University of Alberta

Analysis and Synthesis of Nonuniformly Sampled Systems

by

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To my wife,
for her love and support
and our kids
Subhan, Azka and Alishba.

Abstract

Sampled-data control theory is the mathematical foundation required for the analysis and design of sampled-data systems. An important assumption in the development of conventional sampled-data control theory is that measurement sampling periods are uniform. However, with the widespread use of networked and embedded control systems it is not possible or practically feasible to have constant measurement sampling periods. Consequently, the conventional sampled-data theory needs to be re-evaluated before designing this class of control systems. Motivated by this, this thesis develops mathematical approaches for the analysis and synthesis of sampled-data systems with nonuniform sampling periods.

Two types of variations in sampling period are considered. For the first type, we assume that the measurement is sampled irregularly but the input is updated regularly. For the second type, we assume that both measurement sampling and input updating occur synchronously but with nonuniform intervals. These timing models are general enough to capture many different types of variations in sampling periods.

Both state estimation and control problems are considered. For state estimation, two types of filters are developed: a sampling period dependent for the first type of variations and a robust one for the second type of variations. A novel Markov model of the irregular sampling process together with the theory of Markovian jump systems lead to the design of the first type of filter. The second filter is designed by modelling the nonuniform sampling system as uncertain

feedback system with linear fractional transformation uncertainty. For the control problem, a dynamic robustly stabilizing output feedback controller and a robust $\mathcal{H}_\infty$ controller are developed. Both controllers are designed for the second type of variations in sampling period and analysis is based on modelling the nonuniform sampling system as uncertain feedback system.

All theoretical development in this thesis is in discrete time and design conditions are formulated as linear matrix inequalities (LMI's). Many solvers, such as the LMI toolbox of MATLAB, exist and can solve these convex optimization problems very efficiently. Most constant-parameter filters or controllers designed are easily implemented. The effectiveness of the proposed filters or controllers is demonstrated using simulation results.

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Ghulam Mustafa

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List of Symbols

$\mathbf{R}$	Set of real numbers
$\mathbf{C}$	Set of complex numbers
$\mathbf{Z}$	Set of integers
$\mathbf{Z}^+$	Set of positive integers
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{R}^{n\times m}$	Real n-by-m matrix
$[p_{ij}]$	A matrix with p_{ij} as the i-th row and j-th column element
$\mathrm{diag}(p_1, \ldots, p_n)$	An $n \times n$ diagonal matrix with p_i as its i-th diagonal element
I	Identity matrix
0	Zero matrix
P^T *or* P'	Transpose of P
P^{-1}	Inverse of P
$\mathrm{trace}(P)$	Trace of P
$\lambda(P)$	Eigenvalues of P
$\rho(P)$	Spectral radius of P
$\|P\|$	Norm of P
$\in$	Belongs to
$\subset$	Subset
$\cup$	Union
$\cap$	Intersection
$:=$	Defined as

$P > 0\ (\geq 0)$	P is a positive-definite (semi-definite) matrix
θ_k	Markov chain
$\Box$	End of proof
$\ell_2[0, \infty)$	Space of finite-energy sequences
$\mathcal{H}_2$	H_2-norm of a system
$\mathcal{H}_\infty$	H_∞-norm of a system
$\mathcal{F}_l(\Sigma, \Delta)$	Lower LFT of Σ and Δ
$\mathcal{F}_u(\Sigma, \Delta)$	Upper LFT of Σ and Δ
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	Compact representation of state-space model

List of Acronyms

LTI	Linear time-invariant
LTV	Linear time-varying
LMI	Linear matrix inequality
LFT	Linear fractional transformation
SD	Sampled-data
RS	Robust stability
RP	Robust performance
NCS	Networked control system
KYP	Kalman-Yakobovich-Popov
SG	Small-gain
iff	If and only if

Chapter 1

Introduction

1.1 Motivation

The field of control engineering deals with modifying the behaviour of dynamical systems by manipulating their inputs, to make them work in uncertain environments. Advances in digital technology and the availability of low-cost microprocessors have influenced all areas of science and engineering; and control engineering is no exception. Nearly all control systems constructed today are implemented using digital computers and are referred to as *computer-controlled systems* or *sampled-data systems*. Using computers to implement controllers offer many advantages over analog implementation, such as, the controller can be easily re-programmed, controller is a software program so changes in hardware components do not affect it and delay can be handled easily.

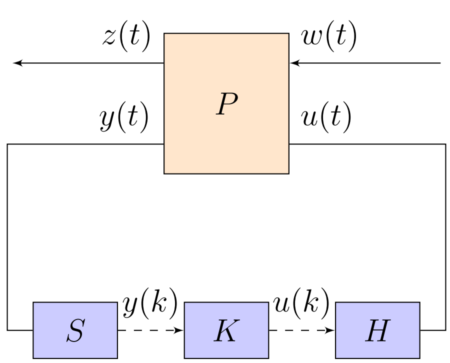

Figure 1.1: Sampled-data feedback control system (solid line: continuous signals, dashed line: discrete signals)

Figure 1.1 shows block diagram of a typical sampled-data control system. In Figure 1.1, P is a physical device or machine whose behaviour is to be controlled; $w(t)$ is the exogenous input which may consist of command inputs, disturbances and measurement noise; $z(t)$ is the signal to be controlled; $u(t)$ is the control input and $y(t)$ is the measured output. The measurement $y(t)$ is sampled by the sampler S and fed to the controller K. The controller is an algorithm running inside a computer, or a microcontroller, processes this measurement and generates a control sequence. This sequence is applied to the input of the plant through a zero-order hold device H. The time instants when the measurement is sampled from the plant are called *sampling instants* and the duration between any two consecutive sampling instants is called *sampling period*.

Sampled-data control theory is the mathematical foundation required for the analysis and design of sampled-data systems. It has been well-developed during the last two-three decades. Three approaches can be used for the controller design: one is the continuous-time approach where a controller is designed in continuous-time and implemented digitally; the second is the discrete-time approach where the plant is discretized and a controller is designed in the discrete-time domain and the third is the direct or sampled-data approach where a discrete-time controller is designed for the continuous-time plant. References [3, 15] give a detailed treatment of continuos-time and discrete-time approaches to the controller design and [9] covers in detail the sampled-data or direct approach.

Sampled-data control theory is based on the assumptions that sampling and hold devices are *synchronized*, e.g. using an external clock, and the measurements are sampled *periodically*. Periodicity plays an important role in the discrete-time and sampled-data approaches to controller design; we get linear time-invariant models, by sampling in discrete-time approach and by lifting in sampled-data approach.

Recent years have seen a rapid development in communication, computation and sensing technologies. This has opened up the possibility to design very large and complex control systems by integrating computation, control and communication.

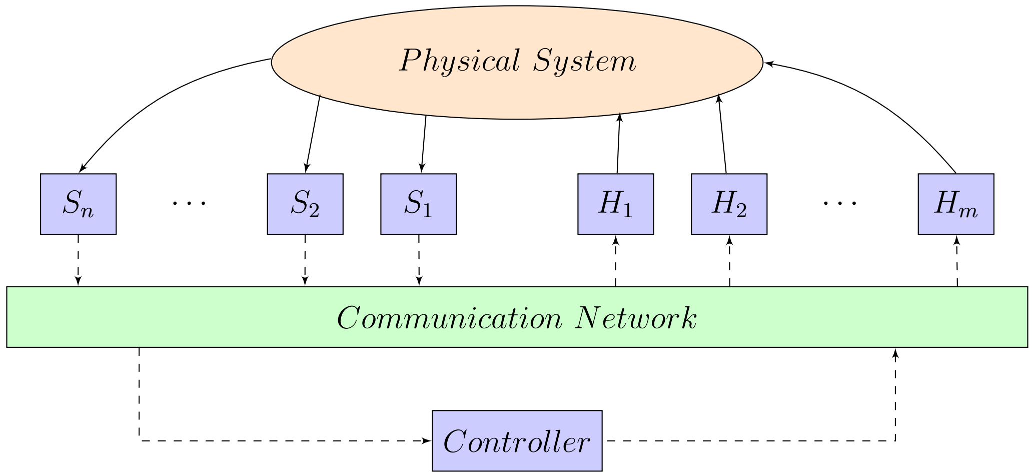

Figure 1.2: Modern sampled-data feedback control system

Figure 1.2 shows a typical configuration of a modern sampled-data control system. A distinctive feature of this configuration is that all information between the controller and the plant is exchanged through a shared communication network. Moreover, the sensors $(S_1, \cdots, S_n)$ and actuators $(H_1, \cdots, H_m)$ could be geographically separated from each other and could be independent nodes on the communication network. Closing the control loop through a communication network has the advantages of reduced wiring cost and ease of maintenance and debugging; however, it makes the analysis and design of control system harder [4, 30, 33, 86]. It may not be possible to guarantee constant sampling periods and synchronized operation of the sampling and hold devices [84].

Timing problems also arise in other control configurations, e.g. embedded systems, where a single computer is used to do many operations including the control. The control task is scheduled along with many other tasks and there is no guarantee of constant sampling periods [33, 77]. Other applications where it is difficult to maintain a constant sampling period include cross-directional control of paper machine systems [58] and brushless DC servo systems [81]

For the aforementioned configurations, if a controller is designed with the assumption of constant sampling period, this could degrade the system performance [67] and may even lead to instability of the closed-loop system [36, 38, 62]. This motivates the development of sampled-data theory where aforementioned assumptions of constant sampling period and synchronized operation of sampler and hold are relaxed. In this thesis, we refer to the class of sampled-data control systems with variations in sampling period as *nonuniformly sampled systems.*

Control systems where control loops are closed through communication networks are called *networked control systems.* Communication networks introduce many imperfections, such as, data packet dropout, limited communication capacity, in addition to the variable transmission periods. There have been a lot of research activities in the control community for the analysis and design of such systems. To learn more about networked control systems, see book [33] and survey article [30].

1.2 Background

The interest in sampled-data control with nonuniform sampling periods can be traced back as early as in 1957 with the PhD thesis of Kalman [34]. The recent wave of interest in the context of networked control systems started with the work of Walsh et al. [74, 75] and Zhang et al. [84–86]. The initial attempts used a continuous-time Lyapunov function to derive sufficient stability conditions. Zhang et al. [85] also showed that stability of the sampled-data closed-loop system can be implied by the stability of the associated discrete-time system by using a Lyapunov function that is decrescent at the sampling instants; they used a randomized algorithm to search for such a Lyapunov function using a grid of sampling periods. A drawback of this approach is that no matter how small the grid is, there is, in general, no guarantee that the sampled-data system will be stable for all possible variations of the sampling period. Improving upon the grid idea, Fujioka [19, 21] took the robustness into account to guarantee stability for all possible variations of the sampling period. The discrete-time system associated with the sampled-data system is time-varying and uncertain because of uncertainly varying sampling periods: Fujioka viewed the variations in the sampling period as perturbations to a nominal sampling period, modelled them as linear fractional transformation uncertainty and used small-gain condition to guarantee robust stability. This approach also opened up the possibility of using other robust control techniques. In [54, 55]

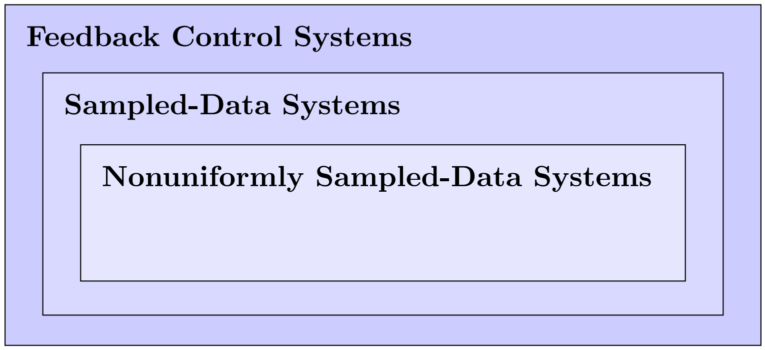

Figure 1.3: Research context

a discrete-time approach using robust linear matrix inequalities was developed and in [31, 32], a discrete-time approach using polytopic modelling was presented.

On the other hand, Fridman et al. [17] followed a delay systems approach. The idea was to model the closed-loop sampled-data system as a continuous-time one with a time-varying delay in the input. This approach was later improved in [16, 46] to take into account the saw-tooth nature of the delay.

Experiments show that discrete-time approaches are, in general, less conservative than the continuous-time approaches. However, most of the approaches are concerned with the state feedback stability analysis and stabilization problem only. A more practical setup to work with is the control with output feedback. Also, not all state variables of a plant are measurable in practice; therefore, it is important to consider the state estimation problem. Moreover, control design that optimize some performance measure should also be considered.

Figure 1.3 shows the context of this research. It falls under the domain of feedback control systems. The area of research is sampled-data systems with specific focus on nonuniformly sampled systems.

1.3 Research Goals and Methods

As pointed out in the previous section, the research on sampled-data control with nonuniform sampling periods is in its infancy. Many problems are still unsolved. The main objective of this research is *"to develop a unified theory for the analysis and synthesis of nonuniformly sampled systems"*. In order to meet the above objective, we set the following goals for this research:

- Most of the existing discrete-time approaches assume the availability of all state variables of the system. In practice, it is not possible to measure all state variables: Therefore, our first goal is to develop a state estimation technique for nonuniformly sampled systems.

- Our next goal will be to extend the existing discrete-time approaches to the more general setup of output feedback control and develop a stability analysis and stabilization method with output feedback.

- Finally, we shall focus on developing a controller design method that takes into account some performance measure of the closed-loop system.

Most of the theoretical development in this research will be in the discrete-time domain. Most of the existing discrete-time approaches assume the availability of all state variables of the system. Our first goal will be to consider the design of a state estimator to estimate the state of the system at nonuniform sampling instants. Stability is a fundamental requirement for the operation of any control system; hence this problem attracts many researchers. In the next step, we shall consider the dynamic output feedback stabilization problem. Another related problem would be to consider the control design with some closed-loop performance specification.

Convex optimization methods have proved to be very useful tool for the analysis and design of control systems. An important feature of these methods is that many control problems can be formulated in a similar way. Many efficient solvers are available that can solve problems of moderate size within reasonable amount of time. In this research, we shall harness the power of these methods to find numerical solutions.

1.4 Thesis Contributions

The research presented in this thesis and the major contributions that distinguish it from other work are listed below:

1. Developed a novel model to describe the nonuniform measurement sampling process using a Markov chain.

2. Using the above model, a method is developed to design a state estimator to estimate the state of the system with nonuniform measurement sampling to minimize the $\mathcal{H}_\infty$ norm of the estimation error system.

3. Using robust control approach, a discrete-time robust $\mathcal{H}_\infty$ filter design procedure is developed.

4. Developed a novel discrete-time LMI-based approach for dynamic output feedback stabilization with nonuniformly sampled measurements.

5. Extended the above-mentioned approach to include $\mathcal{H}_\infty$ performance of the closed-loop system.

1.5 Applications

The proposed mathematical techniques in this thesis are expected to be applicable for a wide range of control systems affected by timing problems. Some potential applications are listed below:

Networked and Embedded Control Systems [30, 33, 74]: In networked and embedded control systems, the communication between different components takes place through a shared communication network. While traversing through the network, the packets may experience delays and even get lost. Therefore, a uniform operation of the sampling and hold devices is not possible.

Web Forming Systems [37, 58]: In web forming systems like paper machines and rolling steel mills, scanning sensors are generally used to measure the cross-directional properties. The measurement time varies with the relative speed of the scanner and the web, resulting in nonuniform measurements.

Brushless DC Servo Motors [57, 81]: In brushless DC servo motors the rotor position is measured through an arrangement of proximity sensors and the measurement time depends on the rotor speed. Therefore the position measurements are nonuniform or aperiodic.

Event Driven Systems [29]: Nonuniform sampling is also employed in event-driven systems where a signal is sampled only when it crosses a certain threshold.

The analysis and design for the aforementioned applications can be carried out within the framework proposed in this thesis. This list of applications is growing with the new systems being developed that combine control, communication and computation.

1.6 Thesis Outline

This thesis has been prepared as per the guidelines from the Faculty of Graduate Studies and Research (FGSR) at the University of Alberta. The rest of the thesis is organized as follows:

Chapter 2 collects the background material which we think is important to understand the contents of the later chapters. It begins with a description of basic setup of nonuniformly sampled-data control systems followed by a description of the timing models we shall use in this thesis. We then list the problems that will be addressed in the rest of the thesis. After that we give a survey of the existing literature related to nonuniformly sampled systems. A detailed survey of the literature related to each problem is given in the introduction of each chapter. We believe that this will facilitate partial reading of the thesis which is expected for a lengthy document. One of the contributions of this research is the Markov modelling of the nonuniform sampling process; therefore, we give a brief introduction to the discrete-time Markovian jump systems. Most theoretical development in this thesis is based on linear matrix inequalities (LMI's). In the last section, we give a brief review of LMI's and their use in control systems.

Chapter 3 is concerned with the design of state estimators. A Markov chain is used to model the nonuniform measurement sampling process and then using this model a method is presented to design an $\mathcal{H}_\infty$ filter. A sampling period dependent, full-order filter structure is considered. Simulation results are given to compare the performance of the sampling period dependent filter with a general time-varying $\mathcal{H}_\infty$ filter which indicate that the proposed filter is better.

Chapter 4 considers the robust control approach for the filter design. Using a linear fractional model of the uncertainty, an algorithm is developed to design a discrete-time, robust $\mathcal{H}_\infty$ filter for the given bounds of variations in sampling period.

Chapter 5 considers the stability analysis and stabilization problems. In this chapter we develop the dynamic output feedback stabilization of nonuniformly sampled systems. A constant-parameter controller is sought. We shall observe that because of the multi-modal approach involved in the analysis, it is very hard to develop LMI conditions for the controller design. Using a linearizing change of variables, we are able to develop LMI conditions which we believe is another important contribution of this reasearch.

Chapter 6 considers the design of a discrete-time robust $\mathcal{H}_\infty$ controller for nonuniformly sampled systems. Extending the stabilization of Chapter 5, the controller design is formulated as a robust stability and performance problem which is, in general, non-convex. We use a DK-type iterative procedure for the controller design.

Finally, Chapter 7 gives a summary of the dissertation and then list some future work.

Chapter 2

Preliminaries

In this chapter we provide the background material which is necessary to understand the contents of the following chapters. We begin with an introduction to the class of nonuniformly sampled systems which is the topic of this thesis and then state the problems being addressed in this research. In the following section, we give a literature review of the existing research related nonuniformly sampled systems. Since we follow a discrete-time approach to deal with nonuniformly sampled systems, fundamental concepts of discrete-time approach to deal with conventional sampled-data systems are reviewed. One of the contributions of this research is the modelling of nonuniform sampling process using a Markov chain and address the state estimation problem using the Markovian jump systems framework. Therefore, the class of Markovian jump systems is also introduced. Finally, a review of LMI's is provided which form the main tool used to solve the problems in this thesis.

2.1 Nonuniformly Sampled Systems

Consider the sampled-data feedback control configuration of Figure 1.1. Let P be a finite-dimensional, linear time-invariant plant with state space model

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{1c} w(t) + B_{2c} u(t), \qquad x(0) = x_0 \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{2.1}$$

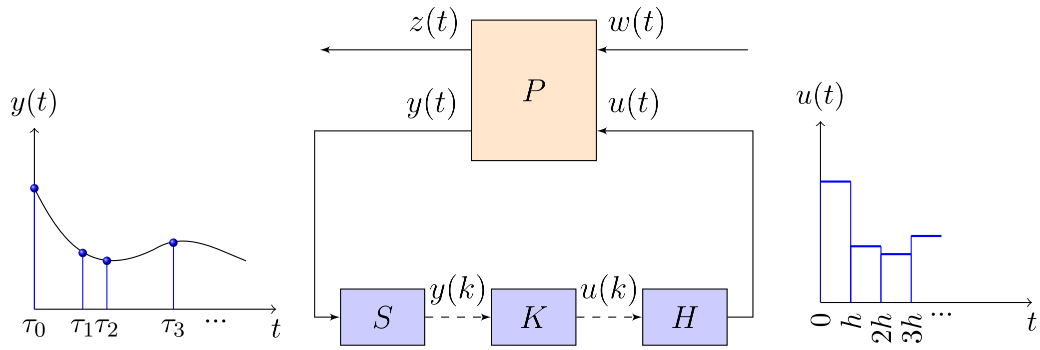

Figure 2.1: Sampled-data feedback control under timing model 1

where $x(t) \in \mathbf{R}^n$ denotes the state of the system, $w(t) \in \mathbf{R}^{m_1}$ is the disturbance input, $u(t) \in \mathbf{R}^{m_2}$ is the control input, $z(t) \in \mathbf{R}^{p_1}$ is the signal to be controlled or estimated and $y(t) \in \mathbf{R}^{p_2}$ is the measurement vector. The matrices A_c, B_{1c}, B_{2c} C_1, C_2, D_{11}, D_{12} and D_{21} are assumed to have compatible dimensions.

The plant P is controlled with a discrete-time controller K where plant data is sampled at non-equidistant time instants. Sampling periods can vary in many different patterns. To have something specific to discuss, we focus on the following two models for sampling period variations.

2.1.1 Timing Models

Timing Model 1. We assume that the input to the plant $u(t)$ is updated at a fast and constant period h, whereas the measurement $y(t)$ is sampled with irregular periods. We also assume that the measurement sampling instants coincide with the input update instants: This means that the measurement sampling periods are integer multiples of the input update period. Figure 2.1 shows a schematic of a sampled-data system under this timing model.

Timing Model 2. Let $\{\tau_k,\ k \geq 0\}$ be a set of uncertain and nonuniformly spaced time instants satisfying

$$0 < h_l \leq \tau_{k+1} - \tau_k \leq h_u < \infty, \quad k = 0, 1, \cdots \tag{2.2}$$

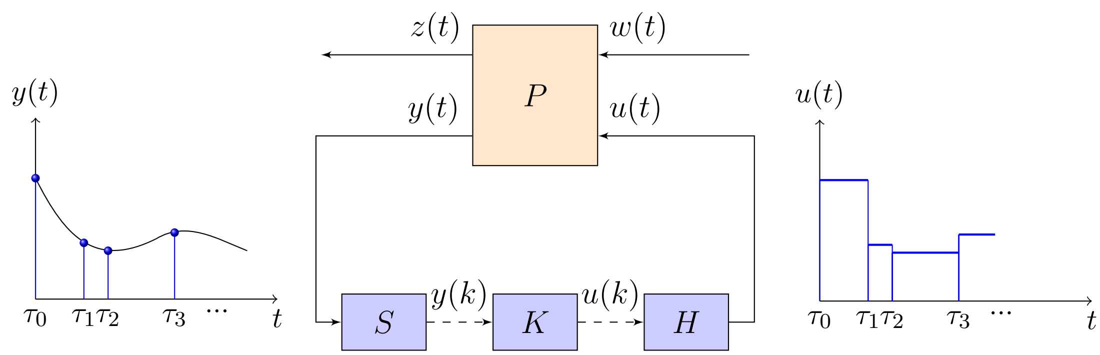

Figure 2.2: Sampled-data feedback control under timing model 2

and

$$\lim_{k\to\infty} \tau_k = \infty. \tag{2.3}$$

The measurement sampling and control updating take place at the uncertain and nonuniform sampling instants τ_k. The only assumption about the sampling instants τ_k's is that the separation between any two consecutive sampling instants is bounded by lower and upper bounds, h_l and h_u, respectively. A configuration of the sampled-data system under this timing model is shown in Figure 2.2.

