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THE UNIVERSITY OF ALBERTA

Generalized Newman-Unti-type Geometric Solutions to the Einstein Gravitational Field Equations for Local Asymptotically Flat Space-Times

by

Frank Kofi Nani

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

IN

THEORETICAL PHYSICS

Department Of Physics

EDMONTON, ALBERTA

Spring 1986

ISBN 0-315-30160-0

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR Frank Kofi Nani

TITLE OF THESIS Generalized Newman-Unti-type Geometric Solutions to the Einstein Gravitational Field Equations for Local Asymptotically Flat Space-Times

DEGREE FOR WHICH THESIS WAS PRESENTED MASTER OF SCIENCE

YEAR THIS DEGREE GRANTED Spring 1986

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DATED 17th OCT. 1985

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Generalized Newman-Unti-type Geometric Solutions to the Einstein Gravitational Field Equations for Local Asymptotically Flat Space-Times submitted by Frank Kofi Nani in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE in THEORETICAL PHYSICS.

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Supervisor

B. A. Campbell

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Co-supervisor

Date 17th Oct. 1935

DEDICATION

I wish to sincerely dedicate this thesis to the living memory of my beloved daddy:

Hieronymus Fidelis Nani

and also to the honour of my ever cherished mommy:

Mansah Akuyonor Nani

for their unlimited love and magnanimity.

ABSTRACT

Using the techniques of Penrose's conformal compactification of space-time, local differential geometric methods, and a few necessary and sufficient basic assumptions on the local asymptotic structure of space-time and its future null boundary $\mathcal{J}^+$, we present a generalized Newman-Unti-type asymptotic series solution for the Riemann tensor components, spin coefficients, metric variables, and the covariant and contravariant metric tensor components for the conformally rescaled space M and the physical space $\hat{M}$. To obtain these solutions, we solved a non-redundant subset of the Newman-Penrose equations [3] which is reformulated into a system of non-linear partial differential equations [9]. These equations are in principle equivalent to the Einstein gravitational field equations. The solutions are first obtained in the conformally rescaled space M which is diffeomorphic to the physical space $\hat{M}$. The manifolds M and $\hat{M}$ have the appropriate causal structure and are assumed to be non-compact, parallelizable and admit spinor structure in the sense of Geroch [13]. Using Penrose's conformal rescaling technique, the solutions in the unphysical space M are transcribed into the physical space $\hat{M}$ in terms of certain freely specifiable components of the Riemann tensor (which constitute the analytic initial data) and certain boundary values of the Newman-Penrose variables on the conformal boundary. The solutions apply to all local asymptotically flat space-times including an arbitrary,

isolated self-gravitating system in a co-ordinate system based on a geodesic, twist-free, expanding null congruence. The time evolution equations are clearly exhibited for the subset of the initial data which do not allow arbitrary time dependence. The curvature function $P(u,\zeta,\bar{\zeta})$ which determines the Riemannian curvature of the 2-surfaces of intersection of the $u=u_0$ (const) hypersurfaces and the local conformal boundary $\mathcal{J}^+$, is left completely arbitrary. In particular, the Bondi-type co-ordinate constraint $P_{,u}=0$ was not imposed. Invoking the theorems of Friedrich [10,11,12], the convergence of the NU-type asymptotic series can be plausibly established for initial data which are assumed analytic.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to Dr. Garry Ludwig for formulating the thesis problem, checking all the cumbersome details and correcting the original manuscript. I offer my profound thanks to Dr. Werner Israel for his encouraging remarks and inspiration. I also express my appreciation to Dr. Hube and Dr. Bruce Campbell and the members of the Oral Examination Committee for their cordiality and goodwill. Finally my special thanks go to Dr. J.A. Kernahan for his advice and support.

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Chapter 1

INTRODUCTION

In the framework of Einstein's theory of gravitation, and other metric theories of gravity, it is believed that a bounded source, e.g. a binary neutron star or a dust cloud, generates a curved spacetime which, at large distances from the source becomes more and more Minkowskian. The gravitational radiation emitted from such an isolated distribution of matter has been one of the most interesting physical phenomena to both experimentalists and theoreticians: The experimentalists hope to make direct measurements of changes in gravitational curvature due to propagation of disturbances emanating from the bounded sources; and at the same time, theoreticians search for closed form analytic solutions to the field equations which will represent a dynamic asymptotically flat space.

Although a mathematically rigorous study of the near zone of the bounded sources is a formidable task due to extremely high field curvatures, substantial progress has been made over the years in the study of the asymptotic region. It is known that gravitational waves from a spatially isolated matter distribution propagate mainly along future-directed outgoing null geodesics in space-time. Hence the investigation of gravitational waves far from their material sources requires an investigation of the structure of space-time asymptotically along such null geodesics. The first successful investigations of this kind were made about two decades ago by Bondi, van der Burg and Metzner [1] for axially symmetric fields, and by Sachs [2] in the general case. In particular, Bondi's work led to the realization that gravitational waves carry mass and this mass at retarded times, decreases monotonically in the presence of radiation. Sachs' peeling theorem describes the asymptotic behaviour of certain components of the metric, the connection and the Riemann curvature tensor, along outgoing null geodesics.

About two decades ago, Newman and Penrose [3] extended the pioneering work of Bondi et al and Sachs, by introducing the null tetrad (from now on referred to as NP) formalism. This led to great simplification in the equations involved in the analysis of the asymptotic structure of gravitational fields. The explicit description of the NP formalism is given in chapter three. The NP formalism has often been used successfully to find new exact solution to the Einstein field equations, both in the vacuum and electrovac case. Shortly after its introduction, the NP formalism was used by Newman and Unti [4] to investigate the asymptotic behaviour of the metric and the spin coefficients for asymptotically flat vacuum solutions. This was later extended to the Einstein-Maxwell case by Kozarzewski [5]. The next major advance in the study of gravitational fields was by Penrose [6,7], who introduced the concept of conformal infinity. Penrose's conformal approach to the asymptotic structure of fields and spacetime, provides a geometric reformulation of the asymptotic flatness condition given in local charts by Bondi et al and Sachs. Combining the NP formalism with Penrose's conformal technique, Ludwig [8] presented an elaborate investigation of the gravitational field for an arbitrary source as regards the asymptotic behaviour of the metric and Riemann tensor components. But this was done in a co-ordinate system $\{u, r, \zeta, \bar{\zeta}\}$ in which the function P (which describes the Riemann curvature of $u=u_0=const$ cuts of future null infininty $\mathcal{J}^+$, does not evolve in time. In an attempt, not only to investigate the asymptotic behaviour but to generate new exact (close form) solutions, Ludwig [9] reformulated the NP equations (which are equivalent, in principle, to the Einstein field equations) into a reduced set of non-linear partial differential equations.

The aim of this present research is to use this restructured system of non-linear partial differential equations as a starting point, and systematically derive a "generalized" asymptotic solution to the Einstein field equations in a co-ordinate system based on a geodesic, twist-free expanding null congruence. The co-ordinate system used is such that the co-ordinate constraint $P_{,u}=0$ is not assumed. The techniques used, include Penrose's conformal rescaling technique and a modified concept of local asymptotic flatness. No other simplifying assumptions are made. The system of non-linear equations is first solved in a conformally rescaled (unphysical) space M. Due to the non-linearities associated with these restructured equations; it is not possible within the scope of this research, to derive the solutions in closed form, in general. The modest purpose of this work is to generate "generalized" Newman-Unti type asymptotic solutions to the Einstein field equations, at a point neighbourhood of the conformal boundary of future null infinity $\mathcal{J}^{+}$. The solution is first presented in a conformally rescaled space M in terms of some freely specifiable initial data set, as an asymptotic power series in Ω. Here, is the conformal factor as well as a member of the co-ordinate chart $x^{i} = [u, \Omega, \zeta, \bar{\zeta}]$ defined in M. Using Penrose's conformal rescaling, the solutions are then transcribed into the physical space $\hat{M}$ (in terms of an alternate set of freely specifiable initial data) as generalized Newman-Unti type asymptotic power series. In $\hat{M}$, rather than using the conformal factor Ω as one of the co-ordinates, a radial co-ordinate r is used which is an affine parameter along the $u = u_0$ (const) hypersurfaces. If analytic data are assumed, the generalized NU-type expansions obtained give, in fact, the first few terms of a convergent power series [10, 11, 12]. It should be noted that the present results are valid for real space-times but could be generalized to complexified spaces with minor changes.

The main new features emerging from this current work include:

1. An approach which is much more systematic and involves only a necessary and sufficient subset of the

Newman-Penrose equations. This subset is equivalent, in principle, to the Einstein field equations. Solutions to these non-linear partial differential equations are presented in terms of certain freely specifiable functions called the initial data. These initial data are clearly exhibited both in the conformally rescaled (unphysical) space as well as the physical space .

2. A presentation of a "generalized" NU-type asymptotic solution to the field equations for an arbitrary gravitational system, in a co-ordiante system which is much more general than previously used. In particular, the constraint $P_{,u}=0$ is not assumed. This leads to additional valuable terms in the solutions.

3. An elaborate solution scheme for the restructured NP equations (or equivalently the field equations) is presented in both the conformally rescaled (unphysical) space as well as the physical space . In a number of cases, the NU-type asymptotic power series are augmented to include more terms than in previous work. These extra high orders could be of use in gravitational wave study and detection in which case there is a need to correlate experimental measurements with theoretically computed values of the asymptotic Riemannian tensor components.

Finally a word on notation. In this presentation, careted quantities refer to the physical space while those without carets refer to the conformally rescaled space .

Superscripts on a variable denote the appropriate coefficient in the expansion of that variable in powers of the conformal factor. The usual symbols are used for the spin coefficients and other NP quantities.

Chapter 2

SPINOR TETRAD ANALYSIS AND THE HOMOMORPHISM

$$\Phi : SL(2,C) \longrightarrow O_{+}^{\uparrow}(1,3)$$

2.1 Spinor and Tetrad algebra

Spinor and tetrad formalisms provide a more compact framework for many computations in curved space time, as compared to the equivalent tensorial formalism. Either the spinor formalism or the null tetrad technique can be used in deriving yet another equivalent formalism, namely the Newman-Penrose formalism [3]. As shown by Geroch [13], every non-compact Riemannian manifold which carries a global field of oriented orthonormal tetrads, admits a spinor structure. The spinor gauge group SL(2,C) is a two-to-one covering homomorphism of the oriented orthonormal tetrad gauge group $O_{+}^{\uparrow}(1,3)$. This group SL(2,C) of unimodular two by two linear transformations acts on a 2-complex dimensional space called the spinor space. The elements of this space are the one-index spinors. Any particular choice of basis in the spinor space is called a spinor dyad. The spinor space S has three other associated spaces; namely, the dual space S^*, the complex conjugate space $\bar{S}$ and the dual complex conjugate space $\bar{S}^*$. The spinor dyads associated with the spaces S, S^*; $\bar{S}$ and $\bar{S}^*$ are labelled respectively as (o^A, ι^A), (o_A, ι_A), $(\bar{o}^{\dot{A}}, \bar{\iota}^{\dot{A}})$, and $(\bar{o}_{\dot{A}}, \bar{\iota}_{\dot{A}})$. The spinors in the dual space S^* are associated with those in S by means of the Levi-Civita [L.C.] symbols defined by

2.01

$$\epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad 2.01$$

Let the spinors ξ^A and ξ_A belong, respectively, to the associated spaces S and S^*. Then, via the L.C. symbols these spinors are related according to

$$\xi^A = \epsilon^{AB}\xi_B \quad , \quad \xi_B = \xi^A \epsilon_{AB} \qquad 2.02$$

Similarly, the relation between $\eta^{\dot{A}}$ and $\eta_{\dot{A}}$ (in their repective associated spaces $\bar{S}$ and $\bar{S}^*$) is described by the expressions:

$$\eta^{\dot{A}} = \epsilon^{\dot{A}\dot{B}}\eta_{\dot{B}} \quad , \quad \eta_{\dot{B}} = \eta^{\dot{A}}\epsilon_{\dot{A}\dot{B}} \qquad 2.03$$

Henceforth, capital indices refer to spinors and run from 0 to 1. Lower case letters refer to tetrad indices and range from 0 to 3. The L.C. symbols are skew-symmetric and hence the order in expressions 2.02 and 2.03 is very important. In particular, the order can be reversed according to the following relation:

$$\xi_A \eta^A = \xi^A \epsilon_{AB} \eta^B = -\eta^A \epsilon_{AB} \xi^B = -\eta_A \xi^A \qquad 2.04$$

This implies in particular that $\xi_A \xi^A = 0$ for any spinor ξ^A. It is readily seen that in terms of the dyads, the L.C. symbols are given by:

$$\begin{aligned} \epsilon^{AB} &= 2\, o^{[A}\iota^{B]} \\ \epsilon_{AB} &= 2\, o_{[A}\iota_{B]} \\ \epsilon^{\dot{A}\dot{B}} &= 2\, \bar{o}^{[\dot{A}}\bar{\iota}^{\dot{B}]} \\ \epsilon_{\dot{A}\dot{B}} &= 2\, \bar{o}_{[\dot{A}}\bar{\iota}_{\dot{B}]} \end{aligned} \qquad 2.05$$

The two dimensional Kronecker delta $\delta_A^{\ B}$ is related to the LC symbol through the relation: $\epsilon_A^{\ B} = \delta_A^{\ B}$

The usual dyad normalization is taken to be

$$o_A \iota^A = -\iota_A o^A = 1 \qquad 2.07$$

All other inner products are identically equal to zero.

We shall now discuss how spinors can linked up with tetrads. Given any Lorentzian manifold M, it is possible to define a two-to-one homomorphism

$$\Phi : SL(2,C) \rightarrow O_+^{\uparrow}(1,3)$$

Under this homomorphism, it is possible to associate with each spinor dyad, a unique normed null tetrad $(k, n, m, \bar{m})$ according to the following scheme:

$$\begin{aligned} k^a &\longleftrightarrow o^A \bar{o}^{\dot{A}} \\ n^a &\longleftrightarrow \iota^A \bar{\iota}^{\dot{A}} \\ m^a &\longleftrightarrow o^A \bar{\iota}^{\dot{A}} \\ \bar{m}^a &\longleftrightarrow \iota^A \bar{o}^{\dot{A}} \end{aligned} \qquad 2.08$$

where the vectors of the null tetrad are related to the orthonormal basis (T^a, X^a, Y^a, Z^a) through the expressions:

$$\begin{aligned} k^a &= 1/\sqrt{2}\,(T^a + Z^a) \\ n^a &= 1/\sqrt{2}\,(T^a - Z^a) \\ m^a &= 1/\sqrt{2}\,(X^a + iY^a) \\ \bar{m}^a &= 1/\sqrt{2}\,(X^a - iY^a) \end{aligned} \qquad 2.09$$

The corresponding inner product relations for the null vectors $(k, n, m, \bar{m})$, follow from those of the spinor dyad (o^A, ι^A). Working them out, using the definitions 2.07 and 2.08, we obtain the following expressions.

$$k_a n^a = -m_a \bar{m}^a = 1 \qquad 2.10$$

All other inner products vanish identically. For the orthonormal basis (T^a, X^a, Y^a, Z^a), the non-vanishing inner products are given by:

$$T_a T^a = -Z_a Z^a = -X_a X^a = -Y_a Y^a = +1 \qquad 2.11$$

If, in addition, we relate the covariant and contravariant metric tensors repectively to the L.C. symbols,

$$\begin{aligned} g_{ab} &= \epsilon_{AB}\,\epsilon_{\dot{A}\dot{B}} \\ g^{ab} &= \epsilon^{AB}\,\epsilon^{\dot{A}\dot{B}} \end{aligned} \qquad 2.12$$

then, we obtain the following relation between the null tetrad and the covariant and contravariant metric tensors:

$$\begin{aligned} g_{ab} &= 2\, k_{(a} n_{b)} - 2\, m_{(a} \bar{m}_{b)} \\ g^{ab} &= 2\, k^{(a} n^{b)} - 2\, m^{(a} \bar{m}^{b)} \end{aligned} \qquad 2.13$$

2.20 Spinor and tetrad gauge transformations

In the previous subsection, we mentioned that the mapping

$$\Phi : SL(2,C) \rightarrow O_+^{\uparrow}(1,3)$$

is a two-to-one homomorphism. We shall continue the discussion more explicitly in this subsection and consider the effects of this homomorphism on the null tetrad. We shall show that it induces a Lorentz transformation and perserve the metric. In the spinor space S, we define the complex linear transformation:

$$\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \qquad 2.14$$

where the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an element of the group SL(2,C) and $\alpha, \beta, \gamma, \delta$ are complex scalars which satisfy the condition $\alpha\delta - \beta\gamma = 1$

For $\alpha \neq 0$, the matrix $A \in SL(2,C)$ can be written as a product of three compatible matrices of the form:

$$A_{①} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad A_{②} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \qquad 2.15$$

$$A_{③} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Written explicitly, the complex linear transformation 2.14 represents three classes Lorentz transformations $L_{①}$, $L_{②}$, and $L_{③}$ given as follows

$$L_{①}: \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \qquad 2.16$$

$$L_{②}: \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \qquad 2.17$$

$$L_{③}: \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \qquad 2.18$$

where $a = a_o e^{i\phi}$ with a_o and ϕ being real quantities while b and c are complex scalars.

The transformation $L_{①}$ corresponds to a boost (an ordinary Lorentz transformation) in the k - n plane and a spatial rotation in the m - $\bar{m}$ plane. The transformation $L_{②}$ is a one-(complex) parameter null rotation about k and leaves the spinor o_A and the null vector k invariant. The transformation $L_{③}$ is a one-(complex) parameter null rotation about the vector n and leaves the spinor and the null vector n invariant. Since the transformations $L_{①}$, $L_{②}$ and

L3 will be employed extensively in obtaining certain desired results in the coming chapters, it is appropriate to discuss them explicitly. Let us begin by considering the action of L1 on the null vectors k, n, m, $\bar{m}$. Under L1, they transform according to the expressions:

$$\begin{pmatrix} k^a \\ n^a \end{pmatrix} \longrightarrow \begin{pmatrix} a_0^2 & 0 \\ 0 & a_0^{-2} \end{pmatrix} \begin{pmatrix} k^a \\ n^a \end{pmatrix} \qquad 2.19a$$

$$\begin{pmatrix} m^a \\ \bar{m}^a \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-2i\phi} & 0 \\ 0 & e^{+2i\phi} \end{pmatrix} \begin{pmatrix} m^a \\ \bar{m}^a \end{pmatrix} \qquad 2.19b$$

We shall show that 2.19a corresponds to a boost in the k-n plane, while 2.19b represents a spatial rotation in the m-$\bar{m}$ plane: If we set $\phi = 0$ in 2.19b and invoke the relation 2.09, we arrive at the following transformation results:

$$\begin{aligned} T^a &\longrightarrow 1/2\left(a_0^2 + a_0^{-2}\right)\left[T^a + \frac{\left(a_0^2 - a_0^{-2}\right)}{\left(a_0^2 + a_0^{-2}\right)} Z^a\right] \\ Z^a &\longrightarrow 1/2\left(a_0^2 - a_0^{-2}\right)\left[T^a + \frac{\left(a_0^2 + a_0^{-2}\right)}{\left(a_0^2 - a_0^{-2}\right)} Z^a\right] \\ X^a &\longrightarrow X^a \\ Y^a &\longrightarrow Y^a \end{aligned} \qquad 2.20$$

The above results 2.20 can be recast into the form:

$$\begin{aligned} T^a &\longrightarrow (1 - v^2)^{-1/2}\left(T^a + vZ^a\right) \\ Z^a &\longrightarrow (1 - v^2)^{-1/2}\left(Z^a + vT^a\right) \\ X^a &\longrightarrow X^a \\ Y^a &\longrightarrow Y^a \end{aligned} \qquad 2.21$$

provided we set $v = \left(a_0^2 - a_0^{-2}\right)/\left(a_0^2 + a_0^{-2}\right)$

The results of 2.21 depict a Lorentz boost in the k-n plane. If we now set $a_0 = 1$ in expression 2.19 and invoke the relation 2.09, we arrive at the following transformation equations

$$\begin{array}{llll} X^a & \cos 2\phi & \sin 2\phi & X^a \\ Y^a & -\sin 2\phi & \cos 2\phi & Y^a \\ T^a & & T^a & \\ Z^a & & Z^a & \end{array}$$

This is a rotation in the m-$\bar{m}$ plane.

Let us next discuss explicitly the effect of the complex linear transformation L(2) on the null tetrad $(k, n, m, \bar{m})$. The required transformation expressions can be worked out by using the relation 2.08 and 2.17. We arrive at the following transformation relations for the tetrad vectors:

$$\begin{array}{lll} k^a & \longrightarrow & k^a \\ m^a & \longrightarrow & ck^a + m^a \\ \bar{m}^a & \longrightarrow & \bar{c}k^a + \bar{m}^a \\ n^a & \longrightarrow & n^a + \bar{c}m^a + c\bar{m}^a + c\bar{c}k^a \end{array} \qquad 2.23$$

Thus, L(2) represents a one-(complex) parameter null rotation about k .

Finally for the complex linear transformation L(3), the null tetrad vectors transform in a manner similar to that for the transformation L(2). But in this case, the roles of k and n and (m and $\bar{m}$) are interchanged; n remains invariant. The results are presented as follows:

$$\begin{array}{lll} n^a & \longrightarrow & n^a \\ m^a & \longrightarrow & m^a + bn^a \\ \bar{m}^a & \longrightarrow & \bar{m}^a + \bar{b}n^a \\ k^a & \longrightarrow & k^a + \bar{b}m^a + b\bar{m}^a + b\bar{b}n^a \end{array} \qquad 2.24$$

Thus L(3) represents a one-(complex) parameter null rotation about the null vector n.

2.[illegible] Spinor and Tetrad Calculus:

In this subsection, we shall discuss briefly the aspects of spinor and tetrad calculus that will be necessary for application in later chapters.

Let $\nabla_{A\dot{A}}$ denote an intrinsic, directional covariant derivative defined on a pseudo-Riemannian manifold M. Given suitably smooth spinor fields $(\xi^{A}, \xi_{A}, \bar{\xi}^{\dot{A}}, \bar{\xi}_{\dot{A}})$ on the manifold M, we may define the action of $\nabla_{A\dot{A}}$, by demanding that not only should it satisfy Leibnitz' rule and linearity, act on scalars fields as a gradient, be torsion-free and preserve the metric, but that it should annihilate the spinors ϵ_{AB} and $\epsilon_{\dot{A}\dot{B}}$ as well. The action of $\nabla_{A\dot{A}}$ on the spinor fields are given by the expression [14]

$$\begin{aligned}
\nabla_{C\dot{C}}\xi^{A} &= \xi^{A}{}_{,C\dot{C}} + \Gamma^{A}_{B C\dot{C}}\,\xi^{B} \\
\nabla_{C\dot{C}}\bar{\xi}^{\dot{A}} &= \bar{\xi}^{\dot{A}}{}_{,C\dot{C}} + \bar{\Gamma}^{\dot{A}}_{\dot{B} C\dot{C}}\,\bar{\xi}^{\dot{B}} \\
\nabla_{C\dot{C}}\bar{\xi}_{\dot{A}} &= \bar{\xi}_{\dot{A},C\dot{C}} - \bar{\Gamma}^{\dot{B}}_{\dot{A} C\dot{C}}\,\bar{\xi}_{\dot{B}} \qquad\qquad 2.25 \\
\nabla_{C\dot{C}}\xi_{A} &= \xi_{A,C\dot{C}} - \Gamma^{B}_{A C\dot{C}}\,\xi_{B}
\end{aligned}$$

The symbol $\Gamma^{A}_{BC\dot{C}}$ denotes the spinor affine connection. It is defined uniquely by requirements that $\nabla_{C\dot{C}}\epsilon_{AB} = \nabla_{C\dot{C}}\epsilon_{\dot{A}\dot{B}} = 0$ It is also possible to introduce in M an equivalent covariant derivative ∇_{a} which acts on vector and tensor fields. It is defined uniquely by the requirement that $\nabla_{a} g = 0$, where g is the metric. We now proceed to expand $\nabla_{A\dot{A}}$ and, equivalently, ∇_{a} in terms of the spinor basis (o, ι) and the null tetrad $(k, n, m, \bar{m})$, repectively. If the components of $\nabla_{A\dot{A}}$ and ∇_{a}

are defined as:

$$\Delta = n^a \nabla_a = \iota^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}$$
$$D = k^a \nabla_a = o^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}$$
$$\delta = m^a \nabla_a = o^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}} \qquad \ldots 2.26a$$
$$\bar{\delta} = \bar{m}^a \nabla_a = \iota^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}$$

then the required expansions for $\nabla_{A\dot{A}}$ and ∇_a in terms of o_A, ι_A and $k_a, n_a, m_a, \bar{m}_a$ respectively are given by the following expressions:

$$\nabla_{A\dot{A}} = o_A \bar{o}_{\dot{A}} \Delta + \iota_A \bar{\iota}_{\dot{A}} D - o_A \bar{\iota}_{\dot{A}} \bar{\delta} - \iota_A \bar{o}_{\dot{A}} \delta$$
$$\nabla_a = k_a \Delta + n_a D - m_a \bar{\delta} - \bar{m}_a \delta \qquad 2.26b$$

We shall now briefly mention other aspects of covariant differentiation such as those involving the components of the Riemann tensor R_{abcd}. Let Ψ_{ABCD}, Λ, and $\Phi_{AB\dot{C}\dot{D}}$ be the irreducible parts of the Riemann tensor R_{abcd} under local SL(2,C) or $O_+^{\uparrow}(1,3)$. Then R_{abcd} can be written explicitly in the form [7]

$$\begin{aligned} R_{abcd} = {} & \Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \\ & + 2\Lambda\{\epsilon_{AC}\epsilon_{BD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\epsilon_{\dot{A}\dot{D}}\epsilon_{\dot{B}\dot{C}}\} \\ & + \epsilon_{AB}\Phi_{CD\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{CD}\Phi_{AB\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}} \end{aligned} \qquad 2.27$$

where, Ψ_{ABCD}, $\Phi_{AB\dot{C}\dot{D}}$ are the spinors form which the Weyl tensor (C_{abcd}) and the trace-free Ricci tennsor (S_{ab}) constructed, repectively.

$$C_{abcd} = \Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \qquad 2.28$$
$$E_{abcd} = \epsilon_{AB}\Phi_{CD\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{CD}\Phi_{AB\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}} \qquad 2.29$$

where C_{abcd} denotes the Weyl conformal curvature tensor and E_{abcd} is defined in expression 3.02, page 19. Also, the curvature scalar $\Lambda = \bar{\Lambda}$ is given by $\Lambda = R/24$.

