \,. \tag{5.86}$$

Replacing ϕ by $\phi+a$ leads to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - (\mu^2+\frac{1}{6}a^2g)a\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}ga\phi^3 - \frac{1}{24}g\phi^4 + \frac{A}{2}(\partial_\mu\phi)^2$$
$$- (B+\frac{1}{6}a^2C)a\phi - \frac{1}{2}(B+\frac{1}{2}a^2C)\phi^2 - \frac{1}{6}aC\phi^2 - \frac{1}{24}C\phi^4 \tag{5.87}$$

where $m^2 = \mu^2+\frac{1}{2}a^2g$. Expanding $Z(\tilde{\phi})$ according to the number of loops, we now proceed to calculate the two-point function relevant to each order of the loop expansion. The tree-level contribution is shown in Fig. 11(a), and gives

$$Z_o = 1 \ . \tag{5.88}$$

The one-loop term is pictured in Fig. 11(b), and contributes

$$\frac{\partial}{\partial p^2}\left\{-\frac{ia^2g^2}{2}\frac{1}{(2\pi)^4}\int\frac{d^dk}{[(p+k)^2-m^2](k^2-m^2)}\right\}_{p^2=0} = \frac{a^2g}{12m^2}\left(\frac{g}{16\pi^2}\right) \ . \tag{5.89}$$

Thus, to one loop, we find

$$Z(x) = 1 + \frac{1}{6}\left(\frac{g}{16\pi^2}\right)\frac{x}{1+x} \tag{5.90}$$

where $x = a^2g/2\mu^2$. Although the renormalization condition $Z(0) = 1$ is automatically satisfied here, ind·ating the wave-function counterterm vanishes to this order, the one-loop contribution to Z is nontrivial.

The two-loop terms are pictured in Fig. 12. Leaving the details of the calculations to the Appendix, we find, from left to right, top to bottom, the two-loop contribution to $Z(\tilde{\phi})$ to consist of the following terms:

$$\frac{\partial}{\partial p^2}\left\{\frac{g^2}{6}\frac{1}{(2\pi)^8}\int\frac{d^dk\, d^d\ell}{(k^2-m^2)(\ell^2-m^2)[(p+k+\ell)^2-m^2]}\right\}_{p^2=0} = \frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left(\frac{1}{\varepsilon}-\gamma-\ell n\, m^2\right) \tag{5.91}$$

$$\frac{\partial}{\partial p^2}\left\{\frac{a^2g^3}{2}\frac{1}{(2\pi)^8}\int\frac{d^dk\, d^d\ell}{(k^2-m^2)^2(\ell^2-m^2)[(p+k)^2-m^2]}\right\}_{p^2=0} = \frac{a^2g^2}{12m^2}\left(\frac{g}{16\pi^2}\right)^2\left(\frac{1}{\varepsilon}-\gamma+1-\ell n\, m^2\right) \tag{5.92}$$

$$\frac{\partial}{\partial p^2}\left\{\frac{a^2g^3}{4}\frac{1}{(2\pi)^8}\int\frac{d^dk\, d^d\ell}{(k^2-m^2)[(p+k)^2-m^2](\ell^2-m^2)[(\ell+p)^2-m^2]}\right\}_{p^2=0}$$

$$= \frac{a^2g}{6m^2}\left(\frac{g}{16\pi^2}\right)^2\left(-\frac{1}{\varepsilon}+\gamma+\ell n\, m^2\right) \tag{5.93}$$

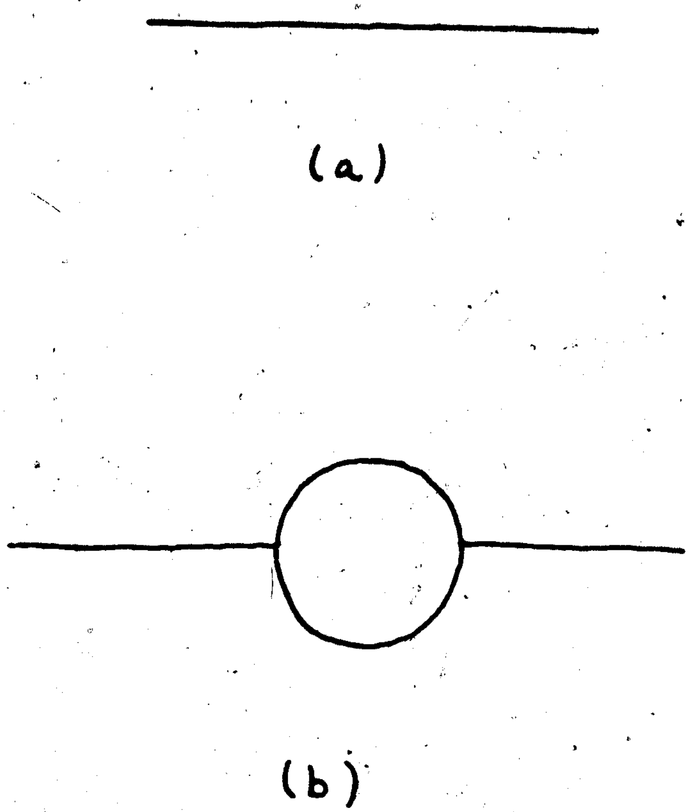


Fig. 11 (a) The zero-loop contribution to $Z(\tilde{\phi})$.
(b) The one-loop contribution to $Z(\tilde{\phi})$.

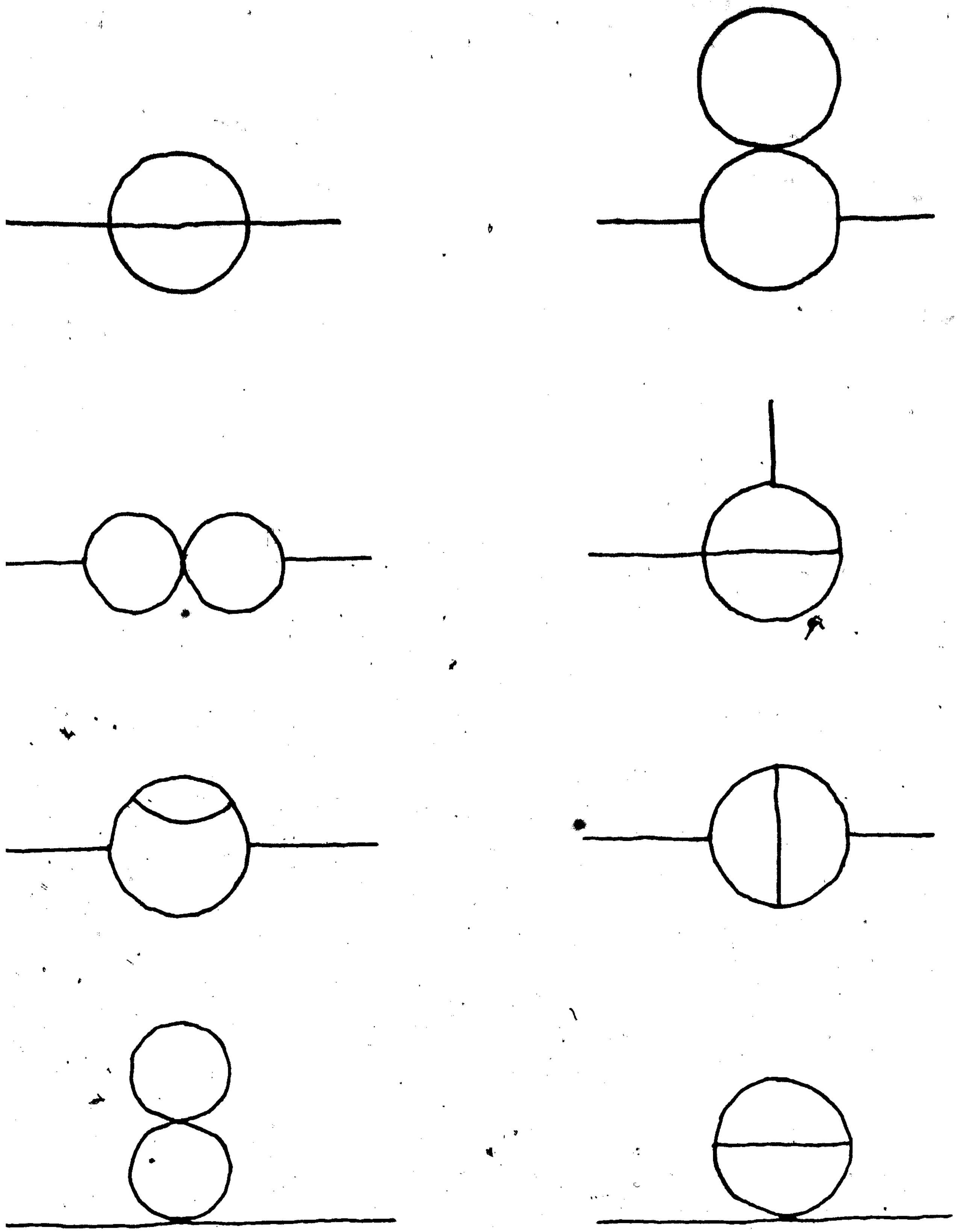


Fig. 12. The two-loop contribution to $Z(\tilde{\phi})$.

