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Integration by Parts and Time Reversal

by
Allanus Hak-Man Tsoi

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University of Alberta
Edmonton

Canada T6G 2G1

Department of Statistics and Applied Probability
Faculty of Science

434 Central Academic Building. Telephone: (403) 492-2052

Professor Robert J. Elliott Telephone: (403) 492-5811
e-mail: RELLIOTT@UALTAVM

March 2, 1990

Faculty of Graduate Studies and Research
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

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Allanus Tsoi has my permission to include material co-authored by both of us in his thesis.

Yours sincerely,

Professor Robert J. Elliott

UNIVERSITY OF ALBERTA

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(Signed)

Permanent Address:
8615 Millwoods Road
Edmonton, Alberta T6K 2Z8

Date: March 2, 1990

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Integration by Parts and Time Reversal** submitted by **Allanus Hak-Man Tsoi** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

R.J. Elliott
Supervisor

A.N. Al-Hussaini

P.M. Hooper

B. Schmuland

V. Zizler

H. Föllmer
External Examiner

Date: 2 March 1990.

TO MICHELLE, MY WIFE

ABSTRACT

Integration by parts and time reversal of stochastic processes have been investigated for some years. In chapter 3 we consider integration by parts for Poisson processes. By using a Girsanov change of measure, we obtain a small ε-perturbation of the rate of the process. This is then compensated by a time change of the process under the new measure. An identity involving the parameter ε is obtained, and the integration by parts formula follows by differentiating with respect to ε and then putting $\varepsilon = 0$. We then utilize this formula to derive a new explicit expression for the integrand that appears in the martingale representation for a Poisson functional.

In chapter 4 we derive integration by parts formulae for functionals of a single jump process. When the state space of the process is Euclidean space, we follow the technique of Norris (1988) by introducing a small ε-perturbation of the state space. We then remove this effect by a Girsanov change of measure, and an integration by parts formula is then obtained by differentiating in ε. When the state space is a general measure space, the above does not work. Instead we consider a small ε-perturbation in the time direction. An integration by parts formula, which involves a time derivative, then follows by differentiating in ε. An expression for the integrand in the martingale representation for functionals of the jump time is also derived.

In chapter 5 we consider time reversal for a standard Poisson process, a point process with Markov intensity, and a point process with a predictable intensity. For a point process N with Markov intensity $h(N_t)$, $H_t = N_t - \int_0^t h(N_s)ds$ is a martingale. We derive the reverse time quasimartingale decomposition of H for

$t \in (0,1]$. For a point process with a predictable intensity, we introduce an analog of the Fréchet derivative for functionals of a Poisson process. We then formulate the integration by parts formula on Poisson space derived in chapter 3 in terms of this derivative, and we utilize this formula to obtain the reverse time decomposition of the point process.

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TABLE OF CONTENTS

Chapter 1

PRELIMINARIES

In this chapter we present a list of basic definitions and theorems which will serve as prerequisites for subsequent chapters. We will take the time index set τ as either $[0, \infty)$ or $[0, 1]$.

1.1. Filtration and Stopping Times.

DEFINITION 1.1.1. *Let $(\Omega, \mathcal{F})$ be a measurable space. A filtration $\{\mathcal{F}_t\}$ of $(\Omega, \mathcal{F})$ is a family of sub-σ-fields $\mathcal{F}_t$, $t \in \tau$, of $\mathcal{F}$ such that if $s \le t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$.*

DEFINITION 1.1.2. *Write $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$. The filtration $\{\mathcal{F}_t\}$ is said to be right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$.*

DEFINITION 1.1.3. *Suppose $(\Omega, \mathcal{F})$ is a measurable space with a filtration $\{\mathcal{F}_t\}_{t\in\tau}$. A random variable $T : \Omega \to \tau$ is said to be a stopping time if for every $t \in \tau$, $\{T \le t\} = \{w : T(w) \le t\} \in \mathcal{F}_t$.*

LEMMA 1.1.4. *Suppose S and T are stopping times. Then $S \wedge T$, $S \vee T$ are stopping times.*

DEFINITION 1.1.5. *Suppose T is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$. Then the σ-field $\mathcal{F}_T$ of events occurring up to time T is the σ-field of events $A \in \mathcal{F}$ such that*

$$A \cap \{T \le t\} \in \mathcal{F}_t \quad \text{for every } t.$$

Proposition 1.1.6. *Suppose S and T are stopping times. If $S \leq T$ a.s., then $\mathcal{F}_S \subseteq \mathcal{F}_T$.*

Definition 1.1.7. *The σ-field $\mathcal{F}_{T-}$ of events strictly prior to the stopping time T is the σ-field generated by $\mathcal{F}_0$ and all sets of the form $A \cap \{t < T\}$, where $t \in [0, \infty]$ and $A \in \mathcal{F}_t$.*

Theorem 1.1.8. *Suppose S and T are stopping times. Then*

(1) $\mathcal{F}_{T-} \subset \mathcal{F}_T$;

(2) T *is* $\mathcal{F}_{T-}$ *measurable;*

(3) if $T \leq S$ a.s., *then* $\mathcal{F}_{T-} \subseteq \mathcal{F}_{S-}$;

(4) for every $A \in \mathcal{F}_S$, $A \cap \{S < T\} \in \mathcal{F}_{T-}$.

Definition 1.1.9. *Suppose $(\Omega, \mathcal{F}, P)$ is a probability space with a filtration $\{\mathcal{F}_t\}_{t \in \tau}$. $\mathcal{F}^P$ will denote the completion of $\mathcal{F}$, and $\mathcal{F}_t^P$, $t \in \tau$, the σ-field generated by $\mathcal{F}_t$ and the P-null sets of $\mathcal{F}^P$. Then $\{\mathcal{F}_t^P\}$ is a filtration on $(\Omega, \mathcal{F}^P, P)$ and is called the completion of the filtration $\{\mathcal{F}_t\}$. A filtration is said to be complete if $\mathcal{F}$ is complete and each $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$.*

Definition 1.1.10. *A stopping time T is said to be predictable if there is a sequence $\{Tn\}$, $n \in \mathbb{N}$, of stopping times such that*

(1) $\{Tn(w)\}$ *is almost surely an increasing sequence in* $[0, \infty)$ *and* $\lim_n Tn(w) = T(w)$ a.s.;

(2) On the set $\{T > 0\}$, $Tn(w) < T(w)$ a.s. *for all* n. *The sequence* $\{Tn\}$ *is said to announce* T.

1.2. Stochastic Processes and Martingales.

Starting from this section we will assume that a probability space $(\Omega, \mathcal{F}, P)$ with a right continuous, complete filtration $\{\mathcal{F}_t\}$, $t \in \tau$, is given. Furthermore, we assume that every stochastic process is real valued.

Definition 1.2.1. *Let $\{X_t\}$, $t \in \tau$, be a stochastic process defined on $(\Omega, \mathcal{F}, P)$. Then $\{X_t\}$ is said to be adapted to $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for each $t \in \tau$.*

Definition 1.2.2. *A stochastic process $\{X_t\}$, $t \in \tau$ is said to be right continuous if for almost all w, the sample path $t \to X_t(w)$ is right continuous. $\{X_t\}$ is said to have left hand limits if the sample paths have left limits. A right continuous process with left limits is said to be a CORLOL process. Similar definitions hold for left continuity and right hand limits.*

Definition 1.2.3. *A stochastic process $\{X_t\}$, $t \in \tau$, is said to be a supermartingale (resp. a submartingale) with respect to the filtration $\{\mathcal{F}_t\}$ if:*

(1) $\{X_t\}$ is adapted to $\{\mathcal{F}_t\}$;

(2) $E[|X_t|] < \infty$, $t \in \tau$;

(3) $E[X_t \mid \mathcal{F}_s] \leq X_s$ a.s. if $t \geq s$ (resp. $E[X_t \mid \mathcal{F}_s] \geq X_s$ a.s. if $t \geq s$).

$\{X_t\}$ is said to be a martingale if it is both a supermartingale and a submartingale.

Definition 1.2.4. *A martingale $\{X_t\}$, $t \in \tau$ is said to be uniformly integrable if the class of random variables $\{X_t,\ t \in \tau\}$ is uniformly integrable.*

Theorem 1.2.5. *Suppose $\{X_t\}$, $t \in [0, \infty)$, is a uniformly integrable martingale.*

Then there is a random variable X_∞ *such that* $\lim_{t\to\infty} X_t(w) = X_\infty(w)$ a.s. *The convergence also takes place in* L^1, *and* $\{X_t\}$, $t \in [0,\infty]$, *is a martingale.*

THEOREM 1.2.6. *If* $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$ *and* Y *is an integrable* $\mathcal{F}_\infty$*-measurable random variable, then* $\{E[Y \mid \mathcal{F}_t]\}$, $t \in [0,\infty]$, *is a uniformly integrable martingale. We can take a right continuous modification* $\{Y_t\}$ *of this martingale, and* $\lim_{t\to\infty} Y_t(w) = Y(w)$ a.s. *and in* L^1.

