NATIONAL LIBRARY
OTTAWA

BIBLIOTHÈQUE NATIONALE
OTTAWA

CANADA

NAME OF AUTHOR....Geoffrey....W....Trott......

TITLE OF THESIS......Analysing...Nonlinear...Systems
......with...the...Volterra...Series.
....

UNIVERSITY...of...Alberta..............................

DEGREE FOR WHICH THESIS WAS PRESENTED.....Ph.D...........

YEAR THIS DEGREE GRANTED.....1971..........................

(Signed).....[signature].............

PERMANENT ADDRESS:
..Elec...Eng...Dept....
..Wollongong...University College
...Wollongong, N.S.W. 2500
AUSTRALIA

DATED...21...June.....1971

NL-91 (10-68)

THE UNIVERSITY OF ALBERTA

ANALYSING NONLINEAR SYSTEMS
WITH THE VOLTERRA SERIES

by

© GEOFFREY WILLIAM TROTT

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

FALL, 1971

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled ANALYSING NONLINEAR SYSTEMS WITH THE VOLTERRA SERIES submitted by GEOFFREY WILLIAM TROTT in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

....
Supervisor

....

....

....

....

....
External Examiner

Date June 21-71

ABSTRACT

A class of nonlinear systems, which can be represented by a feedback system with one or two linear plants, and one single-valued time-invariant nonlinearity, is studied. The Volterra series describing such a system, where the nonlinearity can be represented as a power series, is shown to be unique. Initial conditions on the linear plants are incorporated into the Volterra series in a simple manner. Using this initial condition method, the convergence of the series is assured for inputs containing a step type function. The convergence of the series is investigated, and it is shown that for first and second order systems the series converges, and bounded-input bounded-output stability conditions are obtained for a nonlinearity whose slope lies in the Hurwitz sector. For higher order systems without zeros in their linear plants, it is shown that a contraction can be obtained by use of another norm, which also gives a bound on the output, for a nonlinearity whose slope lies in the Hurwitz sector.

ACKNOWLEDGEMENTS

The author wishes to thank his supervisor, Dr. G.S. Christensen for his guidance and help throughout the course of this research. He would also like to thank Dr. T.D. Rodgers of the Department of Mathematics for the many stimulating criticisms and suggestions offered, and the author's fellow students for all the help and advice freely given.

The financial assistance of the National Research Council (1968-71), and the University of Alberta (1967-68) is also gratefully acknowledged.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF MAJOR SYMBOLS AND ABBREVIATIONS

$\lvert\cdot\rvert$	Absolute value sign.
$\lVert\cdot\rVert$	Norm sign.
$\lVert\cdot\rVert_\lambda$	λ-norm sign.
inc $\lVert\cdot\rVert$	Lipschitz norm = maximum incremental gain of operator.
$\sum_{i=1}^{M} f_i$	The sum of the f_i over i from 1 to M.
$\prod_{i=1}^{M} f_i$	The product of the f_i over i from 1 to M.
$\int_0^t f(\tau)d\tau$	The integral of $f(\tau)$ from 0 to t.
$\frac{df(x)}{dx}$	The derivative of f(x) with respect to x.
—>	Gives or implies.
<—>	Corresponds to.
$\equiv$	Equivalent to.
$\forall$	For all.
$\subset$	Is contained in.
$\in$	Is a member of.
$\rightarrow$	Approaches, as in limit.
C[a,b]	Space of continuous functions over the interval [a,b], complete with the uniform norm.
C_λ[a,b]	Space of continuous functions over the interval [a,b], complete with the λ-norm.
sup	Supremum.
lim	Limit.
C_i^j	Number of ways of choosing i objects from j, without regard for order.
n!	n factorial.

$<$, $\leq$	Less than, less than or equal to.
$>$, $\geq$	Greater than, greater than or equal to.
$=$	Equal to.
$\neq$	Not equal to.
Ⓐ	Association of the two variables.
$*$	Cascade operation.
∞	Infinity.
$\dot{x}$	The derivative of x with respect to time.
$\ddot{x}$	The second derivative of x with respect to time.
iff	If and only if.

CHAPTER I

INTRODUCTION

Nonlinear systems play an important part in the practical physical world. Every transducer or amplifier which is used to make up a system has some inherent nonlinearity especially when the signal it is called upon to handle reaches a sufficiently large magnitude. It would seem that systems could not be analyzed without considering the nonlinearities but fortunately the usual practice is to run the elements of the system such that they approximate linear devices very closely. Then it is possible to analyze and synthesize systems using linear devices recognizing the contraints on the signals which the devices will be called upon to process. However some systems, such as a phase locked loop [11][25]† or a demodulator [29] base their operation upon a nonlinear device and in fact although designers of systems try to avoid using nonlinear devices due to the difficulty in estimating their effect, it may be that the use of such a device would enhance the properties of the system under design. There are also the cases of devices which are linear at small signals for which it is not practical to prevent the device from receiving a large enough signal to drive it into its nonlinear region. A simple case of this type would be an audio amplifier [13][7].

The ability to analyse and even synthesize a nonlinear system would be a valuable asset to the systems design engineer. This ideal is far from being reached due to the complexities involved in even the

---

† Numbers placed within square brackets denote references.

simplest of nonlinear systems.

A nonlinear system is defined as any system for which superposition does not hold. As such it can be a system with a time varying nonlinear element, a time invariant single valued nonlinear element, an element with hysteresis, or any combination of these types of nonlinearity. Ideally, a theory for the analysis of nonlinear systems would be able to handle any nonlinearity, and the linear case would be just a special limiting case in this theory. This ideal too is far from fruition.

There are two areas of interest to the systems engineer about the system under consideration. The first one is whether the system is stable, either asymptotically stable or bounded-input bounded output stable. If the system is stable, then a measure of the performance of the system may be desired. For example, the rise time in response to a step input, or the steady state error of the system, may be important. Usually these two areas are treated by different methods, and there have been quite a few methods developed for different types of systems and different types of nonlinearities. Gibson [12] in his book gives a good introduction to a variety of methods, most of which are limited to systems which can be described by first or second order nonlinear differential equations.

The problem of asymptotic stability has been treated by Liapunov's second method (see [12]) and by Popov [see [1]). These methods give sufficient conditions for stability and in some special cases give necessary conditions too. For these methods the nonlinear function is single valued and contained in a sector $[k_1, k_2]$, and gives a zero output for a zero input (see [1]).

For nonlinear systems asymptotic stability does not guarantee stability with an input (see [12]) so the concept of bounded-input bounded-output stability is investigated. Although other methods such as describing function methods [12] have been used to investigate stability of systems with inputs, it appears that a functional analysis approach to this problem gives very general results quite easily. Also, using a functional series solution to the system equation, called the Volterra series, the system can be represented in terms of its parameters and the input function, for a large class of systems.

V. Volterra [26] introduced the concept of a functional early in this century, and also introduced the type of integral equation with which most engineering systems are described. There have been many mathematicians working on this topic since then, but it was not until N. Wiener in 1942 [27] applied the functional series to the study of a nonlinear electrical circuit problem that engineers saw the possibilities of the theory in the analysis of nonlinear systems. Since then Barrett [2], Brilliant [3], George [11], Zames [29], Parente [18], Christensen [4], Sandberg [22] and Holtzman [14] to mention just a few have investigated the stability of systems (to inputs) and considered the requirements for convergence of the Volterra series. In most cases it is found that the stability limits are very conservative compared to other methods although Zames [30],[31] has proposed a circle theorem which with some restrictions on the nonlinearity gives a condition similar to that for the Popov criterion. However, a system which is bounded-input bounded-output stable may exhibit limit cycle oscillation when the input returns to zero.

All these methods give only sufficient conditions for stability whereas in the linear systems necessary and sufficient conditions follow from the application of the Nyquist or other similar criterion. This has led to the speculation that if the nonlinear function f(y) was replaced by a linear gain ky, and the range of k for which this linear system is stable $[k_1,k_2]$ (the Hurwitz sector) is determined, then the system may be stable for all nonlinear functions f(y) lying in the sector $[k_1,k_2]$ in the (y,f(y)) plane. This conjecture was originally put forward by M. Aizerman [1]. It has been shown to be true for a range of systems, but not true for many others. In fact Willems [28] has found that a counter example to the Aizerman conjecture can be found for almost any sufficiently smooth nonlinearity. However the Hurwitz sector gives an upper bound on the region over which a nonlinearity may be expected to give a stable system.

A more limiting criterion which will be used here is that the slope of the nonlinearity (where it exists) should lie within the range of $[k_1,k_2]$, as defined by the Hurwitz sector for the system.

The second area of interest to the engineer, namely the performance of the system when subjected to different inputs, can be approximated by using the Volterra series. Also the response of the system to initial conditions can be approximated using the same method.

In this thesis a class of systems will be examined using operator notation, where an operator will be signified by an underlined capital letter. The class of systems will be the class of feedback systems that can be represented by up to two linear plants with a single valued time invariant nonlinearity which passes through the origin, and which can be represented arbitrarily accurately by a finite power series. It will be shown with such a system that the Volterra series is a unique series expansion and that there is convergence of the series for nonlinearities whose slopes are within the Hurwitz sector. It will also be shown that for first and second order systems that bounded-input bounded-output stability exists for such nonlinearities. A method of incorporating initial conditions into the series is also discussed, and using this method a means of getting better convergence of the series with step inputs is introduced.

Chapter II introduces the notation and mathematics to be used. Operator algebra is discussed and the general system to be analysed is introduced. The space in which the inputs and outputs belong is then described, along with a weighted norm which is used in the convergence proofs. Some norms of operators are examined and the basic contraction mapping theorem is stated.

Chapter III demonstrates the solution of the system equation using the Volterra series by two different methods, and then a proof is offered to show that these two methods give identical series in the limit. Then a simple way of evaluating the series using transform theory is developed for the general system under study followed by an example.

In Chapter IV the convergence of the series is proved for functions belonging to the L_∞ space. The first and second order systems are shown to give bounded-input bounded-output stability for nonlinearities as mentioned above and for higher order systems the output is shown to be exponentially bounded. There are examples given to show the improvement over some of the other methods.

Non-zero initial conditions are treated in Chapter V. It is shown how these become equivalent to an additional input function, and can easily be incorporated into the Volterra series. Using these results the case of a step input is re-examined and a solution obtained by translation of the equation which gives a better convergence than than given by the straight equation. There are examples worked out and a comparison made.

CHAPTER II

MATHEMATICAL PRELIMINARIES

2.1 Introduction.

In this chapter certain mathematical tools which are used in later chapters will be outlined. The object is not to give the rigorous mathematics behind these tools, but rather to introduce them in a descriptive fashion so that when they are used later there will be no loss of continuity.

Firstly the algebra of working with operators is discussed. This is important in order to express a feedback system in terms of its constituent parts (plants), and is also very necessary to simplify the manipulation of terms of the Volterra series. Then single input feedback systems with one nonlinearity are considered, and a general system is used which can be reduced to several types of systems. The operator equation of this general system is the one that will be analysed in the remainder of the thesis.

Norms play a large role in the functional analysis approach to the stability analysis of nonlinear systems, and the choice of the best norm to use for a particular system is an area of research that has not been looked at very deeply. Two norms are discussed and will be used later along with various methods of describing the size of the effect an operator will have on a function. This is called the gain of the operator but has also been called the norm of the operator. Several examples which will be important are introduced.

Finally the special class of operators which are contraction mappings are introduced and a fixed point theorem is stated along with some boundedness properties of this class.

2.2 Operator Notation.

A physical system which has inputs as functions of time and gives outputs as functions of time will be mathematically modelled as an operator. In this context an operator is a transformation or mapping from a space of functions of time into another such space. For this thesis only operators mapping a space into itself will be dealt with and the space in particular will be a normed linear complete space (a Banach space).

Operators will be denoted by underlined capitals, such as $\underline{H}$, while the functions will be denoted by lower case letters such as x(t), y(t). Laplace transforms of operators and functions will be denoted by capitals which are not underlined such as H(s), X(s), Y(s). The fact that $\underline{H}$ operates on x to give y will be written as

$$y = \underline{H}x(t) = \underline{H}(x) = \underline{H}x \qquad (2.1)$$

As in the case of functions, the set of all functions x for which the operator is valid is the domain of $\underline{H}$, while the set of all y which $\underline{H}(x)$ assumes is called the range of $\underline{H}$.

A numerical subscript to an operator denotes the order of the operator. For example, an operator of order n obeys the following rule where α is a scalar number.

$$\underline{H}_n(\alpha x) = \alpha^n \underline{H}_n(x) \qquad (2.2)$$

A linear operator has an order 1. However, when the order of a system is mentioned this will refer to the order of the differential equation which describes (or can describe) the system.

2.3 Operator Algebra [29], [11].

In order to handle interconnections of operators with relative ease, it is useful to look at the properties of the sum, product and cascade of operators. This helps in relating the behaviour of interconnections of subsystems (such as in feedback systems) to those of the components.

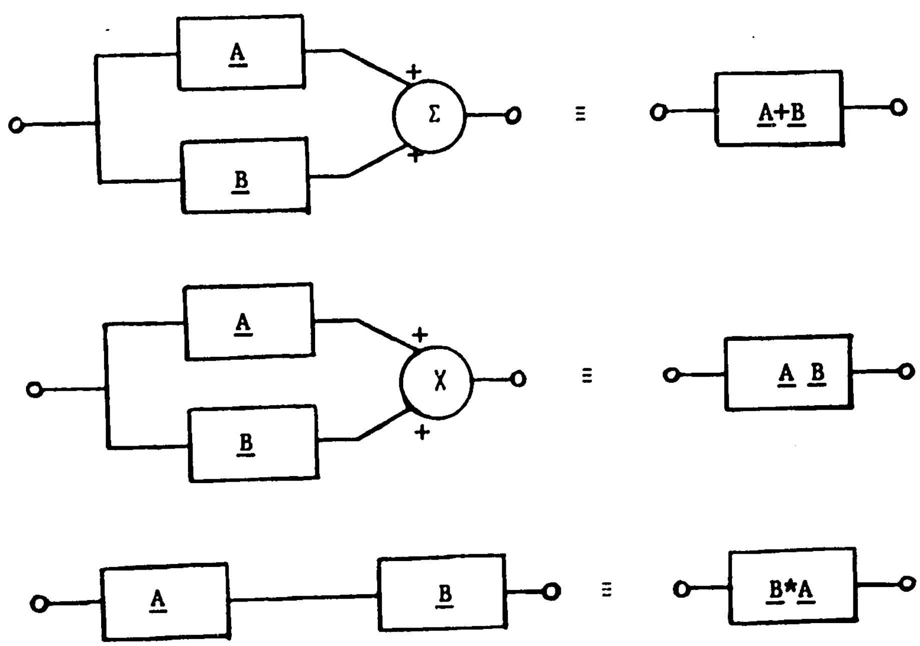

Fig. 2.1 Sum, Product and Cascade of Systems.

The properties of the sum or product of two operators are obvious from Fig. 2.1.

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$

$$(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C}) \qquad (2.3)$$

$$\underline{A}\,\underline{B} = \underline{B}\,\underline{A}$$

$$(\underline{A}\,\underline{B})\underline{C} = \underline{A}(\underline{B}\,\underline{C}) \tag{2.4}$$

The cascade of $\underline{B}$ following $\underline{A}$ is denoted by $\underline{B}*\underline{A}$. It is not a commutative operation but it is associative.

$$(\underline{A}*\underline{B})*\underline{C} = \underline{A}*(\underline{B}*\underline{C}) \tag{2.5}$$

The special cases of linear time invariant operators are commutative in the cascade operation.

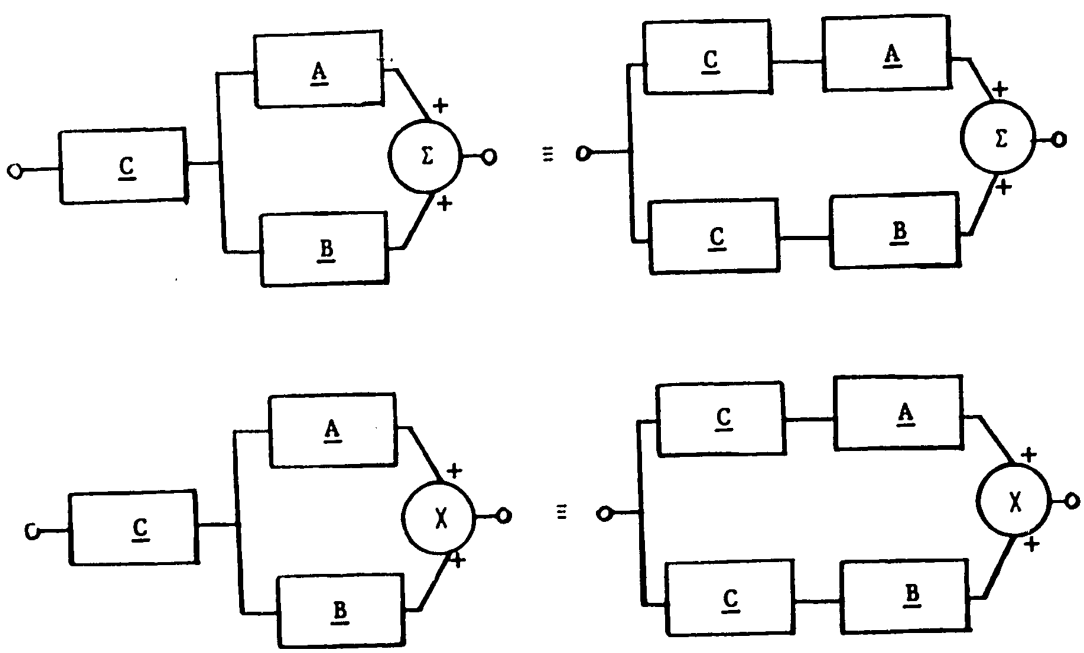

Fig. 2.2 Combination of Cascade with Sum and Product.

