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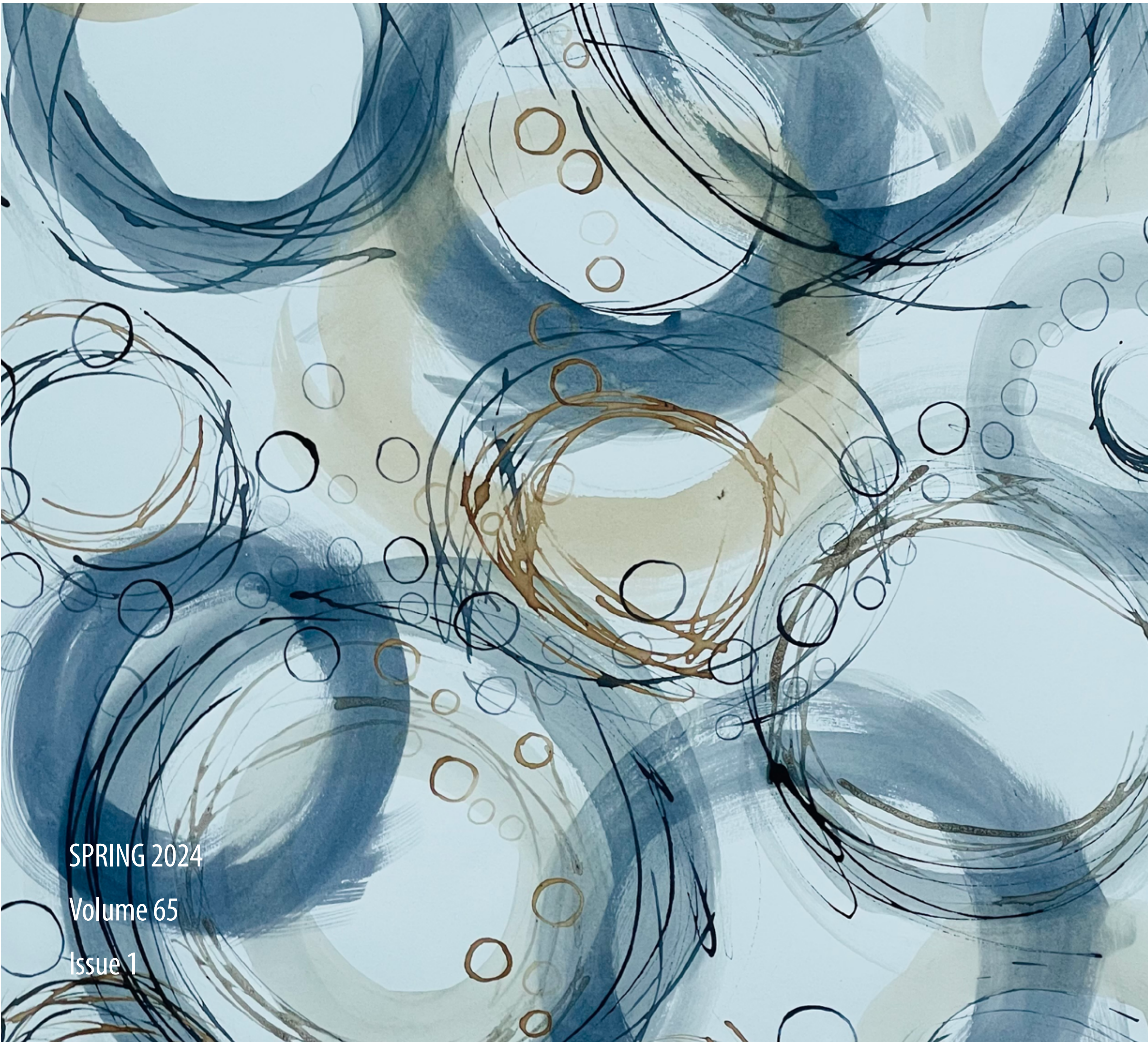

