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THE QUANTUM WITT ALGEBRA AND QUANTIZATIONS OF SOME MODULES OVER THE WITT ALGEBRA

by

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ABSTRACT

In this thesis, we investigate the problem of quantizing the Witt algebra and its representations over complex number field.

In chapter 1, we begin by introducing the quantum Witt algebra, the quantum Virasoro algebra and a q–analogue of the simplest affine Kac–Moody algebra $\hat{s\ell}(2)$. Each of these objects is an example of a new class of non–associative algebras. We call this class of non–associative algebras the quantum Lie algebras, which are defined by using a q–analogue of the Jacobi identity. Next, we introduce the concept of the quantum universal central extension of a quantum Lie algebra and study its basic properties. Finally, we determine the quantum universal central extension of some quantum Lie algebras.

In chapter 2, after introduing quantum flexible algebras, we prove that q–analogues of the two characterizations of the usual Witt algebra hold for the quantum Witt algebra.

In chapter 3, we give our quantization of the enveloping algebra of the Witt algebra; construct the q–analogues of the module of tensor fields over the Witt algebra; study their properties and prove a q–analogue of Kaplansky's theorem concerning the module of tensor fields over the Witt algebra.

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CHAPTER 0. INTRODUCTION

The search for q–analogues of known mathematical objects, which is affectionatelly called the "q–disease", has a long history in mathematics. As early as the 1840's, A.-L.Cauchy, E.Heine and other mathematicians found a q–analogue of the binomial theorem for $|q| < 1$. In 1869, J.Thomae introduced a q–analogue of the gamma function for $0 < q < 1$. The 19th century mathematician F.H.Jackson devoted much of his mathematical career to developing q–analogues of the hypergometric series. Over the past fifteen years, R.Askey, his students and his collaborators have produced a substantial amount of interesting work in the theory of q–hypergeometric series. However, the general area of algebra was not infected by the q–disease until just ten years ago.

In 1981, P.Kulish and N.Reshetikhin [11] introduced the C–associative algebra $U_h(s\ell(2))$ with the generators X^+, X^-, H and the relations:

$$HX^{\pm} - X^{\pm}H = \pm 2X^{\pm},$$
$$X^+X^- - X^-X^+ = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \qquad h \in \mathbb{C},$$

which arose from their study of Quantum Inverse Scattering Method. The associative algebra $U_h(s\ell(2))$ can be regarded as a q–analogue of the enveloping algebra of the 3–dimensional simple Lie algebra $s\ell(2)$ because when $q := e^h$ approaches unity, $U_h(s\ell(2))$ becomes the enveloping algebra of $s\ell(2)$. It was this object that V.Drinfeld [2,3] and M.Jimbo [6,7] recently generalized to the case of finite dimensional simple Lie algebras.

The main development in the representation theory of Lie algebras have been with the finite dimensional Lie algebras, the Kac–Moody algebras and the Virasoro algebra. Although the enveloping algebras of the finite dimensional simple Lie algebras and the Kac–Moody algebras have been quantized and q–analogues of their representations have been found, this is not case for the Virasoro algebra.

It is well–known that the representations of the Virasoro algebra with the center acting trivially, i.e., the representations of the Witt algebra, play a fundamental role in the representation theory of the Virasoro algebra. Therefore, the first problem in developing q–analogues of the representations of the Virasoro algebra is naturally to find q–analogues of the representations of the Witt algebra. The purpose of my thesis is to first address the problem of quantizing the Witt algebra, the Virasoro algebra and their enveloping algebras, and then to investigate possible q–analogues of the representations of the Witt algebra.

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What follows is a summary of the results of my thesis.

Let q be a fixed non-zero complex number with $q^2 \neq 1$ and let $[m]$ denote the "q-integer" $\frac{q^m - q^{-m}}{q - q^{-1}}$, where $m \in \mathbf{Z}$. Our quantization of the Witt algebra and its enveloping algebra originated by considering certain operators, $\{ D_n \mid n \in \mathbf{Z} \}$, called J-derivations on the Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$, where the operators J and $\{ D_n \mid n \in \mathbf{Z} \}$ defined by

$$J : t^n \mapsto q^n t^n,$$
$$\left(\frac{d}{dt}\right)_q : t^n \mapsto [n] t^{n-1},$$
$$D_n : = -t^{n+1} \left(\frac{d}{dt}\right)_q .$$

It turns out that $\mathcal{D} := \oplus_{n \in \mathbf{Z}} \mathbf{C} D_n$ consists of all J-derivations of $\mathbf{C}[t, t^{-1}]$ and the bracket on $\mathcal{D}$, which is defined by

$$(0.1) \qquad [D_m, D_n] := J D_m J^{-1} D_n J - J D_n J^{-1} D_m J \qquad \text{for } m, n \in \mathbf{Z},$$

satisfies

$$[D_m, D_n] = [m - n] D_{m+n}.$$

Clearly, these bracket relations can be obtained from those of the usual Witt algebra by replacing the structure constants $\{ m - n \mid m, n \in \mathbf{Z} \}$ by their q-analogues $\{ [m - n] \mid m, n \in \mathbf{Z} \}$. So it seems reasonable to take $\mathcal{D}$, with the bracket (0.1), as a realization of the q-analogue of the usual Witt algebra. This produces the **quantum Witt algebra**.

Parallel to the fact that the Lie bracket on the Witt algebra satisfies the Jacobi identity, we will prove that the bracket on $\mathcal{D}$ above satisfies the following q-analogue of the Jacobi identity:

$$(0.2) \qquad [[x, y], \sigma(z)] + [[y, z], \sigma(x)] + [[z, x], \sigma(y)] = 0 \qquad \text{for } x, y, z \in \mathcal{D},$$

where $\sigma : \mathcal{D} \to \mathcal{D}$ is a linear map defined by

$$\sigma(D_m) := \frac{q^m + q^{-m}}{2} D_m \qquad \text{for } m \in \mathbf{Z}.$$

We call (0.2) the **quantum Jacobi identity**.

Using the quantum Jacobi identity, we will introduce a class of non-associative algebras called **quantum Lie algebras**, which are a generalization of Lie algebras. In order to discover the q-analogue of the Virasoro algebra, we will define the **quantum universal central extensions** (q–U.C.E.) of quantum Lie algebras and study the q–U.C.E. of the quantum Witt algebra. We will find that if q is not a root of unity, then the q–U.C.E. of the quantum Witt algebra is a 1–dimensional quantum central extension , and this object is defined to be the **quantum Virasoro algebra**.

This completes our summary of the main development in chapter 1.

Before describing the contents of chapter 2, let us recall some interesting results about the Witt algebra. From H.C.Myung's book [14] on Malcev–admissible algebras and I.Kaplansky's paper [9] on the Virasoro algebra, we know that if $A = \oplus_{n\in\mathbf{Z}} A_n$ is a $\mathbf{Z}$–graded algebra over complex number field, then the following are equivalent:

- A is the Witt algebra.
- A is a flexible algebra and the attached minus algebra $(A^-, [\ ,\ ])$ is the Witt algebra.
- A is a Lie algebra, $dim(A_n) \leq 1$, $n \in \mathbf{Z}$ and

$$(A_0 A_1) \neq 0, \quad (A_1 A_{-1}) \neq 0, \quad (A_2 A_{-1}) \neq 0, \quad (A_1 A_{-2}) \neq 0.$$

In chapter 2, after introducing the concept of the **quantum flexible algebras**, we will prove that if q is not a root of unity, then the quantizations of the results above hold for the quantum Witt algebra. It is worth pointing out that throughout this work, the quantum Jacobi identity plays an essential role and we feel that the quantum Lie algebras are a class of non–associative algebras which deserve more attention.

In chapter 3, we will investigate possible q–analogues of the representations of the Witt algebra. The most important representation of the Witt algebra is the one called the module of tensor fields $V_{\alpha\beta}$, where α and β are complex numbers. In 1982, B.L.Feigin and D.B.Fuchs [4] introduced this module and used it in their proof of the Kac's determinant formula. In the same year, I.Kaplansky [9] proved the following remarkble property of the module of tensor fields $V_{\alpha\beta}$:

- If $V = \oplus_{n\in\mathbf{Z}} \mathbf{C} v_n$ is a $\mathbf{Z}$–graded module over the Witt algebra $W = \oplus_{n\in\mathbf{Z}} \mathbf{C} d_n$ and $d_{\pm 1}$ are injective operators on V, then V is isomorphic to the module of tensor fields $V_{\alpha\beta}$ for some complex numbers α and β.

We call this result Kaplansky's theorem. The basic idea in the proof of Kaplansky' theorem has greatly influenced mathematicians and physicists who attacked Kac's conjecture concerning the representations of the Virasoro algebra (see [1],[10] and [12]). Therefore, we consider quantizing Kaplansky theorem an important first step in developing q–analogues of the representations of the Virasoro algebra.

Motivated by our work in chapter 1, we will define the q–analogue of the universal enveloping algebra of the Witt algebra, denoted by $U(W_q)$. Using the operations on the $\mathbb{Z}$-graded modules over the Witt algebra in [4], we will construct two kinds of $U(W_q)$-modules $A(\lambda, \alpha, \beta)$ and $B(\lambda, \alpha, \beta)$, where $\lambda, \alpha, \beta \in \mathbb{C}$ and $\lambda \neq 0$. Both $A(\lambda, \alpha, \beta)$ and $B(\lambda, \alpha, \beta)$ are q-analogues of the module of tensor fields. It turns out that under conditions similar to the ones in Kaplansky's theorem, every $\mathbb{Z}$-graded $U(W_q)$-module $V = \oplus_{n \in \mathbb{Z}} \mathbb{C} v_n$ is isomorphic to either $A(\lambda, \alpha, \beta)$ or $B(\lambda, \alpha, \beta)$ for some $\lambda, \alpha, \beta \in \mathbb{C}$ and $\lambda \neq 0$. This is our quantization of Kaplansky's theorem obtained in chapter 3.

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CHAPTER 1. QUANTUM CENTRAL EXTENSIONS

§1.0. Introduction

The purpose of this chapter is to give one possible solution to some questions proposed by Y.I.Manin in [6]. In section 1, we will construct the quantum Witt algebra W_q from the opertators called J–derivations on the Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$; it turns out that W_q satisfies the quantum Jacobi identity. Other objects which satisfy the quantum Jacobi identity are the quantum Virasoro algebra V_q (where q is not a root of unity and $q \neq 0$) and the quantum Kac–Moody algebra $\widehat{s\ell_q^n}(2)$ (where n is a positive integer and $q^{4n} \neq 0, 1$). All these examples are q–analogues of familiar objects and become those objects when $q \to 1$. In section 2, we will define quantum universal central extension and study its basic properties. In section 3, we will prove that quantum Virasoro algebra V_q and the quantum Kac–Moody algebra $\widehat{s\ell_q^n}(2)$ are the quantum universal central extensions of W_q and $\mathbf{C}[t, t^{-1}] \otimes s\ell_q^n(2)$, respectively.

Throughout the chapter, all vector spaces are assumed to be vector spaces over the complex number field $\mathbf{C}$ and the following notations are used:

- $\mathbf{C}^* := \{ x \in \mathbf{C} | x \neq 0 \}$.
- q is a complex number satisfying $q^2 \neq 0, 1$.
- $o(q) := t$ if q is a primitive t–th root of unity.
- $$T := \begin{cases} o(q), & \text{if } o(q) \text{ is odd,} \\ \dfrac{o(q)}{2}, & \text{if } o(q) \text{ is even.} \end{cases}$$
- $$[m] := \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad < m > := \frac{q^m + q^{-m}}{2}, \qquad \text{where } m \in \mathbf{Z}.$$
- For any $\mathbf{Z}$–graded vector space $V = \oplus_{n \in \mathbf{Z}} V_n$, $J \in Hom(V, V)$ is defined by $J(v_n) := q^n v_n$ for $n \in \mathbf{Z}$ and $v_n \in V_n$.
- $\sigma \in Hom(V, V)$ is defined by $\sigma := \dfrac{J + J^{-1}}{2}$.

§1.1. J–derivations and the quantum Witt algebra

Definition 1.1 Let A be a $\mathbf{Z}$–graded algebra. A linear map $D : A \to A$ is called a J**-derivation** if $D(ab) = D(a)J(b) + J^{-1}(a)D(b)$ for all $a, b \in A$.

Define a family of linear operators D_n on the Laurent polynomial ring $\mathbf{C}[t,t^{-1}]$ by

$$D_n := -t^{n+1}\left(\frac{d}{dt}\right)_q, \qquad \left(\frac{d}{dt}\right)_q : t^m \mapsto [m]t^{m-1},$$

for all $m,n \in \mathbf{Z}$. Set $\mathcal{D} := \oplus_{n\in\mathbf{Z}}\mathbf{C}D_n$, then $\mathcal{D}$ is a $\mathbf{Z}$-graded vector space with the $\mathbf{Z}$-grading $\mathcal{D}_n := \mathbf{C}D_n$.

PROPOSITION 1.1. *With the notations above, we have*

(1) *$\mathcal{D}$ is the vector space formed by all J-derivations of $\mathbf{C}[t,t^{-1}]$.*

(2) *The bracket $[\ ,\ ]$ on $\mathcal{D}$, defined by*

$$[D_m, D_n] := JD_mJ^{-1}D_nJ - JD_nJ^{-1}D_mJ = [m-n]D_{m+n},$$

satisfies the following the **quantum Jacobi identity***:*

$$[[x,y],\sigma(z)] + [[y,z],\sigma(x)] + [[z,x],\sigma(y)] = 0 \tag{1.1}$$

where $x,y,z \in \mathcal{D}$.

PROOF: (1). For $m,n,k \in \mathbf{Z}$, we have

$$\begin{aligned} D_n(t^m \cdot t^k) &= -t^{n+1}\left(\frac{d}{dt}\right)_q (t^{m+k}) = -t^{n+1}[m+k]t^{m+k-1} \\ &= -([m]q^k + [k]q^{-m})t^{n+m+k} \\ &= (-t^{n+1}[m]t^{m-1})(q^kt^k) + (q^{-m}t^m)(-t^{n+1}[k]t^{k-1}) \\ &= D_n(t^m)J(t^k) + J^{-1}(t^m)D_n(t^k), \end{aligned}$$

so D_n is a J-derivation of $\mathbf{C}[t,t^{-1}]$. Conversely, if D is a J-derivation of $\mathbf{C}[t,t^{-1}]$, then $D(t) \in \mathbf{C}[t,t^{-1}]$. Using induction on m, we have that

$$D(t^m) = [m]t^{m-1}D(t) = D(t)\left(\frac{d}{dt}\right)_q (t^m) \qquad \text{for } m \in \mathbf{Z}.$$

Hence, $D = D(t)\left(\frac{d}{dt}\right)_q \in \oplus_{n\in\mathbf{Z}}\mathbf{C}D_n = \mathcal{D}$.

(2). For $k \in \mathbf{Z}$ and $t^k \in \mathbf{C}[t,t^{-1}]$, we have

$$\begin{aligned} JD_mJ^{-1}D_nJ(t^k) &= -q^{m+k}[n+k]D_{m+n}(t^k), \\ JD_nJ^{-1}D_mJ(t^k) &= -q^{n+k}[m+k]D_{m+n}(t^k), \end{aligned}$$

so

$$\begin{aligned}&(JD_mJ^{-1}D_nJ - JD_nJ^{-1}D_mJ)(t^k)\\=&(q^{n+k}[m+k] - q^{m+k}[n+k])D_{m+n}(t^k)\\=&[m-n]D_{m+n}(t^k),\end{aligned}$$

or $JD_mJ^{-1}D_nJ - JD_nJ^{-1}D_mJ = [m-n]D_{m+n}$. In order to prove that the bracket $[\ ,\ ]$ on $\mathcal{D}$ satisfies the quantum Jacobi identity (1.1), it suffices to prove the following:

$$\begin{aligned}&< k > [[D_m, D_n], D_k] + < m > [[D_n, D_k], D_m]\\&\qquad + < n > [[D_k, D_m], D_n] = 0,\end{aligned} \tag{1.2}$$

where $m, n, k \in \mathbf{Z}$. Using $[D_m, D_n] = [m-n]D_{m+n}$, we get

$$\begin{aligned}&< k > [[D_m, D_n], D_k]\\=&< k > [m-n][D_{m+n}, D_k] =< k > [m-n][m+n-k]D_{m+n+k}\\=&(q^k + q^{-k})(q^{2m-k} + q^{k-2m} - q^{2n-k} - q^{k-2n})\frac{D_{m+n+k}}{2(q-q^{-1})^2}\\=&((q^{2m} + q^{-2m}) + (q^{2k-2m} + q^{2m-2k})\\&- (q^{2n} + q^{-2n}) - (q^{2n-2k} + q^{2k-2n}))\frac{D_{m+n+k}}{2(q-q^{-1})^2},\end{aligned}$$

or

$$\begin{aligned}&< k > [[D_m, D_n], D_k]\\=&(< 2m > + < 2m-2k > - < 2n > - < 2n-2k >)\frac{D_{m+n+k}}{(q-q^{-1})^2}.\end{aligned} \tag{1.3}$$

Replacing m, n and k by n, k and m in (1.3) respectively, we get

$$\begin{aligned}&< m > [[D_n, D_k], D_m]\\=&(< 2n > + < 2n-2m > - < 2k > - < 2k-2m >)\frac{D_{m+n+k}}{(q-q^{-1})^2}.\end{aligned} \tag{1.4}$$

Replacing m, n and k by n, k and m in (1.4) respectively, we get

$$\begin{aligned}&< n > [[D_k, D_m], D_n]\\=&(< 2k > + < 2k-2n > - < 2m > - < 2m-2n >)\frac{D_{m+n+k}}{(q-q^{-1})^2}.\end{aligned} \tag{1.5}$$

Now (1.2) follows from (1.3) + (1.4) + (1.5). ∎

Definition 1.2 A $\mathbf{Z}$–graded vector space $L = \oplus_{n\in\mathbf{Z}} L_n$ is called a **quantum Lie algebra** if there is a skew–symmetric bilinear map $[\ ,\ ] : L \times L \to L$ which satisfies the quantum Jacobi identity (1.1) for all $x, y, z \in L$ and preserves $\mathbf{Z}$–grading, i.e. $[L_m, L_n] \subseteq L_{m+n}$ for $m, n \in \mathbf{Z}$.

Let $L = \oplus_{n\in\mathbf{Z}} L_n$ be a quantum Lie algebra. The set $L(\mathbf{Z}) := \{ n \in \mathbf{Z} | L_n \neq 0 \}$ is called the **degree set** of L. A **graded subalgebra** (resp. a **graded ideal**) of L is a subalgebra (resp. an ideal) of L which is, in addition, a $\mathbf{Z}$–graded subspace of the $\mathbf{Z}$–graded vector space L.

Let $L = \oplus_{n\in\mathbf{Z}} L_n$ and $L' = \oplus_{n\in\mathbf{Z}} L'_n$ be quantum Lie algebras. A linear map $f : L \to L'$ is called a **homomorphism of type** r if $f([x, y]) = [f(x), f(y)]$ for $x, y \in L$ and there exists a rational number r such that $rL(\mathbf{Z}) \subseteq \mathbf{Z}$, $f(L_n) \subseteq L'_{rn}$ for $n \in L(\mathbf{Z})$. A homomorphism f of type r is called an **isomorphism of type** r if f is a bijection. A homomorphism of type 1 is also called **$\mathbf{Z}$–grading preserving homomorphism**.

Example 1. By Proposition 1.1(2), the $\mathbf{Z}$–graded vector space $W_q := \oplus_{n\in\mathbf{Z}} \mathbf{C}d_n$ with the $\mathbf{Z}$–grading $(W_q)_n := \mathbf{C}d_n$ becomes a quantum Lie algebra under the following bracket:

$$[d_m, d_n] := [m-n]d_{m+n}, \qquad \text{where } m, n \in \mathbf{Z},$$

which is called the **quantum Witt algebra**.

For a fixed positive integer n, we use $s\ell_q^n(2)$ to denote the three dimensional graded subalgebra $\mathbf{C}d_{-n} \oplus \mathbf{C}d_0 \oplus \mathbf{C}d_n$ of W_q. Note that $s\ell_q^n(2)$ has the $\mathbf{Z}$–grading:

$$\left(s\ell_q^n(2)\right)_k := \mathbf{C}d_k, \qquad \text{where } k = 0, \pm n.$$

If $q^{4n} \neq 1$ and $\lambda \in \mathbf{C}^*$, then the linear map f defined by

$$f(d_n) := \frac{[2n][n]}{[2]}\lambda^{-1}d'_1, \qquad f(d_{-n}) := \lambda d'_{-1}, \qquad f(d_0) := [n]d'_0$$

is an isomorphism of type $\frac{1}{n}$ from $s\ell_q^n(2)$ to $s\ell_q^1(2) := \mathbf{C}d'_{-1} \oplus \mathbf{C}d'_0 \oplus \mathbf{C}d'_1$, where $[d'_i, d'_j] := [i-j]d'_{i+j}$ for $i, j \in \{ 0, \pm 1 \}$.

Example 2. Let q be not a root of unity. We define the **quantum Virasoro algebra** V_q by

$$V_q := \oplus_{n\in\mathbf{Z}} \mathbf{C}d_n \oplus \mathbf{C}c,$$

$$(V_q)_n := \begin{cases} \mathbf{C}d_0 \oplus \mathbf{C}c, & \text{if } n = 0, \\ \mathbf{C}d_n, & \text{if } n \neq 0, \end{cases}$$

$$[d_m, d_n] := [m-n]d_{m+n} + \frac{[m-1][m][m+1]}{[2][3] < m >}\delta_{m+n,o}c,$$

where $m, n \in \mathbb{Z}$. One can check directly that V_q is a quantum Lie algebra.

