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Some Topological and Combinatorial Properties of Amenable Groups and Semigroups

BY

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TO THE MEMORY OF MY GRANDFATHER

ABSTRACT

Let G be a discrete group. A left invariant mean on the Banach space $\ell^\infty(G)$ is a positive linear functional of norm one on $\ell^\infty(G)$, which is invariant under all left translations by elements in G. When a left invariant mean exists, we say that G is left amenable. Left amenability is generalized to discrete semigroups and locally compact groups, where we consider the space $L^\infty(G)$ of all essentially bounded Borel measurable functions. In this thesis we present some results concerning topological and combinatorial aspects of left amenable groups and semigroups.

The first half of the thesis deals with the structure of the set $MTL(G)$ of all left topological invariant means for a locally compact group G, and the set $ML(S)$ of all left invariant means for a discrete semigroup S. We obtain the exact cardinality of the sets $MTL(G)$ and $ML(S)$ and some of their subsets, in terms of the structural properties of G and S. We also prove that the set $MTL(G)$ has no exposed points or G_δ-points if G is not compact, and find necessary and sufficient conditions for the existence of exposed points and G_δ-points of the set $ML(S)$. In doing so, we improve results previously obtained by C. Chou, E. Granirer, M. Klawe, A. Lau, and A. Paterson.

The second half of the thesis concerns the Følner number and Følner-type conditions for a discrete semigroup. The Følner number is a real number between zero and one related to the combinatorial behavior of a semigroup S. It is well known that if the Følner number is zero, then S is left amenable. We prove that there exist left amenable semigroups with Følner number equal to one. Thus we

answer a problem of I. Namioka on the necessity of some Følner-type conditions for a semigroup to be left amenable. We also determine the Følner number for all finite and cancellative semigroups. As a continuation of the work of M. Klawe, we investigate in detail the left amenability and the Følner number of a semidirect product of two semigroups. In particular, we give necessary and sufficient conditions for a semidirect product to be left amenable and to have Følner number zero.

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Preliminaries

I.1. Introduction.

Let G be a (discrete) group and $\ell^\infty(G)$ the Banach space of all bounded real-valued functions on G with the supremum norm. An element μ of $\ell^\infty(G)^*$ is called a mean on $\ell^\infty(G)$ if μ is positive and $\|\mu\| = 1$. For each s in G, we define an operator ℓ_s on $\ell^\infty(G)$ by $\ell_s f(t) = f(st)$, for $f \in \ell^\infty(G)$ and $t \in G$. A mean μ on $\ell^\infty(G)$ is called left invariant if for any $s \in G$ and any $f \in \ell^\infty(G)$, $\mu(\ell_s f) = \mu(f)$. If G admits a left invariant mean on $\ell^\infty(G)$, we say that G is (left) amenable.

This subject originates from the study of Hausdorff [18], Banach [2], and Banach and Tarski [3] on the existence of finitely additive measures on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ which are invariant under all translations and rotations. (The answer, incidentally, is yes in the cases of $\mathbb{R}$ and $\mathbb{R}^2$ and no in the case of $\mathbb{R}^3$.) In 1929, von Neumann [38] made a systematic study of amenable groups. He proved that any solvable group is amenable, and that the free group on two generators is not amenable. Since then, two major generalizations have been made: to semigroups and to locally compact groups. Extensions to semigroups were obtained by Dixmier [11] and Day [8]. General properties of left amenable semigroups are surveyed in [8] and [10]. The early works on locally compact amenable groups are [33] and [21]. Greenleaf [17] is a general reference, and the new comprehensive treatises of Pier [32] and Paterson [31] contain much more up-to-date material on the subject.

In this thesis we investigate two aspects of amenable groups and semigroups. The first part deals with the structure of the set $ML(S)$ of all left invariant means for a left amenable semigroup S and the set $MTL(G)$ of all topological left invariant means for a locally compact amenable group G. Among other things, we are able to obtain the exact cardinality of these sets. The second part concerns Følner numbers and Følner-type conditions for left amenable semigroups. We answer a problem of Namioka on the necessity of some Følner-type conditions for a semigroup to be left amenable.

Chapter I contains some definitions and basic properties of amenable groups and semigroups that we need in this thesis.

We begin Chapter II with a result on the exposed points of the set $ML(S)$ which generalizes results in Chou [4] and Granirer [16]. We prove that the set $ML(S)$ has exposed points if and only if the semigroup S has finite left ideals. The remainder of the chapter is mainly devoted to the study of the cardinality of the set of invariant means. For a non-compact locally compact amenable group, we show that there are $2^{2^{d(G)}}$ topological invariant and inversion invariant means on $L^\infty(G)$, where $d(G)$ denotes the smallest cardinality of a cover of G by compact sets. For a left amenable semigroup S, we define the left thickness $\tau(S)$ of S to be the greatest cardinality of a strongly left thick subset of S, and we prove that $|ML(S)| = 2^{2^{\tau(S)}}$. We also show that $\tau(S)$ is actually equal to the cardinals introduced by Klawe [24] and Paterson [30].

In Chapter III we first summarize the known relations among the various

Følner-type conditions and introduce the Følner number $\varphi(S)$ for a semigroup S. Then we investigate general properties of $\varphi(S)$ and determine $\varphi(S)$ completely for all finite and cancellative semigroups. We also relate $\varphi(S)$ to the cancellation behavior of S by some combinatorial arguments. With the aid of these tools and the amenability results of semidirect products of two semigroups obtained in Klawe [23], we show that none of the Følner-type conditions given by Namioka [29] is necessary for the left amenability of a semigroup. We also obtain necessary and sufficient conditons for a semidirect product to be left amenable or to have Følner number 0.

I.2. Left Amenable Semigroups.

For an arbitrary set X, let $\ell^\infty(X)$ be the Banach space of all bounded real-valued functions on X with the supremum norm. An element $\mu \in \ell^\infty(X)^*$ is called a mean on $\ell^\infty(X)$ if μ is positive and $\|\mu\| = 1$. A countable mean on X is a positive element $\mu \in \ell^1(X)$ with $\|\mu\|_1 = 1$. A countable mean μ is a finite mean if its support, the set $\{x \in X \mid \mu(x) > 0\}$, is finite. Any countable mean, considered as an element of $\ell^\infty(X)^*$, is a mean; and the set of all finite means is w^*-dense in the set of all means on $\ell^\infty(X)$ (see Day [8]).

Let S be a semigroup. A mean μ on $\ell^\infty(S)$ is called left invariant if $\mu(f) = \mu(\ell_s f)$ for all $f \in \ell^\infty(S)$ and $s \in S$, where $\ell_s f \in \ell^\infty(S)$ is defined by $\ell_s f(t) = f(st)$, $t \in S$. When $\ell^\infty(S)$ has a left invariant mean, we say S is left amenable, and denote by $ML(S)$ the set of all left invariant means on $\ell^\infty(S)$. $ML(S)$ is convex and w^*-compact in $\ell^\infty(S)^*$ (cf. [8]).

For a mean μ on $\ell^\infty(S)$ and $s \in S$, we define $s \cdot \mu \in \ell^\infty(S)^*$ by $(s \cdot \mu)f = \mu(\ell_s f)$, $f \in \ell^\infty(S)$. $s \cdot \mu$ is also a mean on $\ell^\infty(S)$, and $(st) \cdot \mu = s \cdot (t \cdot \mu)$ for $s, t \in S$. Every left invariant mean can be expressed as the limit of a net $\{\mu_\lambda\}$ of finite means, and the net $\{\mu_\lambda\}$ satisfies the condition that $s \cdot \mu_\lambda - \mu_\lambda \to 0$ for each s in S in the w^*-topology of $\ell^\infty(G)^*$. A net of finite means with this property is called w^*-convergent to left invariance. Similarly, one can define the convergence in norm to left invariance. Day [8] proved that S is left amenable if and only if there exists a net of finite means w^*-convergent to left invariance. The following lemma guarantees the existence of a net convergent in norm to left invariance.

LEMMA 1.2.1: *Let S be a left amenable semigroup, $\{\mu_\alpha\}_{\alpha \in \Gamma}$ a net of finite means w^*-convergent to left invariance. Then, for any $\alpha \in \Gamma$, any $\varepsilon > 0$, and any $s_1, \ldots, s_n \in S$, there exists a finite mean μ'_α which is a convex combination of elements μ_β, $\beta > \alpha$, such that*

$$\|s_i \cdot \mu'_\alpha - \mu'_\alpha\| < \varepsilon, \quad i = 1, \ldots, n.$$

A proof of Lemma 1.2.1 can be found in Day [8, p. 524].

For subsets A, B of S and $s \in S$, we define $A \cdot B = \{uv : u \in A \text{ and } v \in B\}$, $sA = \{su : u \in A\}$ and $s^{-1}A = \{u \in S : su \in A\}$. We denote $A \cdot A$ by A^2, and so on. χ_A is used to denote the characteristic function of A, and $|A|$ stands for the cardinality of A. $A \backslash B$ denotes the difference set and $A \triangle B$, the symmetric difference of A and B.

When μ is a mean on S, we write $\mu(A)$ for $\mu(\chi_A)$. If $\mu \in ML(S)$ and $s \in S$, we have $\mu(s^{-1}A) = \mu(A)$ since $\ell_s \chi_A = \chi_{s^{-1}A}$, and $\mu(sA) \geq \mu(A)$ since

$s^{-1}(sA) \supset A$. Granirer [14] noticed that since $\mu(sS) = 1$ for $\mu \in ML(S)$, the intersection of finitely many right ideals in S is always nonempty.

The class of left amenable semigroups includes all locally finite groups, solvable groups, and abelian semigroups (see Greenleaf [17] or Hewitt and Ross [19, Section 17]). While all finite groups are left amenable, it is easy to see that a finite semigroup is left amenable if and only if it contains a unique minimal right ideal (see [34]).

Homomorphic images of left amenable semigroups are still left amenable. Also, any subgroup of a left amenable group is left amenable. However, a subsemigroup of a left amenable group need not be left amenable, as shown in Hochster [20]. More generally, we have the following result due to Frey (see Pier [32, Prop. 23.32]).

PROPOSITION 1.2.2. *Let G be a left amenable group and S a subsemigroup of G. Then S is left amenable if and only if S satisfies the finite intersection property for right ideals.*

An important analytic application of left amenable semigroups is their fixed-point property. It appeared first in Day [9].

THEOREM 1.2.3. *Suppose S is a left amenable semigroup of affine mappings on a compact convex subset K of a locally convex space. Then S has a common fixed point in K.*

I.3. Thick Sets and Almost Convergent Functions.

Let A be a subset of a semigroup S. We say that A ~~is left~~ thick in S if for every finite subset F of S, there exists an $s \in S$, such that $Fs \subset A$. Clearly a left ideal of S is left thick. Mitchell [28] obtained the following characterization of left thick subsets.

THEOREM 1.3.1. *If S is a left amenable semigroup, then a subset A of S is left thick in S if and only if there exists $\mu \in ML(S)$ with $\mu(A) = 1$.*

A subset A of S is strongly left thick if for each $B \subset S$ with $|B| < |A|$, the set $A \backslash B$ is left thick in S (see Klawe [22]). A semigroup S is said to be right [left] cancellative if whenever $rs = ts$ [$sr = st$] we have $r = t$. Klawe [22] proved that a right or left cancellative left amenable semigroup is strongly left thick in itself.

A function $f \in \ell^\infty(S)$ is left almost convergent to 1 if for any $\mu \in ML(S)$, $\mu(f) = 1$. We give the following characterization of functions left almost convergent to 1. A proof can be found in Day [10, p. 31].

PROPOSITION 1.3.2. *If S is a left amenable semigroup and $f \in \ell^\infty(S)$, then f is left almost convergent to 1 if and only if for any $\varepsilon > 0$, there exists a finite mean μ such that*

$$\inf_{t\in S}\left\{\sum_s \mu(s)\ell_s f(t)\right\} > 1 - \varepsilon,$$

where the sum is taken for all $s \in S$ with $\mu(s) \neq 0$.

I.4. Locally Compact Amenable Groups.

Let G be a locally compact group, and $L^\infty(G)$ the Banach space of all essentially bounded real-valued Borel measurable functions with respect to the left Haar measure. Two important subspaces of $L^\infty(G)$ are $CB(G)$, the space of all bounded continuous functions, and $UCB(G)$, the space of all bounded uniformly continuous functions.

For a function f defined on G and $s \in G$, we define $\tilde{f}$ by $\tilde{f}(t) = f(t^{-1})$, f^* by $f^*(t) = f(t^{-1})\Delta(t^{-1})$, ${}_sf$ by ${}_sf(t) = f(s^{-1}t)$, and f_s by $f_s(t) = f(ts^{-1})$, where Δ is the modular function of G. For functions f and g defined on G, we define the convolution

$$(g * f)(s) = \int_G g(t)f(t^{-1}s)dt, \quad \forall s \in G.$$

When $f \in L^\infty(G)$ and $g \in L^1(G)$, $g * f$ and $f * \tilde{g}$ are well defined and belong to $L^\infty(G)$. Also we have $(g * f)^* = f^* * g^*$ (see [19]).

A mean on $L^\infty(G)$ is a positive element of norm 1 in $L^\infty(G)^*$. A left invariant mean on $L^\infty(G)$ is a mean μ such that $\mu({}_sf) = \mu(f)$ for all $f \in L^\infty(G)$ and $s \in G$. A topological left invariant mean is a mean μ such that $\mu(g * f) = \mu(f)$ for all $f \in L^\infty(G)$ and $g \in L^1(G)$ with $g \geq 0$ and $\|g\|_1 = 1$. The topological right invariance of a mean μ on $L^\infty(G)$ can be defined as $\mu(f * \tilde{g}) = \mu(f)$. A mean μ is inversion invariant if $\mu(\tilde{f}) = \mu(f)$ for all $f \in L^\infty(G)$. If G admits a topological left invariant mean on $L^\infty(G)$, we say that G is amenable.

It is known that every topological left invariant mean on $L^\infty(G)$ is left invariant, and for each left invariant mean μ on $L^\infty(G)$, there exists a topological left invariant mean μ_1 on $L^\infty(G)$ which coincides with μ on $UCB(G)$. If G is left amenable as a discrete group, then it is amenable. If G is amenable, then there exist topological (two-sided) invariant and inversion invariant means on $L^\infty(G)$. The proofs of these facts can be found in [17] or [32].

The set of all topological left invariant means on $L^\infty(G)$ is denoted by $MTL(G)$. The set of all topological invariant means and the set of all topological invariant inversion invariant means are denoted by $MT(G)$ and $MT^*(G)$, respectively. Each of these three sets is w^*-compact and convex in $L^\infty(G)^*$.

Let $C_{00}(G)$ be the subspace of $L^1(G)$ consisting of all continuous functions with compact support. A net $\{\mu_\lambda\}$ of means on $L^\infty(G)$ defined by functions in $C_{00}(G)$ is said to be convergent to topological left [right] invariance if $g*\mu_\lambda - \mu_\lambda \to 0$ $[\mu_\lambda * \tilde{g} - \mu_\lambda \to 0]$ for all $g \in L^1(G)$ with $g \geq 0$ and $\|g\|_1 = 1$. The following is an analogue of Lemma 1.2.1 (see [8, pp. 523-524] and [17, p. 34]).

LEMMA 1.4.1. *Let G be a locally compact amenable group, $\{\mu_\alpha\}_{\alpha\in\Gamma}$ a net of means defined by functions in $C_{00}(G)$ w^*-convergent to topological left invariance. Then for any $\alpha \in \Gamma$, any $\varepsilon > 0$, and any $g_1, \ldots, g_n \in L^1(G)$ with $g_i \geq 0$, $\|g_i\|_1 = 1$, there exists a mean μ'_α which is a finite convex combination of means μ_β, $\beta > \alpha$, such that*

$$\|g_i * \mu'_\alpha - \mu'_\alpha\| < \varepsilon, \quad i = 1, \ldots, n.$$

The right hand side and two-sided version of Lemma 1.4.1 also remain valid.

CHAPTER II

THE SET OF INVARIANT MEANS

II.1. Introduction.

The study of the cardinality and the structure of the set of invariant means was initiated by Day [8] and Granirer [14]. Most of their work concerns the size of this set. Following the work of Luther [27] and Granirer [14], Klawe [22] finally settled the uniqueness problem for left invariant means on a semigroup. She proved that a left amenable semigroup S has a unique left invariant mean if and only if S contains a unique finite left ideal group. In 1976, Chou [5] proved that for a discrete infinite amenable group G, the cardinality of the set $ML(G)$ is $2^{2^{|G|}}$. Later, Klawe [24] and Paterson [30] obtained various results regarding the size of the set $ML(S)$ for a left amenable semigroup S. Subsequently, Lau [25] and Lau and Paterson [26] solved the uniqueness problem and the cardinality problem for the set $MTL(G)$ for a locally compact amenable group G.