$$\frac{\partial}{\partial p^2}\left\{a^2 g^3 \frac{1}{(2\pi)^8}\int \frac{d^d k\; d^d \ell}{(k^2-m^2)(\ell^2-m^2)[(k+\ell)^2-m^2][(k-p)^2-m^2]}\right\}_{p^2=0}$$

$$= \frac{a^2 g}{3m^2}\left(\frac{g}{16\pi^2}\right)^2 \left(-\frac{1}{\varepsilon} + \gamma - 1 + \ell n\, m^2 - \delta_1\right) \tag{5.94}$$

$$\frac{\partial}{\partial p^2}\left\{\frac{a^4 g^4}{2} \frac{1}{(2\pi)^8}\int \frac{d^d k\; d^d \ell}{(k^2-m^2)^2(\ell^2-m^2)[(k+\ell)^2-m^2][(k-p)^2-m^2]}\right\}_{p^2=0}$$

$$= \frac{a^4 g^2}{12(m^2)^2}\left(\frac{g}{16\pi^2}\right)^2 \left(\frac{1}{\varepsilon} - \gamma + 1 - \ell n\, m^2 - \delta_2\right) \tag{5.95}$$

$$\frac{\partial}{\partial p^2}\left\{\frac{a^4 g^4}{2} \frac{1}{(2\pi)^8}\int \frac{d^d k\; d^d \ell}{(k^2-m^2)(\ell^2-m^2)[(k-\ell)^2-m^2][(\ell+p)^2-m^2][(k+p)^2-m^2]}\right\}_{p^2=0}$$

$$= \frac{a^4 g^4}{12(m^2)^2}\left(\frac{g}{16\pi^2}\right)^2 \delta_3 \tag{5.96}$$

where

$$\delta_1 = -\frac{2}{3} J - \frac{1}{3}, \qquad \delta_2 = \frac{8}{9} J - \frac{17}{36}, \qquad \delta_3 = \frac{2}{9} J + \frac{4}{9}$$

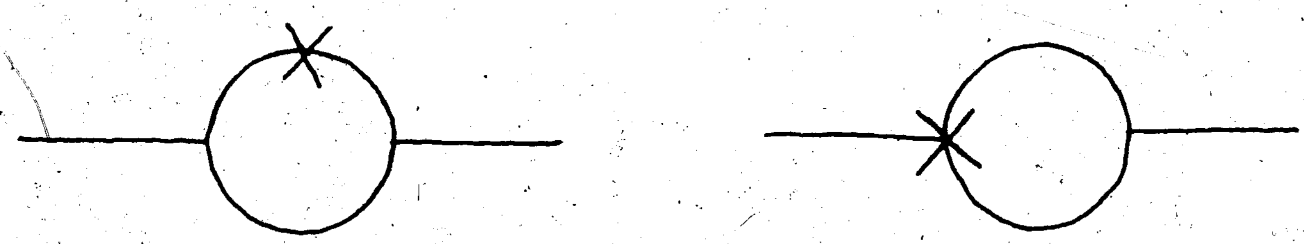
and J is the number given by the transcendental integral

$$J = -\int_0^1 du \frac{\ell n\, u}{u^2-u+1} = \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(1+3n)^2} - \frac{1}{(2+3n)^2}\right)$$

$$= 1.1719536\ldots$$

The last two diagrams of Fig. 12 do not contribute to $Z(\tilde{\phi})$.

In addition to these terms, there will be a contribution from the one-loop mass and coupling constant counterterms, shown in Fig.13(a). The value of these two counterterms can be calculated from the one-point function at the one-loop level, pictured in Fig. 13(b). At this level, the one-point function is, including the counterterm,



(a)

(b)

Fig. 13 (a) The two-loop contribution to $Z(\tilde{\phi})$ from the one-loop counterterms.
(b) The one-point function to the one-loop level.

$$T = (\mu^2 + \frac{1}{6}a^2 g)a + \frac{iag}{2}\int \frac{d^d k}{(2\pi)^4}\frac{1}{k^2 - m^2} + (B_1 + \frac{1}{6}a^2 C_1)a$$

$$= (\mu^2 + \frac{1}{6}a^2 g)a + \frac{ag}{32\pi^2} m^2(-\frac{2}{\varepsilon} + \gamma - 1 + \ln m^2) + (B_1 + \frac{1}{6}a^2 C_1)a \qquad (5.97)$$

If we impose the renormalization conditions (see (5.24))

$$\left.\frac{d^2 V}{d\tilde{\phi}^2}\right|_{\tilde{\phi}=0} = \mu^2 \implies \left.\frac{dT}{da}\right|_{a=0} = \mu^2$$

$$\left.\frac{d^4 V}{d\tilde{\phi}^4}\right|_{\tilde{\phi}=0} = g \implies \left.\frac{d^3 T}{da^3}\right|_{a=0} = g \quad , \qquad (5.98)$$

then we find

$$B_1 = -\frac{\mu^2 g}{32\pi^2}(-\frac{2}{\varepsilon} + \gamma - 1 + \ln \mu^2)$$

$$C_1 = -\frac{3g^2}{32\pi^2}(-\frac{2}{\varepsilon} + \gamma + \ln \mu^2) \qquad (5.99)$$

The contribution due to these counterterms, from left to right in Fig. 13(a), is then

$$\frac{\partial}{\partial p^2}\left\{\frac{ia^2 g^2}{2}\frac{2}{(2\pi)^4}(B_1 + \frac{1}{2}a^2 C_1)\int \frac{d^d k}{[(p+k)^2 - m^2](k^2 - m^2)^2}\right\}_{p^2=0}$$

$$= \frac{a^2 g}{24(m^2)^2}(\frac{g}{16\pi^2})^2\left\{m^2(-\frac{2}{\varepsilon} + 2\gamma - 2 + \ln m^2 + \ln \mu^2)\right.$$

$$\left. + \frac{a^2 g}{2}(-\frac{4}{\varepsilon} + 4\gamma - 1 + 2\ln m^2 + 2\ln \mu^2)\right\} \qquad (5.100)$$

and

$$\frac{\partial}{\partial p^2}\left\{\frac{ia^2 g}{2}\frac{2}{(2\pi)^4}C_1\int \frac{d^d k}{[(p+k)^2 - m^2](k^2 - m^2)}\right\}_{p^2=0} = \frac{a^2 g}{4m^2}(\frac{g}{16\pi^2})^2(\frac{2}{\varepsilon} - 2\gamma - \ln m^2 - \ln \mu^2). \qquad (5.101)$$

Adding the contributions (5.91) to (5.96), (5.100) and (5.101), we find that the poles with coefficients $1/m^2$ and $1/(m^2)^2$ cancel, ensuring the counterterms remain local, leaving us with the two-loop contribution to $Z(\tilde{\phi})$ of

$$Z_2(x) = \frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2 \left\{\frac{1}{\varepsilon} - \gamma - \ell n\, m^2 + \frac{x}{1+x}\left[5\,\ell n(1+x) + \sigma_1\right] + \frac{x^2}{(1+x)^2}\left[-2\,\ell n(1+x) + \sigma_2\right]\right\} \tag{5.102}$$

where

$$x = a^2 g/2\mu^2$$

$$\sigma_1 = -8 - 8\delta_1 = \frac{16}{3} J - \frac{16}{3}$$

$$\sigma_2 = 3 - 4\delta_2 + 4\delta_3 = \frac{20}{3} - \frac{8}{3} J \quad .$$

If we now impose the renormalization condition $Z(0) = 1$, we find the wave function counterterm to be

$$A = Z_3 - 1 = -\frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left(\frac{1}{\varepsilon} - \gamma - \ell n\, \mu^2\right) \quad . \tag{5.103}$$

The function $Z(\tilde{\phi})$ to two loops is then

$$Z(x) = 1 + \frac{1}{6}\left(\frac{g}{16\pi^2}\right)\left(\frac{x}{1+x}\right) + \frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left\{-\ell n(1+x) + \frac{x}{1+x}\left[5\,\ell n(1+x) + \sigma_1\right] + \frac{x^2}{(1+x)^2}\left[-2\,\ell n(1+x) + \sigma_2\right]\right\} \quad . \tag{5.104}$$

Iliopoulos, Itzykson and Martin have also calculated this function to the two-loop level for a ϕ^4 theory using a path-integral approach [I2]. Their result differs from the one here in that they find

$$\sigma_1 = \frac{16}{3} J - 2$$

$$\sigma_2 = \frac{19}{3} - \frac{8}{3} J \quad . \tag{5.105}$$

It is unknown at this time where the source of this discrepency lies.

D. The Renormalization Group

We now turn to a topic that, while not directly connected to the effective potential, nevertheless is one where utilizing the one-point function in finding the counterterms of a theory proves useful. This is in connection with calculating the coefficients of the renormalization group equations [S10, G2]. These equations are a reflection of an observation made when we were studying the massless ϕ^4 theory: as far as physical quantities are concerned, the choice of which point we renormalize at is arbitrary. This can be phrased mathematically in a number of ways [I2], among the most useful being the Callan-Symanzik equations [C1, S12]. For theories with a mass present, however, the Callan-Symanzik equations are inhomogeneous. In many cases, though, one is interested in the high momentum behaviour of Green functions, whereby the mass term can be dropped. The resulting equations are then homogeneous and can be more readily solved. There does exist a set of equations, however, developed by 't Hooft [H9] and Weinberg [W4] (see also [C7, C8, L5]), that are homogeneous even with a mass term present if the counterterms are mass-independent. It is these particular equations that we will be concerned with.