The sum and product are distributive with the cascade as long as the sum or product follow the cascade as shown in Fig. 2.2.

$$(\underline{A} + \underline{B})*\underline{C} = (\underline{A} * \underline{C}) + (\underline{B} * \underline{C})$$

$$(\underline{A}\,\underline{B})*\underline{C} = (\underline{A}*\underline{C})(\underline{B}*\underline{C}) \tag{2.6}$$

The sum operation has a zero operator $\underline{0}$ which corresponds to the open circuit system whose output is zero whatever the input.

$$\underline{0}(x) = 0$$

$$\underline{0} + \underline{A} = \underline{A}$$

$$\underline{A} + (-\underline{A}) = \underline{0} \tag{2.7}$$

$(-\underline{A})$ is called the negative of $\underline{A}$.

The cascade operation has an identity operator $\underline{I}$ which corresponds to the short circuit system whose output always equals the input.

$$\underline{I}(x) = x$$

$$\underline{I}*\underline{A} = \underline{A}*\underline{I} = \underline{A} \tag{2.8}$$

Provided it exists, the inverse of an operator when cascaded before or after the operator gives the identity operator.

$$\underline{A}*\underline{A}^{-1} = \underline{A}^{-1}*\underline{A} = \underline{I} \tag{2.9}$$

The operators used in this thesis can be expressed as a sum of operators of various orders. For example:

$$\underline{A} = \sum_{j=1}^{\infty} \underline{A}_j \tag{2.10}$$

where some of the $\underline{A}_j$ may be the zero operator $\underline{0}$. It will be convenient to consider that the zero operator can have any order. Consider now the effect of operations on the order of an operator.

Clearly addition has no effect on the order. Now if

$$\underline{C}_p(\alpha x) = \underline{A}_j(\alpha x)\ \underline{B}_k(\alpha x) \tag{2.11}$$

$$= \alpha^j \underline{A}_j(x) \alpha^k \underline{B}_k(x)$$

$$= \alpha^{j+k} \underline{A}_j(x)\ \underline{B}_k(x)$$

$$= \alpha^{j+k} \underline{C}_p(x) \tag{2.12}$$

with α a scalar number, then for products

$$p = j+k \tag{2.13}$$

Similarly,

$$\underline{C}p(\alpha x) = \underline{A}_j * \underline{B}_k(\alpha x) \tag{2.14}$$

$$= \underline{A}_j * \alpha^k \underline{B}_k(x)$$

$$= (\alpha^k)^j \underline{A}_j * \underline{B}_k(x)$$

$$= (\alpha^k)^j \underline{C}_p(x) \tag{2.15}$$

and so for cascades,

$$p = kj \tag{2.16}$$

So it can be seen that the effect of multiplication is to sum the two orders while that of cascade is to multiply the two orders.

2.4 Feedback Systems.

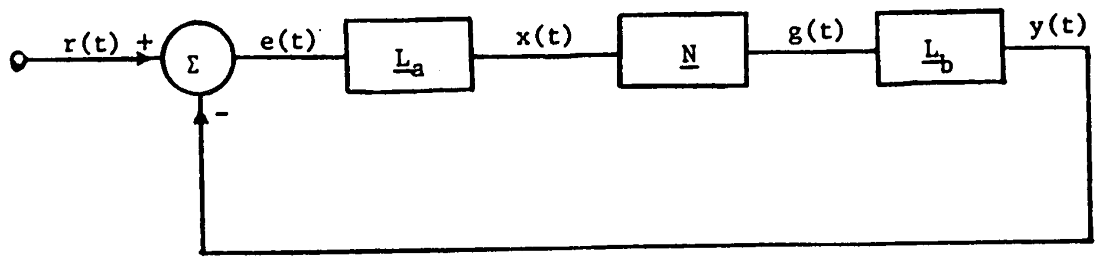

Fig. 2.3 The General Feedback System.

The feedback system of Fig. 2.3 must simultaneously satisfy the following equations.

$$x = \underline{L}_a\,(e)$$

$$g = \underline{N}(x)$$

$$y = \underline{L}_b(g)$$

$$e = r - y \tag{2.17}$$

This is the general system which will be considered for the rest of the thesis containing a single nonlinearity $\underline{N}$ which is single valued and time invariant. $\underline{L}_a$ and $\underline{L}_b$ are linear plants either one of which could be equal to $k\underline{I}$ a linear gain.

Eliminating y, g and e from equations (2.17) gives

$$x = \underline{L}_a\left(r - \underline{L}_b(\underline{N}(x))\right) \tag{2.18}$$

Now assume that x and r are related by an operator

$$x = \underline{H}(r) \tag{2.19}$$

then substituting equation (2.19) into equation (2.18) gives

$$\underline{H}(r) = \underline{L}_a \left(r - \underline{L}_b(\underline{N}(\underline{H}(r)))\right) \tag{2.20}$$

Since this equation holds for all r it may be written in operator form as

$$\underline{H} = \underline{L}_a * (\underline{I} - \underline{L}_b * \underline{N} * \underline{H}) \tag{2.21}$$

Now suppose $\underline{L}_a^{-1}$ exists. Then,

$$\underline{L}_a^{-1} * \underline{H} = \underline{I} - \underline{L}_b * \underline{N} * \underline{H} \tag{2.22}$$

$$\underline{L}_a^{-1} * \underline{H} + \underline{L}_b * \underline{N} * \underline{H} = \underline{I}$$

$$(\underline{L}_a^{-1} + \underline{L}_b * \underline{N}) * \underline{H} = \underline{I} \tag{2.23}$$

$$(\underline{L}_a^{-1} + \underline{L}_b * \underline{N} * \underline{L}_a * \underline{L}_a^{-1}) * \underline{H} = \underline{I}$$

$$(\underline{I} + \underline{L}_b * \underline{N} * \underline{L}_a) * \underline{L}_a^{-1} * \underline{H} = \underline{I}$$

$$\underline{H} = \underline{L}_a * (\underline{I} + \underline{L}_b * \underline{N} * \underline{L}_a)^{-1} \tag{2.24}$$

provided $(\underline{I} + \underline{L}_b * \underline{N} * \underline{L}_a)^{-1}$ exists. Compare equation (2.24) with the feedback equation of linear transform theory for the similar system.

$$H(s) = \frac{L_a(s)}{1 + L_b(s)N(s)L_a(s)} \tag{2.25}$$

assuming $\underline{N}$ to be a linear time invariant operator.

So equation (2.21) is the operator equation of the general system under consideration. The output function could be any of e(t), x(t), g(t) or y(t) which can be obtained from equation (2.21) by using the necessary operators. For example if y(t) was the output and it is desired to express y(t) as

$$y = \underline{G}(r) \tag{2.26}$$

then it follows that

$$\underline{G} = \underline{I_b}*\underline{N}*\underline{H} \tag{2.27}$$

2.5 Functional Representation.

The solution and stability of linear time invariant systems can easily be determined by examination of the roots of the characteristic equation of the system. This method gives necessary and sufficient conditions for stability for such systems. In the case of nonlinear systems, the ideal would be to develop a theory which would enable the solution and stability of the system to be determined for any system. In this theory the linear systems would be a special limiting case. It is thought that using functional analysis techniques offers the best chance of achieving such a general theory.

In engineering it is important to have some way of determining the size of errors or functions. This is taken care of mathematically by introducing norms [19]. The norm of x is written as $||x||$ and must satisfy 3 properties.

1. $||x|| \geq 0$; $||x|| = 0$ iff x=0 almost everywhere.

2. $\| \alpha x \| = |\alpha| \; \| x \|$; where α is a real or complex number.

3. $\| x + y \| \leq \| x \| + \| y \|$ (2.28)

There are no restrictions on how to choose a norm as long as it satisfies the 3 properties above, and so by judicious choice of a norm various properties of a function can be emphasized.

For example, consider the problem of matching an analytic curve to a set of empirical points. If the error is considered to be the distance d_i from the curve to the point i, and a measure of the closeness of the fit is to minimize $\max_i | d_i |$, then there would usually be a different matching of the curve than if $(\sum_i | d_i |^2)^{\frac{1}{2}}$ was minimized.

One of the more useful norms for functions used most frequently in engineering is the uniform norm [46].

$$\| x \| = \sup_t | x(t) | \tag{2.29}$$

The space C[a,b] of continuous functions on the interval [a,b] with the uniform norm becomes a normed linear complete space and so a Banach space. The space C[a,b] will be the one used throughout this thesis. When the uniform norm is used it will be used without subscript.

Now consider the linear space C[a,b] and the norm (see [6])

$$\| x \|_\lambda = \sup_{t \in [a,b]} e^{-\lambda t} | x(t) | \tag{2.30}$$

This norm is equivalent to the uniform norm in that

$$e^{-\lambda b} \| x \| \leq \| x \|_\lambda \leq e^{-\lambda a} \| x \| \qquad ; \; t \in [a,b] \tag{2.31}$$

Since the norms are equivalent, sequences converge in the λ-norm if and only if they converge in the uniform norm. Thus the space $C[a,b]$ with λ-norm is complete and so is a Banach space, which will be called $C_\lambda[a,b]$. Whenever the λ-norm is specifically used it will have the subscript λ.

2.6 Gains of Operators.

A system which is bounded-input bounded-output stable has a bounded output for every finite input. If the system is represented by an operator $\underline{H}$, then for the system to be bounded-input bounded-output stable implies that the gain of $\underline{H}$ is finite for all finite inputs. So if $\underline{H}$ maps functions $x \epsilon C[a,b]$ into $y \epsilon C[a,b]$ then

$$y = \underline{H}x \tag{2.32}$$

$$\| y \| = \| \underline{H}x \| \tag{2.33}$$

where for this discussion either norm can be considered.

Consider the Lipschitz norm of an operator [19].

Definition: The maximum incremental gain of an operator $\underline{H}$ over a set $S \subset D$ where D is the domain of the operator is defined to be the Lipschitz norm of the operator.

$$\text{inc}\| \underline{H} \| = \sup\{\| \underline{H}(x_1) - \underline{H}(x_2) \| / \| x_1 - x_2 \| : x_1 \neq x_2 \epsilon S\} \tag{2.34}$$

<u>Definition</u>: The slope of the operator $\underline{H}$ is defined to be

$$\frac{d\underline{H}}{dx} = \lim_{x_1 \to x} \left(\underline{H}(x_1) - \underline{H}(x) \right) / \ (x_1 - x) \tag{2.35}$$

provided the limit exists.

Clearly it can be seen that for time-invariant operators

$$\text{inc}\| \underline{H} \| = \sup_x \left\| \frac{d\underline{H}}{dx} \right\| \tag{2.36}$$

Now if it is assumed that

$$\underline{H}\ (0) = 0 \tag{2.37}$$

then setting $x_2 = 0$ in definition (2.34) gives

$$\| \underline{H} \| = \sup \{ \| \underline{H}(x_1) \| / \| x_1 \| : \ x_1 \neq 0 \epsilon S \} \tag{2.38}$$

and this will be called the maximum gain of the system.

Since equation (2.38) was derived from equation (2.34) by fixing x_2, then the supremum in equation (2.38) is less than that in equation (2.34), as it is taken over a smaller range. Thus,

$$\text{inc}\| \underline{H} \| \geq \| \underline{H} \| \tag{2.39}$$

From equation (2.38) it can be seen that

$$\| \underline{H}x \| \leq \| \underline{H} \| \, \|x\| \qquad \forall \; x \in S \tag{2.40}$$

Substituting this in equation (2.33) gives

$$\| y \| \leq \| \underline{H} \| \; \| x \| \tag{2.41}$$

and using equation (2.39) a coarser estimate is obtained

$$\| y \| \leq \text{inc} \| \underline{H} \| \; \| x \| \tag{2.42}$$

2.7 Examples of Operators.

(1) Linear Time-invariant Operators.

For this broad class of operators the following applies

$$\begin{aligned} \text{inc} \| \underline{L} \| &= \sup \{ \| \underline{L}(x_1) - \underline{L}(x_2) \| / \| x_1 - x_2 \| \; ; \; x_1 \neq x_2 \in S \} \\ &= \sup \{ \| \underline{L}(x_1 - x_2) \| / \| x_1 - x_2 \| \; ; \; x_1 - x_2 \neq 0 \in S \} \end{aligned} \tag{2.43}$$

$$\text{inc} \; \| \underline{L} \| = \| \underline{L} \| \tag{2.44}$$

Consider the case of a linear convolution operator, which is the linear operator to be used here

$$y(t) = \underline{L}(x) = {}_0\!\int^t h(t-\tau) \; x(\tau) d\tau \tag{2.45}$$

with

$$x(t) = h(t) = 0 \qquad t < 0 \tag{2.46}$$

$$|y(t)| \leq {}_0\int^t |h(t-\tau)||x(\tau)|d\tau$$

$$\sup_t |y(t)| \leq \sup_t |x(t)| \ \sup_t {}_0\int^t |h(\tau)|d\tau$$

$$\|y\| \leq \|x\| \ {}_0\int^\infty |h(\tau)|d\tau \tag{2.47}$$

It has been shown [15], that for these linear operators

$$\|\underline{L}\| = {}_0\int^\infty |h(\tau)|d\tau \tag{2.48}$$

Thus this class of linear systems is stable for all operators which obey the following.

$${}_0\int^\infty |h(\tau)|d\tau < \infty \tag{2.49}$$

Using the λ-norm, the following is obtained from equation (2.45).

$$|y(t)| \leq {}_0\int^t |h(t-\tau)||x(\tau)|e^{-\lambda\tau} \ e^{\lambda\tau} \ d\tau \tag{2.50}$$

$$\leq \sup_{\tau\epsilon[0,T]} |x(\tau)|e^{-\lambda\tau} \sup_{(t-\tau)\epsilon[0,T]} |h(t-\tau)| {}_0\int^t e^{\lambda\tau}d\tau$$

$$\leq \|x\|_\lambda \sup_{t\epsilon[0,T]} |h(t)| \frac{e^{\lambda t}-1}{\lambda} \tag{2.51}$$

$$|y(t)|e^{-\lambda t} \leq \|x\|_\lambda \sup_{t\epsilon[0,T]} |h(t)| \frac{1}{\lambda}$$

Then taking the supremum of the left hand side over $t\epsilon[0,T]$ gives,

$$\|y\|_\lambda \leq \|x\|_\lambda \sup_{t\epsilon[0,T]} |h(t)| \ \lambda^{-1} \tag{2.52}$$

(2) Nonlinear no-memory Time-invariant Operators.

For these cases, the operators can be described by a graph, and the maximum incremental gain is the largest absolute value of the slope of the graph. For example, consider the nonlinearity

$$\underline{N}(x) = \sum_{j=1}^{p} n_j (x)^j \tag{2.53}$$

Then using equation (2.35), it can be seen that

$$\frac{d\underline{N}}{dx} = \sum_{j=1}^{p} n_j j (x)^{j-1} \tag{2.54}$$

and so from equation (2.36)

$$\text{inc} \, ||\underline{N}|| = \sup_{x \in S} \left| \sum_{j=1}^{p} n_j j (x)^{j-1} \right| \tag{2.55}$$

From equation (2.38)

$$||\underline{N}x|| \leq |x| \left\| \sum_{j=1}^{p} n_j (x)^{j-1} \right\| \leq ||x|| \sup_{x \in S} \left\| \sum_{j=1}^{p} n_j (x)^{j-1} \right\|$$

$$||\underline{N}x|| \; / \; ||x|| \leq \sup_{x \in S} \left| \sum_{j=1}^{p} n_j (x)^{j-1} \right| \tag{2.56}$$

(3) Nonlinear Convolution Operators of Order n.

These are of the form

$$\underline{H}_n(x) = {}_0\int^t \cdots {}_0\int^t h_n(t-\tau_1, \ldots, t-\tau_n) x(\tau_1) \ldots x(\tau_n) d\tau_1 \ldots d\tau_n \tag{2.57}$$

It can easily be seen from this equation that $\underline{H}_n$ has order n.

