**University of Alberta**

# ANALYSIS OF A $K$-*PRODUCER* PROBLEM USING THE MONOPOLIST'S PROBLEM FRAMEWORK

by

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# Abstract

We introduce and study a $k$-producer problem which is an extension of the monopolist's problem in economic theory. This models a market with several different monopolist's producing complementary goods. We show that the problem of finding an equilibrium in the $k$-producer problem can be transformed into a system of $k$ monopolist's problems with effective preference functions derived from the original preference function. We find sufficient conditions on the preference function so that the effective preference functions satisfy the generalized single crossing (GSC) property. Analytical and numerical results are discussed for the uni-dimensional case, as well as the economic properties of the model.

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# Chapter 1

# Introduction

Industries, such as rail road transportation and electricity production and distribution are often dominated by single producers, e.g. monopolists. A similar setting can be found in government procurement and optimal taxation. It is common that in such industries the prices set by the corresponding monopolist's are non-linear. Non-linear pricing is often necessary for efficiency. This is the case when the firm's cost per unit of filling or shipping an order varies with the size of the order. Also, non-linear pricing is used as a means of price discrimination that enables a firm with monopoly power to increase its profits by market segmentation where the company would cater towards the high end customers. For example consider a product line of printers, that appeal to different customer segments because more expensive machines have higher rates of output and lower marginal costs. This monopolist's problem of pricing his goods in an optimal way is known as a screening problem.

The monopolist's pricing problem is as follows. He wants to sell goods from a set $Y$ to a set of $X$ consumers and uses the preference $b(x, y)$ that a consumers of type $x \in X$ has for good of type $y \in Y$, the density $f(x)$ of the consumer types and the cost $c(y)$ to set pricing schedules to maximize his profits. The first such problems were formulated and handled by Mussa and Rosen [12]. These models were uni-dimensional, meaning that the monopolist's product types $y$ is one dimensional and the consumer's type $x$ has just one characteristic. Non durable goods such as fuel can be modelled using a uni-dimensional model.

The monopolist's problem is well understood when $X$ and $Y$ are discrete or uni-dimensional [12, 1]. However, uni-dimensional and discrete models do not cover many situations of practical interest. In most cases, pricing of a good depends on more than one characteristic. For example, if we consider a producer of cars, the cars the monopolist produces may differ by qualities such as fuel efficiency, safety, handling, eco-friendliness, etc. Also the consumer's of cars may vary according to their age, income, family size, commuting needs, etc. Therefore, it is natural to study models where the respective dimensions $m$ and $n$ of $X$ and $Y$ are greater than 1. This gives rise to multidimensional screening problems. Some notable publications about multidimensional screening are published by Wilson [17], Armstrong [2], Rochet and Chone [15] and Basov [4].

There are three approaches to the screening problems : direct, dual and Hamiltonian. A detailed review of these approaches can be found in [6]. All three approaches work for the uni-dimensional case, but the direct approach is hard to generalize for multidimensional problems. Rochet and Chone [15] developed the dual approach for multidimensional screening problems. They assumed that $m = n$ and the preference $b(x, y)$ of consumers is linear in type. Basov [4] generalized the dual approach for the case when $n > m$ retaining the linearity assumption. Basov also presented the Hamiltonian approach for the above cases. In general no method is known for solving the case $m > n$. Not much is known when the linearity assumption on $b$ is dropped. Carlier [7] showed there exists a solution to this problem for more general preference functions. However, absence of convexity for general preference functions makes it extremely difficult to characterize a solution for this problem.

We are interested in a situation where there are several monopolist's in the market who are producing complementary goods; these are goods whose use is interrelated with the use of an associated product such that a demand for one (tires, for example) generates demand for the other (gasoline, for example). If the price of one good falls and people buy more of it, they will usually buy more of the complementary good also, whether or not its price also falls. When considering complimentary goods the consumer's preference towards each good is not independent of the other goods, instead it is coupled. Then we can expect that the profit maximizing prices are affected by the prices of the other goods. Then the pricing schedules implemented by all the firms depend on the other product's pricing schedules. For example, consider two firms; one produces cars and the other issues insurance packages, and the consumer is interested in either buying one product or a bundle of both. The producers objective is to maximize his profit, which depends on the bundle that each consumer chooses. Because of the fact that the goods are inter related with each other, we can expect that the profit maximizing prices are going to be affected by the prices of these associated goods. Therefore, the pricing schedules implemented by all the firms depend on the other product's pricing schedules.

The above mentioned problem, which we will call the *k-producer* problem, has not been widely studied. The main topic of interest here is an equilibrium pricing schedule and its properties. In this thesis we use the framework of the monopolist problem to model the general *k-producer* problem and to derive an explicit equilibrium for some special cases of the *k-producer uni-dimensional problem*. The simplest form of this problem is the case where $k = 2$ (here $k$ is the number of producers). For example, consider two firms; one produces cars ($Y_1$) and the other issues insurance packages ($Y_2$) and the consumer ($X$) is interested in either buying one product or a bundle of both, so the consumers' preference function will be of the type $b(x, y_1, y_2)$ and the respective cost functions of the products will be $c_1(y_1)$ and $c_2(y_2)$. Now, we are considering a 2- product problem and the approach that we are proposing breaks this problem into two coupled monopolist's problems, with effective preference functions $b_1(x, y_1)$ and $b_2(x, y_2)$. Then, the 2-product pricing problem comes down to solving a system of two ordinary differential equations. The method can be extended for solving the $k$-producer case. In this thesis, after developing the model, we first establish the required conditions for converting the *k-producer* problem into $k$ coupled monopolist's problems in Proposition 3.A.1 and then Proposition 3.A.2 gives the conditions that our preference function $b(\cdot)$ needs to follow in order for effective preference functions $b_i$ to satisfy the GSC property. (The definition of GSC property is given in chapter 2. For our problem to be tractable, our preference function is required to satisfy this property.) We therefore reduce the problem of finding equilibria to the relatively simple problem of $k$ coupled monopolist's problems. This approach is further exploited in the uni- dimensional case, along with tools of optimal control, to derive a system of ODEs governing the equilibria. We solve these ODEs explicitly (either analytically or numerically) in some special cases.

## 1.A Review of literature

In this section we will present a summary of the previous works done in this field.

First, we will briefly introduce the Mussa and Rosen [12] model for a single product (uni-dimensional continuous model) and then present the results of their model.

Assume a monopolist who faces a continuum of consumers produces a good of quality $y \in Y$. Larger values of $y$ corresponds to a better quality product. The cost of production is assumed to be given by a strictly increasing, convex, twice differentiable function, $c(\cdot)$. Each consumer is interested in consuming at most one unit of the good and has a utility $u(x, y, p(y))$. The consumers also have an outside option of value $u_0(x)$. It is assumed that the consumer's utilities are quasi-linear with respect to $p(y)$:

$$u(x, y, p(y)) = b(x, y) - p(y), \tag{1.1}$$

where $p(y)$ is the price for good of quality $y$. It's assumed that the consumer's type $x$ is distributed on an open, bounded convex set according to a strictly positive, continuous density function $f(.) : X \to \mathbb{R}_+$. Now, to maximize her profits ($Profit = p(y(x)) - c(y(x))$ the monopolist selects a continuous $p$ to solve :

$$\max_{p(\cdot)} \int_X [p(y(x)) - c(y(x))]f(x)\,dx \tag{1.2}$$

$$s.t. \quad y(x) \in \arg\max u(x, y, p(y)) \tag{1.3}$$

$$\max_x u(x, y, p(y)) \geq u_0(x) \tag{1.4}$$

It has been shown that for the above model that its better for the monopolist to discriminate against the low end customers and to focus more on the high end customers. (Exclusion property)

Bunching (grouping consumers of different types and treating them identically) is required sometimes to satisfy the implementability conditions given by 1.3-1.4. However, in most problems bunching is not required if the distribution of types is not too irregular.

Armstrong [2] extended uni-dimensional model proposed by Mussa and Rosen to a multi-dimensional setting. He showed in the multi dimensional model that in general, it's optimal to exclude the low end customers from the market. Armstrong also developed an approach (known as the direct approach) to solve the problem for a class of cases where the implementability conditions were ignored. This method involved following the procedure of the single product case with the use of integration along rays from the origin. An example of the direct approach can be found in [6].

Rochet and Chone [15] showed that the multi dimensional ($m = n$) monopolist's problem for a linear preference function $b(x, y)$ has a unique solution for both the relaxed (Implementability condition is dropped) and the complete case. They also showed bunching is robust in multi dimensional setting unlike in the single product case where bunching could be removed.

Basov [2001] generalized the Rochet and Chone approach for the case when $n \geq m$ retaining the linearity assumption of utilities.

Most of the research in the multi dimensional setting has been done with the assumption that the preference function $b(x, y)$ is linear in types. Carlier [7] formulated the problem for more general set of preference functions, which he defined as $b - convex$ (the definition is given in chapter 3) functions. He also showed there exist a solution for this problem.

$b - convex$ functions generally form a compact but not necessarily convex set. Figalli, Kim and McCann [10] found necessary and sufficient conditions for this set of functions to be convex and make the profit maximization problem faced by the monopolist's into a convex program when $n = m$. They also proved the uniqueness and the stability of the solution.

Pass [13] extended the Figalli, Kim and McCann results for the case $n \neq m$. He showed the necessary condition for the set of b-convex functions to be convex for general values of m and n. This condition is known as b-convexity of Y (Product space). He then proved that when $m > n$ b-convexity of Y implies that the dimensions cannot differ in a meaningful way. The arguments of Carlier, Figalli, Kim and McCann and Pass used tools from optimal transport theory.

Champsaur and Rochet [8] paper studies a market where two firms compete by offering intervals of qualities to the consumers. They made the assumption that there is no bunching. Under that assumption they were able to show an existence of an unique price equilibrium for the duopoly case.

Barelli, Basov, Bugarin and King [3] extended Armstrong's (1996) result on exclusion in multi-dimensional screening models. They made two important contributions in their paper. First they relaxed the strong assumptions Armstrong imposed on preference and consumer types and showed that exclusion is still generic in a less restrictive setting. They also managed to extend the results beyond a monopolistic market in to an oligopoly setting.

Deneckere and Severinov studied a special case of the screening problem, where m=2 and n=1 in their 2004 paper [9]. They managed to reduce the multi-dimensional screening problem to a one-dimensional optimal control problem, whose solution is governed by an ordinary differential equation. They were able to explicitly solve an example of this problem type. The solution to their problem showed that exclusion is not necessary.

The thesis is structured as follows:

In Chapter 2, we discuss the general mathematical background which is needed later, including optimal control theory and multidimensional screening and some of the previous models. In Chapter 3, we introduce the continuous model and the derivation of the model. Here we will derive the multidimensional problem also. In chapter 4, we will solve the problem for $k = 2$ for both discrete and continuous cases. Chapter 5, will be a look at solving $k$-producer case numerically. Chapter 6 is a discussion about the summary of results we obtained and economic aspects of the model.

# Chapter 2

# Preliminaries

## 2.A Optimal Control Problem

**Definition 2.A.1**: *State Variable*

*A state variable is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. Intuitively, the state of a system describes enough about the system to determine its future behaviour.*

**Definition 2.A.2**: *Control Variable*

*A variable qualifies as a control variable if, the variable is subject to the optimizer's choice and the variable will have an effect on the value of the state variable of interest.*

An optimal control problem is the generalization of a calculus of variations problem. It can be used on a problem for which the classical calculus of variations is not applicable. It is an important tool in solving continuous optimization problem of the form:

$$\max_{y(x),u(x)} \int_0^T L(y(x),u(x),x)\,dx$$

$$s.t. \quad y_i'(x) = f_i(y(x),u(x),x), \qquad\qquad i = 1,...,n$$

$$u(x) \in U$$

where $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ , $f_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $y_i'(x)$ denotes the derivative of the function $y_i(x)$. In the above problem $y(x) = (y_1(x),...,y_n(x)) \in \mathbb{R}^n$ and $u(x) = (u_1(x),...,u_m(x)) \in \mathbb{R}^m$ represent the state and control variables respectively, and $x \in [0,\infty)$ and $U$ is a given set in $\mathbb{R}^m$. Assume that $L, f_i, \dfrac{\partial L}{\partial y_j}$ and $\dfrac{\partial f_i}{\partial y_j}$ are continuous with respect to all their arguments for all $i = 1,...,n$ and $j = 1,...,n$. The following theorem gives necessary optimality conditions for the optimal control problem.

**Theorem 2.A.1**: *Pontryagin's Maximum Principle ([14])*

*Let $(y^*(x), u^*(x))$ be optimal for the problem. There exist absolute continuous functions $\lambda(x) = (\lambda_1(x),...,\lambda_n(x)), 0 \leq x \leq T$, such that*

- *for all $u \in U$*

$$H(y^*(x),u,\lambda(x),x) \leq H(y^*(x),u^*(x),\lambda(x),x)$$

*where the Hamiltonian function $H$ is defined as*

$$H(y,u,\lambda,x) = L(y,u,x) + \sum_{i=1}^{n} \lambda_i f_i(y,u,x).$$

- *except at the points of discontinuity of $u^*$,*

$$\frac{\partial H}{\partial y_i}(x, y^*(x), u^*(x), \lambda(x)) = \lambda'_i(x) \quad i = 1, ..., n.$$

- *transversality conditions are satisfied, i.e.,*

$$\lambda(T) = \lambda(0) = 0$$

Moreover if H is a concave function in $x$ and $u$, then the above Pontryagin's Maximum Principle is also sufficient for optimality.