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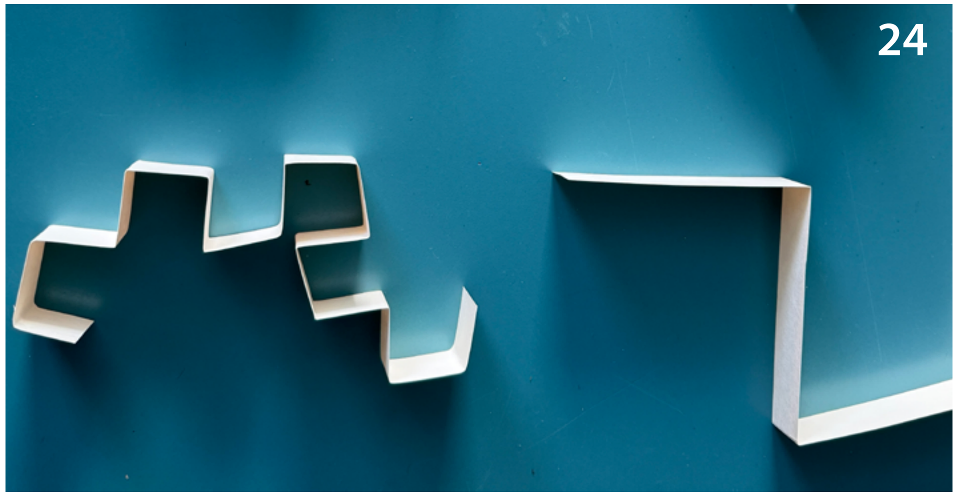

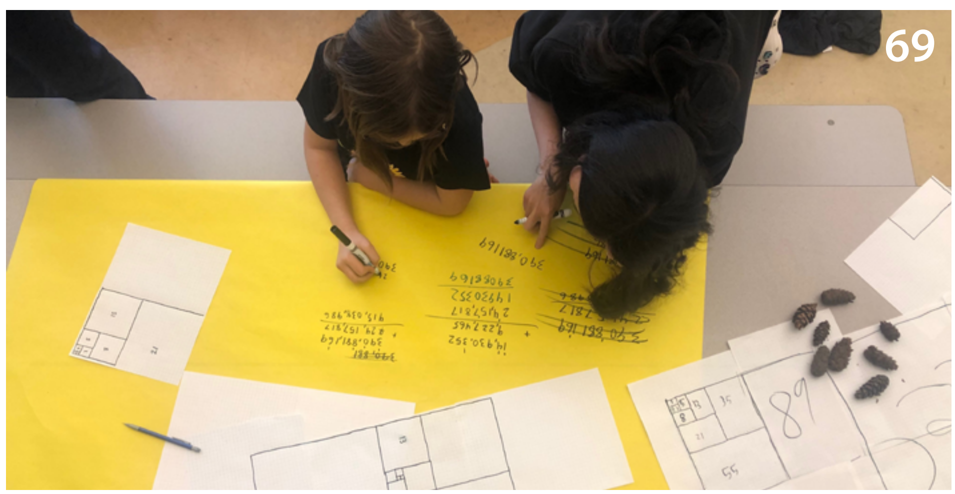

IN EVERY ISSUE

Using Variation to Develop Rich Tasks for the Elementary Classroom

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In a seminal paper, Erlwanger (1975) described the case of Benny, a grade six student learning how to convert between fractions and decimals. Benny had developed a unique procedure for doing so, one he was able to apply consistently and efficiently. Unsurprisingly, Benny demonstrated progress in his coursework and was noted by his teacher to be one of the best students in his class (Erlwanger, 1975, p. 49). The only problem was that Benny's procedure was flawed, based on fundamental misunderstandings of the underlying mathematics, and so he invariably arrived at erroneous answers. This led Erlwanger (1975) to conclude that an "emphasis on instructional objectives and assessment procedures alone may not guarantee an appropriate learning experience for some pupils" (p. 51).

Much has changed in the mathematics classroom since Benny's day, but much remains the same. One constant is an experience likely familiar to most mathematics teachers: imagine you are working with a student to multiply whole numbers. The student seems to demonstrate reasonable progress in some contexts (e.g., 1- by 2-digit numbers), but falters in others (e.g., 2- by 2-digits numbers). A common approach might be to intervene with some additional instruction, say around the idea of using "placeholders" or effectively using the standard algorithm, followed by another round 2- by 2-digit multiplication practice questions (see Ma, 1999). Unfortunately, even if the student eventually arrives at correct answers as a result of the intervention, there is no guarantee of conceptual understanding. So, if a focus on learning objectives and assessment procedures is not sufficient to support conceptual understanding, what is missing?