For the study of the local structure of the w^*-compact convex set $ML(S)$ or $MTL(G)$, it is natural to look for exposed points or G_δ-points (with respect to the w^*-topology) of the set. More generally, we may ask for the size of a smallest neighborhood base for a left invariant mean. Chou [4] proved that if G is a countably infinite amenable group, then $ML(G)$ has no exposed points. Granirer [16] made an intensive study of the structure of subsets of $ML(S)$ for a countable left amenable semigroup S. In particular, he showed that if S is a countable left amenable semigroup, then $ML(S)$ has exposed points (and/or G_δ-

points) if and only if S has finite left ideals. Chou [6] obtained similar results for σ-compact amenable groups. Cf. [V.L. Klee, Jr., Extremal structure of convex sets, II, *Math. Z.* 69 (1958), 90–104] for the definition and basic properties of exposed points.

In this chapter, by comparing the size of nets of means convergent to left invariance and their cluster points, we investigate the cardinality and the geometric structure of the sets $ML(S)$ and $MTL(G)$.

In Section II.2 we characterize the exposed points of $ML(S)$ for an arbitrary left amenable semigroup S as the arithmetic average on minimal finite left ideals. Thus we are able to prove Chou's and Granirer's results without the countability condition.

In Section II.3 we prove an embedding theorem for locally compact groups. If G is a noncompact locally compact group, let $d(G)$ be the smallest cardinality of a cover by compact sets. Then a set of cardinality $2^{2^{d(G)}}$ can be embedded into the set $MT^*(G)$ of all topological invariant and inversion invariant means on $L^\infty(G)$. This improves the result obtained by Lau and Paterson [26].

In Section II.4 we apply the same technique to left amenable semigroups. For a left amenable semigroup S, we define the cardinal

$$\tau(S) = \sup\{|A| : A \subset S \quad \text{is strongly left thick}\}$$

and prove that $|ML(S)| = 2^{2^{\tau(S)}}$. We also show that $\tau(S)$ is in fact equal to Klawe's cardinal

$$\kappa(S) = \min\{|B| : B \subset S, \quad \mu(B) = 1, \quad \forall \mu \in ML(S)\}$$

and Paterson's cardinal

$$\rho(S) = \min\left\{\left|\bigcup_{i=1}^{n} s_i S_i\right| : \{S_1, \ldots, S_n\} \text{ is a partition of } S, \quad s_1, \ldots, s_n \in S\right\}.$$

Some incomplete descriptions of the structure of $ML(S)$ are given in Section II.5. We prove a decomposition theorem for left invariant means on $\ell^\infty(S)$. We also obtain an estimate for the smallest cardinality of a w^*-neighborhood base of a left invariant mean μ in the set $ML(S)$.

II.2. Exposed Points of Left Invariant Means.

In this section, we first prove some lemmas which will be used later, and which are of independent interest as well. Following these lemmas, we establish the main theorems concerning the exposed points of $ML(S)$.

LEMMA 2.2.1. *Let X be an infinite set, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ a net of finite means w^*-convergent to a mean μ. Let κ be an infinite cardinal. If for each subset A of X, $\mu(A) = 0$ whenever $|A| < \kappa$, then $|\Lambda| > \kappa$.*

Proof. Suppose $|\Lambda| = \kappa$. We seek to construct a function $f \in m(X)$ such that $\mu_\lambda(f)$ diverges.

Well order Λ as $\{\lambda_\alpha\}_{\alpha<\kappa}$. We define f by transfinite induction.

Let $\alpha < \kappa$ be an ordinal. Suppose we have defined for each $\beta < \alpha$ a function f_β with range $\{0,1\}$ on a subset A_β of X, satisfying

(1) If β is finite, then A_β is finite. If β is infinite, then $|A_\beta| \le |\beta|$.

(2) $\beta_1 < \beta_2 < \alpha \Rightarrow A_{\beta_1} \subset A_{\beta_2}$ and $f_{\beta_2} \restriction A_{\beta_1} = f_{\beta_1}$.

(3) If $\beta < \alpha$, then there exists $\lambda', \lambda'' > \lambda_\beta$ in Λ, such that the supports of $\mu_{\lambda'}$ and $\mu_{\lambda''}$ are contained in A_β, and $\mu_{\lambda'}(f_\beta) < 1/4$; $\mu_{\lambda''}(f_\beta) > 3/4$.

If α is finite, then $\bigcup_{\beta<\alpha} A_\beta$ is finite. If α is infinite, then $|\bigcup_{\beta<\alpha} A_\beta| \le |\alpha|^2 = |\alpha|$. In both cases $\mu(\bigcup_{\beta<\alpha} A_\beta) = 0$. $\mu_\lambda \xrightarrow{w^*} \mu$ implies that there exists $\lambda' > \lambda_\alpha$ in Λ, such that $\mu_{\lambda'}(X \backslash \bigcup_{\beta<\alpha} A_\beta) > 3/4$. Also since $|\bigcup_{\beta<\alpha} A_\beta \cup \operatorname{supp} \mu_{\lambda'}| < \kappa$, there exists $\lambda'' > \lambda_\alpha$ in Λ such that

$$\mu_{\lambda''}\Big(X \backslash \Big(\bigcup_{\beta<\alpha} A_\beta \cup \operatorname{supp} \mu_{\lambda'}\Big)\Big) > 3/4.$$

Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup \operatorname{supp} \mu_{\lambda'} \cup \operatorname{supp} \mu_{\lambda''}$, and define

$$f_\alpha(S) = \begin{cases} f_\beta(s), & \text{if } s \in A_\beta \text{ for some } \beta < \alpha, \\ 0, & \text{if } s \in \operatorname{supp} \mu_{\lambda'} \backslash \bigcup_{\beta<\alpha} A_\beta, \\ 1, & \text{if } s \in A_\alpha \backslash \Big(\bigcup_{\beta<\alpha} A_\beta \cup \operatorname{supp} \mu_{\lambda'}\Big). \end{cases}$$

It is easy to see that A_α and f_α satisfy conditions (1)–(3).

Now let $f = f_\alpha$ on A_α, $\alpha < \kappa$, and $f = 0$ on $X \backslash \bigcup_{\alpha<\kappa} A_\alpha$. Then $\mu_\lambda(f)$ diverges. In fact

$$\liminf \mu_\lambda(f) \le \frac{1}{4} < \frac{3}{4} \le \limsup \mu_\lambda(f).$$

□

COROLLARY 2.2.2. *Let S be an infinite left amenable right cancellative semigroup, $\{\mu_\lambda\}_{\lambda\in\Lambda}$ a net of finite means w^*-convergent to a left invariant mean μ. Then $|\Lambda| > |S|$.*

Proof. Let $A \subset S$ be such that $|A| < |S|$. Then it is not difficult to see that $\mu(A) = 0$. A proof can be found in [22, Prop. 2.5]. □

LEMMA 2.2.3. *Let S be an infinite left amenable semigroup, μ an extreme point of $ML(S)$. Define the cardinal function $\kappa(\mu) = \min\{|A| : A \subset S \text{ and } \mu(A) = 1\}$. If $\kappa(\mu)$ is infinite, then for each subset B of S, $|B| < \kappa(\mu)$ implies $\mu(B) = 0$.*

Proof. Suppose to the contrary that there is a set $B \subset S$ such that $|B| < \kappa(\mu)$ and $\mu(B) > 0$.

If B is finite, then there is an $s \in S$ with $\mu(\{s\}) > 0$. For any $t \in S$, $\mu(\{ts\}) \geq \mu(\{s\})$. So the left ideal $I = Ss$ of S is finite, and $0 < \mu(I) < 1$ since $\kappa(\mu)$ is infinite. For any $t \in S$, $\mu(tI) \geq \mu(I)$ and $tI \subset I$ imply that $\mu(tI) = \mu(I)$ and $\mu(I \triangle tI) = 0$.

Suppose now B is infinite. Let $x = \sup\{\mu(A) : A \subset S, |A| \leq |B|\}$. By taking a countable union, we can get a subset I of S such that $|I| = |B|$ and $\mu(I) = x$. For any $t \in S$, $\mu(I) \leq \mu(tI) \leq \mu(tI \cup I) \leq x = \mu(I)$ since $|I \cup tI| = |I| = |B|$. So equalities hold everywhere. Thus we also have $0 < \mu(I) < 1$, $\mu(I) = \mu(tI)$ and $\mu(I \triangle tI) = 0$. Denote by $t^{-1}(tI)$ the set $\{s \in S : ts \in tI\}$. It is easy to see that $\mu(I \triangle t^{-1}(tI)) = 0$, since $t^{-1}(tI) \supset I$ and $\mu(t^{-1}(tI)) = \mu(tI) = \mu(I)$.

Let $\mu_1 \in \ell^\infty(S)^*$ be defined by

$$\mu_1(f) = \frac{\mu(f \cdot \chi_I)}{\mu(I)}, \quad f \in \ell^\infty(S).$$

Then μ_1 is positive, of norm 1, and left invariant:

$$\begin{aligned}\mu_1(\ell_t f) &= \frac{\mu((\ell_t f) \cdot \chi_I)}{\mu(I)} = \frac{\mu((\ell_t f) \cdot \chi_{t^{-1}(tI)})}{\mu(I)} \\ &= \frac{\mu(\ell_t(f \cdot \chi_{tI}))}{\mu(I)} = \frac{\mu(f \cdot \chi_{tI})}{\mu(I)} = \frac{\mu(f \cdot \chi_I)}{\mu(I)} = \mu_1(f),\end{aligned}$$

since $\mu(I \triangle t^{-1}(tI)) = 0$ and $\mu(I \triangle tI) = 0$.

Let $\mu_2 = (\mu - \mu(I) \cdot \mu_1)/(1 - \mu(I))$. Then for $f \in \ell^\infty(S)$,

$$\mu_2(f) = \frac{\mu(f) - \mu(f \cdot \chi_I)}{1 - \mu(I)} = \frac{\mu(f \cdot \chi_{S\setminus I})}{\mu(S\setminus I)}.$$

So μ_2 is also in $ML(S)$, and

$$\mu = \mu(I)\mu_1 + (1 - \mu(I))\mu_2$$

is not an extreme point. □

We are now ready to prove our main results. In all cases we shall consider only the w^*-topology on $ML(S)$.

THEOREM 2.2.4. *Let S be a left amenable semigroup, and μ an exposed point of $ML(S)$ (if any). Then μ is a finite mean.*

Proof. Let μ be an extreme point of $ML(S)$ and define $\kappa(\mu)$ as in Lemma 2.2.3. Suppose μ is not a finite mean. Then $\kappa(\mu)$ is infinite. Take $A \subset S$ so that $|A| = \kappa(\mu)$ and $\mu(A) = 1$. Then for any $t \in S$, $\mu(tA \cap A) = 1$ since $\mu(tA) = 1$. In particular $tA \cap A \neq \varnothing$; i.e., there exist $a, b \in A$ with $ta = b$. For fixed $a, b \in A$, let $S_{(a,b)} = \{t \in S : ta = b\}$. Then $\bigcup\{S_{(a,b)} : a, b \in A\} = S$.

Pick $f \in \ell^\infty(S)$ with $\|f\| = 1$, and choose a net $\{\mu_\alpha\}_{\alpha\in\Gamma}$ of finite means w^*-convergent to μ. Then $\{\mu_\alpha\}_{\alpha\in\Gamma}$ is w^*-convergent to left invariance and $\mu_\alpha(f) \to \mu(f)$.

Let Λ be the set of all finite nonempty subsets of $A \times A$, directed by inclusion. Then Λ is a directed set with $|\Lambda| = |A| = \kappa(\mu)$. Take $F = \{(a_i, b_i) : i = 1, \ldots, n\} \in \Lambda$. There exists $\alpha \in \Gamma$ such that for any $\beta > \alpha$, $|\mu_\beta(f) - \mu(f)| < 1/2n$. By the finite intersection property for right ideals, $\bigcap_{i=1}^{n} a_i S \neq \emptyset$. Choose $a \in \bigcap_{i=1}^{n} a_i S$ (a is not necessarily in A), say $a = a_i s_i$, $i = 1, \ldots, n$. By Lemma 1.2.1, there exists a finite mean μ'_α which is a convex combination of elements μ_β, $\beta > \alpha$, such that

$$\|a \cdot \mu'_\alpha - \mu'_\alpha\| < \frac{1}{2n}$$

and

$$\|(b_i s_i) \cdot \mu'_\alpha - \mu'_\alpha\| < \frac{1}{2n}, \quad i = 1, \ldots, n.$$

For $t \in S_{(a_i, b_i)}$, we have

$$\|t \cdot (a \cdot \mu'_\alpha) - a \cdot \mu'_\alpha\| = \|(b_i s_i) \cdot \mu'_\alpha - a \cdot \mu'_\alpha\| < \frac{1}{n}.$$

Also

$$|(a \cdot \mu'_\alpha)(f) - \mu(f)| < \frac{1}{n}.$$

Define $\mu_F = a \cdot \mu'_\alpha$. Then the net $\{\mu_F\}_{F \in \Lambda}$ is w^*-convergent to left invariance and $\lim \mu_F(f) = \mu(f)$. Since μ is an extreme point of $ML(S)$, by Lemma 2.2.3, for any $B \subset S$, $|B| < \kappa(\mu)$ implies $\mu(B) = 0$. By Lemma 2.2.1, $\{\mu_F\}_{F \in \Lambda}$ does not converge to μ since $|\Lambda| = |A| = \kappa(\mu)$. So $\{\mu_F\}_{F \in \Lambda}$ has a w^*-cluster point μ_1 different from μ. Since $\mu_1 \in ML(S)$ and $\mu_1(f) = \mu(f)$, μ is not an exposed point of $ML(S)$. □

For a finite nonempty set $I \subset S$, the arithmetic average on I is the finite mean μ such that for each $a \in I$, $\mu(\{a\}) = 1/|I|$.

THEOREM 2.2.5. *Let S be a left amenable semigroup. Then μ is an exposed point of $ML(S)$ if and only if it is the arithmetic average on a minimal finite left ideal of S.*

Proof. Let I be a minimal finite left ideal of S. By Theorem 1.3.1 there exists $\mu \in ML(S)$ with $\mu(I) = 1$. Since $Ia = I$ for any $a \in I$, I is right cancellative. Also $\mu(aI) = \mu(I)$ implies that $aI = I$ for any $a \in S$. Thus I is left cancellative and in fact a finite group. μ, as the unique invariant mean on I, is the arithmetic average on I. Let f be the characteristic function of I. Then $\mu(f) = 1$. For any $\mu_1 \in ML(S)$, if $\mu_1(f) = \mu_1(I) = 1$, then by the above argument, $\mu = \mu_1$. Thus μ is an exposed point of $ML(S)$. (Remark: Part of the proof is adopted from [14, Thm. 4.1].)

Suppose μ is an exposed point of $ML(S)$. Then μ is a finite mean by Theorem 2.2.4. Let I be the support of μ. For $a \in I$ and $t \in S$, $\mu(\{ta\}) \geq \mu(\{a\}) > 0$, so $ta \in I$. Thus I is a left ideal and it contains a minimal left ideal I_1. If $I \neq I_1$, then as in the proof of Lemma 2.2.3 we have $0 < \mu(I_1) < 1$ and $\mu(I_1 \Delta t I_1) = 0$ for any $t \in S$. These ensure that μ is not an extreme point of $ML(S)$. So I must be a minimal finite left ideal. By the proof of the first part of the theorem, μ is the arithmetic average on I. □

COROLLARY 2.2.6. *For any left amenable semigroup S, $ML(S)$ has exposed points if and only if S has finite left ideals. The number of exposed points of $ML(S)$ is exactly the number of minimal finite left ideals of S.*

COROLLARY 2.2.7. *If S is a right cancellative left amenable infinite semigroup,* *then $ML(S)$ has no exposed points.*

Proof. For any $s \in S$, $|Ss| = |S|$. So S does not have finite left ideals. □

COROLLARY 2.2.8. *If* $\dim(ML(S)) < \infty$, *then S has finite left ideals.*

Proof. If $\dim(ML(S)) < \infty$, then $ML(S)$ is a compact convex subset of a Banach space. So it has exposed points. □

COROLLARY 2.2.9. *Different exposed points of $ML(S)$ are linearly independent.*

Corollary 2.2.6 extends [16, Cor. 4.1]. Corollary 2.2.8 is the main result of Klawe [22].

Suppose S is an infinite left amenable semigroup and K is an invariant subset of βS. Let $M(S,K)$ denote the set of all $\mu \in ML(S)$ with its support contained in K (see [4] for the definitions). Chou [4] proved that if G is a countably infinite amenable group, then $M(G,K)$ has no exposed points. He asked whether this holds for any infinite amenable group. Our Corollary 2.2.7 gives a partial answer to this problem with $K = \beta G$.

Motivated by Granirer [14, Thm. 3.1], we obtain the following generalization.