The components of the Riemann tensor R_{abcd} can be obtained from commutation relations of double differentiation on tetrad (vector fields) or, equivalently, spinor fields, as expressed as follows [3]

$$\nabla_c \nabla_b k_a - \nabla_b \nabla_c k_a = R_{dabc} k^d \qquad 2.30$$

$$\nabla_{(A}{}^{\dot{P}} \nabla_{B)\dot{P}} \xi_C = \Psi_{ABCD} \xi^D + 2 \Lambda \xi_{(A} \epsilon_{B)C} \qquad 2.31$$

$$\nabla_{C(\dot{P}} \nabla^C{}_{\dot{Q})} \xi_A = \Phi_{AB\dot{P}\dot{Q}} \xi^B$$

The equations 2.30 or equivalently 2.31 will be discussed explicitly in the later chapters where they will be thereafter referred to as "Ricci identities". The Bianchi identities are given by the expression [14]

$$\nabla_{[a} R_{bc]de} = 0 \qquad 2.32$$

or, in spinor terms, by the equations

$$\nabla^D{}_{\dot{Y}} \Psi_{ABCD} = \nabla_{(C}{}^{\dot{Z}} \Phi_{AB)\dot{Y}\dot{Z}}$$

$$\nabla^{\dot{B}\dot{Z}} \Phi_{AB\dot{Y}\dot{Z}} = -3 \nabla_{A\dot{Y}} \Lambda \qquad 2.33$$

Chapter 3

THE NEWMAN-PENROSE FORMALISM

The method of spin coefficients, developed by Newman and Penrose [1], has been particularly useful in the theoretical analysis of gravitational fields, in the construction of algebraically special exact solutions of the Einstein field equations, and in numerous other computations in other aspects of curved space-time field theory. We shall now present a detailed discussion of the Newman-Penrose (NP) formalism.

We consider a pseudo-Riemannian 4-space manifold with Lorentzian signature [+ - - -]. Into this space, a system of null tetrad vectors $(k, n, m, \bar{m})$ or, equivalently, a spinor dyad basis (o, ι) is introduced. It is always possible to introduce such a null tetrad locally. Globally, the existence of a null tetrad is equivalent to the existence of a global orthonormal tetrad. As pointed out earlier, such a global tetrad exists if and only if the space-time (assumed non-compact), admits a spinor structure. The normalizations for the spinor dyad basis and the null tetrad vectors were given in the previous chapter.

3.01 The Newman-Penrose variables

In this subsection, we shall discuss explicitly the definitions of the Newman-Penrose variables. These geometric quantities comprise the following:

1. the spin coefficients

2. the Weyl tensor
3. the Ricci tensor
4. the metric variables.

This subsection will be devoted to the discussion of NP variables (1), --- (3), while the metric variables will be considered in the subsection 3.30.

Let us now consider the SPIN COEFFICIENTS. In the NP formalism, the role of the Christoffel symbols is taken over by the twelve quantities called the spin coefficients. These complex scalar functions are defined as follows (either in terms of the dyad or, equivalently, in terms of the tetrad).

$$
\begin{array}{lclcl}
\kappa & = & o^A D o_A & = & m^a D k_a \\
\rho & = & o^A \bar{\delta} o_A & = & m^a \bar{\delta} k_a \\
\sigma & = & o^A \delta o_A & = & m^a \delta k_a \\
\tau & = & o^A \Delta o_A & = & m^a \Delta k_a \\
 & & & & \\
\epsilon & = & o^A D \iota_A & = & 1/2\,(n^a D k_a + m^a D \bar{m}_a) \\
\alpha & = & o^A \bar{\delta} \iota_A & = & 1/2\,(n^a \bar{\delta} k_a + m^a \bar{\delta} \bar{m}_a) \\
\beta & = & o^A \delta \iota_A & = & 1/2\,(n^a \delta k_a + m^a \Delta \bar{m}_a) \\
\gamma & = & o^A \Delta \iota_A & = & 1/2\,(n^a \Delta k_a + m^a \Delta \bar{m}_a) \\
 & & & & \\
\pi & = & \iota^A D \iota_A & = & n^a D \bar{m}_a \\
\lambda & = & \iota^A \bar{\delta} \iota_A & = & n^a \bar{\delta} \bar{m}_a \\
\mu & = & \iota^A \delta \iota_A & = & n^a \delta \bar{m}_a \\
\nu & = & \iota^A \Delta \iota_A & = & n^a \Delta \bar{m}_a
\end{array}
\qquad 3.01
$$

Let us next discuss the NP representation of the Weyl and Ricci curvature spinors and tensors. The Riemann tensor

R_{abcd} can be decomposed, (spinorially or tensorially) in terms of the Weyl spinor (Ψ_{ABCD}), curvature scalar and the Ricci spinor ($\Phi_{AB\dot{A}\dot{B}}$); or equivalently in terms of the Weyl tensor (C_{abcd}), trace free Ricci tensor (S_{ab}) and curvature scalar (R). These decompositions are exhibited as follows:

$$
\begin{aligned}
R_{abcd} &= C_{abcd} + E_{abcd} + 1/12\,(g_{ac}g_{bd} - g_{ad}g_{bc})R \\
E_{abcd} &= 1/2\,(g_{ac}S_{bd} - g_{bc}S_{ad} - g_{ad}S_{bc} + g_{bd}S_{ac}) \\
S_{ab} &= R_{ab} - 1/4\, g_{ab}R
\end{aligned}
\qquad 3.02
$$

Recall expression 2.27 given by:

$$
\begin{aligned}
R_{abcd} &= \Psi_{ABCD}\,\bar{\epsilon}_{\dot{A}\dot{B}}\,\bar{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{AB}\,\epsilon_{CD}\,\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \\
&\quad + 2\Lambda\,(\epsilon_{AC}\,\epsilon_{BD}\,\bar{\epsilon}_{\dot{A}\dot{B}}\,\bar{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{AB}\,\epsilon_{CD}\,\bar{\epsilon}_{\dot{A}\dot{D}}\,\bar{\epsilon}_{\dot{B}\dot{C}}) \\
&\quad + \epsilon_{AB}\,\Phi_{CD\dot{A}\dot{B}}\,\bar{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{CD}\,\Phi_{AB\dot{C}\dot{D}}\,\bar{\epsilon}_{\dot{A}\dot{B}}
\end{aligned}
\qquad 3.03
$$

In the NP formalism, the components of the Weyl tensor or equivalently, the Weyl spinor are represented by the five complex scalars [$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$], defined in terms of Riemann tensor components as follows:

$$
\begin{aligned}
\Psi_0 &= \Psi_{ABCD}\,o^A o^B o^C o^D &&= -C_{abcd}\,k^a m^b k^c m^d \\
\Psi_1 &= \Psi_{ABCD}\,o^A o^B o^C \iota^D &&= -C_{abcd}\,k^a n^b k^c m^d \\
\Psi_2 &= \Psi_{ABCD}\,o^A o^B \iota^C \iota^D &&= -C_{abcd}\,\bar{m}^a n^b k^c m^d \\
\Psi_3 &= \Psi_{ABCD}\,o^A \iota^B \iota^C \iota^D &&= -C_{abcd}\,\bar{m}^a n^b k^c n^d \\
\Psi_4 &= \Psi_{ABCD}\,\iota^A \iota^B \iota^C \iota^D &&= -C_{abcd}\,\bar{m}^a n^b \bar{m}^c n^d
\end{aligned}
\qquad 3.04
$$

Let us consider the trace-free part of the Ricci tensor or, equivalently, Ricci spinor, next. It is defined by [3]

$$\Phi_{AB\dot{A}\dot{B}} = -1/2\,(R_{ab} - 1/4\, g_{ab} R)$$

and determined by six complex-valued scalars $\{\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}, \Phi_{22}\}$ (and their complex conjugates), which are defined as follows, in terms of the Riemann tensor components.

$$\begin{aligned}
\Phi_{00} &= \Phi_{AB\dot{A}\dot{B}}\, o^A o^B \bar{o}^{\dot{A}} \bar{o}^{\dot{B}} &&= -1/2\, R_{ab} k^a k^b \\
\Phi_{01} &= \Phi_{AB\dot{A}\dot{B}}\, o^A o^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} &&= -1/2\, R_{ab} k^a m^b \\
\Phi_{02} &= \Phi_{AB\dot{A}\dot{B}}\, o^A o^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} &&= -1/2\, R_{ab} m^a m^b \\
\Phi_{11} &= \Phi_{AB\dot{A}\dot{B}}\, o^A \iota^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} &&= -1/4\, R_{ab}(k^a n^b + m^a \bar{m}^b) \\
\Phi_{12} &= \Phi_{AB\dot{A}\dot{B}}\, o^A \iota^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} &&= -1/2\, R_{ab} n^a m^b \\
\Phi_{22} &= \Phi_{AB\dot{A}\dot{B}}\, \iota^A \iota^B \bar{\iota}^{\dot{A}} \bar{\iota}^{\dot{B}} &&= -1/2\, R_{ab} n^a n^b
\end{aligned}$$

$$\Phi_{10} = \bar{\Phi}_{01}, \qquad \Phi_{20} = \bar{\Phi}_{02}$$

$$\Phi_{21} = \bar{\Phi}_{12} \qquad 3.05$$

The Ricci scalar R is represented by $\Lambda = \bar{\Lambda} = R/24$.

3.20 Geometric considerations: (the Optical Scalars)

In this subsection, we shall investigate the geometrical meaning of some of the spin coefficients. First of all, we shall discuss briefly the following concepts: null hypersurface, null geodesic, congruence of null geodesics and affine parametrization. A null hypersurface must satisfy the condition

$$g^{ab}\, \partial_a u\, \partial_b u = 0 \qquad 3.06$$

where the symbols ∂_a denote partial differentiation. A vector k^a is null if it satisfies the conditions

$$g^{ab} k^a k^b = k^a k_a = 0 \qquad 3.07$$

A geodesic is a curve whose tangent direction is parallelly propagated along it. An arbitrary curve $x^a = x^a(r)$ with tangent vector $k^\alpha = dx^a/dr$, (where r is an affine parameter), is a geodesic if and only if:

$$\frac{d}{dr} x^{[b} \frac{D}{Dr} \frac{dx^{a]}}{dr} = 0 \qquad 3.08$$

where D/Dr is the absolute (Leibnitz) derivative. An affine parameter is a quantity which can be introduced on each geodesic such that the geodesic assume the preferred form:

$$\frac{D}{Dr} \frac{dx^a}{dr} = 0 \qquad 3.09$$

An affine parameter is preserved by a linear transformation. A congruence of curves is a 3-parameter family of curves of which exactly one passes through each point of the space-time region under consideration. A null geodesic is thus a curve $x^a = x^a(r)$ with null tangent vector $k^a = dx^a/dr$ satisfying the condition:

$$\frac{D}{Dr} \frac{dx^a}{dr} = 0 \qquad 3.09$$

We then define a congruence of null geodesics as a 3-parameter family of null geodesics of which exactly one passes through each point of the space-time region under consideration. The equations of the congruence may be written in the form:

$$x^a = x^a(y^\beta, r) \qquad 3.10$$

Here the parameters y^β label the different curves of the congruence, and r is an affine parameter along each curve.

We shall now consider the geometrical properties of some of the spin coefficents. Most of the spin coefficients can be given useful geometrical interpretations. In this discussion, much emphasis will be attached to three scalar fields called the OPTICAL SCALARS. These scalars, are constructed form the covariant derivative of the null vector k_a. The optical scalars characterize the geometrical properties of the null geodesic congruence to which the null vector field k_a is tangent. We shall now discuss the geometrical properties of the optical scalars explicitly alongside other spin coefficients. The covariant derivative of the null tetrad (vector) k_a is given by the expression:

$$\begin{aligned} k_{a;b} = {} & (\gamma+\bar{\gamma}) k_a k_b + (\epsilon+\bar{\epsilon}) k_a n_b - (\alpha+\bar{\beta}) k_a m_b \\ & - (\bar{\alpha}+\beta) k_a \bar{m}_b - \bar{\tau} m_a k_b - \tau \bar{m}_a k_b \\ & - \bar{\kappa} m_a n_b - \kappa \bar{m}_a n_b + \bar{\sigma} m_a m_b \\ & + \sigma \bar{m}_a \bar{m}_b + \bar{\rho} m_a \bar{m}_b + \rho \bar{m}_a m_b \end{aligned} \qquad 3.11$$

It follows immediately (on contracting 3.11 with k^b and using the prescribed tetrad normalization) that

$$k_{a;b} k^b = D k_a = (\epsilon+\bar{\epsilon}) k_a - \kappa \bar{m}_a - \bar{\kappa} m_a \qquad 3.12$$

If we set =0, then the field of null tetrad (vector) k_a will be tangent to a null geodesic congruence. The geodesics can be affinely parametrized, if under the Lorentz transformation L(I), we make $\epsilon+\bar{\epsilon}=0$. Thus when $K=\epsilon+\bar{\epsilon}=0$, it follows that $Dk_a=0$. It can be shown further that when the conditions

$$\begin{aligned} D k_a &= 0 \\ D n_a &= 0 \\ D m_a &= 0 \\ D \bar{m}_a &= 0 \end{aligned}$$

are satisfied simultaneously, then we have the following results:

$$\kappa = \epsilon + \bar{\epsilon} = \epsilon - \bar{\epsilon} = \pi = 0 \qquad 3.13$$

In particular, if the null tetrad $(k, n, m, \bar{m})$ is parallelly propagated in the k_a direction, the spin coefficients κ, π, ϵ vanish identically. If we substitute the condition $\kappa = \epsilon + \bar{\epsilon} = 0$ into the expression 3.11, and skew-symmetrize the result, we arrive at the following expression:

$$k_{[a;b]} = -(\bar{\alpha} + \beta - \tau)\, k_{[a}\bar{m}_{b]} - (\alpha + \bar{\beta} - \bar{\tau})\, k_{[a}m_{b]} + (\rho - \bar{\rho})\, \bar{m}_{[a}m_{b]} \qquad 3.14$$

and hence:

$$k_{[a;b}k_{c]} = (\rho - \bar{\rho})\, \bar{m}_{[a}m_{b}k_{c]} \qquad 3.15$$

Thus it can be deduced form 3.15 that the congruence of null geodesics will be hypersurface orthogonal (i.e. proportional to the gradient of an arbitrary scalar function u) if and only if $\rho - \bar{\rho} = 0$. This implies that k_a and u are related by the expression

$$k_a = S\, \nabla_a u \qquad 3.16$$

for some constant of proportionality $S(x^a)$. It can be shown (using 3.14) that S the congruence will be equal to the gradient field of u if, in addition to $\rho - \bar{\rho} = 0$, we have the condition

$$\bar{\alpha} + \beta - \tau = 0 \qquad 3.17$$

We shall now define the OPTICAL SCALARS. These quantities are a subset of the spin coefficients and describe the expansion, twist and shear of the null geodesic congruence to which k_a is tangent. The expressions for the optical scalars are exhibited as follows: [14]

Expansion (or divergence) $\Theta = k^{a}_{,a} = -(\rho + \bar{\rho})$

Twist (curl, or rotation) $\omega = \{k_{[a,b]}k^{a,b}\}^{1/2} = -i(\rho - \bar{\rho})$

Shear (or distortion) $|\sigma| = \{\frac{1}{2}k_{(a,b)}k^{a,b} - \frac{1}{2}(k^{a}_{,a})^2\}^{1/2} = (\sigma\bar{\sigma})^{1/2}$ 3.18

If the congruence is changed to that for which the null vectors n constitute a tangent vector field, the roles of the spin coefficient κ, $\epsilon + \bar{\epsilon}$, ρ, σ and τ are taken over, repectively, by $-\nu$, $-(\gamma + \bar{\gamma})$, $-\mu$, $-\lambda$, and $-\pi$.

3.30 The Metric variables

We shall now discuss the metric variables. In order to do so, we need a co-ordinate system and the definition of certain parameters. Let $x^i = [u, \Omega, \zeta, \bar{\zeta}]$ be a local co-ordinate chart such that $u = u_0$ (constant) labels a congruence of null hypersurface, Ω is a parameter along the null geodesic generators of these hypersurfaces and $\zeta + \bar{\zeta}$, $i(\zeta - \bar{\zeta})$, label these generators. The basis of the tangent space $T_p(M)$ of M is given by:

$$\frac{\partial}{\partial x^i} = \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial \Omega}, \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}}\right] \qquad 3.19$$

The intrinsic directional, covariant derivative operators $\nabla_a = [D, \Delta, \delta, \bar{\delta}]$ are tangent vectors to M and can be expanded in terms of the above basis. At the point p in M we define

$$\begin{aligned} D &= a^i \left(\frac{\partial}{\partial x^i}\right)_p \\ \delta &= b^i \left(\frac{\partial}{\partial x^i}\right)_p \\ \bar{\delta} &= \bar{b}^i \left(\frac{\partial}{\partial x^i}\right)_p . \\ \Delta &= d^i \left(\frac{\partial}{\partial x^i}\right)_p \end{aligned} \qquad 3.20$$

where $i \in [0, 1, 2, 3]$. If we parallelly propagate $(\zeta + \bar{\zeta})$ and

(ζ-$\bar{\zeta}$) along the null geodesics with k_a as tangent vector, we shall find that:

$$D\zeta = D\bar{\zeta} = 0 \qquad 3.21$$

This implies that ($\zeta, \bar{\zeta}$) do not change along the chosen null geodesic congruence. Since k_a is taken to be the gradient of a scalar field u, we find the following simplifications:

$$\begin{aligned} \Delta u &= n^a \nabla_a u = n^a k_a = 1 \\ \delta u &= m^a \nabla_a u = m^a k_a = 0 \\ \bar{\delta} u &= \bar{m}^a \nabla_a u = \bar{m}^a k_a = 0 \\ D u &= k^a \nabla_a u = k^a k_a = 0 \end{aligned} \qquad 3.22$$

It follows from conditions 3.21 and 3.22, that expressions 3.20 become:

$$\begin{aligned} D &= a^2 \frac{\partial}{\partial \Omega} \\ \delta &= b^2 \frac{\partial}{\partial \Omega} + b^3 \frac{\partial}{\partial \zeta} + b^4 \frac{\partial}{\partial \bar{\zeta}} \\ \bar{\delta} &= \bar{b}^2 \frac{\partial}{\partial \Omega} + \bar{b}^4 \frac{\partial}{\partial \zeta} + \bar{b}^3 \frac{\partial}{\partial \bar{\zeta}} \\ \Delta &= \frac{\partial}{\partial u} + d^2 \frac{\partial}{\partial \Omega} + d^3 \frac{\partial}{\partial \zeta} + d^4 \frac{\partial}{\partial \bar{\zeta}} \end{aligned} \qquad 3.23$$

Relabelling the coefficients in equations 3.23 and calling them $f, \omega, \xi_1, \xi_2, \bar{\omega}, \bar{\xi}_2, \bar{\xi}_1, U, X, \bar{X}$ respectively, we have the following equations

$$\begin{aligned} D &= f \frac{\partial}{\partial \Omega} \\ \delta &= \omega \frac{\partial}{\partial \Omega} + \xi_1 \frac{\partial}{\partial \zeta} + \xi_2 \frac{\partial}{\partial \bar{\zeta}} \\ \bar{\delta} &= \bar{\omega} \frac{\partial}{\partial \Omega} + \bar{\xi}_2 \frac{\partial}{\partial \zeta} + \bar{\xi}_1 \frac{\partial}{\partial \bar{\zeta}} \\ \Delta &= \frac{\partial}{\partial u} + U \frac{\partial}{\partial \Omega} + X \frac{\partial}{\partial \zeta} + \bar{X} \frac{\partial}{\partial \bar{\zeta}} \end{aligned} \qquad 3.24$$

Here, the function f and U are real, while X, ξ_1, ξ_2, ω are complex-valued. We recall that the directional derivative operators $D, \delta, \bar{\delta}, \Delta$ are also expressible in terms of the null tetrad vectors (k, n, m, $\bar{m}$), according to the following equations.

$$D = k^a \nabla_a$$
$$\Delta = n^a \nabla_a$$
$$\delta = m^a \nabla_a \qquad 3.25$$
$$\bar{\delta} = \bar{m}^a \nabla_a$$

Hence the null tetrad vectors can be expressed as:

$$k^a = (0 , f , 0 , 0)$$
$$n^a = (1 , U , X , \bar{X})$$
$$m^a = (0 , \omega , \xi_1 , \xi_2) \qquad 3.26$$
$$\bar{m}^a = (0 , \bar{\omega} , \bar{\xi}_2 , \bar{\xi}_1)$$

Henceforth, the functions $f, U, \omega, \bar{\omega}, \xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2, X, \bar{X}$ shall be referred to as METRIC VARIABLES. They are related to the space-time metric (line element) ds^2 in M through the following expressions:

$$ds^2 = g^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} = g_{ab} dx^a \otimes dx^b \qquad 3.27$$

where $g^{ab} = 2k^{(a}n^{b)} - 2m^{(a}\bar{m}^{b)}$ is the contravariant metric tensor whose inverse gives the co-variant metric tensor g_{ab}. The explicit evaluation of g^{ab} and g_{ab} in terms of the metric variables is reserved for chapter seven.