These timing models are general enough to capture many patterns for variations in sampling period. Timing model 1 is useful in situations where a state feedback controller is designed ignoring the variations in sampling periods. An observer or a filter can then be designed to provide the state estimate at the control update periods while the measurements are sampled at irregular intervals. Timing model 2 is more general. It naturally includes the case of sampling jitter. Timing models where sampling period variations are assumed to follow a distribution can also be considered as special case of this model.

2.1.2 Problems Addressed in This Thesis

The problems we consider in this thesis are as follows:

Problem 1. As pointed out in Chapter 1 and will also be highlighted in next section, most of the existing discrete-time approaches consider the case of state feedback only. Since, in practice, it is not possible to measure all state variables of the plant, we begin with the design of a state estimator under timing model 1. The state estimator is designed to minimize the $\mathcal{H}_\infty$ norm from disturbance input to estimation error. This problem is solved in Chapter 3.

Problem 2. Timing model 1 seems a bit restrictive because it assumes that the measurement sampling periods are integer multiple of state estimation period. In order to relax this assumption, we consider the design of a state estimator under timing model 2 while still minimizing the $\mathcal{H}_\infty$ norm of the error system. This problem is solved in Chapter 4.

Problem 3. We then focus on the control design problem and consider designing a dynamic controller such that the sampled-data output feedback control system is exponentially stable under timing model 2. This problem is solved in Chapter 5.

Problem 4. We finally consider the γ-suboptimal $\mathcal{H}_\infty$ control problem where we target designing a dynamic controller such that the sampled-data output feedback control system is internally stable under timing model 2 and the discrete-time $\mathcal{H}_\infty$ norm of the closed-loop system from disturbance input to the controlled output is minimized. This problem is solved in Chapter 6.

2.2 Survey of Existing Results

Early results on sampled-data control with variable sampling periods include [1, 2, 12, 34, 38]. Most early work assumed that sampling period variations follow some pattern, e.g., sample periods vary in an interval and those variations are repeated after that interval.

Recent interest in the context of networked control systems started with the work of Walsh et al. [74, 75] and Zhang et al. [84–86], where they mainly developed analytical approaches and studied the stability using a continuous Lyapunov function. Zhang et al. [84] showed that stability of the sampled-data system can be implied by the quadratic stability of the associated discrete-time system, and used a randomized algorithm to search for a discrete time Lyapunov function by using a grid of sampling periods. A grid based approach was also used in [60].

Improving on the grid method, Fujioka [19, 21, 23] took the robustness into account and proposed a stability analysis and stabilization procedure that could guarantee stability for all possible variations of sampling periods. Fujioka's is a robust control approach; he views the variations in the sampling periods as perturbations to a nominal sampling period and models these perturbations as a linear fractional transformation (LFT) uncertainty. Robust quadratic stability, and hence exponential stability of the sampled-data system, can be guaranteed by a simple application of the small-gain theorem. This approach was later extended to the output feedback stability analysis in [20] and the state estimation problem in [50, 51].

Based on the robust control approach, other discrete-time approaches were also presented. In particular, a norm-bounded uncertainty approach was presented in [69–71], an interval matrix approach [39], a discrete-time approach using robust linear matrix inequalities was presented in [54, 55] and an approach using polytopic modelling of the sampled-data systems was presented in [5, 31, 32].

On the other hand, Fridman et al. [17] introduced a delay systems approach for the analysis of nonuniformly sampled systems. The idea is to model the sampled-data system as a continuous-time one with time-varying delay in the input. Then, analysis and synthesis tools from delay systems can be used. The approach was later refined in [16, 22, 46]. The approach has been applied for analysis and synthesis for nonuniformly sampled-data systems in, e.g., [18, 72].

The $\mathcal{H}_\infty$ filtering problem for conventional sampled-data systems using LMI's was considered in [27, 56]. For nonuniformly sampled system, an $\mathcal{H}_\infty$ filtering procedure was developed in [6] using a polytopic modelling of the error system and in [72] using a delay systems approach. We develop two different $\mathcal{H}_\infty$ filtering schemes, one under timing model 1 and the other under timing model 2, and develop LMI conditions that can be very efficiently solved using state-of-the-art LMI solvers. We show the improvement over existing techniques.

2.3 Conventional Sampled-Data Systems

In this section, we review some fundamental concepts of discrete-time approach to the analysis and design of conventional sampled-data systems. The contents of this section are mostly taken from [9].

2.3.1 Step-Invariant Transformation or Zero-Order Hold Discretization

For any two sampling instants $\tau_1 < \tau_2$, the differential equation in (2.1) can be integrated from time τ_1 to time τ_2

$$x(\tau_2) = e^{(\tau_2-\tau_1)A_c}x(\tau_1) + \int_{\tau_1}^{\tau_2} e^{(\tau_2-\tau)A_c}B_{1c}w(\tau)\mathrm{d}\tau + \int_{\tau_1}^{\tau_2} e^{(\tau_2-\tau)A_c}B_{2c}u(\tau)\mathrm{d}\tau.$$

Set $\tau_1 = kh$ and $\tau_2 = (k+1)h$

$$\begin{aligned} x[(k+1)h] = e^{hA_c}x(kh) &+ \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A_c}B_{1c}w(\tau)\mathrm{d}\tau \\ &+ \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A_c}B_{2c}u(\tau)\mathrm{d}\tau. \end{aligned}$$

The control signal $u(\tau)$ is generated by computer and is constant over the k-th sampling period; we also assume that $w(\tau)$ is constant over the k-th sampling period. Defining $x_k := x(kh)$, $w_k := w(kh)$, $u_k := u(kh)$ and using a change of variables to simplify the second and third terms in above equation give

$$x_{k+1} = e^{hA_c}x_k + \int_0^h e^{\tau A_c}B_{1c}\mathrm{d}\tau w_k + \int_0^h e^{\tau A_c}B_{2c}\mathrm{d}\tau u_k$$

Also, sampling the output equations and defining $y_k := y(kh)$ and $z_k := z(kh)$ yields

$$\begin{aligned} z_k &= C_1x_k + D_{11}w_k + D_{12}u_k \\ y_k &= C_2x_k + D_{21}w_k \end{aligned}$$

Therefore, the discrete-time system associated with the continuous-time system in (2.1) is

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_1 w_k + \Gamma_2 u_k \\ z_k &= C_1x_k + D_{11}w_k + D_{12}u_k \\ y_k &= C_2x_k + D_{21}w_k \end{aligned} \tag{2.4}$$

where $\Phi = e^{hA_c}$, $\Gamma_1 = \int_0^h e^{\tau A_c}B_{1c}\mathrm{d}\tau$ and $\Gamma_2 = \int_0^h e^{\tau A_c}B_{2c}\mathrm{d}\tau$. If the sampling period h is constant and known, the matrices Φ, Γ_1 and Γ_2 can be obtained using MATLAB by the command $(\Phi, \begin{bmatrix}\Gamma_1 & \Gamma_2\end{bmatrix}) = \mathrm{c2d}(A_c, \begin{bmatrix}B_{1c} & B_{2c}\end{bmatrix})$.

2.3.2 Controllability

Definition 2.1. The system in (2.4) is said to be controllable if for $w_k \equiv 0$ and every target time $k_1 \geq n > 0$ and every target vector v, there is a control sequence $u_k,\ 0 \leq k \leq k_1$ such that $x_{k_1} = v$, starting from $x_0 = 0$.

- The pair $(\Phi,\, \Gamma_2)$ is controllable iff

$$\operatorname{rank}\begin{bmatrix}\Phi & \Phi\Gamma_2 & \dots & \Phi^{n-1}\Gamma_2\end{bmatrix} = n$$

- The pair $(\Phi,\, \Gamma_2)$ is controllable iff

$$\operatorname{rank}\begin{bmatrix}\Phi - \lambda I & \Gamma_2\end{bmatrix} = n \text{ for each eigenvalue } \lambda \text{ of } \Phi.$$

- $(\Phi,\, \Gamma_2)$ is stabilizable if there exists a matrix F such that the eigenvalues of $\Phi + \Gamma_2 F$ are all inside the open unit disk.

- $(\Phi,\, \Gamma_2)$ is stabilizable iff

$$\operatorname{rank}\begin{bmatrix}\Phi - \lambda I & \Gamma_2\end{bmatrix} = n \text{ for each eigenvalue } \lambda \text{ of } \Phi \text{ with } |\lambda| \geq 1.$$

2.3.3 Observability

Definition 2.2. The pair $(C_2,\, \Phi)$ is observable if the initial state can be computed from the output sequence $\{y_0,\, y_1, \ldots\}$ for some sufficiently large k_1.

- $(C_2,\, \Phi)$ is observable iff

$$\operatorname{rank}\begin{bmatrix} C_2 \\ C_2\Phi \\ \vdots \\ C_2\Phi^{n-1}\end{bmatrix} = n$$

- $(C_2,\, \Phi)$ is defined to be detectable if there exists a matrix H such that all the eigenvalues of $\Phi + HC_2$ are inside the open unit disk.

- $(C_2,\, \Phi)$ is detectable iff

$$\operatorname{rank}\begin{bmatrix}\Phi - \lambda I \\ C_2\end{bmatrix} = n \text{ for each eigenvalue } \lambda \text{ of } \Phi \text{ with } |\lambda| \geq 1.$$

2.3.4 Stability

Definition 2.3. The discrete-time system is stable if, for $w_k \equiv 0$, $u_k \equiv 0$ and for every x_0, $x_k \to 0$ as $k \to \infty$.

Theorem 2.1. *The discrete-time system is said to be quadratically stable if there exists a symmetric matrix $\bar{P} > 0$ such that*

$$\Phi^T \bar{P} \Phi - \bar{P} < 0.$$

Definition 2.4. The discrete-time system is exponentially stable if, for $w_k \equiv 0$, $u_k \equiv 0$ there exist two scalars $\alpha > 0$ and $0 < \beta < 1$ such that

$$\|x_k\| = \alpha e^{\beta k} \|x_0\|.$$

2.3.5 $\mathcal{H}_\infty$ Norm

Let the closed-loop system corresponding to discrete system in (2.4) from the disturbance input w_k to the controlled output z_k has matrices Φ, Γ, C and D. For a given scalar $\gamma > 0$, the $\mathcal{H}_\infty$ norm of the system can be computed using the Kalman-Yakobovich-Popov (KYP) lemma.

Theorem 2.2. *The $\mathcal{H}_\infty$ norm is less than γ if there exists a symmetric matrix $\bar{P} > 0$ such that the following matrix inequality holds:*

$$\begin{bmatrix} \Phi & \Gamma \\ C & D \end{bmatrix}^T \begin{bmatrix} \bar{P} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi & \Gamma \\ C & D \end{bmatrix} - \begin{bmatrix} \bar{P} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0.$$

2.4 Discrete-Time Markovian Jump Systems

Let $\{\theta_k,\, k \geq 0\}$ be a Markov chain taking values in a finite set $\mathbf{S} = \{1, ..., N\}$ with transition probability matrix $P = [p_{ij}]_{i,j \in \mathbf{S}}$. The transition probabilities p_{ij} are defined as

$$Pr[\theta(k+1) = j | \theta(k) = i] = p_{ij}, \quad i, j \in \mathbf{S}$$

with $p_{ij} \geq 0, \forall i, j \in \mathbf{S}$ and $\sum_{j=1}^{N} p_{ij} = 1, \forall i \in \mathbf{S}$.

Consider a discrete-time hybrid system with N modes

$$\begin{aligned} x_{k+1} &= \Phi(\theta_k)x_k + \Gamma(\theta_k)w_k \\ z_k &= C(\theta_k)x_k + D(\theta_k)w_k \end{aligned} \tag{2.5}$$

As usual, x_k represents the state of the system, w_k the noise sequence acting on the system and z_k the output of the system. We assume that all matrices have appropriate dimensions and are functions of the Markov chain θ_k. This type of dynamic systems where mode changes are governed by a Markov chain are called discrete-time Markovian jump systems. For details about Markovian jump systems, see [7, 10]. In the sequel, for notational convenience, we shall denote the matrices associated with i-th mode as $\Phi_i = \Phi(\theta_k = i)$, $\Gamma_i = \Gamma(\theta_k = i)$, $C_i = C(\theta_k = i)$ and $D_i = D(\theta_k = i)$ for $i \in \mathbf{S}$.

2.4.1 Stability

To discuss stability, we consider the following unforced system

$$x_{k+1} = \Phi(\theta_k)x_k \tag{2.6}$$

Definition 2.5. The system in (2.6) is said to be stochastically stable if for all nonzero x_0 and θ_0

$$\sum_{k=0}^{\infty} E(\|x_k\|^2) < \infty.$$

A necessary and sufficient condition to check the stability of (2.6) is given by the following theorem.

Theorem 2.3 ([10]). *System* (2.6) *is stochastically stable if there exist symmetric matrices $P_i > 0,\ i = 1, \ldots, N$ that satisfy*

$$A_i^T \bar{P}_i A_i - P_i < 0, \quad i \in \mathbf{S}$$

where $\bar{P}_i = \sum_{j=1}^{N} p_{ij} P_j$.

2.4.2 $\mathcal{H}_\infty$ Norm

Assume the discrete-time Markovian jump system in (2.5) is stochastically stable and $x_0 = 0$ and let T_{zw} denote the mapping from w_k to z_k . Assume that $w_k \in \ell_2[0, \infty)$ where $\ell_2[0, \infty)$ is the space of square summable sequences with finite $\|.\|_2$-norm. The $\mathcal{H}_\infty$ norm of T_{zw}, denoted as $\|T_{zw}\|_\infty$, is defined as

$$\|T_{zw}\|_\infty := \sup_{\theta_0 \in \mathbf{S}} \sup_{w_k \in \ell_2} \frac{\|z_k\|_2}{\|w_k\|_2}.$$

The $\mathcal{H}_\infty$ norm of T_{zw} can be computed by the following theorem [63].

Theorem 2.4. *Assume T_{zw} is stochastically stable. It satisfies $\|T_{zw}\|_\infty < \gamma$ if there exist symmetric matrices $P_i > 0$ that satisfy*

$$\begin{bmatrix} \Phi_i & \Gamma_i \\ C_i & D_i \end{bmatrix}^T \begin{bmatrix} \bar{P}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_i & \Gamma_i \\ C_i & D_i \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0, \quad i \in \mathbf{S}$$

where $\bar{P}_i = \sum_{j=1}^{N} p_{ij} P_j$.

2.5 Introduction to Linear Matrix Inequalities

A linear matrix inequality (LMI) is a matrix inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{2.7}$$

where $x \in \mathbf{R}^m$ is the variable and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \ldots, m$ are given constant symmetric matrices. The inequality in (2.7) means that $F(x)$ is positive definite, i.e., $v^T F(x) v > 0$ for all nonzero $v \in \mathbf{R}^n$.

A wide variety of problems arising in systems and control theory can be formulated as LMI problems. A feature of LMI problems is they can be solved for several matrix variables. Moreover, structural constraints can be imposed on the matrix variables. In some cases, LMI formulation for control problems remove the restrictions associated with the conventional methods and aid their extension to more general scenarios [73]. For details about LMI methods in control, we refer the reader to [8].

2.5.1 LMI Problems

Most of the LMI problems can be formulated in one of the three standard forms:

1. The feasibility problem
2. The linear objective minimization problem
3. The generalized eigenvalue minimization problem

Feasibility Problem

The feasibility problem consists of determining the variable $x \in \mathbf{R}^m$ such that $F(x) > 0$. An example of feasibility problem is to verify the Lyapunov (or quadratic) stability of a dynamic system.

Linear Objective Minimization Problem

The linear objective minimization problem is an LMI problem of the form

$$\begin{aligned} &\min_{x \in \mathbf{R}^m} \quad C^T x \\ &\text{s.t.} \;\; F(x) > 0 \end{aligned}$$

where C is the vector of coefficients. An example of such a problem is the computation of $\mathcal{H}_\infty$ norm of a dynamic system.

Generalized Eigenvalue Minimization Problem

The generalized eigenvalue minimization problem is of the form

$$\begin{aligned} &\min_{x \in \mathbf{R}^m} \qquad \lambda \\ &\text{s.t.} \;\; F_1(x) > \lambda F_2(x) \end{aligned}$$

The generalized eigenvalue problem is quasi-convex with respect to the parameters x and λ.

2.5.2 Some Useful Results in LMI's

Here we collect some results that are useful while formulating problems as LMI problems.

Schur Complement

The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{2.8}$$

where $Q(x) = Q(x)^T$, $S(x) = S(x)^T$ and $R(x)$ depends affinely on x, is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \tag{2.9}$$

In other words, the set of nonlinear inequalities (2.9) can be represented as the LMI (2.8).

Congruence Transformation

For a given positive definite matrix $Q \in \mathbf{R}^{n\times n}$ and another real matrix $W \in \mathbf{R}^{n\times n}$ such that $\text{rank}(W) = n$, the following inequality holds

$$WQW^T > 0. \tag{2.10}$$

In other words, definiteness of a matrix is invariant under pre- and post-multiplication by a full rank real matrix, and its transpose, respectively. The process from $Q > 0$ to (2.10) using a full rank matrix is called congruence transformation.

Chapter 3

Sampling Period Dependent $\mathcal{H}_\infty$ Filtering*

3.1 Introduction

State space is a very common framework for formulating analysis and synthesis problems for complex control systems. Many state space control design approaches assume that all state variables are available for feedback and provide a state-feedback control strategy. Practically, it is not possible to measure all state variables of the plant, or it is very expensive to measure all the state variables and a state estimator is designed to provide an estimate of the state or a linear combination of the state variables of the system.

Two popular state estimators are the Luenberger observer [42–44] for the noise-free case and the Kalman filter [35, 76] for the Guassian noise. An alternative to Kalman filtering when the noise statistics are unknown is the $\mathcal{H}_\infty$ filter [27, 64, 65].

In this chapter, we study the $\mathcal{H}_\infty$ filtering problem for the sampled-data system under timing model 1 as shown in Figure 2.1 where the input updating period is fast and uniform, but the measurement sampling period is slow and nonuniform. The state estimation problem for this class of systems has been considered in [37, 57] and [81]. In [37], an oversampled Kalman filter is presented. In [57] and [81], a Luenberger type variable sampling rate observer is designed. But, to the authors' best knowledge, the $\mathcal{H}_\infty$ filter design for this class of systems has not yet been considered.

*A version of this chapter has been published in [48]. A shorter version was presented in [47].

On the other hand, Markovian jump systems are an active area of research. These are systems with discrete or continuous dynamics and a Markov chain, governing the transitions between different system modes. The behavior of these systems is primarily determined by the transition probabilities of jumping processes. The $\mathcal{H}_\infty$ filtering problem for these systems has been formulated under many different situations, and with known and unknown transition probabilities, see for instance [13, 14, 82].

In this chapter, a novel approach is used to model the process of nonuniform measurement sampling using a Markov chain. The resulting discrete-time system together with the Markov chain is a Markovian jump system. A mode-dependent, full-order $\mathcal{H}_\infty$ filter is designed. The case of unknown measurement sampling probabilities (or Markov chain's transition probabilities) is also considered. It is noted that Markov chains have already been used, e.g., in networked control systems [68, 78, 80, 83], to model random networked induced delays; however, their use to model the nonuniform sampling process is a new idea.

The rest of this chapter is organized as follows: Section 3.2 presents the formulation of an $\mathcal{H}_\infty$ filtering problem for a nonuniformly sampled system. Section 3.3 describes the main results for the analysis and synthesis of the proposed filter. Simulation results are given in Section 3.4 to show the effectiveness of the proposed approach.

3.2 Problem Formulation

Consider the following discrete-time system

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_1 w_k \\ z_k &= C_1 x_k + D_{11} w_k \end{aligned} \tag{3.1}$$

where $x_k \in \mathbf{R}^n$ is the system state, $w_k \in \mathbf{R}^{m_1}$ is the disturbance input that belongs to $\ell_2[0,\infty)$, and $z_k \in \mathbf{R}^{p_1}$ is the signal to be estimated. System (3.1) is obtained through discretization of the continuous-time system in (2.1) with a fast period h and $u_k \equiv 0$, and is assumed to be stable and detectable. We make the following assumptions about (3.1):

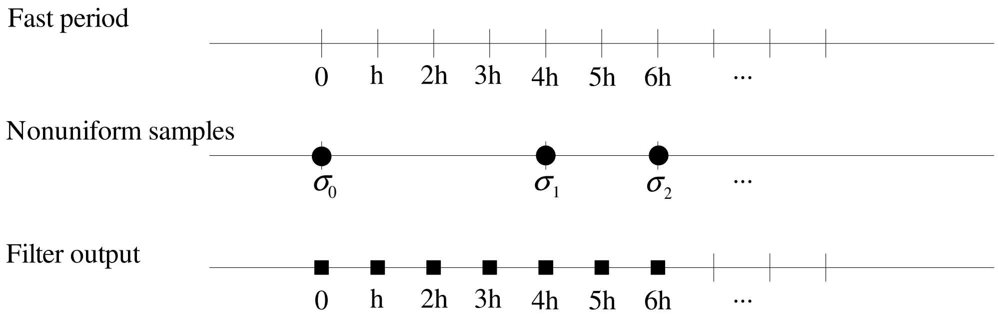

Figure 3.1: State estimation with nonuniform measurements

1. The measurement from the system is sampled only at arbitrary (random) instants σ_i, where $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_i$, and is given by

$$y_{\sigma_i} = C_2 x_{\sigma_i} + D_{21} w_{\sigma_i}, \quad y_{\sigma_i} \in \mathbf{R}^{p_2}.$$

2. The measurement sampling periods $\sigma_{i+1} - \sigma_i$ are integer multiples of the fast period h but otherwise random, that is $\sigma_{i+1} - \sigma_i = (m_i + 1)h$ for some $m_i \in \{0, 1, 2, \cdots, N\}$.

The objective is to design an $\mathcal{H}_\infty$ filter for (3.1) that provides an estimate of the state after every fast period h while taking the measurement at random instants σ_i. This scenario is depicted in Figure 3.1. As shown in Figure 3.1, the filtering system operates at the fast period h, whereas, the measurement is sampled at random instants σ_i, denoted by solid circles. The filter is required to estimate the state at every instant denoted by a solid square.

Remark 3.1. We assume that the measurement sampling period is an integer multiple of the fast period h, however, the system is still a variable sampling one because the measurement sampling period takes values in the set $\{h, \cdots, (N+1)h\}$.

Let $\{\theta_k, k \geq 0\}$ be a Markov chain with state space $\mathbf{S} = \{0, 1, \ldots, N\}$. We use θ_k to model the process of nonuniform measurement sampling. The state transition of the Markov chain is shown in Figure 3.2.

Let y_k be the output of the system in (3.1) at fast periods. In Figure 3.2, M_0 is the state when a new measurement, y_{σ_i}, is sampled and $y_k = y_{\sigma_i}$; M_1 is the state if a new measurement is not sampled after h and the previous measurement, y_{k-1},

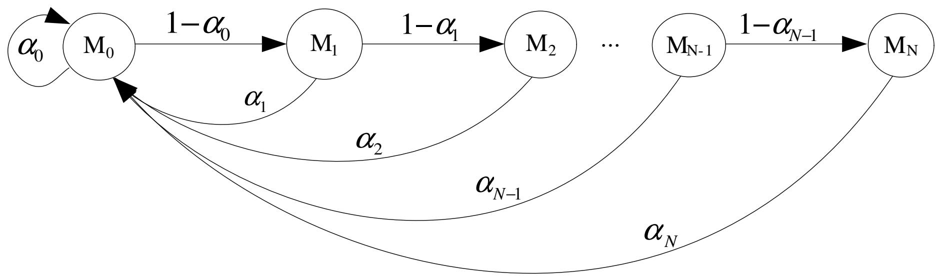

Figure 3.2: State diagram of the Markov chain

is used; M_2 is the state if it is not sampled after $2h$ and, y_{k-2}, is used; and so on. The state M_0 corresponds to $\theta_k = 0$, M_1 to $\theta_k = 1$, and so on.