This article addresses that question by turning to variation theory (Marton, 2015). We argue that one thing missing from Benny's inscriptions and the imagined scenario above is an invitation to notice what is critical to understanding an intended mathematical concept. We think that variation can serve as one such invitation. To this end, we present three task sequences attending to several big ideas in mathematics that were developed as part of an undergraduate course in teaching elementary mathematics, in which one of the authors was an instructor (Josh) and the other authors were students. Each application of variation was based on a rich, cognitively demanding task carefully selected from the mathematics education literature. Developing each task sequence involved identifying an object of learning and its critical features, then enacting patterns of variation, which entailed the contrasting, generalizing, and fusing of those critical features (Marton, 2015). Before turning to these task sequences, we briefly elaborate on the principles and patterns of variation.

What are the Principles of Variation?

In order to understand the principles of variation, we need to clarify how it answers one fundamental question: What is to be learned? As classroom teachers, it is natural in answering this question to point to mathematics content (e.g., fractions), learning outcomes, or curricular competencies. Variation theory, however, answers this question differently. What is to be learned are the aspects of an object of learning. Marton (2015) noted that some aspects (i.e., critical aspects) are needed to make distinctions (e.g., a triangle has three vertices and a square has four), while others (i.e., non-critical aspects) are needed to generalize (e.g., no matter the orientation of a triangle in a plane, it is still a triangle).

When used in this way, critical and non-critical aspects form the basis of the patterns of variation—contrast, generalization, and fusion. Variation theory is predicated on the idea that in learning, we perceive difference against a background of sameness (Kullberg et al., 2017). Moreover, difference must precede sameness (Marton, 2015). This implies that a pattern of variation always begins by using contrast to draw students' noticing to critical aspects of an object of learning. Consider when students begin to work with linear functions. A typical approach is to work through many examples of linear equations, but in offering this degree of sameness, students can miss the critical aspects of linearity. Using contrast would entail having students work with linear and nonlinear functions side-by-side while perhaps holding some other aspects, such as the y-intercept, invariant. In the task sequences below, the decomposition of a number into addends, a number's factors, and the relationship between numerators and denominators in a fraction are all critical aspects for discernment.

Contrast is always followed by generalization. When we generalize, we vary aspects that do not directly focus on an object of learning. To continue with the example in the preceding paragraph, this might mean varying the slopes and y-intercepts of only linear equations. Put another way, we would hold some critical aspects for discernment (linearity) invariant, while varying non-critical aspects for generalization (y-intercepts). This stage might sound as if we are simply reverting back to what we began this paper by critiquing (i.e., many examples of linear equations). But as Marton (2015) noted, the key here is that sameness (generalization) is preceded by difference (contrast). In the task sequences below, the order of addends, whether or not a number is odd or even, and the number of parts in a given fractional relationship are all non-critical aspects for generalization.

The final step is to allow both critical and non-critical to vary together, which is called fusion. In this stage, the learner's noticing is drawn to how aspects of both types may be related and how some aspects directly focused on an object of learning may be discerned while all aspects are varied (Marton, 2015, p. 51). In what follows we apply the principles of variation to three tasks. We conclude with a brief discussion of the challenges and benefits of applying these principles in the mathematics classroom.

Task Sequence #1: Complex Counting

Complex Counting is a task that focuses on the composition and decomposition of numbers (Sci et al., 2016). Students are invited to explore the idea that sets of objects can be decomposed into smaller sets and that smaller sets can be composed to make a larger set. This task requires students to combine pictures of different-size groups of cherries to create a set that contains six cherries. Students are encouraged to come up with various representations of the number six that consist of differing amounts of sets and number of cherries (Sci et al., 2016, p. 436). The task presents the same concept of composition in different ways so that students can make connections between mathematical concepts and ideas (Sci et al., 2016).

The object of learning in this task is the composition and decomposition of numbers and the big ideas consist of the conceptual understanding of counting and adding, which makes it an especially good task for kindergarten and early grades. Students are required to count the number of cherries in a given set, and add them together with other sets of cherries to compose the number six (Figure 1).

