UNIVERSITY OF ALBERTA

Locally Compact Groups and Invariant Means on Their von Neumann Algebras

BY

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1993

ISBN 0-315-88118-6

University of Alberta

Release Form

Name of Author: **Zhiguo Hu**

Title of Thesis:

Locally Compact Groups and Invariant Means

on Their von Neumann Algebras

Degree for which Thesis was Presented: **Doctor of Philosophy**

Year This Degree Granted: **1993**

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UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Locally Compact Groups and Invariant Means on Their von Neumann Algebras** submitted by **Zhiguo Hu** here in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To my Mom and Dad, to Qin and to Kevin

ABSTRACT

Let G be a locally compact group and let $VN(G)$ be the von Neumann algebra generated by the left regular representation of G. In this thesis, we shall investigate the local structure of G at its unit element and the set $TIM(\hat{G})$ of topologically invariant means on $VN(G)$.

We denote by $b(G)$ the smallest cardinality of an open basis at the unit element e of a non-discrete locally compact group G and by μ the first ordinal with $|\mu| = b(G)$. Let $X = \{\alpha; \alpha < \mu\}$ and $\mathcal{F}(X) = \{\phi \in l^\infty(X)^*; \|\phi\| = \phi(1) = 1$ and $\phi(f) = 0$ if $f \in l^\infty(X)$ and $\lim_{\alpha \in X} f(\alpha) = 0\}$. It is shown that $\mathcal{F}(X)$ is a big set, that is, $|\mathcal{F}(X)| = 2^{2^{|X|}}$.

The technique used by Chou of embedding a large set into $TIM(\hat{G})$ is generalized to the non-metrizable case. It is proved that if G is non-discrete, then there exists a one-one map $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ such that $W(\mathcal{F}(X)) \subseteq 2^{TIM(\hat{G})}$. In particular, the exact cardinality of $TIM(\hat{G})$ is obtained for any non-discrete locally compact group G, in terms of the locally structural property of G: $|TIM(\hat{G})| = 2^{2^{b(G)}}$.

In the attempt to achieve results on the cardinality and the structure of the set $TIM(\hat{G})$, we find a very interesting property concerning the local structure of G at e: if G is σ-compact and non-metrizable, then there exists a decreasing net $(N_\alpha)_{\alpha \leq \mu}$ of normal subgroups of G, where μ is the first ordinal satisfying $|\mu| = b(G)$, such that $N_0 = G$, $N_\mu = \{e\}$ and $b(N_\alpha) = b(G)$ for all $\alpha < \mu$; N_α is compact if $\alpha > 0$; $N_\alpha / N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for $\alpha < \mu$; and $N_\gamma = \bigcap_{\alpha < \gamma} N_\alpha$ for each limit ordinal $\gamma \leq \mu$. This improves a result obtained by Lau and Losert.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my heartfelt gratitude to my supervisor, Professor Anthony T. Lau, for his excellent guidance and unremitting encouragement throughout the preparation of this thesis. I indeed count myself fortunate to have worked with some one like him - with a genuine care in general and a unique mathematical insight in particular.

I am also grateful to Professors C. Chou, B. Forrest, E. E. Granirer, V. Losert, P. Milnes and A. L. T. Paterson for their interests in my work and for their helps in various ways.

Special thanks are due to Qin Xu, my husband and colleague, whose continued encouragement and support have meant more than I can say.

Finally, I wish to thank the University of Alberta and the Killam Memorial Scholarship for their financial support.

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CHAPTER 1

INTRODUCTION

The study of the cardinality of the set of invariant means on a group was initiated by Day ([3]) and Granirer ([8]). In 1976, Chou ([1]) showed that for a discrete infinite amenable group G the cardinality of the set $LIM(G)$ of all left invariant means on $l^\infty(G)$ is $2^{2^{|G|}}$. Later, Lau and Paterson ([20]) proved that if G is a noncompact amenable locally compact group, then the set $TLIM(G)$ of all topologically left invariant means on $L^\infty(G)$ has cardinality $2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a covering of G by compact sets. (it is easy to see that when G is compact, $TLIM(G)$ is the singleton containing only the normalized Haar measure of G). Some results on the size of the set $LIM(G) \setminus TLIM(G)$ have been obtained by Granirer ([9]), Rudin ([29]) and Rosenblatt ([26]). See also Yang ([32]) and Miao ([21]) for some recent developments in certain related aspects. More details on the study of the size and the structure of the set of invariant means on groups and semigroups can be found in the books by Pier ([23]) and Paterson ([22])

Let G be a locally compact group, $A(G)$ the Fourier algebra of G, $VN(G)$ the von Neumann algebra defined by the left regular representation $\{\rho, L^2(G)\}$ and $TIM(\hat{G})$ the set of all topologically invariant means on $VN(G)$. The set $TIM(\hat{G})$ was first studied by Dunkl and Ramirez for compact groups. They showed ([4]) that if G is an infinite compact group, then $|TIM(\hat{G})| \geq 2$. Renaud ([25, Theorem 1]) proved that there exists a unique topologically invariant mean on $VN(G)$ when

G is discrete. In ([10, Theorem 1]), Granirer showed the following: if G is non-discrete and second countable (i.e., there is a countable basis for open sets in G), then $TIM(\hat{G})$ is not norm separable. A stronger result was obtained by Chou in ([2, Theorem 3.3]): if G is non-discrete and metrizable, then there exists a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ which embeds a "big subset" (having cardinality 2^c) of $(l^\infty)^*$ into $TIM(\hat{G})$. See also Granirer [13, P.172-173] for the discussion on the set $TIM_p(\hat{G})$ of topologically invariant means on $A_p(G)^*$, where $A_p(G)$ is the Figà-Talamanca-Herz space $(1 < p < \infty)$ and $A_p(G) = A(G)$ if $p = 2$. In particular, he proved that $|TIM_p(\hat{G})| \geq 2^c$ in case G is second countable and non-discrete. Recently, Lau and Losert showed, among many other results, that if $VN(G)$ has a unique topologically invariant mean, then G must be discrete (see [19, Theorem 4.10 and Corollary 4.11]). They actually remedied Renaud's result by using a totally different machinery (as noticed by a number of mathematicians, there is a gap in the proof of [25, Proposition 8], see [19, P.21]).

The main purpose of this thesis is to investigate the cardinality and the structure of the set $TIM(\hat{G})$ of topologically invariant means on the von Neumann algebra $VN(G)$ of a non-discrete locally compact group G. It contains five chapters.

Chapter 2 consists of a summary of notations and preliminary results used throughout this thesis.

In Chapter 3, we apply the set theory to obtain the cardinality of a set of means which will be used to establish a one-one map into $TIM(\hat{G})$. For an initial ordinal μ, let X be the set of all ordinals less than μ with its natural order. We introduce a subset of $l^\infty(X)^*$:

$$\mathcal{F}(X) = \{\phi \in l^\infty(X)^* : \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } f \in c_0(X)\},$$

where $c_0(X) = \{f \in l^\infty(X); \lim_{\alpha \in X} f(\alpha) = 0\}$ and $\mathbf{1}$ is the constant function of value one. This set with $X = \mathbf{N}$ was first considered by Chou [2]. We prove that $\mathcal{F}(X)$ is a big set, that is, $|\mathcal{F}(X)| = 2^{2^{|X|}}$.

Chapter 4 concerns itself with the local structure of G at the unit element e. The main idea of this chapter was motivated by Lau-Losert [19, Lemma 4.8]. Let G be a σ-compact non-metrizable locally compact group and let μ be the initial ordinal satisfying $|\mu| = b(G)$, where $b(G)$ is the smallest cardinality of an open basis at e. We show that there exists a decreasing family $(N_\alpha)_{\alpha \leq \mu}$ of normal subgroups of G such that $N_0 = G$, $N_\mu = \{e\}$ and $b(N_\alpha) = b(G)$ for all $\alpha < \mu$; N_α is compact if $\alpha > 0$; $N_\alpha / N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for $\alpha < \mu$; and $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for each limit ordinal $\gamma \leq \mu$ (Theorem 4.3.1). This interesting property concerning the local structure of G at e plays a key role in our investigation on the set $TIM(\hat{G})$ and its proof constitutes the major technical part of this thesis.

Chapter 5 deals with the set $TIM(\hat{G})$ of topologically invariant means on the von Neumann algebra $VN(G)$ of a non-discrete locally compact group G. We will mainly focus on determining the cardinality of $TIM(\hat{G})$.

In Section 5.2, we generalize Chou's result of embedding a large set into $TIM(\hat{G})$ to the non-metrizable case. Let μ be the initial ordinal with $|\mu| = b(G)$ and $X = \{\alpha ; \alpha < \mu\}$. In case G is σ-compact and non-metrizable, we construct a family of linear isometries $(\pi_j^*)_{j\in J}$ of $l^\infty(X)^*$ into $VN(G)^*$. For each $\phi \in l^\infty(X)^*$, let

$$W_\phi = \{ \text{ all } w^*\text{-cluster points of } (\pi_j^*\phi)_{j\in J} \text{ in } VN(G)^* \}.$$

It is shown that $\{W_\phi\,;\ \phi \in l^\infty(X)^*\}$ is a family of pairwise disjoint non-empty subsets of $VN(G)^*$ and $W_\phi \subseteq TIM(\hat{G})$ if $\phi \in \mathcal{F}(X)$ (Theorem 5.2.4). Consequently, if G is non-discrete, then there exists a one-one map $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ such that $W(\mathcal{F}(X)) \subseteq 2^{TIM(\hat{G})}$ (Theorem 5.2.5).

In Section 5.3, the equality $|TIM(\hat{G})| = 2^{2^{b(G)}}$ is established for a non-discrete locally compact group G (Theorem 5.3.3). If G is abelian and $\hat{G}$ is its dual group, then $A(G)$ can be identified with $L^1(\hat{G})$ and $VN(G)$ with $L^\infty(\hat{G})$; it can be seen that, in this case, $m \in VN(G)^*$ belongs to $TIM(\hat{G})$ if and only if the corresponding mean in $L^\infty(\hat{G})$ is a topologically left invariant mean. Since $b(G) = d(\hat{G})$ (see [17, (24.48)]), our Theorem 5.3.3 coincides with Lau-Paterson's result [20, Theorem 1] for the abelian case.

In Section 5.4, we show that if G is non-discrete, then $TIM(\hat{G})$ contains a subset E with the cardinality $|E| = |TIM(\hat{G})| = 2^{2^{b(G)}}$ such that $\|m_1 - m_2\| = 2$ for m_1, $m_2 \in E$ and $m_1 \neq m_2$. Let $UCB(\hat{G})$ be the space of all uniformly continuous functionals on $A(G)$ and $F(\hat{G})$ be the space of topological almost convergent elements in $VN(G)$. We prove that any norm dense subset of $UCB(\hat{G})/F(\hat{G}) \bigcap UCB(\hat{G})$ has cardinality greater than $b(G)$ when G is non-discrete; in particular, the space $UCB(\hat{G})/F(\hat{G}) \bigcap UCB(\hat{G})$ is not norm separable. We also improve a result obtained by Granirer.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

2.1. Introduction.

This chapter is intended to be a reference for the definitions and notations used throughout the thesis. Section 2.2 is an introduction to locally compact groups, related Banach spaces (algebras) and invariant means, while Section 2.3 deals with some notations and preliminary results on sets and Stone-Čech compactifications.

2.2. Locally Compact Groups, Some Related Banach Spaces and Invariant Means.

Let $\mathbf{C}$ be the complex field. For a Banach space E over $\mathbf{C}$, let E^* denote the Banach space of all bounded linear functionals on E. If $\phi \in E^*$, then the value of ϕ at an element x in E will be written as $\phi(x)$ or $\langle \phi, x \rangle$.

Let G be a locally compact group with unit element e and a fixed left Haar measure λ. The left invariant Haar integral associated with λ will be denoted by $\int_G f(x)\, d\lambda(x)$, or simply by $\int_G f(x)\, dx$.

For $1 \leq p \leq \infty$, let $(L^p(G)\,,\, \|\cdot\|_p)$ be the usual Banach space associated with G and λ. With the inner product

$$\langle f\,,\, g \rangle = \int_G f(x)\overline{g(x)}\, dx\,, \qquad f\,,\, g \in L^2(G)\,,$$

$L^2(G)$ becomes a Hilbert space.

An element $m \in L^\infty(G)^*$ is called a *mean* if

$$\|m\| = 1 \text{ and } m(f) \geq 0 \text{ whenever } f \geq 0 .$$

A mean m is said to be *left invariant* if

$$m({}_xf) = m(f) , \qquad \text{for } x \in G \text{ and } f \in L^\infty(G) ,$$

where ${}_xf$ is the left translation of f by x, i.e., $({}_xf)(y) = f(xy)$, $y \in G$. A mean m is said to be *topologically left invariant* if

$$m(\phi * f) = m(f) ,$$

for $f \in L^\infty(G)$ and $\phi \in L^1(G)$ with $\phi \geq 0$ and $\|\phi\|_1 = 1$, where

$$(\phi * f)(x) = \int_G \phi(t) f(t^{-1}x)\, dt , \qquad x \in G .$$

Let $LIM(G)$ ($TLIM(G)$) denote the set of (topologically) left invariant means on $L^\infty(G)$. When $LIM(G)$ (or $TLIM(G)$) is nonempty, G is said to be *amenable*.