Consider now a bound on $\underline{H}_n(x)$.

$$|\underline{H}_n(x)| \leq {}_0\int^t \cdots {}_0\int^t |h_n(t-\tau_1,\ldots,t-\tau_n)||x(\tau_1)|\ldots|x(\tau_n)|d\tau_1\ldots d\tau_n$$

$$\leq \|x\|^n {}_0\int^\infty \cdots {}_0\int^\infty |h_n(\tau_1,\ldots,\tau_n)|d\tau_1\ldots d\tau_n \qquad (2.58)$$

Taking the supremum over t of each side gives

$$\|\underline{H}_n(x)\| \leq \|x\|^n {}_0\int^\infty \cdots {}_0\int^\infty |h_n(\tau_1,\ldots,\tau_n)|d\tau_1\ldots d\tau_n \qquad (2.59)$$

Thus for absolute convergence of the series given by

$$\underline{H}(x) = \sum_{j=1}^{\infty} \underline{H}_j(x) \qquad (2.60)$$

it follows that

$$\|\underline{H}(x)\| \leq \sum_{j=1}^{\infty} \|\underline{H}_j(x)\|$$

$$\leq \sum_{j=1}^{\infty} \|x\|^j {}_0\int^\infty \cdots {}_0\int^\infty |h_j(\tau_1,\ldots,\tau_j)|d\tau_1\ldots d\tau_j \qquad (2.61)$$

A sufficient condition that the series in equation (2.61) should converge, is that $\|x\|$ be small enough, as long as

$${}_0\int^\infty \cdots {}_0\int^\infty |h_n(\tau_1,\ldots,\tau_n)|d\tau_1\ldots d\tau_n \leq Q < \infty \qquad (2.62)$$

The computation of the gains of these operators is more difficult than for the other types of operators considered, and it is better to try and find the gain of the total operator $\underline{H}$ by some other means.

2.8 Contraction Mapping

Let E be a Banach space and let $\underline{H}$ be an operator mapping an open subset U of E into U; suppose there is a constant α , $0<\alpha<1$, such that for every x_1, x_2 in U

$$\| \underline{H}(x_1) - \underline{H}(x_2) \| \leq \alpha \| x_1 - x_2 \| \tag{2.63}$$

or using the notation used here

$$\text{inc} \|\underline{H}\| = \alpha \tag{2.64}$$

then $\underline{H}$ is called a contraction operator on U.

If it can be shown that an operator is a contraction then there are many theorems available [20][21] to estimate the fixed point of the operator, and the error bound of the solution obtained by iteration, as well as a bound on the solution.

<u>Theorem</u> [21] Let $\underline{H}$ be a contraction on U in a Banach space E with $\text{inc} \|\underline{H}\| = \alpha$. Suppose there is a sphere $S:\{\| x-x_0\| \leq a\}$ that is contained in U and that

$$\| \underline{H}(x_0) - x_0 \| < (1-\alpha)a \tag{2.65}$$

Then there is a unique fixed point x' in S, that is, a function x' for which

$$\underline{H}(x') = x' \tag{2.66}$$

and x' is the limit of the sequence $\{x_n\}$ generated by

$$x_{n+1} = \underline{H}(x_n) \tag{2.67}$$

and the estimate of error is given by

$$\| x' - x_n \| \leq \alpha^n (1-\alpha)^{-1} \| x_1 - x_o \| \qquad (2.68)$$

Proof: (See [21] p.37).

Comment: If the operator is a contraction over the whole space E then clearly a is as large as needed and so the condition given in equation (2.65) does not give any restriction. Also equation (2.65) can be satisfied by choosing x_o such that the left hand side is small enough. This condition makes sure that during the iteration, the iterands are always contained in U.

If the operator $\underline{H}$ is in an equation such as

$$x(t) = y(t) + \underline{H}\ (x) \qquad (2.69)$$

then it is possible to bound x provided $\underline{H}$ is a contraction.

$$\| x \| \leq \| y \| + \| \underline{H}\, x \|$$

by the triangle inequality of norms.

$$\| x \| \leq \| y \| + \| \underline{H} \| \, \| x \| \qquad (2.70)$$

by the use of equation (2.40). Then rearranging equation (2.70) gives

$$\| x \| \leq \| y \| (1 - \| \underline{H} \|)^{-1} \qquad (2.71)$$

provided

$$\| \underline{H} \| < 1 \qquad (2.72)$$

But since $\underline{H}$ is a contraction it follows that

$$\| \underline{H} \| \leq \text{inc} \| \underline{H} \| = \alpha < 1 \tag{2.73}$$

So for an equation such as (2.69) where the operator $\underline{H}$ has been shown to be a contraction in a particular norm and over a space then a bounded function y(t) implies a bounded solution x(t). This then is the basis for the bounded-input bounded-output type of stability.

CHAPTER III

UNIQUENESS OF THE VOLTERRA SERIES

3.1 Introduction.

The Volterra series is a part way solution to the problem of finding an explicit relationship between the output and input of a nonlinear system. It does give such a relationship but in the form of a (usually) infinite series whose terms are in general rather difficult to evaluate. Also the series is able to be formed only for a class of nonlinearities which can be expressed as a power series. There are two properties of the series which are important; does the series converge and is it bounded?

Firstly, two methods of calculating the series are examined and it is proved that these two methods give the same series. Then methods are shown for evaluating the terms of the series and an example used to illustrate the computations and also to introduce the next chapters which will deal with the convergence and boundedness problems for the general class of systems under consideration.

3.2 The Volterra Series by Iteration.

Consider the system equation (2.21) with

$$\underline{L}_a * \underline{L}_b = \underline{L} \tag{3.1}$$

Then

$$\underline{H} = \underline{L}_a - \underline{L}*\underline{N}*\underline{H} \tag{3.2}$$

This equation can be solved by iteration, assuming for the moment that the series converges and is bounded.

Let

$$\underline{H}^o = \underline{L}_a$$

be the starting approximation. Then

$$\underline{H}^1 = \underline{L}_a - \underline{L}*\underline{N}*\underline{L}_a$$

$$\underline{H}^2 = \underline{L}_a - \underline{L}*\underline{N}*(\underline{L}_a - \underline{L}*\underline{N}*\underline{L}_a)$$

$$\underline{H}^3 = \underline{L}_a - \underline{L}*\underline{N}*(\underline{L}_a - \underline{L}*\underline{N}*(\underline{L}_a - \underline{L}*\underline{N}*\underline{L}_a)) \tag{3.3}$$

$$\underline{H}^{n+1} = \underline{L}_a - \underline{L}*\underline{N}*\underline{H}^n \tag{3.4}$$

where $\underline{H}^n$ is the nth iterate. Now let

$$\underline{N}x = \sum_{j=2}^{M} n_j(x)^j \tag{3.5}$$

then substituting in the starting approximation gives

$$\underline{H}^1 = \underline{L}_a - \underline{L}*\sum_{j=2}^{M} n_j(\underline{L}_a)^j = \sum_{j=1}^{M} \underline{H}_j^1 \tag{3.6}$$

where the $\underline{H}_j{}^1$ are given by

$$\underline{H}_1^1 = \underline{L}_a$$

$$\underline{H}_j^1 = -\underline{L}*n_j(\underline{L}_a)^j \qquad 2\leq j\leq M \tag{3.7}$$

Substituting equations (3.7) into equation (3.4)

$$\underline{H}^2 = \underline{L}_a - \underline{L}^* \sum_{j=2}^{M} n_j \left(\sum_{i=1}^{M} \underline{H}_i^1 \right)^j = \sum_{j=1}^{M^2} \underline{H}_j^2 \tag{3.8}$$

Now $\left(\sum_{i=1}^{M} \underline{H}_i^1 \right)^j$ can be written in ascending order of operators, where $\underline{H}_j$ denotes the operator of order j[16].

$$\left(\sum_{i=1}^{M} \underline{H}_i^1 \right)^j = \sum_{i=1}^{M} \underline{H}_i^{1(j)} + \sum_{i=M+j}^{Mj} \underline{Z}_i^1 \tag{3.9}$$

where $\underline{H}_i^{1(j)}$ is of order $i + j - 1$ and is given by

$$\underline{H}_i^{1(j)} = \sum_{p_{j-1}=1}^{i} \sum_{p_{j-2}=1}^{p_{j-1}} \cdots \sum_{p_1=1}^{p_2} \underline{H}_{p_1}^1 \underline{H}_{p_2-p_1+1}^1 \cdot \underline{H}_{i-p_{j-1}+1}^1 \tag{3.10}$$

This expansion gives all the combinations of $\underline{H}_i^1$'s which when multiplied together give the $\underline{H}_i^1$ of order i+j-1. The $\underline{Z}_i^1$ of equation (3.9) are operators of higher order of the expansion which cannot be expressed in the form of equation (3.10). Substituting equation (3.9) into equation (3.8) gives

$$\sum_{j=1}^{M^2} \underline{H}_j^2 = \underline{L}_a - \underline{L}^* \sum_{j=2}^{M} n_j \left(\sum_{i=1}^{M} \underline{H}_i^{1(j)} + \sum_{i=M+j}^{Mj} \underline{Z}_i^1 \right) \tag{3.11}$$

$$= \underline{L}_a - \sum_{j=2}^{M} n_j \sum_{i=1}^{M} \underline{L}^* \underline{H}_i^{1(j)} - \underline{L}^* \sum_{j=2}^{M} n_j \sum_{i=M+j}^{Mj} \underline{Z}_i^1 \tag{3.12}$$

$$\underline{H}_1^2 = \underline{L}_a \tag{3.13}$$

$$\underline{H}_2^2 = - n_2 \underline{L}^* (\underline{H}_1^1)^2 = - n_2 \underline{L}^* (\underline{L}_a)^2 \tag{3.14}$$

$$\underline{H}_3^2 = -\underline{L}^*[n_2(\underline{H}_1^1\,\underline{H}_2^1 + \underline{H}_2^1\,\underline{H}_1^1) + n_3(\underline{H}_1^1)^3\,]$$

$$= 2n_2^2\underline{L}^*(\underline{L}_a(\underline{L}^*(\underline{L}_a)^2)) - n_3\underline{L}^*(\underline{L}_a)^3 \tag{3.15}$$

$$\underline{H}_j^2 = -\sum_{k=2}^{j} n_k \underline{L}^*\underline{H}_{j-k+1}^{1(k)} \qquad 2\leq j\leq M \tag{3.16}$$

There are M^2 of the $\underline{H}_j^2$ terms and by the nth iteration there would be M^n terms. Some of these terms cease to change after a certain number of iterations, and so just add to the work without providing any additional information. For example, in the case above the terms $\underline{H}_1^1$ and $\underline{H}_1^2$ are equal as are $\underline{H}_2^1$ and $\underline{H}_2^2$ but the third order terms differ as shown in equation (3.15) where another factor has been introduced over that for $\underline{H}_3^1$.

3.3 The Volterra Series by Substitution.

Consider the same equation (3.2) with $\underline{N}$ as defined in equation (3.5). Then

$$\underline{H} = \underline{L}_a - \underline{L}^*\sum_{i=2}^{M} n_i(\underline{H})^i \tag{3.17}$$

Now let $\underline{H}$ be a Volterra series of the form

$$\underline{H} = \sum_{j=1}^{\infty} \underline{H}_j \tag{3.18}$$

where $\underline{H}_j$ is the nonlinear convolution operator of order j as defined in Section 2.7. Then substituting equation (3.18) into equation (3.17) gives

$$\sum_{j=1}^{\infty} \underline{H}_j = \underline{L}_a - \underline{L}^*\sum_{i=2}^{M} n_i(\sum_{j=1}^{\infty}\underline{H}_j)^i \tag{3.19}$$

Now using equation (3.10) to express the sum raised to a power

$$\left(\sum_{j=1}^{\infty} \underline{H}_j\right)^i = \sum_{j=1}^{\infty} \underline{H}_j^{(i)} \tag{3.20}$$

where $\underline{H}_j^{(i)}$ has order i+j-1. Using equation (3.20) to simplify the notation in equation (3.19), gives

$$\sum_{j=1}^{\infty} \underline{H}_j = \underline{L}_a - \underline{L}* \sum_{i=2}^{M} n_i \sum_{j=1}^{\infty} \underline{H}_j^{(i)} \tag{3.21}$$

$$= \underline{L}_a - \sum_{j=1}^{\infty} \sum_{i=2}^{M} n_i \underline{L}*\underline{H}_j^{(i)}$$

$$= \underline{L}_a - \sum_{j=2}^{\infty} \sum_{k=2}^{\min(M,j)} n_k \underline{L}*\underline{H}_{j-k+1}^{(k)} \tag{3.22}$$

by rearranging the terms and changing the order of summation. Now using the rules for cascade sum and product as outlined in Section 2.3 and equating orders on each side of equation (3.22).

$$\underline{H}_1 = \underline{L}_a$$

$$\underline{H}_j = -\sum_{k=2}^{\min(M,j)} n_k \underline{L}*\underline{H}_{j-k+1}^{(k)} \qquad j \geq 2 \tag{3.23}$$

$$\underline{H}_2 = -\, n_2 \underline{L}*(\underline{L}_a)^2 \tag{3.24}$$

$$\underline{H}_3 = 2n_2^{\,2}\, \underline{L}*(\underline{L}_a(\underline{L}*(\underline{L}_a)^2)) - n_3 \underline{L}*(\underline{L}_a)^3 \tag{3.25}$$

By comparing equations (3.25) and (3.15) it is seen that they are equal. However will they remain equal through more iterations? This is answered in the next section.

Evaulation of the terms of equation (3.23) give as many terms of the Volterra series as is desired.

3.4 The Uniqueness of the Volterra Series [23]

Consider now what has been done in the previous two sections. For the iteration case an initial estimate was chosen for the solution. Let

$$x = \underline{H}(r) \tag{3.26}$$

and then

$$x = \underline{L}_a(r) - \underline{L}*\underline{N}(x) \tag{3.27}$$

Now take the initial estimate to be

$$x_o = \underline{L}_a(r) \tag{3.28}$$

Then from equation (3.27) using iteration

$$x_1 = \underline{L}_a(r) - \underline{L}*\underline{N}(x_o) \tag{3.29}$$

$$x_{n+1} = \underline{L}_a(r) - \underline{L}*\underline{N}(x_n) \tag{3.30}$$

So that the limit of the sequence $\{x_n\}$

$$x' = \lim_{n\to\infty} x_n = \underline{L}_a(r) - \lim_{n\to\infty} \underline{L}*\underline{N}(x_n) \tag{3.31}$$

For the substitution case it was assumed that

$$x' = \lim_{n\to\infty} \sum_{j=1}^{n} \underline{H}_j(r) \tag{3.32}$$

assuming the limit exists. Substituting in equation (3.27)

$$x' = \underline{L}_a(r) - \underline{L}*\underline{N}\left(\lim_{n\to\infty} \sum_{j=1}^{n} \underline{H}_j(r)\right) \tag{3.33}$$

$$= \underline{L}_a(r) - \underline{L}*\underline{N}(x') \tag{3.34}$$

So it is necessary to show there exists a unique solution to equation (3.27) so that

$$\lim_{n\to\infty} \underline{L}*\underline{N}(x_n) = \underline{L}*\underline{N}(\lim_{n\to\infty} x_n) \tag{3.35}$$

Now if $\underline{L}*\underline{N}$ obeys a Lipschitz condition that is

$$\| \underline{L}*\underline{N}(x_n) - \underline{L}*\underline{N}(x_{n-1}) \| \leq K \| x_n - x_{n-1} \| \tag{3.36}$$

where K is a real positive constant, and if K<1 over a region U, then the contraction mapping theorem of Section 2.8 can be applied to the system.

So there exists a unique solution to equation (3.17) if the conditions of the theorem hold, and consequently equation (3.35) is true.

It has been shown that the first few terms of the two series calculated by the two methods are in fact identical and it is found that as the iteration is carried on further,more terms do equal those calculated by substitution. The problem with the iteration method is the large number of terms which must be carried through each iteration. The easiest method of evaluating the series is to use the substitution method to get terms up to a certain order and then perhaps do one iteration.

3.5 Evaluation of the Kernels of the Volterra Series

There are two main methods of evaluating the kernels of the series. The first uses time domain analysis by using the integral expression for the general convolution operator. Then the evaluation of the kernels involves integrations. For example consider equations (3.24), (3.25) with

$$\underline{L}_a(r) = \int_0^t f(t-\tau)\, r\,(\tau)d\tau \tag{3.37}$$

$$\underline{L}(r) = {}_0\int^t g(t-\tau)\ r(\tau)d\tau \tag{3.38}$$

and

$$f(t) = g(t) = r(t) = 0 \quad \text{for} \quad t < 0 \tag{3.39}$$

as is found in all practical systems. Then it can be seen that

$$\underline{H}_1(r) = {}_0\int^t f(t-\tau)\ r(\tau)d\tau = {}_0\int^t h(t-\tau_1)r(\tau_1)d\tau_1 \tag{3.40}$$

$$\underline{H}_2(r) = {}_0\int^t {}_0\int^t h_2(t-\tau_1, t-\tau_2)\ r(\tau_1)r(\tau_2)d\tau_1 d\tau_2 \tag{3.41}$$

$$= -n_2\ {}_0\int^t g(t-\tau)d\tau\ \left({}_0\int^\tau f(t-\tau_1)\ r(\tau_1)d\tau_1\right)^2 \tag{3.42}$$

$$= -n_2\ {}_0\int^t {}_0\int^\tau {}_0\int^\tau g(t-\tau)\ f(\tau-\tau_1)f(\tau-\tau_2)r(\tau_1)$$

$$r(\tau_2)d\tau_1 d\tau_2 d\tau$$

Now using equation (3.39) the integrals can be expanded

$$\underline{H}_2(r) = -n_2\ {}_0\int^t {}_0\int^t {}_0\int^t g(t-\tau)f(\tau-\tau_1)f(\tau-\tau_2)d\tau\ r(\tau_1)r(\tau_2)d\tau_1 d\tau_2 \tag{3.43}$$

Thus it can be seen that

$$h_2(t-\tau_1,\ t-\tau_2) = -n_2\ {}_0\int^t g(t-\tau)f(\tau-\tau_1)f(\tau-\tau_2)d\tau \tag{3.44}$$

and similar integrations can be used to evaluate the higher order kernels. However, as in the linear case, the integrations become somewhat easier if they are done in a transform domain.