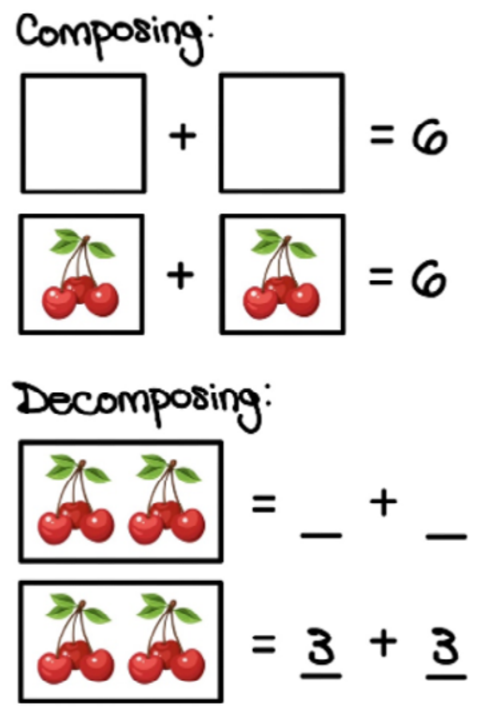

Figure 1

They will also take the total number of cherries and break it into various smaller sets of cherries. Students will explore numerous ways of representing numbers and become familiar with multiple ways to compose and decompose numbers.

For contrast, a critical aspect that can be varied is the composition and decomposition of a number. In order for students to develop an understanding of the process of composition, they must also be shown counter-examples of decomposition. Students will compose numbers by combining smaller sets of cherries to equal a total of six and will decompose the total number of six cherries into smaller sets. Under composing, two sets of three cherries are represented to equal a total of six cherries. Under decomposing, the total number of cherries is six and is broken apart into two groups of three cherries.

For generalization, a non-critical aspect that can be varied is the combinations in which a number can be composed. Students will be representing the number six using sets of various quantities (Figure 2).

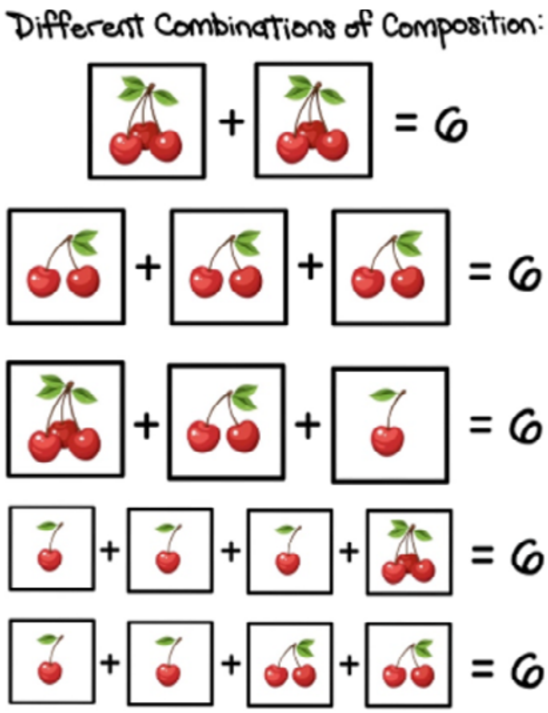

Figure 2

For fusion, students will combine the ideas they have developed previously in the task sequence. Students are invited to compose and decompose numbers using a variety of sets and number of cherries (Figure 3).