Let $VN(G)$ be the von Neumann algebra defined by the left regular representation $\{\rho, L^2(G)\}$ of G, i.e., the closure of the linear span of $\{\rho(a) ; a \in G\}$ in the weak operator topology on $\mathcal{B}(L^2(G))$, where $\mathcal{B}(L^2(G))$ is the Banach algebra of all bounded linear operators on $L^2(G)$ and $\rho(a)f(x) = f(a^{-1}x)$, $x \in G$, $f \in L^2(G)$. The weak operator topology on $\mathcal{B}(L^2(G))$ is the topology generated by the family $\{P_{f,g} ; f, g \in L^2(G)\}$ of seminorms on $\mathcal{B}(L^2(G))$, where

$$P_{f,g}(T) = |\langle Tf , g\rangle| , \qquad \text{for } T \in \mathcal{B}(L^2(G)) .$$

Let $A(G)$ be the Fourier algebra of G, consisting of all functions of the form $f * \check{g}$, where $f, g \in L^2(G)$ and $\check{g}(x) = \overline{g(x^{-1})}$.

If $\phi = f * \check{g} \in A(G)$, then ϕ can be regarded as an ultraweakly continuous functional on $VN(G)$ defined by

$$\phi(T) = \langle Tf \, , \, g \rangle \, , \qquad \text{for } T \in VN(G) \, .$$

The ultraweak topology on $\mathcal{B}(L^2(G))$ is the topology generated by the following seminorms:

$$T \longmapsto \Big| \sum_{i=1}^{\infty} \langle Tf_i \, , \, g_i \rangle \Big| \, , \qquad \text{for } T \in \mathcal{B}(L^2(G)) \, ,$$

where $f_1, f_2, \cdots$ and $g_1, g_2, \cdots$ run through $L^2(G)$ with $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|^2 < \infty$.

Furthermore, as shown by P. Eymard in [6, P.210 and P.218], each ultraweakly continuous functional on $VN(G)$ is of the form $f * \check{g}$ with $f, g \in L^2(G)$.

Therefore, $A(G)$ is the predual of $VN(G)$, i.e., $A(G)^* = VN(G)$. In particular, the w^*-topology (i.e., the $\sigma(VN(G) \, , \, A(G))$-topology) and the weak operator topology on $VN(G)$ coincide.

Also, $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup \{ |\phi(T)| \; ; \; T \in VN(G)$ and $\|T\| \leq 1 \}$ forms a commutative Banach algebra.

There is a natural action of $A(G)$ on $VN(G)$ given by

$$\langle u \cdot T \, , \, v \rangle \; = \; \langle T \, , \, uv \rangle \, , \qquad \text{for } u \, , v \in A(G) \, , \, T \in VN(G) \, .$$

Under this action, $VN(G)$ becomes a Banach $A(G)$-module. For more details on the algebras $VN(G)$ and $A(G)$, see Eymard [6].

An $m \in VN(G)^*$ is called a *topologically invariant mean* on $VN(G)$, if

(i) $\|m\| = \langle m , I \rangle = 1$, where $I = \rho(e)$ denotes the identity operator,

(ii) $\langle m , u \cdot T \rangle = \langle m , T \rangle$ for $T \in VN(G)$ and $u \in A(G)$ with $u(e) = 1$.

Let $TIM(\hat{G})$ be the set of all topologically invariant means on $VN(G)$. It is known that $TIM(\hat{G})$ is a non-empty w^*-compact convex subset of $VN(G)^*$ (see Renaud [25] for a further discussion).

Let $C(G)$ denote the Banach space of bounded continuous complex-valued functions on G with the supremum norm and $C_{oo}(G)$ denote all functions in $C(G)$ with compact support, where the support of a continuous function u on G is the closure of the set $\{x \in G \; ; \; u(x) \neq 0\}$.

The support of an element $f \in L^2(G)$ is defined by saying that $x \notin$ supp f if and only if there exists a neighbourhood V of x such that $\langle f, v \rangle = 0$ for all $v \in C_{oo}(G)$ with supp $v \subseteq V$. The support of an operator $T \in VN(G)$ is defined by saying that $x \notin$ supp T if and only if there exists a neighbourhood U of e such that $x \notin$ supp (Tu) for all $u \in C_{oo}(G)$ with supp $u \subseteq U$ (see [15, P.117]). An equivalent definition for supp T is that $x \in$ supp T if and only if $u \cdot T = \mathbf{0}$ implies $u(x) = 0$ for all $u \in A(G)$ (see [6, Proposition 4.4] or [14, P.119]).

Let $UCB(\hat{G})$ denote the norm closure of $A(G) \cdot VN(G)$. Then $UCB(\hat{G})$ is a C^*-subalgebra and an $A(G)$-submodule of $VN(G)$ (see [12]) which coincides with the norm closure of $\{T \in VN(G) \; ; \text{supp } T \text{ is compact }\}$. In case G is abelian, $UCB(\hat{G})$ is isometrically algebra isomorphic to the algebra of bounded uniformly continuous functions on the dual group $\hat{G}$ of G. For this reason, operators in $UCB(\hat{G})$ are called uniformly continuous functionals on $A(G)$ (see [11]). The C^*-algebra $UCB(\hat{G})$ and its relationship with other C^*-subalgebras of $VN(G)$ have been studied by Granirer in [11] and [12] and by Lau in [18]. By the definitions of $TIM(\hat{G})$ and $UCB(\hat{G})$, each element m in $TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$.

Dunkl-Ramirez in [5] called $\{T \in VN(G);\, u \longmapsto u \cdot T$ is a weakly compact operator of $A(G)$ into $VN(G)\}$ the space of weakly almost periodic functionals of $A(G)$ and denoted it by $W(\hat{G})$. It turns out that $W(\hat{G})$ is a self-adjoint closed $A(G)$-submodule of $VN(G)$ which coincides with the space of weakly almost periodic functions in $L^\infty(\hat{G})$ when G is abelian (see [5] for more details).

Chou [2] used $F(\hat{G})$ to denote the space of all $T \in VN(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as m runs through $TIM(\hat{G})$ and called $F(\hat{G})$ the space of topological almost convergent elements in $VN(G)$. We can easily check that $F(\hat{G})$ is a norm closed self-adjoint $A(G)$-submodule of $VN(G)$.

It is known that $W(\hat{G})$ has a unique topologically invariant mean (see [5] and [11]). In particular, this gives that $W(\hat{G}) \subseteq F(\hat{G})$. The above inclusion was also obtained by Chou using his results on characterizations of $F(\hat{G})$. See Chou [2] for more information on $F(\hat{G})$.

Let ϕ_1 and ϕ_2 be two positive definite functions in $A(G)$. We say that ϕ_1 is *orthogonal to* ϕ_2 if $\|\phi_1 - \phi_2\| = \|\phi_1\| + \|\phi_2\|$ (see [30, P.31]).

A net $(\phi_\alpha)_{\alpha\in\Lambda}$ in $A(G)$ is said to be ***topologically convergent to invariance*** if

$$\lim_\alpha \|v\phi_\alpha - \phi_\alpha\| = 0\,, \quad \text{for } v \in A(G) \text{ with } v(e) = 1\,.$$

Let $b(G)$ be the smallest cardinality of an open basis at the unit element e of a locally compact group G. When G is abelian and $\hat{G}$ is the dual group of G, Hewitt and Stromberg showed that $b(G) = d(\hat{G})$, the smallest cardinality of a covering of $\hat{G}$ by compact sets (see [16] and [17, (24.48)]).

2.3. Sets and Stone-Čech Compactifications.

For any two sets A and B, $A \setminus B$ denotes their difference, 1_A denotes the characteristic function of A as a subset of the underlying set or locally compact group, 2^A is the set of all functions from A to $\{0,1\}$, and $|A|$ is the cardinality of A. Then $|2^A| = 2^{|A|}$, the cardinality of the set of all subsets of A. So we also use 2^A to denote the set of all subsets of A.

When α is an ordinal number, $|\alpha|$ means the cardinality of the set $\{\beta\ ;\ \beta$ is an ordinal and $\beta < \alpha\}$. An ordinal α is called an *initial ordinal* if $|\alpha|$ is infinite and $\beta < \alpha$ implies $|\beta| < |\alpha|$ (see [27, P.271]).

Lemma 2.3.1. *Let α be an initial ordinal. If β and $\gamma > 0$ are ordinals such that $\beta + \gamma = \alpha$, then $\gamma = \alpha$.*

Proof. Since $\gamma > 0$, $\beta < \beta + \gamma = \alpha$ (see [27, P.193]). Then $|\beta| < |\alpha|$ because α is an initial ordinal. But $|\beta| + |\gamma| = |\alpha|$. It follows that $|\gamma| = |\alpha|$. Also, $\gamma \leq \beta + \gamma = \alpha$ (see [27, P.193]). Therefore, $\gamma = \alpha$.

□

If X is a set, let $l^\infty(X)$ be the Banach space of all bounded complex-valued functions on X with the supremum norm. It is well-known that if $\phi \in l^\infty(X)^*$, then any two of the following three conditions implies the remaining one.

(*i*) $\|\phi\| = 1$.

(*ii*) $\phi(\mathbf{1}) = 1$.

(*iii*) $\phi \geq 0$, that is, $\phi(f) \geq 0$ for all non-negative $f \in l^\infty(X)$.

Where $\mathbf{1}$ is the constant function of value one. When $\phi \in l^\infty(X)^*$ has any two of the above properties, we call ϕ a *mean* on $l^\infty(X)$.

Let X be a directed set. We define a subset of means on $l^\infty(X)$ as following:

$$\mathcal{F}(X) = \{\phi \in l^\infty(X)^* \ ; \ \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } f \in c_o(X)\} ,$$

where $c_o(X) = \{f \in l^\infty(X);\ \lim_{\alpha \in X} f(\alpha) = 0\}$. This set with $X = \mathbf{N}$, the set of all positive integers, was first considered by Chou when he introduced the technique to embed a large set into $TIM(\hat{G})$ (see [2]). Yang in [32] studied the case $X = \Lambda(Y)$, the set of all non-empty finite subsets of an infinite set Y directed by inclusion.

When X is a directed set, a *tail* in X is defined by

$$T_\alpha = \{\beta \in X \ ; \ \beta \geq \alpha\} , \qquad \alpha \in X .$$

Therefore, $\phi \in \mathcal{F}(X)$ if and only if ϕ is a mean on $l^\infty(X)$ and $\phi(1_{T_\alpha}) = 1$ for all $\alpha \in X$.

If X is a set (with the discrete topology), βX denotes the Stone-Čech compactification of X. Then $l^\infty(X)$ is isometrically isomorphic to $C(\beta X)$, the Banach space of all bounded continuous complex-valued functions on βX with the supremum norm.

Thus βX can be identified with the spectrum of $l^\infty(X)$, i.e., the set of all nonzero multiplicative linear functionals on $l^\infty(X)$ with the Gelfand topology (see, say, [31, Proposition 4.5, P.18]). In this way, each $x \in X$ is identified with the evaluation $\hat{x}$ on $l^\infty(X)$ at x, i.e., $\hat{x}(f) = f(x)$ for $f \in l^\infty(X)$.

On the other hand, βX can also be obtained by "fixing" the free ultrafilters on X, that is, $\beta X = \{$ all ultrafilters on $X\}$ with $\{Z^* \ ; \ Z \subseteq X\}$ as a base for closed subsets of βX, where $Z^* = \{\phi \in \beta X \ ; \ Z \in \phi\}$ (see [7, P.86-87]). Now, every $x \in X$ corresponds to the fixed ultrafilter ϕ_x on X containing $\{x\}$, i.e., $\phi_x = \{E \ ; \ x \in E \subseteq X\}$. Either way of the above embeddings will be used later.

When $X = \mathbf{N}$, Chou in [2] pointed out that $\beta\mathbf{N} \setminus \mathbf{N} \subseteq \mathcal{F}(\mathbf{N})$. For the general case, we have

Lemma 2.3.2. *Let X be a directed set. If $\phi \in \beta X$ and ϕ contains $\{T_\alpha \ ; \ \alpha \in X\}$, then $\phi \in \mathcal{F}(X)$.*

Proof. Let $\phi \in \beta X$ and ϕ contains $\{T_\alpha \ ; \ \alpha \in X\}$. Since ϕ is in the spectrum of $l^\infty(X)$, ϕ is a mean on $l^\infty(X)$. It is known that $E \in \phi$ if and only if $\phi(1_E) = 1$. Now, for each $\alpha \in X$, $T_\alpha \in \phi$ and hence $\phi(1_{T_\alpha}) = 1$. Therefore, $\phi \in \mathcal{F}(X)$.

□

CHAPTER 3

THE SUBSET $\mathcal{F}(X)$ OF MEANS ON $l^\infty(X)$

3.1. Introduction.

For a directed set X, let $\mathcal{F}(X)$ be the subset of $l^\infty(X)^*$ defined as Section 2.3. If $X = \mathbf{N}$, then $|\mathcal{F}(\mathbf{N})| = 2^c = 2^{2^{|\mathbf{N}|}}$, since $\beta\mathbf{N} \setminus \mathbf{N} \subseteq \mathcal{F}(\mathbf{N})$ (see Chou [2, P.208]), where c is the cardinality of the continuum. When $X = \Lambda(Y)$, the set of all non-empty finite subsets of an infinite set Y directed by inclusion, Yang proved in [32, Lemma 2.1] that $|\mathcal{F}(X)| = 2^{2^{|X|}}$ if $l^\infty(X)$ is the real Banach space.

