56

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AUTHOR – AUTEUR

Full Name of Author – Nom complet de l'auteur

MARTIN JONES STRAUCH

Date of Birth – Date de naissance

16-9-62

Canadian Citizen – Citoyen canadien

☐ Yes Oui ☒ No Non

Country of Birth – Lieu de naissance

ENGLAND

Permanent Address – Résidence fixe

BILTON ROAD, RUGBY, WARWICKSHIRE, ENGLAND

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Name of Supervisor – Nom du directeur de thèse

DR. N. TOMCZAK-JAEGERMANN

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TENSOR PRODUCTS OF BANACH SPACES WITH UNCONDITIONAL BASES

by

MARTIN STRAUCH

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(Signed) M. Strauch

PERMANENT ADDRESS:

55 Bilton Road, Rugby

Warwickshire, CV22 7AN

England

DATED 25th June 1985

THE UNIVERSITY OF ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled . . . .
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. . . . . . N. Tomczak-Jaegermann . . . .
Supervisor

. . . Anthony T. Lau . . .

. . . . Vadim Kaftal . . . .

. . . J. E. Lewis . . . . .

. . . Douglas Hollis . . .

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ABSTRACT

We introduce and study the notions of upper and lower row, column and projection estimates for tensor products of Banach spaces with unconditional bases. We relate these notions to upper and lower estimates satisfied by Banach lattices. In particular, we have a duality property, a renorming theorem and a geometrical property which correspond to similar results from Banach lattice theory. We also give a convergence theorem for tensor products (concerning the Kadec-Klee property). Finally, we show that if a Banach space X has an unconditional basis and $g\ell(X \otimes_{\vee} \ell_2)$ is finite then X is isomorphic to ℓ_1.

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Lastly I must mention my parents, I thank them for being just that.

TABLE OF CONTENTS

INTRODUCTION

The notions of upper and lower estimates were introduced independently by T. Shimogaki [18] and T. Figiel and W.B. Johnson [3] for Banach lattices (or Banach spaces with unconditional bases). Their definitions were derived from convexity and concavity conditions and they play an important role in studying the connection between type and convexity and between cotype and concavity. These notions depend heavily on the structure of Banach lattices. In general, tensor products of Banach spaces lack the kind of structure necessary for similar notions. However by restricting ourselves to certain special Banach spaces we are able to make some progress. To be specific, we require all the Banach spaces involved to have unconditional bases. The majority of the thesis is then concerned with the definition of upper and lower row, column and projection estimates (see Section Two for definitions), and the properties of tensor products satisfying these estimates. These notions are indeed analogous in some ways to those of upper and lower estimates for Banach lattices. We also give a convergence theorem (concerning the Kadec-Klee property) for tensor products satisfying certain estimates.

A tensor product of two Banach spaces inherits a certain degree of structure from those two spaces. It is natural to try to determine just how much structure is inherited. S. Kwapien and A. Pelczynski showed in [9] that, in general, even the tensor product of two Banach spaces with unconditional bases fails to have an unconditional basis. The study of local unconditional structure is a natural consideration for tensor products. Y. Gordon and D.R. Lewis [4,5,10] did much work in this area, especially for tensor products with the least and greatest crossnorms (see Section One for definitions). C. Schütt [17] showed that for any tensor norm on a tensor product of finite dimensional Banach spaces, in order to study unconditional structure it is enough to consider a certain basis. G. Pisier [15] studied the problem of tensor products of Banach spaces being isomorphic to subspaces of Banach spaces with unconditional bases.

In Section One we present definitions from the general theory of Banach spaces. Special attention is given to the notion of unconditional basis, which we use throughout the thesis. We also give definitions from tensor product theory. There is no absolutely standard set of definitions, the ones we give are consistent and sufficient for our purposes.

We introduce in Section Two the upper and lower row, column and projection estimates for tensor products of Banach spaces with unconditional bases. These notions are in some ways analogous to upper and lower estimates in Banach lattice theory, but they are certainly weaker. The majority of Section Two is devoted to a study of tensor products satisfying these estimates. Our theory developed in this part of the thesis follows a very similar course to the one taken by the theory of upper and lower estimates for Banach lattices. We end Section Two with examples of spaces satisfying upper and lower projection estimates.

We consider the Kadec-Klee property for tensor products in Section Three. Using a lemma from [19] adapted for tensor products we give a condition for a tensor product to have the Kadec-Klee property. In particular, we show that for $1 < p, q < \infty$, $\ell_p \otimes_\nu \ell_q$ has the property (where ν is the greatest norm).

In Section Four we consider the local unconditional structure of tensor products. More specifically we show that if a Banach space X has an unconditional basis and $g\ell(X \otimes_{\vee} \ell_2)$ is finite (see Section Four for the definition of the $g\ell$ constant), then X is isomorphic to ℓ_1. This gives a characterization of ℓ_1 among Banach spaces with unconditional bases.

Finally, in Section Five, we present some as yet unanswered problems which arise from the notions considered in this thesis.

SECTION ONE

STANDARD DEFINITIONS

In general our notation for the theory of Banach spaces is standard [12]. One exception is that we denote the action of a functional x^* on elements x by $\langle x,x^* \rangle$. In most of the following it does not matter if the field of scalars is real or complex, we shall indicate whenever we restrict ourselves to the reals. We make some definitions for general Banach spaces before going on to discuss tensor products and norms on tensor products.

For any two Banach spaces X and Y we denote the *Banach-Mazur distance* between X and Y by $d(X,Y)$, that is,

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\},$$
$$d(X,Y) = \infty \text{ if } X \text{ and } Y \text{ are not isomorphic.}$$

A basis $\{e_i\}$ of a Banach space X is called *unconditional* if there exists a constant C such that for every $x = \Sigma a_i e_i \in X$,

$$\|\sum_i \varepsilon_i a_i e_i\|_X \le C\|x\|_X \quad \text{for every } \varepsilon_i = \pm 1 \text{ for } i = 1,2,\ldots .$$

The smallest such constant C is called the unconditional basis constant of the basis $\{e_i\}$ and is denoted $\mathrm{unc}(\{e_i\})$. The *unconditional basis constant* of a Banach space X is defined as

$$\mathrm{unc}(X) = \inf\{\mathrm{unc}(\{e_i\}) : \{e_i\} \text{ is a basis of } X\} .$$

We put $\mathrm{unc}(X) = \infty$ if X has no unconditional basis. A basis $\{e_i\}$

is called 1-unconditional if $\mathrm{unc}(\{e_i\}) = 1$. If X has an unconditional basis $\{e_i\}$, then there exists an equivalent norm $|||\cdot|||$ on X so that $\{e_i\}$ is 1-unconditional with respect to $|||\cdot|||$. This renorming is defined by taking

$$|||x||| = \sup\{\|\sum \varepsilon_i a_i e_i\|_X : \varepsilon_i = \pm 1 \text{ for } i = 1,2,\ldots\}$$

$$\text{for every } x = \sum a_i e_i \text{ in } X.$$

Given an unconditional basis $\{e_i\}_{i=1}^{\infty}$ of X and any subset E of the integers, we define the projection $P_E : X \to \mathrm{span}[e_i]_{i\in E}$ by,

$$P_E\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i\in E} a_i e_i \quad \text{for} \quad \sum_{i=1}^{\infty} a_i e_i \quad \text{in } X.$$

These projections are called the natural projections associated to the unconditional basis $\{e_i\}_{i=1}^{\infty}$ and have the property that $\sup_E \|P_E\| \le \mathrm{unc}(\{e_i\})$.

Given an unconditional basis $\{e_i\}$, two vectors $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i$ are disjointly supported if $x_i y_i = 0$ for each i.

Let $1 < p < \infty$. A Banach space with an unconditional basis $\{e_i\}_{i=1}^{\infty}$ is said to satisfy an *upper* or *lower* p-*estimate* if there exists a constant M such that for all finite sequences $\{x_i\}_{i=1}^{n}$ of disjointly supported vectors we have,

$$\left\|\sum_{i=1}^{n} x_i\right\|_X \le M\left(\sum_{i=1}^{n} \|x_i\|_X^p\right)^{1/p}, \text{ respectively,}$$

$$\left\|\sum_{i=1}^{n} x_i\right\|_X \ge M^{-1}\left(\sum_{i=1}^{n} \|x_i\|_X^p\right)^{1/p}.$$

In either case the least such constant M is called the estimate constant. For spaces satisfying such estimates we have the following theorem [12], which we shall use later.

Theorem. For $1 < p < \infty$, X satisfies an upper (respectively lower) p-estimate if and only if X^* satisfies a lower (respectively upper) p*-estimate, where $1/p + 1/p^* = 1$, with equal estimate constants.

We now define the tensor product of two Banach spaces and introduce a variety of norms on it. Let X and Y be Banach spaces. Following Schatten [16] we introduce the symbols $\otimes$ and $+$, and for finite sequences $\{x_i\}_{i=1}^n \subset X$, $\{y_i\}_{i=1}^n \subset Y$ and scalars $\{a_i\}_{i=1}^n$, we construct formal expressions

$$a_1(x_1 \otimes y_1) + a_2(x_2 \otimes y_2) + \ldots + a_n(x_n \otimes y_n) = \sum_{i=1}^n a_i(x_i \otimes y_i) .$$

On these expressions we introduce an equivalence relation $\sim$ subject to

i) $\sum_{i=1}^n x_i \otimes y_i \sim \sum_{i=1}^n x_{\pi(i)} \otimes y_{\pi(i)}$ for all permutations π .

ii) $(x_1' + x_1'') \otimes y_1 + \sum_{i=2}^n x_i \otimes y_i \sim x_1' \otimes y_1 + x_1'' \otimes y_1 + \sum_{i=2}^n x_i \otimes y_i$.

ii) $x_1 \otimes (y_1' + y_1'') + \sum_{i=2}^n x_i \otimes y_i \sim x_1 \otimes y_1' + x_1 \otimes y_1'' + \sum_{i=2}^n x_i \otimes y_i$.

iii) $\sum_{i=1}^n (a_i x_i) \otimes y_i \sim \sum_{i=1}^n x_i \otimes (a_i y_i) \sim \sum_{i=1}^n a_i (x_i \otimes y_i)$ for all scalar a_i .

Two expressions are said to be equivalent if one can be transformed into the other by a finite number of successive applications of i), ii), ii)', and iii).

We denote by $X \otimes Y$ the linear space of all equivalence classes of expressions of the form $z = \sum_{i=1}^{n} a_i(x_i \otimes y_i)$, $(\{x_i\}_{i=1}^{n} \subset X$, $\{y_i\}_{i=1}^{n} \subset Y$, scalars $\{a_i\}_{i=1}^{n})$ with the natural operations of addition and scalar multiplication.