The Laplace transform in the linear case changes convolution integrals into algebraic multiplications. For the nonlinear case, multidimensional transforms have been introduced [11], which allow the nth order convolution to become an algebraic multiplication. These multidimensional transforms are defined in a similar way to the Laplace transform [8]. If $f(t_1,\ldots,t_n)$ is an nth order kernel or function and $F(s_1,\ldots,s_n)$ is its multidimensional transform, then

$$F(s_1,s_2,\ldots,s_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(t_1,\ldots,t_n)e^{-s_1t_1\cdots-s_nt_n}dt_1\ldots dt_n \tag{3.45}$$

$$f(t_1,\ldots t_n) = \left(\frac{1}{2\pi j}\right)^n \int_{\sigma_1-j\infty}^{\sigma_1+j\infty}\cdots\int_{\sigma_n-j\infty}^{\sigma_n+j\infty} F(s_1,\ldots,s_n)e^{s_1t_1+\cdots+s_nt_n}ds_1\ldots ds_n \tag{3.46}$$

$$j = \sqrt{-1}$$

Many of the properties of the Laplace transform have similar properties in the multidimensional transforms. A few of the more useful properties to the present application will be listed.

<u>Convolution</u>: In a linear system, convolution in the time domain corresponds to multiplication in the transform domain. This also holds in the case of a multidimensional convolution. Such a convolution would be given by

$$\hat{x}(t_1,\ldots,t_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(t_1-\tau_1,\ldots,t_n-\tau_n)r(\tau_1)\ldots r(\tau_n)d\tau_1\ldots d\tau_n \tag{3.47}$$

For this case it can easily be seen by taking transforms that

$$\hat{X}(s_1,\ldots,s_n) = H_n(s_1,\ldots,s_n)\, R(s_1)\ldots R(s_n) \tag{3.48}$$

This is not quite what is required as x should really be a function only of t but this can be accomplished in equation (3.47) by letting

$$t_1 = t_2 = t_3 = \ldots\ldots = t_n = t \tag{3.49}$$

This is known as association of variables, and for rational transform functions it is possible to perform the operation in the transform domain by inspection, with the help of a table of examples [11][25].

<u>Cascades of Systems</u>: Two types of cascades will be used:

a) A linear system $L_1(s)$ followed by an n-dimensional system with transform $H_n(s_1,\ldots,s_n)$. The transform of the cascade is

$$\underline{H}_n * \underline{L}_1 \leftrightarrow H_n(s_1,\ldots\ldots,s_n) L_1(s_1)\ldots\ldots L_1(s_n) \tag{3.50}$$

b) An n-dimensional system $H_n(s_1,\ldots\ldots,s_n)$ followed by a linear system $L_1(s)$.

$$\underline{L}_1 * \underline{H}_n \leftrightarrow L_1(s_1 + s_2 + \ldots\ldots + s_n)\; H_n(s_1,\ldots,s_n) \tag{3.51}$$

Now consider equations (3.23) again

$$\underline{H}_1 = \underline{L}_a$$

$$\underline{H}_j = - \sum_{k=2}^{\min(M,j)} n_k\, \underline{L} * \underline{R}^{(k)}_{j-k+1} \tag{3.52}$$

Let the transforms of $\underline{L}_a$ and $\underline{L}$ be $L_a(s)$ and $L(s)$ respectively. Let the transform of $\underline{H}_j$ be $H_j(s_1,\ldots,s_j)$. Then it follows from the above that

$$H_1(s) = L_a(s) \tag{3.53}$$

$$H_2(s_1,s_2) = -n_2\, L(s_1+s_2)\, L_a(s_1)\, L_a(s_2) \tag{3.54}$$

$$H_3(s_1,s_2,s_3) = -2n_2\, L(s_1+s_2+s_3)H_1(s_1)H_2(s_2,s_3) - n_3L(s_1+s_2+s_3)H_1(s_1)H_1(s_2)H_1(s_3) \tag{3.55}$$

$$= L(s_1+s_2+s_3)\left(2n_2^2\, L_a(s_1)L(s_2+s_3)L_a(s_2)L_a(s_3) - n_3L_a(s_1)L_a(s_2)L_a(s_3)\right)$$

$$H_3(s_1,s_2,s_3) = L(s_1+s_2+s_3)L_a(s_1)L_a(s_2)L_a(s_3)\left(2n_2^{\,2}L(s_2+s_3)-n_3\right) \tag{3.56}$$

The expressions become quite lengthy as the order of the kernel increases although the basic simplicity remains. However, the association of variables where required, although only needing simple steps, becomes quite cumbersome.

Once the transforms of the kernels are calculated then many types of inputs can be treated including classes of random inputs [25]. This method is a generalization of the method of Poincare(see [12] p.198) and also that used by Doetsch [8], in which the solution is found for a particular input. For sinusoidal inputs the transforms of the kernels can be used to give the steady state sinusoidal output of the system and the harmonics [11]. This can be useful for distortion analysis of systems. Obtaining the response to a step input of some magnitude can give the speed of response of the system and the series is quite accurate for the first transient response of the system.

3.6 Example of Volterra Series Calculation

Consider the general system of Fig. 2.3 with zero initial conditions and

$$\underline{L}_b = \underline{I} \tag{3.57}$$

$$\underline{N}(x) = \varepsilon(x)^3 \tag{3.58}$$

$$\underline{L}_a(x)' = \underline{L}(x) = {}_o\int^t e^{-a(t-\tau)}x(\tau)d\tau \tag{3.59}$$

Then the operator equation of the system becomes

$$x(t) = \underline{L}(r) - \varepsilon\underline{L}(x^3) \tag{3.60}$$

Now using equations (3.57), (3.58), (3.59)with equations (3.23) gives

$$\underline{H}_1 = \underline{L}$$

$$\underline{H}_j = -\varepsilon\ \underline{L}*\underline{H}^{(3)}_{j-2} \qquad j\geq 2 \tag{3.61}$$

Now assuming that

$$\underline{H}_o = \underline{0}$$

and grouping like terms together for convenience it follows that

$$\underline{H}_2 = \underline{0} \tag{3.62}$$

$$\underline{H}_3 = -\varepsilon\underline{L}*\underline{H}_1^{(3)} = -\varepsilon\underline{L}*(\underline{H}_1)^3 = -\varepsilon\underline{L}*(\underline{L})^3 \tag{3.63}$$

$$\underline{H}_4 = \underline{0} \tag{3.64}$$

$$\underline{H}_5 = -\varepsilon\underline{L}*\underline{H}_3^{(3)} = -\varepsilon\underline{L}*3(\underline{H}_1)^2\underline{H}_3$$

$$= -\varepsilon\underline{L}*3(\underline{L})^2(-\varepsilon)\underline{L}*(\underline{L})^3$$

$$\underline{H}_5 = 3\varepsilon^2\underline{L}*[(\underline{L})^2(\underline{L}*(\underline{L})^3)] \tag{3.65}$$

$$\underline{H}_6 = \underline{0} \tag{3.66}$$

$$\underline{H}_7 = -\ \varepsilon\underline{L}*\underline{H}_5^{(3)} = -\ \varepsilon\underline{L}*[3(\underline{H}_1)^2\underline{H}_5 + 3\underline{H}_1(\underline{H}_3)^2]$$

$$\underline{H}_7 = -3\ \varepsilon^3\underline{L}*\{3(\underline{L})^2(\underline{L}*[(\underline{L})^2(\underline{L}*(\underline{L})^3)]) + \underline{L}[\underline{L}*(\underline{L})^3]^2\} \tag{3.67}$$

Expressing these operators in their transforms gives

$$H_1(s) = \frac{1}{s+a} \tag{3.68}$$

$$H_{2n}(s_1,\ldots,s_{2n}) = 0 \tag{3.69}$$

$$H_3(s_1,s_2,s_3) = -\varepsilon \frac{1}{s_1+s_2+s_3+a} \quad \frac{1}{(s_1+a)(s_2+a)(s_3+a)} \tag{3.70}$$

$$H_5(s_1,s_2,s_3,s_4,s_5) = \frac{3\varepsilon^2}{(s_1+s_2+s_3+s_4+s_5+a)}\left\{\frac{1}{(s_1+a)(s_2+a)(s_3+s_4+s_5+a)}\right.$$

$$\left.\frac{1}{(s_3+a)(s_4+a)(s_5+a)}\right\}$$

$$= \frac{3\ \varepsilon^2}{((\sum_{i=1}^{5} s_i)+a)} \left\{ \prod_{i=1}^{5} \frac{1}{(s_i+a)}\right\} \frac{1}{(s_3+s_4+s_5+a)} \tag{3.71}$$

$$H_7(s_1,\ldots,s_7) = \frac{-3\varepsilon^3}{((\sum_{i=1}^{7} s_i)+a)} \left\{ \prod_{i=1}^{7} \frac{1}{(s_i+a)}\right\} \left(\frac{3}{(s_3+s_4+s_5+s_6+s_7+a)(s_5+s_6+s_7+a)} + \right.$$

$$\left. \frac{1}{(s_2+s_3+s_4+a)(s_5+s_6+s_7+a)} \right) \tag{3.72}$$

Let $r(t)$ be a step function, and let $\hat{x}_n(t_1,\ldots,t_n)$ be the inverse of the transform $\hat{X}_n(s_1,\ldots,s_n)$. Then

$$r(t) = k\ u(t) \tag{3.73}$$

$$R(s) = \frac{k}{s} \tag{3.74}$$

$$\hat{X}_n(s_1,\ldots,s_n) = H_n(s_1,\ldots,s_n)R(s_1)\ldots R(s_n) \tag{3.75}$$

Now using equations (3.55) to (3.59)

$$\hat{X}_1(s) = \frac{k}{s(s+a)} = \frac{k}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right) \tag{3.76}$$

$$\hat{x}_1(t) = \frac{k}{a}\left(u(t) - e^{-at}\right) \tag{3.77}$$

This is the linear approximation to the system.

$$\hat{X}_3(s_1,s_2,s_3) = -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3 \frac{a}{(s_1+s_2+s_3+a)}\left(\frac{1}{s_1} - \frac{1}{s_1+a}\right)\left(\frac{1}{s_2} - \frac{1}{s_2+a}\right)\left(\frac{1}{s_3} - \frac{1}{s_3+a}\right)$$

$$= -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3\left(\frac{a}{s_1+s_2+s_3+a}\right)\left[\frac{1}{s_1s_2} - \frac{1}{s_1(s_2+a)} - \frac{1}{s_2(s_1+a)} + \frac{1}{(s_1+a)(s_2+a)}\right]\left(\frac{1}{s_3} - \frac{1}{s_3+a}\right) \tag{3.78}$$

Now using the inspection technique to associate variables, first s_1 and s_2 are associated.

$$\hat{X}_3(s_1,s_2,s_3)\Big|_{s_1 Ⓐ s_2} = -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3\left(\frac{a}{s_1+s_3+a}\right)\left(\frac{1}{s_1} - \frac{2}{s_1+a} + \frac{1}{s_1+2a}\right)\left(\frac{1}{s_3} - \frac{1}{s_3+a}\right) \tag{3.79}$$

Associating s_1 and s_3 now gives

$$\hat{X}_3(s_1,s_3)\,|\,s_1 \textcircled{A} s_3 = -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3\left(\frac{a}{s_1+a}\right)\left(\frac{1}{s_1} - \frac{3}{s_1+a} + \frac{3}{s_1+2a} - \frac{1}{s_1+3a}\right) \tag{3.80}$$

$$\hat{X}_3(s) = -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3\left(\frac{1}{s} + \frac{3/2}{s+a} - \frac{3a}{(s+a)^2} - \frac{3}{s+2a} + \frac{1/2}{s+3a}\right) \tag{3.81}$$

$$\hat{x}_3(t) = -\frac{\varepsilon}{a}\left(\frac{k}{a}\right)^3\left(u(t) + e^{-at}\left(\frac{3}{2} - 3at\right) - 3e^{-2at} + \frac{1}{2}e^{-3at}\right) \tag{3.82}$$

In a similar fashion,

$$\hat{X}_5(s_1,\ldots,s_5) = 3\left(\frac{\varepsilon}{a}\right)^2\left(\frac{k}{a}\right)^5\left(\frac{a}{\left(\sum_{i=1}^{5} s_i\right)+a}\right)\left(\frac{a}{s_3+s_4+s_5+a}\right)\prod_{i=1}^{5}\left(\frac{1}{s_i} - \frac{1}{s_i+a}\right) \tag{3.83}$$

and associating variables gives

$$\hat{X}_5(s) = 3\left(\frac{\varepsilon}{a}\right)^2\left(\frac{k}{a}\right)^5\frac{a}{s+a}\left\{\frac{1}{s} - \frac{1/2}{s+a} - \frac{3a}{(s+a)^2} - \frac{5}{s+2a} + \frac{6a}{(s+2a)^2} + \frac{8}{s+3a} - \frac{3a}{(s+3a)^2} - \frac{4}{s+4a} + \frac{1/2}{s+5a}\right\} \tag{3.84}$$

$$\hat{x}_5(t) = \left(\frac{\varepsilon}{a}\right)^2\left(\frac{k}{a}\right)^5\{3u(t) + e^{-at}[6.125 - 1.5at - 4.5(at)^2] - e^{-2at}[3+18at] - e^{-3at}[9.75 - 4.5at] + 4e^{-4at} - 0.375e^{-5at}\} \tag{3.85}$$

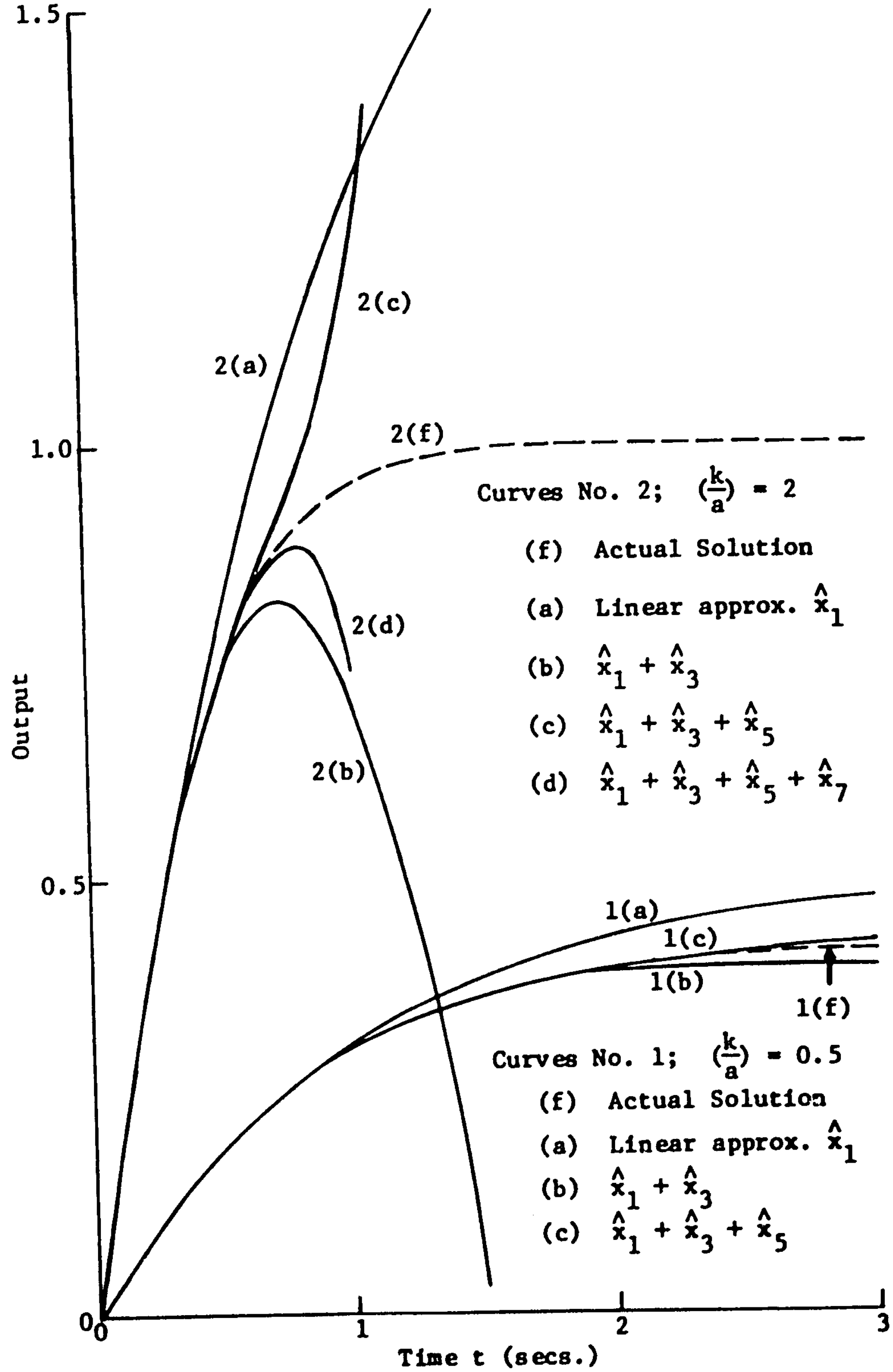

Fig 3.1 Approximations to System

and similarly for $\hat{x}_7$ except that here the expression must be split into a sum of two simpler parts in order that the association of variables may be carried out by inspection.

$$\hat{x}_7(t) = -(\frac{\varepsilon}{a})^3(\frac{k}{a})^7\{12u(t)+e^{-at}[26.7375+6.375at-11.25(at)^2 -4.5(at)^3] +e^{-2at}[3-54at-54(at)^2]-e^{-3at}[36.5625+ 78.75at-20.25(at)^2]-e^{-4at}[23-48at]+e^{-5at}[22.3125 -5.625at]-4.8e^{-6at}+.3125e^{-7at}\} \quad (3.86)$$

These results are plotted in Fig. 3.1 for the following parameters along with the exact solution

$$a = (\frac{\varepsilon}{a}) = 1 \quad (3.87)$$

$$(\frac{k}{a}) = 0.5 \text{ and } 2 \quad (3.88)$$

It is seen that there is good convergence for small time intervals and the larger number of terms used the better the convergence. However for the $(\frac{k}{a} = 2)$ curve the solutions diverge wildly past a certain time while for the other curve convergence appears to be for all time.