Example 3. If A is an associative algebra and $L = \oplus_{n\in\mathbb{Z}} L_n$ a quantum Lie algebra, then $A \otimes L$ becomes a quantum Lie algebra under the bracket defined by

$$[a \otimes x, b \otimes y] := ab \otimes [x, y], \qquad \text{where } a, b \in A,\ x, y \in L,$$

and the $\mathbb{Z}$-grading:$(A \otimes L)_n := A \otimes L_n$ for $n \in \mathbb{Z}$. In particular,$\mathbb{C}[t, t^{-1}] \otimes s\ell_q^n(2)$ is a quantum Lie algebra.

Example 4. Let n be a fixed positive integer. If $q^{4n} \neq 1$, then we define the simplest **quantum Kac–Moody algebra** $\widehat{s\ell_q^n}(2)$ as follows:

$$\widehat{s\ell_q^n}(2) := \mathbb{C}[t, t^{-1}] \otimes s\ell_q^n(2) \oplus \mathbb{C}c,$$

$$\left(\widehat{s\ell_q^n}(2)\right)_k := \begin{cases} 0, & \text{if } k \neq 0, \pm n, \\ \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}d_k, & \text{if } k = \pm n, \\ \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}d_0 \oplus \mathbb{C}c, & \text{if } k = 0, \end{cases}$$

$$\begin{aligned} [u \otimes d_0, v \otimes d_0] &: = -Res\left(u\frac{dv}{du}\right)c, \\ [u \otimes d_0, v \otimes d_{\pm n}] &: = \mp[n]uv \otimes d_{\pm n}, \\ [u \otimes d_n, v \otimes d_{-n}] &: = [2n]uv \otimes d_0 + \frac{[2n]}{[n] < n >}Res\left(u\frac{dv}{dt}\right)c, \\ [u \otimes d_n, v \otimes d_n] &: = [u \otimes d_{-n}, v \otimes d_{-n}] := [c, \widehat{s\ell_q^n}(2)] := 0, \end{aligned}$$

where $u, v \in \mathbb{C}[t, t^{-1}]$, Res and $\frac{d}{dt}$ are the usual residue and derivation, respectively. Using the following well-known fact:

$$Res\left(uv\frac{dw}{dt}\right) + Res\left(vw\frac{du}{dt}\right) + Res\left(wu\frac{dv}{dt}\right) = 0,$$

where $u, v, w \in \mathbb{C}[t, t^{-1}]$, it is easy to check that $\widehat{s\ell_q^n}(2)$ is a quantum Lie algebra.

A quantum Lie algebra L is called **graded simple** (resp. **simple**) if $[L, L] \neq 0$ and L does not have any graded ideals (resp. ideals) which are different from $\{0\}$ and L.

PROPOSITION 1.2. *For quantum Witt algebra W_q, we have*

(1) *W_q is graded simple $\Longleftrightarrow$ q is not a root of unity or $o(q) \neq 4$.*
(2) *W_q is simple $\Longleftrightarrow$ q is not a root of unity.*

PROOF: (1). $\Leftarrow$: Let $0 \neq N$ be a graded ideal of W_q. We have two cases to discuss.

- If q is not a root of unity, then

$$d_m \in N \qquad \text{for some } m \in \mathbf{Z}.$$

Since $N \ni [d_m, d_{-m}] = [2m]d_0$, we have

$$d_0 \in N. \tag{1.6}$$

Therefore, $N \ni [d_k, d_0] = [k]d_k$, for $k \in \mathbf{Z}$. This implies that

$$d_k \in N \qquad \text{for } k \in \mathbf{Z} \setminus \{0\}. \tag{1.7}$$

It follows from (1.6) and (1.7) that $N = W_q$.

- If q is a root of unity and $o(q) \neq 4$, then $[2] \neq 0$ and

$$d_{kT+i} \in N \qquad \text{for some } k \in \mathbf{Z} \text{ and } 0 \leq i \leq T-1.$$

If $2i \notin T\mathbf{Z}$, then $[2i] \neq 0$ and

$$N \ni [d_{kT+i}, d_{-kT-i}] = [2kT + 2i]d_0 = [2i]d_0,$$

which implies (1.6).

If $2i \in T\mathbf{Z}$, then $[2i+1] \neq 0$ and

$$N \ni [d_{kT+i}, d_{-kT-i}] = [2kT + 2i + 1]d_1 = [2i+1]d_1.$$

So

$$d_1 \in N \qquad \text{and} \qquad N \in [d_1, d_{-1}] = [2]d_0,$$

which also implies (1.6).

Hence, we proved that (1.6) always holds.

In order to prove that $N = W_q$, it suffices to prove

$$d_{mT+r} \in N \qquad \text{for } m \in \mathbf{Z} \text{ and } 0 \leq r \leq T-1 \tag{1.8}$$

If $r \neq 0$, then, by (1.6)

$$N \ni [d_{mT+r}, d_0] = [mT + r]d_{mT+r} = q^{mT}[r]d_{mT+r}.$$

So

$$d_{mT+r} \in N \qquad \text{for } m \in \mathbf{Z} \text{ and } 1 \leq r \leq T-1. \tag{1.9}$$

If $r = 0$, then $d_{mT+1} \in N$ by (1.9). Hence,

$$N \ni [d_{mT+1}, d_{-1}] = [mT+2]d_{mT} = q^{mT}[2]d_{mT},$$

i.e., $d_{mT} \in N$ for $m \in \mathbf{Z}$.

(1). $\Rightarrow$: If $o(q) = 4$, then $N := \oplus_{n \in \mathbf{Z}} \mathbf{C} d_{2n+1}$ is a graded ideal of W_q, so W_q is not graded simple.

(2). $\Leftarrow$: Let $0 \neq N$ be an ideal of W_q and let

$$0 \neq x = a_1 d_{m_1} + a_2 d_{m_2} + \cdots + a_\ell d_{m_\ell} \in N, \tag{1.10}$$

where $a_i \in \mathbf{C}^*$, $m_i \in \mathbf{Z}$ and $m_i \neq m_j$ if $1 \leq i \neq j \leq \ell$.

We choose $x \in N \setminus \{0\}$ such that ℓ in (1.10) is minimal. Then $\ell = 1$. In fact, if $\ell \geq 2$, we have

$$N \ni [d_{m_1}, x] = a_2[m_1 - m_2]d_{m_2} + \cdots + a_\ell[m_1 - m_\ell]d_{m_\ell},$$
$$a_i[m_1 - m_i] \neq 0 \qquad \text{for } 2 \leq i \leq \ell,$$

which contradicts to the choice of ℓ.

Therefore, we have proved that there exists some $d_m \in N$. As above, this implies that $N = W_q$.

(2). $\Rightarrow$: If q is a root of unity, then $N(a_i, k_i, 0 \leq i \leq \ell)$ defined by

$$N(a_i, k_i, 0 \leq i \leq \ell) := \bigoplus_{\substack{k \in \mathbf{Z} \\ 0 \leq s \leq T-1}} \mathbf{C}\left(\sum_{i=0}^{\ell} a_i d_{(k_i+k)T+s}\right)$$

is an ideal which is different from $\{0\}$ and W_q, where ℓ is a positive integer, $a_0, a_1, \ldots, a_\ell \in \mathbf{C}^*$, $k_0, k_1, \ldots, k_\ell \in \mathbf{Z}$, $k_i - k_j \in 2\mathbf{Z}$ and $k_i \neq k_j$, for $0 \leq i < j \leq \ell$. Therefore, W_q is not simple. ∎

§1.2. Quantum central extensions

In this section, we assume that all homomorphisms betwteen quantum Lie algebras are $\mathbf{Z}$–grading preserving homomorphisms.

An exact sequence of quantum Lie algebras

$$0 \longrightarrow \mathcal{C} \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } L \longrightarrow 0 \tag{2.1}$$

is said to be a **quantum extension** of L by C, which is denoted by $\lfloor i, \pi \rfloor$. This quantum extension is said to be a **quantum central extension** if $[i(C), E] = 0$. A **morphism** of the quantum extension (2.1) to another extension

$$0 \longrightarrow C' \xrightarrow{i'} E' \xrightarrow{\pi'} L \longrightarrow 0 \tag{2.2}$$

is a pair (ψ, φ) of quantum Lie algebra homomorphisms such that the diagram

$$\begin{array}{ccccc} C & \xrightarrow{i} & E & \xrightarrow{\pi} & L \\ \psi\downarrow & & \varphi\downarrow & & \| \\ C' & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & L \end{array}$$

is commutative. A quantum central extension (2.1) of L is called **quantum universal central extension** if, given any quantum central extension (2.2) of L, there is a unique morphism from (2.1) to (2.2). Two quantum extensions E and E' of L by C are **equivalent** if there is an isomorphism $\varphi : E \simeq E'$ making the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \varphi\downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & E' & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Let V be a $\mathbb{Z}$–graded vector space and L a quantum Lie algebra. A skew–symmetric bilinear maps $\alpha : L \times L \to V$ is said to be a **quantum 2–cocycle** of L with coeffients in V if α satisfies the following conditions:

$$\alpha(L_n, L_m) \subseteq V_{n+m}, \tag{2.3}$$

$$\alpha([x, y], \sigma(z)) + \alpha([y, z], \sigma(x)) + \alpha([z, x], \sigma(y)) = 0 \tag{2.4}$$

for all $n, m \in \mathbb{Z}$ and $x, y, z \in L$.

Let $Z_q^2(L, V)$ be the space of all quantum 2–cocycles of L with coefficients in V and $B_q^2(L, V)$ the space of all skew–symmetric bilinear map $\alpha : L \times L \to V$ satisfying (2.3) and

$$\alpha(x, y) = f([x, y]) \tag{2.5}$$

for $x, y \in L$ and some $\mathbb{Z}$–grading preserving linear map $f : L \to V$. It is clear that $B_q^2(L, V) \subseteq Z_q^2(L, V)$. The quotient $H_q^2(L, V) := \dfrac{Z_q^2(L, V)}{B_q^2(L, V)}$ is called the **quantum second cohomology group** of L with coefficients in V.

Let (2.1) be a quantum central extention of L by $\mathcal{C}$, where $[\mathcal{C},\mathcal{C}]=0$. Then the **equivalent class** $[\lfloor i,\pi\rfloor]$ containing the quantum central extension (2.1) is defined to be the set of all quantum central extentions of L by $\mathcal{C}$ which are equivalent to (2.1). Let $CE_q(L,\mathcal{C})$ denote the set of all equivalent classes of quantum central extensions of L by $\mathcal{C}$.

Thus, if L and $\mathcal{C}$ are quantum Lie algebras with $[\mathcal{C},\mathcal{C}]=0$, then

$$H_q^2(L,\mathcal{C})=\{\,[\alpha]:=\alpha+B_q^2(L,\mathcal{C})|\alpha\in Z_q^2(L,\mathcal{C})\,\},$$
$$CE_q(L,\mathcal{C})=\left\{[\lfloor i,\pi\rfloor]\quad|\quad \begin{array}{l}\lfloor i,\pi\rfloor \text{ is a quantum central exten-}\\ \text{sion of } L \text{ by } \mathcal{C}\end{array}\right\}.$$

Now we construct two maps F and G between $H_q^2(L,\mathcal{C})$ and $CE_q(L,\mathcal{C})$ as follows:

• $F: H_q^2(L,\mathcal{C})\to CE_q(L,\mathcal{C})$ is defined by $F([\alpha])=[\lfloor i_\alpha,\pi_\alpha\rfloor]$ where

$$(2.6)\qquad 0\longrightarrow \mathcal{C}\xrightarrow{\;i_\alpha\;}E_\alpha\xrightarrow{\;\pi_\alpha\;}L\longrightarrow 0$$

is defined by

$$E_\alpha:=\oplus_{n\in\mathbf{Z}}(E_\alpha)_n,\qquad \text{where}(E_\alpha)_n:=L_n\oplus\mathcal{C}_n,$$

$$[(x,u),(y,v)]:=([x,y],\alpha(x,y)),$$

$$i_\alpha:u\mapsto(0,u),\qquad \pi_\alpha:(x,u)\mapsto x$$

for $x,y\in L$ and $u,v\in\mathcal{C}$.

• $G: CE_q(L,\mathcal{C})\to H_q^2(L,\mathcal{C})$ is defined by

$$G([\lfloor i,\pi\rfloor])=[\alpha],\qquad \alpha_0=i\alpha,$$

$$\alpha_0(x,y):=[\beta(x),\beta(y)]-\beta([x,y])\qquad \text{for } x,y\in L,$$

where $\beta:L\to E$ is a $\mathbf{Z}$–grading preserving linear map satisfying $\pi\beta=id|_L$.

Using the notations above, we have

Proposition 2.1. *The map F is a bijection with inverse G.*

Proof: What we need to check is the following:
(i) F is well–defined;
(ii) G is well–defined;
(iii) $FG=1$ and $GF=1$.

(i). For $x, y, z \in L$ and $u, v, w \in \mathcal{C}$, we have

$$\begin{aligned}
&[[(x,u),(y,v)],\sigma((z,w))] + [[(y,v),(z,w)],\sigma((x,u))] \\
&\qquad + [[(z,w),(x,u)],\sigma((y,v))] \\
=&[([x,y],\alpha(x,y)),(\sigma(z),\sigma(w))] + [([y,z],\alpha(y,z)),(\sigma(x),\sigma(u))] \\
&\qquad + [([z,x],\alpha(z,x)),(\sigma(y),\sigma(v))] \\
=&([[x,y],\sigma(z)],\alpha([x,y],\sigma(z))) + ([[y,z],\sigma(x)],\alpha([y,z],\sigma(x))) \\
&\qquad + ([[z,x],\sigma(y)],\alpha([z,x],\sigma(y))) \\
=&0
\end{aligned}$$

Also, it is easy to check that

$$\begin{aligned}
[(E_\alpha)_m,(E_\alpha)_n] \subseteq (E_\alpha)_{m+n} \qquad &\text{for } m,n \in \mathbf{Z}, \\
[(x,u),(y,v)] = -[(y,v),(x,u)] \qquad &\text{for } x,y \in L \text{ and } u,v \in \mathcal{C}.
\end{aligned}$$

Therefore, E_α is a quantum Lie algebra. It is obvious that i_α is injective and π_α is surjective. So we have proved that (2.6) is a quantum central extension of L by $\mathcal{C}$.

Next, we prove that (2.6) is independent of the choice of α.

Suppose that α and $\alpha' \in Z_q^2(L,\mathcal{C})$ and $\alpha' - \alpha \in B_q^2(L,\mathcal{C})$, i.e., there exists a $\mathbf{Z}$-grading preserving linear map $f : L \to \mathcal{C}$ such that

$$(\alpha' - \alpha)(x,y) = f([x,y]) \qquad \text{for } x,y \in L.$$

We define $\varphi : E_\alpha \to E_{\alpha'}$ by

$$\varphi : (x,u) \mapsto (x, u + f(x)), \qquad \text{where } x \in L \text{ and } u \in \mathcal{C}.$$

On one hand, if we denote the bracket on E_α and $E_{\alpha'}$ by $[\ ,\ ]_\alpha$ and $[\ ,\ ]_{\alpha'}$, respectively, then, for $x, y \in L$ and $u, v \in \mathcal{C}$,

$$\begin{aligned}
\varphi([(x,u),(y,v)]_\alpha) &= \varphi([x,y],\alpha(x,y)) \\
&= ([x,y],\alpha(x,y) + f([x,y])) \\
&= ([x,y],\alpha'(x,y)) \\
&= [(x,u),(y,v)]_{\alpha'} \\
&= [(x,u+f(x)),(y,v+f(y))]_{\alpha'} \\
&= [\varphi((x,u)),\varphi((y,v))]_{\alpha'}.
\end{aligned}$$

It follows that φ is an isomorphism from the quantum Lie algebra E_α to the quantum Lie algebra $E_{\alpha'}$.

On the other hand, φ makes the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i_\alpha} & E_\alpha & \xrightarrow{\pi_\alpha} & L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & C & \xrightarrow{i_{\alpha'}} & E_{\alpha'} & \xrightarrow{\pi_{\alpha'}} & L & \longrightarrow & 0 \end{array}$$

Therefore, $\lfloor i_\alpha, \pi_\alpha \rfloor$ and $\lfloor i_{\alpha'}, \pi_{\alpha'} \rfloor$ are equivalent.
This proves (i).

(ii). Fix a $\mathbb{Z}$-grading preserving linear map $\beta : L \to E$ which satisfies $\pi\beta = id|_L$. Define $\alpha_0 : L \times L \to E$ by

$$\alpha_0(x,y) := [\beta(x), \beta(y)] - \beta([x,y]), \qquad \text{for } x, y \in L. \tag{2.7}$$

Then $\alpha_0(x,y) \in Ker(\pi) = Im(i)$. Using the injectivity of i, we can construct $\alpha : L \times L \to C$ such that $\alpha_0 = i\alpha$. For $x, y, z \in L$,

$$\begin{aligned} &\alpha_0([x,y], \sigma(z)) + \alpha_0([y,z], \sigma(x)) + \alpha_0([z,x], \sigma(y)) \\ =&([[\beta(x), \beta(y)], \sigma(\beta(z))]) + ([[\beta(y), \beta(z)], \sigma(\beta(x))]) \\ &+ ([[\beta(z), \beta(x)], \sigma(\beta(y))]) \\ &- \beta([[x,y], \sigma(z)] + [[y,z], \sigma(x)] + [[z,x], \sigma(y)]) = 0, \end{aligned}$$

by (2.7) and the fact: $\sigma\beta = \beta\sigma$. This implies that α is a quantum 2–cocycle.

Now we prove that the equivalent class $[\alpha]$ constructed from β is independent of the choice of β. Suppose $\beta' : L \to E$ is another $\mathbb{Z}$-grading preserving linear map satisfying $\pi\beta' = id|_L$. Using the same procedure as above, we can construct α_0' and a quantum 2–cocycle α' such that

$$\alpha_0' = i\alpha', \qquad \alpha_0'(x,y) = [\beta'(x), \beta'(y)] - \beta'([x,y]),$$

where $x, y \in L$. That $Im(\beta' - \beta) \in Ker(\pi) = Im(i)$ produces a $\mathbb{Z}$-grading preserving linear map $f : L \to C$ satisfying $\beta' - \beta = if$. It follows that

$$\begin{aligned} &i(\alpha - \alpha')(x,y) = (\alpha_0 - \alpha_0')(x,y) \\ =&[(\beta - \beta')(x), \beta(y)] + [\beta'(x), (\beta - \beta')(y)] + if([x,y]) = if([x,y]). \end{aligned}$$

Thus $(\alpha - \alpha')((x,y)) = f([x,y])$, i.e. $\alpha - \alpha' \in B_q^2(L, C)$, or $[\alpha] = [\alpha']$.

(iii). Let $[\lfloor i, \pi \rfloor] \in CE_q(L, C)$, then $FG([\lfloor i, \pi \rfloor]) = [\lfloor i_\alpha, \pi_\alpha \rfloor]$, where $\lfloor i, \pi \rfloor$ and $\lfloor i_\alpha, \pi_\alpha \rfloor$ denote the quantum central extensions (2.1) amd (2.6), respectively.

In order to prove that $\lfloor i, \pi \rfloor$ and $\lfloor i_\alpha, \pi_\alpha \rfloor$ are equivalent, we define $\varphi : E_\alpha \to E$ by

$$\varphi((x,u)) := \beta(x) + i(u), \qquad \text{where } x \in L \text{ and } u \in C.$$

Then, for $x, y \in L$ and $u, v \in C$, we have

$$\begin{aligned}
&\varphi([(x,u),(y,v)]) = \varphi([x,y],\alpha(x,y)) \\
=&\beta([x,y]) + i\alpha(x,y) \\
=&\beta([x,y]) + \alpha_0(x,y) = [\beta(x),\beta(y)] \\
=&[\beta(x) + i(u), \beta(y) + i(v)] \\
=&[\varphi((x,u)), \varphi((y,v))].
\end{aligned}$$

This proves that φ is a $\mathbb{Z}$–grading preserving homomorphism.

Furthermore, the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{i_\alpha} & E_\alpha & \xrightarrow{\pi_\alpha} & L & \longrightarrow & 0 \\
 & & \| & & \varphi\downarrow & & \| & & \\
0 & \longrightarrow & C & \xrightarrow{i} & E & \xrightarrow{\pi} & L & \longrightarrow & 0
\end{array}$$

In fact, for $x \in L$ and $u \in C$, we have

$$\varphi i_\alpha(u) = \varphi((0,u)) = \beta(0) + i(u) = i(u)$$

and

$$\pi\varphi((x,u)) = \pi(\beta(x) + i(u)) = x = \pi_\alpha((x,u)).$$

It follows that φ is a $\mathbb{Z}$–grading preserving isomomorphism. This means that $\lfloor i, \pi \rfloor$ and $\lfloor i_\alpha, \pi_\alpha \rfloor$ are equivalent, i.e., $[\lfloor i, \pi \rfloor] = [\lfloor i_\alpha, \pi_\alpha \rfloor]$.

Thus, we have proved that $FG = 1$.

Similarly, one can check $GF = 1$. ∎

PROPOSITION 2.2. *Let L be a quantum Lie algebra. If $[L, L] = L$, then L has a quantum universal central extension.*

PROOF: Let $L = \oplus_{n \in \mathbb{Z}} L_n$ be a quantum Lie algebra. We define the q–analogue of the usual second homology group of L with trivial coefficients by $H_2^q(L, \mathbb{C}) := \dfrac{\wedge^2(L)}{I}$, where I is the $\mathbb{Z}$–graded subspace of $\wedge^2(L)$ spanned by all elements of the form

$$[x,y] \wedge \sigma(z) + [y,z] \wedge \sigma(x) + [z,x] \wedge \sigma(y),$$

where $x, y, z \in L$.