There is a natural identification of $X \otimes Y$ with the space of finite rank operators from X^* into Y, given in the following way: an expression $\sum_{i=1}^{n} a_i(x_i \otimes y_i)$ represents the operator T defined by

$$T(x^*) = \sum_{i=1}^{n} a_i \langle x_i, x^* \rangle y_i \quad \text{for } x^* \text{ in } X^* .$$

It can be easily checked that if two expressions are equivalent then the corresponding operators are equal. In particular, an expression $x \otimes y$ for $x \in X$ and $y \in Y$ represents the operator determined by

$$(x \otimes y)x^* = \langle x, x^* \rangle y \quad \text{for each } x^* \text{ in } X^* .$$

Definition [16]. A real-valued function α acting on $X \otimes Y$ will be called a *crossnorm* if,

I) For every $z \in X \otimes Y$, $\alpha(z) \geq 0$; $\alpha(z) = 0 \iff z = 0$.

For every $z \in X \otimes Y$ and for all scalars a, $\alpha(az) = |a|\alpha(z)$.

For all $y, z \in X \otimes Y$, $\alpha(y + z) \leq \alpha(y) + \alpha(z)$.

II) For all $x \in X$, $y \in Y$, $\alpha(x \otimes y) = \|x\|_X \|y\|_Y$.

We denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ under the crossnorm α.

Definition [16]. For a crossnorm α on $X \otimes Y$ we define the *associate norm* α' on $X^* \otimes Y^*$ by, for every $\sum_{j=1}^{m} x_j^* \otimes y_j^*$ in $X^* \otimes Y^*$,

$$\alpha'\left(\sum_{j=1}^{m} x_j^* \otimes y_j^*\right) = \sup\left\{\left|\sum_{j=1}^{m}\sum_{i=1}^{n} \langle x_i, x_j^*\rangle\langle y_i, y_j^*\rangle\right|\right\}$$

where the supremum is taken over all $\sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$ with $\alpha(\sum_{i=1}^{n} x_i \otimes y_i) \leq 1$. For two crossnorms α and β with $\alpha(z) \leq \beta(z)$ for all z in $X \otimes Y$, clearly $\alpha'(y) \geq \beta'(y)$ for all $y \in X^* \otimes Y^*$.

Let u be a member of $X^* \otimes X$, $u = \sum_{j=1}^{n} a_j^* \otimes b_j$ where $a_1^*, \ldots, a_n^*$ are in X^*, $b_1, \ldots, b_n$ are in X. Then the trace of u is defined by

$$tr(u) = \sum_{j=1}^{n} \langle b_j, a_j^*\rangle.$$

The trace of u does not depend on the choice of equivalent tensor representations of u. Trace is a linear functional on $X^* \otimes X$.

Observe that for every element z of $X^* \otimes Y^*$ there is a corresponding functional ϕ_z on $X \otimes Y$ defined by

$$\phi_z(u) = tr(zu) = \langle u, z\rangle.$$

Since $|\phi_z(u)| \leq \alpha(u)\alpha'(z)$, $\phi_z \in (X \otimes_\alpha Y)^*$ and $\|\phi_z\| \leq \alpha'(z)$. This correspondence induces an isometric isomorphism between $X^* \otimes_{\alpha'} Y^*$ and $(X \otimes_\alpha Y)^*$ if at least one of X and Y is of finite dimension.

Definition [16]. The *least crossnorm* λ is defined for $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ as,

$$\lambda(z) = \sup \{ |\langle z, x^* \otimes y^* \rangle| : x^* \in X^*,\ y^* \in Y^*;\ \|x^*\|, \|y^*\| \leq 1 \},$$

$$= \sup \left\{ \left| \sum_{i=1}^{n} \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| : x^* \in X^*,\ y^* \in Y^*;\ \|x^*\|, \|y^*\| \leq 1 \right\}.$$

Then $\lambda(z)$ represents the operator norm of the operator $z : X^* \to Y$ determined by the expression $\sum_{i=1}^{n} x_i \otimes y_i$ and $X \otimes_\lambda Y$ is isometrically isomorphic to a closed subspace of the compact operators from X^* to Y ($X \otimes_\lambda Y$ is isometrically isomorphic to the space of compact operators from X^* to Y if and only if Y has the approximation property [12]).

Given a crossnorm α on $X \otimes Y$, the associate norm α' is a crossnorm on $X^* \otimes Y^*$ if and only if $\alpha(z) \geq \lambda(z)$ for every z in $X \otimes Y$. For, if there exists a z in $X \otimes Y$ such that $\alpha(z) \leq \lambda(z)$, then, by the definitions of the associate norm and the least crossnorm, there exists $x^* \in X^*$, $y^* \in Y^*$ with $\alpha(z)\|x^*\|\|y^*\| < |(x^* \otimes y^*)(z)| \leq \alpha'(x^* \otimes y^*)\alpha(z)$. Therefore, $\alpha'(x^* \otimes y^*) > \|x^*\|\|y^*\|$ and α' is not a crossnorm on $X^* \otimes Y^*$. Conversely, by the definition of the associate norm, we have $\alpha'(x^* \otimes y^*) \geq \|x^*\|\|y^*\|$ for every $x^* \in X^*$, $y^* \in Y^*$. Now suppose $\alpha \geq \lambda$, then $\alpha' \leq \lambda'$. For any $\sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$,

$$\left| (x^* \otimes y^*) \left(\sum_{i=1}^{n} x_i \otimes y_i \right) \right| \leq \|x^*\|\|y^*\| \lambda \left(\sum_{i=1}^{n} x_i \otimes y_i \right)$$

therefore we have,

$$\alpha'(x^* \otimes y^*) \leq \lambda'(x^* \otimes y^*) \leq \|x^*\|\|y^*\|$$

and so α' is a crossnorm on $X^* \otimes Y^*$.

Definition. The *greatest crossnorm* ν is defined for $z \in X \otimes Y$ as,

$$\nu(z) = \inf\left\{ \sum_{i=1}^{n} \|x_i\|_X \|y_i\|_Y : \sum_{i=1}^{n} x_i \otimes y_i = z \right\}.$$

If α is any crossnorm on $X \otimes Y$ then $\alpha(z) \leq \nu(z)$ for all z in $X \otimes Y$. The associate of ν is λ, i.e. $X^* \otimes_{\nu'} Y^* = X^* \otimes_{\lambda} Y^*$, also we have, $(X \otimes_{\lambda} Y)^* = X^* \otimes_{\nu} Y^*$ and $(X \otimes_{\nu} Y)^* = L(Y,X^*)$ for all Banach spaces X and Y.

Given a tensor product $X \otimes Y$ and two arbitrary Banach spaces X_1 and Y_1, for every $S \in L(X,X_1)$, $T \in L(Y,Y_1)$ there exists a unique operator $S \otimes T : X \otimes Y \to X_1 \otimes Y_1$ defined for $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ by,

$$(S \otimes T)(z) = \sum_{i=1}^{n} (Sx_i) \otimes (Ty_i).$$

Definition. A crossnorm α on $X \otimes Y$ will be called a *tensor norm* if,

III) For any $S \in L(X)$, $T \in L(Y)$ we have,

$$\alpha\big((S \otimes T)(z)\big) \leq \|S\| \alpha(z) \|T\| \quad \text{for every } z \in X \otimes Y.$$

IV) $\lambda(z) \leq \alpha(z) \leq \nu(z)$ for every $z \in X \otimes Y$.

Both λ and ν are tensor norms and the associate of a tensor norm on $X \otimes Y$ is a tensor norm on $X^* \otimes Y^*$.

Definition. A real-valued function α defined on all pairs (X,Y) of Banach spaces is called an *ideal norm* if it is a tensor norm on each tensor product $X \otimes Y$ and if for arbitrary Banach spaces X_1 and Y_1, we have, for all $S \in L(X,X_1)$, $T \in L(Y,Y_1)$,

$$\alpha\big((S \otimes T)(z)\big) \leq \|S\|\alpha(z)\|T\| \text{ for every } z \text{ in } X \otimes_\alpha Y .$$

Both λ and ν are ideal norms. Other examples of ideal norms will be given in later sections.

SECTION TWO

UPPER AND LOWER ESTIMATES IN TENSOR PRODUCTS

In this section we introduce the definitions of upper and lower p-row, column and projection estimates for tensor products of Banach spaces with unconditional bases. These notions are very similar to those of upper and lower p-estimates for Banach lattices [3]. We prove several results about tensor products satisfying these estimates, following a similar course to the one taken by the theory of upper and lower p-estimates as presented in [12]. More specifically, we have a duality theorem, a renorming theorem and a theorem concerning the existence of isomorphic copies of ℓ_1^n or ℓ_∞^n on rows or columns which are very similar to the corresponding theorems for Banach lattices.

Recall (c.f. Section 1) that given an unconditional basis $\{e_i\}_{i=1}^\infty$ for some Banach space, for a subset E of the integers, we denote by P_E the natural projection from the space onto $\text{span}[e_i]_{i\in E}$. The identity will be denoted by I.

In the following $\{e_i\}_{i=1}^\infty$ and $\{f_j\}_{j=1}^\infty$ will be normalized unconditional bases for X and Y respectively and α will be a tensor norm on $X \otimes Y$.

Definition. Let $1 < p < \infty$. The tensor product $X \otimes_\alpha Y$ is said to satisfy, respectively

i) a lower p-row estimate,

ii) a lower p-column estimate,

iii) a lower p-projection estimate,

if there exists a constant M_r, M_c or M such that for all $z \in X \otimes_\alpha Y$ and for all finite partitions of the integers $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^m$, we have, respectively,

$$\text{i)} \quad M_r \alpha(z) \geq \left(\sum_{i=1}^n \alpha\big((P_{E_i} \otimes I)(z)\big)^p \right)^{1/p}$$

$$\text{ii)} \quad M_c \alpha(z) \geq \left(\sum_{j=1}^m \alpha\big((I \otimes P_{F_j})(z)\big)^p \right)^{1/p}$$

$$\text{iii)} \quad M\alpha(z) \geq \left(\sum_{i=1}^n \sum_{j=1}^m \alpha\big((P_{E_i} \otimes P_{F_j})(z)\big)^p \right)^{1/p} .$$

Similarly for $1 < p < \infty$ we say that the tensor product $X \otimes_\alpha Y$ satisfies, repsectively,

i)' an upper p-row estimate,

ii)' an upper p-column estimate,

iii)' an upper p-projection estimate,

if there exists a constant M^r, M^c or M such that for all $z \in X \otimes_\alpha Y$ and all finite partitions of the integers $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^m$, we have, respectively,

$$\text{i)'} \quad \alpha(z) \leq M^r \left(\sum_{i=1}^n \alpha\big((P_{E_i} \otimes I)(z)\big)^p \right)^{1/p}$$

$$\text{ii)'} \quad \alpha(z) \leq M^c \left(\sum_{j=1}^m \alpha\big((I \otimes P_{F_j})(z)\big)^p \right)^{1/p}$$

$$\text{iii)'} \quad \alpha(z) \leq M \left(\sum_{i=1}^n \sum_{j=1}^m \alpha\big((P_{E_i} \otimes P_{F_j})(z)\big)^p \right)^{1/p} .$$

In each case the smallest possible constant is called the estimate constant.

Note that satisfying an upper p-projection estimate is equivalent to satisfying both an upper p-row estimate and an upper p-column estimate with $M \leq M^c M^r$ and $M \geq \max(M^c, M^r)$. Similarly in the case of the lower p-projection estimate.