3.40 The Newman-Penrose equations

The Newman-Penrose equations can be divided into three classes:

(i) a) The commutator equations

b) The metric equations

(ii) The "Ricci identities"

(iii) The "Bianchi identities"

Let us consider the commutator equations first. These commutator equations are derived from the action of commutators of the intrinsic directional derivative operators $\{D, \Delta, \delta, \bar{\delta}\}$ on an arbitrary scalar field Φ. The resulting equations are:

$$
\begin{aligned}
[\Delta D - D\Delta]\phi &= [(\gamma+\bar{\gamma})D + (\epsilon+\bar{\epsilon})\Delta - (\tau+\bar{\pi})\bar{\delta} - (\bar{\tau}+\pi)\delta]\phi \\
[\delta D - D\delta]\phi &= [(\bar{\alpha}+\beta-\bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho}+\epsilon-\bar{\epsilon})\delta]\phi \\
[\delta\Delta - \Delta\delta]\phi &= [-\bar{\nu}D + (\tau-\bar{\alpha}-\beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu-\gamma+\bar{\gamma})\delta]\phi \\
[\bar{\delta}\delta - \delta\bar{\delta}]\phi &= [(\bar{\mu}-\mu)D + (\bar{\rho}-\rho)\Delta - (\bar{\alpha}-\beta)\bar{\delta} - (\bar{\beta}-\alpha)\delta]\phi
\end{aligned}
\qquad 3.28
$$

We shall now discuss the procedure for deriving the METRIC EQUATIONS. We start with the NP commutator equations 3.28, substitute for the differential operators using the expressions of 3.24; and finally replace Φ, in turn by each member of the set $x^i = [u, \Omega, \zeta, \bar{\zeta}]$. We need these metric equations only with the following simplifications:

$$
\begin{aligned}
\kappa = \pi = \epsilon &= 0 \\
\tau - (\bar{\alpha}+\beta) &= 0 \\
\rho - \bar{\rho} &= 0
\end{aligned}
\qquad 3.29
$$

Thus the metric equations in the space M are as follows:

$$
\begin{aligned}
DU &= \Delta f + \tau\bar{\omega} + \bar{\tau}\omega - f(\gamma+\bar{\gamma}) && M_1 \\
D\omega &= \delta f + \sigma\bar{\omega} - \tau f && M_2 \\
DX &= \tau\bar{\xi}_2 + \bar{\tau}\xi_1 && M_3 \\
D\xi_1 &= \sigma\bar{\xi}_2 + \rho\xi_1 && M_4 \\
D\xi_2 &= \sigma\bar{\xi}_1 + \rho\xi_2 && M_5 \\
\delta U - \Delta\omega &= (\mu-\gamma+\bar{\gamma})\omega + \bar{\lambda}\bar{\omega} - \bar{\nu}f && M_6 \\
\delta\bar{\omega} - \bar{\delta}\omega &= (\bar{\beta}-\alpha)\omega + (\bar{\alpha}-\beta)\bar{\omega} + f(\mu-\bar{\mu}) && M_7
\end{aligned}
$$

$$\bar\delta\xi_1 - \delta\bar\xi_2 = (\alpha-\bar\beta)\xi_1 + \beta-\bar\alpha)\bar\xi_2 \qquad M_8$$

$$\delta\chi - \Delta\xi_1 = (\mu-\gamma+\bar\gamma)\xi_1 + \bar\lambda\bar\xi_2 \qquad M_9$$

$$\bar\delta\chi - \Delta\xi_2 = \bar\lambda\bar\xi_1 + (\mu-\gamma+\bar\gamma)\xi_2 \qquad M_{10} \qquad 3.30$$

The equations M_1 - M_5 constitute the RADIAL metric equations, while M_6-M_{10} give the NON-RADIAL metric equations. In the NP formalism, the "Ricci identities" arrive from the relation:

$$v_{a;b;c} - v_{a;c;b} = R_{dabc}v^d \qquad 3.31$$

where v_a is an arbitrary vector field and R_{abcd} is the Riemann tensor. Recall that the equivalent spinor formulation is given by the equations:

$$\nabla_{(A}{}^{\dot P}\nabla_{B)\dot P}\xi_C = \Psi_{ABCD}\xi^D + 2\Lambda\xi_{(A}\epsilon_{B)C}$$

$$\nabla_{C(\dot P}\nabla^C{}_{\dot Q)}\xi_A = \Phi_{AB\dot P\dot Q}\xi^B \qquad 3.32$$

Explicit evaluation of 3.31 or equivalently 3.32, taking all possible contractions with null tetrad vectors and spinor dyads repectively yields the NP version of the Ricci identity. The Ricci identities are given by the expressions [3]

$$D\rho - \bar\delta\kappa = (\rho^2+\sigma\bar\sigma) + (\epsilon+\bar\epsilon)\rho - \bar\kappa\tau - \kappa(3\alpha+\bar\beta-\pi) + \Phi_{00} \qquad R_1$$

$$D\sigma - \delta\kappa = (\rho+\bar\rho)\sigma + (3\epsilon-\bar\epsilon)\sigma - (\tau-\bar\pi+\bar\alpha+3\beta)\kappa + \Psi_0 \qquad R_2$$

$$D\tau - \Delta\kappa = (\tau+\bar\pi)\rho + (\bar\tau+\pi)\sigma + (\epsilon-\bar\epsilon)\tau - (3\gamma+\bar\gamma)\kappa + \Psi_1 + \Phi_{01} \qquad R_3$$

$$D\alpha - \bar\delta\epsilon = (\rho+\bar\epsilon-2\epsilon)\alpha + \beta\bar\sigma - \bar\beta\epsilon - \kappa\lambda - \bar\kappa\gamma + (\epsilon+\rho)\pi + \Phi_{10} \qquad R_4$$

$$D\beta - \delta\epsilon = (\alpha+\pi)\sigma + (\bar\rho-\bar\epsilon)\beta - (\mu+\gamma)\kappa - (\bar\alpha-\bar\pi)\epsilon + \Psi_1 \qquad R_5$$

$$D\gamma - \Delta\epsilon = (\tau+\bar\pi)\alpha + (\bar\tau+\pi)\beta - (\epsilon+\bar\epsilon)\gamma - (\gamma+\bar\gamma)\epsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11} \qquad R_6$$

$$D\lambda - \bar\delta\pi = (\rho\lambda+\bar\sigma\mu) + \pi^2 + (\alpha-\bar\beta)\pi - \nu\bar\kappa - (3\epsilon-\bar\epsilon)\lambda + \Phi_{20} \qquad R_7$$

$$D\mu - \bar{\delta}\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda \qquad R_8$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21} \qquad R_9$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4 \qquad R_{10}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01} \qquad R_{11}$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda + \Phi_{11} \qquad R_{12}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \qquad R_{13}$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + [\tau - 3\beta - \bar{\alpha}]\nu + \Phi_{22} \qquad R_{14}$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \qquad R_{15}$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02} \qquad R_{16}$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda \qquad R_{17}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \epsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3 . \qquad R_{18}$$

Finally let us briefly discuss the Bianchi identities as represented in the NP formalism. They are given by:

$$R_{ab[cd;e]} = 0 \qquad 3.33$$

The equivalent spinor representation is given by:

$$\nabla^{D}_{\dot{G}} \Psi_{ABCD} = \nabla_{(C}{}^{\dot{H}} \phi_{AB)\dot{G}\dot{H}}$$

$$\nabla^{A\dot{G}} \phi_{AB\dot{G}\dot{H}} = -3 \nabla_{B\dot{H}} \Lambda \qquad 3.34$$

The spin coefficient formulation of the "Bianchi identities" need not be considered separatedly since it can be obtained from the "Ricci identitites" as compatibility requirements. That means, taking certain combinations of $(D, \Delta, \delta, \bar{\delta})$-derivatives of the "Ricci identities" produces the "Bianchi identities". Equivalently, taking all possible contractions of the tensor version of the Bianchi identity 3.33 with all null tetrad vectors [or equivalently taking all possible contractions of 3.34 with all spinor dyads] yields the NP version of this identity. The explicit equations are given by Pirani [16].

3.5c The NP variables and Lorentz transformations

In the previous chapter we discussed explicitly the complex linear transformation:

$$\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} o^A \\ \iota^A \end{pmatrix}$$

and showed that it induces Lorentz transformations L①, L② and L③. In this subsection, we shall discuss the effects of these local Lorentz transformations on the Newman-Penrose (NP) variables.

Let us start with the effect of L① on the NP variables and the covariant directional derivative operators. Using the definitions for the NP variables it is quite straightforward to work out the following results:

<u>spin coefficients</u>

$$\begin{array}{lll} \kappa \to a_o^4 e^{2i\phi}\kappa & \sigma \to a_o^2 e^{4i\phi}\sigma & \rho \to a_o^2 \rho \\ \tau \to e^{2i\phi}\tau & \nu \to a_o^{-4} e^{-2i\phi}\nu & \lambda \to a_o^{-2} e^{-4i\phi}\lambda \\ \mu \to a_o^{-2}\mu & \pi \to e^{-2i\phi}\pi & \epsilon \to a_o^2(\epsilon + D\ln a_o + iD\phi) \end{array}$$

$$\beta \to e^{2i\phi}(\beta + \delta\ln a_o + i\delta\phi) \qquad \gamma \to a_o^{-2}(\gamma + \Delta\ln a_o + i\Delta\phi) \qquad 3.35$$

$$\alpha \to e^{-2i\phi}(\alpha + \bar\delta\ln a_o + i\delta\phi)$$

<u>Weyl tensor components</u>

$$\begin{array}{ll} \Psi_0 \to a_o^4 e^{4i\phi}\Psi_0 & \Psi_1 \to a_o^2 e^{2i\phi}\Psi_1 \\ \Psi_2 \to \Psi_2 & \Psi_3 \to a_o^{-2} e^{-2i\phi}\Psi_3 \\ \Psi_4 \to a_o^{-4} e^{-4i\phi}\Psi_4 & \end{array} \qquad 3.36$$

<u>Ricci tensor components</u>

$$\begin{array}{lll} \phi_{00} \to a_o^4\phi_{00} & \phi_{01} \to a_o^2 e^{2i\phi}\phi_{01} & \phi_{02} \to e^{4i\phi}\phi_{02} \\ \phi_{11} \to \phi_{11} & \phi_{12} \to a_o^{-2} e^{2i\phi}\phi_{12} & \phi_{22} \to a_o^{-4}\phi_{22} \end{array} \qquad 3.38$$

Components of ∇_a

$$D \to a_0^2 D \qquad \delta \to e^{i\Phi}\delta \qquad \bar{\delta} \to e^{-i\Phi}\bar{\delta} \qquad \Delta \to a_0^{-2}\Delta$$

The results for the metric variables will be discussed in chapter seven.

Similarly, the transformation expressions for the NP variables and components of ∇_a under the Lorentz transformation (L2) are given by:

Spin coefficients

$$\kappa \to \kappa \qquad \sigma \to \bar{c}\kappa + \sigma \qquad \rho \to c\kappa + \rho \qquad \epsilon \to c\kappa + \epsilon$$

$$\tau \to c\bar{c}\kappa + c\sigma + \bar{c}\rho + \tau \qquad \alpha \to c^2\kappa + c\epsilon + c\rho + \alpha$$

$$\beta \to c\bar{c}\kappa + c\sigma + \bar{c}\epsilon + \beta \qquad \pi \to c^2\kappa + 2c\epsilon + \pi + Dc$$

$$\gamma \to c^2\bar{c}\kappa + c^2\sigma + c\bar{c}(\rho + \epsilon) + c(\tau + \beta) + \bar{c}\alpha + \gamma$$

$$\mu \to c^2\bar{c}\kappa + c^2\sigma + 2c\bar{c}\epsilon + 2c\beta + \bar{c}\pi + \mu + \bar{c}Dc + \delta c$$

$$\lambda \to c^3\kappa + c^2(\rho + 2\epsilon) + 2c\alpha + c\pi + \lambda + cDc + \bar{\delta}c$$

$$\begin{aligned}\nu \to\; & c^3\bar{c}\kappa + c^3\sigma + c^2\bar{c}(\rho + 2\epsilon) + c^2(\tau + 2\beta) + \bar{c}c(2\alpha + \pi) \\ & + c(2\gamma + \mu) + \bar{c}\lambda + \nu + c\bar{c}Dc + c\delta c + \bar{c}\bar{\delta}c \\ & + \Delta c\end{aligned} \qquad 3.39$$

Weyl tensor components

$$\Psi_0 \to \Psi_0 \qquad \Psi_1 \to c\Psi_0 + \Psi_1$$

$$\Psi_2 \to c^2\Psi_0 + 2c\Psi_1 + \Psi_2$$

$$\Psi_3 \to c^3\Psi_0 + 3c^2\Psi_1 + 3c\Psi_2 + \Psi_3 \qquad 3.40$$

$$\Psi_4 \to c^4\Psi_0 + 4c^3\Psi_1 + 6c^2\Psi_2 + 4c\Psi_3 + \Psi_4$$

Ricci tensor components

$$\Phi_{00} \to \Phi_{00} \qquad \Phi_{01} \to \bar{c}\Phi_{00} + \Phi_{01}$$

$$\Phi_{02} \to \bar{c}^2\Phi_{00} + 2\bar{c}\Phi_{01} + \Phi_{02}$$

$$\Phi_{11} \to c\bar{c}\Phi_{00} + c\Phi_{01} + \bar{c}\Phi_{10} + \Phi_{11}$$

$$\Phi_{12} \to \bar{c}^2c\Phi_{00} + 2c\bar{c}\Phi_{01} + \bar{c}^2\Phi_{10} + c\Phi_{02} + 2\bar{c}\Phi_{11} + \Phi_{12} \qquad 3.41$$

$$\Phi_{22} \to c^2\bar{c}^2\Phi_{00} + 2c^2\bar{c}\Phi_{01} + 2\bar{c}^2 c\Phi_{10} + c^2\Phi_{02} + \bar{c}^2\Phi_{20}$$
$$+ 4c\bar{c}\Phi_{11} + 2c\Phi_{12} + 2\bar{c}\Phi_{21} + \Phi_{22}$$

<u>components of ∇_a</u>

$$D \to D \qquad \delta \to \bar{c}D + \delta \qquad \bar{\delta} \to cD + \bar{\delta}$$

$$\Delta \to c\bar{c}D + c\delta + \bar{c}\bar{\delta} + \Delta \tag{3.42}$$

Finally, let us consider the effect of the transformation L③ on the NP variables. Under the action of L③, the NP variables transform in a manner analogous to that for L②. We simply make the following interchanges:

$$o_A \to i\iota_A \qquad \iota_A \to i o_A \qquad b \to c$$

$$D \leftrightarrow \Delta \qquad \delta \leftrightarrow \bar{\delta}$$

$$\Psi_1 \leftrightarrow \Psi_3 \qquad \Psi_0 \leftrightarrow \Psi_4 \qquad \Psi_2 \leftrightarrow \Psi_2$$

$$\kappa \leftrightarrow -\nu \qquad \sigma \leftrightarrow -\lambda \qquad \rho \leftrightarrow -\mu$$

$$\tau \leftrightarrow -\pi \qquad \epsilon \leftrightarrow -\gamma \qquad \beta \leftrightarrow -\alpha$$

$$\Phi_{00} \leftrightarrow \Phi_{22} \qquad \Phi_{01} \leftrightarrow \Phi_{21} \qquad \Phi_{02} \leftrightarrow \Phi_{20}$$

$$\Phi_{11} \leftrightarrow \Phi_{11}$$

Chapter 4

THE CONFORMAL TECHNIQUE

When the Einstein gravitational field theory is applied to an isolated self-gravitating source there is a need of a conceptual framework for computing basic physical parameters such as mass, stress-energy, curvature of spacetime, gravitational radiation and gravitational multipoles at points located in the asymptotic region of the source. The computation of these physical properties involve integrations over 2-dimensional space-like surfaces. In order to take into account most of the gravitational effects produced by the source, these 2-surfaces must be chosen to lie at infinity. Let us consider the question of choice of integration surface. This problem arises in all classical theories based on a non-Abelian gauge group, which in this case is the Lorentz group acting in the tangent space at each point in the manifold. We have already stated that the 2-surfaces should lie at infinity. Now, space-like infinity is not a good candidate if we wish to discuss the dynamic evolution of the source. This is so because the mass defined at space-like infinity includes the energy contained in incoming and outgoing radiation, and this remains constant. We therefore choose the smooth 2-surfaces of integration to lie at null infinity. These two-surfaces are variously called cuts, or cross-sections. Each cut defines a unique outgoing hypersurface, generated by null rays which leave it orthogonally and enter the interior of the the space-time.

Since gravitational waves emanating from a spatially isolated distribution of matter, propagate mainly along future-directed outgoing null geodesic in space time, the investigation of gravitational waves far from their material sources must require an analysis of the structures of space-time asymptotically along such null geodesics. We shall now look at the concept of null infinity in more detail.

4.10 The concept of conformal null infinity and asymptotic flatness.

The concept of null infinity and asymptotic flatness are the main entities of the conformal rescaling technique introduced by Penrose [6]. With the aid of Penrose's conformal techniques, the asymptotic behaviour of such quantities as the components of the metric tensor (covariant and contravariant), the Riemann tensor, and the spin coefficients can be determined in a suitable frame in space-time. Let us now discuss explicitly the concept of conformal null infinity.

Let $(\hat{M}, \hat{g})$ be an orientable, Hausdorff, C^{∞} space-time manifold, equipped with a regular suitably smooth Lorentzian covariant metric

$$\hat{g} = \hat{g}_{ab}\, d\hat{x}^{a} \otimes d\hat{x}^{b}$$

of signature [+ - - -]. If the topological structure of $(\hat{M}, \hat{g})$ is suitable, it is possible to adjoin more point-sets to this spacetime. This can be done in such a way that the resulting rescaled metric g extends smoothly to these new points. We then form what is referred to as the UNPHYSICAL space $(\mathcal{M}, g)$, Let Ω be a suitable smooth (C^{∞}) and everywhere positive scalar function defined on the physical space-time $(\hat{\mathcal{M}}, \hat{g})$. This function, now called the CONFORMAL FACTOR, can be smoothly extended in $(\hat{\mathcal{M}}, \hat{g})$ but is zero-valued at the extra points affixed to $(\hat{\mathcal{M}}, \hat{g})$. This implies that, the extra points located in the unphysical space-time $(\mathcal{M}, g)$ are at infinity. The boundary of $(\hat{\mathcal{M}}, \hat{g})$, as constituted by these extra points is called CONFROMAL INFINITY, denoted by the symbol $\mathcal{J}$. For global requirements on an asymptotically flat space-time, we require that the conformal boundary be the disjoint union of two subsets, namely past null infinity ($\mathcal{J}^{-}$) and future null ($\mathcal{J}^{+}$) each of topology $S^2 \times \mathbb{R}$. Although, time-like and space-like geodesics, fields of non-zero rest mass, and gravitational interactions are NOT invariant under conformal rescalings, many important physical quantities, are -- such as null geodesics, null hypersurfaces, electromagnetic interactions and zero rest mass free fields. The main advantage of the conformal rescaling technique is that it makes possible a very convenient and co-ordinate free definition of asymptotic flatness in general relativity.

Let us now present a detailed discussion of the concept of asymptotic flatness. There are several interesting approaches to the GLOBAL and LOCAL conditions of asymptotic flatness by many authors including Penrose [7,15], Ludwig [8], Ashtekar and Dray [17], Persides [18], Newman and Tod [19]. But we shall limit the present discussion to only LOCAL conditions for asymptotic flatness which are necessary and sufficient for the current research. These local conditions are as follows:

Consider an orientable, non-compact Riemannian 4-dimensional manifold which is endowed with a Lorentzian metric of signature [+ - - -]. To this physical space time we shall attach a future oriented conformal boundary and define a conformal factor and a diffeomorphism

$$\phi : \hat{M} \rightarrow M$$

where M is the rescaled (unphysical manifold equipped with metric g. Then the physical space-time $(\hat{M}, \hat{g})$ is said to be locally asymptotically flat when the following conditions are satisfied:

AF1.

The conformally rescaled covariant metric tensor g_{ab} is related to the physical covariant metric tensor through the expression:

$$\hat{g}_{ab} = \Omega^{-2} g_{ab} \qquad 4.01$$

AF2.

The conformal factor Ω is a positive real-valued scalar function defined on $\hat{M}$. On the conformal boundary $\mathcal{J}^+$, we have the following conditions for Ω :

$$\Omega = 0\ ,\ \nabla_a \Omega \neq 0\ ,\ (\nabla_a \Omega)(\nabla^a \Omega) = 0$$

Thus $\nabla_a \Omega$ is non-zero and null on $\mathscr{I}^+$.

AF3.

The conformally rescaled Weyl tensor (C_{abcd}) vanishes on the conformal boundary $\mathscr{I}^+$.

AF4.

The conformally rescaled Ricci tensor (R_{ab}) remains finite and smooth on $\mathscr{I}^+$.

The conditions AF1 and AF2 are weak and do not require the conformal boundary $\mathscr{I}^+$ to have complete generators. These conditions are satisfied by a class of stationary spacetimes including Schwarzschild, Kerr and certain Weyl solutions. Also, there exist several physically interesting non-stationary spacetimes such as: the Robinson-Trautman solutions, the C-metric, and the Vaidya radiating metric which satisfy these conditions.

4.20 <u>The application of the conformal rescaling techniques to Tensor and Spinor fields</u>.

In this subsection, we shall use the Penrose conformal rescaling technique to work out the transformation laws for tensor and spinor functions under the diffeomorphism $\phi : \hat{\mathcal{M}} \rightarrow \mathcal{M}$. All tensor and spinor fields will be assumed to be of differentiability class C^∞.

Let $\hat{\nabla}_{A\dot{A}}$ be the pseudo-Riemannian connection defined on $\hat{\mathcal{M}}$ and determined uniquely by the metric $\hat{g}$. Similarly let

$\nabla_{A\dot{A}}$ be the corresponding connection defined on $\mathcal{M}$ and determined uniquely by the metric g. The transformation laws (properties) for the metric, the dyad, the Levi-Civita symbols and the connection $\nabla_{A\dot{A}}$ are given by the following expressions [7]

$$ds = \Omega\, d\hat{s}$$
$$g_{ab} = \Omega^2 \hat{g}_{ab} \qquad 4.03$$

$$\begin{aligned} o_A &= \hat{o}_A & o^A &= \Omega^{-1}\hat{o}^A \\ \iota_A &= \Omega\,\hat{\iota}_A & \iota^A &= \hat{\iota}^A \\ \epsilon_{AB} &= \Omega\,\hat{\epsilon}_{AB} & \epsilon_{\dot{A}\dot{B}} &= \Omega\,\hat{\epsilon}_{\dot{A}\dot{B}} \\ \epsilon^{AB} &= \Omega^{-1}\hat{\epsilon}^{AB} & \epsilon^{\dot{A}\dot{B}} &= \Omega^{-1}\hat{\epsilon}^{\dot{A}\dot{B}} \end{aligned} \qquad 4.04$$

$$\nabla_{A\dot{A}} X = \hat{\nabla}_{A\dot{A}} X \qquad 4.05;$$

Here X is any arbitrary, smooth scalar field. Now, for the spinor fields $(\xi^A, \xi_A, \eta^{\dot{A}}, \eta_{\dot{A}})$, we have the following transformation laws:[7]

$$\begin{aligned} \nabla_{A\dot{A}}\xi^B &= \hat{\nabla}_{A\dot{A}}\xi^B + \epsilon_A{}^B\,\Omega^{-1}\xi^C\,\nabla_{C\dot{A}}\Omega \\ \nabla_{A\dot{A}}\xi_B &= \hat{\nabla}_{A\dot{A}}\xi_B - \Omega^{-1}\xi_A\,\nabla_{B\dot{A}}\Omega \\ \nabla_{A\dot{A}}\eta^{\dot{B}} &= \hat{\nabla}_{A\dot{A}}\eta^{\dot{B}} + \epsilon_{\dot{A}}{}^{\dot{B}}\,\eta^{\dot{C}}\,\Omega^{-1}\,\nabla_{A\dot{C}}\Omega \\ \nabla_{A\dot{A}}\eta_{\dot{B}} &= \hat{\nabla}_{A\dot{A}}\eta_{\dot{B}} - \eta_{\dot{A}}\,\Omega^{-1}\,\nabla_{A\dot{B}}\Omega \end{aligned} \qquad 4.06$$

The components of $\nabla_{A\dot{A}}$ i.e. $D, \Delta, \delta, \bar{\delta}$, when acting on scalar fields, transform according to:

$$D = \Omega^{-1}\hat{D} \qquad \delta = \Omega^{-1}\hat{\delta}$$

$$\bar{\delta} = \Omega^{-1}\hat{\bar{\delta}} \qquad \Delta = \hat{\Delta} \qquad 4.07$$

We shall now present the explicit relation between the Ricci tensors $\hat{R}_{ab}$ $[\hat{g}_{ab}]$ in $\hat{M}$ and R_{ab} $[g_{ab}]$ in M. We start with the definition of the Riemann tensor given by

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = R^d{}_{cab} k_d \qquad 4.08.$$

We then perform all the necessary contractions with vectors, and invoke the following relation :

$$\nabla_a k_b = \hat{\nabla}_a k_b + 2\Omega^{-1} k_{(a}\nabla_{b)}\Omega - \Omega^{-1} g_{ab} k^c \nabla_c \Omega \qquad 4.09$$

We then perform the covariant differentiations, followed by the following specifications:

$$\hat{P}_{ab} = \tfrac{1}{2}\hat{R}_{ab} - \tfrac{1}{12}\hat{R}\hat{g}_{ab}$$

$$P_{ab} = \tfrac{1}{2} R_{ab} - \tfrac{1}{12} R g_{ab} \qquad 4.10$$

After some lengthy calculations, we arrive at the desired conformal rescaling expression given by:

$$\hat{P}_{ab} = P_{ab} - \Omega^{-1}\nabla_a\nabla_b\Omega + \tfrac{1}{2}\Omega^{-2} g_{ab} g^{cd}(\nabla_c\Omega)(\nabla_d\Omega) \qquad 4.11$$

The curvature information not contained in $\hat{P}_{ab}$ or, equivalently, in $\hat{R}_{ab}$ is exhibited by the conformal curvature tensor

$$\hat{C}^{ab}{}_{cd} = \hat{R}^{ab}{}_{cd} + 4\,\hat{P}_{[c}{}^{[a}\delta_{d]}{}^{b]} \qquad 4.12$$

which transforms according to the law:

$$\hat{C}^{ab}{}_{cd} = \Omega^2 C^{ab}{}_{cd} \qquad 4.13$$

The tensor $\hat{P}_{ab}$ contains the same information as the Ricci tensor $\hat{R}_{ab}$. The quantities P_{ab}, g_{ab}, $\nabla_a\Omega$, $\nabla_a\nabla_b\Omega$ all remain finite and continous on the conformal boundary $\mathcal{J}^+$. The tensor $\hat{P}_{ab}$ can be decomposed into its trace-free part ($\Phi_{AB\dot{A}\dot{B}}$) and trace (Λ) whose transformation laws under rescaling are respectively:

$$\text{(i)}\quad \Omega\hat{\Phi}_{AB\dot{A}\dot{B}} = \Omega\Phi_{AB\dot{A}\dot{B}} + \nabla_a\nabla_b\Omega - 1/4\, g_{ab}\nabla_c\nabla^c\Omega \qquad 4.14a$$

$$\text{where}\quad \hat{\Phi}_{AB\dot{A}\dot{B}} = 1/4\,\hat{P}\hat{g}_{ab} - \hat{P}_{ab} \qquad 4.14b$$

$$\text{and (ii)}\quad \hat{\Lambda} = \Omega^2\Lambda - 1/4\,\Omega\nabla_c\nabla^c\Omega + 1/2(\nabla_c\Omega)(\nabla^c\Omega)$$

$$\text{where}\quad \hat{\Lambda} = 1/24\,\hat{R} \qquad 4.15$$

4.03 <u>Conformal rescaling technique and the Newman-Penrose (NP) variables.</u>

In this subsection, we shall discuss the explicit conformal rescaling expressions for the spin coefficients, for the components of the Weyl and Ricci tensors.