THEOREM 2.2.10. *If $ML(S)$ has exposed points, then it is the w^*-closed convex hull of all its exposed points.*

Proof. Suppose $ML(S)$ has exposed points. Then S has finite left ideals. Let $\{I_\alpha\}$ be the class of all its minimal finite left ideals and $A = \bigcup I_\alpha$. Then A is a right ideal of S since for any $s \in S$, $I_\alpha s$ is also a minimal left ideal. For any

$\mu \in ML(S)$, $\mu(A) = 1$. Thus μ is the w^*-limit of a net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ of finite means with supports in A. For each $\lambda \in \Lambda$, define

$$\mu'_\lambda = \sum_\alpha \mu_\lambda(I_\alpha)\varphi_\alpha,$$

where φ_α is the arithmetic average on I_α. Then μ'_λ is a convex combination of some φ_α. Take a minimal finite left ideal $I_0 = \{a_1, \ldots, a_n\}$. For any I_α and any $a \in I_\alpha$, it is easy to see that $\sum_{i=1}^n a_i \cdot \mu_\lambda(a) = \mu_\lambda(I_\alpha)$. So $\mu'_\lambda = n^{-1}\sum_{i=1}^n a_i \cdot \mu_\lambda$. Since $\{\mu_\lambda\}$ converges to left invariance, it follows that $\{\mu'_\lambda\}$ converges to μ in the w^*-topology. □

COROLLARY 2.2.11. *(Granirer-Klawe Theorem. See [22].) For any left amenable semigroup S, $\dim(ML(S)) = n$ if and only if S contains exactly n minimal finite left ideals.*

Proof. If S has n minimal finite left ideals, then $ML(S)$ has n exposed points. By Corollary 2.2.9, $\dim(ML(S)) \geq n$. By Theorem 2.2.10, $ML(S)$ is the convex hull of those exposed points. So $\dim(ML(S)) = n$.

On the other hand, if $\dim(ML(S)) = n$, by Corollary 2.2.8, S has finite left ideals. Again by Corollary 2.2.9, S has only finitely many minimal finite left ideals. That this number is n follows readily from the proof of the first part. □

Some more results on the structure of $ML(S)$ will be presented in Section II.5.

II.3. The Set $MTL(G)$.

In this section, we consider the problem of embedding a large set (both in the topological as well as the set theoretical sense) into the set $MTL(G)$ of all topological left invariant means. Throughout this section G will be a noncompact locally compact amenable group. The set Φ we define here is also used in the next section.

Suppose Λ is a directed set, and $\ell^\infty(\Lambda)$ is the Banach space of all bounded real-valued functions on Λ, with the supremum norm. $\ell^\infty(\Lambda)$ can be considered as the set of all bounded nets in $\mathbb{R}$ with directed set Λ. Define

$$\Phi = \{\varphi \in \ell^\infty(\Lambda)^* : \varphi(x) \le \limsup_{\lambda\in\Lambda} x(\lambda), \quad \forall x \in \ell^\infty(\Lambda)\}.$$

An equivalent condition for $\varphi \in \Phi$ is that φ is positive, $\|\varphi\| = 1$, and $\varphi(x) = \lim_{\lambda\in\Lambda} x(\lambda)$ if the limit exists.

Chou [6] considered the relation between the set Φ when $\Lambda = \mathbb{N}$ and the set of all topological invariant means on the von Neumann algebra $VN(G)$ of a first countable locally compact group G. In our applications Λ is always the set of all nonempty finite subsets of some infinite set X, directed by inclusion. We denote it by $\Lambda(X)$.

LEMMA 2.3.1. *If $\Lambda = \Lambda(X)$ for an infinite set X, then $|\Phi| = 2^{2^{|X|}}$.*

Proof. The proof follows Rudin [36, Thm. 1.3]. The Stone-Čech compactification $\beta\Lambda$ of Λ can be considered as a subset of $\ell^\infty(\Lambda)^*$. For $\lambda \in \Lambda$, define $S_\lambda = \{\lambda' \in \Lambda : \lambda \subset \lambda'\}$. Then $\{S_\lambda\}_{\lambda\in\Lambda}$ is a filter base on Λ. For $\varphi \in \beta\Lambda$, $\varphi \in \Phi$

if φ contains $\{S_\lambda\}_{\lambda\in\Lambda}$. Since $|\Lambda| = |X|$, it suffices to show that there are $2^{2^{|X|}}$ ultrafilters on Λ containing $\{S_\lambda\}_{\lambda\in\Lambda}$.

Since $|S_\lambda| = |X|$ for each $\lambda \in \Lambda$, we can define inductively for each $\lambda \in \Lambda$ a finite subset A_λ of S_λ, such that $|A_\lambda| = 2^{2^{|\lambda|}}$, and $\lambda \neq \lambda' \Rightarrow A_\lambda \cap A_{\lambda'} = \emptyset$. Label the elements of A_λ by ordered $2^{|\lambda|}$-tuples $(x_1, x_2, \ldots, x_{2^{|\lambda|}})$, where $x_i = 0$ or 1. Let E_i be the subset of A_λ consisting of the $2^{|\lambda|}$-tuples which have $x_i = 0$. Thus if we let $E_i^0 = E_i$, $E_i^1 = A_\lambda \backslash E_i$, then $\bigcap_{i=1}^{2^{|\lambda|}} E_i^{\varepsilon_i} \neq \emptyset$ for any choice $\varepsilon_i \in \{0,1\}$. Denote the sets E_i, $i = 1, 2, \ldots, 2^{|\lambda|}$, by $E(h)$, where h is a map from λ into $\{0,1\}$.

For each map $f : X \to \{0,1\}$, define $B(f) = \bigcup\{E(f \restriction \lambda) : \lambda \in \Lambda\}$. Suppose $f_1, \ldots, f_n, f_{n+1}, \ldots, f_m$ are different functions from X into $\{0,1\}$, and $\lambda \in \Lambda$. Then there exists $\lambda' \supset \lambda$ such that the restrictions $f_i \restriction \lambda'$ are different. So

$$E(f_1 \restriction \lambda') \cap \cdots \cap E(f_n \restriction \lambda') \cap (A_{\lambda'} \backslash E(f_{n+1} \restriction \lambda'))$$

$$\cap \cdots \cap (A_{\lambda'} \backslash E(f_m \restriction \lambda')) \neq \emptyset.$$

It follows that

$$B(f_1) \cap \cdots \cap B(f_n) \cap (\Lambda \backslash B(f_{n+1}))$$

$$\cap \cdots \cap (\Lambda \backslash B(f_m)) \cap S_\lambda \neq \emptyset.$$

And hence for any map $F : 2^X \to \{0,1\}$, the collection

$$\{B(f)^{F(f)} : f \in 2^X\} \cup \{S_\lambda : \lambda \in \Lambda\},$$

where $B(f)^0 = B(f)$ and $B(f)^1 = \Lambda \backslash B(f)$, is a filter base. Thus we have $2^{2^{|X|}}$ different ultrafilters containing $\{S_\lambda\}_{\lambda\in\Lambda}$. □

Now we deal with the embedding of the set Φ into the set $MTL(G)$ of all topological left invariant means. Our main results concern the cardinality of $MT^*(G)$, $MT(G)\backslash MT^*(G)$, and $MTL(G)\backslash MT(G)$.

Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be a net of means on $L^\infty(G)$ defined by functions in $C_{00}(G)$, and suppose $\{\mu_\lambda\}_{\lambda\in\Lambda}$ converges to topological left invariance and the family $\{\text{supp } \mu_\lambda\}$ is discrete in G; i.e., for any $s \in G$, there is a neighborhood U of s, meeting at most one set in the family. Let H be the w^*-closed convex hull of the set of all w^*-cluster points of $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in $L^\infty(G)^*$, and Φ defined for the directed set Λ as above. Then H is a nonempty w^*-compact subset of $MTL(G)$. The following lemma, which we shall need, is a generalization of Theorem 3.3 in Chou [6]:

LEMMA 2.3.2. *Let Φ and H be defined as above. Then there exists a linear isometry of $\ell^\infty(\Lambda)^*$ into $L^\infty(G)^*$ which maps Φ w^*-homeomorphically onto H.*

Proof. Let $\pi : L^\infty(G) \to \ell^\infty(\Lambda)$ be defined by

$$\pi(f)(\lambda) = \mu_\lambda(f), \qquad f \in L^\infty(G), \quad \lambda \in \Lambda.$$

Obviously π is linear, positive, and $\|\pi\| = 1$. Choose $x \in \ell^\infty(\Lambda)$. Define $f_x \in L^\infty(G)$ as follows: first on each set supp μ_λ, define f_x to be $x(\lambda)$. Since the family $\{\text{supp } \mu_\lambda\}_{\lambda\in\Lambda}$ is discrete, f_x is well defined and continuous on $\bigcup_{\lambda\in\Lambda}$ supp μ_λ, and the set $\bigcup_{\lambda\in\Lambda}$ supp μ_λ is closed. Thus f_x can be extended continuously to G with its range contained in $[-\|x\|, \|x\|]$. Since $\pi(f_x) = x$ and $\|f_x\| = \|x\|$, the dual map π^* is a linear isometry from $\ell^\infty(\Lambda)^*$ into $L^\infty(G)^*$.

Now let $\varphi \in \Phi$. Then for any $f \in L^\infty(G)$,

$$\pi^*(\varphi)(f) = \varphi(\pi(f)) \le \limsup_{\lambda\in\Lambda} \pi(f)(\lambda)$$

$$= \limsup_{\lambda\in\Lambda} \mu_\lambda(f).$$

Thus we can find a w^*-cluster point μ of $\{\mu_\lambda\}_{\lambda\in\Lambda}$, such that $\pi^*(\varphi)(f) \le \mu(f)$. Since $\mu \in H$ and H is w^*-closed convex, by the Hahn–Banach theorem, $\pi^*(\varphi) \in H$.

Notice that Φ is w^*-compact and convex in $\ell^\infty(\Lambda)^*$, and that π^* is w^*-continuous. Therefore, in order to finish the proof, we need only to show that every w^*-cluster point of $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is in $\pi^*(\Phi)$. Let μ be a w^*-cluster point of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Define $\varphi \in \ell^\infty(\Lambda)^*$ as follows: For $x \in \ell^\infty(\Lambda)$, let f_x be as in the first part of the proof, and let $\varphi(x) = \mu(f_x)$. It is easy to see that φ is well-defined, linear and positive. Also $\|\varphi\| = 1$ since μ is a w^*-cluster point of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Let $x \in \ell^\infty(\Lambda)$ be such that $\lim_{\lambda\in\Lambda} x(\lambda) = 0$. Then for any $\varepsilon > 0$, there exists $\lambda_0 \in \Lambda$, such that $\lambda_0 < \lambda$ implies $|x(\lambda)| < \varepsilon$. Choose $\lambda > \lambda_0$ so that $|\mu(f_x) - \mu_\lambda(f_x)| < \varepsilon$. Thus $|\mu(f_x)| < 2\varepsilon$ since $\mu_\lambda(f_x) = x(\lambda)$. This implies that $\varphi(x) = \mu(f_x) = 0$, and hence $\varphi \in \Phi$. Finally, for any $f \in L^\infty(G)$, let $x = \pi(f)$. Then for each $\lambda \in \Lambda$, $\mu_\lambda(f) = \mu_\lambda(f_x)$. Thus $\pi^*(\varphi)(f) = \varphi(\pi(f)) = \varphi(x) = \mu(f_x) = \mu(f)$. So $\pi^*(\varphi) = \mu$, and this completes the proof of Lemma 2.3.2. □

Let $d(G)$ be the smallest cardinality of a cover of G by compact sets, as in Lau and Paterson [26]. It is shown therein that $|MTL(G)| = 2^{2^{d(G)}}$. It is easy to see that if G is not compact $d(G)$ is just the Lindelöf number; i.e., the smallest cardinal $L(G)$ such that any open cover of G has a subcover of cardinality $\leq L(G)$.

LEMMA 2.3.3. *Let A be a Borel subset of G. If A can be written as the union of $< d(G)$ compact subsets of G, then $\mu(A) = 0$ for any $\mu \in MTL(G)$.*

Proof. We may assume $d(G)$ is infinite. For $x \in G$, $xA \cap A \neq \emptyset \Leftrightarrow x \in AA^{-1}$. $AA^{-1} \neq G$ since it is the union of $< d(G)$ compact sets. So there exists $x \in G$ such that $xA \cap A = \emptyset$. This means that for any $\mu \in MTL(G)$, $\mu(A) \leq 1/2$. By induction we see that $\mu(A) = 0$. □

Our next theorem generalizes Chou's results for σ-compact groups in [6]. Results in this format appear first in Granirer [16]. Let X be a compact cover of G with $|X| = d(G)$, and let $\Lambda = \Lambda(X)$, the directed set of all finite subsets of X. Let Φ be defined for Λ as before.

THEOREM 2.3.4. *Let G be a noncompact amenable group, and $\mu_0 \in MTL(G)$. Let F be a subset of $L^\infty(G)$ with $|F| \leq d(G)$. Define*

$$M = \{\mu \in MTL(G) : \mu(f) = \mu_0(f), \forall f \in F\}.$$

Then there exists a linear isometry π^* of $\ell^\infty(\Lambda)^*$ into $L^\infty(G)^*$, such that $\pi^*(\Phi) \subset M$. Furthermore, if μ_0 is also topological right invariant or inversion invariant, then π^* can be chosen to map Φ into

$$M' = \{\mu \in MT(G) : \mu(f) = \mu_0(f), \quad \forall f \in F\}$$

or

$$M'' = \{\mu \in MT^*(G) : \mu(f) = \mu_0(f), \quad \forall f \in F\},$$

respectively.

Proof. We assume that $F = \{f_x\}_{x \in X}$ and $\|f_x\| = 1$ for all $x \in X$. Choose a net $\{\mu_\gamma\}$ of means defined by functions in $C_{00}(G)$, such that $\{\mu_\gamma\}$ converges to μ_0 in the w^*-topology. If μ_0 is inversion invariant, we may suppose it is also the case for each μ_γ, since then $(\mu_\gamma + \mu_\gamma^*)/2$ is also w^*-convergent to μ_0. By Lemma 1.4.1, we may obtain a net of means in $C_{00}(G)$, denoted still by $\{\mu_\gamma\}$, convergent strongly to topological left (and right) invariance (if μ_0 is also right invariant), and satisfying the condition that $\mu_\gamma(f) \to \mu_0(f)$, for each $f \in F$.

Let U be a symmetric compact neighborhood of e in G. We proceed to construct a net $\{\mu_\gamma\}$ of means with the directed set $\Lambda = \Lambda(X)$, satisfying the following properties:

i) Each μ_λ is defined by a function in $C_{00}(G)$;

ii) If $\lambda \neq \lambda'$, then $U \cdot \text{supp}\, \mu_\lambda \cap U \cdot \text{supp}\, \mu_{\lambda'} = \emptyset$;

iii) For each $\lambda \in \Lambda$ and $x \in \lambda$, $|\mu_\lambda(f_x) - \mu_0(f_x)| < 1/|\lambda|$;

iv) For each $\lambda \in \Lambda$ and $s \in \bigcup \lambda$ (union of finitely many elements in X), $\|{}_s\mu_\lambda - \mu_\lambda\| < 1/|\lambda|$;

v) If μ_0 is also right invariant, then for each $\lambda \in \Lambda$ and $s^{-1} \in \bigcup \lambda$, $\|(\mu_\lambda)_s - \mu_\lambda\| < 1/|\lambda|$;

vi) If μ_0 is inversion invariant, so is each μ_λ.