We consider a duality property similar to the one for upper and lower p-estimates satisfied by a Banach space X with unconditional basis $\{e_i\}_{i=1}^{\infty}$. The problem arises that the sequence $\{e_i^*\}_{i=1}^{\infty}$ of coordinate functionals associated with $\{e_i\}_{i=1}^{\infty}$ is a basis of X^* if and only if $\{e_i\}_{i=1}^{\infty}$ is a shrinking basis [12]. However it is always an unconditional basis of $X' = \text{span}[e_i^*]_{i=1}^{\infty}$. By restricting X^* to X' we ensure that all the natural projections are adjoints of the natural projections associated to the unconditional basis of X. Now we have the following proposition.

<u>Proposition 1</u>. Let $1 < p < \infty$. $X \otimes_{\alpha} Y$ satisfies an upper, respectively lower, p-row, column or projection estimate if and only if $X' \otimes_{\alpha'} Y'$ satisfies a lower, respectively upper, p^*-row, column or projection estimate $(1/p + 1/p^* = 1)$. Moreover, the estimate constants are equal.

<u>Proof</u>. Suppose that $X \otimes_{\alpha} Y$ satisfies a lower p-column estimate. For any $u \in X \otimes_{\alpha} Y$, $v \in X' \otimes_{\alpha'} Y'$ and for any finite partition of the integers $\{F_j\}_{j=1}^{m}$,

$$|\langle u,v\rangle| = \left|\langle u, \sum_{j=1}^{m} (I \otimes P_{F_j})(v)\rangle\right|$$

$$\leq \sum_{j=1}^{m} |\langle (I \otimes P_{F_j})(u) , (I \otimes P_{F_j})(v)\rangle|$$

$$\leq \sum_{j=1}^{m} \alpha\big((I \otimes P_{F_j})(u)\big)\alpha'\big((I \otimes P_{F_j})(v)\big)$$

so by Hölder's inequality and the lower p-column estimate,

$$|\langle u,v\rangle| \leq \left(\sum_{j=1}^{m} \alpha\big((I \otimes P_{F_j})(u)\big)^{p}\right)^{1/p}\left(\sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(v)\big)^{p^*}\right)^{1/p^*}$$

$$\leq M_c\alpha(u)\left(\sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(v)\big)^{p^*}\right)^{1/p^*} .$$

Therefore, by the definition of the associate norm α',

$$\alpha'(v) \leq M_c\left(\sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(v)\big)^{p^*}\right)^{1/p^*} .$$

Thus if $X \otimes_\alpha Y$ satisfies a lower p-column estimate with estimate constant M_c then $X' \otimes_{\alpha'} Y'$ satisfies an upper p^*-column estimate with estimate constant less than or equal to M_c .

Now suppose that $X \otimes_\alpha Y$ satisfies an upper p-column estimate. Let $\{F_j\}_{j=1}^{m}$ be any finite partition of the integers and for $\omega \in X' \otimes_{\alpha'} Y'$ fix ξ_j, $j = 1,\ldots,m$ such that, $\sum_{j=1}^{m} \xi_j^p = 1$ and

$$\left\{\sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(\omega)\big)^{p^*}\right\}^{1/p^*} = \sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(\omega)\big)\xi_j .$$

Let $u_j \in X \otimes_\alpha Y$ be such that,

$$\alpha(u_j) = \xi_j \quad \text{and} \quad \langle u_j, (I \otimes P_{F_j})(\omega)\rangle = \alpha'\big((I \otimes P_{F_j})(\omega)\big)\xi_j$$

for $j = 1,\ldots,m$. Since,

$$\alpha\big((I \otimes P_{F_j})(u_j)\big)\alpha'\big((I \otimes P_{F_j})(\omega)\big) \geq \langle (I \otimes P_{F_j})(u_j), (I \otimes P_{F_j})(\omega)\rangle$$
$$= \alpha'\big((I \otimes P_{F_j})(\omega)\big)\xi_j,$$

we have,

$$\xi_j \leq \alpha\big((I \otimes P_{F_j})(u_j)\big) \leq \alpha(u_j) = \xi_j, \text{ so,}$$
$$\alpha\big((I \otimes P_{F_j})(u_j)\big) = \xi_j = \alpha(u_j).$$

Put $u = \sum_{j=1}^{m} (I \otimes P_{F_j})(u_j)$. Then,

$$\alpha(u) \leq M^c \left\{ \sum_{j=1}^{m} \xi_j^p \right\}^{1/p} = M^c .$$

Thus,

$$\left(\sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(\omega)\big)^{p^*} \right)^{1/p^*} = \sum_{j=1}^{m} \alpha'\big((I \otimes P_{F_j})(\omega)\big)\xi_j$$
$$= \sum_{j=1}^{m} \langle u_j, (I \otimes P_{F_j})(\omega)\rangle$$
$$= \langle u, \sum_{j=1}^{m} (I \otimes P_{F_j})(\omega)\rangle$$

$$\leq \alpha(u)\alpha'(\omega)$$
$$\leq M^c \ \alpha'(\omega)$$

Therefore if $X \otimes_\alpha Y$ satisfies an upper p-column estimate with estimate constant M^c then $X' \otimes_{\alpha'} Y'$ satisfies a lower p*-column estimate with estimate constant less than or equal to M^c.

Since $X'' \otimes_{\alpha''} Y'' \supseteq X \otimes_\alpha Y$, if $X' \otimes_{\alpha'} Y'$ satisfies an upper p*-column estimate then $X \otimes_\alpha Y$ satisfies a lower p-column estimate, with estimate constant less than or equal to that of $X' \otimes_{\alpha'} Y'$. Together with the first part of the proof this shows that the estimate constants must be equal.

The case of $X \otimes_\alpha Y$ satisfying an upper or lower p-row estimate is similar, the case of $X \otimes_\alpha Y$ satisfying an upper or lower p-projection estimate follows from the respective p-row and column estimates.

□

Let α be a crossnorm on $X \otimes Y$. We say that the tensor product $X_1 \otimes_\beta Y_1$ is equivalent to $X \otimes_\alpha Y$ if,

1) X_1 and Y_1 are renormings of X and Y respectively.

2) β is a crossnorm on $X_1 \otimes Y_1$ such that there exist constants a and b with,

$$a\alpha(z) \leq \beta\big((Id_1 \otimes Id_2)(z)\big) \leq b\alpha(z) \quad \text{for every} \quad z \in X \times_\alpha Y,$$

where $Id_1 : X \to X_1$, $Id_2 : Y \to Y_1$ are the formal identities induced by the renormings.

Proposition 2. Let α be a tensor norm. If $X \otimes_\alpha Y$ satisfies an upper or lower p-row or column estimate then there exists an equivalent tensor product (in the sense defined above), $X_1 \otimes_\beta Y_1$, which satisfies the same estimate with estimate constant equal to one. ∎

Proof. We show the case of $X \otimes_\alpha Y$ satisfying a lower p-column estimate, the other cases are similar.

If $X \otimes_\alpha Y$ satisfies a lower p-column estimate then for every $y \in Y$ and for all finite partitions of the integers, $\{F_i\}_{i=1}^n$, we have,

$$M_C \|y\|_Y \geq \left\{ \sum_{i=1}^n \|P_{F_i} y\|_Y^p \right\}^{1/p}$$

where M_C is the lower p-column estimate constant of $X \otimes_\alpha Y$.

Let Y_1 represent Y renormed with the norm $|||\cdot|||$, where, for all $y \in Y$,

$$|||y||| = \sup\left\{\left(\sum_{i=1}^n \|P_{F_i} y\|_Y^p\right)^{1/p} : \{F_i\}_{i=1}^n \text{ is a finite partition}\right\}.$$

Then $\|y\|_Y \leq |||y||| \leq M_C \|y\|_Y$ for all $y \in Y$ and also, $|||y||| \geq (\sum_{i=1}^n |||P_{F_i} y|||^p)^{1/p}$ for all finite partitions of the integers $\{F_i\}$ and all $y \in Y$.

Let $Id : Y_1 \to Y$ be the formal identity induced by the renorming.

Renorm $X \otimes_\alpha Y$ with the norm β_1 such that for all $z \in X \otimes_\alpha Y$,

$$\beta_1(z) = \sup\left\{\left(\sum_{i=1}^n \alpha\big((I \otimes P_{F_i})(z)\big)^p\right)^{1/p} : \{F_i\}_{i=1}^n \text{ is a finite partition}\right\}.$$

Then $\alpha(z) \leq \beta_1(z) \leq M_c\alpha(z)$ for all $z \in X \otimes_\alpha Y$ though β_1 is not necessarily a crossnorm on $X \otimes Y$.

Now consider the real-valued function β on $X \otimes Y_1$ defined by,

$$\beta(z) = \beta_1\big((I \otimes Id)(z)\big) \text{ for all } z \in X \otimes Y_1 .$$

We claim that β is a crossnorm on $X \otimes Y_1$ such that $X \otimes_\beta Y_1$ is equivalent to $X \otimes_\alpha Y$.

I) Clearly i) $\beta(z) \geq 0$ for all $z \in X \otimes Y_1$, $\beta(z) = 0 \Longleftrightarrow z = 0$.

ii) $\beta(az) = |a|\beta(z)$ for all $z \in X \otimes Y_1$ and all scalars a.

For y and z in $X \otimes Y_1$, put,

$$\xi = (I \otimes Id)(y) \text{ and } \zeta = (I \otimes Id)(z).$$

Then by the triangle inequality for α and the ℓ_p-norm,

$$\begin{aligned}\beta(y+z) &= \beta_1(\xi+\zeta) \\ &= \sup\left\{\sum_{i=1}^{n} \alpha\big((I \otimes P_{F_i})(\xi+\zeta)\big)^p\right\}^{1/p} \\ &\leq \sup\left\{\sum_{i=1}^{n} \big(\alpha((I \otimes P_{F_i})(\xi) + \alpha((I \otimes P_{F_i})(\zeta))\big)^p\right\}^{1/p} \\ &\leq \sup\left\{\sum_{i=1}^{n} \alpha\big((I \otimes P_{F_i}(\xi)\big)^p\right\}^{1/p} \\ &\quad + \sup\left\{\sum_{i=1}^{n} \alpha\big((I \otimes P_{F_i})(\zeta)\big)^p\right\}^{1/p} \\ &= \beta_1(\xi) + \beta_1(\zeta) = \beta(y) + \beta(z).\end{aligned}$$

Hence (iii).

II) Let $x \in X$, $y \in Y_1$. Then,

$$\begin{aligned}\beta(x \otimes y) &= \beta_1(x \otimes Idy) \\ &= \sup\left\{\sum_{i=1}^{n} \alpha\left(x \otimes P_{F_i}(Idy)\right)^p\right\}^{1/p} \\ &= \sup\left\{\sum_{i=1}^{n} \|x\|_X^p \|P_{F_i}(Idy)\|_Y^p\right\}^{1/p} \\ &= \sup\left\{\sum_{i=1}^{n} \|P_{F_i}(Idy)\|_Y^p\right\}^{1/p} \|x\|_X \\ &= \|x\|_X |||y||| .\end{aligned}$$

Thus β is a crossnorm on $X \otimes Y_1$ and by the definition of β_1, $X \otimes_\beta Y_1$ is equivalent to $X \otimes_\alpha Y$. We now show that $X \otimes_\beta Y_1$ satisfies the lower p-column estimate with estimate constant equal to one.