## 2.B Screening Problem

Screening is a contracting problem with hidden information (asymmetric information). The uninformed party (principal) offers a contract to the informed party (agent). Some examples of screening problems are :

- Insurance

  Insuree knows her risk, insurer does not and insurer offers several packages with different premiums and deductibles.

- Pricing

  Buyer knows her valuation of the product, seller does not and seller offers different qualities at different prices, or quantity discounts.

When there is asymmetric information in the market, screening can involve incentives that encourage the better informed to self-select or self-reveal.

## 2.C The Monopolist's Screening Model

Consider a situation where a monopolist produces a good with $n$ quality dimensions, which can be captured by a vector $y = (y_1, y_2, ..., y_n) \in Y \subset \mathbb{R}^n$. For example if the monopolist produces cars, then $y_1$ could be the maximal speed, $y_2$ the safety ratings, $y_3$ the engine efficiency and so on. Furthermore, assume that the consumers who are interested in consuming the monopolist's product have an unobservable type $x = (x_1, x_2, ..., x_n) \in X \subset \mathbb{R}^m$. The consumer types $x$ are differentiated by qualities such as $x_1$, which could be the income, $x_2$ the age, $x_3$ the social status and so on. It is assumed that each consumer is interested in consuming at most one good the monopolist has to offer.

By purchasing a good of quality $y$ a consumer receives a utility of

$$u(x, y, p(y))$$

where $p : Y \to \mathbb{R}$ denotes the price a consumer has to pay for the good, note here that $p(\cdot)$ is the monopolist's control mechanism and, in fact depends only on $x$ through $y$. The consumers also have an outside option of value $u_0(x)$ which is also known as the reservation utility : utility that the consumer obtains if he decides to opt out and pursue other opportunities. For example, buyer of a car has the outside option of buying a non-luxury car and he will only buy a luxury car only if the utility value of a luxury car is greater than the outside option value. The consumers' strive to maximize their utility. Denote by $y(x) \in \arg\max_x (u(x, y, p(y)))$ the good that maximizes the consumer's (of type $x$) utility.

We assume that the consumer's type $x$ is distributed according to an open, bounded continuous density function $f(.) : X \to \mathbb{R}_+$ and that $u$ is a continuous function, strictly increasing on both $x$ and $y$ and strictly decreasing in $p$. Moreover, we assume that $u(x, y, p)$ is twice continuously differentiable in $y$ and $x$ and both $u(x, y, .)$ and $\nabla u(x, y, .)$ are analytic.

Denote by $c(y) : Y \to \mathbb{R}$ the amount the monopolist has to pay to produce a good with quality characteristics $y$. The consumer has the option of not buying a good; we represent this with a null good $y^0 \in Y$, which is a good the monopolist offers at cost (i.e. $p(y^0) = c(y^0)$).

Now, to maximize her profits the monopolist selects a continuous $p$ to solve

$$\max_{p(\cdot)} \int_X [p(y(x)) - c(y(x))] f(x)\, dx \tag{2.1}$$

$$s.t. \quad y(x) \in \arg\max u(x, y, p(y)) \tag{2.2}$$

$$\max_y u(x, y, p(y)) \geq u_0(x) \tag{2.3}$$

Usually it is assumed that the consumer's utilities are quasi-linear with respect to $p(y)$:

$$u(x, y, p(y)) = b(x, y) - p(y) \tag{2.4}$$

Where $b(x, y)$ is the preference function a consumer $x$ has for good $y$. Then $u_0(x)$ (utility of the null good) is:

$$u_0(x) = b(x, y^0) - c(y^0)$$

which is the utility consumer $x$ derives from opting out of good $y$ (recall that $p(y^0) = c(y^0)$).

From now on this form of the consumer utility is assumed. We will also assume for simplicity that m = n, although most of the results presented later can be generalized to cases where $n \geq m$.

The problem given by (2.1)-(2-3) is very difficult to work with, as $y(x)$ in the functional depends indirectly on the control $p(y)$ and it is not clear at this point that $p(y)$ determines $y(x)$ uniquely. Now to proceed with the analysis of the model, we wish to change the variables in order to get rid of the uncomfortable constraints. Therefore, we introduce the indirect utility function, the consumer's surplus S(x):

$$S(x) = \max_{y \in Y}(b(x, y) - p(y)) \tag{2.5}$$

Thus, S(x)($\geq u_0(x)$) is the surplus of a consumer type $x$ who chooses the bundle $y$ that maximizes his utility. Using this on some $y$ that maximizes consumer's utility one can solve,

$$p(y(x)) = b(x, y(x)) - S(x) \tag{2.6}$$

It is shown in Carlier (2001) [7] that S is continuous, almost everywhere differentiable and satisfies the envelope conditions:

$$\nabla S(x) = \nabla_x b(x, y(x))$$

The continuity of $S$ implies that the optimal tariff will be continuous, proof can be found in [7].

Our reformulation will also require the following definitions.

**Definition 2.C.1**: An allocation $y(\cdot) : X \to Y$ is called implementable if there exists a continuous function $p(\cdot) : Y \to \mathbb{R}$ such that

$$y(x) \in \arg\max_{x \in \mathbb{R}^n_+} b(x,y) - p(y)$$

for any $x \in X$.

The question whether or not an allocation is implementable can be answered with the Theorem 2.C.1 below, for which we need some definitions.

**Definition 2.C.2**: The function $S^b(y)$ defined by

$$S^b(y) = \max_{x \in X}(b(x,y) - S(x))$$

is called the b-conjugate of S(x).

**Definition 2.C.3**: The function $S^{bb}(x)$ defined by

$$S^{bb}(x) = \max_{y \in Y}(b(x,y) - S^b(y))$$

is called the b-bi conjugate of S(x).

Functions of the form given by Definition 2.C.3 are known as *b*-convex functions.

**Definition 2.C.4**: $b(x,y)$ is said to satisfy the generalized single-crossing (GSC) property for all $x \in X$ if,

$$\nabla_x b(x,y_1) = \nabla_x b(x,y_2) \Rightarrow y_1 = y_2$$

**Theorem 2.C.1**: *Assume* $b(x, y)$ *is continuous in both arguments continuously differentiable in* $x$ *and satisfies GSC. An allocation* $y(x)$ *and surplus* $S(x)$ *are implementable if and only if the following conditions hold.*

- $S(x)$ *is continuous and a.e. differentiable.*
- $y(x)$ *is upper hemicontinuous*[1] *and the envelope condition holds a.e.*
- $S(x) = S^{bb}(x)$.

For the proof see [7]. We can now reformulate the original problem given by (2.1)-(2.3) (with quasi-linear utilities) using $S$ rather than $p$.

The profit of the monopolist is :

$$p(y(x)) - c(y(x)) \tag{2.7}$$

for an allocation $y(x)$ which maximizes utility of consumers of type $x$. Then using surplus function S(x) given by (2.5) and (2.6) we can rewrite (2.7) as:

$$b(x, y) - S(x) - c(y(x)) \tag{2.8}$$

Now we can rewrite (2.1) as:

$$\max_{y} \int_{X} [(b(x, y) - S(x) - c(y(x)))]f(x)\,dx \tag{2.9}$$

and by noting that the implementability condition (given by (2.2)) of $y(x)$ can be replaced by an envelope condition of $S(x)$ and the condition three of

[1] A correspondence $f : X \rightrightarrows Y$ is upper hemicontinuous if it has a closed graph and the image of $f$ is compact.

Theorem 2.C.1. Furthermore, a function $S_0(x)$ can be defined by $S_0(x) = b_0(x, y^0) - p_0(y^0(x))^2$ where $b_0 - p_0$ represents the utility of the outside option to the consumer and thus the multidimensional screening problem takes the form:

$$\max_{y(\cdot)} \int_X [(b(x, y(x)) - S(x) - c(y(x)))] f(x)\, dx \tag{2.10}$$

$$s.t. \nabla S(x) = \nabla_x b(x, y(x)) \tag{2.11}$$

$$S(x) = S^{bb}(x) \tag{2.12}$$

$$S(x) \geq S_0(x) \tag{2.13}$$

Here we are maximizing over the set of $b$-convex functions and $y(x)$ is uniquely determined by condition (2.11) in terms of S(x). Carlier [7] proved the existence of at least one solution to the above problem. By dropping constraint (2.12) in the complete problem, it boils down to a problem of calculus of variations with inequality constraints and this is known as the relaxed problem. The relaxed problem is an optimal control problem with a state variable $S$ and a vector of control variables $x$. Basov [6] proved that under certain quite general assumptions the relaxed problem has a unique solution. The relaxed version of the multidimensional screening problem was first introduced by Wilson [17] and Armstrong [2].

The complete problem is very difficult to work with and to solve. A full solution to the complete problem is known for the uni-dimensional case (i.e. $m = n = 1$) [6]. Rochet and Chone [15] presented an approach (for linear

[2] $p(y^0))$ represents the price of the outside option.

$b(x, y) = xy$) when $m = n > 1$ and their approach was generalized for the case $n \geq m$ by Basov [4]. In general no method is known for the case $m > n$.

## 2.D Solving the problem in 1-dimension (Hamiltonian Approach)

The Hamiltonian approach for the monopolist's problem was first developed by Basov for the special case considered by Rochet and Chone, i.e. the case when the utilities are linear in types:

$$b(x, y) =< x, y >= \sum_{i=1}^{m} x_i y_i$$

Basov later generalized the approach to handle more general $b()$ [5] . Interpreting the relaxed problem as a problem of control theory, we can define the Hamiltonian for the problem given by (2.10),(2.11),(2.12) and (2.13).

$$H(S, y, x, \lambda) = (b(x, y) - S(x) - c(y))f(x) + < \lambda, \nabla_x b(x, y) > + \eta(S(x) - S_0(x))$$

Now the first order optimality conditions for the relaxed problem can be stated as,

**Theorem 2.D.1**: *Suppose the surplus function $S^*(\cdot)$ solves the relaxed problem. Then there exists continuously differentiable vector function $\lambda : X \to \mathbb{R}^m$ and continuous function $\eta : X \to \mathbb{R}_+$ and continuous a.e. differentiable function $S(\cdot)$ such that $S^*(x) = max(S(x), S_0(x))$ and the following first* *order conditions hold:*

$$\boldsymbol{\nabla} \cdot \lambda = -\frac{\partial H}{\partial S} a.e.on \Omega \tag{2.14}$$

$$< \lambda, b >= 0 a.e.on \partial\Omega \tag{2.15}$$

$$\eta \geq 0, S(x) \geq S_0(x) \tag{2.16}$$

$$\eta(S(x) - S_0(x)) = 0 \tag{2.17}$$

$$x \in \arg\max H(S, y, x, \lambda) \tag{2.18}$$

The first equation in the theorem governs the evolution of the co state vector. The next equation is a straightforward generalization of the transversality condition, the third and fourth equations are the complementary slackness condition and the fifth equation is Pontryagin's maximum principle.

We will end this chapter by presenting a solved example of the monopolist's problem in the uni-dimensional case [6]. Assume that the consumer's preference is given by :

$$b(x, y) = xy$$

where $x$ is distributed uniformly on $(0, 1)$ and the monopolist's cost is given by :

$$c(y) = \frac{y^2}{2}$$

and the value of the outside option is zero (i.e. $u_0(x) = 0$). Then the problem is of the following form.

$$\max \int_0^1 xy(x) - S(x) - \frac{y^2(x)}{2}\, dx$$

$$s.t. S'(x) = y(x)$$

$$S(0) = 0$$

$$S(x) \text{ is convex}$$

Note that convexity is equivalent to $y'(x) = S''(x) \geq 0$.

The Hamiltonian for the problem is :

$$H(S, y, x, \lambda) = xy - S(x) - \frac{y^2}{2} + \lambda y$$

The first order conditions are :

$$\lambda'(x) = -\frac{\partial H}{\partial x} = 1 \tag{2.19}$$

$$\lambda(1) = 0 \tag{2.20}$$

$$\frac{\partial H}{\partial y} = x - y + \lambda(x) = 0 \tag{2.21}$$

and it can be shown using (2.19) and (2.20) that $\lambda(x) = x - 1$. Then y(x) is given by

$$y(x) = 2x - 1$$

$$y(x) = \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2} \\ 0 & \text{if } x \leq \frac{1}{2} \end{cases}$$

$y(x)$ is increasing in $x$ and therefore implementable. To find $p(y)$ we will integrate the envelope condition $S'(x) = y$.

$$S(x) = \begin{cases} x^2 - x + c & \text{if } x \geq \dfrac{1}{2} \\ c - \frac{1}{4} & \text{if } x \leq \dfrac{1}{2} \end{cases}$$

where constant $c$ is found using $S(0) = 0$, which implies $c = \frac{1}{4}$. Therefore,

$$p(y(x)) = xy(x) - S(x) = x^2 - \frac{1}{4}$$

Using y(x) we can obtain price in terms of $y$.

$$p(y) = \frac{1}{4}(y^2 + 2y)$$

# Chapter 3

# The Model

We are interested in a situation where there are several monopolists in the market who are producing complementary goods. A complementary good is a good whose use is inter related with the use of an associated good such that a demand for one (tires, for example) generates demand for the other (gasoline, for example). In the model that we are considering each consumer's preference depends on the bundle of goods he assembles, one from each producer. The consumer's objective is to find the best bundle that maximizes his utility[3] and each producer's objective is to maximize his profit depending on the bundle that the consumer's choose. Because of the fact that the goods are inter related with each other, we can expect that the profit maximizing prices are going to be affected by the prices of these associated goods. Then the pricing schedules implemented by all the firms depend on the other product's pricing schedules.