Let us begin with the spin coefficients. The conformal rescaling expressions for the spin coefficients can be obtained by using the following procedure:

We start with the definitions of the spin coefficients in $\hat{M}$ and similarly in M (using the spinor formulation, since it is much easier to work with than the equivalent null tetrad approach). We then invoke the transformation laws for the spinor dyad as given by 4.05. The covariant differentiations are related by means of the transformation laws stated in 4.06. Finally we simplify the results using the definitions of the directional derivatives $D, \Delta, \delta, \bar{\delta}$ in M and the careted versions in $\hat{M}$. We obtain the following results for conformal rescaling of the spin coefficients under the diffeomorphism $\phi: \hat{M} \rightarrow M$.

$$
\begin{array}{lll}
\hat{\kappa} = \Omega^3 \kappa & \hat{\sigma} = \Omega^2 \sigma & \hat{\epsilon} = \Omega^2 \epsilon \\
\hat{\beta} = \Omega \beta & \hat{\lambda} = \lambda & \hat{\nu} = \Omega^{-1} \nu \\
\hat{\rho} = \Omega^2 (\rho + D \ln \Omega) & , & \hat{\tau} = \Omega (\tau + \delta \ln \Omega) \\
\hat{\pi} = \Omega (\pi - \bar{\delta} \ln \Omega) & & \hat{\alpha} = \Omega (\alpha + \bar{\delta} \ln \Omega) \\
\hat{\mu} = \mu - \Delta \ln \Omega & & \gamma = \gamma + \Delta \ln \Omega
\end{array}
$$

Let us now derive the conformal rescaling expression for the components of the Ricci tensor. We start with the conformal transformation law for the trace-free Ricci spinor. We then contract Φ_{ABAB} alternately with all the possible products of $(o, \bar{o}, \iota, \bar{\iota})$ and invoke the transformation laws for the spinor dyads and the derivative operator. We simplify results using the definitions of the spin coefficients and the components of the Ricci tensor in $\hat{M}$ and M. Finally, we simplify the results further by invoking the following previously established co-ordinate specifications:

$$
\begin{array}{lll}
D\Omega = f & \delta\Omega = w & \Delta\Omega = U \\
D\zeta = D\bar{\zeta} = 0 & & \bar{\delta}\Omega = \bar{w}
\end{array}
\qquad (4.17)
$$

The conformal rescaling expression for the components of the trace-free Ricci tensor under the diffeomorphism $\phi : \hat{M} \rightarrow M$ are then found to be:

$$
\begin{aligned}
\hat{\Phi}_{00} &= \Omega^4 \Phi_{00} + \Omega^3 [Df - (\epsilon + \bar{\epsilon}) f + \kappa \bar{w} + \bar{\kappa} w] \\
\hat{\Phi}_{01} &= \Omega^3 \Phi_{01} + \Omega^2 [Dw - \bar{\pi} f + (\bar{\epsilon} - \epsilon) w + \kappa U] \\
\hat{\Phi}_{02} &= \Omega^2 \Phi_{02} + \Omega [\delta w - \bar{\lambda} f + (\bar{\alpha} - \beta) w + \sigma U] \\
\hat{\Phi}_{11} &= \Omega^2 \Phi_{11} + \tfrac{1}{2} \Omega [DU + \bar{\delta} w - \bar{\mu} f - \bar{\pi} \bar{w} + (\bar{\beta} - \alpha - \pi) w \\
&\quad + (\rho + \epsilon + \bar{\epsilon}) U]
\end{aligned}
$$

$$\hat{\Phi}_{12} = \Omega\Phi_{12} + \delta U - \mu\omega - \bar{\lambda}\bar{\omega} + (\bar{\alpha} + \beta)U$$

$$\hat{\Phi}_{22} = \Phi_{22} + \Omega^{-1}[\Delta U - \nu\omega - \bar{\nu}\bar{\omega} + (\gamma + \bar{\gamma})U] .$$

$$\hat{\Phi}_{10} = \bar{\hat{\Phi}}_{10} \qquad \bar{\hat{\Phi}}_{20} = \bar{\hat{\Phi}}_{02} \qquad \hat{\Phi}_{21} = \bar{\hat{\Phi}}_{12} . \qquad 4.18$$

Similarly, using the transformation law for the trace of the Ricci tensor (Λ) given by equation 4.15 and using the procedure just described, we arrive at the relation:

$$\Lambda = \Omega^{2}\Lambda + \xi U - \omega\bar{\omega} + \tfrac{1}{2}\Omega[\bar{\delta}\omega - DU - \bar{\mu}\xi + \bar{\pi}\bar{\omega} + (\pi + \bar{\beta} - \alpha)\omega + (\rho - \epsilon - \bar{\epsilon})U] \qquad 4.19$$

Finally, by a similar procedure, we derive the conformal rescaling expressions for components of the Weyl tensor. We start with the transformation law for the Weyl spinor given by:

$$\hat{\Psi}_{ABCD} = \Omega^{-2}\Psi_{ABCD}$$

We then contract $\hat{\Psi}_{ABCD}$, alternately using all the possible products of $(o, \bar{o}, \iota, \bar{\iota})$. We simplify results using the transformation laws for the spinor dyads and invoke the definitions of the components of the Weyl tensor in M and $\hat{M}$. Then the desired conformal rescaling expressions for the components of the Weyl tensor under the differomorphism $\phi: \hat{M} \to M$ are found to be as follows:

$$\hat{\Psi}_0 = \Omega^4\Psi_0 \qquad \hat{\Psi}_1 = \Omega^3\Psi_1 \qquad \hat{\Psi}_2 = \Omega^2\Psi_2$$

$$\hat{\Psi}_3 = \Omega\Psi_3 \qquad \hat{\Psi}_4 = \Psi_4$$

Chapter 5

CHOICE OF A PREFERRED FRAME.

In this chapter, we shall discuss in detail [illegible] up of a local FRAME on the conformal boundary and propagate it to the interior of the space $\mathcal{M}-\mathcal{I}^+$. The term frame refers to a set of four quantities, namely: a dyad tetrad system, conformal factor Ω, a co-ordinate chart, and a function P of the co-ordinates. These quantities are closely related such that, generally any change effected on one will automatically induce changes in the other three. The members of the FRAME will be discussed explicitly in the succeeding subsections of this chapter. It is appropriate now to state that the function defines the curvature of the 2-surface of intersection of the u=contant hypersurfaces with the conformal boundary $\mathcal{I}^+$. The choice of a frame is non-unique and we will choose certain "preferred" frames. The freedom left in the choice of such a frame is called the generalized Newman-Unti freedom (to be discussed later). Each different choice of frame (accompanied by the appropriate change in the freely specifiable data) will in general, change the k-direction and lead to a different tube of the same space, -- a tube that does, however, end on the same neighbourhood of the same point of $\mathcal{I}^+$ as the original tube. Globally the new frame expresses the same spacetime in a frame based on a different congruence of twist-free and expanding null geodesics. We shall begin the frame set-up by considering the choice of the conformal factor and the dyad/tetrad. Our starting point is the neighbourhood of a point on the conformal boundary $\mathscr{I}^+$. We postulate only local properties for $\mathscr{I}^+$. These include the requirements that $\mathscr{I}^+$ need not necessarily have complete cuts nor possess complete generators nor be a null hypersurface of topology $S^2 \times \mathbb{R}$. This means that the piece of local spacetime we shall "build" will not necessarily be asymptotically flat in the global sense.

5.1 Choice of conformal factor and dyad/tetrad

We consider a point $x_\circ$ located on the conformal boundary $\mathscr{I}^+$ of the spacetime to be constructed and let $\mathcal{H}(x_\circ)$ be the neighbourhood of $x_\circ$. Starting from $\mathcal{H}(x_\circ)$, we shall construct a piece of spacetime consisting of a tube of null geodesics arriving at this neighbourhood from the choosen κ-direction. For this piece of spacetime, our local $\mathscr{I}^+$ is future null finity. Assume that on this $\mathscr{I}^+$, the conformal factor and the conformally rescaled (unphysical) Weyl tensor (C_{abcd}) vanish identically and the unphysical Ricci tensor (R_{ab}) remains regular and finite; and in addition, $\nabla_a\Omega$ is a non-zero null vector. On our local $\mathscr{I}^+$, we define the null vector n to be tangent to the generators of $\mathscr{I}^+$ and defined by the expression:

$$n_a = -\nabla_a \Omega \Big|_{\mathscr{I}^+} \qquad 5.0.1a$$

or equivalently, using spinors, we have

$$n_{A\dot{A}} = -\nabla_{A\dot{A}}\Omega \,\Big|_{\mathcal{J}^+} \qquad 5.01$$

The negative sign is chosen to imply that n is a future oriented null vector. We shall set up the FRAME in such a way that the only spin coefficients which might be non-zero on $\mathcal{J}^+$ are σ, $\bar{\alpha} - \beta$, μ, and $\gamma + \bar{\gamma}$. We shall call such a frame PREFERRED. These preferred frames form a representation of the generalized Newman-Unti group. Let us briefly discuss the conformal factor Ω. We can deduce from the conditions of local asymptotic flatness that the conformal factor Ω is non-unique. Any two possible choices and are related by

$$\Omega' = \Theta\,\Omega \qquad 5.02$$

where Θ is a regular strictly positive scalar function defined everywhere on $\mathcal{M}$, including the conformal boundary of future null infinity. Thus a new choice of Ω, leads to another preferred frame in which

$$n'_a = \Theta\, n_a \,\Big|_{\mathcal{J}^+} \qquad 5.03$$

Further discussion of the conformal factor is reserved for later subsections. Let us now analyse the geometrical properties of the generators of $\mathcal{J}^+$ as deduced from the n_a null geodesic congruence. When we apply the conditions of AF 2 to the trace-free part of the Ricci tensor $\hat{R}_{ab}$, we arrive at the following result, which is valid on $\mathcal{J}^+$

$$\nabla_a \nabla_b \Omega = 1/4\, g_{ab} \nabla_c \nabla^c \Omega \qquad 5.04$$

The conditions 5.01 and 5.04 imply that the generators of $\mathcal{J}^+$ are shear-free, twist-free but not necessarily divergenceless. These conclusions are deduced from the following analysis: The shear, twist, and divergence of the generators of $\mathcal{J}^+$ are defined respectively by the expressions:

$$(\lambda\bar{\lambda})^{1/2} = 1/2\left\{n_{(a,b)}n^{a,b} - (n^a{}_{;a})^2\right\}^{1/2}$$

$$i(\mu - \bar{\mu}) = \left\{n_{[a;b]}n^{a;b}\right\}^{1/2}$$

$$(\mu + \bar{\mu}) = n^a{}_{;a} \qquad 5.05$$

Now, the general expression for the covariant differentiation of the tangent vector n_a is given by

$$\begin{aligned} n_{a;b} = {} & \nu m_a k_b - \lambda m_a m_b - \mu m_a \bar{m}_b + \pi m_a n_b \\ & + \bar{\nu}\bar{m}_a k_b - \bar{\lambda}\bar{m}_a\bar{m}_b - \bar{\mu}\bar{m}_a m_b + \bar{\pi}\bar{m}_a n_b \\ & - (\gamma+\bar{\gamma}) n_a k_b + (\alpha+\bar{\beta}) n_a m_b + (\bar{\alpha}+\beta) n_a \bar{m}_b \qquad 5.06a \\ & - (\epsilon+\bar{\epsilon}) n_a n_b \end{aligned}$$

Restricting equation 5.06a to the conformal boundary $\mathcal{J}^+$ and using the expressions

$$n_a = -\nabla_a \Omega\big|_{\mathcal{J}^+}$$

$$\nabla_a\nabla_b\Omega = 1/4\, g_{ab}\nabla_c\nabla^c\Omega\big|_{\mathcal{J}^+}$$

$$g_{ab} = 2k_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$$

we obtain the following result on $\mathcal{J}^+$

$$n_{a;b} = -1/4\left\{k_a n_b + n_a k_b - m_a\bar{m}_b - \bar{m}_a m_b\right\}\nabla_c\nabla^c\Omega \qquad 5.06$$

It follows from equations 5.05 and 5.06, that on the conformal boundary $\mathcal{J}^+$:

$$\lambda = \nu = \bar{\alpha}+\beta = \bar{\lambda} = 0$$

$$\mu = \bar{\mu} = -(\gamma+\bar{\gamma}) = -1/4\,\nabla_c\nabla^c\Omega$$

Hence the shear $(\lambda\bar{\lambda})$ and the twist $i(\mu-\bar{\mu})$ of the generators of $\mathscr{J}^+$ vanish identically but the divergence, given by $-1/2\nabla_c\nabla^c\Omega$ can only vanish by an appropriate choice of the conformal factor Ω.

We shall now use the considerable amount of freedom we have in the choice of frame towards further simplication. We define Q to be a positive definite scalar function on the conformal boundary $\mathscr{J}^+$. Then we perform these two operations -- (T_1 followed by T_2)

T_1: Rescale the metric with

conformal factor $\Theta = a\bar{a}$

T_2: Perform the spinor transformation

$$\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} o^A \\ \iota^A \end{pmatrix}$$

where $a = a_o e^{i\phi}$

We can solve the following equations

$$\bar{\delta}\{\ln(Q\bar{a})\} = \alpha$$

$$\Delta\{\ln(Q\bar{a})\} = \gamma \qquad 5.08$$

on $\mathscr{J}^+$ for the parameters a_o and ϕ. We can show that two expressions of 5.08 are compatible on $\mathscr{J}^+$ by taking the $\bar{\delta}-\Delta$ mixed derivatives and using the NP equations R_{10} and the resutls of 5.07, in addition to the fact that the unphysical Weyl spinor Ψ_{ABCD} and its dyad frame components vanish on $\mathscr{J}^+$. By using the results of the transformation LD, in addition to the results of conformal transformation of the

NP variables, we can deduce that the previously established tetrad conditions:

$$l_A \bar{l}_{\dot{A}} = -\nabla_{A\dot{A}}\Omega$$

$$\lambda = \bar{\lambda} = \nu = \bar{\alpha} + \beta = 0$$

remain invariant under the operations T, and T'. We also find that α, β, γ, and μ are given by the expressions

$$\alpha = -\bar{\beta} = \bar{\delta}\ln Q$$

$$\gamma = -\tfrac{1}{2}\mu = \Delta \ln Q \qquad 5.09$$

The next strategy is to explore and exploit all the remaining freedom in the choice of the spinor dyad (o^A, ι^A) as a field on $\mathscr{J}^+$. The spinor o^A and hence the tetrad vector k^a is pinned down to some extent as a field on $\mathscr{J}^+$ by requiring that

$$\mathrm{Im}\,\rho' = 0$$

$$\tau' = 0 \qquad 5.10$$

That this can be done, follows from the transformation laws for ρ and τ under the Lorentz transformation L(3) (discussed earlier in chapter 3.00). This leads to the equations

$$\mathrm{Im}\{\bar{\delta}b - 2b\alpha - \rho\} = 0$$

$$\tau + 2b\gamma - \Delta b = 0 \qquad 5.11$$

That the two equations of 5.11 are compatible on $\mathscr{J}^+$ is seen by working out the mixed derivatives

$$\bar{\delta}\{\Delta b - \tau - 2b\gamma\} = 0$$

$$\Delta\{\mathrm{Im}(\bar{\delta}b - 2b\alpha - \rho)\} = 0 \qquad 5.12$$

and applying the NP commutator equations. The imaginary part of ρ is now zero on $\mathscr{J}^+$. The real part can made to vanish identically in $\mathcal{M}$ by means of conformal rescaling. This is done by solving the equation

$$\rho' + \bar{\rho}' = 2\, D \ln \theta$$

subject to $\theta = 1$ on $\mathcal{J}^+$ 5.13

5.20 Choice of co-ordinate system

In this subsection, we shall discuss how to set up another element of the frame, namely the co-ordinate system. Let $x_0 \in \mathcal{J}^+$ and consider a neighbourhood of x_0. From each point of this neighbourhood, a null geodesic emanates into the interior of $\mathcal{M}$ in the k_a direction, of our null tetrad. We now extend our dyad to the interior points of $\mathcal{M}$ by propagating the dyad chosen on $\mathcal{J}^+$ parallely along these null geodesics. As a consequence, the spin coefficients κ, π, and ϵ vanish identically.

Proposition: This null geodesic congruence is hypersurface orthogonal. In fact, k_a is equal to the gradient of some function, say u. (i.e. $k_a = \nabla_a u$).

Proof: Since κ, ϵ, π vanish identically, the NP equation R_1 is given by

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \qquad 5.14$$

The real and imaginary parts of equation 5.14 are given respectively by

$$D(\rho+\bar{\rho}) = \bar{\rho}^2 + \rho^2 + 2\bar{\sigma}\sigma + 2\Phi_{00} \qquad 5.15$$

$$D(\rho-\bar{\rho}) = (\rho+\bar{\rho})(\rho-\bar{\rho}) \qquad 5.16$$

But in $\mathcal{M}$, we have $\rho+\bar{\rho}=0$, thus $D(\rho-\bar{\rho})=0$ which means that $\mathrm{Im}\,\rho$ is constant along the tube of null geodesics in $\mathcal{M}$ (in the k_a-direction). But $Im\,\rho$ is zero on $\mathcal{J}^+$ (shown earlier). Hence ρ vanishes identically in $\mathcal{M}$. Therefore the congruence is hypersurface orthogonal; and as an additional result, we have found that

$$\phi_{00} = -\sigma\bar{\sigma} \qquad 5.17$$

Now, as deduced form the previous chapter, the null vector k_a is equal to the gradient fo a scalar function say u iff:

$$\kappa = \epsilon + \bar{\epsilon} = 0 \qquad \rho - \bar{\rho} = 0$$
$$\bar{\alpha} + \beta - \tau = 0 \qquad 5.18$$

It remains to show that: $\bar{\alpha} + \beta - \tau = 0$.

From the NP equations R_1, R_4 and R_5 we deduce that:

$$D(\bar{\alpha} + \beta - \tau) = \sigma(\bar{\alpha} + \beta - \tau) \qquad 5.19$$

Hence $\bar{\alpha} + \beta - \tau = 0$ in since it vanishes identically on $\mathcal{J}^+$. Hence $k_a = \nabla_a u$. This completes the proof.

We now discuss how to set up the co-ordinate system and choose the curvature function P. We start from the the neighbourhood of a point $x_0 \in \mathcal{J}^+$ on the conformal boundary $\mathcal{J}^+$. We consider a tube of null geodesics arriving at this neighbourhood from the chosen k_a-direction. For this piece of space-time (to be constructed), our $\mathcal{J}^+$ is future "null infinity". We propagate the tetrad parallely along the null geodesics and consequently $\kappa = \pi = \epsilon = 0$ identically: The geodesics of $\mathcal{M}$ arriving at this neighbourhood from the k_a direction were shown to be hypersurface orthogonal and k_a was shown to be given by the expression $k_a = \nabla_a u$. The co-ordinatization is done as follows: We choose u as the first co-ordinate. Now, each null hypersurface

$u=u_0$ constant cuts the conformal boundary $\mathscr{I}^+$ in a two surface. We choose one such 2-surface and introduce the co-ordinate $(\zeta, \bar{\zeta})$ on it. We then extend this co-ordinate pair $(\zeta, \bar{\zeta})$ to the entire $\mathscr{I}^+$ by demanding that they remain constant along the generators of $\mathscr{I}^+$. This means that on $\mathscr{I}^+$:

$$\Delta \zeta = \Delta \bar{\zeta} = 0 \qquad 5.19$$

The co-ordinates $(u, \zeta, \bar{\zeta})$ can used to specify the parameter P, which determines the curvature of the 2-surfaces formed by the intersection of $u=u_0$ hypersurfaces and $\mathscr{I}^+$. We shall prefer to leave $P(u, \zeta, \bar{\zeta})$ an arbitrary function of $(u, \zeta, \bar{\zeta})$ as in [4]. Further discussion on will come later. We further extend $(\zeta, \bar{\zeta})$ into all of M by imposing the constraint that they remain constant along the null geodesic congruence. This implies that

$$D\zeta = D\bar{\zeta} = 0 \quad \text{in} \quad M \qquad 5.20$$

We have now defined a co-ordinate system $x^i = [u, \Omega, \zeta, \bar{\zeta}]$ everywhere along the tube of null geodesics arriving at the neighbourhood of the point $x_0 \in \mathscr{I}^+$, (in the k_a direction), if we choose the conformal factor Ω as the remaining co-ordinate. This is feasible, at least in a neighbourhood of $\mathscr{I}^+$, since on $\mathscr{I}^+$ we have

$$D\Omega = k^a \nabla_a \Omega = -k^a n_a = -1 \qquad 5.21$$

As stated in a previous section, the metric variables consist of the set $[f, \omega, \xi_1, \xi_2, X$ and $U]$. The variables f and U are real-valued scalar functions while the rest of the set i.e. (ω, ξ_1, ξ_2) are complex-valued scalar functions. All the metric variables and all the other NP variables (i.e.

components of the Riemann tensor and the spin coefficients are assumed to be analytic and regular in the conformal factor. We shall now discuss the curvature function P, more explicitly. We define P as follows:

$$P(u,\zeta,\bar{\zeta}) = -\tfrac{1}{2}\xi_1 \quad \text{on } \mathcal{J}^+ \qquad 5.22$$

The choice of $(\zeta, \bar{\zeta})$ on one $u=u_0$ hypersurface was arbitrary; but if we choose $(\zeta, \bar{\zeta})$ such that

$$\delta\bar{\zeta} = 0 \quad \text{on } \mathcal{J}^+ \qquad 5.23$$

then the metric variable ξ_2 will vanish identically on $\mathcal{J}^+$ and we find that:

$$\xi_2 = 0 \quad , \quad \delta = \xi_1 \frac{\partial}{\partial\zeta} \quad \text{on } \mathcal{J}^+ \qquad 5.24$$

At this juncture, it is possible to establish a relation between the functions Q and P mentioned in the choice of FRAME. This can be done as follows: Invoke the NP commutator equations and replace the scalar function Φ alternately by the co-ordinates $\zeta, \bar{\zeta}$ subject to the conditions:

$$\xi_1 = -2P(u,\zeta,\bar{\zeta}) \;,\; \xi_2 = 0 \quad \text{on } \mathcal{J}^+$$