The system we shall study is the familiar ϕ^4 theory. In terms of the bare quantities, the Lagrange density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_o)^2 - \frac{1}{2} m_o^2 \phi_o^2 - \frac{1}{4!} g_o \phi_o^4 \quad . \tag{5.106}$$

Let us rewrite this in terms of the renormalized quantities

$$\begin{aligned} \phi_o &= Z_3^{\frac{1}{2}} \phi \\ m_o^2 &= Z_2 Z_3^{-1} m^2 \\ g_o &= Z_1 Z_3^{-2} g \quad . \end{aligned} \tag{5.107}$$

The Lagrange density now reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4 + \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4 , \qquad (5.108)$$

where we have defined the counterterms to be

$$\begin{aligned} A &= Z_3 - 1 \\ B &= (Z_2 - 1)m^2 \\ C &= (Z_1 - 1)g , \end{aligned} \qquad (5.109)$$

and we assume Z_1, Z_2 and Z_3 do not depend on m^2.

The renormalization prescription adjusts the counterterms in an unambiguous way so as the ultraviolet divergences are eliminated. These divergences will be poles at $d = 4$ when dimensional regularization is used. In this scheme it is convenient to introduce a unit of mass, μ, into the definition of the coupling constant g so as g remains dimensionless in arbitrary dimensions. We then expand Z_3, m_o^2 and g_o in the following manner:

$$\begin{aligned} Z_3 &= 1 + \sum_{j,k} \frac{c_{ij}\, g^k}{(4-d)^j} \\ m_o^2 &= \left\{1 + \sum_{j,k} \frac{b_{jk}\, g^k}{(4-d)^j}\right\} m^2 \\ g_o &= \mu^{4-d}\left\{g + \sum_{j,k} \frac{a_{jk}\, g^k}{(4-d)^j}\right\} . \end{aligned} \qquad (5.110)$$

Thus, for all d, g is dimensionless and m has mass dimension one. In what follows, we will assume the coefficients a_{jk}, b_{jk} and c_{jk} do not depend on the mass m^2.

Let us now consider the renormalized effective action $\Gamma(\phi, m, g, \mu)$. This will be related to the unrenormalized one $\Gamma_u(\tilde{\phi}_o, m_o(d), g_o(d), d)$ by the following:

$$\Gamma(\tilde{\phi},m,g,\mu) = \lim_{d\to 4} \Gamma_u(Z_3^{\frac{1}{2}}\tilde{\phi},m_o(d),g_o(d),d) \quad . \tag{5.111}$$

Now, differentiating this equation with respect to μ, keeping everything else fixed, gives

$$(\mu \frac{\partial}{\partial\mu}+\mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}+\mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m})\Gamma = \frac{\partial(Z_3^{\frac{1}{2}}\tilde{\phi})}{\partial\mu} \frac{\delta\Gamma_\mu}{\delta(Z_3^{\frac{1}{2}}\tilde{\phi})} \quad . \tag{5.112}$$

This then leads to the equation

$$(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-\gamma_m m\frac{\partial}{\partial m}-\frac{1}{2}\gamma_z\tilde{\phi}\frac{\delta}{\delta\tilde{\phi}})\Gamma(\tilde{\phi},m,g,\mu) = 0 \ , \tag{5.113}$$

where the renormalization group equation coefficients are defined by

$$\beta = \mu\frac{\partial g}{\partial\mu}$$

$$\gamma_m = \mu\frac{\partial}{\partial\mu}\ln(\frac{m_o}{m})$$

$$\gamma_z = \mu\frac{\partial}{\partial\mu}\ln Z_3 \tag{5.114}$$

where it is understood that the limit as d tends to 4 is to be taken. We wish to apply equation (5.113) to the effective potential $V(\tilde{\phi})$ now. By dimensional analysis, we have

$$(\mu\frac{\partial}{\partial\mu}+m\frac{\partial}{\partial m}+\tilde{\phi}\frac{\partial}{\partial\tilde{\phi}})\ V = 4V \quad . \tag{5.115}$$

Thus, equation (5.113) becomes

$$[4+\beta\frac{\partial}{\partial g}-\frac{1}{2}(2+\gamma_m)m\frac{\partial}{\partial m}-\frac{1}{2}(2+\gamma_z)\tilde{\phi}\frac{\partial}{\partial\tilde{\phi}}]V(\tilde{\phi},m,g) = 0 \ . \tag{5.116}$$

We can express this in terms of the dimensionless quantity

$$G = \frac{1}{am^2}\left.\frac{\partial V}{\partial\tilde{\phi}}\right|_{\tilde{\phi}=a} = \frac{T}{am^2} \quad , \tag{5.117}$$

whereby equation (5.116) becomes

$$[(2+\gamma_z)a^2 \frac{\partial}{\partial a^2} + (2+\gamma_m)m^2 \frac{\partial}{\partial m^2} - \beta \frac{\partial}{\partial g} + \gamma_m + \gamma_z] G(a^2, m^2, g) = 0. \quad (5.118)$$

This equation is typical of a renormalization group equation for the one-point function for all renormalizable scalar theories. There will be one β function for each coupling constant, one γ_m function for each mass, and one γ_z function for each field. In general, then, there will be a system of coupled differential equations.

The degree of usefulness of the renormalization group equations varies from model to model. Having found β, γ_m and γ_z in some suitable approximation scheme, one may then solve the homogeneous equation (5.118) to find $G(a^2, m^2, g)$. Typically, G found by a loop expansion contains logarithms of a^2, and the expansion is reliable only so long as these logarithms are small. It is this restriction that the "improved" G found by solving the renormalization group equation may not be subject to: this G may be valid over a larger range of a^2 than that found in a loop expansion. Note that these equations do not allow us to escape the restriction that the coupling constant be small.

A similar derivation carried out on the Green functions of the theory leads to renormalization group equations for these functions. These equations contain the same β, γ_m and γ_z functions that appear in (5.118). The equations govern how the Green functions behave as the momentum is varied, and the "improved" Green functions in this case may be more reliable over a larger range of momentum than those found in a loop expansion (see [W4, C[illegible]

To solve these equations one must then find a suitable approximation for the functions β, γ_m and γ_z. A typical approximation is the loop expansion. Finding the one-point function in the theory where the field (or fields) is shifted by a constant amount in fact provides a fairly efficient determination of mass and coupling constant counterterms (as the one-point function necessarily has zero external momentum, the wave-function renormalization counterterm must be determined independently). Having found these counterterms, the renormalization group coefficients are then readily calculated. As an illustration of the method, we will calculate these coefficients for an $O(n)$-symmetric ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{2} m^2 (\phi^a \phi^a) - \frac{1}{4!} g(\phi^a \phi^a)^2 + \frac{A}{2}(\partial_\mu \phi^a)^2 - \frac{1}{2} B(\phi^a \phi^a) - \frac{1}{4!} C(\phi^a \phi^a)^2 \tag{5.119}$$

with $a = 1, \ldots n$. We now perform the shift $\phi_1 \to \phi_1 + a$, resulting in

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu \phi_1^2) + \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\kappa^2 \phi_1^2 - \frac{1}{2}\tau^2 \phi_i^2 - (m^2 + \frac{1}{6}a^2 g) a\phi_1 \\ & - \frac{1}{6} a g \phi_1 (\phi_1^2 + \phi_i^2) - \frac{1}{24} g(\phi_1^2 + \phi_i^2)^2 - (B + \frac{1}{6}a^2 C) a\phi_1 - \frac{1}{2}(B + \frac{1}{2}a^2 C)\phi_1^2 \\ & - \frac{1}{2}(B + \frac{1}{6}a^2 C)\phi_i^2 - \frac{1}{6} a C \phi_1 (\phi_1^2 + \phi_i^2) - \frac{1}{24} C(\phi_1^2 + \phi_i^2)^2 + \frac{A}{2}(\partial_\mu \phi^a)^2 \end{aligned} \tag{5.120}$$

where

$$i = 2, \ldots n$$

$$\kappa^2 = m^2 + \frac{1}{2} a^2 g \quad .$$

$$\tau^2 = m^2 + \frac{1}{6} a^2 g \quad .$$

We will now calculate the one-point function for ϕ_1 but, in keeping with the requirement that Z_1, Z_2 and Z_3 in the expansion (5.107) are mass-independent, we have to use a different renormalization scheme than that employed previously in Section B. The tree level contribution, shown in Fig. 14(a), gives

$$T_o = (m^2 + \frac{1}{6} a^2 g) a \quad . \tag{5.121}$$

The one-loop terms are seen in Fig. 14(b), and contribute

$$\frac{iag\mu^{\varepsilon}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\kappa^2}=\frac{ag}{32\pi^2}(m^2+\frac{a^2g}{2})[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\kappa^2}{4\pi\mu^2})] \quad (5.122)$$

and

$$\frac{iag\mu^{\varepsilon}}{2}\frac{n-1}{3}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\tau^2}=\frac{ag}{32\pi^2}\frac{n-1}{3}(m^2+\frac{a^2g}{6})[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\tau^2}{4\pi\mu^2})] \quad (5.123)$$

To the one-loop level, then, we have, including the counterterms,

$$T=T_o+T_1=(m^2+\frac{1}{6}a^2g)a+\frac{ag}{32\pi^2}\Big\{m^2[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\kappa^2}{4\pi\mu^2})]$$
$$+m^2(\frac{n-1}{3})[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\tau^2}{4\pi\mu^2})]+\frac{a^2g}{2}[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\kappa^2}{4\pi\mu^2})]$$
$$+\frac{a^2g}{2}(\frac{n-1}{3})[-\frac{2}{\varepsilon}+\gamma-1+\ln(\frac{\tau^2}{4\pi\mu^2})]\Big\}+(B_1+\frac{1}{6}a^2C_1)a\ . \quad (5.124)$$

Now, in order to make Z_1, Z_2 and Z_3 in (5.107) independent of m^2, we may choose our renormalization program to be the minimal subtraction scheme [H8]: the counterterms are adjusted so as to cancel the poles at $d=4$ only. Using this scheme, we obtain

$$B_1=(\frac{g}{16\pi^2})(1+\frac{n-1}{3})\frac{1}{\varepsilon}m^2 \quad (5.125)$$

$$C_1=3(\frac{g}{16\pi^2})(1+\frac{n-1}{9})\frac{1}{\varepsilon}g\ . \quad (5.126)$$

Although the minimal subtraction scheme does not correspond to renormalizing at one particular point, it is a consistent renormalization scheme and is found to be convenient in these types of calculations.