Given $\varepsilon > 0$, for any finite partition of the integers $\{F_i\}_{i=1}^n$ and any $z \in X \otimes_\beta Y_1$, for each $i = 1,\ldots,n$ there exists a finite partition $\{F_{ik}\}_{k=1}^{n_i}$ of F_i so that

$$\beta_1\left((I \otimes P_{F_i})(\zeta)\right)^p \leq \sum_{k=1}^{n_i} \alpha\left((I \otimes P_{F_{ik}})(\zeta)\right)^p + \varepsilon/n ,$$

where $\zeta = (I \otimes Id)z$.

Then we have,

$$\sum_{i=1}^{n} \beta\big((I \otimes P_{F_i})(z)\big)^p = \sum_{i=1}^{n} \beta_1\big((I \otimes P_{F_i})(\zeta)\big)^p$$

$$\leq \sum_{i=1}^{n} \left\{ \sum_{k=1}^{n_i} \alpha\big((I \otimes P_{F_{ik}})(\zeta)\big)^p + \frac{\varepsilon}{n} \right\}$$

$$= \varepsilon + \sum_{i=1}^{n} \sum_{k=1}^{n_i} \alpha\big((I \otimes P_{F_{ik}})(\zeta)\big)^p$$

$$\leq \varepsilon + \beta_1(\zeta)^p$$

$$= \varepsilon + \beta(z)^p .$$

Since $\varepsilon > 0$ is arbitrary, $X \otimes_\beta Y_1$ satisfies the upper p-column estimate with estimate constant equal to one.

□

Remark. In the definition of the upper and lower estimates for $X \otimes_\alpha Y$, we require α to be a tensor norm so that the projections $\{P_{E_i} \otimes P_{F_j}\}_{i=1}^{n}\,_{j=1}^{m}$ are uniformly bounded. We were unable to prove that the crossnorm β in the above proposition is infact a tensor norm. However the projections involved will be uniformly bounded because of the equivalence of the tensor products $X \otimes_\alpha Y$ and $X_1 \otimes_\beta Y_1$, where α is a tensor norm. We were also unable to prove a similar proposition for a tensor produce $X \otimes_\alpha Y$ satisfying an upper or a lower projection estimate.

For $1 \leq p \leq \infty$, a Banach lattice which is both p-convex and p-concave is isomorphic to $L_p(\mu)$ [12]. We have the following proposition.

Proposition 3. For $1 < p < \infty$, if $X \otimes_\alpha Y$ satisfies both an upper and a lower p-projection estimate with estimate constants M_1 and M_2 respectively, then $d(X \otimes_\alpha Y, \ell_p) \le M_1 M_2$.

Proof. Let $\{u_k\}$ be the unit vector basis of ℓ_p. Let $\{w_k\}$ be the sequence of elements of $X \otimes_\alpha Y$ defined by $w_k = e_i \otimes f_j$ where for $k = m^2 + s$, $(s = 1,\dots,2m+1;\ m = 0,1,\dots)$ we put

$$i = \begin{cases} m+1 & \text{for } 1 \le s \le m \\ s-m & \text{for } m+1 \le s \le 2m+1 \end{cases}$$

$$j = \begin{cases} s & \text{for } 1 \le s \le m \\ m+1 & \text{for } m+1 \le s \le 2m+1 \end{cases}$$

Then $\{w_k\}$ is a basis for $X \otimes_\alpha Y$ [9].

Define $A : \ell_p \to X \otimes_\alpha Y$ by $A(u_k) = w_k$.

For a finite sequence $\{z_k\}_{k=1}^n$, by the upper p-projection estimate,

$$\alpha\left(A\left(\sum_{k=1}^n z_k u_k\right)\right) \le M_1 \left(\sum_{k=1}^n |z_k|^p\right)^{1/p} = M_1 \left\| \sum_{k=1}^n z_k u_k \right\|_{\ell_p}.$$

By the lower p-projection estimate,

$$\left\| A^{-1}\left(\sum_{k=1}^n z_k w_k\right)\right\|_{\ell_p} = \left(\sum_{k=1}^n |z_k|^p\right)^{1/p} \le M_2\, \alpha\left(\sum_{k=1}^n z_k w_k\right).$$

Therefore if $z = \{z_k\} \in \ell_p$ is a finite sequence, then,

(1) $$\frac{1}{M_2} \|z\|_{\ell_p} \le \alpha\left(\sum_k z_k w_k\right) \le M_1 \|z\|_{\ell_p}.$$

So if $\|z\|_{\ell_p} = (\sum_k |z_k|^p)^{1/p} < \infty$ then $\sum_k z_k w_k$ converges in $X \otimes_\alpha Y$ and, $\alpha(Az) \leq M_1 \|z\|_{\ell_p}$.

If $a = \sum_k a_k w_k$ is in $X \otimes_\alpha Y$ then it follows from (1) that the sequence $\{a_k\}$ is Cauchy in ℓ_p. Thus,

$$\frac{1}{M_2} \|A^{-1} a\|_{\ell_p} \leq \alpha(a).$$

Therefore $d(X \otimes_\alpha Y, \ell_p) \leq \|A\| \|A^{-1}\| = M_1 M_2$. □

Johnson showed in [7] that a Banach space X with an unconditional basis satisfies a non-trivial upper or lower estimate if and only if X does not contain uniformly isomorphic copies of ℓ_1^n, respectively ℓ_∞^n. The same result is true for Banach lattices [12,18]. We prove a similar result for tensor products of Banach spaces with 1-unconditional bases, using the argument given in [12]. We show that $X \otimes_\alpha Y$ satisfies non-trivial upper and lower row estimates if and only if $X \otimes_\alpha Y$ does not have uniformly isomorphic copies of ℓ_1^n, respectively ℓ_∞^n, on blocks of rows, with a similar result for upper and lower column estimates.

<u>Theorem 1</u>. Let X and Y have 1-unconditional bases.

i) There does not exist a $p < \infty$ so that $X \otimes_\alpha Y$ satisfies a lower p-row, respectively column estimate if and only if for every $\varepsilon > 0$ and all integers n there exists a z in $X \otimes_\alpha Y$ together with a finite partition of the integers $\{E_i\}_{i=1}^n$ such that

$$\max_{1 \leq i \leq n} |a_i| \leq \alpha\left(\sum_{i=1}^n a_i (P_{E_i} \otimes I)(z)\right) \leq (1+\varepsilon) \max_{1 \leq i \leq n} |a_i|.$$

respectively,

$$\max_{1\le i\le n} |a_i| \le \alpha\left(\sum_{i=1}^{n} a_i (I \otimes P_{E_i})(z)\right) \le (1+\varepsilon) \max_{1\le i\le n} |a_i|,$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

ii) There does not exist a $p > 1$ so that $X \otimes_\alpha Y$ satisfies an upper p-row, respectively column, estimate if and only if for every $\varepsilon > 0$ and all integers n there exists a $z \in X \otimes_\alpha Y$ together with a finite partition of the integers $\{E_i\}_{i=1}^n$ such that,

$$(1-\varepsilon) \sum_{i=1}^{n} |a_i| \le \alpha\left(\sum_{i=1}^{n} a_i (P_{E_i} \otimes I)(z)\right) \le \sum_{i=1}^{n} |a_i|$$

respectively,

$$(1-\varepsilon) \sum_{i=1}^{n} |a_i| \le \alpha\left(\sum_{i=1}^{n} a_i (I \otimes P_{E_i})(z)\right) \le \sum_{i=1}^{n} |a_i|$$

for every choice of scalars $\{a_i\}_{i=1}^n$.

Proof. By a duality argument i) implies ii). We show i) for the case of a lower p-column estimate.

Assume that $X \otimes_\alpha Y$ satisfies no lower p-column estimate for $p < \infty$. For any finite partition of the integers $\{F_i\}_{i=1}^n$ and for $z \in X \otimes_\alpha Y$ put $z_i = (I \otimes P_{F_i})(z)$, $i = 1,\ldots,n$. Then for all $N = 1,2,\ldots$, let β_N be the smallest constant such that for all such sequences $\{z_i\}_{i=1}^n$,

$$\inf_{1\le i\le N} \alpha(z_i) \le \beta_N \left(\sum_{i=1}^{N} z_i\right).$$

Clearly, $1 = \beta_1 \ge \cdots \ge 0$.

Let $\{z_{i,j}\}_{i=1,j=1}^{N\quad M}$ be a double sequence of the above form.

Then,

$$\inf_{1\le i\le N} \alpha(z_{i,j}) \le \beta_N \alpha\left(\sum_{i=1}^{N} z_{i,j}\right) \quad \text{for every} \quad j = 1,\ldots,M$$

and,

$$\inf_{1\le j\le M} \alpha\left(\sum_{i=1}^{N} z_{i,j}\right) \le \beta_M \alpha\left(\sum_{i=1}^{N} \sum_{j=1}^{M} z_{i,j}\right).$$

So,

$$\inf_{\substack{1\le i\le N\\ 1\le j\le M}} \alpha(z_{i,j}) \le \beta_M\beta_N \alpha\left(\sum_{i=1}^{N} \sum_{j=1}^{M} z_{i,j}\right).$$

Thus $\beta_{MN} \le \beta_M\beta_N$.

Assume that $\beta_k < 1$ for some integer $k > 1$ and put

$$\gamma = -\log \beta_k/\log k.$$

Let n be an arbitrary integer and choose j so that

$$k^j \le n \le k^{j+1}.$$

Then,

$$\beta_n \le \beta_{k^j} \le (\beta_k)^j = 1/k^{\gamma j} \le k^\gamma/n^\gamma.$$

Thus there exists a constant $k < \infty$ and a $\gamma > 0$ such that $\beta_n \leq k/n^\gamma$ for every n.