To obtain an estimation on the cardinality of the set $TIM(\hat{G})$, we need to establish a one-one map from $\mathcal{F}(X)$ into $TIM(\hat{G})$ for some directed set X. Throughout this chapter, μ will be an initial ordinal and X denote the set $\{\beta\ ;\ \beta$ is an ordinal and $\beta < \mu\}$ with its natural order. We shall show that $\mathcal{F}(X)$ is a big set in $l^\infty(X)^*$, that is, $|\mathcal{F}(X)| = 2^{2^{|X|}}$.

3.2. The Cardinality of $\mathcal{F}(X)$.

To prove the equality $|\mathcal{F}(X)| = 2^{2^{|X|}}$, we begin with a technical lemma which provides us a family $\mathcal{Y}$ of functions in 2^X such that $|\mathcal{Y}| = 2^{|X|}$ and any two functions in $\mathcal{Y}$ are not cofinal.

Lemma 3.2.1. *There exists a family $\{f_i\ ;\ i \in I\} \subseteq 2^X$ such that $|I| = 2^{|X|}$ and*

$$f_i\,|_{T_\alpha} \neq f_j\,|_{T_\alpha}\,, \qquad \text{for } i\,, j \in I \text{ with } i \neq j\,,\ \alpha \in X\,,$$

where $f|_A$ is the restriction of function f to the set A.

Proof. Case (i). Assume that $2^{|\alpha|} < 2^{|X|}$ for all $\alpha \in X$.

For each pair $f, g \in 2^X$, we define $f \sim g$ if there exists an element $\alpha \in X$ such that $f|_{T_\alpha} = g|_{T_\alpha}$. Then "$\sim$" is an equivalent relation on 2^X. Let $f \in 2^X$. We put $[f] = \{g \in 2^X \; ; \; g \sim f\}$, the equivalent class containing f. Let I be the set of all such equivalent classes. Then $2^X = \bigcup \{[f] \; ; \; [f] \in I\}$.

Fix an $f \in 2^X$. We have that $[f] = \bigcup_{\alpha \in X} F_\alpha$, where $F_\alpha = \{g \in 2^X \; ; \; g|_{T_\alpha} = f|_{T_\alpha}\}$. Since $|F_\alpha| = 2^{|\alpha|}$, then $|[f]| \leq \sum_{\alpha \in X} 2^{|\alpha|}$. This is true for every $f \in 2^X$. Hence,

$$\begin{aligned} 2^{|X|} &= |2^X| = \sum_{[f] \in I} |[f]| \\ &\leq \sum_{[f] \in I} \Big(\sum_{\alpha \in X} 2^{|\alpha|}\Big) = |I| \Big(\sum_{\alpha \in X} 2^{|\alpha|}\Big) \\ &= max\,\Big(|I|\,, \sum_{\alpha \in X} 2^{|\alpha|}\Big). \end{aligned}$$

By König-Zermelo's inequality (see [27, P.313]), we have

$$\sum_{\alpha \in X} 2^{|\alpha|} < \prod_{\alpha \in X} 2^{|X|} = (2^{|X|})^{|X|} = 2^{|X|}. \tag{3.2.1}$$

Obviously, $|I| \leq 2^{|X|}$. Consequently,

$$2^{|X|} \leq max\,\Big(|I|\,, \sum_{\alpha \in X} 2^{|\alpha|}\Big) \leq 2^{|X|}. \tag{3.2.2}$$

Now (3.2.1) and (3.2.2) combined give $|I| = 2^{|X|}$. For each equivalent class $i \in I$, we choose an $f_i \in i$. Then the family $\{f_i \; ; \; i \in I\}$ satisfies the requirement.

Case (ii). Assume that $2^{|\alpha|} = 2^{|X|}$ for some $\alpha \in X$.

Let $\alpha_o = min\{\beta \ ; \ \beta \in X \text{ and } 2^{|\beta|} = 2^{|X|}\}$. Then α_o is a limit ordinal. By the generalized division algorithm (see [24, P.177]), there exists a unique pair of ordinals η and ε such that $\mu = \alpha_o\eta + \varepsilon$ and $\varepsilon < \alpha_o$. Note that $\varepsilon < \alpha_o < \mu$ and $\alpha_o(\zeta + 1) = \alpha_o\zeta + \alpha_o$ for any ordinal ζ. By Lemma 2.3.1, $\varepsilon = 0$ and η has to be a limit ordinal.

Let ω be the initial ordinal satisfying $|\omega| = 2^{|X|}$. Let $I = \{i \ ; \ i$ is an ordinal and $i < \omega\}$. Then $|I| = |\omega| = 2^{|X|}$. In the following we inductively construct a family $(X_i)_{i<\omega}$ of subsets of X such that

$$X_i \bigcap T_\alpha \neq X_j \bigcap T_\alpha, \qquad \text{for } i, j < \omega \text{ with } i \neq j,\ \alpha \in X. \tag{3.2.3}$$

Let $i_o < \omega$. Assume that we have chosen a family $(X_i)_{i<i_o}$ of subsets of X satisfying (3.2.3). Recall that $\mu = \alpha_o\eta$. For every $\xi < \eta$, let S_ξ be the segment of ordinals between $\alpha_o\xi$ and $\alpha_o(\xi+1)$, i.e.,

$$S_\xi = \{\alpha \ ; \ \alpha_o\xi \leq \alpha < \alpha_o(\xi + 1)\}.$$

Since $\alpha_o\xi_1 < \alpha_o\xi_2$ if and only if $\xi_1 < \xi_2$ (see [27, P.200]), then $\{S_\xi \ ; \ \xi < \eta\}$ is pairwise disjoint.

Furthermore, $X = \bigcup_{\xi<\eta} S_\xi$ and, for every $\alpha \in X$, there exists a $\xi < \eta$ such that $S_\xi \subseteq T_\alpha$. Now, $|S_\xi| = |\alpha_o|$ and hence $2^{|S_\xi|} = 2^{|\alpha_o|} = 2^{|X|}$ for all $\xi < \eta$. But we have that $|\{X_i \bigcap S_\xi \ ; \ i < i_o\}| \leq |i_o| < 2^{|X|}$ for $\xi < \eta$. Consequently, for each $\xi < \eta$, there exists a set $B_\xi \subseteq S_\xi$ such that $B_\xi \notin \{X_i \bigcap S_\xi \ ; \ i < i_o\}$. Let $X_{i_o} = \bigcup_{\xi<\eta} B_\xi$.

Then $X_{i_0} \bigcap S_\xi \neq X_i \bigcap S_\xi$ if $i < i_0$ and $\xi < \eta$. Hence $X_{i_0} \bigcap T_\alpha \neq X_i \bigcap T_\alpha$ for all $i < i_0$ and $\alpha \in X$.

Therefore, the family $\{X_i \; ; \; i \leq i_0\}$ has property (3.2.3). By transfinite induction, we obtain a family $(X_i)_{i<\omega}$ of subsets of X satisfying (3.2.3).

Finally, for each $i \in I$, let $f_i : X \longrightarrow \{0, 1\}$ be the characteristic function of X_i. Then $\{f_i \; ; \; i \in I\}$ has the required property.

□

Remark 3.2.2. Under the generalized continuum hypothesis (GCH, for short), $a < b$ implies that $2^a < 2^b$, where a and b are any two cardinal numbers. In the above Lemma 3.2.1, $|\alpha| < |\mu| = |X|$ for all $\alpha \in X$, since μ is an initial ordinal. Thus, if the GCH is assumed, we always have that $2^{|\alpha|} < 2^{|X|}$ for all $\alpha \in X$, this is just the case (i) in the above proof. To avoid using the GCH, we have to consider case (ii) in our proof as well.

Now we are ready to show that $\mathcal{F}(X)$ is big set in $l^\infty(X)^*$.

Proposition 3.2.3. $|\mathcal{F}(X)| = 2^{2^{|X|}}$.

Proof. Obviously, $|\mathcal{F}(X)| \leq |l^\infty(X)^*| = 2^{2^{|X|}}$. By Lemma 2.3.2, it suffices to show that there are $2^{2^{|X|}}$ many ultrafilters on X containing $\{T_\alpha \; ; \; \alpha \in X\}$. We now follow an argument of Rudin [28, Theorem 1.3] (see also the proof of [32, Lemma 2.1]).

Let $\Lambda = \Lambda(X)$ be the set of all non-empty finite subsets of X. In the following, we shall construct a family $\{A_\tau \,;\, \tau \in \Lambda\}$ of subsets of X satisfying

(i) $|A_\tau| = 2^{2^{|\tau|}}$;

(ii) if $\tau \neq \tau'$, then $A_\tau \bigcap A_{\tau'} = \emptyset$;

(iii) if $\alpha \in A_\tau$, then $\alpha \geq max(\tau)$, where $max(\tau) = max\{\beta \,;\, \beta \in \tau\}$.

Since $|\Lambda| = |X| = |\mu|$, we can write $\Lambda = \{\tau_i \,;\, i < \mu\}$. Let $i_0 < \mu$. Assume that we have defined a family $\{A_{\tau_i} \,;\, i < i_0\}$ of subsets of X satisfying $(i) - (iii)$. Let $B = \bigcup_{i<i_0} A_{\tau_i}$. Then $|B| < |\mu|$, since each A_{τ_i} is finite and $|i_0| < |\mu|$. Let α be the unique ordinal satisfying $max(\tau_{i_0}) + \alpha = \mu$ (see [27, P.194]). Since $\alpha \neq 0$, $\alpha = \mu$, by Lemma 2.3.1. In particular, $|\alpha| = |\mu| > |B|$. But $|\mu|$ is infinite. So, we can choose a finite set $A_{\tau_{i_0}} \subseteq \{\beta \,;\, max(\tau_{i_0}) \leq \beta < \mu\} \setminus B$ with $|A_{\tau_{i_0}}| = 2^{2^{|\tau_{i_0}|}}$. Clearly, the family $\{A_{\tau_i} \,;\, i \leq i_0\}$ has properties $(i) - (iii)$. By transfinite induction, we have constructed a family $\{A_{\tau_i} \,;\, i < \mu\} = \{A_\tau \,;\, \tau \in \Lambda\}$ of subsets of X satisfying $(i) - (iii)$.

For each $\tau \in \Lambda$, label the elements of A_τ by ordered $2^{|\tau|}$-tuples $(x_1,\, x_2,\, \cdots,\, x_{2^{|\tau|}})$ with $x_i \in \{0,\, 1\}$. Let E_i be the subset of A_τ consisting of the $2^{|\tau|}$-tuples which have $x_i = 0$. If we let $E_i^0 = E_i$ and $E_i^1 = A_\tau \setminus E_i$, then $\bigcap_{i=1}^{2^{|\tau|}} E_i^{\epsilon_i}$ is not empty for any choice of $\epsilon_i \in \{0,\, 1\}$, since $(\epsilon_1,\, \epsilon_2,\, \cdots,\, \epsilon_{2^{|\tau|}}) \in \bigcap_{i=1}^{2^{|\tau|}} E_i^{\epsilon_i}$. Denote the sets E_i, $i = 1,\, 2,\, \cdots,\, 2^{|\tau|}$, by $E(h)$, where h is a map from τ to $\{0,\, 1\}$.

Let $\mathcal{Y} = \{f_j \,;\, j \in I\} \subseteq 2^X$ be the same family of functions as in Lemma 3.2.1. For each $f \in \mathcal{Y}$, we define

$$B(f) \;=\; \bigcup \{E(f\,|_\tau) \,;\, \tau \in \Lambda\},$$

where $f\,|_\tau$ is the restriction of f to the set τ.

Suppose that $f_1, \cdots, f_n, f_{n+1}, \cdots, f_m$ are distinct functions in $\mathcal{Y}$ and $\alpha \in X$. Since $f_1\,|_{T_\alpha}, \cdots, f_n\,|_{T_\alpha}, f_{n+1}\,|_{T_\alpha}, \cdots, f_m\,|_{T_\alpha}$ are different (by Lemma 3.2.1), there exists an element $\tau \in \Lambda$ such that $\tau \subseteq T_\alpha$ and $f_1\,|_\tau, \cdots, f_n\,|_\tau, f_{n+1}\,|_\tau, \cdots, f_m\,|_\tau$ are different. Hence the above argument gives

$$E(f_1\,|_\tau) \bigcap \cdots \bigcap E(f_n\,|_\tau) \bigcap (A_\tau \setminus E(f_{n+1}\,|_\tau) \bigcap \cdots \bigcap (A_\tau \setminus E(f_m\,|_\tau) \neq \emptyset.$$

By $\tau \subseteq T_\alpha$ and property (iii), $A_\tau \subseteq T_\alpha$. Therefore,

$$E(f_1\,|_\tau) \bigcap \cdots \bigcap E(f_n\,|_\tau) \bigcap (A_\tau \setminus E(f_{n+1}\,|_\tau) \bigcap \cdots \bigcap (A_\tau \setminus E(f_m\,|_\tau) \bigcap T_\alpha \neq \emptyset.$$

Note that $\{A_\tau\ ;\ \tau \in \Lambda\}$ is pairwise disjoint. It follows that

$$B(f_1) \bigcap \cdots \bigcap B(f_n) \bigcap (X \setminus B(f_{n+1})) \bigcap \cdots \bigcap (X \setminus B(f_m)) \bigcap T_\alpha \neq \emptyset.$$

Hence for any map $F : \mathcal{Y} \longrightarrow \{0, 1\}$, the collection

$$\{B(f)^{F(f)}\ ;\ f \in \mathcal{Y}\} \bigcup \{T_\alpha\ ;\ \alpha \in X\},$$

where $B(f)^0 = B(f)$ and $B(f)^1 = X \setminus B(f)$, generates a filter base. Consequently, we have $2^{|\mathcal{Y}|} = 2^{2^{|X|}}$ different ultrafilters on X containing $\{T_\alpha\ ;\ \alpha \in X\}$. This completes the proof of the proposition.