We now come to the two-loop terms. We begin by defining the following integrals:

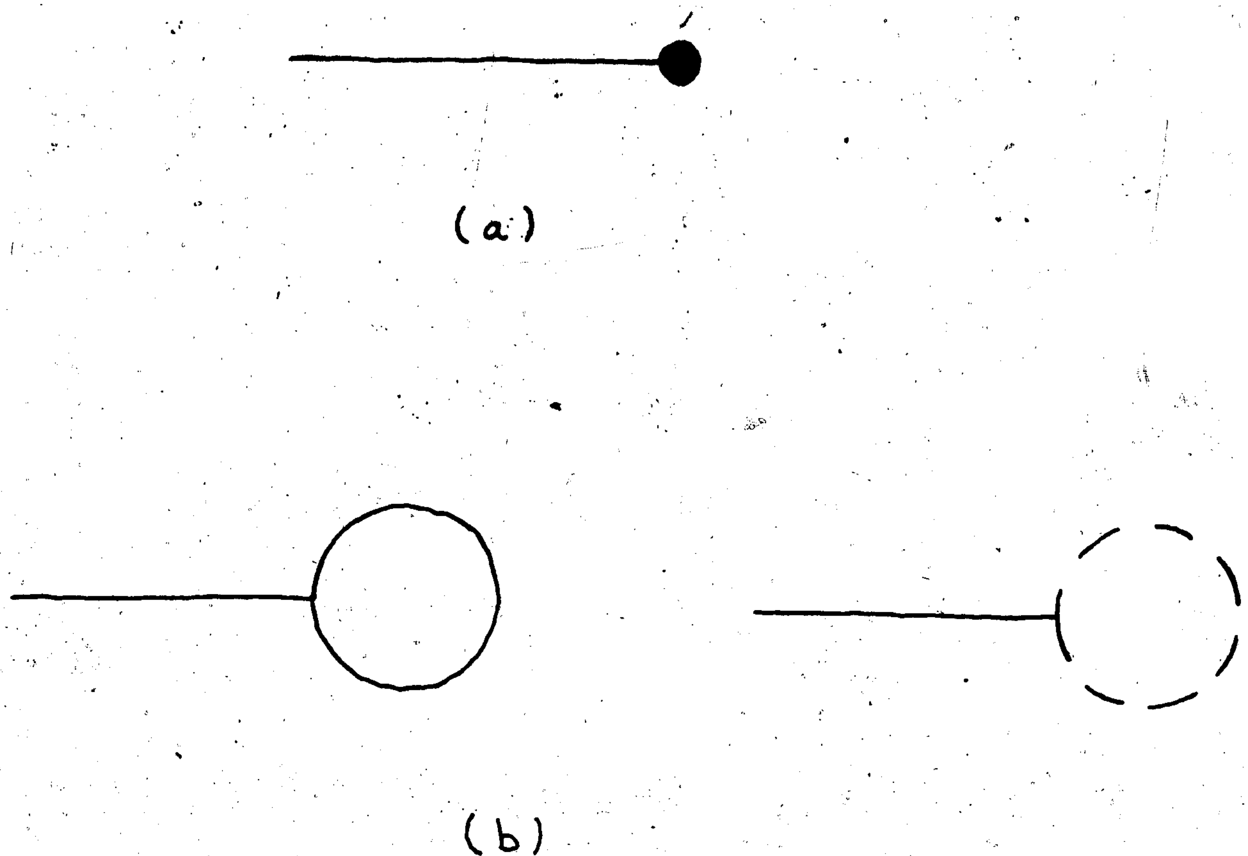


Fig. 14 (a) The tree-level approximation to T.
(b) The one-loop contributions to T. A solid line denotes ϕ_1, a dashed one ϕ_i.

$$I(x,y) = -\frac{(ag)g\mu^{2\varepsilon}}{4(2\pi)^{2d}} \int \frac{d^d k \, d^d p}{(k^2-x)^2(p^2-y)} \tag{5.127}$$

$$J(x,y) = -\frac{(ag)g\mu^{2\varepsilon}}{6(2\pi)^{2d}} \int \frac{d^d k \, d^d p}{(p^2-x)(k^2-x)[(p+k)^2-y]} \tag{5.128}$$

$$K(x,y,z) = -\frac{(ag)^3 \mu^{2\varepsilon}}{4(2\pi)^{2d}} \int \frac{d^d k \, d^d p}{(p^2-x)^2(k^2-y)[(p+k)^2-z]} \; . \tag{5.129}$$

The two-loop terms then become the following. Fig. 15(a) gives, from left to right,

$$I(\kappa^2,\kappa^2) + \frac{n-1}{3} I(\kappa^2,\tau^2) + \frac{n-1}{9} I(\tau^2,\kappa^2) + \frac{n-1}{3}\left(1+\frac{n-2}{3}\right) I(\tau^2,\tau^2)$$

$$= -\frac{ag^2}{2^{10}\pi^4}\left(-\frac{2}{\varepsilon}+\gamma-1\right)\left(\frac{2}{\varepsilon}-\gamma\right)\left\{\kappa^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon} + \frac{n-1}{3}\tau^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\right.$$

$$\left. + \frac{n-1}{9}\kappa^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon/2} + \frac{n-1}{9}(2+n-1)\tau^2\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon}\right\} \; . \tag{5.130}$$

Fig. 15(b) leads to

$$J(\kappa^2,\kappa^2) + (1+1+1)\frac{n-1}{9} J(\tau^2,\kappa^2)$$

$$= -\frac{ag^2}{2^9\pi^4}\left(-\frac{1}{\varepsilon}+\gamma-1\right)\left(\frac{2}{\varepsilon}+1\right)\left\{\kappa^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon} + \frac{n-1}{9}\left[2\tau^2\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon}\right.\right.$$

$$\left.\left. + \kappa^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon}\right]\right\} \; . \tag{5.131}$$

Fig. 15(c) contributes

$$K(\kappa^2,\kappa^2,\kappa^2) + (1+1)\frac{n-1}{27} K(\tau^2,\tau^2,\kappa^2) + \frac{n-1}{9} K(\kappa^2,\tau^2,\tau^2)$$

$$= -\frac{a^3 g^3}{2^{10}\pi^4}\left(-\frac{1}{\varepsilon}+\gamma\right)\left(\frac{2}{\varepsilon}+1\right)\left\{\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon} + \frac{2(n-1)}{27}\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon}\right.$$

$$\left. + \frac{n-1}{9}\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon}\right\} \; . \tag{5.132}$$