[3]If the consumer's preference is given by $b(x, y_1, ..., y_k)$ , where consumer types $x \in X \subseteq \mathbb{R}^n$ and product type $y_i \in Y_i \subseteq \mathbb{R}^{m_i}$. Also the prices for product $i$ is given by $p_i(y_i)$, then the utility function $u(x, y_1, ..., y_k, p_1, ..., p_k)$ is equal to :

$$u(x, y_1, ..., y_k, p_1, ..., p_k) = b(x, y_1, ..., y_k) - \sum_{i=1}^{k} p_i(y_i)$$

For example, consider two firms, one producing cars and the other issuing insurance packages. The consumer is interested in either buying one product or a bundle of both. If the consumer decides to buy a high end car he will look for an insurance package with more coverage and if he decides to buy a low end car he might go for a low end insurance package, so the consumer's preference towards one good is affected by the associated good.

What we are interested in is to see whether the above mentioned problem can be modelled in a similar way to the monopolist model. In this chapter we will introduce our model for the *k-producer* problem. First we will look at the general model for the multi-dimensional case.

## 3.A Multi-dimensional case

Suppose there are $k$ - producers in the market where, the $i^{th}$ producer produces a good with $m_i$ quality dimensions, which can be captured by a vector $Y_i \subseteq \mathbb{R}^{m_i}$, and the consumer's type $x$ is distributed on $X \subseteq \mathbb{R}^n$ according to a positive, continuous density function $f(\cdot)$. Now we will define some functions that are required to set up the model.

The cost for producer $i$ to produce good $i$ is given by $c_i(y_i)$, where $c_i : Y_i \to \mathbb{R}$ is a increasing and differentiable function.

Producer $i$ has the ability to choose a pricing function $p_i : Y_i \to \mathbb{R}$ for his product. Here $p_i(y_i)$ represents the price that the producer charges for the good $y_i$.

In our problem consumer's are going to buy a bundle of the $k$ goods that are offered and we will define $b : X \times Y_1 \times \cdots \times Y_k \to \mathbb{R}$ as the consumer's preference function. Here $b(x, y_1, y_2, ..., y_k)$ represents the preference that a consumer's of type $x$ has towards a bundle of goods $(y_1, y_2, ..., y_k)$ (or set of allocations of the $k$ goods).

The consumer's objective is to find the bundle that maximizes his utility. The utility of a consumer of type $x$ is given by function $u : X \times Y_1 \times \cdots \times Y_k \to \mathbb{R}$,

$$u(x, y_1, y_2, ..., y_k, p_1, ..., p_k) = b(x, y_1, y_2, ..., y_k) - \sum_{i=1}^{k} p_i(y_i)$$

That is, consumer's choose to buy the bundle of goods $(y_1, y_2, ..., y_k)$ with $y_i(x) \in \arg\max u(x, y_1, y_2, ..., y_k, p_1, ..., p_k)$. Now each producer's goal is to maximize his profit according to the bundle that the consumer's choose. We will call $Profit_i$ the resulting profits for the $i^{th}$ producer when each of the k producer's have set their prices. $Profit_i$ depend on the set of $(p_1, ..., p_k)$ prices. We now introduce $y_i^0$ , which is the null good for good of type $y_i$. We assume $c(y_i^0) = p(y_i^0) = 0$. This represents the consumers option not to buy a good, and the monopolist's obligation not to charge him if he does so.

The profit that the $i^{th}$ producer gains from customer type $x$ is given by :

$$p_i(y_i(x)) - c_i(y_i(x))$$

$$y_i(x) \in \arg\max u(x, y_1, y_2, ..., y_k, p_1, ..., p_k)$$

Then, the total profit of the $i^{th}$ producer gains from all the customer types $x \in X$ is :

$$Profit_i = \int_X [p_i(y_i(x)) - C_i(y_i(x))] f(x) dx \tag{3.1}$$

We can write $k$ total profit functions of the form (3.1) for the $k$ producers. Then our problem boils down to finding the $k$ set of pricing functions that would maximize each producer's total profit. This set of pricing functions are an equilibrium in our model. An equilibrium in our model is a situation where none of the producer's has an incentive to change his prices. We now present the definition of an equilibrium in the model in a formal way.

**Definition 3.A.1**: An equilibrium set of prices $(\bar{p_1}, ..., \bar{p_k})$ is such that, for all $i$, $p_i \to Profit_i(\bar{p_1}, ..., \overline{p_{i-1}}, p_i, \overline{p_{i+1}}, ..., \bar{p_k})$ is maximized at $p_i = \bar{p_i}$.

Therefore, it's not beneficial for any producer to move away from the equilibrium prices.

Next, we will define the surplus function which we are going to use in Proposition 3.A.1.

**Definition 3.A.2**: The function $S(x)$ is defined by,

$$S(x) = \max_{y_1,\dots,y_k} \{b(x, y_1, y_2, ..., y_k) - \sum_{i=1}^{k} p_i(y_i)\}$$

is called the surplus function.

The consumers also has the choice of opting out of buying any one of the goods $y_i$ in the bundle. We will represent the opt out option using $S_0(x; p_1, ..., p_k)$ which is defined as follows :

$$S_0(x; p_1, ..., p_k) = \max_{y_1,\dots,y_k} \{b(x, y_1, ..., y_i^0, ..., y_k) - \sum_{j \neq i} p_j(y_j)\},$$

where $y_i^0$ represents that the consumer decided to opt out of buying product $y_i$. $S_0$ represents the best surplus consumer $x$ can obtain without buying good of type $y_i$.

The surplus value of consumer's choice will be greater than the value of the opt out surplus. i.e.

$$S(x) \geq S_0(x)$$

**Proposition 3.A.1**: *Suppose the producer's j fix their prices $p_j(y_j)$ for all $j \neq i$. Then the $i^{th}$ monopolist's problem of maximizing his profits is equivalent to the classical monopolist problem with an effective preference :*

$$v_{p_1,p_2,\ldots,p_k}(x, y_i) = \max_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k} \{b(x, y_1, y_2 \ldots, y_k) - \sum_{j \neq i} p_j(y_j)\}$$

*Proof.* We want to maximize the $i^{th}$ producers profit for the allocation set $(y_1(x), y_2(x), ..., y_k(x))$, which maximizes the consumers utility. Then our problem is to maximize :

$$Profit_i = \int_X [p_i(y_i(x)) - c_i(y_i(x))]f(x)dx \tag{3.2}$$

$$s.t. \quad (y_1(x), y_2(x), ..., y_k(x)) \in \arg\max_{y_1,y_2,\ldots,y_k} \{b(x, y_1, y_2 \ldots, y_k) - \sum_{j=1}^{k} p_j(y_j)\} \tag{3.3}$$

for $i = 1, 2, ..., k$

subject to $p_i(y_i^0) = 0$, which is the opt out condition.

Instead of tackling problem given by (3.2)-(3.3), where we are considering all $k$ allocations, we can simplify it into k-monopolist problems by fixing the prices $p_j(y_j)$ for all $j \neq i$ and by introducing $v_{p_1,p_2,\ldots,p_k}$ :

$$v_{p_1,p_2,\ldots,p_k}(x, y_i) = \max_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k} \{b(x, y_1, y_2 \ldots, y_k) - \sum_{j \neq i} p_j(y_j)\}$$

The corresponding monopolist's problem is :

$$Profit_i = \int_X [p_i(y_i(x)) - c_i(y_i(x))] f(x) dx \tag{3.4}$$

$$s.t. \quad y_i(x) \in \arg\max_{y_i} \{v_{p_1,p_2,\ldots,p_k}(x, y_i) - p_i(y_i)\} \tag{3.5}$$

The opt out condition is $p_i(y_i^0) = 0$, so it's clear that the opt out condition is the same for both problems.

Now the transformed problem given by (3.4)-(3.5) is of the form of the classical problem. We can write $k$ such problems for the $k$ producers.

Note that the functions to be maximized (3.2) and (3.4) are the same for both the classical model and the $k$-producer model. If we can show that the constraints (3.3) and (3.5) are equivalent, then the reduction is possible.

First lets suppose (3.5) is true.

We need to find $y_j(x)$ for $j \neq i$ that satisfies (3.3).

Choose $y_j(x), \quad j \neq i$. So that

$$v_{p_1,p_2,\ldots,p_k}(x, y_i(x)) = b(x, y_1(x), y_2(x) \ldots, y_k(x)) - \sum_{j \neq i} p_j(y_j(x))$$

By assumption $y_i(x) \in \arg\max_{y_i} \{v_{p_1,p_2,\ldots,p_k}(x, y_i(x)) - p_i(y_i(x))\}$. So,

$$b(x, y_1(x), y_2(x) ..., y_k(x)) - \sum p_j(y_j(x)) = v_{p_1,p_2,...,p_k}(x, y_i(x)) - p_i(y_i(x))$$

and

$$v_{p_1,p_2,...,p_k}(x, y_i(x)) - p_i(y_i(x)) \geq v_{p_1,p_2,...,p_k}(x, z_i) - p_i(z_i)$$

for any $z_i$. Note that,

$$v_{p_1,p_2,...,p_k}(x, z_i) - p_i(z_i) \geq b(x, z_1, z_2 ..., z_k) - \sum p_j(z_j) \qquad (*)$$

for any $z_j, \quad j \neq i$. Therefore by $(*)$, we get that (3.5) implies (3.3).

Now suppose (3.3) is true. For an allocation $\{y_1(x), y_2(x) ..., y_k(x)\}$ we will show $y_i(x) \in \arg\max_{y_i}\{v_{p_1,p_2,...,p_k}(x, y_i) - p_i(y_i)\}$. We have,

$$b(x, y_1(x), y_2(x) ..., y_k(x)) - \sum p_j(y_j(x)) \geq b(x, z_1, z_2 ..., z_k) - \sum p_j(z_j) \qquad (**)$$

for all $z_1, z_2 ..., z_k$. Then rewriting $b(x, y_1(x), y_2(x) ..., y_k(x)) - \sum p_j(y_j(x))$ as $v_{p_1,p_2,...,p_k}(x, y_i(x)) - p_i(y_i(x))$ and using equation $(**)$, we get the following equation maximizing $(**)$ over $z_j$, $j \neq i$,

$$b(x, y_1(x), y_2(x) ..., y_k(x)) - \sum_{j \neq i} p_j(y_j(x)) - p_i(y_i(x)) = v_{p_1,p_2,...,p_k}(x, y_i(x)) - p_i(y_i(x))$$

$$v_{p_1,p_2,...,p_k}(x, y_i(x)) - p_i(y_i(x)) \geq v_{p_1,p_2,...,p_k}(x, z_i(x)) - p_i(z_i(x))$$

for all $z_i$. This completes the proof. □

Now we can define effective surplus function using $v_{p_1,p_2,\ldots,p_k}$.

**Definition 3.A.3**: The effective surplus function $S_v(x)$ is defined by,

$$S_v(x) = \max_{y_i}\{v_{p_1,p_2,\ldots,p_k}(x, y_i) - p_i(y_i)\}$$

where,

$$v_{p_1,p_2,\ldots,p_k}(x, y_i) = \max_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k}\{b(x, y_1, y_2\ldots, y_k) - \sum_{j\neq i} p_j(y_j)\}$$

We are only going to consider special set of preference functions in our model. Proposition 3.A.2 give the properties those functions need to have to satisfy the GSC property. The GSC property is important because it uniquely determines $y_i(x)$ in terms of $S(x)$.

**Proposition 3.A.2**: *Suppose $X, Y_i \subseteq \mathbb{R}^n$ for $i = 1, 2, \ldots, k$, $b : X \times Y_1 \times \cdots \times Y_k \to \mathbb{R}$ and $b(x, y_1, y_2\ldots, y_k) = h(x+y_1+y_2+\cdots+y_k)$ , where $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth, strictly convex function. Then $v_{p_1,p_2,\ldots,p_k}(x, y_i) = v_{p_1,p_2,\ldots,p_k}(x+y_i)$ is strictly convex and satisfies the generalized single crossing property.*

*Proof.*

$$v_{p_1,p_2,\ldots,p_k}(x, y_i) = \max_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k}\{b(x, y_1, y_2\ldots, y_k) - \sum_{j\neq i} p_j(y_j)\} \tag{3.6}$$

$$= \max_{y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k}\{h(x + y_1 + y_2 + \cdots + y_k) - \sum_{j\neq i} p_j(y_j)\} \tag{3.7}$$

$$= v_{p_1,p_2,\ldots,p_k}(x + y_i) \tag{3.8}$$

Because $h(x+y_1+y_2+\cdots+y_k)$ is strictly convex, it implies that $v_{p_1,p_2,\ldots,p_k}(x, y_i) =$

$v_{p_1,p_2,\ldots,p_k}(x+y_i)$ is also strictly convex, as a maximum of convex functions [16].

To show that $v_{p_1,p_2,\ldots,p_k}(x, y_i) = v_{p_1,p_2,\ldots,p_k}(x+y_i)$ satisfies the GSC property (Definition 2.C.4), we use the fact that a gradient of a convex function, $z \to \nabla v(z)$ is one to one [16]. Now if we consider $(x + y^*)$ and $(x + y^{**})$, we have

$$\nabla_x v_{p_1,p_2,\ldots,p_k}(x + y^*) = \nabla_x v_{p_1,p_2,\ldots,p_k}(x + y^{**}) \Rightarrow x + y^* = x + y^{**}$$

Then we can simply cancel out the $x$ to get the required result.

$$x + y^* = x + y^{**} \Rightarrow y^* = y^{**}$$

Therefore $v_{p_1,p_2,\ldots,p_k}(x, y_i)$ satisfies the GSC property. □

Let's consider a preference function $b(\cdot)$ of the form in Proposition 3.A.2. Then the effective preference function satisfies the GSC property. The use of an effective surplus function converts this problem into $k$ identical monopolist problems as stated in Proposition 3.A.1. Let us look at the problem in terms of product $i$.