□

CHAPTER 4

σ-COMPACT NON-METRIZABLE GROUPS

4.1. Introduction.

Let G be a locally compact group with unit element e and $b(G)$ be the smallest cardinality of an open basis at e defined as in Section 2.2. In this chapter, we shall present an important property of a σ-compact non-metrizable locally compact group G concerning its local structure at e: if G is σ-compact and non-metrizable, then there exists a decreasing net $(N_\alpha)_{\alpha \leq \mu}$ of normal subgroups of G, where μ is the first ordinal satisfying $|\mu| = b(G)$, such that $N_0 = G$, $N_\mu = \{e\}$ and $b(N_\alpha) = b(G)$ for all $\alpha < \mu$; N_α is compact if $\alpha > 0$; $N_\alpha / N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for $\alpha < \mu$; and (iv) $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma \leq \mu$. This property is very crucial for our investigation on the set $TIM(\hat{G})$ and is interesting in itself.

We begin the chapter with two results about closed subgroups. The first one is a simple application of the Kakutani-Kodaira Theorem, while the second one deals with the relation between $b(N)$ and $b(G)$, where N is a closed subgroup of G.

This chapter is motivated by Lau-Losert [19, Lemma 4.8].

4.2. Closed Subgroups.

In this section, we shall show some facts about closed subgroups which will be used to prove the main result of this chapter.

Lemma 4.2.1. *Let G be a σ-compact locally compact group. Let N be a closed normal subgroup of G and U an open neighborhood of e. Then there exists a compact normal subgroup M of G such that $M \subseteq N \bigcap U$ and N/M is metrizable.*

Proof. By the Kakutani-Kodaira Theorem (see [17, (8.7)]), there exists a compact normal subgroup K of G such that G/K is metrizable and $K \subseteq U$. Let $M = K \bigcap N$. Then M is a compact normal subgroup of G and $M \subseteq N \bigcap U$. Note that $N/M \cong NK/K \subseteq G/K$. Therefore, N/M is metrizable.

□

Lemma 4.2.2. *Let G be a locally compact group and N be a closed subgroup of G. Let $\aleph$ be a cardinal number. If N is an intersection of no more than $\aleph$ open subsets of G, then $b(G) \leq \aleph b(N)$.*

Proof. Choose a set I with $|I| = \aleph$. Since G is a normal topological space (i.e., any two disjoint closed subsets of G can be separated by two disjoint open subsets of G), by the assumption, we can write $N = \bigcap_{i \in I} A_i$, where each A_i is a closed subset of G and $N \subseteq \overset{\circ}{A_i}$ (the interior of A_i in G).

Also we choose a set J with $|J| = b(N)$ such that $\{B_j \bigcap N \ ; \ j \in J\}$ is a neighborhood basis at e in N, where each B_j is a compact neighborhood of e in G. We can assume that $B_j \subseteq K$ for $j \in J$, where K is a fixed compact subset of G.

Let $\Lambda(I)$ (resp. $\Lambda(J)$) be the set of all non-empty finite subsets of I (resp. J). For any $\xi \in \Lambda(I)$ and $\eta \in \Lambda(J)$, denote $A_\xi = \bigcap_{i \in \xi} A_i$ and $B_\eta = \bigcap_{j \in \eta} B_j$. Then

A_ξ and B_η are neighborhoods of e in G.

We claim that $\{A_\xi \bigcap B_\eta\ ;\ \xi \in \Lambda(I),\ \eta \in \Lambda(J)\}$ is a neighborhood basis at e in G. Assume that there exists a neighborhood U of e in G such that $A_\xi \bigcap B_\eta \not\subseteq U$ for all $\xi \in \Lambda(I)$ and $\eta \in \Lambda(J)$. Choose an element $x_{\xi,\eta} \in (A_\xi \bigcap B_\eta) \setminus U$ for each pair $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$.

We direct $\Lambda(I)$ and $\Lambda(J)$ by counter inclusion (i.e., $\zeta_1 \leq \zeta_2$ if and only if $\zeta_2 \subseteq \zeta_1$), and direct $\Lambda(I) \times \Lambda(J)$ by $(\xi_1, \eta_1) \leq (\xi_2, \eta_2)$ if and only if $\xi_1 \leq \xi_2$ and $\eta_1 \leq \eta_2$. Then, the net $(x_{\xi,\eta})_{(\xi,\eta)\in\Lambda(I)\times\Lambda(J)}$ in K has a cluster point, say, $x \in K$. By the direction on $\Lambda(I) \times \Lambda(J)$ and the compactness of $A_\xi \bigcap B_\eta$, we have that $x \in A_\xi \bigcap B_\eta$ for all $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$. Consequently,

$$x \in \bigcap_{(\xi,\eta)\in\Lambda(I)\times\Lambda(J)} (A_\xi \bigcap B_\eta) = \bigcap_{\eta\in\Lambda(J)} (N \bigcap B_\eta) = \bigcap_{j\in J} (N \bigcap B_j) = \{e\},$$

i.e., $x = e$. But U is a neighborhood of e in G and $x_{\xi,\eta} \notin U$ for all $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$. This contradicts the fact that $x = e$ is a cluster point of $(x_{\xi,\eta})_{(\xi,\eta)\in\Lambda(I)\times\Lambda(J)}$. It follows that $\{A_\xi \bigcap B_\eta\ ;\ \xi \in \Lambda(I),\ \eta \in \Lambda(J)\}$ is a neighborhood basis at e in G.

Since $|J| = b(N)$, $|J| = 1$ or $|J|$ is infinite. In any case, we have that $|\Lambda(J)| = |J| = b(N)$. So,

$$b(G) \leq |\Lambda(I) \times \Lambda(J)| = |\Lambda(I)||\Lambda(J)| = |\Lambda(I)|b(N).$$

If $\aleph$ is infinite, then $|\Lambda(I)| = |I| = \aleph$ and hence $b(G) \leq \aleph b(N)$ by the above inequality. If $\aleph$ is finite, then N is an open subgroup of G and now $b(N) = b(G)$. Therefore, we always have that $b(G) \leq \aleph b(N)$.

□

4.3. The Local Structure of σ-Compact Non-Metrizable Groups.

The main result of this chapter is contained in the following theorem.

Theorem 4.3.1. *Let G be a σ-compact non-metrizable locally compact group with unit element e. Then there exists a limit ordinal μ and a decreasing family $(N_\alpha)_{\alpha\leq\mu}$ of normal subgroups of G (i.e., $\alpha \leq \beta$ implies $N_\alpha \supseteq N_\beta$) such that*

(*i*) $N_0 = G$ *and* $N_\mu = \{e\}$;

(*ii*) N_α *is compact for each* $\alpha > 0$;

(*iii*) $N_\alpha/N_{\alpha+1}$ *is metrizable but* $N_{\alpha+1} \neq N_\alpha$ *for all* $\alpha < \mu$;

(*iv*) $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ *for every limit ordinal* $\gamma \leq \mu$;

(*v*) $b(N_\alpha) = b(G)$ *for all* $\alpha < \mu$.

Furthermore, μ is minimal among all such families and μ is the initial ordinal satisfying $|\mu| = b(G)$.

Proof. Let d be the initial ordinal satisfying $|d| = b(G)$. Then d is a limit ordinal. Let $\{O_\alpha \,;\, \alpha < d\}$ be an open basis at e in G.

Let $N_0 = G$. By Lemma 4.2.1, there exists a compact normal subgroup N_1 of G such that $N_1 \subseteq N_0 \bigcap O_0$ and N_0/N_1 is metrizable.

Let $d_\circ < d$. Assume that we have chosen a decreasing family $(N_\alpha)_{\alpha<d_\circ}$ of normal subgroups of G such that N_α is compact for each $0 < \alpha < d_0$, $N_{\alpha+1} \subseteq N_\alpha \bigcap O_\alpha$ and $N_\alpha/N_{\alpha+1}$ is metrizable if $\alpha + 1 < d_\circ$, and $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma < d_\circ$. If $d_\circ$ is a limit ordinal, then we put $N_{d_\circ} = \bigcap_{\alpha<d_\circ} N_\alpha$. If $d_\circ = \beta+1$

(such β is unique), then, by Lemma 4.2.1, we choose N_{d_o} to be the compact normal subgroup of G such that $N_{d_o} \subseteq N_\beta \bigcap O_\beta$ and N_β / N_{d_o} is metrizable. By transfinite induction, we get a decreasing family $(N_\alpha)_{\alpha<d}$ of normal subgroups of G such that $N_0 = G$, N_α is compact for all $0 < \alpha < d$, $N_{\alpha+1} \subseteq N_\alpha \bigcap O_\alpha$ and $N_\alpha / N_{\alpha+1}$ is metrizable for $\alpha < d$, and $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma < d$.

Now

$$\bigcap_{\alpha<d} N_\alpha \subseteq \bigcap_{\alpha<d} O_\alpha = \{e\},$$

so, $\bigcap_{\alpha<d} N_\alpha = \{e\}$. Let $N_d = \{e\}$. Then $N_d = \bigcap_{\alpha<d} N_\alpha$.

We claim that for each $0 < \alpha < d$, N_α is an intersection of no more than $|\alpha|\aleph_0$ open subsets of G, where $\aleph_0$ is the first infinite cardinal number. This is true for $\alpha = 1$ because N_1 is a G_δ-set in G (i.e., N_1 is an intersection of countably many open subsets of G, since G/N_1 is metrizable). Let $d_o < d$. Assume that the above statement is true for all $0 < \alpha < d_o$. If d_o is a limit ordinal, then $N_{d_o} = \bigcap_{\alpha<d_o} N_\alpha$ and hence, by the inductive assumption, N_{d_o} is an intersection of no more than $|d_o|^2\aleph_0 = |d_o|\aleph_0$ open subsets of G. If $d_o = \beta + 1$ for some $\beta < d$, then N_{d_o} is a G_δ-set in N_β, since N_β/N_{d_o} is metrizable. By the assumption that N_β is an intersection of no more than $|\beta|\aleph_0$ open subsets of G, N_{d_o} is an intersection of no more than $|\beta|\aleph_0^2 = |\beta|\aleph_0 = |d_o|\aleph_0$ open subsets of G. By transfinite induction, our assertion follows.

Let $0 < \alpha < d$. By the above claim and Lemma 4.2.2, we have

$$b(G) \leq (|\alpha|\aleph_0) b(N_\alpha). \tag{4.3.1}$$

Since d is the initial ordinal satisfying $|d| = b(G) > \aleph_0$, then $|\alpha| < b(G)$ and hence

$$|\alpha|\aleph_0 = max(|\alpha|, \aleph_0) < b(G). \tag{4.3.2}$$

Now (4.3.1) and (4.3.2) combined give

$$b(G) \leq (|\alpha|\aleph_0)b(N_\alpha)$$

$$= max(|\alpha|\aleph_0, b(N_\alpha)) = b(N_\alpha),$$

i.e., $b(G) \leq b(N_\alpha)$. Conversely, $b(N_\alpha) \leq b(G)$, since N_α is a subgroup of G. Therefore, $b(N_\alpha) = b(G)$ for all $\alpha < d$.

We conclude that $(N_\alpha)_{\alpha \leq d}$ is a decreasing family of normal subgroups of G satisfying

(i) $N_0 = G$ and $N_d = \{e\}$;

(ii) N_α is compact for each $\alpha > 0$;

$(iii)'$ $N_\alpha / N_{\alpha+1}$ is metrizable for all $\alpha < d$;

(iv) $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma \leq d$;

(v) $b(N_\alpha) = b(G)$ for all $\alpha < d$.

Let μ be the minimal ordinal among all such families. We see that μ has to be a limit ordinal. In fact, assume that $\mu = \nu+1$ (ν is infinite, since G is non-metrizable). Then $N_\nu = N_\nu/N_\mu$ is metrizable, contradicting the fact that $b(N_\nu) = b(G) > \aleph_0$. It follows that μ is a limit ordinal. By passing to an appropriate subfamily, we can achieve that $N_{\alpha+1} \neq N_\alpha$ for all α. The ordinal type of this subfamily will be still μ, by minimality. Note that $(i) - (v)$ implies (i), (ii), $(iii)'$, (iv) and (v). Consequently, μ is minimal among all families satisfying $(i) - (v)$.