We arrive at the following expressions

$$\delta \ln(P/Q^2) = 0$$

$$\bar{\delta} \ln(P/Q^2) = 0$$

$$\Delta \ln(P/Q^2) = 0 \qquad 5.25$$

This leads to the deduction that:

$$P = c(\zeta)Q^2 \qquad 5.26$$

The function $c(\zeta)$ can be put to unity by means of the

Newman-Unti freedom which we shall discuss explicitly in the next subsection. Hence P and Q are real scalar functions related by the expression

$$P = Q^2 \qquad 5.27$$

The boundary values of the metric variables are obtained as follows: Recall that metric variables $[f, w, \xi_1, \xi_2, U, X]$ are defined as the coefficients of the relations

$$D = f\frac{\partial}{\partial\Omega}$$

$$\delta = w\frac{\partial}{\partial\Omega} + \xi_1\frac{\partial}{\partial\zeta} + \xi_2\frac{\partial}{\partial\bar{\zeta}}$$

$$\delta = \bar{w}\frac{\partial}{\partial\Omega} + \bar{\xi}_2\frac{\partial}{\partial\zeta} + \bar{\xi}_1\frac{\partial}{\partial\bar{\zeta}}$$

$$\Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial\Omega} + X\frac{\partial}{\partial\zeta} + \bar{X}\frac{\partial}{\partial\bar{\zeta}}$$

The values of the metric variables on the conformal boundary $\mathcal{J}^+$ are obtained by subjecting the above equations to the following specifications on $\mathcal{J}^+$

(i) $\delta\Omega = \Delta\Omega = 0$

(ii) $\Delta\zeta = \Delta\bar{\zeta} = 0$

(iii) $D\Omega = -1$

(iv) $\delta\bar{\zeta} = 0 \qquad 5.28$

The condition 5.28 (i-iv) have all been discussed earlier. Thus on the conformal boundary $\mathcal{J}^+$, the metric variables take the following values.

$$f^0 = -1 \qquad w^0 = U^0 = X^0 = 0$$

$$\xi_2^0 = 0 \qquad \xi_1^0 = -2P(u,\zeta,\bar{\zeta}) \qquad 5.29$$

The zero super-script is used to denote the fact that the variable is evaluated on $\mathcal{J}^+$. Henceforth, the value of any NP variable B on $\mathcal{J}^+$ is denoted by B^0. As stated earlier B is regular and analytic in Ω and can be expanded as an Ω-series to any desired order:

$$B = B^0 + B^{(1)}\Omega + B^{(2)}\Omega^2 + B^{(3)}\Omega^3 + \ldots + B^{(n-1)}\Omega^{n-1} + O(\Omega^n)$$

The metric variable f can be specified further if we use the following equations:

$$\nabla_a \nabla_b \Omega = 1/4\, g_{ab} \nabla_c \nabla^c \Omega \big|_{\mathcal{J}^+},$$

$$D = f \frac{\partial}{\partial \Omega}$$

where $Dk^a = 0$

$$D\Omega = f \qquad 5.31a$$

We discover that $D^2\Omega = 0$ which implies that the function f must have the form

$$f = -1 + f^{(2)}\Omega^2 + f^{(3)}\Omega^3 + O(\Omega^4) \qquad 5.31b$$

5.30 <u>The generalized Newman-Unti freedom</u>

We shall now discuss the generalized Newman-Unti freedom. In the previous subsection we stated that the term FRAME refers to a set of four, comprising a co-ordinate system, conformal factor, curvature function $P(u,\zeta,\bar{\zeta})$ and dyad/tetrad system all tied together in the way described. We shall now discuss the asymptotic forms of the finite transformations which preserve all the relations derived during the choice of a preferred frame. These transformations are elements of the generalized Newman-Unti group. The freedom left in the choice of a preferred frame is called the generalized Newman-Unti freedom. An element of the generalized NU group will change this frame to another, i.e. it will in general, change the coordinate system, the null tetrad, the conformal factor and the curvature function $P(u, \zeta, \bar{\zeta})$. The generalized NU group acts locally and we do not require the generators of $\mathscr{J}^+$ to be complete, nor do we require complete cross-sections.

We shall now discuss how to change from one "preferred" frame to another. Let $x^i = [u, \Omega, \zeta, \bar{\zeta}]$ be the local co-ordinate chart of one such frame on a neighbourhood of a point $x_0 \in \mathscr{J}^+$. Let $x'^i = [u', \Omega', \zeta', \bar{\zeta}']$ be another. It can be shown that they must be related by

$$
\begin{aligned}
u' &= G(u, \zeta, \bar{\zeta}) + O(\Omega) \\
\zeta' &= \zeta'(\zeta) + O(\Omega) \\
\Omega' &= \Theta\, \Omega + O(\Omega^2)
\end{aligned}
\qquad 5.35
$$

where $\Theta = G_{,u}$ and G is an arbitrary smooth function. This change in co-ordinates and conformal factor must be accompanied by a boost (with parameter a_0) in the k-n plane of the tetrad and a spatial rotation (with parameter ϕ) in the m-$\bar{m}$ plane subject to the conditions:

$$
\begin{aligned}
\Theta &= a_0^2 \\
\frac{\partial \zeta'}{\partial \zeta} &= \Theta\, P' P^{-1} e^{-2i\phi}
\end{aligned}
$$

This must be followed immediately by a null rotation of the tetrad about the null vector n with parameter b given by

$$b = 2\Theta^{-1} P \frac{\partial G}{\partial \zeta} e^{2i\phi}$$

We now present a brief summary of the boundary conditions on the metric variables, spin-coefficients, components of the Riemann tensor and the curvature function $P(u, \zeta, \bar{\zeta})$. For each preferred frame, which is unique up to the Newman-Unti group, we have

$$\kappa = \pi = \rho = 0$$

$$\tau - (\bar{\alpha} + \beta) \doteq 0 \quad \text{and} \quad \Phi_{00} + \sigma\bar{\sigma} = 0 \qquad 5.38$$

identically in $\mathcal{M}$. On the conformal boundary $\mathcal{J}^{+}$, the metric variables U, X, ω, ξ_2 and the unphysical Weyl spinor Ψ_{ABCD} components vanish, as do all the spin coefficients but, possibly $\sigma, \alpha, \beta, \gamma$ and μ. In addition,

$$\alpha^{\circ} = 1/2\, \bar{\delta}^{\circ} \ln P = -\bar{\beta}^{\circ}$$

$$2\gamma^{\circ} = -\mu^{\circ} = \Delta^{\circ} \ln P$$

and $P(u, \zeta, \bar{\zeta})$ is an arbitrary real positive function on $\mathcal{J}^{+}$. Also we have the following results

$$f^{\circ} = -1 \qquad f^{(1)} = 0$$

$$\xi_1^{\circ} = -2P(u, \zeta, \bar{\zeta})$$

Chapter 6

THE TECHNIQUES FOR GENERATING SYSTEMATIC SOLUTIONS TO THE EINSTEIN FIELD EQUATIONS IN THE NEWMAN-PENROSE FORMALISM:

In this section, we shall discuss explicitly how to restructure the NP equations into a non-redundant, necessary and sufficient system of non linear differential equations which can enable us construct a local solution to Einstein field equations. In other words, using the system of restructured NP equations we shall discuss how to construct a piece of physical space-time (metric) in the neighbourhood of a point on the conformal boundary $\mathscr{I}^+$. These transformed solutions of the Einstein field equations will be valid in some neighbourhood of $\mathscr{I}^+$. Beyond this neighbourhood, there is a possibility of formation of caustics and co-ordinate singularities; but once such a local solution is obtained, it can be analysed rigorously as to the range of validity of the co-ordinate system as well as its asymptotic flatness and possible global extensions. Before we describe the technique of generating systematic solutions to the field equations we have to reformulate the relevant system of NP equations. The equations at our disposal are:

(i) The metric equations $(M_1 - M_{10})$

(ii) The "Ricci identities" $(R_1 - R_{18})$

(iii) The transformation equations for the Ricci tensor under the conformal rescaling $(L_{12} - L_{18})$

A complete solution to the equations listed in (i)-(iii) will include not only the local spacetime metric via the metric variables, but also the spin-coefficients, and the components of the Riemann tensor. These solutions are first of all obtained in the unphysical space $\mathcal{M}$ by using either the integration technique or, equivalently, the Penrose conformal technique. The solutions in the unphysical space $\mathcal{M}$ are then transcribed into the physical space $\hat{\mathcal{M}}$ in terms of an alternate set of freely specifiable functions which now constitute the appropriate initial data set. We shall first present the reformulation scheme for the NP equations.

6.10 The Ludwig reformulation scheme for the NP equations:

The NP equations (comprising the metric equations, "Ricci identities", "Bianchi identities", and conformal rescaling expressions for the NP variables) contain a number of redundant equations which must be discarded when seeking an efficient and systematic method for generating solutions to the Einstein field equations. Ludwig [9] devised an elegant technique for reformulating the NP equations into a compact system of necessary and sufficient non-linear differential equations. The benefits from this approach include the following:

1. All redundant subsets of the metric equations and the "Ricci identities" are consequently eliminated.
2. It enables a systematic derivation of known exact solutions such as Schwarzchild, Vaidya and, Robinson-Trautmann; it also makes possible the generation of asymptotic solutions from which we can

recover known solutions (Newman-Unti [4], Ludwig [8], etc.) in a much more systematic manner.

3. The freely specifiable initial data are clearly exhibited even in the case of an arbitrary gravitational source.

4. The resulting system of non-linear partial differential equations can be scrutinized for possible simplifying assumptions that might lead to new exact solutions to the Einstein field equations in closed form.

We now give a detailed description of the reformulation procedure for the space M. The relevant expressions at our disposal are the metric equations $(M_1 - M_{10})$, the "Ricci identities" $(R_1 - R_{..})$; the explicit structure for the directional covariant derivative operator $\nabla_a (= D, \delta, \bar{\delta}, \Delta)$, thee conformal rescaling equations for the components of the Riemann tensor and the tetrad conditions:

$$\kappa = \pi = \epsilon = \rho = \bar{\rho} = 0 \qquad \tau = (\bar{\alpha} + \beta)$$

Let us start with the procedure for the metric variables and their complex conjugates. From the radial metric equations, we derive the following system of differential equations:

$$L_1 : \frac{\partial}{\partial \Omega} \begin{pmatrix} \xi_1 \\ \bar{\xi}_2 \end{pmatrix} = \mathfrak{f}^{-1} \begin{pmatrix} 0 & \sigma \\ \bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \bar{\xi}_2 \end{pmatrix}$$

and its complex conjugate

$$\frac{\partial}{\partial \Omega} \begin{pmatrix} \bar{\xi}_1 \\ \xi_2 \end{pmatrix} = \mathfrak{f}^{-1} \begin{pmatrix} 0 & \bar{\sigma} \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_1 \\ \xi_2 \end{pmatrix}$$

$$
L_2: \quad \frac{\partial}{\partial\Omega}\begin{bmatrix} w \\ \bar{w} \\ \tau \\ \bar{\tau} \end{bmatrix} = f^{-1}\begin{Bmatrix} \frac{\partial f}{\partial\Omega} & \sigma & -f & 0 \\ \bar{\sigma} & \frac{\partial f}{\partial\Omega} & 0 & -f \\ 2f^{-1}\sigma\bar{\sigma} & \frac{\partial\sigma}{\partial\Omega} & 0 & 0 \\ \frac{\partial\bar{\sigma}}{\partial\Omega} & 2f^{-1}\sigma\bar{\sigma} & 0 & 0 \end{Bmatrix}\begin{bmatrix} w \\ \bar{w} \\ \tau \\ \bar{\tau} \end{bmatrix}
$$

$$
+ f^{-1}\begin{Bmatrix} \xi_1 \frac{\partial f}{\partial\zeta} + \xi_2 \frac{\partial f}{\partial\bar{\zeta}} \\ \bar{\xi}_1 \frac{\partial f}{\partial\bar{\zeta}} + \bar{\xi}_2 \frac{\partial f}{\partial\zeta} \\ 2\phi_{01} + \bar{\xi}_1 \frac{\partial\sigma}{\partial\bar{\zeta}} + \bar{\xi}_2 \frac{\partial\sigma}{\partial\zeta} + 2\sigma\left\{ \frac{\partial\bar{\xi}_1}{\partial\bar{\zeta}} + \frac{\partial\bar{\xi}_2}{\partial\zeta} - \bar{\xi}_1 \frac{\partial\ln\xi}{\partial\bar{\zeta}} - \bar{\xi}_2 \frac{\partial\ln\xi}{\partial\zeta} \right\} \\ 2\phi_{10} + \xi_1 \frac{\partial\bar{\sigma}}{\partial\zeta} + \xi_2 \frac{\partial\bar{\sigma}}{\partial\bar{\zeta}} + 2\bar{\sigma}\left\{ \frac{\partial\xi_1}{\partial\zeta} + \frac{\partial\xi_2}{\partial\bar{\zeta}} - \xi_1 \frac{\partial\ln\xi}{\partial\zeta} - \xi_2 \frac{\partial\ln\xi}{\partial\bar{\zeta}} \right\} \end{Bmatrix}
$$

where $\xi = \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2$

$$
L_3: \quad \frac{\partial}{\partial\Omega} X = f^{-1}(\tau\bar{\xi}_2 + \bar{\tau}\xi_1), \quad \frac{\partial}{\partial\Omega}\bar{X} = f^{-1}(\bar{\tau}\xi_2 + \tau\bar{\xi}_1)
$$

$$
L_4: \quad \frac{\partial}{\partial\Omega}\begin{pmatrix} U \\ \gamma+\bar{\gamma} \end{pmatrix} = f^{-1}\begin{pmatrix} \frac{\partial f}{\partial\Omega} & -f \\ 2f^{-1}\sigma\bar{\sigma} & 0 \end{pmatrix}\begin{pmatrix} U \\ \gamma+\bar{\gamma} \end{pmatrix}
$$

$$
+ f^{-1}\begin{pmatrix} \tau\bar{w} + \bar{\tau}w + \Delta' f \\ 2\delta\alpha + 2\bar{\delta}\bar{\alpha} - 6\alpha\bar{\alpha} - 2\beta\bar{\beta} + 2\tau\bar{\tau} - 12\Lambda + (\Psi_2 + 2\Lambda)' + (\Psi_2 + 2\Lambda)' \end{pmatrix}
$$

where $\Delta' = \Delta - U\frac{\partial}{\partial\Omega}$; $(\Psi_2 + 2\Lambda)' = \Psi_2 + 2\Lambda - (U/f)\sigma\bar{\sigma}$

The equations L_1-L_4 can now be solved, in principle, for the metric variables ($\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$), ($w, \bar{w}$), ($X, \bar{X}$), and U, repectively. The solutions are obtained in in terms of the freely specifiable data set. D_1: $f, \sigma, \Lambda, \phi_{01}$, and the boundary conditions on future conformal null infinity $\mathcal{J}^+$. We can also obtain solutions for ($\tau, \bar{\tau}$) using equation L_2. Similarly the equation L_4 can be solved for ($\gamma, \bar{\gamma}$). We shall now derive expressions for the other spin coefficients using the non-radial metric equations. The spin coefficients [α, β, γ, μ, ν, λ] and their complex conjugates are defined in this scheme by the following expressions:

$$L5: \alpha - \bar{\beta} = \xi^{-1}\{\bar{\xi}_1(\bar{\delta}\xi_1 - \delta\bar{\xi}_2) - \bar{\xi}_2(\bar{\delta}\xi_2 - \delta\bar{\xi}_1)\}$$

$$L6 \quad \nu = f^{-1}\{\Delta\bar{\omega} - \bar{\delta}U + \bar{\omega}(\bar{\mu} + \gamma - \bar{\gamma}) + \lambda\omega\}$$

$$L7 \quad \lambda = \xi^{-1}\{\bar{\xi}_1(\bar{\delta}X - \Delta\bar{\xi}_2) - \bar{\xi}_2(\bar{\delta}\bar{X} - \Delta\bar{\xi}_1)\}$$

$$L8 \quad \mu - \bar{\mu} = f^{-1}\{\delta\bar{\omega} - \bar{\delta}\omega + \omega(\alpha - \bar{\beta}) + \bar{\omega}(\beta - \bar{\alpha})\}$$

$$L9 \quad \mu + \bar{\mu} = \xi^{-1}\{\xi_1(\bar{\delta}\bar{X} - \Delta\bar{\xi}_1) - \xi_2(\bar{\delta}X - \Delta\bar{\xi}_2) + \bar{\xi}_1(\delta X - \Delta\xi_1) - \bar{\xi}_2(\delta\bar{X} - \Delta\xi_2)\}$$

$$L10: 2(\bar{\gamma} - \gamma) = -(\mu - \bar{\mu}) + \xi^{-1}\{\bar{\xi}_1(\delta X - \Delta\xi_1) - \bar{\xi}_2(\delta\bar{X} - \Delta\xi_2) - \xi_1(\bar{\delta}\bar{X} - \Delta\bar{\xi}_1) + \xi_2(\bar{\delta}X - \Delta\bar{\xi}_2)\}$$

where $\xi = \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2$

Next, using a non-redundant subset of the "Ricci identities" (R, - R, ,), the Ricci tensor components are defined in the reformulation scheme by the following expressions and their complex conjugates:

$$L11: \quad \phi_{00} = -\sigma\bar{\sigma}$$

$$L12: \quad \phi_{01} = \Psi_1 - \bar{\delta}\sigma - \sigma(\bar{\beta} - 3\alpha)$$

$$L13: \quad \phi_{11} = -3\Lambda + \delta\alpha + \bar{\delta}\bar{\alpha} - 3\alpha\bar{\alpha} - \beta\bar{\beta}$$

$$L14: \quad \phi_{22} = \bar{\delta}\bar{\nu} - \Delta\bar{\mu} - \bar{\mu}^2 - \lambda\bar{\lambda} - \bar{\mu}(\gamma + \bar{\gamma}) - \bar{\nu}(\bar{\tau} - 3\bar{\beta} - \alpha)$$

$$L15: \quad \phi_{02} = \delta\tau - \Delta\sigma - \mu\sigma - \tau(\tau + \beta - \bar{\alpha}) - \sigma(\bar{\gamma} - 3\gamma)$$

$$L16: \quad \phi_{12} = \delta\gamma - \Delta\beta - \mu\tau + \sigma\nu - \beta(\bar{\gamma} - \gamma + \mu) - \alpha\bar{\lambda}$$

$$\phi_{10} = \bar{\phi}_{01} \qquad \phi_{21} = \bar{\phi}_{12} \qquad \phi_{20} = \bar{\phi}_{02}$$

Similarly, using the remaining non-redundant subset of the Ricci identities, we obtain the following definitions for the components of the Weyl tensor.

L_{17} $\Psi_0 = D\dot{\sigma}$

L_{18} $\Psi_1 = \Phi_{01} + \bar{\delta}\sigma + \sigma(\bar{\beta} - 3\alpha)$

L_{19} $\Psi_2 = -2\Lambda + \bar{\delta}\tau - \sigma\lambda - \tau(\bar{\tau} + \alpha - \bar{\beta})$

L_{20} $\Psi_3 = \bar{\delta}\gamma - \Delta\alpha - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau})$

L_{21} $\Psi_4 = \bar{\delta}\nu - \Delta\lambda - (\mu + \bar{\mu})\lambda + \lambda(\bar{\gamma} - 3\gamma) + \nu(3\alpha + \bar{\beta} - \bar{\tau})$

We also need the conformal rescaling expressions for the Riemann tensor components under the diffeomorphism,

$$\phi : \hat{M} \rightarrow M$$

Using the tetrad condition $\kappa = \pi = \epsilon = 0$, $\rho = \bar{\rho} = 0$ and $\tau - (\bar{\alpha} + \beta) = 0$, we obtain the required conformal rescaling expressions from the results discussed in chapter four.

L_{22}: $\hat{\Phi}_{00}\Omega^{-3} = \Omega\Phi_{00} + Df$

L_{23}: $\hat{\Phi}_{01}\Omega^{-2} = \Omega\Phi_{01} + D\omega$

L_{24}: $\hat{\Phi}_{02}\Omega^{-1} = \Omega\Phi_{02} + \delta\omega - \bar{\lambda}f + (\bar{\alpha} - \beta)\omega + \sigma U$

L_{25}: $\hat{\Phi}_{11}\Omega^{-1} = \Omega\Phi_{11} + 1/2\{DU + \bar{\delta}\omega - \bar{\mu}f + (\bar{\beta} - \alpha)\omega\}$

L_{26}: $\hat{\Phi}_{12} = \Omega\Phi_{12} + \delta U - \mu\omega - \bar{\lambda}\bar{\omega} + \tau\omega$

L_{27}: $\hat{\Phi}_{22} = \hat{\Phi}_{22} + \Omega^{-1}\{\Delta U - \nu\omega - \bar{\nu}\bar{\omega} + (\gamma + \bar{\gamma})U\}$

L_{28}: $\hat{\Lambda} = \Omega^2\Lambda + fU - \omega\bar{\omega} + 1/2\Omega\{\bar{\delta}\omega - D\omega - \bar{\mu}f$
$+ (\bar{\beta} - \alpha)\omega\}$

L_{29}: $\hat{\Psi}_0 = \Omega^4\Psi_0 \quad \hat{\Psi}_1 = \Omega^3\Psi_1 \quad \hat{\Psi}_2 = \Omega^2\Psi_2$

L_{30}: $\hat{\Psi}_3 = \Omega\Psi_3 \quad \hat{\Psi}_4 = \Psi_4$

The corresponding results for the spin coefficients are given explicitly in chapter four while those for the metric variables will be worked out comprehensively in the next chapter. Using the reformulated equations L_1 -- L_{11}, we, in principle, obtain solutions for the Einstein field equations and hence construct a piece of (unphysical) space time in the neighbourhood of a point on the conformal boundary. Then, invoking the conformal rescaling expression for the components of the Riemann tensor and other relevant NP variables subject to the Lorentz gauge invariant transformations L①, L② and L③, we transcribe the solutions in the unphysical space M into the physical space $\hat{M}$. These solutions, at this instance, are in terms of the freely specifiable initial data set D_1: ξ, σ, ϕ_{01}, $\overset{\circ}{\Lambda}$ and ρ. However, there are at least two problems to be resolved. First, the freely specifiable data set D_1 are variables in the unphysical space M not $\hat{M}$. Second, and most importantly the Ricci tensor of the spacetime generated with data set D_1, has no specific property. Furthermore, if we impose certain constraints on these free data, such as the vacuum conditions, they cease to be freely specifiable and satisfy some complicated constraint equations. We shall rectify these drawbacks by re-expressing this free data set D_1 in terms of an alternate free data set D_2 of well-behaved functions involving the components of the physical Ricci tensor. However, there still remains the major difficulty. It is seemingly impossible, in general, to generate closed form solutions to all the non-linear systems of differential equations. Nevertheless, we can generate generalized

Newman-Unti type asymptotic expansion solutions for the field equations in the neighbourhood of a point $x_0 \in \mathscr{I}^+$; and then transcribe results into the physical space $\hat{M}$. To do this, we assume analyticity and regularity of each NP variable in terms of the conformal factor Ω. Then for each NP variable, we simply substitute a power series in Ω into the reformulated equations. We then solve for the expansion coefficients to a high enough order. These results initially obtained in M, are then transcribed into $\hat{M}$ in terms of an alternate, appropriate, freely specifiable initial data set D_1. The final Newman-Unti type asymptotic expansion solution in $\hat{M}$ is given in terms of a parameter r instead of Ω. This r is an affine parameter along the null geodesics in the u=constant hypersurfaces. The explicit choice of r will be discussed in the next chapter.

Chapter 7

GENERALIZED ASYMPTOTIC SOLUTIONS TO THE EINSTEIN FIELD EQUATIONS:

In the previous chapter, we discussed comprehensively the techniques required for solving the restructured NP equations at our disposal. We shall now use these techniques to generate generalized Newman-Unti-type asymptotic expansions which are solutions to the Einstein field equations. The solutions are initially obtained in the rescaled space M and then transcribed into the physical space $\hat{M}$. The co-ordinate system in $\hat{M}$ is based on a twist-free, expanding null geodesic congruence while the corresponding co-ordinate system in the rescaled space M is a twist-free, non-expanding null geodesic congruence as described in chapter five.

7.10 Solution in the rescaled space M

In this subsection we shall present expressions for all NP variables including the covariant and contravariant metric tensors in the rescaled space M. The main advantages derived from the derivation of the solution in the unphysical space M (before their ultimate transcription into the physical space $\hat{M}$), include the following:

1. In the unphysical space M, there are simplifying frame conditions such as the vanishing of the expansion parameter (ρ) along the null congruence. This condition and others simplify the cumbersome differential

equations somewhat.