Let $P = [p_{ij}]$ is the transition probability matrix of θ_k, where $p_{ij} = Pr\{\theta_{k+1} = j|\theta_k = i\}$. Let α_i be the probability that a new measurement is sampled after i-th fast period; given that it was not sampled after $(i-1)$-th period, then, $1-\alpha_i$ is the probability that a new measurement is not sampled. In terms of α_i, the transition probability matrix P can be written as:

$$P = \begin{bmatrix} \alpha_0 & 1-\alpha_0 & 0 & \cdots & 0 \\ \alpha_1 & 0 & 1-\alpha_1 & \cdots & 0 \\ \vdots & & & \ddots & \\ \alpha_{N-1} & 0 & 0 & \dots & 1-\alpha_{N-1} \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.2}$$

Remark 3.2. An important issue with the use of Markov chain is the knowledge of transition probabilities. While for some systems it may be possible to determine them experimentally, for many, it may not be. An example of how to model these probabilities for a networked control system can be found in [70].

Consider a mode-dependent filter of the following form

$$\begin{aligned} \hat{x}_{k+1} &= A_f(\theta_k)\hat{x}_k + B_f(\theta_k)\tilde{y}_k \\ \hat{z}_k &= C_f(\theta_k)\hat{x}_k + D_f(\theta_k)\tilde{y}_k \end{aligned} \tag{3.3}$$

where $\hat{x}_k \in \mathbf{R}^n$ is the filter state, $\tilde{y}_k \in \mathbf{R}^{p_1}$ is the filter input and $\hat{z}_k \in \mathbf{R}^{p_2}$ is the estimated signal. $A_f(\theta_k), B_f(\theta_k), C_f(\theta_k)$, and $D_f(\theta_k)$ are the filter parameters to be designed. For notational convenience, we shall use a subscript i with the matrix name when $\theta_k = i$.

Since the measurement from the system is sampled nonuniformly, the input to the filter $\tilde{y}_k$ takes values in the set $\{y_{k-i} : i \in \mathbf{S}\}$ depending upon the length of the current sampling period. For example, if the current measurement sampling period is h, then $\tilde{y}_k = y_k$ and $\theta_k = i = 0$. Define $\bar{x}_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T$, $\bar{w}_k = \begin{bmatrix} w_k^T & w_{k-1}^T & \ldots & w_{k-N}^T \end{bmatrix}^T$, $\tilde{z}_k = z_k - \hat{z}_k$, the error system will be

$$
\begin{aligned}
\bar{x}_{k+1} &= \begin{bmatrix} \Phi & 0 \\ B_{f_0}C_2 & A_{f_0} \end{bmatrix} \bar{x}_k + \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ B_{f_0}D_{21} & 0 & \ldots & 0 \end{bmatrix} \bar{w}_k, \\
&= \begin{bmatrix} \Phi & 0 \\ 0 & A_{f_0} \end{bmatrix} \bar{x}_k + \begin{bmatrix} 0 & 0 \\ B_{f_0}C_2 & 0 \end{bmatrix} \bar{x}_{k-0} + \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ B_{f_0}D_{21} & 0 & \ldots & 0 \end{bmatrix} \bar{w}_k, \\
\tilde{z}_k &= \begin{bmatrix} C_1 & -C_{f_0} \end{bmatrix} \bar{x}_k + \begin{bmatrix} -D_{f_0}C_2 & 0 \end{bmatrix} \bar{x}_{k-0} + \begin{bmatrix} D_{11} - D_{f_0}D_{21} & 0 & \ldots & 0 \end{bmatrix} \bar{w}_k.
\end{aligned}
$$

However, if the measurement sampling period is not h, then $\tilde{y}_k = y_{k-1}$, $\theta_k = i = 1$, and the error system will be

$$
\begin{aligned}
\bar{x}_{k+1} &= \begin{bmatrix} \Phi & 0 \\ 0 & A_{f_1} \end{bmatrix} \bar{x}_k + \begin{bmatrix} 0 & 0 \\ B_{f_1}C_2 & 0 \end{bmatrix} \bar{x}_{k-1} + \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ 0 & B_{f_1}D_{21} & \ldots & 0 \end{bmatrix} \bar{w}_k, \\
\tilde{z}_k &= \begin{bmatrix} C_1 & -C_{f_1} \end{bmatrix} \bar{x}_k + \begin{bmatrix} -D_{f_1}C_2 & 0 \end{bmatrix} \bar{x}_{k-1} + \begin{bmatrix} D_{11} & -D_{f_1}D_{21} & \ldots & 0 \end{bmatrix} \bar{w}_k.
\end{aligned}
$$

In general for any $i \in \mathbf{S}$, the error dynamics can be represented by the following Markov jump delay system

$$
\begin{aligned}
\bar{x}_{k+1} &= \mathcal{A}_i \bar{x}_k + \mathcal{A}_{d_i} \bar{x}_{k-i} + \mathcal{B}_i \bar{w}_k, \\
\tilde{z}_k &= \mathcal{C}_i \bar{x}_k + \mathcal{C}_{d_i} \bar{x}_{k-i} + \mathcal{D}_i \bar{w}_k,
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
&\mathcal{A}_i = \begin{bmatrix} \Phi & 0 \\ 0 & A_{f_i} \end{bmatrix}, \qquad \mathcal{A}_{d_i} = \begin{bmatrix} 0 & 0 \\ B_{f_i}C_2 & 0 \end{bmatrix}, \\
&\mathcal{B}_0 = \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ B_{f_0}D_{21} & 0 & \ldots & 0 \end{bmatrix}, \quad \ldots, \quad \mathcal{B}_N = \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & B_{f_N}D_{21} \end{bmatrix}, \\
&\mathcal{C}_i = \begin{bmatrix} C_1 & -C_{f_i} \end{bmatrix}, \quad \mathcal{C}_{d_i} = \begin{bmatrix} -D_{f_i}C_2 & 0 \end{bmatrix}, \\
&\mathcal{D}_0 = \begin{bmatrix} D_{11} - D_{f_0}D_{21} & 0 & \ldots & 0 \end{bmatrix}, \quad \ldots, \quad \mathcal{D}_N = \begin{bmatrix} D_{11} & 0 & \ldots & -D_{f_N}D_{21} \end{bmatrix}.
\end{aligned}
$$

The goal of this paper is to design a mode-dependent filter of the form in (3.3) for the system in (3.1) such that the error system in (3.4) is stochastically stable and has an $\mathcal{H}_\infty$ disturbance attenuation level γ.

Remark 3.3. The error system can also be modeled as a delay free system by defining an augmented state. A conference version of this paper using the augmented state approach can be found in [47].

Definition 3.1. The error system in (3.4) is said to be stochastically stable in the mean square sense if, for $\bar{w}_k \equiv 0$ and every initial condition $\bar{x}_i, i = -1, -2, \ldots, -N$, $\theta_0 \in \mathbf{S}$, we have

$$E\left\{\sum_{k=0}^{\infty} \|\bar{x}_k\|_2^2 | \bar{x}_i, \theta_0\right\} < \infty.$$

Definition 3.2. For nonzero $\bar{w}_k \in \ell_2[0, \infty]$ and a given constant $\gamma > 0$, the error system in (3.4) is said to be stochastically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ if it is stochastically stable and under zero initial conditions,

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|_2^2\} \leq \gamma^2 \sum_{k=0}^{\infty} \|\bar{w}_k\|_2^2$$

holds.

Remark 3.4. The assumption of zero initial conditions while defining the $\mathcal{H}_\infty$ performance index for LTI systems is a standard one in the control literature. This makes the definition compatible to that using transfer functions. However, if we are interested in the $\mathcal{H}_\infty$ performance from nonzero initial conditions, then, assuming $\bar{x}_0$ as the initial state of the system, it can be defined as:

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|_2^2\} \leq \gamma^2 \{\sum_{k=0}^{\infty} \|\bar{w}_k\|_2^2 - \bar{x}_0^T R \bar{x}_0\},$$

where $R > 0$ is a given weighting matrix for the initial state. The analysis and synthesis equations for the filter can then be modified accordingly. An approach to the $\mathcal{H}_\infty$ filter design for Markov jump linear systems with a nonzero initial state can be found in [13].

3.3 $\mathcal{H}_\infty$ Filter Analysis and Design

This section presents the main results for the $\mathcal{H}_\infty$ filtering analysis and design. LMI conditions are derived to ensure stochastic stability and $\mathcal{H}_\infty$ performance of the error system, with or without the knowledge of transition probabilities. The derivations in Lemma 1 and 2 are motivated by the results in [68, 82], however, a Lyapunov functional dependent on sampling periods is considered.

3.3.1 $\mathcal{H}_\infty$ Performance Analysis with Known Probabilities

Lemma 3.1. *Given a scalar $\gamma > 0$, the filtering error system in (3.4) is stochastically stable in the mean-square sense with $\mathcal{H}_\infty$ performance level γ, if there exist matrices $P_i > 0$ and $Q_{r,i} > 0$ for all $i \in \mathbf{S}$ and $r \in \Psi = \{1, 2, \ldots, N\}$ such that the following matrix inequalities*

$$\begin{bmatrix} -\bar{P}_i & 0 & \bar{P}_i\tilde{\mathcal{A}}_i & \bar{P}_i\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{3.5}$$

hold, where $\bar{P}_i = \sum_{j=0}^{N} p_{ij} P_j$, and

$$\begin{aligned}
\tilde{\mathcal{A}}_i &= \begin{bmatrix} \mathcal{A}_i + \epsilon_0 \mathcal{A}_{d_0} & \epsilon_1 \mathcal{A}_{d_1} & \ldots & \epsilon_N \mathcal{A}_{d_N} \end{bmatrix} \\
\tilde{\mathcal{B}}_i &= \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ \epsilon_0 B_{f_0} D_{21} & \epsilon_1 B_{f_1} D_{21} & \ldots & \epsilon_N B_{f_N} D_{21} \end{bmatrix} \\
\tilde{\mathcal{C}}_i &= \begin{bmatrix} \mathcal{C}_i + \epsilon_0 \mathcal{C}_{d_0} & \epsilon_1 \mathcal{C}_{d_1} & \ldots & \epsilon_N \mathcal{C}_{d_N} \end{bmatrix} \\
\tilde{\mathcal{D}}_i &= \begin{bmatrix} D_{11} - \epsilon_0 D_{f_0} D_{21} & -\epsilon_1 D_{f_1} D_{21} & \ldots & -\epsilon_N D_{f_N} D_{21} \end{bmatrix} \\
\Upsilon_i &= \mathrm{diag}(Q_{1,j} - P_i, Q_{2,j} - Q_{1,i}, \ldots, Q_{N,j} - Q_{N-1,i}, -Q_{N,i}) \\
\epsilon_i &= \begin{cases} 1, & \text{if } \theta_k = i \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. We first prove the stochastic stability in the mean-square sense for the error system in (3.4). With $\bar{w}_k \equiv 0$, the equation in (3.4) becomes

$$\begin{aligned} \bar{x}_{k+1} &= \mathcal{A}_i \bar{x}_k + \mathcal{A}_{d_i} \bar{x}_{k-i}, \\ \tilde{z}_k &= \mathcal{C}_i \bar{x}_k + \mathcal{C}_{d_i} \bar{x}_{k-i}. \end{aligned}$$

Consider the following Lyapunov functional

$$\bar{V}(\bar{x}_k, k) = \bar{x}_k^T P_i \bar{x}_k + \sum_{r=1}^{N} \bar{x}_{k-r}^T Q_{r,i} \bar{x}_{k-r}, \quad \forall\, i \in \mathbf{S}, \tag{3.6}$$

where P_i and $Q_{r,i}$ satisfy (3.5). Let $\varphi(k) \triangleq \{\bar{x}_{k-i},\ i = 1, \ldots, N\}$, then for $\theta_k = i$

and $\theta_{k+1} = j$, we have

$$
\begin{aligned}
E\{\Delta\bar{V}(\bar{x}_k, k)\} &\triangleq E\{\bar{V}(\bar{x}_{k+1}, k+1)|\varphi(k), \theta_k\} - \bar{V}(\bar{x}_k, k) \\
&= E\{\bar{x}_{k+1}^T \sum_{j=0}^{N} p_{ij} P_j \bar{x}_{k+1}\} + \sum_{r=1}^{N} \bar{x}_{k+1-r}^T Q_{r,j} \bar{x}_{k+1-r} \\
&\quad - \bar{x}_k^T P_i \bar{x}_k - \sum_{r=1}^{N} \bar{x}_{k-r}^T Q_{r,i} \bar{x}_{k-r} \\
&= E\{\bar{x}_{k+1}^T \bar{P}_i \bar{x}_{k+1}\} + \bar{x}_k^T (Q_{1,j} - P_i)\bar{x}_k + \bar{x}_{k-1}^T (Q_{2,j} - Q_{1,i})\bar{x}_{k-1} + \ldots \\
&\quad + \bar{x}_{k-N+1}^T (Q_{N,j} - Q_{N-1,i})\bar{x}_{k-N+1} + \bar{x}_{k-N}^T (-Q_{N,i})\bar{x}_{k-N} \\
&= E\{(\mathcal{A}_i \bar{x}_k + \mathcal{A}_{d_i} \bar{x}_{k-i})^T \bar{P}_i (\mathcal{A}_i \bar{x}_k + \mathcal{A}_{d_i} \bar{x}_{k-i})\} + \bar{x}_k^T (Q_{1,j} - P_i)\bar{x}_k \\
&\quad + \bar{x}_{k-1}^T (Q_{2,j} - Q_{1,i})\bar{x}_{k-1} + \ldots + \bar{x}_{k-N+1}^T (Q_{N,j} - Q_{N-1,i})\bar{x}_{k-N+1} \\
&\quad + \bar{x}_{k-N}^T (-Q_{N,i})\bar{x}_{k-N}.
\end{aligned}
$$

Define $\xi_k = \begin{bmatrix} \bar{x}_k^T & \bar{x}_{k-1}^T & \ldots & \bar{x}_{k-N}^T \end{bmatrix}^T$, then

$$
E\{\Delta\bar{V}(\bar{x}_k, k)\} = \xi_k^T (\tilde{\mathcal{A}}_i^T \bar{P}_i \tilde{\mathcal{A}}_i + \Upsilon_i)\xi_k.
$$

By Schur complement, (3.5) guarantees that $\tilde{\mathcal{A}}_i^T \bar{P}_i \tilde{\mathcal{A}}_i + \Upsilon_i < 0$, this means that the system is stochastically mean-square stable. For $\bar{w}_k \in \ell_2[0, \infty)$

$$
\begin{aligned}
E\{\tilde{z}_k^T \tilde{z}_k\} - \gamma^2 \bar{w}_k^T \bar{w}_k &= E\{(\mathcal{C}_i \bar{x}_k + \mathcal{C}_{d_i} \bar{x}_{k-i} + \mathcal{D}_i \bar{w}_k)^T (\mathcal{C}_i \bar{x}_k + \mathcal{C}_{d_i} \bar{x}_{k-i} + \mathcal{D}_i \bar{w}_k)\} - \gamma^2 \bar{w}_k^T \bar{w}_k \\
&= \xi_k^T \tilde{\mathcal{C}}_i^T \tilde{\mathcal{C}}_i \xi_k + \xi_k^T \tilde{\mathcal{C}}_i^T \tilde{\mathcal{D}}_i \bar{w}_k + \bar{w}_k^T \tilde{\mathcal{D}}_i^T \tilde{\mathcal{C}}_i \xi_k + \bar{w}_k^T (\tilde{\mathcal{D}}_i^T \tilde{\mathcal{D}}_i - \gamma^2 I)\bar{w}_k.
\end{aligned}
$$

Therefore,

$$
E\{\tilde{z}_k^T \tilde{z}_k\} - \gamma^2 \bar{w}_k^T \bar{w}_k + E\{\Delta\bar{V}(\bar{x}_k, k)\} = \eta_k^T \phi \eta_k, \tag{3.7}
$$

where $\eta_k = \begin{bmatrix} \xi_k^T & \bar{w}_k^T \end{bmatrix}^T$, and

$$
\phi = \begin{bmatrix} \tilde{\mathcal{A}}_i & \tilde{\mathcal{B}}_i \\ \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \end{bmatrix}^T \begin{bmatrix} \bar{P}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{A}}_i & \tilde{\mathcal{B}}_i \\ \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \end{bmatrix} + \begin{bmatrix} \Upsilon_i & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.
$$

Inequality (3.5) ensures that $\phi < 0$. Thus, from (3.7)

$$
E\{\tilde{z}_k^T \tilde{z}_k\} < \gamma^2 \bar{w}_k^T \bar{w}_k - E\{\Delta\bar{V}(\bar{x}_k, k)\}. \tag{3.8}
$$

Summing both sides of (3.8), we get

$$
\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|_2^2\} < \sum_{k=0}^{\infty} \gamma^2 \|\bar{w}_k\|_2^2 + \bar{V}_0 - E\{(\bar{V}(\infty)\}.
$$

Under zero initial conditions $\bar{V}_0 = 0$, and $E\{\bar{V}(\infty)\} \geq 0$, therefore,

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|_2^2\} \leq \sum_{k=0}^{\infty} \gamma^2 \|\bar{w}_k\|_2^2.$$

Thus, (3.5) is a sufficient condition to ensure stability and $\mathcal{H}_\infty$ disturbance attenuation for the filtering error system in (3.4). This completes the proof. $\square$

3.3.2 $\mathcal{H}_\infty$ Performance Analysis with Unknown Probabilities

In this section, we establish a condition for the stochastic stability and $\mathcal{H}_\infty$ performance of the error system in (3.4) when the measurement sampling probabilities α_i are partially or completely unknown.

Lemma 3.2. *Given a scalar $\gamma > 0$, the filtering error system in (3.4) is stochastically stable in the mean-square sense with $\mathcal{H}_\infty$ performance level γ, if there exist matrices $P_i > 0$ and $Q_{r,i} > 0$ for all $i \in \mathbf{S}$ and $r \in \Psi = \{1, 2, \ldots, N\}$ such that the following matrix inequalities*

$$\Theta_i = \begin{bmatrix} -\frac{1}{\pi_{K,i}} P_{K,i} & 0 & \frac{1}{\pi_{K,i}} P_{K,i} \tilde{\mathcal{A}}_i & \frac{1}{\pi_{K,i}} P_{K,i} \tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{3.9}$$

$$\Theta_{ij} = \begin{bmatrix} -\bar{P}_j & 0 & \bar{P}_j \tilde{\mathcal{A}}_i & \bar{P}_j \tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall j \in \mathbf{S}_{UK,i}, \tag{3.10}$$

hold, where $P_{K,i} = \sum_{j \in \mathbf{S}_{K,i}} p_{ij} P_j$, $\pi_{K,i} = \sum_{j \in \mathbf{S}_{K,i}} p_{ij}$, $\mathbf{S}_{K,i} \triangleq \{j | p_{ij} \text{ is known}\}$, *and* $\mathbf{S}_{UK,i} \triangleq \{j | p_{ij} \text{ is unknown}\}$.

Proof. Rewriting (3.5) as

$$\sum_{j\in\mathbf{S}_{K,i}} p_{ij}\begin{bmatrix} -P_j & 0 & P_j\tilde{\mathcal{A}}_i & P_j\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} + \sum_{j\in\mathbf{S}_{UK,i}} p_{ij}\begin{bmatrix} -P_j & 0 & P_j\tilde{\mathcal{A}}_i & P_j\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0$$

$$= \pi_{K,i}\begin{bmatrix} -\frac{1}{\pi_{K,i}}P_{K,i} & 0 & \frac{1}{\pi_{K,i}}P_{K,i}\tilde{\mathcal{A}}_i & \frac{1}{\pi_{K,i}}P_{K,i}\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}$$

$$+ (1-\pi_{K,i})\begin{bmatrix} -P_j & 0 & P_j\tilde{\mathcal{A}}_i & P_j\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0$$

$$= \pi_{K,i}\Theta_i + (1-\pi_{K,i})\Theta_{ij}$$

Inequalities (3.9) and (3.10) ensure that $\Theta_i < 0$ and $\Theta_{ij} < 0$, for all $i \in \mathbf{S}$ and $j \in \mathbf{S}_{UK,i}$. This implies that the system will be stable and will have $\mathcal{H}_\infty$ attenuation level γ when the sampling probabilities are completely or partially unknown. □

It can be observed that Lemma 2 is reduced to Lemma 1, when all the probabilities are known. We also note that there is cross-coupling between matrix product terms of different operation modes. This makes it difficult to use Lemma 1 and Lemma 2 for the filter design. A slack matrix approach given in [82] can be used to remove this coupling.

Lemma 3.3. *Given $\gamma > 0$, the error system in (3.4) is stochastically stable with guaranteed $\mathcal{H}_\infty$ performance level γ if there exist matrices $P_i > 0, Q_{r,i} > 0$ and R_i for all $i \in \mathbf{S}$ such that the following matrix inequalities*

$$\begin{bmatrix} \Lambda_j - R_i - R_i^T & 0 & R_i\tilde{\mathcal{A}}_i & R_i\tilde{\mathcal{B}}_i \\ * & -I & \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \\ * & * & \Upsilon_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{3.11}$$

hold, where $\Lambda_j = \bar{P}_i$ for (3.5)*, $\Lambda_j = \frac{1}{\pi_{K,i}}P_{K,i}$ for* (3.9)*, and $\Lambda_j = P_j$ for* (3.10).

Proof. See [82]. □

3.3.3 $\mathcal{H}_\infty$ Filter Design

The following theorem gives sufficient conditions for the existence and synthesis of a mode-dependent $\mathcal{H}_\infty$ filter.

