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WEIGHTED HARDY-LITTLEWOOD-SOBOLEV INEQUALITY ON THE UNIT SPHERE

by

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Abstract

One of the main aims in this thesis is to establish analogues of the classical Hardy-Littlewood-Sobolev (HLS) inequality for weighted orthogonal polynomial expansions (WOPEs) on the unit sphere, the unit ball and the simplex. An optimal condition for which this inequality holds is obtained. Classical proofs of the optimality of this inequality on the usual Euclidean spaces rely on the dilation operators and do not seem applicable in our setting, where dilations are not available. The crucial ingredients in our proofs in this thesis are a series of new sharp pointwise estimates for some important kernel functions that appear naturally in the WOPEs. These estimates are more difficult to establish, and will be useful for some other problems in WOPEs.

The HLS inequality for the first order fractional integral operator has been playing important roles in many applications. The second part in this thesis proves an equivalent version of the first order HLS inequality, which involves the tangent gradient and the difference operators. This equivalent version has the advantages that it is much simpler and much easier to deal with in applications. While the main tool for the proof of this equivalent version is the Calderon-Zygmund decomposition, the details are much more involved. Of particular importance in our proof is an elegant decomposition of the second order differential-difference operator associ-

ated with the WOPEs, discovered in this thesis. It turns out that this decomposition is very useful in several other problems, such as the uncertainty principle of the WOPEs.

The main results of this thesis have many interesting applications in N-widths, embedding of function spaces and approximation theory.

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Contents

Notation

$\mathbb{R}^d$	the d dimensional Euclidean space
$\mathbb{S}^{d-1}$	the $(d-1)$ dimensional unite sphere
$\langle\cdot,\cdot\rangle$	the regular inner product in $\mathbb{R}^d$
$\|\cdot\|$	the Euclidean norm in $\mathbb{R}^d$
$\lvert\cdot\rvert$	the absolute norm in $\mathbb{R}^d$, $\lvert x\rvert = \lvert x_1\rvert + \lvert x_2\rvert + \cdots + \lvert x_d\rvert$
$\bar{x}$	the absolute form in $\mathbb{R}^d$, $\bar{x} = (\lvert x_1\rvert, \cdots, \lvert x_d\rvert)$
$L^p(\mathbb{S}^{d-1})$	the L^p space on $\mathbb{S}^{d-1}$ about the regular Lebegue measure
$\|\cdot\|_p$	the regular norm on $L^p(\mathbb{S}^{d-1})$
$\mathbb{Z}_2^d$	the set of $\{\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_d) : \varepsilon_i \in \{0,1\}, i = 1, 2\cdots, d\}$
σ_ε	the reflect operator on $\mathbb{R}^d$, $\sigma_\varepsilon : x \mapsto x - 2\langle x, \varepsilon\rangle\varepsilon$
$\sim$	the asymptotically equivalence.
$\lesssim$	the asymptotically less relationship

Chapter 1

Introduction

The classical Hardy-Littlewood-Sobolev (HLS) inequality on $\mathbb{R}^d$ states that if $\alpha > 0$ and $1 < p \le q < \infty$, then the inequality

$$\|(-\Delta)^{-\alpha/2} f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

holds if and only if $\alpha = d(\frac{1}{p} - \frac{1}{q})$, where Δ denotes the usual Laplace operator on $\mathbb{R}^d$, and $(-\Delta)^{-\frac{\alpha}{2}}$ denotes the fractional power of $(-\Delta)$ (see [StWe, Ch V.]). This inequality has been playing crucial roles in many areas of mathematics, such as harmonic analysis, approximation theory, partial differential equations, and numerical analysis, to name a few (see [CoLi],[St],[TaWe] and the references therein). For example, the well known Sobolev embedding theorem follows directly from this inequality [So]. Due to the importance of the HLS inequality, it has been established in many other different settings, where fractional order integrals are defined in terms of orthogonal expansions, see, for instance, ([To, pp.150-156]) for the Fourier series expansions on the d-torus, [Wa] for the spherical harmonic expansions on the unit sphere.

The main purpose of this thesis is to establish an analogue of the HLS inequality for the weighted orthogonal polynomial expansions (WOPEs) on the unit sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \ \|x\| = 1\}$, and several other related domains, such as the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \ \|x\| \le 1\}$ and the simplex $T^d := \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_j \geqslant 0, \ \ j = 1, \cdots, d. \ |x| \le 1\}$. Here and throughout the paper, $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^d$, and we write $|x| := \sum_{j=1}^d |x_j|$ for $x \in \mathbb{R}^d$.

To be more precise, we need to describe some necessary notations first. Throughout the paper, d is a positive integer and all functions are assumed to be real and Lebesgue measurable on their underlying domains. We denote by $d\sigma(x)$ the usual rotation invariant measure on $\mathbb{S}^{d-1}$ normalized by $\int_{\mathbb{S}^{d-1}} d\sigma(x) = 1$. The weight functions that we will consider on $\mathbb{S}^{d-1}$ are product functions given by

$$h_\kappa^2(x) := \prod_{j=1}^{d} |x_j|^{2\kappa_j}, \tag{1.0.1}$$

where $\kappa := (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ and $\kappa_{\min} := \min_{1\le j\le d} \kappa_j \ge 0$.

Given $1 \le p \le \infty$, we denote by $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ the usual L^p- space with L^p-norm $\|\cdot\|_{\kappa,p}$ being defined with respect to the measure $d\mu_\kappa := h_\kappa^2(x)\, d\sigma(x)$ on $\mathbb{S}^{d-1}$. Thus, $\|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} |f(x)|^p h_\kappa^2(x) d\sigma(x)\right)^{1/p}$ for $1 \le p < \infty$, with the usual change when $p = \infty$. We will simply write $L^p(h_\kappa^2)$ for the space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ whenever the underlying domain is understood and there is no confusion from the context.

Let Π_n^d denote the space of all real spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$; namely, the restrictions to $\mathbb{S}^{d-1}$ of all real algebraic polynomials in d variables of total degree at most n. We denote by $\mathcal{H}_j^d(h_\kappa^2)$ the space of all real spherical polynomials of degree j that are orthogonal to spherical polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2)$. In other words, $\mathcal{H}_j^d(h_\kappa^2)$ is the orthogonal complement of Π_{j-1}^d in the Hilbert space Π_j^d equipped with the inner product

$$\langle f, g\rangle_\kappa := \int_{\mathbb{S}^{d-1}} f(x)g(x)h_\kappa^2(x)\, d\sigma(x), \quad f, g \in L^2(h_\kappa^2),$$

where it is agreed that $\Pi_{-1}^d = \{0\}$. Each function in $\mathcal{H}_j^d(h_\kappa^2)$ is called a spherical h-harmonic of degree j. In the case when $\kappa = 0$ (i.e., the unweighted case), $\mathcal{H}_j^d(h_\kappa^2)$ coincides with the space of usual spherical harmonics of degree j, which had been studied extensively in previous literatures (see, for instance, [WaLi, DaXu2] and the references therein). The theory of h-harmonics was developed by Dunkl (see [Du1, DuXu]) for a family of weight functions invariant under a finite reflection group, of which h_κ in (2.2.1) is the example of the group $\mathbb{Z}_2^d$. One of the difficulties for the study of spherical h-harmonics comes from the fact that the weight functions h_κ^2 contain zeros on the underlying domain $\mathbb{S}^{d-1}$, near which much more delicate analysis is required.

Since the set of all spherical polynomials is dense in $L^2(h_\kappa^2)$, the usual theory of Hilbert space shows that the space $L^2(h_\kappa^2)$ has an orthogonal decomposition $L^2(h_\kappa^2) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j^d(h_\kappa^2)$. This means that each function $f \in L^2(h_\kappa^2)$ has an orthogonal expansion in spherical h-harmonics, $f = \sum_{n=0}^{\infty} \mathrm{proj}_n(h_\kappa^2; f)$, converging in the norm of $L^2(h_\kappa^2)$, where $\mathrm{proj}_n(h_\kappa^2; f)$ denotes the orthogonal projection of f onto the space $\mathcal{H}_n^d(h_\kappa^2)$. Since $\mathbb{S}^{d-1}$ is compact and the weight function h_κ^2 is integrable over $\mathbb{S}^{d-1}$, it can be easily seen that the projection operator $\mathrm{proj}_n(h_\kappa^2)$ extends to a bounded operator on the whole space $L^1(h_\kappa^2)$ for each fixed n. Thus, associated with each function $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$, there is a WOPE $\sum_{j=0}^\infty \mathrm{proj}_j(h_\kappa^2; f)$.

It turns out that each space $\mathcal{H}_j^d(h_\kappa^2)$ coincides with the eigenvector space of a self-adjoint, nonnegative definite, second-order differential-difference operator $\Delta_{\kappa,0}$, an analogue Laplace Beltrami operator, corresponding to the eigenvalue $-j(j+2\lambda_\kappa)$, where $\lambda_\kappa := \frac{d-2}{2} + |\kappa|$. In other words, $f \in \mathcal{H}_j^d(h_\kappa^2)$ if and only if $f \in C^2(\mathbb{S}^{d-1})$ and $\Delta_{\kappa,0} f = -j(j+2\lambda_\kappa)f$. The explicit expression of the operator $\Delta_{\kappa,0}$ will be given in Theorem 2.3.6 in the next section. In the case of $\kappa = 0$, $\Delta_{\kappa,0}$ is simply the usual Laplace-Beltrami operator on $\mathbb{S}^{d-1}$, which was very well studied (see [WaLi]). In general, this operator is self-adjoint, semi-positive definite, but unbounded on $L^2(h_\kappa^2)$. More importantly, the eigenvalue expansions of every $f \in L^2(h_\kappa^2)$ coincides with the above mentioned WOPE of f. Thus, for each spherical polynomial f and every positive integer m, we have

$$(-\Delta_{\kappa,0})^m f = \sum_{j=0}^\infty (j(j+2\lambda_\kappa))^m \,\mathrm{proj}_j(h_\kappa^2; f),$$

where there are only finitely many nonzero terms in the above sum. Indeed, motivated by the above discussion we introduce the fractional integral operator $I_\kappa^\alpha := (-\Delta_{\kappa,0})^{-\alpha/2}$ for each $\alpha > 0$ in a distributional sense via $\mathrm{proj}_0(h_\kappa^2; I_\kappa^\alpha f) = 0$, and

$$\mathrm{proj}_j(h_\kappa^2; I_\kappa^\alpha f) = (j(j+2\lambda_\kappa))^{-\alpha/2}\,\mathrm{proj}_j(h_\kappa^2; f), \quad j = 1, 2, \cdots .$$

Furthermore, it can be shown that for each $\alpha > 0$ and $f \in L^p(h_\kappa^2)$ with $1 \le p < \infty$,

$$I_\kappa^\alpha f = \sum_{j=1}^\infty (j(j+2\lambda_\kappa))^{-\alpha/2}\,\mathrm{proj}_j(h_\kappa^2; f), \tag{1.0.2}$$

with the infinite series on the right hand side converging in the norm of $L^p(h_\kappa^2)$.

One of the main purposes in this paper is to show the following HLS inequality for the above mentioned WOPEs: If $1 < p < q < \infty$ and $\alpha > 0$, then the weighted HLS inequality,

$$\|I_\kappa^\alpha f\|_{\kappa,q} \leqslant C\|f\|_{\kappa,p}, \quad \forall f \in L^p(h_\kappa^2), \tag{1.0.3}$$

holds if and only if $\alpha \geq (2\sigma_k + 1)(\frac{1}{p} - \frac{1}{q})$, where $\sigma_\kappa := \lambda_\kappa - \kappa_{\min}$ and $\kappa_{\min} := \min_{1\leq j\leq d} \kappa_j$. Note that this inequality leads immediately to a sharp Sobolev embedding theorem, which has many important applications in approximation theory (see [CaQu]).

The proof of (1.0.3) relies largely on delicate pointwise estimate of the convolution kernels of I_κ^α, which were not known before. It should be pointed out that the sharpness of the classical HLS inequality for functions on $\mathbb{R}^d$ are normally proved via rescaling the functions on the underlying domain and using some symmetry of the Fourier transform, which do not seem to work in our setting where dilation is no longer available.

Of particular interest is the case when $\alpha = 1$, where we can rewrite (1.0.3) equivalently in the following form: For $1 < p < q < \infty$ and $(2\sigma_k + 1)(\frac{1}{p} - \frac{1}{q}) \leq 1$,

$$\|f\|_{\kappa,q} \leq C\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p}. \tag{1.0.4}$$

Indeed, this is the most frequently used version of the HLS inequality in practice, where only $f \in C^1(\mathbb{S}^{d-1})$ is assumed. Note that $(-\Delta_{\kappa,0})^{1/2}$ is always well defined on a dense subset of $L^2(h_\kappa^2)$ as the operator $-\Delta_{\kappa,0}$ is semi-positive definite.

A problem with (1.0.4) is that the fractional derivative $(-\Delta_{\kappa,0})^{\frac{1}{2}}$ that is a global operator rather than a local operator, which makes it much more difficult to deal with in practice and some applications in approximation theory. In other words, evaluation of the derivative $(-\Delta_{\kappa,0})^{\frac{1}{2}} f$ at a point $x_0 \in \mathbb{S}^{d-1}$ for a given $f \in C^1(\mathbb{S}^{d-1})$ depends not only on the restriction of f to a small neighbourhood of x_0, but also on the behavior of f on the complement of this neighbourhood. To avoid such a difficulty, in the unweighted case, for the usual Laplace-Beltrami operator Δ_0, one normally uses the following equivalent version of the HLS inequality: for $1 < p < q < \infty$ and $(d-1)(\frac{1}{p} - \frac{1}{q}) \leq 1$,

$$\|f\|_q \leq C\|\nabla_0 f\|_p, \tag{1.0.5}$$

due to the equivalence relation (see, for instance, [DaDiHu])

$$\|(-\Delta_0)^{\frac{1}{2}} f\|_p \sim \|\nabla_0 f\|_p, \quad 1 < p < \infty, \tag{1.0.6}$$

where $\|\cdot\|_p$ denotes the L^p-norm defined with respect to the Lebesgue $d\sigma(x)$ on $\mathbb{S}^{d-1}$, and ∇_0 denotes the tangential gradient given by

$$\nabla = \frac{1}{r}\nabla_0 + \xi\frac{\partial}{\partial r}, \quad x = r\xi,\, \xi \in \mathbb{S}^{d-1},$$

where $\nabla_0 = (\partial_1, \partial_2, \cdots, \partial_d)$ is the usual gradient operator. In comparison with (1.0.4), the tangential gradient ∇_0 in (1.0.5) is a local operator, and therefore, is much easier to handle in applications.

Our next purpose is to prove a weighted analogue of (1.0.6), which, in turn, will imply a weighted analogue of (1.0.5). Our main result in this direction asserts that if $1 < p < \infty$, then

$$\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p} \sim \|\nabla_0 f\|_{\kappa,p} + \max_{1\le i\le d} \|E_i f\|_{\kappa,p}, \tag{1.0.7}$$

where

$$E_i f(x) = \frac{f(\sigma_i x) - f(x)}{x_i}, \tag{1.0.8}$$

$$\sigma_i x = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_d), \quad i = 1, \dots, d. \tag{1.0.9}$$

As a result, we deduce from (1.0.4) that for $1 < p < q < \infty$ and $(2\sigma_k + 1)(\frac{1}{p} - \frac{1}{q}) \le 1$,

$$\|f\|_{\kappa,q} \le C\|\nabla_0 f\|_{\kappa,p} + C\max_{1\le i\le d} \|E_i f\|_{\kappa,p}. \tag{1.0.10}$$

The proof of (1.0.7), however, turns out to be rather involved and difficult. It relies on several very delicate estimates of some kernels, as well as the following elegant formula pointed out first in this thesis:

$$\langle \Delta_{\kappa,0} f, g\rangle_\kappa = \int_{\mathbb{S}^{d-1}} \Big[\nabla_0 f \cdot \nabla_0 g\Big] h_\kappa^2(x)\, d\sigma(x) + \sum_{i=1}^d \kappa_i \langle E_i f, E_i g\rangle_\kappa. \tag{1.0.11}$$

In fact, this last formula plays crucial roles in our proof of (1.0.7). Since it enhances our understanding of the WOPEs on $\mathbb{S}^{d-1}$, we believe that it will have some other important applications as well in the future. Clearly, compared with the fractional derivative on the right hand side of (1.0.4), the inequality (1.0.7) looks simpler and is more convenient to use in applications. For example, the terms on the right hand side of (1.0.7) are clearly computable, while the fractional derivative in (1.0.4) is much harder to compute if at all possible.