THEOREM 1.2.7. *(Optimal Stopping) If* $\{X_t\}$, $t \in [0,\infty]$, *is a right continuous martingale, and* S *and* T *are two stopping times such that* $S \le T$ a.s., *then*

$$X_S = E[X_T \mid \mathcal{F}_S] \quad \text{a.s.}$$

DEFINITION 1.2.8. *Let* $\mathcal{H}$ *denote the family of subsets of* $[0,\infty)\times\Omega$ *containing all sets of the form* $\{0\}\times F_0$ *and* $(s,t]\times F$, *where* $F_0 \in \mathcal{F}_0$ *and* $F \in \mathcal{F}_s$ *for* $s < t$ *in* $[0,\infty)$. *The* σ*-field* $\mathcal{P}$ *generated by* $\mathcal{H}$ *is called the predictable* σ*-field.*

THEOREM 1.2.9. *The predictable* σ*-field* $\mathcal{P}$ *is generated by the family of left-continuous, adapted processes.*

DEFINITION 1.2.10. *A process* $\{X_t\}$ *is said to be predictable if the map* $X : [0,\infty)\times\Omega \to \mathbb{R}$ *is measurable with respect to the predictable* σ*-field* $\mathcal{P}$.

NOTATION 1.2.11. *Led* $\mathcal{B} = \mathcal{B}([0,\infty))$ *be the Borel* σ*-field of* $[0,\infty)$. *Write* $B(\mathcal{B}\times\mathcal{F})$ *for the space of bounded,* $\mathcal{B}\times\mathcal{F}$ *measurable processes, and* $B(\mathcal{P})$ *as the space of bounded predictable processes.*

THEOREM 1.2.12. *(Projection Theorem) There is a unique linear order preserving projection* Π_p *of* $B(\mathcal{B}\times\mathcal{F})$ *onto* $B(\mathcal{P})$ *such that for* $X \in B(\mathcal{B}\times\mathcal{F})$ *and for every* *predictable stopping time* T,

$$E[X_T I_{\{T<\infty\}}] = E[(\Pi_P X)_T I_{\{T<\infty\}}].$$

$\Pi_P(X)$ *is called the predictable projection of* X.

THEOREM 1.2.13. *Suppose* $X \in B(\mathcal{B} \times \mathcal{F})$. *Then for any predictable stopping time* T,

$$\Pi_P(X)_T I_{\{T<\infty\}} = E[X_T I_{\{T<\infty\}} \mid \mathcal{F}_{T-}].$$

DEFINITION 1.2.14. *A* $\mathcal{B}([0,\infty)) \times \mathcal{F}$ *measurable stochastic process* $\{A_t\}$, $t \in [0,\infty)$, *with values in* $[0,\infty)$, *is called an increasing process if almost every sample path* $A\cdot(w)$ *is right continuous and increasing.*

DEFINITION 1.2.15. $\mathcal{V}^+$ *will denote the family of processes* $\{A_t\}$ *which is increasing and adapted to the filtration* $\{\mathcal{F}_t\}$. $\mathcal{V}_0^+$ *will denote those processes* $\{A_t\} \in \mathcal{V}^+$ *such that* $A_0 = 0$.

DEFINITION 1.2.16. $\mathcal{V} = \mathcal{V}^+ - \mathcal{V}^+$ *is the set of processes, each of which is the difference of two elements of* $\mathcal{V}^+$.

DEFINITION 1.2.17. $\mathcal{A}^+$ *denotes the set of integrable increasing processes, that is, the set of increasing processes* $\{A_t\}$ *adapted to the filtration* $\{\mathcal{F}_t\}$, *such that* $E[A_\infty] < \infty$. $\mathcal{A}_0^+$ *will denote the set of processes* $\{A_t\} \in \mathcal{A}^+$ *with* $A_0 = 0$.

DEFINITION 1.2.18. *A right continuous uniformly integrable supermartingale* $\{X_t\}$ *is said to be of class* D *if the set of random variables* $\{X_T\}$, *for* T *any stopping time, is uniformly integrable.*

THEOREM 1.2.19. *Suppose $\{X_t\}$ is a right continuous supermartingale of class D. Then there exists a unique predictable increasing process $\{A_t\} \in \mathcal{A}_0^+$ such that the process $M_t = X_t + A_t$ is a uniformly integrable martingale.*

THEOREM 1.2.20. *Suppose $\{X_t\}$, $t \in [0, \infty]$, is a right continuous supermartingale. Then $\{X_t\}$ has a unique decomposition of the form*

$$X_t = M_t - A_t$$

where $\{A_t\}$ is an increasing predictable process, $A_0 = 0$ a.s., and there is an increasing sequence $\{Tn\}$ of stopping times such that $\lim_{n \to \infty} Tn = \infty$ a.s., and each process $\{M_t^{Tn}\}$ is a uniformly integrable martingale. The above decomposition is called the Doob–Meyer decomposition of the supermartingale $\{X_t\}$.

DEFINITION 1.2.21. *A martingale $\{X_t\}$, $t \in [0, \infty)$, is called a square integrable martingale if*

$$\sup_t E[X_t^2] < \infty.$$

$\mathcal{H}^2$ will denote the class of square integrable martingales.

NOTATION 1.2.22. *If $\mathcal{C}$ is some family of processes, then $\mathcal{C}_{\text{loc}}$ will denote the family of processes which are locally in $\mathcal{C}$. That is, $\{Y_t\} \in \mathcal{C}_{\text{loc}}$ if there is an increasing sequence of stopping times $\{Tn\}$ such that $\lim_n Tn = \infty$ a.s., and that each stopped process $\{Y_t^{Tn}\} = \{Y_{t \wedge Tn}\}$ is in $\mathcal{C}$.*

If $\mathcal{C}$ is any class of processes, $\mathcal{C}_0$ will denote the set of $X \in \mathcal{C}$ with $X_0 = 0$ a.s.

DEFINITION 1.2.23. *$\mathcal{M}$ will denote the class of uniformly integrable martingales. We shall write $\mathcal{L}$ for the class $(\mathcal{M}_{\mathrm{loc}})_0$.*

DEFINITION 1.2.24. *Two local martingales $M, N \in \mathcal{M}_{\mathrm{loc}}$ are orthogonal if their product $MN = \{M_t N_t\}$ is in $\mathcal{L}$.*

DEFINITION 1.2.25. *$\mathcal{H}^{2,c} \subseteq \mathcal{H}^2$ will denote the space of continuous square integrable martingales. $\mathcal{H}^{2,d}$ is the subspace orthogonal $\mathcal{H}^{2,c}$. Martingales in $\mathcal{H}^{2,d}$ are said to be purely discontinuous.*

THEOREM 1.2.26. *For any $M \in \mathcal{H}^2$, there is a unique decomposition of M of the form*

$$M = M^c + M^d,$$

where $M^c \in \mathcal{H}_0^{2,c}$, and $M^d \in \mathcal{H}^{2,d}$.

THEOREM 1.2.27. *Suppose $M \in \mathcal{L}$. Then there is a unique decomposition*

$$M = M^c + M^d$$

where M^c is a continuous local martingale and M^d is a totally discontinuous local martingale.

1.3. Quadratic Variation Processes.

DEFINITION 1.3.1. *Suppose $M \in \mathcal{H}^2$. Then the predictable quadratic variation of M, denoted by $\langle M, M\rangle$, is the unique predictable increasing process in $\mathcal{A}^+$ given by the Doob–Meyer decomposition of the supermartingale*

$$X_t = E[M_\infty^2 \mid \mathcal{F}_t] - M_t^2.$$

PROPOSITION 1.3.2. *$M_t^2 - \langle M, M\rangle_t$ is a martingale, and $\langle M, M\rangle_0 = M_0^2$.*

NOTATION 1.3.3. *For any process $\{X_t\}$ having left limits, write*

$$\Delta X_t = X_t - X_{t-}.$$

DEFINITION 1.3.4. *For $M \in \mathcal{H}^2$, define*

$$[M, M]_t = \langle M^c, M^c\rangle_t + \sum_{s\leq t} \Delta M_s^2.$$

$[M, M]_t$ is called the optional quadratic variation of M.

DEFINITION 1.3.5. *Suppose $M, N \in \mathcal{H}^2$. Define*

$$\langle M, N\rangle = \frac{1}{2}(\langle M + N, M + N\rangle - \langle M, M\rangle - \langle N, N\rangle).$$

DEFINITION 1.3.6. *Suppose $M, N \in \mathcal{H}^2$. Define*

$$[M, N]_t = \langle M^c, N^c\rangle_t + \sum_{s\leq t} \Delta M_s \Delta N_s.$$

PROPOSITION 1.3.7. *$MN - \langle M, N\rangle$ is a martingale, and $M_0N_0 = \langle M, N\rangle_0$.*

PROPOSITION 1.3.8. *$MN - [M, N]$ is a martingale, and $M_0N_0 = [M, N]_0 = \Delta M_0 \Delta N_0$.*

PROPOSITION 1.3.9. *If M is a continuous local martingale, then $M \in \mathcal{H}^2_{\text{loc}}$.*

DEFINITION 1.3.10. *Suppose $M \in \mathcal{H}^2_{\text{loc}}$. So there is an increasing sequence of stopping times $\{Tn\}$ such that $\lim_n Tn = \infty$ a.s., and $M(n) \in \mathcal{H}^2$, where*

$$M(n)_t = M_t^{Tn}.$$

From Theorem 1.2.26,

$$M(n) = M(n)^c + M(n)^d$$

is unique. If $Tn \leq Tm$, then $M(n)^c_t = M(m)^c_t$, and $M(n)^d_t = M(m)^d_t$ for $t \leq Tn(w)$. Since the predictable quadratic variation process is unique, so

$$\langle M(n), M(n)\rangle_t = \langle M(m), M(m)\rangle_t \quad \text{for } t \leq Tn(w).$$