The profit function of product $i$ is calculated as follows,

$$Profit_i = \int\limits_X [p_i(y_i(x)) - c_i(y_i(x))]f(x)dx \tag{3.9}$$

Now, using the definition of effective consumer surplus function we can write an expression for $p_i(y_i)$, for an allocation $y_i(\cdot)$ which maximizes the utility of consumer's of type $x$ as:

$$p_i(y_i) = v_{p_1,p_2,\ldots,p_k}(x, y_i) - S_v(x) \tag{3.10}$$

We can rewrite equation (3.9) using (3.10) as:

$$Profit_i = \int\limits_X [v_{p_1,p_2,\ldots,p_k}(x, y_i(x)) - S_v(x) - C_i(y_i(x))]f(x)dx$$

Similarly, we can get $k$ such profit functions for the $k$ producer's.

We know that the effective preference function $v_{p_1,p_2,\ldots,p_k}(x, y_i)$ satisfies the GSC property, therefore we can use the theorem 2.C.1 to set the required implementability conditions for the allocations. Then the general *k-producer* model (KPM) takes the form :

$$Profit_i = \int\limits_X [v_{p_1,p_2,\ldots,p_k}(x, y_i) - S_v(x) - C_i(y_i(x))]f(x)dx \tag{3.11}$$

$$s.t. \quad \nabla S_v(x) = \nabla_x v_{p_1,p_2,\ldots,p_k}(x, y_i(x)) \tag{3.12}$$

$$S_v(x) \geq S_0(x; p_1, \ldots p_k) \tag{3.13}$$

where $i = 1, ..., k$. The results in Carlier [7] then ensures existence of an optimal $S$ for each fixed $p_1, ..., p_k$.

The equations of type (3.11) is much easier to analyse than (3.2) because (3.11) can become a convex problem under some assumptions for the $v_{p_1,p_2,\ldots,p_k}(x, y_i)$. The conditions that $v_{p_1,p_2,\ldots,p_k}(x, y_i)$ need to satisfy for the profit problem to be convex is given in Figalli, Kim and McCann (2011) [10]. If one could impose conditions on $b()$ which ensure $v_{p_1,p_2,\ldots,p_k}$ satisfies the conditions given by [10], for any $p_1, ..., p_k$, one might hope to apply fixed point theorems to deduce existence of an equilibrium. This seems difficult. The analysis of the $k - producer$ problem is very difficult for the multi-dimensional case. It's significantly easier in the 1-dimensional case and in the next section we discuss the method to solve the $k - producer$ problem in 1-dimension.

## 3.B 1-dimensional case

We present a proposition below that we will use to model the 1-dimensional case. Take $Y_i, X = [0, 1]$.

**Proposition 3.B.1**: *If,*

$\frac{\partial^2 b}{\partial x \, \partial y_i} > 0 \quad \textit{for all } i$

$\frac{\partial^2 b}{\partial y_i \, \partial y_j} > 0 \quad \textit{for all } i \neq j$

*Then,*

$\frac{\partial^2 v_{p_1, p_2, \ldots, p_k}}{\partial x \, \partial y_i} > 0 \quad \textit{for all } i$

*Proof.*

$$v_{p_1, p_2, \ldots, p_k}(x, y_i) = \max_{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k} \{b(x, y_1, y_2 ..., y_k) - \sum_{j \neq i} p_j(y_j)\} \tag{3.14}$$

For $j \neq i$, choose $y_j(x, y_i) \in \arg\max\{b(x, y_1, y_2 ..., y_k) - \sum_{j \neq i} p_j(y_j)\}$. Now, taking the derivative of $v_{p_1, p_2, \ldots, p_k}$ w.r.t $x$ and $y_i$,

$$\frac{\partial v_{p_1, p_2, \ldots, p_k}(x, y_i)}{\partial x} = \frac{\partial b(x, y_1(x, y_i), ..., y_k(x, y_i))}{\partial x} \tag{3.15}$$

by the well known envelope condition. Differentiating and using the chain rule, we obtain,

$$\frac{\partial^2 v_{p_1,p_2,\dots,p_k}(x,y_i)}{\partial x\,\partial y_i} = \frac{\partial^2 b}{\partial x\,\partial y_i} + \sum_{j\neq i} \frac{\partial^2 b}{\partial x\,\partial y_j}\frac{\partial y_j}{\partial y_i} \tag{3.16}$$

By assumption both $\frac{\partial^2 b}{\partial x\,\partial y_i}$ and $\frac{\partial^2 b}{\partial y_i\,\partial y_j}$ are positive, so if we can show $\frac{\partial y_j}{\partial y_i}$ is also positive then our proof is complete. By maximality of the $y_j(x, y_i)$,

$$\frac{\partial b(x, y_1(x,y_i), \dots, y_k(x,y_i))}{\partial y_j} - \frac{\partial p_j(y_j(x,y_i))}{\partial y_j} = 0 \tag{3.17}$$

Differentiating with respect to $y_i$ yields,

$$\frac{\partial^2 b}{\partial y_j\,\partial y_i} + \frac{\partial^2 b}{\partial y_j^2}\frac{\partial y_j}{\partial y_i} - \frac{\partial^2 p_j}{\partial y_j^2}\frac{\partial y_j}{\partial y_i} = 0 \Rightarrow \frac{\partial y_j}{\partial y_i} = \frac{-\frac{\partial^2 b}{\partial y_j\,\partial y_i}}{\frac{\partial^2 b}{\partial y_j^2} - \frac{\partial^2 p_j}{\partial y_j^2}} \tag{3.18}$$

The denominator is negative because of maximality, therefore $\frac{\partial y_j}{\partial y_i}$ is positive.

□

The condition $\frac{\partial^2 b}{\partial x\,\partial y_i} > 0$ of Proposition 3.B.1 is the equivalent of the GSC property in the 1-dimensional case and it's also known as the Spence- Mirrlees [11] condition.

If $\frac{\partial^2 b}{\partial x\,\partial y_i} > 0$, then $b(\cdot)$ is supermodular and $b(\cdot)$ is supermodular if and only if $b_y$ increases in x. Using supermodularity we get the following equations for $x > x^*$ :

$$b_y(x,y) \geq b_y(x^*,y)$$

$$b(x, y+\epsilon) - b(x,y) \geq b(x^*, y+\epsilon) - b(x^*, y) \qquad (***)$$

where $\epsilon > 0$.

From the equation $(***)$, it's clear that a higher end consumer has a greater preference for a high end good over a low end one than lower end consumer does. Therefore, 1-dimensional Spence- Mirrlees condition implies that higher end consumers are more likely to go for a high end good rather than a low end good.

By similar reasoning the new condition $\frac{\partial^2 b}{\partial y_i \partial y_j} > 0$ that we propose in Proposition 3.B.2 implies that if the consumer buys a one type of high end good he is more likely to pair it with a high end complimentary good rather than pairing it with a low end complimentary good. This seems quite natural economically.

Now, we can present the 1-dimensional model as follows:

$$Profit_i = \int_X [v_{p_1,p_2,...,p_k} - S_v(x) - c_i(y_i(x))]f(x)dx \tag{3.19}$$

$$s.t. \quad S_v'(x) = \frac{\partial v_{p_1,p_2,...,p_k}(x, y_i)}{\partial x}, \quad S_v(0) = 0 \tag{3.20}$$

where $i = 1, ..., k$

We can apply the Hamiltonian approach introduced in setion 2.D of chapter 2 to solve (KPM).

The Hamiltonian for product 1 is :

$$H = [v_{p_1,p_2,...,p_k}(x, y_1) - S_v(x) - c_1(y_1(x))]f(x) + \lambda(x)\frac{\partial v_{p_1,p_2,...,p_k}(x, y_1)}{\partial x}$$

The first order conditions are,

$$\lambda'(x) = -\frac{\partial H}{\partial x} = f(x) \tag{3.21}$$

$$\lambda(1) = 0 \tag{3.22}$$

$$\frac{\partial H}{\partial y_1} = [\frac{\partial v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial y_1} - c_1'(y_1(x))]f(x) + \lambda(x)\frac{\partial^2 v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial x\,\partial y_1} = 0 \tag{3.23}$$

We can show that $\lambda(x) = F(x) - 1$ by using (3.21) and(3.22). Here $F(x)$ is the cumulative distribution function of $f(x)$. Using the first order conditions with $\lambda(x)$ we end up with the following equation :

$$[\frac{\partial v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial y_1} - c_1'(y_1(x))]f(x) + \lambda(x)\frac{\partial^2 v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial x\,\partial y_1} = 0 \tag{3.24}$$

$$c_1'(y_1(x)) = \frac{\partial v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial y_1} - [\frac{1-F(x)}{f(x)}]\frac{\partial^2 v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial x\,\partial y_1} \tag{3.25}$$

We can similarly get equations of type (3.25) for each producer. Solving those system of $k$ partial differential equations on any interval where the monotonicity constraints don't bind, we can find an equilibrium of allocations $(\overline{y_1(x)}$ , $\overline{y_2(x)}$,...,$\overline{y_k(x)})$ for the given model. Generally, its difficult to solve this problem for general utility and cost functions. We present the steps for a solution to a special case under certain assumptions. We will present it as a proposition.

**Proposition 3.B.2**: *Our assumptions are,*

- *The consumer types and product types are distributed according to density function $f(\cdot)$.*
- *The preference function is of the form : $b(x, y_1, y_2, ..., y_k) = xy_1 + xy_2 +$*

$... + xy_k + y_1y_2 + y_1y_3 + ... + y_1y_k + ... + y_{k-1}y_k.$

*Then the equilibrium of allocations* $(\overline{y_1(x)}\ ,\ \overline{y_2(x)},...,\overline{y_k(x)})$ *for the given model satisfies the following system of* $k$ *ODEs:*

$$c_j'(\overline{y_j(x)}) = x + \sum_{i \neq j} \overline{y_i(x)} + \frac{(F(x) - 1)}{f(x)}[1 + \frac{\sum_{i \neq j} \overline{y_i'(x)}}{1 + \overline{y_j'(x)}}] \tag{3.26}$$

*where* $i = 1, ..., k.$

*whenever the constraint* $y_i(x) \geq 0$ *is non binding.*

*Proof.* The preference function is :

$$b(x, y_1, y_2, ..., y_k) = xy_1 + xy_2 + ... + xy_k + y_1y_2 + y_1y_3 + ... + y_1y_k + ... + y_{k-1}y_k$$

Then the effective preference function satisfies the Spence - Mirrlees condition [11] (Proposition 3.B.1). Now we can use the Hamiltonian approach to solve (KPM).

The Hamiltonian for product 1 is:

$$H = [v_{p_1,p_2,...,p_k}(x, y_1) - S_v(x) - c_1(y_1(x))]f(x) + \lambda(x)\frac{\partial v_{p_1,p_2,...,p_k}(x, y_1)}{\partial x}$$

The first order conditions are,

$$\frac{\partial \lambda}{\partial x} = -\frac{\partial H}{\partial x} = f(x) \tag{3.27}$$

$$\lambda(1) = 0 \tag{3.28}$$

$$\frac{\partial H}{\partial y_1} = [\frac{\partial v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial y_1} - c_1'(y_1(x))]f(x) + \lambda(x)\frac{\partial^2 v_{p_1,p_2,\ldots,p_k}(x,y_1)}{\partial x\,\partial y_1} = 0 \tag{3.29}$$

Now, we show that all $y_i$'s are functions of $(x+y_1)$ type.

$$v_{p_1,p_2,\ldots,p_k}(x,y_1) = \max_{y_2,\ldots,y_k}\{b(x,y_1,y_2,...,y_k) - \sum_{j\neq 1} p_j(y_j)\}$$

$$= \max_{y_2,\ldots,y_k}\{xy_1 + xy_2 + ... + xy_k + y_1y_2 + y_1y_3 + ... + y_1y_k + ... + y_{k-1}y_k - \sum_{j\neq 1} p_j(y_j)\}$$

$$= xy_1 + \max_{y_2,\ldots,y_k}\{xy_2 + ... + xy_k + y_1y_2 + y_1y_3 + ... + y_1y_k + ... + y_{k-1}y_k - \sum_{j\neq 1} p_j(y_j)\}$$

$$= xy_1 + \max_{y_2,\ldots,y_k}\{y_2(x+y_1+y_3+...+y_k) + \cdots + y_{k-1}(x+y_1+y_k) + y_k(x+y_1) - \sum_{j\neq 1} p_j(y_j)\}$$

Note the function inside the max is a function only of $x+y_1$. Therefore, the maximizing $y_j(x,y_1)$, for $j \geq 2$, is also a function of $x+y_1$. Therefore, $y_j(x,y_1) = y_j(x+y_1)$.