If the derivatives with respect to time are removed from the system differential equation, a nonlinear algebraic equation is obtained. Attempting to solve this algebraic equation by an iterative process such as Newton's method, will fail for larger $(\frac{k}{a})$ (such as $\frac{k}{a} = 2$), and the successive iterates will diverge in much the same way as the solutions here have. This explains why the convergence at steady state is so poor in these types of examples.

CHAPTER IV

A LARGER REGION OF CONTRACTION

4.1 Introduction

One of the major problems in the analysis of nonlinear systems is to obtain realistic bounds on the nonlinearity to insure that the system will be stable. Or the other way of tackling the problem is to find or modify the linear part of the system to give a stable system with the given nonlinearity [30], [31]. Each of these approaches has been used and the one to be investigated here is the former.

A method is presented which gives good results for first and second order systems (meaning systems whose linear plant can be represented by first or second order differential equations), but does not appear to work for higher order systems. For these systems another norm is introduced which then gives the required contraction but does not allow any conclusions to be made about the stability of the system. However it does put a bound on the output for a specified time interval and so implies convergence of the Volterra series over that time interval.

4.2 First Order Systems

Consider the system of equation (3.2) where

$$L(s) = \frac{b}{s+a} \tag{4.1}$$

$$\alpha < \frac{d\underline{N}(x)}{dx} < \beta \qquad \text{for all } x \in C[0,\infty) \tag{4.2}$$

where α and β are real constant to be determined such that the sector $[\alpha,\beta]$ is the largest possible for stability of the system. The system equation is

$$x(t) = \underline{L}_a(r) - \underline{L}*\underline{N}(x) \tag{4.3}$$

$$\underline{L}_a*\underline{L}_b = \underline{L} \tag{4.4}$$

In order to keep the system and analysis as general as possible let

$$\underline{L}_a(r) = w(t) \tag{4.5}$$

$$x(t) = w(t) - \underline{L}*\underline{N}(x) \tag{4.6}$$

Now expressing equation (4.6) in its differential form using equation (4.1) gives

$$\dot{x} + ax = \dot{w} + aw - b\underline{N}(x) \tag{4.7}$$

Add bux to each side of equation (4.7) where u is a real number.

$$\dot{x} + (a+bu)x = \dot{w} + aw + b(ux - \underline{N}(x)) \tag{4.8}$$

Now returning to the integral operator form of equation (4.8),

$$x(t) = \underline{H}(x) = w(t) - u\underline{K}(w) + \underline{K}*(ux-\underline{N}(x)) \tag{4.9}$$

where

$$K(s) = \frac{b}{s+a+bu} \tag{4.10}$$

The Lipschitz constant of $\underline{H}$ as defined in equation (4.9), is obtained as follows.

$$\|\underline{H}(x_1)-\underline{H}(x_2)\| = \|\underline{K}*(u(x_1-x_2)-(x_1-x_2)\frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2})\|$$

$$\text{for all } x_1 \neq x_2 \in C[0,\infty) \qquad (4.11)$$

$$\|\underline{H}(x_1)-\underline{H}(x_2)\| \leq \|\underline{K}\| \, \|x_1-x_2\| \sup_{x_1 \neq x_2 \in C[0,\infty)} \left| u- \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \right| \qquad (4.12)$$

$\|\underline{K}\|$ can easily be evaluated.

$$\|\underline{K}\| = {}_0\int^\infty b \; e^{-(a+bu)t} dt = \frac{b}{a+bu} \qquad (4.13)$$

From equations (4.12) and (4.13), there exists a contraction if,

$$\frac{b}{a+bu} \sup_{x_1 \neq x_2 \in C[0,\infty)} \left| u- \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \right| \leq \gamma < 1 \qquad (4.14)$$

So, by removing the absolute value signs, equation (4.14) becomes,

$$-\gamma u -\gamma\frac{a}{b} \leq -u + \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \leq \gamma u + \gamma\frac{a}{b} \qquad ; \qquad x_1 \neq x_2 \in C[0,\infty)$$

$$-\gamma\frac{a}{b} + u(1-\gamma) \leq \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \leq \gamma\frac{a}{b} + u(1+\gamma) \qquad (4.15)$$

If $\underline{N}(x)$ is assumed to be differentiable, and letting $\gamma \to 1$ with the consequential removal of the equality signs, equation (4.15) becomes,

$$-\frac{a}{b} < \frac{d\underline{N}(x)}{dx} < 2u + \frac{a}{b} \qquad (4.16)$$

Since u can be made as large as is needed, the following condition is obtained.

$$-\frac{a}{b} < \frac{d\underline{N}(x)}{dx} \leq Q < \infty \qquad \forall\ x \epsilon C[0,\infty) \tag{4.17}$$

For this system, these limits $[-\frac{a}{b}, \infty)$ define the Hurwitz sector, over which the system with $\underline{N}(x)$ being a linear gain, is asymptotically stable. Since the linear case is to be included within the general nonlinearity, then this sector is the maximum possible for stability of the system.

Now consider a bound on the output of the nonlinear system.

$$x(t) = (\underline{I} - u\underline{K})w(t) + \underline{K}*(ux - \underline{N}(x)) \tag{4.18}$$

Taking norms on both sides and rearranging gives,

$$||x|| \leq ||\underline{I}-u\underline{K}||\ ||w|| + ||\underline{K}|| \sup_{x \neq 0 \epsilon C[0,\infty)} \left| u - \frac{\underline{N}(x)}{x} \right| ||x|| \tag{4.19}$$

It has been shown that for $\underline{N}$ obeying equation (4.17), and for sufficiently large u, then

$$||\underline{K}|| \sup_{x_1 \neq x_2 \epsilon C[0,\infty)} \left| u - \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \right| \leq \gamma < 1 \tag{4.20}$$

and along with the assumption that

$$\underline{N}(0) = 0 \tag{4.21}$$

these imply that

$$||\underline{K}|| \sup_{x_1 \neq 0 \epsilon C[0,\infty)} \left| u - \frac{\underline{N}(x_1)}{x_1} \right| \leq ||\underline{K}|| \sup_{x_1 \neq x_2 \epsilon C[0,\infty)} \left| u - \frac{\underline{N}(x_1)-\underline{N}(x_2)}{x_1-x_2} \right| \leq \gamma < 1 \tag{4.22}$$

Substituting equation (4.22) into equation (4.19)

$$\| x \| \leq \| \underline{I} - u\underline{K} \| \| w \| + \gamma \| x \|$$

$$\| x \| \leq (1-\gamma)^{-1} \| \underline{I} - u\underline{K} \| \| w \| \tag{4.23}$$

$$\leq (1-\gamma)^{-1} \| \underline{I} - u\underline{K} \| \| \underline{L}_a \| \| r \| \tag{4.24}$$

$$\| I - u\underline{K} \| \leq 1 + ub \int_0^{\infty} e^{-(a+ub)t} dt = 1 + \frac{ub}{a+ub}$$

$$\leq 2 \tag{4.25}$$

$$\| \underline{L}_a \| \leq \max (1, a^{-1}, a^{-1}b, b) < \infty \tag{4.26}$$

by the definition of the system.

Clearly then equation (4.25) implies bounded-input bounded-output stability for all first order systems with a single nonlinearity which obeys

$$\underline{N}(0) = 0$$

$$-\frac{a}{b} < \frac{d\underline{N}(x)}{dx} \leq Q < \infty \quad ; \quad \forall\, x \in C\,[0,\infty) \tag{4.27}$$

4.3 Second Order Systems

Proposition: Consider the systems which can be described by the operator equation

$$x = \underline{L}_a(r) - \underline{L} * \underline{N}(x) \tag{4.28}$$

where the Laplace transform of $\underline{L}$ is given by

$$L(s) = \frac{b_1 s+b_2}{s^2+a_1 s+a_2} \quad (4.29)$$

and $\underline{L}_a$ is a linear plant of order less than or equal to 2 such that

$$\| L_a \| < \infty \quad (4.30)$$

and nonlinearity $\underline{N}(x)$ is defined by

$$\underline{N}(x) = \sum_{j=2}^{M} n_j (x)^j \quad (4.31)$$

$$\alpha < \frac{d\underline{N}x}{dx} = \sum_{j=2}^{M} j n_j (x)^{j-1} < \beta \quad (4.32)$$

where $[\alpha,\beta]$ is the Hurwitz sector for the linear system with $\underline{N}(x)$ replaced by a linear gain; then equation (4.28) has a unique solution and is bounded-input bounded-output stable.

The Hurwitz sector for second order systems, with all the coefficients in equation (4.29) positive, is the sector $[\alpha,\infty)$, where

$$\alpha = -\min \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \quad (4.33)$$

If some of the coefficients in equation (4.29) are negative, then the β of equation (4.32) may be constrained to be less than infinity.

In the proof of the proposition, complex valued functions will be introduced. Their absolute value will be taken to be the square root of the sum of the squares of the real and imaginary parts of the function.

<u>Proof</u>: Define w(t) to be

$$w(t) = \underline{L}_a(r) \tag{4.34}$$

Then equation (4.28) becomes

$$x(t) = w(t) - \underline{L}*\underline{N}(x) \tag{4.35}$$

and expressing this equation in its differential form gives,

$$\ddot{x}+a_1\dot{x}+a_2x = \ddot{w}+a_1\dot{w}+a_2w - (b_1 \frac{d}{dt} +b_2)\underline{N}(x) \tag{4.36}$$

Now add to both sides $(2c-a_1)\dot{x}+(2c^2-a_2)x$, where c is a real positive number.

$$\ddot{x}+2c\dot{x}+2c^2x = \ddot{w}+a_1\dot{w}+a_2w + \frac{d}{dt} [(2c-a_1)x-b_1\underline{N}(x)] + (2c^2-a_2) x - b_2\underline{N}(x) \tag{4.37}$$

Define two new variables.

$$x_1 = x-w$$

$$x_2 = \dot{x}_1 + (2c-a_1)w - [(2c-a_1)x-b_1\underline{N}(x)] \tag{4.39}$$

then the following matrix equation can be written.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2c^2 & -2c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ 2cb_1-b_2 \end{pmatrix} \underline{N}(x) + \begin{pmatrix} 2c-a_1 \\ 2ca_1-2c^2-a_2 \end{pmatrix} (x-w) \tag{4.40}$$

The system matrix can be diagonalized using the Van der Monde matrix.

The eigenvalues of the matrix are $-c(1\pm j)$, where $j=\sqrt{-1}$.

$$P = \begin{pmatrix} 1 & 1 \\ -c(1-j) & -c(1+j) \end{pmatrix} \qquad (4.41)$$

$$P^{-1} = \begin{pmatrix} \frac{1}{2}(1-j) & -\frac{j}{2c} \\ \frac{1}{2}(1+j) & \frac{j}{2c} \end{pmatrix} \qquad (4.42)$$

Now transform equation (4.40), using

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad (4.43)$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -c(1-j) & 0 \\ 0 & -c(1+j) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}(b_1-j([b_2/c]-b_1)) \\ \frac{1}{2}(b_1+j([b_2/c]-b_1)) \end{pmatrix}\underline{N}(x) + \begin{pmatrix} \frac{1}{2}(2c-a_1-j(a_1-[a_2/c])) \\ \frac{1}{2}(2c-a_1+j(a_1-[a_2/c])) \end{pmatrix}(x-w) \qquad (4.44)$$

and from equations (4.34), (4.41), and (4.38),

$$x_1 = x-w = z_1+z_2 \qquad (4.45)$$

Substituting equation (4.45) into the expression for $\frac{d\underline{N}(x)}{dx}$ given in equation (4.32),

$$\frac{d\underline{N}(x)}{dx} = \sum_{j=2}^{M} jn_j(z_1+z_2+w)^{j-1} \qquad (4.46)$$

It can easily be seen that

$$\left.\frac{d\underline{N}(x)}{dz_1}\right|_{\substack{z_2\\ \text{const.}}} = \left.\frac{d\underline{N}(x)}{dz_2}\right|_{\substack{z_1\\ \text{const.}}} = \frac{d\underline{N}(x)}{dx} \tag{4.47}$$

Consider equation (4.44) broken into its two separate equations, and consider the equation in z_1 first.

$$\dot{z}_1 = \tfrac{1}{2}\big(2c-a_1-j(a_1-[a_2/c])\big)(z_2-w)+\tfrac{1}{2}\big(-a_1-j(a_1-2c-[a_2/c])\big)z_1-$$

$$\tfrac{1}{2}\big(b_1-j([b_2/c]-b_1)\big)\underline{N}(x) \tag{4.48}$$

Now define real constants λ_1, λ_2, λ_3, λ_4, and add $(e-jf)z_1$ to each side of equation (4.48), where e and f are real numbers, such that

$$\dot{z}_1+(e-jf)z_1 = \tfrac{1}{2}\big(2c-a_1-j(a_1-[a_2/c])\big)(z_2-w) + \big(e-\lambda_1-j(f-\lambda_2)\big)z_1 -$$

$$(\lambda_3-j\lambda_4)\underline{N}(x) \tag{4.49}$$

and putting equation (4.49) into operator notation gives,

$$z_1 = \underline{H}(z_1) = \tfrac{1}{2}\big(2c-a_1-j(a_1-[a_2/c])\big)\underline{K}(z_2-w) + \underline{K}\big([e-\lambda_1-j(f-\lambda_2)]$$

$$z_1-(\lambda_3-j\lambda_4)\underline{N}(x)\big) \tag{4.50}$$

Finding the Lipschitz constant of the operator $\underline{H}$ gives,

$$\|\underline{H}(z_{11})-\underline{H}(z_{12})\| \le \|\underline{K}\|\ \|z_{11}-z_{12}\| \sup_{z_1\in C^*[0,\infty)} \Big| e-\lambda_1-\lambda_3\frac{d\underline{N}(x)}{dz_1} -$$

$$j\Big(f-\lambda_2-\lambda_4\frac{d\underline{N}(x)}{dz_1}\Big)\Big| \tag{4.51}$$

where $C^*[0,\infty)$ is the space of continuous complex functions. Then using equation (4.47) and noting that $\frac{d\underline{N}(x)}{dz_1}$ is a real function as well as the following

$$\|\underline{K}\| = {}_0\int^\infty |\exp(-e+jf)t|\,dt = {}_0\int^\infty \exp(-et)dt = \frac{1}{e} \tag{4.52}$$

gives

$$\|\underline{H}(z_{11}) - \underline{H}(z_{12})\| = \frac{1}{e} \sup_{z_1} \left((e-\lambda_1-\lambda_3 \frac{d\underline{N}(x)}{dx})^2 + (f-\lambda_2-\lambda_4 \frac{d\underline{N}(x)}{dx})^2\right)^{\frac{1}{2}} \|z_{11} - z_{12}\| \tag{4.53}$$

For contraction to hold for $\underline{H}$, it is required that

$$\frac{1}{e} \sup_{z_1} \left((e-\lambda_1-\lambda_3 \frac{d\underline{N}(x)}{dx})^2 + (f-\lambda_2-\lambda_4 \frac{d\underline{N}(x)}{dx})^2\right)^{\frac{1}{2}} \leq \gamma < 1 \tag{4.54}$$

and this implies that

$$(\frac{d\underline{N}(x)}{dx})^2 (\lambda_3^2+\lambda_4^2) - 2(\frac{d\underline{N}(x)}{dx}) (\lambda_3(e-\lambda_1)+\lambda_4(f-\lambda_2)) \leq \gamma^2 e^2-(e-\lambda_1)^2-(f-\lambda_2)^2$$

$$\text{for all } z_1 \in C^*[0,\infty) \tag{4.55}$$

Completing the square and taking the square root and rearranging gives

$$\lambda_3(e-\lambda_1)+\lambda_4(f-\lambda_2)-\left((\lambda_3^2+\lambda_4^2)(\gamma^2e^2-(e-\lambda_1)^2-(f-\lambda_2)^2)+(\lambda_3(e-\lambda_1)+\lambda_4(f-\lambda_2))^2\right)^{\frac{1}{2}}$$

$$\leq (\lambda_3^2+\lambda_4^2) \frac{d\underline{N}(x)}{dx} \leq \lambda_3(e-\lambda_1) + \lambda_4(f-\lambda_2) + \left((\lambda_3^2+\lambda_4^2)(\gamma^2e^2-(e-\lambda_1)^2-\right.$$

$$\left.(f-\lambda_2)^2) + (\lambda_3(e-\lambda_1) + \lambda_4(f-\lambda_2))^2\right)^{\frac{1}{2}} \tag{4.56}$$

Letting $\gamma \to 1$ and removing equalities makes equation (4.56) of the form

$$\alpha < \frac{d\underline{N}(x)}{dx} < \beta \tag{4.57}$$

Now if $\underline{N}(x)$ were a linear gain, then for stability of the linear system so formed, it would be required that

$$\alpha = -\min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \tag{4.58}$$

and substituting this in equation (4.56) gives

$$\beta = -\alpha + \frac{f4cb_2+(e-f)4c^2b_1+2(b_2-b_1c)(a_1c-a_2-2c^2)-2a_1b_1c^2}{2b_1^2c^2+b_2^2-2b_1b_2c} \tag{4.59}$$

$$e(a_1+\alpha b_1) = \left(f-c+\frac{1}{2}\left(a_1+\alpha b_1-\alpha\frac{b_2}{c}-\frac{a_2}{c}\right)\right)^2+\frac{1}{4}\left(\alpha b_1+a_1\right)^2 \tag{4.60}$$

The equation in z_2 is just the complex conjugate of that in z_1 and so can be treated in like manner to give the same conditions; namely that contraction holds for all $\underline{N}(x)$ obeying condition (4.57) with α and β defined in equations (4.58), (4.59) and (4.60). So by the contraction mapping theorem there exist unique functions z_1^* and z_2^* satisfying equation (4.44) and so there exists a unique $x(t)$ defined by equation (4.45) which is a solution of equation (4.28).