Let $\alpha : L \times L \to H_2^q(L, \mathbb{C})$ be the map defined by

$$\alpha : (x,y) \mapsto x \wedge y + I \qquad \text{for } x, y \in L.$$

Then $\alpha \in Z_q^2(L, H_2^q(L, \mathbf{C}))$. Let $F([\alpha])$ be the quantum central extension:

$$(2.8) \qquad 0 \longrightarrow H_2^q(L, \mathbf{C}) \xrightarrow{i_\alpha} E_\alpha \xrightarrow{\pi_\alpha} L \longrightarrow 0$$

as constructed prior to Proposition 2.1.

Suppose that

$$(2.9) \qquad 0 \longrightarrow C' \xrightarrow{i'} E' \xrightarrow{\pi'} L \longrightarrow 0$$

is any quantum central extension of L. We denote the bracket on E' by $[\ ,\ ]'$. Let us write E' as a direct sum of subspaces $\oplus_{n\in\mathbf{Z}}(L'_n \oplus i'(C'_n))$, where $L' := \oplus_{n\in\mathbf{Z}} L'_n$ is a preimage of L under π'. Identifying L with L' and C' with $i'(C')$, we have $E' = \oplus_{n\in\mathbf{Z}}(E')_n$, where $(E')_n = L_n \oplus (C')_n$.

Then we have a quantum 2–cocycle $\beta : L \times L \to C'$ with

$$[x, y]' = [x, y] + \beta(x, y) \qquad \text{for } x, y \in L.$$

It follows that there is an induced map $\beta : \wedge^2(L) \to C'$ vanishing on the subspace $I \subseteq \wedge^2(L)$ defined above. This allows us to define a linear map $\psi : H_2^q(L, \mathbf{C}) \to C'$ by

$$\psi(\alpha(x, y)) = \beta(x, y), \qquad \text{where } x, y \in L.$$

Next, we define the linear map $\varphi : E_\alpha \to E'$ by

$$\varphi((x, u)) = x + \psi(u), \qquad \text{where } x \in L \text{ and } u \in H_2^q(L, \mathbf{C}).$$

Then the following diagram is commutative:

$$\begin{array}{ccccc}
H_2^q(L, \mathbf{C}) & \xrightarrow{i_\alpha} & E_\alpha & \xrightarrow{\pi_\alpha} & L \\
\psi\downarrow & & \varphi\downarrow & & \Vert \\
C' & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & L
\end{array}$$

In fact, for $x \in L$ and $u \in H_2^q(L, \mathbf{C})$, we have

$$\pi'\varphi((x, u)) = \pi'(x + \psi(u)) = x = \pi_\alpha((x, u))$$

and

$$\varphi i_\alpha(\alpha(x, y)) = \varphi((0, \alpha(x, y))) = \psi\alpha(x, y) = i'\psi(\alpha(x, y)).$$

It follows that $\pi'\varphi = id \cdot \pi_\alpha$ and $\varphi i_\alpha = i'\psi$.

If $x, y \in L$ and $u, v \in H_2^q(L, \mathbf{C})$, then

$$
\begin{aligned}
\varphi([(x,u),(y,v)]) &= \varphi(([x,y], \alpha(x,y))) \\
&= [x,y] + \psi\alpha(x,y) \\
&= [x,y] + \beta(x,y) = [x,y]' \\
&= [x + \psi(u), y + \psi(v)]' = [\varphi((x,u)), \varphi((y,v))].
\end{aligned}
$$

So φ is a $\mathbf{Z}$-grading preserving homomorphism.
This proves that (ψ, φ) is a morphism from (2.8) to (2.9).

Let $E := [E_\alpha, E_\alpha]$. Since $[L, L] = L$, $\pi_\alpha(E) = L$ and $E_\alpha = E + Im(i_\alpha)$. Thus

$$E = [E_\alpha, E_\alpha] = [E + Im(i_\alpha), E + Im(i_\alpha)] = [E, E].$$

Let $C := E \cap Im(i_\alpha)$. We will prove that

$$0 \longrightarrow C \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } L \longrightarrow 0$$

is the quantum universal central extension of L, where i is the imbedding map and π is the restriction of π_α to E.

Indeed, given any other quantum central extension (2.9) of L, we have the following commutative diagram by what we have proved:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\ i\ } & E & \xrightarrow{\ \pi\ } & L & \longrightarrow & 0 \\
 & & \psi_0 \downarrow & & \varphi_0 \downarrow & & \| & & \\
0 & \longrightarrow & H_2^q(L, \mathbf{C}) & \xrightarrow{\ i_\alpha\ } & E_\alpha & \xrightarrow{\ \pi_\alpha\ } & L & \longrightarrow & 0 \\
 & & \psi \downarrow & & \varphi \downarrow & & \| & & \\
0 & \longrightarrow & C' & \xrightarrow{\ i'\ } & E' & \xrightarrow{\ \pi'\ } & L & \longrightarrow & 0
\end{array}
$$

where φ_0 is the imbedding maps and ψ_0 is defined by

$$\psi_0 : i_\alpha(u) \mapsto u \qquad \text{for } i_\alpha(u) \in C \text{ and } u \in H_2^q(L, \mathbf{C})$$

Hence, $(\psi\psi_0, \varphi\varphi_0)$ is a morphism from $\lfloor i, \pi \rfloor$ to $\lfloor i', \pi' \rfloor$.
Suppose that (ψ', φ') is any other morphism from $\lfloor i, \pi \rfloor$ to $\lfloor i', \pi' \rfloor$, i.e.,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\ i\ } & E & \xrightarrow{\ \pi\ } & L & \longrightarrow & 0 \\
 & & \psi' \downarrow & & \varphi' \downarrow & & \| & & \\
0 & \longrightarrow & C' & \xrightarrow{\ i'\ } & E' & \xrightarrow{\ \pi'\ } & L & \longrightarrow & 0
\end{array}
$$

Then, for all $z \in E$, we have $\pi'(\varphi\varphi_0)(z) = \pi(z) = \pi'\varphi'(z)$ and

$$(\varphi\varphi_0)(z) - \varphi'(z) \in Ker(\pi') = i'(C').$$

Now let $x, y \in E$, then

$$\begin{aligned}(\varphi\varphi_0)([x,y]) &= [(\varphi\varphi_0)(x), (\varphi\varphi_0(y)] \\ &= [((\varphi\varphi_0)(x) - \varphi'(x)) + \varphi'(x), ((\varphi\varphi_0)(y) - \varphi'(y)) + \varphi'(y)] \\ &= [\varphi'(x), \varphi'(y)] = \varphi'([x,y]).\end{aligned}$$

It follows from $E = [E, E]$ that

$$\varphi\varphi_0 = \varphi'$$

and

$$i'(\psi\psi_0) = (\varphi\varphi_0)i = \varphi i = i'\psi'$$

or

$$\psi\psi_0 = \psi'$$

because i' is injective.

Thus, we have proved that for any quantum central extension $\lfloor i', \pi' \rfloor$ of L, there exists a unique morphism $(\psi\psi_0, \varphi\varphi_0)$ from $\lfloor i, \pi \rfloor$ to $\lfloor i', \pi' \rfloor$. Therefore, $\lfloor i, \pi \rfloor$ is the quantum universal central extension of L. ∎

§1.3. The quantum universal central extension of W_q and $\mathbb{C}[t, t^{-1}] \otimes s\ell_q^n(2)$

In this section, we will study the quantum universal central extension of the quantum Witt algebra W_q and the quantum Lie algebra $\mathbb{C}[t, t^{-1}] \otimes s\ell_q^n(2)$, respectively.

THEOREM 3.1. *If q is not a root of unity, then the quantum Virasoro algebra V_q is the quantum universal central extension of the quantum Witt algebra W_q.*

PROOF: Let

$$0 \longrightarrow C \longrightarrow L \longrightarrow W_q \longrightarrow 0$$

be an arbitrary quantum central extension of W_q with corresponding quantum 2–cocycle α. By (2.4), we have

$$\alpha([d_m, d_n], \sigma(d_k)) + \alpha([d_n, d_k], \sigma(d_m)) + \alpha([d_k, d_m], \sigma(d_n)) = 0,$$

or

$$
\begin{aligned}
&< k > [m-n]\alpha(d_k, d_{m+n}) + < m > [n-k]\alpha(d_m, d_{n+k}) \\
&\quad + < n > [k-m]\alpha(d_n, d_{k+m}) = 0
\end{aligned} \tag{3.1}
$$

Letting $k = 0$ in (3.1), we get

$$
[m-n]\alpha(d_0, d_{m+n}) + [m+n]\alpha(d_m, d_n) = 0 \tag{3.2}
$$

Setting $k = 1$ and $n = -m-1$ in (3.1), we obtain

$$
\begin{aligned}
&< m+1 > [m-1]\alpha(d_{m+1}, d_{-m-1}) \\
&= < 1 > [-2m-1]\alpha(d_1, d_{-1}) + < m > [m+2]\alpha(d_m, d_{-m})
\end{aligned} \tag{3.3}
$$

Set

$$
\ell_n := \begin{cases} d_n - \dfrac{1}{[n]}\alpha(d_0, d_n), & \text{for } n \neq 0, \\ d_0 + \dfrac{1}{[2]}\alpha(d_1, d_{-1}), & \text{for } n = 0. \end{cases}
$$

On one hand, if $m + n \neq 0$, then

$$
\begin{aligned}
[\ell_m, \ell_n] &= [d_m, d_n] = [m-n]d_{m+n} + \alpha(d_m, d_n) \\
&= [m-n]\left(\ell_{m+n} + \frac{1}{[m+n]}\alpha(d_0, d_{m+n})\right) + \alpha(d_m, d_n) \\
&= [m-n]\ell_{m+n} + \left(\frac{[m-n]}{[m+n]}\alpha(d_0, d_{m+n}) + \alpha(d_m, d_n)\right).
\end{aligned}
$$

By (3.2), we have

$$
[\ell_m, \ell_n] = [m-n]\ell_{m+n} \qquad \text{if } m+n \neq 0. \tag{3.4}
$$

On the other hand,

$$
\begin{aligned}
[\ell_m, \ell_{-m}] &= [d_m, d_{-m}] = [2m]d_0 + \alpha(d_m, d_{-m}) \\
&= [2m]\left(\ell_0 - \frac{1}{[2]}\alpha(d_1, d_{-1})\right) + \alpha(d_m, d_{-m}) \\
&= [2m]\ell_0 + \left(-\frac{[2m]}{[2]}\alpha(d_1, d_{-1}) + \alpha(d_m, d_{-m})\right).
\end{aligned} \tag{3.5}
$$

Now we define $\beta : W_q \times W_q \to C$ by

$$
\beta(d_m, d_n) = \begin{cases} 0, & \text{if } m+n \neq 0; \\ -\dfrac{[2m]}{[2]}\alpha(d_1, d_{-1}) + \alpha(d_m, d_{-m}), & \text{if } m+n = 0. \end{cases}
$$

It follows from (3.4) and (3.5) that

$$[\ell_m, \ell_n] = [m-n]\ell_{m+n} + \beta(d_m, d_n). \tag{3.6}$$

Note β is a quantum 2-cocycle with $\beta(d_1, d_{-1}) = 0$. Hence, by (3.3)

$$< m+1 > [m-1]\beta(d_{m+1}, d_{-m-1}) =< m > [m+2]\beta(d_m, d_{-m}).$$

This implies that

$$\beta(d_m, d_{-m}) = \frac{[m-1][m][m+1]}{[2][3] < m >} < 2 > \beta(d_2, d_{-2}), \qquad m \geq 2.$$

Setting $c =< 2 > \beta(d_2, d_{-2})$, (3.6) becomes

$$[\ell_m, \ell_n] = [m-n]\ell_{m+n} + \frac{[m-1][m][m+1]}{[2][3] < m >}\delta_{m+n,0}c,$$

which proves that V_q is the quantum universal central extension of W_q. ∎

THEOREM 3.2. *If $q^{4n} \neq 1$, then $\widehat{s\ell^n_q}(2)$ is the quantum universal central extension of $\mathbf{C}[t, t^{-1}] \otimes s\ell^n_q(2)$.*

PROOF: Set $ux := u \otimes x$ for $u \in \mathbf{C}[t, t^{-1}]$ and $x \in s\ell^n_q(2)$. Let

$$0 \longrightarrow \mathcal{C} \longrightarrow L \xrightarrow{\ \pi\ } \mathbf{C}[t, t^{-1}] \otimes s\ell^n_q(2) \longrightarrow 0$$

be an arbitrary quantum central extension of $\mathbf{C}[t, t^{-1}] \otimes s\ell^n_q(2)$ and $a' \in L$ such that $\pi(a') = a$, then $[a', b']$ is independent of the choice of the preimages of a and b, where $a, b \in \mathbf{C}[t, t^{-1}] \otimes s\ell^n_q(2)$.

For $u \in \mathbf{C}[t, t^{-1}]$, we define $ud'_{\pm n}$ and ud'_0 in L by

$$ud'_{\pm n} := \mp\frac{1}{[n]}[d'_0, (ud_{\pm n})'], \qquad ud'_0 := \frac{1}{[2n]}[d'_n, (ud_{-n})'] \tag{3.7}$$

Set

$$\{u, v\} := [(ud_0)', (vd_0)'] \in \mathcal{C}, \qquad u, v \in \mathbf{C}[t, t^{-1}] \tag{3.8}$$

Then we have

$$\pi(ud'_0) = \frac{1}{[2n]}[\pi(d'_n), \pi((ud_{-n})')] = \frac{1}{[2n]}[d_n, ud_{-n}] = ud_0,$$

and

$$\pi(ud'_{\pm n}) = \mp\frac{1}{[n]}[\pi(d'_0), \pi((ud_{\pm n})')] = \mp\frac{1}{[n]}[d_0, ud_{\pm n}] = ud_{\pm n}.$$

It follows that

$$[ud'_0, vd'_0] = \{u, v\}, \tag{3.9}$$

$$ud'_{\pm n} = \mp\frac{1}{[n]}[d'_0, ud'_{\pm n}]. \tag{3.10}$$

Since

$$\pi([vd'_0, ud'_{\pm n}]) = [\pi(vd'_0), \pi(ud'_{\pm n})]$$
$$= [vd_0, ud_{\pm n}] = \mp[n]uvd_{\pm n},$$

so we get

$$[vd'_0, ud'_{\pm n}] - (\mp[n]uvd'_{\pm n}) \in \mathcal{C}. \tag{3.11}$$

From

$$[vd'_0, [d'_0, ud'_{\pm n}]] + [d'_0, [ud'_{\pm n}, vd'_0]] + < n > [ud'_{\pm n}, [vd'_0, d'_0]] = 0,$$

we get, by (3.8) and (3.9)

$$[vd'_0, [d'_0, ud'_{\pm n}]] = [d'_0, [vd'_0, ud'_{\pm n}]]. \tag{3.12}$$

Using (3.10), (3.11) and (3.12),

$$[vd'_0, ud'_{\pm n}] = [vd'_0, \mp\frac{1}{[n]}[d'_0, ud'_{\pm n}]]$$
$$= \mp\frac{1}{[n]}[vd'_0, [d'_0, ud'_{\pm n}]] = \mp\frac{1}{[n]}[d'_0, [vd'_0, ud'_{\pm n}]]$$
$$= \mp\frac{1}{[n]}[d'_0, \mp[n]uvd'_{\pm n}] = [d'_0, uvd'_{\pm n}] = \mp[n]uvd'_{\pm n}.$$

This proves that

$$[vd'_0, ud'_{\pm n}] = \mp[n]uvd'_{\pm n}. \tag{3.13}$$

In particular, let $u = 1$ in (3.13), we have

$$[vd'_0, d'_{\pm n}] = \mp[n]vd'_{\pm n}. \tag{3.14}$$

From

$$< n > [[ud'_0, d'_n], vd'_{-n}] + [[d'_n, vd'_{-n}], ud'_0] + < n > [[vd'_{-n}, ud'_0], d'_n] = 0$$

and

$$[d'_n, vd'_{-n}] = [2n]vd'_0, \tag{3.15}$$

we get

$$[[ud'_0, d'_n], vd'_{-n}] = [[ud'_0, vd'_{-n}], d'_n] + \frac{[2n]}{< n >}[ud'_0, vd'_0]. \tag{3.16}$$

So, by (3.9), (3.13), (3.14) and (3.16)

$$\begin{aligned}
[ud'_n, vd'_{-n}] &= [-\frac{1}{[n]}[ud'_0, d'_n], vd'_{-n}] \\
&= -\frac{1}{[n]}[[ud'_0, d'_n], vd'_{-n}] \\
&= -\frac{1}{[n]}[[ud'_0, vd'_{-n}], d'_n] - \frac{[2n]}{[n] < n >}[ud'_0, vd'_0] \\
&= -\frac{1}{[n]}[[n]uvd'_{-n}, d'_n] - \frac{[2n]}{[n] < n >}\{u, v\} \\
&= [d'_n, uvd'_{-n}] - \frac{[2n]}{[n] < n >}\{u, v\}.
\end{aligned}$$

It follows from (3.15) that

$$[ud'_n, vd'_{-n}] = [2n]uvd'_0 - \frac{[2n]}{[n] < n >}\{u, v\} \tag{3.17}$$

Similarly, after using quantum Jacobi identity for $(d'_0, ud'_{\pm n}, vd'_{\pm n})$, we obtain (3.18)

$$< n > [[d'_0, ud'_{\pm n}], vd'_{\pm n}] + [[ud'_{\pm n}, vd'_{\pm n}], d'_0] + < n > [[vd'_{\pm n}, d'_0], ud'_{\pm n}] = 0.$$

Since $\pi([ud'_{\pm n}, vd'_{\pm n}]) = [\pi(ud'_{\pm n}), \pi(vd'_{\pm n})] = [ud_{\pm n}, vd_{\pm n}] = uv[d_{\pm n}, d_{\pm n}] = 0$, $[ud'_{\pm n}, vd'_{\pm n}] \in \mathcal{C}$. It follows from (3.18) that

$$[[d'_0, ud'_{\pm n}], vd'_{\pm n}] = [[d'_0, vd'_{\pm n}], ud'_{\pm n}]. \tag{3.19}$$

We have, by (3.10) and (3.19)

$$\begin{aligned}
[ud'_{\pm n}, vd'_{\pm n}] &= \mp\frac{1}{[n]}[[d'_0, ud'_{\pm n}], vd'_{\pm n}] \\
&= \mp\frac{1}{[n]}[[d'_0, vd'_{\pm n}], ud'_{\pm n}] \\
&= \mp\frac{1}{[n]}[\mp[n]vd'_{\pm n}, ud'_{\pm n}] \\
&= [vd'_{\pm n}, ud'_{\pm n}] = -[ud'_{\pm n}, vd'_{\pm n}],
\end{aligned}$$

i.e.,

$$[ud'_{\pm n}, vd'_{\pm n}] = 0. \tag{3.20}$$

Finally, applying the quantum Jacobi identity to (ud'_0, vd'_n, wd'_{-n}) gives that

$$\{u, vw\} + \{v, wu\} + \{w, uv\} = 0.$$

By the Lamma (2.16) in [3],

$$\{u, v\} = -Res\left(u\frac{dv}{dt}\right)\{t, t^{-1}\}, \qquad u, v \in \mathbb{C}[t, t^{-1}]. \tag{3.21}$$

Now the Theorem 2.4 follows from (3.9), (3.13), (3.17), (3.20) and (3.21) ∎

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CHAPTER 2. THE CHARACTERIZATIONS OF THE QUANTUM WITT ALGEBRA

§2.0. Basic definitions and main theorem

In [1] and [3], two different characterizations of the Witt algebra were found by I.Kaplansky and M.L.Tomber, respectively. In this chapter, we prove that the quantizations of the two characterizations hold for the quantum Witt algebra.

The following notations will be used throughout the paper:

- $\mathbf{Z}$:= the set of all integers.
- $\mathbf{C}$:= the set of all complex numbers.
- $[m] := \dfrac{q^m - q^{-m}}{q - q^{-1}}$, where $m \in \mathbf{Z}$, $q \in \mathbf{C} \setminus \{0\}$ and q is not a root of unity.
- $[0]! := 1$ and $[m]! := [m][m-1]\ldots[2][1]$ for $m \in \mathbf{Z}$ and $m > 0$.
- $< m >:= \dfrac{q^m + q^{-m}}{2}$, where $m \in \mathbf{Z}$.
- For any $\mathbf{Z}$-graded vector space $\oplus_{n\in\mathbf{Z}} V_n$, $J \in Hom(V, V)$ is defined by $J(v_n) := q^n v_n$ for $n \in \mathbf{Z}$ and $v_n \in V_n$.
- $\sigma := \dfrac{J + J^{-1}}{2}$.
- Let $A = \oplus_{n\in\mathbf{Z}} A_n$ be a $\mathbf{Z}$-graded algebra over $\mathbf{C}$ with mutiplication denoted by xy for $x, y \in A$. As usual, we define the $\mathbf{Z}$-graded algebras $(A^-, [\ ,\ ])$ and $(A^+, \circ)$ by

$$A^{\pm} := \oplus_{n\in\mathbf{Z}} (A^{\pm})_n, \qquad \text{where } (A^{\pm})_n := A_n,$$

$$\begin{aligned} [x, y] : &= xy - yx, \\ x \circ y : &= \frac{xy + yx}{2}, \end{aligned}$$

where $x, y \in A$. Let

$$\begin{aligned} (x, y, z)_q : &= (xy)\sigma(z) - \sigma(x)(yz), \\ J_q(x, y, z) : &= (xy)\sigma(z) + (yz)\sigma(z) + (zx)\sigma(y). \end{aligned}$$

where $x, y, z \in A$.

Definition Let A be a $\mathbf{Z}$-graded algebra over $\mathbf{C}$ with multiplication denoted by xy.

(1). A is called a **quantum flexible algebra** if $(x, y, x)_q = 0$ for $x, y \in A$.