Using the envelope condition on $v$, we can obtain the following derivative,

$$\frac{\partial v_{p_1,p_2,\ldots,p_k}}{\partial x} = y_1 + \sum_{i=2}^{k} y_i(x+y_1)$$

Now differentiating with respect to $y_1$ yields,

$$\frac{\partial^2 v_{p_1,p_2,\ldots,p_k}}{\partial x\,\partial y_1} = 1 + \sum_{i=2}^{k} y_i'(x+y_1)$$

and

$$\frac{\partial v_{p_1,p_2,\ldots,p_k}}{\partial y_1} = x + \sum_{i=2}^{k} y_i(x+y_1)$$

We can show that $\lambda(x) = F(x) - 1$ by using (3.27) and(3.28). Using the first order conditions with $\lambda(x)$ and above obtained derivatives, we end up with the following equation :

$$(c_1'(y_1(x)))f(x) = [x + \sum_{i=2}^{k} y_i(x+y_1)]f(x) + (F(x)-1)[1 + \sum_{i=2}^{k} y_i'(x+y_1)] \tag{3.30}$$

$$c_1'(y_1(x)) = x + \sum_{i=2}^{k} y_i(x+y_1) + \frac{(F(x)-1)}{f(x)}[1 + \sum_{i=2}^{k} y_i'(x+y_1)] \tag{3.31}$$

Suppose equilibrium occurs at $(\overline{y_1(x)}\ ,\ \overline{y_2(x)},\ldots,\overline{y_k(x)})$. If the consumer $x$ chooses $\overline{y_1(x)}$, then this implies that he will choose $y_2(x+\overline{y_1(x)})$ and will be $y_2(x+\overline{y_1(x)}) = \overline{y_2(x)}$ at equilibrium.

$$y_1(x) = \overline{y_1(x)} \tag{3.32}$$

$$y_i(x+\overline{y_1(x)}) = \overline{y_i(x)} \tag{3.33}$$

differentiating (3.33) we get,

$$y_i'(x+\overline{y_1(x)}) = \frac{\overline{y_i'(x)}}{1+\overline{y_1'(x)}} \tag{3.34}$$

Using the equations (3.32)-(3.34) we can rewrite equation (3.31).

$$c_1'(\overline{y_1(x)}) = x + \sum_{i=2}^{k} \overline{y_i(x)} + \frac{(F(x)-1)}{f(x)}[1 + \frac{\sum_{i=2}^{k} \overline{y_i'(x)}}{1+\overline{y_1'(x)}}] \tag{3.35}$$

Because of the symmetric nature of this problem we can similarly get another (k-1) equations. In general,

$$c_j'(\overline{y_j(x)}) = x + \sum_{i\neq j} \overline{y_i(x)} + \frac{(F(x)-1)}{f(x)}[1 + \frac{\sum_{i\neq j} \overline{y_i'(x)}}{1+\overline{y_j'(x)}}] \tag{3.36}$$

where $i = 1, ..., k$. $\square$

If $c_i(y_i) = c_1(y_1)$ for all $i = 1, ..., k$. Then we can assume $\overline{y_i(x)} = y_1(x)$ for $i = 2, ..., k$. Also let $f(x) = 1 \quad \text{on} \quad [0, 1]$. Then,

$$c_1'(\overline{y_1(x)}) = 2x - 1 + (k-1)\overline{y_1(x)} + (x-1)\frac{(k-1)\overline{y_1'(x)}}{1+\overline{y_1'(x)}}, \tag{3.37}$$

so finding the $k$-product equilibrium boils down to solving a ordinary differential equation of type (3.37). In general its difficult to find an analytic solution when $k > 2$. In chapter 5 we present some numerical results obtained using Runge-Kutta method for our proposed model.

# Chapter 4

# Analytical Results

## 4.A Discrete Case

In this section we are going to present the solution for the k=2 case under certain assumptions.

Our assumptions are as follows,

- The consumer types are distributed uniformly.
- The consumers has only two choices either to buy the product $i$ (represented by 1) or to refrain from buying the product $i$ (represented by 0).
- The costs are fixed.

### 4.A.1 Setting up the problem

The consumer types $x$ are distributed uniformly in $[0, 1]$. The product $i$'s allocation type $y_i$ can take only two values 0 and 1 (i.e. $y_i \in \{0, 1\}$ ). The cost function of producer $i$ is of the form, $c_i(y_i) = C_i$ when $y_i = 1$ and $c_i(y_i) = 0$ when $y_i = 0$. Similarly we define the pricing function $p_i(y_i)$ as $p_i(y_i) = P_i$ when $y_i = 1$ and $p_i(y_i) = 0$ when $y_i = 0$ , for $i = 1, 2$.

Here $C_1, C_2, P_1, P_2 \in \mathbb{R}_+$. We are going to consider a special case of the consumers preference function of type,

$$b(x, y_1, y_2) = xy_1 + xy_2 + y_1y_2$$

Then, the utility function $u(x, y_1, y_2)$ of the consumer is :

$$u(x, y_1, y_2) = xy_1 + xy_2 + y_1y_2 - p_1(y_1) - p_2(y_2)$$

Depending on the consumers choice, the surplus function $S(x)$ consists of 4 distinct cases.

$$S(x) = \max \begin{cases} 2x + 1 - P_1 - P_2 & \text{if consumer buys both products.} \\ x - P_1 & \text{if consumer buys product 1.} \\ x - P_2 & \text{if consumer buys product 2.} \\ 0 & \text{if consumer buys neither.} \end{cases}$$

The surplus of the consumer type $x$ would be the maximum of these four options.

The profit function of product $i$ is given by the integral,

$$\int_x^1 (p_i(y_i(x)) - c_i(y_i(x))\, dx \quad \text{where } i = 1, 2.$$

### 4.A.2 Finding equilibrium candidates

Now, that we have set-up the problem, the next step is to find candidates for $P_1$ and $P_2$ that will give us an equilibrium. Suppose $P_2 - 1 \leq P_1 \leq P_2 + 1$. Then, the maximum surplus occurs when the consumer buys both products. Therefore,

$$2x + 1 - P_1 - P_2 \geq 0 \Rightarrow x \geq \frac{P_1 + P_2 - 1}{2}$$

$$\begin{aligned} \text{Profit of product 1} &= \int_{\frac{P_1+P_2-1}{2}}^{1} (P_1 - C_1)\, dx &\quad (4.1)\\ &= (P_1 - C_1)(\frac{3 - P_1 - P_2}{2}) &\quad (4.2) \end{aligned}$$

$$\begin{aligned} \text{Profit of product 2} &= \int_{\frac{P_1+P_2-1}{2}}^{1} (P_2 - C_2)\, dx &\quad (4.3)\\ &= (P_2 - C_2)(\frac{3 - P_1 - P_2}{2}) &\quad (4.4) \end{aligned}$$

Now, to find the profit maximizing value of $P_1$, we fix the value of $P_2$ and obtain the first derivative of equation (4.2) with respect to $P_1$. Then, to find the profit maximizing value of $P_2$, we fix the value of $P_1$ and obtain the first derivative of equation (4.4) with respect to $P_2$.

$$\frac{\mathrm{d}}{\mathrm{d}P_1}((P_1 - C_1)(\frac{3 - P_1 - P_2}{2})) = 0 \Rightarrow P_1 = \frac{C_1 - P_2 + 3}{2} \quad (4.5)$$

$$\frac{\mathrm{d}}{\mathrm{d}P_2}((P_2 - C_2)(\frac{3 - P_1 - P_2}{2})) = 0 \Rightarrow P_2 = \frac{C_2 - P_1 + 3}{2} \tag{4.6}$$

Solving equations (4.5) and (4.6) we find our first set of candidates for an equilibrium.

$$P_1 = \frac{2C_1 - C_2 + 3}{3} \tag{4.7}$$
$$P_2 = \frac{2C_2 - C_1 + 3}{3} \tag{4.8}$$

Now suppose $P_1 > P_2 + 1$. Then for values of $x$ between $P_2$ and $P_1 - 1$ the maximum surplus occurs when consumer buys product 2 and for $x$ values greater than $P_1 - 1$ the maximum surplus occurs when consumer buys both products. Therefore,

$$\text{Profit of product 1} = \int_{P_1-1}^{1} (P_1 - C_1)\,dx \tag{4.9}$$
$$= (P_1 - C_1)(2 - P_1) \tag{4.10}$$

$$\text{Profit of product 2} = \int_{P_2}^{1} (P_2 - C_2)\,dx \tag{4.11}$$
$$= (P_2 - C_2)(1 - P_2) \tag{4.12}$$

To find the profit maximizing values of $P_1$ and $P_2$ we obtain the first derivative of equations (4.10) and (4.12) with respect to $P_1$ and $P_2$ respectively.

$$\frac{\mathrm{d}}{\mathrm{d}P_1}((P_1 - C_1)(2 - P_1)) = 0 \Rightarrow P_1 = \frac{C_1 + 2}{2} \tag{4.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}P_2}((P_2 - C_2)(1 - P_2)) = 0 \Rightarrow P_2 = \frac{C_2 + 1}{2} \tag{4.14}$$

Equations (4.13) and (4.14) give the second candidate for an equilibrium.

Now suppose $P_1 < P_2 - 1$. Then for values of $x$ between $P_1$ and $P_2 - 1$ the maximum surplus occurs when consumer buys product 1 and for $x$ values greater than $P_2 - 1$ the maximum surplus occurs when the consumer buys both products. Therefore,

$$\text{Profit of product 1} = \int_{P_1}^{1} (P_1 - C_1)\, dx \tag{4.15}$$

$$= (P_1 - C_1)(1 - P_1) \tag{4.16}$$

$$\text{Profit of product 2} = \int_{P_2-1}^{1} (P_2 - C_2)\, dx \tag{4.17}$$

$$= (P_2 - C_2)(2 - P_2) \tag{4.18}$$

to find the profit maximizing values of $P_1$ and $P_2$ we obtain the first derivative of the equations (4.17) and (4.18) with respect to $P_1$ and $P_2$ respectively.

$$\frac{\mathrm{d}}{\mathrm{d}P_1}((P_1 - C_1)(1 - P_1)) = 0 \Rightarrow P_1 = \frac{C_1 + 1}{2} \tag{4.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}P_2}((P_2 - C_2)(2 - P_2)) = 0 \Rightarrow P_2 = \frac{C_2 + 2}{2} \tag{4.20}$$

The equations (4.19) and (4.20) gives the third candidate for an equilibrium. The integrals (4.1) , (4.3) , (4.9) , (4.11) , (4.15) and (4.17) are defined when $0 \leq P_1, P_2 \leq 2$ and $0 \leq P_1 + P_2 \leq 3$

### 4.A.3 Verifying the equilibrium candidates

The final step is to check whether the candidates we found are actual equilibriums or not. First, let's write the profit function of product 1 fixing the value of $P_2$ and the profit function of product 2 fixing the value of $P_1$.

$$Profit_1(P_1, P_2) = \begin{cases} (P_1 - C_1)(1 - P_1) & \text{if } P_1 \leq P_2 - 1 \\ (P_1 - C_1)(\dfrac{3 - P_1 - P_2}{2}) & \text{if } P_2 - 1 \leq P_1 \leq P_2 + 1 \\ (P_1 - C_1)(2 - P_1) & \text{if } P_1 \geq P_2 + 1 \end{cases}$$

$$Profit_2(P_1, P_2) = \begin{cases} (P_2 - C_2)(2 - P_2) & \text{if } P_1 \leq P_2 - 1 \\ (P_2 - C_2)(\dfrac{3 - P_1 - P_2}{2}) & \text{if } P_2 - 1 \leq P_1 \leq P_2 + 1 \\ (P_2 - C_2)(1 - P_2) & \text{if } P_1 \geq P_2 + 1 \end{cases}$$

The profit maximizing $P_1$ and $P_2$ values for each case is given below.

The $P_1$ which maximizes $Profit_1(P_1, P_2) = \begin{cases} \dfrac{C_1+1}{2} & \text{if } P_1 \leq P_2 - 1 \\ \dfrac{2C_1 - C_2 + 3}{3} & \text{if } P_2 - 1 \leq P_1 \leq P_2 + 1 \\ \dfrac{C_1+2}{2} & \text{if } P_1 \geq P_2 + 1 \end{cases}$

The $P_2$ which maximizes $Profit_2(P_1, P_2) = \begin{cases} \dfrac{C_2+2}{2} & \text{if } P_1 \leq P_2 - 1 \\ \dfrac{2C_2 - C_1 + 3}{3} & \text{if } P_2 - 1 \leq P_1 \leq P_2 + 1 \\ \dfrac{C_2+1}{2} & \text{if } P_1 \geq P_2 + 1 \end{cases}$

Now, for any candidate $\{P_1^*, P_2^*\}$ to be an equilibrium, we need to check whether $P_1^*$ is the maximizer for $Profit_1(P_1, P_2^*)$ and vice- versa.

First, we will check the equilibrium given by (4.7) and (4.8) i.e. $\{\frac{2C_1 - C_2 + 3}{3}, \frac{2C_2 - C_1 + 3}{3}\}$. Let's fix the value of $P_2$ as $\frac{2C_2 - C_1 + 3}{3}$. Then or all 3 cases we can find a new inequality using the fix $P_2$ value and the maximizer values of $P_1$ in the respective case.

$$P_1 = \frac{C_1+1}{2} \text{ and } P_1 \leq P_2 - 1 \Rightarrow 4C_2 - 5C_1 \geq 3 \tag{4.21}$$

$$P_1 = \frac{2C_1 - C_2 + 3}{3} \text{ and } P_2 - 1 \leq P_1 \leq P_2 + 1 \Rightarrow -1 \leq C_1 - C_2 \leq 1 \tag{4.22}$$

$$P_1 = \frac{C_1+2}{2} \text{ and } P_1 \geq P_2 + 1 \Rightarrow 4C_2 - 5C_1 \leq -6 \tag{4.23}$$

from the above three inequalities it's clear that if (4.21) is true then (4.23) is false and vice versa. We get the Corresponding $P_1$ value of the equilibrium candidate when (4.22) is true. Suppose inequality (4.22) is true then (4.21) is true if $C_1 \leq 1$ and (4.23) is true if $C_1 \geq 2$. Therefore when $-1 \leq C_1 - C_2 \leq 1$

and $1 < C_1 < 2$ only (4.22) occurs and $P_1 = \dfrac{2C_1 - C_2 + 3}{3}$.