A second way to establish a weighted analogue of (1.0.5) is to use the differential-difference operators,

$$\mathcal{D}_{i,j} := x_i\mathcal{D}_j - x_j\mathcal{D}_i,$$

which were recently introduced by Yuan Xu [Xu3] in the study of uncertainty principle of the WOPEs. These operators appear very naturally in the setting of Dunkl analysis, where the operators $\mathcal{D}_i := \partial_i + E_i$, $i = 1, \cdots, d$ are normally used to replace the usual partial derivatives D_i, $1 \leq i \leq d$. In fact, they are the weighted analogues of the following angular derivatives, which have recently been found to be very useful in approximation theory on the sphere [DaXu4]:

$$D_{i,j} := x_i\partial_j - x_j\partial_i.$$

Properties about the operators $D_{i,j}$ and $\mathcal{D}_{i,j}$ can be found in Section 1.8 [DaXu2], and [Xu3] respectively. One of the very important properties of the operators $\mathcal{D}_{i,j}$ is that they are invariant on the spaces $\mathcal{H}_n^d(h_\kappa^2)$, $n = 0, 1, \ldots$ of spherical h-harmonics; that is, $\mathcal{D}_{i,j}P \in \mathcal{H}_n^d(h_\kappa^2)$, for any $P \in \mathcal{H}_n^d(h_\kappa^2)$. As the last result on the sphere, we shall prove that for all $1 < p < \infty$, $f \in C^1(\mathbb{S}^{d-1})$ satisfying that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)d\sigma(x)$,

$$\|(-\Delta_{\kappa,0})^{1/2}f\|_{\kappa,p} \sim \max_{1\leq i<j\leq d} \|\mathcal{D}_{i,j}f\|_{\kappa,p}. \tag{1.0.12}$$

Therefore, we deduce that for $1 < p < q < \infty$ and $(2\sigma_k + 1)(\frac{1}{p} - \frac{1}{q}) \leq 1$, and

$$\|f\|_{k,q} \leq C \max_{1\leqslant i<j\leqslant d} \|\mathcal{D}_{i,j}f\|_{\kappa,p}. \tag{1.0.13}$$

The proof of (1.0.12) turns out to be much easier than that of (1.0.10). On the other hand, however, the terms on the right hand side of the inequality (1.0.10) may be more convenient to deal with and appear more frequently in spherical harmonic analysis.

One reason why we consider WOPEs on the sphere is that it allows us to establish similar weighted results on some other related domains, such as the unit ball $\mathbb{B}^d$, and the standard simplex T^d of $\mathbb{R}^d$. In fact, we will also establish analogues of the weighted HLS inequalities (Theorem 3.1.1), with respect to the weights

$$W_\kappa^B(x) := \prod_{j=1}^d |x_j|^{2\kappa_j}(1 - \|x\|^2)^{\kappa_{d+1}-1/2}, \text{ for } \kappa \in [0, \infty)^{d+1}$$

on the unit ball $\mathbb{B}^d$, and with respect to the weights

$$W_\kappa^T(x) := \prod_{j=1}^{d} x_j^{\kappa_j - 1/2}(1 - |x|)^{k_{d+1}-1/2}, \text{ for } \kappa \in [0, \infty)^{d+1}$$

on the simplex T^d. In most cases, the results on these domains can be deduced directly from the corresponding weighted results on the sphere.

In addition, the notation we introduce in this and next chapter will keep throughout the context. The constants in the thesis are denoted by $c, c_1, \cdots$ and $C, C_1, \cdots$ which may vary at every occurrence.

Chapter 2

Preliminaries

To better describe our results, in the chapter we shall introduce some needed preliminaries and standard notion which will be valid throughout the rest of this report.

2.1 Orthogonal Polynomials

In this section, we will introduce some well known result about orthogonal polynomials, which one can refer to [Sz] and [DuXu] for the details.

Theorem 2.1.1. Suppose that X is a compact subset in $\mathbb{R}^d$ with a finite measure μ and a set of linearly independent functions, $\{f_i, i = 1, 2, \cdots\}$ in $L^2(X, \mu)$. Then there exist a set of functions $\{D_j(x) : j \in \mathbb{N}\}$ which is orthogonal in $L^2(X, \mu)$ and satisfies that

$$\mathrm{span}\{D_j(x) : 1 \leqslant j \leqslant n\} = \mathrm{span}\{f_j(x) : 1 \leqslant j \leqslant n\}.$$

In fact, we can choose D_n's in the way $g_{i,j} = \langle f_i, f_j \rangle$, for $i, j \in \mathbb{N}$

$$D_n(x) := \det \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n-1,1} & g_{n-1,2} & \cdots & g_{n-1,n} \\ f_1(x) & f_2(x) & \cdots & f_n(x) \end{bmatrix},$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^(X, \mu)$.

In particular, we let X be an interval $[a, b]$ and μ be a probability measure supported on $[a, b]$ such that $\int_a^b |x|^n d\mu(x) < \infty$, for all n and $\{1, x, x^2, \cdots\}$ is linearly independent in $L^2_{[a,b]}(\mu)$. Applying the above discussion to the basis $f_j(x) = x^{j-1}$, the orthonormal polynomials are defined by

$$p_n(x) = (d_{n+1} d_n)^{-1/2} D_{n+1}(x),$$

where $d_n = \langle D_n, D_n \rangle$. Further more, what is worth to point out is that every p_n is a polynomial with degree n and satisfies that

$$\int_a^b p_n(x)q(x)d\mu(x) = 0,$$

for any polynomials q of degree $< n$.

For $n \geqslant 0$, the polynomial $xp_n(x)$ is of degree $n+1$ and can be expressed by $\{p_j : j \leqslant n+1\}$, but more is true.

Theorem 2.1.2. For the case of orthonormal polynomials,

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x),$$

where $a_n = (d_n d_{n+2}/d_{n+1}^2)^{1/2}$ and $b_n = \int_a^b xp_n^2(x)d\mu(x)$.

With these formulae one can easily find the reproducing kernel for polynomials of degree $\leqslant n$, the Christoffel-Darboux formula:

Proposition 2.1.3. For $n \geqslant 1$,

$$\begin{aligned}\sum_{j=0}^{n} p_j(x)p_j(y) &= \frac{k_n}{k_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y},\\ \sum_{j=0}^{n} p_j(x)^2 &= \frac{k_n}{k_{n+1}}(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x))\end{aligned}$$

with k_n being the leading coefficient of p_n.

Next, we give out two important examples, which will play a key role in the context.

Example 2.1.4. Thy are called ultraspherical polynomials. For a parameter $\lambda > -\frac{1}{2}$, the weight function $\mu(x) = (1-x^2)^{\lambda-\frac{1}{2}}$ on $-1 < x < 1$. Let

$$P_n^\lambda(x) = \frac{(-1)^n}{2^n(\lambda+\frac{1}{2})_n}(1-x^2)^{1/2-\lambda}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2},\ n = 0,1,2\cdots.$$

Then $\{P_n^\lambda(x)\}_{n=0}^\infty$ satisfy the properties below.

1 $\int_{-1}^1 q(x)P_n^\lambda(x)(1-x^2)^{\lambda-1/2}dx = \int_{-1}^1 \frac{d^n}{dx^n}q(x)(1-x^2)^{n+\lambda-1/2}dx$, for any polynomial $q(x)$.

2 $\int_{-1}^1 x^m P_n^\lambda(x)(1-x^2)^{\lambda-1/2}dx = 0$ for $0 \leqslant m < n$.

3 $P_{n+1}^{\lambda}(x) = \frac{2(n+\lambda)}{n+2\lambda} x P_n^{\lambda}(x) - \frac{n}{n+2\lambda} P_{n-1}^{\lambda}(x)$.

4 For $n \geqslant 1$,

$$\frac{d}{dx} P_n^{\lambda}(x) = \frac{n(n+2\lambda)}{1+2\lambda} P_{n-1}^{\lambda+1}(x).$$

For $n \geqslant 0, \lambda > 0$, a key tool *Gegenbauer polynomials* $C_n^{\lambda}(x)$ can given by

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!} P_n^{\lambda}(x).$$

Then we have

$$\frac{d}{dx} C_n^{\lambda}(x) = 2\lambda C_{n+1}^{\lambda+1}(x) \text{ and } C_n^{\lambda}(1) = \frac{(2\lambda)_n}{n!},$$

where $(a)_n$ denotes that $(a)_n = a(a+1)\cdots(a+n-1)$.

Example 2.1.5. For parameters $\alpha, \beta > -1$, the weight function is $(1-x)^{\alpha}(1+x)^{\beta}$ on $-1 \leq x \leq 1$, i.e. $d\mu = (1-x)^{\alpha}(1+x)^{\beta} dx$. The *Jocobi polynomials* $P_n^{(\alpha,\beta)}(x)$ are the orthogonal polynomials with respect to the basis $f_j = x^j$ and $P_n^{(\alpha,\beta)}(1) = \frac{(\beta+1)_n}{n!}$. Then the orthogonality relations are found similarly to the Gegenbauer polynomials, precisely,

1 $\int_{-1}^{1} x^m P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx = 0$ for $0 \leqslant m < n$.

2 $P_n^{(\alpha+1,\beta)}(t) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{j=0}^{n} \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(t)$.

3 For $n \geqslant 1$,

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.1.1}$$

Of course the Jacobi weight includes the Gegenbauer weight as a special case. In terms of the usual notation the relation is

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x) = \frac{\Gamma(2\lambda+n)}{\Gamma(\lambda+\frac{1}{2}+n)} P_n^{\alpha-\frac{1}{2},\beta-\frac{1}{2}}.$$

2.2 Spherical Harmonic Analysis

Let $d \geqslant 2$ be an integer, and $\mathbb{S}^{d-1}$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^d$ with the usual Lebegue measure $d\sigma(x)$ and norm $\|\cdot\|$. As it is well known, spherical harmonics are the restrictions of harmonic homogeneous polynomials to the sphere. Further more, they can be considered as an application of orthogonal polynomials of several variables with $X = \mathbb{S}^{d-1}$, $d\mu = d\sigma$ and the basis being $f_\alpha = x^\alpha$, for $\alpha \in \mathbb{N}^d$.

However, now we introduce more general case in terms of the weight functions that are product functions on $\mathbb{S}^{d-1}$ given by

$$h_\kappa^2(x) := \prod_{j=1}^{d} |x_j|^{2\kappa_j}, \tag{2.2.1}$$

where $\kappa := (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ and $\kappa_{\min} := \min_{1\le j\le d} \kappa_j \ge 0$.

More precisely, first of all, throughout this thesis all functions are assumed to be real and Lebesgue measurable on their underlying domains. Given $1 \le p \le \infty$, we denote by $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ the usual L^p- space with L^p-norm $\|\cdot\|_{\kappa,p}$ being defined with respect to the measure $h_\kappa^2(x)\, d\sigma(x)$ on $\mathbb{S}^{d-1}$. Thus, $\|f\|_{\kappa,p} := \big(\int_{\mathbb{S}^{d-1}} |f(x)|^p h_\kappa^2(x) d\sigma(x)\big)^{1/p}$ for $1 \le p < \infty$, with the usual change when $p = \infty$. We will simply write $L^p(h_\kappa^2)$ for the space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ whenever the underlying domain is understood and no confusion is possible from the context.

Consider the Hilbert space $L^2(h_\kappa^2)$ with the inner product $\langle \cdot, \cdot \rangle_\kappa$ given by

$$\langle f, g \rangle_\kappa := \int_{\mathbb{S}^{d-1}} f(x)g(x)d\mu_\kappa(x), \quad \text{for} \quad f, g \in L^2(h_\kappa; \mathbb{S}^{d-1}).$$

Let Π_n^d denote the space of all real spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$; namely, the restrictions to $\mathbb{S}^{d-1}$ of all real algebraic polynomials in d variables of total degree at most n . We denote by $\mathcal{H}_j^d(h_\kappa^2)$ the space of all real spherical polynomials of degree j that are orthogonal to spherical polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2)$. In other words, $\mathcal{H}_j^d(h_\kappa^2)$ is the orthogonal complement of Π_{j-1}^d in $L^2(h_\kappa^2)$, where it is agreed that $\Pi_{-1}^d = \{0\}$. Each function in $\mathcal{H}_j^d(h_\kappa^2)$ is called a spherical h-harmonic of degree j. In the case when $\kappa = 0$ (i.e., the unweighted case), $\mathcal{H}_j^d(h_\kappa^2)$ coincides with the space of usual spherical harmonics of degree j, which had been studied extensively in previous literatures (see, for instance, [WaLi] and [DaXu2]). Then the Fourier analysis of continuous functions on the weighted unit sphere is performed as an orthogonal decomposition $L^2(h_\kappa^2) = \bigoplus_{j=0}^\infty \mathcal{H}_j^d(h_\kappa^2)$.

2.3 Dunkl Theory

In the late 1980s, C.F.Dunkl introduced Dunkl operators in a series of papers ([Du1]-[Du5]), which became a key tool in the study of special functions with reflection symmetries.

To begin the theory, we at first define *reflections* with respect to vectors.

Definition 2.3.1. Given a nonzero vector $\alpha \in \mathbb{R}^d$, the reflection $\sigma_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is a map defined by

$$\sigma_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

for any $x \in \mathbb{R}^d$.

Then σ_α satisfies the following useful properties.

Properties 2.3.2. σ_α is orthogonal, i.e., $\langle \sigma_\alpha(x), \sigma_\alpha(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^d$; and $\det \sigma_\alpha = -1$

Besides reflections, we also introduce the concept of a *reflection group*. Let

$$O(d, \mathbb{R}) = \{f : \mathbb{R}^d \to \mathbb{R}^d \text{ linear and } \langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle \text{ for all } \alpha, \beta\}$$

be the *orthogonal group* of $\mathbb{R}^d$.

Now we are in the position to point out the definition of *root system* and reflection group.

Definition 2.3.3. Let $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ be a finite set. Then $\mathcal{R}$ is called a root system, if for all $\alpha \in \mathcal{R}$,

1 $\mathcal{R} \bigcap L_\alpha = \{\pm\alpha\}$, where $L_\alpha = \{c\alpha : c \in \mathbb{R}\}$;

2 $\sigma_\alpha(\mathcal{R}) = \mathcal{R}$.

The subgroup $G(\mathcal{R}) \subset O(d, \mathbb{R})$ which is generated by the reflections $\{\sigma_\alpha : \alpha \in \mathcal{R}\}$ is call the *reflection group* (or *Coxeter-group*) associated with $\mathcal{R}$. The dimension of $\text{span}\{\mathcal{R}\}$ is called the rank of $\mathcal{R}$. Since each root system can be written as a disjoint union $\mathcal{R} = \mathcal{R}_+ \cup (-\mathcal{R}_+)$. We call such a set $\mathcal{R}_+$ a *positive root subsystem*. Of course, its choice is not unique.

The following is an example to understand the theory better. It is also the case we will study in the context.

Example 2.3.4. Let the system $\mathcal{R} = \{\pm e_i : i = 1, 2, \cdots, d\}$, where e_i's are the standard basic vectors of $\mathbb{R}^d$. Then the reflection group $G(\mathcal{R})$ is generated by the sign changes $\sigma_i : e_i \mapsto -e_i$, $i = 1, 2, \cdots, d$. The group of sign changes is isomorphic to $\mathbb{Z}_2^d$. The corresponding root system has rank d. A positive system $\mathcal{R}_+ = \{e_i : i = 1, 2 \cdots, d\}$.

From now on we fix $\mathcal{R}$ to be a root system in $\mathbb{R}^d$, normalized in the sense that $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \mathcal{R}$; and let G be the reflection group generated by $\{\sigma_\alpha, \alpha \in \mathcal{R}\}$. Then the *Dunkl operators* defined below can be considered as an extension of the usual partial derivatives in terms of reflections.

Definition 2.3.5. Let $\kappa : \mathcal{R} \to \mathcal{R}$ be a function on the root system $\mathcal{R}$ and invariant under the natural action of G. Then for a fixed normal vector $\xi \in \mathbb{R}^d$, the Dunkl operator $\mathcal{D}_\xi := \mathcal{D}_\xi(\kappa)$ is defined by

$$\mathcal{D}_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle},$$

for $f \in C^1(\mathbb{R}^d)$, where ∂_ξ is the directional derivative corresponding to ξ.

For the particular case when the root system $\mathcal{R} = \{\pm e_i : i = 1, 2, \cdots, d\}$, κ is equal to a d-dimensional vector a in the sense of $\kappa(\pm e_i) = a_i$. At this moment, κ is invariant under the group actions of G and we denote such vector a as κ as well for convenience. Then

$$\mathcal{D}_i f(x) := \mathcal{D}_{e_i} f(x) = \partial_i f(x) + \sum_{i=1}^{d} \kappa_i \frac{f(x) - f(\sigma_i x)}{x_i}, \quad x \in \mathbb{R}^d.$$

Of particular importance, we define Δ_κ , an analogue of Laplace operator, by

$$\Delta_\kappa := \sum_{j=1}^{d} \mathcal{D}_j^2.$$

Of particular, in the case $\mathbb{Z}_2^d$, we shall need the following essential Theorem (for more detail see [DaXu2], Lemma 7.1.8).

Theorem 2.3.6. Given a fixed $\kappa \geqslant 0$, in the spherical-polar coordinates $x = r\xi, r > 0, \xi \in \mathbb{S}^{d-1}$, then we have that,

$$\Delta_\kappa = \frac{d^2}{dr^2} + \frac{2\lambda_\kappa + 1}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_{\kappa,0},$$

where, with Δ_0 denoting the usual Laplace-Beltrami operator,

$$\Delta_{\kappa,0} f = \frac{1}{h_\kappa}[\Delta_0(f h_\kappa) - f\Delta_0 h_\kappa] - \sum_{j=1}^{d} \frac{\kappa_j}{\xi} E_j f,$$

where $E_j f(\xi) = \frac{f(\xi)-f(\sigma_i \xi)}{\xi}$, for $\xi \in \mathbb{S}^{d-1}$.

Further more, the h-spherical harmonics are eigenfunctions of the $\Delta_{h,0}$, that is,

$$\Delta_{\kappa,0} Y_n^h(\xi) = -n(n+2\lambda_\kappa) Y_n^h(\xi), \quad \forall\, Y_n^h \in \mathcal{H}_n^d(h_\kappa^2),\, \xi \in \mathbb{S}^{d-1}.$$

Another aspect of Dunkl theory we need to emphasis is Dunkl's intertwining operator. It was first shown in [Du4] that for non-negative multiplicity functions, the associated commutative algebra of Dunkl operators is intertwined with the algebra of usual partial differential operatos by a unique linear and homogeneous isomorphism on polynomials. More precisely, the following theorem is formulated.