By the same procedure as above, we can prove that for each $0 < \alpha < \mu$, N_α is an intersection of no more than $|\alpha|\aleph_0$ open subsets of G. Since $\bigcap_{\alpha<\mu} N_\alpha = N_\mu = \{e\}$, $\{e\}$ is an intersection of no more than $|\mu|^2\aleph_0 = |\mu|$ open subsets of G. Applying Lemma 4.2.2 to $N = \{e\}$, we get that $b(G) \leq |\mu| b(N) = |\mu|$. But $\mu \leq d$, by the minimality of μ, and $|d| = b(G)$. Therefore, $|\mu| = b(G)$ and hence $\mu = d$, i.e., μ is the initial ordinal satisfying $|\mu| = b(G)$. This completes the proof of the proposition.

□

Remark 4.3.2. The basic idea used in constructing $(N_\alpha)_{\alpha\leq\mu}$ is essentially the same as that used in Lau-Losert [19, Lemma 4.8]. The net $(N_\alpha)_{\alpha\leq\lambda}$ there possesses property $(i) - (iv)$. Here, for our purpose, we begin with showing the existence of the family of subgroups of G satisfying $(i) - (v)$. Hence the result is strengthened in the following two related aspects, which are important in the sequel.

(1) The limit ordinal μ is totally determined by the local structure of the σ-compact non-metrizable group G (μ is actually the first ordinal satisfying $|\mu| = b(G)$).

(2) $b(N_\alpha) = b(G)$ for all $\alpha < \mu$ (this property reflects, in some sense, that each compact normal subgroup N_α in this net has the same "non-metrizability" as G does).

CHAPTER 5

THE SET OF INVARIANT MEANS ON $VN(G)$

5.1. Introduction.

Chou showed in [2, Theorem 3.2] the existence of a mutually orthogonal sequence in $A(G)$ which is topologically convergent to invariance for a non-discrete metrizable locally compact group G. By making use of such a sequence, he proved that if G is non-discrete and metrizable, then there exists a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ which embeds the large set $\mathcal{F}(\mathbf{N})$ into $TIM(\hat{G})$ ([2, Theorem 3.3]).

We begin this chapter with showing the existence of a net (u_α^j) in $A(G)$, where G is a σ-compact and non-metrizable locally compact group, such that (u_α^j) is topologically convergent to invariance and $(u_\alpha^j)_\alpha$ is mutually orthogonal for each fixed j. Then we construct a family of linear isometries of $l^\infty(X)^*$ into $VN(X)^*$, where $X = \{\alpha\,;\,\alpha < \mu\}$ and μ is the first ordinal satisfying $|\mu| = b(G)$. This family of linear isometries can be used to set up some correspondence between $\mathcal{F}(X)$ and $TIM(\hat{G})$.

Finally, we prove that if G is a non-discrete locally compact group, then there exists a one-one map $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ such that $W(\mathcal{F}(X)) \subseteq 2^{TIM(\hat{G})}$, and the equality $|TIM(\hat{G})| = 2^{2^{b(G)}}$ is established for a non-discrete group G.

Also, some structural results on the set $TIM(\hat{G})$ and some related subspaces of $VN(G)$ are obtained in terms of the local structural property of G.

5.2. The Generalization of Chou's Theorems to the Non-Metrizable Case.

Let G be a σ-compact non-metrizable locally compact group. Let $(N_\alpha)_{\alpha\le\mu}$ be the decreasing family of normal subgroups of G as in Theorem 4.3.1. By the properties of $(N_\alpha)_{\alpha\le\mu}$, we can define a family $(P_\alpha)_{\alpha<\mu}$ of projections in $VN(G)$ as in the proof of Lau-Losert [19, Theorem 4.10]. Let $P_0 = \mathbf{0} \in VN(G)$. For $0<\alpha<\mu$, let $P_\alpha \in VN(G)$ be the central projection defined by convolution with the normalized Haar measure λ_α of N_α. More explicitly, $P_\alpha : L^2(G) \longrightarrow L^2(G/N_\alpha)$ $(\subseteq L^2(G))$ is given by

$$(P_\alpha f)(x) = \int_{N_\alpha} f(t^{-1}x)\, d\lambda_\alpha(t), \qquad f\in L^2(G),\ 0<\alpha<\mu,$$

where $L^2(G/N_\alpha)$ is the subspace of $L^2(G)$ consisting of all functions in $L^2(G)$ which are constant on the cosets of N_α (see [6, (3.23)]).

Now $(P_\alpha)_{\alpha<\mu}$ is an increasing net of projections in $VN(G)$, i.e., $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\alpha<\beta<\mu$. Define

$$Q_\alpha = P_{\alpha+1} - P_\alpha, \qquad \alpha<\mu.$$

Then $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $VN(G)$, that is,

$$Q_\alpha Q_\beta = \begin{cases} Q_\alpha & \text{if } \alpha=\beta, \\ \mathbf{0} & \text{if } \alpha\ne\beta. \end{cases}$$

We begin this section with a technical lemma.

Lemma 5.2.1. *Let G be a σ-compact and non-metrizable locally compact group and $(N_\alpha)_{\alpha\le\mu}$ be the decreasing family of normal subgroups of G as in Theorem 4.3.1. Let*

$(Q_\alpha)_{\alpha<\mu}$ be the orthogonal net of projections in $VN(G)$ defined as above. If U is a neighborhood of the unit element e of G and $\alpha < \mu$, then there exists an $f \in L^2(G)$ such that $\|f\|_2 = 1$, $supp\, f \subseteq UN_\alpha$, and $Q_\alpha f = f$.

Proof. Since $N_{\alpha+1} \subsetneq N_\alpha$, there exists an $x_o \in N_\alpha$ such that $x_o N_{\alpha+1} \bigcap N_{\alpha+1} = \emptyset$. By the compactness of $N_{\alpha+1}$, there exists a neighborhood V of e in G such that

$$x_o N_{\alpha+1} V \bigcap N_{\alpha+1} V = \emptyset. \tag{5.2.1}$$

We can assume that $V \subseteq U$ and V is compact.

Let $g = 1_{N_{\alpha+1}V}$. Then $g \in L^2(G)$, $\operatorname{supp} g \subseteq VN_{\alpha+1} \subseteq UN_{\alpha+1}$, and g is constant on the cosets of $N_{\alpha+1}$, i.e., $g \in L^2(G/N_{\alpha+1})$. If $\alpha = 0$, then $Q_0 g = P_1 g = g$, since $g \in L^2(G/N_1)$. Now $g/\|g\|_2$ satisfies the requirements. In the following we assume that $\alpha > 0$ and we shall show that $g \notin L^2(G/N_\alpha)$.

Assume that $g \in L^2(G/N_\alpha)$. Then there exists an $h \in L^2(G)$ such that h is constant on the cosets of N_α and $g = h$ a.e.. Now $g = 1$ on $N_{\alpha+1}V$ and $g = 0$ on $N_\alpha V \setminus N_{\alpha+1}V$. Hence, there exist measurable subsets $W_1 \subseteq N_{\alpha+1}V$ and $W_2 \subseteq N_\alpha V \setminus N_{\alpha+1}V$ such that

$$\lambda(W_1) = \lambda(N_{\alpha+1}V), \tag{5.2.2}$$

$$\lambda(W_2) = \lambda(N_\alpha V \setminus N_{\alpha+1}V), \tag{5.2.3}$$

and $h = 1$ on W_1, $h = 0$ on W_2. Therefore, $h = 1$ on $N_\alpha W_1$ and $h = 0$ on $N_\alpha W_2$, since h is constant on the cosets of N_α. It follows that $N_\alpha W_1 \bigcap N_\alpha W_2 = \emptyset$. On the other hand, we have

$$\begin{aligned} \lambda(x_oW_1 \bigcap W_2) &= \lambda(x_oN_{\alpha+1}V \bigcap W_2) && \text{(by (5.2.2))} \\ &= \lambda(x_oN_{\alpha+1}V \bigcap (N_\alpha V \setminus N_{\alpha+1}V)) && \text{(by (5.2.3))} \\ &= \lambda(x_oN_{\alpha+1}V) && \text{(by (5.2.1))} \\ &= \lambda(N_{\alpha+1}V) \geq \lambda(V) > 0. \end{aligned}$$

In particular, $x_oW_1 \bigcap W_2 \neq \emptyset$ and hence $N_\alpha W_1 \bigcap N_\alpha W_2 \neq \emptyset$, a contradiction. We conclude that $g \notin L^2(G/N_\alpha)$.

Let $f = Q_\alpha g$ $(= (P_{\alpha+1} - P_\alpha)g = g - P_\alpha g$, since $g \in L^2(G/N_{\alpha+1}))$. Then $f \in L^2(G)$ and $f \neq \mathbf{0}$ in $L^2(G)$. Now

$$Q_\alpha f = Q_\alpha{}^2 g = Q_\alpha g = f,$$

i.e., $Q_\alpha f = f$. Also,

$$\begin{aligned} (P_\alpha g)(x) &= \int_{N_\alpha} g(t^{-1}x)\, d\lambda_\alpha(t) \\ &= \int_{N_\alpha} 1_{N_{\alpha+1}V}(t^{-1}x)\, d\lambda_\alpha(t) \\ &= \lambda_\alpha(N_\alpha \bigcap x(N_{\alpha+1}V)^{-1}), \qquad x \in G. \end{aligned}$$

Then $(P_\alpha g)(x) = 0$ if $x \notin N_\alpha N_{\alpha+1}V = N_\alpha V$. This gives $\operatorname{supp}(P_\alpha g) \subseteq N_\alpha V$. But $\operatorname{supp} g \subseteq N_{\alpha+1}V \subseteq N_\alpha V$. Consequently,

$$\operatorname{supp} f = \operatorname{supp}(g - P_\alpha g) \subseteq N_\alpha V = VN_\alpha \subseteq UN_\alpha.$$

Replacing f by $f/\|f\|_2$, we complete the proof of the lemma.

□

Let G, $(N_\alpha)_{\alpha\leq\mu}$ and $(Q_\alpha)_{\alpha<\mu}$ be the same as in Lemma 5.2.1. Let J be a set with $|J| = b(G)$, where $b(G)$ is the smallest cardinality of an open basis at $e \in G$ defined as in Section 2.2. Let $\{U_j\,;\, j \in J\}$ be an open basis at e. For each $j \in J$, we choose a symmetric neighborhood V_j of e such that $V_j{}^2 \subseteq U_j$. If $\alpha < \mu$ and $j \in J$, then, by Lemma 5.2.1, there exists an $f_\alpha^j \in L^2(G)$ such that $\|f_\alpha^j\|_2 = 1$, $\operatorname{supp} f_\alpha^j \subseteq V_j N_\alpha$, and $Q_\alpha f_\alpha^j = f_\alpha^j$. Let

$$u_\alpha^j \;=\; f_\alpha^j * \tilde{f}_\alpha^j\,, \qquad\qquad \alpha < \mu\,,\; j \in J.$$

Then it is easy to see that $u_\alpha^j \in A(G)$, $\|u_\alpha^j\| = u_\alpha^j(e) = 1$, and

$$\operatorname{supp} u_\alpha^j \subseteq (V_j N_\alpha)(V_j N_\alpha)^{-1} = V_j N_\alpha N_\alpha{}^{-1} V_j{}^{-1} = V_j{}^2 N_\alpha \subseteq U_j N_\alpha\,,$$

i.e., $\operatorname{supp} u_\alpha^j \subseteq U_j N_\alpha$.

Fix $j \in J$. We have that $\|u_\alpha^j - u_\beta^j\| \leq \|u_\alpha^j\| + \|u_\beta^j\| = 2$ for α, $\beta < \mu$. Note that $(Q_\gamma)_{\gamma<\mu}$ is an orthogonal net of projections in $VN(G)$, $u_\gamma^j = f_\gamma^j * \tilde{f}_\gamma^j$ and $Q_\gamma f_\gamma^j = f_\gamma^j$.

It follows that

$$u_\alpha^j(Q_\beta) \;=\; \langle Q_\beta f_\alpha^j\,,\, f_\alpha^j\rangle \;=\; \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{5.2.4}$$

But $\|Q_\alpha - Q_\beta\| = 1$ if α, $\beta < \mu$ and $\alpha \neq \beta$, since $(Q_\alpha)_{\alpha<\mu}$ is orthogonal. So we get

$$\|u_\alpha^j - u_\beta^j\| \;\geq\; |(u_\alpha^j - u_\beta^j)(Q_\alpha - Q_\beta)| \;=\; |u_\alpha^j(Q_\alpha - Q_\beta) - u_\beta^j(Q_\alpha - Q_\beta)| \;=\; 2.$$

Consequently, $\|u_\alpha^j - u_\beta^j\| = \|u_\alpha^j\| + \|u_\beta^j\| = 2$ for all α, $\beta < \mu$ with $\alpha \neq \beta$, that is, $(u_\alpha^j)_{\alpha<\mu}$ is a mutually orthogonal net in $A(G)$.