Now fix the value of $P_1$ as $\dfrac{2C_1 - C_2 + 3}{3}$. Then, for all 3 cases we can find a new inequality using the fix $P_1$ value and the maximizer values of $P_2$ in the respective case.

$$P_2 = \frac{C_2 + 2}{2} \text{ and } P_1 \leq P_2 - 1 \Rightarrow 4C_1 - 5C_2 \leq -6 \tag{4.24}$$

$$P_2 = \frac{2C_2 - C_1 + 3}{3} \text{ and } P_2 - 1 \leq P_1 \leq P_2 + 1 \Rightarrow -1 \leq C_1 - C_2 \leq 1 \tag{4.25}$$

$$P_2 = \frac{C_1 + 1}{2} \text{ and } P_1 \geq P_2 + 1 \Rightarrow 4C_1 - 5C_2 \geq 3 \tag{4.26}$$

from the above three inequalities it's clear that if (4.24) is true then (4.26) is false and vice versa. Suppose inequality (4.25) is true then (4.24) is true if $C_2 \geq 2$ and (4.26) is true if $C_2 \leq 1$. Therefore when $-1 \leq C_1 - C_2 \leq 1$ and $1 < C_2 < 2$ only (4.24) occurs and $P_2 = \dfrac{2C_2 - C_1 + 3}{3}$.

So under the conditions $-1 \leq C_1 - C_2 \leq 1$ and $1 < C_1, C_2 < 2$, the first candidate $\{\dfrac{2C_1 - C_2 + 3}{3}, \dfrac{2C_2 - C_1 + 3}{3}\}$ is in fact an equilibrium of the problem.

To verify the second candidate given by (4.13) and (4.14), we fix the value of $P_2$ as $\dfrac{C_2 + 1}{2}$ and apply the same procedure. Then if,

$$P_1 = \frac{C_1 + 1}{2} \text{ and } P_1 \leq P_2 - 1 \Rightarrow C_1 - C_2 \leq -2 \tag{4.27}$$

$$P_1 = \frac{2C_1 - C_2 + 3}{3} \text{ and } P_2 - 1 \leq P_1 \leq P_2 + 1 \Rightarrow -9 \leq 4C_1 - 5C_2 \leq 3 \tag{4.28}$$

$$P_1 = \frac{C_1 + 2}{2} \text{ and } P_1 \geq P_2 + 1 \Rightarrow C_1 - C_2 \geq 1 \tag{4.29}$$

It's clear that cases given by (4.27) and (4.29) both will not be satisfied simultaneously. It can be shown that when $C_1 - C_2 \geq 1$ and $C_1 < 2$ only (4.29) is true and $P_1 = \frac{C_1 + 2}{2}$.

Now fix the value of $P_1$ as $\frac{C_1 + 2}{2}$. Then if

$$P_2 = \frac{C_2 + 2}{2} \text{ and } P_1 \leq P_2 - 1 \Rightarrow C_1 - C_2 \leq -2 \tag{4.30}$$

$$P_2 = \frac{2C_2 - C_1 + 3}{3} \text{ and } P_2 - 1 \leq P_1 \leq P_2 + 1 \Rightarrow -6 \leq 5C_1 - 4C_2 \leq 6 \tag{4.31}$$

$$P_2 = \frac{C_2 + 1}{2} \text{ and } P_1 \geq P_2 + 1 \Rightarrow C_1 - C_2 \geq 1 \tag{4.32}$$

It can be shown that when $C_1 - C_2 \geq 1$ and $C_1 > 2$ only (4.32) is true and

$P_2 = \dfrac{C_2+1}{2}$. Then the second candidate $\{\dfrac{C_1+2}{2}, \dfrac{C_2+1}{1}\}$ is an equilibrium if the following conditions $C_1 - C_2 \geq 1$ , $C_1 < 2$ and $C_1 > 2$ are satisfied. However, last two conditions presents a contradiction. Therefore second candidate is not an equilibrium. Similarly we can show that the third candidate given by (4.19) and(4.20) is also not an equilibrium of the problem.

So when $-1 \leq C_1 - C_2 \leq 1$ and $1 < C_1, C_2 < 2$ the first candidate $\{\dfrac{2C_1 - C_2 + 3}{3}, \dfrac{2C_2 - C_1 + 3}{3}\}$ is an equilibrium of the problem. For other values of $C_1, C_2$ we could not find an equilibrium.

We were able to show for this example that under the conditions $-1 \leq C_1 - C_2 \leq 1$ and $1 < C_1, C_2 < 2$ the first candidate $\{\dfrac{2C_1 - C_2 + 3}{3}, \dfrac{2C_2 - C_1 + 3}{3}\}$ is in fact an equilibrium of the problem.

## 4.B Continuous Case

In this section we present an example for the uni-dimensional $k - producer$ model for the $k = 2$ case under certain assumptions. The assumptions are as follows,

- The consumer types $(x)$ are distributed uniformly.(i.e. $f(\cdot) = 1$ on $[0, 1]$)
- The set $Y_i \in [0, 1]$
- The cost functions of the products are of the form $\dfrac{y_i^2}{2}$.
- The preference function is of the form : $b(x, y_1, y_2) = xy_1 + xy_2 + y_1 y_2$.

### 4.B.1 Setting up the problem

The cost function of Product $i$ is given by $c_i(y_i) = \frac{y_i^2}{2}$ and $p_i(y_i)$ is the pricing function of Product $i$.

We are going to consider a special case of the consumers preference function of type,

$$b(x, y_1, y_2) = xy_1 + xy_2 + y_1y_2$$

Let $S(x)$ be the effective surplus function.

$$S(x) = \max_{y_1}\{v(x, y_1) - p_1(y_1)\}$$

where,

$$\begin{aligned} v(x, y_1) &= \max_{y_2}\{b(x, y_1, y_2) - p_2(y_2)\} \\ &= \max_{y_2}\{xy_1 + xy_2 + y_1y_2 - p_2(y_2)\} \\ &= xy_1 + \max_{y_2}\{y_2(x + y_1) - p_2(y_2)\} \end{aligned}$$

The use of effective surplus functions converts this problem into 2 identical monopolist problems (by Proposition 3.A.1). First, let's look at the problem in terms of product 1.

The profit function of product 1 is calculated as follows,

$$\text{Profit of product 1} = \int_0^1 [p_1(y_1(x)) - c_1(y_1(x))]dx$$
$$= \int_0^1 [v(x, y_1(x)) - S(x) - c_1(y_(x))]dx$$

$s.t.\ S'(x) = v_x(x, y_1(x))\ ,\ S(0) = 0$

This problem is of the same type as Proposition 3.B.1, so we will use it to get the following equations. Suppose an equilibrium occurs at $\{y_1\overline{(x)}\ ,\ y_2\overline{(x)}\}$. Then from Proposition 3.B.2,

$$y_1\overline{(x)} = 2x - 1 + y_2\overline{(x)} + (x-1)\frac{y_2'\overline{(x)}}{1 + y_1'\overline{(x)}} \tag{4.33}$$

$$y_2\overline{(x)} = 2x - 1 + y_1\overline{(x)} + (x-1)\frac{y_1'\overline{(x)}}{1 + y_2'\overline{(x)}} \tag{4.34}$$

### 4.B.2 Finding the equilibrium and pricing schemes

Now, to find $\{y_1\overline{(x)}\ ,\ y_2\overline{(x)}\}$ we use the ordinary differential equations given by (4.33) and (4.34). Because of the symmetric nature of the problem we will look for a solution the assumption $y_1\overline{(x)}= y_2\overline{(x)}$. Then, both (4.33) and (4.34) end up been the same equation as,

$$y_1\overline{(x)} = 2x - 1 + y_1\overline{(x)} + (x-1)\frac{y_1'\overline{(x)}}{1 + y_1'\overline{(x)}}$$

Then, by isolating $y_1'\overline{(x)}$ we end up with the following ODE problem to solve.

$$\overline{y_1'(x)} = \frac{1-2x}{3x-2}, \quad \overline{y_1(0)} = 0 \tag{4.35}$$

As the solution $\overline{y_1'(x)}$ is negative on $(0, \frac{1}{2})$, the $\overline{y_1'(x)} \geq 0$ constraint binds here. Therefore, we set $\overline{y_1(x)}$ as zero in the interval $(0, \frac{1}{2})$. Now, to find $\overline{y_1(x)}$ for $x > \frac{1}{2}$ we solve the ODE (4.35) with the condition $\overline{y_1(0.5)} = 0$ , which results in,

$$\int d\overline{y_1(x)} = \int \frac{1-2x}{3x-2}dx \Rightarrow \overline{y_1(x)} = -\frac{1}{9}\ln(2-3x) - \frac{2x}{3} + c,$$

using the condition $\overline{y_1(0.5)} = 0$, we can determine the value of $c$.

$$c = \frac{1}{9}\ln(0.5) + \frac{1}{3}$$

Then $\overline{y_1(x)}$ is of the form :

$$\overline{y_1(x)} = \begin{cases} 0 & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ -\dfrac{1}{9}\ln(2-3x) - \dfrac{2x}{3} + \dfrac{1}{9}\ln(0.5) + \dfrac{1}{3} & \text{if } \dfrac{1}{2} < x < \hat{x} \end{cases}$$

when $x = \hat{x}$, $\bar{y_1}(\hat{x}) = 1$. We can see from the graph below that $\hat{x}$ is close to 0.66.

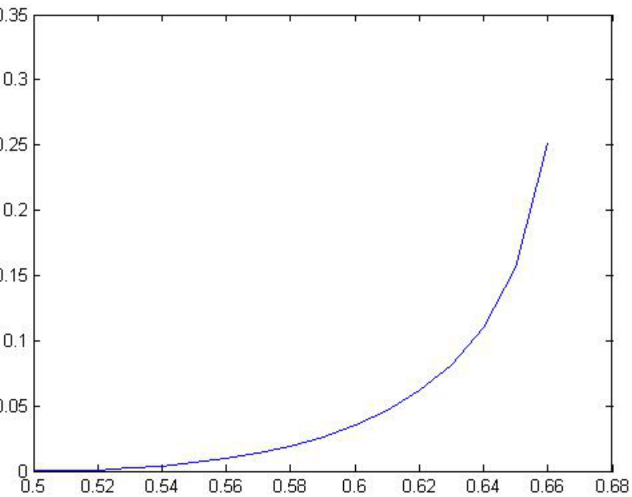


Figure 4.1: Graph of $y_1(x)$ on the interval $(0.5, 0.66)$

Next we find S(x) using the following ODE and $\overline{y_1(x)}$.

$$S'(x) = \overline{y_1(x)} + \overline{y_2(x)} = 2\overline{y_1(x)}, S(0.5) = 0 \tag{4.36}$$

Then,

$$S(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -\frac{2x^2}{3} + \frac{8x}{9} + \frac{2}{9}(x-1)\ln(2-3x) + \frac{4}{27}\ln(4-6x) - \frac{5}{18} & \text{if } \frac{1}{2} < x < \hat{x} \end{cases}$$

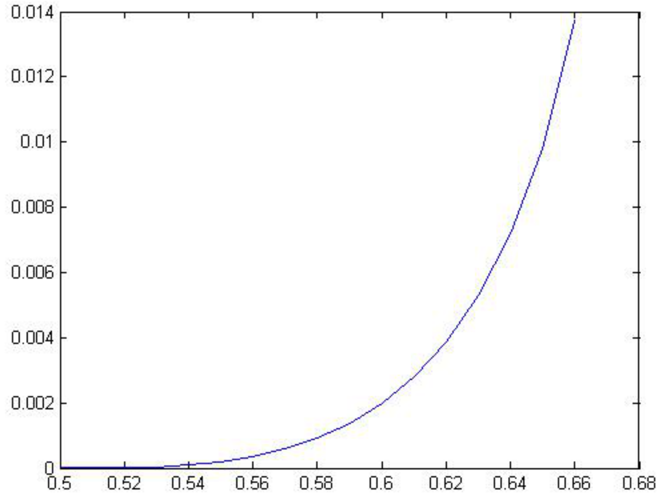


Figure 4.2: Graph of $S(x)$ on the interval $(0.5, 0.66)$

Because of the symmetric nature of the problem we use $p_1(y_1(x)) = p_2(y_2(x))$ to find the pricing schemes.

$$p_1(y_1(x)) = p_2(y_2(x)) = \frac{1}{2}[y_1\overline{(x)}(2x + y_1\overline{(x)}) - S(x)] \tag{4.37}$$

We can invert the monotone function $y_1(x)$ and substitute the inverse function for $x$ in in (4.37) to find $p_1(y_1)$. The pricing schemes obtained by (4.37) is feasible in interval $(0, \hat{x})$.

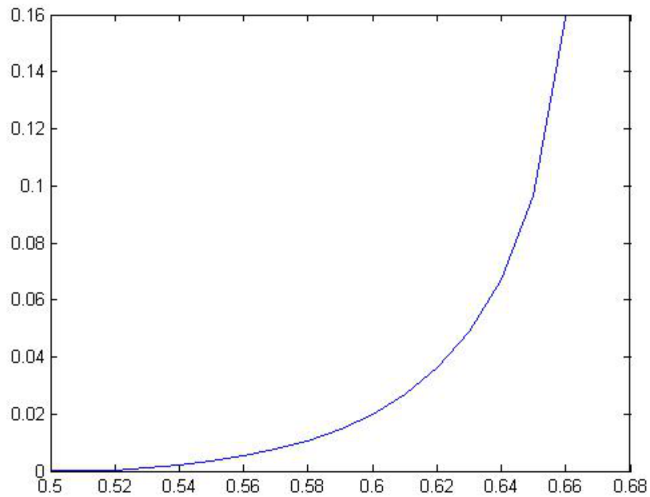


Figure 4.3: Graph of $p_1(y(x))$ on the interval $(0.5, 0.66)$

It's clear from the solution that certain group of customer types ($x \in (0, 0.5)$) are excluded from the model. This economically means that the producers are not serving the low end customers, instead are focusing on the high end customers. This property seems to persist in general for any $k$. We will discuss more about the exclusion property in Chapter 6.