Theorem 2.3.7. For a fixed $\kappa \geqslant 0$, there exists a unique linear isomorphism, " intertwining operator", V_κ such that

$$V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \quad V_\kappa|_{\mathcal{P}_0} = id \quad \text{and} \quad \mathcal{D}_\xi V_\kappa = V_\kappa \partial_\xi, \text{ for all}\, \xi \in \mathbb{R}^d.$$

Particularly, recall that if $\{Y_m^h : m = 1, 2, \cdots, M\}$ with $M := \dim \mathcal{H}_j^d(h_\kappa^2)$, is an orthonormal basis of $\mathcal{H}_j^d(h_\kappa^2)$, then the projection operator proj_j from $L^2(h_\kappa^2, \mathbb{S}^{d-1})$ onto $\mathcal{H}_j^d(h_\kappa^2)$ satisfies that

$$\mathrm{proj}_j\, f(x) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(y) Z_j^\kappa(x,y) h_\kappa^2(y) d\sigma(y) \tag{2.3.1}$$

for any $f \in L^2(h_\kappa; \mathbb{S}^{d-1})$, with

$$Z_j^\kappa(x,y) = \sum_{m=1}^{M} Y_m^h(x) Y_m^h(y), \qquad \forall x, y \in \mathbb{S}^{d-1}. \tag{2.3.2}$$

A key point is that the kernel $Z_j^\kappa(x,y)$ can be deduced via intertwining operators as following formulas of Yuan Xu (see [Xu] and [Xu2] for details)

$$Z_j^\kappa(x,y) = \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa[C_j^{\lambda_\kappa}(\langle \cdot, y\rangle)](x), \tag{2.3.3}$$

where $C_j^{\lambda_\kappa}$ denotes the Gengenbauer Polynomial with degree j and indice λ_κ as defined previously and the intertwining operator V_κ is given explicitly by

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1t_1, x_2t_2, \cdots, x_dt_d) \prod_{j=1}^{d} (1+t_j)(1-t_j^2)^{\kappa_j - 1} dt, \quad (2.3.4)$$

where $c_\kappa = c_{\kappa_1} c_{\kappa_2} \cdots c_{\kappa_d}$ with $c_\mu = \frac{\Gamma(\mu+1/2)}{\sqrt{\pi}\Gamma(\mu)}$. If any one of κ_i's equals 0, then the formula holds in the following sense:

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1} dt = \frac{f(1)+f(-1)}{2}.$$

2.4 Singular Integrals on Homogeneous Spaces

In this section, we shall extent some well-known classical results of harmonic analysis to the more general setting of homogeneous spaces, which are guaranteed on the weighted unit sphere as a consequence. For more detail of proof below, one can refer to [St].

Definition 2.4.1. Given a measure space $(X, \mathbb{B}, \mu)$ with a metric ρ, it is called homogeneous space, if all open balls $B(x,r) := \{y \in X : \rho(x,y) < r\}, x \in X, r > 0$ are measurable with positive finite measure, and that one has the doubling property there exists a positive constant C such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x,r)),$$

for any $x \in X, r > 0$. In addition, the best constant C for which this last inequality holds is called the doubling constant of μ.

Then its Hardy-Littlewood(HL) Maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y), \quad \forall x \in X.$$

Similarly, one can easily extend the Vitali-type and Whitney-type covering lemma to this setting

Lemma 2.4.2. For any measurable subset $E \subset X$, if $\mathbb{B}$ is a finite cover of E, collecting of open balls, then there exist a set $\{B_1, B_2, \cdots, B_m\} \subset \mathbb{B}$ such that

$$\mu(E) \leqslant C \sum_{j=1}^{m} \mu(B_j),$$

for some positive constant C.

This yields the maximal inequality

$$\|\mu\{x \in X : |M(f)(x)| > \alpha\} \lesssim \frac{\|f\|_{L^p(X)}}{\alpha},$$

for any positive number α, and

$$\|Mf\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$$

for $1 < p \leqslant \infty$.

Lemma 2.4.3. Let $G \subsetneqq X$ be a nonempty open set. Then there exist a collection of open balls $\{B_k\}_{k\in\mathbb{N}}$ such that

1 B_k's are disjoint with each other.

2 there exists $c_1 > 1$ such that $G = \bigcup c_1 B_k$.

3 there exists $c_2 > 1$ such that $G^c \bigcap c_2 B_k \neq \emptyset$, for any $k \in \mathbb{N}$.

With this, we have the Calderòn-Zygmund decomposition and Singular integrals on the general setting.

Theorem 2.4.4. Let $f \in L^1(X)$ and $\alpha > \frac{1}{\mu(X)} \int_X |f| d\mu$. Then there exist a decomposition of f, $f = g + \sum_j b_j$ and a sequence of disjoint balls $\{B_j\}$ so that

1 $|g(x)| \leqslant c\alpha$ a.e. $x \in X$.

2 Each b_j is supported in $3B_j$, and

$$\int_X b_j d\mu = 0, \quad \int_{B_j} |b_j| d\mu \leqslant c\alpha\mu(B_j).$$

3 $\sum_j \mu(B_j) \leqslant c\frac{\|f\|_{L^1(X)}}{\alpha}$.

Theorem 2.4.5. Let T be an operator in the form

$$(Tf)(x) = \int_X K(x,y)f(y)d\mu(y),$$

and bounded on $L^q(X)$ with norm A; that is

$$\|Tf\|_{L^q(X)} \leqslant A\|f\|_{L^q(x)}, \quad \forall\, f \in L^q(X).$$

Moreover, if K satisfies that for some constant $c > 1$,

$$\int_{B(z,c\delta)^c} |K(x,y) - K(x,z)|d\mu(x) \leqslant A, \quad \forall\, y \in B(z,\delta), \tag{2.4.1}$$

for all $y \in X$, $\delta > 0$. Then the operator T is bounded in L^q norm on $L^p \cap L^q$ for $1 < p < q$; that is

$$\|Tf\|_p \leqslant A\|f\|_p, \quad \text{for} \quad f \in L^p \cap L^q.$$

In addition, it is necessary to point out the following remarks.

1 T can be extended to L^q uniquely and keep the boundedness, since $L^p \cap L^q$ is dense in L^q;

2 If there is an upper bound for the radius of all of balls in X, then the condition " for any $\delta > 0$" can be deduced to " for $0 < \delta < \delta_0$ with some $\delta_0 > 0$";

3 The domain in the integral of (2.4.1) can be replaced as well by a measurable set D^c with $\mu(D) \leqslant c\sum_j \mu(B_j)$.

2.5 Cesàro means

In this subsection, we will talk about some facts about Cesàro means in terms of the spherical h-harmonic, which will be a key tool of our following proof. For more details, one can refer to [DaXu2] and [DaXu5].

Definition 2.1. For $\delta > 0$, the Cesàro means of the spherical function f are defined by

$$S_n^\delta(h_\kappa^2; f) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \mathcal{P}_j^\kappa f,$$

where A_j^δ denotes as

$$A_j^\delta = \left(\begin{array}{c} \delta + j \\ j \end{array} \right) = \frac{(\delta+j)(\delta+j-1)\cdots(\delta+1)}{j!}.$$

Theorem 2.5.1 ([DaXu2],corollary 8.1.2)**.** If $\delta > \sigma_\kappa$, then for $f \in L^p(h_\kappa, \mathbb{S}^{d-1})$ and $1 \leqslant p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$,

$$\sup_n \|S_n^\delta(h_\kappa^2; f)\|_{p,\kappa} \leqslant c\|f\|_{p,\kappa}.$$

Consider $S_n^\delta(h_\kappa^2; f)$ as a convolution:

$$S_n^\delta(h_\kappa^2; f) = f * K_n^\delta(h_\kappa^2),$$

then the kernel $K_n^\delta(h_\kappa^2; x, y)$ is the Cesàro means of $Z_j^{\lambda_\kappa}(x, y)$

$$K_n^\delta(h_\kappa^2; x, y) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-k}^\delta Z_j^{\lambda_\kappa}(x, y),$$

which has the following pointwise estimate.

Theorem 2.5.2 ([DaXu2], Theorem 8.1.1)**.** For any $x, y \in \mathbb{S}^{d-1}$,

$$|K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y)| \leqslant cn^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-\beta(\delta)},$$

where $\beta(\delta) := \min\{d+1, \delta - \sigma_\kappa + d\}$.
Further more, for any $\delta > \sigma_\kappa$

$$\int_{\mathbb{S}^{d-1}} |K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) d\sigma(y) \leqslant C, \tag{2.5.1}$$

where C is a constant independent of n.

Theorem 2.5.3 ([DaXu2],B.1.13)**.** Let

$$S_n^\ell(u) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \frac{j+\lambda}{\lambda} C_j^\lambda(u),$$

then for $\ell > 2\lambda + 1$,

$$0 \leqslant S_n^\ell(u) \leqslant cn^{-1}(1 - u + n^{-2})^{\lambda+1}.$$

2.6 Difference operators

Let f be a function defined on $\mathbb{R}$. For $r = 1, 2, \cdots$, we define the difference operator Δ^r by

$$\Delta^0 f(x) = f(x), \quad \Delta f(x) = f(x) - f(x+1), \quad \Delta^r = \Delta^{r-1}(\Delta f(x)). \tag{2.6.1}$$

For a sequence $\{a_n\}_{n=0}^{\infty}$, the difference operator $\Delta^r a_n$ is defined as $\Delta^r f(n)$, where $f(n) = a_n$.

Proposition 2.6.1. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence converging to 0 and $\{f_j\}_{j=1}^{\infty}$ be a functional sequence. Then

$$\sum_{j=1}^{\infty} a_j f_j(t) = \sum_{j=1}^{\infty} \Delta^{\ell+1} a_j A_j^{\ell} K_j^{\ell}(t), \tag{2.6.2}$$

where $A_j^{\ell} = \left(\begin{array}{c} \ell + j \\ j \end{array} \right)$, $K_j^{\ell}(t) = \frac{1}{A_j^{\ell}} \sum_{k=0}^{j} A_{j-k}^{\ell} f_k(t)$.

Proof. Using summation by parts repeatedly, we can write

$$\sum_{j=1}^{\infty} a_j f_j(t) = \sum_{j=1}^{\infty} \Delta^{\ell+1} a_j \sum_{k_{\ell+1}}^{j} \sum_{k_\ell=1}^{k_{\ell+1}} \cdots \sum_{k_1=1}^{k_2} f_{k_1}(t).$$

By induction on ℓ, We claim that

$$\sum_{k_{\ell+1}}^{j} \sum_{k_\ell=1}^{k_{\ell+1}} \cdots \sum_{k_1=1}^{k_2} f_{k_1}(t) = \sum_{k=1}^{j} A_{j-k}^{\ell} f_k(t). \tag{2.6.3}$$

First, it is simple to see that (2.6.3) holds for $\ell = 0$.
Next, We assume that (2.6.3) is true for $\ell \leqslant n$.
Then

$$\begin{aligned}
&\sum_{k_{n+1}}^{j} \sum_{k_n=1}^{k_{n+1}} \cdots \sum_{k_1=1}^{k_2} f_{k_1}(t) = \sum_{k_{n+1}=1}^{j} \sum_{k=1}^{k_{n+1}} A_{k_{n+1}-k}^{n-1} f_k(t) \\
=& \sum_{k=1}^{j} \sum_{k_{n+1}=k}^{j} A_{k_{n+1}-k}^{n-1} f_k(t) = \sum_{k=1}^{j} \sum_{m=0}^{j-k} A_m^{n-1} f_k(t) = \sum_{k=1}^{j} A_{j-k}^{n} f_k(t)
\end{aligned}$$

where the last equality is followed from the fact $\sum_{m=0}^{s} A_m^n = A_s^{n+1}$. $\square$

2.7 the Littlewood-Paley theory

As we know, the Littlwood Paley theory introduce a way to express and quantify orthogonality of the Fourier transform on L^p function spaces. As a performance of Fourier analysis on the unite sphere $\mathbb{S}^{d-1}$ with the weight h_κ^2, h-spherical harmonics provide a tool to extend the Littlewood Paley theory (see [DaXu2, Chapter 3]).

Theorem 2.7.1 (Littlewood-Paley Inequality)**.** Let m be the smallest positive integer greater than $d/2$. If θ is a compactly supported function in $C^m[0,\infty]$ with $\operatorname{supp}\theta \subset (a,b)$ for some $0 < a < b < \infty$, then for all $f \in L^p(\mathbb{S}^{d-1})$ with $1 < p < \infty$,

$$\|(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2)^{1/2}\|_p \leqslant \|f\|_p,$$

where

$$\Delta_{\theta,j}(f) := \sum_{n=0}^{\infty} \theta(\frac{n}{2^j}) \operatorname{proj}_n(f),\ j = 1, 2, \cdots \tag{2.7.1}$$

and c depends only on p,d,a and b. If, in addition,

$$0 < A_1 \leqslant \sum_{j=0}^{\infty} |\theta(2^{-j}t)|^2 \leqslant A_2 < \infty,\ \forall t > 0,$$

and $\int_{\mathbb{S}^{d-1}} f(x) d\sigma(x)$, then

$$\|(\sum_{j=0}^{\infty} |\Delta_{\theta,j} f|^2)^{1/2}\|_p \sim \|f\|_p.$$

Theorem 2.7.2 (Fefferman-Stein)**.** If $1 < p, q < \infty$ and $\{f_j\}$ is a sequence of functions on X, then

$$\| \left(\sum_{j=0}^{\infty} |M_\mu f_j|^q \right)^{1/q} \|_p \lesssim \| \left(\sum_{j=0}^{\infty} |f_j|^q \right)^{1/q} \|_p,$$

where $(X, d\mu)$ is a measurable space and M_μ is the Hardy-Littlewood maximum function correspondingly.

Theorem 2.7.3 (Weighted Nikolskii's Inequalities [DaWa],Lemma 2.3)**.** Let $0 < p < q \leqslant \infty$, then for any $g \in \prod_n^d$

$$\|g\|_{q,\kappa} \leqslant Cn^{(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})}\|g\|_{p,\kappa},$$

where C depends only on p, q and κ.

Chapter 3

Hardy-Littlewood-Sobolev Inequality on the Unite Sphere

In this chapter, we shall first formulate results of pointwise estimete that will be indispensable in much of our future work. Next, the first main result, HLS inequality and its necessary conditions, on the weighted unit sphere will be introduced and proved in detail. Finally, the analogue conclusions on the corresponding weighted unit ball and simplex will be builded up.

Let $\eta \in C^{3\ell-1}[0,\infty)$ supported in $[0,2]$ with the jth order derivative $\eta^{(j)}(0)=0$ for $j=1,2,\cdots,3\ell-2$. We define

$$L_N(t) := \sum_{j=1}^{\infty} \eta(\frac{j}{N})\frac{j+\lambda}{\lambda}C_j^{\lambda}(t),\ t\in[-1,1].$$

In fact, L_N can be seen as a partial summation operator. For our purpose one of aspects of its importance arises form the following fact.

Theorem 3.0.4 ([DaXu2] Theorem 2.6.5)**.** Let ℓ be a positive integer. For any $\theta \in [0,\pi]$ and $N \in \mathbb{N}$,

$$|L_N^{(r)}(\cos\theta)| \leqslant c_\ell \|\eta^{(3\ell-1+2r)}\|_\infty N^{2\lambda+1+2r}(1+N\theta)^{-\ell},\ j=0,1,\cdots.$$

The proof of the fact is not involved in the context, but it will be best to present it in an appendix.

Lemma 3.0.5. Let Ψ be a polynomial on $[-1,1]$ satisfying that for any positive number $\ell > 2\lambda_\kappa + 1$, there exists a constant C which only depends on ℓ and d such that

$$|\Psi(\cos\theta)| \leq Cn^{2\lambda_\kappa+1+r}(1+n\theta)^{-\ell},$$

for any r. Then for any $x, y \in \mathbb{S}^{d-1}$, there exists a positive number ℓ_0 such that

$$\left|V_\kappa\Big[\Psi_n(\langle y,\cdot\rangle)\Big](x)\right| \le C_{\ell_0}\frac{n^{d-1+r}(1+n\rho(\bar{x},\bar{y}))^{-\ell_0}}{\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{2\kappa_j}}. \tag{3.0.1}$$

Furthermore,

$$\left|V_\kappa\Big[\Psi_n(\langle x,\cdot\rangle)\Big](y)\right| \le C'\frac{n^{r}(1+n\widetilde{\rho}(x,y))^{-\ell_0}}{U_{n^{-1}}(x)+U_{n^{-1}}(y)+U(\bar{x},\bar{y})}, \tag{3.0.2}$$

and

$$\left|V_\kappa\Big[\Psi_n(\langle x,\cdot\rangle)\Big](y)\right| \le C'\frac{n^{r}(1+n\widetilde{\rho}(x,y))^{-\ell_0}}{U(\bar{x},\bar{y})}, \tag{3.0.3}$$

where $U_r(x)$ denotes as

$$U_r(x) = \int_{\rho(x,y)\le r} h_\kappa^2(y)d\sigma(y)$$

and $U(x,y) = U_{\rho(x,y)}(x)$.