(2). A is called a **quantum Lie algebra** if $xy = -yx$ and the following **quantum Jacobi identity** holds:

$$J_q(x, y, z) = 0 \qquad \text{for } x, y \text{ and } z \in A.$$

One example of quantum Lie algebras is the **quantum Witt algebra** W_q defined by

$$\begin{aligned} W_q &:= \oplus_{n\in\mathbf{Z}}(W_q)_n, \qquad &\text{where } (W_q)_n := \mathbf{C}d_n, \\ d_m d_n &:= [m-n]d_{m+n}, \qquad &\text{where } m, n \in \mathbf{Z}. \end{aligned}$$

Another interesting example of a quantum Lie algebra is the **quantum Virasoro algebra** $V_q = \oplus_{n\in\mathbf{Z}}(V_q)_n$ defined by

$$(V_q)_n := \begin{cases} \mathbf{C}d_0 \oplus \mathbf{C}c, & \text{if } n = 0, \\ \mathbf{C}d_n, & \text{if } n \neq 0, \end{cases}$$

$$\begin{aligned} d_m d_n &:= [m-n]d_{m+n} + \frac{[m-1][m][m+1]}{[2][3] < m >}\delta_{m+n,0}c, \\ c d_m &:= d_m c := 0, \end{aligned}$$

where $m, n \in \mathbf{Z}$.

Remark. If we define a family of linear opertors d_n on the Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$ by

$$d_n := -t^{n+1}\left(\frac{d}{dt}\right)_q, \qquad \left(\frac{d}{dt}\right)_q : t^m \mapsto [m]t^{m-1}$$

for all $m, n \in \mathbf{Z}$, then the $\mathbf{Z}$-graded vector space $\oplus_{n\in\mathbf{Z}}\mathbf{C}d_n$ over $\mathbf{C}$ becomes the quantum Witt algebra under the following multiplication:

$$d_m \cdot d_n := Jd_mJ^{-1}d_nJ - Jd_nJ^{-1}d_mJ$$

Now we can state the main theorem.

THROREM. *Let $A = \oplus_{n\in\mathbf{Z}}A_n$ be a $\mathbf{Z}$-graded algebra over $\mathbf{C}$. Then the following are equivalent:*

(1) *A is the quantum Witt algebra.*
(2) *A is a quantum flexible algebra and $(A^-, [\ ,\ ])$ is the quantum Witt algebra.*
(3) *A is a quantum Lie algebra, $dim(A_n) \leq 1$ for all $n \in \mathbf{Z}$ and*

$$(A_0A_1) \neq 0, \quad (A_1A_{-1}) \neq 0, \quad (A_2A_{-1}) \neq 0, \quad (A_{-2}A_1) \neq 0.$$

Because it is clear that (1)⇒(2) and (1)⇒(3), we only need to prove that (2)⇒(1) and (3)⇒(1).

§2.1. The proof of the theorem: (2)⇒(1)

Using the same argument as in [2], we can prove:

LEMMA. *Let A be a $\mathbb{Z}$-graded algebra and $(a), (b), (c), (a'), (b')$ denote the following statements:*
(a) A is quantum flexible.
(b) $(x, y, z)_q + (z, y, x)_q = 0$ for $x, y, z \in A$.
(c) $[\sigma(x), y \circ z] = [x, y] \circ \sigma(z) + \sigma(y) \circ [x, z]$ for $x, y, z \in A$.
(a') A is quantum flexible and $(A^-, [\ ,\ ])$ is a quantum Lie algebra.
(b') $[\sigma(x), yz] = [x, y]\sigma(z) + \sigma(y)[x, z]$ for $x, y, z \in A$.
Then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(a') \Leftrightarrow (b')$.

PROOF: $(a) \Rightarrow (b)$: Since A is quantum flexible, we have

$$((x + z)y)\,\sigma(x + z) = \sigma(x + z)\,(y(x + z)), \qquad \text{where } x, y, z \in A,$$

i.e.

$$\begin{aligned}&(xy)\sigma(x) + (xy)\sigma(z) + (zy)\sigma(x) + (zy)\sigma(z)\\ &= \sigma(x)(yx) + \sigma(x)(yz) + \sigma(z)(yx) + \sigma(z)(yz),\end{aligned}$$

or

$$((xy)\sigma(z) - \sigma(x)(yz)) + ((zy)\sigma(x) - \sigma(z)(yx)) = 0, \tag{2.1}$$

which is

$$(x, y, z)_q + (z, y, x)_q = 0, \qquad \text{where } x, y, z \in A.$$

$(b) \Rightarrow (c)$: It follows from (b) that

$$(x, y, z)_q + (z, y, x)_q + (x, z, y)_q + (y, z, x)_q = (y, x, z)_q + (z, x, y)_q,$$

where $x, y, z \in A$, or

$$\begin{aligned}&(xy)\sigma(z) - \sigma(x)(yz) + (zy)\sigma(x) - \sigma(z)(yx)\\ &\quad + (xz)\sigma(y) - \sigma(x)(zy) + (yz)\sigma(x) - \sigma(y)(zx)\\ &= (yx)\sigma(z) - \sigma(y)(xz) + (zx)\sigma(y) - \sigma(z)(xy).\end{aligned}$$

This implies that

$$(2.2) \qquad [\sigma(x), y \circ z] = [x,y] \circ \sigma(z) + \sigma(y) \circ [x,z] \qquad \text{for } x,y,z \in A.$$

$(c) \Rightarrow (a)$: Let $z = x$ in (2.2), we get

$$[\sigma(x), y \circ x] = [x,y] \circ \sigma(x),$$

or

$$\begin{aligned}&\sigma(x)(yx) + \sigma(x)(xy) - (yx)\sigma(x) - (xy)\sigma(x)\\ &= (xy)\sigma(x) - (yx)\sigma(x) + \sigma(x)(xy) - \sigma(x)(yx).\end{aligned}$$

It follows that

$$\sigma(x)(yx) = (xy)\sigma(x) \qquad \text{for } x,y \in A.$$

$(a') \Rightarrow (b')$: Let $x,y,z \in A$. On one hand, by $(a) \Rightarrow (c)$, we have

$$(2.3) \qquad [\sigma(x), y \circ z] = [x,y] \circ \sigma(z) + \sigma(y) \circ [x,z].$$

On the other hand, since $(A^-, [\ ,\ ])$ is a quantum Lie algebra, we also have

$$[\sigma(x), [y,z]] + [\sigma(y), [z,x]] + [\sigma(z), [x,y]] = 0,$$

or

$$(2.4) \qquad [\sigma(x), \frac{1}{2}[y,z]] = \frac{1}{2}[[x,y]\sigma(z)] + \frac{1}{2}[\sigma(y), [x,z]].$$

It follows from (2.3)+(2.4) that

$$\begin{aligned}[\sigma(x), yz] &= [\sigma(x), y \circ z + \frac{1}{2}[y,z]] = [\sigma(x), y \circ z] + [\sigma(x), \frac{1}{2}[y,z]]\\ &= ([x,y] \circ \sigma(z) + \frac{1}{2}[[x,y], \sigma(z)]) + (\sigma(y) \circ [x,z] + \frac{1}{2}[\sigma(y), [x,z]])\\ &= [x,y]\sigma(z) + \sigma(y)[x,z].\end{aligned}$$

$(b') \Rightarrow (a')$: Let $x = z$ in (b'), we get

$$[\sigma(x), yx] = [x,y]\sigma(x),$$

or

$$\sigma(x)(yx) - (yx)\sigma(x) = (xy)\sigma(x) - (yx)\sigma(x),$$

i.e.,

$$\sigma(x)(yx) = (xy)\sigma(x) \qquad \text{for } x, y \in A.$$

This proves that A is a quantum flexible algebra. Therefore, (c) holds for the algebra A.

Next, we prove that $(A^-, [\ ,\ ])$ is a quantum Lie algebra. For $x, y, z \in A$, we have, by (c) and (b')

$$\begin{aligned}
\frac{1}{2}[\sigma(x), [y, z]] &= [\sigma(x), \frac{1}{2}[y, z]] \\
&= [\sigma(x), yz - y \circ z] = [\sigma(x), yz] - [\sigma(x), y \circ z] \\
&= ([x, y]\sigma(z) + \sigma(y)[x, z]) - ([x, y] \circ \sigma(z) + \sigma(y) \circ [x, z]) \\
&= ([x, y]\sigma(z) - [x, y] \circ \sigma(z)) + (\sigma(y)[x, z] - \sigma(y) \circ [x, z]) \\
&= \frac{1}{2}[[x, y], \sigma(z)] + \frac{1}{2}[\sigma(y), [x, z]].
\end{aligned}$$

This implies that

$$[[x, y], \sigma(z)] + [[y, z], \sigma(x)] + [[z, x], \sigma(y)] = 0,$$

i.e., $(A^-, [\ ,\ ])$ is a quantum Lie algebra. ∎

Now we begin to prove: (2)⇒(1).

By the Lemma above, we know that (c) and (b') hold for the algebra A in the theorem. Because $(A^-, [\ ,\ ])$ is the quantum Witt algebra, we can choose a $\mathbf{C}$–basis $\{ e_n \mid n \in \mathbf{Z} \}$ of A such that

$$\begin{aligned}
A &= \oplus_{n \in \mathbf{Z}} A_n, \qquad \text{where } A_n = \mathbf{C}e_n, \\
[e_n, e_m] &= e_n e_m - e_m e_n = [n - m]e_{n+m}, \qquad \text{where } m, n \in \mathbf{Z}, \\
e_m e_n &\in \mathbf{C}e_{m+n}.
\end{aligned}$$

In particular, $e_0 e_0 = a e_0$ for some $a \in \mathbf{C}$.

Using (b') to the triple e_n, e_0 and e_m, we get

$$(2.5) \qquad < n > [e_n, e_0 e_m] = < m > [n] e_n e_m + [n - m] e_0 e_{n+m} \qquad \text{for } n, m \in \mathbf{Z}.$$

Let $m = 0$ in (2.5), then

$$< n > a[n] e_n = [n](e_n e_0 + e_0 e_n).$$

Hence,

$$(2.6) \qquad < n > a e_n = e_n e_0 + e_0 e_n \qquad \text{for } n \in \mathbf{Z} \setminus \{0\}.$$

On the other hand,

$$[n]e_n = e_n e_0 - e_0 e_n \qquad \text{for } n \in \mathbb{Z}. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$e_n e_0 = \frac{<n> a + [n]}{2} e_n \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}. \tag{2.8}$$

$$e_0 e_n = \frac{<n> a - [n]}{2} e_n \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}. \tag{2.9}$$

Let $m = -n \neq 0$ in (2.5), we get

$$<n> [e_n, e_0 e_{-n}] =<-n> [n]e_n e_{-n} + [2n]e_0 e_0.$$

By (2.9), the equation above becomes

$$\frac{<n>^2 a + <n> [n] - 2a}{2}[2n]e_0 =<n> [n]e_n e_{-n} \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}. \tag{2.10}$$

Computing $\frac{<n>^2 a + <n> [n] - 2a}{2}[2n][n]e_n$ by using (2.8), (2.10) and (b'), we have

$$\begin{aligned}
&\frac{<n>^2 a + <n> [n] - 2a}{2}[2n][n]e_n \\
&= \frac{<n>^2 a + <n> [n] - 2a}{2}[2n][e_n, e_0] \\
&= [e_n, <n> [n]e_n e_{-n}] = [n][\sigma(e_n), e_n e_{-n}] \\
&=<n> [n][2n]e_n e_0 \\
&= \frac{1}{2} <n> [n][2n](<n> a + [n])e_n \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}.
\end{aligned}$$

It follows from $\frac{1}{2}[n][2n] \neq 0$ that $a = 0$. Therefore, we have proved that

$$e_n e_0 = -e_0 e_n = \frac{1}{2}[n]e_n \qquad \text{for } n \in \mathbb{Z}.$$

Hence,

$$e_n \circ e_0 = 0 \qquad \text{for } n \in \mathbb{Z}. \tag{2.11}$$

Now we prove that

$$e_m \circ e_n = 0 \qquad \text{for } m, n \in \mathbb{Z}.$$

In fact, we can assume that $m \neq 0$ by (2.11). (c) in the Lemma implies that

$$[\sigma(e_m), e_0 \circ e_n] = [e_m, e_0] \circ \sigma(e_n) + \sigma(e_0) \circ [e_m, e_n].$$

Using (2.11), this gives us the following equation

$$0 = [m] < n > e_m \circ e_n,$$

or

$$e_m \circ e_n = 0.$$

Finally, we get

$$e_m e_n = \frac{1}{2}[e_m, e_n] + e_m \circ e_n = \frac{1}{2}[e_m, e_n] = \frac{1}{2}[m-n]e_{m+n}.$$

Let $d_n := 2^{1-n}e_n$ for $n \in \mathbf{Z}$, then

$$A = \oplus_{n\in\mathbf{Z}} A_n, \qquad \text{where } A_n = \mathbf{C}d_n,$$
$$d_m d_n = [m-n]d_{m+n}, \qquad \text{where } m, n \in \mathbf{Z},$$

i.e., A is the quantum Witt algebra.

§2.2. The proof of the theorem: (3)⇒(1)

As usual, we can choose $0 \neq d_i \in A_i$, where $i = 0, \pm 1$, such that

(3.1) $$d_i d_j = [i-j]d_{i+j} \qquad \text{for } i, j = 0, \pm 1.$$

Since $A_2 A_{-1} \neq 0$, $A_2 \neq 0$. Hence, $A_2 = \mathbf{C}u_2$ where $u_2 \neq 0$. Let $d_{-1}u_2 = td_1$ for some $t \in \mathbf{C}$, then $t \neq 0$. Let $d_2 := -\frac{[3]u_2}{t}$, then

(3.2) $$d_2 \neq 0, \qquad d_{-1}d_2 = -[3]d_1.$$

For $x \in A$, we define $ad(x) \in End(A)$ by

$$(ad(x))(a) := xa \qquad \text{for } a \in A.$$

For $k \geq 2$, we define

(3.3) $$d_k := \frac{(-1)^k (ad(d_1))^{k-2}(d_2)}{[k-2]!}.$$

It is clear that

$$d_1 d_{k-1} = [2-k]d_k \qquad \text{for } k \geq 0. \tag{3.4}$$

Now we use induction on k to prove the following facts:

$$d_k \neq 0, \qquad d_0 d_k = -[k]d_k, \qquad d_{-1}d_k = -[k+1]d_{k-1} \qquad \text{for } k \geq 2. \tag{3.5}$$

If $k = 2$, then that $d_2 \neq 0$ and $d_{-1}d_2 = -[3]d_1$ follow from (3.2). Let $d_0 d_2 = a_2 d_2$. It follows from $J_q(d_0, d_{-1}, d_2) = 0$ that $a_2 = -[2]$.

Assume that (3.5) is true for $k-1$ where $k \geq 3$, so there exist $a_k, b_k \in \mathbb{C}$ such that

$$d_0 d_k = a_k \dot{a}_k, \qquad d_{-1}d_k = b_k d_{k-1}.$$

Using (3.4), the induction assumption and $J_q(d_0, d_1, d_{k-1}) = 0$, we have

$$\begin{aligned} 0 &= d_0(d_1 d_{k-1}) + <1> d_1(d_{k-1}d_0) + <k-1> d_{k-1}(d_0 d_1) \\ &= [2-k]d_0 d_k + <1> [k-1]d_1 d_{k-1} - <k-1> d_{k-1}d_1 \\ &= [2-k]a_k d_k + <1> [k-1][2-k]d_k - <k-1> [k-2]d_k. \end{aligned}$$

Since $k \geq 3$, $[2-k] \neq 0$. Hence, we get

$$a_k d_k = (- <1> [k-1] - <k-1>)d_k. \tag{3.6}$$

Similarly, it follows from $J_q(d_{-1}, d_1, d_{k-1}) = 0$ that

$$<1> [k-2]b_k d_{k-1} = (- <1> [k-3][k] - <k-1> [2][k-1])d_{k-1}. \tag{3.7}$$

Since $d_{k-1} \neq 0$, it follows from (3.7) that

$$\begin{aligned} <1> [k-2]b_k &= - <1> [k-3][k] - <k-1> [2][k-1] \\ &= - <1> [k-2][k+1]. \end{aligned}$$

So $b_k = -[k+1] \neq 0$. Hence,

$$d_{-1}d_k = -[k+1]d_{k-1} \neq 0 \Longrightarrow d_k \neq 0.$$

Going back to (3.6), we have

$$a_k = - <1> [k-1] - <k-1> = -[k].$$

Therefore, (3.5) holds.

By (3.1) , (3.4) and (3.5), we have proved that

$$d_i d_k = [i-k]d_{i+k} \qquad \text{for } i = 0, \pm 1 \text{ and } k \geq -1. \tag{3.8}$$

Our next step is to prove that (3.8) is also true for $i = 2$, i.e.

$$d_2 d_k = [2-k]d_{2+k} \qquad \text{for } k \geq -1. \tag{3.9}$$

Let $d_2 d_k = c_k d_{2+k}$ where $c_k \in \mathbf{C}$ and $k \geq -1$. Using (3.8) and $J_q(d_2, d_{-1}, d_k) = 0$, we have

$$\begin{aligned} -<1>[k+3]c_k d_{k+1} &= <1> c_k d_{-1} d_{k+2} = <1> d_{-1}(d_2 d_k) \\ &= <2> d_2(d_{-1}d_k) + <k> d_k(d_2 d_{-1}) \\ &= -<2>[k+1]d_2 d_{k-1} + <k>[3]d_k d_1 \\ &= -<2>[k+1]c_{k-1}d_{k+1} + <k>[3][k-1]d_{k+1}, \end{aligned}$$

i.e.,

$$-<1>[k+3]c_k = -<2>[k+1]c_{k-1} + <k>[3][k-1].$$

By (3.8), $c_{-1} = [3]$. Assume that $c_{k-1} = [3-k]$, then

$$\begin{aligned} -<1>[k+3]c_k &= -<2>[k+1][3-k] + <k>[3][k-1] \\ &= -<1>[k+3][2-k]. \end{aligned}$$

So $c_k = [2-k]$. By induction, (3.9) is true.

Now we can prove that

$$d_m d_n = [m-n]d_{m+n} \qquad \text{for } m, n \geq -1. \tag{3.10}$$

Because of (3.8), we can assume that $2 \leq m < n$. We use induction on m. (3.9) tells us that (3.10) is true for $m = 2$. Assume that $m \geq 3$ and (3.10) is true for $m-1$. From $J_q(d_1, d_{m-1}, d_n) = 0$, we get

$$<n>[2-m]d_m d_n = <n>[2-m][m-n]d_{m+n}.$$

Since $m \geq 3$, $[2-m] \neq 0$. It follows that $d_m d_n = [m-n]d_{m+n}$.

Therefore, we have proved that

- *Fact 1.* The subalgebra $\oplus_{n\geq -1} A_n$ of A has a C–basis

$$\{\, d_n \mid n \geq -1, d_n \in A_n \,\}$$

such that

$$A_n = \mathbf{C}d_n \neq 0, \qquad d_m d_n = [m-n]d_{m+n} \qquad \text{for } m,n \geq -1.$$

Similarly, if we choose $0 \neq d_{-2} \in A_{-2}$ such that $d_1 d_{-2} = [3]d_1$ and define

$$d_k := \frac{(ad(d_{-1}))^{-k-2}(d_{-2})}{[-k-2]!} \qquad \text{for } k \leq -2.$$

Then we can prove the following:

- *Fact 2.* The subalgebra $\oplus_{n\leq 1} A_n$ of A has a $\mathbf{C}$–basis

$$\{\, d_n \mid n \leq 1, d_n \in A_n \,\}$$

such that

$$A_n = \mathbf{C}d_n \neq 0, \qquad d_m d_n = [m-n]d_{m+n} \qquad \text{for } m,n \leq 1.$$

Furthermore, we have

$$d_2 d_{-2} = [4]d_0. \tag{3.11}$$

In fact, let $d_2 d_{-2} = r d_0$, where $r \in \mathbf{C}$, then

$$J_q(d_1, d_{-2}, d_3) = 0 \Longrightarrow d_4 d_{-2} = [6]d_2. \tag{3.12}$$

$$J_q(d_1, d_{-2}, d_4) = 0 \quad \text{and} \quad (3.12) \Longrightarrow d_5 d_{-2} = [7]d_3. \tag{3.13}$$

$$J_q(d_1, d_{-2}, d_2) = 0 \Longrightarrow <2> d_3 d_{-2} = (<2> [3]^2 - <1> r)d_1. \tag{3.14}$$

Using $J_q(d_2, d_3, d_{-2}) = 0$, (3.13) and (3.14), we get

$$(<1> + <3> [3])r = <2> ([7] + [3]^2),$$

or $<2> [4]r = <2> [4]^2$, i.e., $r = [4]$.

In order to complete the proof, we have to prove that

$$d_m d_{-n} = [m+n]d_{m-n} \qquad \text{for } m,n \in \mathbf{Z}. \tag{3.15}$$

By fact 1 and fact 2, we can assume that $m \geq 2$ and $n \geq 2$. Now we use induction on m to prove (3.15).

• $m = 2$, in which case, we have to prove

$$d_2 d_{-n} = [2+n] d_{2-n} \qquad \text{for } n \geq 2. \tag{3.16}$$

Let $d_2 d_{-n} = g_n d_{2-n}$, then $g_2 = [4]$ by (3.11). Assuming that $g_n = [2+n]$, it follows from $J_q(d_{-1}, d_2, d_{-n}) = 0$ that

$$\begin{aligned} <2> [n-1] g_{n+1} &= <1> [n-3] g_n + <n> [3][n+1] \\ &= <2> [n-1][n+3]. \end{aligned}$$

This implies that $g_{n+1} = [n+3]$. Hence, we have proved (3.16).

• Assume that (3.15) holds for $2 \leq m \leq k-1$, then we will prove that (3.15) also holds for $m = k$. In fact, using $J_q(d_1, d_{k-1}, d_{-n}) = 0$, we have

$$<n> [2-k] d_k d_{-n} = <n> [2-k][k+n] d_{k-n}.$$

It follows from $<n> [2-k] \neq 0$ that $d_k d_{-n} = [k+n] d_{k-n}$.