We can, therefore, define the predictable quadratic variation process of $M \in \mathcal{H}^2_{\text{loc}}$ as the unique process $\langle M, M\rangle \in \mathcal{A}^+_{\text{loc}}$ such that

$$\langle M, M\rangle^{Tn}_t = \langle M(n), M(n)\rangle_t.$$

Similarly, the optional quadratic variation process of $M \in \mathcal{H}^2_{\text{loc}}$ is the process $[M, M] \in \mathcal{A}^+_{\text{loc}}$ such that

$$[M, M]^{Tn}_t = \langle M(n)^c, M(n)^c\rangle_t + \sum_{s \leq t\wedge Tn} \Delta M(n)^2.$$

PROPOSITION 1.3.11. *Let $M \in \mathcal{L}$ and $M = M^c + M^d$ as in Theorem 1.2.27. Then the optional quadratic variation of M is the increasing process*

$$[M, M]_t = \langle M^c, M^c\rangle_t + \sum_{s\leq t} \Delta M^2_s.$$

If $M, N \in \mathcal{L}$, define

$$[M, N]_t = \langle M^c, N^c\rangle_t + \sum_{s\leq t} \Delta M_s \Delta N_s.$$

1.4. Stochastic Integrals with Respect to Martingales.

DEFINITION 1.4.1. *Let Λ be the collection of those processes (H_t) having the property that there exists a sequence of real numbers*

$$0 = t_0 < t_1 < \cdots < t_n < \ldots,$$

$\lim_n t_n \uparrow \infty$, *and a sequence of random variables* $\{H_i(w)\}_{i=0}^{\infty}$ *such that* H_i *is* $\mathcal{F}_{t_i}$-*measurable,* $\sup_i \|H_i\|_\infty < \infty$, *and*

$$H_t(w) = \begin{cases} H_0(w) & \text{if } t = 0 \\ H_i(w) & \text{if } t \in (t_i, t_{i+1}]. \end{cases}$$

DEFINITION 1.4.2. *For* $X \in \mathcal{H}^2$, *set*

$$\|X\|_T = E[X_T^2]^{1/2}$$

$$\|X\| = \sum_{n=1}^{\infty} 2^{-n}(\|X\|_n \wedge 1).$$

DEFINITION 1.4.3. *For* $M \in \mathcal{H}^2$, *let* $L^2(M)$ *denote the class of predictable processes* $\{H_t\}$ *such that for every* $T > 0$,

$$(\|H\|_{2,T}^M)^2 = E\Big[\int_0^T H_s^2 d\langle M, M\rangle_s\Big] < \infty.$$

For $H \in L^2(M)$, *set*

$$\|H\|_2^M = \sum_{n=1}^{\infty} 2^{-n}(\|H\|_{2,n}^M \wedge 1).$$

LEMMA 1.4.4. *Λ is dense in $L^2(M)$ with respect to the metric $\|\cdot\|_2^M$.*

DEFINITION 1.4.5. *For* $H \in \Lambda$ *of the form*

$$H_t = H_0 I_{\{t=0\}}(t) + \sum_{i=0}^{\infty} H_i I_{(t_i, t_{i+1}]}(t)$$

define

$$\int_0^t H_s dM_s = \sum_{i=0}^{n-1} H_i(M_{t_{i+1}} - M_{t_i})$$

$$+ H_n(M_t - M_{t_n}) \quad \text{for } t_n \le t \le t_{n+1},\ n = 1, 2, \ldots$$

Then

$$\Big(\int_0^t H_s dM_s\Big)_{t\ge 0} \in \mathcal{H}^2,$$

and

$$\Big\| \int_0^{\cdot} H_s dM_s \Big\| = \|H\|_2^M.$$

Using this isometry, $H \in \Lambda \to \int_0^{\cdot} H_s dM_s \in \mathcal{H}^2$ *is extended to* $H \in L^2(M) \to \int_0^{\cdot} H_s dM_s \in \mathcal{H}^2$. $\int_0^{\cdot} H_s dM_s$ *is called the stochastic integral of* $H \in L^2(M)$ *with respect to* $M \in \mathcal{H}^2$.

PROPOSITION 1.4.6. *If* $M, N \in \mathcal{H}^2$, $H \in L^2(M)$, $K \in L^2(N)$, *then for* $t > s \ge 0$,

$$E\Big[\int_s^t H_r dM_r \int_s^t K_r dN_r \mid \mathcal{F}_s\Big] = E\Big[\int_s^t H_r K_r d\langle M, N\rangle_r \mid \mathcal{F}_s\Big].$$

DEFINITION 1.4.7. *Suppose* $M \in \mathcal{H}^2_{\text{loc}}$. *Let* $L^2_{\text{loc}}(M)$ *be the class of predictable processes* $\{H_t\}$ *such that there exists a sequence of stopping times* $\{\sigma_n\}$ *such that* $\sigma_n \uparrow \infty$ a.s., and

$$E\Big[\int_0^{T\wedge\sigma_n} H_s^2 d\langle M, M\rangle_s\Big] < \infty$$

for every $T > 0$ and $n = 1, 2, \ldots$

DEFINITION 1.4.8. *Suppose* $M \in \mathcal{H}^2_{\text{loc}}$, $H \in L^2_{\text{loc}}(M)$. *Then we may choose a sequence of stopping times* $\{\sigma_n\}$ *such that* $\sigma_n \uparrow \infty$ a.s., $M^{\sigma_n} \in \mathcal{H}^2$, and

$$E\Big[\int_0^{T\wedge\sigma_n} H_s^2 d\langle M, M\rangle_s\Big] < \infty$$

for every $T > 0$ *and* $n = 1, 2, \ldots$.

Hence for $H_t^n = H_t I_{\{t \le \sigma_n\}}$ and $M(n) = M^{\sigma_n}$, we can define $\int_0^{\cdot} H_s^n dM(n)_s$, and for $m < n$,

$$\int_0^t H_s^m dM(m)_s = \int_0^{t \wedge \sigma_m} H_s^n dM(n)_s.$$

Thus there exists a unique process $\int_0^{\cdot} H_s dM_s$ such that

$$\int_0^{t \wedge \sigma_n} H_s dM_s = \int_0^t H_s^n dM(n)_s, \quad n = 1, 2, \ldots$$

and

$$\int_0^{\cdot} H_s dM_s \in \mathcal{H}_{\text{loc}}^2.$$

$\int_0^{\cdot} H_s dM_s$ is called the stochastic integral of $H \in L_{\text{loc}}^2(M)$ with respect to $M \in \mathcal{H}_{\text{loc}}^2$.

1.5. Semi-martingale and the Differentiation Rule.

DEFINITION 1.5.1. *An adapted process* $\{X_t\}_{t \ge 0}$ *is a semi-martingale if it has a decomposition of the form*

$$X_t = X_0 + M_t + A_t$$

where $M \in \mathcal{L}$ *and* $A \in \mathcal{V}_0$. *Write*

$$X^c = M^c.$$

Then X^c *is called the continuous martingale part of* X.

DEFINITION 1.5.2. *Suppose* $X = X_0 + M + A$ *is a semi-martingale, and that* H *is a predictable, locally bounded process. Define*

$$\int_0^t H_s dX_s = H_0 X_0 + \int_0^t H_s dM_s + \int_0^t H_s dA_s.$$

DEFINITION 1.5.3. *Suppose X is a semi-martingale. Then the optional quadratic variation of X is the process*

$$[X,X]_t = \langle X^c, X^c\rangle_t + \sum_{s\leq t} \Delta X_s^2.$$

If Y is a second semi-martingale, define

$$[X,Y] = \langle X^c, Y^c\rangle_t + \sum_{s\leq t} \Delta X_s \Delta Y_s.$$

THEOREM 1.5.4. *Suppose X is a semi-martingale and F a twice continuously differentiable function. Then $F(X)$ is a semi-martingale, and*

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-})dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X^c, X^c\rangle_s$$
$$+ \sum_{0<s\leq t} (F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s).$$

PROPOSITION 1.5.5. *If X and Y are semi-martingales, then the product XY is a semi-martingale, and*

$$X_t Y_t = \int_{]0,t]} X_{s-}dY_s + \int_{]0,t]} Y_{s-}dX_s + [X,Y]_t.$$

Chapter 2

INTRODUCTION TO INTEGRATION BY PARTS: THE APPROACHES OF BISMUT AND NORRIS

Before describing Bismut's approach we first present the simple situation of a transformed Brownian motion, as described by Williams [8].

Let $\{B_t : t \in [0,1]\}$ be a one-dimensional canonical Brownian motion defined on $(C[0,1], \mathcal{A}, (\mathcal{A}_t), P)$ starting at 0. For u, a bounded predictable process, and for $\varepsilon \in \mathbb{R}$, define a new measure Q^ε on $(C[0,1], \mathcal{A})$ by

$$\begin{aligned} \frac{dQ^\varepsilon}{dP}\Big|_{\mathcal{A}_t} &= \eta_t^\varepsilon \\ &= \exp\left\{\varepsilon \int_0^t u_s dB_s - \frac{1}{2}\varepsilon^2 \int_0^t u_s^2 ds\right\}. \end{aligned} \tag{2.1}$$

Then from the Girsanov's Theorem, the process

$$B_t^\varepsilon = B_t - \varepsilon \int_0^t u_s ds$$

is a Brownian motion under Q^ε (see [5]).