Firstly, well-order Λ by $\{\lambda_\alpha\}_{\alpha<d(G)}$. Let $\alpha < d(G)$ be an ordinal and suppose for each $\beta < \alpha$ we have defined a mean μ_{λ_β} satisfying i) – vi). Write $A_\alpha = \bigcup_{\beta<\alpha} \text{supp}\, \mu_{\lambda_\beta}$. Then the set $U^3 \cdot A_\alpha \cdot U^3$ is the union of $< d(G)$ compact sets. So $\mu_0(U^3 \cdot A_\alpha \cdot U^3) = 0$ by Lemma 2.3.3. Take a function $\varphi \in C_{00}(G)$ such that its support is contained in U and $\varphi = \varphi^*$, $\varphi \geq 0$, and $\|\varphi\|_1 = 1$. Following Hulanicki's proof for Reiter's conditions ([17, pp. 44–45]), we choose a small symmetric neighborhood E of e in G such that

(2.3.1) $$\|\psi_E * \varphi - \varphi\|_1 < \varepsilon \quad \text{and} \quad \|{}_s\varphi - \varphi\|_1 < \varepsilon, \quad \forall s \in E,$$

where $\varepsilon > 0$ is any given number and ψ_E is the normalized characteristic function of E. Select $\{s_1, \ldots, s_n\} \subset \bigcup \lambda_\alpha$ so that $\bigcup_{i=1}^n s_i E \supset \bigcup \lambda_\alpha$. Since the net $\{\mu_\gamma\}$ is convergent strongly to topological left invariance, there exists μ_γ such that

$$\begin{aligned}
&\|\psi_{s_iE} * \mu_\gamma - \mu_\gamma\| < \varepsilon, \quad i = 1, \ldots, n, \\
&\|\varphi * \mu_\gamma - \mu_\gamma\| < \varepsilon, \\
&|\mu_\gamma(f_x) - \mu_0(f_x)| < \varepsilon, \quad \forall x \in \lambda_\alpha, \text{ and} \\
&\mu_\gamma(U^3 \cdot A_\alpha \cdot U^3) < \varepsilon.
\end{aligned} \tag{2.3.2}$$

The last inequality is a consequence of Lemma 2.3.3. Let μ' be the restriction of μ_γ to $G \backslash U^3 \cdot A_\alpha \cdot U^3 = B$; i.e.,

$$\mu'(f) = \frac{\mu_\gamma(f \cdot \chi_B)}{\mu_\gamma(B)}, \quad f \in L^\infty(G).$$

It is easy to see that $\|\mu' - \mu_\gamma\| < \varepsilon' < 2\varepsilon/(1-\varepsilon)$. Let $\mu_{\lambda_\alpha} = \varphi * \mu'$. Then, as shown in [17, p. 45], $\|{}_s\mu_{\lambda_\alpha} - \mu_{\lambda_\alpha}\| < 11\varepsilon'$ for all $s \in \bigcup \lambda_\alpha$. Also, $|\mu_{\lambda_\alpha}(f_x) - \mu_0(f_x)| < 5\varepsilon'$ for $x \in \lambda_\alpha$. Thus μ_{λ_α} satisfies iii) and iv) if ε is chosen properly. Since μ_{λ_α} is continuous and supp $\mu_{\lambda_\alpha} \subset U \cdot$ supp μ_γ, $\mu_{\lambda_\alpha} \in C_{00}(G)$. It is easy to see that supp $\mu_{\lambda_\alpha} \cap U^2 \cdot A_\alpha = \emptyset$, so ii) is satisfied. Now suppose μ_0 is also right invariant. Then in the above argument, we add to (2.3.1) the conditions

$$\|\varphi * \psi_E - \varphi\|_1 < \varepsilon/m \quad \text{and} \quad \|\varphi_s - \varphi\|_1 < \varepsilon/m, \quad \forall s \in E,$$

where $m = \max\{\Delta(s) : s \in \bigcup \lambda_\alpha\}$. Since $\{\mu_\gamma\}$ is also strongly convergent to topological right invariance, we have in (2.3.2) the requirements

$$\begin{aligned}
&\|\mu_\gamma * \tilde{\psi}_{s_iE} - \mu_\gamma\| < \varepsilon, \quad i = 1, \ldots, n, \text{ and} \\
&\|\varphi * \mu_\gamma * \varphi - \mu_\gamma\| < \varepsilon.
\end{aligned}$$

In this case if we let $\mu_{\lambda_\alpha} = \varphi * \mu' * \varphi$, it follows that μ_{λ_α} satisfies i) – v). Finally, we suppose that μ_0 is inversion invariant. Then A_α is a symmetric set since each supp μ_{λ_β}, $\beta < \alpha$, is symmetric. Thus $\mu_\gamma = \mu_\gamma^* \Rightarrow \mu' = \mu'^* \Rightarrow \mu_{\lambda_\alpha}^* = \varphi^* * \mu'^* * \varphi^* = \varphi * \mu' * \varphi = \mu_{\lambda_\alpha}$.

The net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is convergent to topological left invariance, as proved in Hulanicki [21, p. 96]. By ii) it is easy to see that the family {supp μ_λ} is discrete. Thus by Lemma 2.3.2, there exists a linear isometry $\pi^* : \ell^\infty(\Lambda)^* \to L^\infty(G)^*$, such that $\pi^*(\Phi) \subset M$. When μ_0 is also topological right invariant, the net we constructed is also convergent to topological right invariance, so $\pi^*(\Phi) \subset M'$. The last statement of the theorem is now obvious. □

COROLLARY 2.3.5. *There is a subset of $MT^*(G)$ linearly isometric to Φ. In particular* $|MT^*(G)| = 2^{2^{d(G)}}$.

Proof. The first part is obvious by the existence of topological invariant and inversion invariant means on $L^\infty(G)$. This together with Lemma 2.3.1 imply the inequality $|MT^*(G)| \geq 2^{2^{d(G)}}$. The other inequality $|MTL(G)| \leq 2^{2^{d(G)}}$ was proved by Lau and Paterson [26, Thm. 1]. □

COROLLARY 2.3.6. *If $MTL(G) \neq MT(G)$, then $MTL(G)\backslash MT(G)$ contains a subset linearly isometric to Φ. In particular* $|MTL(G)\backslash MT(G)| = 2^{2^{d(G)}}$.

Proof. Suppose $\mu_0 \in MTL(G)\backslash MT(G)$. Then there exist $f \in L^\infty(G)$ and $g \in L^1(G)$ with $g \geq 0$, $\|g\|_1 = 1$, such that $\mu_0(f) \neq \mu_0(f * \tilde{g})$. Let $F = \{f, f * \tilde{g}\}$ in Theorem 2.3.4. Then the set $M \subset MTL(G)\backslash MT(G)$. □

COROLLARY 2.3.7. *If $MT(G) \neq MT^*(G)$, then $MT(G)\backslash MT^*(G)$ contains a subset linearly isometric to Φ. In particular* $|MT(G)\backslash MT^*(G)| = 2^{2^{d(G)}}$.

Proof. Suppose $\mu_0 \in MT(G)\backslash MT^*(G)$. Then there exists $f \in L^\infty(G)$ such that $\mu_0(f) \neq \mu_0(\tilde{f})$. Let $F = \{f, \tilde{f}\}$ in Theorem 2.3.4. Then the set $M' \subset MT(G)\backslash MT^*(G)$. □

COROLLARY 2.3.8. *If G is not compact, then $MTL(G)$, $MT(G)$ and $MT^*(G)$ do not contain any point which is the intersection of $d(G)$ many w^*-open subsets. In particular, they do not have any weak* G_δ-points or w^*-exposed points.*

Proof. Suppose $\mu_0 \in MTL(G)$ is the intersection of $d(G)$ many w^*-open subsets of $MTL(G)$. Then there exists a set $\{g_\alpha\}_{\alpha<d(G)} \subset L^\infty(G)$, such that

$$\{\mu_0\} = \{\mu \in MTL(G) : \mu(g_\alpha) = \mu_0(g_\alpha),\ \forall \alpha < d(G)\}.$$

By Theorem 2.3.4, the set on the right hand side is not a singleton. In fact, it has cardinality $2^{2^{d(G)}}$. □

Our Corollary 2.3.5 improves the main theorem in Lau and Paterson [26]. Corollaries 2.3.6 and 2.3.7 are partial generalizations of Theorem 3 and Theorem 4 of Rosenblatt and Talagrand [35]. Corollary 2.3.8 extends Chou's results in [6, §5] to any noncompact amenable group.

Our next result offers some information about the structure of the sets $MTL(G)$, $MT(G)$, and $MT^*(G)$. Let X be a set such that $|X| = d(G)$, where $d(G)$ is defined as before. Let $\Lambda = \Lambda(X)$. Suppose $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is a net of means on $L^\infty(G)$ defined by functions in $C_{00}(G)$, such that the family $\{\text{supp } \mu_\lambda\}$ is discrete. Such a net is called left (two-sided) fundamental if the net $\{\mu_\lambda\}$ is convergent to topological left (two-sided) invariance. It is called inversion fundamental if it is two-sided fundamental and $\mu_\lambda = \mu_\lambda^*$ for all $\lambda \in \Lambda$.

PROPOSITION 2.3.9. *Let G be an amenable locally compact noncompact group. Then the sets $MTL(G)$, $MT(G)$ and $MT^*(G)$ are the w^*-closures of the sets of w^*-cluster points of all left, two-sided, and inversion fundamental nets on G, respectively.*

Proof. We give a proof for the set $MTL(G)$. The proofs in the remaining cases are similar. Let $\mu_0 \in MTL(G)$ and $f_1, \ldots, f_n \in L^\infty(G)$. Then by Theorem 2.3.4, there is a left fundamental net on G such that any w^*-cluster point μ of the net satisfies $\mu(f_i) = \mu_0(f_i)$, $i = 1, \ldots, n$. □

II.4. Cardinality of $ML(S)$.

In this section we intend to prove an analogue of Theorem 2.3.4 for semigroups. We also show that Klawe's result on the cardinality of $ML(S)$ in [24] remains correct and further answer a question posed by Paterson in [30]. Throughout this section, S will denote an infinite left amenable semigroup.

Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be a net of finite means on $\ell^\infty(S)$ with mutually disjoint supports in S, and convergent to left invariance. Let H be the w^*-closed convex hull of the set of all w^*-cluster points of $\{\mu_\lambda\}_{\lambda\in\Lambda}$, and let Φ be defined for Λ as before.

LEMMA 2.4.1. *There exists a linear isometry of $\ell^\infty(\Lambda)^*$ into $\ell^\infty(S)^*$ which maps Φ w^*-homeomorphically onto H.*

Proof. Indeed, the proof of Lemma 2.3.2 can be carried through since we have not used any property related to the group structure of G therein. □

If μ is a left invariant mean on $\ell^\infty(S)$, we define the cardinality $\kappa(\mu) = \min\{|A| : A \subset S,\ \mu(A) = 1\}$, as in Section II.2. We say that μ is a pure κ-mean if $\kappa = \kappa(\mu)$ is infinite and for $A \subset S$, $|A| < \kappa \Rightarrow \mu(A) = 0$. In Lemma 2.2.3 we proved that if μ is an extreme point of $ML(S)$ and $\kappa(\mu)$ is infinite, then μ is a pure $\kappa(\mu)$-mean.

We are now ready to prove the promised analogue of Theorem 2.3.4, which generalizes Granirer's results on countable left amenable semigroups in [16, §II].

Let A be an infinite strongly left thick subset of S. Suppose μ_0 is such that for each $B \subset A$ with $|B| < |A|$, $\mu_0(A \backslash B) = 1$. The existence of μ_0 is proved in Lemma 2.4.7. Let $\Lambda = \Lambda(A \times A)$ and define Φ as before. Let $\{f_s\}_{s \in A}$ be a subset of $\ell^\infty(S)$, and define

$$M = \{\mu \in ML(S) : \mu(A) = 1,\ \mu(f_s) = \mu_0(f_s),\ \forall s \in A\}.$$

THEOREM 2.4.2. *There exists a linear isometry from $\ell^\infty(\Lambda)^*$ into $\ell^\infty(S)^*$, which maps Φ into M.*

Proof. The proof is in some sense a refinement of Theorem 2.2.4. For each pair $a, b \in A$, define $S_{(a,b)} = \{t \in S : ta = b\}$. Then $\bigcup\{S_{(a,b)} : a, b \in S\} = S$, as proved in Theorem 2.2.4. Suppose $\|f_s\| = 1$ for all $s \in A$.

Well order Λ by $\{\lambda_\alpha\}_{\alpha < |A|}$. Let $\alpha < |A|$ be an ordinal and suppose for each $\beta < \alpha$ we have defined a finite mean μ_{λ_β}, satisfying

i) $\mu_{\lambda_\beta}(A) = 1$ for each $\beta < \alpha$;

ii) If $\beta < \gamma < \alpha$, then μ_{λ_β} and μ_{λ_γ} have disjoint supports;

iii) If $\beta < \alpha$ and $\lambda_\beta = \{(a_i, b_i)\}_{i=1}^n$, then

$$\|t \cdot \mu_{\lambda_\beta} - \mu_{\lambda_\beta}\| < \frac{1}{n}, \quad \forall t \in S_{(a_i, b_i)}, \quad i = 1, \ldots, n;$$

iv) If $\beta < \alpha$ and $\lambda_\beta = \{(a_i, b_i)\}_{i=1}^n$, then

$$|\mu_{\lambda_\beta}(f_{a_i}) - \mu_0(f_{a_i})| < \frac{1}{n}, \quad i = 1, \ldots, n.$$

Now write $\lambda_\alpha = \{(a_i, b_i)\}_{i=1}^n$. Since S has the finite intersection property for right ideals, $\bigcap_{i=1}^n a_i S \neq \emptyset$. Thus we can choose $a \in S$, such that $a = a_i s_i$ for some $s_i \in S$, $i = 1, \ldots, n$. Let $A_\alpha = \bigcup_{\beta<\alpha} \operatorname{supp} \mu_{\lambda_\beta}$ and $B = A \backslash A_\alpha$. Then $\mu_0(B) = 1$ since $|A_\alpha| < |A|$. By Lemma 1.2.1, we can find a finite mean μ with its support contained in B, such that

$$\|a \cdot \mu - \mu\| < \frac{1}{8n}, \quad \|(b_i s_i) \cdot \mu - \mu\| < \frac{1}{8n}, \quad i = 1, \ldots, n,$$

and

$$|\mu(f_{a_i}) - \mu_0(f_{a_i})| < \frac{1}{2n}, \quad i = 1, \ldots, n.$$

Then we have $(a \cdot \mu)(B) > (8n-1)/8n$, and for each $t \in S_{(a_i, b_i)}$,

$$\|t \cdot (a \cdot \mu) - (a \cdot \mu)\| = \|(b_i s_i) \cdot \mu - a \cdot \mu\| < \frac{1}{4n}.$$

Define μ_{λ_α} to be the restriction of $a \cdot \mu$ to B:

$$\mu_{\lambda_\alpha}(f) = \frac{(a \cdot \mu)(f \cdot \chi_B)}{(a \cdot \mu)(B)}, \quad f \in \ell^\infty(S).$$

Then it is easy to see that

$$\|\mu_{\lambda_\alpha} - a\cdot\mu\| < \frac{\frac{1}{8n}}{1-\frac{1}{8n}} + \frac{1}{8n} < \frac{2}{7n}.$$

Thus for each $t \in S_{(a_i,b_i)}$, $i = 1,\ldots,n$,

$$\|t\cdot\mu_{\lambda_\alpha} - \mu_{\lambda_\alpha}\| \le \|t\cdot(\mu_{\lambda_\alpha} - a\cdot\mu)\| + \|t\cdot(a\cdot\mu) - a\cdot\mu\|$$

$$+ \|a\cdot\mu - \mu_{\lambda_\alpha}\| < \frac{2}{7n} + \frac{1}{4n} + \frac{2}{7n} < \frac{1}{n}.$$

Also

$$|\mu_{\lambda_\alpha}(f_{a_i}) - \mu_0(f_{a_i})| \le |\mu_{\lambda_\alpha}(f_{a_i}) - (a\cdot\mu)(f_{a_i})|$$

$$+ |(a\cdot\mu)(f_{a_i}) - \mu(f_{a_i})| + |\mu(f_{a_i}) - \mu_0(f_{a_i})|$$

$$< \frac{2}{7n} + \frac{1}{8n} + \frac{1}{2n} < \frac{1}{n},$$

and the support of μ_{λ_α} is contained in B.

The net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ converges to left invariance, and the means μ_λ have mutually disjoint supports in A. Thus by Lemma 2.4.1, there exists a linear isometry from $\ell^\infty(\Lambda)^*$ into $\ell^\infty(S)^*$, which maps Φ onto H, the w^*-closed convex hull of all w^*-cluster points of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Finally, it is easily seen that $H \subset M$. □

COROLLARY 2.4.3. *If $A \subset S$ is infinite and strongly left thick, then A supports $2^{2^{|A|}}$ left invariant means.*

Proof. This follows from Lemma 2.3.1. □

COROLLARY 2.4.4. *(Klawe [24]). If S is infinite and strongly left thick in itself, then $|ML(S)| = 2^{2^{|S|}}$.*

We shall see later that the converse of Corollary 2.4.4 is also true.

COROLLARY 2.4.5. *If $\mu_0 \in ML(S)$ is a pure κ-mean, then μ_0 is not the intersection of κ many w^*-open subsets of $ML(S)$.*

Proof. This proof is entirely analogous to that of Corollary 2.3.8. □

From this corollary we deduce that any pure κ-mean on $\ell^\infty(S)$ with κ infinite is not a G_δ-point of $ML(S)$. A complete characterization of weak* G_δ-points of $ML(S)$ is given in Section II.5.