2. One is able to specify explicitly the boundary conditions at future null infinity.
3. It also allows a systematic derivation and transcription of the appropriate initial data set from M to $\hat{M}$.

At our disposal are the reformulated non-linear partial differential equations for the NP variables. We then substitute Ω-series expansions for each NP variable into the restructured equations L_1--L_{21} and solve for the (NP variable) coefficients to a suitable order. Initially the solutions in M are obtained in terms of the free data set $D_1:(\xi, \sigma, \phi_{01}, \Lambda, P)$. We then use the conformal rescaling equations for the Ricci tensor (i.e. equation L_{22}-L_{24}) to derive an alternate set of appropriate free data D_2.

This free data set D_2 is specified as follows:

(i) $\hat{\phi}_{02}$, $\hat{\phi}_{11}$, ϕ_{21}, ϕ_{22} as functions of $(u, \Omega, \zeta, \bar{\zeta})$

(ii) P ; σ° as functions of $(u, \zeta, \bar{\zeta})$

(iii) Ψ_0 $\hat{\phi}_{01} - \hat{\phi}^{(4)}_{01}\Omega^4$, $\hat{\Lambda} - \hat{\Lambda}^{(3)}\Omega^3$, $\hat{\phi}_{00}$
as functions of $(\Omega, \zeta, \bar{\zeta})$

(iv) $\mathrm{Re}\,\hat{\Psi}^{(1)}_2$, $\Psi^{(1)}_1$ as functions of $(\zeta, \bar{\zeta})$

The results in the rescaled (unphysical) space M and consequently the physical space $\hat{M}$ are re-written using the differential operator called $\eth$ (edth). This technique enables us to express the solutions in a more compact form. Now, let us discuss briefly how the $\eth$-operator is defined: Consider an arbitrary 2-surface of intersection of the u=constant hypersurfaces with the conformal boundary $\mathscr{I}^+$. Let the metric on this surface be in the conformally flat form; i.e.

$$ds^2 = \frac{d\zeta\, d\bar{\zeta}}{P^2} \qquad 7.01$$

Any well-behaved function η defined on the two surface, which under Lorentz transform L(1) behaves as

$$\eta' = e^{si\phi}\eta \qquad 7.02$$

is said to have spin weight s. The operators $\eth$ and $\bar{\eth}$ are then defined as follows:

$$\begin{aligned} \eth\eta &= -\delta^{\circ}\eta - 2s\bar{\alpha}^{\circ}\eta \\ \bar{\eth}\eta &= -\bar{\delta}^{\circ}\eta + 2s\bar{\alpha}^{\circ}\eta \end{aligned} \qquad 7.03$$

where η is any spin weight s function. The operators $\eth$ and $\bar{\eth}$ have the property of raising and lowering, respectively, the spin weights by unity. In particular, the quantities $\eth\eta$ and $\bar{\eth}\eta$ have respectively the spin weights $s+1$ and $s-1$. The commutation relation for the edth operators is given by the expression:

$$(\bar{\eth}\eth - \eth\bar{\eth})\eta = 2s\eta\,\eth\bar{\eth}\ln P \qquad 7.04$$

After a somewhat lengthy calculation in which we use all the previously enumerated techniques, we obtain a solution in $\mathcal{M}$ for NP variables and the covariant and contravariant metric tensors, in terms of the free data set D_1. We also use the following abbreviations:

$$\hat{\Psi}_3^{(1)} = \eth\lambda^{①} + \bar{\eth}\Phi_{11}^{0}, \quad \lambda^{①} = \dot{\bar{\sigma}}^{0} + \mu^{0}\bar{\sigma}^{0}$$

$$\Phi_{11}^{0} = \eth\bar{\eth}\ln P, \quad \alpha^{0} = -\bar{\beta}^{0} = -\frac{\partial P}{\partial\bar{\zeta}}, \quad 2\gamma^{0} = -\mu^{0} = \dot{P}P^{-1} \tag{7.05}$$

$$\hat{\bar{\Psi}}_2^{(1)} - \hat{\Psi}_2^{(1)} = \dot{\bar{\sigma}}^{0}\sigma^{0} - \bar{\sigma}^{0}\dot{\sigma}^{0} - \bar{\eth}^{2}\sigma^{0} + \eth^{2}\bar{\sigma}^{0}$$

The solutions are as follows:

The metric variables:

$$\begin{aligned} U = {} & \mu^{0}\Omega + \Phi_{11}^{0}\Omega^{2} + \Big[-\tfrac{2}{3}\hat{\Phi}_{11}^{(3)} + \tfrac{1}{2}\Psi_2^{(1)} + \tfrac{1}{2}\bar{\Psi}_2^{(1)} + \mu^{0}|\sigma^{0}|^{2} \\ & + \tfrac{1}{2}(|\sigma^{0}|^{2})_{,u}\Big]\Omega^{3} + \Big[3\hat{\Lambda}^{(4)} - \tfrac{1}{3}\eth\bar{\eth}\hat{\Phi}_{00}^{(4)} + \tfrac{1}{2}\hat{\Phi}_{00}^{(4)}\Phi_{11}^{0} + \tfrac{1}{2}|\sigma^{0}|^{2}\Phi_{11}^{0} \\ & + \hat{\Phi}_{11}^{(4)} + \tfrac{3}{4}|\eth\bar{\sigma}^{0}|^{2} + \tfrac{1}{6}\bar{\eth}\Psi_1^{(1)} + \tfrac{1}{6}\eth\bar{\Psi}_1^{(1)}\Big]\Omega^{4} + O(\Omega^{5}) \end{aligned}$$

$$\begin{aligned} X = {} & -P\eth\bar{\sigma}^{0}\Omega^{2} + P\Big[-\tfrac{2}{3}\bar{\Psi}_1^{(1)} + \tfrac{1}{3}\sigma^{0}\bar{\eth}\bar{\sigma}^{0} + 2\bar{\sigma}^{0}\bar{\eth}\sigma^{0}\Big]\Omega^{3} \\ & - P\Big[2|\sigma^{0}|^{2}\eth\bar{\sigma}^{0} + \tfrac{1}{2}\hat{\Phi}_{00}^{(4)}\eth\bar{\sigma}^{0} - \tfrac{5}{6}\bar{\sigma}^{0}\Psi_1^{(1)} + \tfrac{7}{12}\bar{\sigma}^{0}\bar{\sigma}^{0}\eth\sigma^{0} \\ & + \hat{\Phi}_{10}^{(5)} + \tfrac{1}{4}\eth\bar{\Psi}_0^{(1)} + \tfrac{2}{3}\bar{\sigma}^{0}\eth\hat{\Phi}_{00}^{(4)} - \tfrac{1}{3}\bar{\eth}\hat{\Phi}_{00}^{(5)}\Big]\Omega^{4} + O(\Omega^{5}) \end{aligned}$$

$$f = -1 - \Big[\tfrac{1}{2}|\sigma^{0}|^{2} + \tfrac{1}{2}\hat{\Phi}_{00}^{(4)}\Big]\Omega^{2} - \tfrac{1}{3}\hat{\Phi}_{00}^{(5)}\Omega^{3} + O(\Omega^{4})$$

$$\begin{aligned} \omega = {} & \tfrac{1}{2}\bar{\eth}\sigma^{0}\Omega^{2} + \Big[\tfrac{1}{3}\Psi_1^{(1)} - \tfrac{1}{3}\bar{\sigma}^{0}\eth\sigma^{0} - \sigma^{0}\eth\bar{\sigma}^{0} - \tfrac{1}{6}\eth\hat{\Phi}_{00}^{(4)}\Big]\Omega^{3} \\ & + \Big[\tfrac{3}{8}\sigma^{0}\sigma^{0}\bar{\eth}\bar{\sigma}^{0} + |\sigma^{0}|^{2}\bar{\eth}\sigma^{0} + \tfrac{1}{2}\sigma^{0}\bar{\eth}\hat{\Phi}_{00}^{(4)} - \tfrac{1}{4}\eth\hat{\Phi}_{00}^{(5)} - \tfrac{1}{4}\sigma^{0}\bar{\Psi}_1^{(1)} \\ & + \tfrac{1}{2}\hat{\Phi}_{00}^{(4)}\bar{\eth}\sigma^{0} + \tfrac{1}{2}\hat{\Phi}_{01}^{(5)} + \tfrac{1}{8}\bar{\eth}\Psi_0^{(1)}\Big]\Omega^{4} + O(\Omega^{5}) \end{aligned}$$

$$\begin{aligned} \xi_1 = {} & -2P - |\sigma^{0}|^{2}P\Omega^{2} + P\Big[\tfrac{1}{3}|\sigma^{0}|^{2}\hat{\Phi}_{00}^{(4)} + \tfrac{1}{4}|\sigma^{0}|^{4} + \tfrac{1}{12}\sigma^{0}\bar{\Psi}_0^{(1)} \\ & + \tfrac{1}{4}\bar{\sigma}^{0}\Psi_0^{(1)}\Big]\Omega^{4} + O(\Omega^{5}) \end{aligned}$$

$$\begin{aligned} \xi_2 = {} & 2\sigma^{0}P\Omega + P\Big[-\tfrac{1}{3}\Psi_0^{(1)} - \tfrac{1}{3}\sigma^{0}\hat{\Phi}_{00}^{(4)}\Big]\Omega^{3} - \tfrac{1}{6}P\Big[\sigma^{0}\hat{\Phi}_{00}^{(5)} \\ & + \Psi_0^{(2)}\Big]\Omega^{4} + O(\Omega^{5}) \end{aligned}$$

• Spin coefficients:

$$\nu = -\bar{\eth}\mu^0 \Omega + \left[-\tfrac{1}{2}\bar{\eth}\phi^0_{11} - \tfrac{1}{2}\hat{\Psi}^{(1)}_3 + \tfrac{1}{2}\bar{\sigma}^0 \eth \mu^0 \right]\Omega^2$$
$$+ \left[-\tfrac{1}{2}\bar{\eth}\hat{\Psi}^{(1)}_2 - \tfrac{2}{3}\hat{\phi}^{(3)}_{21} + \tfrac{4}{9}\bar{\eth}\hat{\phi}^{(3)}_{11} - \tfrac{1}{6}\bar{\eth}\hat{\bar{\Psi}}^{(1)}_2 - \tfrac{1}{6}\bar{\sigma}^0 \hat{\Psi}^{(1)}_3 \right.$$
$$\left. + \tfrac{1}{2}\bar{\sigma}^0 \eth \phi^0_{11} - \tfrac{1}{6}\lambda^{(1)}\bar{\eth}\bar{\sigma}^0 - \tfrac{1}{6}\sigma^0 \bar{\eth}\lambda^{(1)} + \tfrac{1}{6}\hat{\phi}^{(4)}_{00}\bar{\eth}\mu^0 \right]\Omega^3 + O(\Omega^4)$$

$$\mu = \mu^0 - \tfrac{1}{2}\bar{\eth}^2\sigma^0 \Omega^2 + \left[\tfrac{2}{3}\sigma^0 \bar{\eth}\eth\bar{\sigma}^0 + \tfrac{2}{3}|\sigma^0|^2 \phi^0_{11} - \tfrac{1}{3}\bar{\eth}\Psi^{(1)}_1 \right.$$
$$\left. + \tfrac{3}{2}|\eth\bar{\sigma}^0|^2 + \tfrac{1}{2}\bar{\sigma}^0 \eth\bar{\eth}\sigma^0 + \tfrac{1}{6}|\eth\sigma^0|^2 \right]\Omega^3 + O(\Omega^4)$$

$$\lambda = \lambda^{(1)}\Omega + \left[-\tfrac{1}{2}\bar{\eth}\eth\bar{\sigma}^0 + \bar{\sigma}^0\phi^0_{11} \right]\Omega^2 + \left[\tfrac{1}{2}\bar{\sigma}^0 \bar{\eth}^2\sigma^0 - \tfrac{1}{2}\hat{\phi}^{(4)}_{20} \right.$$
$$- \tfrac{1}{3}\bar{\sigma}^0 \hat{\phi}^{(3)}_{11} + \tfrac{1}{12}\bar{\eth}^2\hat{\phi}^{(4)}_{00} + \tfrac{1}{6}\sigma^0 \bar{\eth}^2\bar{\sigma}^0 + \tfrac{2}{3}(\bar{\eth}\sigma^0)(\eth\bar{\sigma}^0)$$
$$\left. - \tfrac{1}{6}\eth\bar{\Psi}^{(1)}_1 + \tfrac{1}{2}\bar{\sigma}^0 \hat{\Psi}^{(1)}_2 + \tfrac{1}{2}|\sigma^0|^2\lambda^{(1)} + \tfrac{1}{2}\bar{\sigma}^0 \eth^2\bar{\sigma}^0 \right]\Omega^3 + O(\Omega^4)$$

$$\gamma = -\tfrac{1}{2}\mu^0 - \phi^0_{11}\Omega + \left[-\tfrac{3}{2}\hat{\Psi}^{(1)}_2 - \sigma^0\lambda^{(1)} + \tfrac{1}{2}\alpha^0 \bar{\eth}\sigma^0 + \tfrac{1}{2}\bar{\eth}^2\sigma^0 \right.$$
$$\left. - \tfrac{1}{2}\bar{\alpha}^0 \eth\bar{\sigma}^0 - \tfrac{1}{2}\eth^2\bar{\sigma}^0 \right]\Omega^2 + \left[\alpha^0 \left(\tfrac{1}{3}\Psi^{(1)}_1 - \tfrac{1}{6}\bar{\sigma}^0 \eth\sigma^0 - \sigma^0 \eth\bar{\sigma}^0 \right) \right.$$
$$- \tfrac{1}{2}\bar{\eth}\Psi^{(1)}_1 + \bar{\alpha}^0 \left(-\tfrac{1}{3}\bar{\Psi}^{(1)}_1 + \tfrac{1}{6}\sigma^0 \bar{\eth}\bar{\sigma}^0 + \bar{\sigma}^0 \bar{\eth}\sigma^0 \right) - \tfrac{1}{6}\eth\bar{\Psi}^{(1)}_1$$
$$- |\eth\bar{\sigma}^0|^2 - 2\hat{\phi}^{(4)}_{11} - 4\hat{\Lambda}^{(4)} + \tfrac{1}{2}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} - \tfrac{5}{6}\hat{\phi}^{(4)}_{00}\phi^0_{11}$$
$$\left. - \tfrac{1}{2}|\sigma^0|^2 \phi^0_{11} \right]\Omega^3 + O(\Omega^4)$$

$$\beta = -\bar{\alpha}^0 + \sigma^0 \alpha^0 \Omega + \left[-\tfrac{1}{2}|\sigma^0|^2 \bar{\alpha}^0 + \tfrac{1}{2}\sigma^0 \eth\bar{\sigma}^0 - \tfrac{1}{2}\Psi^{(1)}_1 \right]\Omega^2$$
$$+ \left[\tfrac{1}{6}\sigma^0 \bar{\Psi}^{(1)}_1 - \tfrac{1}{2}|\sigma^0|^2 \bar{\eth}\sigma^0 + \tfrac{1}{6}\Psi^{(1)}_0 \alpha^0 - \phi^{(5)}_{01} - \tfrac{1}{3}\bar{\eth}\Psi^{(1)}_0 \right.$$
$$- \tfrac{1}{6}\sigma^0\sigma^0 \bar{\eth}\bar{\sigma}^0 - \tfrac{5}{6}\hat{\phi}^{(4)}_{00}\bar{\eth}\sigma^0 - \tfrac{2}{3}\sigma^0 \bar{\eth}\hat{\phi}^{(4)}_{00} + \tfrac{1}{3}\eth\hat{\phi}^{(5)}_{00}$$
$$\left. + \tfrac{1}{6}\sigma^0\alpha^0 \hat{\phi}^{(4)}_{00} \right]\Omega^3 + O(\Omega^4)$$

$$\sigma = \sigma^0 - \tfrac{1}{2}\Psi^{(1)}_0 \Omega^2 - \tfrac{1}{3}\Psi^{(2)}_0 \Omega^3 + O(\Omega^4)$$

$$
\begin{aligned}
\alpha = {} & \alpha^{0} + [\bar{\sigma}^{0}\bar{\alpha}^{0} - \eth\bar{\sigma}^{0}]\Omega + [\tfrac{1}{2}|\sigma^{0}|^{2}\alpha^{0} - \tfrac{1}{2}\bar{\tau} \\
& + \tfrac{1}{2}\sigma^{0}\bar{\eth}\bar{\sigma}^{0} + \tfrac{3}{2}\bar{\sigma}^{0}\bar{\eth}\sigma^{0}]\Omega^{2} + [\tfrac{1}{2}\bar{\sigma}^{0}\Psi_{1}^{(1)} - \tfrac{1}{2}\bar{\sigma}^{0}\bar{\sigma}^{0}\eth\sigma^{0} \\
& - \tfrac{3}{2}|\sigma^{0}|^{2}\eth\bar{\sigma}^{0} - \hat{\Phi}_{10}^{(5)} - \tfrac{1}{6}\eth\bar{\Psi}_{0}^{(1)} - \tfrac{2}{3}\hat{\Phi}_{00}^{(4)}\eth\bar{\sigma}^{0} - \tfrac{2}{3}\bar{\sigma}^{0}\eth\hat{\Phi}_{00}^{(4)} \\
& + \tfrac{1}{3}\bar{\eth}\hat{\Phi}_{00}^{(5)} - \tfrac{1}{6}\bar{\alpha}^{0}\bar{\sigma}^{0}\hat{\Phi}_{00}^{(4)} - \tfrac{1}{6}\bar{\Psi}_{0}^{(1)}\bar{\alpha}^{0}]\Omega^{3} + O(\Omega^{4})
\end{aligned}
$$

$$
\begin{aligned}
\tau = {} & -\bar{\eth}\sigma^{0}\Omega + [-\Psi_{1}^{(1)} + \tfrac{1}{2}\bar{\sigma}^{0}\eth\sigma^{0} + 2\sigma^{0}\eth\bar{\sigma}^{0}]\Omega^{2} \\
& + [-2|\sigma^{0}|^{2}\bar{\eth}\sigma^{0} - 2\hat{\Phi}_{01}^{(5)} - \tfrac{1}{2}\bar{\eth}\Psi_{0}^{(1)} - \tfrac{2}{3}\sigma^{0}\sigma^{0}\bar{\eth}\bar{\sigma}^{0} \\
& - \tfrac{3}{2}\hat{\Phi}_{00}^{(4)}\bar{\eth}\sigma^{0} - \tfrac{4}{3}\sigma^{0}\bar{\eth}\hat{\Phi}_{00}^{(4)} + \tfrac{2}{3}\eth\hat{\Phi}_{00}^{(5)} + \tfrac{2}{3}\sigma^{0}\bar{\Psi}_{1}^{(1)}]\Omega^{3} \\
& + O(\Omega^{4})
\end{aligned}
$$

<u>Weyl tensor components</u>

$$
\begin{aligned}
\Psi_{0} &= \Psi_{0}^{(1)}\Omega + \Psi_{0}^{(2)}\Omega^{2} + O(\Omega^{3}) \\
\Psi_{1} &= \Psi_{1}^{(1)}\Omega + [3\hat{\Phi}_{01}^{(5)} + \bar{\eth}\Psi_{0}^{(1)} + \tfrac{5}{2}\hat{\Phi}_{00}^{(4)}\bar{\eth}\sigma^{0} \\
&\quad + 2\sigma^{0}\bar{\eth}\hat{\Phi}_{00}^{(4)} - \eth\hat{\Phi}_{00}^{(5)}]\Omega^{2} + O(\Omega^{3}) \\
\Psi_{2} &= \hat{\Psi}_{2}^{(1)}\Omega + [\bar{\eth}\Psi_{1}^{(1)} - \tfrac{1}{2}\eth\bar{\eth}\hat{\Phi}_{00}^{(4)} + 4\hat{\Lambda}^{(4)} + 2\hat{\Phi}_{11}^{(4)} \\
&\quad + \Phi_{11}^{0}\hat{\Phi}_{00}^{(4)}]\Omega^{2} + O(\Omega^{3}) \\
\Psi_{3} &= \hat{\Psi}_{3}^{(1)}\Omega + [\bar{\eth}\hat{\Psi}_{2}^{(1)} + \hat{\Phi}_{21}^{(3)} - \tfrac{2}{3}\bar{\eth}\hat{\Phi}_{11}^{(3)}]\Omega^{2} + O(\Omega^{3}) \\
\Psi_{4} &= [\bar{\eth}^{2}\mu^{0} - \lambda^{(1)}_{,u} - 2\mu^{0}\lambda^{(1)}]\Omega + \bar{\eth}\hat{\Psi}_{3}^{(1)}\Omega^{2} + O(\Omega^{3})
\end{aligned}
$$

Ricci tensor components

$$\phi_{00} = -|\sigma^{\circ}|^2 + \left[\tfrac{1}{2}\sigma^{\circ}\bar{\Psi}_0^{(1)} + \tfrac{1}{2}\bar{\sigma}^{\circ}\Psi_0^{(1)} \right]\Omega^2$$
$$+ \left[\tfrac{1}{3}\sigma^{\circ}\bar{\Psi}_0^{(2)} + \tfrac{1}{3}\bar{\sigma}^{\circ}\Psi_0^{(2)} \right]\Omega^3 + O(\Omega^4)$$

$$\phi_{01} = \bar{\eth}\sigma^{\circ} + \left[\Psi_1^{(1)} - \bar{\sigma}^{\circ}\eth\sigma^{\circ} - 3\sigma^{\circ}\eth\bar{\sigma}^{\circ} \right]\Omega + \left[3\hat{\phi}_{01}^{(5)} \right.$$
$$+ \tfrac{9}{2}|\sigma^{\circ}|^2\bar{\eth}\sigma^{\circ} + \tfrac{1}{2}\bar{\eth}\Psi_0^{(1)} + \tfrac{3}{2}\sigma^{\circ}\sigma^{\circ}\bar{\eth}\bar{\sigma}^{\circ} + \tfrac{5}{2}\hat{\phi}_{00}^{(4)}\bar{\eth}\sigma^{\circ}$$
$$\left. + 2\sigma^{\circ}\bar{\eth}\hat{\phi}_{00}^{(4)} - \eth\hat{\phi}_{00}^{(5)} - \sigma^{\circ}\bar{\Psi}_1^{(1)} \right]\Omega^2 + O(\Omega^3)$$

$$\phi_{11} = \phi_{11}^{\circ} + \left[\tfrac{3}{2}\hat{\Psi}_2^{(1)} - \tfrac{1}{2}\bar{\eth}^2\sigma^{\circ} + \tfrac{3}{2}\sigma^{\circ}\lambda^{①} + \eth^2\bar{\sigma}^{\circ} \right]\Omega$$
$$+ \left[-\tfrac{1}{4}|\eth\bar{\sigma}^{\circ}|^2 - \tfrac{3}{4}\sigma^{\circ}\bar{\eth}\eth\bar{\sigma}^{\circ} - \tfrac{3}{4}\bar{\sigma}^{\circ}\eth\bar{\eth}\sigma^{\circ} + 3\hat{\phi}_{11}^{(4)} \right.$$
$$+ 6\hat{\Lambda}^{(4)} - \frac{3}{\cancel{4}}\cancel{\eth}\,\bar{\eth}\hat{\phi}_{00}^{(4)} + \tfrac{3}{2}\hat{\phi}_{00}^{(4)}\phi_{11}^{\circ} + \tfrac{1}{2}|\sigma^{\circ}|^2\phi_{11}^{\circ} - \tfrac{1}{4}|\eth\sigma^{\circ}|^2$$
$$\left. + \tfrac{1}{2}\bar{\eth}\Psi_1^{(1)} + \tfrac{1}{2}\eth\bar{\Psi}_1^{(1)} \right]\Omega^2 + O(\Omega^3)$$

$$\phi_{02} = -\dot{\sigma}^{\circ} - 2\mu^{\circ}\sigma^{\circ} + \left[\eth\bar{\eth}\sigma^{\circ} - 2\sigma^{\circ}\phi_{11}^{\circ} \right]\Omega$$
$$+ \left[\tfrac{3}{2}\hat{\phi}_{02}^{(4)} + \sigma^{\circ}\hat{\phi}_{11}^{(3)} - \tfrac{3}{2}\sigma^{\circ}\hat{\Psi}_2^{(1)} - \tfrac{5}{2}\sigma^{\circ}\eth^2\bar{\sigma}^{\circ} - \tfrac{1}{4}\eth^2\hat{\phi}_{00}^{(4)} \right.$$
$$- \tfrac{1}{2}\hat{\phi}_{00}^{(4)}\bar{\lambda}^{①} + \tfrac{1}{2}\Psi_0^{(1)}\mu^{\circ} - \tfrac{1}{2}\bar{\sigma}^{\circ}\eth^2\sigma^{\circ} + \tfrac{1}{2}\eth\Psi_1^{(1)} - 2(\eth\bar{\sigma}^{\circ})(\eth\sigma^{\circ})$$
$$\left. - \tfrac{3}{2}\sigma^{\circ}\sigma^{\circ}\lambda^{①} - \tfrac{1}{2}|\sigma^{\circ}|^2\bar{\lambda}^{①} \right]\Omega^2 + O(\Omega^3)$$

$$\phi_{22} = -\dot{\mu}^{\circ} + \left[\bar{\eth}\eth\mu^{\circ} + 2\mu^{\circ}\phi_{11}^{\circ} \right]\Omega + \left[\tfrac{1}{2}\eth\hat{\Psi}_3^{(1)} + \tfrac{1}{2}\bar{\eth}\hat{\bar{\Psi}}_3^{(1)} \right.$$
$$+ \tfrac{3}{2}\mu^{\circ}\hat{\Psi}_2^{(1)} + \tfrac{3}{2}\mu^{\circ}\hat{\bar{\Psi}}_2^{(1)} - \tfrac{1}{2}\bar{\sigma}^{\circ}\eth^2\mu^{\circ} - \tfrac{1}{2}\sigma^{\circ}\bar{\eth}^2\mu^{\circ}$$
$$\left. - \dot{\bar{\sigma}}^{\circ}\dot{\sigma}^{\circ} + \mu^{\circ}\mu^{\circ}|\sigma^{\circ}|^2 \right]\Omega^2 + O(\Omega^3)$$

$$\Lambda = \left[-\tfrac{1}{2}\hat{\Psi}_2^{1} + \tfrac{1}{2}\bar{\eth}^2\sigma^0 - \tfrac{1}{2}\sigma^0\lambda^{(1)} \right]\Omega + \left[-\hat{\Phi}_{11}^{(4)} - 9/4\,|\eth\bar{\sigma}^0|^2 \right.$$
$$- \tfrac{3}{4}\sigma^0\bar{\eth}\eth\bar{\sigma}^0 - \tfrac{3}{4}\bar{\sigma}^0\eth\bar{\eth}\sigma^0 - 2\hat{\Lambda}^{(4)} + \tfrac{1}{4}\eth\bar{\eth}\hat{\Phi}_{00}^{(4)} - \tfrac{1}{2}\hat{\Phi}_{00}^{(4)}\Phi_{11}^0$$
$$\left. - \tfrac{3}{2}|\sigma^0|^2\Phi_{11}^0 - 1/4\,|\eth\sigma^0|^2 \right]\Omega^2 + O(\Omega^3)$$