Choose $p < \infty$ such that $p^\gamma > 1$. For a finite partition of the integers $\{F_i\}_{i=1}^n$ and for $z \in X \otimes_\alpha Y$, let $\{z_k\}$ be a permutation of $\{(I \otimes P_{F_i})(z)\}$ such that

$$\alpha(z_1) \geq \cdots \geq \alpha(z_n).$$

Then, for every $1 \leq j \leq n$,

$$\alpha(z_j) = \inf_{1 \leq i \leq j} \alpha(z_i) \leq \beta_j \alpha\left(\sum_{i=1}^{j} z_i\right) \leq k'\alpha\left(\sum_{i=1}^{n} z_i\right)/j^\gamma .$$

Consequently,

$$\left(\sum_{j=1}^{n} \alpha(z_j)^p\right)^{1/p} \leq k'\alpha\left(\sum_{j=1}^{n} z_j\right)\left(\sum_{j=1}^{\infty} 1/j^{p\gamma}\right)^{1/p} .$$

This shows that $X \otimes_\alpha Y$ satisfies a lower p-column estimate, contrary to our assumption. Hence $\beta_N = 1$ for every N. Then for every $\varepsilon > 0$, for all positive integers n, there exists a $z \in X \otimes_\alpha Y$ together with a finite partition of the integers $\{F_i\}_{i=1}^n$ such that,

$$1 \leq \inf_{1 \leq i \leq n} \alpha\left((I \otimes P_{F_i})(z)\right) \leq \alpha\left(\sum_{i=1}^{n} (I \otimes P_{F_i})(z)\right) < 1 + \varepsilon .$$

Then for any choice of scalars $\{a_i\}_{i=1}^n$ we have,

$$\max_{1\le i\le n} |a_i| \le \alpha\left(\sum_{i=1}^{n} a_i (I \otimes P_{F_i})(z)\right) \le \max_{1\le i\le n} |a_i|\, \alpha\left(\sum_{i=1}^{n} (I \otimes P_{E_i})(z)\right)$$

$$\le (1+\varepsilon) \max_{1\le i\le n} |a_i| .$$

The converse is trivial. □

For $1 < p \le 2$ and $q \ge 2$ a Banach space X is of type p or cotype q if there exists a constant k so that for every finite sequence $\{x_i\}_{i=1}^{n} \subset X$,

(A) $$\int_0^1 \left\| \sum_{i=1}^{n} r_i(t) x_i \right\|_X dt \le k \left(\sum_{i=1}^{n} \|x_i\|_X^p \right)^{1/p}$$

respectively,

(B) $$\int_0^1 \left\| \sum_{i=1}^{n} r_i(t) x_i \right\|_X dt \ge k^{-1} \left(\sum_{i=1}^{n} \|x_i\|_X^q \right)^{1/q} ,$$

where $\{r_i\}_{i=1}^{\infty}$ is the sequence of Rademacher functions,

$$r_i(t) = \text{sign} \sin 2^i \pi t, \quad \text{for } t \in [0,1].$$

The smallest constant k satisfying (A) or (B) is called the type p constant of X (denoted $T_p(X)$), respectively the cotype q constant of X (denoted $C_q(X)$).

If for $1 < p < 2 < q$ a Banach lattice satisfies an upper p and a lower q-estimate then it is of type p and of cotype q [12,3]. These are lattices which satisfy a lower 2-estimate without being of cotype 2. We were unable to prove similar results for tensor products.

However a Banach lattice of type p and cotype q satisfies an upper p-estimate and a lower q-estimate, [12] and we have a similar proposition for tensor products.

Proposition 4. i) If $X \otimes_\alpha Y$ has type p then $X \otimes_\alpha Y$ satisfies an upper p-projection estimate.

ii) If $X \otimes_\alpha Y$ has cotype q then $X \otimes_\alpha Y$ satisfies a lower q-projection estimate.

Proof. i) By the definition of type p, for any finite partition of the integers $\{E_i\}_{i=1}^m$ and for any $z \in X \otimes_\alpha Y$ we have,

$$\alpha\left(\sum_{i=1}^m (P_{E_i} \otimes I)(z)\right) \leq \text{unc}(X) \int_0^1 \alpha\left(\sum_{i=1}^m r_i(t)(P_{E_i} \otimes I)(z)\right) dt$$

$$\leq \text{unc}(X) T_p(X \otimes_\alpha Y) \left(\sum_{i=1}^m \alpha\left(P_{E_i} \otimes I)(z)\right)^p\right)^{1/p},$$

where $T_p(X \otimes_\alpha Y)$ is the type p constant of $X \otimes_\alpha Y$. Thus $X \otimes_\alpha Y$ satisfies an upper p-row estimate with estimate constant less than or equal to $\text{unc}(X)T_p(X \otimes_\alpha Y)$. By a similar argument, $X \otimes_\alpha Y$ satisfies an upper p-column estimate with estimate constant less than or equal to $\text{unc}(Y)T_p(X \otimes_\alpha Y)$. Therefore $X \otimes_\alpha Y$ satisfies an upper p-projection estimate with estimate constant less than or equal to $\text{unc}(X)\text{unc}(Y)\,[T_p(X \otimes_\alpha Y)]^2$.

ii) By the definition of cotype q, for any finite partition of the integers $\{E_i\}_{i=1}^m$ and for any $z \in X \otimes_\alpha Y$, we have,

$$\left(\sum_{i=1}^{m} \alpha\left((P_{E_i} \otimes I)(z)\right)^q\right)^{1/q} \le \int_0^1 \alpha\left(\sum_{i=1}^{m} r_i(t)(P_{E_i} \otimes I)(z)\right) C_q(X \otimes_\alpha Y)$$

$$\le \operatorname{unc}(X) C_q(X \otimes_\alpha Y)\, \alpha\left(\sum_{i=1}^{m} (P_{E_i} \otimes I)(z)\right).$$

Thus $X \otimes_\alpha Y$ satisfies a lower q-row estimate with estimate constant less than or equal to $\operatorname{unc}(X)C_q(X \otimes_\alpha Y)$. By a similar argument to the one used in i), we can show that $X \otimes_\alpha Y$ satisfies a lower q-projection estimate with estimate constant less than or equal to $\operatorname{unc}(X)\operatorname{unc}(Y)[C_q(X \otimes_\alpha Y]^2$. □

We now give examples of spaces which satisfy either an upper p-projection estimate or a lower q-projection estimate. We also determine the projection estimates satisfied by $\ell_p \otimes_\lambda \ell_q$ and $\ell_p \otimes_\nu \ell_q$ for $1 < p, q < \infty$.

Example. 1) If X satisfies an upper p-estimate with estimate constant M_X and Y satisfies an upper q-estimate with estimate constant M_Y, then $X \otimes_\lambda Y$ satisfies an upper r-projection estimate with estimate constant M, where $r = \min(p,q)$, $M \le M_X M_Y$.

2) If X satisfies a lower p-estimate with estimate constant M_X and Y satisfies a lower q-estimate with estimate constant M_Y, then $X \otimes_\nu Y$ satisfies a lower s-projection estimate with estimate constant M, where $s = \max(p,q)$, $M \le M_X M_Y$.

Proof. If both X and Y satisfy upper estimates then neither contain n-dimensional subspaces uniformly isomorphic to ℓ_1^n's [7]. Then the unconditional bases of X and Y are shrinking [6], i.e. in the notation of Proposition 2, $X^* = X'$, $Y^* = Y'$. Therefore by Proposition 2, 1) $\Rightarrow$ 2). We show 1).

Let $a \in X^*$, $\|a\|_{X^*} \leq 1$ and let $z \in X \otimes_\lambda Y$. For all finite partitions of the integers $\{E_i\}_{i=1}^n$, $\{F_j\}_{j=1}^m$,

$$\|(a \otimes I)(z)\|_Y \leq M_Y \left(\sum_{j=1}^m \|(a \otimes P_{F_j})(z)\|_Y^q \right)^{1/q}$$

$$\leq M_Y \left\{ \sum_{j=1}^m \left(\sum_{i=1}^n \|(aP_{E_i} \otimes P_{F_j})(z)\|_Y \right)^q \right\}^{1/q}$$

$$\leq M_Y \left\{ \sum_{j=1}^m \left(\sum_{i=1}^n \|P_{E_i} a\|_{X^*} \lambda\left((P_{E_i} \otimes P_{F_j})(z)\right) \right)^q \right\}^{1/q}$$

Since X satisfies an upper p-estimate, X^* satisfies a lower p^*-estimate, so for any finite partition of the integers $\{E_i\}_{i=1}^n$ we have, $1 \geq \|a\|_{X^*} \geq M_X^{-1} \{\sum_{i=1}^n \|P_{E_i} a\|_{X^*}^{p^*}\}^{1/p^*}$: So by Holder's inequality,

$$\|(a \otimes I)(z)\|_Y \leq M_Y M_X \left\{ \sum_{j=1}^m \left(\sum_{i=1}^n \lambda\left((P_{E_i} \otimes P_{F_j})(z)\right)^p \right)^{q/p} \right\}^{1/q} .$$

Thus,

$$\lambda(z) \leq M \left\{ \sum_{i=1}^n \sum_{j=1}^m \lambda\left((P_{E_i} \otimes P_{F_j})(z)\right)^r \right\}^{1/r} .$$

where $r = \min(p,q)$, $M \leq M_X M_Y$.

□

<u>Corollary</u>. 1) $\ell_p \otimes_\lambda \ell_q$ satisfies an upper r-projection estimate with

$r = \min(p,q)$, estimate constant equal to one.

2) $\ell_p \otimes_\nu \ell_q$ satisfies a lower s-projection estimate with $s = \max(p,q)$, estimate constant equal to one.

Remark. For any tensor norm α, $\ell_p \otimes_\alpha \ell_q$ does not satisfy an upper r-projection estimate or a lower s-projection estimate where $r > \min(p,q)$ and $s < \max(p,q)$. For if it were to satisfy an upper r-projection estimate with $r > \min(p,q)$ then, by the definition of a tensor norm, ℓ_p and ℓ_q would both have to satisfy upper r-estimates. This is impossible since for any $x \in \ell_p$ and any finite partition of the integers $\{E_i\}_{i=1}^n$, $\|x\|_{\ell_p} = \left(\sum_{i=1}^n \|P_{E_i}x\|_{\ell_p}^p\right)^{1/p}$. Similarly for the lower s-projection estimate.

SECTION THREE

CONVERGENCE IN TENSOR PRODUCTS. THE KADEC-KLEE PROPERTY

A Banach space $(X, \|\cdot\|)$ is said to have the Kadec-Klee property (also known as the Radon-Riesz property or property (H)) if:

For any sequence $\{x_n\}$ in X with x in X so that $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$.

All locally uniformly convex spaces have this property [13].

In this section we give a condition for a tensor product $X \otimes_\alpha Y$ to satisfy the Kadec-Klee property. The method we use is based on ideas from [19] and [1].

<u>Theorem</u>. Let $X \otimes_\alpha Y$ satisfy a lower p-projection estimate with estimate constant equal to one, for some p, $1 < p < \infty$. Then $X \otimes_\alpha Y$ has the Kadec-Klee property.

For any Banach space with an unconditional basis let P_n be the projection of the space onto the subspace spanned by the first n basic vectors. Put $Q_n = I - P_n$.