In [24] Maria Klawe defined the cardinal

$$\kappa(S) = \min\{|B| : B \subset S, \ \mu(B) = 1 \text{ for all } \mu \in ML(S)\}$$

and gave a proof that $|ML(S)| = 2^{2^{\kappa(S)}}$. However, there is a gap in her proof as pointed out in Paterson [30]. Define the left thickness of a semigroup S to be

$$\tau(S) = \sup\{|A| : A \subset S, \ A \text{ is strongly left thick}\}.$$

LEMMA 2.4.6. *Suppose that $\kappa(S)$ is infinite and let $A \subset S$ be such that $|A| = \kappa(S)$ and $\mu(A) = 1$ for all $\mu \in ML(S)$. Then A is strongly left thick.*

Proof. Let $B \subset S$ with $|B| < \kappa(S)$. It is enough to prove that there exists $\mu \in ML(S)$ such that $\mu(B) = 0$. Let $r(B) = \inf\{\mu(B) : \mu \in ML(S)\}$. Since the map $\mu \to \mu(B)$ is w^*-continuous on $ML(S)$, we can find $\mu \in ML(S)$ such that $\mu(B) = r(B)$. Thus we may assume on the contrary that $r(B) > 0$.

If B is finite, then S contains finite left ideals. And since $\kappa(S)$ is infinite, there are infinitely many minimal finite left ideals in S as shown in Section II.2. So there exists a minimal finite left ideal I of S disjoint from B. The arithmetic average μ_I on I is a left invariant mean on $\ell^\infty(S)$ such that $\mu_I(B) = 0$, which is impossible.

If B is infinite, define $r = \sup\{r(C) : B \subset C \subset S, |C| = |B|\}$, where $r(C)$ is defined in the same way as $r(B)$. By taking a countable union, we can get a set $C \subset S$, with $C \supset B$, $|C| = |B|$, and $r(C) = r$. Since $|C| < \kappa(S)$, there is $\mu \in ML(S)$ such that $\mu(C) < 1$. Thus $r < 1$. Let $\Lambda = \Lambda(S)$, the directed set of all finite subsets of S. Take $\lambda = \{t_1, \ldots, t_n\} \in \Lambda$. Then $r(t_1 C \cup t_2 C \cup \cdots \cup t_n C \cup C) = r(C)$, since $r(C)$ is the maximum. Take $\mu_\lambda \in ML(S)$ such that $\mu_\lambda(t_1 C \cup \cdots \cup t_n C \cup C) = r(C)$. Then since $\mu_\lambda(C) \geq r(C)$, we have $\mu_\lambda(t_i C \cup C) = \mu_\lambda(C)$, $i = 1, \ldots, n$. Let μ be a w^*-cluster point of the net $\{\mu_\lambda\}$. Then $\mu \in ML(S)$, and for every $t \in S$, $\mu(C) = \mu(tC) = \mu(C \cup tC) = r(C) < 1$. Following Lemma 2.2.3, we let $\mu' \in ML(S)$ be defined by

$$\mu'(f) = \frac{\mu(f \cdot \chi_{S \setminus C})}{\mu(S \setminus C)}, \quad f \in \ell^\infty(S).$$

Then $\mu'(C) = 0$. This completes our proof of Lemma 2.4.6. □

LEMMA 2.4.7. *Suppose $A \subset S$ is infinite. Then A is strongly left thick if and only if there exists $\mu \in ML(S)$ such that μ is a pure $|A|$-mean and $\mu(A) = 1$. In particular $r(S) \leq \kappa(S)$.*

Proof. If there exists a pure $|A|$-mean $\mu \in ML(S)$ such that $\mu(A) = 1$, then it is easy to see that A is strongly left thick in S.

Conversely, let Γ be the directed set consisting of all subsets B of A with $|B| < |A|$, directed by inclusion. For each $B \in \Gamma$, there exists $\mu_B \in ML(S)$ such that $\mu_B(A\backslash B) = 1$, since A is strongly left thick. Let μ be a w^*-cluster point of the net $\{\mu_B\}_{B\in\Gamma}$. Then μ satisfies our requirements. $\square$

THEOREM 2.4.8. *If $\kappa(S)$ is infinite, then $\tau(S) = \max\{|A| : A$ is strongly left thick in $S\} = \kappa(S)$.*

THEOREM 2.4.9. *Suppose S is a left amenable semigroup such that $ML(S)$ is infinite dimensional. Then $|ML(S)| = 2^{2^{\tau(S)}} = 2^{2^{\kappa(S)}}$.*

Proof. By the definition of $\kappa(S)$, we know that $|ML(S)| \le |\ell^\infty(A)^*| = 2^{2^{\kappa(S)}}$, where $A \subset S$ is such that $|A| = \kappa(S)$ and $\mu(A) = 1$ for all $\mu \in ML(S)$. Since A is strongly left thick in S, by Corollary 2.4.3, $|ML(S)| \ge 2^{2^{|A|}} = 2^{2^{\tau(S)}}$. $\square$

Theorem 2.4.9 ensures that Klawes's assertion in [24] is indeed correct.

Paterson defined in [30] the cardinal

$$\rho(S) = \min\Big\{\Big|\bigcup_{i=1}^{n} s_i S_i\Big| : n \ge 1,\ \{S_1,\ldots,S_n\} \text{ is a partition of } S,\ s_1,\ldots,s_n \in S\Big\}$$

and proved for some special cases that $\rho(S) = \kappa(S)$. Our next result shows that this equality holds for all S such that $ML(S)$ is infinite dimensional.

THEOREM 2.4.10. *If S is a left amenable semigroup such that $ML(S)$ is infinite dimensional, then $\rho(S) = \kappa(S) = \tau(S)$.*

Proof. Let $A \subset S$ be such that $|A| = \kappa(S)$ and $\mu(A) = 1$ for all $\mu \in ML(S)$. Then A is strongly left thick in S by Lemma 2.4.6. Let $\{S_1,\ldots,S_n\}$ be a partition of S, and $s_1,\ldots,s_n \in S$. Then for any $\mu \in ML(S)$, $\mu(\bigcup_{i=1}^n s_i S_i) > 0$. So $A\backslash\bigcup_{i=1}^n s_i S_i$ is not left thick, and hence $|\bigcup_{i=1}^n s_i S_i| \ge |A| = \kappa(S)$.

On the other hand, since the characteristic function χ_A is left almost convergent to 1, by Proposition 1.3.2, for any $\varepsilon > 0$, we can find a finite mean μ, such that

$$\inf_{t\in S}\left\{\sum_s \mu(s)\ell_s\chi_A(t)\right\} = \inf_{t\in S}\left\{\sum_s \mu(s)\chi_{s^{-1}A}(t)\right\} > 1-\varepsilon.$$

This implies that there are elements $s_1,\ldots,s_n \in S$ such that $\bigcup_{i=1}^n s_i^{-1}A = S$. Since $\bigcup s_i(s_i^{-1}A) \subset A$, we see that $|A| \geq \rho(S)$. □

COROLLARY 2.4.11. *(Paterson). If $ML(S)$ is infinite dimensional, then* $|ML(S)| = 2^{2^{\rho(S)}}$.

Let A be a strongly left thick subset of S. We proved in Theorem 2.4.2 that there is a net $\{\mu_\lambda\}_{\lambda\in\Lambda(A)}$ of finite means with mutually disjoint supports contained in A, convergent to left invariance. We call such a net a fundamental net on A. We are now ready to prove an analogue of Proposition 2.3.9. Let M be the subset of $ML(S)$ consisting of all $\mu \in ML(S)$ such that μ i[illegible]re $|A|$-mean and $\mu(A) = 1$.

PROPOSITION 2.4.12. *M is the w^*-closure of all w^*-cluster points of fundamental nets on A.*

Proof. Let $\{\mu_\lambda\}$ be a fundamental net on A, and $B \subset S$ with $|B| < |A|$. Since there are at most $|B|$ many μ_λ in the net with their supports intersecting B, we can find $\lambda \in \Lambda(A)$ such that $\lambda' > \lambda$ implies supp $\mu_{\lambda'} \cap B = \emptyset$. So $\mu_{\lambda'}(B) = 0$. This implies that any w^*-cluster point of $\{\mu_\lambda\}$ is contained in M.

The other inclusion can be proved as in Proposition 2.3.9. □

II.5. Structure of $ML(S)$.

Let S be a left amenable semigroup. Granirer [14] proved that S admits a left invariant countable mean on $\ell^\infty(S)$ if and only if S has finite left ideals. Now we consider another extreme case. An element $\mu \in ML(S)$ is called purely infinite if $\mu(F) = 0$ for any finite subset F of S.

THEOREM 2.5.1. *(a) Any left invariant mean on $\ell^\infty(S)$ is a convex combination of a countable mean and a purely infinite mean in $ML(S)$. (b) Any purely infinite mean in $ML(S)$ is a convex combination of countably many mutually singular elements in $ML(S)$, each of which is a pure κ-mean for some infinite cardinal κ.*

Proof. Suppose $\mu \in ML(S)$, and define $r = \sup\{\mu(F) : F \text{ is finite}\}$. Obviously, $\mu \in \ell^1(S) \Leftrightarrow r = 1$, and μ is purely infinite $\Leftrightarrow r = 0$. Suppose $0 < r < 1$. Choose finite subsets F_n of S such that $\mu(F_n) \to r$. For any $t \in S$, since $\mu(F_n \cup tF_n) \to r$ as $n \to \infty$, $\mu(F_n \Delta tF_n) \to 0$. This implies that $\mu(F_n \Delta t^{-1} F_n) \to 0$. So if we let μ_n be the finite mean defined by the restriction of μ to F_n, then $\|\mu_n - t \cdot \mu_n\| \to 0$, $\forall t \in S$. Also it is easy to see that $\{\mu_n\}$ is a Cauchy sequence in $\ell^1(S)$ (also in $\ell^\infty(S)^*$). Thus the limit μ' of $\{\mu_n\}$ is a left invariant countable mean on $\ell^\infty(S)$. Let $\mu'' = (1-r)^{-1} \cdot (\mu - r \cdot \mu')$. Then μ'' is purely infinite, and $\mu = r \cdot \mu' + (1-r)\mu''$.

Now suppose μ is a purely infinite element in $ML(S)$ and is not a pure κ-mean for any cardinal κ. Let $\kappa = \min\{|A| : A \subset S,\ \mu(A) > 0\}$. Then κ is an infinite cardinal, since μ is purely infinite. Let $r = \sup\{\mu(A) : A \subset S,\ |A| = \kappa\}$. As shown in Lemma 2.2.3, there exists $B \subset S$, such that $|B| = \kappa$ and $\mu(B) = r$.

Since μ is not a pure κ-mean, $0 < r < 1$. The restriction of μ to B is also a left invariant mean, and is a pure κ-mean. Now an induction on the set of all countable ordinals will give us a finite or countably infinite decomposition $\mu = \sum_i \alpha_i \mu_i$, where $\alpha_i > 0$, $\sum \alpha_i = 1$, and each $\mu_i \in ML(S)$ is a pure κ_i-mean for some infinite cardinal κ_i. Also $i \neq j \Rightarrow \kappa_i \neq \kappa_j$. This implies that the means μ_i are mutually singular. □

COROLLARY 2.5.2. *$ML(S)$ is the norm closed convex hull of all countable means and all pure κ-means in $ML(S)$.*

COROLLARY 2.5.3. *The set of all purely infinite elements in $ML(S)$ is the w^*-closed convex hull of w^*-cluster points of all fundamental nets on infinite strongly left thick sets.*

The next theorem dwells on the local structure of $ML(S)$. It generalizes Corollary 2.4.5.

THEOREM 2.5.4. *Let $\mu \in ML(S)$. If $\mu = \alpha_0\mu_0 + \sum_i \alpha_i \mu_i$ is a decomposition as in Theorem 2.5.1, where μ_0 is countable and each μ_i is a pure κ_i-mean. If $\alpha_i > 0$, $i \neq 0$, then μ is not the intersection of κ_i many w^*-open subsets of $ML(S)$.*

Proof. Write $\mu = \alpha_i\mu_i + \alpha'\mu'$, where $\mu' \in ML(S)$ and $\alpha_i + \alpha' = 1$. Suppose on the contrary that there exists a family $\{f_\beta\}_{\beta<\kappa_i} \subset \ell^\infty(S)$, such that

$$\{\mu\} = \{\overline{\mu} \in ML(S) : \overline{\mu}(f_\beta) = \mu(f_\beta),\ \forall \beta < \kappa_i\}.$$

This implies that

$$\{\mu_i\} = \{\overline{\mu} \in ML(S) : \overline{\mu}(f_\beta) = \mu_i(f_\beta),\ \forall \beta < \kappa_i\}.$$

This is impossible by Corollary 2.4.5. □

COROLLARY 2.5.5. *(a) μ is a weak* G_δ-point of $ML(S)$ if and only if $\mu \in ML(S) \cap \ell^1(S)$. (b) $ML(S)$ has weak* G_δ-points if and only if S has finite left ideals. In this case the set of weak* G_δ-points of $ML(S)$ is the norm closed convex hull of the set of all w^*-exposed points of $ML(S)$.*

Proof. By the previous theorem a G_δ-point of $ML(S)$ is a countable mean. Also it is obvious that a countable mean in $ML(S)$ is a G_δ-point. The second statement then follows from Granirer [14]. The last statement is a consequence of Theorem 2.2.5. □

It is interesting to compare this corollary with Theorem 2.2.10, which asserts that $ML(S)$ itself is the w^*-closed convex hull of all its w^*-exposed points. We now give a generalization of this fact.

THEOREM 2.5.6. *Let $\kappa = \min\{|A| : A \subset S$ and is left thick$\}$. Then $ML(S)$ is the w^*-closed convex hull of all elements $\mu \in ML(S)$ with $\kappa(\mu) = \kappa$.*

Proof. When κ is finite, this is Theorem 2.2.10. So suppose κ is infinite. We shall prove that in fact $ML(S)$ is the w^*-closure of all pure κ-means.

Take $\mu_0 \in ML(S)$ and $f \in \ell^\infty(S)$. By virtue of the Hahn-Banach theorem, it is enough to find $\mu \in ML(S)$, such that μ is a pure κ-mean and $\mu(f) = \mu_0(f)$. Choose a left thick subset A of S with $|A| = \kappa$. As in the proof of Theorem 2.2.4, we can define a net $\{\mu_\lambda\}_{\lambda \in \Lambda}$ of finite means with the directed set $\Lambda = \Lambda(A \times A)$, convergent to left invariance, and such that $\mu_\lambda(f) \to \mu_0(f)$. Any w^*-cluster point of the net satisfies our requirements. □

CHAPTER III

FØLNER NUMBERS AND FØLNER-TYPE CONDITIONS

III.1. Introduction.

Let S be a semigroup. Consider the following Følner-type conditions on S:

(A) There exists a number k, $0 < k < 1$, such that for any elements $s_1, \ldots, s_n$ of S (not necessarily distinct), there is a finite subset A of S, satisfying

$$\frac{1}{n}\sum_{i=1}^{n}|A \backslash s_i A| \leq k|A|.$$

(B) Given any finite subset F of S, and any number $\varepsilon > 0$, there exists a finite subset A of S, such that for each $s \in F$,

$$|A \backslash sA| \leq \varepsilon |A|.$$

We call condition (A) the weak Følner condition (WFC) and condition (B), as in [1] and [23], the strong Følner condition (SFC). When S is a group, Følner [12] proved that both WFC and SFC are equivalent to the amenability of S. Frey [13] introduced the condition FC, which is equivalent to SFC when S is left cancellative (see [1]):

(FC) Given any finite subset F of S, and any number $\varepsilon > 0$, there exists a finite subset A of S, such that for each $s \in F$,

$$|sA \backslash A| \leq \varepsilon |A|.$$

He proved that if S is left amenable, then FC holds, but the converse is not true (see Namioka [29] for an elegant proof of this fact). In general, SFC is sufficient for the left amenability (LA) of S (cf. [1], also [29]); however, it is not necessary (see Klawe [23] for an example). Also WFC is not sufficient for LA (see Namioka [29] and also see our Theorem 3.2.3. In 1964, Namioka gave two sufficient conditions stronger than WFC. We will refer to them as the weak and strong Namioka-Følner conditions.

($WNFC$) There exists a number k, $0 < k < 1$, such that for any elements $s_1, \ldots, s_n$; $s'_1, \ldots, s'_n$ of S, there is a finite subset A of S satisfying

$$\frac{1}{n}\sum_{i=1}^{n} |s_i A \cap s'_i A| \geq k|A|.$$

($SNFC$) There exists a number k, $0 < k < 1/2$, such that for any elements $s_1, \ldots, s_n$ of S, there is a finite subset A of S satisfying

$$\frac{1}{n}\sum_{i=1}^{n} |A \backslash s_i A| \leq k|A|.$$

Namioka [29] proved that $SNFC$ implies $WNFC$ and $WNFC$ implies LA. In fact he showed that if $SNFC$ holds for k then $WNFC$ holds for $1-2k$. Also it is easy to see that if $WNFC$ holds for k, then $SNFC(WFC)$ holds for $1-k$. Namioka [29, p. 26] posed the problem whether those conditions are necessary; i.e., whether LA implies $WNFC$ or $SNFC$.