Proof. By theorem 3.0.4, we have,

$$|\Psi(\cos\theta)| \leqslant C2^{s(2\lambda_\kappa+1-\alpha)}(1+2^s\theta)^{-\ell} \tag{3.0.4}$$

which implies that

$$\begin{aligned}|V_\kappa(\Psi(<\cdot,y>))(x)| &\leqslant C\int_{[-1,1]^d}|\Psi(\sum_{j=1}^{d}t_jx_jy_j)|\prod_{j=1}^{d}(1+t_j)(1-t_j^2)^{\kappa_j-1}dt\\ &\le C_\ell\int_{[-1,1]^d}n^{2\lambda_\kappa+1+r}(1+n\theta)^{-\ell}\prod_{j=1}^{d}(1-t_j^2)^{k_j-1}dt\end{aligned}$$

by setting $\cos\theta = \sum_{j=1}^{d} t_jx_jy_j$. Noticing that

$$\theta^2 \backsim 1-\cos\theta = 1-\sum_{j=1}^{d}t_jx_jy_j$$

and

$$1-\sum_{j=1}^{d}|x_jy_j|+\sum_{j=1}^{d}|x_jy_j|(1-|t_j|)\sim\rho(\bar{x},\bar{y})^2+\sum_{j=1}^{d}|x_jy_j|(1-|t_j|),$$

we have that there exists a constant c such that for any $t\in[-1,1]^d$

$$\big[\rho(\bar{x},\bar{y})^2+\sum_{j=1}^{d}|x_jy_j|(1-|t_j|)\big]^{\frac{1}{2}}\le c\theta.$$

Thus

$$\begin{aligned}
&|V_\kappa(\Psi(<\cdot,y>))(x)|\\
\le& C_\ell n^{2\lambda_\kappa+1+r}\int_{[0,1]^d}\left(1+n\big[\rho(\bar{x},\bar{y})^2+\sum_{j=1}^{d}|x_jy_j|(1-t_j)\big]^{\frac{1}{2}}\right)^{-\ell}\prod_{j=1}^{d}(1-t_j)^{\kappa_j-1}dt\\
\le& n^{2\lambda_\kappa+1+r}\int_{[0,1]^d}\prod_{j=1}^{d}\Big[1+n\rho(\bar{x},\bar{y})+n\big(|x_jy_j|(1-t_j)\big)^{\frac{1}{2}}\Big]^{-\frac{\ell}{d}}(1-t_j)^{\kappa_j-1}dt\\
\le& n^{2\lambda_\kappa+1+r}\prod_{j=1}^{d}\int_0^{n|x_jy_j|^{\frac{1}{2}}}\Big[1+n\rho(\bar{x},\bar{y})+m\Big]^{-\frac{\ell}{d}}\frac{m^{2\kappa_j-1}}{n^{2\kappa_j}|x_jy_j|^{\kappa_j}}dm\\
\le& n^{d-1+r}\prod_{j=1}^{d}|x_jy_j|^{-\kappa_j}\int_0^{n|x_jy_j|^{\frac{1}{2}}}\Big[1+n\rho(\bar{x},\bar{y})+m\Big]^{-\frac{\ell}{d}}m^{2\kappa_j-1}dm
\end{aligned}$$

If we denote $\int_0^{na}\Big[1+nb+m\Big]^{-\ell}m^{2k-1}dm$ as M, for some positive numbers a,b,k, then by choosing ℓ so large that $\ell>2d|\kappa|$ and breaking up the integral in the parts where $m<(1+nb)$ and $m>(1+nb)$, we claim that

$$M\le c(1+nb)^{-\ell+2k}[a^{-2k}+n^{2k}+b^{-2k}]a^{2k}$$

because that

$$\int_0^{(1+nb)}\Big[1+nb+m\Big]^{-\ell}m^{2k-1}dm\le c_1(1+nb)^{-\ell}n^{2k}a^{2k}\le(1+nb)^{-\ell+2k}b^{-2k}a^{2k},$$

and

$$\int_{(1+nb)}^{\infty}\Big[1+nb+m\Big]^{-\ell}m^{2k-1}dm\le c_2(1+nb)^{-\ell+2k}.$$

Therefore, we obtain an estimate

$$
\begin{aligned}
&|V_\kappa(\Psi(<\cdot,y>))(x)| && (3.0.5)\\
\le& c_\ell n^{d-1+r}[1+n\rho(\bar{x},\bar{y})]^{-\frac{\ell}{d}+2|\kappa|}\prod_{j=1}^{d}\Big[|x_jy_j|^{-\kappa_j}+n^{2\kappa_j}+\rho(\bar{x},\bar{y})^{-2\kappa_j}\Big]. && (3.0.6)
\end{aligned}
$$

For given $x,y\in\mathbb{S}^{d-1}$, if let j be such that $|x_j|>4\rho(\bar{x},\bar{y})$,

$$|y_j|-|x_j|\le|\bar{x}-\bar{y}|\le 4\rho(\bar{x},\bar{y})<4|x_j|,$$

and

$$|x_j|>4\rho(\bar{x},\bar{y})>2|\bar{x}-\bar{y}|\ge 2|x_j|-2|y_j|,$$

so

$$|x_j|\sim|y_j|;$$

otherwise if $|x_j|\le 4\rho(\bar{x},\bar{y})$,

$$|y_j|\le|x_j|+|x_j-y_j|\le 4\rho(\bar{x},\bar{y})+2\rho(\bar{x},\bar{y}).$$

Then we divide the product in (3.0.5) into two parts, $J_1=\{j=1,\cdots,d:|x_j|>4\rho(\bar{x},\bar{y})\}$ and $J_2=\{j=1,\cdots,d:|x_j|\le 4\rho(\bar{x},\bar{y})\}$. Using the discussion above, we obtain that

$$
\begin{aligned}
&\prod_{j=1}^{d}\Big[|x_jy_j|^{-\kappa_j}+n^{2\kappa_j}+\rho(\bar{x},\bar{y})^{-2\kappa_j}\Big]\le c_1\prod_{j=1}^{d}\Big[|x_jy_j|^{\frac{1}{2}}+n^{-1}+\rho(\bar{x},\bar{y})\Big]^{-2\kappa_j}\\
\le& c_2\prod_{j\in J_1}\Big[|x_j|+n^{-1}\Big]^{-2\kappa_j}\prod_{j\in J_2}\Big[n^{-1}+\rho(\bar{x},\bar{y})\Big]^{-2\kappa_j}\\
\le& c_3\prod_{j=1}^{d}\Big[|x_j|+n^{-1}+\rho(\bar{x},\bar{y})\Big]^{-2\kappa_j}
\end{aligned}
$$

Thus we obtain the estimate (3.0.1). For (3.0.2) and (3.0.3), it just need to notice the estimate of the area of a cap, given by Y. Xu in [DaXu2], that is

$$U(x,\theta)\sim\theta^{d-1}\prod_{j=1}^{d}[|x_j|+\theta]^{2\kappa_j}.$$

□

Noting that if we break up the sum in the kernel of I_κ^α in the sense that

$$K_\alpha(x,y)=\sum_{s=0}^{\infty}\sum_{j=1}^{\infty}\varphi(\frac{j}{2^s})(j(j+2\lambda_\kappa))^{-\alpha/2}\frac{\lambda_\kappa+j}{\lambda_\kappa}V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x,\cdot\rangle)\Big](y),$$

where φ is supported on $[\frac{1}{2},2]$ and of class $C^\infty[0,\infty)$. Then by setting

$$\varphi_s(t):=2^{-s\alpha}\varphi(t)(t(t+2^{-s+1}\lambda_\kappa))^{-\alpha/2},$$

and applying the above lemma 3.0.1 to $\sum_{j=1}^{\infty}\varphi_s(\frac{j}{2^s})\frac{\lambda_\kappa+j}{\lambda_\kappa}V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x,\cdot\rangle)\Big](y)$, we obtain that

$$|K_\alpha(x,y)|\lesssim\sum_{s=0}^{\infty}\frac{2^{-s\alpha}(1+2^s\rho(\bar{x},\bar{y}))^{-\ell}}{U(\bar{x},\bar{y})}\lesssim\frac{\rho(\bar{x},\bar{y})^\alpha}{U(\bar{x},\bar{y})},\tag{3.0.7}$$

for any $x,y\in\mathbb{S}^{d-1}$, since $\|\varphi_s^{(3\ell-1)}\|_\infty\le c2^{-\alpha}$.

3.1 HLS on the unit sphere

After establishing the pointwise estimate of the kernel of I_κ^α, we are on the position to formulate our first main result: The Hardy-Littlewood-Sobolev theorem of fractional integration.

Theorem 3.1.1. Let $\alpha>0$, $1\leqslant p<q<\infty$ and $\sigma_\kappa:=\frac{d-2}{2}+|\kappa|-\kappa_{min}$.

a If $p>1$ and $\alpha\geqslant(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})$, then

$$\|I_\kappa^\alpha f\|_{q,\kappa}\leqslant C\|f\|_{\kappa,p},\quad\forall f\in L^p(h_\kappa^2;\mathbb{S}^{d-1})\tag{3.1.1}$$

b If $p=1$ and in addition, $\alpha>(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})$, then (3.1.1) is still true.

c If $p=1$ and $\alpha=(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})$, then the inequality in part(a) will be replaced by

$$\mu_\kappa\{x\in\mathbb{S}^{d-1}:|I_\kappa^\alpha f(x)|>\lambda\}\leqslant c\frac{\|f\|_{q,\kappa}}{\lambda},$$

for any $\lambda>0$. That is, the mapping I_κ^α is of "weak-type" $(1,q)$.

Proof. Using the estimate (3.0.7), in view of the identity (1.0.2), we have that

$$\begin{aligned} I_\kappa^\alpha f(x) &= \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} f(y)K(x,y)h_\kappa^2(y)d\sigma(y) \\ &\lesssim \int_{\mathbb{S}^{d-1}} \frac{|f(y)|\rho(\bar{x},\bar{y})^\alpha}{U(\bar{x},\bar{y})}d\mu(y), \end{aligned}$$

where $d\mu(y) = h_\kappa^2(y)d\sigma(y)$.
Let us decompose K as $K_1 + K_\infty$, where

$$K_1(x,y) = K(x,y) \quad \text{if } \rho(\bar{x},\bar{y}) \leqslant \delta \quad K_1(x,y) = 0 \quad \text{if } \rho(\bar{x},\bar{y}) > \delta \tag{3.1.2}$$

$$K_\infty(x,y) = K(x,y) \quad \text{if } \rho(\bar{x},\bar{y}) > \delta \quad K_\infty(x,y) = 0 \quad \text{if } \rho(\bar{x},\bar{y}) \leqslant \delta \tag{3.1.3}$$

where δ is a fixed positive constant which will be determined later. First for a fixed point $x \in \mathbb{S}^{d-1}$, we consider

$$\begin{aligned} I_1(x) :&= \int_{\mathbb{S}^{d-1}} |f(y)K_1(x,y)|d\mu_\kappa(y) \\ &= \sum_{k=0}^{\infty}\int_{2^{-k-1}\leqslant\rho(\bar{x},\bar{y})<2^{-k}\delta} \frac{|f(y)|\rho(\bar{x},\bar{y})^\alpha}{U(\bar{x},\bar{y})}d\mu(y) \\ &\leqslant \sum_{k=0}^{\infty}\frac{(2^{-k}\delta)^\alpha}{U(\bar{x},2^{-k-1}\delta)}\int_{\rho(\bar{x},\bar{y})<2^{-k}\delta}|f(y)|d\mu(y) \\ &\leqslant c_1\sum_{\varepsilon\in\mathbb{Z}_2^d}\delta^\alpha M_\kappa f(\varepsilon x) \end{aligned}$$

For the other part, by Holder's Inequality, we have

$$\begin{aligned} I_2(x) :&= \int_{\mathbb{S}^{d-1}} |f(y)K_\infty(x,y)|d\mu_\kappa(y) \\ &\leqslant \|f\|_{\kappa,p}\left\{\int_{\rho(\bar{x},\bar{y})\geqslant\delta}\rho(\bar{x},\bar{y})^{(\alpha-(d-1))p'}\prod_{j=1}^{d}\Big(\frac{|y_j|}{(|x_j|+\rho(\bar{x},\bar{y}))^{p'}}\Big)^{2\kappa_j}d\sigma(y)\right\}^{\frac{1}{p'}} \\ &\leqslant c_2\|f\|_{\kappa,p}\left[\int_{\rho(\bar{x},\bar{y})\geqslant\delta}\rho(\bar{x},\bar{y})^{(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q}-1)p'+(2|\kappa|-2\kappa_{\min})}d\sigma(y)\right]^{1/p'} \\ &\leqslant c_3\|f\|_{\kappa,p}\sum_{\varepsilon\in\mathbb{Z}_d^2}\Big[\int_\delta^\infty\theta^{-\frac{p'}{q}(2\sigma_\kappa+1)-1}d\theta\Big]^{1/p'} \\ &= C\|f\|_{\kappa,p}\delta^{-\frac{2\sigma_\kappa+1}{q}}. \end{aligned}$$

Now, let

$$\tilde{\delta} = \Big(\frac{\|f\|_{\kappa,p}}{\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x)}\Big)^{\frac{p}{2\sigma_\kappa+1}}.$$

If $\tilde{\delta} \geqslant \pi$, then

$$|\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x)| \lesssim \|f\|_{\kappa,p}.$$

At the moment, take $\delta = \pi$, then $I_2 = 0$, and

$$|I_\kappa^\alpha f(x)| \lesssim \|f\|_{\kappa,p}.$$

Otherwise let $\delta = \tilde{\delta}$, then

$$|I_\kappa^\alpha f(x)| \lesssim \|f\|_{\kappa,p}^{(1-\frac{p}{q})}(\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x))^{\frac{p}{q}}. \tag{3.1.4}$$

Thus for any $x \in \mathbb{S}^{d-1}$,we can obtain that

$$|I_\kappa^\alpha f(x)| \lesssim \|f\|_{\kappa,p} + \|f\|_{\kappa,p}^{(1-\frac{p}{q})}(\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x))^{\frac{p}{q}}.$$

Since M_κ is strong (p,p) operator for $p > 1$, and $h_\kappa^2(x)d\sigma(x)$ is invariable for $\mathbb{Z}_2^d$,

$$\|I_\kappa^\alpha f\|_{q,\kappa}^q \leqslant C\|f\|_{\kappa,p}^{q-p}\|f\|_{\kappa,p}^p + \|f\|_{\kappa,p} = C\|f\|_{\kappa,p}^q.$$

Finally for $\alpha > (2\sigma_\kappa + 1)(\frac{1}{p} - \frac{1}{q})$, there exists $q' \in (1,\infty)$ such that $q' > q$ and

$$\alpha = (2\sigma_\kappa + 1)(\frac{1}{p} - \frac{1}{q'}).$$

Then by the above what we have shown and Holder Inequality, we obtain that

$$\|I_\kappa^\alpha f\|_{q,\kappa} \lesssim \|I_\kappa^\alpha f\|_{\kappa,q'} \leqslant c\|f\|_{\kappa,p}.$$

Thus part(a) of the theorem is proved.

From the above proof, we can see that there is a cut-off function $\varphi \in C^\infty(\mathbb{R})$ such that $\chi_{[1,2]} \leqslant \varphi \leqslant \chi_{[1,4]}$ and for any $x \in \mathbb{S}^{d-1}$,

$$
\begin{aligned}
I_\kappa^\alpha f(x) &= \sum_{j=1}^\infty \sum_{m=1}^\infty \phi(\frac{m}{2^j})(m(m+2\lambda_\kappa))^{-\alpha/2}\operatorname{proj}_m f(x) \\
&= \sum_{j=1}^\infty \sum_{m=1}^\infty [\Delta^{\ell+1}\phi(\frac{m}{2^j})(m(m+2\lambda_\kappa))^{-\alpha/2}] A_m^\ell S_m^\ell(h_\kappa^2; f)
\end{aligned}
$$

where the last identity is from the formula (2.6.2) and $S_m^\ell(h_\kappa^2; f)$ is the Cesaro means of spherical function f. In fact, we can choose φ as $\eta(\frac{x}{2}) - \eta(x)$ where η is defined in proof of Lemma. Noting that

$$
|\Delta^{\ell+1}\phi(\frac{m}{N})| \sim N^{-\ell-1} \quad \text{and} \quad A_m^\ell \sim m^\ell, \tag{3.1.5}
$$

then for any integer $\ell > \sigma_\kappa$, there exists a constant c only depending on α, κ, q, d such that

$$
\begin{aligned}
\|I_\kappa^\alpha f\|_{\kappa,q} &\leqslant c_1 \sum_{j=1}^\infty 2^{-j(\ell+1)} \sum_{2^j \leqslant m < 2^{j+2}} m^{-\alpha+\ell} \|S_m^\ell(h_\kappa^2; f)\|_{\kappa,q} \\
&\leqslant c_2 \sum_{j=1}^\infty 2^{-j(\ell+1)} \sum_{2^j \leqslant m < 2^{j+2}} m^{-\alpha+\ell+(2\sigma_\kappa+1)(1-\frac{1}{q})} \|S_m^\ell(h_\kappa^2; f)\|_{\kappa,1} \\
&\leqslant c_3 \sup_m \|S_m^\ell(h_\kappa^2; f)\|_{\kappa,1} \sum_{j=1}^\infty 2^{-j(\alpha-(2\sigma_\kappa+1)(1-\frac{1}{q}))} \\
&\leqslant c\|f\|_{\kappa,1},
\end{aligned}
$$

where the second inequality is followed from weighted Nikolskii Theorem given by F.Dai and Y.Xu [DaXu5], and the third one is just a special case $p = 1$ in theorem 2.5.1.

Finally, we point out two remarks. First is the method in part (b) still works on the case $\alpha > (2\sigma_\kappa + 1)(\frac{1}{p} - \frac{1}{q})$ by just replacing 1 as p in the proof; however, it will fail on the case $\alpha = (2\sigma_\kappa+1)(\frac{1}{p} - \frac{1}{q})$, since the series $\sum_{j=1}^\infty 2^{-j(\alpha-(2\sigma_\kappa+1)(1-\frac{1}{q}))}$ will be divergent at this moment. Second, because Hardy Littlewood Maximal function is only weak type for L^1, it is impossible to apply the method in part (a) to part (b).