<u>Lemma 1</u>. Let α be a tensor norm on $X \otimes Y$. If $\{z_n\} \subset X \otimes_\alpha Y$ and $z \in X \otimes_\alpha Y$ so that $z_n \to z$ weakly, then there exists an increasing sequence of finite rank projections $\{S_k\}$, with the identity as the strong limit and,

$$\alpha\big((S_k \times S_k)(z_k) - z\big) \to 0 \quad \text{as} \quad k \to \infty .$$

Proof. Fix $\{z_n\}$ in $X \otimes_\alpha Y$ and z in $X \otimes_\alpha Y$ so that $z_n \to z$ weakly. Let j be a positive integer. There exist $\{x_i\}$ in X and $\{y_i\}$ in Y such that $z = \sum_i x_i \otimes y_i$; choose N_j so that for $z' = \sum_{i=1}^{N_j} x_i \otimes y_i$ we have,

$$\alpha(z - z') \leq 2^{-(j+1)}/\big(1 + \mathrm{unc}(\{e_i\})\ \mathrm{unc}(\{f_j\})\big)\,.$$

Then,

$$\alpha\big((P_n \otimes P_n)(z) - z\big)$$

$$\leq \alpha\big((P_n \otimes P_n)(z - z')\big) + \alpha(z - z') + \alpha\big((P_n \otimes P_n)(z') - z'\big)$$

$$\leq \| P_n \otimes P_n \| \alpha(z - z') + \alpha(z - z') + \alpha\big((P_n \otimes P_n)(z') - z'\big)$$

$$\leq 2^{-(j+1)} + \sum_{i=1}^{N_j} \alpha\big((P_n \otimes P_n)(x_i \otimes y_i) - x_i \otimes y_i\big)$$

$$= 2^{-(j+1)} + \sum_{i=1}^{N_j} \alpha\big((Q_n \otimes P_n)(x_i \otimes y_i) + (P_n \otimes Q_n)(x_i \otimes y_i) + (Q_n \otimes Q_n)(x_i \otimes y_i)\big)$$

$$\leq 3N_j \max_{1 \leq i \leq N_j} \{\alpha\big((Q_n \otimes P_n)(x_i \otimes y_i)\big), \alpha\big((P_n \otimes Q_n)(x_i \otimes y_i)\big), \alpha\big((Q_n \otimes Q_n)(x_i \otimes y_i)\big)\} + 2^{-(j+1)}$$

$$= 3N_j \max_{1 \leq i \leq N_j} \{ \| Q_n x_i \|_X \| P_n y_i \|_Y ,\ \| P_n x_i \|_X \| Q_n y_i \|_Y ,\ \| Q_n x_i \|_X \| Q_n y_i \|_Y \} + 2^{-(j+1)}.$$

Since each term (except $2^{-(j+1)}$) tends to zero as n tends to infinity, there exists $n_1(j)$ such that,

(1) $\alpha\big((P_n \otimes P_n)(z) - z\big) < 1/2^j$ for all $n \geq n_1(j)$.

By the assumed weak convergence,

$$(x^* \otimes y^*)(z_n - z) \to 0 \text{ for all } x^* \text{ in } X^* \text{ and } y^* \text{ in } Y^*.$$

In particular,

$$(x^*\otimes y^*)[(P_{n_1(j)}\otimes P_{n_1(j)})(z_n - z)] = \big(P^*_{n_1(j)}(x^*)\otimes P^*_{n_1(j)}(y^*)\big)(z_n - z) \to 0$$

for all x^* in X^*, y^* in Y^*.

Write,

$$(P_{n_1(j)} \otimes P_{n_1(j)})(z_n) = \sum_{i=1}^{n_1(j)} \sum_{k=1}^{n_1(j)} z_n(i,k)e_i \otimes f_k ,$$

$$(P_{n_1(j)} \otimes P_{n_1(j)})(z) = \sum_{i=1}^{n_1(j)} \sum_{k=1}^{n_1(j)} z(i,k)e_i \otimes f_k .$$

By considering the functionals $e_i^* \otimes f_k^*$ $i, k = 1,\ldots,n_1(j)$, the weak convergence of z_n to z gives $z_n(i,k) \to z(i,k)$ for each $i, k = 1,\ldots,n_1(j)$. So there exists $n_2(j) \geq n_1(j)$ such that for each $i, k = 1,\ldots,n_1(j)$, we have

$$|z_n(i,k) - z(i,k)| < (Mn_1^2(j)2^{j+1})^{-1} \quad \text{for } n \geq n_2(j),$$

where $M = \max_{1\leq i,k\leq n_1(j)} \alpha(e_i \otimes f_k)$. Additionally we may assume that for $j = 1,2,\ldots,$ we have $n_2(j) > n_2(j$ Then we have,

$$\alpha\big((P_{n_1(j)} \otimes P_{n_1(j)})(z_n - z)\big) \le \sum_{i,k=1}^{n_1(j)} \alpha\big((z_n(i,k) - z(i,k))e_i \otimes f_k\big)$$

(2)

$$< 1/2^{j+1} \quad \text{for} \quad n \ge n_2(j) .$$

Put $S_k = P_1$ for $1 \le k < n_2(1)$ and

$S_k = P_{n_1(j)}$ for $n_2(j) \le k < n_2(j+1)$ $j = 1,2,\ldots.$

Then by (1) and (2) for $j = 1,2,\ldots$ and $k \ge n_2(j)$ we have,

$$\alpha\big((S_k \otimes S_k)(z_k) - z\big) < 2/2^j .$$

This concludes the proof. □

Proof of Theorem. Let $\{z_n\} \subset X \otimes_\alpha Y$ and let $z \in X \otimes_\alpha Y$ be such that $\alpha(z_n) \to \alpha(z)$ and $z_n \to z$ weakly. By the lemma there exists an increasing sequence of finite rank projections $\{S_k\}_{k=1}^\infty$, such that

(3) $$\alpha\big((S_k \otimes S_k)(z_k) - z\big) \to 0 \quad \text{as} \quad k \to \infty .$$

Put $T_k = I - S_k$ for $k = 1,2,\ldots.$ By the lower p-projection estimate for $X \otimes_\alpha Y$ we have, for every $k = 1,2,\ldots$,

$$\alpha(z_k)^p \ge \alpha\big((S_k \otimes S_k)(z_k)\big)^p + \alpha\big((S_k \otimes T_k)(z_k)\big)^p + \alpha\big((T_k \otimes S_k)(z_k)\big)^p + \alpha\big((T_k \otimes T_k)(z_k)\big)^p$$

$$\ge \alpha\big((S_k \otimes S_k)(z_k)\big)^p + \{\alpha\big((S_k \otimes T_k)(z_k)\big) + \alpha\big((T_k \otimes S_k)(z_k)\big) + \alpha\big((T_k \otimes T_k)(z_k)\big)\}^p/3^{p-1}$$

So,

$$3^{P-1}\{\alpha(z_k)^P - \alpha\big((S_k \otimes S_k)(z_k)\big)^P\}^{1/P} \geq \alpha\big((S_k \otimes T_k)(z_k)\big) + \alpha\big((T_k \otimes S_k)(z_k)\big) + \alpha\big((T_k \otimes T_k)(z_k)\big).$$

Then,

$$\begin{aligned}\alpha(z_k - z) &\leq \alpha\big((S_k \otimes S_k)(z_k) - z\big) + \alpha\big((S_k \otimes T_k)(z_k)\big) + \alpha\big((T_k \otimes S_k)(z_k)\big) \\ &\quad + \alpha\big((T_k \otimes T_k)(z_k)\big) \\ &\leq \alpha\big((S_k \otimes S_k)(z_k) - z\big) + 3^{P-1}\{\alpha(z_k)^P - \alpha\big((S_k \otimes S_k)(z_k)\big)^P\}^{1/P},\end{aligned}$$

for every $k = 1,2,\ldots$. By (3),

$$\alpha\big((S_k \otimes S_k)(z_k) - z\big) \to 0 \quad \text{as} \quad k \to \infty.$$

We also have,

$$\alpha\big((S_k \otimes S_k)(z_k)\big) \to \alpha(z) \quad \text{as} \quad k \to \infty.$$

By our assumptions, $\alpha(z_k) \to \alpha(z)$, so

$$\{\alpha(z_k)^P - \alpha\big((S_k \otimes S_k)(z_k)\big)^P\}^{1/P} \to 0.$$

Hence, $\alpha(z_k - z) \to 0$ as $k \to \infty$. □

<u>Corollary</u>. For $1 < p, q < \infty$, $\{z_n\}_{n=1}^{\infty} \subset \ell_p \otimes_\nu \ell_q$ and $z \in \ell_p \otimes_\nu \ell_q$ with,

i) $\nu(z_n') \to \nu(z)$

ii) $z_n \to z$ weakly,

we have $\nu(z_n - z) \to 0$ as $n \to \infty$.

In [1] Arazy showed that $S_1 = \ell_2 \otimes_\nu \ell_2$ satisfies the Kadec-Klee property. This corollary generalizes that result.

SECTION FOUR

THE $g\ell$-CONSTANT OF $X \otimes_\nu \ell_2$

For this section the space of scalars will be the real numbers.

In their paper [9], Kwapien and Pelczynski showed that for $1 \leq p, q < \infty$ and $1/p + 1/q \geq 1$, none of the spaces $\ell_p \otimes_\lambda \ell_q$, $\ell_{p^*} \otimes_\nu \ell_{q^*}$ is isomorphic to a subspace of a Banach space with an unconditional basis. They also showed that if $\{e_n\}$ is a complete orthonormal system in ℓ_2 and α is a tensor norm on $\ell_2 \otimes \ell_2$ then $\{e_i \otimes e_j\}$ is an unconditional basis of $\ell_2 \otimes_\alpha \ell_2$ if and only if α is equivalent to the Hilbert-Schmidt norm.

A notion naturally related to the existence of an unconditional basis is that of local unconditional structure.

Definition [5,2]. We say that a Banach space X has local unconditional structure if there is a constant k such that for each finite dimensional subspace $E \subset X$ there is a finite dimensional Banach space F and operators $S \in L(E,F)$, $T \in L(F,X)$ with $TS|_E$ the identity on E such that $\|T\|\|S\| \operatorname{unc}(F) \leq k$. We put $\operatorname{lust}(X) = \inf k$.

We now introduce two Banach spaces of operators, the operators that factor through L_p spaces and the p-absolutely summing operators [8,14]. These play an important rôle in the study of local unconditional structure.

Let $1 \leq p \leq \infty$. For an operator u from X to Y, we say that u is L_p-*factorable* ($u \in \Gamma_p(X,Y)$) if and only if there is a measure μ

and operators v from X to $L_p(\mu)$ and w from $L_p(\mu)$ to Y^{**} such that $iu = wv$ where i is the canonical embedding of Y into Y^{**}. Then we put

$$\gamma_p(u) = \inf\{\|v\|\|w\| : wv \text{ is such a factorization}\}.$$

It may be shown that the L_p-factorable operators from X to Y together with the norm γ_p form a Banach space [8].

Let $1 \leq p < \infty$. For an operator u from X to Y, we say that u is p-*absolutely summing* $(u \in \Pi_p(X,Y))$ if and only if there exists a constant k such that for any finite sequence $\{x_i\}_{i=1}^{n}$ in X we have,

$$\left(\sum_{i=1}^{n} \|u(x_i)\|_Y^p\right)^{1/p} \leq k \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p\right)^{1/p}.$$

The p-absolutely summing norm of u, $\pi_p(u)$ is the infimum of all such k, and it is easily shown that $(\Pi_p(X,Y), \pi_p)$ is a Banach space.