The following diagram summarizes the known implications among the various Følner-type conditions for a semigroup.

$$\begin{array}{ccccccc} SFC & \Rightarrow & SNFC & \Rightarrow & WNFC & \Rightarrow & WFC \\ & & & \not\nwarrow\!\!\not\searrow & \Downarrow & \not\nearrow & \\ & & & & LA & \Rightarrow & FC \end{array}$$

In Section III.2, we define the Følner number $\varphi(S)$ for an arbitrary semigroup S and investigate some general properties of $\varphi(S)$. In particular, we determine $\varphi(S)$ completely for all finite semigroups and cancellative semigroups. In Section III.3 we obtain, by some combinatorial computations, two inequalities for $\varphi(S)$ related to the cancellation behavior of S, one of which is the main tool used in §III.4 to solve Namioka's problem.

In Section III.4, based on Klawe's work on semidirect products in [23], we are able to show that there exists a left amenable semigroup not satisfying even WFC, thus answering Namioka's problem. We also give some necessary and sufficient conditions for a semidirect product to be left amenable.

The last section of this chapter is devoted to the Følner number of a semidirect product of two semigroups satisfying SFC. We prove that the Følner number for these semigroups is either 0 or 1, and obtain necessary and sufficient conditions for the number to be 0.

III.2. Følner Numbers.

In this section we give a formula for Følner numbers of finite semigroups related to the number of minimal right ideals. Then we show that the Følner number of a cancellative semigroup S is 0 or 1 according as S is left amenable or not.

We follow Wong [40] in defining the Følner number of a semigroup. Let S be a semigroup and $0 < k \leq 1$. We say that S has property (F_k) if for any $s_1, \ldots, s_n \in S$ (not necessarily distinct), there is a finite (nonempty) subset A of

S such that

$$\frac{1}{n}\sum_{i=1}^{n}|A\backslash s_iA| \leq k|A|.$$

The Følner number of S is defined by

$$\varphi(S) = \inf\{k : 0 < k \leq 1 \quad \text{and } S \text{ has property } (F_k)\}.$$

$\varphi(S)$ is well-defined since every semigroup has property (F_1).

By the definition we can see that $WFC \Leftrightarrow \varphi(S) < 1$ and $SNFC \Leftrightarrow \varphi(S) < 1/2$. Also it is easy to see that SFC implies $\varphi(S) = 0$. Our first result deals with the converse.

PROPOSITION 3.2.1. *Let S be a semigroup. If $\varphi(S) = 0$, then S satisfies SFC.*

Proof. Let $F = \{s_1, \ldots, s_n\}$ be any finite subset of S, and $\varepsilon > 0$. Since $\varphi(S) = 0$, there exists a finite subset A of S, such that

$$\frac{1}{n}\sum_{i=1}^{n}|A\backslash s_iA| \leq \frac{\varepsilon}{n}|A|.$$

Therefore $|A\backslash s_iA| \leq \varepsilon|A|$ for all i, $1 \leq i \leq n$. □

PROPOSITION 3.2.2. *Let S be a semigroup. If there are n disjoint right ideals $I_1, \ldots, I_n$ in S, then*

$$\varphi(S) \geq \frac{n-1}{n}.$$

Proof. Pick $s_i \in I_i$ for $i = 1, \ldots, n$. For any finite subset A of S, the sets s_iA are mutually disjoint. So

$$\sum_{i=1}^{n}|A \cap s_iA| \leq |A|.$$

This implies that

$$\frac{1}{n}\sum_{i=1}^{n}|A\backslash s_iA| \geq \frac{n-1}{n}|A|.$$

□

THEOREM 3.2.3. *If S is a finite semigroup, then $\varphi(S) = 1 - 1/n$, where n is the number of minimal right ideals of S.*

Proof. By Proposition 3.2.2, $\varphi(S) \geq 1 - 1/n$. On the other hand, let $I_1, \ldots, I_n$ be the n minimal right ideals of S, and $A = \bigcup_{i=1}^{n} I_i$. Since any two minimal right ideals in a finite semigroup have the same cardinality, we have $|A| = n|I_1|$. For any $s \in S$, sA is a right ideal, so it contains a minimal right ideal I_i. Thus $|A \cap sA| \geq |I_i| = n^{-1}|A|$, and $|A\backslash sA| \leq (1 - 1/n)|A|$. □

COROLLARY 3.2.4. *For a finite semigroup S, the following are equivalent:*

(1) S is left amenable;

(2) $\varphi(S) = 0$ (S satisfies SFC);

(3) $\varphi(S) < 1/2$ (S satisfies $SNFC$).

Proof. A finite semigroup is left amenable if and only if it contains a unique minimal right ideal (see [34]). □

COROLLARY 3.2.5 ([29]). *There are semigroups which satisfy WFC but are not left amenable.*

COROLLARY 3.2.6. *Let S be a semigroup, h a homomorphism of S onto a finite semigroup. Then $\varphi(S) \geq \varphi(h(S))$.*

Proof. If $h(S)$ has n minimal right ideals, then S admits at least n disjoint right ideals. By Proposition 3.2.2, $\varphi(S) \geq 1 - 1/n = \varphi(h(S))$. □

It is well known that a homomorphic image of a left amenable semigroup is also left amenable. It would be desirable to have Corollary 3.2.6 hold for arbitrary h. Unfortunately, this is not true in general. An example where $\varphi(S) = 0$ but $\varphi(h(S)) = 1$ is given in Section III.4.

If G is a group, then $\varphi(G) = 0$ or 1 according as G is amenable or not [40, Thm. 2.2(3)]. This is also true for cancellative semigroups. In other words, the Følner number of a cancellative semigroup never takes values other than 0 and 1.

THEOREM 3.2.7. *If S is a cancellative semigroup, then $\varphi(S) = 0$ or 1 according as S is left amenable or not.*

Proof. If S is left amenable, the[illegible] $= 0$ since SFC, in this case, is equivalent to FC (see [1]).

Suppose S is not left amenable.

CASE (I). *S has two disjoint right ideals I_1 and I_2.*

Choose $s_1 \in I_1$ and $s_2 \in I_2$. $s_1 I_1$ and $s_1 I_2$ are disjoint right ideals contained in I_1. Also $s_2 I_1$ and $s_2 I_2$ are disjoint right ideals contained in I_2. Thus we obtain four disjoint right ideals. Proceeding inductively, we can find, for any positive integer n, 2^n disjoint right ideals in S. By Proposition 3.2.2, $\varphi(S) = 1$.

CASE (II). *Any two right ideals of S have nonempty intersection.*

By Dubreil's theorem [7, p. 36]), S can be embedded into a group G, such that

$$G = \{xy^{-1} : x, y \in S\}.$$

By Proposition 1.2.2, G is not amenable. Hence $\varphi(G) = 1$. Suppose $\varphi(S) < k < 1$, and consider $x_1y_1^{-1}, x_2y_2^{-1}, \ldots, x_ny_n^{-1} \in G$, where $x_i, y_i \in S$. We prove first that there exists an element $s \in S$ such that $x_1y_1^{-1}s, \ldots, x_ny_n^{-1}s$ are all in S. By induction, suppose that there exists $s' \in S$, such that $x_1y_1^{-1}s', \ldots, x_{n-1}y_{n-1}^{-1}s' \in S$. By the structure of G, $y_n^{-1}s'$ can be written as ab^{-1}, where $a, b \in S$. Let $s = s'b$. Then $x_iy_i^{-1}s = (x_iy_i^{-1}s')b \in S$ for $i \le n-1$, and $x_ny_n^{-1}s = x_na \in S$.

Write $s_i = x_iy_i^{-1}s$. Notice that $s_is^{-1} = x_iy_i^{-1}$ for $1 \le i \le n$. By the assumption $\varphi(S) < k < 1$, there is a finite subset A of S, such that

$$\frac{1}{n}\sum_{i=1}^{n} |A \backslash s_iA| \le k|A|.$$

It follows that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} |(A \cup sA)\backslash s_is^{-1}(A \cup sA)| &= |A \cup sA| - \frac{1}{n}\sum_{i=1}^{n} |(A \cup sA) \cap s_is^{-1}(A \cup sA)| \\
&\le |A \cup sA| - \frac{1}{n}\sum_{i=1}^{n} |A \cap s_iA| \\
&= |A \cup sA| - |A| + \frac{1}{n}\sum_{i=1}^{n} |A\backslash s_iA| \\
&\le |A \cup sA| - (1-k)|A| \\
&\le |A \cup sA| - \frac{1-k}{2}|A \cup sA| \\
&= \frac{1+k}{2}|A \cup sA|.
\end{aligned}$$

This means that $\varphi(G) \le (1+k)/2 < 1$, which contradicts the fact that $\varphi(G) = 1$. $\square$

COROLLARY 3.2.8. *For a cancellative semigroup S, the following are equivalent:*

(i) S is left amenable;

(ii) $\varphi(S) = 0$ (S satisfies SFC);

(iii) $\varphi(S) < 1$ (S satisfies WFC).

Let S be a semigroup having the finite intersection property for right ideals; i.e., any two right ideals of S have nonempty intersection (e.g. any left amenable semigroup has this property). We can define an equivalence relation R on S by

$$sRt \Leftrightarrow \exists x \in S, \quad sx = tx.$$

The set $S/(R)$ of the R-equivalence classes forms a right cancellative semigroup — the right cancellative quotient of S. We refer to [15] for more details about the semigroup $S/(R)$. Whenever $S/(R)$ exists, S is left amenable if and only if $S/(R)$ is left amenable [39], and $\varphi(S) = 0$ if and only if $\varphi(S/(R)) = 0$ ([1] and [23]).

THEOREM 3.2.9. *Let S be a semigroup with the finite intersection property for right ideals. Then $\varphi(S) \leq \varphi(S/(R))$.*

Proof. This follows from the proof of Theorem 4 in Argabright and Wilde [1]. □

We are unable to prove equality in Theorem 3.2.9. This of course raises the question as to whether strict inequality can hold.

III.3. Følner Number and Left Cancellation.

For a right cancellative semigroup S, $\varphi(S) = 0$ if and only if S is left amenable and left cancellative ([1] and [23]). In this section we shall see that $\varphi(S)$ really depends on the left cancellativity of S. The first result provides a link between $\varphi(S)$ and the size of left cancellative classes.

THEOREM 3.3.1. *Let S be a right cancellative semigroup. If there exist distinct elements $s_1, s_2, \ldots, s_{2n}$ of S, and $r \in S$, such that*

$$rs_1 = rs_2 = \cdots = rs_{2n},$$

then $\varphi(S) \geq 1/3 - 1/6n$.

Proof. Suppose S has property (F_k) for some $k \in (0,1]$ (see beginning of Section III.2). We will prove that $k \geq 1/3 - 1/6n$. By (F_k) we know that there exists a finite subset A of S such that

$$\text{(3.3.1)} \qquad \frac{1}{3n}\Big(n|A\backslash rA| + \sum_{i=1}^{2n} |A\backslash s_i A|\Big) \leq k|A|.$$

Define $f : S \to \mathbb{Z}^+$ by $f = \sum_{i=1}^{2n} \chi_{s_i^{-1}A}$, where $s_i^{-1}A \subset S$ is the set of all $x \in S$ such that $s_i x \in A$. Let $W_j = \{a \in A : f(a) = j\}$ for $0 \leq j \leq 2n$. Let $T_0 = A$ and

$$T_j = \{y \in A : y = s_i a \text{ for some } a \in \bigcup_{m=j}^{2n} W_m \text{ and } i \in \{1, \ldots, 2n\}\},$$

for $j = 1, \ldots, 2n$. Finally, let $S_j = T_j \backslash T_{j+1}$, $j = 0, 1, \ldots, 2n-1$ and $S_{2n} = T_{2n}$.

Since $S_j \subset \left(\bigcup_{i=1}^{2n} s_i W_j\right) \cap A$, it is not difficult to see that $|W_j| \geq j^{-1}|S_j|$, for $j \geq 1$, by the definition of f. Thus we have

$$
\begin{aligned}
\sum_{i=1}^{2n} |A \backslash s_i A| &\geq \sum_{i=1}^{2n} |A \backslash s_i^{-1} A| \\
&= 2n|A| - \sum_{i=1}^{2n} |A \cap s_i^{-1} A| \\
&= 2n|A| - \sum_{a \in A} f(a) \qquad (3.3.2) \\
&= 2n \sum_{j=0}^{2n} |W_j| - \sum_{j=1}^{2n} j|W_j| \\
&\geq \sum_{j=1}^{2n} (2n-j)|W_j| \geq \sum_{j=1}^{2n} \frac{2n-j}{j} |S_j|.
\end{aligned}
$$

Also since $T_1 = A \cap \bigcup_{i=1}^{2n} s_i A$, $S_0 \subset A \backslash s_i A$ for all $i = 1, \ldots, 2n$. Thus we have the inequality

$$\sum_{i=1}^{2n} |A \backslash s_i A| \geq 2n|S_0|,$$

and hence by (3.3.2),

$$\sum_{i=1}^{2n} |A \backslash s_i A| \geq n|S_0| + \frac{1}{2} \sum_{j=1}^{2n} \frac{2n-j}{j} |S_j|. \tag{3.3.3}$$

Now consider $|A \backslash rA|$. We claim that for $j \geq 1$,

$$\left| rS_j \Big\backslash \bigcup_{m=j+1}^{2n} rS_m \right| \leq \frac{1}{j} |S_j|. \tag{3.3.4}$$

Suppose $x \in rS_j \backslash \bigcup_{m=j+1}^{2n} rS_m$. Then there is $s \in S_j$ with $x = rs$, where $s = s_{i_0} a$ for some i_0 and $a \in A$ with $f(a) = j$. Here the equality holds since $s \notin T_{j+1}$. Thus there are j distinct s_i such that $s_i a \in A$. Also, since S is right cancellative, these $s_i a$ are distinct. Moreover, since $rs_i a = rs_{i_0} a = x \notin \bigcup_{m=j+1}^{2n} rS_m = rT_{j+1}$, all the $s_i a$ are in S_j. We have thus proved that for any $x \in rS_j \backslash \bigcup_{m=j+1}^{2n} rS_n$, there are at least j elements $s \in S_j$, such that $rs = x$. This gives (3.3.4). Summing up for $j = 0, 1, \ldots, 2n$, we obtain

$$|rA| \leq |rS_0| + \sum_{j=1}^{2n} \frac{1}{j}|S_j|$$
$$\leq |S_0| + \sum_{j=1}^{2n} \frac{1}{j}|S_j|,$$

and

$$(3.3.5) \qquad |A \backslash rA| \geq |A| - |rA| \geq \sum_{j=1}^{2n} \left(1 - \frac{1}{j}\right)|S_j|.$$

Finally, from (3.3.1), (3.3.3) and (3.3.5),

$$k|A| \geq \frac{1}{3n}\left(n|A \backslash rA| + \sum_{i=1}^{2n} |A \backslash s_i A|\right)$$
$$\geq \frac{1}{3n}\left(n \sum_{j=1}^{2n} (1 - \frac{1}{j})|S_j| + n|S_0| + \frac{1}{2}\sum_{j=1}^{2n} \frac{2n-j}{j}|S_j|\right)$$
$$= \frac{1}{3n}\left(n|S_0| + \sum_{j=1}^{2n} (n - \frac{1}{2})|S_j|\right)$$
$$\geq \frac{1}{3n}\sum_{j=0}^{2n} (n - \frac{1}{2})|S_j| = (\frac{1}{3} - \frac{1}{6n})|A|;$$

i.e., $k \geq 1/3 - 1/6n$. □

It can be seen from the above proof that for an arbitrary semigroup S, the same result also holds under the additional condition that $s_1, \ldots, s_{2n}$ belong to different right cancellative classes. In other words, for any $a \in S$, $i \neq j$ implies $s_i a \neq s_j a$.

COROLLARY 3.3.2. *For any semigroup S, if $\varphi(S) \neq 0$, then $\varphi(S) \geq 1/6$.*

Proof. We may assume that S is left amenable. By Lemma 2.1 in [23], there exist $r, s, t \in S$ with $rs = rt$ but $sx \neq tx$ for any $x \in S$. Now our theorem applies with $n = 1$. □

If there is a subset in S having a sort of "uniform cancellation property", we can get a much sharper inequality for $\varphi(S)$ which will be used to solve Namioka's problem.

THEOREM 3.3.3. *Suppose S is a right cancellative semigroup. If there exists a finite subset F in S with the following properties:*

(i) $|F| = n \geq 2$,

(ii) $\forall r, s, t \in F,\ \ rs = rt$,

(iii) $\forall r_1, r_2 \in F,\ \ \forall s, t \in S,\ \ r_1 s = r_1 t \Leftrightarrow r_2 s = r_2 t$,

then $\varphi(S) \geq 1 - 1/n$.

We divide the proof into a series of lemmas.

LEMMA 3.3.4. *For any positive integer $m \geq 2$, the set F^m also has properties (i) – (iii).*

Proof. (i) Take $r \in F$. Then $F^m = Fr^{m-1}$ by (ii). But $|Fr^{m-1}| = n$ since S is right cancellative.