Theorem 3.1. *Consider system in (3.1) with $\gamma > 0$ be a given constant. If there exist matrices $P_{1i} > 0, P_{3i} > 0, Q_{1r,i} > 0, Q_{3r,i} > 0$, and P_{2i}, $Q_{2r,i}$, X_i, Y_i, Z_i, A_{F_i}, B_{F_i}, C_{F_i}, D_{F_i}, for all $i \in \mathbf{S}, r \in \Psi$, such that the following matrix inequalities*

$$\begin{bmatrix} \Pi_1 & 0 & \Pi_2 & \Pi_3 \\ * & -I & \Pi_4 & \Pi_5 \\ * & * & \bar{\Upsilon}_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{3.12}$$

hold, where

$$\begin{aligned}
\Pi_1 &= \begin{bmatrix} \Lambda_{1j} - X_i - X_i^T & \Lambda_{2j} - Y_i - Z_i^T \\ * & \Lambda_{3j} - Y_i - Y_i^T \end{bmatrix}, \\
\Pi_2 &= \begin{bmatrix} X_i\Phi + \epsilon_0 B_{F_0} C_2 & A_{F_i} & \epsilon_1 B_{F_1} C_2 & 0 & \dots & \epsilon_N B_{F_N} C_2 & 0 \\ Z_i\Phi + \epsilon_0 B_{F_0} C_2 & A_{F_i} & \epsilon_1 B_{F_1} C_2 & 0 & \dots & \epsilon_N B_{F_n} C_2 & 0 \end{bmatrix}, \\
\Pi_3 &= \begin{bmatrix} X_i\Gamma_1 + \epsilon_0 B_{F_0} D_{21} & \epsilon_1 B_{F_1} D_{21} & \dots & \epsilon_N B_{F_N} D_{21} \\ Z_i\Gamma_1 + \epsilon_0 B_{F_0} D_{21} & \epsilon_1 B_{F_1} D_{21} & \dots & \epsilon_N B_{F_N} D_{21} \end{bmatrix}, \\
\Pi_4 &= \begin{bmatrix} C_1 - \epsilon_0 D_{F_0} C_2 & -C_{F_i} & -\epsilon_1 D_{F_1} C_2 & 0 & \dots & -\epsilon_N D_{F_N} C_2 & 0 \end{bmatrix}, \\
\Pi_5 &= \begin{bmatrix} D_{11} - \epsilon_0 D_{F_0} D_{21} & -\epsilon_1 D_{F_1} D_{21} & \dots & -\epsilon_N D_{F_N} D_{21} \end{bmatrix}, \\
\Lambda_{zj} &\triangleq \begin{cases} \frac{1}{\pi_{K,i}} P_{K,zj}, & \text{if } j \in \mathbf{S}_{K,i} \\ P_{zj}, & \text{if } j \in \mathbf{S}_{UK,i} \end{cases}, \quad z = 1, 2, 3
\end{aligned}$$

$\bar{\Upsilon}_i$ is the partitioning of Υ_i, and ϵ_i are defined in (3.5)*. Then, the filter parameters achieving the desired γ are given by $A_{fi} = Y_i^{-1} A_{Fi}$, $B_{fi} = Y_i^{-1} B_{Fi}$, $C_{fi} = C_{Fi}$ and $D_{fi} = D_{Fi}$.*

Proof. Partition the matrices $P_i, Q_{r,i}$ and R_i in (3.11) as

$$P_i = \begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix}, \quad Q_{r,i} = \begin{bmatrix} Q_{1r,i} & Q_{2r,i} \\ * & Q_{3r,i} \end{bmatrix}, \quad R_i = \begin{bmatrix} X_i & Y_i \\ Z_i & Y_i \end{bmatrix},$$

and replace $Y_i A_{fi}$ with A_{Fi}, $Y_i B_{fi}$ with B_{Fi}, C_{fi} with C_{Fi}, and D_{fi} with D_{Fi}, to obtain (3.12). □

Remark 3.5. It is remarked that the analysis and design of an $\mathcal{H}_2$ filter can be carried out on similar lines. The interested reader is referred to [10, Chapter 5] for further details.

Remark 3.6. To find the best $\mathcal{H}_\infty$ performance index γ, set $\delta = \gamma^2$ and minimize δ subject to (3.12).

Remark 3.7. We also remark that there exist other studies that consider filtering problem with uncertainty in the measurement channel, such as filtering with multiple packet dropouts in networked control systems [53, 59]. The uncertain observations are modelled using a Bernoulli's variable and the error system is a stochastic system. The Bernoulli's model can also be used to model the nonuniform measurement sampling process and the system can be modelled as a switched system with two modes. It is note that the proposed Markov model is a generalization of the Bernoulli's model to N modes. Another advantage of the Markov model is that by choosing different transition probabilities, the effect of longer sampling periods can be effectively taken into account while designing the filter.

3.4 Simulation Results

Consider system (3.1) with

$$\Phi = \begin{bmatrix} 0.2 & 0.05 \\ -0.02 & 0.3 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5 & -0.7 \end{bmatrix}, \quad D_{11} = 0,$$
$$C_2 = \begin{bmatrix} 1 & 0.6 \end{bmatrix}, \quad D_{21} = 0.3.$$

In the first experiment we show how the $\mathcal{H}_\infty$ performance of the proposed filter measured by γ in (3.12) improves as the probabilities of the measurement sampling periods increase. Assume the measurement is randomly sampled after h, $2h$, or $3h$, i.e., $N = 2$. The transition probability matrix for the Markov chain is

$$P = \begin{bmatrix} \alpha_0 & 1-\alpha_0 & 0 \\ \alpha_1 & 0 & 1-\alpha_1 \\ 1 & 0 & 0 \end{bmatrix},$$

where α_0 is the probability that the measurement is sampled after h, α_1 is the probability that it is sampled after $2h$, given that it was not sampled after h, and α_3 is the probability it is sampled after $3h$. Note that we set $\alpha_3 = 1$ to indicate that the measurement will certainly be sampled after $3h$. Figure 3.3 shows the $\mathcal{H}_\infty$ performance as a function of α_0 and α_1. It can be seen that the performance of the filter improves as the probabilities approach to 1.

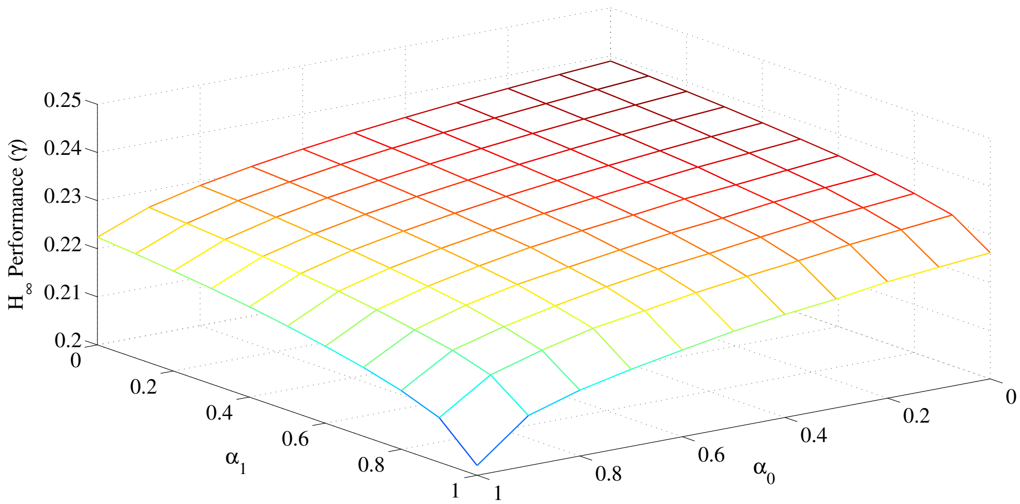

Figure 3.3: $\mathcal{H}_\infty$ performance as a function of α_0 and α_1

In the second experiment we show how the performance of the proposed filter is affected when the measurement sampling probabilities are completely or partially unknown. Let $\alpha_0 = 0.8$ and $\alpha_1 = 0.9$, the $\mathcal{H}_\infty$ performance assuming different levels of knowledge about the measurement sampling probabilities is given in Table 3.1. It can be seen in Table 3.1 that as the knowledge about the probabilities increases, the $\mathcal{H}_\infty$ performance improves.

It is known that a general time-varying or steady-state $\mathcal{H}_\infty$ filter can be designed for the discrete-time system in (3.1). For that, we can write the system in (3.1) as

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_1 w_k, \\ z_k &= C_1 x_k + D_{11} w_k, \\ y_k &= C_2(k) x_k + D_{21}(k) w_k, \end{aligned} \tag{3.13}$$

Table 3.1: $\mathcal{H}_\infty$ performance with varying knowledge of α_0 and α_1

α_0	α_1	Minimum $\mathcal{H}_\infty$ performance (γ)
0.8	0.9	0.2186
unknown	0.9	0.2269
0.8	unknown	0.2331
unknown	unknown	0.2390

where

$$C_2(k) = \begin{cases} C_2, & \text{if } k = \sigma_i \\ 0, & \text{otherwise} \end{cases}, \quad D_{21}(k) = \begin{cases} D_{21}, & \text{if } k = \sigma_i \\ 0, & \text{otherwise} \end{cases}.$$

The time-varying $\mathcal{H}_\infty$ filter is given as

$$\begin{aligned} \hat{x}_{k+1} &= \Phi \hat{x}_k + K(k)(y_k - C_2(k)\hat{x}_k), \\ \hat{z}_k &= C_1 \hat{x}_k. \end{aligned}$$

where $K(k)$ is a time-varying filter gain which can be computed by solving the difference Riccati equation associated with the time-varying system in (3.13). Similarly, a steady-state filter can be designed by assuming that the measurement is sampled after each fast period, i.e., $C_2(k) = C_2$ and $D_{21}(k) = D_{21}$, and solving the algebraic Riccati equation associated with this system. The reader is referred to [66] or [65] for details.

It is remarked that the proposed Markov jump model is more general than the time-varying discrete-time model in (3.13). The Markov model can incorporate naturally any knowledge about the measurement sampling probabilities. Secondly, the parameters of the proposed filter are computed offline, which is important for systems where computational resources are restricted such as networked and embedded control systems.

Let the measurement sampling probabilities be $\alpha_0 = 0.5$ and $\alpha_1 = 0.7$. Using Theorem 3.1, we can design a mode-dependent filter that gives an upper bound on the $\mathcal{H}_\infty$ performance as $\gamma = 0.2276$. For the same probabilities, the time-varying $\mathcal{H}_\infty$ filter gives an upper bound on the $\mathcal{H}_\infty$ performance as $\gamma = 0.2525$.

Let $w_k = w_{1k} + w_{2k}$ where $w_{1k} = 2e^{-0.02k} \sin(0.1\pi k)$ and w_{2k} is defined as

$$w_{2k} = \begin{cases} -1.5, & \text{for } 31 \leq k \leq 60 \\ 1.5, & \text{for } 101 \leq k \leq 130 \\ 0, & \text{otherwise.} \end{cases}$$

Figure 3.4 shows a comparison of the filtering error responses of the proposed mode-dependent $\mathcal{H}_\infty$ filter and the time-varying $\mathcal{H}_\infty$ filter with zero initial conditions, i.e., $\bar{x}_0 = 0$. It can be seen that the proposed filter attenuates the disturbance more effectively.

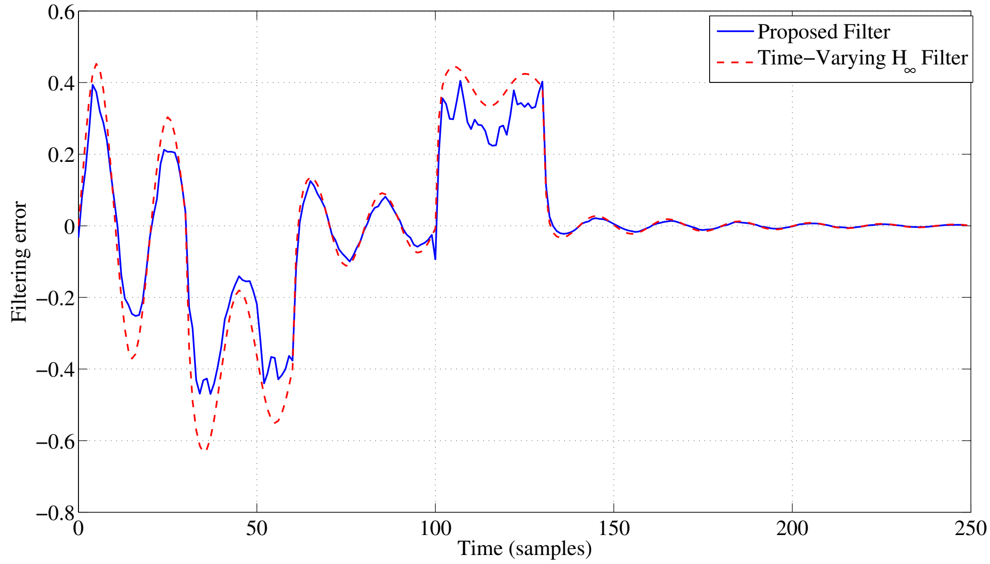

Figure 3.4: Filtering error response

We also did a Monte Carlo simulation with 100 runs to compare the performance of two optimal filters. From the simulation data, we computed the average of the maximum magnitudes of the estimation error for the mode-dependent filter as 0.5334 and for the time-varying $\mathcal{H}_\infty$ filter as 0.8862. We also computed the average value of the ratio of the 2-norms of the estimation error and the disturbance input: 0.1970 for the proposed filter and 0.2846 for the time-varying $\mathcal{H}_\infty$ filter. This means the proposed filter performs about 30% better than the time-varying $\mathcal{H}_\infty$ filter.

In another experiment, we increase the measurement sampling probabilities to $\alpha_0 = 0.8$ and $\alpha_1 = 0.95$ while keeping all other parameters the same as above and redesign the filters. From the simulation data, we find the average values of the maximum magnitudes of the estimation errors as 0.4694 and 0.7019 for the two filters while the ratios of the 2-norms are 0.1816 and 0.2769. The performance of the proposed filter is about 34% better than the time-varying $\mathcal{H}_\infty$ filter. These experiments clearly demonstrate the superiority of the proposed filter over the time-varying $\mathcal{H}_\infty$ filter.

3.5 Conclusion

The design of a sampling period dependent, full-order $\mathcal{H}_\infty$ filter for a class of nonuniformly sampled systems is presented. The nonuniform measurements are modelled by a Markov chain and the resulting filtering error system is a Markov jump delay system. The existence condition for the filter is presented in terms of LMI's for the cases of known and unknown sampling probabilities. The effectiveness of the proposed approach is demonstrated through simulation results.

Chapter 4

Robust $\mathcal{H}_\infty$ Filtering*

4.1 Introduction

As discussed in previous chapter, many control algorithms are state based; however, measuring all the state variables of a system is either not possible or feasible. State estimators are designed to provide an estimate of the state variables. In the previous chapter, we developed a method to design a sampling period dependent $\mathcal{H}_\infty$ filter. We considered the class of sampled-data systems where measurements are sampled at nonuniform sampling instants; however, the state is required to be estimated at uniform and faster instants. Assuming that the measurement sampling periods are integer multiples of the state estimation, a Markov model of the nonuniform sampling process led to the design of a filter using the Markovian jump systems approach.

In practice, variations in sampling period can be more arbitrary and the assumption that nonuniform measurement sampling periods are integer multiples of the state estimation period may be restrictive. In this chapter, we relax this assumption and aim at designing an estimator to estimate the state at time instants synchronized with the measurement sampling instants. A linear fractional transformation modelling of the variations in sampling period enable us design a robust discrete-time $\mathcal{H}_\infty$ filter.

The problem of $\mathcal{H}_\infty$ filter design for discrete-time systems with uniform sampling has been well-studied, see, for instance, [27, 79]. For the nonuniform sampling

*A version of this chapter has been published in [51]. A shorter version was presented in [50].

case, the $\mathcal{H}_\infty$ filtering problem has been considered in [6, 72]. In [72], the problem is treated in continuous time using the input-delay approach. In [6], a robust discrete-time $\mathcal{H}_\infty$ filter is designed; the filter design, however, requires the solution of bilinear matrix inequalities.

This chapter studies the filtering problem using a linear fractional transformation (LFT) approach and the filter design procedure is presented in terms of linear matrix inequalities (LMI's). This idea was presented by the authors in [50]; however, the scope of the analysis and design has been extended to robust performance. In order to achieve robust performance, the $\mathcal{H}_\infty$ norm of the error system is minimized for both the disturbance as well as uncertainty channels. A DK-type iterative procedure is proposed to apply fixed D-scaling to reduce the conservatism. Simulation results and a comparison study with the existing result show that the proposed approach is effective.

The rest of this chapter is organized as follows: In Section 4.2 we formulate the robust $\mathcal{H}_\infty$ filtering problem. Some preliminary results are given in Section 4.3. The main results for the analysis and design of the $\mathcal{H}_\infty$ filter are presented in Section 4.4. A numerical example is given in Section 4.5 to demonstrate the effectiveness of the proposed approach.

4.2 Problem Statement

Consider the sampled-data system configuration as shown in Figure 2.2 where P is a stable, continuous linear time-invariant plant described by

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{1c} w(t), \quad x(0) = 0, \\ y(t) &= C_2 x(t) + D_{21} w(t), \\ z(t) &= C_1 x(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$ is the system state, $w(t) \in \mathbf{R}^{m_1}$ is the disturbance, $y(t) \in \mathbf{R}^{p_1}$ is the measured output, and $z(t) \in \mathbf{R}^{p_2}$ is the signal to be estimated. A_c, B_{1c}, C_1, C_2, and D_{21} are matrices of compatible dimensions. Note that (4.1) is obtained from (2.1) by taking $u(t) \equiv 0$ and $x_0 = 0$ for the filtering problem.

The measurement $y(t)$ from the system is sampled when $t = \tau_k$ where $\{\tau_k : k \geq$

$0\}$ is a set of arbitrary sampling instants with properties

$$\tau_0 = 0, \quad \text{and} \quad 0 < h_l \le \tau_{k+1} - \tau_k \le h_u < \infty, \tag{4.2}$$

for given h_l and h_u. Note that (4.2) implies $\lim_{k\to\infty} \tau_k = \infty$.

Let h_k denote the k-th sampling period, namely, $h_k := \tau_{k+1} - \tau_k$, a discrete-time equivalent of (4.1) at the sampling instants τ_k is given as

$$\begin{aligned} x_{k+1} &= \Phi(h_k)x_k + \Gamma_1(h_k)w_k, \\ y_k &= C_2 x_k + D_{21} w_k, \\ z_k &= C_1 x_k, \end{aligned} \tag{4.3}$$

where $x_k := x(\tau_k)$, $w_k := w(\tau_k)$, $y_k := y(\tau_k)$, $z_k := z(\tau_k)$, and

$$\Phi(h_k) := e^{h_k A_c}, \quad \Gamma_1(h_k) := \int_0^{h_k} e^{(h_k-\eta)A_c} \mathrm{d}\eta B_{1c}.$$

Consider a discrete-time filter of the form

$$\begin{aligned} \hat{x}_{k+1} &= A_f \hat{x}_k + B_f y_k, \\ \hat{z}_k &= C_f \hat{x}_k + D_f y_k, \end{aligned} \tag{4.4}$$

where $\hat{x}_k$ and $\hat{z}_k$ are estimates of x_k and z_k, respectively. Define $\bar{x}_k^T = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}$ and $e_k = z_k - \hat{z}_k$; using (4.3) and (4.4), the error system can be written as

$$\begin{aligned} \bar{x}_{k+1} &= \bar{A}(h_k)\bar{x}_k + \bar{B}(h_k)w_k, \\ e_k &= \bar{C}\bar{x}_k + \bar{D}w_k, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \bar{A}(h_k) = \begin{bmatrix} \Phi(h_k) & 0 \\ B_f C_2 & A_f \end{bmatrix}, \quad \bar{B}(h_k) = \begin{bmatrix} \Gamma_1(h_k) \\ B_f D_{21} \end{bmatrix}, \\ \bar{C} = \begin{bmatrix} C_1 - D_f C_2 & -C_f \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} -D_f D_{21} \end{bmatrix}. \end{aligned}$$

The goal is to design a filter of the form in (4.4) for the system in (4.1) such that the error system in (4.5) is asymptotically stable with an $\mathcal{H}_\infty$ performance level $\gamma > 0$.

Definition 4.1. The error system in (4.5) is asymptotically stable if, for $w_k \equiv 0$ and $\bar{x}_0 \neq 0$, $\bar{x}_k \to 0$ as $k \to \infty$.

Definition 4.2. For $w_k \neq 0$, the error system is said to have an $\mathcal{H}_\infty$ performance level $\gamma > 0$ if

$$\|e_k\|_2 \le \gamma \|w_k\|_2.$$

4.3 Preliminaries

In order to analyze the $\mathcal{H}_\infty$ performance of the error system we give the following lemma.

Lemma 4.1. *Given* $0 < h_l < h_u < \infty$, $\gamma > 0$, *and the filter parameters* A_f, B_f, C_f *and* D_f, *the error system in* (4.5) *is asymptotically stable with an* $\mathcal{H}_\infty$ *performance level* γ *if there exists a symmetric matrix* $\bar{P} > 0$ *such that the following matrix inequality*

$$\begin{bmatrix} -\bar{P} & 0 & \bar{A}(h_k)\bar{P} & \bar{B}(h_k) \\ * & -I & \bar{C}\bar{P} & \bar{D} \\ * & * & -\bar{P} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.6}$$

holds for all $h_k \in \begin{bmatrix} h_l & h_u \end{bmatrix}$.

This is an extension of the discrete, time-invariant $\mathcal{H}_\infty$ filtering lemma [27] to the time-varying case.

The difficulty in applying Lemma 4.1 is that (4.6) has to hold for infinite many values of sampling periods $h_k \in [h_l\ h_u]$. The challenge is to convert it to a numerically tractable form.

In [21], Fujioka proposed a stability robustness idea to construct a grid $\mathcal{G}$ such that if the matrix inequality in (4.6) holds for the finite number of sampling periods in the grid, it will hold for all sampling periods in $[h_l\ h_u]$. We follow this idea to test the condition in (4.6) for the filter design. For this, we need the following lemma.

Lemma 4.2. *The error system in* (4.5) *can be re-configured as in Figure 4.1, where*

$$\Sigma(h_0): \qquad \begin{aligned} \bar{x}_{k+1} &= \bar{A}(h_0)\bar{x}_k + \bar{B}_1\xi_k + \bar{B}(h_0)w_k, \\ \eta_k &= \bar{C}_1(h_0)\bar{x}_k + \bar{D}_1(h_0)w_k, \\ e_k &= \bar{C}\bar{x}_k + \bar{D}w_k, \end{aligned} \tag{4.7}$$

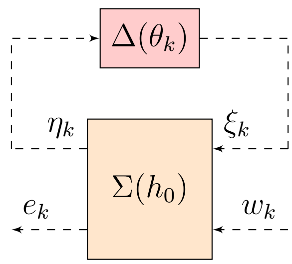

Figure 4.1: LFT representation of the error system

$\xi_k = \Delta(\theta_k)\eta_k$, *and* $\Delta(\theta_k) = \int_0^{\theta_k} e^{\eta A_c} \mathrm{d}\eta$. *The matrices in* (4.7) *are*

$$\bar{A}(h_0) = \begin{bmatrix} \Phi(h_0) & 0 \\ B_f C_2 & A_f \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \bar{B}(h_0) = \begin{bmatrix} \Gamma_1(h_0) \\ B_f D_{21} \end{bmatrix},$$
$$\bar{C}_1(h_0) = \begin{bmatrix} A_c \Phi(h_0) & 0 \end{bmatrix}, \quad \bar{D}_1(h_0) = A\Gamma_1(h_0) + B_{1c}.$$

Proof. Fix $h_k = h_0 + \theta_k$, from (4.3) we can write

$$\Phi(h_0 + \theta_k) = e^{\theta_k A_c}\Phi(h_0) = (I + \Delta(\theta_k)A_c)\Phi(h_0)$$

and

$$\begin{aligned} \Gamma_1(h_0 + \theta_k) &= \int_0^{h_0} e^{(h_0+\theta_k-\eta)A_c} B_{1c} \mathrm{d}\eta + \int_{h_0}^{h_0+\theta_k} e^{(h_0+\theta_k-\eta)A_c} B_{1c} \mathrm{d}\eta \\ &= (I + \Delta(\theta_k)A_c)\Gamma_1(h_0) + \Delta(\theta_k)B_{1c}. \end{aligned}$$

Re-write the matrices in error system (4.5) using these expressions and define $\eta_k = \bar{C}_1(h_0)\bar{x}_k + \bar{D}_1(h_0)w_k$ to get (4.7). □

The mapping from $w_k \to e_k$ is

$$e_k = \mathcal{F}_u(\Sigma, \Delta)w_k = [\Sigma_{22} + \Sigma_{21}\Delta(\theta_k)(I - \Sigma_{11}\Delta(\theta_k))^{-1}\Sigma_{12}]w_k,$$

where $\{\Sigma_{i,j} | \, i, j = 1, 2\}$ are the transfer matrices of the associated channels in Figure. 4.1. To ensure stability in the presence of variations of sampling periods, it is required that $\{\Sigma_{i,j} | \, i, j = 1, 2\}$ be stable and

$$\|\Sigma_{11}\Delta(\theta_k)\|_\infty < 1. \tag{4.8}$$

One can easily find a scalar α such that $\alpha > \|\Sigma_{11}\|_\infty$. Therefore, the system will be robustly stable as long as $\|\Delta(\theta_k)\|_\infty \leq \frac{1}{\alpha}$. Thus, in order to ensure robust stability, we need to bound $\|\Delta(\theta_k)\|$. There can be many different bounds for $\Delta(\theta_k)$, one such bound is given in the following lemma.