Let $X = \{\alpha\,;\, \alpha \text{ is an ordinal and } \alpha < \mu\}$ directed by its natural order. Direct J by $i < j$ if and only if $U_j \subseteq U_i$, and $J \times X$ by $(i, \alpha) \le (j, \beta)$ if and only if $i \le j$ and $\alpha \le \beta$. A few of properties of the net $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ is summarized in the following theorem.

Theorem 5.2.2. *Under the same assumptions as above, the net* $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ *has the following properties.*

(*i*) $u_\alpha^j \in A(G)$, $\|u_\alpha^j\| = u_\alpha^j(e) = 1$ *and* $\operatorname{supp} u_\alpha^j \subseteq U_j N_\alpha$ *for all* $(j, \alpha) \in J \times X$.

(*ii*) *For each fixed* $j \in J$, $(u_\alpha^j)_{\alpha\in X}$ *is a mutually orthogonal net in* $A(G)$, *that is,*

$$\|u_\alpha^j - u_\beta^j\| = \|u_\alpha^j\| + \|u_\beta^j\| = 2\,, \quad \text{for all } \alpha\,,\, \beta < \mu \text{ with } \alpha \ne \beta\,.$$

(*iii*) $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ *is topologically convergent to invariance, that is, if* $v \in A(G)$ *and* $v(e) = 1$, *then*

$$\lim_{(j,\alpha)\in J\times X} \|vu_\alpha^j - u_\alpha^j\| \;=\; 0\,.$$

Proof. By the above argument, we only need to show that the net $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ possesses property (iii). Our proof follows Renaud [25, Proposition 3].

Let $\epsilon > 0$ and K be a compact neighborhood of e in G. Then there exists a $u \in A(G)$ such that $u = 1$ on K. Now, $(v - u)(e) = 0$. Since points are synthesis for $A(G)$ (see [6, (4.11) Corollary 2]), there exists a $w \in A(G) \bigcap C_{00}(G)$ such that $\|(v - u) - w\| < \epsilon$ and $w = 0$ on some neighborhood U of e.

Note that $\operatorname{supp} u_\alpha^j \subseteq U_j N_\alpha$. If $U_j N_\alpha \subseteq U \bigcap K$, then $uu_\alpha^j = u_\alpha^j$ and $wu_\alpha^j = 0$.

Hence for such u_α^j, we have

$$\begin{aligned} \|vu_\alpha^j - u_\alpha^j\| &= \|(v-u-w)u_\alpha^j\| \\ &= \|v-u-w\|\|u_\alpha^j\| \\ &= \|v-u-w\| < \epsilon. \end{aligned}$$

Therefore, by the direction on $J \times X$, we only have to show that there exists an element $(j_o, \alpha_o) \in J \times X$ such that $U_{j_o} N_{\alpha_o} \subseteq U \bigcap K$. Choose a neighborhood V of e such that $V^2 \subseteq U \bigcap K$. Since $(N_\alpha)_{0<\alpha<\mu}$ is a decreasing net of compact subgroups of G and $\bigcap_{0<\alpha<\mu} N_\alpha = \{e\}$, there exists an $\alpha_o < \mu$ such that $N_{\alpha_o} \subseteq V$. Let $j_o \in J$ be such that $U_{j_o} \subseteq V$. Then $U_{j_o} N_{\alpha_o} \subseteq V^2 \subseteq U \bigcap K$..

□

Remark 5.2.3. Recall that if N is a compact normal subgroup of a locally compact group G, then $A(G/N)$ embeds into $A(G)$ (corresponding to the subspace of all N- periodic functions in $A(G)$, see [6, Proposition (3.25)]). In our case, now $\bigcup_{0<\alpha<\mu} A(G/N_\alpha)$ is norm dense in $A(G)$ (since $\bigcup_{0<\alpha<\mu} L^2(G/N_\alpha)$ is norm dense in $L^2(G)$). For a fixed $0 < \alpha < \mu$, $(u_\alpha^j)_{j\in J}$ may not be topologically convergent to invariance. However, since N_α is synthesis for $A(G)$ (see [15, p.94]), we still can show that $(u_\alpha^j)_{j\in J}$ is topologically convergent to invariance "for $A(G/N_\alpha)$", that is,

$$\lim_{j\in J} \|vu_\alpha^j - u_\alpha^j\| = 0$$

for all $v \in A(G)$ with $v = 1$ on N_α. But this fact will not be needed in the sequel.

For a directed set X, let $\mathcal{F}(X)$ be the subset of means on $l^\infty(X)$ defined as in Section 2.3. Chou in [2] showed that if G is a non-discrete metrizable locally compact group, then there exists a mutually orthogonal sequence in $A(G)$ which is topologically convergent to invariance. Using such a sequence, he constructed a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ which embeds the large set $\mathcal{F}(\mathbf{N})$ into $TIM(\hat{G})$ (see [2, Theorem 3.3]). Recall that the net $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ in Theorem 5.2.2 is topologically convergent to invariance and $(u_\alpha^j)_{\alpha\in X}$ is mutually orthogonal for each fixed $j \in J$. Thus in case G is non-metrizable, although we can not set up one linear isometry embedding a big set into $TIM(\hat{G})$, we still have the following weaker version of Chou's results obtained by modifying his technique.

Theorem 5.2.4. *Let G be a σ-compact non-metrizable locally compact group and $(N_\alpha)_{\alpha\le\mu}$ be the decreasing family of normal subgroups of G as in Theorem 4.3.1. Let $X = \{\alpha\,;\, \alpha < \mu\}$ with its natural order and $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ be the same net in $A(G)$ as in Theorem 5.2.2. For every j in J, define $\pi_j : VN(G) \longrightarrow l^\infty(X)$ by*

$$\pi_j(T)(\alpha) \;=\; \langle T\,,\, u_\alpha^j\rangle, \qquad T \in VN(G)\,,\ \alpha \in X.$$

Then

(a) for each $j \in J$, π_j is a positive linear mapping of $VN(G)$ onto $l^\infty(X)$ with $\|\pi_j\| = 1$ and the conjugate π_j^* is a linear isometry of $l^\infty(X)^*$ into $VN(G)^*$;

(b) for each $\phi \in l^\infty(X)^*$, if we let

$$W_\phi \;=\; \{\,all\ w^*\text{-cluster points of } (\pi_j^*\phi)_{j\in J} \text{ in } VN(G)^*\,\},$$

then $W_\phi \neq \emptyset$, $W_\phi \subseteq TIM(\hat{G})$ *if* $\phi \in \mathcal{F}(X)$, *and the family* $\{W_\phi\,;\ \phi \in l^\infty(X)^*\}$ *is pairwise disjoint.*

Proof. (a) Fix $j \in J$. Clearly, π_j is linear, $\pi_j(I)$ is the constant function of value one, and $\pi_j(T) \geq \mathbf{0}$ if $T \geq \mathbf{0}$. If $T \in VN(G)$ and $\alpha \in X$, then $|\pi_j(T)(\alpha)| = |\langle T\,,\, u_\alpha^j\rangle| \leq \|T\|\|u_\alpha^j\| = \|T\|$ Therefore, $\|\pi_j\| = 1$. To see that π_j is onto and π_j^* is an isometry, we only have to show that for each $f \in l^\infty(X)$, there exists a $T \in VN(G)$ such that $\pi_j(T) = f$ and $\|T\| = \|f\|_\infty$.

Let $\Lambda = \Lambda(X)$ be the set of all non-empty finite subsets of X directed by inclusion. Let $f \in l^\infty(X)$. For each $\tau \in \Lambda$, let $S_\tau = \sum_{\alpha\in\tau} f(\alpha)Q_\alpha$. Since $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $VN(G)$ and $f \in l^\infty(X)$, then

$$\|S_\tau\| \leq \|f\|_\infty, \qquad \text{for all } \tau \in \Lambda,$$

and the net $(S_\tau)_{\tau\in\Lambda}$ is convergent in the weak operator topology to an operator $T \in VN(G)$ with $\|T\| \leq \|f\|_\infty$.

Recall that on $VN(G)$ the $\sigma(VN(G)\,,\,A(G))$-topology coincides the weak operator topology. Consequently, by $u_\alpha^j \in A(G)$ and formula (5.2.4), we get

$$\begin{aligned} \pi_j(T)(\alpha) &= \langle T\,,\, u_\alpha^j\rangle = \lim_{\tau\in\Lambda}\langle S_\tau\,,\, u_\alpha^j\rangle \\ &= \lim_{\tau\in\Lambda}\sum_{\beta\in\tau} f(\beta)\langle Q_\beta\,,\, u_\alpha^j\rangle \\ &= f(\alpha), \qquad\qquad \text{for all } \alpha \in X, \end{aligned}$$

i.e., $\pi_j(T) = f$. In particular, $\|f\|_\infty \leq \|\pi_j\|\|T\| = \|T\|$, and hence $\|T\| = \|f\|_\infty$.

This completes the proof of (a).

An interesting fact here is that the above operator T is independent of the choice of j in J, that is, given $f \in l^\infty(X)$, there exists a "common" $T \in VN(G)$ such that

$$\|T\| = \|f\|_\infty \text{ and } \pi_j(T) = f, \text{ for all } j \in J.$$

We need this fact later.

(b) Let $\phi \in l^\infty(X)^*$. Since $\|\pi_j^*\phi\| = \|\phi\|$ for all $j \in J$ and the unit ball in $VN(G)^*$ is w^*-compact, then the net $(\pi_j^*\phi)_{j\in J}$ must have a w^*- cluster point in $VN(G)^*$. So, $W_\phi \neq \emptyset$.

Let $\phi \in \mathcal{F}(X)$ and $F \in W_\phi$. Then there exists a subnet $(\pi_{j'}^*\phi)_{j'}$ of $(\pi_j^*\phi)_{j\in J}$ such that $\pi_{j'}^*\phi \longrightarrow F$ in the $\sigma(VN(G)^*, VN(G))$-topology. Now,

$$\|F\| \leq \liminf_{j'} \|\pi_{j'}^*\phi\| = \|\phi\| = 1,$$

and

$$\langle F, I\rangle = \lim_{j'} \langle \pi_{j'}^*\phi, I\rangle = \lim_{j'} \langle \phi, \pi_{j'}(I)\rangle = \phi(\mathbf{1}) = 1,$$

where $\mathbf{1}$ is the constant function of value one. Therefore, $\|F\| = \langle F, I\rangle = 1$. Let $T \in VN(G)$ and $v \in A(G)$ with $v(e) = 1$. Then

$$\langle F, v\cdot T - T\rangle = \lim_{j'} \langle \pi_{j'}^*\phi, v\cdot T - T\rangle = \lim_{j'} \langle \phi, \pi_{j'}(v\cdot T - T)\rangle. \qquad (5.2.5)$$

By Theorem 5.2.2, $\lim_{j',\alpha} \|vu_\alpha^{j'} - u_\alpha^{j'}\| = 0$. Thus, we get

$$\begin{aligned} \lim_{j',\alpha} \pi_{j'}(v\cdot T - T)(\alpha) &= \lim_{j',\alpha} \langle v\cdot T - T, u_\alpha^{j'}\rangle \\ &= \lim_{j',\alpha} \langle T, vu_\alpha^{j'} - u_\alpha^{j'}\rangle = 0. \end{aligned}$$

So, given $\epsilon > 0$, there exists j'_o and α_o such that

$$|\pi_{j'}(v \cdot T - T)(\alpha)| < \epsilon, \qquad \text{for all } (j', \alpha) \geq (j'_o, \alpha_o). \tag{5.2.6}$$

Since $\phi \in \mathcal{F}(X)$, then (5.2.6) implies that

$$|\langle \phi, \pi_{j'}(v \cdot T - T) \rangle| \leq \epsilon, \qquad \text{for all } j' \geq j'_o. \tag{5.2.7}$$

Consequently, (5.2.5) and (5.2.7) combined give

$$\langle F, v \cdot T - T \rangle = \lim_{j'} \langle \phi, \pi_{j'}(v \cdot T - T) \rangle = 0,$$

i.e., $\langle F, v \cdot T \rangle = \langle F, T \rangle$ for all $T \in VN(G)$, $v \in A(G)$ with $v(e) = 1$. We conclude that $W_\phi \subseteq TIM(\hat{G})$ for all $\phi \in \mathcal{F}(X)$.

Let $\phi_1, \phi_2 \in l^\infty(X)^*$ be two different elements. Assume that $F \in W_{\phi_1} \bigcap W_{\phi_2}$. Let $f \in l^\infty(X)$. By the fact mentioned at the end of the proof of (a), there exists a "common" $T \in VN(G)$ such that

$$\pi_j(T) = f, \qquad \text{for all } j \in J.$$

Then $\langle \phi_1, f \rangle = \langle \phi_1, \pi_j(T) \rangle = \langle \pi_j^* \phi_1, T \rangle$ for all $j \in J$. Similarly, we have that $\langle \phi_2, f \rangle = \langle \pi_j^* \phi_2, T \rangle$ for all $j \in J$. By taking limits on subnets, we thus get that $\langle \phi_1, f \rangle = \langle F, T \rangle$ and $\langle \phi_2, f \rangle = \langle F, T \rangle$, i.e., $\langle \phi_1, f \rangle = \langle \phi_2, f \rangle$. This is true for all $f \in l^\infty(X)$. It follows that $\phi_1 = \phi_2$, contradicting the fact that $\phi_1 \neq \phi_2$. Therefore, $W_{\phi_1} \bigcap W_{\phi_2} = \emptyset$ for all $\phi_1, \phi_2 \in l^\infty(X)^*$ with $\phi_1 \neq \phi_2$. This completes the proof of the theorem.