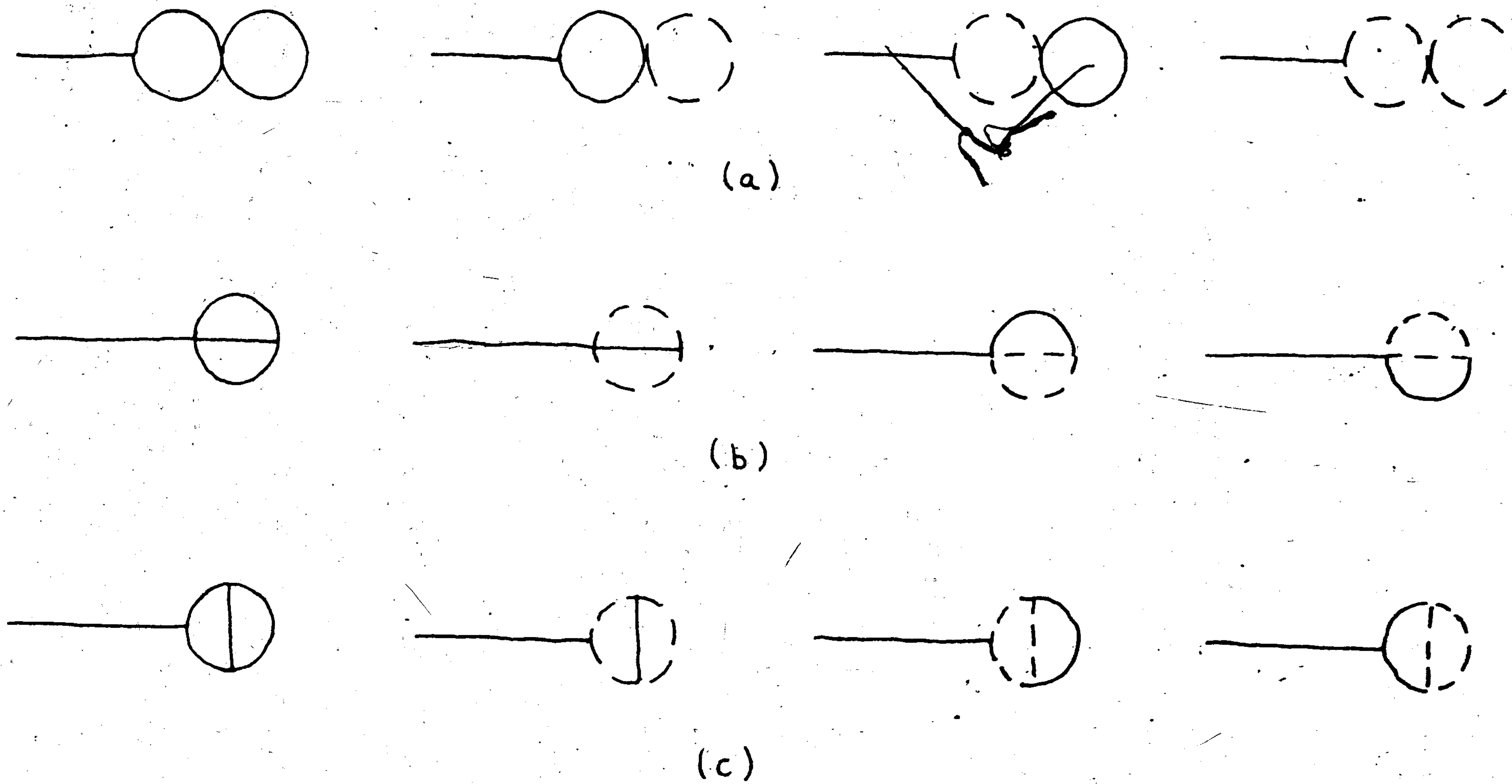


Fig. 15. The two-loop contribution to T.

These integrals are of the same form as those encountered in the massless ϕ^4 theory of Section B, the nontrivial ones again being found in the Appendix.

In addition to these terms, we have the contributions coming from the one-loop counterterms, pictured in Fig. 16. Fig. 16(a) gives

$$\frac{iaC_1\mu^\varepsilon}{2}\int\frac{d^3k}{(2\pi)^d}\frac{1}{k^2-\kappa^2}=\frac{aC_1}{2^5\pi^2}\left(-\frac{2}{\varepsilon}+\gamma-1\right)\kappa^2\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon/2} \tag{5.133}$$

and

$$\frac{iaC_1\mu^\varepsilon}{2}\;\frac{n-1}{3}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\tau^2}=\frac{aC_1}{2^5\pi^2}\left(-\frac{2}{\varepsilon}+\gamma-1\right)\frac{n-1}{3}\,\tau^2\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon/2}. \tag{5.134}$$

Fig. 16(b) contributes

$$\frac{iag\mu^\varepsilon}{2}\left(B_1+\frac{a^2C_1}{2}\right)\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\kappa^2)^2}=-\frac{ag}{2^5\pi^2}\left(B_1+\frac{a^2C_1}{2}\right)\left(\frac{2}{\varepsilon}-\gamma\right)\left(\frac{\kappa^2}{4\pi\mu^2}\right)^{-\varepsilon/2} \tag{5.135}$$

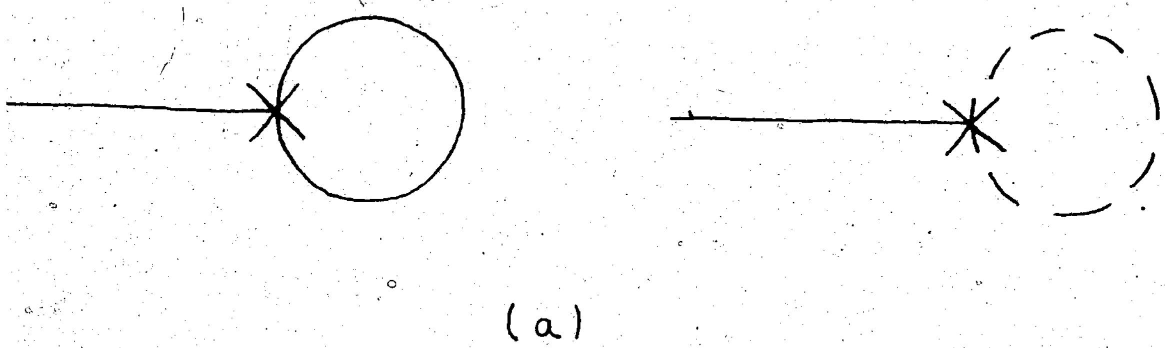
and

$$\frac{iag\mu^\varepsilon}{2}\left(B_1+\frac{a^2C_1}{6}\right)\frac{n-1}{3}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\tau^2)^2}$$
$$=-\frac{ag}{2^5\pi^2}\left(B_1+\frac{a^2C_1}{6}\right)\frac{n-1}{3}\left(\frac{2}{\varepsilon}-\gamma\right)\left(\frac{\tau^2}{4\pi\mu^2}\right)^{-\varepsilon/2} \tag{5.136}$$

with B_1 and C_1 found in (5.125) and (5.126).

The entire two-loop contribution, the sum of (5.130)-(5.136), then becomes, including the counterterm,

$$T_2=-\frac{ag^2}{2^9\pi^4}\left\{\frac{2}{\varepsilon^2}\left[2+n-1+\frac{(n-1)^2}{9}\right]m^2-\frac{1}{\varepsilon}\left[1+\frac{n-1}{3}\right]m^2\right.$$
$$\left.+\frac{6}{\varepsilon^2}\left[1+\frac{n-1}{9}\right]^2\frac{a^2g}{2}-\frac{2}{\varepsilon}\left[1+\frac{5(n-1)}{27}\right]\frac{a^2g}{2}\right\}$$
$$+\text{finite terms}+\left(B_2+\frac{1}{6}a^2C_2\right)a\ . \tag{5.137}$$



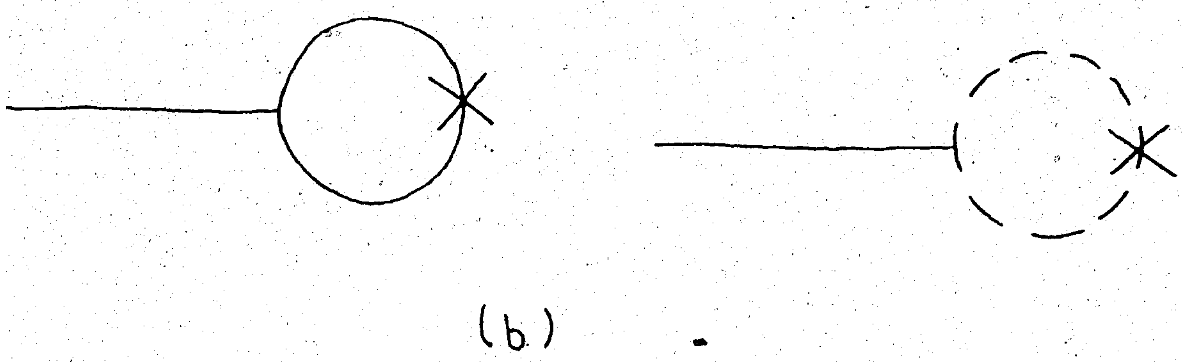


Fig. 16. The two-loop contribution from the one-loop counterterms.

Again we see that the $(1/\varepsilon)\ell n(\kappa^2/4\pi\mu^2) + (1/\varepsilon)\ell n(\tau^2/4\pi\mu^2)$ terms present in each of (5.104)-(5.110) have cancelled in the sum, ensuring that the counterterms remain local. Minimal subtraction then yields

$$B_2 = \left(\frac{g}{16\pi^2}\right)^2 \left\{\frac{1}{\varepsilon^2}\left[2+n-1+\frac{(n-1)^2}{9}\right] - \frac{1}{2\varepsilon}\left[1+\frac{n-1}{3}\right]\right\} m^2 \qquad (5.138)$$

and

$$C_2 = 3\left(\frac{g}{16\pi^2}\right)^2 \left\{\frac{3}{\varepsilon^2}\left[1+\frac{n-1}{9}\right]^2 - \frac{1}{\varepsilon}\left[1+\frac{5(n-1)}{27}\right]\right\} g \quad . \qquad (5.139)$$

Including the one-loop counterterms, (5.125) and (5.126), we have then to two loops,

$$B = \left\{\left(\frac{g}{16\pi^2}\right)\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon} + \left(\frac{g}{16\pi^2}\right)^2 \left[\left(2+n-1+\frac{(n-1)^2}{9}\right)\frac{1}{\varepsilon^2} - \frac{1}{2}\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}\right]\right\} m^2 \qquad (5.140)$$

and

$$C = \left\{3\left(\frac{g}{16\pi^2}\right)\left(1+\frac{n-1}{9}\right)\frac{1}{\varepsilon} + 3\left(\frac{g}{16\pi^2}\right)^2 \left[3\left(1+\frac{n-1}{9}\right)^2\frac{1}{\varepsilon^2} - \left(1+\frac{5(n-1)}{27}\right)\frac{1}{\varepsilon}\right]\right\} g \quad . \qquad (5.141)$$