Lemma 4.3 ([40]). *For a given matrix $A_c \in \mathbf{R}^{n\times n}$ and $t \geq 0$, we have*

$$\|e^{A_c t}\| \leq e^{\mu(A_c)t},$$

where $\mu(A_c)$ is the logarithmic norm of A associated with the 2-norm, and is given by

$$\mu(A_c) = \lambda_{\max}\left(\frac{A_c + A_c^*}{2}\right).$$

For nominal performance, it is required that, in addition to (4.8), we have $\|\Sigma_{22}\|_\infty \leq \gamma$. A filter design procedure to achieve nominal performance was given in [50]. For robust performance, it is required that, in addition to (4.8), we have

$$\|\mathcal{F}_u(\Sigma, \Delta)\|_\infty \leq \gamma. \tag{4.9}$$

4.4 Main Results

4.4.1 Analysis

In this section, we state the main theorem to analyze the robust stability and $\mathcal{H}_\infty$ performance of the error system.

Theorem 4.1. *Given $h_0 > 0$, $\gamma > 0$, and the filter parameters A_f, B_f, C_f and D_f, the error system in (4.5) is robustly stable for all $h_k \in \mathcal{H}(h_0, \alpha)$ if there exists a symmetric matrix $\bar{P} > 0$ such that (4.9) and (4.8) hold. Here $\alpha = \|\Sigma_{11}\|_\infty$ and the interval $\mathcal{H}(h_0, \alpha)$ is defined as*

$$\mathcal{H}(h_0, \alpha) := (\underline{h}, \overline{h}) \cap (0, \infty), \tag{4.10}$$

where $\underline{h}$ and $\overline{h}$ are given as follows:

$$
\begin{aligned}
&L1) \text{ if } \mu(-A_c) = 0, \quad \underline{h} = h_0 - \alpha^{-1},\\
&L2) \text{ elseif } \mu(-A_c) \leq -\alpha, \quad \underline{h} = -\infty,\\
&L3) \text{ else } \quad \underline{h} = h_0 - \frac{1}{\mu(-A_c)}\log(1 + \alpha^{-1}\mu(-A_c)).\\
&U1) \text{ if } \mu(A_c) = 0, \quad \overline{h} = h_0 + \alpha^{-1},\\
&U2) \text{ elseif } \mu(A_c) \leq -\alpha, \quad \overline{h} = \infty\\
&U3) \text{ else } \quad \overline{h} = h_0 + \frac{1}{\mu(A_c)}\log(1 + \alpha^{-1}\mu(-A_c)).
\end{aligned}
$$

Proof. Take minimal realizations of the system in (4.1) and the filter in (4.4). If matrix inequality (4.6) is satisfied, this means there exists a symmetric and positive-definite matrix $\bar{P}$ such that $\rho(\bar{A}(h_0)) < 1$. Since, Σ_{ij}, $i, j = 1, 2$, have the same A-matrix, they are stable.

The interval in (4.10) can be determined using Lemma 4.3 following steps given in [21, Proof of Theorem 1]. What we show is (4.8) holds for all $h_k \in \begin{bmatrix} h_l & h_u \end{bmatrix}$. We prove that (4.8) holds for all $h_k \in \begin{bmatrix} h_0 & h_u \end{bmatrix}$, i.e. for $U1$, $U2$ and $U3$ in (4.10). The proof for $L1$, $L2$ and $L3$ for $h_k \in \begin{bmatrix} h_l & h_0 \end{bmatrix}$ can be similarly derived.

From Lemma 4.3, we have

$$\|\Delta(\theta_k)\| \leq \int_0^{\theta_k} \|e^{A_c\eta}\| \mathrm{d}\eta \leq \int_0^{\theta_k} e^{\mu(A_c)\eta} \mathrm{d}\eta$$

when $\theta_k \geq 0$.

If $\mu(A_c) = 0$, then

$$\|\Delta(\theta_k)\| \leq \theta_k,$$

and (4.8) will hold as long as

$$\alpha\theta_k < 1,$$

which is the case of $U1$.

Now, if $\mu(A_c) \neq 0$, then

$$\|\Delta(\theta_k)\| = \frac{e^{\mu(A_c)\theta_k} - 1}{\mu(A_c)}.$$

Now, if $\mu(A_c) < 0$, the right hand side in above equation goes to $\frac{-1}{\mu(A_c)}$ when θ_k goes to ∞. The condition in (4.8) will hold if

$$\frac{-\alpha}{\mu(A_c)} \leq 1,$$

which is the case of $U2$.

Now, consider the case of $\mu(A_c) \neq 0$ and $\frac{-\alpha}{\mu(A_c)} > 1$. In this case, the condition in (4.8) will hold for all $\theta_k > 0$ if

$$\alpha \frac{e^{\mu(A_c)\theta_k} - 1}{\mu(A_c)} \leq 1.$$

Since $1 + \alpha^{-1}\mu(A_c) > 0$ here, we get two cases:

Case A: If $\mu(A_c) > 0$, then

$$\mu(A_c)\theta_k \leq \log(1 + \alpha^{-1}\mu(A_c)).$$

Case B: If $\mu(A_c) < 0$, then

$$\mu(A_c)\theta_k \geq \log(1 + \alpha^{-1}\mu(A_c)).$$

Therefore, we get

$$\theta_k \geq \frac{1}{\mu(A_c)} \log(1 + \alpha^{-1}\mu(A_c)),$$

which is the case of $U3$.

$\square$

A direct use of Theorem 4.1 could be conservative because of the small-gain type condition in (4.8). This conservatism can be reduced by using a multi-model approach: Define a grid of sampling intervals $h_i > 0$ $(i = 1, 2, \cdots, N)$, and apply Theorem 4.1 for each h_i. The error system will, then, be robustly stable with $\mathcal{H}_\infty$ performance γ for all sampling periods given by

$$h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i).$$

4.4.2 Design

In this section, we discuss the $\mathcal{H}_\infty$ filter design.

Theorem 4.2. *Given $h_i > 0$ $(i = 1, 2, \cdots, N)$ and $\gamma = \mathrm{diag}(\gamma_1, \gamma_2) > 0$, if there exist symmetric and positive-definite matrices $Z \in \mathbf{R}^{n\times n}$, $Y \in \mathbf{R}^{n\times n}$, and matrices $F \in \mathbf{R}^{n\times p_1}$, $G \in \mathbf{R}^{p_2\times n}$, $D_F \in \mathbf{R}^{p_2\times p_1}$ and $Q \in \mathbf{R}^{n\times n}$ such that the small-gain* *condition in* (4.8) *and the LMI*

$$
\begin{bmatrix}
-Z & -Z & 0 & 0 & Z\Phi(h_i) & Z\Phi(h_i) & \frac{1}{d_\xi}Z & Z\Gamma(h_i) \\
* & -Y & 0 & 0 & Y\Phi(h_i)+FC+Q & Y\Phi(h_i)+FC_2 & \frac{1}{d_\xi}Y & Y\Gamma(h_i)+FD_{21} \\
* & * & -I & 0 & d_\xi A\Phi(h_i) & d_\xi A_c\Phi(h_i) & 0 & d_\xi A\Gamma(h_i)+d_\xi B_{1c} \\
* & * & * & -I & C_1-D_FC_2-G & C_1-D_FC_2 & & -D_FD_{21} \\
* & * & * & * & -Z & -Z & 0 & 0 \\
* & * & * & * & * & -Y & 0 & 0 \\
* & * & * & * & * & * & -\gamma_1^2 I & 0 \\
* & * & * & * & * & * & * & -\gamma_2^2 I
\end{bmatrix} < 0. \tag{4.11}
$$

hold for all $h_i > 0$*, then the error system in* (4.5) *is robustly stable with* $\mathcal{H}_\infty$ *performance level* γ *for all*

$$
h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i)
$$

with filter parameters

$$
\begin{aligned}
A_f &= -Y^{-1}Q(I-Y^{-1}Z)^{-1}, \quad B_f = -Y^{-1}F, \\
C_f &= G(I-Y^{-1}Z)^{-1}, \quad D_f = D_F.
\end{aligned} \tag{4.12}
$$

Proof. The LMI in (4.11) is obtained through a congruence transformation on matrix inequality (4.6) with a fixed D-scaling, $D_\xi = \mathrm{diag}(d_\xi I, I)$. For that, we take

$$
\bar{P} := \begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}, \quad \bar{P}^{-1} := \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix}
$$

where X, $\hat{X}$, Y, and $\hat{Y}$ are symmetric and positive-definite matrices. Define

$$
J_1 := \begin{bmatrix} X^{-1} & Y \\ 0 & V^T \end{bmatrix},
$$

and perform a congruence transformation on (4.6) with $J = \mathrm{diag}(J_1, I, J_1, I)$. From the definitions of $\bar{P}$ and $\bar{P}^{-1}$, we note that $XY + UV^T = I$ and $XV + U\hat{Y} = 0$. Using these relations, defining $Z := X^{-1}$, $F := VB_f$, $Q := VA_fU^TZ$, $G := C_fU^TZ$, and replacing h_k with h_i, we get (4.11). □

We have some remarks about Theorem 4.2.

Remark 4.1. The condition in (4.11) is based on $\|D_\xi \mathcal{F}_u(\Sigma, \Delta) D_\xi^{-1}\|_\infty \leq \gamma$ that applies a fixed D-scaling to the uncertainty channel. This is a very standard approach for the robust synthesis problems [87]. The selection of the matrix D_ξ is a trial-and-error process. We select the value of d_ξ by performing a search on a grid $\mathcal{G}_\xi$.

Remark 4.2. The condition in (4.11) redefines the $\mathcal{H}_\infty$ performance γ as $\gamma = \text{diag}(\gamma_1, \gamma_2)$. This allows us to make a trade-off between the length of variation in the sampling intervals (i.e. magnitude of uncertainty) and the $\mathcal{H}_\infty$ performance from the disturbance input to the estimation error.

Now we present a procedure for the filter design using Theorem 4.2 such that $[h_l, h_u] \subseteq \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i)$.

Procedure 4.1. Robust $\mathcal{H}_\infty$ Filter Design for Nonuniformly Sampled Systems

---

Given $0 < h_l < h_u < \infty$, $d_\xi \in \mathcal{G}_\xi$ and a large positive integer N_0

0. Initialization: $\mathcal{G} \leftarrow \{(h_l + h_u)/2\}$

1. if $\#\mathcal{G} \geq N_0$, stop without obtaining a filter. Here $\#\mathcal{G}$ denotes the number of elements in the grid $\mathcal{G}$.

2. Minimize

$$(1 - a)\delta_1 + a\delta_2$$

subject to (4.11) for all $h_i's$ where h_i is the i^{th} smallest element in $\mathcal{G}$, $0 \leq a \leq 1$ is the relative weight on the performance of the two channels, $\delta_1 = \gamma_1^2$ and $\delta_2 = \gamma_2^2$.

3. If

$$[h_l, h_u] \subseteq \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \alpha_i),$$

The error system in (4.5) will be robustly stable with $\mathcal{H}_\infty$ performance $\gamma_2 = \sqrt{\delta_2}$ with the filter parameters given by (4.12). Stop. Here

$$\alpha_i := \|\Sigma_{11}(h_i)\|_\infty.$$

4. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \bigcup \{(L_j + U_j)/2\}$$

for all j where L_j and U_j are determined so that

$$\bigcup_{j=1}^{M} (L_j, U_j) = (h_l, h_u) \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$
$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M$$

are satisfied. Where $M \leq \#\mathcal{G} + 1$. Go to step **1**.

5. Search for a value of d_ξ in $\mathcal{G}_\xi$ that minimizes γ_2.

Remark 4.3. Step 1 in the algorithm is introduced to avoid numerical issues when $\#\mathcal{G}$ is too large.

4.5 Numerical Example

Consider the following parameters for the plant in (4.1)

$$A_c = \begin{bmatrix} -b/J & K_T/J \\ -K_q/L_a & -R_a/L_a \end{bmatrix}, \qquad B_{1c} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_{21} = 0.$$

The values of the constants are $b = 0.1$Nms, $J = 0.01\text{kgm}^2/\text{s}^2$, $K_T = K_q = 0.01$Nm/A, $R_a = 1\Omega$, and $L_a = 0.5$H. This system was considered in [6] where the authors designed a robust $\mathcal{H}_\infty$ filter for $h_l = 0.001$ and $h_u = 0.099$ with $\mathcal{H}_\infty$ performance $\gamma = 1.8174$. Following Procedure 4.1 with $a = 0.9$ and $d_\xi = 0.512$, we can find a filter with $\gamma_2 = 1.2638$ with grid $\mathcal{G} = \{0.05\}$. The filter parameters are

$$A_f = \begin{bmatrix} 0.0960 & -0.1632 \\ -0.4189 & 0.7090 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.0598 \\ 0.0559 \end{bmatrix},$$
$$C_f = \begin{bmatrix} -1.9726 & 3.0004 \end{bmatrix}, \quad D_f = \begin{bmatrix} 1.1157 \end{bmatrix}.$$

Let $w_k = 2\exp(-0.01k)\sin(0.02\pi k)$ and $\bar{x}_0 = 0$. Figure 4.2 shows a plot of the estimation error and disturbance input. We observe that the disturbance is effectively attenuated.

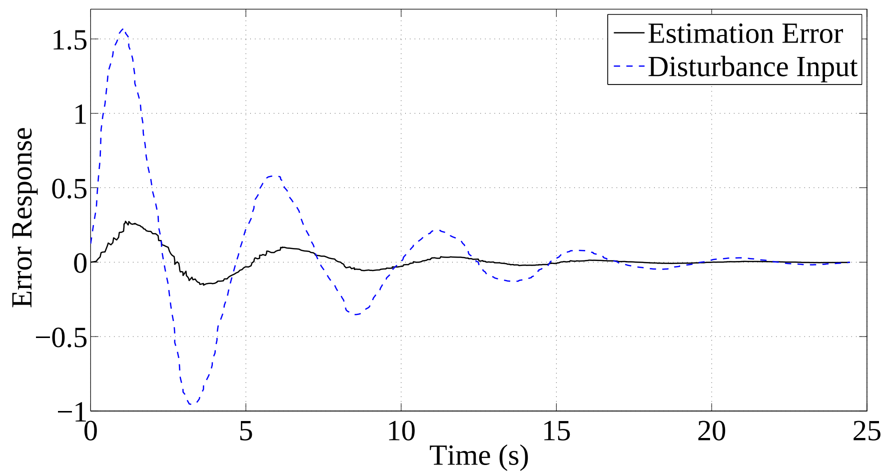

Figure 4.2: Estimation error response

4.6 Concluding Remarks

This chapter presents a discrete-time, robust $\mathcal{H}_\infty$ filter design procedure for systems whose sampling periods vary between a lower and an upper bound. Re-configuring the time-varying error system as an uncertain system with linear fractional transformation uncertainty, a robust filter design procedure is presented. The designed filter ensures robust stability and performance of the error system for all possible variations of sampling periods within the given bounds. The effectiveness of the approach is demonstrated through a comparison with an existing result.

Chapter 5

Dynamic Output Feedback Stabilization*

5.1 Introduction

In this chapter, we focus on the control problem. Stability is a fundamental requirement for the safe operation of all control systems. It is noted that most of the existing discrete-time approaches pertaining to nonuniformly sampled systems consider the case of state-feedback only whereas a more practical setup to work is the output feedback case. A stability analysis problem with dynamic output feedback controller was considered in [20] and a procedure was developed to find a Lyapunov function to conclude the quadratic stability of the associated discrete-time system and, hence, the exponential stability of the sampled-data system. The purpose of this chapter is to extend the results of [20] to the synthesis of dynamic controllers for nonuniformly sampled-data systems. It is remarked that the extension is not trivial as the approach uses a grid of sampling periods together with robustness to conclude stability. The design variables are shared among all instances of plant model and getting LMI conditions to design a robust controller is difficult. Motivated by the linearization technique introduced in [28], a design procedure for dynamic output feedback stabilization of nonuniformly sampled-data systems is developed. Numerical experiments show the effectiveness of the proposed approach.

The rest of this chapter is organized as follows: In Section 5.2 the problem

*A version of this chapter has been submitted for publication in [49]

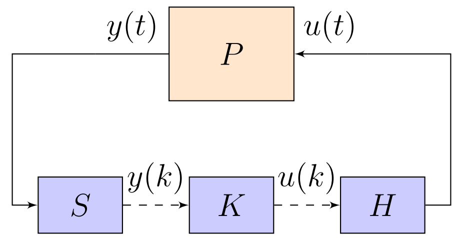

Figure 5.1: Nonuniformly sampled-data feedback control system

is formulated; the stability robustness analysis is given in Section 5.3. The main results for the controller design are developed in Section 5.4 and, finally, simulation results are given in Section 5.5.

5.2 Nonuniformly Sampled-Data Feedback Control

Consider a sampled-data feedback control configuration as shown in Figure 5.1. Let P be a finite-dimensional, linear time-invariant plant with state space model

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{2c} u(t), \qquad x(0) = x_0 \\ y(t) &= C_2 x(t) \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbf{R}^n$ denotes the state of the system, $u(t) \in \mathbf{R}^{m_2}$ is the control input and $y(t) \in \mathbf{R}^{p_2}$ is the measurement vector. The matrices A_c, B_{2c} and C_2 are assumed to have compatible dimensions.

The measurement $y(t)$ is sampled at uncertain and nonuniformly spaced time instants τ_k, i.e. $y(k) = y(\tau_k)$, satisfying

$$0 < h_l \leq \tau_{k+1} - \tau_k \leq h_u < \infty, \quad k = 0, 1, \cdots, \tag{5.2}$$

and

$$\lim_{k \to \infty} \tau_k = \infty. \tag{5.3}$$

The control input $u(t)$ to the system is generated using a zero-order-hold H synchronized with the sampler S

$$u(t) = u(k), \quad t \in [\tau_k,\ \tau_{k+1})$$

where $u(k)$ is determined using a finite-dimensional, discrete, linear constant-parameter controller K with state space model

$$\begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K y(k) \\ u(k) &= C_K x_K(k) \end{aligned} \tag{5.4}$$

where $x_K(k) = x_K(\tau_k)$ is the controller state and $u(k) = u(\tau_k)$ is the controller output. The closed-loop system is governed by

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{2c} C_K x_K(k), \quad t \in [\tau_{k+1}, \tau_k) \\ x_K(k+1) &= A_K x_K(k) + B_K y(k) \\ y(k) &= C_2 x(\tau_k), \quad k = 0, 1, \cdots \end{aligned} \tag{5.5}$$

The problem we consider in this chapter is as follows:

Problem: Design the controller parameters (A_K, B_K, C_K) such that the sampled-data output feedback control system (5.5) is exponentially stable for all sampling periods $h_k \in [h_l \ h_u]$ where $h_k = \tau_{k+1} - \tau_k$ is the k-th sampling period.

The stability of (5.5) can be studied in discrete time domain. For that, bring in a zero-order-hold discretization of the plant P and define $\xi(k) = \begin{bmatrix} x(\tau_k) & x_K(\tau_k) \end{bmatrix}^T$ as the state of the closed-loop system at the sampling instants τ_k, evolving as

$$\begin{aligned} \xi(k+1) &= \begin{bmatrix} A(h_k) & B_2(h_k) C_K \\ B_K C_2 & A_K \end{bmatrix} \xi(k), \\ &= \Phi(h_k) \xi(k), \qquad k = 0, 1, \cdots \end{aligned} \tag{5.6}$$

where $A(h_k) = e^{h_k A_c}$ and $B_2(h_k) = \int_0^{h_k} e^{(h_k - \eta) A_c} B_{2c} \mathrm{d}\eta$. The following lemma from [20, 85] establishes a connection between the exponential stability of (5.5) and the quadratic stability of (5.6).

Lemma 5.1. *Given the controller parameters (A_K, B_K, C_K), h_l and h_u, the sampled-data feedback closed-loop system* (5.5) *is exponentially stable if the uncertain discrete-time system in* (5.6) *is quadratically stable, that is, if there exists a symmetric matrix $\tilde{P} > 0$ such that the following inequality*

$$\Phi(h_k)^* \tilde{P} \Phi(h_k) - \tilde{P} < 0 \tag{5.7}$$

holds for all $h_k \in [h_l \ h_u]$.

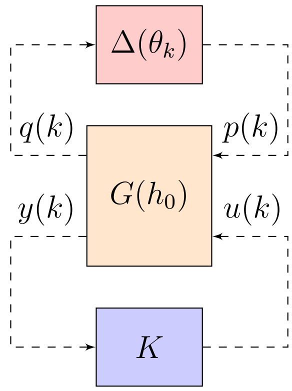

Figure 5.2: Uncertain sampled-data feedback system

The condition in (5.7) needs to be verified for infinite many values of h_k which is numerically impossible. By exploiting the robustness, an algorithm was developed in [20] to conclude the quadratic stability of (5.6) by solving (5.7) for a finite number of sampling periods in a grid.

5.3 Robust Stability Analysis

In this section, we develop a condition for quadratic stability of (5.6) for given controller parameters (A_k, B_K, C_K). This is achieved by viewing the variations in sampling period as perturbations to a nominal sampling period and modelling the perturbations as a linear fractional transformation uncertainty.