Define

$$\underline{N}'(z_1) = \left(e-\lambda_1-j(f-\lambda_2)\right)z_1-(\lambda_3-j\lambda_4)\,\underline{N}(x) \tag{4.61}$$

and then equation (4.50) becomes

$$z_1 = \underline{H}z_1 = v_1\,\underline{K}\,(z_2-w) + \underline{K}*\underline{N}'(z_1) \tag{4.62}$$

In a similar way

$$z_2 = \underline{H}'z_2 = v_2\,\underline{K}'\,(z_1-w) + \underline{K}'*\underline{N}''(z_2) \tag{4.63}$$

where v_1 and v_2 are complex numbers. From the previous results on the contraction it is known that

$$\|\,\underline{K}*\underline{N}'\,\| \leq \text{inc}\,\|\,\underline{K}*\underline{N}'\,\| \leq \gamma_1 < 1 \tag{4.64}$$

$$\|\,\underline{K}'*\underline{N}''\| \leq \text{inc}\,\|\,\underline{K}'*\underline{N}''\| \leq \gamma_2 < 1 \tag{4.65}$$

Taking norms in equation (4.62)

$$\|\,z_1\,\| \leq |v_1|\,\|\underline{K}\|\;\|\,z_2-w\,\| + \|\,\underline{K}*\underline{N}'\|\;\|\,z_1\| \tag{4.66}$$

$$\|\,z_1\,\| \leq |v_1|\,\|\underline{K}\|\;(1-\gamma_1)^{-1}(\,\|z_2\| + \|\,w\,\|\,) \tag{4.67}$$

Similarly from equation (4.63)

$$\|\,z_2\,\| \leq |v_2|\,\|\underline{K}'\|\,(1-\gamma_2)^{-1}\,(\,\|z_1\| + \|\,w\|\,) \tag{4.68}$$

Now substituting equation (4.68) into equation (4.67)

$$\|z_1\| \leq |v_1|\,\|\underline{K}\|\;(1-\gamma_1)^{-1}\,\|\,w\|\;(1 + |v_2|\;\|\,\underline{K}'\|\,(1-\gamma_2)^{-1}) +$$

$$|v_1\,v_2|\,\|\underline{K}\|\,\|\underline{K}'\|\,(1-\gamma_1)^{-1}(1-\gamma_2)^{-1}\,\|\,z_1\| \tag{4.69}$$

Using equation (4.52) and also

$$\|\underline{K}'\| = {}_0\int^{\infty} |\exp(-et-jft)|dt = {}_0\int^{\infty} \exp(-et)dt = \frac{1}{e} \qquad (4.70)$$

gives

$$\|z_1\| \left(1-|v_1 v_2| e^{-2}[(1-\gamma_1)(1-\gamma_2)]^{-1}\right) \leq D\|w\| \qquad (4.71)$$

where D is a finite positive number. Now e can be made as large as necessary, so that there exists an E, such that e≥E implies;

$$1-|v_1 v_2|[(1-\gamma_1)(1-\gamma_2)]^{-1} e^{-2} \geq \delta^{-1} > 0 \qquad (4.72)$$

Thus equation (4.71) and equation (4.72) give

$$\|z_1\| \leq \delta D \|w\| \qquad (4.73)$$

and putting equation (4.73) into equation (4.68) gives

$$\|z_2\| \leq |v_2|[e(1-\gamma_2)]^{-1}(1+\delta D) \|w\| \qquad (4.74)$$

Equations (4.73) and (4.74) imply bounded-input bounded-output stability for the system described in the proposition. It can be seen from equations (4.59) and (4.60) that the sector [α,β] can be expanded to the Hurwitz sector by increasing e sufficiently.

Examples of Sector Calculations

a) $b_1=0 \quad \longrightarrow \quad \alpha= -\frac{a_2}{b_2}$

Let $c= \frac{a_1}{2}$, then

$$e= \frac{1}{a_1} f^2 + \frac{a_1}{4}$$

and this implies

$$-\frac{a_2}{b_2} < \frac{d\underline{N}(x)}{dx} < 2\frac{a_1}{b_2}f - \frac{a_2}{b_2} \tag{4.75}$$

and clearly as $e\to\infty$, so does $(2\frac{a_1}{b_2}f - \frac{a_2}{b_2})$

b) $b_2 = 0 \quad \to \quad \alpha = -\frac{a_1}{b_1}$

Let $c = \sqrt{\frac{a_2}{2}}$ then

$f = \sqrt{2a_2}$

and this implies

$$-\frac{a_1}{b_1} < \frac{d\underline{N}(x)}{dx} < \frac{2}{b_1}e - \frac{a_1}{b_1} \tag{4.76}$$

and this, as before, can be expanded to cover the Hurwitz sector.

4.4 Higher Order Systems [24]

The diagonalization procedure used for the second order systems, and which was a generalization of the method used for the first order systems, does not appear to give the larger region for third and higher order systems. For these cases it is necessary to introduce a new norm, the λ- norm, and the space $C_\lambda[a,b]$ of continuous functions on [a,b] will become the space $C_\lambda[0,T]$.

Consider the same system equation as before

$$x = \underline{L}_a(r) - \underline{L}*\underline{N}(x) \tag{4.77}$$

with

$$||\underline{L}_a|| < \infty \tag{4.78}$$

$$L(s) = \frac{1}{s^m + a_1 s^{m-1} + \ldots + a_m} \qquad m \geq 3 \tag{4.79}$$

$$\underline{N}(x) = \sum_{i=2}^{M} n_i (x)^i \tag{4.80}$$

$$\alpha < \frac{d\underline{N}(x)}{dx} = \sum_{i=2}^{M} n_i i (x)^{i-1} < \beta \tag{4.81}$$

$$x(t) = w(t) - \underline{L} * \underline{N}(x) \tag{4.82}$$

Add to each side of equation (4.82) $(c-a_m)\underline{L}(x)$, where c is a real positive number.

$$(\underline{I}+(c-a_m)\underline{L})x(t) = w(t) + \underline{L}((c-a_m)x-\underline{N}(x)) \tag{4.83}$$

$$x(t) = (\underline{I}+(c-a_m)\underline{L})^{-1}w(t) + (\underline{I}+(c-a_m)\underline{L})^{-1} * \underline{L}((c-a_m)x-\underline{N}(x)) \tag{4.84}$$

Now taking Laplace transforms of $(\underline{I}+(c-a_m)\underline{L})^{-1}$ gives,

$$(I+(c-a_m)L)^{-1}(s) = \frac{s^m + a_1 s^{m-1} + \ldots + a_m}{s^m + a_1 s^{m-1} + \ldots + c} \tag{4.85}$$

$$\rightarrow \qquad (I+(c-a_m)L)^{-1}(s)L(s) = \frac{1}{s^m + a_1 s^{m-1} + \ldots + c} = K(s) \tag{4.86}$$

$$x(t) = \underline{H}(x) = (\underline{I}+(c-a_m)\underline{L})^{-1}w(t) + \underline{K}((c-a_m)x-\underline{N}(x)) \tag{4.87}$$

The Lipschitz constant in the λ-norm is now to be determined.

$$|\underline{H}(x_1)-\underline{H}(x_2)| = |\underline{K}\{(c-a_m)(x_1-x_2) - [\underline{N}(x_1)-\underline{N}(x_2)]\}| \tag{4.88}$$

where $\underline{K}$ is an integral operator with kernel k(t), as given in equation (4.86).

$$|\underline{H}(x_1) - \underline{H}(x_2)| \le {}_o\int^t |k(t-\tau)||(c-a_m)(x_1-x_2)(\tau) - [\underline{N}(x_1)-\underline{N}(x_2)]|d\tau \quad (4.89)$$

$$\le \sup_{t\in[0,T]} |k(t)| {}_o\int^t \sup_{t\in[0,T]} |(c-a_m)(x_1-x_2) - [\underline{N}(x_1)- \underline{N}(x_2)]|d\tau \quad (4.90)$$

$$\le K_o\int^t \sup_{\tau\in[0,T]} |x_1-x_2|e^{-\lambda\tau}e^{\lambda\tau} \sup_{x\in C_\lambda[0,T]} \left|c-a_m- \frac{d\underline{N}(x)}{dx}\right| d\tau$$

$$\le K \|x_1-x_2\|_\lambda \sup_{x\in C_\lambda[0,T]} \left|c-a_m - \frac{d\underline{N}(x)}{dx}\right| \frac{e^{\lambda t}-1}{\lambda} \quad (4.91)$$

where K is defined to be $\sup_{t\in[0,T]} |k(t)|$. From equation (4.91), rearranging gives the following.

$$|\underline{H}(x_1) - \underline{H}(x_2)|e^{-\lambda t} \le K \lambda^{-1} \sup_{x\in C_\lambda[0,T]} \left|c-a_m- \frac{d\underline{N}(x)}{dx}\right| \|x_1-x_2\|_\lambda$$

$$\Rightarrow \quad \|\underline{H}(x_1) - \underline{H}(x_2)\|_\lambda \le K \lambda^{-1} \sup_{x\in C_\lambda[0,T]} \left|c-a_m- \frac{d\underline{N}(x)}{dx}\right| \|x_1-x_2\|_\lambda \quad (4.92)$$

For contraction to hold, the following condition must be satisfied.

$$K \lambda^{-1} \sup_{x\in C_\lambda[0,T]} \left|c-a_m- \frac{d\underline{N}(x)}{dx}\right| \le \gamma < 1 \quad (4.93)$$

$$\rightarrow \qquad c-a_m - \frac{\gamma\lambda}{K} \leq \frac{d\underline{N}(x)}{dx} \leq c-a_m + \frac{\gamma\lambda}{K} \tag{4.94}$$

Let $\gamma \rightarrow 1$ and remove the equality and set $\lambda = cK$ gives

$$-a_m < \frac{d\underline{N}(x)}{dx} < 2c-a_m \tag{4.95}$$

Now for the inverse operator $(\underline{I}+(c-a_m)\underline{L})^{-1}$ to exist, such that its norm is finite, c is restricted to a range of values such that the zeros of the polynomial $s^m+a_1s^{m-1}+\ldots+c$ are all in the left half of the s plane. It is found that for equation (4.95) to cover the Hurwitz sector, c does in fact lie in the range to give a bounded inverse operator.

Consider now a bound on the output

$$\| x \|_\lambda \leq \| (\underline{I} + (c-a_m)\,\underline{L})^{-1}\|_\lambda \| w \|_\lambda + \text{inc}\|\underline{H}\|_\lambda \| x \|_\lambda \tag{4.96}$$

It has been shown above that for $\underline{N}$ satisfying equation (4.95) then

$$\text{inc}\,\|\underline{H}\|_\lambda \leq \gamma < 1 \tag{4.97}$$

$$\rightarrow \qquad \| x\|_\lambda \leq (1-\gamma)^{-1} \| (\underline{I} + (c-a_m)\,\underline{L})^{-1}\|_\lambda \, \| w \|_\lambda \tag{4.98}$$

Equation (4.98) implies bounded-input bounded-output stability in the space $C_\lambda[0,T]$. However this does not mean that $|x(t)|$ is not an increasing function of time but that it is increasing no faster than $De^{\lambda\tau}$ where D is a constant.

By the contraction mapping theorem there exists a unique solution to the equation (4.77) in the space $C_\lambda[0,T]$. This solution can be obtained by the Volterra series which converges on the interval $[0,T]$.

4.5 Comments and Comparisons

All the previous results hold if the nonlinearity contains a linear term. Consider the nonlinearity to be of the form $n_1 x + \underline{N}(x)$ where $\underline{N}(x)$ is as defined previously. Then equation (4.3) becomes

$$x(t) = \underline{L}_a(r) - n_1\underline{L}(x) - \underline{L}*\underline{N}(x) \tag{4.99}$$

$$(\underline{I} + n_1\underline{L})x = \underline{L}_a(r) - \underline{L}*\underline{N}(x)$$

$$x(t) = (\underline{I} + n_1\underline{L})^{-1}*\underline{L}_a(r) - (\underline{I} + n_1\underline{L})^{-1}*\underline{L}*\underline{N}(x) \tag{4.100}$$

provided $(\underline{I} + n_1\underline{L})^{-1}$ exists. It can be seen that (4.100) is of the same form as equation (4.99) with $\underline{L}_a$ replaced by $(\underline{I} + n_1\underline{L})*\underline{L}_a$ and $\underline{L}$ replaced by $(\underline{I} + n_1\underline{L})^{-1}*\underline{L}$. As defined here, this transformation does not increase the order of the system.

Although the method used for higher order systems does not give stability of the output of the system, the systems do appear to be bounded-input bounded-output stable for the conditions imposed on the nonlinearity. This has been noticed by computing solutions to various systems. Also systems with zeros could be handled by this method but as stability bounds are not obtained these systems are not shown here.

For comparison with other results, Barrett [2] considered the system

$$\ddot{x} + a_1\dot{x} + a_2 x + \varepsilon x^3 = y(t) \tag{4.101}$$

and found a bound on x for which the solution was bounded-input bounded-output stable.

$$\|x\| < (3\varepsilon H)^{-\frac{1}{2}} \tag{4.102}$$

where

$$H = \begin{cases} \frac{1}{a_2}(1+\exp[-a_1(4a_2-a_1^2)^{-\frac{1}{2}}])/(1-\exp[-a_1(4a_2-a_1^2)^{-\frac{1}{2}}]) & ; \quad a_1^2 < 4a_2 \\ \frac{1}{a_2} & ; \quad a_1^2 \geq 4a_2 \end{cases} \quad (4.103)$$

From Section 4.3 using equation (4.101)

$$\frac{dN(x)}{dx} = 3\varepsilon\, x^2$$

$$\alpha = -a_2$$

$$\beta = 2a_1 f - a_2 \qquad f \text{ as large as needed} \qquad (4.104)$$

These results give two bounds, one for a positive ε and another for a negative ε .

$$\varepsilon > 0 \quad \rightarrow \quad \|x\| < \left(\frac{2a_1 f - a_2}{3\varepsilon}\right)^{\frac{1}{2}} < \infty$$

$$\varepsilon < 0 \quad \rightarrow \quad \|x\| < \left(\left|\frac{a_2}{3\varepsilon}\right|\right)^{\frac{1}{2}} \qquad (4.105)$$

The results obtained by Barrett were also obtained by Lepschy, Marchesini and Picci [17] using a comparison method while a slightly smaller region was obtained by Christensen [4], using contraction mapping and fixed point theorems.

CHAPTER V

INITIAL CONDITIONS AND THE VOLTERRA SERIES

5.1 Introduction

All of the examples with the Volterra series used up to here have assumed zero initial conditions. In this chapter the inclusion of initial conditions of the differential equation describing the linear plant will be discussed and a method for incorporating the initial conditions into the series will be developed. This method, coupled with the previous method for evaluating the series, gives a relatively simple method of determining the Volterra series for non zero initial conditions.

Using the initial condition method, it is then shown how to treat a step input to give a quickly converging series. In practice, any input which will tend to a non zero average value can be treated in this way.

5.2 Inclusion of Initial Conditions in the Volterra Series [5]

Consider the general system in Fig. 5.1.