Therefore, we have proved that (3.15). In other words, A is the quantum Witt algebra.

Although the q-analogue of the Theorem 1 in [1] holds for the quantum Witt algebra, the q-analogue of the Theorem 2 in [1] does not hold for the q-analogue of the enveloping algebra of the Witt algebra. In fact, q-deformations of the module of the tensor fields over the Witt algebra are not unique up to isomorphism. We are going to discuss them in chapter 3.

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CHAPTER 3. QUANTIZATIONS OF THE MODULE OF TENSOR FIELDS OVER THE WITT ALGEBRA

§3.0. Introdution

The representation $V_{\alpha\beta}$ of the Witt algebra on the space of "the tensor fields" of the form $P(z)z^{\alpha}(dz)^{\beta}$ is usually called the module of tensor fields over the Witt algebra. Here α and β are complex numbers and $P(z)$ is an arbitrary polynomial in z and z^{-1}. The module $V_{\alpha\beta}$ of the Witt algebra plays a very important role in the representation theory of the Virasoro algebra. In 1982, I.Kaplansky proved in [3] that if $\oplus_{n\in\mathbf{Z}}\mathbf{C}v_n$ is a $\mathbf{Z}$-graded module of the Witt algebra $W = \oplus_{n\in\mathbf{Z}}\mathbf{C}d_n$ and $d_{\pm1}$ are injective operators on V, then V is isomorphic to the module $V_{\alpha\beta}$ of tensor fields for some $\alpha, \beta \in \mathbf{C}$. We call this result Kaplansky's Throrem. The main purpose of this chapter is to prove a q-analogue of Kaplansky's Theorem.

Throughout this chapter, we assume that

- All vector spaces are the vector spaces over the complex number field $\mathbf{C}$.
- $\mathbf{C}^* := \{ x \in \mathbf{C} \mid x \neq 0 \}$.
- q is a complex number satisfying $q^2 \neq 0, 1$.
- $\ell n(z)$ is the principal value of the logarithm.
- $q^{\alpha} := e^{\alpha \ell n(q)}$ for $\alpha \in \mathbf{C}$.
- $[\alpha] := \dfrac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}}$ for $\alpha \in \mathbf{C}$.

In section 1, after defining q-analogue $U(W_q)$ of the envoloping algebra of the Witt algebra, we will construct two kinds of $U(W_q)$-modules $A(\lambda, \alpha, \beta)$ and $B(\lambda, \alpha, \beta)$ by using a version of the operations over $\mathbf{Z}$-graded modules of the Witt algebra introduced by B.L.Feigin and D.B.Fuchs in [1], where $(\lambda, \alpha, \beta) \in \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}$. Both $A(\lambda, \alpha, \beta)$ and $B(\lambda, \alpha, \beta)$ become the module of tensor fields over the Witt algebra when $q \to 1$. In section 2, we will find the necessary and sufficient conditions for $X(\lambda, \alpha, \beta) \simeq Y(\lambda', \alpha', \beta')$ (where $X, Y \in \{ A, B \}$) and study the reducibility and unitarity of $X(\lambda, \alpha, \beta)$. In section 3, we will prove a q-analogue of Kanplansky's theorem.

§3.1. The construction of $U(W_q)$–modules $X(\lambda, \alpha, \beta)$

Based on Proposition 1.1 in chapter 1, we introduce the following definition:

Definition 1.1 q–analogue $U(W_q)$ of the envoloping algebra of the Witt algebra is defined as the associative algebra with generators $\{ J^{\pm 1}, d_m \mid m \in \mathbf{Z} \}$ and the following relations:

$$(1.1) \qquad JJ^{-1} = J^{-1}J = 1, \qquad Jd_mJ^{-1} = q^m d_m,$$

$$(1.2) \qquad q^m d_m d_n J - q^n d_n d_m J = [m-n]d_{m+n},$$

where $m, n \in \mathbf{Z}$.

Definition 1.2 A $U(W_q)$–module V is called a $\mathbf{Z}$–graded module if $V = \oplus_{n\in\mathbf{Z}} V_n$ and $d_m(v_n) \in V_{m+n}$ for $m, n \in \mathbf{Z}$.

For every $\lambda \in \mathbf{C}^*$, we define an algebra isomorphism $\varphi(\lambda)$ of $U(W_q)$ as follows:

$$\varphi(\lambda) : J^{\pm 1} \mapsto \lambda^{\pm 1} J^{\pm 1}, \qquad d_m \mapsto \lambda^{-1} d_m \qquad \text{for } m \in \mathbf{Z}.$$

If $V = \oplus_{n\in\mathbf{Z}} V_n$ is a $\mathbf{Z}$-graded $U(W_q)$-module with $J(v_n) = q^n v_n$ for all $n \in \mathbf{Z}$ and $v_n \in V_n$, then we can construct three more modules from V: **contragradient module** $\bar{V} := \oplus_{n\in\mathbf{Z}} (\bar{V})_n$, **adjoint module** $V^* := \oplus_{n\in\mathbf{Z}} (V^*)_n$ and **inverted module** $V^\circ := \oplus_{n\in\mathbf{Z}} (V^\circ)_n$, where

$$\begin{aligned} (\bar{V})_n &:= Hom(V_n, \mathbf{C}), & J \mid (\bar{V})_n &:= q^n \cdot id; \\ (V^*)_n &:= Hom(V_{-n}, \mathbf{C}), & J \mid (V^*)_n &:= q^{-n} \cdot id; \\ (V^\circ)_n &:= V_{-n}, & J \mid (V^\circ)_n &:= q^{-n} \cdot id \end{aligned}$$

and the definitions of the operators d_m on $\bar{V}$, V^* and V° are the same as in [1].

It is easy to check that $\bar{V}$ is a $\mathbf{Z}$–graded $U(W_q)$–module and V^*, as well as V°, is a $\mathbf{Z}$–graded $U(W_{q^{-1}})$–module. As $U(W_q)$–modules, $(V^*)^\circ \simeq \bar{V}$.

In particular, if $V = \oplus_{k\in\mathbf{Z}} \mathbf{C} v_k$ is a $\mathbf{Z}$–graded $U(W_q)$–module with the natural $\mathbf{Z}$–grading and the following module action on V:

$$(1.3) \qquad J(v_k) := q^k v_k, \qquad d_n(v_k) := a(q,n,k) v_{n+k}$$

where $n, k \in \mathbf{Z}$ and $a(q,n,k) \in \mathbf{C}$, then we can describe the contragradient module $\bar{V}$, the adjoint module V^* and the inverted module V° as follows:

$$(1.4) \qquad \bar{V} = \oplus_{k\in\mathbf{Z}} \mathbf{C} v_k, \qquad J(v_k) = q^k v_k, \qquad d_n(v_k) = a(q,-n,n+k) v_{n+k};$$

$$(1.5) \qquad V^* = \oplus_{k\in\mathbf{Z}} \mathbf{C} v_k, \qquad J(v_k) = q^{-k} v_k, \qquad d_n(v_k) = -a(q,n,-n-k) v_{n+k};$$

(1.6) $V^\circ = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad J(v_k) = q^{-k}v_k, \qquad d_n(v_k) = -a(q,-n,-k)v_{n+k}.$

For $\alpha,\beta \in \mathbf{C}$, set

(1.7) $$a(q,n,k) := -\left([k+\alpha]q^\alpha + [n+1][\beta]q^{n+k}\right),$$

where $n,k \in \mathbf{Z}$. Then (1.3) and (1.7) define a $U(W_q)$–module action on $V(\alpha,\beta) := \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k$.

Let us check (1.2), i.e.

(1.8) $q^m d_m d_n(v_k) - q^n d_n d_m(v_k) = [m-n]d_{m+n}J^{-1}(v_k) \qquad$ for $m,n,k \in \mathbf{Z}$.

Let $q^m d_m d_n(v_k) = A_{m,n,k}v_{m+n+k}$, then $q^n d_n d_m(v_k) = A_{n,m,k}v_{m+n+k}$. By (1.7), we have

$$\begin{aligned} A_{m,n,k} = {} & q^{m+2\alpha}[k+\alpha][n+k+\alpha] + q^{2m+n+k+\alpha}[k+\alpha][m+1][\beta] \\ & + q^{m+n+k+\alpha}[n+1][\beta][n+k+\alpha] \\ & + q^{2m+2n+2k}[n+1][m+1][\beta]^2 \end{aligned}$$

It follows that

$$\begin{aligned} & q^m d_m d_n(v_k) - q^n d_n d_m(v_k) = (A_{m,n,k} - A_{n,m,k})v_{m+n+k} \\ = & ([k+\alpha](q^{m+2\alpha}[n+k+\alpha] - q^{n+2\alpha}[m+k+\alpha]) \\ & \quad + q^{m+n+k}[\beta]((q^{m+\alpha}[k+\alpha][m+1] + q^\alpha[n+1][n+k+\alpha]) \\ & \quad - (q^{n+\alpha}[k+\alpha][n+1] + q^\alpha[m+1][m+k+\alpha])))v_{m+n+k} \\ = & ([k+\alpha](-q^{-k}q^\alpha[m-n]) \\ & \quad + q^{m+n+k}[\beta](-q^{-k}[m-n][m+n+1]))v_{m+n+k} \\ = & -[m-n]([k+\alpha]q^\alpha + [m+n+1][\beta]q^{m+n+k})q^{-k}v_{m+n+k} \\ = & [m-n]d_{m+n}J^{-1}(v_k), \end{aligned}$$

so (1.8) is true.

By the discussion above, we can construct the contragradient module $\bar{V}(\alpha,\beta)$ from (1.4). By (1.7),

$$\begin{aligned} & -a(q,-n,n+k) = [n+k+\alpha]a^\alpha + [-n+1][\beta]q^k \\ = & ([n+k]q^\alpha + [\alpha]q^{-n-k})q^\alpha + [\beta]\frac{q^{1-n} - q^{n-1}}{q-q^{-1}}q^k \\ = & \frac{q^{n+k} - q^{-n-k}}{q-q^{-1}}q^{2\alpha} + \frac{q^{2\alpha}-1}{q-q^{-1}}q^{-n-k} + \frac{[\beta]q}{q-q^{-1}}q^{-n+k} - \frac{[\beta]q^{-1}}{q-q^{-1}}q^{n+k} \\ = & \frac{q^{-n+k}}{q-q^{-1}}((q^{2n} - q^{-2k})q^{2\alpha} + (q^{2\alpha}-1)q^{-2k} + [\beta]q - [\beta]q^{-1}q^{2n}) \\ = & \frac{q^{-n+k}}{q-q^{-1}}(-q^{-2k} + (q^{2\alpha} - [\beta]q^{-1})q^{2n} + [\beta]q). \end{aligned}$$

Hence, the contragradient module $\bar{V}(\alpha,\beta)$ can be described as

$$\bar{V}(\alpha,\beta) := \oplus_{k\in\mathbf{Z}}\mathbf{C}u_k, \qquad J(u_k) := q^k u_k,$$

(1.9)

$$\begin{aligned} d_n(u_k) &:= -([n+k+\alpha]q^{\alpha} + [1-n][\beta]q^k)u_{n+k} \\ &= -\frac{q^{-n+k}}{q-q^{-1}}(-q^{-2k} + (q^{2\alpha} - [\beta]q^{-1})q^{2n} + [\beta]q)u_{n+k}. \end{aligned}$$

If we replace q by q^{-1} in (1.5), then we get a $\mathbf{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(1)}$:

$$V(\alpha,\beta)^{(1)} := \oplus_{k\in\mathbf{Z}}\mathbf{C}u_k, \qquad J(u_k) := q^k u_k,$$

(1.10)

$$\begin{aligned} d_n(u_k) &:= -a(q^{-1}, n, -n-k)u_{n+k} \\ &= ([-n-k+\alpha]q^{-\alpha} + [n+1][\beta]q^k)u_{n+k} \\ &= -\frac{q^{-n+k}}{q-q^{-1}}(-q^{-2k} + (q^{-2\alpha} - q[\beta])q^{2n} + q^{-1}[\beta])u_{n+k}. \end{aligned}$$

Similarly, if we replace q by q^{-1} in (1.6), then, by (1.7)

$$\begin{aligned} a(q^{-1}, -n, -k) &= -([-k+\alpha]q^{-\alpha} + [-n+1][\beta]q^{n+k}) \\ &= [k-\alpha]q^{-\alpha} + [n-1][\beta]q^{n+k} \\ &= \frac{q^k}{q-q^{-1}}\left(-q^{-2k} + q^{-1}[\beta]q^{2n} + (q^{-2\alpha} - q[\beta])\right). \end{aligned}$$

Choose $\alpha', \beta' \in \mathbf{C}$ such that

$$q[\beta'] = q^{-2\alpha} - q[\beta] \quad \text{and} \quad q^{2\alpha'} - q^{-1}[\beta'] = q^{-1}[\beta],$$

then we get a $\mathbf{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(2)}$ as follows :

$$V(\alpha,\beta)^{(2)} := \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad J(v_k) := q^k v_k,$$

(1.11)

$$\begin{aligned} d_n(v_k) &:= -([n+k+\alpha]q^{n+\alpha} + [1-n][\beta]q^{n+k})v_{n+k} \\ &= -\frac{q^k}{q-q^{-1}}(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2n} + q[\beta])v_{n+k}. \end{aligned}$$

Finally, let us rewrite the $\mathbf{Z}$-graded $U(W_q)$- module as:

$$V(\alpha,\beta) := \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad J(v_k) := q^k v_k,$$

(1.12)

$$\begin{aligned} d_n(v_k) &:= -([k+\alpha]q^{\alpha} + [1+n][\beta]q^{n+k})v_{n+k} \\ &= -\frac{q^k}{q-q^{-1}}(-q^{-2k} + q[\beta]q^{2n} + (q^{2\alpha} - q^{-1}[\beta]))v_{n+k}. \end{aligned}$$

Let

$$v_k := q^{-k}u_k \qquad \text{for } k \in \mathbf{Z}.$$

By (1.9) and (1.11)

$$\bar{V}(\alpha,\beta) = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k = V(\alpha,\beta)^{(2)}. \tag{1.13}$$

If we choose α', $\beta' \in \mathbf{C}$ such that

$$q[\beta'] = q^{-2\alpha} - q[\beta] \quad \text{and} \quad q^{2\alpha'} - q^{-1}[\beta'] = q^{-1}[\beta],$$

then, by (1.10) and (1.12)

$$V(\alpha,\beta)^{(1)} = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k = V(\alpha',\beta'). \tag{1.14}$$

(1.13) and (1.14) implies that the contruction which produces the module $\bar{V}(\alpha,\beta)$ (resp. $V(\alpha,\beta)^{(1)}$) does not take us out of the class of the $\mathbf{Z}$-graded $U(W_q)$-module $V(\alpha,\beta)^{(2)}$ (resp. $V(\alpha,\beta)$).

Hence, for any $(\lambda,\alpha,\beta) \in \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}$, we can construct two kinds of $\mathbf{Z}$-graded $U(W_q)$-module $A(\lambda,\alpha,\beta)$ and $B(\lambda,\alpha,\beta)$ by using (1.11), (1.12) and $\varphi(\lambda)$ as follows (where $n, k \in \mathbf{Z}$):

$$\begin{aligned} A(\lambda,\alpha,\beta) : &= \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad J(v_k) := \lambda q^k v_k, \\ d_n(v_k) : &= -\lambda^{-1}\left([k+\alpha]q^{\alpha} + [1+n][\beta]q^{n+k}\right)v_{n+k} \\ &= -\frac{\lambda^{-1}q^k}{q-q^{-1}}\left(-q^{-2k} + q[\beta]q^{2n} + (q^{2\alpha} - q^{-1}[\beta])\right)v_{n+k} \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} B(\lambda,\alpha,\beta) : &= \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad J(v_k) := \lambda q^k v_k, \\ d_n(v_k) : &= -\lambda^{-1}\left([n+k+\alpha]q^{n+\alpha} + [1-n][\beta]q^{n+k}\right)v_{n+k} \\ &= -\frac{\lambda^{-1}q^k}{q-q^{-1}}\left(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2n} + q[\beta]\right)v_{n+k} \end{aligned} \tag{1.16}$$

§3.2. The properties of $U(W_q)$-modules $X(\lambda,\alpha,\beta)$

In this section, we assume that q is not a root of unity.

Let X be A or B, we define

$$c\ell(X) := \{\, X(\lambda, \alpha, \beta) \mid (\lambda, \alpha, \beta) \in \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}\,\}.$$

A $U(W_q)$–module V is said to be in $c\ell(X)$ if $V \simeq X(\lambda, \alpha, \beta)$ (as $U(W_q)$–modules) for some $X(\lambda, \alpha, \beta) \in c\ell(X)$.

PROPOSITION 2.1. *For any fixed* $h \in \mathbf{Z}$, $X(\lambda, \alpha, \beta) \simeq X(\lambda q^h, \alpha + h, \beta')$ *as* $U(W_q)$*–modules, where* $\beta' \in \mathbf{C}$ *with* $[\beta'] = [\beta]q^{2h}$.

PROOF: Let

$$X(\lambda, \alpha, \beta) = \oplus_{k \in \mathbf{Z}} \mathbf{C} v_k \quad \text{and} \quad X(\lambda q^h, \alpha + h, \beta') = \oplus_{k \in \mathbf{Z}} \mathbf{C} u_k.$$

We define a linear map $\varphi : X(\lambda, \alpha, \beta) \to X(\lambda q^h, \alpha + h, \beta')$ by

$$\varphi : v_k \mapsto u_{k-h} \qquad \text{for } k \in \mathbf{Z}.$$

It is clear that φ is bijective. Now we prove that φ preserves the $U(W_q)$–module actions.

From

$$\begin{aligned} \varphi J(v_k) &= \varphi(\lambda q^k v_k) = \lambda q^k \varphi(v_k) \\ &= \lambda q^k u_{k-h} = (\lambda q^h)(q^{k-h} u_{k-h}) \\ &= J(u_{k-h}) = J\varphi(v_k), \end{aligned}$$

we get that $\varphi J(v_k) = J\varphi(v_k)$.

Let

$$\beta' := \frac{\ell n([\beta]q^{2h} + \sqrt{[\beta]^2 q^{4h}(q - q^{-1})^2 + 4}) - \ell n(2)}{\ell n q},$$

then $[\beta'] = [\beta]q^{2h}$.

If $X = A$, then, by (1.15)

$$\begin{aligned} \varphi d_n(v_k) &= -\frac{\lambda^{-1} q^k}{q - q^{-1}}(-q^{-2k} + q[\beta]q^{2n} + (q^{2\alpha} - q^{-1}[\beta]))\varphi(v_{n+k}) \\ &= -\frac{\lambda^{-1} q^k}{q - q^{-1}}(-q^{-2k} + q[\beta]q^{2n} + (q^{2\alpha} - q^{-1}[\beta]))u_{n+k-h} \\ &= -\frac{(\lambda q^h)^{-1} q^{k-h}}{q - q^{-1}}(-q^{-2(k-h)} + q[\beta']q^{2n} + (q^{2(\alpha+h)} - q^{-1}[\beta']))u_{n+k-h} \\ &= d_n(u_{k-h}) = d_n\varphi(v_k). \end{aligned}$$

If $X = B$, then, by (1.16)

$$\begin{aligned}\varphi d_n(v_k) &= -\frac{\lambda^{-1}q^k}{q-q^{-1}}(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2n} + q[\beta])\varphi(v_{n+k}) \\ &= -\frac{\lambda^{-1}q^k}{q-q^{-1}}(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2n} + q[\beta])u_{n+k-h} \\ &= -\frac{(\lambda q^h)^{-1}q^{k-h}}{q-q^{-1}}(-q^{-2(k-h)} + (q^{2(\alpha+h)} - q^{-1}[\beta'])q^{2n} + q[\beta'])u_{n+k-h} \\ &= d_n(u_{k-h}) = d_n\varphi(v_k).\end{aligned}$$

This proves Proposion 2.1. ∎

PROPOSITION 2.2. *If $q \in \mathbf{R}$, then $U(W_q)$ has an antilinear anti-involution θ such that $\theta(J^{\pm 1}) := J^{\pm 1}$ and $\theta(d_n) := d_{-n}$ for all $n \in \mathbf{Z}$.*

PROOF: Let F be the free associative algebra on the set $\{\, \underline{J^{\pm 1}}, \underline{d_m} \mid m \in \mathbf{Z} \,\}$. Let I be the ideal of F generated by

$$\underline{J}\cdot\underline{J^{-1}} - 1, \qquad \underline{J^{-1}}\cdot\underline{J} - 1, \qquad \underline{Jd_mJ^{-1}} - q^m\underline{d_m},$$
$$q^m\underline{d_m}\cdot\underline{d_nJ} - q^n\underline{d_n}\cdot\underline{d_mJ} - [m-n]\underline{d_{m+n}},$$

where $m, n \in \mathbf{Z}$. Then $U(W_q) := \dfrac{F}{I}$.