Before we can express m_o^2 in terms of m^2 and g_o in terms of g, however, we must find the wave-function renormalization counterterm Z_3 (see (5.107)). This will be done by demanding that the following be finite (see (5.23)):

$$\left.\frac{\partial\tilde{\Gamma}^{(2)}(p^2)}{\partial p^2}\right|_{p^2=0} , \qquad (5.142)$$

where $\tilde{\Gamma}^{(2)}(p^2)$ is the two-point function in the unshifted theory (5.108). To the two-loop level, the only graphs we need consider are those shown in Fig. 17. Thus, we have to make finite the following:

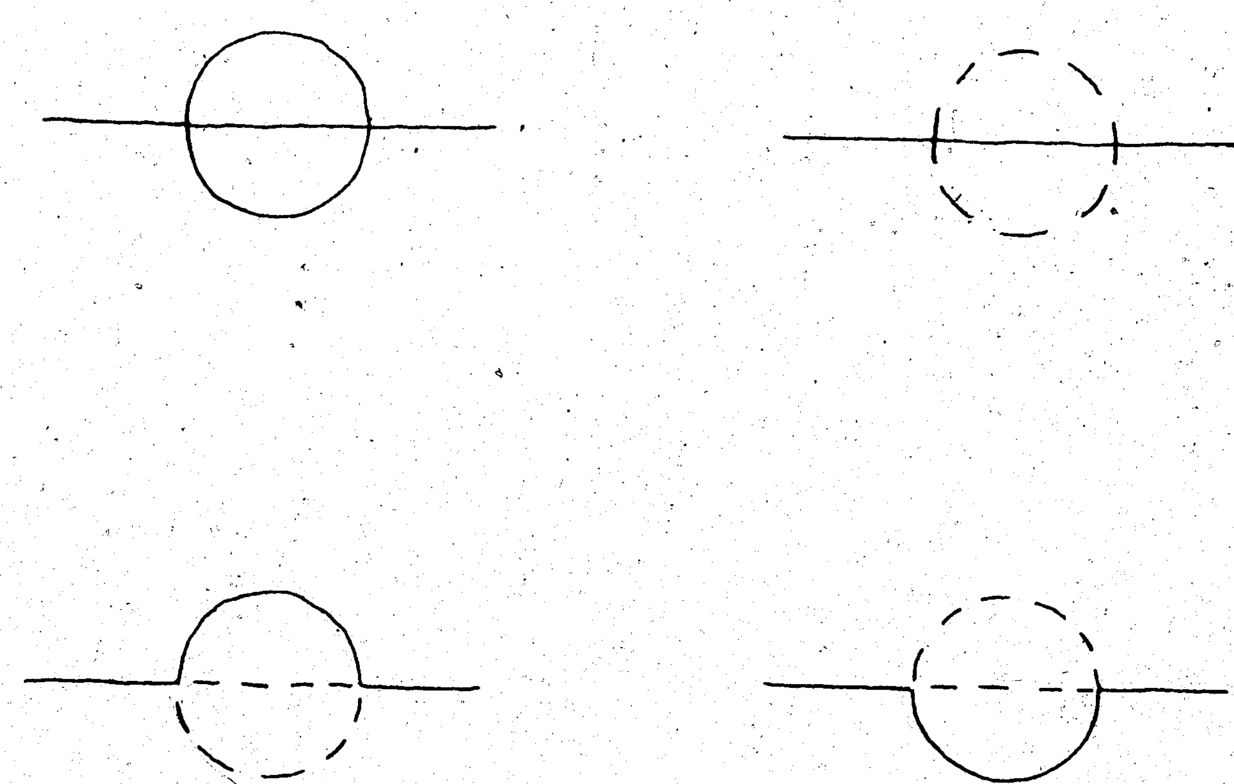


Fig. 17. Graphs needed to determine Z_3 up to the two-loop level.

$$\frac{\partial}{\partial p^2}\left\{+\frac{g^2\mu^{2\varepsilon}}{6(2\pi)^{2d}}\int\frac{d^dk\,d^d\ell}{(k^2-\kappa^2)(\ell^2-\kappa^2)[(p+k+\ell)^2-\kappa^2]}\right.$$

$$\left.+(1+1+1)\frac{g^2\mu^{2\varepsilon}}{6(2\pi)^{2d}}\frac{n-1}{9}\int\frac{d^dk\,d^d\ell}{(k^2-\kappa^2)(\ell^2-\tau^2)[(p+k+\ell)^2-\kappa^2]}\right\}\Bigg|_{p^2=0}$$

$$=\frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}+\text{finite terms}\ . \qquad (5.143)$$

For details of this last calculation, see the Appendix. In the minimal subtraction scheme, then, the wave-function renormalization counterterm is determined to be

$$A = Z_3 - 1 = -\frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}\ . \qquad (5.144)$$

Therefore, using (5.107) and (5.109), the expansions of the bare in terms of renormalized quantities, equations (5.110), read, to two loops,

$$Z_3 = 1-\frac{1}{12}\left(\frac{g}{16\pi^2}\right)^2\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}$$

$$m_o^2 = Z_2Z_3^{-1}m^2 = \left(1+\frac{B}{m^2}\right)Z_3^{-1}m^2$$

$$=\left\{1+\left(\frac{g}{16\pi^2}\right)\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}+\left(\frac{g}{16\pi^2}\right)^2\left[\left(2+n-1+\frac{(n-1)^2}{9}\right)\frac{1}{\varepsilon^2}\right.\right.$$

$$\left.\left.-\frac{5}{12}\left(1+\frac{n-1}{3}\right)\frac{1}{\varepsilon}\right]\right\}m^2$$

$$g_o = Z_1Z_3^{-2}g = \left(1+\frac{C}{g}\right)Z_3^{-2}g$$

$$=\left\{1+3\left(\frac{g}{16\pi^2}\right)\left(1+\frac{n-1}{9}\right)\frac{1}{\varepsilon}+\left(\frac{g}{16\pi^2}\right)^2\left[9\left(1+\frac{n-1}{9}\right)^2\frac{1}{\varepsilon^2}\right.\right.$$

$$\left.\left.-\frac{17}{6}\left(1+\frac{3(n-1)}{17}\right)\frac{1}{\varepsilon}\right]\right\}g\mu^{\varepsilon}\ . \qquad (5.145)$$

Using the definitions (5.114), we then find the renormalization group coefficients to be

$$\beta = \mu \frac{\partial g}{\partial \mu} = -(1 - g \frac{\partial}{\partial g}) a_1$$

$$= [3(1 + \frac{n-1}{9})(\frac{g}{16\pi^2}) - \frac{17}{3}(1 + \frac{3(n-1)}{17})(\frac{g}{16\pi^2})^2] g$$

$$\gamma_m = \mu \frac{\partial}{\partial \mu} \ell n(\frac{m_o}{m}) = -g \frac{\partial b_1}{\partial g}$$

$$= -(1 + \frac{n-1}{3})(\frac{g}{16\pi^2}) + \frac{5}{6}(1 + \frac{n-1}{3})(\frac{g}{16\pi^2})^2$$

$$\gamma_z = \mu \frac{\partial}{\partial \mu} \ell n \, Z_3 = -g \frac{\partial C_1}{\partial g}$$

$$= \frac{1}{6}(1 + \frac{n+1}{3})(\frac{g}{16\pi^2})^2 , \tag{5.146}$$

where a_1, b_1 and c_1 denote respectively a_{1k}, b_{1k} and c_{1k}, summed over k, in the expansions (5.110). Note that factors like γ and $\ell n\ 4\pi$, present in the intermediate steps due to our particular choice of dimensional regularization, have disappeared in the final expressions (5.144). These results agree with the $n = 1$ case found in [I2] and [C7], where the two- and four-point functions are evaluated explicitly to find the counterterms (they may also be obtained using what is called the background field method, which is a powerful technique in obtaining the counterterms for a theory; see [D1, H4, H10, A2, I1]).

We have seen in this chapter that much information about a theory can be obtained by studying the theory where the fields are shifted by a constant. The effective potential can be calculated fairly efficiently, as can the function $Z(\tilde{\phi})$. As well, the counterterms and hence the coefficients of the renormalization group equations can also be obtained in a relatively efficient manner. The use of the shifted theory simplifies considerably the calculations done here as opposed to more direct methods like, for instance, calculating the effective potential or $Z(\tilde{\phi})$

by an infinite summation of graphs or obtaining counterterms by a direct evaluation of the relevant n-point functions. This is especially true if there are more than one field, mass and coupling constant present, where just the combinatorics involved in the calculations can become quite difficult using the more direct methods. The methods used in the calculations done here involve essentially the same amount of labour as that found in the path-integral approach, at times with some more algebra, but are done by dealing directly with the Lagrange density.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Let us just give a brief summary of the major points of the preceding chapters.

We started by considering two models that are quite important in present-day particle physics: the Maxwell field and the abelian Higgs model. A covariant quantization of either one leads to the presence of unphysical degrees of freedom, this being related to the occurrence of a dipole field in both with a suitable gauge-fixing term. An indefinite metric quantization removes these unphysical modes from the physical space by giving them either negative or zero norm, the physical space necessarily containing only positive norm states. However, after noticing only part of the whole space could be physical, we inquired into whether or not it was possible to eliminate the unphysical modes in a positive definite metric Hilbert space by some suitable criterion. Just as an indefinite metric quantization showed that it was possible to quantize these models without the introduction of nonlocal, noncovariant fields, as in a Coulomb-type gauge formalism, we wondered if the introduction of negative and zero norm states was necessary in a local covariant formulation.