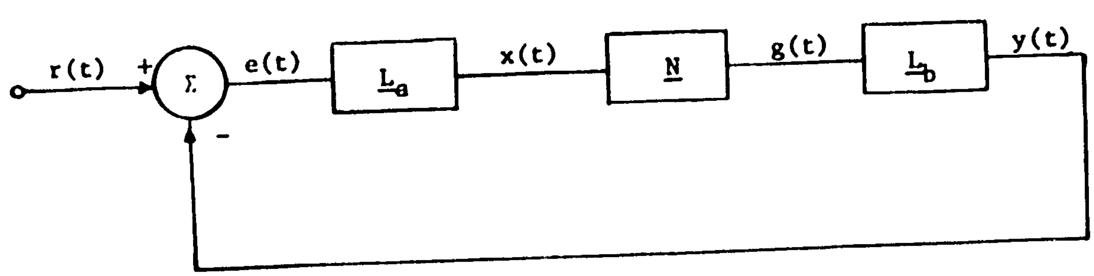

Fig. 5.1 The General System

where $\underline{L}_a$ and $\underline{L}_b$ are linear operators, and $\underline{N}$ is a single-valued time-invariant nonlinearity. Let

$$\underline{L}_a(s) = \frac{c_1 s^{p-1}+\ldots+c_p}{s^p+a_1 s^{p-1}+\ldots+a_p} \tag{5.1}$$

$$\underline{L}_b(s) = \frac{d_1 s^{q-1}+\ldots+d_q}{s^q+b_1 s^{q-1}+\ldots+b_q} \tag{5.2}$$

$$\underline{N}(x) = \sum_{i=2}^{M} n_i(x)^i \tag{5.3}$$

and let $\underline{L}_a$ and $\underline{L}_b$ have initial conditions $\{x(0+),x^{(1)}(0+),\ldots,x^{(p-1)}(0+)\}$ and $\{y(0+),y^{(1)}(0+),\ldots,y^{(q-1)}(0+)\}$ respectively, where $y(0+)$ is the value of $y(t)$ just after $t=0$, and $y^{(q)}(t)$ is the q'th derivative of $y(t)$.

Consider first the equation

$$y(t) = \underline{L}_b(g) \tag{5.4}$$

The solution to this equation, including initial conditions, is given in Laplace transforms by,[8]

$$Y(s) = \frac{\sum_{i=0}^{q-1}\sum_{j=1}^{q-i} s^i b_{q-i-j} y^{(j-1)}(0+)}{s^q+b_1 s^{q-1}+\ldots+b_q} + L_b(s)G(s) \tag{5.5}$$

where the following definitions are made.

$$b_0 = 1$$

$$y^{(0)}(0+) = y(0+) \tag{5.6}$$

Equation (5.5) can be written in the time domain as

$$y(t) = y_o(t) + \underline{L}_b(g) \tag{5.7}$$

where $y_o(t)$ is the initial condition function given by the inverse transform of the appropriate function in equation (5.5).

In a similar fashion the equation

$$x(t) = \underline{L}_a(e) \tag{5.8}$$

can be treated

$$X(s) = \frac{\sum_{i=0}^{p-1} \sum_{j=1}^{p-i} s^i a_{p-i-j} \, x^{(j-1)}(0+)}{s^p + a_1 s^{p-1} + \ldots\ldots + a_p} + L_a(s)E(s) \tag{5.9}$$

where similarly

$$a_0 = 1 \text{ and } \quad x^{(0)}(0+) = x(0+) \tag{5.10}$$

Taking the inverse Laplace transform gives

$$x(t) = x_o(t) + \underline{L}_a(e) \tag{5.11}$$

Now expressing the system output as x(t) and the input r(t) gives

$$x(t) = x_o(t) + \underline{L}_a\big(r(t) - y_o(t) - \underline{L}_b * \underline{N}(x)\big) \tag{5.12}$$

$$= x_o(t) + \underline{L}_a\big(r(t) - y_o(t)\big) - \underline{L} * \underline{N}(x) \tag{5.13}$$

where as before

$$\underline{L} = \underline{L}_a * \underline{L}_b \tag{5.14}$$

Equation (5.13) is the equation representing the general system under consideration with initial conditions on the linear plants. The effect of initial conditions is to modify the input function. Clearly all the previous results obtained about stability of the system and the convergence of the Volterra series will also hold in this case for the modified input function.

In the evaluation of the Volterra series the modified input function could be expressed as

$$w(t) = x_o(t) + \underline{L}_a\left(r(t) - y_o(t)\right) \tag{5.15}$$

and then the output could be expressed as a function of this new function $w(t)$.

$$x(t) = w(t) - \underline{L}*\underline{N}(x) \tag{5.16}$$

$$x(t) = \sum_{i=1}^{\infty} \underline{H}_i(w) \tag{5.17}$$

This appears to simplify the kernels of the series a little but in effect just removes the factor $\prod_{k=1}^{i} L_a(s_k)$ from the expression for each $H_i(s_1,\ldots s_i)$.

The case where $\underline{N}(x)$ contains a linear term can be treated the same way as was suggested in Section 4.5.

5.3 Example of Initial Conditions

Consider the system of Fig. 5.1 with

$$L_a(s) = \frac{1}{s} \tag{5.18}$$

$$L_b(s) = \frac{k}{s+1} \tag{5.19}$$

$$\underline{N}(x) = x + \varepsilon x^3 \tag{5.20}$$

and with input and initial conditions of

$$r(t) = 0 \tag{5.21}$$

$$x(0+) = x^o \; ; \quad y(0+) = y^o \tag{5.22}$$

Then from equations (5.5) and (5.9),

$$X_o(s) = \frac{x^o}{s} \tag{5.23}$$

$$Y_o(s) = \frac{y^o}{s+1} \tag{5.24}$$

$$W(s) = \frac{x^o}{s} - \frac{y^o}{s(s+1)} = \frac{x^o s + x^o - y^o}{s(s+1)} \tag{5.25}$$

and substituting $\underline{N}(x)$ in equation (5.16) gives

$$x(t) = w(t) - \underline{L}(x) - \varepsilon\underline{L}(x^3) \tag{5.26}$$

$$x(t) = (\underline{I} + \underline{L})^{-1} w(t) - \varepsilon(\underline{I} + \underline{L})^{-1} * \underline{L}(x^3) \tag{5.27}$$

Now evaluating $(\underline{I}+\underline{L})^{-1}$ in Laplace transforms

$$(I + L)^{-1}(s) = \frac{s(s+1)}{s^2+s+k} \tag{5.28}$$

$$\rightarrow \quad (I + L)^{-1}(s)\, W(s) = \frac{x^o s + x^o - y^o}{s^2 + s + k} = W'(s) \tag{5.29}$$

$$(I + L)^{-1}(s)\, L(s) = \frac{k}{s^2 + s + k} = K(s) \tag{5.30}$$

Equation (5.27) now becomes

$$x(t) = w'(t) - \varepsilon \underline{K}(x^3) \tag{5.31}$$

and expressing x(t) as a series function of w'(t) gives

$$x(t) = \underline{H}(w') = \sum_{j=1}^{\infty} \underline{H}_j(w') \tag{5.32}$$

As before, substituting equation (5.32) into equation (5.31) and equating orders,

$$\underline{H}_1 = \underline{I} \tag{5.33}$$

$$\underline{H}_3 = -\varepsilon \underline{K}^*(\underline{I})^3 \tag{5.34}$$

$$\underline{H}_5 = 3\varepsilon^2 \underline{K}^*(\underline{I})^2 \underline{K}^*(\underline{I})^3 \tag{5.35}$$

$$\underline{H}_{2n} = \underline{0} \tag{5.36}$$

and so on. As before let

$$\hat{X}_n(s_1,\ldots,s_n) = H_n(s_1,\ldots,s_n)\, W'(s_1)\ldots\ldots W'(s_n) \tag{5.37}$$

then evaluating the $\hat{X}_n(s_1,\ldots,s_n)$ and associating variables gives

$$\hat{X}_1(s) = \frac{x^o s + x^o - y^o}{s^2 + s + k} = \frac{x^o(s+\frac{1}{2}) + \frac{x^o}{2} - y^o}{(s+\frac{1}{2})^2 + k - \frac{1}{4}} \tag{5.38}$$

$$\hat{X}_3(s_1,s_2,s_3) = \frac{-\varepsilon k}{(s_1+s_2+s_3)^2 + (s_1+s_2+s_3) + k} \prod_{i=1}^{3} \frac{x^o s_i + x^o - y^o}{s_i^2 + s_i + k}$$

(5.39)

$$\hat{X}_3(s) = -\varepsilon k \left\{ \frac{A_1(s+\frac{1}{2}) + A_2}{(s+\frac{1}{2})^2 + k - \frac{1}{4}} + \frac{A_3(s+\frac{3}{2}) + A_4}{(s+\frac{3}{2})^2 + k - \frac{1}{4}} + \frac{A_5(s+\frac{3}{2}) + A_6}{(s+\frac{3}{2})^2 + 9(k-\frac{1}{4})} \right\} \tag{5.40}$$

where A_1, A_2, A_3, A_4, A_5, A_6 are all real numbers depending on x^o, y^o and k.

5.4 Modifications of the Series for Step Inputs.

Consider the example worked out in Section 3.6 of a system with a step input. In this example it was found that with inputs of a small magnitude there was good convergence of the series for all time. However, as the input became larger the series converged for a finite time interval but diverged quite rapidly after that. It is known from computer studies that this system is quite stable for all inputs and so the question is raised of whether it is possible to transform the equation so that the series converges for all inputs. A method for achieving this is outlined below which uses a transformation and the initial value method of the previous sections.

The system to be considered is that of Fig. 5.1 .

$$x(t) = x_o(t) + \underline{L}_a \left(r(t) - y_o(t)\right) - \underline{L}*\underline{N}(x) \tag{5.41}$$

with $\underline{L}_a$, $\underline{L}_b$ and $\underline{N}$ as defined in equations (5.1), (5.2), (5.3). $x_o(t)$ and $y_o(t)$ are initial condition functions as defined in Section 5.2 and

$$\underline{L} = \underline{L}_a * \underline{L}_b \tag{5.42}$$

Assume both linear operators are stable, so that the initial condition functions tend to zero as t increases.

$$\lim_{t\to\infty} x_o(t) = 0$$

$$\lim_{t\to\infty} y_o(t) = 0$$

$$\lim_{t\to\infty} \underline{L}_a(y_o(t)) = 0 \qquad (5.43)$$

Also assume that for an input function

$$r(t) = Uu(t) \qquad (5.44)$$

that the output is bounded, and that the output reaches a steady state value. Then,

$$\|x\| \leq K_1$$

$$\rightarrow \qquad \|\underline{N}x\| \leq K_2 \qquad (5.45)$$

$$\lim_{t\to\infty} x(t) = x_f \qquad (5.46)$$

Let

$$\underline{L}(x) = {}_0\int^t h(\tau)x(t-\tau)d\tau \qquad (5.47)$$

and from the assumptions of stability of the linear operators, then

$$|h(t)| < K_3\exp(-\nu t) \qquad \nu>0 \qquad (5.48)$$

Consider the limit as $t\to\infty$ of $\underline{L}*\underline{N}(x)$.

$$\underline{L}*\underline{N}(x) = {}_0\int^t h(\tau)\underline{N}(x(t-\tau))d\tau \qquad (5.49)$$

It is assumed that both h(t) and x(t) are equal to zero for all $t<0$, so that equation (5.49) may be written

$$\underline{L}*\underline{N}(x) = {}_0\int^{\infty} h(\tau)\underline{N}(x(t-\tau))d\tau \qquad (5.50)$$

and then

$$\lim_{t\to\infty} \underline{L}*\underline{N}(x) = {}_0\int^{\infty} h(\tau)\left(\lim_{t\to\infty} \underline{N}(x(t-\tau))\right)d\tau \qquad (5.51)$$

since both $|h(t)|$ and $|\underline{N}(x)|$ are bounded. $\underline{N}(x)$ is single valued and continuous, so that using equation (5.46), equation (5.51) becomes

$$\begin{aligned}\lim_{t\to\infty} \underline{L}*\underline{N}(x) &= {}_0\int^{\infty} h(\tau)\underline{N}(x_f)d\tau \\ &= \underline{N}(x_f)\,{}_0\int^{\infty} h(\tau)d\tau \qquad (5.52)\end{aligned}$$

Now ${}_0\int^{\infty} h(\tau)d\tau$ exists, and so using Laplace transform theory [8] to evaluate the integral gives,

$$\begin{aligned}{}_0\int^{\infty} h(\tau)d\tau &= \lim_{t\to\infty} {}_0\int^{t} h(\tau)d\tau \\ &= \lim_{s\to 0} sL(s)/s \\ &= \lim_{s\to 0} L(s) \qquad (5.53)\end{aligned}$$

Incorporating this result into equation (5.41), and taking the limit as $t\to\infty$ on both sides, gives

$$\lim_{t\to\infty} x(t) = x_f = \lim_{s\to 0} sL_a(s)U/s - \underline{N}(x_f)\lim_{s\to 0} L(s)$$

$$x_f = U \lim_{s \to 0} L_a(s) - \underline{N}(x_f) \lim_{s \to 0} L(s) \tag{5.54}$$

If the algebraic equation (5.54) has a real solution for x_f then this is the final value of x(t). So now in equation (5.41) substitute for x a function $z(t) + x_f$. Then z(t) is a function whose steady state value is zero and has an initial value given by

$$z(0+) = x(0+) - x_f = x^o - x_f \tag{5.55}$$

$$\dot{z}(0+) = \dot{x}(0+) \tag{5.56}$$

and so on for the initial values of the higher derivatives. Making these substitutions gives

$$z(t) + x_f = z_o(t) + U \lim_{s \to 0} L_a(s) - \underline{L}_a y_o(t) - \underline{L}*\underline{N}(z(t) + x_f) \tag{5.57}$$

$$= z_o(t) - \underline{L}_a\, y_o(t) + U \lim_{s \to o} L_a(s) - \underline{N}(x_f) \lim_{s \to o} L(s) - \underline{L}*\underline{N}'(z)$$

$$\rightarrow \quad z(t) = z_o(t) - \underline{L}_a y_o(t) - \underline{L}*\underline{N}'(z) \tag{5.58}$$

where $z_o(t)$ is the initial condition for x(t) with x(0+) replaced by $z(0+) + x_f$, and $\underline{N}'(z)$ can be written as

$$\underline{N}'(z) = \sum_{i=2}^{M} n_i \sum_{j=1}^{i} C_j^i \; z^j \; x_f^{\,i-j} \tag{5.59}$$

and where

$$C_j^i = i! \; (j!(i-j)!)^{-1} \tag{5.60}$$

This new nonlinearity contains a linear term which can be removed and placed with the linear plant.

$$z(t) = z_o(t) - \underline{L}_a\, y_o(t) - \underline{L}* \sum_{i=2}^{M} n_i \; C_1^i\, z\, x_f^{\,i-1}$$

$$- \underline{L}* \sum_{i=2}^{M} n_i \sum_{j=2}^{i} C_j^i \; z^j\, x_f^{\,i-j} \qquad (5.61)$$

$$(\underline{I} + \sum_{i=2}^{M} n_i \; i x_f^{\,i-1} \underline{L})\; z(t) = z_o(t) - \underline{L}_a y_o(t) - \underline{L}*\underline{N}''(z) \qquad (5.62)$$

$$\Rightarrow \qquad z(t) = \underline{K}^{-1}\, z_o(t) - (\underline{K}^{-1} * \underline{L}_a) y_o(t) - \underline{K}^{-1} * \underline{L} * \underline{N}''(z) \qquad (5.63)$$

where

$$\underline{N}''(z) = \sum_{i=2}^{M} n_i \sum_{j=2}^{i} C_j^i\, z^j\, x_f^{\,i-j} = \sum_{i=2}^{M} z^i \sum_{j=i}^{M} n_j\, C_i^j \;\; x_f^{\,j-i} \qquad (5.64)$$

$$\underline{K} = \underline{I} + \sum_{i=2}^{M} n_i \; i\; x_f^{\,i-1}\; \underline{L} \qquad (5.65)$$

and $\underline{K}^{-1} * \underline{L}$ is now the new linear system. It can be seen from the expression for $N''(z)$ that the greatest power of the nonlinearity has not increased but more terms have appeared at lower powers.

$z(t)$ can be calculated from equation (5.63) in the form of a Volterra series.

$$z(t) = \sum_{n=1}^{\infty} \underline{H}_n \; (z_o(t) - \underline{L}_a \; y_o(t)\;) \qquad (5.66)$$

and then $x(t)$ can be found from

$$x(t) = x_f + z(t) = x_f + \sum_{n=1}^{\infty} \underline{H}_n (z_o(t) - \underline{L}_a y_o(t)\;) \qquad (5.67)$$

and it can be seen that x_f plays the role of a zero order operator.

5.5 Example

Consider the same example as in Section 3.6.