By now the part (b) is complete.

To begin proving part (c), we construct a sequence of functions which makes (3.1.1) destroyed. Without loss of generality, we suppose that $\kappa_1 = \min_j \kappa_j$ for a given positive vector $\kappa \in \mathbb{R}^d$. Let $f_N(x) = \sum_{j=1}^{N} Z_j^\kappa(x, e_1)$ with $e_1 := (1, 0, \cdots, 0)$ for any integer N and $x \in \mathbb{S}^{d-1}$, then

$$\begin{aligned}
I_\kappa^\alpha f_N(x) &= \sum_{j=1}^{N} (j(j+2\lambda_\kappa))^{-\alpha/2} Z_j^\kappa(x, e_1) \\
&= V_\kappa[\sum_{j=1}^{N} (j(j+2\lambda_\kappa))^{-\alpha/2} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(\langle \cdot, e_1 \rangle)](x)
\end{aligned}$$

Choose a cut-off function $\eta \in C^\infty(\mathbb{R})$ with $\chi_{[0,1]} \leqslant \eta \leqslant \chi_{[0,2]}$ and let

$$L_N(t_1) := \sum_{j=1}^{\infty} \eta(\frac{j}{N})(j(j+2\lambda_\kappa))^{-\alpha/2} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(x_1 t_1),$$

Then a pointwise lower bound estimate is obtained as the following

$$\begin{aligned}
\lim_{N\to\infty} |I_\kappa^\alpha f_N(x)| &= \lim_{N\to\infty} V_\kappa[\sum_{j=1}^{\infty} \eta(\frac{j}{N})(j(j+2\lambda_\kappa))^{-\alpha/2} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(\langle \cdot, e_1 \rangle)](x) \\
&= c_1 \int_{[-1,1]^d} \sum_{j=1}^{\infty} (j(j+2\lambda_\kappa))^{-\alpha/2} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(t_1 x_1) \prod_{j=1}^{d} (1+t_j)(1-t_j^2)^{\kappa_j - 1} dt \\
&\geqslant c_2 \int_0^1 (1 - t_1 x_1)^{(\alpha - 2\lambda_\kappa - 1)/2} (1-t_1)^{k_1 - 1} dt_1 \\
&\geqslant c_2 \int_0^{1-|x_1|} (1-|x_1|)^{(\alpha - 2\lambda_\kappa - 1)/2} t^{\kappa_1 - 1} dt \\
&= c_2 (\sqrt{1-|x_1|})^{\alpha - (2\sigma_\kappa + 1)}
\end{aligned}$$

where the second identity is garanteed by Lebesgue Dominated Convergence Theorem(LDCT) through checking

$$\begin{aligned}
|L_N(t_1)| &= \sum_{j=1}^{2N} [\Delta^{\ell+1} \eta(\frac{j}{N})](j(j+\lambda_\kappa))^{-\alpha/2} A_j^\ell S_j^\ell(x_1 t_1) \\
&\lesssim \sum_{j=1}^{2N} N^{-(\ell+1)} j^{-\alpha} j^{-\ell} j^{-1} (1 - x_1 t_1 + j^{-2})^{-(\lambda_\kappa + 1)} \\
&\lesssim (1-|x_1|)^{-(\lambda_\kappa + 1)} \sum_{j=1}^{\infty} j^{-\alpha - 1} < \infty
\end{aligned}$$

by using summation by parts formula and theorm 2.5.3 in the first and second step respectively, and the third one is implied in [AsWa] given by Richard Askey and Stephen Wainger; that is

$$\sum_{j=1}^{\infty}(j(j+2\lambda_\kappa))^{-\alpha/2}\frac{j+\lambda_\kappa}{\lambda_\kappa}C_j^{\lambda_\kappa}(t)\backsim(1-t)^{(\alpha-2\lambda_\kappa-1)/2}.$$

Next, considering the subdomain of $\mathbb{S}^{d-1}$

$$\Omega:=\{x\in\mathbb{S}^{d-1}:\frac{1}{2}\leqslant x_1\leqslant 1\}$$

we have that:

$$\begin{aligned}
\|\lim_{N\to\infty}I_\kappa^\alpha f_N\|_{q,\kappa}^q &\geqslant c_3\int_\Omega \rho(x,e_1)^{(\alpha-(2\sigma_\kappa+1))q}\prod_{j=1}^{d}|x_j|^{2\kappa_j}d\sigma(x)\\
&=c_3\int_0^{\frac{\pi}{3}}\int_{\mathbb{S}^{d-2}}\theta^{(\alpha-(2\sigma_\kappa+1))q}\sin^{d-2}\theta\prod_{j=2}^{d}|\xi_j|^{2\kappa_j}\sin^{2\kappa_j}\theta d\sigma(\xi)d\theta\\
&=c_3\int_0^{\frac{\pi}{3}}\theta^{(\alpha-(2\sigma_\kappa+1))q+d-2+2|\kappa|-2\kappa_1}d\theta.
\end{aligned}$$

Since $\alpha=(2\sigma_\kappa+1)(1-\frac{1}{q})$,

$$\|\lim_{N\to\infty}I_\kappa^\alpha f_N\|_{q,\kappa}^q\gtrsim\int_0^{\frac{\pi}{3}}\theta^{-1}d\theta=\infty.$$

Then, by Fatou lemma,

$$\liminf_{n\to\infty}\|I_\kappa^\alpha f_n\|_{\kappa,q}\geqslant\|\liminf_{n\to\infty}I_\kappa^\alpha f_n\|_{q,\kappa}=\infty.$$

On the other hand, however, using summation by parts repeatedly, we can obtain that:

$$f_N(x)=\sum_{j=1}^{\infty}[\Delta^{\ell+1}\eta(\frac{j}{N})]A_j^\ell K_j^\ell(h_\kappa^2;x,e_1),$$

which implies that

$$\|f_N\|_{\kappa,1}\lesssim N^{-\ell-1}\sum_{j=1}^{2N}j^\ell\|K_j^\ell(h_\kappa^2;\cdot,e_1)\|_{1,\kappa}\leqslant C,$$

where the last equality is followed from the fact (2.5.1).

To prove the weak-type(1,1) inequality, we shall import the estimate (3.1.4);

that is

$$|I_\kappa^\alpha f(x)| \lesssim \|f\|_{\kappa,p}^{(1-\frac{p}{q})} (\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\sigma_\varepsilon x))^{\frac{p}{q}},$$

since it is deduced without depending on the condition $p > 1$.

Then using the special case $p = 1$, for any $\lambda > 0$,

$$\begin{aligned}
&\mu_\kappa\{x \in \mathbb{S}^{d-1} : |I_\kappa^\alpha f(x)| > \lambda\}\\
\leqslant&\mu_\kappa\{x \in \mathbb{S}^{d-1} : \|f\|_{1,\kappa}^{1-\frac{1}{q}} (\sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x))^{\frac{1}{q}} > \lambda\}\\
=&\mu_\kappa\{x \in \mathbb{S}^{d-1} : \sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa f(\varepsilon x) > \frac{\lambda^q}{\|f\|_{1,\kappa}^{q-1}}\}\\
\leqslant&\sum_{\varepsilon\in\mathbb{Z}_2^d} \mu_\kappa\{M_\kappa f(\varepsilon x) > \frac{\lambda^q}{2^d\|f\|_{1,\kappa}^{q-1}}\}\\
\leqslant&2^{2d}\frac{\|f\|_{1,\kappa}^q}{\lambda^q}
\end{aligned}$$

where the last step is guaranteed by invariance of $d\mu_\kappa$. □

3.2 Necessity of conditions in HLS Theory

After showing the sufficient conditions for the boundedness of I_κ^α from $L^p(h_\kappa^2)$ to $L^p(h_\kappa^2)$, we shall see below that they are also necessary in some degree.

Theorem 3.2.1. If the inequality

$$\|I_\kappa^\alpha f\|_{q,\kappa} \leqslant C\|f\|_{\kappa,p}, \quad \forall f \in L^p(h_\kappa^2; \mathbb{S}^{d-1}),$$

holds for some α and $1 < p < q \leqslant \infty$,Then one must have that

$$\alpha \geqslant (2\sigma_\kappa + 1)(\frac{1}{p} - \frac{1}{q}).$$

Proof. Choose cut-off function $\phi \in C^\infty[0, \infty)$ such that $\chi_{[1,2]} \leqslant \phi \leqslant \chi_{[1,4]}$. For an integer N and $\kappa = (\kappa_1.\kappa_2, \cdots, \kappa_d) \in [0, \infty)^d$, without loss the general we suppose that $\kappa_1 = \kappa_{\min}$ and define:

$$f_N(x) := \sum_{j=0}^{\infty} \phi(\frac{j}{N}) Z_j^\kappa(x, e_1),\ \forall x = (x_1, x_2, \cdots, x_d) \in \mathbb{S}^{d-1},$$

Noting the fact(one can refer to [Sz]) that

$$Z_j^\kappa(x,e_1) = \frac{j+\lambda_\kappa}{\lambda_\kappa} c_{\kappa_1} \int_{-1}^{1} C_j^{\lambda_\kappa}(x_1 t_1)(1+t_1)(1-t_1)^{\kappa_1 - 1} dt_1 \\ = \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1), \tag{3.2.1}$$

and

$$\|C_j^{(\lambda,\mu)}\|_\infty = C_j^{(\lambda,\mu)}(1) \sim j^{2\lambda-1},$$

we have that

$$\|f_N\|_{\kappa,\infty} = \|\sum_{j=0}^{\infty} \phi(\frac{j}{N}) \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{(\lambda_\kappa-\kappa_1,\kappa_1)}\|_\infty \leqslant \sum_{j=N}^{4N} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{(\lambda_\kappa-\kappa_1,\kappa_1)}(1) \\ = c_1 \sum_{j=N}^{4N} \frac{j+\lambda_\kappa}{\lambda_\kappa} j^{2\lambda_\kappa - 2\kappa_1 - 1} = c_2 N^{2\sigma_\kappa + 1}.$$

Let Δ denote the difference operator defined in (6.0.1), and K_j^ℓ be the Cesaro (C,ℓ) means of the sequence $\frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^{(\lambda_\kappa - \kappa_1, \kappa_1)}(t)$, namely

$$K_j^\ell(t) = \frac{1}{A_j^\ell} \sum_{k=0}^{j} A_{j-k}^\ell \frac{\lambda_\kappa + k}{\lambda_\kappa} C_k^{(\lambda_\kappa - \kappa_1, \kappa_1)}(t),\ t \in [-1,1],$$

where $A_n^m = \left(\begin{array}{c} m+n \\ n \end{array} \right) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$.
Using summation by parts $\ell + 1$ times, we can obtain that:

$$f_N(x) = \sum_{j=1}^{\infty} [\Delta^{\ell+1} \phi(\frac{j}{N})] A_j^\ell K_j^\ell(h_\kappa^2; x, e_1),$$

Then by theorem 2.5.2 with $\ell > \sigma_\kappa$,

$$\|f_N\|_{1,\kappa} \lesssim N^{-\ell-1} \sum_{j=N}^{2N} j^\ell \|K_j^\ell(h_\kappa^2; \cdot, e_1)\|_{1,\kappa} \leqslant C,$$

where C is independent of N.
Hence the $L^p(h_\kappa^2)$ estimate can be obtained as following

$$\|f_N\|_{\kappa,p} \leqslant \|f_N\|_{\kappa,1}^{\frac{1}{p}} \|f_N\|_{\kappa,\infty}^{1-\frac{1}{p}} \backsim N^{(2\sigma_\kappa+1)(1-\frac{1}{p})}. \tag{3.2.2}$$

Next we will find out the $L^q(h_\kappa^2)$ estimate of $I_\kappa^\alpha f_N$. To do it, we begin with the special and crucial case $q=2$. With the summation expression (2.3.2) of $Z_j^\kappa(\cdot,\cdot)$ and the orthonormal relationship between the basis $\{Y_m^{h,j}: m=1,2,\cdots,M\}$ of $\mathcal{H}_j^d(h_\kappa^2)$, we can obtain that

$$\begin{aligned}
\|I_\kappa^\alpha f_N\|_{2,\kappa}^2 &= C\sum_{j=1}^{\infty}\phi^2(\frac{j}{N})(j(j+2\lambda_\kappa))^{-\alpha}\sum_{m=1}^{M}[Y_m^{h,j}(e_1)]^2\\
&= C\sum_{j=1}^{\infty}\phi^2(\frac{j}{N})(j(j+2\lambda_\kappa))^{-\alpha}Z_j^\kappa(e_1,e_1)\\
&= C\sum_{j=1}^{\infty}\phi^2(\frac{j}{N})(j(j+2\lambda_\kappa))^{-\alpha}\frac{\lambda_\kappa+j}{\lambda_\kappa}C_j^{(\lambda_\kappa-\kappa_1,\kappa_1)}(1)\\
&= C\sum_{j=1}^{\infty}\phi^2(\frac{j}{N})j^{-2\alpha}j^{2\lambda_\kappa-2\kappa_1}
\end{aligned}$$

Thus,

$$\sum_{j=N}^{2N}j^{-2\alpha+2\lambda_\kappa-2\kappa_1}\lesssim\|I_\kappa^\alpha f_N\|_{2,\kappa}^2\lesssim\sum_{j=N}^{4N}j^{-2\alpha+2\lambda_\kappa-2\kappa_1},$$

which implies that

$$\|I_\kappa^\alpha f_N\|_{2,\kappa}\backsim N^{-\alpha+\frac{2\sigma_\kappa+1}{2}}.$$

Similarly with the method we just use in the estimate of $\|f_N\|_{\infty,\kappa}$ and $\|f_N\|_{1,\kappa}$, we have that

$$\|I_\kappa^\alpha f_N\|_{\infty,\kappa}\backsim N^{-\alpha+2\sigma_\kappa+1}\qquad\|I_\kappa^\alpha f_N\|_{1,\kappa}\lesssim N^{-\alpha}.$$

Then for $1<q<2$,

$$N^{-\alpha+\frac{2\sigma_\kappa+1}{2}}\backsim\|I^{\alpha_\kappa}f_N\|_{2,\kappa}\leqslant\|I_\kappa^\alpha f_N\|_{q,\kappa}^{\frac{q}{2}}\|I_\kappa^\alpha f_N\|_{\infty,\kappa}^{1-\frac{q}{2}}\lesssim\|I_\kappa^\alpha f_N\|_{q,\kappa}^{\frac{q}{2}}N^{(1-\frac{q'}{2})(-\alpha+2\sigma_\kappa+1)},$$

which implies that

$$\|I_\kappa^\alpha f_N\|_{q,\kappa}\gtrsim N^{-\alpha+(2\sigma_\kappa+)(1-\frac{1}{q})}. \tag{3.2.3}$$

On the other hand, for the case $q>2$,

$$N^{-\alpha+\frac{2\sigma_\kappa+1}{2}}\backsim\|I_\kappa^\alpha f_N\|_{2,\kappa}\leqslant\|I_\kappa^\alpha f_N\|_{q,\kappa}^{\frac{q'}{2}}\|I_\kappa^\alpha f_N\|_{\kappa,1}^{1-\frac{q'}{2}}\lesssim N^{-\alpha(1-\frac{q'}{2})}\|I_\kappa^\alpha f_N\|_{q,\kappa}^{\frac{q'}{2}},$$

which also implies that (3.2.3). Finally combining (3.2.3), (3.2.2) with the hypothesis (3.2.1), we can get for arbitrary large number N,

$$N^{-\alpha+(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})} \lesssim 1.$$

which implies that

$$\alpha \geqslant (2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q}).$$

$\square$

3.3 HLS on unit ball and simplex

In this chapter, we shall work out the similar conclusions for orthogonal expansions on the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leqslant 1\}$ with respect to the weight function

$$W_\kappa^B(x) := \prod_{j=1}^d |x_j|^{2\kappa_j}(1-\|x\|^2)^{\kappa_{d+1}-1/2},\ \kappa_j \geqslant 0, \tag{3.3.1}$$

and on the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_j \geqslant 0, j = 1, 2, \cdots, d, |x| \leqslant 1\}$ with respect to the weight function

$$W_\kappa^T(x) := \prod_{j=1}^d x_j^{\kappa_j-1/2}(1-|x|)^{k_{d+1}-1/2}. \tag{3.3.2}$$

That is, corresponding operator $I_\kappa^{\alpha,\mathbb{B}^d}$ and $I^{\alpha,\mathbb{T}^d}$ satisfy the inequality

$$\|Qf\|_{\kappa,q} \leqslant C\|f\|_{\kappa,p}, \quad \forall f \in L^p(h_\kappa^2),$$

for $1 < p < q < \infty$ and $\alpha \geqslant (2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})$ or $\alpha > (2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})$ and $p = 1$, where Q is $I_\kappa^{\alpha,\mathbb{B}^d}$ or $I^{\alpha,\mathbb{T}^d}$.

To prove it, it is enough to build a bridge to connect the unit ball $\mathbb{B}^d$ or the simplex $\mathbb{T}^d$ with the unit sphere $\mathbb{S}^d$ on which the inequality has been studied totally.