Define an antilinear map $\theta : F \to U(W_q)$ by

$$\theta(\underline{J^{\pm 1}}) := J, \qquad \theta(\underline{d_m}) := d_{-m}$$
$$\theta : a\underline{x_1}\cdot\underline{x_2}\dots\underline{x_t} \mapsto \bar{a}x_t\dots x_2x_1,$$

where $a \in \mathbf{C}$, $x_i \in \{\, J^{\pm 1}, d_m \mid m \in \mathbf{Z} \,\}$, $x_i := \underline{x_i} + I$ and t is a natural number. Then

$$\theta(uv) = \theta(v)\theta(u) \qquad \text{for } u, v \in F.$$

If $q \in \mathbf{R}$, then $[m] \in \mathbf{R}$ for $m \in \mathbf{Z}$. So

$$\begin{aligned}\theta(q^m\underline{d_m}\cdot\underline{d_nJ} - q^n\underline{d_n}\cdot\underline{d_mJ}) &= q^m\theta(\underline{J})\theta(\underline{d_n})\theta(\underline{d_m}) - q^n\theta(\underline{J})\theta(\underline{d_m})\theta(\underline{d_n}) \\ &= q^m(Jd_{-n})d_{-m} - q^n(Jd_{-m})d_{-n} \\ &= q^m(q^{-n}d_{-n}J)d_{-m} - q^n(q^{-m}d_{-m}J)d_{-n} \\ &= q^{m-n}d_{-n}(Jd_{-m}) - q^{n-m}d_{-m}(Jd_{-n}) \\ &= q^{m-n}d_{-n}(q^{-m}d_{-m}J) - q^{n-m}d_{-m}(q^{-n}d_{-n}J) \\ &= q^{-n}d_{-n}d_{-m}J - q^{-m}d_{-m}d_{-n}J \\ &= [m-n]d_{-(m+n)} = [m-n]\theta(\underline{d_{m+n}}) \\ &= \theta([m-n]\underline{d_{m+n}}),\end{aligned}$$

or

$$\theta(q^m \underline{d_m} \cdot \underline{d_n J} - q^n \underline{d_n} \cdot \underline{d_m J} - [m-n]\underline{d_{m+n}}) = 0.$$

Similarly, we have

$$\theta(\underline{J} \cdot \underline{J^{-1}} - 1) = \theta(\underline{J^{-1}} \cdot \underline{J} - 1) = \theta(\underline{J d_m J^{-1}} - q^m \underline{d_m}) = 0.$$

Therefore, θ vanishes on I. This induces an antilinear anti-homomorphism $\theta : U(W_q) \to U(W_q)$ such that

$$\theta(J_{\pm 1}) = J^{\pm 1} \quad \text{and} \quad \theta(d_n) = d_{-n} \quad \text{for } n \in \mathbf{Z}.$$

It is clear that $\theta^2 = 1$. So, θ is an anlinear anti-involution. ▮

Definition 2.1 Let $q \in \mathbf{R}$. A $U(W_q)$-module V is **unitary** with respect to θ if there is an Hermitian form $< . \mid . >$ on V such that

$$< v \mid v >> 0 \qquad \text{for } v \in V \text{ and } v \neq 0,$$

$$(*) \qquad < x(u) \mid v > = < u \mid \theta(x) v > \qquad \text{for } u, v \in V \text{ and } x \in U(W_q).$$

An Hermitian form $< . \mid . >$ satisfying (*) is called a **contravariant form.**

Let $X(\alpha, \beta) := X(1, \alpha, \beta)$, then the following proposition is clear:

PROPOSITION 2.3. *For $(\lambda, \alpha, \beta) \in \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}$, we have*

1. *$X(\lambda, \alpha, \beta)$ is reducible if and only if $X(\alpha, \beta)$ is reducible.*
2. *If $q \in \mathbf{R}$, then $X(\lambda, \alpha, \beta)$ is unitary with respect to θ if and only if $\lambda \in \mathbf{R}$, $\lambda \neq 0$ and $X(\alpha, \beta)$ is unitary with respect to θ.* ▮

Now we prove

PROPOSITION 2.4. *Let $X, Y \in \{ A, B \}$. Then*

1. *$X(\lambda, \alpha, \beta) \simeq Y(\lambda_1, \alpha_1, \beta_1) \iff$ there exist some $h \in \mathbf{Z}$ such that $\lambda = \lambda_1 q^h$ and some $\mathbf{Z}$-grading preserving isomorphism φ such that $\varphi : X(\alpha, \beta) \simeq Y(\alpha', \beta')$, where $\alpha' = \alpha_1 + h$ and $[\beta'] = [\beta_1] q^{2h}$.*
2. *Every submodule of $X(\lambda, \alpha, \beta)$ respects the $\mathbf{Z}$-grading of $X(\lambda, \alpha, \beta)$.*

PROOF: (1). $\Rightarrow$: Let

$$X(\lambda, \alpha, \beta) = \oplus_{k \in \mathbf{Z}} \mathbf{C} v_k, \qquad Y(\lambda_1, \alpha_1, \beta_1) = \oplus_{k \in \mathbf{Z}} \mathbf{C} u_k,$$

$$\psi : X(\lambda, \alpha, \beta) \simeq Y(\lambda_1, \alpha_1, \beta_1), \qquad \psi(v_k) = a_{j_1} u_{j_1} + \cdots + a_{j_r} u_{j_r},$$

where $a_{j_s} \in \mathbb{C}^*$ and $j_s \neq j_t$ if $s \neq t$. That $\psi J(v_k) = J\psi(v_k)$ gives that $\lambda = \lambda_1 q^{j_s - k}$ for all $1 \leq s \leq r$. Because q is not a root of unity, $r = 1$. It follows that

$$\psi(v_k) = a_{f(k)} u_{f(k)}, \qquad \text{where } f(k) \in \mathbb{Z}.$$

Since $q^{f(k)-k} = \dfrac{\lambda}{\lambda_1}$, $f(k) - k = f(k') - k'$ for all $k, k' \in \mathbb{Z}$. Let $h := f(k) - k$, then

$$\psi(v_k) = a_{k+h} u_{k+h} \qquad \text{for } k \in \mathbb{Z}.$$

By Proposition 2.1, $\eta : Y(\lambda_1, \alpha_1, \beta_1) \simeq Y(\lambda_1 q^h, \alpha', \beta')$, where $\alpha' = \alpha_1 + h$ and $[\beta'] = [\beta_1] q^{2h}$. Let $\varphi := \eta\psi$, then φ preserves the $\mathbb{Z}$-grading and $\varphi : X(\lambda, \alpha, \beta) \simeq Y(\lambda, \alpha', \beta')$. Using the automorphism $\varphi(\lambda)$ of $U(W_q)$, we get that $\varphi : X(\alpha, \beta) \simeq Y(\alpha', \beta')$.

$\Leftarrow$: If $X(\alpha, \beta) \simeq Y(\alpha', \beta')$, then

$$\begin{aligned} &X(\lambda, \alpha, \beta) \simeq Y(\lambda, \alpha', \beta') \\ &= Y(\lambda_1 q^h, \alpha_1 + h, \beta') \simeq Y(\lambda_1, \alpha_1, \beta_1). \end{aligned}$$

(2) Let V be a submodule of $X(\lambda, \alpha, \beta) = \oplus_{k \in \mathbb{Z}} \mathbb{C} v_k$.
If $v \in V$, then

$$v = a_{j_1} v_{j_1} + \ldots a_{j_r} v_{j_r},$$

where $a_{j_s} \in \mathbb{C}^*$ and $j_s \neq j_t$ if $1 \leq s \neq t \leq r$.

Applying the operator J to $v, J(v), \ldots, J^{r-2}(v)$, respectively, we get a system with respect to $a_{j_1} v_{j_1}, a_{j_2} v_{j_2}, \ldots, a_{j_r} v_{j_r}$:

$$\begin{aligned} v &= a_{j_1} v_{j_1} + a_{j_2} v_{j_2} + \cdots + a_{j_r} v_{j_r}, \\ \lambda^{-1} J(v) &= q^{j_1}(a_{j_1} v_{j_1}) + q^{j_2}(a_{j_2} v_{j_2}) + \cdots + q^{j_r}(a_{j_r} v_{j_r}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda^{-(r-1)} J^{r-1}(v) &= (q^{j_1})^{r-1}(a_{j_1} v_{j_1}) + (q^{j_2})^{r-1}(a_{j_2} v_{j_2}) + \cdots + (q^{j_r})^{r-1}(a_{j_r} v_{j_r}). \end{aligned}$$

The determinant of its coefficients matrix is a Vandermonde determinant with respect to $q^{j_1}, q^{j_2}, \ldots, q^{j_r}$. Since $q^{j_s} \neq q^{j_t}$ for $1 \leq s \neq t \leq r$, the determinant is not zero. Thus each of the $a_{j_s} v_{j_s}$ $(s = 1, 2, \ldots, r)$ can be expressed as a linear combination of $v, \lambda^{-1} J(v), \ldots, \lambda^{-(r-1)} J^{r-1}(v)$, i.e., $a_{j_s} v_{j_s} \in V$, or $v_{j_s} \in V$, proving (2). ∎

The proposition above tells us that if q is not a root of unity, then in order to study the properties of the $U(W_q)$-module $X(\lambda, \alpha, \beta)$, it suffices to study the properties of the $U(W_q)$-module $X(\alpha, \beta)$.

PROPOSITION 2.5. *Let* $(\alpha,\beta) \in \mathbf{C}\times\mathbf{C}$. *Then* $B(\alpha,\beta) \in c\ell(A) \iff q^{2\alpha+1} + (q^4-1)[\beta] \neq 0$.

PROOF: $\Leftarrow$: Since $q^{2\alpha+1} + (q^4-1)[\beta] \neq 0$, we can find $(\alpha',\beta') \in \mathbf{C}\times\mathbf{C}$ such that

$$q^{2\alpha'} - q^{-1}[\beta'] = q[\beta] \quad \text{and} \quad q[\beta'] = q^{2\alpha} - q^{-1}[\beta].$$

Hence, $B(\alpha,\beta) = A(\alpha',\beta') \in c\ell(A)$ by (1.15) and (1.16).

$\Rightarrow$: If $B(\alpha,\beta) \in c\ell(A)$, then , by Proposition 2.4, there exists a $\mathbf{Z}$-grading preserving isomorphism φ such that

$$\varphi : A(\alpha',\beta') \simeq B(\alpha,\beta) \qquad \text{for some } (\alpha',\beta') \in \mathbf{C}\times\mathbf{C}.$$

Set

$$A(\alpha',\beta') = \oplus_{k\in\mathbf{Z}}\mathbf{C}v'_k, \qquad B(\alpha,\beta) = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k,$$
$$\varphi(v'_k) = a_k v_k, \qquad \text{where } a_k \in \mathbf{C},$$

and

$$(2.1) \qquad a_{\alpha\beta} := q^{2\alpha} - q^{-1}[\beta], \quad b_{\alpha,\beta} := a_{\alpha,\beta} - 1, \quad c_\beta := q[\beta] - 1$$

Using (1.15), (1.16) and $\varphi d_n(v'_k) = d_n\varphi(v'_k)$, we have

$$(2.2) \qquad (-x + q[\beta']y + a_{\alpha'\beta'})a_{n+k} = (-x + a_{\alpha\beta}y + q[\beta])a_k,$$

where

$$x := q^{-2k} \qquad \text{and} \qquad y := q^{2n}.$$

It follows from (2.2) that

$$(2.3) \qquad (b_{\alpha'\beta'}x + q[\beta']y)a_{n+k} = (c_\beta x + a_{\alpha\beta}y)a_0,$$

$$(2.4) \qquad (b_{\alpha'\beta'}x + q[\beta'])a_k = (c_\beta x + a_{\alpha\beta})a_0.$$

Mutiplying the both sides of (2.2) by

$$(b_{\alpha'\beta'}x + q[\beta']y)(b_{\alpha'\beta'}x + q[\beta']),$$

we get by using (2.3) and (2.4)

$$\begin{aligned}&(-x + q[\beta']y + a_{\alpha'\beta'})(c_\beta x + a_{\alpha\beta}y)(b_{\alpha'\beta'}x + q[\beta'])\\ &= (-x + a_{\alpha\beta}y + q[\beta])(b_{\alpha'\beta'}x + q[\beta']y)(c_\beta x + a_{\alpha\beta}).\end{aligned}$$

Comparing the coefficients of x^2y and xy^2 gives us the following identities:

$$-a_{\alpha\beta}b_{\alpha'\beta'} + q[\beta']c_\beta b_{\alpha'\beta'} = -q[\beta']c_\beta + a_{\alpha\beta}b_{\alpha'\beta'}c_\beta, \tag{2.5}$$

$$[\beta']a_{\alpha\beta}b_{\alpha'\beta'} = [\beta']c_\beta a_{\alpha\beta}. \tag{2.6}$$

Let $n = 0$ in (2.2), we have

$$q[\beta'] + a_{\alpha'\beta'} = q[\beta] + a_{\alpha\beta}. \tag{2.7}$$

- If $a_{\alpha\beta} = 0$, then $[\beta] = q^{2\alpha+1}$ by (2.1). Hence,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = q^{2\alpha+5} \neq 0.$$

- If $a_{\alpha\beta} \neq 0$ and $[\beta'] \neq 0$, then, by (2.1) and (2.6), we get

$$q^{2\alpha'} - q^{-1}[\beta'] = q[\beta]. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$q^{2\alpha} - q^{-1}[\beta] = q[\beta']. \tag{2.9}$$

Using (2.8) and (2.9), we have that

$$\begin{aligned} 0 \neq q^{2\alpha'} &= q^{-1}[\beta'] + q[\beta] \\ &= q^{-2}(q^{2\alpha} - q^{-1}[\beta]) + q[\beta] \\ &= q^{-3}(q^{2\alpha+1} + (q^4 - 1)[\beta]), \end{aligned}$$

so $q^{2\alpha+1} + (q^4 - 1)[\beta] \neq 0$.

- If $a_{\alpha\beta} \neq 0$ and $[\beta'] = 0$, then $b_{\alpha'\beta'} \neq 0$ by (2.4). It follows from (2.5) that $c_\beta = -1$, i.e. $[\beta] = 0$. Hence,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = q^{2\alpha+1} \neq 0. \quad \blacksquare$$

PROPOSITION 2.6. *Let φ be a $\mathbf{Z}$–grading preserving linear map and $\alpha, \beta, \alpha', \beta' \in \mathbf{C}$. Then $\varphi : A(\alpha, \beta) \simeq A(\alpha', \beta')$ if and only if one of the following conditions holds:*

(1) $q^{2\alpha} = q^{2\alpha'}$ and $[\beta] = [\beta']$;
(2) $(q^{2\alpha} - 1)(q^{2\alpha'} - 1) \neq 0$ and $[\beta] = [\beta'] = 0$;
(3) $q^{2\alpha+1} = [\beta] = q^{2\alpha'-1}$, $[\beta'] = 0$ and $q^{2\alpha'} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}$;
(4) $q^{2\alpha'+1} = [\beta'] = q^{2\alpha-1}$, $[\beta] = 0$ and $q^{2\alpha} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}$.

PROOF: Assume that $\varphi : A(\alpha,\beta) \simeq A(\alpha',\beta')$. Let

$$A(\alpha,\beta) := \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k, \qquad A(\alpha',\beta') := \oplus_{k\in\mathbf{Z}}\mathbf{C}v'_k,$$
$$\varphi(v_k) := a_k v'_k, \qquad \text{where } k\in\mathbf{Z} \text{ and } a_k\in\mathbf{C}^*,$$

and $a_{\alpha\beta}$, $b_{\alpha\beta}$ defined by (2.1).

Using (1.15), $\varphi d_n(v_k) = d_n\varphi(v_k)$ and the same argument as the one in the proof of Proposition 2.5, we have

$$(b_{\alpha\beta}x + q[\beta])a_k = (b_{\alpha'\beta'}x + q[\beta'])a_0, \tag{2.10}$$

$$\begin{aligned}&(-x+q[\beta]y+a_{\alpha\beta})(b_{\alpha\beta}x+q[\beta])(b_{\alpha'\beta'}x+q[\beta']y)\\ =&(-x+q[\beta']y+a_{\alpha'\beta'})(b_{\alpha'\beta'}x+q[\beta'])(b_{\alpha\beta}x+q[\beta]y),\end{aligned} \tag{2.11}$$

where $x := q^{-2k}$ and $y := q^{2n}$.

Comparing the coefficients of xy, y^2, x and y in (2.11), we get

$$q[\beta]^2 b_{\alpha'\beta'} + [\beta']a_{\alpha\beta}b_{\alpha\beta} = q[\beta']^2 b_{\alpha\beta} + [\beta]a_{\alpha'\beta'}b_{\alpha'\beta'}, \tag{2.12}$$
$$[\beta]^2[\beta'] = [\beta']^2[\beta], \tag{2.13}$$
$$[\beta]a_{\alpha\beta}b_{\alpha'\beta'} = [\beta']a_{\alpha'\beta'}b_{\alpha\beta}, \tag{2.14}$$
$$[\beta][\beta']a_{\alpha\beta} = [\beta][\beta']a_{\alpha'\beta'}. \tag{2.15}$$

- If $[\beta][\beta'] \neq 0$, then (1) follows from (2.13) and (2.15).
- If $[\beta] = [\beta'] = 0$, then, by (2.10)

$$(q^{2\alpha}-1)a_k = (q^{2\alpha'}-1)a_0.$$

So $q^{2\alpha}-1 = 0 \Leftrightarrow q^{2\alpha'}-1=0$, i.e., either (1) holds, or (2) holds.

- If $[\beta]\neq 0$ and $[\beta']=0$, then $[\beta]a_{\alpha\beta}b_{\alpha'\beta'} = 0$ by (2.14). Since $b_{\alpha'\beta'}\neq 0$ by (2.10), $a_{\alpha\beta}=0$. Hence,

$$q^{2\alpha+1} = [\beta] = q^{2\alpha'-1}, \tag{2.16}$$

where the last equation in (2.16) follows from (2.12).

By (2.10), we have

$$(-q^{-2k}+q^{2\alpha'})a_k = (b_{\alpha\beta}x+q[\beta])a_k = b_{\alpha'\beta'}xa_0 \neq 0$$

for all $k \in \mathbf{Z}$. This implies that

$$q^{2\alpha'} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}. \tag{2.17}$$

(2.16) and (2.17) implies that (3) in Proposition 2.6 holds.

• If $[\beta] = 0$ and $[\beta'] \neq 0$, then $[\beta']a_{\alpha'\beta'}b_{\alpha\beta} = 0$ by (2.14). Since $b_{\alpha\beta} \neq 0$ by (2.10), $a_{\alpha'\beta'} = 0$. Hence,

$$q^{2\alpha'+1} = [\beta'] = q^{2\alpha-1}, \tag{2.18}$$

where the last equation in (2.18) follows from (2.12).

It follows from (2.10) that

$$q^{2\alpha} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}.$$

By (2.18) and (2.19), (4) in Proposition 2.6 holds.

Conversely, if one of the conditions in Proposition 2.6 holds, then the $\mathbf{Z}$–grading preserving linear map φ defined by

$$\varphi(v_k) := \frac{(q^{2\alpha'} - q^{-1}[\beta'] - 1)q^{-2k} + q[\beta']}{(q^{2\alpha} - q^{-1}[\beta] - 1)q^{-2k} + q[\beta]} v'_k \qquad \text{for } k \in \mathbf{Z}$$

is an isomorphism from $A(\alpha, \beta)$ to $A'(\alpha', \beta')$. ∎

Proposition 2.7. *Let φ be a $\mathbf{Z}$-grading preserving linear map and $\alpha, \beta, \alpha', \beta' \in \mathbf{C}$. Then $\varphi : B(\alpha, \beta) \simeq B(\alpha', \beta')$ if and only if one of the following conditions holds:*

(1) $q^{2\alpha} = q^{2\alpha'}$ and $[\beta] = [\beta']$;
(2) $q^{2\alpha} = q^{-1}[\beta]$, $q^{2\alpha'} = q^{-1}[\beta']$ *and* $(q[\beta] - 1)(q[\beta'] - 1) \neq 0$;
(3) $q^{2\alpha+1} = [\beta] = q^{2\alpha'-1}$, $[\beta'] = 0$ *and* $q^{2\alpha'} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}$;
(4) $q^{2\alpha'+1} = [\beta] = q^{2\alpha-1}$, $[\beta] = 0$ *and* $q^{2\alpha} \notin \{ q^{2k} \mid k \in \mathbf{Z} \}$.

Proof: Assume that $\varphi : B(\alpha, \beta) \simeq B(\alpha'\beta')$. Let

$$B(\alpha, \beta) := \oplus_{k \in \mathbf{Z}} \mathbf{C} v_k, \qquad B(\alpha', \beta') := \oplus_{k \in \mathbf{Z}} \mathbf{C} v'_k,$$
$$\varphi(v_k) := a_k v'_k, \qquad \text{where } k \in \mathbf{Z} \text{ and } a_k \in \mathbf{C}^*,$$

and $a_{\alpha\beta}$, c_β are defined by (2.1).

Using (1.16), $\varphi d_n(v_k) = d_n \varphi(v_k)$ and the same argument as the one in the proof of Proposition 2.5, we have

$$(c_\beta x + a_{c\beta})a_k = (c_{\beta'} x + a_{\alpha'\beta'})a_0. \tag{2.20}$$

$$
\begin{aligned}
&(-x + a_{\alpha\beta}y + q[\beta])(c_\beta x + a_{\alpha\beta})(c_{\beta'}x + a_{\alpha'\beta'}y) \\
&=(-x + a_{\alpha'\beta'}y + q[\beta'])(c_{\beta'}x + a_{\alpha'\beta'})(c_\beta x + a_{\alpha\beta}y).
\end{aligned} \tag{2.21}
$$

Comparing the coefficients of x^2, xy, y^2, x and y in (2.21), we get

$$-c_{\beta'}a_{\alpha\beta} + q[\beta]c_\beta c_{\beta'} = -c_\beta a_{\alpha'\beta'} + q[\beta']c_\beta c_{\beta'}, \tag{2.22}$$

$$c_{\beta'}a_{\alpha\beta}^2 + q[\beta]c_\beta a_{\alpha'\beta'} = c_\beta a_{\alpha'\beta'}^2 + q[\beta']c_{\beta'}a_{\alpha\beta}, \tag{2.23}$$

$$a_{\alpha\beta}^2 a_{\alpha'\beta'} = a_{\alpha'\beta'}^2 a_{\alpha\beta}, \tag{2.24}$$

$$[\beta]a_{\alpha\beta}c_{\beta'} = [\beta']a_{\alpha'\beta'}c_\beta, \tag{2.25}$$

$$[\beta]a_{\alpha\beta}a_{\alpha'\beta'} = [\beta']a_{\alpha\beta}a_{\alpha'\beta'}. \tag{2.26}$$

- If $a_{\alpha\beta}a_{\alpha'\beta'} \neq 0$, then (1) in Propositiopn 2.7 follows from (2.24) and (2.26).
- If $a_{\alpha\beta} = a_{\alpha'\beta'} = 0$, then, by (2.20)

$$(q[\beta] - 1)a_k = (q[\beta'] - 1)a_0.$$

So $(q[\beta] - 1) = 0 \Leftrightarrow q[\beta'] - 1 = 0$, i.e., either (1) holds, or (2) holds.