□

Recall that if A is a set, then 2^A denotes the set of all subsets of A. The above theorem together with the embedding results for $VN(G)^*$ will yield the following

Theorem 5.2.5. *Let G be a non-discrete locally compact group. Let μ be the initial ordinal satisfying $|\mu| = b(G)$ and $X = \{\alpha\,;\,\alpha < \mu\}$ with its natural order. Then there exists a one-one map $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ such that*

(i) $W(\phi) \neq \emptyset$ for all $\phi \in l^\infty(X)^*$;

(ii) $W(\phi_1) \bigcap W(\phi_2) = \emptyset$ if $\phi_1, \phi_2 \in l^\infty(X)^*$ and $\phi_1 \neq \phi_2$;

(iii) $W(a\phi) = aW(\phi)$ and $W(\phi_1 + \phi_2) \subseteq W(\phi_1) + W(\phi_2)$ for all ϕ, ϕ_1, $\phi_2 \in l^\infty(X)^*$ and $a \in \mathbf{C}$;

(iv) $W(\phi) \subseteq TIM(\hat{G})$ if $\phi \in \mathcal{F}(X)$.

Proof. When G is metrizable, this corollary is a consequence of Chou [2, Theorem 3.3]. In the following we assume that G is non-metrizable.

If G is σ-compact, let $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ be defined by $W(\phi) = W_\phi$, where $W_\phi \subseteq VN(G)^*$ is the same as in Theorem 5.2.4 (b). Then W satisfies (i), (ii) and (iv). It is easy to check that W also satisfies (iii).

In the general case (G not necessarily σ-compact), let $G_\circ$ be a compactly generated open subgroup of G. Let $t : A(G_\circ) \longrightarrow A(G)$ be the extension map defined by $tv = \overset{\circ}{v}$, where $\overset{\circ}{v} = v$ on $G_\circ$ and 0 outside $G_\circ$. Then, by Granirer [10, Theorem 3], t^{**} is a linear isometry of $VN(G_\circ)^*$ into $VN(G)^*$ and $t^{**}(TIM(\hat{G}_\circ)) = TIM(\hat{G})$. Note that now $G_\circ$ is σ-compact and non-metrizable and $b(G_\circ) = b(G)$. We let

$W_1 : l^\infty(X)^* \longrightarrow 2^{VN(G_o)^*}$ be the map given in the previous paragraph. Define $W = \widehat{t^{**}} \circ W_1$, where $\widehat{t^{**}} : 2^{VN(G_o)^*} \longrightarrow 2^{VN(G)^*}$ is the map generated by t^{**}, i.e., $\widehat{t^{**}}(\mathcal{E}) = \{t^{**}F ;\, F \in \mathcal{E}\}$ for all $\mathcal{E} \subseteq VN(G_o)^*$. Then $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ has properties $(i) - (iv)$.

□

Corollary 5.2.6. *Let G be a non-discrete locally compact group. Then*

$$|TIM(\hat{G})| \geq 2^{2^{b(G)}}.$$

Proof. Let μ be the initial ordinal with $|\mu| = b(G)$ and $X = \{\alpha ;\, \alpha < \mu\}$. Let $W : l^\infty(X)^* \longrightarrow 2^{VN(G)^*}$ be the one-one map in Theorem 5.2.5. Then, by the properties (i), (ii) and (iv) and Proposition 3.2.3, we have

$$|TIM(\hat{G})| \geq |\mathcal{F}(X)| = 2^{2^{|X|}} = 2^{2^{b(G)}}.$$

□

5.3. The Cardinality of $TIM(\widehat{G})$.

To obtain the exact cardinality of $TIM(\hat{G})$, we need two more technical lemmas.

Lemma 5.3.1. *Let G be a non-discrete locally compact group and K be a compact subset of G. Let*

$$C_K(G) = \{f ;\, f \in C_{oo}(G) \text{ and } \operatorname{supp} f \subseteq K\}.$$

Then there exists a subset $\mathcal{L}$ of $L^2(G)$ such that $|\mathcal{L}| \le b(G)$ and $\mathcal{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$.

Proof. Choose a set J with $|J| = b(G)$. Let $\{U_j\,;\, j \in J\}$ be an open basis at the unit element e of G. Since K is compact, for each fixed $j \in J$, there exist $x_1^j, \cdots, x_{n_j}^j \in K$ such that $K \subseteq \bigcup_{k=1}^{n_j} x_k^j U_j$.

Let $\mathcal{E}_\circ$ be the set of all such sets $x_k^j U_j \bigcap K$, $j \in J$ and $k = 1, \cdots, n_j$. Then $|\mathcal{E}_\circ| \le |J| = b(G)$ (since $b(G)$ is infinite), and $\mathcal{E}_\circ$ is a basis for open sets in K (with the relative topology). Let

$$\mathcal{E} = \{E\,;\, E = \bigcup_{k=1}^{n} H_k \text{ for some } H_1, \cdots, H_n \in \mathcal{E}_\circ\}.$$

Then we still have $|\mathcal{E}| \le b(G)$.

Define

$$\mathcal{L} = \{\sum_{k=1}^{n} a_k 1_{E_k}\,;\, a_k \in \mathbf{Q}_c,\ E_k \in \mathcal{E},\ k = 1, \cdots, n\},$$

where $\mathbf{Q}_c = \{a + ib \in \mathbf{C}\,;\, a, b \text{ are rationales}\}$. Then $\mathcal{L} \subseteq L^2(G)$ and $|\mathcal{L}| \le b(G)$, since $\mathbf{Q}_c$ is countable and $|\mathcal{E}| \le b(G)$ (with $b(G)$ infinite).

We claim that $\mathcal{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$. We can assume that $\lambda(K) > 0$. Let $f \in C_K(G)$ with $\|f\|_\infty > 0$ and let $\epsilon > 0$. Then there exists a partition $\{F_k\,;\, k = 1, \cdots, n\}$ of $\operatorname{supp} f$ ($\subseteq K$) such that each F_k is measurable and

$$|f(x) - f(y)| < \delta_1, \qquad \text{for } x, y \in F_k,\ k = 1, \cdots, n,$$

where $\delta_1 = \epsilon\,(4\lambda(K)^{1/2})^{-1}$. By the density of $\mathbf{Q}_c$ in $\mathbf{C}$, for each k, we can choose an

$a_k \in \mathbf{Q}_c$ such that

$$|a_k| \leq \|f\|_\infty \text{ and } |f(x) - a_k| < 2\delta_1, \text{ for } x \in F_k. \tag{5.3.1}$$

Fix $1 \leq k \leq n$. By the regularity of the left Haar measure λ of G, there exist an open set O_k and a compact set M_k such that $M_k \subseteq F_k \subseteq O_k$ and

$$\lambda(O_k \setminus M_k) < \delta_2, \tag{5.3.2}$$

where $\delta_2 = \epsilon^2(2n\|f\|_\infty)^{-2}$. Note that $\mathcal{E}_o$ is a basis for open sets in K, $O_k \bigcap K$ is open in K and M_k is compact. Then there exist $H_1^k, \cdots, H_m^k \in \mathcal{E}_o$ such that $M_k \subseteq \bigcup_{l=1}^m H_l^k \subseteq O_k \bigcap K$. Let $E_k = \bigcup_{l=1}^m H_l^k$. Then $E_k \in \mathcal{E}$. Now $M_k \subseteq E_k \subseteq O_k$ and $M_k \subseteq F_k \subseteq O_k$. Hence (5.3.2) implies

$$\lambda(E_k \triangle F_k) \leq \lambda(O_k \setminus M_k) < \delta_2, \tag{5.3.3}$$

where $E_k \triangle F_k$ is the symmetric difference of E_k and F_k. Let

$$g = \sum_{k=1}^n a_k 1_{E_k}.$$

Then $g \in \mathcal{L}$. Recall that $f = \sum_{k=1}^n f 1_{F_k}$. Hence,

$$\begin{aligned}
|f - g| &= |\sum_{k=1}^n (f 1_{F_k} - a_k 1_{E_k})| \leq \sum_{k=1}^n |f 1_{F_k} - a_k 1_{E_k}| \\
&= \sum_{k=1}^n |f - a_k| 1_{F_k \bigcap E_k} + \sum_{k=1}^n |f| 1_{F_k \setminus E_k} + \sum_{k=1}^n |a_k| 1_{E_k \setminus F_k} \\
&\leq 2\delta_1 \sum_{k=1}^n 1_{F_k \bigcap E_k} + \|f\|_\infty \sum_{k=1}^n 1_{F_k \triangle E_k} \qquad \text{(by (5.3.1))} \\
&\leq 2\delta_1 1_K + \|f\|_\infty \sum_{k=1}^n 1_{F_k \triangle E_k}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|f-g\|_2 &\leq 2\delta_1\|1_K\|_2 + \|f\|_\infty \sum_{k=1}^{n} \|1_{F_k\triangle E_k}\|_2 \\
&= 2\delta_1\lambda(K)^{1/2} + \|f\|_\infty \sum_{k=1}^{n} \lambda(F_k\triangle E_k)^{1/2} \\
&< \frac{\epsilon}{2} + \|f\|_\infty n\,\delta_2^{1/2} \qquad \text{(by (5.3.3))} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

i.e., $\|f-g\|_2 < \epsilon$. It follows that $\mathcal{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$.

□

Lemma 5.3.2. *Let G be a non-discrete locally compact group and V be a compact subset of G. Let*

$$A_V(G) = \{v\,;\, v \in A(G) \text{ and } \operatorname{supp} v \subseteq V\}.$$

Then there exists a subset $\mathcal{S}$ of $A(G)$ such that $|\mathcal{S}| \leq b(G)$ and $\mathcal{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_V(G)$.

Proof. Choose a compact neighbourhood K of e such that $V \subseteq K$. Define

$$A_1 = \operatorname{span}\{f * \tilde{g},\, f,\, g \in C_{oo}(G),\ \operatorname{supp} f \subseteq K,\ \operatorname{supp} g \subseteq K\},$$

where span E means the linear span of E.

Assume that there exists a $v \in A_V(G)$ such that $v \notin \overline{A_1}$ (the norm closure of A_1 in $A(G)$). Then by the Hahn-Banach Theorem, there exists a $T \in VN(G) = A(G)^*$

such that $\langle T, v\rangle \neq 0$ but $\langle T, f * \tilde{g}\rangle = \langle Tf, g\rangle = 0$ for all $f,\ g \in C_{oo}(G)$ with $\operatorname{supp} f \subseteq K$ and $\operatorname{supp} g \subseteq K$. By the definition of supp T, we have that $K \subseteq G \setminus \operatorname{supp} T$, i.e., $\operatorname{supp} T \subseteq G \setminus K \subseteq G \setminus V$. Note that $v \in A_V(G)$ and hence $\operatorname{supp} v \subseteq V$. It follows that $\operatorname{supp} v \bigcap \operatorname{supp} T = \emptyset$. By [6, Proposition (4.6) and (4.8)], $v \cdot T = \mathbf{0}$. We choose a $u \in A(G)$ such that $u = 1$ on V. Then $vu = v$, and hence

$$0 = \langle v \cdot T, u\rangle = \langle T, vu\rangle = \langle T, v\rangle,$$

contradicting the fact that $\langle T, v\rangle \neq 0$. We conclude that $A_V(G) \subseteq \overline{A_1}$.

Let

$$C_K(G) = \{f \,;\, f \in C_{oo}(G) \text{ and } \operatorname{supp} f \subseteq K\}.$$

By Lemma 5.3.1, there exists an $\mathcal{L} \subseteq L^2(G)$ such that $|\mathcal{L}| \leq b(G)$ and $\mathcal{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$.

Define

$$\mathcal{S} = \{\sum_{i=1}^{n} a_i f_i * \tilde{g}_i \,;\, a_i \in \mathbf{Q}_c,\ f_i,\ g_i \in \mathcal{L},\ i = 1, \cdots, n\},$$

where $\mathbf{Q}_c$ is the same dense subset of $\mathbf{C}$ as in the proof of Lemma 5.3.1. Then $|\mathcal{S}| \leq b(G)$, since $\mathbf{Q}_c$ is countable and $|\mathcal{L}| \leq b(G)$. Since $\mathcal{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$ and $\mathbf{Q}_c$ is dense in $\mathbf{C}$, by the definition of $A(G)$, $\mathcal{S}$ is $\|\cdot\|_{A(G)}$-dense in A_1. Recall that $A_V(G) \subseteq \overline{A_1}$. Therefore, $\mathcal{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_V(G)$.

□

We are now ready to find out the precise cardinality of $TIM(\hat{G})$ for any non-discrete locally compact group G.

Theorem 5.3.3. *Let G be a non-discrete locally compact group. Let $b(G)$ be the smallest cardinality of an open basis at the unit element e of G. Then*

$$|TIM(\hat{G})| = 2^{2^{b(G)}}.$$

Proof. By Corollary 5.2.6, we only have to show that $|TIM(\hat{G})| \leq 2^{2^{b(G)}}$.