We found that a positive metric quantization was possible, and led to the correct number of physical degrees of freedom, but care must be taken. At the free-field level, the criterion used to select out a physical space was that, on this space, Poincaré transformations must be implemented unitarily so as to form a symmetry group. If one then imposed the requirement that observables be gauge invariant, one is then led to the usual interpretations of these models. However, since at the free-field level there is no compelling reason to make such a postulate, we next studied the two models coupled to external c-number sources. We then found that if one did not impose gauge invariance, the S operator would not be unitary on the physical space. In order to conserve probabilities, then, one must postulate gauge invariance, and the resulting models contain the expected number of physical degrees of freedom. We thus see the important role gauge invariance plays in making these models physically sensible when the positive metric is used, and how the concepts of pseudounitarity and pseudohermiticity translate from an indefinite metric space into a Hilbert space.

We next went beyond the tree level and considered quantum corrections to the effective action, the generator of one-particle-irreducible functions in field theory. This functional is useful in studies of spontaneous breakdown of symmetry, which occurs in the Higg's model. We were concerned mainly with the first two terms of the expansion of the effective action in coordinate space: the effective potential $V(\tilde{\phi})$ and the function $Z(\tilde{\phi})$. We showed that these two functions can be evaluated perturbatively fairly efficiently by considering the theory where the fields are shifted by a constant amount. The same shifting also leads to a relatively efficient evaluation of the counterterms of a theory, and hence to the renormalization group equation coefficients. The methods used simplify these calculations considerably, especially where more than one field, mass and coupling constant is present, where just the combinatorics involved in more direct methods are fairly formidable. Although these calculations can also be performed using the path-integral formalism, at times with less algebra, one deals here directly with the Lagrange density.

As a final note, we would like to point out some problems with the effective potential. As Jackiw showed [J1, D2], the effective potential in the presence of gauge fields is gauge dependent (see also [F2, F3]). This would seem to cast doubt on any physical prediction made using the effective potential. For example, Coleman and Weinberg [C3] showed that a symmetry, present at the tree level, could be broken by radiative corrections. If this mechanism were to work in the Weinberg-Salam model, say, then it would lead to an elimination of one free parameter in that model and then to a definite prediction of the mass ratio of the Higg's particle to another particle [C3, M1]. However, if this mass ratio were to be gauge dependent, such a prediction would be meaningless. Fortunately, it turns out that, for a simplified model of this mechanism, the gauge dependence cancels out for the predicted physical mass ratio up to the two-loop level [K1]. Is this a general result? Another curious result in the presence of gauge fields is that differentiating the one-point function in the shifted theory m times is not equivalent to adding m zero momentum external legs, except in the Landau gauge [W3]. Thus, the effective potential is not simply related to the one-point function by a single differentiation of the potential. However, Weinberg showed that, using only the one-point function and self-energy insertions, mass corrections are indeed gauge invariant [W3]. There appears to be something in the formalism that guarantees physical results to be gauge invariant, even though in intermediate steps using either the one-point function or the effective potential explicit gauge dependence appears. Whether or not this occurs to all orders is not known. Finally, we end with a note of caution. It can be proved on general grounds that the effective potential must be convex everywhere

[S1, I2, F6]. Since the first few terms of the loop expansion of the Higg's potential violates this property, and indeed becomes complex in some regions, there appears some doubt as to the consistency of a perturbative approximation to this model. It remains to be seen whether or not this problem affects significantly the phenomenological aspect of models using the Higg's potential.

APPENDIX

In this Appendix we will present some of the details of the calculation of the nontrivial integrals encountered in chapter V. We will need the following formulas [H7]:

$$\int \frac{d^d k}{(k^2-Q)^m} = i\pi^{d/2} (-1)^m \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} Q^{\frac{d}{2}-m}$$

$$\Gamma(1-\frac{d}{2}) = -\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon) \qquad \ldots \epsilon = 4-d$$

$$x\Gamma(x) = \Gamma(1+x)$$

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m}} = \frac{\Gamma(\alpha_1+\ldots+\alpha_m)}{\Gamma(\alpha_1)\ldots(\alpha_m)} \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{m-2}} dx_{m-1}$$

$$\times \frac{x_{m-1}^{\alpha_1-1} (x_{m-2}-x_{m-1})^{\alpha_2-1} \ldots (1-x_1)^{\alpha_m-1}}{[a_1 x_{m-1} + a_2(x_{m-2}-x_{m-1}) + \ldots + a_m(1-x_1)]^{\alpha_1+\ldots+\alpha_m}} . \qquad (A1)$$

In the following we freely drop polynomials that can be absorbed in the two-loop counterterms.

(a)
$$I_1 = \int \frac{d^d k \, d^d \ell}{(k^2-\mu^2)(\ell^2-\mu^2)[(k+\ell)^2-\mu^2]} = 2 \int d^d k \, d^d \ell \int_0^1 dx \int_0^x \frac{dy}{D^3} \qquad (A2)$$

where

$$D = [\ell + \frac{k(1-x)}{1-x+y}]^2 (1-x+y) + k^2 (\frac{x + xy - x^2 - y^2}{1-x+y}) - \mu^2 .$$

Carrying out the momentum integrations, we arrive at

$$I_1 = \pi^d(-\frac{1}{\varepsilon}+\gamma-1)\int_0^1 dx\int_0^x \frac{dy}{(x+xy-x^2-y^2)^{d/2}}(\mu^2)^{d-3}$$

$$= \pi^d(-\frac{1}{\varepsilon}+\gamma-1)\int_0^1 dx\,dy\,dz\;\frac{\delta(x+y+z-1)}{(xy+yz+zx)^{d/2}}\,(\mu^2)^{d-3}\;. \qquad \text{(A3)}$$

This remaining integral diverges at $x=-y$, $y=-z$, and $z=-x$ for $d=4$. To isolate the pole, let

$$x = \beta\rho \quad , \quad y = \beta(1-\rho) \quad , \quad z = 1-\beta\;. \qquad \text{(A4)}$$

The pole at $x=-y$ is then at $\beta=0$. For small β, we have

$$\int_0^1 dx\,dy\,dz\;\frac{\delta(x+y+z-1)}{(xy+yz+zx)^{d/2}} \sim \int_0^1 \beta d\beta\int_0^1 d\rho\;(\beta)^{-d/2} \sim \frac{2}{\varepsilon}\;. \qquad \text{(A5)}$$

Since the integral is symmetric, the total singularity is $6/\varepsilon$. To evaluate the finite part, it is necessary to subtract

$$\frac{1}{(x+y)^{d/2}} + \frac{1}{(y+z)^{d/2}} + \frac{1}{(z+x)^{d/2}} \qquad \text{(A6)}$$

from the integrand in (A5). We then have

$$I_1 = \pi^d(-\frac{1}{\varepsilon}+\gamma-1)\left\{\frac{6}{\varepsilon}+\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\right.$$

$$\left.\times\left[\frac{1}{(xy+yz+zx)^2}-\frac{1}{(x+y)^2}-\frac{1}{(y+z)^2}-\frac{1}{(z+x)^2}\right]\right\}(\mu^2)^{d-3}$$

$$= \pi^d(-\frac{1}{\varepsilon}+\gamma-1)\left\{\frac{6}{\varepsilon}+\int_0^1 d\rho\int_0^1 \beta d\beta\left[\frac{1}{\beta^2\{1+\beta[\rho(1-\rho)-1]\}^2}-\frac{1}{\beta^2}\right.\right.$$

$$\left.\left.-\frac{1}{(1-\beta\rho)^2}-\frac{1}{[1-\beta(1-\rho)]^2}\right]\right\}(\mu^2)^{d-3}$$

$$= \pi^d(-\frac{1}{\varepsilon}+\gamma-1)(\frac{6}{\varepsilon}+3)(\mu^2)^{d-3} \qquad \text{(A7)}$$

(b) $$I_2 = \int \frac{d^dk\, d^d\ell}{(k^2-m^2)(\ell^2-m^2)[(k+\ell)^2-\mu^2]}$$

$$= \frac{\pi^d}{3}\left(-\frac{1}{\varepsilon}+\gamma-1\right)\iiint_0^1 dx\, dy\, dz\; \frac{\delta(x+y+z-1)}{(xy+yz+zx)^{d/2}}$$

$$\times \left\{[\mu^2x+m^2(1-x)]^{d-3}+[\mu^2y+m^2(1-y)]^{d-3}+[\mu^2z+m^2(1-z)]^{d-3}\right\}$$

$$= \pi^d\left[-\frac{1}{\varepsilon}+\gamma-1\right)\left(\frac{2}{\varepsilon}+1\right)[2(m^2)^{d-3}+(\mu^2)^{d-3}] \qquad \text{(A8)}$$

(c) $$I_3 = \int \frac{d^dk\, d^d\ell}{(k^2-\mu^2)^2(\ell^2-\mu^2)[(k+\ell)^2-\mu^2]}$$

$$= \frac{1}{3}\frac{\partial}{\partial\mu^2} I_1$$

$$= \pi^d\left(-\frac{1}{\varepsilon}+\gamma\right)\left(\frac{2}{\varepsilon}+1\right)(\mu^2)^{d-4} \qquad \text{(A9)}$$