Now we briefly consider a case where our cost functions are in linear form instead of quadratic form. Let's define $c_1(y_1)$ and $c_1(y_1)$ as :

$$c_1(y_1) = \alpha y_1$$

$$c_2(y_2) = \beta y_2$$

The preference function is the same as the above considered example. Using (3.36) we can write the following equations :

$$\alpha = 2x - 1 + \overline{y_2(x)} + (x-1)\frac{\overline{y_2'(x)}}{1+\overline{y_1'(x)}} \tag{4.38}$$

$$\beta = 2x - 1 + \overline{y_1(x)} + (x-1)\frac{\overline{y_1'(x)}}{1+\overline{y_2'(x)}} \tag{4.39}$$

If we assume $\alpha = \beta$ and using $\overline{y_1(x)} = \overline{y_2(x)}$, we can rewrite the equations (4.38) and (4.39) as a single equation.

$$\alpha(1+\overline{y_1'(x)}) = (2x-1)(1+\overline{y_1'(x)}) + \overline{y_1(x)}(1+\overline{y_1'(x)}) + (x-1)\overline{y_1'(x)} \tag{4.40}$$

Simplifying (4.40) gives us :

$$(2x + \overline{y_1(x)} - 1 - \alpha) + (3x + \overline{y_1(x)} - 2 - \alpha)\overline{y_1'(x)} = 0 \tag{4.41}$$

The figure 4.4 represents the numerical form of the solution to equation (4.41) when $\alpha = 0.5$.

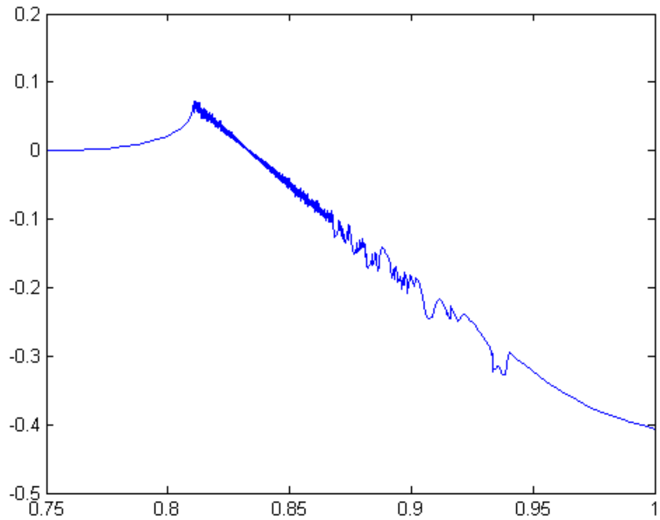


Figure 4.4: The graph of $y_1(x)$ on the interval $(0.75, 1)$ for $\alpha = 0.5$

The actual solution should be non decreasing in the interval $[0, 1]$, and will have the following form,

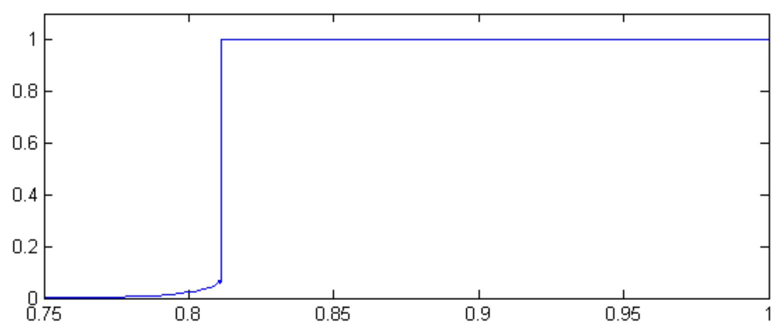


Figure 4.5: The actual graph of $y_1(x)$ on the interval $(0.75, 1)$ for $\alpha = 0.5$

Therefore, it's possible to assume that we can extend our approach of the $k$-producer model to include general cost functions.

We end this chapter by considering an example with a different type of the consumers' preference function,

$$b(x, y_1, y_2) = xy_1y_2$$

.

The cost functions are in quadratic form. (i.e. $c_i(y_i) = \frac{y_1^2}{2}$)

This preference function follows the Spence- Mirrlees condition, because $x, y_i \in [0, 1]$.

Let $S(x)$ be the effective surplus function.

$$S(x) = \max_{y_1}\{v(x, y_1) - p_1(y_1)\}$$

where,

$$\begin{aligned} v(x, y_1) &= \max_{y_2}\{b(x, y_1, y_2) - p_2(y_2)\} \\ &= \max_{y_2}\{xy_1y_2 - p_2(y_2)\} \\ &\max_{y_2}\{y_2(xy_1) - p_2(y_2)\} \qquad\qquad (*) \end{aligned}$$

The use of an effective surplus function converts this problem into 2 identical monopolist problems (by Proposition 3.A.1). The maximizing $y_2(x, y_1)$ in $(*)$ is a function of the product $xy_1$. First, let's look at the problem in terms of product 1.

The profit function of product 1 is calculated as follows,

$$\text{Profit of product } 1 = \int_0^1 [p_1(y_1(x)) - c_1(y_1(x))]dx$$
$$= \int_0^1 [v(x, y_1(x)) - S(x) - c_1(y_(x))]dx$$

$s.t.\ S'(x) = v_x(x, y_1(x))\ ,\ S(0) = 0$

This problem is again of same type as Proposition 3.B.1, so we will use it to get the following equations. Suppose an equilibrium occurs at $\{y_1\overline{(x)}$ , $y_2\overline{(x)}\}$. Then from Proposition 3.B.2, we get the following 1st order equations:

$$\frac{\partial \lambda}{\partial x} = -\frac{\partial H}{\partial x} = 1 \tag{4.42}$$
$$\lambda(1) = 0 \tag{4.43}$$
$$\frac{\partial H}{\partial y_1} = [\frac{\partial v(x, y_1)}{\partial y_1} - c_1'(y_1(x))]f(x) + \lambda(x)\frac{\partial^2 v(x, y_1)}{\partial x\, \partial y_1} = 0 \tag{4.44}$$

now we will give out the calculations required to get the derivatives in (4.44).

by the envelope condition,

$$\frac{\partial v}{\partial x} = y_1 y_2(x y_1)$$

now differentiating with respect to $y_1$ yields,

$$\frac{\partial^2 v}{\partial x\, \partial y_1} = y_2(x y_1) + y_1 \frac{\partial y_2(x y_1)}{\partial y_1}$$

so at the equilibrium,

$$y_1(x) = \overline{y_1(x)}$$

$$y_2(xy_1) = \overline{y_2(x)}$$

$$\frac{\partial y_2}{\partial y_1} = y_2'(x\bar{y_1})(\overline{y_1(x)} + x\overline{y_1'(x)})$$

now we can obtain the required ODE similar to the first example to find the solution.

# Chapter 5

# Numerical Results

When $k \geq 3$ it is very difficult to solve the ODEs from chapter 4 analytically, so we need to use numerical methods in order to get an idea about the form of the solution. In this chapter we present some numerical results of the $k$-producer model using the Runge-Kutta method for the type of problems discussed in Proposition 3.B.2.

Let's consider an example when k=3. We will make the following assumptions,

- The consumer types are distributed uniformly on $X = [0,1]$. (i.e $f(\cdot) = 1$ on $[0,1]$ )
- The set $Y_i = [0,1]$.
- The cost functions of the products are of the form $\frac{y_i^2}{2}$.
- The preference function is of the form : $b(x, y_1, y_2, y_3) = xy_1 + xy_2 + xy_3 + y_1y_2 + y_1y_3 + y_2y_3$

This model is of the form of the Proposition 3.B.2. Suppose equilibrium occurs at $(\overline{y_1(x)}, \overline{y_2(x)}, \overline{y_3(x)})$. Then using Proposition 3.B.2 we end up with the following equations :

$$\overline{y_j(x)} = 2x - 1 + \sum_{i \neq j} \overline{y_i(x)} + (x-1)\frac{\sum_{i \neq j} \overline{y_i'(x)}}{1 + \overline{y_j'(x)}} \tag{5.1}$$

where $j = 1, 2, 3$.

Assume $\overline{y_i(x)} = y_1(x)$ for $i = 2, 3$. Then,

$$\overline{y_1(x)} = 2x - 1 + 2\overline{y_1(x)} + (x-1)\frac{2\overline{y_1'(x)}}{1 + \overline{y_1'(x)}} \tag{5.2}$$

$$\overline{y_1'(x)} = \frac{1 - 2x - \overline{y_1(x)}}{4x - 3 + \overline{y_1(x)}}, \quad \overline{y_1(0.5)} = 0 \tag{5.3}$$

So finding the 3-product equilibrium boils down to solving the ordinary differential equation given by (5.3). The reason for setting $\overline{y_1(0.5)} = 0$ is the same as we discussed in chapter 4. Now we present a graph of the numerical solution of (5.3), using the Runge-Kutta method.

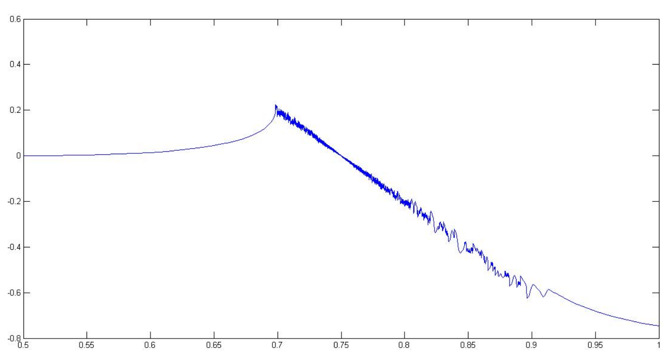


Figure 5.1: Graph of $(y_1(x))$ on the interval $(0.5, 1)$ for $k = 3$

From the graph it's clear that our allocation is increasing upto 0.7 and

$\overline{y_1(x)}$ will be equal to 1 near 0.7 ( recall our products $y_i \in [0,1]$ ). We want our allocations to be increasing in the whole interval, so for values $x \in (0.7, 1]$ are set to be equal to the value $\overline{y_1(x)} = 1$. That would make our allocation $\overline{y_1(x)}$ increasing in the whole interval, which is what we require. Then the graph would have the following form.

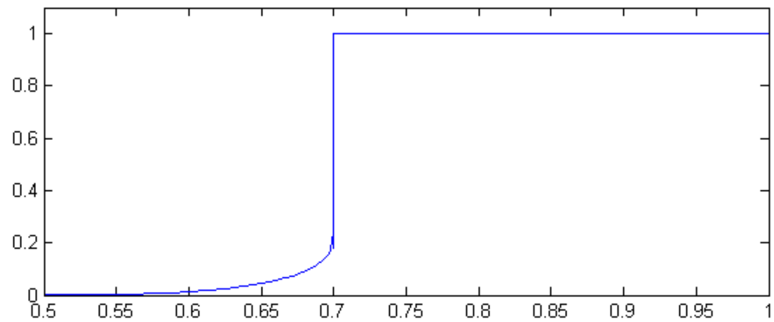


Figure 5.2: The actual graph of $(y_1(x))$ on the interval $(0.5, 1)$ for $k = 3$

Next, we look at what happens to $\overline{y_1(x)}$ when we increase the number of producers $(k)$ in the market.

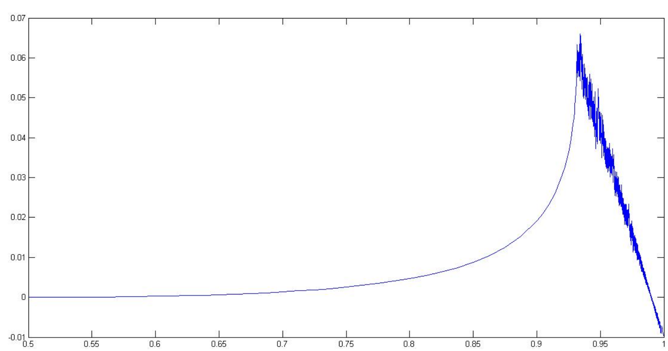

Figure 5.3: Graph of $(y_1(x))$ on the interval $(0.5, 1)$ for $k = 100$

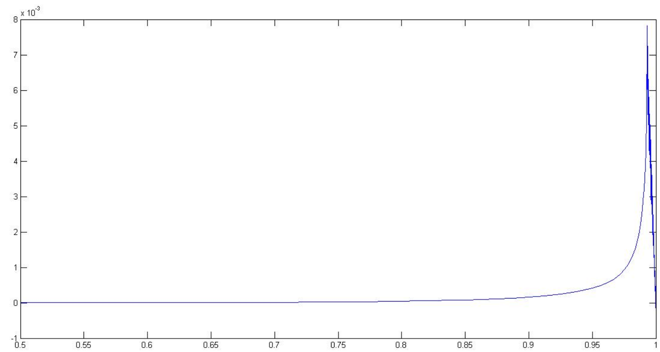


Figure 5.4: Graph of $(y_1(x))$ on the interval $(0.5, 1)$ for $k = 10000$

It's clear from the above figures that as the number of producers increase, the interval where $\overline{y_1(x)}$ is strictly increasing becomes larger as well.