Let g be a strongly differentiable function on $C[0,1]$. Thus for fixed $y \in C[0,1]$, there exists a bounded linear functional, denoted by dg^y, called the Fréchet derivative of g at y, defined on $C[0,1]$, and with values in $\mathbb{R}$, such that

$$g(y+z) - g(y) = dg^y(z) + o(\|z\|),$$

where $\|z\| = \sup_t |z(t)|$. By the Riesz Representation Theorem, there is a signed measure μ_g^y on $[0,1]$ which satisfies

$$dg^y(z) = \int_0^1 z(t)\mu_g^y(dt).$$

Suppose, furthermore, that g is uniformly Lipschitz. Then from the above discussion, we have

$$\begin{aligned} E[g(B)] &= E^{Q^\varepsilon}[g(B^\varepsilon)] \\ &= E\Big[\eta_1^\varepsilon g(B. - \varepsilon \int_0^\cdot u_s ds\Big]. \end{aligned} \tag{2.2}$$

where E^{Q^ε} denotes that the expectation is taken with respect to Q^ε. Differentiating (2.2) with respect to ε and then putting $\varepsilon = 0$, we obtain

$$E\Big[g(B)\int_0^1 u_s dB_s\Big] - E\Big[dg^B\Big(\int_0^\cdot u_s ds\Big)\Big] = 0,$$

so that

$$\begin{aligned} E\Big[g(B)\int_0^1 u_s dB_s\Big] &= E\Big[\int_0^1 \int_0^t u_s ds \mu_g^B(dt)\Big] \\ &= E\Big[\int_0^1 u_s \mu_g^B(s,1] ds\Big]. \end{aligned} \tag{2.3}$$

The above integration by parts formula is related to the martingale representation for Brownian functionals as follows: first we have the martingale representation:

$$g(B) = E[g(B)] + \int_0^1 \gamma_s dB_s \tag{2.4}$$

for some predictable process γ_s. Then

$$\begin{aligned} E\Big[g(B)\int_0^1 u_s dB_s\Big] &= E\Big[\int_0^1 u_s dB_s \int_0^1 \gamma_s dB_s\Big] \\ &= E\Big[\int_0^1 \gamma_s u_s ds\Big]. \end{aligned} \tag{2.5}$$

Now let C_s^* be the predictable projection of $\mu_g^B(s,1]$. Then

$$C_s^* = E[\mu_g^B(s,1] \mid \mathcal{A}_s] \text{ a.e.}$$

Therefore,

$$E[u_s\mu_g^B(s,1]] = E\big[E[u_s\mu_g^B(s,1] \mid \mathcal{A}_s]\big]$$

$$= E[u_sC_s^*]. \tag{2.6}$$

Hence from (2.3), (2.5) and (2.6), we obtain the relation

$$E\Big[\int_0^1 \gamma_s u_s ds\Big] = E\Big[\int_0^1 C_s^* u_s ds\Big]. \tag{2.7}$$

Since (2.7) holds for any bounded predictable process u_s, and since γ_s and C_s^* are predictable, we have the explicit martingale representation due to Clark [4]

$$\gamma_s = E[\mu_g^B(s,1] \mid \mathcal{A}_s]$$

or

$$g(B) = E[g(B)] + \int_0^1 E[\mu_g^B(s,1] \mid \mathcal{A}_s]dB_s. \tag{2.8}$$

We will now consider a more general diffusion following Bismut [2]. Let Ω denote the space $C(\mathbb{R}^+;\mathbb{R}^m)$ of continuous functions defined on $\mathbb{R}^+$ with values in $\mathbb{R}^m$. If $w \in C(\mathbb{R}^+;\mathbb{R}^m)$, $w_t = (w_t^1,\dots,w_t^m)$ denotes the trajectory of w. Let $\mathcal{F}_t$ be the right continuous, complete σ-field generated by $\sigma\{w_s : s \le t\}$. Suppose P is the Brownian measure on Ω with $P(w_0 = 0) = 1$.

Let $X_0, X_1, \dots, X_m$ be a family of $m+1$ vector fields defined on $\mathbb{R}^d$ with values in $\mathbb{R}^d$, which are C^∞, bounded with bounded differentials of all orders.

Consider the stochastic differential equation

$$dx = X_0(x)dt + X_i(x)\cdot dw^i$$

$$x(0) = x \tag{2.9}$$

where dw^i denotes the Stratonovitch differential of w^i, or the equivalent equation

$$dx = \Big(X_0 + \frac{1}{2}\,\frac{\partial X_i}{\partial x}\,X_i\Big)(x)dt + X_i(x)\cdot \delta w^i$$

$$x(0) = x \tag{2.10}$$

where δw^i is the Itô differential of w^i.

For every $x \in \mathbb{R}^d$, (2.9) has a unique solution which is continuous a.s. (see [6]). Moreover, we can consider the stochastic flows associated with (2.9). That is, we consider the mapping

$$(w,t,x) \to \varphi_t(w,x)$$

which satisfies:

(a) $\varphi_t(w,x)$ is measurable in the variable w, and continuous in the variable (t,x);

(b) For any $x \in \mathbb{R}^d$, $t \to \varphi_t(w,x)$ is the essentially unique solution of (2.9) (see [6]). Moreover, $\varphi_t(w,x)$ satisfies the following:

  (i) Almost surely, for every $t \geq 0$, $\varphi_t(w,x)$ is a C^∞ diffeomorphism of $\mathbb{R}^d$ onto $\mathbb{R}^d$.

  (ii) The differentials $\frac{\partial^m}{\partial x^m}\,\varphi_t(w,x)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}^d$.

  (iii) For any $x \in \mathbb{R}^d$, $Z_t = \frac{\partial}{\partial x}\,\varphi_t(w,x)$ is the unique solution of the stochastic differential equation

$$dZ = \frac{\partial X_0}{\partial x}\,(x_t)Zdt + \frac{\partial X_i}{\partial x}(x_t)Z\cdot dw^i$$

$$Z(0) = I. \tag{2.11}$$

(iv) For any $x \in \mathbb{R}^d$, $Z'_t = \left[\frac{\partial \varphi_t}{\partial x}(w,x)\right]^{-1}$ is the unique solution of the equation

$$dZ' = -Z' \frac{\partial X_0}{\partial x}(x_t)dt - Z' \frac{\partial X_i}{\partial x}(x_t) \cdot dw^i$$

$$Z'(0) = I. \tag{2.12}$$

(v) For any $T > 0$, $R > 0$, $1 \le p < \infty$, $0 \le |m| < \infty$, the random variables

$$\sup_{\substack{0 \le t \le T \\ |x| \le R}} \left| \frac{\partial^m}{\partial x^m} \varphi_t(w,x) \right|, \qquad \sup_{\substack{0 \le t \le T \\ |x| \le R}} \left| \left(\frac{\partial \varphi_t}{\partial x}(w,x) \right)^{-1} \right|$$

are in L_p.

The above results are proved in [3].

If L is an adapted locally integrable process with values in $\mathbb{R}^d$, if $z_0 \in \mathbb{R}^d$, and if z_t is defined by

$$z_t = z_0 + \int_0^t L_s ds,$$

then $\varphi_t(w, z_t)$ is a continuous semi-martingale given by the following generalized differentiation rule (see [1]):

$$\varphi_t(w,z_t) = z_0 + \int_0^t X_0(\varphi_s(w,z_s))ds + \int_0^t X_i(\varphi_s(w,z_s))dw_s^i$$
$$+ \int_0^t \frac{\partial \varphi_s}{\partial x}(w,z_s)dz_s. \tag{2.13}$$

Let $u = (u^1, \ldots, u^m)$ be defined on $\mathbb{R}^d$ with values in $\mathbb{R}^m$, which is C^∞ and bounded, with all its derivatives of polynomial growth. Consider the stochastic differential equation

$$dX_t^1 = X_0(X_t^1)dt + X_i(X_t^1)[dw^i + u^i(x_t)dt]$$

$$X_0^1 = x. \tag{2.14}$$

Let z_t^u be the unique solution to the differential equation

$$dz^u = \left[\frac{\partial \varphi_t}{\partial x}(w, z^u)\right]^{-1} [X_i(\varphi_t(w, z^u))u^i(x_t)]dt$$

$$z^u(0) = x. \tag{2.15}$$

Then it follows from the generalized differentiation rule (2.13) and (2.15) that $\varphi_t(w, z_t^u)$ is the essentially unique solution of (2.14).

If g is a bounded function defined on $C([0,T]; \mathbb{R}^d)$ with values in $\mathbb{R}$, which is continuous and strongly differentiable, let dg^y be the differential of g at y for $y \in C([0,T]; \mathbb{R}^d)$. Let $\mu^y(t)$ be the finite measure on $[0,T]$ that appears in the Riesz Representation of dg^y. That is, $\mu^y(t)$ satisfies the relation:

$$\langle dg^y, z\rangle = \int_0^T \langle z_t, d\mu^y(t)\rangle$$

for $z \in C([0,T]; \mathbb{R}^d)$.