The system was

$$\underline{N}(x) = \varepsilon x^3$$

$$\underline{L}_a(s) = \frac{1}{s+a}$$

$$\underline{L}_b(s) = 1$$

$$x(0+) = 0$$

$$r(t) = k\,u(t) \qquad (5.68)$$

Equation (5.41) becomes

$$x(t) = \underline{L}_a(r) - \varepsilon \underline{L}_a(x^3) \qquad (5.69)$$

Taking the limit as $t \to \infty$.

$$x_f = (k/a) - (\varepsilon/a)\,(x_f)^3 \qquad (5.70)$$

$$x_f^{\,3} + (a/\varepsilon)\,x_f - (k/\varepsilon) = 0 \qquad (5.71)$$

From the solution of cubic equations, it is known that

$$x_f = (k/\varepsilon)^{\frac{1}{3}} \left([0.5 + 0.5(1 + 4a^3/27k^2\varepsilon)^{\frac{1}{2}}]^{\frac{1}{3}} + [0.5 - 0.5(1 + 4a^3/27k^2\varepsilon)^{\frac{1}{2}}]^{\frac{1}{3}} \right) \qquad (5.72)$$

which is the only real solution as long as

$$1 + 4a^3/27k^2\varepsilon > 0 \qquad (5.73)$$

For stability of the linear system a>o and it then depends on ε for conditions (5.73) to hold. In fact all that is needed is

$$\varepsilon > -4a^3/27k^2 \tag{5.74}$$

Now let

$$x(t) = z(t) + x_f \tag{5.75}$$

and substituting in equation (5.69) and cancelling terms

$$z(t)+x_f = z_o(t)+(\frac{k}{a}) - (\frac{\varepsilon}{a})x_f^3 - \varepsilon \underline{L}_a(z^3+3x_f z^2+3x_f^2 z)$$

(5.76)

$$z(t) = z_o(t) - 3\varepsilon x_f^2 \underline{L}_a(z) - \varepsilon \underline{L}_a (z^3 + 3x_f z^2)$$

$$z(t) = (\underline{I} + 3\varepsilon x_f^2 \underline{L}_a)^{-1} z_o(t) - \varepsilon(\underline{I} + 3\varepsilon x_f^2 \underline{L}_a)^{-1} * \underline{L}_a(z^3+3x_f z^2)$$

(5.77)

For this case using equation (5.9)

$$Z_o(s) = \frac{-x_f}{s+a} \tag{5.78}$$

$$(I + 3\varepsilon x_f^2 L_a)^{-1}(s) = \frac{s+a}{s+a+3\varepsilon x_f^2} \tag{5.79}$$

Let

$$w(t) = (\underline{I} + 3\ \varepsilon x_f^2\ \underline{L}_a)^{-1}\ z_o(t) \tag{5.80}$$

$$\rightarrow\quad W(s) = \frac{-x_f}{s+a+3\varepsilon x_f^2} \tag{5.81}$$

and

$$\underline{K} = (\underline{I} + 3\varepsilon\, x_f^{\,2}\ \underline{L}_a)^{-1} * \underline{L}_a$$

$$\rightarrow \qquad K(s) = \frac{1}{s+a+3\varepsilon x_f^{\,2}} \tag{5.82}$$

For convenience in notation, let

$$\alpha = a + 3\varepsilon x_f^{\,2} \tag{5.83}$$

$$\gamma = 3\varepsilon x_f^{\,2}/\ \alpha < 1 \tag{5.84}$$

Then solving equation (5.77) for z(t) in the form of a Volterra series

$$z(t) = \sum_{j=1}^{\infty} \hat{z}_j(t) \tag{5.85}$$

$$\hat{z}_1(t) = w(t) = -x_s e^{-at}$$

$$\hat{z}_2(t) = -x_s(\gamma/6)\ (5e^{-\alpha t} - 6e^{-2\alpha t} + e^{-3\alpha t})$$

$$\hat{z}_3(t) = -x_s(\gamma^2/72)(47e^{-\alpha t} - 120e^{-2\alpha t} + 102e^{-3\alpha t} - 32e^{-4\alpha t} + 3e^{-5\alpha t})$$

$$\hat{z}_4(t) = -x_s(\gamma^3/2160)(1069e^{-\alpha t} - 4320e^{-2\alpha t} + 6855e^{-3\alpha t}$$

$$-5360e^{-4\alpha t} + 2115e^{-5\alpha t} - 384e^{-6\alpha t} + 25e^{-7\alpha t}) \tag{5.86}$$

These results were obtained by methods outlined before but grouping terms with the same power of x_s or γ under the same order of operator. The results are plotted in Fig. 5.2 where it can be seen that there is good convergence of the series. Compared to Fig. 3.1 there is more error for t

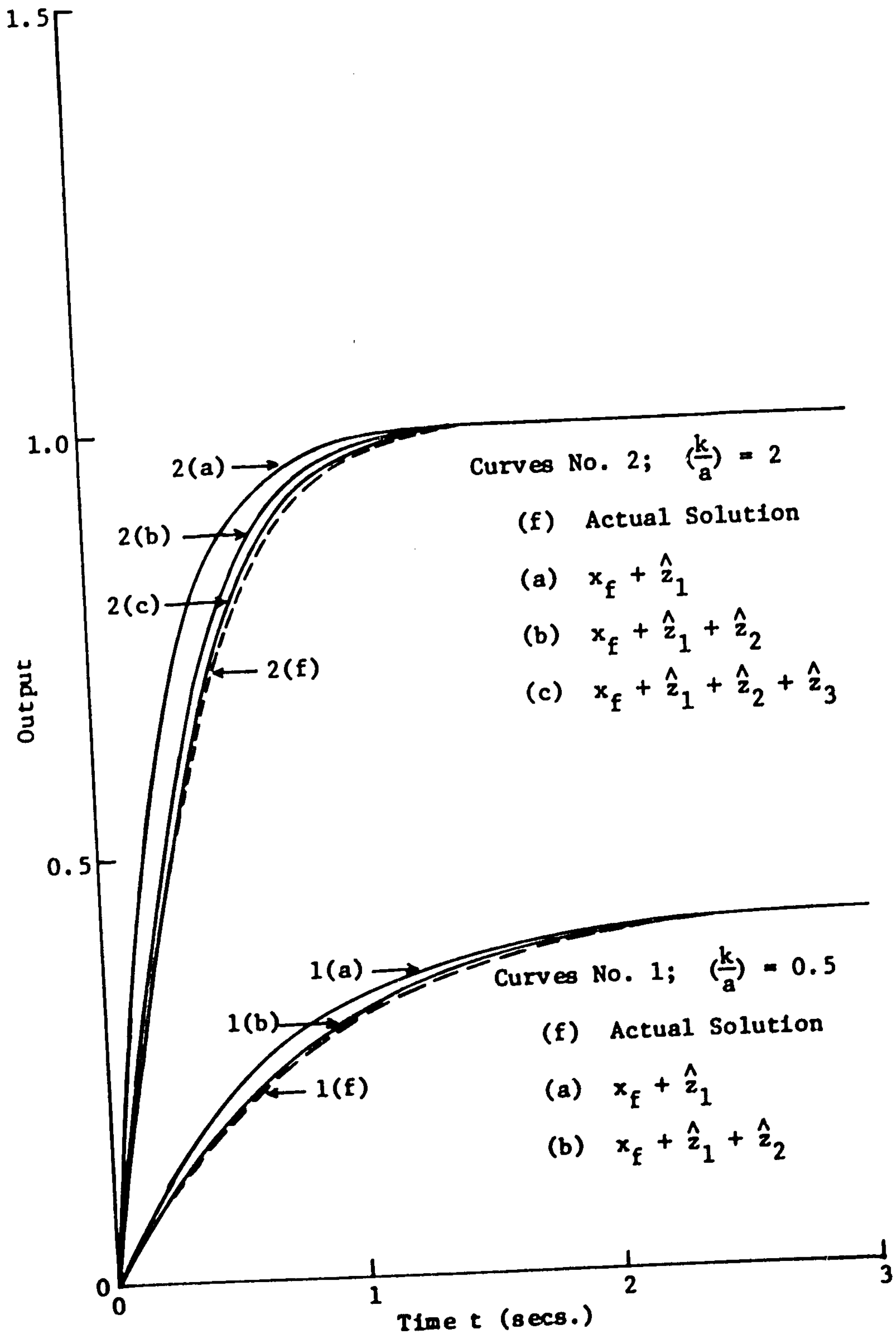

Fig 5.2 Step Input Results

small but there is good convergence here at all inputs while the solution shown in Fig.3.1 does not converge for all time t and all inputs.

CHAPTER VI

COMMENTS AND CONCLUSIONS

6.1 The Volterra Series

The Volterra series is a very useful tool for the nonlinear systems engineer. It can be used for a large class of systems whose nonlinearity can be represented with sufficient accuracy as a finite power series, and some derivatives of this power series, when these derivatives can be incorporated as zeros of the linear plant. This is quite a large class of systems, as physical nonlinearities are usually not too abrupt, and input signals to systems are usually limited in magnitude by physical considerations.

The main problem with the series, as with all types of power series solutions, is to obtain convergence of the series. For small inputs and the types of systems considered here, the series always converges but as the signals increase in magnitude, the series begins to diverge as is shown in the example in Chapter III. This does not mean that the system is not stable however, and if it can be shown that the system is in fact stable then the system may be able to be transformed to a new equivalent system whose Volterra series does converge. This has been done for the case of the step input in Chapter V.

Another method of evaluating the Volterra series by differentiating the series and substituting into the differential equation has been used by Flake [10], which should be able to handle all the types of nonlinearities that contain sums of powers and products of derivaties. A special method for including initial conditions is also given but there is more work involved in calculating the series for these methods.

Using the kernels of the series, it is possible to analyse the response of the system to quite a few inputs which engineers are interested in. For sinusoidal inputs the distortion terms are quite readily calculated substituting $s = \pm jw$ into the transforms of the kernels [11]. For initial conditions or step inputs then the methods developed here give good convergence of the series and for impulses the kernels themselves give the output function.

One way of synthesizing a nonlinear system would be to use a general nonlinearity and work out several terms of the series. Then the coefficients of the powers of the nonlinearity could be chosen to give a desired result or close to it.

6.2 Stability Analysis

The methods of analysis used here to investigate stability depend on the functional analysis approach using contraction mapping. This approach combines well with the Volterra series and gives useful results. For instance, over the region in which the series converges absolutely, if the initial condition function is an exponentially damped function then by consideration of the series it can be seen that the output will also be exponentially damped and so the system is asymptotically stable [5].

For the expanded region of convergence as found here there is no guarantee of asymptotic stability in general. It appears from other work (see [1]) that all systems of first and second order with nonlinearities as used here which are stable at the origin are asymptotically stable over the whole region of bounded-input bounded-output stability.

The problems arise with higher order systems with zeros in the linear plant. It should be possible to determine the asymptotic stability bounds using the functional approach with or without the Volterra series and more work is needed in this area.

6.3 Conclusions

In this thesis a class of nonlinear systems is considered. It consists of up to two linear plants and one single-valued time-invariant nonlinearity which can be represented to a sufficient degree of accuracy by a finite power series. With this class of systems the easiest way to calculate the series was demonstrated with and without initial conditions and a transformation introduced to guarantee convergence of a stable system with a step input.

The bounded-input bounded-output stability of this class of systems was investigated and for first and second order systems a sufficient condition was found that the slope of the nonlinearity should be contained in the Hurwitz sector of the system with the nonlinearity replaced by a linear gain. For higher order systems, a bound was found on the output from which stability could not be assumed, but it has been found from computed results that for nonlinearities with slopes within the Hurwitz sector, the systems were bounded-input bounded-output stable so that more investigation is needed in this area.

Also some work is needed on the prediction of limit cycles along with the bounded-input bounded-output stability. If it can be shown that a system is bounded-input bounded-output stable and no limit cycle can exist then the system must also be asymptotically stable. This is especially important for the higher order systems.

In summary this thesis shows the use to which the Volterra series can be put in the analysis of nonlinear systems. The Volterra series was shown to be unique so that it can be calculated the way that is easiest and a proposed method of calculation was set out. A simple way of including initial conditions in the series is introduced and it is used to give faster convergence for step input functions. The contraction region has been expanded for a large class of systems to give a convergent series for a finite time interval. Then using a diagonalization technique on first and second order systems bounded-input bounded-output stability is obtained for a nonlinearity whose slope lies in the Hurwitz sector of the system. Although this has been done by other methods, it is now possible to find a solution to the system.

BIBLIOGRAPHY

1. M.A. Aizerman and F.R. Gantmacher, "Absolute Stability of Regulator Systems", San Francisco: Holden-Day, 1964

2. J.F. Barrett, "The Use of the Volterra Series to find the Region of Stability of a Nonlinear Differential Equation", International Journal of Control, Vol. 1 No. 3, pp. 209-216, March 1965.

3. M.B. Brilliant, "Theory of the Analysis of Nonlinear Systems", MIT Research Laboratory of Electronics Technical Report 345, March 1958.

4. G.S. Christensen, "On the Convergence of the Volterra Series", IEEE Transactions on Automatic Control, Vol. AC-13, No. 6, pp. 736-737, December 1968.

5. G.S. Christensen and G.W. Trott, "On the Inclusion of Initial Conditions in the Volterra Series", International Journal of Control, Vol. 12, No. 5, pp. 835-838, November 1970.

6. S.C. Chu and J.B. Diaz, "A Fixed Point Theorem for <<In The Large>> Applications of the Contraction Principle", Atti Della Accademia Delle Scienze Di Torino, Vol. 99, pp. 351-363, 1964-1965.

7. C.A. Desoer, "Nonlinear Distortion in Feedback Amplifiers", IRE Transactions on Circuit Theory, Vol. CT-9, No. 1, pp.2-6 March 1962.

8. G. Doetsch, "Guide to the Applications of Laplace Transforms", London: Van Nostrand, 1961.

9. R.E. Fitts, "Two Counter-examples to Aizerman's Conjecture", IEEE Transactions on Automatic Control, Vol. AC-11, No. 3, pp. 553-556, July 1966.

10. R.H. Flake, "Volterra Series Representation of Nonlinear Systems", Transactions of the American Institute of Electrical Engineers, Part II; Applications and Industry, Vol. 81, No. 6, pp. 330-335, January 1963.

11. D.A. George, "Continuous Nonlinear Systems", MIT Research Laboratory of Electronics Technical Report 355, July 1959.

12. J.E. Gibson, "Nonlinear Automatic Control", New York: McGraw-Hill 1963

13. J.M. Holtzman, "Nonlinear Distortion in Feedback Systems", Bell System Technical Journal, Vol. 47, No. 4, pp. 503-509, April 1968.

14. J.M. Holtzman, "Nonlinear System Theory: A Functional Analysis Approach", Englewood Cliffs, N.J.: Prentice Hall, 1970.

15. A.N. Kolmogorov and S.V. Fomin, "Functional Analysis", Vol. 1, Rochester, N.Y.: Graylock Press, 1957.

16. Y.H. Ku and A.A. Wolf, "Volterra-Wiener Functionals for Nonlinear Systems", Journal of Franklin Institute, Vol. 281, No. 1, pp. 9-26, January 1966.

17. A. Lepschy, G. Marchesini and G. Picci, "A Functional Approach to the Stability of Nonlinear Systems by Use of the Comparison Theorem", International Journal of Control, Vol. 11, No. 2, pp. 293-302, February 1970.

18. R.B. Parente, "Functional Analysis of Systems Characterized by Nonlinear Differential Equations", MIT Ph.D. Dissertation, 1965.

19. W.A. Porter, "Modern Foundations of Systems Engineering", New York: MacMillan, 1966.

20. T.L. Saaty, "Modern Nonlinear Equations", New York: McGraw-Hill, 1967.

21. T.L. Saaty and J. Bram, "Nonlinear Mathematics", New York: McGraw-Hill, 1964.

22. I.W. Sandberg, "On the Response of Nonlinear Control Systems to Periodic Input Signals", Bell System Technical Journal, Vol. 43, pp. 911-926, May 1964.

23. G.W. Trott and G.S. Christensen, "On the Uniqueness of the Volterra Series", IEEE Transactions on Automatic Control, Vol. AC-14, No. 6, pp. 759-760, December 1969.

24. G.W. Trott and G.S. Christensen, "A Larger Region of Contraction for the Volterra Series", International Journal of Control, to be published.

25. H. Van Trees, "Functional Techniques for the Analysis of the Nonlinear Behaviour of Phase Locked Loops", Proceedings of IEEE, Vol. 52, No. 8, pp. 894-911, August 1964.

26. V. Volterra, "Theory of Functionals and of Integral and Integro-Differential Equations", New York: Dover, 1959.

27. N. Wiener, "Response of a Nonlinear Device to Noise", MIT Radiation Laboratory Technical Report No. 129, April 1942.

28. J.C. Willems, "Perturbation Theory for the Analysis of Instability in Nonlinear Feedback Systems", Fourth Annual Allerton Conference on Circuit and System Design, 1968.

29. G. Zames, "Nonlinear Operators for System Analysis", MIT Research Laboratory of Electronics Technical Report 370, August 1960.

30. G. Zames, "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems - Part I: Conditions Derived Using Concepts of Loop Gain, Conicity, and Positivity", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, pp. 228-238, April 1966.

31. G. Zames, "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems - Part II: Conditions Involving Circles in the Frequency Plane and Sector Nonlinearities", IEEE Transactions on Automatic Control, Vol. AC-11, No. 3, pp. 465-476, July 1966.