- If $a_{\alpha\beta} = 0$ and $a_{\alpha'\beta'} \neq 0$, then

$(2.20) \Rightarrow c_\beta \neq 0.$
$(2.25) \Rightarrow [\beta'] = 0.$
$(2.23) \Rightarrow q[\beta]c_\beta a_{\alpha'\beta'} = c_\beta a_{\alpha'\beta'}^2 \Rightarrow q[\beta] = a_{\alpha'\beta'} = q^{2\alpha'} \Rightarrow [\beta] = q^{2\alpha'-1}.$
$a_{\alpha\beta} = 0 \Rightarrow [\beta] = q^{2\alpha+1}.$

Furthermore, by (2.20)

$$0 \neq c_\beta x a_k = (c_{\beta'}x + a_{\alpha'\beta'})a_0 = (-q^{-2k} + q^{2\alpha'})a_0 \Rightarrow q^{2\alpha'} \neq q^{-2k},$$

where $k \in \mathbf{Z}$, i.e.,

$$q^{2\alpha'} \notin \{\, q^{2k} \mid k \in \mathbf{Z} \,\}.$$

So (3) holds in this case.

- If $a_{\alpha\beta} \neq 0$ and $a_{\alpha'\beta'} = 0$, then

$(2.20) \Rightarrow c_{\beta'} \neq 0.$
$(2.25) \Rightarrow [\beta'] = 0.$
$(2.23) \Rightarrow c_{\beta'}a_{\alpha\beta}^2 = q[\beta']c_{\beta'}a_{\alpha\beta} \Rightarrow q^{2\alpha} = a_{\alpha\beta} = q[\beta'] \Rightarrow [\beta'] = q^{2\alpha-1}.$
$a_{\alpha'\beta'} = 0 \Rightarrow q^{2\alpha} \notin \{\, q^{2k} \mid k \in \mathbf{Z} \,\}.$
$(2.20) \Rightarrow q^{2\alpha} \notin \{\, q^{2k} \mid k \in \mathbf{Z} \,\}.$

Hence, (4) is true.

Conversely, if one of the conditions in Proposition 2.7 holds, then the $\mathbf{Z}$–grading preserving linear map φ defined by

$$\varphi(v_k) := \frac{(q[\beta'] - 1)q^{-2k} + (q^{2\alpha'} - q^{-1}[\beta'])}{(q[\beta] - 1)q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])} v'_k \qquad \text{for } k \in \mathbf{Z}$$

is an isomorphism from $B(\alpha, \beta)$ to $B(\alpha', \beta')$. ∎

PROPOSITION 2.7. *For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$, we have*

(1) *$A(\alpha, \beta)$ is reducible $\Leftrightarrow$ either $q^{2\alpha} - q^{2t} = [\beta] = 0$ or $q^{2\alpha+1} = [\beta] = q^{-2t-1}$ for some $t \in \mathbf{Z}$.*

(2) *If $B(\alpha, \beta) \notin c\ell(A)$, then $B(\alpha, \beta)$ is irreducible.*

PROOF: (1).$\Rightarrow$: Let N be a non–zero submodule of $A(\alpha, \beta)$ with $N \neq A(\alpha, \beta)$. Then N is a direct sum of some of the 1–dimensional subspace $\mathbf{C}v_j$. Let $v_t \in N$.

• If there exist two distinct vectors v_m and v_n $(m \neq n)$ in $A(\alpha, \beta) \setminus N$, then, by (1.15)

$$N \ni d_{m-t}(v_t) = -\frac{q^t}{q - q^{-1}}(-q^{-2t} + q[\beta]q^{2(m-t)} + (q^{2\alpha} - q^{-1}[\beta]))v_m,$$

$$N \ni d_{n-t}(v_t) = -\frac{q^t}{q - q^{-1}}(-q^{-2t} + q[\beta]q^{2(n-t)} + (q^{2\alpha} - q^{-1}[\beta]))v_n.$$

Hence,

$$-q^{-2t} + q[\beta]q^{2(m-t)} + (q^{2\alpha} - q^{-1}[\beta]) = 0, \tag{2.27}$$

$$-q^{-2t} + q[\beta]q^{2(n-t)} + (q^{2\alpha} - q^{-1}[\beta]) = 0. \tag{2.28}$$

It follows from (2.27)–(2.28) that $[\beta] = 0$. Going back to (2.27), we get that $q^{2\alpha} = q^{-2t}$ and t is unique. In other words, $N = \mathbf{C}v_t$ in this case.

• If $N = \oplus_{k \in \mathbf{Z}, k \neq t} \mathbf{C}v_k$, then we choose $v_m, v_n \in N$ with $m \neq n$. By (1.15),

$$N \ni d_{t-m}(v_m) = -\frac{q^m}{q - q^{-1}}(-q^{-2m} + q[\beta]q^{2(t-m)} + (q^{2\alpha} - q^{-1}[\beta]))v_t,$$

$$N \ni d_{t-n}(v_n) = -\frac{q^n}{q - q^{-1}}(-q^{-2n} + q[\beta]q^{2(t-n)} + (q^{2\alpha} - q^{-1}[\beta]))v_t.$$

Since $v_t \notin N$, we have to have

$$-q^{-2m} + q[\beta]q^{2(t-m)} + (q^{2\alpha} - q^{-1}[\beta]) = 0, \tag{2.29}$$

$$-q^{-2n} + q[\beta]q^{2(t-n)} + (q^{2\alpha} - q^{-1}[\beta]) = 0. \tag{2.30}$$

It follows from (2.29)–(2.30) that

$$(q^{-2m} - q^{-2n})(q^{2t+1}[\beta] - 1) = 0,$$

or

$$q^{2t+1}[\beta] = 1.$$

Going back to (2.29), we get

$$q^{2\alpha+1} = [\beta].$$

(1).$\Leftarrow$: If $q^{2\alpha} - q^{-2t} = [\beta] = 0$, then $\mathbf{C}v_t$ is a submodule of $A(\alpha,\beta)$. If $q^{2\alpha+1} = [\beta] = q^{-2t-1}$, then $\oplus_{\substack{k\in\mathbf{Z}\\ k\neq t}}\mathbf{C}v_k$ is a submodule of $A(\alpha,\beta)$. Therefore, $A(\alpha,\beta)$ is always reducible in these cases.

(2). If $B(\alpha,\beta)\notin c\ell(A)$, then, by Proposition 2.5,

$$q^{2\alpha+1} + (q^4 - 1)[\beta] = 0. \tag{2.31}$$

Assume that $N \neq 0$ is a submodule of $B(\alpha,\beta)$, it follows from Proposition 2.4 that $N = \oplus_{k\in S}\mathbf{C}v_k$ for some non–empty subset S of $\mathbf{Z}$.

• If there exists $v_m \notin N$ and $v_n \notin N$ with $m \neq n$, then by (1.[illegible]

$$N \ni d_{m-k}(v_k) = -\frac{q^k}{q-q^{-1}}\left(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2(m-k)} + q[\beta]\right)v_m,$$
$$N \ni d_{n-k}(v_k) = -\frac{q^k}{q-q^{-1}}\left(-q^{-2k} + (q^{2\alpha} - q^{-1}[\beta])q^{2(n-k)} + q[\beta]\right)v_n,$$

where $v_k \in N$. So the coefficients of v_m and v_n have to be zero. This implies that $q^{2\alpha} - q^{-1}[\beta] = 0$. Going back to (2.31), we get that $[\beta] = 0$, which contradicts (2.31).

• If $\mathbf{Z}\setminus S = \{s\}$, then we can choose $v_m \in N$ and $v_n \in N$ with $m \neq n$. As above, it follows from $d_{s-m}(v_m) \in N$ and $d_{s-n}(v_n)\in N$ that $[\beta] = 0$, which is impossible. ∎

For $t \in \mathbf{Z}$, we define

$$A_t(\alpha,\beta) = \begin{cases} \dfrac{A(\alpha,\beta)}{\mathbf{C}v_t}, & \text{if } q^{2\alpha} - q^{-2t} = [\beta] = 0; \\ \oplus_{\substack{k\in\mathbf{Z}\\ k\neq t}}\mathbf{C}v_k, & \text{if } q^{2\alpha+1} = [\beta] = q^{-2t-1}. \end{cases}$$

Then, $A_t(\alpha,\beta)$ is an irreducible $U(W_q)$–module by the proof of (1) in Proposition 2.7.

Proposition 2.8. *Let $q \in \mathbf{R}$, then with respect to the antilinear anti-involution θ of $U(W_q)$,*

(1) *$A(\alpha,\beta)$ is unitary $\Longleftrightarrow$ $q > 0$ and $q^{2\alpha} = q^{-1}[\beta] + q\overline{[\beta]}$.*
(2) *$A_t(\alpha,\beta)$ and $B(\alpha,\beta)$ are not unitary, where $B(\alpha,\beta) \notin c\ell(A)$.*

Proof: (1). $\Rightarrow$: Assume that $A(\alpha,\beta) = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k$ is unitary and $<.\mid.>$ the contravariant form on V. So

$$< d_m(v_k) \mid v_\ell >=< v_k \mid d_{-m}(v_\ell) > \qquad \text{for } m,k,\ell \in \mathbf{Z}.$$

Let $n := \ell := m + k$, then we have, by (1.15)

$$\begin{aligned} &q^k\left(-q^{-2k} + q[\beta]q^{2n-2k} + (q^{2\alpha} - q^{-1}[\beta])\right) < v_n \mid v_n > \\ (2.32)\qquad &= q^n\left(-q^{-2n} + q\overline{[\beta]}q^{-2n+2k} + (\overline{q^{2\alpha}} - q^{-1}\overline{[\beta]})\right) < v_k \mid v_k >, \end{aligned}$$

where $n,k \in \mathbf{Z}$. Let $k = 0$ in (2.32), we get

$$(2.33)\qquad (q[\beta]q^{2n} + b_{\alpha\beta}) < v_n \mid v_n >= q^n(\overline{c_\beta}q^{-2n} + \overline{a_{\alpha\beta}}) < v_0 \mid v_0 >,$$

where $a_{\alpha\beta}$, $b_{\alpha,\beta}$ and c_β are defined by (2.1).

It follows from (2.32) and (2.33) that

$$\begin{aligned} &(-x + q[\beta]xy + a_{\alpha\beta})(b_{\alpha\beta}x + q[\beta])(\overline{a_{\alpha\beta}}y + \overline{c_\beta}) \\ (2.34)\qquad &= (-x + \overline{a_{\alpha\beta}}xy + q\overline{[\beta]})(\overline{c_\beta}x + \overline{a_{\alpha\beta}})(q[\beta]y + b_{\alpha\beta}), \end{aligned}$$

where $x := q^{-2k}$ and $y := q^{2n}$. Comparing the coefficients of x^2y, xy and y, we get

$$(2.35)\qquad -\overline{a_{\alpha\beta}}b_{\alpha\beta} + q[\beta]\overline{c_\beta}b_{\alpha\beta} = -q[\beta]\overline{c_\beta} + \overline{c_\beta}\overline{a_{\alpha\beta}}b_{\alpha\beta},$$

$$(2.36)\qquad q^2[\beta]^2\overline{c_\beta} + a_{\alpha\beta}\overline{a_{\alpha\beta}}b_{\alpha\beta} = \overline{a_{\alpha\beta}}^2b_{\alpha\beta} + q^2[\beta]\overline{[\beta]}\overline{c_\beta},$$

$$(2.37)\qquad [\beta]a_{\alpha\beta}\overline{a_{\alpha\beta}} = q[\beta]\overline{[\beta]}\overline{a_{\alpha\beta}}.$$

Suppose that $[\beta] = 0$, then $a_{\alpha\beta} = q^{2\alpha} \neq 0$, $c_\beta = -1$ and $b_{\alpha\beta} \neq 0$ by (2.1) and (2.33). It follows from (2.36) that $a_{\alpha\beta} = \overline{a_{\alpha\beta}}$. So (2.33) becomes that

$$(q^{2\alpha} - 1) < v_n \mid v_n >= q^n(q^{2\alpha} - q^{-2n}) < v_0 \mid v_0 > \qquad \text{for } n \in \mathbf{Z}.$$

This implies that $f(n) := \dfrac{q^{2\alpha} - q^{4n}}{q^{2\alpha} - 1} > 0$ for all $n \in \mathbf{Z}$, which is impossible because $f(n)f(-n) < 0$ for large $n > 0$. Therefore, we have proved that $[\beta] \neq 0$.

Similarly, we can prove that $a_{\alpha\beta} \neq 0$ by using (2.35).

Going back to (2.37), we have $a_{\alpha\beta} = q\overline{[\beta]}$, i.e. $q^{2\alpha} = q^{-1}[\beta] + q\overline{[\beta]}$.

Finally, choose an odd $n_0 \in \mathbf{Z}$ such that $q[\beta]q^{2n_0} + b_{\alpha\beta} \neq 0$, then (2.33) gives that

$$< v_{n_0} \mid v_{n_0} >= q^{-n_0} < v_0 \mid v_0 >,$$

which imllies that $q > 0$.

$\Leftarrow$: Define a Hermitiam form $< . \mid . >$ on V by

$$< v_n \mid v_m >:= \delta_{nm} q^{-n} \qquad \text{for all } n, m \in \mathbf{Z}.$$

It is easy to check that $< . \mid . >$ is a contravariant form.

(2). Assume that $A_t(\alpha, \beta)$ is unitary, then, using the same argument as above, we can get

$$q^{2\alpha} = q^{-1}[\beta] + q\overline{[\beta]},$$

which is impossible because we have either $q^{2\alpha} - q^{2t} = [\beta] = 0$ or $q^{2\alpha+1} = [\beta] = q^{-2t-1}$. This proves that $A_t(\alpha, \beta)$ is not unitary.

Finally, we prove that if $B(\alpha, \beta) \notin c\ell(A)$, then $B(\alpha, \beta)$ is not unitary.

Suppose that $B(\alpha, \beta)$ has a contravariant form $< .|. >$ and $< v|v >> 0$ for $0 \neq v \in B$, then

$$(2.38) \qquad < d_m(v_k)|v_\ell >=< v_k|d_{-m}(v_\ell) > \qquad \text{for } m, k, \ell \in \mathbf{Z}.$$

By (1.16) and Proposition 2.5,

$$(2.39) \qquad d_n(v_j) = -\frac{q^k}{q - q^{-1}}(-q^{-2k} + q^3[\beta]q^{2n} + q[\beta])v_{n+j} \qquad \text{for } n, j \in \mathbf{Z}.$$

Let $n := \ell := m + k$, then, by (2.38) and (2.39)

$$\begin{aligned} &q^k(-q^{-2k} + q^3[\beta]q^{2n-2k} + q[\beta]) < v_n|v_n > \\ =&q^n(-q^{-2n} + q^3\overline{[\beta]}q^{-2n+2k} + q\overline{[\beta]}) < v_k|v_k > . \end{aligned}$$

Let $n = k$ in (2.40), we get

$$q^3[\beta] + q[\beta] = q^3\overline{[\beta]} + q\overline{[\beta]} \quad \text{or} \quad [\beta] = \overline{[\beta]}.$$

Hence, (2.40) becomes

$$
\begin{aligned}
&q^k(-q^{-2k} + q^3[\beta]q^{2n-2k} + q[\beta]) < v_n|v_n > \\
&=q^n(-q^{-2n} + q^3[\beta]q^{-2n+2k} + q[\beta]) < v_k|v_k > .
\end{aligned} \tag{2.41}
$$

Using (2.41) and the same argument as the one in the proof of (1), we have

$$
\begin{aligned}
&(-x + q^3[\beta]xy + q[\beta])(q[\beta]y + (q^3[\beta] - 1))((q[\beta] - 1)x + q^3[\beta]) \\
=&(q[\beta]xy - x + q^3[\beta])((q^3[\beta] - 1)x + q[\beta])(q^3[\beta]y + (q[\beta] - 1)),
\end{aligned}
$$

where $x := q^{-2k}$ and $y := q^{2n}$. Comparing the constants in the equation above, we conclude that

$$q[\beta] \cdot (q^3[\beta] - 1) \cdot q^3[\beta] = q^3[\beta] \cdot q[\beta] \cdot (q[\beta] - 1).$$

Since $[\beta] \neq 0$, $q^3 = q$, i.e., $q^2 = 1$, which is impossible. ∎

§3.3. q–analogue of Kaplansky's theorem

LEMMA 3.1. *Let q be not a root of unity, then for all integers n and all positive integers s, we have in $U(W_q)$*

$$
\begin{aligned}
d_n d_{-n}^s = q^{-2ns} d_{-n}^s d_n + q^{-sn}\frac{[2n][sn]}{[n]} d_{-n}^{s-1} d_0 J^{-1} \\
+ [sn][(s-1)n] d_{-n}^{s-1} J^{-2}.
\end{aligned}
$$

PROOF: We use induction on s. It is clear that the Lemma is true for $s = 1$. Now we assume that the Lemma is true for s, then

$$
\begin{aligned}
d_n d_{-n}^{s+1} &= (d_n d_{-n}^s) d_{-n} \\
&= (q^{-2ns} d_{-n}^s d_n + q^{-sn}\frac{[2n][sn]}{[n]} d_{-n}^{s-1} d_0 J^{-1} \\
&\quad + [sn][(s-1)n] d_{-n}^{s-1} J^{-2}) d_{-n} \\
&= q^{-2ns} d_{-n}^s (q^{-2n} d_{-n} d_n + q^{-n}[2n] d_0 J^{-1}) \\
&\quad + [sn][(s-1)n] q^{2n} d_{-n}^s J^{-2} \\
&\quad + q^{-sn}\frac{[2n][sn]}{[n]} d_{-n}^{s-1} q^n (q^{-n} d_{-n} d_0 + [n] d_{-n} J^{-1}) J^{-1} \\
&= q^{-2n(s+1)} d_{-n}^{s+1} d_n + q^{-(s+1)n}(q^{-sn}[n] + q[sn])\frac{[2n]}{[n]} d_{-n}^s d_0 J^{-1} \\
&\quad + (q^{-(s-1)n}[2n] + q^{2n}[(s-1)])[sn] d_{-n}^s J^{-2} \\
&= q^{-2n(s+1)} d_{-n}^{s+1} d_n + q^{-(s+1)n}\frac{[(s+1)n][2n]}{[n]} d_{-n}^s d_0 J^{-1} \\
&\quad + [(s+1)n][sn] d_{-n}^s J^{-2}.
\end{aligned}
$$

This proves the Lemma. ∎

THEOREM 3.2. *Let q be not a root of unity and $V = \oplus_{k\in\mathbf{Z}}\mathbf{C}v_k$ a $\mathbf{Z}$-graded $U(W_q)$-module with $J(v_k) \in \mathbf{C}v_k$ for $k \in \mathbf{Z}$. If d_1 and d_{-1} are injective operators on V and*

$$(Jd_1d_{-1}J - Jd_{-1}d_1J)(v_0) \neq \frac{1}{q-q^{-1}}v_0,$$

then $V \simeq A(\lambda,\alpha,\beta)$ or $V \simeq B(\lambda,\alpha,\beta)$ for some $(\lambda,\alpha,\beta) \in \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}$.

PROOF: Since $J(v_k) \in \mathbf{C}v_k$, $d_1(v_k) \neq 0$ and $Jd_1J^{-1}(v_k) = qd_1(v_k)$, there exists some $\lambda \in \mathbf{C}^*$ such that $J(v_k) = \lambda q^k v_k$ for all $k \in \mathbf{Z}$. Using the automorphism $\varphi(\lambda^{-1})$, we can assume that $\lambda = 1$, in which case, we will prove that either $V \simeq A(\alpha,\beta)$ or $V \simeq B(\alpha,\beta)$ for some $(\alpha,\beta) \in \mathbf{C}\times\mathbf{C}$.