For $\ell \in \mathbb{R}$, let us consider the stochastic differential equation

$$dx_t^\ell = X_0(x_t^\ell)dt + X_i(x_t^\ell)(dw^i + \ell u^i(x_t)dt)$$

$$x_0^\ell = x. \tag{2.16}$$

From the above discussion, we know that $\varphi_t(w, z_t^{\ell u})$ is the essentially unique solution of (2.16), where $z^{\ell u}$ is the solution of the differential equation

$$dz^{\ell u} = \left[\frac{\partial \varphi_t}{\partial x}(w, z^{\ell u})\right]^{-1} [X_i(\varphi_t(w, z^{\ell u}))]\ell u^i(x_t)dt$$

$$z^{\ell u}(0) = x. \tag{2.17}$$

On the other hand, let v_t^ℓ be the adapted process defined by

$$v_t^\ell = w_t + \int_0^t \ell u(x_s) ds.$$

Consider the family of exponentials $Z_T^{\ell u}$ defined by

$$Z_T^{\ell u} = \exp\Big\{ - \int_0^T \langle \ell u(x_s), \delta w \rangle - \frac{1}{2} \int_0^T |\ell u(x_s)|^2 ds \Big\}. \tag{2.18}$$

Define a new measure Q^ℓ by

$$\frac{dQ^\ell}{dP}\Big|_{\mathcal{F}_T} = Z_T^{\ell u}.$$

Then by the Girsanov formula ([5]), v_t^ℓ is a Brownian motion under the measure Q^ℓ. Hence under Q^ℓ, $\varphi.(w,x)$ is the unique solution to equation (2.16). Thus it follows that

$$E^P[g(\varphi.(w, z_.^{\ell u}(w)))] = E^P[g(\varphi.(w,x)) Z_T^{\ell u}(w)]. \tag{2.19}$$

Here g is any bounded function defined on $C([0,T]; \mathbb{R}^d)$ with values in $\mathbb{R}$, which is continuous and strongly differentiable. By differentiating (2.19) with respect to ℓ, and then putting $\ell = 0$, we obtain the integration by parts formula

$$E\Big[g(\varphi.(w,x)) \int_0^T u^i(x_s) \delta w^i\Big]$$
$$= E\Big[\int_0^T u^i(x_s) ds \Big\langle X_i(\varphi_s(w,x)), \int_{[s,T]} \frac{\partial \varphi_s}{\partial x}(w,x) \Big[\frac{\partial \varphi_v}{\partial x}(w,x)\Big]^{-1} d\mu^{\varphi.(w,x)}(v) \Big\rangle\Big]. \tag{2.20}$$

We now consider the derivation of an integration by parts formula for jump processes due to Norris ([7]).

Let x_t be the solution to the stochastic differential equation

$$dx_t = X(x_{t-})dt + Y(x_{t-}, y)(\mu - \nu)(dy, dt)$$

$$x_0 = x \in \mathbb{R}^d. \tag{2.21}$$

where μ is a Poisson random measure on $E \times [0, \infty)$, ν is the compensator of μ, and is of the form $\nu(dy, dt) = G(dy)dt$. We will assume that $E = \mathbb{R}^d \backslash \{0\}$. Furthermore, G is a Radon measure on E and there is an open set $E' \subseteq E$ and a function $g \in C^1(E')$ with

$$G(dy) = g(y)dy$$

$$g > 0 \text{ on } E'.$$

We assume for now that the coefficients X and Y in (2.21) satisfy:

(i) $X, Y(\cdot, y)$ are C^1, $Y(x, \cdot)$ is C^1 on E'; $X(x)$, $DX(x)$, $Y(x,y)$, $D_1Y(x,y)$ are uniformly bounded, and $D_2Y(x,y)$ is bounded on $\mathbb{R}^d \times K'$ for each compact $K' \subseteq E'$;

(ii) supp $Y \subseteq \mathbb{R}^d \times K$ for some compact $K \subseteq E$.

These conditions ensure, in particular, that x_t has only finitely many jumps in any interval $0 \leq t \leq T$, and is between jumps just the solution of a first order ODE.

Let $v(t, y)$ be a predictable function defined on $[0, \infty) \times E$ with values in $\mathbb{R}^d$. Assume that

(i) $v(t, \cdot)$ is C^1 for each $0 \leq t < \infty$; v and D_2v are uniformly bounded.

(ii) supp $v(\cdot, \cdot) \subseteq [0, \infty) \times K'$ for some compact $K' \subseteq E'$.

We will use the function $v(\cdot,\cdot)$ to perturb the measure μ, and v is called the perturbation.

For small $h \in \mathbb{R}$, define

$$\theta^h(t,y) = y + v(t,y)h.$$

A perturbed random measure μ^h is then defined by

$$\int_0^t \int_E \phi(s,y)\mu^h(dy,ds) = \int_0^t \int_E \phi(s,\theta^h(s,y))\mu(dy,ds). \tag{2.22}$$

Notice that in (2.22) only the state space is perturbed, and the times at which the jumps occur are preserved.

Set

$$\lambda^h(t,y) = \begin{cases} \det D_2\theta^h(t,y)\dfrac{g(\theta^h(t,y))}{g(y)} & y \in K' \\ 1 & y \notin K'. \end{cases}$$

Now consider the martingale X_t defined by

$$X_t = \int_0^t \int_E (\lambda^h(s,y) - 1)(\mu - \nu)(dy,ds)$$

and define the family of exponentials Z_t^h by

$$\begin{aligned} Z_t^h &= \exp\Big(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t\Big) \prod_{0\le s\le t}(1+\Delta X_s)e^{-\Delta X_s} \\ &= \exp\Big(\int_0^t \int_E \log\lambda^h(s,y)\mu(dy,ds) - \int_0^t \int_E (\lambda^h(s,y)-1)\nu(dy,ds)\Big) \end{aligned} \tag{2.23}$$

Then

$$dZ_t^h = Z_{t-}^h(\lambda^h(t,y) - 1)(\mu - \nu)(dy,ds). \tag{2.24}$$

So Z_t^h is a martingale, $E[Z_t^h] = 1$, and we may define a new probability measure P^h by

$$\frac{dP^h}{dP}\Big|_{\mathcal{F}_t} = Z_t^h.$$

We will now show that, under P^h, μ^h has the original law of μ. It suffices to check for test function $\phi \in L'(\mu)$ and for

$$U_t^h = \exp\Big\{\int_0^t \int_E \phi(s,y)\mu^h(dy,ds)\Big\} Z_t^h$$

$$= \exp\Big\{\int_0^t \int_E \phi(s,\theta^h(s,y))\mu(dy,ds)\Big\} Z_t^h$$

that $E[U_t^h]$ does not depend on h. Write

$$Y_t = \exp\Big\{\int_0^t \int_E \phi(s,\theta^h(s,y))\mu(dy,ds)\Big\}.$$

By the differentiation rule,

$$U_t^h = 1 + \int_0^t \int_E Y_{s-} dZ_s^h + \int_0^t \int_E Z_{s-}^h dY_s + [Y, Z^h]_t.$$

But

$$\int_0^t \int_E Z_{s-}^h dY_s = \int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1]\mu(dy,ds)$$

$$[Y, Z^h]_t = \sum_{0 \le s \le t} \Delta Y_s \Delta Z_s^h$$

$$= \int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1][\lambda^h(s,y) - 1]\mu(dy,ds).$$

Thus

$$U_t^h = 1 + \int_0^t \int_E Y_{s-} dZ_s^h + \int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1]\lambda^h(s,y)\mu(dy,ds).$$

Hence

$$
\begin{aligned}
E[U_t^h] &= 1 + E\Big[\int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1]\lambda^h(s,y)\mu(dy,ds)\Big] \\
&= 1 + E\Big[\int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1]\lambda^h(s,y)\nu(dy,ds)\Big] \\
&= 1 + E\Big[\int_0^t \int_E U_{s-}^h [\exp\{\phi(s,\theta^h(s,y))\} - 1] g(\theta^h(s,y)) \det D_2\theta^h(s,y) dy ds\Big] \\
&= 1 + \int_0^t \int_{E'} E[U_{s-}^h][\exp(\phi(s,y)) - 1] g(y) dy ds \qquad (2.25)
\end{aligned}
$$

by the Jacobian formula in $\mathbb{R}^d$. Since (2.25) determines $E[U_t^h]$ uniquely, therefore, in particular, $E[U_t^h]$ does not depend on h.

Next, let us consider the perturbed process X_t^h defined by

$$
dX_t^h = X(X_{t-}^h)dt + Y(X_{t-}^h, y)(\mu^h - \nu)(dy, dt) \qquad (2.26)
$$

$$
x_0 = x \in \mathbb{R}^d.
$$

It follows that the law of X_t^h under P^h does not depend on h, so for all differentiable function $f \in L^2(P)$, we have

$$
\begin{aligned}
E[f(x_t)] &= E^h[f(X_t^h)] \\
&= E[f(X_t^h) Z_t^h]. \qquad (2.27)
\end{aligned}
$$

By differentiating (2.27) with respect to h, and putting $h = 0$, we then obtain the integration by parts formula

$$
E[Df(x_t)Dx_t] + E\Big[f(x_t)\int_0^t \int_E \frac{\operatorname{div}(g \cdot v)(s,y)}{g(y)} (\mu - \nu)(dy, ds)\Big] = 0. \qquad (2.28)
$$

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Chapter 3

INTEGRATION BY PARTS FOR POISSON PROCESSES

3.1. Introduction.

As described in Chapter 2, Bismut obtains an integration by parts formula for a diffusion by considering a small perturbation of the trajectories and then compensating for this by using a Girsanov change of measure (see also [1]). A Poisson process is a counting process, and all jumps are of unit size. Consequently, a perturbation of the trajectories of the kind considered by Bismut does not make sense. Instead we consider below a Girsanov change of measure which alters the rate of the Poisson process by a small amount. This is then compensated by considering a time change of the process under the new measure. An identity involving the perturbation parameter ε is obtained, and the integration by parts formula follows by differentiating with respect to ε and putting $\varepsilon = 0$. The case where the function depends only on finitely many jumps is discussed first, and the general case, for a functional of the Poisson process over the time interval $[0, 1]$, is then deduced.