Some of the NP variables listed as free data [illegible] allow arbitrary u (time) dependence. In particular, their appropriate time evolution (u-development) equations are given by the following:

$$\dot{\Psi}_0^{(1)} = 3\bar{\sigma}^0\hat{\Psi}_2^{(1)} - \eth\Psi_1^{(1)} + 3\hat{\Phi}_{02}^{(4)} + 2\sigma^0\hat{\Phi}_{11}^{(3)} - 1/2\,\eth^2\hat{\Phi}_{00}^{(4)} - \dot{\sigma}^0\hat{\Phi}_{00}^{(4)}$$
$$- 3\mu^0\Psi_0^{(1)} - \mu^0\sigma^0\hat{\Phi}_{00}^{(4)}$$
$$\dot{\hat{\Psi}}_2^{(1)} = -\eth\hat{\Psi}_3^{(1)} + \hat{\Phi}_{22}^{(2)} + 2/3\,\dot{\hat{\Phi}}_{11}^{(3)} + 2\mu^0\hat{\Phi}_{11}^{(3)} - 3\mu^0\hat{\Psi}_2^{(1)} + \sigma^0\hat{\Psi}_4^{(1)}$$
$$\dot{\Psi}_1^{(1)} = 2\hat{\Phi}_{12}^{(3)} + 2\sigma^0\hat{\Psi}_3^{(1)} - \eth\hat{\Psi}_2^{(1)} - 4/3\,\eth\hat{\Phi}_{11}^{(3)} - 3\mu^0\Psi_1^{(1)}$$
$$\dot{\hat{\Phi}}_{00}^{(4)} = -4\hat{\Phi}_{11}^{(3)} - 2\hat{\Phi}_{00}^{(4)}\mu^0$$
$$\dot{\hat{\Phi}}_{00}^{(5)} = 12\hat{\Lambda}^{(4)} - 3\mu^0\hat{\Phi}_{00}^{(5)} - \eth\bar{\eth}\hat{\Phi}_{00}^{(4)} - 2\hat{\Phi}_{00}^{(4)}\Phi_{11}^0$$

We shall now present solutions for the unphysical space-time metric ds^2. The unphysical metric is given in the rescaled space M by the relation:

$$ds^2 = g^{ab}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^b} = g_{ab}\,dx^a\otimes dx^b$$

Since $\quad g^{ab} = 2\,k^{(a}n^{b)} - 2\,m^{(a}\bar{m}^{b)}$

the contravariant metric tensor g^{ab} has the following components with $a, b \in \{0, 1, 2, 3\}$

$$g^{00} = g^{02} = g^{03} = 0 \qquad g^{01} = \xi$$

$$g^{11} = 2(\xi U - \omega\bar{\omega}) \qquad g^{12} = \xi X - \omega\bar{\xi}_2 - \bar{\omega}\xi$$

$$g^{22} = -2\xi_1\bar{\xi}_2 \qquad g^{23} = -(\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2)$$

$$g^{13} = \bar{g}^{12} \qquad g^{33} = \bar{g}^{22}$$

Substituting the solutions of the metric variables into the component equations of g^{ab} give the solutions for the unphysical contravariant metric tensor. The metric (line element) ds^2 is then obtained by substituting results of 7.07 into equation 7.06. We now present solutions for the unphysical cotravariant metric tensor components in $\mathcal{M}$:

Contravariant metric tensor components:

$$g^{01} = -1 - \tfrac{1}{2}\left[\,|\sigma^{o}|^2 + \hat{\Phi}^{(4)}_{00}\right]\Omega^2 - \tfrac{1}{3}\hat{\Phi}^{(5)}_{00}\Omega^3 + O(\Omega^4)$$

$$\begin{aligned} g^{11} = {} & -2\mu^{o}\Omega - 2\Phi^{o}_{11}\Omega^2 + \left[-3|\sigma^{o}|^2\mu^{o} - \hat{\Phi}^{(4)}_{00}\mu^{o} + \tfrac{4}{3}\hat{\Phi}^{(3)}_{11}\right. \\ & \left. - \hat{\Psi}^{(1)}_2 - \hat{\bar{\Psi}}^{(1)}_2 - (|\sigma^{o}|^2)_{,u}\right]\Omega^3 + \left[-\tfrac{2}{3}\hat{\Phi}^{(5)}_{00}\mu^{o} - 2|\sigma^{o}|^2\Phi^{o}_{11}\right. \\ & - 2\hat{\Phi}^{(4)}_{00}\Phi^{o}_{11} - 2|\eth\bar{\sigma}^{o}|^2 - 6\hat{\Lambda}^{(4)} + \tfrac{2}{3}\eth\bar{\eth}\hat{\Phi}^{(4)}_{00} - 2\hat{\Phi}^{(4)}_{11} \\ & \left. - \tfrac{1}{3}\bar{\eth}\Psi^{(1)}_1 - \tfrac{1}{3}\eth\bar{\Psi}^{(1)}_1\right]\Omega^4 + O(\Omega^5) \end{aligned}$$

$$\begin{aligned} g^{23} = {} & -4P^2 - 8|\sigma^{o}|^2 P\Omega^2 + P^2\left[\tfrac{8}{3}|\sigma^{o}|^2\hat{\Phi}^{(4)}_{00} + \tfrac{4}{3}\bar{\sigma}^{o}\Psi^{(1)}_{o}\right. \\ & \left. + \tfrac{4}{3}\sigma^{o}\bar{\Psi}^{(1)}_{o}\right]\Omega^4 + O(\Omega^5) \end{aligned}$$

$$g^{12} = 2P\eth\bar{\sigma}^0\Omega^2 + P\left[\tfrac{4}{3}\bar{\Psi}_1^{(1)} - \sigma^0\bar{\eth}\bar{\sigma}^0 - 5\bar{\sigma}^0\bar{\eth}\sigma^0 - \tfrac{1}{3}\bar{\eth}\hat{\phi}_{00}^{(4)}\right]\Omega^3$$
$$+ P\left[7|\sigma^0|^2\eth\bar{\sigma}^0 + 2\hat{\phi}_{00}^{(4)}\eth\bar{\sigma}^0 - 2\bar{\sigma}^0\Psi_1^{(1)} + 2\bar{\sigma}^0\bar{\sigma}^0\eth\sigma^0\right.$$
$$\left.+ 2\hat{\phi}_{10}^{(5)} + \tfrac{1}{2}\eth\bar{\Psi}_0^{(1)} + 2\bar{\sigma}^0\eth\hat{\phi}_{00}^{(4)} - \tfrac{5}{6}\bar{\eth}\hat{\phi}_{00}^{(5)}\right]\Omega^4 + O(\Omega^5)$$

$$g^{22} = 8\bar{\sigma}^0P^2\Omega + P^2\left[-\tfrac{4}{3}\bar{\Psi}_0^{(1)} - \tfrac{4}{3}\bar{\sigma}^0\hat{\phi}_{00}^{(4)} + 4|\sigma^0|^2\bar{\sigma}^0\right]\Omega^3$$
$$+ P^2\left[-\tfrac{2}{3}\hat{\phi}_{00}^{(5)}\bar{\sigma}^0 - \tfrac{2}{3}\bar{\Psi}_0^{(2)}\right]\Omega^4 + O(\Omega^5)$$

$$g^{00} = g^{02} = g^{03} = 0$$

$$g^{13} = \overline{g^{12}} \qquad g^{33} = \overline{g^{22}}$$

Having obtained solutions for the components of the contravariant metric tensor g^{ab} in $\mathcal{M}$, we shall now work out the much desired components of the covariant metric tensor g_{ab} in $\mathcal{M}$. The components of g_{ab} are obtained by inverting the non-singular matrix:

$$g^{ab} = \begin{pmatrix} 0 & g^{01} & 0 & 0 \\ g^{10} & g^{11} & g^{12} & g^{13} \\ 0 & g^{21} & g^{22} & g^{23} \\ 0 & g^{31} & g^{32} & g^{33} \end{pmatrix} \qquad 7.08$$

In particular, the components of the covariant metric tensor g_{ab} in $\mathcal{M}$ are given by the expressions:

$$g_{01} = f^{-1} \qquad g_{02} = -\tfrac{1}{16}P^{-4}f^{-1}\left[g^{13}g^{32} - g^{12}g^{33}\right]$$

$$g_{00} = -f^{-2}g^{11} - f^{-1}\left[g_{02}g^{12} + \bar{g}_{02}\bar{g}^{12}\right]$$

$$g_{22} = -\tfrac{1}{16}P^{-4}g^{33} \qquad g_{23} = \tfrac{1}{16}P^{-4}g^{23} \qquad 7.09$$

where $g_{03} = \bar{g}_{02} \quad g_{33} = \bar{g}_{22}$

$g_{11} = g_{12} = g_{13} = 0$

After some algebraic simplifications we obtain the following results for the components of the covariant metric tensor g_{ab} in $\mathcal{M}$.

Covariant metric tensor components:

$$g_{01} = -1 + \tfrac{1}{2}[\,|\sigma^o|^2 + \hat{\phi}^{(4)}_{00}]\Omega^2 + \tfrac{1}{3}\hat{\phi}^{(5)}_{00}\Omega^3 + O(\Omega^4)$$

$$g_{00} = 2\mu^o\Omega + 2\phi^o_{11}\Omega^2 + [\mu^o|\sigma^o|^2 - \mu^o\hat{\phi}^{(4)}_{00} + (|\sigma^o|)_{,u} - \tfrac{4}{3}\hat{\phi}^{(3)}_{11} + \hat{\Psi}^{(1)}_2 + \bar{\hat{\Psi}}^{(1)}_2]\Omega^3 + [-\tfrac{2}{3}\hat{\phi}^{(5)}_{00}\mu^o + 6\hat{\Lambda}^{(4)} - \tfrac{2}{3}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} + 2\hat{\phi}^{(4)}_{11} + \tfrac{1}{3}\bar{\eth}\Psi^{(1)}_1 + \tfrac{1}{3}\eth\bar{\Psi}^{(1)}_1]\Omega^4 + O(\Omega^5)$$

$$g_{22} = -\tfrac{1}{2}\sigma^o P^{-2}\Omega + \tfrac{1}{4}P^{-2}[\tfrac{1}{3}\Psi^{(1)}_o + \tfrac{1}{3}\sigma^o\hat{\phi}^{(4)}_{00} - |\sigma^o|^2\sigma^o]\Omega^3 + \tfrac{1}{24}P^{-2}[\hat{\phi}^{(5)}_{00}\sigma^o + \Psi^{(2)}_o]\Omega^4 + O(\Omega^5)$$

$$g_{02} = -\tfrac{1}{2}P^{-1}\bar{\eth}\sigma^o\Omega^2 + P^{-1}[-\tfrac{1}{3}\Psi^{(1)}_1 + \tfrac{1}{4}\eth|\sigma^o|^2 + \tfrac{1}{12}\eth\hat{\phi}^{(4)}_{00}]\Omega^3 + P^{-1}[-\tfrac{1}{4}\hat{\phi}^{(4)}_{00}\bar{\eth}\sigma^o - \tfrac{1}{6}\sigma^o\bar{\Psi}^{(1)}_1 - \tfrac{1}{2}\hat{\phi}^{(5)}_{01} - \tfrac{1}{8}\bar{\eth}\Psi^{(1)}_o - \tfrac{1}{3}\sigma^o\bar{\eth}\hat{\phi}^{(4)}_{00} + \tfrac{5}{24}\eth\hat{\phi}^{(5)}_{00}]\Omega^4 + O(\Omega^5)$$

$$g_{23} = -\tfrac{1}{4}P^{-2} - \tfrac{1}{2}P^{-2}|\sigma^o|^2\Omega^2 + P^{-2}[\tfrac{1}{6}|\sigma^o|^2\hat{\phi}^{(4)}_{00} + \tfrac{1}{12}\bar{\sigma}^o\Psi^{(1)}_o + \tfrac{1}{12}\sigma^o\bar{\Psi}^{(1)}_o]\Omega^4 + O(\Omega^5)$$

$$g_{13} = g_{12} = g_{11} = 0$$

$$g_{03} = \bar{g}_{02} \qquad g_{33} = \bar{g}_{22}$$

7.20 Solution to the Field Equations in the physical space $\hat{M}$

We shall now discuss the transcription of the solution from the rescaled space M into the physical space $\hat{M}$. Let us first analyse the co-ordinate system in M to see if it is suitable in the physical space $\hat{M}$. Since null hypersurfaces, and null geodesics remain invariant under conformal rescaling, we find that the co-ordinates $(u, \zeta, \bar{\zeta})$ can be used in M as well as in $\hat{M}$. But instead of Ω, we introduce a new co-ordinate r which we define as an affine parameter along the null geodesics in the u=const. hypersurfaces. Thus our co-ordinates in M are related to those in $\hat{M}$ by the expressions:

$$\begin{aligned} \Omega &= \Omega(r, u, \zeta, \bar{\zeta}) \\ u &= \hat{u}, \quad \zeta = \hat{\zeta} \\ \bar{\zeta} &= \hat{\bar{\zeta}} \end{aligned} \qquad 7.10$$

where the function $(r, u, \zeta, \bar{\zeta})$ is yet to be specified. In particular, the form of the function $\Omega(r, u, \zeta, \bar{\zeta})$ is determined by solving the equation:

$$\frac{\partial \Omega}{\partial r} = \Omega^2 f \qquad 7.11$$

Since we are interested in asymptotic expansion-type solutions it is more convenient that Ω be obtained as a series in the affine parameter r. We find that

$$\Omega = r^{-1} - f^{(2)} r^{-3} - \tfrac{1}{2} f^{(3)} r^{-4} + O(r^{-5}) \qquad 7.12$$

where we have eliminated the coefficient of r^{-2} by a suitable choice of origin of affine parameter:

$$r \longrightarrow r + R(u, \zeta, \bar{\zeta})$$

Let us now analyse the tetrad system to see if it is suitable. We discover that the tetrad conditions $\hat{\kappa} = \hat{\pi} = \hat{\epsilon} = 0$ $\in \hat{M}$ are not directly satisfied since $\hat{\pi}$ does not vanish after the conformal rescaling:

$$o^A = \Omega^{-1} \hat{o}^A$$
$$\iota^A = \hat{\iota}^A \qquad 7.13$$

Thus, the spin frame $[o^A, \iota^A]$ associated with $[\hat{o}^A, \hat{\iota}^A]$ under conformal rescaling is not suitable. However, if we accompany the conformal rescaling by a null rotation about the tetrad vector $k_a (= o_A \bar{o}_{\dot{A}})$ with parameter c obtained by solving

$$\hat{D} c = \bar{\omega} . \qquad 7.14$$

then (with respect to the new frame $[\hat{o}'^A, \hat{\iota}'^A]$), we now maintain the desired tetrad conditions

$$\hat{\kappa}' = \hat{\pi}' = \hat{\epsilon}' = 0 \qquad 7.15$$

The explicit analytic form of c is obtained by solving 7.14. We obtain the following result.

$$\begin{aligned} c = & -\tfrac{1}{2}\, \eth \bar{\sigma}^\circ \Omega + [-\tfrac{1}{6} \bar{\Psi}_1^{(1)} + \tfrac{1}{6} \sigma^\circ \bar{\eth} \bar{\sigma}^\circ + \tfrac{1}{2} \bar{\sigma}^\circ \bar{\eth} \sigma^\circ \\ & + \tfrac{1}{12} \bar{\eth} \hat{\Phi}_{00}^{(4)}] \Omega^2 + [-\tfrac{1}{8} \bar{\sigma}^\circ \bar{\sigma}^\circ \eth \sigma^\circ - \tfrac{1}{4} |\sigma^\circ|^2 \eth \bar{\sigma}^\circ \\ & - \tfrac{1}{6} \bar{\sigma}^\circ \eth \hat{\Phi}_{00}^{(4)} + \tfrac{1}{12} \bar{\eth} \hat{\Phi}_{00}^{(5)} + \tfrac{1}{2} \bar{\sigma}^\circ \Psi_1^{(1)} - \tfrac{1}{12} \hat{\Phi}_{00}^{(4)} \eth \bar{\sigma}^\circ \\ & - \tfrac{1}{6} \hat{\Phi}_{10}^{(5)} - \tfrac{1}{24} \eth \bar{\Psi}_0^{(1)}] \Omega^3 + O(\Omega^4) \end{aligned} \qquad 7.16$$

Under this new change of frame the differential operator in $\hat{M}$ take the forms:

$$\hat{D}' = \frac{\partial}{\partial r} \qquad \hat{\delta}' = \hat{\omega}' \frac{\partial}{\partial r} + \hat{\xi}_1 \frac{\partial}{\partial \zeta} + \hat{\xi}_2 \frac{\partial}{\partial \bar{\zeta}}$$

$$\hat{\Delta}' = \frac{\partial}{\partial u} + \hat{U}' \frac{\partial}{\partial \Omega} + \hat{X}' \frac{\partial}{\partial \zeta} + \hat{\bar{X}}' \frac{\partial}{\partial \bar{\zeta}}$$

$$\hat{\bar{\delta}}' = \hat{\bar{\omega}}' \frac{\partial}{\partial r} + \hat{\bar{\xi}}_2' \frac{\partial}{\partial \zeta} + \hat{\bar{\xi}}_1' \frac{\partial}{\partial \bar{\zeta}} \tag{7.17}$$

We shall now work out the transformation expressions of the metric variables from M to $\hat{M}$, under a conformal rescaling and a null rotation about k_a. We obtain the following results:

1. $$\hat{\omega}' = \hat{\omega} + \bar{c}$$
$$\hat{\omega} = \Omega^{-1} f^{-1} [\, \omega - \xi_1 \frac{\partial}{\partial \zeta} \Omega - \xi_2 \frac{\partial}{\partial \bar{\zeta}} \Omega \,]$$

2. $$\hat{U}' = -c\bar{c} + c\hat{\omega} + \bar{c}\hat{\bar{\omega}} + U$$
$$\hat{U} = \Omega^{-2} f^{-1} [\, U - \frac{\partial}{\partial u} \Omega - X \frac{\partial}{\partial \zeta} \Omega - \bar{X} \frac{\partial}{\partial \bar{\zeta}} \Omega \,]$$

3. $$\hat{\xi}_1' = \hat{\xi}_1$$
$$\hat{\xi}_1 = \Omega \xi_1$$

4. $$\hat{\xi}_2' = \hat{\xi}_2$$
$$\hat{\xi}_2 = \Omega \xi_2$$

5. $$\hat{X}' = \hat{X} + c\hat{\xi}_1 + \bar{c}\hat{\bar{\xi}}_2$$
$$\hat{X} = X$$

In deriving the above expressions for the metric variables, we use the following relations between the differential operators in M and $\hat{M}$:

$$\hat{D} = \Omega^{-2} \hat{D}'$$

$$\hat{\delta} = \Omega^{-1} [\, \hat{\delta}' - \bar{c} \hat{D}' \,]$$

$$\hat{\bar{\delta}} = \Omega^{-1}[\hat{\bar{\delta}}' - c\hat{D}']$$
$$\hat{\Delta} = \hat{\Delta}' - c\hat{\delta}' - \bar{c}\hat{\bar{\delta}}' + c\bar{c}\hat{D}' \qquad 7.18$$

the primes will from now on be dropped for convenience. Similarly, the relation between the rest of the NP variables in $\mathcal{M}$ relative to $[o^A, \iota^A]$ and those in $\hat{\mathcal{M}}$ relative to $[\hat{o}^{A'}, \hat{\iota}^{A'}]$ can be worked out using the transformation relations worked out in chapter three. We now present the (generalized Newman-Unti-type) asymptotic expansion solutions to the field equations: (In effect, we are transcribing the solutions in the unphysical space $\mathcal{M}$ into the physical space $\hat{\mathcal{M}}$ using the techniques just discussed).

metric variables

$$\hat{\xi}_1 = -2Pr^{-1} - P[2|\sigma^0|^2 + \hat{\phi}^{(4)}_{00}]r^{-3} - \frac{1}{3}P\hat{\phi}^{(5)}_{00}r^{-5} + O(r^{-6})$$

$$\hat{\xi}_2 = 2\sigma^0 Pr^{-2} + P[2|\sigma^0|^2\sigma^0 - \frac{1}{3}\Psi^{(1)}_0 + \frac{5}{3}\sigma^0\hat{\phi}^{(4)}_{00}]r^{-4} + P[-\frac{1}{6}\Psi^{(2)}_0 + \frac{1}{2}\sigma^0\hat{\phi}^{(5)}_{00}]r^{-5} + O(r^{-6})$$

$$\hat{X} = -\frac{1}{3}P[\bar{\Psi}^{(1)}_1 + \frac{1}{2}\bar{\eth}\hat{\phi}^{(4)}_{00}]r^{-3} - \frac{1}{12}P[2\bar{\sigma}^0\eth\hat{\phi}^{(4)}_{00} - 4\bar{\sigma}^0\Psi^{(1)}_1 + 2\eth\bar{\Psi}^{(1)}_0 + 8\hat{\phi}^{(4)}_{00}\eth\bar{\sigma}^0 + 8\hat{\phi}^{(5)}_{10} + 2\bar{\eth}\hat{\phi}^{(5)}_{00}]r^{-4} + O(r^{-5}).$$

$$\hat{U} = -\mu^0 r - \phi^0_{11} + [-\frac{1}{2}\hat{\Psi}^{(1)}_2 - \frac{1}{2}\hat{\bar{\Psi}}^{(1)}_2 - \frac{4}{3}\hat{\phi}^{(3)}_{11}]r^{-1} + [-\frac{1}{6}\eth\bar{\Psi}^{(1)}_1 - \frac{1}{6}\bar{\eth}\Psi^{(1)}_1 + \frac{1}{6}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} - \hat{\phi}^{(4)}_{11} \quad \hat{N}^{(4)} - \frac{1}{3}\phi^0_{11}\hat{\phi}^{(4)}_{00}]r^{-2} + O(r^{-3})$$

$$\hat{w} = -\bar{\eth}\sigma^{\circ} r^{-1} + \left[-\tfrac{1}{4}\eth\hat{\Phi}^{(4)}_{00} - \tfrac{1}{2}\Psi^{(1)}_{1} + \sigma^{\circ}\eth\bar{\sigma}^{\circ} \right] r^{-2}$$
$$+ \left[-\tfrac{1}{6}\sigma^{\circ}\bar{\eth}\hat{\Phi}^{(4)}_{00} + \tfrac{1}{3}\sigma^{\circ}\bar{\Psi}^{(1)}_{1} - |\sigma^{\circ}|^{2}\bar{\eth}\sigma^{\circ} - \tfrac{7}{6}\hat{\Phi}^{(4)}_{00}\bar{\eth}\sigma^{\circ} \right.$$
$$\left. - \tfrac{1}{6}\bar{\eth}\Psi^{(1)}_{0} - \tfrac{2}{3}\hat{\Phi}^{(5)}_{01} + \tfrac{1}{6}\eth\hat{\Phi}^{(5)}_{00} \right] r^{-3} + O(r^{-4})$$