Lemma 5.2. *Given a nominal sampling period $h_0 > 0$ such that $\theta_k = h_k - h_0$, the sampled-data system in Figure 5.1 can be reconfigured as an uncertain sampled-data feedback system as shown in Figure 5.2, where*

$$G(h_0) = \left[\begin{array}{c|cc} A(h_0) & I & B_2(h_0) \\ \hline C_1(h_0) & 0 & D_{12}(h_0) \\ C_2 & 0 & 0 \end{array}\right],$$

$$\Delta(\theta_k) = \int_0^{\theta_k} e^{\eta A_c} \mathrm{d}\eta, \qquad q = \Delta(p)$$

$$C_1(h_0) = A_c A(h_0), \quad D_{12}(h_0) = A_c B_2(h_0) + B_{2c}.$$

Proof. Fix $h_k = h_0 + \theta_k$, from (5.6) we can write

$$A(h_0 + \theta_k) = e^{\theta_k A_c} A(h_0) = (I + \Delta(\theta_k) A_c) A(h_0)$$

and

$$\begin{aligned}B_2(h_0+\theta_k) &= \int_0^{h_0} e^{(h_0+\theta_k-\eta)A_c}B_{2c}\mathrm{d}\eta + \int_{h_0}^{h_0+\theta_k} e^{(h_0+\theta_k-\eta)A_c}B_{2c}\mathrm{d}\eta \\ &= e^{\theta_k A_c}B_2(h_0) + \Delta(\theta_k)B_{2c} \\ &= (I+\Delta(\theta_k)A_c)B_2(h_0) + \Delta(\theta_k)B_{2c}.\end{aligned}$$

Defining $q(k) = C_1(h_0)x(k) + D_{12}(h_0)u(k)$, the proof easily follows. □

For given controller parameters, the uncertain system in Figure 5.2 reduces to a feedback connection of an uncertain, time-varying operator $\Delta(\theta_k)$ and a linear constant-parameter system $\Sigma(h_0) = \mathcal{F}_l(G(h_0), K) = \Psi(h_0)(zI - \Phi(h_0))^{-1}\Gamma$, where $\Psi(h_0) = \begin{bmatrix} C_1(h_0) & D_{12}(h_0)C_K \end{bmatrix}$, $\Gamma = [\mathrm{I}\ 0]^T$ and $\Phi(h_0)$ as defined in (5.6). The symbol $\mathcal{F}_l$ denotes the lower star product of $G(h_0)$ and K. The robust stability of $\Sigma(h_0)$ can then be verified as a simple application of the small-gain theorem

$$\alpha\|\Delta\| < 1 \tag{5.8}$$

where $\alpha \geq \|\Sigma(h_0)\|_\infty$. Therefore, stability robustness of $\Sigma(h_0)$ can be verified by bounding $\Delta(\theta_k)$. Many different bounds can be defined for $\Delta(\theta_k)$, one such bound is given by the following lemma.

Lemma 5.3 ([40]). *For a given matrix $A_c \in \mathbf{R}^{n\times n}$ and $t \geq 0$, we have*

$$\|e^{A_c t}\| \leq e^{\mu(A_c)t},$$

where $\mu(A_c)$ is the logarithmic norm of A associated with the 2-norm, and is given by

$$\mu(A_c) = \lambda_{\max}\left(\frac{A_c + A_c^*}{2}\right).$$

The quadratic stability of (5.6) can be verified by invoking the following theorem.

Theorem 5.1. *Given $h_i > 0$ $(i = 1, 2, \cdots, N)$, $\Phi(h_i)$, Γ and $\Psi(h_i)$, if there exist a symmetric matrix $P > 0$ and α_i $(i = 1, 2, \cdots, N)$ satisfying N matrix inequalities*

$$\begin{bmatrix} P & \Phi(h_i)P & \Gamma & 0 \\ * & P & 0 & P\Psi(h_i)' \\ * & * & I & 0 \\ * & * & * & \alpha_i I \end{bmatrix} > 0, \tag{5.9}$$

then (5.7) *is satisfied with* $\tilde{P} = P^{-1}$ *for all* $h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$ *where the intervals* $\mathcal{H}(h_i, \sqrt{\alpha_i})$ *are defined as*

$$\mathcal{H}(h_i, \sqrt{\alpha_i}) := (\underline{h}, \overline{h}) \cap (0, \infty), \tag{5.10}$$

where

$$\begin{aligned}
&L1) \text{ if } \mu(-A_c) = 0, \quad \underline{h} = h_i - (\sqrt{\alpha_i})^{-1}, \\
&L2) \text{ elseif } \mu(-A_c) \leq -\sqrt{\alpha_i}, \quad \underline{h} = -\infty, \\
&L3) \text{ else } \underline{h} = h_i - \frac{1}{\mu(-A_c)} \log(1 + (\sqrt{\alpha_i})^{-1}\mu(-A_c)). \\
&U1) \text{ if } \mu(A_c) = 0, \quad \overline{h} = h_i + (\sqrt{\alpha_i})^{-1}, \\
&U2) \text{ elseif } \mu(A_c) \leq -\sqrt{\alpha_i}, \quad \overline{h} = \infty \\
&U3) \text{ else } \overline{h} = h_i + \frac{1}{\mu(A_c)} \log(1 + (\sqrt{\alpha_i})^{-1}\mu(-A_c)).
\end{aligned}$$

Proof. See [20]. □

5.4 Robust Controller Design

In this section, we present the main theorem for the dynamic controller design. In order to convert the inequality in (5.9) into design LMI, one can use the partitions of P and P^{-1} as in [61]; however, the design variables appear as product terms with plant parameters $A(h_i)$ and $B_2(h_i)$. These nonlinear product terms need to be absorbed somewhere to get LMI conditions. The following two properties could be useful for that [28].

Property 1. For square matrices G and R of compatible dimensions with R being symmetric and positive-definite, i.e., $R > 0$, the following inequality holds:

$$G' R^{-1} G \geq G + G' - R \tag{5.11}$$

Proof. The proof follows from the fact that $(G - R)' R^{-1} (G - R) \geq 0$. □

Property 2. For matrices H, G non-singular, symmetric Q and symmetric $R > 0$ of compatible dimensions, if the following LMI

$$\begin{bmatrix} H + H' - Q & G' \\ G & R \end{bmatrix} > 0 \tag{5.12}$$

holds, satisfy the constraint $Q < H^{'}G^{-1}RG^{'^{-1}}H$.

Proof. Applying Schur complement to the lower diagonal yields $Q < H + H^{'} - G^{'}R^{-1}G$. From (5.11), $H^{'}G^{-1}RG^{'^{-1}}H \geq H + H^{'} - G^{'}R^{-1}G$, thus the result follows. $\square$

To convert the condition in (5.9) into synthesis LMI, partition the matrices P and P^{-1} as

$$P = \begin{bmatrix} X & U \\ U' & \hat{X} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & V \\ V' & \hat{Y} \end{bmatrix}. \tag{5.13}$$

where X, $\hat{X}$, Y and $\hat{Y}$ are symmetric and positive definite matrices. Defining

$$T = \begin{bmatrix} Y & I \\ V' & 0 \end{bmatrix}$$

and performing congruence transformation on (5.9) with $\mathrm{diag}(T, T, I, I)$ gives

$$\begin{bmatrix} T^{'}PT & T^{'}\Phi(h_i)PT & T^{'}\Gamma & 0 \\ * & T^{'}PT & 0 & T^{'}P\Psi(h_i)^{'} \\ * & * & I & 0 \\ * & * & * & \alpha_i I \end{bmatrix} > 0$$

where

$$T^{'}PT = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}, \quad T^{'}\Gamma = \begin{bmatrix} Y \\ I \end{bmatrix}$$

$$T^{'}\Phi(h_i)PT = \begin{bmatrix} YA(h_i) + FC_2 & YL_i + FC_2X \\ & +VA_KU^{'} \\ A(h_i) & L_i \end{bmatrix}$$

$$\Psi(h_i)PT = \begin{bmatrix} C_1(h_i) & C_1(h_i)X + D_{12}(h_i)L \end{bmatrix}$$

$$F = VB_K, \quad L = C_KU^{'}$$

$$L_i = A(h_i)X + B_2(h_i)L$$

To recover the controller parameter A_K, the term YL_i in $T^{'}\Phi(h_i)PT$ can be absorbed using the property in (5.12). Now, we give the main theorem for the synthesis of dynamic controllers for nonuniformly sampled-data systems.

Theorem 5.2. *Given $h_i > 0$ ($i = 1, 2, \cdots, N$), if there exist symmetric matrices $X > 0$, $Y > 0$, $Q > 0$, $R > 0$, matrices F, L, M and scalars α_i such that the* *following $N+1$ matrix inequalities*

$$\begin{bmatrix} Q & I \\ * & R \end{bmatrix} > 0 \tag{5.14}$$

$$\begin{bmatrix} Y & I & YA(h_i)+FC_2 & M & Y & 0 & 0 & Y \\ * & X & A(h_i) & L_i & I & 0 & 0 & 0 \\ * & * & Y & I & 0 & C_1(h_i)' & 0 & 0 \\ * & * & * & X & 0 & XC_1(h_i)'+L'D_{12}(h_i)' & L_i'+G' & 0 \\ * & * & * & * & I & 0 & 0 & 0 \\ * & * & * & * & * & \alpha_i I & 0 & 0 \\ * & * & * & * & * & * & Q & 0 \\ * & * & * & * & * & * & * & R \end{bmatrix} > 0, \tag{5.15}$$

hold, then the uncertain discrete-time system in (5.6) *will be stabilized by the controller parameterized by*

$$\begin{aligned} A_K &= V^{-1}(M+YG-FC_2X)(U')^{-1} \\ B_K &= V^{-1}F \\ C_K &= L(U')^{-1} \end{aligned} \tag{5.16}$$

for all $h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$ *where the intervals* $\mathcal{H}(h_i, \sqrt{\alpha_i})$ *are defined in* (5.10).

Remark 5.1. The matrices U and V do not appear in the LMI conditions in (5.14) and (5.15). One of them can be chosen freely to satisfy $VU' = I - XY$. For example, choosing $V = V' = Y$ gives $U' = Y^{-1} - X$.

Remark 5.2. Even though Theorem 5.2 is developed using the linear fractional modelling of the closed-loop system, it can be used for controller design for other discrete-time models, such as the polytopic models [31, 32].

Proof. Note that (5.14) together with (5.15) ensures that $R > Q^{-1} > 0$, this means (5.15) holds if R is replaced by Q^{-1}. Also for each i, it is verified that

$$\begin{bmatrix} Q \\ 0 \\ 0 \\ (L_i+G)' \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} Q & 0 & 0 & L_i+G & 0 & 0 \end{bmatrix} \geq 0. \tag{5.17}$$

Replacing R by Q^{-1}, performing Schur complement w.r.t the last two rows and columns of (5.15) and adding (5.17) after pre- and post-multiplying first row and column with Y, we get

$$\begin{bmatrix} Y & I & YA(h_i)+FC_2 & M+YL_i & Y & 0 \\ * & X & A(h_i) & L_i & I & 0 \\ * & * & Y & I & 0 & C_1(h_i) \\ * & * & * & X & 0 & XC_1(h_i)' + L^{'}D_{12}(h_i)^{'} \\ * & * & * & * & I & 0 \\ * & * & * & * & * & \alpha_i I \end{bmatrix} > 0,$$

which together with the transformations (5.13) and (5.16) yields

$$\begin{bmatrix} T^{'}PT & T^{'}\Phi(h_i)PT & T^{'}\Gamma & 0 \\ * & T^{'}PT & 0 & T^{'}P\Psi(h_i)^{'} \\ * & * & I & 0 \\ * & * & * & \alpha_i I \end{bmatrix} > 0$$

which is equivalent to (5.9). □

Once we fix a grid of sampling periods, a controller can be designed using Theorem 5.2. The following procedure can be used to generate a grid and controller design.

Procedure 5.1. Robust Dynamic Controller Design for Nonuniformly Sampled-Data Feedback Systems

---

Given $0 < h_l < h_u < \infty$ and a large positive integer N_0

0. Initialization: $\mathcal{G} \leftarrow \{(h_l + h_u)/2\}$

1. if $\#\mathcal{G} \geq N_0$, stop without designing a controller. Here $\#\mathcal{G}$ denotes the number of elements in the grid $\mathcal{G}$.

2. Minimize

$$\sum_{i=1}^{\#\mathcal{G}} \beta_i$$

subject to (5.14) and (5.15), with α_i replaced with β_i, for all $h_i's$ where h_i is the i-th smallest element in $\mathcal{G}$.

3. If $[h_l\ h_u] \subseteq \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \sqrt{\alpha_i})$. The sampled-data closed-loop system will be exponentially stabilized by controller parameters as defined in (5.16). Stop. Here

$$\alpha_i := \lambda_{\max}(R_i - S_i'(Q_i - P)^{-1}S_i) + \epsilon$$

where ϵ is a small positive number and

$$P = \begin{bmatrix} Y & V \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & U \end{bmatrix}^{-1},$$

$$\begin{bmatrix} Q_i & S_i \\ S_i' & R_i \end{bmatrix} = \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix}'.$$

4. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \bigcup \{(L_j + U_j)/2\}$$

for all j where L_j and U_j are determined so that

$$\bigcup_{j=1}^{M} (L_j, U_j) = (h_l, h_u) \backslash \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$
$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M$$

are satisfied, where $M \leq \#\mathcal{G} + 1$. Go to step **1**.

Remark 5.3. The number N_0 is introduced to avoid the numerical difficulties which may happen if the sampling period is too small.

5.5 Simulation Results and Discussion

In this section, we give some numerical examples to demonstrate the applicability and effectiveness of the proposed approach. The simulations were done on a computer with Mac OS X (10.7.3), Intel Core 2 Duo 2.4 GHz CPU, MATLAB 7.13 and Yalmip [41].

5.5.1 Example 1

Consider the following parameters for the plant in (5.1)

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

This plant was considered in [52] where a continuous-time observer-based controller with sampled measurements was designed using the delay systems approach. An anticipative controller was shown to stabilize this plant for $h_k \in [0,\ 0.976]$.

Using the proposed control design procedure (Procedure 5.1) with $[h_l\ h_u] = [0.1,\ 0.976]$, we can find a controller that exponentially stabilize this plant with parameters

$$A_K = \begin{bmatrix} -0.7412 & 0.2696 \\ -1.3114 & 0.7802 \end{bmatrix}, \quad B_K = \begin{bmatrix} 1.7312 \\ 1.3192 \end{bmatrix}, \quad C_K = \begin{bmatrix} -0.1858 & -6.0434 \end{bmatrix}.$$

The search took 7.13 s with $\#\mathcal{G} = 5$, where $\#\mathcal{G} = 5$ denotes the number of elements in the grid. To avoid unbounded solutions, we restricted the search for the entries of X, Q and R to be less than 10^3. Next, we search for a controller that could maximize h_u. We can find a numerically reliable solution for $h_k \in [0.1,\ 2.7]$s within 37 seconds and $\#\mathcal{G} = 32$. In fact, controllers can be designed to tolerate even larger variations in sampling period by relaxing the size of entries of X, Q and R and with increased computation time.

5.5.2 Example 2

Consider the linearized model of an unstable batch reactor

$$A_c = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} \quad B_{2c} = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Walsh et al. [74] verified the stability of this system for a given controller for $h_k \in [0,\ 10^{-5}]$ when the loop is closed through a network. However, their simulations revealed the closed-loop system stablity for $h_k \in [0,\ 0.08]$s.

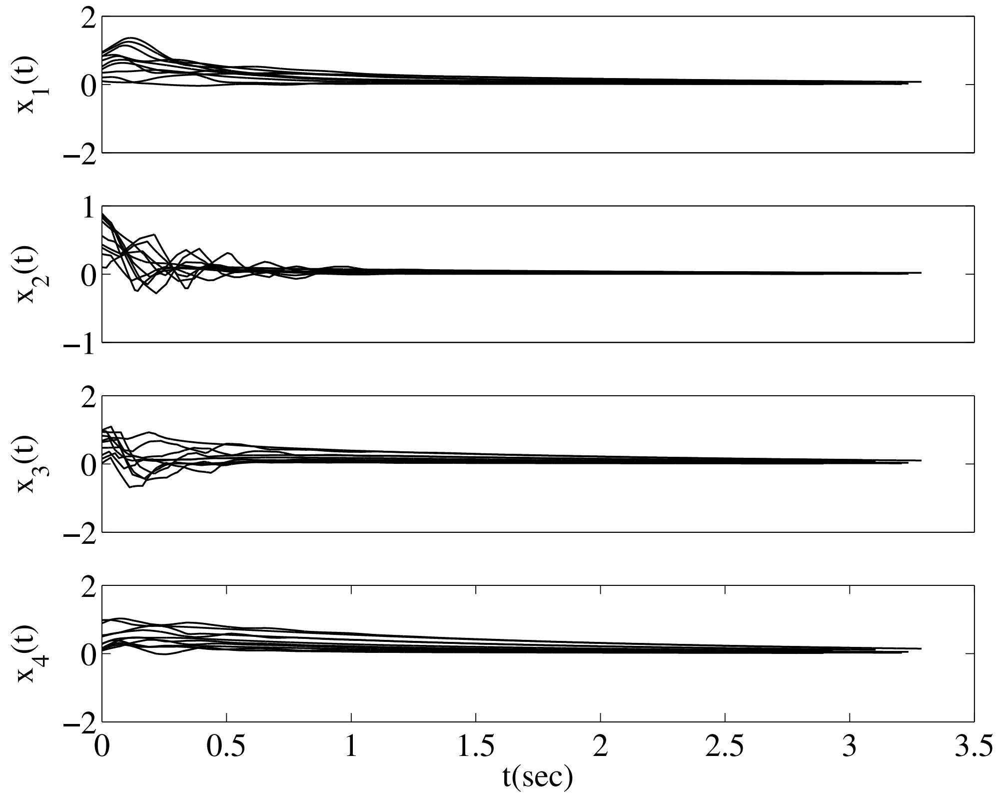

Figure 5.3: Zero-input responses of controlled batch reactor

We use Procedure 5.1 to design a stabilizing controller for $h_l = 0.01$ and $h_u = 0.08$. We can find one within 12.43 s with only four elements in the grid. The controller parameters are

$$A_K = \begin{bmatrix} 0.2057 & -1.1969 & -0.6483 & 0.6621 \\ 0.1624 & -0.3541 & 0.1845 & -0.1749 \\ -0.4482 & -7.8337 & 0.2425 & 0.5180 \\ -0.1179 & -8.1502 & -0.2873 & 1.0528 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0.8282 & 1.3003 \\ -0.0902 & 1.1779 \\ 0.4620 & 8.0967 \\ 0.1664 & 8.5772 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 1.0408 & -1.3895 & 0.8838 & -1.0210 \\ 3.1953 & -0.6524 & 2.6950 & -2.6282 \end{bmatrix}.$$

Figure 5.3 shows 10 initial value responses for arbitrary initial conditions and sampling periods taking values in the interval [0.01, 0.08]. We observe convergence of all the states of the controlled system.

These examples show that the proposed design method is effective.

5.6 Conclusions

This chapter developed a numerical procedure for the stabilization of nonuniformly sampled-data systems via dynamic output feedback controllers. The analysis and design were carried out in the discrete-time domain. Simulation results and a comparison with the other approaches showed the effectiveness of the proposed approach. Most of the existing discrete-time approaches dealing with nonuniformly sampled-data systems consider the case of state feedback only. We hope that the results presented in this chapter will help advance the state of research to a more general and higher level with output feedback control. The work can be extended in many ways, such as, control design with performance. Also, the analysis is based on a single quadratic Lyapunov function approach, another extension could be to use a switched Lyapunov function [26].

Chapter 6

Discrete-Time $\mathcal{H}_\infty$ Control

6.1 Introduction

In the previous chapter, we considered the dynamic output feedback stabilization problem and developed a method to robustly stabilize the closed-loop sampled-data system for all possible variations of sampling period. In this chapter, we extend the stabilization method to include the performance of closed-loop system. We use $\mathcal{H}_\infty$ norm of the closed-loop system as a performance measure and develop a discrete-time method for $\mathcal{H}_\infty$ controller design. In the past, the $\mathcal{H}_\infty$ controller design for nonuniformly sampled systems was addressed using the delay systems approach only. To the author's best knowledge, discrete-time $\mathcal{H}_\infty$ control of these systems has not yet been considered. The problem is challenging because the closed-loop sampled-data system is time-varying and aperiodic.

The $\mathcal{H}_\infty$ control for nonuniformly sampled systems using the delay systems approach was considered in [72] where a continuous-time controller with sampled inputs was designed. The designed controller is time-varying when implemented in discrete time, and dependent on uncertain sampling period. It is desirable to design a controller that can be easily implemented. This paper presents such a robust controller by following the discrete-time linear factional transformation approach [21]. Sufficient LMI conditions are developed. The resulting controller is guaranteed to maintain stability and performance for all variations of the sampling period within the given bounds.

The rest of this chapter is organized as follows: In Section 6.2 the problem is

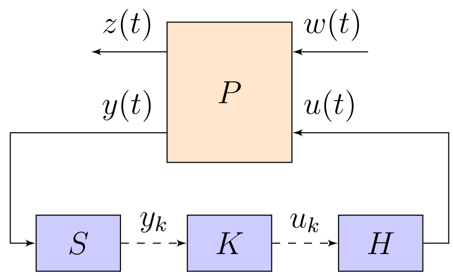

Figure 6.1: Nonuniformly sampled-data feedback $\mathcal{H}_\infty$ control system

formulated. The robust stability and performance analysis is given in Section 6.3. The main results for the robust controller design are developed in Section 6.4. Finally, a numerical example is given in Section 6.5 to demonstrate the effectiveness of the proposed method.

6.2 Problem Formulation

Consider a sampled-data feedback control configuration as shown in Figure 6.1. Let P be a finite-dimensional, linear time-invariant plant with state space model

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{1c} w(t) + B_{2c} u(t), \qquad x(0) = x_0 \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{6.1}$$

where $x(t) \in \mathbf{R}^n$ denotes the state of the system, $w(t) \in \mathbf{R}^{m_1}$ is the disturbance input, $u(t) \in \mathbf{R}^{m_2}$ is the control input, $z(t) \in \mathbf{R}^{p_1}$ is the controlled output and $y(t) \in \mathbf{R}^{p_2}$ is the measured output. The matrices A_c, B_{1c}, B_{2c}, C_1, C_2, D_{11}, D_{12} and D_{21} are assumed to have compatible dimensions.

The measurement $y(t)$ is sampled at uncertain and nonuniformly spaced time instants τ_k, i.e. $y_k = y(\tau_k)$, satisfying

$$0 < h_l \leq \tau_{k+1} - \tau_k \leq h_u < \infty \tag{6.2}$$

and

$$\lim_{k\to\infty} \tau_k = \infty.$$

The control input $u(t)$ to the system is generated using a zero-order-hold H synchronized with the sampler S

$$u(t) = u_k, \quad t \in [\tau_k,\ \tau_{k+1})$$

where u_k is determined using a finite-dimensional, discrete, linear constant-parameter controller K with state space model

$$\begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K y_k \\ u_k &= C_K x_K(k) \end{aligned} \tag{6.3}$$

where $x_K(k) := x_K(\tau_k)$ is the controller state and $u_k := u(\tau_k)$ is the controller output. Assuming $w(t)$ to be piece-wise constant, the zero-order hold equivalent of (6.1) at the sampling instants is given by

$$\begin{aligned} x_{k+1} &= A(h_k)x_k + B_1(h_k)w_k + B_2(h_k)u_k \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k \\ y_k &= C_2 x_k + D_{21} w_k \end{aligned} \tag{6.4}$$

where $A(h_k) = e^{h_k A_c}$, $B_i(h_k) = \int_0^{h_k} e^{(h_k-\eta)A_c} B_i \mathrm{d}\eta$, $i = 1, 2$, $x_k := x(\tau_k)$, $w_k := w(\tau_k)$, $z_k := z(\tau_k)$ and $y_k := y(\tau_k)$. Defining $\bar{x}_k = \begin{bmatrix} x_k & x_K(k) \end{bmatrix}^T$ and using (6.3) and (6.4), the dynamics of the closed-loop system at the sampling instants can be written as

$$\begin{aligned} \bar{x}_{k+1} &= \begin{bmatrix} A(h_k) & B_2(h_k)C_K \\ B_K C_2 & A_K \end{bmatrix} \bar{x}_k + \begin{bmatrix} B_1(h_k) \\ B_K D_{21} \end{bmatrix} w_k \\ z_k &= \begin{bmatrix} C_1 & D_{12} C_K \end{bmatrix} \bar{x}_k + D_{11} w_k \end{aligned} \tag{6.5}$$

Our goal in this chapter is to design the controller parameters A_K, B_K and C_K such that the closed-loop system in (6.5) is internally stable and

$$\|z_k\|_2 \leq \gamma \|w_k\|_2 \tag{6.6}$$

for all sampling periods $h_k \in [h_l\ h_u]$ where $h_k = \tau_{k+1} - \tau_k$ is the k-th sampling period. The disturbance signal w_k is assumed to be unknown but with bounded energy, i.e. $w_k \in \ell_2[0,\ \infty)$.