There is a close relation between integration by parts formulae and martingale representation results. It is well known that any uniformly integrable martingale on the sigma fields generated by a Poisson process can be represented as a stochastic integral with respect to the associated martingale. The integrand can be obtained by considering one jump at a time (though the precise form given in equation

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1. A version of this chapter has been submitted for publication. Robert J. Elliott and Allanus H. Tsoi.

(3.2.12) does not appear to be in the literature). What is interesting is that the integration by parts method gives an alternative expression for this integrand, which does involve a derivative of the functional of the process. The equality of these two expressions is verified in the appendix when the functional depends on finitely many jump times. This expression for the integrand is similar to that obtained by Clark [2] for functionals of Brownian motion.

3.2. Preliminaries and Martingale Representation.

DEFINITION 3.2.1. *Let $N = (N_t,\ t \geq 0)$ be a purely discontinuous, adapted, increasing process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, all of whose jumps are equal to 1. Let $T_1, T_2, \ldots, T_n \ldots$ be the jump times of N. Then N is a Poisson process if the random variables $T_1, T_2 - T_1, \ldots, T_n - T_{n-1}, \ldots,$ are exponentially distributed with parameter 1, and are independent of $\mathcal{F}_0, \mathcal{F}_{T_1}, \ldots, \mathcal{F}_{T_{n-1}}, \ldots$ respectively.*

The following is the characterization of a Poisson process due to P. Levy (see for example [3]).

PROPOSITION 3.2.2. *Suppose N is a counting process, as above and $\{\mathcal{F}_t,\ t \geq 0\}$ is its right continuous, complete filtration. Then N is a Poisson process if both $Q_t = N_t - t$ and $Q_t^2 - t$ are $\{\mathcal{F}_t\}$ martingales.*

The following martingale representation result is well known, but the explicit form (3.2.12) does not appear to be in the literature.

Let N be a Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with jump times $T_1, \ldots, T_n, \ldots$. We shall write $T_0 = 0$. Let $G(T_1, \ldots, T_n, \ldots)$ be an integrable function of $T_1, \ldots, T_n, \ldots$. Consider the martingale M defined by:

$$M_t := E[G(T_1, \ldots, T_n, \ldots) \mid \mathcal{F}_t]. \tag{3.2.1}$$

For $n \geq 1$, write

$$\begin{aligned} \ell^{n-1}(T_n) &= M_{T_n} - M_{T_{n-1}} \\ &= E[G \mid \mathcal{F}_{T_n}] - E[G \mid \mathcal{F}_{T_{n-1}}]. \end{aligned} \tag{3.2.2}$$

Then for each $n \geq 1$ and for $t \geq T_{n-1}$, the process:

$$\begin{aligned} M_t^{(n)} &:= E[\ell^{n-1}(T_n) \mid \mathcal{F}_t] \\ &= M_{T_n \wedge t} - M_{T_{n-1} \wedge t} \end{aligned} \tag{3.2.3}$$

is an $(\mathcal{F}_t)$ martingale starting at time T_{n-1}.

THEOREM 3.2.3. *There exists a predictable process $\{g_s,\ s \geq 0\}$ such that the martingale M defined by (3.2.1) has the representation:*

$$M_t = E[G] + \int_0^t g_s dQ_s \tag{3.2.4}$$

where $Q_t = N_t - t$.

Moreover, for $T_{n-1} < s \leq T_n$,

$$\begin{aligned} g_s &= g^{n-1}(s) \\ &= \ell^{n-1}(s) + \int_{]T_{n-1}, s]} \ell^{n-1}(u) e^{s-u} du. \end{aligned} \tag{3.2.5}$$

Proof. Since for each $n \geq 1$, the process $\{M_t^{(n)}\}$ defined by (3.2.3) is a martingale starting at time T_{n-1}, we can use the method given in [3] to obtain the representation:

$$M_t^{(n)} = M_{T_n \wedge t} - M_{T_{n-1} \wedge t}$$

$$= \int_{]T_{n-1},t]} g^{n-1}(s) dq_s^{n-1} \qquad t \geq T_{n-1} \tag{3.2.6}$$

where

$$q_t^{n-1} = I_{t \geq T_n} - ((t \wedge T_n) - T_{n-1}) \tag{3.2.7}$$

and

$$g^{n-1}(s) = \ell^{n-1}(s) + \int_{]T_{n-1},s]} \ell^{n-1}(u) e^{s-u} du. \tag{3.2.8}$$

Now for $t \geq T_n$, (3.2.6) and (3.2.7) give:

$$M_{T_n \wedge t} - M_{T_{n-1} \wedge t} = \int_{]T_{n-1},t]} g^{n-1}(s) dI_{s \geq T_n} - \int_{]T_{n-1},t]} g^{n-1}(s) d(s \wedge T_n - T_{n-1})$$
$$= g^{n-1}(T_n) - \int_{T_{n-1}}^{T_n} g^{n-1}(s) ds.$$

Certainly, $M_t - M_0 = \sum_{1 \leq n < \infty} (M_{T_n \wedge t} - M_{T_{n-1} \wedge t})$. Hence for $T_{n-1} \leq t < T_n$,

$$M_t - M_0 = \sum_{1 \leq i \leq n-1} (M_{T_i} - M_{T_{i-1}}) + (M_t - M_{T_{n-1}})$$
$$= \sum_{1 \leq i \leq n-1} \int_{]T_{i-1},T_i]} g^{i-1}(s) dq_s^{i-1} + \int_{]T_{n-1},t]} g^{n-1}(s) dq_s^{n-1}$$
$$= \sum_{i=1}^{n-1} g^{i-1}(T_i) - \Big(\sum_{i=1}^{n-1} \int_{T_{i-1}}^{T_i} g^{i-1}(s) ds + \int_{T_{n-1}}^{t} g^{n-1}(s) ds \Big).$$

Letting $g_s = g^{n-1}(s)$ for $T_{n-1} \le s < T_n$, for $T_{n-1} \le t < T_n$,

$$M_t - M_0 = \int_0^t g_s dN_s - \int_0^t g_s ds$$
$$= \int_0^t g_s dQ_s.$$

□

Remarks 3.2.4. (i) By definition,

$$\ell^{n-1}(s) = E[G \mid T_1, \dots, T_{n-1}, T_n = s] - E[G \mid \mathcal{F}_{T_{n-1}}]. \tag{3.2.9}$$

(ii) The representation for M can also be written as

$$M_t = E[G] + \sum_{n=1}^{\infty} \int_{T_{n-1}\wedge t}^{T_n \wedge t} g^{n-1}(s) dQ_s,$$

so that letting $t \to \infty$, we get:

$$G(T_1, \dots, T_n, \dots) = E[G] + \sum_{n=1}^{\infty} \int_{T_{n-1}}^{T_n} g^{n-1}(s) dQ_s. \tag{3.2.10}$$

(iii) If H is a function which depends on a finite number of the jump times $T_1, \dots, T_n$, a similar proof gives:

$$H(T_1, \dots, T_n) = E[H] + \sum_{i=1}^{n} \int_{T_{i-1}}^{T_i} g^{i-1}(s) dQ_s \tag{3.2.11}$$

where the g^i, $i = 0, 1, \dots, n-1$, are given by (3.2.8), and

$$\ell^i(s) = E[H(T_1, \dots, T_n) \mid T_1, \dots, T_{i-1},\ T_i = s] - E[H(T_1, \dots, T_n) \mid \mathcal{F}_{T_{i-1}}].$$

The following gives another expression for the g^n.