(ii) This follows from the fact that $r_1 \ldots r_m r'_1 \ldots r'_m = r_1^{2m}$ for $r_1, \ldots, r_m, r'_1, \ldots, r'_m \in F$.

(iii) For $r_1 \ldots r_m$ and $r'_1 \ldots r'_m \in F^m$, and $s, t \in S$, if

$$r_1 \ldots r_m s = r_1 \ldots r_m t,$$

then

$$r_1' \dots r_m' s = r_1' r_1^{m-1}\, s = r_1' r_1^{m-1}\, t = r_1' \dots r_m' t$$

by (iii), since $r_1 r_1^{m-1}\, s = r_1 \dots r_m s = r_1 \dots r_m t = r_1 r_1^{m-1}\, t$. □

Now let A be a finite subset of S. Given a positive integer m, we define an equivalence relation $\sim_m$ on A by

$$s \sim_m t \Leftrightarrow \exists r \in F^m, \quad \text{such that} \quad rs = rt.$$

By (iii) this defines an equivalence relation. An equivalence class for the relation $\sim_m$ will be called a class of level m. Denote by N_m the total number of classes of level m in A. Since $s \sim_m t \Rightarrow s \sim_{m+1} t$, each class of level $m+1$ is the disjoint union of some classes of level m, and

$$|A| \geq N_1 \geq N_2 \geq \dots .$$

Denote by

$$k_m = \frac{1}{n} \sum_{r \in F^m} \frac{|A \backslash rA|}{|A|}.$$

LEMMA 3.3.5. *For any (nonempty) finite subset A of S, if $k_m < 1 - 1/n$, then*

$$N_m - N_{2m} > \frac{1}{4}\Big(1 - \frac{1}{n} - k_m\Big)^2 |A|.$$

Proof. Define a function $f : S \to \mathbb{Z}^+$ by $f = \sum_{r \in F^m} \chi_{rA}$. We have $0 \leq f(s) \leq n$, and the average of f on A is given by

$$\begin{aligned}\frac{1}{|A|} \sum_{s \in A} f(s) &= \frac{1}{|A|} \sum_{r \in F^m} |rA \cap A| \\ &= \frac{1}{|A|} \sum_{r \in F^m} (|A| - |A \backslash rA|) \\ &= n - k_m n = (1 - k_m)n.\end{aligned}$$

Let δ be any real number greater than 1, and let $A_1 = \{s \in A : f(s) > (1-k_m)n/\delta\}$ and $A_2 = A \setminus A_1$. Then

$$\begin{aligned}(1-k_m)n|A| = \sum_{s\in A} f(s) &= \sum_{s\in A_1} f(s) + \sum_{s\in A_2} f(s) \\ &\le n|A_1| + \frac{(1-k_m)n}{\delta}|A|.\end{aligned}$$

So

$$|A_1| \ge \left(1 - \frac{1}{\delta}\right)(1-k_m)|A|. \tag{3.3.6}$$

Let C be a class of level m. Then for any $r \in F^m$, $|rC| = 1$. Furthermore, if $s \sim_m t$ and $r_1, r_2 \in F^m$, then $r_1 s \sim_m r_2 t$, by (ii). Thus $(F^m \cdot C) \cap A$ is contained in a single class of level m (which may be empty).

Suppose that there exists $s \in C$ with $f(s) > 0$. Then $s \in r_i A$ for distinct $r_1, r_2, \ldots, r_{f(s)} \in F^m$. In other words, there exist $f(s)$ classes $C_1, C_2, \ldots, C_{f(s)}$ of level m with $r_i C_i = \{s\}$. It is easy to see that these C_i are disjoint. By (ii), these classes are contained in the same class $\overline{C}$ of level $2m$. For a class C' of level m such that $(F^m \cdot C') \cap A \neq \emptyset$, $C' \subset \overline{C}$ if and only if $(F^m \cdot C') \cap A \subset C$. For, let $t_1 \in C'$ and $r \in F^m$ be such that $rt_1 \in A$, and $t_2 \in C_1 \subset \overline{C}$. Then

$$\begin{aligned}(F^m \cdot C') \cap A \subset C \Leftrightarrow rt_1 \in C &\Leftrightarrow rt_1 \sim_m rt_2 \\ &\Leftrightarrow r^2 t_1 = r^2 t_2 \Leftrightarrow t_1 \sim_{2m} t_2 \\ &\Leftrightarrow C' \subset \overline{C}.\end{aligned}$$

This means that the map $C \to \overline{C}$ is independent of the choice of s and it is 1–1.

For every class C of level m for which $\overline{C}$ is defined, let $V(\overline{C})$ be the number of classes of level m contained in $\overline{C}$. Then for any $r \in F^m$, $|r\overline{C}| = V(\overline{C})$. So $\sum_{s \in C} f(s) \leq n \cdot V(\overline{C})$ by the definition of f, and

$$|C \cap A_1| < n \cdot V(\overline{C}) \Big/ \frac{(1-k_m)n}{\delta} = \frac{\delta \cdot V(\overline{C})}{1-k_m}. \tag{3.3.7}$$

If $C \cap A_1 \neq \emptyset$, then there exists an $s \in C$ with $f(s) > (1-k_m)n/\delta$. So $V(\overline{C}) \geq f(s) > (1-k_m)n/\delta$. Thus by (3.3.7),

$$\begin{aligned} \frac{V(\overline{C})-1}{|C \cap A_1|} &> \frac{V(\overline{C})-1}{\frac{\delta \cdot V(\overline{C})}{1-k_m}} = \frac{1-k_m}{\delta}\left(1 - \frac{1}{V(\overline{C})}\right) \\ &> \frac{1-k_m}{\delta}\left[1 - \frac{\delta}{(1-k_m)n}\right]. \end{aligned} \tag{3.3.8}$$

And then from (3.3.6) and (3.3.8)

$$\begin{aligned} N_m - N_{2m} &\geq \sum_{\overline{C}} (V(\overline{C}) - 1) \\ &\geq \sum \{V(\overline{C}) - 1 \mid C \cap A_1 \neq \emptyset\} \\ &> \sum_{C} |C \cap A_1| \frac{1-k_m}{\delta}\left[1 - \frac{\delta}{(1-k_m)n}\right] \\ &= \frac{1-k_m}{\delta}\left[1 - \frac{\delta}{(1-k_m)n}\right] |A_1| \\ &\geq \frac{\delta - 1}{\delta^2}(1-k_m)^2 \left[1 - \frac{\delta}{(1-k_m)n}\right] |A|. \end{aligned}$$

Let $\delta = 2\left(1 - \frac{1}{(1-k_m)n+1}\right)$. Then we obtain

$$N_m - N_{2m} > \frac{1}{4}\left(1 - k_m - \frac{1}{n}\right)^2 |A|.$$

□

Proof of Theorem 3.3.3. Suppose $\varphi(S) < 1 - 1/n$. Then WFC holds for some $k < 1 - 1/n$. By WFC, for any positive integer ℓ, there exists a finite subset

A of S such that

$$\frac{1}{(\ell+1)n}\sum_{i=0}^{\ell}\sum_{r\in F^{2^i}}|A\backslash rA|\le k|A|.$$

Adopting the above notations, we have

$$\frac{1}{\ell+1}\sum_{i=0}^{\ell}k_{2^i}\le k,\quad \text{or}\quad \frac{1}{\ell+1}\sum_{i=0}^{\ell}(1-\frac{1}{n}-k_{2^i})\ge 1-\frac{1}{n}-k.$$

Then

$$\begin{aligned}|A| &\ge \sum_{i=0}^{\ell}(N_{2^i}-N_{2^{i+1}})\ge \sum\left\{N_{2^i}-N_{2^{i+1}}\ :0\le i\le \ell,\ k_{2^i}<1-\frac{1}{n}\right\}\\ &>\sum\left\{\frac{1}{4}(1-\frac{1}{n}-k_{2^i})^2|A|:0\le i\le\ell,\ k_{2^i}<1-\frac{1}{n}\right\}\\ &\ge\frac{|A|}{4}\cdot\frac{1}{\ell+1}\left[\sum\left\{1-\frac{1}{n}-k_{2^i}\ :0\le i\le\ell,\ k_{2^i}<1-\frac{1}{n}\right\}\right]^2\\ &\ge\frac{|A|}{4}\cdot\frac{1}{\ell+1}\left[\sum_{i=0}^{\ell}\left(1-\frac{1}{n}-k_{2^i}\right)\right]^2\\ &\ge\frac{|A|}{4}(\ell+1)(1-\frac{1}{n}-k)^2,\end{aligned}$$

or for any $\ell>0$,

$$\frac{1}{4}(\ell+1)\left(1-\frac{1}{n}-k\right)^2<1.$$

This is a contradiction since $1-1/n-k>0$, and the proof is complete. □

COROLLARY 3.3.6. *Let S be a right cancellative semigroup. If there exists a finite subset F of S satisfying conditions (i)–(iii) of Theorem 3.3.3, then S does not satisfy $SNFC$.*

COROLLARY 3.3.7. *Let S be a right cancellative semigroup. If there exists a sequence $\{F_n\}$ of finite subsets of S satisfying conditions (ii) and (iii) of Theorem 3.3.3 and $|F_n|\to\infty$, then S does not satisfy WFC.*

REMARK 3.3.8. The conclusion $\varphi(S) \geq 1 - 1/n$ is the best possible. For, consider the semigroup $\{a_1, \ldots, a_n\}$ with the operation $a_i a_j = a_i$. It is easy to check that this semigroup, with F equal to itself, satisfies all the conditions of Theorem 3.3.3, and $\varphi(S) = 1 - 1/n$ by Theorem 3.2.3.

For later applications we need a slightly different version of Theorem 3.3.3.

THEOREM 3.3.9. *Let S be a semigroup with the finite intersection property for right ideals. If S has a finite subset F with the following properties:*

(i) $|F| = n \geq 2$,

(ii) $\forall r, s, t \in F, \; rs = rt$,

(iii)' $\forall r_1, r_2 \in F, \; \forall s, t \in S, \; r_1 s R r_1 t \Leftrightarrow r_2 s R r_2 t$,

(iv) Different elements of F belong to different right cancellative classes;

i.e., $\forall r_1, r_2 \in F, \; r_1 R r_2 \Rightarrow r_1 = r_2$,

then $\varphi(S) \geq 1 - 1/n$. (See the last part of section III.2 for the relation R.)

To prove Theorem 3.3.9, we need to change the equivalence relation $\sim_m$ into $\sim'_m$ defined by

$$s \sim'_m t \Leftrightarrow \exists r \in F^m, \quad rsRrt$$

in the proof of Theorem 3.3.3. The rest of the proof works with little modification.

III.4. Semidirect Products and Left Amenability.

For a semigroup U, we denote by $\mathrm{End}(U)$ the semigroup of all endomorphisms of U. Similarly, $\mathrm{Inj}(U)$ and $\mathrm{Sur}(U)$ will be the semigroups of all injective or surjective endomorphisms of U, respectively. And $\mathrm{Aut}(U) = \mathrm{Inj}(U) \cap \mathrm{Sur}(U)$.

Let U and T be two semigroups, ρ a homomorphism of T into $\mathrm{End}(U)$. The semidirect product of U by T (with respect to ρ) is the set $U \times T$ associated with the multiplication $\langle u, a\rangle\langle v, b\rangle = \langle u\rho_a(v), ab\rangle$, denoted by $U \times_\rho T$. It is also a semigroup.

Maria Klawe [23] initiated the study of semidirect products for amenable semigroups. For convenience, we collect some of her results here (Propositions 3.4.1–3.4.5).

PROPOSITION 3.4.1. *If U and T are right cancellative, so is $S = U \times_\rho T$. If U and T are left cancellative, then S is left cancellative iff $\rho(T) \subset \mathrm{Inj}(U)$.*

PROPOSITION 3.4.2. *If U and T are left amenable and $\rho(T) \subset \mathrm{Sur}(U)$, then $S = U \times_\rho T$ is also left amenable.*

PROPOSITION 3.4.3. *If $S = U \times_\rho T$ is left amenable, then U and T are left amenable.*

PROPOSITION 3.4.4. *If U and T satisfy SFC and $\rho(T) \subset \mathrm{Aut}(T)$, then $S = U \times_\rho T$ also satisfies SFC.*

PROPOSITION 3.4.5. *If $S = U \times_\rho T$ satisfies SFC, then U and T also satisfy SFC.*

From those results one can see that if U and T are two left amenable cancellative semigroups, $\rho : T \to \mathrm{Sur}(U)$ a homomorphism such that $\rho(T) \not\subset \mathrm{Inj}(U)$, then $S = U \times_\rho T$ is left amenable, right cancellative, but not left cancellative. So it does not satisfy SFC (see [23] or our Theorem 3.3.1). The following example is due to Klawe.

EXAMPLE 3.4.6 ([23]). Let U be the free abelian semigroup generated by the elements $\{u_i : i = 0, 1, 2, \ldots\}$, and T an infinite cyclic semigroup with generator a. Define $\rho : T \to \mathrm{Sur}(U)$ by $\rho_a(u_i) = u_{i-1}$ if $i \geq 1$ and $\rho_a(u_0) = u_0$. Since $\rho_a \notin \mathrm{Inj}(U)$, the semidirect product $S = U \times_\rho T$ is left amenable but does not satisfy SFC.

In the remaining part of this section, we will use Klawe's example 3.4.6 to solve both Namioka's problem and Klawe's problem on the homomorphic image of a semigroup with SFC. Then we will give some necessary and sufficient conditions for a semidirect product to be left amenable.

PROPOSITION 3.4.7. *There exist left amenable semigroups with Følner number equal to 1. So none of $SNFC$, $WNFC$ or WFC is necessary for a semigroup to be left amenable.*

Proof. Klawe's example S is left amenable and right cancellative. Let $F_n = \{\langle u_0^{j-1} u_1^{n-j}, a\rangle : j = 1, \ldots, n\}$, where $u^0 u^n$ is understood to be u^n. Then F_n satisfies conditions (i)–(iii) of Theorem 3.3.3 with $|F_n| = n$. So $\varphi(S) = 1$. (This can also be obtained directly from Theorem 3.5.1.) □

Klawe [23] asked whether homomorphic images of semigroups satisfying SFC also satisfy SFC. We now show that Klawe's example is a homomorphic image of some semigroup having SFC.

PROPOSITION 3.4.8. *There exists a semigroup X and a homomorphism h from X such that $\varphi(X) = 0$ and $\varphi(h(X)) = 1$.*

Proof. Let Y be the free abelian semigroup generated by $\{u_i : i \in \mathbb{Z}\}$, U, T and ρ as in Example 3.4.6. Define $\tau : T \to \mathrm{Aut}(Y)$ by $\tau_a(u_i) = u_{i-1}$, for $i \in \mathbb{Z}$. Let $X = Y \times_\tau T$. Then $\varphi(X) = 0$ by Proposition 3.4.4. Define a homomorphism $h' : Y \to U$ by

$$h'(u_i) = \begin{cases} u_i, & i \geq 1; \\ u_0, & i \leq 0. \end{cases}$$

Note that $h' \circ \tau_a = \rho_a \circ h'$. Now define $h : X \to S = U \times_\rho T$ by $h(\langle x, a^n \rangle) = \langle h'(x), a^n \rangle$. Then

$$\begin{aligned} h(\langle x, a^n \rangle \langle y, a^m \rangle) &= h(\langle x\tau_{a^n}(y), a^{n+m} \rangle) \\ &= \langle h'(x)h'(\tau_{a^n}(y)), a^{n+m} \rangle = \langle h'(x)\rho_{a^n}(h'(y)), a^{n+m} \rangle \\ &= \langle h'(x), a^n \rangle \langle h'(y), a^m \rangle = h(\langle x, a^n \rangle) h(\langle y, a^m \rangle). \end{aligned}$$

So h is a homomorphism of X onto S. By Proposition 3.4.7, $\varphi(S) = 1$. □

Among other properties of S, we point out that any left amenable subsemigroup of S has Følner number either 0 or 1, and any finitely generated left amenable subsemigroup of S is abelian. The proofs are omitted.

Now we give two necessary and sufficient conditions for a semidirect product to be left amenable. In the next section we will give necessary and sufficient conditions for a semidirect product to satisfy SFC.

THEOREM 3.4.9. *Let U and T be two left amenable semigroups, $\rho : T \to End(U)$ a homomorphism. Then the following are equivalent:*

(i) $S = U \times_\rho T$ is left amenable;

(ii) $S = U \times_\rho T$ has the finite intersection property for right ideals;

(iii) $\forall u \in U,\ \forall a \in T,\ u\rho_a(U) \cap \rho_a(U) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii). This is a well-known fact.

(ii) $\Rightarrow$ (iii). Take $u \in U,\ a \in T$. By (ii), $\langle u, a\rangle S \cap \langle \rho_a(u), a\rangle S \neq \emptyset$. This implies that $u\rho_a(U) \cap \rho_a(u)\rho_a(U) = u\rho_a(U) \cap \rho_a(uU) \neq \emptyset$.