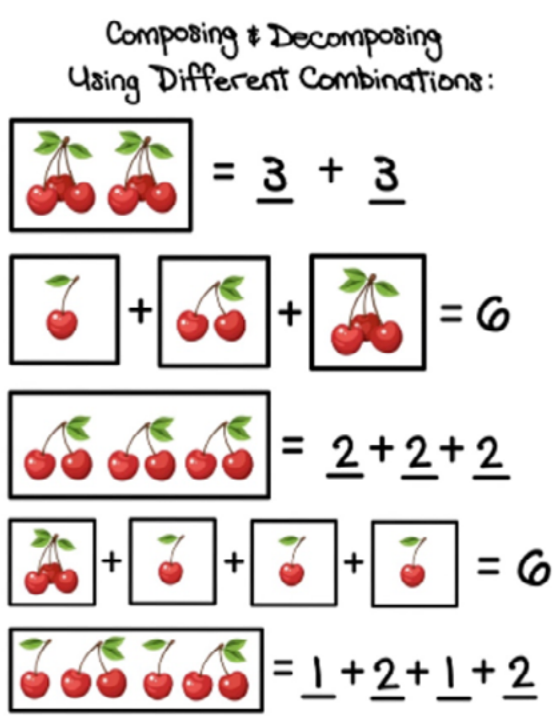

Figure 3

The first example is of decomposition. The total number of six cherries and is broken apart into two sets of three cherries. The second example shows composition. It shows three sets of cherries, the first set having one cherry, the second set having two cherries, and the third set having three cherries. Adding these sets togethers totals to six cherries. This final task allows students to notice the similarities and differences among the processes of composing and decomposing as well as understanding that a number can be composed and/or decomposed in various ways.

Task Sequence #2: Sorting Rectangles Using Prime and Composite Numbers

Sorting Rectangles is a task that involves students building rectangular arrays using coloured tiles for the numbers 1 to 25 ("Sorting Rectangles," 2011). In this task, students construct as many arrays as they can for a specific number of tiles and then create pictorial representations of the resulting rectangles on grid paper. Students then record the number of rows and columns for each rectangle in a table to display the factors of each number. Students can work on this task in small groups, pairs, or individually. In the task presented here, we modified the original Sorting Rectangles task to focus on developing students' understanding of composite and prime numbers.

Number is a big idea in mathematics (Charles, 2005) and this task's focus on composite and prime numbers particularly suits Grades 3-4 students. During this task, teachers should aim to draw students' attention to the idea that that composite numbers have at least two rectangular arrays, whereas prime numbers only have a single array. In what follows, we use the patterns of variation to focus students' attention on this big idea of number.

As we know, prime numbers have only two factors (1 and the number itself), while composite numbers have more than two factors. The number of factors thus constitutes a critical aspect of the task. To attend to this critical aspect, we draw students' attention to the relationship between the number of arrays they construct for an assigned number and the factors recorded in the rows and columns of their tables.

To effect the principle of contrast, teachers can present composite numbers (i.e., numbers with two or more arrays) and prime numbers (i.e., numbers with only a single array) simultaneously. Presenting both prime and composite numbers as examples illustrates differences in factors and can help students discern that composite numbers have more than two factors.

To generalize, one must now identify a non-critical aspect of number in this context. For example, one such non-critical aspect is the size of the number. Consider the following sequences of prime and composite numbers: 2, 11, 17 and 4, 12, 20. For each sequence, a critical aspect is held constant (number of factors), while a non-critical aspect is varied. This generalization highlights that the characteristic of a number being composite with multiple factors is not dependent on the size of the number or number of tiles assigned in the task. Figure 4 shows three examples of two composite and one prime number.

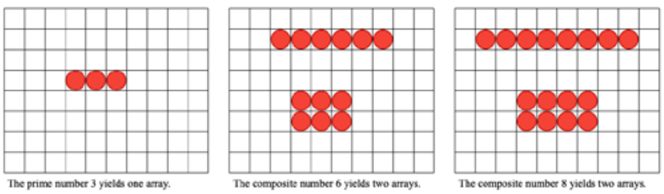

Figure 4

To effect fusion, teachers can direct students' attention to the relationship between critical and non-critical aspects by varying them simultaneously. For example, teachers can point students towards critical aspects through contrast by emphasizing how composite numbers have three or more factors and prime numbers have only two factors. Changing both critical and non-critical aspects of the task can make salient to students how multiple factors of composite numbers do not depend on number size.

Task Sequence #3: Equivalent Fractions Using Circle Models

Wessman-Enzinger and Hofer (2020) investigated the notion of unconventional units through the topic of equivalent fractions. Students are first given a fraction represented in a physical circle model, then asked to represent a given fraction in different ways. In many cases, students come to realize they are limited in the number of representations they can make if the whole is defined as only a single circle. For example, for the fraction 3/8, only two representations are possible if the whole that they are using remains as one circle, since the model does not include 1/16 pieces (see Figure 5). This means that for students to come up with a third representation they must be flexible in their thinking and redefine the whole as something other than the single-circle.