PROPOSITION 3.2.5. *The g^n which appear in (3.2.8) can be expressed as:*

$$g^{n-1}(t) = E[G \mid T_1, \ldots, T_{n-1},\ T_n = t] - e^{t-T_{n-1}} E[I_{T_n > t} G \mid \mathcal{F}_{T_{n-1}}]. \qquad (3.2.12)$$

Proof. By (3.2.8),

$$\begin{aligned}
g^{n-1}(t) &= \ell^{n-1}(t) + e^t \int_{]T_{n-1},t]} E[G \mid T_1, \ldots, T_{n-1},\ T_n = u] e^{-u} du \\
&\quad - e^t \int_{]T_{n-1},t]} E[G \mid \mathcal{F}_{T_{n-1}}] e^{-u} du \\
&= \ell^{n-1}(t) + e^{t-T_{n-1}} \int_{]T_{n-1},t]} E[G \mid T_1, \ldots, T_{n-1},\ T_n = u] e^{-(u-T_{n-1})} du \\
&\quad - E[G \mid \mathcal{F}_{T_{n-1}}] e^t (e^{-T_{n-1}} - e^{-t}) \\
&= E[G \mid T_1, \ldots, T_{n-1},\ T_n = t] \\
&\quad + e^{t-T_{n-1}} \int_{]T_{n-1},t]} E[G \mid T_1, \ldots T_{n-1},\ T_n = u] e^{-(u-T_{n-1})} du \\
&\quad - e^{t-T_{n-1}} E[G \mid \mathcal{F}_{T_{n-1}}]. \qquad (3.2.13)
\end{aligned}$$

But

$$\begin{aligned}
E[G \mid \mathcal{F}_{T_{n-1}}] &= E[E[G \mid \mathcal{F}_{T_n}] \mid \mathcal{F}_{T_{n-1}}] \\
&= E[(I_{T_n > t} + I_{T_n \le t}) E[G \mid \mathcal{F}_{T_n}] \mid \mathcal{F}_{T_{n-1}}] \\
&= E[I_{T_n > t} E[G \mid \mathcal{F}_{T_n}] \mid \mathcal{F}_{T_{n-1}}] \\
&\qquad + \int_{]T_{n-1},t]} E[G \mid T_1, \ldots, T_{n-1},\ T_n = u] e^{-(u-T_{n-1})} du. \qquad (3.2.14)
\end{aligned}$$

Hence from (3.2.13) and (3.2.14), we get (3.2.12). □

Remarks 3.2.6. (i) For a function $H(T_1, \dots, T_n)$ the g^i which appear in (3.2.11) can be expressed as:

$$g^{i-1}(t) = E[H(T_1, \dots, T_n) \mid T_1, \dots, T_{i-1},\ T_i = t]$$

$$- \, e^{t-T_{i-1}} E[I_{T_i > t} H(T_1, \dots, T_n) \mid \mathcal{F}_{T_{i-1}}]. \qquad (3.2.15)$$

(ii) The integrands in the representations of functionals of the form $G(T_1 \wedge 1, T_2 \wedge 1, \dots)$ are given by the same expression (3.2.12).

3.3. Time Change of a Poisson Process.

As in Section 3.2, let N be a Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Throughout the rest of this chapter we let $\{u_t,\ t \geq 0\}$ be a real predictable process satisfying:

(i) $\{u_t,\ t \geq 0\}$ is positive and a.s. bounded, $|u_t| \leq B$ a.s. say.

(ii) There exists a bounded interval, say, $[0, b]$, such that $u_s(w) = 0$ if $s \notin [0, b]$, a.s.

For $\varepsilon > 0$, consider the martingale:

$$X_t := \int_0^t \varepsilon u_s dQ_s$$

$$= \sum_{0 \leq s \leq t} \varepsilon u_s \Delta N_s - \int_0^t \varepsilon u_s ds. \qquad (3.3.1)$$

Define the family of exponentials

$$\Lambda_t^\varepsilon := \exp(X_t - \frac{1}{2}\langle X^c, X^c \rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

$$= \prod_{0 \leq s \leq t} (1 + \varepsilon u_s \Delta N_s) \exp\Big(- \int_0^t \varepsilon u_s ds\Big). \qquad (3.3.2)$$

Then $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ satisfies the equation:

$$\Lambda_t^\varepsilon = 1 + \int_0^t \Lambda_{s-}^\varepsilon dX_s$$
$$= 1 + \int_0^t \Lambda_{s-}^\varepsilon \varepsilon u_s dQ_s \qquad (3.3.3)$$

and $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is a martingale.

LEMMA 3.3.1. *$\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is a uniformly integrable martingale. Hence $\Lambda_\infty^\varepsilon$ exists and a new probability measure P^ε can be defined by*

$$\frac{dP^\varepsilon}{dP} = \Lambda_\infty^\varepsilon.$$

Proof. It suffices to show that the martingale $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is square integrable.

Recall u vanishes outside the interval $[0, b]$ and $|u_s| < B$ a.s. By (3.3.3) and Itô's rule,

$$(\Lambda_t^\varepsilon)^2 = 1 + 2\int_0^t \Lambda_{s-}^\varepsilon d\Lambda_s^\varepsilon + \sum_{0\leq s\leq t} (\Lambda_{s-}^\varepsilon \varepsilon u_s \Delta N_s)^2$$
$$= 1 + 2\int_0^t \Lambda_{s-}^\varepsilon d\Lambda_s^\varepsilon + \int_0^t (\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2 dQ_s + \int_0^t (\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2 ds.$$

For $0 \leq t \leq b$,

$$E[(\Lambda_t^\varepsilon)^2] = 1 + \int_0^t E[(\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2] ds$$
$$\leq 1 + \varepsilon^2 B^2 \int_0^t E[(\Lambda_s^\varepsilon)^2] ds.$$

So by Gronwall's inequality,

$$E[(\Lambda_t^\varepsilon)^2] \leq \exp(\varepsilon^2 B^2 t)$$
$$\leq \exp(\varepsilon^2 B^2 b) \qquad 0 \leq t \leq b.$$

And $\Lambda_t^\varepsilon = \Lambda_b^\varepsilon$ for $t > b$. Hence the martingale $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is square integrable.

$\Lambda_\infty^\varepsilon > 0$ a.s. and $E[\Lambda_\infty^\varepsilon] = 1$ so we can define a new probability measure P^ε by putting

$$\frac{dP^\varepsilon}{dP} = \Lambda_\infty^\varepsilon. \tag{3.3.4}$$

Then the process $\{Q_t^\varepsilon\}$ defined by

$$Q_t^\varepsilon := N_t - \int_0^t (1 + \varepsilon u_s) ds \tag{3.3.5}$$

is an $(\mathcal{F}_t)$ martingale under P^ε (see [4]).

Now define

$$\phi_\varepsilon(t) := \int_0^t (1 + \varepsilon u_s) ds. \tag{3.3.6}$$

Let $\psi_\varepsilon(t) = \phi_\varepsilon^{-1}(t)$. Then $\psi_\varepsilon(\phi_\varepsilon(t)) = t$ so

$$\psi_\varepsilon(t) = \int_0^t \frac{1}{1 + \varepsilon u_{\psi_\varepsilon(s)}}\, ds. \tag{3.3.7}$$

LEMMA 3.3.2. *Let $\mathcal{F}_t^\varepsilon = \mathcal{F}_{\psi_\varepsilon(t)}$. Then the process $\{N_t^\varepsilon,\ t \geq 0\}$ defined by:*

$$N_t^\varepsilon := N_{\psi_\varepsilon(t)} \tag{3.3.8}$$

is a Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t^\varepsilon), P^\varepsilon)$.

Proof. Since $Q_t^\varepsilon = N_t - \phi_\varepsilon(t)$ is an $(\mathcal{F}_t)$ martingale under P^ε, so $N_{\psi_\varepsilon(t)} - t$ is an $(\mathcal{F}_t^\varepsilon)$ martingale under P^ε. Let $Y_t^\varepsilon = N_{\psi_\varepsilon(t)} - t$. By Itô's rule,

$$\begin{aligned}
(Y_t^\varepsilon)^2 &= 2\int_0^t Y_{s-}^\varepsilon dY_s^\varepsilon + [Y^\varepsilon, Y^\varepsilon]_t \\
&= 2\int_0^t Y_{s-}^\varepsilon dY_s^\varepsilon + \sum_{s \leq t} (\Delta N_{\psi_\varepsilon(s)})^2 \\
&= 2\int_0^t Y_{s-}^\varepsilon dY_s^\varepsilon + N_{\psi_\varepsilon(t)}.
\end{aligned}$$

Hence $(Y_t^\varepsilon)^2 - t$ is also an $(\mathcal{F}_t^\varepsilon)$ martingale under P^ε. Therefore, using Levy's characterization $\{N_t^\varepsilon,\ t \geq 0\}$ is Poisson.

3.4. Integration by Parts.

Suppose G is a function of the first n jump times $T_1, \ldots, T_n$ of a Poisson process N. Since $\phi_\varepsilon(t) = \psi_\varepsilon^{-1}(t)$, if T_i is the i-th jump time of $\{N_t\}$, then $\phi_\varepsilon(T_i)$ is the i-th jump time of the process $\{N_{\psi_\varepsilon(t)}\}$. Changing the rate of the point process by a Girsanov transformation, and then changing the time scale of the process, we have the following result:

THEOREM 3.4.1. *Let $G(T_1, \ldots, T_n)$ be bounded with bounded first partial derivatives. Then*

$$E\Big[\Big(\int_0^\infty u_s dQ_s\Big) G(T_1, \ldots, T_n)\Big] = -E\Big[\sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big]. \quad (3.4.1)$$

Proof. By the results in Section 3.3, because $N_{\psi_\varepsilon(t)}$ is a Poisson process under P^ε with jump times $\phi_\varepsilon(T_i)$; consequently

$$E[G(T_1, \ldots, T_n)] = E^\varepsilon[G(\phi_\varepsilon(T_1), \ldots, \phi_\varepsilon(T_n))]$$

$$= E[\Lambda_\infty^\varepsilon G(\phi_\varepsilon(T_1), \ldots, \phi_\varepsilon(T_n))] \quad (3.4.2)$$

where $E^\varepsilon[\ ]$ denotes that expectation is taken with respect to P^ε. Differentiating (3.4.2) with respect to ε, and then setting $\varepsilon = 0$, we get:

$$E\Big[\Lambda_{\infty|\varepsilon=0}^\varepsilon \frac{d}{d\varepsilon} G(\phi_\varepsilon(T_1), \ldots, \phi_\varepsilon(T_n))_{|\varepsilon=0}\Big]$$

$$+ E\Big[\Big(\frac{d}{d\varepsilon} \Lambda_\infty^\varepsilon\Big)_{|\varepsilon=0} G(\phi_\varepsilon(T_1), \ldots, \phi_\varepsilon(T_n))_{|\varepsilon=0}\Big] = 0. \quad (3.4.3)$$

From (3.3.3) and the definition of $\Lambda^{\varepsilon}_{\infty}$,

$$\frac{d\Lambda^{\varepsilon}_{\infty}}{d\varepsilon}_{|\varepsilon=0} = \int_0^{\infty} u_s dQ_s.$$

Noting the definition of ϕ_{ε}, (3.4.3) becomes (3.4.1) and the proof is complete.