(d) $$I_4 = \int \frac{d^dk\, d^d\ell}{(k^2-m^2)^2(\ell^2-m^2)[(k+\ell)^2-\mu^2]}$$

$$= \frac{1}{2}\frac{\partial}{\partial m^2} I_2$$

$$= \pi^d\left(-\frac{1}{\varepsilon}+\gamma\right)\left(\frac{2}{\varepsilon}+1\right)(m^2)^{d-4} \qquad \text{(A10)}$$

(e) $$I_5 = \int \frac{d^dk\, d^d\ell}{(k^2-\mu^2)^2(\ell^2-m^2)[(k+\ell)^2-m^2]}$$

$$= \frac{\partial}{\partial\mu^2} I_2$$

$$= \pi^d\left(-\frac{1}{\varepsilon}+\gamma\right)\left(\frac{2}{\varepsilon}+1\right)(\mu^2)^{d-4} \qquad \text{(A11)}$$

(f) $$I_6 = \int \frac{d^d k \, d^d \ell}{(k^2-m^2)(\ell^2-m^2)[(p+k+\ell)^2-m^2]}$$

$$= \pi^d\left(-\frac{1}{\varepsilon}+\gamma-1\right)\int_0^1 dx\,dy\,dz\,\frac{\delta(x+y+z-1)}{(xy+yz+zx)^{d/2}}\left[m^2-\frac{p^2xyz}{xy+yz+zx}\right]^{d-3} \quad (A12)$$

$$\left.\frac{\partial I_6}{\partial p^2}\right|_{p^2=0} = -\pi^d\left(-\frac{1}{\varepsilon}+\gamma-1\right)(1-\varepsilon)(m^2)^{-\varepsilon}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{xyz}{(xy+yz+zx)^3}$$

$$= \frac{\pi^d}{2}\left(\frac{1}{\varepsilon}-\gamma-\ln m^2\right) \quad (A13)$$

(g) $$I_7 = \int \frac{d^d k \, d^d \ell}{(k^2-m^2)(\ell^2-m^2)[(p+k+\ell)^2-\mu^2]}$$

$$= \frac{\pi^d}{3}\left(-\frac{1}{\varepsilon}+\gamma-1\right)\int_0^1 dx\,dy\,dz\,\frac{\delta(x+y+z-1)}{(xy+yz+zx)^{d/2}}\left\{\left[\mu^2x+m^2(1-x)-\frac{p^2xyz}{xy+yz+zx}\right]^{d-3}\right.$$

$$\left.+\left[\mu^2y+m^2(1-y)-\frac{p^2xyz}{xy+yz+zx}\right]^{d-3}+\left[\mu^2z+m^2(1-z)-\frac{p^2xyz}{xy+yz+zx}\right]^{d-3}\right\} \quad (A14)$$

$$\left.\frac{\partial I_7}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{2}\left(\frac{1}{\varepsilon}-\gamma-\ldots\right) \quad (A15)$$

(h) $$I_8 = \int \frac{d^d k \, d^d \ell}{(k^2-m^2)(\ell^2-m^2)[(k+\ell)^2-m^2][(k-p)^2-m^2]}$$

$$= -\pi^d\left(\frac{1}{\varepsilon}-\gamma\right)\int_0^1 dx\int_0^x dy\int_0^y dz\,\frac{1}{(x+xy-x^2-y^2)^{d/2}}\left\{m^2-\frac{p^2z(x+xy-x^2-y^2-z+zy)}{x+xy-x^2-y^2}\right\}^{-\varepsilon} \quad (A17)$$

$$\left.\frac{\partial I_8}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{m^2}(-1+\varepsilon\gamma)(m^2)^{-\varepsilon}\int_0^1 dx\,dy\,dz\,dw\;\delta(x+y+z+w-1)$$

$$\times\frac{z[(x+y)w+xy]}{[(x+y)(w+z)+xy]^{1+d/2}} \quad (A18)$$

The remaining integral diverges for $d = 4$; we set

$$x = \beta u \qquad z = (1-\beta)v$$

$$y = \beta(1-u) \qquad w = (1-\beta)(1-v) \tag{A19}$$

isolate the singularity, and then subtract the appropriate term to find the finite part. We find, setting $a = u^2 - u + 1$,

$$\left.\frac{\partial I_8}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{m^2}(-1+\epsilon\gamma)(m^2)^{-\epsilon}\left\{\frac{1}{3\epsilon}+\int_0^1 du\, dv\, d\beta\left[\frac{(1-\beta)^2}{\beta}\left(\frac{v}{(1-a\beta)^2}-\frac{v^2(1-\beta)}{(1-a\beta)^3}\right)\right.\right.$$

$$\left.\left.-\frac{v(1-v)}{\beta}\right]\right\}$$

$$= \frac{\pi^d}{3m^2}\left(-\frac{1}{\epsilon}+\gamma-1+\ln m^2-\delta_1\right) \tag{A20}$$

where

$$\delta_1 = \int_0^1 du\left\{\left(\frac{3u^2-3u+1}{(u^2-u+1)^3}\right)\ln u - \frac{u(1-u)}{(u^2-u+1)^2}\right\} = -\frac{1}{3}-\frac{2}{3}J\,.$$

Here, J is the number given by the transcendental integral

$$J = -\int_0^1 du\,\frac{\ln u}{u^2-u+1} = \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(1+3n)^2}-\frac{1}{(2+3n)^2}\right)$$

$$= 1.1719536\ \ldots.. \tag{A21}$$

(i) $$I_9 = \int\frac{d^dk\, d^d\ell}{(k^2-m^2)^2(\ell^2-m^2)[(k+\ell)^2-m^2][(k-p)^2-m^2]}$$

$$= \pi^d(1+\epsilon-\epsilon\gamma)\int_0^1 dx\int_0^x dy\int_0^y \frac{dz(y-z)}{(x+xy-x^2-y^2)^{d/2}}\left\{m^2-\frac{p^2z[x+xy-x^2y^2-z+zy]}{x+xy-x^2-y^2}\right\}^{d-5} \tag{A22}$$

$$\left.\frac{\partial I_9}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{(m^2)^2}(1+\varepsilon-\varepsilon\gamma)(m^2)^{-\varepsilon}\int_0^1 dz\,dy\,dz\,dw\,\delta(x+y+z+w-1)$$

$$\times\frac{wz[(x+y)w+xy]}{[(x+y)(w+z)+xy]^{1+d/2}}$$

$$= \frac{\pi^d}{6(m^2)^2}\left(\frac{1}{\varepsilon}-\gamma+1-\ln m^2-\delta_2\right) \qquad \text{(A23)}$$

where

$$\delta_2 = \int_0^1 du\left\{\left(\frac{u^3(u-1)^3(u^2-u+4)}{(u^2-u+1)^4}\right)\ln u + \frac{u^6-3u^5+5u^3+u^2}{4(u^2-u+1)^3}\right\}$$

$$= \frac{8}{9}J - \frac{17}{36}$$

(j) $$I_{10} = \int\frac{d^dk\,d^d\ell}{(k^2-m^2)(\ell^2-m^2)[(k-\ell)^2-m^2][(k+p)^2-m^2][(\ell+p)^2-m^2]}$$

$$= \pi^d(1+\varepsilon-\varepsilon\gamma)\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z dw\,\frac{1}{[y+yw-y^2-w^2]^{d/2}}$$

$$\times\left\{m^2-p^2\left[\frac{[y(1-x+z-w)-(z-w)^2][y(1-y+w)-w^2]-[y(1-x)+w(z-w)]^2}{y[y(1-y+w)-w^2]}\right]\right\}^{d-5} \qquad \text{(A24)}$$

$$\left.\frac{\partial I_{10}}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{(m^2)^2}(1+\varepsilon-\varepsilon\gamma)(m^2)^{-\varepsilon}\int_0^1 dx\,dy\,dz\,dw\,dv\;\delta(x+y+z+w+v-1)$$

$$\times\frac{zy(x+v)+vx(w+y)+w(z+y)(v+x)}{[(x+y+z)(w+v)+z(x+y)]^{1+d/2}} \qquad \text{(A25)}$$

To evaluate the remaining integral (which is finite for $d=4$), let

$$x = (1-\beta)v \qquad w = \beta(1-u)\alpha$$

$$y = (1-\beta)(1-v) \qquad v = \beta(1-u)(1-\alpha)$$

$$z = \beta u \qquad \text{(A26)}$$

We then obtain

$$\left.\frac{\partial I_{10}}{\partial p^2}\right|_{p^2=0} = \frac{\pi^d}{(2m^2)^2}\int_0^1 d\beta\, du\, dv\, d\alpha\ \beta^2(1-\beta)(1-u)$$

$$\times\ \frac{2(1-\beta)^2\beta v(1-v)+2\beta^2(1-u)^2\alpha(1-\alpha)(\beta u+1-\beta)+\beta^2(1-\beta)u(1-u)}{[(1-\beta)\beta+\beta^2 u(1-u)]^3}$$

$$= \frac{\pi^d}{6(m^2)^2}\,\delta_3 \qquad \text{(A.27)}$$

where

$$\delta_3 = \int_0^1 du\left\{\left(\frac{(u-1)^3(2u^2+1)(u^2+1)}{(u^2-u+1)^4}\right)\ell n[u(1-u)] - \frac{(1-u)(3u^4-10u^3+2)}{2(u^2-u+1)^3}\right\}$$

$$= \frac{2}{9}\,J + \frac{4}{9}$$

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