Let us consider the following three mappings which plays a role like bridge from $\mathbb{S}^{d-1}$ to $\mathbb{B}^d$:

$$\begin{aligned}
\phi_1 :& \mathbb{B}^d \to \mathbb{S}^d_{\perp} \\
& (x_1, x_2, \cdots, x_d) \mapsto (x_1, x_2, \cdots, x_d, \sqrt{1-\|x\|^2}) \\
\phi_2 :& \mathbb{B}^d \to \mathbb{S}^d_{\top} \\
& (x_1, x_2, \cdots, x_d) \mapsto (x_1, x_2, \cdots, x_d, -\sqrt{1-\|x\|^2}) \\
T :& \mathbb{S}^d \to \mathbb{B}^d \\
& (x_1, x_2, \cdots, x_d, x_{d+1}) \mapsto (x_1, x_2, \cdots, x_d)
\end{aligned}$$

It is to point that the Jacobi matrix of each ϕ_j, $j = 1, 2$, is a $(d+1) \times d$ matrix,

$$\begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1 \\
\frac{-x_1}{\sqrt{1-\|x\|^2}} & \frac{-x_2}{\sqrt{1-\|x\|^2}} & \cdots & \frac{-x_d}{\sqrt{1-\|x\|^2}}
\end{pmatrix}.$$

Then we claim that

Lemma 3.3.1. Given a function $f \in L^1(W_\kappa^B; \mathbb{B}^d)$, for a.e. $x \in \mathbb{S}^d$,

$$(I_\kappa^{\alpha,\mathbb{B}^d} f)(Tx) = I_\kappa^\alpha (f \circ T)(x).$$

Proof. From the definitions of these operators, it suffices to show the reproducing kernel $P_n(W_\kappa^B; x, y)$ satisfies that for any $x, y \in \mathbb{B}^d$,

$$P_n(W_\kappa^{\mathbb{B}}; x, y) = Z_n^\kappa((x, \pm x_{d+1}), (y, y_{d+1})) + Z_n^\kappa((x, \pm x_{d+1}), (y, -y_{d+1})) \quad (3.3.3)$$

which implies for $x \in \mathbb{S}^d$,

$$\mathrm{proj}_n^{\alpha,B} f(Tx) = \mathrm{proj}_n^\alpha (f \circ T)(x) \quad \forall n \in \mathbb{N}.$$

However, (3.3.3) can be guaranteed by the following check. Obviously, the left side part in (3.3.3) is a polynomial of degree n. Furthermore, for any polynomial p with degree n, which is orthogonal with any polynomial of lower degree,

$$
\begin{aligned}
&\int_{\mathbb{B}^d} p(y)\left[Z_n^\kappa((x,\pm x_{d+1}),(y,y_{d+1}))+Z_n^\kappa((x,\pm x_{d+1}),(y,-y_{d+1}))\right]W_\kappa^B(y)dy\\
=&\int_{\mathbb{S}^d_\perp} p(Ty)Z_n^\kappa((x,\pm x_{d+1}),y)h_\kappa^2(y)d\sigma(y)+\int_{\mathbb{S}^d_\top} p(Ty)Z_n^\kappa((x,\pm x_{d+1}),y)h_\kappa^2(y)d\sigma(y)\\
=&\int_{\mathbb{S}^d} p(Ty)Z_n^\kappa((x,\pm x_{d+1}),y)h_\kappa^2(y)d\sigma(y)\\
=&p(T(x,\pm x_{d+1}))=p(x)
\end{aligned}
$$

□

Applying the above lemma, the HLS inequality can be deduced immediately by

$$
\begin{aligned}
\int_{\mathbb{B}^d}|I_\kappa^{\alpha,B}f(x)|^qW_\kappa^B(x)dx=&\int_{\mathbb{S}^d}|I_\kappa^\alpha(f\circ T)x)|^qh_\kappa^2(x)d\sigma(x)\\
\leqslant& c\int_{\mathbb{S}^d}|fT(x)|^ph_\kappa^2(x)d\sigma(x)\\
=&2c\int_{\mathbb{B}^d}|f(x)|^pW_\kappa^B(x)dx,
\end{aligned}
$$

where c is the same constant in Theorem 3.1.1.

Similarly, between $\mathbb{S}^d$ and $\mathbb{T}^d$ we define the following mappings:

$$
\begin{aligned}
\phi_\varepsilon:&\mathbb{T}^d\to\mathbb{S}^d_\varepsilon\\
&(x_1,x_2,\cdots,x_d)\mapsto\sigma_\varepsilon(\sqrt{x_1},\sqrt{x_2},\cdots,\sqrt{x_d},\sqrt{1-|x|})\\
G:&\mathbb{S}^d\to\mathbb{T}^d\\
&(x_1,x_2,\cdots,x_d,x_{d+1})\mapsto(x_1^2,x_2^2,\cdots,x_d^2)
\end{aligned}
$$

Then the analogue of (3.3.3) is that: for any fixed $\varepsilon_0\in\mathbb{Z}_2^{d+1}$, the reproducing kernel $P_n(W_\kappa^T;x,y)$ satisfies

$$
P_n(W_\kappa^T;x,y)=\sum_{\varepsilon\in\mathbb{Z}_2^{d+1}}Z_{2n}^\kappa(\psi_{\varepsilon_0}(x),\psi_\varepsilon(y)).
$$

And the proof is also same as that of Lemma 3.3.1, expect for one slight change: At this moment, we divide the sphere into 2^{d+1} parts, where $\mathbb{S}^d_\varepsilon=\{\sigma_\varepsilon\bar{x}:x\in\mathbb{S}^d\}$

for each $\varepsilon \in \mathbb{Z}_2^{d+1}$. Thus combining the uniform transform

$$\int_{\mathbb{T}^d} |f(x)|^p W_\kappa^T(x)dx = \frac{1}{2^{d+1}} \int_{\mathbb{S}^d} |f(G(x))|^p h_\kappa^2(x)d\sigma(x), \quad f \in L^p(\mathbb{T}^d, W_\kappa^T), \tag{3.3.4}$$

deduces the following theorem directly.

Theorem 3.3.2. For any $x \in \mathbb{S}^d$ and $f \in L^1(W_\kappa^T; \mathbb{T}^d)$,

$$\operatorname{proj}_n^{\alpha,T} f(Gx) = \operatorname{proj}_{2n}^{\alpha}(f \circ G)(x)$$

and

$$\operatorname{proj}_{2n+1}^{\alpha}(f \circ G) = 0,$$

for any $n \in \mathbb{N}$. Moreover,

$$I_\kappa^{\alpha,T} f = I_\kappa^{\alpha}(f \circ G),$$

and

$$\|I_\kappa^{\alpha,T} f\|_{q,\kappa,T} \leqslant 2^{d+1} c \|f\|_{p,\kappa,T}, \quad \forall f \in L^p(W_\kappa^T; \mathbb{T}^d),$$

for the same parameters p, q, α in the case of $\mathbb{B}^d$ and constant c in Theorem 3.1.1.

Chapter 4

Decomposition of Generalized Laplace-Beltrami Operator

In the HLS theory, it is most concerned to people when $\alpha = 1$. In particular, at this moment, the inequality can be rewritten as

$$\|f\|_{\kappa,p} \leq C\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,q},$$

for certain proper p, q. Motivated by this discussion, in this chapter, we shall introduce two versions of decomposition for the operator, Laplace-Beltramic $(\Delta_{\kappa,0})$. These lead to a practical replacement of the operator, $(-\Delta_{\kappa,0})^{1/2}$, in the sense of the equivalence of the $L^p(h_\kappa^2)$ norm.

4.1 the 1st Version of Decomposition

At first, we shall recall definitions of every related operators. Given a positive vector $\kappa \in \mathbb{R}^d$, for $1 \leqslant j \leqslant d$,

$$\mathcal{D}_j := \partial_j + \kappa_j E_j \quad \text{and} \quad E_j f(x) := \frac{f(x) - f(\sigma_j x)}{x_j}.$$

For $1 \leqslant i \neq j \leqslant d$,

$$D_{i,j} := x_i \partial_j - x_j \partial_i, \quad E_{i,j} := x_i \kappa_j E_j - x_j \kappa_i E_i$$

and

$$\mathcal{D}_{i,j} := x_i \mathcal{D}_j - x_j \mathcal{D}_i = D_{i,j} + E_{i,j}.$$

We also need the following facts that can be found in [DaXu2],

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j \quad \text{and} \quad \mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i.$$

The next proposition gives a decomposition for $\Delta_{\kappa,0}$.

Proposition 4.1.1. For $\xi \in \mathbb{S}^{d-1}$ and $f \in C^2(\mathbb{S}^{d-1})$,

$$\Delta_{\kappa,0} f(\xi) = \sum_{1 \le i < j \le d} \mathcal{D}_{i,j}^2 f(\xi) + Tf(\xi), \tag{4.1.1}$$

where

$$\begin{aligned} Tf(x) := {} & (d-2) \sum_{i=1}^{d} \kappa_i (f(\xi) - f(\sigma_i \xi)) \\ & + 2 \sum_{1 \le i < j \le d} \kappa_i \kappa_j (f(\xi) - f(\sigma_i \sigma_j \xi)). \end{aligned} \tag{4.1.2}$$

Proof. After some straightforward calculation, we obtain that

$$\begin{aligned} E_{i,j}^2(f) = & -2\kappa_i \kappa_j (f - f\sigma_i \sigma_j) \\ D_{i,j} E_{i,j}(f)(x) = & x_i^2 \partial_j E_j + x_j^2 \partial_i E_i - [\kappa_i (f - f\sigma_i) + \kappa_j (f - f\sigma_j)] \\ & - [x_j \kappa_i (\partial_j f - \partial_j f(\sigma_i x)) + x_i \kappa_j (\partial_i f - \partial_i f(\sigma_j x))] \\ E_{i,j} D_{i,j}(f)(x) = & x_i^2 E_j \partial_j + x_j^2 E_i \partial_i \\ & - [x_i \kappa_j (\partial_i f(\sigma_j x) + \partial_i f) + x_j \kappa_i (\partial_j f(\sigma_i x) + \partial_j f)] \end{aligned}$$

Then it follows that

$$\begin{aligned} & \sum_{1 \leqslant i < j \leqslant d} (D_{i,j} E_{i,j} + E_{i,j} D_{i,j}) f(x) \\ = & 2 \sum_{1 \leqslant i < j \leqslant d} \left(\frac{\kappa_j x_i^2}{x_j} \partial_j + \frac{\kappa_i x_j^2}{x_i} \partial_i \right) f(x) - 2 \sum_{1 \leqslant i < j \leqslant d} (x_i \kappa_j \partial_i + x_j \kappa_i \partial_j) f(x) \\ & - \sum_{i=1}^{d} \frac{\kappa_i E_i f(x)}{x_i} - (d-2) \sum_{i=1}^{d} \kappa_i (f - f\sigma_i)(x) \\ = & 2 \sum_{1 \leqslant i < j \leqslant d} \left(\frac{x_j \kappa_i}{x_i} - \frac{x_i \kappa_j}{x_j} \right) D_{i,j} f(x) - \sum_{i=1}^{d} \frac{\kappa_i E_i f(x)}{x_i} - (d-2) \sum_{i=1}^{d} \kappa_i (f - f\sigma_i)(x). \end{aligned}$$

From the facts

$$\sum_{1 \leqslant i < j \leqslant d} \mathcal{D}_{i,j}^2 = \sum_{1 \leqslant i < j \leqslant d} (D_{i,j}^2 + E_{i,j}^2 + D_{i,j} E_{i,j} + E_{i,j} D_{i,j})$$

and

$$\Delta_{\kappa,0}(f)(x) = \sum_{1\leqslant i<j\leqslant d} h_\kappa^{-2} D_{i,j} h_\kappa^2 D_{i,j}(f)(x) - \sum_{j=1}^{d} \frac{\kappa_i E_i f(x)}{x_j}$$

$$= \sum_{1\leqslant i<j\leqslant d} D_{i,j}^2 f(x) + 2 \sum_{1\leqslant i<j\leqslant d} \left(\frac{x_j\kappa_i}{x_i} - \frac{x_i\kappa_j}{x_j}\right) D_{i,j} f(x) - \sum_{j=1}^{d} \frac{\kappa_i E_j f(x)}{x_j},$$

we shall get the decomposition (4.1.1). □

Before deriving the equivalent relationship, we still need the following two lemmas. The inner product we will mention below always means the weighted form, that is,

$$\langle f, g\rangle = \int_{\mathbb{S}^{d-1}} f(x)g(x)h_\kappa^2(x)d\sigma(x),$$

for certain pair of appropriate functions f, g and a fixed positive vector κ.

Lemma 4.1.2. Suppose $f, g \in L^2(\mathbb{S}^{d-1})$, then for any pair of i, j, with $1 \leq i < j \leq d$,

$$\langle \mathcal{D}_{i,j} f, g\rangle = -\langle f, \mathcal{D}_{i,j} g\rangle. \tag{4.1.3}$$

Further more, if $P \in \mathcal{H}_\kappa^d(\mathbb{S}^{d-1})$, so is $\mathcal{D}_{i,j}P$.

Proof. At first, we fix a positive vector $\kappa \in \mathbb{R}^d$ with $\kappa_j > 1$ for $j = 1, 2, \cdots, d$ so that $h_\kappa^2(x)$ is a continuously differentiable function. Then

$$D_{i,j} h_\kappa^2(x) = 2(\frac{\kappa_j x_i}{x_j} - \frac{\kappa_i x_j}{x_i})h_\kappa^2,$$

which follows that

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} (D_{i,j}f)(x)g(x)h_\kappa^2(x)d\sigma(x)\\
&= - \int_{\mathbb{S}^{d-1}} f(x)(D_{i,j}g)(x)h_\kappa^2(x)d\sigma(x) - \int_{\mathbb{S}^{d-1}} f(x)g(x)(D_{i,j}h_\kappa^2)(x)d\sigma(x)\\
&= - \langle f, D_{i,j} g\rangle - 2\int_{\mathbb{S}^{d-1}} f(x)g(x)\left(\frac{\kappa_j x_i}{x_j} - \frac{\kappa_i x_j}{x_i}\right) h_\kappa^2(x)d\sigma(x)
\end{aligned}$$

On the other hand, we consider the operator $E_{i,j}$.

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} (E_{i,j}f)(x)g(x)h_\kappa^2(x)d\sigma(x)\\
=&\kappa_j \int_{\mathbb{S}}^{d-1} x_i \frac{f(x)-f(\sigma_j x)}{x_j}\frac{g(x)+g(\sigma_j x)}{2}h_\kappa^2(x)d\sigma(x)\\
&-\kappa_i \int_{\mathbb{S}}^{d-1} x_j \frac{f(x)-f(\sigma_i x)}{x_i}\frac{g(x)+g(\sigma_i x)}{2}h_\kappa^2(x)d\sigma(x)\\
=&\kappa_j \int_{\mathbb{S}}^{d-1} x_i f(x)\frac{g(x)+g(\sigma_j x)}{x_j}h_\kappa^2(x)d\sigma(x) - \kappa_i \int_{\mathbb{S}}^{d-1} x_j f(x)\frac{g(x)+g(\sigma_i x)}{x_i}h_\kappa^2(x)d\sigma(x)\\
=&-\langle f, E_{i,j}g\rangle + 2\int_{\mathbb{S}^{d-1}} f(x)g(x)\left(\frac{\kappa_j x_i}{x_j} - \frac{\kappa_i x_j}{x_i}\right)h_\kappa^2(x)d\sigma(x).
\end{aligned}$$

Next, since $\mathcal{D}_{i,j} = D_{i,j} + E_{i,j}$ and analytic continuation , we obtain the desired identity (4.1.3).

Finally given any $P \in \mathcal{H}_\kappa^n(\mathbb{S}^{d-1})$ and $Q \in \prod_{n-1}^d(\mathbb{S}^{d-1})$, then $\mathcal{D}_{i,j}Q \in \prod_{n-1}^d(\mathbb{S}^{d-1})$ and

$$\langle \mathcal{D}_{i,j}P, Q\rangle = -\langle P, \mathcal{D}_{i,j}Q\rangle = 0.$$

So $\mathcal{D}_{i,j}P \in \mathcal{H}_\kappa^n(\mathbb{S}^{d-1})$ as well. □

Lemma 4.1.3. Let T be the operator as defined in Proposition 4.1.1. Then T is a positive bounded operator, i.e. for any function $f \in L^p(\mathbb{S}^{d-1})$,

$$\langle Tf, f\rangle \geq 0.$$

Further more,

$$\|T(f)\|_{\kappa,p} \leq c_1\|(-\Delta_{\kappa,0})^{1/2}f\|_{\kappa,p} + c_2 \max_{1\leq i<j\leq d}\|\mathcal{D}_{i,j}f\|_{\kappa,2},$$

where c_1 can be chosen as small as we need.