Both π_p and γ_p are ideal norms. The norms π_1 and γ_∞ are in trace duality, for an operator u from X to Y,

(1) $$\pi_1(u) = \sup\{|\mathrm{tr}(vu)| : v : Y \to X,\ \gamma_\infty(v) \leq 1\}.$$

In [5] Gordon and Lewis proved that for an operator u from X to ℓ_2,

(2) $$\gamma_1(u) \leq \mathrm{lust}(X)\pi_1(u).$$

They then used (2) to show that for $1 < p, q \leq \infty$ neither $\ell_p \otimes_\vee \ell_q$ nor $\ell_{p^*} \otimes_\lambda \ell_{q^*}$ nor their duals, biduals etc. have local unconditional structure. Lewis also proved in [10] that $\ell_2 \otimes_\alpha \ell_2$ has local unconditional structure only if α is equivalent to the Hilbert-Schmidt norm. It has been shown [5,16] that if $\max(p,r,2) < s$ or $\max(p^*,s,2) < r$ then $\Pi_p(\ell_r,\ell_s)$ does not have local unconditional structure. However for $1 \leq p \leq s \leq 2 \leq r \leq p^*$, $\Pi_p(\ell_r,\ell_s)$ does have local unconditional structure.

Formula (2) leads to the following definitions of the $g\ell$-constant of a Banach space:

The *$g\ell$-constant* of a Banach space X is defined as,

$$g\ell(X) = \sup\{\gamma_1(u) : u \in \pi_1(X,\ell_2)\ ;\ \pi_1(u) \leq 1\}\ .$$

By (2), $g\ell(X) \leq \text{lust}(X)$. The $g\ell$-constant has been a most useful means of proving that certain spaces do not have local unconditional structure.

In [4] Gordon showed that for Banach spaces X and Y, $g\ell(X_k \otimes_\lambda Y_k) \to \infty$ for all increasing sequences $\{X_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ of finite dimensional subspaces of X and Y respectively, if and only if X and Y do not contain subspaces uniformly isomorphic to ℓ_∞^n's for $n = 1,2,\ldots$. Then $g\ell(X \otimes_\lambda Y)$ is finite (for infinite dimensional Banach spaces X and Y) if and only if at least one of X and Y contains subspaces uniformly isomorphic to ℓ_∞^n's. It follows that $g\ell(X \otimes_\vee Y)$ is finite if and only if at least one of X and Y contains subspaces uniformly isomorphic to ℓ_1^n's. We give an elementary proof that if X is a Banach space with an unconditional basis such that $g\ell(X \otimes_\vee \ell_2)$ is finite, then X is isomorphic to ℓ_1.

Theorem. If X is a Banach space with an unconditional basis such that $g\ell(X \otimes_\nu \ell_2)$ is finite then X is isomorphic to ℓ_1 and $d(X \otimes_\nu \ell_2, \ell_1) \le 16[k_G\, g\ell(X \otimes_\nu \ell_2)]^2 [\mathrm{unc}(X)]^3$, (where k_G is Grothendieck's constant).

We use four lemmas.

Let X be an infinite dimensional Banach space with the unconditional basis $\{e_i\}_{i=1}^\infty$. Let $\{\phi_i\}_{i=1}^\infty$ be the standard unit vector basis of ℓ_2. An operator T in $L(X,\ell_2)$ is called diagonal if for each i, $Te_i = t_i\phi_i$, where $\{t_i\}_{i=1}^\infty$ is some sequence of scalars.

Our first lemma is a version of Theorem 4.2 in [11], where it was shown that if there exists a k so that $\pi_1(T) \le k\|T\|$ for all operators T in $L(X,Y)$ then X is isomorphic to ℓ_1 and Y is isomorphic to ℓ_2.

Lemma 1. Let X be an infinite dimensional Banach space with the unconditional basis $\{e_i\}_{i=1}^\infty$. Put $X_n = \mathrm{span}[e_i]_{i=1}^n$. If there exists a constant k so that for every n and every diagonal operator T in $L(X_n, \ell_2^n)$ we have,

$$\pi_1(T) \le k\|T\|, \tag{3}$$

then X is isomorphic to ℓ_1, and $d(X,\ell_1) \le k^2(\mathrm{unc}\{e_i\})^3$.

Proof. Let n be any integer and let $\{\mu_i\}_{i=1}^n$ be a sequence of positive numbers such that $\sum_{i=1}^n \mu_i^2 = 1$. Define $T: X_n \to \ell_2^n$ by

$$Tx = \sum_{i=1}^n \mu_i a_i \phi_i \quad \text{for} \quad x = \sum_{i=1}^n a_i e_i \quad \text{in } X.$$

Let ρ be the unconditional basis constant of the basis $\{e_i\}_{i=1}^{\infty}$.

Then, for $i = 1,2,\ldots,$

$$|a_i| = \frac{1}{2}\left\| \sum_{j=1}^{n} a_j e_j + \sum_{j\neq i} (-a_j)e_j + a_i e_i \right\|_X \leq \frac{1}{2}\left[\left\| \sum_{i=1}^{n} a_i e_i \right\|_X + \rho \left\| \sum_{i=1}^{n} a_i e_i \right\|_X\right]$$

$$\leq \rho \left\| \sum_{i=1}^{n} a_i e_i \right\|_X$$

hence,

$$\|Tx\|_2 = \left(\sum_{i=1}^{n} |a_i \mu_i|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} \mu_i^2\right)^{\frac{1}{2}} \sup_{1\leq i\leq n} |a_i| \leq \rho \|x\|_X ,$$

for all x in X_n.

Consequently $\|T\| \leq \rho$ and by (3) $\pi_1(T) \leq \rho k$.

Since $\left\| \sum_{i=1}^{n} \varepsilon_i a_i e_i \right\|_X \leq \rho \|x\|_X$ for every $x = \sum_{i=1}^{n} a_i e_i \in X$ and for every choice of $\varepsilon_i = \pm 1$, $i = 1,\ldots,n$, we get by the definition of $\pi_1(T)$ that,

$$\sum_{i=1}^{n} |a_i|\mu_i = \sum_{i=1}^{n} \|Ta_i e_i\|_2 \leq \pi_1(T) \sup_{\substack{x^*\in X^* \\ \|x^*\|\leq 1}} \left(\sum_{i=1}^{n} |a_i||\langle e_i,x^*\rangle|\right)$$

$$= \pi_1(T) \sup_{\substack{x^*\in X^* \\ \|x^*\|\leq 1}} \left(\sum_{i=1}^{n} \varepsilon_i a_i \langle e_i,x^*\rangle\right),$$

where $\varepsilon_i a_i \langle e_i,x^*\rangle = |a_i \langle e_i,x^*\rangle|$, $i = 1,\ldots,n$. Thus,

$$\sum_{i=1}^{n} |a_i|\mu_i \le \pi_1(T) \sup_{\substack{x^*\in X^* \\ \|x^*\|\le 1}} \langle \sum_{i=1}^{n} \varepsilon_i a_i e_i , x^* \rangle$$

$$= \pi_1(T) \| \sum_{i=1}^{n} \varepsilon_i a_i e_i \|_X \le \pi_1(T)\rho\|x\|_X .$$

Now choose $\{\mu_i\}_{i=1}^n$ such that $\sum_{i=1}^n \mu_i^2 = 1$ and $\sum_{i=1}^n |a_i|\mu_i = (\sum_{i=1}^n |a_i|^2)^{1/2}$. Then,

$$(\sum_{i=1}^{n} |a_i|^2)^{1/2} = \sum_{i=1}^{n} |a_i|\mu_i \le \rho^2 k\|x\|_X \quad \text{for} \quad x = \sum_{i=1}^{n} a_i e_i \in X_n .$$

Define the diagonal operator $S : X_n \to \ell_2^n$ by $Sx = \sum_{i=1}^n a_i\phi_i$ for $x = \sum_{i=1}^n a_i e_i \in X_n$. Then,

$$\|Sx\|_2 = (\sum_{i=1}^{n} |a_i|^2)^{1/2} \le \rho^2 k\|x\|_X$$

and hence $\pi_1(S) \le \rho^2 k^2$. Consequently, by the definition of $\pi_1(S)$,

$$\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} \|Sa_i e_i\|_2 \le \pi_1(S) \sup_{\varepsilon_i = \pm 1} \|\sum_{i=1}^{n} \varepsilon_i a_i e_i\|_X$$

(4)

$$\le \pi_1(S)\rho\|x\|_X \le \rho^3 k^2\|x\| .$$

Therefore by the triangle inequality for the norm of X and by (4), for each n and for every $x = \sum_{i=1}^n a_i e_i \in X_n$ we have,

$$\|x\|_X \le \sum_{i=1}^{n} |a_i| \le \rho^3 k^2\|x\|_X .$$

This proves that X is isomorphic to ℓ_1 and $d(X,\ell_1) \leq \rho^3 k^2$. □

Lemma 2. Let $\{e_i\}_{i=1}^{\infty}$, $\{f_j\}_{j=1}^{\infty}$ be unconditional bases for X and Y respectively. Then for any ideal norm α,

$$\alpha\left(\sum_i a_{i,i} e_i \otimes f_i\right) \leq c\alpha\left(\sum_{i,j} a_{i,j} e_i \otimes f_j\right)$$

for any sequence of scalars $\{a_{i,j}\}_{i,j=1}^{\infty}$, where $c = \mathrm{unc}(\{e_i\})\mathrm{unc}(\{f_j\})$.

Proof. Let X_1 and Y_1 represent X and Y renormed so that $\{e_i\}$ and $\{f_j\}$ are 1-unconditional. Let $Id_1 : X \to X_1$ and $Id_2 : Y \to Y_1$ be the formal identities induced by these renormings. Since α is an ideal norm, for any $\xi \in X_1 \otimes_\alpha Y_1$, we have

$$\alpha(z) \leq \alpha(\xi) \leq c\alpha(z) \quad \text{where} \quad z = (Id_1^{-1} \otimes Id_2^{-1})(\xi)$$

and c is the product of the unconditional constants of the bases $\{e_i\}$ and $\{f_j\}$ in X and Y respectively. Thus we need only show the lemma for 1-unconditional bases $\{e_i\}$ and $\{f_j\}$.

For any sequence of scalars $\{a_{i,j}\}_{i,j=1}^{\infty}$, let $\{\varepsilon_i\}_{i=1}^{\infty}$ be the sequence such that $\varepsilon_1 = -1$ and $\varepsilon_i = 1$ for $i = 2,3,\ldots$. Then,

$$\alpha\left(a_{1,1}e_1\otimes f_1 + \sum_{i,j\geq 2} a_{i,j}e_i\otimes f_j\right) = \alpha\left(\sum_{i,j} a_{i,j}e_i\otimes f_j + \sum_{i,j} \varepsilon_i\varepsilon_j a_{i,j} e_i\otimes f_j\right)/2$$

$$\leq \frac{1}{2}\left\{\alpha\left(\sum_{i,j} a_{i,j}e_i\otimes f_j\right) + \alpha\left(\sum_{i,j} \varepsilon_i\varepsilon_j a_{i,j}e_i\otimes f_j\right)\right\}$$

$$= \alpha\left(\sum_{i,j} a_{i,j}e_i\otimes f_j\right)$$

In a similar way, for any k we have

$$\alpha\Big(\sum_{i=1}^{k} a_{i,i}e_i\otimes f_i + \sum_{i,j\ge k+1} a_{i,j}e_i\otimes f_j\Big)$$
$$\le \alpha\Big(\sum_{i=1}^{k-1} a_{i,i}e_i\otimes f_i + \sum_{i,j\ge k} a_{i,j}e_i\otimes f_j\Big).$$

Thus we get by induction,

$$\alpha\Big(\sum_i a_{i,i}e_i\otimes f_i\Big) \le \alpha\Big(\sum_{i,j} a_{i,j}e_i\otimes f_j\Big)$$

as required. □

For $1 \le p, q \le \infty$ and any n we denote by $id_{p,q}$ formal identities from ℓ_p^n to ℓ_q^n. We often write id for $id_{2,2}$.