(iii) $\Rightarrow$ (i). For each $a \in T$, define a linear operator P_a on $\ell^\infty(U)$ by $P_a g(u) = g(\rho_a(u))$ for $g \in \ell^\infty(U)$ and $u \in U$. Each P_a induces a dual operator P_a^* on $\ell^\infty(U)^*$ given by $P_a^*\psi(g) = \psi(P_a g)$ for $\psi \in \ell^\infty(U)^*$ and $g \in \ell^\infty(U)$. Obviously, if ψ is a mean on $\ell^\infty(U)$, $P_a^*\psi$ is also a mean on $\ell^\infty(U)$. Suppose ψ is a left invariant mean on $\ell^\infty(U)$, $v \in U$. By (iii), there are $x, y \in U$, such that $v\rho_a(x) = \rho_a(y)$. We have

$$\begin{aligned} P_a^*\psi(\ell_v g) &= \psi(P_a(\ell_v g)) = \psi(\ell_x P_a(\ell_v g)) \\ &= \psi(P_a(\ell_{v\rho_a(x)}\, g)) = \psi(P_a(\ell_{\rho_a(y)}\, g)) \\ &= \psi(\ell_y(P_a(g)) = \psi(P_a g) = P_a^*\psi(g). \end{aligned}$$

Thus $P_a^*\psi$ is also a left invariant mean. As in the proof of [23, Lemma 3.3 and Prop. 3.4], the map $a \to P_a^*$ is a representation of T in the set of linear mappings on the set $ML(U)$ of all left invariant means on $\ell^\infty(U)$. Since $ML(U)$ is w^*-compact and convex, by Theorem 1.2.3, there exists $\psi \in ML(U)$ with $P_a^*\psi = \psi$ for each $a \in T$. For each $f \in \ell^\infty(S)$ define $\bar{f} \in \ell^\infty(T)$ by $\bar{f}(a) = \psi(f_a)$, where $f_a \in \ell^\infty(U)$ is defined as $f_a(u) = f(u, a)$. Choose $\nu \in ML(T)$ and define $\mu \in \ell^\infty(S)^*$ by $\mu(f) = \nu(\bar{f})$. It follows by routine computation that μ is a left invariant mean on S (see [23, Prop. 3.4]). So S is left amenable. □

COROLLARY 3.4.10. *Let U and T be two left amenable semigroups, $\rho : T \to$*

$End(U)$ a homomorphism. If for any $a \in T$, $\rho_a(U)$ contains a right ideal of U, then $S = U \times_\rho T$ is left amenable.

Proof. Take $u \in U$ and $a \in T$. Since $\rho_a(U)$ contains a right ideal, $u\rho_a(U)$ also contains a right ideal. U, as a left amenable semigroup, has the finite intersection property for right ideals. Therefore $u\rho_a(U) \cap \rho_a(U) \neq \emptyset$. □

EXAMPLE 3.4.11. We give some applications of Theorem 3.4.9 and Corollary 3.4.10.

(i) Let $U = \{q \in \mathbb{Q} : q \geq 1\}$ with the usual addition. $T = \{r \in \mathbb{Q} : r \geq 1\}$ with the usual multiplication. The action of T on U is given by the relation $\rho_r(q) = rq$, $r \in T$, $q \in U$. Since for any $r \in T$, $\rho_r(U) = \{q \in U : q \geq r\}$ is an ideal in U, by Corollary 3.4.10, $S = U \times_\rho T$ is left amenable.

(ii) Let $\mathbb{Q}^+$ be the set of nonnegative rationals, and $\mathbb{Z}^+$ the set of nonnegative integers, with the usual addition. Let $U = \mathbb{Q}^+ \oplus \mathbb{Z}^+$, T the infinite cyclic semigroup generated by a. Define $\rho_a(\langle r, n\rangle) = \langle r + n, n\rangle$. Then $\rho_a(U)$ does not contain any ideal of U. But by Theorem 3.4.9, $S = U \times_\rho T$ is still left amenable.

III.5. Semidirect Products and Følner-Type Conditions.

For left cancellative semigroups, finite semigroups, and abelian semigroups, SFC, $SNFC$ and $WNFC$ are all equivalent (to left amenability). It is natural to ask whether these conditions are equivalent in general. In this section we will prove that for a semidirect product of two semigroups satisfying SFC, they are equivalant (to $LA + WFC$).

If a semigroup S has the finite intersection property for right ideals and its right cancellative quotient semigroup $S/(R)$ is left cancellative, we say S satisfies Sorenson's condition. See [37] or [23] for Sorenson's conjecture. It is known that S satisfies SFC if and only if S is left amenable and satisfies Sorenson's condition (cf. [1] and [23]).

Let U be a semigroup with the finite intersection property for right ideals, and $h \in \text{End}(U)$. Since sRt implies $h(s)Rh(t)$, h can be reduced to $\bar{h} \in \text{End}(U/(R))$, defined by $\bar{h}(\bar{s}) = \overline{h(s)}$. And for $h_1, h_2 \in \text{End}(U)$, $\bar{h}_1 \circ \bar{h}_2 = \overline{h_1 \circ h_2}$. If $\rho : T \to \text{End}(U)$ is a homomorphism from another semigroup T into $\text{End}(U)$, then we can define $\bar{\rho} : T \to \text{End}(U/(R))$ by $\bar{\rho}_a = \overline{\rho_a}$. $\bar{\rho}$ is also a homomorphism.

THEOREM 3.5.1. *Let U and T be two semigroups where U satisfies Sorenson's condition. Suppose $\rho : T \to End(U)$ is a homomorphism such that $\bar{\rho}(T) \not\subset Inj(U/(R))$. Then the semidirect product $S = U \times_\rho T$ is either not left amenable or $\varphi(S) = 1$. In both cases S does not satisfy $WNFC$.*

Proof. For convenience we write $\sim$ for the right cancellative relation R on U. Sorenson's condition implies that for all $u, v, w \in U$, $wu \sim wv \Rightarrow u \sim v$.

Assume that S is left amenable and $\bar{\rho}(T) \not\subset \text{Inj}(U/(R))$. Then there exists $a \in T$ and $u, v \in U$ such that $u \not\sim v$ but $\rho_a(u) \sim \rho_a(v)$.

We claim that for any positive integer n, there are two elements $u_n, v_n \in U$ such that $\rho_{a^{n-1}}(u_n) \not\sim \rho_{a^{n-1}}(v_n)$ but $\rho_{a^n}(u_n) = \rho_{a^n}(v_n)$.

Select $w \in U$ with $\rho_a(u)w = \rho_a(v)w$. Since S is left amenable, by Theorem 3.4.9, $wU \cap \rho_a(U) \neq \emptyset$. Choose $w' \in U$ with $\rho_a(w') \in wU$. Then $\rho_a(uw') =$

$\rho_a(vw')$, and $uw' \not\sim vw'$, since $u \not\sim v$. Let $u_1 = uw'$ and $v_1 = vw'$.

Suppose $n \geq 2$. Again since S is left amenable, $\langle u_1, a^{n-1}\rangle S \cap \langle v_1, a^{n-1}\rangle S \neq \emptyset$. Therefore, $u_1\rho_{a^{n-1}}(U) \cap v_1\rho_{a^{n-1}}(U) \neq \emptyset$. Choose $w', w'' \in U$ so that

$$(3.5.1) \qquad u_1\rho_{a^{n-1}}(w') = v_1\rho_{a^{n-1}}(w'').$$

Since $u_1 \not\sim v_1$, $\rho_{a^{n-1}}(w') \not\sim \rho_{a^{n-1}}(w'')$. Applying ρ_a to both sides of (3.5.1), we get $\rho_a(u_1)\rho_{a^n}(w') = \rho_a(v_1)\rho_{a^n}(w'') = \rho_a(u_1)\rho_{a^n}(w'')$. Sorenson's condition on U ensures that $\rho_{a^n}(w') \sim \rho_{a^n}(w'')$. By the same argument as in the previous paragraph, we can find $w \in U$, with $\rho_{a^n}(w'w) = \rho_{a^n}(w''w)$, and also $\rho_{a^{n-1}}(w'w) \not\sim \rho_{a^{n-1}}(w''w)$. Let $u_n = w'w$ and $v_n = w''w$.

Define

$$F_n = \{\langle w_1 w_2 \ldots w_n, a^n\rangle \in S : w_i = u_i \text{ or } v_i\}.$$

Then F_n satisfies conditions (i), (ii), (iii)′ and (iv) in Theorem 3.3.9 with $|F_n| = 2^n$, as we will show.

(i) and (iv). We prove by induction that any two different words $w_1 w_2 \ldots w_n$ are not in the same right cancellative class of U. This implies (i) and (iv).

Suppose this is true for $n = k-1 \geq 1$. Denote by

$$F_k' = \{w_1 w_2 \ldots w_{k-1} u_k : w_i = u_i \text{ or } v_i\},$$

and

$$F_k'' = \{w_1 w_2 \ldots w_{k-1} v_k : w_i = u_i \text{ or } v_i\}.$$

By the inductive assumption and the fact $ac \sim bc \Rightarrow a \sim b$, each set F_k' or F_k'' satisfies our requirement. Let $w_1 \ldots w_{k-1} u_k \in F_k'$ and $w_1' \ldots w_{k-1}' v_k \in F_k''$. If they are in the same right cancellative class, then

$$\rho_{a^{k-1}}(u_1)\rho_{a^{k-1}}(u_2)\ldots\rho_{a^{k-1}}(u_{k-1})\rho_{a^{k-1}}(u_k)$$

$$= \rho_{a^{k-1}}(w_1 w_2 \ldots w_{k-1} u_k)$$

$$\sim \rho_{a^{k-1}}(w_1' w_2' \ldots w_{k-1}' v_k)$$

$$= \rho_{a^{k-1}}(u_1)\rho_{a^{k-1}}(u_2)\ldots\rho_{a^{k-1}}(u_{k-1})\rho_{a^{k-1}}(v_k).$$

Since U satisfies Sorenson's condition, we have

$$\rho_{a^{k-1}}(u_k) \sim \rho_{a^{k-1}}(v_k).$$

This contradicts our choice of u_k and v_k.

(ii) This follows from the fact that

$$\rho_{a^n}(w_1 w_2 \ldots w_n) = \rho_{a^n}(u_1)\rho_{a^n}(u_2)\ldots\rho_{a^n}(u_n).$$

(iii)′ For $s \in S$, write $s = \langle P_1(s), P_2(s)\rangle$. Suppose $r_1, r_2 \in F_n$ and $s, t \in S$ are such that $\exists x \in S,\ r_1 sx = r_1 tx$. Equivalently we have

$$P_1(r_1)\rho_{a^n}(P_1(sx)) = P_1(r_1)\rho_{a^n}(P_1(tx)), \tag{3.5.2}$$

and

$$a^n P_2(sx) = a^n P_2(tx), \tag{3.5.3}$$

by the definition of semidirect products. By Sorenson's condition, there exists $w \in U$ such that $\rho_{a^n}(P_1(sx))w = \rho_{a^n}(P_1(tx))w$. Theorem 3.4.9 implies that $wU \cap \rho_{a^n P_2(sx)}(U) \neq \emptyset$. Thus there exists $w' \in U$ such that

$$\rho_{a^n}(P_1(sx))\rho_{a^n P_2(sx)}(w') = \rho_{a^n}(P_1(tx))\rho_{a^n P_2(tx)}(w'), \tag{3.5.4}$$

since $a^n P_2(sx) = a^n P_2(tx)$. Let $y = x\langle w', a\rangle$. Then it is easy to check that $\rho_{a^n}(P_1(sy)) = \rho_{a^n}(P_1(ty))$ and $a^n P_2(sy) = a^n P_2(ty)$, by (3.5.4) and (3.5.3). It follows that $r_2 sy = r_2 ty$.

As a left amenable semigroup, S has the finite intersection property for right ideals. So by Theorem 3.3.9, $\varphi(S) = 1$. □

COROLLARY 3.5.2. *Let U and T be two semigroups where U satisfies SFC and T is left amenable. Suppose $\rho : T \to End(U)$ is a homomorphism satisfying condition (iii) in Theorem 3.4.9 and such that $\bar{\rho}(T) \not\subset Inj(U/(R))$. Then the semidirect product $S = U \times_\rho T$ is left amenable and $\varphi(S) = 1$; i.e., S does not satisfy WFC.*

Proof. By Theorem 3.4.9, S is left amenable. □

This corollary gives a large class of counterexamples for Namioka's problem.

Now we turn to conditions under which S satisfies SFC.

Let U and T be two semigroups satisfying SFC, and $\rho : T \to \mathrm{End}(U)$ a homomorphism. Suppose $S = U \times_\rho T$ is left amenable, and $\bar{\rho}(T) \subset \mathrm{Inj}(U/(R))$. Note that these conditions are necessary for S to satisfy SFC by Proposition 3.4.5 and Theorem 3.5.1.

Let $\langle u, a\rangle$, $\langle v, b\rangle \in S$, and suppose that there exists $\langle w, c\rangle \in S$, such that $\langle w, c\rangle\langle u, a\rangle = \langle w, c\rangle\langle v, b\rangle$; i.e., $w\rho_c(u) = w\rho_c(v)$ and $ca = cb$. Since U and T satisfy Sorenson's condition, there is an $x \in U$ and a $d \in T$, such that

$$\rho_c(u)x = \rho_c(v)x \quad \text{and} \quad ad = bd. \tag{3.5.5}$$

$\rho_c(u) \sim \rho_c(v)$ and $\bar{\rho}(U) \subset \mathrm{Inj}(U/(R))$ imply $u \sim v$. So there exists $x_1 \in U$ with $ux_1 = vx_1$. Since S is left amenable, $x_1U \cap \rho_{ad}(U) \neq \emptyset$ by Theorem 3.4.9. Hence we can find $x_2 \in U$ such that $u\rho_{ad}(x_2) = v\rho_{ad}(x_2) = v\rho_{bd}(x_2)$, or

$$u\rho_a(\rho_d(x_2)) = v\rho_b(\rho_d(x_2)). \tag{3.5.6}$$

It follows from (3.5.5) and (3.5.6) that

$$\langle u, a\rangle\langle \rho_d(x_2), d\rangle = \langle v, b\rangle\langle \rho_d(x_2), d\rangle.$$

Thus we have proved that S satisfies Sorenson's condition. But S is left amenable, so we obtain the following result.

LEMMA 3.5.3. *Let U and T be two semigroups satisfying SFC, and $\rho : T \to End(U)$ a homomorphism. If $\bar{\rho}(T) \subset \mathrm{Inj}(U/(R))$ and condition (iii) of Theorem 3.4.9 holds for ρ, then $S = U \times_\rho T$ satisfies SFC.*

Summing up the above results, we obtain the main theorem of this section.

THEOREM 3.5.4. *Let U and T be two semigroups satisfying SFC, $\rho : T \to End(U)$ a homomorphism. Let $S = U \times_\rho T$ be the semidirect product. Then the following are equivalent:*

(1) S satisfies SFC;

(2) S satisfies $SNFC$;

(3) S satisfies $WNFC$;

(4) S is left amenable and satisfies WFC;

(5) $\bar{\rho}(T) \subset \mathrm{Inj}(U/(R))$ and for all $u \in U$ and $a \in T$, $u\rho_a(U) \cap \rho_a(U) \neq \emptyset$.

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from the diagram of implications in section III.1. Also (4) $\Rightarrow$ (5) is an application of Theorem 3.4.9 and Theorem 3.5.1; (5) $\Rightarrow$ (1) is the previous lemma. □

If U and T are cancellative, then $\bar{\rho} = \rho$ and $U/(R) = U$, and moreover, the left amenability of U and T is equivalent to SFC. By Proposition 3.4.3, this is a consequence of each of (1), (2), (3) or (4).

COROLLARY 3.5.5. *Let U and T be two cancellative semigroups, and $\rho : T \to End(U)$ a homomorphism. Let $S = U \times_\rho T$ be the semidirect product. Then the following are equivalent:*

(1) S satisfies SFC;

(2) S satisfies $SNFC$;

(3) S satisfies $WNFC$;

(4) S is left amenable and satisfies WFC;

(5) U and T are left amenable, $\rho(T) \subset Inj(U)$, and for all $u \in U$ and $a \in T$, $u\rho_a(U) \cap \rho_a(U) \neq \emptyset$.

PROBLEM 3.5.6. Is there any left amenable semigroup S such that $0 < \overset{\circ}{\varphi}(S) < 1$? If not, then all the conditions SFC, $SNFC$, $WNFC$ and $LA + WFC$ are equivalent. We know that such an example cannot be finite, or abelian, or left cancellative, or a semidirect product of those "better" semigroups. Our Section III.3 is aimed at exploring this direction. But we can only get a lower bound of 1/6 (Corollary 3.3.2).

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