Set

$$d_0(v_0) = av_0, \qquad d_1d_{-1}(v_j) = x_jv_j, \qquad d_{-1}d_1(v_j) = y_jv_j,$$

where $a, x_j, y_j \in \mathbf{C}$ and $j \in \mathbf{Z}$. We consider the system (i) with respect to α and β:

$$-([\alpha]q^\alpha + [\beta]) = a, \tag{3.1}$$

$$[\alpha]q^\alpha([\alpha-1]q^\alpha + [2][\beta]) = x_0 \tag{3.2}$$

and the system (ii) with respect to α and β:

$$\begin{aligned} -([\alpha]q^\alpha + [\beta]) &= a, \\ [\alpha+1]q^\alpha([\alpha]q^\alpha + [2][\beta]q) &= y_0. \end{aligned} \tag{3.3}$$

First, we assume that there exist α and β such that (i) holds. Using induction on j and (3.1) gives us

$$d_0(v_j) = -([\alpha+j]q^\alpha + [\beta]q^j)v_j \qquad \text{for } j \in \mathbf{Z}. \tag{3.4}$$

Since $(qd_1d_{-1}J - q^{-1}d_{-1}d_1J)(v_j) = [2]d_0v_j$, we have by (3.4)

$$q^{j+1}x_j - q^{j-1}y_j = -[2][\alpha+j]q^\alpha - [2][\beta]q^j \qquad \text{for } j \in \mathbf{Z}. \tag{3.5}$$

Furthermore, computing $d_1d_{-1}d_1(v_{j-1})$ in two ways produces the following relation between x_j and y_j:

$$x_j = y_{j-1} \qquad \text{for } j \in \mathbf{Z}. \tag{3.6}$$

Going back to (3.5), we get

$$q^{j+1}x_j - q^{j-1}x_{j+1} = -[2][\alpha+j]q^\alpha - [2][\beta]q^j \qquad \text{for } j \in \mathbf{Z}. \tag{3.7}$$

Now we claim that

$$x_j = [\alpha + j]q^\alpha([\alpha + j - 1]q^\alpha + [2][\beta]q^j) \qquad \text{for } j \in \mathbf{Z}. \tag{3.8}$$

By (3.2), (3.8) is true for $j = 0$. Assume that (3.8) is true for j, then (3.8) is also true for $j \pm 1$. For example, let us prove that (3.8) is true for $j + 1$. By (3.7),

$$\begin{aligned} x_{j+1} &= q^2 x_j + [2][\alpha + j]q^{\alpha - j + 1} + [2][\beta]q \\ &= q^2[\alpha + j]q^\alpha([\alpha + j - 1]q^\alpha + [2][\beta]q^j) + [2][\alpha + j]q^{\alpha - j + 1} + [2][\beta]q \\ &= [\alpha + j]q^{2\alpha}(q^2[\alpha + j - 1] + [2]q^{-\alpha - j + 1}) \\ &\qquad + [2][\beta]q^\alpha([\alpha + j]q^{2+j} + q^{1-\alpha}) \\ &= [\alpha + j]q^{2\alpha}[\alpha + j + 1] + [2][\beta]q^\alpha q^{j+1}[\alpha + j + 1] \\ &= [\alpha + j + 1]q^\alpha([\alpha + j]q^\alpha + [2][\beta]q^{j+1}). \end{aligned}$$

Hence, (3.8) is true for all $j \in \mathbf{Z}$ by induction.

Let $j = 1$ in(3.8), we get (3.3). So we have proved that if α and β satisfy (i), then α and β also satisfy (ii).

Similarly, we can prove that if α and β satisfy (ii), then α and β also satisfy (i).

Now we prove that either (i) has a solution or (ii) has a solution.

Choose $\beta, \beta' \in \mathbf{C}$ such that

$$(-a - [\beta])((-a - [\beta])q^{-1} - 1 + [2][\beta]) = x_0, \tag{3.9}$$

$$((-a - [\beta'])q + 1)((-a - [\beta']) + [2][\beta']q) = y_0. \tag{3.10}$$

Suppose that both (i) and (ii) has no solution, then we have to have

$$-\left(\frac{q^{2\alpha} - 1}{q - q^{-1}} + [\beta]\right) \neq a \qquad \text{for all } \alpha \in \mathbf{C}; \tag{3.11}$$

$$-\left(\frac{q^{2\alpha'} - 1}{q - q^{-1}} + [\beta']\right) \neq a \qquad \text{for all } \alpha' \in \mathbf{C}. \tag{3.12}$$

In fact, if $-\left(\frac{q^{2\alpha} - 1}{q - q^{-1}} + [\beta]\right) = a$ for some $\alpha \in \mathbf{C}$, then α and β satisfies (3.1). Hence, by (3.9)

$$\begin{aligned} x_0 &= [\alpha]q^\alpha([\alpha]q^\alpha \cdot q^{-1} - 1 + [2][\beta]) \\ &= [\alpha q^\alpha([\alpha - 1]q^\alpha + [2][\beta]), \end{aligned}$$

i.e., α and β also satisfies (3.2). So (α, β) is a solution of (i), which contradicts our assumption. This proves (3.11).

Similarly, if $-\left(\frac{q^{2\alpha'}-1}{q-q^{-1}}+[\beta']\right)=a$ for some $\alpha' \in \mathbf{C}$, then

$$-([\alpha']q^{\alpha'}+[\beta'])=a$$

and, by (3.10)

$$\begin{aligned} y_0 &= ([\alpha']q^{\alpha'}\cdot q+1)([\alpha']q^{\alpha'}+[2][\beta']q) \\ &= [\alpha'+1]q^{\alpha'}([\alpha']q^{\alpha'}+[2][\beta']q). \end{aligned}$$

Hence, (α', β') is a solution of (ii), which is impossible by our assumption. So (3.12) holds.

It follows from (3.11) and (3.12) that

$$[\beta]=[\beta']=\frac{1}{q-q^{-1}}-a. \tag{3.13}$$

By (3.9) and (3.13), we have

$$\begin{aligned} x_0 &= (-a-[\beta])(-aq^{-1}-1+([2]-q^{-1})[\beta]) \\ &= (-a-[\beta])(-aq^{-1}-1+q[\beta]) \\ &= -\frac{1}{q-q^{-1}}\left(-aq^{-1}-1+q\left(\frac{1}{q-q^{-1}}-a\right)\right) \\ &= -\frac{1}{q-q^{-1}}\left(-a(q+q^{-1})+\left(-1+\frac{q}{q-q^{-1}}\right)\right) \\ &= -\frac{1}{q-q^{-1}}\left(-a(q+q^{-1})+\frac{q^{-1}}{q-q^{-1}}\right) \\ &= \frac{1}{(q-q^{-1})^2}(a(q^2-q^{-2})-q^{-1}), \end{aligned}$$

i.e.,

$$a(q^2-q^{-2})-q^{-1}=(q-q^{-1})^2x_0. \tag{3.14}$$

By (3.10) and (3.13), we have

$$\begin{aligned} y_0 &= ((-a-[\beta'])q+1)(-a+([2]q-1)[\beta']) \\ &= \left(-\frac{q}{q-q^{-1}}+1\right)\left(-a+q^2\left(\frac{1}{q-q^{-1}}-a\right)\right) \\ &= \frac{q^{-1}}{q-q^{-1}}\left(a(q^2+1)-\frac{q^2}{q-q^{-1}}\right) \\ &= \frac{1}{(q-q^{-1})^2}(a(q^2-q^{-2})-q), \end{aligned}$$

i.e.,

$$a(q^2 - q^{-2}) - q = (q - q^{-1})^2 y_0. \tag{3.15}$$

(3.14)–(3.15) gives us

$$q - q^{-1} = (q - q^{-1})^2 (x_0 - y_0),$$

or

$$\frac{1}{q - q^{-1}} = x_0 - y_0,$$

which contradicts the assumption in Theorem 3.2. This proves that either (i) has a solution or (ii) has a solution. Therefore there exists $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that (i) holds.

Using (3.4) and (3.8), we can choose a basis of V, say $\{ v_k \mid k \in \mathbf{Z} \}$, such that

$$d_n(v_j) = -([\alpha + j]q^\alpha + [n+1][\beta]q^{n+j})v_{n+j}, \tag{3.16}$$

where $n = 0, \pm 1$ and $j \in \mathbf{Z}$.

For $j \in \mathbf{Z}$, set

$$d_2(v_j) := e(j)v_{j+2}, \qquad d_{-2}(v_j) := g(j)v_{j-2},$$

$$e(j) := f(j) - ([j + \alpha]q^\alpha + [3][\beta]q^{j+2}), \tag{3.17}$$

$$g(j) := h(j) - ([j + \alpha]q^\alpha - [\beta]q^{j-2}), \tag{3.18}$$

where $e(j), g(j), f(j), h(j) \in \mathbf{C}$.

Using (3.16) and following identities:

$$(q^2 d_2 d_{-1} J - q^{-1} d_{-1} d_2 J)(v_j) = [3] d_1(v_j),$$
$$(q^{-2} d_{-2} d_1 J - q d_1 d_{-2} J)(v_j) = -[3] d_{-1}(v_j),$$

we get

$$q^{\alpha+j-1}[\alpha + j + 2]e(j) - q^{\alpha+j+2}[\alpha + j]e(j-1) = -[3]([\alpha + j]q^\alpha + [2][\beta]q^{j+1}),$$

$$\begin{aligned} q^{j+1}&([\alpha + j - 2]q^\alpha + [2][\beta]q^{j-1})g(j) \\ &- q^{j-2}([\alpha + j]q^\alpha + [2][\beta]q^{j+1})g(j+1) = [3][\alpha + j]q^\alpha. \end{aligned}$$

It follows from (3.17) and (3.18) that

$$[\alpha+j+2]f(j)=q^3[\alpha+j]f(j-1),$$
$$([\alpha+j]q^\alpha+[2][\beta]q^{j+1})h(j+1)=q^3([\alpha+j-2]q^\alpha+[2][\beta]q^{j-1})h(j).$$

These identities imply that

(3.19)
$$f(j)=\frac{q^{3j}[\alpha+1][\alpha+2]}{[\alpha+j+1][\alpha+j+2]}f(0),$$
(3.20)
$$h(j)=\frac{q^{3j}([\alpha-1]q^\alpha+[2][\beta])([\alpha-2]q^\alpha+[2][\beta]q^{-1})}{([\alpha+j-2]q^\alpha+[2][\beta]q^{j-1})([\alpha+j-1]q^\alpha+[2][\beta]q^j)}h(0),$$

where $j\in\mathbf{Z}$. Note, that denominators in (3.19) and (3.20) are non zero follows from (3.16) and $d_{\pm1}(v_j)\neq 0$ for all $j\in\mathbf{Z}$.

Let $z:=q^{-j}[j]$. We can rewrite (3.17)—(3.20) as follows:

(3.21)
$$q^{-j}e(j-2)=q^{-j}f(j-2)-(q^2z+[\alpha-2]q^\alpha+[3][\beta]).$$
(3.22)
$$q^{-j}g(j)=q^{-j}h(j)-(z+[\alpha]q^\alpha-[\beta]q^{-2}).$$
(3.23)
$$q^{-j}f(j-2)=\frac{q^{-2}[\alpha+1][\alpha+2]f(0)}{(q^{3-\alpha}z+[\alpha+1]-[2]q^{1-\alpha})(q^{2-\alpha}z+[\alpha+2]-[2]q^{-\alpha})}.$$
(3.24)
$$q^{-j}h(j)=\frac{([\alpha-1]q^\alpha+[2][\beta])([\alpha-2]q^\alpha+[2][\beta]q^{-1})h(0)}{(qz+[\alpha-1]q^\alpha+[2][\beta])(q^2z+[\alpha-2]q^\alpha+[2][\beta]q^{-1})}.$$

where $j\in\mathbf{Z}$.

By Lemma 3.1 and a direct computation, we can get

(3.25) $$q^{-j}e(j-2)\cdot q^{-j}g(j)=q^2z^2+c_1z+c_2 \qquad \text{for large even } j,$$

where c_1 and c_2 are complex numbers, which are independent of j.

Using (3.21)—(3.24), we have

(3.26) $$q^{-j}e(j-2)=-q^2\frac{R_1}{R_2},\qquad q^{-j}g(j)=-\frac{T_1}{T_2}$$

where

$$
\begin{aligned}
R_1 :&= (z + [\alpha-2]q^{\alpha-2} + [3][\beta]q^{-2})(z + [\alpha+1]q^{\alpha-3} - [2]q^{-2})\times \\
&\quad \times (z + [\alpha+2]q^{\alpha-2} - [2]q^{-2}) \\
&\quad - q^{2\alpha-9}[\alpha+1[\alpha+2]f(0), \\
R_2 :&= (z + [\alpha+1]q^{\alpha-3} - [2]q^{-2})(z + [\alpha+2]q^{\alpha-2} - [2]q^{-2}), \\
T_1 :&= (z + [\alpha]q^{\alpha} - [\beta]q^{-2})(z + [\alpha-1]q^{\alpha-1} + [2][\beta]q^{-1})\times \\
&\quad \times (z + [\alpha-2]q^{\alpha-2} + [2][\beta]q^{-3}) \\
&\quad - q^{-3}([\alpha-1]q^{\alpha} + [2][\beta])([\alpha-2]q^{\alpha} + [2][\beta]q^{-1})h(0), \\
T_2 :&= (z + [\alpha-1]q^{\alpha-1} + [2][\beta]q^{-1})(z + [\alpha-2]q^{\alpha-2} + [2][\beta]q^{-3}).
\end{aligned}
$$

(3.25) implies that as the polynomials with respect to z, we have

$$R_2T_2 \quad \text{divides} \quad R_1T_1 \tag{3.27}$$

Now we have two cases to discuss:

• Case 1. $f(0)g(0) = 0$, in which case, either $f(0) = 0$ or $g(0) = 0$. If $f(0) = 0$, then (3.27) becomes

$$T_2 \quad \text{divides} \quad (z + [\alpha-2]q^{\alpha-2} + [3][\beta]q^{-2})T_1.$$

It follows that

$$-\frac{q^3([\alpha-1]q^{\alpha} + [2][\beta])([\alpha-2]q^{\alpha} + [2][\beta]q^{-1})h(0)}{T_2} \tag{3.28}$$

is a polynomial of z, hence, it is zero. Since $d_1(v_{-1}) \neq 0$ and $d_1(v_{-2}) \neq 0$, the coefficient of $h(0)$ in (3.28) is not zero. So we have to have $h(0) = 0$.

Similarly, if $h(0) = 0$, then we also have $f(0) = 0$.

Therefore, $f(0)g(0) = 0$ implies that $f(0) = g(0) = 0$. By (3.17)—(3.20), (3.16) is also true for $n = \pm 2$ and $j \in \mathbf{Z}$. This proves that $V = A(\alpha, \beta)$ because $U(W_q)$ is generated by $\{ J^{\pm 1}, d_0, d_{\pm 1}, d_{\pm 2} \}$.

• Case 2. $f(0)g(0) \neq 0$. Since $d_{\pm 1}(v_j) \neq 0$ for all $j \in \mathbf{Z}$, the coefficients of $f(0)$ and $g(0)$ in R_1 and T_1 are non-zero. It follows from (3.27) that R_2 divides T_1 and T_2 divides R_1, i.e.

$$T_1 = (z + [\alpha+1]q^{\alpha-3} - [2]q^{-2})(z + [\alpha+2]q^{\alpha-2} - [2]q^{-2})(z + G)$$

$$R_1 = (z + [\alpha-1]q^{\alpha-1} + [2][\beta]q^{-1})(z + [\alpha-2]q^{\alpha-2} + [2][\beta]q^{-3})(z + H)$$

where $G, H \in \mathbf{C}$. Comparing the coefficients of z^2, we get

$$G = [\alpha - 2]q^{\alpha-2} + [3][\beta]q^{-2}, \qquad H = q^{\alpha}[\alpha] - [\beta]q^{-2}.$$

Going back to (3,26), we have

$$\begin{aligned} &q^{-j}e(j) \\ &= -\frac{(z + [\alpha+2]q^{\alpha+2} - [\beta]q^2)(z + [\alpha+1]q^{\alpha+1} + [2][\beta]q^3)}{(z + [\alpha+1]q^{\alpha+1})(z + [\alpha+2]q^{\alpha+2})} \times \\ &\quad \times (z + [\alpha]q^{\alpha} + [2][\beta]q), \end{aligned} \tag{3.29}$$

$$\begin{aligned} &q^{-j}g(j) \\ &= -\frac{(z + [\alpha+1]q^{\alpha-3} - [2]q^{-2})(z + [\alpha+2]q^{\alpha-2} - [2]q^{-2})}{(z + [\alpha-1]q^{\alpha-1} + [2][\beta]q^{-1})(z + [\alpha-2]q^{\alpha-2} + [2][\beta]q^{-3})} \times \\ &\quad \times (z + [\alpha-2]q^{\alpha-2} + [3][\beta]q^{-2}) \end{aligned} \tag{3.30}$$

for large even j. In particular, the rational fuction $q^{-j}e(j)$ of z and the rational fuction of the right side of (3.29) take the same values at infinitely many different points:

$$\{\, q^{-j}[j] \mid \text{for large even } j \,\}$$

It follows that (3.29) is true for all $j \in \mathbf{Z}$.
Similarly, (3.30) is also true for all $j \in \mathbf{Z}$.

Now we choose $a_0 = 1$ and $a_j \in \mathbf{C}^*$ for $j \in \mathbf{Z}$ such that

$$\frac{a_{j+k}}{a_{j+k+1}} = \frac{z + [\alpha+k+1]q^{\alpha+k+1}}{z + [\alpha+k]q^{\alpha+k} + [2][\beta]q^{1+2k}} \qquad \text{for } j,k \in \mathbf{Z}. \tag{3.31}$$

Set $u_j := a_j v_j$, we get

$$\begin{aligned} d_n(u_j) &= -q^j(z + [\alpha+n]q^{\alpha+n} + [1-n][\beta]q^n)u_{n+j} \\ &= -([\alpha+n+j]q^{\alpha+n} + [1-n][\beta]q^{n+j})u_{n+j} \end{aligned} \tag{3.32}$$

for $n = 0, \pm 1, \pm 2$ and $j \in \mathbf{Z}$.

For example, let us check that (3.32) is true for $n = 2$ and all $j \in \mathbf{Z}$. By (3.31), we have

$$\frac{a_j}{a_{j+2}} = \frac{(z + [\alpha+1]q^{\alpha+1})(z + [\alpha+2]q^{\alpha+2})}{(z + [\alpha]q^{\alpha} + [2][\beta]q)(z + [\alpha+1]q^{\alpha+1} + [2][\beta]q^3)} \tag{3.33}$$

(3.29) and (3.33) imply that

$$d_2(u_j) = a_j(v_j) = a_j e(j) v_{j+2} = q^j \frac{a_j}{a_{j+2}} q^{-j} e(j) u_{j+2}$$
$$= -q^j \frac{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})}{(z+[\alpha]q^{\alpha}+[2][\beta]q)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)} \times$$
$$\times \frac{(z+[\alpha+2]q^{\alpha+2}-[\beta]q^2)(z+[\alpha+1]q^{\alpha+1}+[2][\beta]q^3)}{(z+[\alpha+1]q^{\alpha+1})(z+[\alpha+2]q^{\alpha+2})} \times$$
$$\times (z+[\alpha]q^{\alpha}+[2][\beta]q) u_{j+2}$$
$$= -q^j (z+[\alpha+2]q^{\alpha+2}-[\beta]q^2) u_{j+2}.$$

Therefore, $V = \oplus_{n \in \mathbf{Z}} \mathbf{C} u_n = B(\alpha, \beta)$ by (3.32). ∎

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CONCLUSIONS AND SOME OPEN PROBLEMS

After finishing this thesis, I become aware that a group of physicists have also begun to study q-deformations of the Virasoro algebra in their papers: [1], [3], [4], [5], [6] and [9]. Comparing these physicists' papers with my thesis, I find the following facts:

1. In spite of physicists have several methods of constructing the quantum Witt algebra, none of these methods is the same as my methord of constructing the quantum Witt algebra.

2. Although the authors of [1], of [4] and I independently discovered the q deformation of the Virasoro algebra under the condition that q is not a root of unity, the methods of computing the q-analogue of the Virasoro algebra central term are the same one, which is a q-version of the method of computing the Virasoro algebra central term in [2].

3. Physicists' work in [1], [3], [4], [5], [6] and [9] only overlap some work in Chapter 1 of this thesis.

4. Let $U_q(W)$ be the associative algebra generated by $\{ d_m \mid m \in \mathbf{Z} \}$ with relations:

$$q^{m-n} d_m d_n - q^{n-m} d_n d_m = [m-n] d_{m+n} \qquad \text{for } m, n \in \mathbf{Z}.$$

This algebra was construcred by T. L. Curtright and C. K. Zachos in [6]. Using Definition 1.2 in Chapter 3 to define a $\mathbf{Z}$-graded $U_q(W)$-module, I note that $V = \oplus_{n \in \mathbf{Z}} V_n$ is a $\mathbf{Z}$-graded $U_q(W)$-module if and only if V is a $\mathbf{Z}$-graded $U(W_q)$-module with $J \mid V_n = q^n \cdot id$. So we can restate those results in Chapter 3 in terms of the algebra $U_q(W)$.

The facts above show that the work in this thesis has physics background. Recently, I have found that quantizing the Kac Conjecture proved by C.Martin and A.Piard in [8] is in connection with q-Gamma functions. Therefore, studying the quantization of the representation theory of the Virasoro algebra is closely related to both physics and some other areas in mathematics. The work in Chapter 3 is just a beginning on quantizing the representation theory of the Virasoro algebra, but it make me to believe that the work produced in developing the q-analogue of the representation theory of the Virasoro algebra will interest both physicists and mathematicians. In addition to this, studying the structure of quantum Lie algebras is also a quite interesting problem. I would like to conclude my thesis by proposing the following open problems about the structure of quantum Lie algebras:

Problem 1: Classify all complex finite dimensional $\mathbf{Z}$-graded simple quantum Lie algebras.

Problem 2: Let $L = \oplus_{n\in\mathbf{Z}} L_n$ be a quantum Lie algebra. If $q^4 \neq 1$, then L_{-1} is a module over the Lie algebra L_0. Therefore, one natural question is that finding the analogue of the Theorem in [7] for quantum Lie algebras. In other words, find all complex infinite dimensional $\mathbf{Z}$-graded simple quantum Lie algebras $L = \oplus_{n\in\mathbf{Z}} L_n$ which satisfy the following conditions:

(1) $dim(L_n) < \infty, \qquad$ for all $n \in \mathbf{Z}$.

(2) $\overline{\lim}_{n\to\infty} \dfrac{\ell n(dim(L_n))}{\ell n \mid n \mid} < \infty.$

(3) $L_{-1} \oplus L_0 \oplus L_1$ generates L and the mondule L_{-1} over Lie algebra L_0 is irreducible.

Problem 3: Is the quantum Witt algebra the unique complex infinite dimensional graded simple quantum Lie algebra $L = \oplus_{n\in\mathbf{Z}} L_n$ satisfying the condisions (1) and (2) above if $q^r \neq 1$ for r=3,4,6,8 and $L_{-1} \oplus L_0 \oplus L_1$ does not generates L?

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