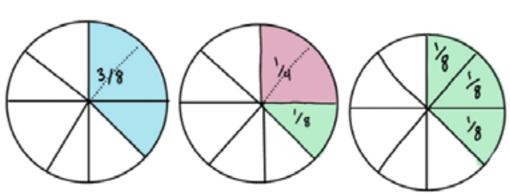

Figure 5

Fractions tend to be daunting for many students. A task such as this has the potential to prompt a deeper understanding and recognition of connections to procedures and builds conceptual understanding about fractions (Wessman-Enzinger & Hofer, 2020). Through this task students engage deeply with fractions and are being given opportunities to reason and explore connections between concepts, which are important curricular competencies (British Columbia Ministry of Education, 2019). Moreover, the task occasions engagement with several big ideas in mathematics, such as numbers, equivalence, and comparison (Charles, 2005). For these reasons, this is an ideal task for students in grades 3 to 5.

The object of learning in this task is the relationship between parts and wholes in equivalent fractions. In order for students to understand that a fractional relationship can be represented in numerous ways, they must notice both that different numerators for a given whole reflect different part-whole relationships and that a given part-whole relationship can be expressed using different numerators and denominators. Using variation theory, we can begin by contrasting between pairs of fractions that are equivalent (e.g., 3/4 and 6/8) and not equivalent (3/4 and 7/8). They will be asked to notice that the area being covered on each circle is the same for some pairs but not others. Students will be asked to notice that although the denominator for both fractions remains the same, the fractional relationship changes.

To generalize, students can be shown that how a whole is represented does not necessarily change the part-whole relationship. Students will be asked to represent equivalent fractions in multiple ways and will find that there are only two concrete representations possible when using the physical circle model. To reinforce the idea that what is defined as a whole is not a critical aspect of equivalent fractions, the students will be shown the same fraction but represented using two circles as the whole.

For fusion, students can be given a fraction and shown a variety of representations and asked which are equivalent to the given fraction (Figure 6).

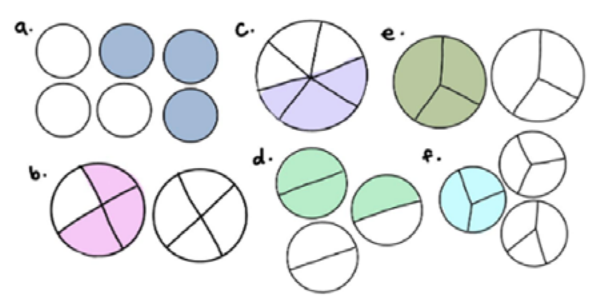

Figure 6

In these representations, both critical (i.e., the parts of a whole/ numerator) and non-critical (i.e., how the whole is represented) aspects will be varied. This will require students to make sure the number of parts of the different representations are equivalent to the given fraction and require understanding that the whole of a fraction can be represented in a number of ways.

Successes and Challenges of Using Variation to Develop Pedagogical Expertise

We end by considering the potential impacts of using variation theory for developing teaching expertise. The three task sequences presented here were initially developed as part of an assignment in an introductory course for pre-service elementary mathematics teachers. The purpose of the assignment was two-fold. First, it was to gain experience in carefully selecting rich tasks from the mathematics education literature, as reflected in each task sequence's origins (Sorting Rectangles, 2011; Sci et al., 2016; Wessman-Enzinger & Hofer, 2020). Second, it was to use the principles of variation to develop pre-service teachers' awareness of the important mathematics in each task. This involved clearly identifying an object of learning, as well as aspects of the tasks that we wanted students to notice. Ultimately, and following Marton (2015), we were interested in better understanding what could be made possible for students to learn through each task.

As pre-service teachers, some of the authors faced significant challenges. One in particular was identifying the critical aspects of an object of learning in a given task. This often required significant reflection on the specific groups of learners and anticipation of potential responses to the task. Furthermore, it took time to carefully plan and design these lessons, since once the critical and non-critical aspects were identified, they needed to be varied in an intentional way. But these authors also found variation to be an aid them in being intentional in their pedagogical decisions. Using variation requires teachers to plan specific, well-crafted examples, not just randomly selected questions from a textbook, and to present them in deliberate sequence to draw students' noticing to the big mathematical ideas. We hope the task sequences we present here can help to support new and experienced teachers alike in their mathematics classrooms.

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