Lemma 3. Let X be an n-dimensional Banach space, with a 1-unconditional basis $\{e_i\}_{i=1}^n$. Let $u : X \to \ell_2^n$ be a diagonal operator, then,

$$\pi_1(u) \le \sup|tr(uv\ id_{2,\infty})|$$

where the supremum is taken over all diagonal operators $v : \ell^n \to X$ with $\|v\| \le k_G$ (k_G is Grothendieck's constant).

Proof. Let $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ be the unit vector bases for ℓ_2^n and ℓ_1^n respectively. It follows from (1) that, for an operator u from X to ℓ_2^n, we have by trace duality,

$$\pi_1(u) = \sup\{|tr(uw)| : w : \ell_2^n \to X,\ \gamma_\infty(w) \le 1\}.$$

Now since u is diagonal and since γ_∞ is an ideal norm, by Lemma 2 for every operator w from ℓ_2^n to X there is a diagonal operator w' from ℓ_2^n to X such that $|tr(uw)| = |tr(uw')|$ and $\gamma_\infty(w) \geq \gamma_\infty(w')$. Therefore in the formula for $\pi_1(u)$ we need only consider diagonal operators w from ℓ_2^n to X. Now $\gamma_\infty(w) = \gamma_1(w^*)$, so factor w^* as,

$$w^* : X^* \underset{S}{\rightarrow} L_1(\mu) \underset{T}{\rightarrow} \ell_2^n$$

for some measure μ. By Grothendieck's theorem [11], $\pi_1(T) \leq k_G\|T\|$, thus we have

$$\pi_1(w^*) = \pi_1(TS) \leq \|S\|\pi_1(T) \leq k_G\|S\|\|T\|.$$

Taking the infimum over all factorizations $w^* = TS$ it follows that,

$$\pi_1(w^*) \leq k_G\gamma_1(w^*) = k_G\gamma_\infty(w).$$

For $y = \sum_{i=1}^n y_i e_i^*$ in [illegible]ite $w^*(y) = \sum_{i=1}^n w_i y_i \phi_i$ in ℓ_2^n.

Factor w^* as,

$$w^* : X^* \underset{\Delta}{\rightarrow} \ell_1^n \xrightarrow[id_{1,2}]{} \ell_2^n$$

where $\Delta(y) = \sum_{i=1}^n w_i y_i \psi_i$ in ℓ_1^n.

We want to show that $\|\Delta\| \leq k_G\gamma_\infty(w)$. Now for $y = \sum_{i=1}^n y_i e_i^*$ in X^* we have

(5)
$$\|y\|_{X^*} = \sup_{\varepsilon_i = \pm 1} \|\sum_{i=1}^{n} \varepsilon_i y_i e_i^*\|_{X^*}$$

$$= \sup_{\varepsilon_i = 1} \sup_{\substack{x \in X \\ \|x\| \le 1}} |\sum_{i=1}^{n} \langle \varepsilon_i y_i e_i^*, x \rangle|$$

$$= \sup_{\substack{x \in X \\ \|x\| \le 1}} \sum_{i=1}^{n} |\langle y_i e_i^*, x \rangle|.$$

Set $x_i = y_i e_i^*$ for $i = 1,\ldots,n$. By the definition of the 1-summing norm and (5) we have

$$\|\Delta y\|_{\ell_1^n} = \sum_{i=1}^{n} |w_i y_i| = \sum_{i=1}^{n} \|w^* x_i\|$$

$$\le \pi_1(w^*)\|y\|_{X^*}$$

$$\le k_G \gamma_\infty(w)\|y\|_{X^*}$$

and the estimate for $\|\Delta\|$ follows.

So for every diagonal operator w from ℓ_2^n to X with $\gamma_\infty(w) \le 1$ there exists a diagonal operator $v = \Delta^*$ from ℓ_∞^n to X with $\|v\| \le k_G$ such that $tr(uw) = tr(uv\, id_{2,\infty})$ for any diagonal operator u from X to ℓ_2^n. Then,

$$\pi_1(u) \le \sup|tr(uv\, id_{2,\infty})|$$

where the supremum is taken over all diagonal operators v from ℓ_∞^n to X with $\|v\| \le k_G$. □

Our next lemma is Theorem 2.2 from [5].

Lemma 4. The inclusion map of $\ell_1^n \otimes_\lambda \ell_1^n$ into $HS(\ell_2^n)$ has π_1-norm at most 2. The inclusion map of $\ell_1^n \otimes_\nu \ell_2^n$ into $HS(\ell_2^n)$ has π_1-norm at most $2\sqrt{n}$.

In [5] Theorem 2.2 the Khintchine constant $\sqrt{3}$ is used (rather than $\sqrt{2}$). We use the best Khintchine constant, $\sqrt{2}$, due to Szarek [20].

Proof of the Theorem. Since X has an unconditional basis, X is isomorphic to a space X_1 with a 1-unconditional basis and $d(X,X_1) \leq unc(X)$. In this case we have $g\ell(X_1 \otimes_\nu \ell_2) \leq unc(X) g\ell(X \otimes_\nu \ell_2)$ and so we prove the theorem for $unc(X) = 1$.

For any integer n let $X_n = span[e_i]_{i=1}^n$, where $\{e_i\}_{i=1}^\infty$ is a 1-unconditional basis. Then the subspace $X_n \otimes_\nu \ell_2^n$ of $X \otimes_\nu \ell_2$ is complemented by a norm one projection, and therefore $g\ell(X_n \otimes_\nu \ell_2^n) \leq g\ell(X \otimes_\nu \ell_2)$.

Let $w = uv\, id_{2,\infty}$ where u is a diagonal operator from X_n to ℓ_2^n and v and $id_{2,\infty}$ are as in Lemma 3, so that

$$\pi_1(u) \leq \sup_{\|v\| \leq k_G} |tr(uv\, id_{2,\infty})| .$$

Then we have,

$$w : \ell_2^n \xrightarrow[id_{2,\infty}]{} \ell_\infty^n \underset{v}{\rightarrow} X_n \underset{u}{\rightarrow} \ell_2^n .$$

Consider the following factorization of $w \otimes id$,.

$$HS(\ell_2^n) \xrightarrow[id_{2,\infty} \otimes id_{2,\infty}]{} \ell_\infty^n \otimes_\nu \ell_\infty^n \xrightarrow[v \otimes id_{\infty,2}]{} X_n \otimes_\nu \ell_2^n \xrightarrow[u \otimes id]{} \ell_2^n \otimes_\nu \ell_2^n \xrightarrow[id \otimes id]{} HS(\ell_2^n)$$

where $HS(\ell_2^n)$ is the space of Hilbert-Schmidt operators on ℓ_2^n.

Now it can be shown that

(6) $$tr(w \otimes id) = tr(w)tr(id) = n\ tr(w).$$

Put $T = (v \otimes id_{\infty,2})(id_{2,\infty} \otimes id_{2,\infty})$; $S = (id \otimes id)(u \otimes id)$. Then $w \otimes id = ST$. By the trace duality of the π_1 and γ_∞ norms, by (6) and by $\gamma_\infty(T) = \gamma_1(T^*)$ we have,

$$n\ tr(w) \leq \gamma_\infty(T)\pi_1(S)$$

$$\leq \gamma_1(T^*)\|u\|\|id\|\pi_1(id \otimes id).$$

From Lemma 4, $\pi_1(id \otimes id) \leq 2\sqrt{n}$, so,

$$\sqrt{n}\ tr(w) \leq \gamma_1(T^*)\ \ 2\ \|u\|.$$

Factor T^* as,

$$X_n^* \otimes_\lambda \ell_2^n \xrightarrow[v^* \otimes id_{2,1}]{} \ell_1^n \otimes_\lambda \ell_1^n \xrightarrow[id_{1,2} \otimes id_{1,2}]{} HS(\ell_2^n).$$

Now $\gamma_1(T^*) \leq g\ell(X_n^* \otimes_\lambda \ell_2^n)\pi_1(T^*) \leq g\ell(X \otimes_\nu \ell_2)\|v^*\|\|id_{2,1}\|\pi_1(id_{1,2} \otimes id_{1,2})$.

From Lemma 4, $\pi_1(id_{1,2} \otimes id_{1,2}) \leq 2$, so

$$\gamma_1(T^*) \leq 2\sqrt{n}\ \|v\| g\ell(X \otimes_\nu \ell_2).$$

$$\sqrt{n}\ \mathrm{tr}(w) \leq 4\sqrt{n}\ \|u\| g\ell(X \otimes_\nu \ell_2) \|v\| \ , \quad \text{so,}$$

$$\pi_1(u) \leq 4k_G \|u\| g\ell(X \otimes_\nu \ell_2) .$$

Therefore by Lemma 1, X is isomorphic to ℓ_1 and

$$d(X,\ell_1) \leq 16[k_G\ g\ell(X \otimes_\nu \ell_2)]^2 . \qquad \square$$

SECTION FIVE

PROBLEMS

In Section Two we mentioned the connection between type and upper estimates and between cotype and lower estimates for Banach lattices. A Banach lattice of type p or cotype q satisfies an upper p, respectively lower q-estimate, and we proved a similar result for tensor products. However for Banach lattices there are some results in the opposite direction, are there similar results for tensor products? To be specific:

<u>Problem 1</u>. Let $1 < p < 2 < q$ and suppose that the tensor product $X \otimes_\alpha Y$ satisfies an upper p-projection estimate and a lower q-projection estimate. Then is $X \otimes_\alpha Y$ of type p and cotype q?

Let T be a compact operator acting in ℓ_2 and let $s(T) = \{s_j(T)\}$ be the sequence of eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$. Suppose that E is a symmetric sequence space with norm $\|\cdot\|$, then the corresponding unitary ideal S_E is defined by

$$S_E = \{T \text{ compact} : s(T) \in E\}$$

and for T in S_E, $\|T\|_E = \|s(T)\|$. Now every compact operator acting in ℓ_2 has a tensor representation, so:

<u>Problem 2</u>. Does the space S_E satisfy an upper p or a lower q-projection estimate if and only if the corresponding symmetric sequence space E satisfies an upper p-estimate, respectively a lower q-estimate? (Assuming $1 < p \le 2$, $2 \le q < \infty$).

[Since S_E contains a subspace isomorphic to E one direction is trivial.] We conjecture that problem 2 is true.

Our last problem is a generalization of the theorem given in Section Four.

Problem 3. Let X and Y be Banach spaces with unconditional bases and let $g\ell(X \otimes_{\vee} Y)$ be finite. Then is at least one of X and Y isomorphic to ℓ_1 ?

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