Let U and V be two compact neighbourhoods of e in G such that $U \subsetneq V$. We choose two functions $u_\circ$ and $v_\circ$ in $A(G)$ such that $u_\circ(e) = 1$, $v_\circ = 1$ on U, $\operatorname{supp} u_\circ \subseteq U$ and $\operatorname{supp} v_\circ \subseteq V$. Then $u_\circ = u_\circ v_\circ$. Let

$$\mathcal{B} = \{u_\circ \cdot T\,;\, T \in VN(G)\}.$$

Then $\mathcal{B}$ is a subspace of $VN(G)$, and each $m \in TIM(\hat{G})$ is determined by its value on $\mathcal{B}$, by the definition of $TIM(\hat{G})$. Hence we have

$$|TIM(\hat{G})| \leq c^{|\mathcal{B}|}, \tag{5.3.4}$$

where c is the cardinality of the continuum.

In the following we shall prove that $|\mathcal{B}| \leq c^{b(G)}$. Let $T \in VN(G)$ and $v \in A(G)$. Then

$$\langle u_\circ \cdot T\,,\, v\rangle = \langle T\,,\, u_\circ v\rangle = \langle T\,,\, u_\circ v_\circ v\rangle = \langle u_\circ \cdot T\,,\, v_\circ v\rangle. \tag{5.3.5}$$

Now $v_\circ v \in A(G)$ with support contained in V. Define

$$A_V(G) = \{v \in A(G)\,;\, \operatorname{supp} v \subseteq V\}.$$

Then, by (5.3.5), each $u_\circ \cdot T \in \mathcal{B}$ is determined by its value on $A_V(G)$. By Lemma 5.3.2, there exists an $\mathcal{S} \subseteq A(G)$ such that $|\mathcal{S}| \leq b(G)$ and $\mathcal{S}$ is $\|\cdot\|_{A(G)}$-dense in

$A_V(G)$. Hence each $u_o \cdot T \in \mathcal{B}$ is determined by its value on $\mathcal{S}$. Consequently,

$$|\mathcal{B}| \leq c^{|\mathcal{S}|} \leq c^{b(G)}. \tag{5.3.6}$$

Finally, (5.3.4) and (5.3.6) combined give

$$|TIM(\hat{G})| \leq c^{|\mathcal{B}|} \leq c^{c^{b(G)}} = 2^{2^{b(G)}},$$

since $b(G)$ is infinite.

□

Remark 5.3.4. Lau and Paterson showed that if G is a non-compact amenable locally compact group, then $|TLIM(G)| = 2^{2^{d(G)}}$, where $TLIM(G)$ is the set of all topologically left invariant means on $L^\infty(G)$ and $d(G)$ is the smallest cardinality of a covering of G by compact sets (see [20, Theorem 1]). When G is abelian and $\hat{G}$ is the dual group of G, $A(G)$ can be identified with $L^1(\hat{G})$ (by Fourier transform) and $VN(G)$ with $L^\infty(\hat{G})$; each $f \in L^\infty(\hat{G})$ can be regarded as a multiplication operator on $L^2(\hat{G})$ which is isomorphic to $L^2(G)$ by Plancherel's theorem. Under these identifications, the module action of $L^1(\hat{G})$ on $L^\infty(\hat{G})$ is just the usual convolution. Consequently, $m \in VN(G)^*$ belongs to $TIM(\hat{G})$ if and only if the corresponding mean on $L^\infty(\hat{G})$ is a topologically left invariant mean. In particular, $|TIM(\hat{G})| = |TLIM(\hat{G})|$. Now $b(G) = d(\hat{G})$ (see [17, (24.48)]). Therefore, when G is abelian, our Theorem 5.3.3 coincides with Lau-Paterson's result.

5.4. Some Results on Structures.

For a locally compact group G, let $b(G)$ be the smallest cardinality of an open basis at the unit element e of G defined as before. The format of the following two theorems is due to Chou [2]. He discussed the case when G is metrizable.

Theorem 5.4.1. *If G is a non-discrete locally compact group, then $TIM(\hat{G})$ contains a subset E such that $|E| = |TIM(\hat{G})| = 2^{2^{b(G)}}$ and if $m_1, m_2 \in E$ and $m_1 \neq m_2$, then $\|m_1 - m_2\| = 2$. In particular, $TIM(\hat{G})$ is not norm separable.*

Proof. When G is metrizable, this is shown by Chou (see [2, Corollary 3.5]).

In the following we assume that G is non-metrizable. By Granirer [10, Theorem 3], we may assume that G is σ-compact. Let μ be the limit ordinal associated with G as in Theorem 4.3.1, $X = \{\alpha ; \alpha < \mu\}$ with its natural order and $\mathcal{F}(X)$ the subset of $l^\infty(X)^*$ defined as in Section 2.3. Let

$$\mathcal{A} = \{\phi \in \beta X ; \phi \text{ contains } \{T_\alpha ; \alpha \in X\}\},$$

where βX is the Stone-Čech compactification of the discrete set X and T_α is a tail in X as in Section 2.3. Then, by Lemma 2.3.2 and the proof of Proposition 3.2.3, $\mathcal{A} \subseteq \mathcal{F}(X)$ and $|\mathcal{A}| = 2^{2^{|X|}} = 2^{2^{b(G)}}$.

Let $\phi_1, \phi_2 \in \mathcal{A}$ with $\phi_1 \neq \phi_2$. Then $\|\phi_1 - \phi_2\| = 2$, since $\phi_1, \phi_2 \in \beta X$. Let $\psi_1 \in W_{\phi_1}$ and $\psi_2 \in W_{\phi_2}$, where W_ϕ is the non-empty subset of $TIM(\hat{G})$ defined for each $\phi \in \mathcal{F}(X)$ as in Theorem 5.2.4. Then, there exist subnets $(\pi^*_{j_1})_{j_1}$ and $(\pi^*_{j_2})_{j_2}$ of $(\pi^*_j)_{j \in J}$, where $(\pi^*_j)_{j \in J}$, is the net of linear maps associated with G as in

Theorem 5.2.4, such that

$$\pi_{j_1}^* \phi_1 \longrightarrow \psi_1 \quad \text{and} \quad \pi_{j_2}^* \phi_2 \longrightarrow \psi_2$$

in the $\sigma(VN(G)^*, VN(G))$-topology. Since $\|\psi_1\| = \|\psi_2\| = 1$, $\|\psi_1 - \psi_2\| \leq 2$.

On the other hand, if $f \in l^\infty(X)$ with $\|f\|_\infty = 1$, then, by the fact mentioned in the proof of Theorem 5.2.4, there exists a "common" $T \in VN(G)$ such that $\|T\| = \|f\|_\infty = 1$ and $\pi_j(T) = f$ for all $j \in J$.

Hence, we get

$$\begin{aligned} \|\psi_1 - \psi_2\| &\geq |\langle \psi_1 - \psi_2 \,, T \rangle| \\ &= \lim_{j_1, j_2} |\langle \pi_{j_1}^* \phi_1 - \pi_{j_2}^* \phi_2 \,, T \rangle| \\ &= \lim_{j_1, j_2} |\langle \phi_1 \,, \pi_{j_1}(T) \rangle - \langle \phi_2 \,, \pi_{j_2}(T) \rangle| \\ &= |\langle \phi_1 \,, f \rangle - \langle \phi_1 \,, f \rangle| \\ &= |\langle \phi_1 - \phi_2 \,, f \rangle| \,, \end{aligned}$$

that is,

$$\|\psi_1 - \psi_2\| \geq |\langle \phi_1 - \phi_2 \,, f \rangle| \,, \quad \text{for all } f \in l^\infty(X) \text{ with } \|f\|_\infty = 1 \,.$$

It follows that $\|\psi_1 - \psi_2\| \geq \|\phi_1 - \phi_2\| = 2$. Consequently, $\|\psi_1 - \psi_2\| = 2$ for $\psi_1 \in W_{\phi_1}$ and $\psi_2 \in W_{\phi_2}$.

For each $\phi \in \mathcal{A}$, choose a $\psi \in W_\phi$. Let E be the set of all such ψ. Then $|E| = |\mathcal{A}| = 2^{2^{b(G)}}$ and $\|m_1 - m_2\| = 2$ for all $m_1, m_2 \in E$ with $m_1 \neq m_2$.

□

Recall that $F(\hat{G})$ is the space of all $T \in VN(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as m runs through $TIM(\hat{G})$. Also, each $m \in TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$ and $W(\hat{G}) \subseteq F(\hat{G})$. Thus Theorem 5.3.3 will yield the following

Theorem 5.4.2. *Let G be a non-discrete locally compact group. If $\mathcal{B}$ is a norm dense subset of the quotient space $VN(G)/F(\hat{G})$ [or $UCB(\hat{G})/F(\hat{G})\bigcap UCB(\hat{G})$], then*

$$|\mathcal{B}| > b(G).$$

In particular, the quotient spaces $VN(G)/F(\hat{G})$, $UCB(\hat{G})/F(\hat{G})\bigcap UCB(\hat{G})$ and $UCB(\hat{G})/W(\hat{G})\bigcap UCB(\hat{G})$ are not norm separable.

Proof. Assume that $\mathcal{B}$ is norm dense in $VN(G)/F(\hat{G})$. Then there exists a subset $\mathcal{D}$ of $VN(G)$ such that $|\mathcal{D}| = |\mathcal{B}|$ and the set

$$\mathcal{E} = \{T + S\,;\, T \in \mathcal{D} \text{ and } S \in F(\hat{G})\}$$

is norm dense in $VN(G)$. Thus each $m \in TIM(\hat{G})$ is determined by its value on $\mathcal{E}$.

Fix an $m_o \in TIM(\hat{G})$. We have

$$m(T+S) = m(T) + m(S) = m(T) + m_o(S),$$

for all $m \in TIM(\hat{G})$, $T \in \mathcal{D}$ and $S \in F(\hat{G})$. Therefore, each $m \in TIM(\hat{G})$ is determined by its value on $\mathcal{D}$. Consequently, we have

$$|TIM(\hat{G})| \leq c^{|\mathcal{D}|} = c^{|\mathcal{B}|} = 2^{\aleph_0|\mathcal{B}|}, \tag{5.4.1}$$

where c is the cardinality of the continuum and $\aleph_0$ is the first infinite cardinal number.

On the other hand, by Theorem 5.3.3,

$$|TIM(\hat{G})| = 2^{2^{b(G)}} > 2^{b(G)}. \tag{5.4.2}$$

Now (5.4.1) and (5.4.2) combined give

$$\aleph_0|\mathcal{B}| > b(G).$$

But $b(G) \geq \aleph_0$, since G is non-discrete. Therefore, $|\mathcal{B}| > b(G)$.

Similarly, we can prove the $UCB(\hat{G})/F(\hat{G})\bigcap UCB(\hat{G})$ case, since each $m \in TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$, by the definitions of $TIM(\hat{G})$ and $UCB(\hat{G})$.

□

If $u \in A(G)$ with $u(e) = 1$, let

$$u^{\perp} = \{T \in VN(G)\,;\, u \cdot T = \mathbf{0}\}.$$

If $T \in u^{\perp}$ and $m \in TIM(\hat{G})$, then $m(T) = m(u{\cdot}T) = m(\mathbf{0}) = 0$. Hence, $u^{\perp} \subseteq F(\hat{G})$. Note that $W(\hat{G}) \subseteq F(\hat{G})$. By the same procedure as in the proof of Theorem 5.4.2, we can also prove the following

Theorem 5.4.3. *Let G be a non-discrete locally compact group. Let $u \in A(G)$ with $u(e) = 1$ and X be a subspace of $VN(G)$ such that $UCB(\hat{G})$ is contained in the norm closure of $W(\hat{G}) + u^{\perp} + X$. If $X_\circ$ is a norm dense subset of X, then $|X_\circ| > b(G)$.*

In particular, X is not norm separable.

Proof. Let $m \in TIM(\hat{G})$. Since $m(T) = 0$ for all $T \in u^{\perp}$ and m is determined by its value on $UCB(\hat{G})$, by the assumptions, m is determined by its value on $W(\hat{G}) + X$. Fix an $m_\circ \in TIM(\hat{G})$. Then we have

$$m(T+S) = m(T) + m(S) = m_\circ(T) + M(S),$$

for all $T \in W(\hat{G})$ and $S \in X$.

Therefor, each $m \in TIM(\hat{G})$ is determined by its value on X and hence by its value on $X_\circ$. Thus we have

$$|TIM(\hat{G})| \leq c^{|X_\circ|} = 2^{\aleph_0 |X_\circ|}.$$

On the other hand,

$$|TIM(\hat{G})| = 2^{2^{b(G)}} > 2^{b(G)},$$

by Theorem 5.3.3. Thus,

$$\aleph_0 |X_\circ| > b(G) \geq \aleph_0 .$$

It follows that $|X_\circ| > b(G)$.

□

Remark 5.4.4. (a) Theorem 5.4.3 actually improves Granirer [11, Theorem 12]. Under the same assumptions, he showed only that X is not separable.

(b) If the generalized continuum hypothesis is assumed, then the conclusions in Theorem 5.4.2 and Theorem 5.4.3 can be strengthened as $|\mathcal{B}| \geq 2^{b(G)}$ and $|X_\circ| \geq 2^{b(G)}$, respectively.

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