Proof. To prove the positivity of T, it suffices to show

$$\langle f - f\sigma_i, f\rangle \leq 0.$$

However, since $2f(x)f(\sigma_i x) \leq f^2(x) + f^2(\sigma_i x)$,

$$\langle f - f\sigma_i, f\rangle \geq \frac{1}{2}\int (f - f\sigma_i)^2 d\mu_\kappa \geq 0.$$

Further more, since the operator $(-\Delta_{\kappa,0})^{1/2}$ is self-adjoint and the identity (4.1.3),

$$\|(-\Delta_{\kappa,0})^{1/2}f\|_{\kappa,2} = \langle(-\Delta_{\kappa,0})f, f\rangle \tag{4.1.4}$$

$$= \sum_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,2} - \langle Tf, f\rangle \le c \max_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,2}. \tag{4.1.5}$$

Next,we fix a cut-off function $\eta \in C^\infty[0,\infty)$ such that $\chi_{[0,1]} \le \eta \le \chi_{[0,2]}$. Then we define

$$V_n^\kappa(f) = \sum_{j=0}^{\infty} \eta(\frac{j}{n}) \operatorname{proj}_j f.$$

About the operator V_n^κ, as a consequence of Theorem 7.1 in [**?**], we assert that there exists a constant c only depending on p, n and κ such that

$$\|f - V_n^\kappa f\|_{\kappa,p} \le cn^{-r}\|(-\Delta_{\kappa,0})^{r/2}f\|_{\kappa,p},$$

for $1 \le p \le \infty$ and $r > 0$; and $\|V_n^\kappa f\|_{\kappa,2} \sim \|V_n^\kappa f\|_{\kappa,p}$, since $V_n^\kappa(L^1(\mathbb{S}^{d-1}))$ is a finite dimensional polynomial space. Then for a temporarily fixed natural number n, we have that

$$\|V_n^\kappa f\|_{\kappa,2} \leqslant c\|(-\Delta_{\kappa,0})^{1/2}V_n^\kappa f\|_{\kappa,2} \leqslant c \max_{1\leqslant i<j\leqslant d} \|\mathcal{D}_{i,j}V_n^\kappa f\|_{\kappa,2}$$

$$=c \max_{1\leqslant i<j\leqslant d} \|V_n^\kappa \mathcal{D}_{i,j}f\|_{\kappa,2} \leqslant c' \max_{1\leqslant i<j\leqslant d} \|V_n^\kappa \mathcal{D}_{i,j}f\|_{\kappa,p} \leqslant c'' \max_{1\leqslant i<j\leqslant d} \|\mathcal{D}_{i,j}f\|_{\kappa,2}$$

Finally, by the boundedness of T, we obtain

$$\|Tf\|_{\kappa,p} \le c\|f - V_n^\kappa f\|_{\kappa,p} + \|T(V_n^\kappa f)\|_{\kappa,p}$$

$$\le cn^{-1}\|(-\Delta_{\kappa,0})^{1/2}f\|_{\kappa,p} + c' \max_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,2}$$

where c is independent of n.

□

Now we give out a relation between $(-\Delta_{\kappa,0})^{1/2}$ and $\mathcal{D}_{i,j}$'s.

Theorem 4.1.4. If $1 < p < \infty$ and $f \in C^1(\mathbb{S}^{d-1})$, then

$$\max_{1\leqslant i<j\leqslant d} \|\mathcal{D}_{i,j}f\|_{\kappa,p} \le C\|(-\Delta_{\kappa,0})^{1/2}f\|_{\kappa,p}.$$

Proof. At first, it can be seen that it suffices to show that

$$\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-1/2}f\|_{\kappa,p} \le C\|f\|_{\kappa,p},$$

for any $1 \le i \ne j \le d$ and p, f as we assumed above. Then we fix a pair of i, j, $1 \le i < j \le d$ temporarily and choose $\theta \in C^\infty[0,\infty)$ satisfying that

$$\operatorname{supp}\theta \subset (\frac{1}{2}, 2) \quad \text{and} \quad \sum_{j=0}^{\infty}\theta(\frac{x}{2^j}) = 1, \ \forall x > 0;$$

and $\Delta_{\theta,j}$ which is defined as (2.7.1),

$$f_n := \Delta_{\theta,j}f = \sum_{j=0}^{\infty}\theta(\frac{j}{2^n})\operatorname{proj}_j f.$$

Then by the Little-Paly Inequality 2.7.1, we can obtain that

$$\begin{aligned}\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}f\|_{\kappa,p} &\sim \|(\sum_{n=0}^{\infty}|\Delta_{\theta,n}\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}f|^2)^{\frac{1}{2}}\|_{\kappa,p} \\ &= \|(\sum_{n=0}^{\infty}|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}f_n|^2)^{\frac{1}{2}}\|_{\kappa,p},\end{aligned}$$

where the last identity is from the fact that $\mathcal{D}_{i,j}P \in \mathcal{H}_\kappa^d(\mathbb{S}^{d-1})$, for $P \in \mathcal{H}_\kappa^d(\mathbb{S}^{d-1})$. We rewrite the operator $\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}$ as

$$\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}f_n = \int_{\mathbb{S}^{d-1}} f_n(y)KK_n(x,y)h_\kappa^2(y)d\sigma(y),$$

then its associated kernel KK_n is that

$$KK_n(x,y) = \mathcal{D}_{i,j}V_\kappa[G_n(\langle\cdot,y\rangle)](x) = (x_iy_j - x_jy_i)V_\kappa[G_n'(\langle\cdot,y\rangle)](x),$$

with

$$G_n(t) = \sum_{j=0}^{\infty}\theta(\frac{j}{2^n})(j(j+2\lambda_\kappa))^{-\frac{1}{2}}\frac{j+\lambda_\kappa}{\lambda_\kappa}C_j^{\lambda_\kappa}(t).$$

According to Lemma 3.0.5

$$
\begin{aligned}
|KK_n(x,y)| &\lesssim |x_iy_j - x_jy_i| \frac{2^n(1+2^n\rho(\bar{x},\bar{y}))^{-\ell}}{U(\bar{x},\bar{y})} \\
&\lesssim \frac{(1+2^n\rho(\bar{x},\bar{y}))^{-\ell+1}}{U(\bar{x},\bar{y})},
\end{aligned}
$$

since $|x_iy_j - x_jy_i| \leqslant \rho(\bar{x},\bar{y})$. Then we can claim that,

$$
|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-1/2} f_n(x)| \le c \sum_{\varepsilon \in Z_2^d} M_\kappa f_n(\varepsilon x),
$$

where M_κ is the Hardy-Littlewood maximum function about the measure μ_κ, since ℓ can be chosen large enough. Therefore, using Fefferman-Stein theorem 2.7.2 and Littlewood-Paly Inequality 2.7.1 again. One can have that

$$
\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-1/2} f\|_{\kappa,p} \leqslant C_d \|(\sum_{n=0}^{\infty} |M_\kappa f_n|^2)^{1/2}\|_{\kappa,,p} \leqslant C_d \|(\sum_{n=0}^{\infty} |f_n|^2)^{1/2}\|_{\kappa,p} \leqslant C_d' \|f\|_{\kappa,p}.
$$

□

Now, we shall derive the converse inequalities by using the decomposition (Proposition 4.1.1) and the basic duality property. Together with the above inequality, this we formulate as a corollary.

Corollary 4.1.5. If $1 < p < \infty$ and $f \in C^1(\mathbb{S}^{d-1})$, then

$$
\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p} \sim \max_{1 \leqslant i < j \leqslant d} \|\mathcal{D}_{i,j} f\|_{\kappa,p}.
$$

Proof. It suffices to show one way of the above inequality, that is,

$$
\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} \le C \max_{1 \le i < j \le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p}. \tag{4.1.6}
$$

Given a function f as assumed and $g \in L^{p'}(h_\kappa^2; \mathbb{S}^{d-1})$, $\frac{1}{p} + \frac{1}{p'} = 1$, such that $\int_{\mathbb{S}^{d-1}} g(y) h_\kappa^2(y)\, d\sigma(y) = 0$, otherwise we may consider the function $\widetilde{g}(x) = g(x) - \int_{\mathbb{S}^{d-1}} g(y) h_\kappa^2(y)\, d\sigma(y)$, we have that

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} (-\Delta_{\kappa,0})^{\frac{1}{2}} f(y) g(y) h_\kappa^2(y)\, d\sigma(y) \\
=&\int_{\mathbb{S}^{d-1}} (-\Delta_{\kappa,0}) f(y) (-\Delta_{\kappa,0})^{-\frac{1}{2}} g(y) h_\kappa^2(y)\, d\sigma(y) \\
=&\sum_{1\le i<j\le d} \int_{\mathbb{S}^{d-1}} \mathcal{D}_{i,j} f(y) \Big[\mathcal{D}_{i,j} (-\Delta_{\kappa,0})^{-\frac{1}{2}} g(y)\Big] h_\kappa^2(y)\, d\sigma(y) \\
&\quad - \int_{\mathbb{S}^{d-1}} T f(y) (-\Delta_{\kappa,0})^{-\frac{1}{2}} g(y) h_\kappa^2(y)\, d\sigma(y) \\
\le& \sum_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p} \|\mathcal{D}_{i,j} (-\Delta_{\kappa,0})^{-\frac{1}{2}} g\|_{\kappa,p'} + C\|Tf\|_{\kappa,p} \|(-\Delta_{\kappa,0})^{-\frac{1}{2}} g\|_{\kappa,p'} \\
\le& c_1 \max_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p} + c_2 \|Tf\|_{\kappa,p} \le c_1' \max_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p} + \frac{1}{2} \|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p},
\end{aligned}
$$

where the last inequality is from Lamma 4.1.3.

Now taking the supremum for the right hand side of above inequality as f ranges over all functions in $L^2(\mathbb{S}^{d-1}) \cap L^{p'}(\mathbb{S}^{d-1})$ with $\|g\|_{\kappa,p'} \le 1$, we obtain therefore the desired result (4.1.6). □

4.2 the 2nd Version of Decomposition

In this section, we shall introduce another decomposition of the h-harmonic Laplace-Beltrami operator $\Delta_{\kappa,0}$, which lead to a far reaching and practical replacement of the operator $(-\Delta_{\kappa,0})^{1/2}$ in the $L^p(\mathbb{S}^{d-1})$ norm sense, for $p > 1$.

Before going any further, we shall make a few comments that will help to clarify the meaning of the operator $D_{i,j}$ and E_j. First, for any fixed $1 \le i < j \le d$, the definition of $D_{i,j}$ is independent of any extension of a spherical function. To understand it, we adapt the polar coordinates on the (x_i, x_j) plane, that is, $(x_i, x_j) = (r_{i,j} \cos\theta_i, r_{i,j} \sin\theta_j)$. Then it follows that

$$
\frac{\partial}{\partial x_i} = \cos\theta_{i,j} \frac{\partial}{\partial r_{i,j}} - \frac{\sin\theta_{i,j}}{r} \frac{\partial}{\partial \theta_{i,j}}
$$

and

$$
\frac{\partial}{\partial x_j} = \sin\theta_{i,j} \frac{\partial}{\partial r_{i,j}} + \frac{\cos\theta_{i,j}}{r} \frac{\partial}{\partial \theta_{i,j}},
$$

which implies that

$$D_{i,j} = x_i\partial_j - x_j\partial_i = \frac{\partial}{\partial\theta_{i,j}}.$$

This is another way of saying that, $D_{i,j}$ is just the angular derivative with respect to the (x_i, x_j) plane.

Next, from the above discussion it is not hard to assert that

Proposition 4.2.1. For any given $f, g \in C^1(\mathbb{S}^{d-1})$ and $1 \le i < j \le d$,

$$\int_{\mathbb{S}^{d-1}} (D_{i,j}f)(x)g(x)d\sigma(x) = -\int_{\mathbb{S}^{d-1}} f(x)(D_{i,j}g)(x)d\sigma(x).$$

The following proposition shows the decomposition of the operator $\Delta_{\kappa,0}$ in terms of $D_{i,j}$'s and E_j's in the sense of inner product.

Proposition 4.2.2. For $f, g \in C^2(\mathbb{S}^{d-1})$,

$$\langle(-\Delta_{\kappa,0})f, g\rangle = \sum_{1\le i<j\le d} \langle D_{i,j}f, D_{i,j}g\rangle + \sum_{i=1}^{d} \frac{\kappa_i}{2}\langle E_i f, E_i g\rangle.$$

In particular, this implies that

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,2}^2 = \sum_{1\le i<j\le d} \|D_{i,j}f\|_{\kappa,2}^2 + \sum_{i=1}^{d} \frac{\kappa_i}{2}\|E_i f\|_{\kappa,2}^2 \tag{4.2.1}$$

$$= \|\nabla_0 f\|_{\kappa,2}^2 + \sum_{i=1}^{d} \frac{\kappa_i}{2}\|E_i f\|_{\kappa,2}^2. \tag{4.2.2}$$

Proof. First, note that

$$g(x) = \frac{g(x) + g(\sigma_j x)}{2} + \frac{g(x) - g(\sigma_j x)}{2},$$

with the first term on the right of this equation being even in x_j and the second term being odd in x_j. Hence, for any $\varepsilon > 0$,

$$\begin{aligned} &2\int_{\{x\in\mathbb{S}^{d-1}:|x_j|\ge\varepsilon\}} \frac{f(x) - f(\sigma_j x)}{x_j^2} g(x)h_\kappa^2(x)\,d\sigma(x) \\ &= \int_{\{x\in\mathbb{S}^{d-1}:|x_j|\ge\varepsilon\}} \frac{f(x) - f(\sigma_j x)}{x_j}\frac{g(x) - g(\sigma_j x)}{x_j}h_\kappa^2(x)\,d\sigma(x). \end{aligned}$$

The desired formula then follows by letting $\varepsilon \to 0$. $\square$

Next, with the aid of Calderon-Zygmund singular integral and the estimate, Lemma 3.0.5) , we will establish the estimation of $E_j(-\Delta_{\kappa,0})^{1/2}$, which leads to the relation between $\Delta_{\kappa,0}$ and $D_{i,j}$ instantly.

Theorem 4.2.3. Given any $1 < p \leqslant \infty$ and $j = 1, 2, \cdots, d$

$$\|E_j(-\Delta_{\kappa,0})^{-\frac{1}{2}} f\|_p \lesssim \|f\|_p, \tag{4.2.3}$$

for any $f \in L_p(\mathbb{S}^{d-1})$.

Proof. For the simplest case $p = 2$, it can be seen from (4.2.1) directly. However, the $L^p(\mathbb{S}^{d-1})$ inequalities, when $p \neq 2$, will be obtained as a corollary of the theory of singular integrals as given in §2.4.

Now, let $j = 1, \cdots, d$ and $0 < r < \arccos \frac{1}{\sqrt{d}}$, and keep it temporarily fixed. Define the associated kernel Kel_j of the operator $E_j(-\Delta_{\kappa,0})^{-1/2}$ by

$$\left[E_j(-\Delta_{\kappa,0})^{-\frac{1}{2}}\right](f)(x) = \int_{\mathbb{S}^{d-1}} f(y)\,\mathrm{Kel}_j(x,y) h_\kappa^2(y) d\sigma(y),$$

for any $f \in L^2(\mathbb{S}^{d-1}) \cap \prod_n^d(\mathbb{S}^{d-1})$ with all integer n. Then it suffices to show that for any $y, z \in \mathbb{S}^{d-1}$ with $\rho(y,z) < r$,

$$\int_{\rho(\bar{x},\bar{z})>2r} |\,\mathrm{Kel}_j(x,y) - \mathrm{Kel}_j(x,z)| h_\kappa^2(y) d\sigma(x) \leqslant C,$$

Without lose of generality, we will just consider the case $j = 1$. As we presented in Chapter 3,

$$(-\Delta_{\kappa,0})^{-\frac{1}{2}} f(x) = \sum_{s=1}^{\infty} \int_{\mathbb{S}^{d-1}} f(y) K_s(x,y) d\mu(y),$$

where

$$K_s(x,y) = \sum_{j=1}^{\infty} \psi_s(\frac{j}{2^s})(j(j+2\lambda_\kappa))^{-\frac{1}{2}} \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa[C_j^{\lambda_\kappa}(\langle\cdot, y\rangle)](x)$$

with $\psi \in C^\infty$, $\operatorname{supp}\psi \subset [\frac{1}{2}, 1]$ and $\|\psi^{(\ell)}\|_\infty \leqslant C2^{-s}$. Then using integral by parts,

$$KK_s(x,y) := E_1^{(x)} K_s(x,y) = \sum_{j=1}^{\infty} \psi_s(\frac{j}{2^s}) \frac{j+\lambda_\kappa}{\lambda_\kappa} \tilde{V}_\kappa^{(1)}[C_j^{\lambda_\kappa}(\langle\cdot, y\rangle)](x),$$

where

$$\tilde{V}_\kappa^{(1)}(x,y) = \int_{[-1,1]^d} \frac{1}{x_1}\left[C_j^{\lambda_\kappa}(\sum_{k=1}^d x_k y_k t_k) - C_j^{\lambda_\kappa}(\sum_{k=2}^d x_k y_k t_k - x_1 y_1 t_1)\right]$$
$$\times \prod_{n=1}^d (1-t_n^2)^{\kappa_n -1}(1+t_n)dt$$
$$= \int_{[-1,1]^d} \frac{y_1}{\kappa_1}(C_j^{\lambda_\kappa})'(\sum_{k=1}^d x_k y_k t_k)(1-t_1)\prod_{n=1}^d (1-t_n^2)^{\kappa_n-1}(1+t_n)dt.$$

It follows that

$$E_j(-\Delta_{\kappa,0})^{-\frac{1}{2}} f = \sum_{s=1}^\infty \int_{\mathbb{S}^{d-1}} f(y) KK_s(x,y) d\mu(y).$$

Then given a fixed point $z \in \mathbb{S}^{d-1}$, without loss of generality, we assume that $|z_d| = \max\{|z_j| : j = 1, 2, \cdots, d\}$ and $z_d > 0$. Now we define

$$M(x,y') := y_1 (C_j^{\lambda_\kappa})'(\sum_{k=1}^d x_k y_k t_k), \quad \text{for any} \quad y' = (y_1, \cdots, y_{d-1}) \in \mathbb{B}^{d-1},$$

where $y_d = \sqrt{1 - y_1^2 - \cdots - y_{d-1}^2}$. Consider the function

$$\phi(c) := M(x, (1-c)z' + cy').$$

By the intermidiate theorem, for any $y, z \in \mathbb{S}^{d-1}$ and $\rho(y,z) < \arccos(\frac{1}{\sqrt{d}})$there exists $c \in (0,1)$ such that

$$\begin{aligned} &M(x,y') - M(x,z') = \phi(1) - \phi(0) = \phi'(c) \\ =&(y_1 - z_1)(C_j^{\lambda_\kappa})'(\sum_{k=1}^d x_k \xi_k t_k) + \xi_1 (C_j^{\lambda_\kappa})''(\sum_{k=1}^d x_k \xi_k t_k)\sum_{n=1}^d T_n(y,z) x_n t_n \end{aligned}$$

where $\xi' = (1-c)z' + cy'$, $\xi_d > 0$ such that $\xi = (\xi', \xi_d) \in \mathbb{S}^{d-1}$,

$$\begin{aligned} T_n(y,z) &= y_n - z_n, \quad \text{for,} \quad n = 1, 2, \cdots, d-1; \\ T_d(y,z) &= \frac{(y_d - z_d)((1-c)z_d + cy_d) + (1-2c)(1-\cos\rho(z,y))}{\sqrt{((1-c)z_d + cy_d)^2 + 2c(1-c)(1-\cos\rho(z,y))}} \end{aligned}$$