The results obtained by transcribing the spin coefficients from the unphysical space $\mathcal{M}$ to the physical space $\hat{\mathcal{M}}$ are as follows:

<u>spin coefficients</u>:

$$\hat{\alpha} = \alpha^{\circ} r^{-1} + \bar{\sigma}^{\circ}\bar{\alpha}^{\circ} r^{-2} + \left[-\tfrac{1}{4}\bar{\eth}\hat{\Phi}^{(4)}_{00} + |\sigma^{\circ}|^{2}\alpha^{\circ} \right.$$
$$\left. + \tfrac{1}{2}\alpha^{\circ}\hat{\Phi}^{(4)}_{00} \right] r^{-3} + O(r^{-4})$$

$$\hat{\beta} = -\bar{\alpha}^{\circ} r^{-1} - \sigma^{\circ}\alpha^{\circ} r^{-2} + \left[-\tfrac{1}{2}\Psi^{(1)}_{1} - \bar{\alpha}^{\circ}|\sigma^{\circ}|^{2} \right.$$
$$\left. - \tfrac{1}{2}\bar{\alpha}^{\circ}\hat{\Phi}^{(4)}_{00} \right] r^{-3} + O(r^{-4})$$

$$\hat{\rho} = -r^{-1} - \left[|\sigma^{\circ}|^{2} + \hat{\Phi}^{(4)}_{00} \right] r^{-3} - \tfrac{1}{2}\hat{\Phi}^{(5)}_{00} r^{-4} + O(r^{-5})$$

$$\hat{\tau} = \left[-\tfrac{1}{4}\eth\hat{\Phi}^{(4)}_{00} - \tfrac{1}{2}\Psi^{(1)}_{1} \right] r^{-3} + \left[-\tfrac{1}{3}\bar{\eth}\Psi^{(1)}_{0} + \tfrac{1}{6}\sigma^{\circ}\bar{\Psi}^{(1)}_{1} \right.$$
$$\left. - \tfrac{7}{12}\sigma^{\circ}\bar{\eth}\hat{\Phi}^{(4)}_{00} - \tfrac{4}{3}\hat{\Phi}^{(4)}_{00}\bar{\eth}\sigma^{\circ} - \tfrac{4}{3}\hat{\Phi}^{(5)}_{01} + \tfrac{1}{3}\eth\hat{\Phi}^{(5)}_{00} \right] r^{-4} + O(r^{-5})$$

$$\hat{\sigma} = \sigma^{\circ} r^{-2} + \left[|\sigma^{\circ}|^{2}\sigma^{\circ} + \sigma^{\circ}\hat{\Phi}^{(4)}_{00} - \tfrac{1}{2}\Psi^{(1)}_{0} \right] r^{-4}$$
$$+ \left[-\tfrac{1}{3}\Psi^{(2)}_{0} + \tfrac{1}{3}\sigma^{\circ}\hat{\Phi}^{(5)}_{00} \right] r^{-5} + O(r^{-6})$$

$$\hat{\gamma} = \tfrac{1}{2}\mu^{\circ} + \left[-\tfrac{1}{2}\hat{\Psi}^{(1)}_{2} - \tfrac{2}{3}\hat{\Phi}^{(3)}_{11} \right] r^{-2} + \left[-\tfrac{1}{3}\bar{\eth}\Psi^{(1)}_{1} - \tfrac{1}{6}\bar{\alpha}^{\circ}\bar{\Psi}^{(1)}_{1} \right.$$
$$+ \tfrac{1}{6}\alpha^{\circ}\Psi^{(1)}_{1} - \hat{\Phi}^{(4)}_{11} - \hat{\Lambda}^{(4)} + \tfrac{1}{6}\eth\bar{\eth}\hat{\Phi}^{(4)}_{00} + \tfrac{1}{12}\alpha^{\circ}\eth\hat{\Phi}^{(4)}_{00}$$
$$\left. - \tfrac{1}{12}\bar{\alpha}^{\circ}\bar{\eth}\hat{\Phi}^{(4)}_{00} - \tfrac{1}{3}\Phi^{\circ}_{11}\hat{\Phi}^{(4)}_{00} \right] r^{-3} + O(r^{-4})$$

$$\hat{\mu} = -\phi^{0}_{11} r^{-1} + \left[\tfrac{2}{3}\hat{\phi}^{(3)}_{11} - \hat{\Psi}^{(1)}_{2} - \sigma^{0}\lambda^{①} \right] r^{-2}$$
$$+ \left[-\tfrac{1}{2}\bar{\eth}\Psi^{(1)}_{1} + \tfrac{1}{4}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} - 3\hat{\Lambda}^{(4)} - \hat{\phi}^{(4)}_{11} - \phi^{0}_{11}\hat{\phi}^{(4)}_{00} - |\sigma^{0}|^{2}\phi^{0}_{11} \right] r^{-3}$$
$$+ O(r^{-4})$$

$$\hat{\lambda} = \lambda^{①} r^{-1} + \bar{\sigma}^{0}\phi^{0}_{11} r^{-2} + \left[\tfrac{1}{2}\bar{\sigma}^{0}\hat{\Psi}^{(1)}_{2} - \tfrac{1}{2}\hat{\Phi}^{(4)}_{20} + \tfrac{1}{2}\mu^{0}\bar{\sigma}^{0}\hat{\phi}^{(4)}_{00} \right.$$
$$\left. - \tfrac{1}{3}\bar{\sigma}^{0}\hat{\phi}^{(3)}_{11} + |\sigma^{0}|^{2}\lambda^{①} + \tfrac{1}{2}\dot{\bar{\sigma}}^{0}\hat{\phi}^{(4)}_{00} \right] r^{-3} + O(r^{-4})$$

$$\hat{\nu} = -\bar{\eth}\mu^{0} - \hat{\Psi}^{(1)}_{3} r^{-1} + \left[\tfrac{1}{3}\bar{\eth}\hat{\phi}^{(3)}_{11} - \hat{\phi}^{(3)}_{21} - \tfrac{1}{2}\bar{\eth}\hat{\Psi}^{(1)}_{2} \right] r^{-2}$$
$$+ O(r^{-3})$$

The solution for the Riemann tensor (ie. Weyl and Ricci tensor) components under this scheme is as follows:

Weyl tensor components

$$\hat{\Psi}_{0} = \Psi^{(1)}_{0} r^{-5} + \Psi^{(2)}_{0} r^{-6} + O(r^{-7})$$

$$\hat{\Psi}_{1} = \Psi^{(1)}_{1} r^{-4} + \left[3\hat{\phi}^{(5)}_{01} + \bar{\eth}\Psi^{(1)}_{0} + 4\hat{\phi}^{(4)}_{00}\bar{\eth}\sigma^{0} \right.$$
$$\left. + 2\sigma^{0}\bar{\eth}\hat{\phi}^{(4)}_{00} - \eth\hat{\phi}^{(5)}_{00} \right] r^{-5} + O(r^{-6})$$

$$\hat{\Psi}_{2} = \hat{\Psi}^{(1)}_{2} r^{-3} + \left[\bar{\eth}\Psi^{(1)}_{1} - \tfrac{1}{2}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} + 4\hat{\Lambda}^{(4)} \right.$$
$$\left. + 2\hat{\phi}^{(4)}_{11} + \phi^{0}_{11}\hat{\phi}^{(4)}_{00} \right] r^{-4} + O(r^{-5})$$

$$\hat{\Psi}_{3} = \hat{\Psi}^{(1)}_{3} r^{-2} + \left[\bar{\eth}\hat{\Psi}^{(1)}_{2} + \hat{\phi}^{(3)}_{21} - \tfrac{2}{3}\bar{\eth}\hat{\phi}^{(3)}_{11} \right] r^{-3}$$
$$+ O(r^{-4})$$

$$\hat{\Psi}_{4} = \left[\bar{\eth}^{2}\mu^{0} - \lambda^{①}_{,u} - 2\mu^{0}\lambda^{①} \right] r^{-1} + \left[\bar{\eth}\hat{\Psi}^{(1)}_{3} \right] r^{-2}$$
$$+ O(r^{-3})$$

Ricci tensor components:

$$\hat{\Phi}_{00} = \hat{\Phi}^{(4)}_{00} r^{-4} + \hat{\Phi}^{(5)}_{00} r^{-5} + O(r^{-6})$$

$$\hat{\Phi}_{01} = \tfrac{1}{2}\eth\hat{\Phi}^{(4)}_{00} r^{-4} + \hat{\Phi}^{(5)}_{01} r^{-5} + O(r^{-6})$$

$$\hat{\Phi}_{02} = \hat{\Phi}^{(4)}_{02} r^{-4} + O(r^{-5})$$

$$\hat{\Phi}_{11} = \hat{\Phi}^{(3)}_{11} r^{-3} + \hat{\Phi}^{(4)}_{11} r^{-4} + O(r^{-5})$$

$$\hat{\Phi}_{12} = \hat{\Phi}^{(3)}_{12} r^{-2} + O(r^{-3})$$

$$\hat{\Phi}_{22} = \hat{\Phi}^{(2)}_{22} r^{-2} + O(r^{-3})$$

$$\hat{\Lambda} = -\tfrac{1}{3}\hat{\Phi}^{(3)}_{11} r^{-3} + \hat{\Lambda}^{(4)} r^{-4} + O(r^{-5})$$

The freely-specifiable data set D, given by expression 7.01, the abbreviations stated in equations 7.05 as well as the u-development (time dependent) equation for the freely specifiable components of the Riemann tensor given by equations 7.06, are applicable both in the unphysical space $\mathcal{M}$ and the physical space $\hat{\mathcal{M}}$. We shall now present solutions for the contravariant metric tensor $\hat{g}^{ab}$ in the physical space $\hat{\mathcal{M}}$. In order to do so, we may use the conformal rescaling expression:

$$\hat{g}^{ab} = \Omega^2 g^{ab} \qquad 7.19$$

But we must accompany the conformal rescaling with a co-ordinate transformation:

$$CT : [u, r, \zeta, \bar{\zeta}] \longrightarrow [u, \Omega, \zeta, \bar{\zeta}] \qquad 7.20$$

Thus, the required transcription formula for the contravariant metric tensor is given by:

$$\hat{g}^{a'b'} = \Omega^2 \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{b'}}{\partial x^{b}} \, g^{ab} \qquad 7.21$$

An equivalent but less glamourous way is to use the equation 7.07 with some modifications: In particular if we set $f=1$ and replace the metric variables by their values in $\hat{M}$, we shall obtain the following component equations for $\hat{g}^{a'b'}$:

$$\hat{g}'^{00} = \hat{g}'^{02} = \hat{g}'^{03} = 0 \qquad \hat{g}'^{01} = 1$$

$$\hat{g}'^{11} = 2[\hat{U}' - \hat{\omega}'\hat{\bar{\omega}}'] \qquad \hat{g}'^{12} = \hat{X}' - \hat{\omega}'\hat{\bar{\xi}}' - \hat{\bar{\omega}}'\hat{\xi}'_1$$

$$\hat{g}'^{22} = -2\,\hat{\xi}'_1\hat{\bar{\xi}}'_2 \qquad \hat{g}'^{23} = -[\hat{\xi}'_1\hat{\bar{\xi}}' + \hat{\xi}'_2\hat{\bar{\xi}}'_2]$$

$$\hat{g}'^{13} = \hat{\bar{g}}'^{12} \qquad \hat{g}'^{33} = \hat{\bar{g}}'^{22} \qquad 7.22$$

The primes will be dropped.

If we use either 7.21 or 7.22, we obtain the following solutions for the contravariant metric tensor components in the physical space $\hat{M}$

$$\begin{aligned}\hat{g}^{11} = {} & -2\phi^{0}_{11} - 2\mu^{0} r + \left[-\hat{\Psi}^{(1)}_2 - \hat{\bar{\Psi}}^{(1)}_2 - \tfrac{8}{3}\hat{\phi}^{(3)}_{11}\right] r^{-1} \\ & + \left[-\tfrac{1}{3}\eth\bar{\Psi}^{(1)}_1 - \tfrac{1}{3}\bar{\eth}\Psi^{(1)}_1 + \tfrac{1}{3}\eth\bar{\eth}\hat{\phi}^{(4)}_{00} - 2\hat{\Lambda}^{(4)} - \tfrac{2}{3}\phi^{0}_{11}\hat{\phi}^{(4)}_{00} \right. \\ & \left. - 2|\eth\bar{\sigma}^{0}|^2 - 2\hat{\phi}^{(4)}_{11}\right] r^{-2} + O(r^{-3})\end{aligned}$$

$$\hat{g}^{00} = \hat{g}^{02} = \hat{g}^{03} = 0 \qquad \hat{g}^{01} = 1$$

$$\hat{g}^{12} = -2P\eth\bar{\sigma}^{0}r^{-2} + P\left[-\frac{4}{3}\bar{\Psi}_{1}^{(1)} - \frac{2}{3}\bar{\eth}\hat{\phi}_{00}^{(4)} + 4\bar{\sigma}^{0}\bar{\eth}\sigma^{0}\right]r^{-3}$$
$$+ P\left[2\bar{\sigma}^{0}\Psi_{1}^{(1)} - \frac{1}{2}\eth\bar{\Psi}_{0}^{(1)} - 4\hat{\phi}_{00}^{(4)}\eth\bar{\sigma}^{0} - 2\hat{\phi}_{10}^{(5)} + \frac{1}{2}\bar{\eth}\hat{\phi}_{00}^{(5)}\right.$$
$$\left.- 6|\sigma^{0}|^{2}\eth\bar{\sigma}^{0}\right]r^{-4} + O(r^{-5})$$

$$\hat{g}^{22} = 8\bar{\sigma}^{0}P^{2}r^{-3} + \frac{8}{3}P^{2}\left[6|\sigma^{0}|^{2}\bar{\sigma}^{0} - \frac{1}{2}\bar{\Psi}_{0}^{(1)} + 4\hat{\phi}_{00}^{(4)}\bar{\sigma}^{0}\right]r^{-5}$$
$$+ \frac{8}{3}P^{2}\left[5\hat{\phi}_{00}^{(5)}\bar{\sigma}^{0} - \bar{\Psi}_{0}^{(2)}\right]r^{-6} + O(r^{-7})$$

$$\hat{g}^{23} = -4P^{2}r^{-2} - 4P^{2}\left[3|\sigma^{0}|^{2} + \hat{\phi}_{00}^{(4)}\right]r^{-4}$$
$$- \frac{4}{3}P^{2}\hat{\phi}_{00}^{(5)}r^{-5} + O(r^{-6})$$

$$\hat{g}^{33} = \bar{\hat{g}}^{22} \qquad \hat{g}^{13} = \bar{\hat{g}}^{12}$$

The covariant metric tensor in the physical space $\hat{M}$ is obtained by using the relation:

$$\hat{g}_{a'b'} = \Omega^{-2}\frac{\partial x^{a}}{\partial x^{a'}}\frac{\partial x^{b}}{\partial x^{b'}}g_{ab} \qquad 7.23$$

Equivalently, we can obtain the components of $\hat{g}_{a'b'}$ by inverting the matrix

$$(\hat{g}^{ab}) = \begin{pmatrix} 0 & \hat{g}^{01} & 0 & 0 \\ \hat{g}^{10} & \hat{g}^{11} & \hat{g}^{12} & \hat{g}^{13} \\ 0 & \hat{g}^{21} & \hat{g}^{22} & \hat{g}^{23} \\ 0 & \hat{g}^{31} & \hat{g}^{32} & \hat{g}^{33} \end{pmatrix}$$

We obtain the following results:

covariant metric tensor components

$$\hat{g}_{00} = 2\phi_{11}^{0} + 2\mu^{0}r + \left[\frac{8}{3}\hat{\phi}_{11}^{(3)} + \hat{\bar{\Psi}}_{2}^{(1)} + \hat{\Psi}_{2}^{(1)}\right]r^{-1}$$
$$+ \left[\frac{1}{3}\bar{\eth}\Psi_{1}^{(1)} + \frac{1}{3}\eth\bar{\Psi}_{1}^{(1)} - \frac{1}{3}\eth\bar{\eth}\hat{\phi}_{00}^{(4)} + 2\hat{\Lambda}^{(4)} + 2\hat{\phi}_{11}^{(4)}\right.$$
$$\left.+ \frac{2}{3}\hat{\phi}_{00}^{(4)}\phi_{11}^{0}\right]r^{-2} + O(r^{-3}).$$

$$\hat{g}_{02} = -\tfrac{1}{2} P^{-1} \bar{\eth} \sigma^{0} + \tfrac{1}{2} P^{-1} \left[-\tfrac{2}{3} \Psi_{1}^{(1)} - \tfrac{1}{3} \eth \hat{\phi}^{(4)}_{00} \right] r^{-1}$$

$$+ \tfrac{1}{2} P^{-1} \left[-\hat{\phi}^{(4)}_{00} \bar{\eth} \sigma^{0} - \tfrac{1}{3} \sigma^{0} \bar{\Psi}_{1}^{(1)} - \hat{\phi}^{(5)}_{01} - \tfrac{1}{4} \bar{\eth} \Psi_{0}^{(1)} \right.$$

$$\left. - \tfrac{2}{3} \sigma^{0} \bar{\eth} \hat{\phi}^{(4)}_{00} + \tfrac{1}{4} \eth \hat{\phi}^{(5)}_{00} \right] r^{-2} + O(r^{-3})$$

$$\hat{g}_{22} = -\tfrac{1}{2} \sigma^{0} P^{-2} r + \tfrac{1}{12} P^{-2} \left[4 \sigma^{0} \hat{\phi}^{(4)}_{00} + \Psi_{0}^{(1)} \right] r^{-1}$$

$$+ \tfrac{1}{8} P^{-2} \left[\hat{\phi}^{(5)}_{00} \sigma^{0} + \tfrac{1}{3} \Psi_{0}^{(2)} \right] r^{-2} + O(r^{-3})$$

$$\hat{g}_{23} = -\tfrac{1}{4} P^{-2} r^{2} + \tfrac{1}{4} P^{-2} \left[-|\sigma^{0}|^{2} + \hat{\phi}^{(4)}_{00} \right]$$

$$+ \tfrac{1}{12} \hat{\phi}^{(5)}_{00} P^{-2} r^{-1} + O(r^{-2}).$$

$$\hat{g}_{01} = 1 \qquad \hat{g}_{11} = \hat{g}_{12} = \hat{g}_{13} = 0$$

$$\hat{g}_{33} = \bar{\hat{g}}_{22} \qquad \hat{g}_{03} = \bar{\hat{g}}_{02} \tag{7.24}$$

The space-time metric (or line element) $d\hat{s}^2$ is therefore given by

$$d\hat{s}^2 = \hat{g}_{00}\, du\, du + 2 \hat{g}_{01}\, du\, dr + 2 \hat{g}_{02}\, du\, d\zeta$$

$$+ 2 \hat{g}_{03}\, du\, d\bar{\zeta} + \hat{g}_{22}\, d\zeta\, d\zeta + 2 \hat{g}_{23}\, d\zeta\, d\bar{\zeta}$$

$$+ \hat{g}_{33}\, d\bar{\zeta}\, d\bar{\zeta} \tag{7.25}$$

where the components of $\hat{g}_{ab}$ are as given by the expressions of 7.24.

Chapter 8

CONCLUDING REMARKS

In the foregoing section we have presented "generalized" NU-type asymptotic solutions to the Einstein field equations. These results are applicable, for example, to an isolated self-gravitating source in a co-ordinate system based on an arbitrary geodesic, twist-free expanding null congruence. The co-ordinate constraint $P_{,u}=0$ was not assumed in course of the computations but could be applied to the final results if desired. Thus our results contain additional terms to those obtained in previous work [8] in which $P_{,u}=0$ was imposed. The NU-type asymptotic expansions for the metric and Riemann tensor components and other NP quantities are actually solutions to the Einstein field equations only after we replace the Ricci tensor components appearing in the free data set by the appropriate components of the stress-energy tensor.

The current generalized NU-type asymptotic solution applies to all arbitrary sources whose fields show Sachs' "peeling-off" property -- e.g. the Weyl curvature tensor decays according to the relation

$$\Psi_n = O(r^{-5+n}) \qquad \text{where} \quad n \in (0,1,2,3,4)$$

The results apply to those fields of arbitrary spins such as vector fields (photons), spinor fields (neutrinos), and tensor fields (gravitinos). Our generalized line element (space-time metric) in the physical space $\hat{M}$ is given by where the covariant metric tensor components $\hat{g}_{ab}$ are given [illegible] expressions 7.24. If the constraint $R_{,u} = 0$ is imposed on the solutions in $\hat{M}$, we shall obtain the results for the NP variables as given by Ludwig [8]. The corresponding line element (metric) will take the form, given by Ludwig [8] with a change of co-ordinate chart to $\{u, r, \theta, \phi\}$ where the co-ordinates are related by $\zeta = e^{i\phi}\cot\theta/2$.

In the case of an Einstein Maxwell field, our expressions reduce to those of Exton et al [21] except for notation and conventions and some additional terms contributed by the constraint $R_{,u} = 0$. In this case, $\hat{\Phi}^{(4)}_{00}$, $\hat{\Phi}^{(5)}_{00}$, and $\hat{\Lambda}$ vanish identically but not $\hat{\Phi}^{(6)}_{00}$: The asymptotically flat vacuum solutions of Newman and Unti [4] are also recovered when the vacuum conditions are imposed on our results. For a well-behaved neutrino field as source, we have $\hat{\Lambda} = \hat{\Phi}^{(4)}_{00} = 0$ and $\hat{\Phi}^{(5)}_{00} \neq 0$; whereas for a well-behaved scalar field, $\hat{\Phi}^{(3)}_{11}$ and $\hat{\Phi}^{(4)}_{00}$ do not vanish identically. Under certain algebraically special conditions, the "generalized" line element 7.25 can be manipulated to derive known solutions such as Schwarzschild, Reissner-Nordstrom and Vaidya. We shall briefly discuss the Schwarzchild and Reissner-Nordstrom cases: For the Schwarzchild vacuum metric in standard co-ordinates, we set the shear σ^{0} and all Riemann tensor components appearing in the free data set equal to zero except for the mass parameter [illegible] which we defined as

[illegible]

In addition, the $u=u_0$ (const.) cuts of $\mathscr{J}^+$ are taken to be spherical and as such

$$P = 1/2\,(1 + \zeta\bar{\zeta})$$

In order to obtain the Reissner-Nordstrom metric, we set all Riemann tensor components appearing in the free data set and the shear σ^0 equal to zero except the mass parameter $\mathrm{Re}\,\hat{\Psi}_2^{(1)}$ and $\hat{\Phi}_{11}$. As above, we define $\mathrm{Re}\,\hat{\Psi}_2^{(1)} = 2M$ and $P = (1 + \zeta\bar{\zeta})$. In addition we make the defintion

$$\hat{\Phi}_{11} = 2\,\phi_1\bar{\phi}_1$$

where is the only non-vanishing component of the electric field.

If we complexify $\mathscr{J}^+$, we can obtain complex space-times in the same manner that real space-times are obtained. The generalized NU-type asymptotic solutions will look formally the same as in the real case presented in this research. But NP variables which are real in the real space-time become complex and variables which are complex conjugates become independent upon complexification. Notationally, the bar over such a variable is replaced by a tilde . Thus it is possible to modify our current results in the real space $\hat{M}$ such as to apply to complex space as well.

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