We will conclude this chapter by considering a case where the cost functions are not identical to each other. Suppose $k = 2$ and the cost functions are,

$$c_1(y_1) = \frac{y_1^2}{2}$$

$$c_2(y_2) = y_2^2$$

The preference function is same as the above considered example. Using (3.36) we can write the following equations :

$$y_1 = 2x - 1 + \overline{y_2(x)} + (x-1)\frac{\overline{y_2'(x)}}{1 + \overline{y_1'(x)}} \tag{5.4}$$

$$2y_2 = 2x - 1 + \overline{y_1(x)} + (x-1)\frac{\overline{y_1'(x)}}{1+\overline{y_2'(x)}} \tag{5.5}$$

We can solve the ODE system given by (5.4) and (5.5) numerically using matlab. The resulting solution has the following form :

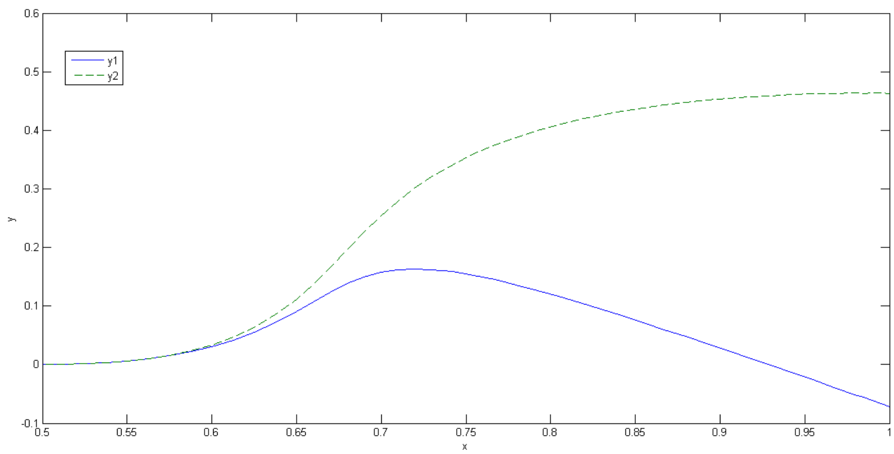


Figure 5.5: The graph of $(y_1(x), y_2(x))$ on the interval $(0.5, 1)$ for $k = 2$

We want both $y_1$ and $y_2$ to be non decreasing in $[0, 1]$, so for $x$ values greater than 0.725 (approximate value) we set all $y_1$ values to be equal to $y_1(0.725)$. Now we can use the fact that $y_1(> 0.725) = constant$ and equations (5.4) and (5.5) to find the actual form of the solution.

Then the solution would be of the following form,

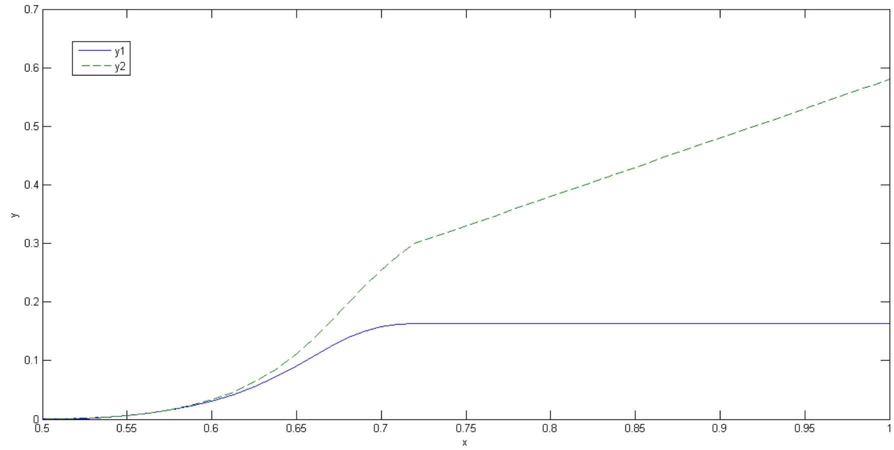


Figure 5.6: The actual solution form graph of $(y_1(x), y_2(x))$ on the interval $(0.5, 1)$ for $k = 2$

# Chapter 6

# Discussion of Economic Properties and Summary

In this chapter we discuss exclusion and partial exclusion properties of our model and present an example showing these two properties. We conclude this chapter by presenting a summary of the thesis.

## 6.A Exclusion in the model

Consider the model we proposed in Proposition 3.B.2 with any general type of cost functions $c_i(\cdot)$. Let $f(\cdot) = 1$ on $[0, 1]$. Then using equation (3.36) we get the following set of equations :

$$c_j'(y_j\overline{(x)}) = 2x - 1 + \sum_{i \neq j} y_i\overline{(x)} + (x-1)\frac{\sum_{i \neq j} y_i'\overline{(x)}}{1 + y_j'\overline{(x)}} \tag{6.1}$$

where $i = 1, ..., k$.

When $x = 0$ , $\overline{y_i(x)} = 0$ for $i = 1, ..., k$. Subbing in those values in equation (6.1) results in :

$$\frac{\sum_{i \neq j} \overline{y_i'(0)}}{1 + \overline{y_j'(0)}} = -1 - c_j'(0) \tag{6.2}$$

where $i = 1, ..., k$.

The allocations $\overline{y_i'(x)}$'s need to be increasing ( i.e. $\overline{y_i'(x)} \geq 0$). Economically it's natural to assume $c_i(\cdot)$'s are increasing functions, in which case the right hand side of (6.2) is always negative. Therefore the $\overline{y_i'(0)}$'s can't all be positive at the same time. We can see that for small $x$ and all $i$ we must have $y_i'\overline{(x) = 0}$, so $y_i\overline{(x) = 0}$ on a small interval. Because of that we need to set :

$$\overline{y_i(x)} = 0 \quad \text{for } x \in [0, a_1], \tag{6.3}$$

we can find the value of $a_1$ by subbing (6.3) in equation (6.1). Then,

$$a_1 = \frac{1 + c_j'(0)}{2} \tag{6.4}$$

where $j = 1, ..., k$.

The equation (6.4) presents us with $k$ such candidates for $a_1$, and the minimum of those $k$ candidates is the value of $a_1$. For $x > a_1$, some of the $\overline{y_i'(x)}$ can be positive.

Economically this means that it's always optimal to exclude low end customers from the model (i.e customer types($x$) that fall under the range $[0, a_1]$).

We saw from the example we presented at the end of chapter 2 that exclusion occurs in classical monopolist's problem also. Exclusion seems to be generic for higher dimensions, but depends on $b$ in one dimensions (at least for the classical problem).

Next we introduce a property that is unique to our model.

## 6.B Partial Exclusion

Suppose the $a_1$ value comes from the $r^{th}$ producers equation. Then, for values between $[a_1, a_2]$ (for some $a_2$ to be determined.), the $r^{th}$ producer follows the following ODE :

$$c_r'(\overline{y_r(x)}) = 2x - 1 \tag{6.5}$$

$$\overline{y_r(a_1)} = 0 \tag{6.6}$$

Then, by solving 6.5-6.6 we can find $\overline{y_r(x)}$. Then, to find $a_2$ we solve the following $k-1$ ODE problems.

$$c_j'(\overline{y_j(a_2)}) = 2x - 1 + \overline{y_r(x)} + (x-1)\overline{y_r'(x)} \tag{6.7}$$

$$\overline{y_j(a_2)} = 0 \tag{6.8}$$

where $j \neq r$.

The equations (6.7)-(6.8) presents us with $k-1$ such candidates for $a$, and the minimum of those $k$ candidates is the value of $a_2$.

This implies that the customer types $x$ that fall in $[a_1, a_2]$ range will only buy product $r$. If we extend this method we expect that in general that customer types $x$ that fall in the next interval $[a_2, a_3]$ will buy the $r^{th}$ product and another one and so on.

Therefore, in our model certain customer types will buy certain types of goods only. We call this phenomena partial exclusivity.

In the examples we solve, a positive fraction of consumers always buys the highest possible quality bundle. The numerical results indicates that as $k$ gets large, this fraction gets smaller.

Next we will present an example that shows the partial exclusion phenomena.

Suppose $k = 3$ and cost functions are as follows :

$$c_1(\bar{y_1}) = \frac{\bar{y_1^2}}{2}$$
$$c_2(\bar{y_2}) = \frac{\bar{y_2^2}}{2} + \frac{\bar{y_2}}{2}$$
$$c_3(\bar{y_3}) = \frac{\bar{y_3^2}}{2} + \frac{\bar{y_3}}{3}$$

Then,

$$c_1'(\bar{y_1}) = \bar{y_1}$$
$$c_2'(\bar{y_2}) = \bar{y_2} + \frac{1}{2}$$
$$c_3'(\bar{y_3}) = \bar{y_3} + \frac{1}{3}$$

Now, using (6.4), we can find three candidates for the $a_1$. Let $p\hat{r}o1, p\hat{r}o2$ and $p\hat{r}o3$ be those values, that respectively coming from product 1, product 2 and product 3.

$$\hat{pro1} = \frac{1 + c_1'(0)}{2} = \frac{1}{2}$$
$$\hat{pro2} = \frac{1 + c_2'(0)}{2} = \frac{3}{4}$$
$$\hat{pro3} = \frac{1 + c_3'(0)}{2} = \frac{2}{3}$$

Then $a_1 = min\{\hat{pro1}, \hat{pro2}, \hat{pro3}\} = \frac{1}{2}$. Therefore for $y_i\overline{(x)} = 0 \quad \text{for } x \in [0, 0.5]$. Then for some interval, $[0.5, a_2]$ customers will only buy product 1. Now we can find an equation for $y_1\overline{(x)}$ as given below.

$$y_1\overline{(x)} = 2x - 1 \quad \text{for } x \in [0.5, a_2]$$
$$y_1\overline{(0.5)} = 0$$

Next, we want to find the candidates for values of $a_2$, which are found by solving the following equations obtained using (6.1).

$$c_2'(0) = \frac{1}{2} = 2a_2 - 1 + y_1\overline{(a_2)} + y_1'\overline{(a_2)}(a_2 - 1) \Rightarrow \frac{1}{2} = 2a_2 - 1 + 2a_2 - 1 + 2(a_2 - 1) \tag{6.9}$$

$$c_3'(0) = \frac{1}{3} = 2a_2 - 1 + y_1\overline{(a_2)} + y_1'\overline{(a_2)}(a_2 - 1) \Rightarrow \frac{1}{3} = 2a_2 - 1 + 2a_2 - 1 + 2(a_2 - 1) \tag{6.10}$$

Solutions for (6.9) and (6.10) are $\frac{3}{4}$ and $\frac{13}{18}$ respectively. So $a_2$ is the minimum of those two values, which is $\frac{13}{18}$. This implies that for values of $x \in [\frac{1}{2}, \frac{13}{18}]$, only product 1 is bought. For $x > \frac{13}{18}$, $y_3'\overline{(x)}$ is positive implies that customers who falls in an interval $[\frac{13}{18}, a_3]$ will buy both product 1 and 3.

To find value of $a_3$, first we need to find expressions for $\overline{y_1(x)}$ and $\overline{y_3(x)}$ in $x \in [\frac{1}{2}, \frac{13}{18}]$. We do that by the solving following system of equations:

$$c_1'(\overline{y_1(x)}) = \overline{y_1(x)} = 2x - 1 + \overline{y_3(x)} + \frac{\overline{y_3'(x)}}{1 + \overline{y_1'(x)}}(x-1) \tag{6.11}$$

$$c_3'(\overline{y_3(x)}) = \overline{y_3(x)} + \frac{1}{3} = 2x - 1 + \overline{y_1(x)} + \frac{\overline{y_1'(x)}}{1 + \overline{y_3'(x)}}(x-1) \tag{6.12}$$

(6.11) and (6.12) are difficult to solve analytically. However, it's clear that by following this method we can clearly see the partial exclusion phenomena in our model.

## 6.C Summary

In this thesis, we presented the $k$- producer model, a problem that has not been widely studied. We were able to model the $k$-producer model using the same framework as in the classical monopolist model. First we transformed the $k$-producer model into a system of $k$ monopolist problems. This transformation was achieved using an effective preference function $v_{p_1,p_2,\ldots,p_k}$ which is defined in Proposition 3.A.1. Then we were able to identify the properties that the preference functions should have to satisfy the GSC property, which are stated in Proposition 3.A.2.( The preference functions need to satisfy the GSC property in order to have solution for our problem.)

Using those propositions we were able to form the general $k$-producer model as given by (3.11)-(3.12). Even though we were able to model the problem in a multi-dimensional setting, the analysis of the multi-dimensional model is extremely difficult to carry out. The 1-dimensional case was more easier to handle and we were able to find the system of $k$ ODEs (as stated in Proposition 3.B.2) which give us a set of equilibrium allocations $(y_1\overline{(x)}\ ,\ y_2\overline{(x)},...,y_k\overline{(x)})$ .

Generally, for $k > 2$ it's difficult to find an analytical solution for the $k$-producer model even for the 1-dimensional case, as the system of ODEs is highly non-linear. The exclusion property of the classical monopolist problem is preserved in our model also. We were able to discover an interesting economic phenomena in our model, which is the partial exclusion of the consumer types in different ranges. This property is unique to our model.

The main contributions of this thesis were to identify the conditions on $b$ under which each monopolist's maximization can be reduced to a classical monopolist's problem (which is at least reasonably tractable, as the the GSC property is satisfied by the preference functions, even though the preference functions vary depending on the other monopolist's prices). Finding an equilibrium is then equivalent to simultaneously solving these problems. In one dimension, this result is further exploited to derive a system of ODEs governing the equilibrium.

Our analysis was restricted to some special types of cost and preference functions. The $k$-producer model for general cost and utility functions needs to be further investigated. Also multi-dimensional $k$-producer model is a largely open problem that could yield interesting results.

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