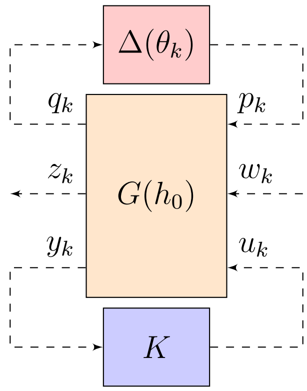

Figure 6.2: Uncertain sampled-data feedback system

6.3 Robust $\mathcal{H}_\infty$ Performance Analysis

In this section, we develop a method for robust stability and performance analysis of (6.5) for given controller parameters A_k, B_K and C_K. In the sequel, we show that the nonuniformly sampled-data system can be reconfigured as an uncertain sampled-data feedback system by viewing the variations in sampling period as perturbations to a nominal sampling period and modelling the perturbations as linear fractional transformation uncertainty. The robust analysis and design tools can then be applied.

Lemma 6.1. *Given a nominal sampling period $h_0 > 0$ such that $\theta_k = h_k - h_0$, the sampled-data system in Figure 6.1 can be reconfigured as an uncertain sampled-data feedback system as shown in Fgure 6.2, where*

$$
\begin{aligned}
G(h_0) &= \left[\begin{array}{c|cccc} A(h_0) & I & B_1(h_0) & B_2(h_0) \\ \hline C_{1\Delta}(h_0) & 0 & D_{11\Delta}(h_0) & D_{12\Delta}(h_0) \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{array}\right], \\
\Delta(\theta_k) &= \int_0^{\theta_k} e^{\eta A_c} \mathrm{d}\eta, \qquad q_k = \Delta(p_k) \\
C_{1\Delta}(h_0) &= A_c A(h_0), \quad D_{11\Delta}(h_0) = A_c B_1(h_0) + B_{1c} \\
D_{12\Delta}(h_0) &= A_c B_2(h_0) + B_{2c}.
\end{aligned}
$$

Proof. Fix $h_k = h_0 + \theta_k$, from (6.5) we can write

$$A(h_0 + \theta_k) = e^{\theta_k A_c} A(h_0) = (I + \Delta(\theta_k) A_c) A(h_0)$$

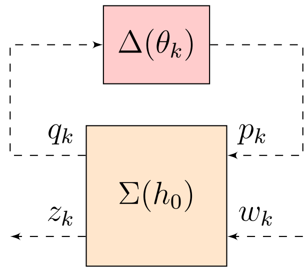

Figure 6.3: Uncertain closed-loop system

and

$$
\begin{aligned}
B_i(h_0+\theta_k) &= \int_0^{h_0} e^{(h_0+\theta_k-\eta)A_c} B_{ic} \mathrm{d}\eta \\
&+ \int_{h_0}^{h_0+\theta_k} e^{(h_0+\theta_k-\eta)A_c} B_{ic} \mathrm{d}\eta \\
&= e^{\theta_k A} B_i(h_0) + \Delta(\theta_k) B_{ic} \\
&= (I + \Delta(\theta_k)A_c)B_i(h_0) + \Delta(\theta_k)B_{ic}, \qquad i = 1, 2.
\end{aligned}
$$

Defining $q_k = C_{1\Delta}(h_0)x_k + D_{11\Delta}(h_0)w_k + D_{12\Delta}(h_0)u_k$, the proof easily follows. $\square$

For given controller parameters, the uncertain system in Figure 6.2 reduces to a feedback connection of an uncertain, time-varying operator $\Delta(\theta_k)$ and a linear constant-parameter system $\Sigma(h_0) = \mathcal{F}_l(G(h_0), K)$ as shown in Figure 6.3, where

$$
\Sigma(h_0) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A}(h_0) & \bar{B}_1(h_0) & \bar{B}_2(h_0) \\ \hline \bar{C}_1(h_0) & \bar{D}_{11}(h_0) & \bar{D}_{12}(h_0) \\ \bar{C}_2(h_o) & \bar{D}_{21}(h_0) & \bar{D}_{22}(h_0) \end{array}\right]
$$

$$
\begin{aligned}
\bar{A}(h_0) &= \begin{bmatrix} A(h_0) & B_2(h_0)C_K \\ B_K C_2 & A_K \end{bmatrix} \\
\bar{B}_1(h_0) &= \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \bar{B}_2(h_0) = \begin{bmatrix} B_1(h_0) \\ B_K D_{21} \end{bmatrix} \\
\bar{C}_1(h_0) &= \begin{bmatrix} C_{1\Delta}(h_0) & 0 \end{bmatrix}, \quad \bar{D}_{11}(h_0) = 0, \quad \bar{D}_{12}(h_0) = D_{11\Delta}(h_0) \\
\bar{C}_2(h_0) &= \begin{bmatrix} C_1 & D_{12}C_K \end{bmatrix}, \quad \bar{D}_{21}(h_0) = 0, \quad \bar{D}_{22}(h_0) = D_{11}.
\end{aligned}
$$

The robust stability of $\Sigma(h_0)$ can then be verified as a simple application of the small-gain theorem

$$
\alpha \|\Delta\| < 1 \tag{6.7}
$$

where $\alpha \geq \|\Sigma_{11}(h_0)\|_\infty$. Therefore, stability robustness of $\Sigma(h_0)$ can be verified by bounding $\Delta(\theta_k)$. Many different bounds can be defined for $\Delta(\theta_k)$, one such bound is given by the following lemma.

Lemma 6.2. [40] *For a given matrix $A_c \in \mathbf{R}^{n\times n}$ and $t \geq 0$, we have*

$$\|e^{A_c t}\| \leq e^{\mu(A_c)t},$$

where $\mu(A_c)$ is the logarithmic norm of A_c associated with the 2-norm, and is given by

$$\mu(A_c) = \lambda_{\max}\left(\frac{A_c + A_c^*}{2}\right).$$

For robust performance, in addition to (6.7) it is required that $\|\Sigma(h_0)\|_\infty$ be minimized. Therefore, the robust stability and performance analysis problem can be solved using the scaled-small gain condition [61, 87]:

$$\inf_{D_s} \|D_s \Sigma(h_0) D_s^{-1}\| \tag{6.8}$$

where $D_s \in \mathbf{R}^{(n+p1)\times(n+m_1)}$ is a scaling matrix with structure

$$D_s = \begin{bmatrix} \beta I & \\ & I \end{bmatrix}$$

The robust stability and performance can be verified by invoking the following theorem.

Theorem 6.1. *Given $h_i > 0$ $(i = 1, 2, \cdots, N)$, $\Phi(h_i)$, $\Gamma(h_i)$, $\Psi(h_i)$ and $\Upsilon(h_i)$, if there exists a symmetric matrix $P > 0$ and δ_i $(i = 1, 2, \cdots, N)$ satisfying N matrix inequalities*

$$\begin{bmatrix} P & \Phi(h_i)P & \Gamma(h_i) & 0 \\ * & P & 0 & P\Psi(h_i)' \\ * & * & I & \Upsilon(h_i) \\ * & * & * & \delta_i I \end{bmatrix} > 0, \tag{6.9}$$

then the closed-loop sampled-data systems is robustly stable with $\mathcal{H}_\infty$ performance $\sqrt{\gamma}$ for all $h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$ where $\delta_i I = \mathrm{diag}(\alpha_i I,\ \gamma I)$ and the matrices $\Phi(h_i)$, $\Gamma(h_i)$, $\Psi(h_i)$ and $\Upsilon(h_i)$ are defined as

$$\Phi(h_i) = \bar{A}(h_i), \qquad \Gamma(h_i) = \begin{bmatrix} \bar{B}_1(h_i) & \bar{B}_2(h_i) \end{bmatrix}$$
$$\Psi(h_i) = \begin{bmatrix} \bar{C}_1(h_i) \\ \bar{C}_2(h_i) \end{bmatrix}, \qquad \Upsilon(h_i) = \begin{bmatrix} \bar{D}_{11}(h_i) & \bar{D}_{12}(h_i) \\ \bar{D}_{21}(h_i) & \bar{D}_{22}(h_i) \end{bmatrix}$$

and the intervals $\mathcal{H}(h_i,\sqrt{\alpha_i})$ *are defined as*

$$\mathcal{H}(h_i,\sqrt{\alpha_i}) := (\underline{h},\overline{h}) \cap (0,\infty), \tag{6.10}$$

where

$$\begin{aligned}
&L1) \text{ if } \mu(-A_c)=0, \quad \underline{h}=h_i-(\sqrt{\alpha_i})^{-1},\\
&L2) \text{ elseif } \mu(-A_c)\le -\sqrt{\alpha_i}, \quad \underline{h}=-\infty,\\
&L3) \text{ else } \underline{h}=h_i-\frac{1}{\mu(-A_c)}\log(1+(\sqrt{\alpha_i})^{-1}\mu(-A_c)).\\
&U1) \text{ if } \mu(A_c)=0, \quad \overline{h}=h_i+(\sqrt{\alpha_i})^{-1},\\
&U2) \text{ elseif } \mu(A_c)\le -\sqrt{\alpha_i}, \quad \overline{h}=\infty\\
&U3) \text{ else } \overline{h}=h_i+\frac{1}{\mu(A_c)}\log(1+(\sqrt{\alpha_i})^{-1}\mu(-A_c)).
\end{aligned}$$

Proof. If the inequality in (6.9) is satisfied, then

$$P^T\Phi(h_i)P-P<0$$

holds for all $h_i \in \mathcal{G}$. Since all subsystems $\Sigma_{ij},\ i,j=1,2$ share the same Φ matrix, they will be stable.

The disturbance attenuation factor δ_i is re-defined to allow trade-off between the range of variations in the sampling period and the attenuation factor for w_k.

The definition of intervals $\mathcal{H}(h_i,\sqrt{\alpha_i})$ follows similar steps as in [20] and is given below. We show that (6.7) holds for all $h_k \in \begin{bmatrix} h_l & h_u \end{bmatrix}$. We only prove that (6.7) holds for all $h_k \in \begin{bmatrix} h_0 & h_u \end{bmatrix}$, i.e. for $U1$, $U2$ and $U3$ in (6.10); the proof for $L1$, $L2$ and $L3$ for $h_k \in \begin{bmatrix} h_l & h_0 \end{bmatrix}$ can be similarly derived.

From Lemma 6.2, we have

$$\|\Delta(\theta_k)\| \le \int_0^{\theta_k} \|e^{A_c\eta}\| \mathrm{d}\eta \le \int_0^{\theta_k} e^{\mu(A_c)\eta}\mathrm{d}\eta$$

when $\theta_k \ge 0$.

If $\mu(A_c)=0$, then

$$\|\Delta(\theta_k)\| \le \theta_k,$$

and (6.7) will hold as long as

$$\alpha\theta_k < 1,$$

which is the case of $U1$.

Now, if $\mu(A_c) \neq 0$, then

$$\|\Delta(\theta_k)\| = \frac{e^{\mu(A_c)\theta_k} - 1}{\mu(A_c)}.$$

Now, if $\mu(A_c) < 0$, the right hand side in above equation goes to $\frac{-1}{\mu(A_c)}$ when θ_k goes to ∞. The condition in (6.7) will hold if

$$\frac{-\alpha}{\mu(A_c)} \leq 1,$$

which is the case of $U2$.

Now, consider the case of $\mu(A_c) \neq 0$ and $\frac{-\alpha}{\mu(A_c)} > 1$. In this case, the condition in (6.7) will hold for all $\theta_k > 0$ if

$$\alpha \frac{e^{\mu(A_c)\theta_k} - 1}{\mu(A_c)} \leq 1.$$

Since $1 + \alpha^{-1}\mu(A_c) > 0$ here, we get two cases:

Case A: If $\mu(A_c) > 0$, then

$$\mu(A_c)\theta_k \leq \log(1 + \alpha^{-1}\mu(A_c)).$$

Case B: If $\mu(A_c) < 0$, then

$$\mu(A_c)\theta_k \geq \log(1 + \alpha^{-1}\mu(A_c)).$$

Therefore, we get

$$\theta_k \geq \frac{1}{\mu(A_c)} \log(1 + \alpha^{-1}\mu(A_c)),$$

which is the case of $U3$. This completes the proof. □

6.4 Robust $\mathcal{H}_\infty$ Controller Design

This section is devoted to the robust controller design. Since the analysis conditions involve an application of the scaled small-gain theorem, robust controller design is based on a D-K type iterative procedure. Also, since the above theorem involves a multi-modal approach, some nonlinear product terms need to be absorbed in order to find a constant-parameter controller. The following two properties could be useful [28].

Property 3. For square matrices G and symmetric $R > 0$ of compatible dimensions, the following inequality holds:

$$G^{'}R^{-1}G \geq G + G^{'} - R \tag{6.11}$$

Proof. The proof follows from the fact that

$$(G - R)'R^{-1}(G - R) \geq 0.$$

□

Property 4. For matrices H, G non-singular, symmetric Q and symmetric $R > 0$ of compatible dimensions such that the following LMI

$$\begin{bmatrix} H + H^{'} - Q & G^{'} \\ G & R \end{bmatrix} > 0 \tag{6.12}$$

holds, then $Q < H^{'}G^{-1}RG^{'^{-1}}H$.

Proof. Applying Schur complement to the lower diagonal yields $Q < H + H^{'} - G^{'}R^{-1}G$. From (6.11), $H^{'}G^{-1}RG^{'^{-1}}H \geq H + H^{'} - G^{'}R^{-1}G$, thus the result follows.

□

To convert the condition in (6.9) into synthesis LMI, partition the matrices P and P^{-1} as

$$P = \begin{bmatrix} X & U \\ U^{'} & \hat{X} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & V \\ V^{'} & \hat{V} \end{bmatrix}. \tag{6.13}$$

where X, $\hat{X}$, Y and $\hat{Y}$ are symmetric and positive definite matrices. Defining

$$T = \begin{bmatrix} Y & I \\ V^{'} & 0 \end{bmatrix}$$

and performing congruence transformation on (6.9) with $\mathrm{diag}(T, T, I, I)$ gives

$$\begin{bmatrix} T^{'}PT & T^{'}\Phi(h_i)PT & T^{'}\Gamma(h_i) & 0 \\ * & T^{'}PT & 0 & T^{'}P\Psi(h_i)^{'} \\ * & * & I & \Upsilon(h_i) \\ * & * & * & \delta_i I \end{bmatrix} > 0$$

where

$$
\begin{aligned}
T^{'}PT &= \begin{bmatrix} Y & I \\ I & X \end{bmatrix} \\
T^{'}\Gamma &= \begin{bmatrix} Y & YB_1(h_i) + FD_{21} \\ I & B_1(h_i) \end{bmatrix} \\
T^{'}\Phi(h_i)PT &= \begin{bmatrix} YA(h_i) + FC_2 & YL_i + FC_2X \\ & +VA_KU^{'} \\ A(h_i) & L_i \end{bmatrix} \\
T^{'}P\Psi^{'}(h_i) &= \begin{bmatrix} C_{1\Delta}^{'}(h_i) & C_1^{'}(h_i) \\ XC_{1\Delta}^{'}(h_i) + L^{'}D_{12\Delta}^{'}(h_i) & XC_1^{'}(h_i) + L^{'}D_{12}^{'}(h_i) \end{bmatrix} \\
\Upsilon(h_i) &= \begin{bmatrix} 0 & D_{11\Delta}(h_i) \\ 0 & D_{11} \end{bmatrix} \\
F &= VB_K, \quad L = C_KU^{'} \\
L_i &= A(h_i)X + B_2(h_i)L
\end{aligned}
$$

To recover the controller parameter A_K, the term YL_i in $T^{'}\Phi(h_i)PT$ can be absorbed using the property in (6.12). Now, we give the main theorem for the synthesis of dynamic controllers for nonuniformly sampled systems.

Theorem 6.2. *Given $h_i > 0$ $(i = 1, 2, \cdots, N)$, if there exist symmetric matrices $X > 0$, $Y > 0$, $Q > 0$, $R > 0$, matrices F, L, M and scalars γ_i and μ_i such that the following $N + 1$ matrix inequalities*

$$
\begin{bmatrix} Q & I \\ * & R \end{bmatrix} > 0 \tag{6.14}
$$

$$
\begin{bmatrix}
Y & I & YA(h_i)+FC_2 & M & Y & YB_1(h_i)+FD_{21} & 0 & 0 & 0 & Y \\
* & X & A(h_i) & L_i & I & B_1(h_i) & 0 & 0 & 0 & 0 \\
* & * & Y & I & 0 & 0 & C_{1\Delta}^{'}(h_i) & C_1(h_i)^{'} & 0 & 0 \\
* & * & * & X & 0 & 0 & XC_{1\Delta}^{'}(h_i)+L^{'}D_{12\Delta}^{'}(h_i) & XC_1(h_i)'+L^{'}D_{12}(h_i)^{'} & L_i^{'}+G^{'} & 0 \\
* & * & * & * & I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & I & D_{11\Delta}^{'}(h_i) & D_{11}^{'} & 0 & 0 \\
* & * & * & * & * & * & \alpha_i I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \gamma I & 0 & 0 \\
* & * & * & * & * & * & * & * & Q & 0 \\
* & * & * & * & * & * & * & * & * & R
\end{bmatrix} > 0, \tag{6.15}
$$

hold, then the closed-loop system in (6.5) *will be internally stable with* $\mathcal{H}_\infty$ *performance* $\sqrt{\gamma}$ *with controller parameters*

$$
\begin{aligned}
A_K &= V^{-1}(M+YG-FC_2X)(U^{'})^{-1} \\
B_K &= V^{-1}F \\
C_K &= L(U^{'})^{-1}
\end{aligned} \tag{6.16}
$$

for all $h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \sqrt{\alpha_i})$ *where the intervals* $\mathcal{H}(h_i, \sqrt{\alpha_i})$ *are defined in* (6.10).

Remark 6.1. The matrices U and V do not appear in the LMI conditions in (6.14) and (6.15). One of them can be chosen freely to satisfy $VU^{'} = I - XY$. For example, choosing $V = V^{'} = Y$ gives $U^{'} = Y^{-1} - X$.

Remark 6.2. Even though Theorem 6.2 is developed using the linear fractional modelling of the closed-loop system, it can be used for $\mathcal{H}_\infty$ controller design for other discrete-time models, such as the polytopic models [31].

Proof. Note that (6.14) together with (6.15) ensures that $R > Q^{-1} > 0$, this means

(6.15) holds if R is replaced by Q^{-1}. Also for each i, it is verified that

$$\begin{bmatrix} Q \\ 0 \\ 0 \\ (L_i+G)' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} Q & 0 & 0 & L_i+G & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0 \tag{6.17}$$

Replacing R by Q^{-1}, performing Schur complement w.r.t the last two rows and columns of (6.15) and adding (6.17) after pre- and post-multiplying first row and column by Y, we get

$$\begin{bmatrix} Y & I & YA(h_i)+FC_2 & M+YL_i & Y & YB_1(h_i)+FD_{21} & 0 & 0 \\ * & X & A(h_i) & L_i & I & B_1(h_i) & 0 & 0 \\ * & * & Y & I & 0 & 0 & C_{1\Delta}'(h_i) & C_1(h_i)' \\ * & * & * & X & 0 & 0 & XC_{1\Delta}'(h_i)+L'D_{12\Delta}'(h_i) & XC_1(h_i)'+L'D_{12}(h_i)' \\ * & * & * & * & I & 0 & 0 & 0 \\ * & * & * & * & * & I & D_{11\Delta}'(h_i) & D_{11}' \\ * & * & * & * & * & * & \alpha_i I & 0 \\ * & * & * & * & * & * & * & \gamma I \end{bmatrix} > 0,$$

which together with the transformations (6.13) and (6.16) yield

$$\begin{bmatrix} T'PT & T'\Phi(h_i)PT & T'\Gamma(h_i) & 0 \\ * & T'PT & 0 & T'P\Psi(h_i)' \\ * & * & I & \Upsilon(h_i) \\ * & * & * & \delta_i I \end{bmatrix} > 0$$

which is equivalent to (6.9). $\square$

A difficulty in applying Theorem 6.2 is how to select the sampling periods in the grid such that if (6.14) and (6.15) hold for elements in the grid will imply that they hold for all $h_k \in \begin{bmatrix} h_l & h_u \end{bmatrix}$. In the following, we give a procedure to generate such a grid for the robust $\mathcal{H}_\infty$ controller design.

Procedure 6.1. Robust $\mathcal{H}_\infty$ Controller Design for Nonuniformly Sampled Systems

---

Given $0 < h_l < h_u < \infty$ and a large positive integer N_0

0. Initialization: $\mathcal{G} \leftarrow \{(h_l + h_u)/2\}$

1. If $\#\mathcal{G} \geq N_0$, stop without designing a controller. Here $\#\mathcal{G}$ denotes the number of elements in the grid $\mathcal{G}$.

2. Minimize

$$\sum_{i=1}^{\#\mathcal{G}} a_1\alpha_i + a_2\gamma$$

subject to (6.14) and (6.15) for all $h_i's$ where h_i is the i^{th} smallest element in $\mathcal{G}$ and a_1 and a_2 are the weighting factors.

3. If

$$[h_l, h_u] \subseteq \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \mu_i),$$

the sampled-data closed-loop system will be internally stabilized by controller parameterin in (6.16) with $\mathcal{H}_\infty$ performance $\sqrt{\gamma}$. Stop. Here

$$\mu_i := \lambda_{\max}(R_i - S_i'(Q_i - P)^{-1}S_i) + \epsilon$$

where ϵ is a small positive number and

$$P = \begin{bmatrix} Y & V \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & U \end{bmatrix}^{-1},$$

$$\begin{bmatrix} Q_i & S_i \\ S_i' & R_i \end{bmatrix} = \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi(h_i) & \Gamma \\ \Psi(h_i) & 0 \end{bmatrix}'$$

4. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \bigcup \{(L_j + U_j)/2\}$$

for all j where L_j and U_j are determined so that

$$\bigcup_{j=1}^{M}(L_j, U_j) = (h_l, h_u)\bigcup_{i=1}^{\#\mathcal{G}}\mathcal{H}(h_i, \sqrt{\mu_i}),$$
$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M$$

are satisfied, where $M \leq \#\mathcal{G} + 1$. Go to step **1**.

5. Search for the scaling parameter β to minimize γ.

Remark 6.3. The number N_0 is introduced to avoid numerical difficulties which may happen if sampling periods are too small.

6.5 Numerical Example

In this section, we give a numerical example to compare the proposed approach with the delay systems approach. The design was carried out on a computer with Mac OS X (10.7.3), Intel Core 2 Duo 2.4 GHz CPU, MATLAB 7.13 and Yalmip [41]. The following parameters for the plant in (6.1) were considered:

$$A_c = \begin{bmatrix} 0 & 1 \\ -16 & 4.8 \end{bmatrix}, \qquad B_{1c} = B_{2c} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_{11=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_{21} = 0.1$$

This plant was considered in [72] where three different continuous-time controllers with sampled input were designed using the delay systems approach. Their type 2 controller uses a zero-order hold mechanism, similar to our controller structure achieves an $\mathcal{H}_\infty$ performance level $\gamma = 339.8$. It is important to note that their controller is time-varying when implemented in discrete time.