It is to point out that the fact

$$\rho(\bar{x},\bar{\xi})\sim\rho(\bar{x},\bar{y})\sim\rho(\bar{x},\bar{z})\quad\text{for}\quad\rho(\bar{x},\bar{z})>2\rho(\bar{y},\bar{z})$$

and that $|T_d(y,z)|\leqslant\sqrt{d}|y_d-z_d|$. Then when $|x_1|>4\rho(\bar{x},\bar{\xi})$ implying that $|x_1|\sim|\xi_1|$, without loss of generality, we suppose that x_i, y_i, $i=1,\cdots,d$, have the same sign, otherwise we consider $\tilde{x}$ such that $\tilde{x}_i=\operatorname{sign}(y_i)x_i$, for $i=1,\cdots,d$. Then from the Lemma 3.0.5, we obtain that

$$\begin{aligned}
I_2(s):=&\sum_{j=2}^{\infty}\psi(\frac{j}{2^s})\sum_{i=1}^{d}x_i\xi_1T_i(y,z)\int_{[-1,1]^d}(C_j^{\lambda_\kappa})''(\sum_{k=1}^{d}x_k\xi_kt_k)t_i(1-t_1)\\
&\times\prod_{n=1}^{d}(1-t_n^2)^{\kappa_n-1}(1+t_n)dt\\
\leqslant&c\sum_{n\sim2^s}|\sum_{i=1}^{d}x_iT_i(y,z)||\xi_1|\frac{n^d(1+n\rho(\bar{x},\bar{\xi}))^{-\ell_0}}{\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{\xi})+n^{-1})^{2\kappa_j}}\frac{|x_1\xi_1|^{-1/2}}{(|x_1|+\rho(\bar{x},\bar{\xi})+2^{-s})}\\
\leqslant&c'\frac{\rho(y,z)2^{sd}(1+2^s\rho(\bar{x},\bar{z}))^{-\ell}}{\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+2^{-s})^{2\kappa_j}}\\
\leqslant&c''\frac{\rho(y,z)2^s(1+2^s\rho(\bar{x},\bar{z}))^{-\ell}}{U(\bar{x},\bar{z})+U_{2^{-s}}(\bar{z})+U_{2^{-s}}(\bar{x})},
\end{aligned}$$

where the last second inequality is based on the fact

$$|\sum_{i=1}^{d}x_iT_i(y,z)|\leqslant c\rho(y,z)\rho(\bar{x},\bar{y}).$$

On the other hand, if $|x_1|\leqslant4\rho(\bar{x},\bar{\xi})$ which implies that $|\xi_1|\leqslant6\rho(\bar{x},\bar{\xi})$, then

$$
\begin{aligned}
I_2(s) :=& \sum_{j=2}^{\infty} \psi(\frac{j}{2^s}) \sum_{i=1}^{d} x_i \xi_1 T_i(y,z) \int_{[-1,1]^d} (C_j^{\lambda_\kappa})''(\sum_{k=1}^{d} x_k \xi_k t_k) t_i (1-t_1) \\
& \times \prod_{n=1}^{d} (1-t_n^2)^{\kappa_n - 1}(1+t_n) dt \\
\leqslant & c \sum_{n \sim 2^s} |\sum_{i=1}^{d} x_i T_i(y,z)||\xi_1| \frac{n^d (1+n\rho(\bar{x},\bar{y}))^{-\ell_0}}{\prod_{j=1}^{d}(|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{2\delta_j}} \\
\leqslant & C \frac{\rho(y,z) 2^{sd} (1+2^s \rho(\bar{x},\bar{z}))^{-\ell}}{\prod_{j=1}^{d}(|x_j| + \rho(\bar{x},\bar{y}) + 2^{-s})^{2\kappa_j}} \frac{\rho(\bar{x},\bar{y})^2}{(|x_1| + \rho(\bar{x},\bar{y}) + 2^{-s})^2} \\
\leqslant & C \frac{\rho(y,z) 2^s (1+2^s \rho(\bar{x},\bar{z}))^{-\ell}}{U(\bar{x},\bar{z}) + U_{2^{-s}}(\bar{z}) + U_{2^{-s}}(\bar{x})}.
\end{aligned}
$$

Similarly, by letting

$$
I_1(s) := \sum_{j=1}^{\infty} \psi(\frac{j}{2^s}) \frac{j+\lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^d} (y_1 - z_1)(C_j^{\lambda_\kappa})'(\sum_{k=1}^{d} x_k \xi_k t_k)(1-t_1) \prod_{n=1}^{d} (1-t_n^2)^{\kappa_n - 1}(1+t_n) dt
$$

and the Theorem 3.0.5 given in 3.1, we have

$$
|I_1(s)| \leq C \frac{\rho(y,z) 2^s (1+2^s \rho(\bar{x},\bar{\xi}))^{-\ell}}{U(\bar{x},\bar{\xi}) + U_{2^{-s}}(\bar{\xi}) + U_{2^{-s}}(\bar{x})},
$$

Finally, for any $y, z \in \mathbb{S}^{d-1}$ the desired inequality can be guaranteed directly since

$$
\begin{aligned}
& \int_{\rho(\bar{x},\bar{z}) \geqslant \delta r} |\operatorname{Kel}_1(x,y) - \operatorname{Kel}_1(x,z)| d\mu(x) \leqslant C \int_{\rho(\bar{x},\bar{z}) \geqslant \delta r} \sum_{s=1}^{\infty} [I_1(s) + I_2(s)] d\mu(x) \\
\leqslant & C \int_{\rho(\bar{x},\bar{z}) \geqslant \delta r} \frac{\rho(y,z)}{\rho(\bar{x},\bar{z}) U(\bar{x},\bar{z})} d\mu(x) \leqslant C'
\end{aligned}
$$

By the (4.2.1), we can see that (4.2.3) is true for the case $p = 2$. Then according to the Caldron-Zygmund theorem, (4.2.3) is true for $1 < p \leqslant 2$. To show the part of $p > 2$, we just need to consider the adjoint operator of $E_j(\Delta_{\kappa,0})^{-\frac{1}{2}}$, whose kernel is

$$
KK^*(x,y) = \sum_{s=1}^{\infty} E_1^{(y)} K_s(x,y).
$$

Using the same method, it can be proved as well. Hence, by now we finish the prove of theorem 4.2.3.

For $2 < p < \infty$, we will exploit the duality between L^p and L^q, $1/p + 1/q = 1$, and the fact that the theorem is proved for L^q. Observe the following : if a funtion ψ is locally integrable and if $sup|\int \psi\varphi dx| = A < \infty$, where the sup is taken over all continuous φ with compact support which verify $\|\varphi\| \leqslant 1$, then $\psi \in L^q$ and $\|\psi\| = A$. This being so, take $f \in L^1 \bigcap L^p, (2 < p < \infty)$, and φ of the type described above. $\square$

The following corollary is a direct result of Theorem4.1.5 and Theorem4.2.3 due to the relationship $\mathcal{D}_{i,j} = D_{i,j} + E_{i,j}$ and triangle inequality.

Corollary 4.2.4. For any $1 < p < \infty$,

$$\|(\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p} \sim \max_{1\leqslant i<j\leqslant d} \|D_{i,j} f\|_{\kappa,p} + \max_{i=1,\cdots,d} \|E_i f\|_{\kappa,p}.$$

Consequently, the HLS inequalities can be expressed in another way.

Corollary 4.2.5. Given a positive vector κ and a pair of p, q such that $1 < p < q < \infty$ and

$$(2\sigma_\kappa + 1)(\frac{1}{p} - \frac{1}{q}) \leq 1,$$

then for $f \in L^p(\mathbb{S}^{d-1}) \cap L^q(\mathbb{S}^{d-1})$,

$$\|f\|_{\kappa,q} \lesssim \|\nabla_0 f\|_{\kappa,p} + \max_i \|E_i f\|_{\kappa,p},$$

where the operators ∇_0, E_i are defined as above.

Chapter 5

Future Work

By now, our results are just associated with the weight function

$$h_\kappa^2(x) = \prod_{j=1}^{d} |x_j|^{2\kappa_j}.$$

Next, we expect to extend them on more general weight

$$w_\kappa^2(x) = \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2\kappa_\alpha},$$

in terms of a root system R.

Chapter 6

Appendix

Lemma 6.0.6. Let ℓ be a positive integer and $\eta \in C^{3\ell-1}[0,\infty)$ with $\mathrm{supp}\eta \subset [0,2]$ and $\eta^{(j)}(0) = 0$ for $j = 1, 2, \cdots, 3\ell - 2$. We consider function

$$G_n^{(\alpha,\beta)}(t) = \sum_{k=0}^{\infty} \eta(\frac{k}{n}) \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(t).$$

Then if $\alpha \le \beta \le -1/2$ and $\theta \in [0, \frac{\pi}{2}]$, for a positive integer j,

$$|G_n^{(j)}(\cos\theta)| \le c_{\ell,j} \|\eta^{(3\ell-1)}\|_\infty n^{2\alpha+2j+2}(1+n\theta)^{-\ell}.$$

Proof. From the fact

$$\frac{d}{dt} P_n^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)},$$

for $j \le n$, we have

$$\frac{d^j}{dt^j} P_n^{(\alpha,\beta)}(t) = \frac{(n+\alpha+\beta+1)\cdots(n+\alpha+\beta+j)}{2^j} P_{n-j}^{(\alpha+j,\beta+j)}(t).$$

This implies that

$$\begin{aligned} G_n^{(j)}(t) &= \sum_{k=j}^{\infty} \eta(\frac{k}{n}) \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+j+1)}{\Gamma(k+\beta+1)} P_{k-j}^{(\alpha+j,\beta+j)}(t) \\ &= \sum_{k=0}^{\infty} \eta(\frac{k+j}{n}) \frac{(2k+\alpha+\beta+2j+1)\Gamma(k+\alpha+\beta+2j+1)}{\Gamma(k+\beta+j+1)} P_k^{(\alpha+j,\beta+j)}(t) \end{aligned}$$

Then using summation by parts over and over again, we have that

$$
\begin{aligned}
G_n^{(j)}(t) &= \sum_{k=0}^{\infty} a_{n,1}(k) \sum_{s=0}^{k} \frac{(2s+\alpha+\beta+2j+1)\Gamma(s+\alpha+\beta+2j+1)}{\Gamma(s+\beta+j+1)} P_s^{(\alpha+j,\beta+j)}(t) \\
&= \sum_{k=0}^{\infty} a_{n,1}(k) \frac{\Gamma(s+\alpha+\beta+2j+2)}{\Gamma(s+\beta+j+1)} P_k^{(\alpha+j+1,\beta+j)}(t) \\
&= \sum_{k=0}^{\infty} a_{n,2}(k) \sum_{s=0}^{k} \frac{(2s+\alpha+\beta+2j+2)\Gamma(s+\alpha+\beta+2j+2)}{\Gamma(s+\beta+j+1)} P_s^{(\alpha+j+1,\beta+j)}(t) \\
&= \sum_{k=0}^{\infty} a_{n,2}(k) \frac{\Gamma(k+\alpha+\beta+2j+3)}{\Gamma(k+\beta+j+1)} P_k^{(\alpha+j+2,\beta+j)}(t) \\
&= \cdots \\
&= \sum_{k=0}^{\infty} a_{n,\ell}(k) \frac{\Gamma(k+\alpha+\beta+2j+\ell+1)}{\Gamma(k+\beta+j+1)} P_k^{(\alpha+j+\ell,\beta+j)}(t)
\end{aligned}
$$

where $a_{n,0}(s) = (2s+\alpha+\beta+2j+1)\eta(\frac{j+s}{n})$ and

$$
a_{n,\ell+1}(s) = \frac{a_{n,\ell}(s)}{2s+\alpha+\beta+2j+\ell+1} - \frac{a_{n,\ell}(s+1)}{2s+\alpha+\beta+2j+\ell+3};
$$

and the 2nd, 4th identities are from the relation

$$
\frac{\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\beta+1)} P_k^{(\alpha+1,\beta)}(t) = \sum_{s=1}^{k} \frac{(2s+\alpha+\beta+1)\Gamma(s+\alpha+\beta+1)}{\Gamma(s+\beta+1)} P_s^{(\alpha,\beta)}(t).
$$

We claim that for the fixed ℓ and $m+p \le q$,

$$
|a_{n,q}^{(m)}(s)| \le c_{\ell,j}(1+s)^{-m-2p+1}\Big(\frac{1+s}{n}\Big)^{2\ell-1}\|\eta^{(\ell+m)}\|_\infty. \qquad (6.0.1)
$$

Assuming it is true, by letting $m=0$ and $p=q=\ell$, we get that

$$
|a_{n,\ell}(s)| \le c_{\ell,j}\|\eta^{(\ell+m)}\|_\infty n^{-2\ell+1}.
$$

Adding the fact

$$
|P_n^{\alpha,\beta}(\cos\theta)| \le cn^{-1/2}(n^{-1}+\theta)^{-\alpha-1/2}(n^{-1}+\pi-\theta)^{-\beta-1/2},
$$

and the assumption $\theta \in [0, \pi/2]$, we obtain that if $2n > \theta^{-1}$,

$$\begin{aligned}
|G_n^{(j)}(\cos\theta)| &\leq c\|\eta^{(3\ell-1)}\|_\infty n^{-2\ell+1} \sum_{k=0}^{2n} k^{\alpha+j+\ell} k^{-1/2}(k^{-1}+\theta)^{-(\alpha+j+\ell)-1/2} \\
&\leq c\|\eta^{(3\ell-1)}\|_\infty n^{-2\ell+1} \Big[\sum_{k=0}^{\theta^{-1}} k^{2\alpha+2j+2\ell} + \sum_{k=\theta^{-1}}^{2n} k^{\alpha+j+\ell-1/2}\theta^{-(\alpha+j+\ell)-1/2}\Big] \\
&\leq c\|\eta^{(3\ell-1)}\|_\infty n^{-2\ell+1} \Big[\theta^{-(2\alpha+2j+2\ell+1)} + n^{\alpha+j+\ell+1/2}\theta^{-(\alpha+j+\ell)-1/2}\Big] \\
&\leq c\|\eta^{(3\ell-1)}\|_\infty n^{-2\ell+1}(\theta^{-1}+n)^{(2\alpha+2j+2\ell+1)}(1+n\theta)^{-(\alpha+j+\ell)-1/2} \\
&\leq c\|\eta^{(3\ell-1)}\|_\infty n^{(2\alpha+2j+2)}(1+n\theta)^{-(\alpha+j+\ell)-1/2}
\end{aligned}$$

Now we return to prove the claim we used. First of all, we can see that

$$a_{n,1}(s) = \eta(\frac{s+j}{n}) - \eta(\frac{s+j+1}{n}).$$

So take derivative about s m times,

$$\begin{aligned}
|a_{n,1}^{(m)}(s)| &= |-\int_0^1 \frac{d^{m+1}}{dt^{m+1}}\eta(\frac{s+j+t}{n})dt| \\
&= n^{-m-1}\int_0^1 |\frac{d^{m+1}\eta}{dt^{m+1}}(\frac{s+j+t}{n})|dt \\
&\leq n^{-m-1}(\frac{s+j+1}{n})^{2\ell-1}\|\eta^{(m+2\ell)}\|_{[0,\frac{s+j+1}{n}]} \\
&\leq c(1+s)^{-m-2p+1}(\frac{s+1}{n})^{2\ell-1}\|\eta^{(m+2\ell)}\|_{[0,\frac{s+j+1}{n}]}
\end{aligned}$$

where $q = 1$ and $p \leq q$. By induction, we suppose (6.0.1) is true for q and show so it is for $q+1$. Since

$$a_{n,q+1}(s) = \int_0^1 \frac{d}{dt}\frac{a_{n,q}(s+t)}{2s+2t+\alpha+\beta+q+2j+1}dt,$$

$$
\begin{aligned}
|a^{(m)}_{n,q+1}(s)| &\leq \int_0^1 \max_{k_1+k_2=m+1} a^{(k_1)}_{n,\ell}(s+t)(1+s)^{-k_2-1}dt \\
&\leq c\|\eta^{(m+2\ell)}\|_{[0,\frac{s+j+1}{n}]}(1+s)^{-k_2-1}\int_0^1 (1+s+t)^{-k_1-2p+1}(\frac{s+1+t}{n})^{2\ell-1}dt \\
&\leq c\|\eta^{(m+2\ell)}\|_{[0,\frac{s+j+1}{n}]}(1+s)^{-m-2p-1}(\frac{s+1}{n})^{2\ell-1} \\
&= c\|\eta^{(m+2\ell)}\|_{[0,\frac{s+j+1}{n}]}(1+s)^{-m-2p'+1}(\frac{s+1}{n})^{2\ell-1}
\end{aligned}
$$

where $p \leq q$ and $p' = p+1 \leq q+1$. So we conclude the claim and thus complete the proof. $\square$

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