Using the proposed control design procedure (Procedure 6.1) with $[h_l,\ h_u] = [0.1,\ \pi/25]$, we can find an $\mathcal{H}_\infty$ robust controller with parameters

$$A_K = \begin{bmatrix} -1.2088 & 0.0456 \\ -14.8297 & -0.1877 \end{bmatrix}, B_K = \begin{bmatrix} -1.9923 \\ -10.8497 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 0.7193 & 0.8117 \end{bmatrix},$$

that can achieve an $\mathcal{H}_\infty$ performance level $\gamma = 308.7116$. The search took 7.098 s with $\#\mathcal{G} = 5$, $\beta = 37$, $a_1 = 0.1$ and $a_2 = 0.9$.

We can not say that the proposed controller is better than the one in [72] because we consider discrete-time $\mathcal{H}_\infty$ performance and the authors in [72] consider continuous-time $\mathcal{H}_\infty$ performance. However, it is obvious that being constant-parameter, the proposed controller is easier to implement.

6.6 Conclusions

This chapter developed a numerical procedure for the robust $\mathcal{H}_\infty$ controller design for nonuniformly sampled systems. Because of the variations in sampling period, the problem converts to robust control design. A DK type iterative procedure is developed to design a controller that guarantees internal stability and $\mathcal{H}_\infty$ performance of the closed-loop systems for all possible variations of sampling period. A comparison with the existing result is also given to demonstrate effectiveness of the proposed approach.

Chapter 7

Summary and Future Work

Because of the advantages offered by computers, most modern control systems are constructed using digital computers, micro-controllers or microprocessors. The conventional sampled-data control theory assumes that measurement sampling and control updating occur at uniform sampling intervals which is a reasonable assumption for many control applications. However, with the widespread use of networked and embedded control systems, many control loops are closed through a shared communication medium where constant sampling periods are not possible. If not taken into account during the design of control systems, variations in sampling period can degrade the performance and can even lead to instability of the control system.

Motivated by this, this thesis explores analysis and synthesis methods for nonuniformly sampled systems. Two main approaches have been proposed in the literature to address the analysis and synthesis of nonuniformly sampled systems. One is the delay systems approach where nonuniformly sampled systems are modelled as continuous-time one with time-varying delay in input. The analysis and synthesis methods developed for time-delay systems can then be used. The other is the discrete-time approach where nonuniformly sampled systems are modelled as uncertain discrete-time system. The discrete-time methods developed for nonuniformly sampled systems include the linear fractional transformation approach, a polytopic modelling approach and a robust linear matrix inequalities approach. Numerical experiments in different references show that the discrete-time approaches are, in general, less conservative than the delay systems approach.

This thesis follows a discrete-time linear fractional transformation approach for most theoretical development.

Sampling periods can vary in variety of different ways. To have something specific to discuss, two types of variations in sampling periods were considered. Firstly, it was assumed that the state estimation or control updating occurs at fast and uniform time intervals but the measurements are sampled at slow and nonuniform intervals. We called it Timing Model 1. Secondly, it was assumed that both sampling and hold devices operate at nonuniform sampling intervals but are synchronized and called Timing Model 2. To make Timing Model 2 more realistic, sampling periods were assumed to be bounded by a lower and an upper bound. These timing models were described in detail in **Chapter 2**.

In **Chapter 3**, a sampling period dependent, full-order $\mathcal{H}_\infty$ filter was designed with the assumption of first type of variations in the sampling period. The process of nonuniform measurement sampling was modelled using a Markov chain. The estimation error system together with the Markov chain was modelled as a Markovian jump system and filter design was formulated as a convex optimization problem. The effectiveness of the proposed filter over a time-varying $\mathcal{H}_\infty$ filter designed using the Riccati equation approach was demonstrated through simulation results. An advantage of the proposed filter is it can be designed offline and switching among different modes can be implemented at fast sampling period.

Chapter 4 developed a robust discrete-time $\mathcal{H}_\infty$ filtering with second type of variations in sampling interval. A constant-parameter structure was considered which is advantageous from implementation point of view. The error system was reconfigured as an uncertain discrete-time system with a linear fractional transformation uncertainty. A concrete algorithm was developed for the filter design. The algorithm can provide a filter in a finite number of iterations. A comparison with filters developed using the polytopic and delay systems approaches showed that the proposed filter is more effective in attenuating disturbance input.

Chapters 5 and **6** considered the control problem. Extending the existing discrete-time approaches, Chapter 5 proposed a method to design a dynamic output feedback controller. The designed controller has constant parameters and is easier to implement; it guarantees the exponential stability of the sampled-data systems under Timing Model 2. Chapter 6 designed an internally stabilizing dynamic controller that also minimized the discrete-time $\mathcal{H}_\infty$ norm of the closed-loop system. The problem was treated as robust control design problem and a DK-type iterative procedure was developed. Both the stabilization and $\mathcal{H}_\infty$ control methods employed a grid of sampling periods for controller design. Getting LMI conditions with the Lyapunov matrix shared among different instances of the plant was not a trivial task. A linearization scheme was used to formulate LMI design conditions.

All the mathematical methods in this thesis are developed in discrete time, and design problems are formulated as LMI problems. Many efficient solvers exist that can solve LMI problems of reasonable size in a finite time. A major advantage of the designed filters and controllers is that they can be easily implemented. It is, however, remarked that discrete-time approaches are not free from conservatism. Discrete-time approaches ignore the inter-sample behaviour which may not be desirable for many real-time control systems.

7.1 Major Thesis Contributions

This thesis is concerned with the analysis and synthesis of nonuniformly sampled systems and the main contributions of this thesis are summarized below:

- **Chapter 3** presented a novel Markov model for nonuniform sampling and developed a method for a mode-dependent $\mathcal{H}_\infty$ filtering design.

- **Chapter 4** presented a novel robust discrete-time filter using the linear fractional transformation approach.

- **Chapter 5** presented for the first time a convex optimization procedure to design a discrete-time dynamic output feedback controller that exponential stabilize the sampled-data closed-loop system for all possible variations of sampling period.

- **Chapter 6** extended the stabilizing control method to minimize the discrete-time $\mathcal{H}_\infty$ norm of the closed-loop system.

7.2 Directions for Future Work

There is still scope for further exploration. The following are some suggested areas that could be pursued in future research:

- **Switched Lyapunov Function**

  Most of the analysis results in this thesis assume quadratic Lyapunov functions. In a recent study [25], it was shown that there is a gap between the exponential stability of the sampled-data system and the robust quadratic stability of the associated discrete-time system and exponential stability of the discrete-time system, instead of quadratic stability, was used to imply the exponential stability of the sampled-data system. The conservatism reduction was shown using a numerical example. The exponential stability of the discrete-time system can be verified using a switched Lyapunov function [11]. A difficulty with this approach is how to model the uncertain discrete-time system as a switched system. One way is to let the switching happens at the sampling periods in the grid; however, it dramatically increases the computational burden as the number of sampling periods in the grid can be large. In [24], a method is proposed to construct a bimodal switched Lyapunov function for the state-feedback stabilization problem. The extension to the general case of N-mode switching is still an open problem.

- **Integral Quadratic Constraints**

  Since discrete-time system associated with nonuniformly sampled system is time-varying and uncertain, most of the analysis results in this thesis employed a small-gain or scaled small-gain condition to conclude robust stability and performance of the closed-loop system. The conservativeness of the small-gain condition can be reduced by considering other conditions, such as, the integral quadratic constraints. Integral quadratic constraints can be used to formulate the analysis and design problems [45] .

- **Sampled-Data Approach to Controller Design**

  This thesis followed a discrete-time approach to the filter or control design. The filters or controllers are designed using a discretization of the continuous-time plant. For the conventional sampled-data control, the associated discrete-time system is time-invariant and the analysis and design are much easier. For the nonuniformly sampled systems, the associated discrete-time systems are time-varying and uncertain and a robust discrete-time approach can be used. A disadvantage of this approach is it completely ignores the inter-sample behaviour of the system.

The sampled-data approach considers the discrete-time controller design for the continuous-time plant. For the case of uniform sampling, the sampled-data system can be lifted to an infinite-dimensional functional space and a fictitious linear, time-invariant, discrete-time model can be used for the analysis and synthesis problems. For the nonuniform sampling case, we believe that an uncertain discrete-time model can be used by following the idea of nonuniform lifting [46]; however, the problem is much harder because the operators associated with nonuniformly sampled-data systems are time-varying.

Another method is to use the linear differential inclusion formalism as in [32] to develop design conditions using quasi-quadratic Lyapunov functions that can consider the inter-sample behaviour as well.

Bibliography

[1] P. Albertos and A. Crespo. Real-time control of non-uniformly sampled systems. *Control Eng. Pract.*, 7(4):445–68, 1999.

[2] P. Albertos and J. Salt. Digital regulators redesign with irregular sampling. In *11th IFAC Triennial World Congress*, pages 465–69, Tallin, Estonia, USSR, 1990.

[3] K. J. Åström and B. Wittenmark. *Computer-Controlled Systems*. Prentice Hall, third edition, 1997.

[4] J. Baillieul and P. J. Antsaklis. Control and communication challenges in networked real-time systems. *Proc. IEEE*, 95(1):9 –28, 2007.

[5] A. Balluchi, P. Murrieri, and A. L. Sangiovanni-Vincentelli. Controller synthesis on non-uniform and uncertain discrete-time domains. In *Hybrid Systems: Computation and Control*. New York: Springer, 2005.

[6] R. A. Borges, R. C. L. F. Oliveira, C. T. Abdallah, and P. L. D. Peres. $\mathcal{H}_\infty$ filtering of networked systems with time-varying sampling rates. *Proc. ACC*, pages 3372–3377, 2009.

[7] E. K. Boukas. *Stochastic switching systems: Analysis and design*. Birkhäuser, Berlin, 2005.

[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[9] T. Chen and B. Francis. *Optimal Sampled-Data Control Systems*. Springer, London, 1995.

[10] O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markov Jump Linear Systems*. Springer, 2005.

[11] J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Trans. Autom. Control*, 47(11):1883–1887, 2002.

[12] W. L. de Koning, L. G. van Willigenburg, B. Agrotechnion, and E. mail Gerard. Randomized digital optimal control. In Farokh Marvasti, editor, *Nonuniform Sampling Theory and Practice*, Information Technology: Transmission, Processing, and Storage, chapter 12, pages 519–542. Springer, 2001.

[13] C. E. de Souza and M. D. Fragoso. $\mathcal{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters. *Proc. IEEE CDC*, 3:2181–2186, 1997.

[14] C. E. de Souza, A. Trofino, and K. A. Barbosa. Mode-independent $\mathcal{H}_\infty$ filters for Markovian jump linear systems. *IEEE Trans. Autom. Control*, 51(11):1837–1841, 2006.

[15] G. F. Franklin, J. D. Powell, and M. L. Workman. *Digital Control of Dynamic Systems*. Ellis-Kagle Press, 1200 Pilarcitos Ave, Half Moon Bay, CA 94019, third edition, 1997.

[16] E. Fridman. A refined input delay approach to sampled-data control. *Automatica*, 46(2):421–427, 2010.

[17] E. Fridman, A. Seuret, and J.-P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441 – 1446, 2004.

[18] E. Fridman, U. Shaked, and V. Suplin. Input/output delay approach to robust sampled-data $\mathcal{H}_\infty$ control. *Systems & Control Letters*, 54(3):271–282, 2005.

[19] H. Fujioka. Stability analysis for a class of networked/embedded control systems: A discrete-time approach. In *Proc. IEEE ACC*, pages 4997–5002, 2008.

[20] H. Fujioka. Stability analysis for a class of networked/embedded control systems: output feedback case. In *17th IFAC World Congress*, pages 4210–4215, 2008.

[21] H. Fujioka. A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices. *IEEE Trans. Autom. Control*, 54(10):2440–2445, 2009.

[22] H. Fujioka. Stability analysis of systems with aperiodic sample-and-hold devices. *Automatica*, 45(3):771–775, 2009.

[23] H. Fujioka and T. Nakai. Stabilizing systems with aperiodic sampling-and-hold devices: state feedback case. *IET Control Theory Appl.*, 4:262–272, 2010.

[24] H. Fujioka and T. Nakai. Constructing a bimodal switched Lyapunov function for non-uniformly sampled-data feedback systems. In *Proc. IEEE ACC*, pages 2228–2233, 2011.

[25] H. Fujioka, T. Nakai, and L. Hetel. A switched Lyapunov function approach to stability analysis of non-uniformly sampled-data systems. In *Proc. IEEE ACC*, pages 1803–1804, 2010.

[26] H. Fujioka and Y. Oishi. A switched lyapunov function approach to stability analysis of non-uniformly sampled-data systems with robust LMI techniques. In *Proc. 8th IEEE Asian Control Conference*, pages 1487–1491, 2011.

[27] J. C. Geromel, J. Bernussou, G. Garcia, and M.C. De Oliveira. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ robust filtering for discrete-time linear systems. *SIAM J. Control Optim.*, 38(5):1353–1368, 2000.

[28] J. C. Geromel, R. H. Korogui, and J. Bernussou. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ robust output feedback control for continuous time polytopic systems. *IET Control Theory Appl.*, 1(5):1541–1549, 2007.

[29] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. Van Den Bosch. Analysis of event-driven controllers for linear systems. *Int. J. Control*, 81(4):571–590, 2008.

[30] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proc. IEEE*, 95(1):138–162, 2007.

[31] L. Hetel, J. Daafouz, and C. Iung. Analysis and control of LTI and switched systems in digital loops via an event-based modelling. *Int. J. Control*, 81(7):1125 – 1138, 2008.

[32] L. Hetel, A. Kruszewski, W. Perruquetti, and J. Richard. Discrete and intersample analysis of systems with aperiodic sampling. *IEEE Trans. Autom. Control*, 56(7):1696–1701, 2011.

[33] D. Hristu-Varsakelis and W. S. Levine (Editors). *Handbook of Networked and Embedded Control Systems*. Birkhäuser, 2005.

[34] R. E. Kalman. *Analysis and Synthesis of Linear Dynamical Systems Operating on Randomly Sampled Data*. PhD thesis, Columbia University, New York, NY, 1957.

[35] R. E. Kalman. A new approach to linear filtering and prediction problems. *ASME Trans. - Journal basic Eng.*, 82(1):35–45, 1960.

[36] R. E. Kalman and J. E. Bertram. A unified approach to the theory of sampling systems. *J. Franklin Inst.*, 267(5):405 – 436, 1959.

[37] M. Krucinski, C. Cloet, M. Tomizuka, and R. Horowitz. Asynchronous observer for a copier paper path. *Proc. IEEE CDC*, 3:2611–2612, 1998.

[38] H. Kushner and L. Tobias. On the stability of randomly sampled systems. *IEEE Trans. Autom. Control*, 14(4):319 – 324, 1969.

[39] F. Liu, Y. Yao, F. He, and S. Chen. Stability analysis of networked control systems with time-varying sampling periods. *J. Control Theory Appl.*, 6:22–25, 2008.

[40] C. V. Loan. The sensitivity of the matrix exponential. *SIAM J. Numer. Anal.*, 14:971–981, 1977.

[41] J. Löfberg. Yalmip: A toolbox for modeling and optimization in MATLAB. In *Proc. Computer-Aided Control System Design*, 2004.

[42] D. G. Luenberger. Observing the state of a linear system. *IEEE Trans. Military Electron.*, 8(2):74 –80, 1964.

[43] D. G. Luenberger. Observers for multivariable systems. *IEEE Trans. Autom. Control*, 11(2):190 – 197, 1966.

[44] D. G. Luenberger. An introduction to observers. *IEEE Trans. Autom. Control*, 16(6):596 – 602, 1971.

[45] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Autom. Control*, 42(6):819 –830, 1997.

[46] L. Mirkin. Some remarks on the use of time-varying delay to model sample-and-hold circuits. *IEEE Trans. Autom. Control*, 52(6):1109–1112, 2007.

[47] G. Mustafa and T. Chen. H_∞ filtering for nonuniformly sampled systems. In *23rd Canadian Conference on Electrical and Computer Engineering (CCECE)*, pages 1–4. IEEE, 2010.

[48] G. Mustafa and T. Chen. H_∞ filtering for nonuniformly sampled systems: A markovian jump systems approach. *Systems & Control Letters*, 60(10):871 – 876, 2011.

[49] G. Mustafa and T. Chen. Stabilization of non-uniformly sampled-data systems via dynamic output feedback control. *Submitted for publication in IET Control Theory Appl.*, 2012.

[50] G. Mustafa, T. Chen, and H. Fujioka. Robust H_∞ filtering for nonuniformly sampled systems. In *Proc. 50th IEEE CDC and European Control Conf.*, pages 1269 –1273, 2011.

[51] G. Mustafa, T. Chen, and H. Fujioka. Robust H_∞ filter design for nonuniformly sampled systems. In Li Qiu, Jie Chen, Tetsuya Iwasaki, and Hisaya Fujioka, editors, *Developments in Control Theory Towards Glocal Control*, volume 76 of *Control Engineering*, chapter 3, pages 17–26. Inst of Engineering & Technology, 2012.

[52] P. Naghshtabrizi and J.P. Hespanha. Designing an observer-based controller for a network control system. In *Proc. 44th IEEE CDC and European Control Conf.*, pages 848 – 853, 2005.

[53] N. Nahi. Optimal recursive estimation with uncertain observation. *IEEE Trans. Inf. Theory*, 15(4):457 – 462, 1969.

[54] Y. Oishi and H. Fujioka. Stability and stabilization of aperiodic sampled-data control systems: An approach using robust linear matrix inequalities. In *Proc. CDC/CCC*, pages 8142–8147. IEEE, 2009.

[55] Y. Oishi and H. Fujioka. Stability and stabilization of aperiodic sampled-data control systems using robust linear matrix inequalities. *Automatica*, 46(8):1327 – 1333, 2010.

[56] R. M. Palhares and P. L. D. Peres. Optimal filtering schemes for linear discrete-time systems - an LMI approach. In *Int. Symposium Industrial Electronics*, volume 3, pages 1120–1125, 1997.

[57] A. M. Phillips and M. Tomizuka. Multirate estimation and control under time-varying data sampling with applications to information storage devices. *Proc. ACC*, 6:4151–4155, 1995.

[58] M. F. Sagfors and H. T. Toivonen. $\mathcal{H}_\infty$ and LQG control of asynchronous sampled-data systems. *Automatica*, 33(9):1663–68, 1997.

[59] M. Sahebsara, T. Chen, and S.L. Shah. Optimal filtering with random sensor delay, multiple packet dropout and uncertain observations. *Int. J. Control*, 80(2):292–301, 2007.

[60] A. Sala. Computer control under time-varying sampling period: An LMI gridding approach. *Automatica*, 41(12):2077 – 2082, 2005.

[61] C. Scherer. Theory of Robust Control. Delft University of Technology, The Netherlands, 2001.

[62] M. Schinkel and W.-H. Chen. Control of sampled-data systems with variable sampling rate. *Int. J. Syst. Sci.*, 37(9):609–618, 2006.

[63] P. Seiler and R. Sengupta. A bounded real lemma for jump systems. *IEEE Trans. Autom. Control*, 48(9):1651 – 1654, 2003.

[64] U. Shaked and Y. Theodor. $\mathcal{H}_\infty$-optimal estimation: a tutorial. In *Proc. 31st IEEE CDC*, pages 2278 –2286 vol.2, 1992.

[65] X. Shen and L. Deng. Game theory approach to discrete $\mathcal{H}_\infty$ filter design. *IEEE Trans. Signal Process.*, 45:1092–1095, 1997.

[66] D. Simon. *Optimal State Estimation: Kalman, $\mathcal{H}_\infty$, and Nonlinear Approaches.* Wiley-Interscience, 2006.

[67] J. Skaf and S. Boyd. Analysis and synthesis of state-feedback controllers with timing jitter. *IEEE Trans. Autom. Control*, 54(3):652 –657, March 2009.

[68] H. Song, L. Yu, and W.-A. $\mathcal{H}_\infty$ filtering of network-based systems with random delay. *Signal Process.*, 89(4):615–622, 2009.

[69] Y. S. Suh. Stability and stabilization of nonuniform sampling systems. *Automatica*, 44(12):3222–3226, 2008.

[70] Y. S. Suh. Stability analysis of time-varying sampling systems with scheduling dependent probabilistic bound information. *Proc. ICCAS-SICE*, pages 2196–2199, 2009.

[71] Y. S. Suh. Stability and stabilization of nonuniform sampling systems using a matrix bound of a matrix exponential. In *Proc. 5th Int. Conf. Emerg. Intelligent Comput. Tech. Appl.*, pages 1059–1066. Springer-Verlag, 2009.

[72] V. Suplin, E. Fridman, and U. Shaked. Sampled-data $\mathcal{H}_\infty$ control and filtering: Nonuniform uncertain sampling. *Automatica*, 43(6):1072–1083, 2007.

[73] M. C. Turner, G. Herrmann, and I. Postlethwaite. An introduction to linear matrix inequalities in control. Technical Report 02-04, Control and Instrumentation Research Group, Department of Engineering, University of Leicester, Leicester, LE1 7RH, U.K., 2004.

[74] G. C. Walsh, H. Ye, and L. Bushnell. Stability analysis of networked control systems. *IEEE Trans. Contr. Syst. Technol.*, 10(3):438–446, 2002.

[75] G.C. Walsh, Hong Ye, and L. Bushnell. Stability analysis of networked control systems. In *Proc. ACC*, volume 4, pages 2876 –2880 vol.4, 1999.

[76] G. Welch and G. Bishop. An introduction to the Kalman filter. *University of North Carolina at Chapel Hill, Chapel Hill, NC*, 7(1), 1995.

[77] B. Wittenmark, J. Nilsson, and M. Törngren. Timing problems in real-time control systems. In *Proc. ACC*, 1995.

[78] L. Xiao, A. Hassibi, and J.P. How. Control with random communication delays via a discrete-time jump system approach. In *Proc. IEEE ACC*, volume 3, pages 2199 –2204 vol.3, 2000.

[79] L. Xie, L. Lu, D. Zhang, and H. Zhang. Improved robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering for uncertain discrete-time systems. *Automatica*, 40:873—880, 2004.

[80] S. Xu, T. Chen, and J. Lam. Robust $\mathcal{H}_\infty$ filtering for uncertain markovian jump systems with mode-dependent time delays. *IEEE Trans. Autom. Control*, 48(5):900 – 907, 2003.

[81] J. Y. Yen, Y. L. Chen, and M. Tomizuka. Variable sampling rate controller design for brushless DC motor. *Proc. IEEE CDC*, 1:462–467, 2002.

[82] L. Zhang and E.-K. Boukas. Mode-dependent $\mathcal{H}_\infty$ filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities. *Automatica*, 45(6):1462–1467, 2009.

[83] L. Zhang, Y. Shi, T. Chen, and B. Huang. A new method for stabilization of networked control systems with random delays. *IEEE Trans. Autom. Control*, 50(8):1177–1181, 2005.

[84] W. Zhang. *Stability Analysis of Networked Control Systems*. PhD thesis, Case Western Reserve University, 10900 Euclid Ave., Cleveland, Ohio 44106, 2001.

[85] W. Zhang and M. Branicky. Stability of networked control systems with time-varying transmission period. In *Conf. Commun. Contr. Comput.*, 2001.

[86] W. Zhang, M. S. Branicky, and S. M. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21:84–99, 2001.

[87] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, 1996.