□

Remark 3.4.2. Consider a function H of the form $H(T_1 \wedge 1, \dots, T_n \wedge 1)$ where H is bounded and has bounded first derivatives. Applying Theorem 3.4.1 to $G(T_1, \dots, T_n) = H(T_1 \wedge 1, \dots, T_n \wedge 1)$ and noting that

$$\frac{\partial}{\partial t_i} G(T_1, \dots, T_n) = \frac{\partial}{\partial t_i} H(T_1 \wedge 1, \dots, T_n \wedge 1) I_{T_i \leq 1},$$

we have the following:

Corollary 3.4.3. *If $H(T_1 \wedge 1, \dots, T_n \wedge 1)$ is bounded and has bounded first derivatives, then*

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big) H(T_1 \wedge 1, \dots, T_n \wedge 1)\Big]$$
$$= -E\Big[\sum_{i=1}^{n} \frac{\partial}{\partial t_i} H(T_1 \wedge 1, \dots, T_n \wedge 1) \int_0^{T_i} u_s ds I_{T_i \leq 1}\Big]. \quad (3.4.4)$$

Remark 3.4.4. Recall the martingale representation (3.2.11):

$$G(T_1, \dots, T_n) = E[G] + \sum_{i=1}^{n} \int_{T_{i-1}}^{T_i} g^{i-1}(s) dQ_s \quad (3.4.5)$$

or

$$G(T_1, \dots, T_n) = E[G] + \int_0^{T_n} g_s dQ_s, \quad (3.4.6)$$

where

$$g_s = g^{i-1}(s) \qquad \text{for} \qquad T_{i-1} \leq s < T_i.$$

If we substitute (3.4.6) into the left hand side of (3.4.1), we get

$$\begin{aligned}
&E\Big[\Big(\int_0^\infty u_s dQ_s\Big)\Big(E[G] + \int_0^{T_n} g_s dQ_s\Big)\Big] \\
&= E[G]E\Big[\int_0^\infty u_s dQ_s\Big] + E\Big[\int_0^\infty u_s dQ_s \int_0^{T_n} g_s dQ_s\Big] \\
&= E\Big[\int_0^{T_n} g_s dQ_s E\Big[\int_0^\infty u_s dQ_s \mid \mathcal{F}_{T_n}\Big]\Big] \\
&= E\Big[\int_0^{T_n} g_s dQ_s \int_0^{T_n} u_s dQ_s\Big] \\
&= E\Big[\int_0^{T_n} u_s g_s ds\Big] = E\Big[\int_0^\infty u_s g_s ds\Big] \qquad (3.4.7)
\end{aligned}$$

where $g_s = 0$ for $s \geq T_n$.

Also, if we consider the measure μ defined by

$$\mu(dt) = \sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n)\delta_{T_i}(dt).$$

Then the right hand side of (3.4.1) can be written

$$\begin{aligned}
&-E\Big[\sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big] \\
&= -E\Big[\int_0^\infty \int_0^t u_s ds \mu(dt)\Big] \\
&= -E\Big[\int_0^\infty \int_0^\infty I_{0 \leq s \leq t < \infty} u_s ds \mu(dt)\Big] \\
&= -E\Big[\int_0^\infty \mu[s, \infty) u_s ds\Big] \\
&= -E\Big[\int_0^\infty \sum_{i=1}^n I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n) u_s ds\Big]. \qquad (3.4.8)
\end{aligned}$$

Let $C_s = \sum_{i=1}^{n} I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n)$. Then there exists a predictable projection C^* of C, such that for each s,

$$C_s^* = E[C_s \mid \mathcal{F}_{s-}] \qquad \text{a.s.}$$

Also for any predictable process $\{u_s,\ s \geq 0\}$,

$$\begin{aligned} E[u_s C_s] &= E[u_s E[C_s \mid \mathcal{F}_{s-}]] \\ &= E[u_s C_s^*]. \end{aligned} \tag{3.4.9}$$

Let $\mathcal{H}$ be the family of subsets of $[0,\infty) \times \Omega$ of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $s < t$. Recall that the predictable σ-field is generated by $\mathcal{H}$. Taking $u = I_{\{0\}\times F_0}$ or $u = I_{(s,t]\times F}$, then u satisfies the hypothesis in Section 3.3, so (3.4.7), (3.4.8) and (3.4.9) hold for these u. Also because of (3.4.9), on comparing (3.4.7) and (3.4.8), we have

$$E\Big[\int_0^\infty u_s g_s ds\Big] = -E\Big[\int_0^\infty u_s C_s^* ds\Big]$$

holds for all u which are indicators of sets in $\mathcal{H}$. Since $\mathcal{H}$ generates the predictable σ-field and the processes g and C^* are predictable, therefore we have proved the following result:

PROPOSITION 3.4.5.

$$g_s = -E\Big[\sum_{i=1}^{n} I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \tag{3.4.10}$$

Now if we recall (3.2.11)

$$H(T_1 \wedge 1, \ldots, T_n \wedge 1) = E[H] + \sum_{i=1}^{n} \int_{T_{i-1}\wedge 1}^{T_i \wedge 1} g^{i-1}(s) dQ_s \tag{3.4.11}$$

or

$$H(T_1 \wedge 1, \ldots, T_n \wedge 1) = E[H] + \int_0^{T_n \wedge 1} g_s dQ_s \tag{3.4.12}$$

where $g_s = g^{i-1}(s)$ for $T_{i-1} \wedge 1 \le s < T_i \wedge 1$.

An argument similar to the above shows that

$$g_s = -E\Big[\sum_{i=1}^{n} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i}(T_1 \wedge 1, \ldots, T_n \wedge 1) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \tag{3.4.13}$$

The form of g given in Section 3.2 and that given in (3.4.13) are at first sight rather different. A direct proof of their equality is sketched in the Appendix. Next we have the following integration by parts formula:

THEOREM 3.4.6. *Suppose $G = G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots)$ is a bounded function and its first partial derivatives are all bounded by a constant $K > 0$. Then*

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big) G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots)\Big]$$
$$= -\sum_{i=1}^{\infty} E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) \int_0^{T_i} u_s ds I_{T_i \le 1}\Big]. \tag{3.4.14}$$

Proof. First note that for each $M > 0$, the partial sum

$$\sum_{i=1}^{M} E[I_{T_i \le 1}] = \sum_{i=1}^{M} P(N_1 \ge i) \le 4e^{-1},$$

so that by hypothesis, the right hand side of (3.4.14) is finite. For each $n \ge 1$, define

$$H^n(T_1, \ldots, T_n) := E[G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) \mid \mathcal{F}_{T_n}].$$

Then

$$H^n(T_1, \ldots, T_n) = E[G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots,$$
$$(T_n + S_{n+1} + \cdots + S_{n+i}) \wedge 1, \ldots) \mid \mathcal{F}_{T_n}]$$
$$= E^S[G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots,$$
$$(T_n + S_{n+1} + \cdots + S_{n+i}) \wedge 1, \ldots)], \tag{3.4.15}$$

where $S_k = T_k - T_{k-1}$ for $k \geq 1$, and the last expectation E^S in (3.4.15) is taken only over the random variables $S_{n+1}, \ldots, S_{n+i}, \ldots$, and the $T_1, \ldots, T_n$ are given. From (3.4.1),

$$E\Big[\Big(\int_0^\infty u_s dQ_s\Big) H^n(T_1, \ldots, T_n)\Big] = -\sum_{i=1}^{n-1} E\Big[\frac{\partial H^n}{\partial t_i}(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big] - E\Big[\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) \int_0^{T_n} u_s ds\Big]. \tag{3.4.16}$$

And from (3.4.15),

$$\begin{aligned}\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) &= E^S\Big[\frac{\partial}{\partial t_n} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots)\Big] \\ &= E^S\Big[\sum_{i=n}^{\infty} \frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots) I_{T_i \leq 1}\Big] \\ &= \sum_{i=n}^{\infty} E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big].\end{aligned}$$

Hence

$$\begin{aligned}&E\Big[\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) \int_0^{T_n} u_s ds\Big] \\ &= E\Big[\int_0^{T_n} u_s ds \sum_{i=n}^{\infty} E\Big[\frac{\partial G}{\partial t_i}(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big] \\ &\to 0 \qquad \text{as } n \to \infty\end{aligned} \tag{3.4.17}$$

by the hypotheses on $\{u_s\}$ and G.

Also for $1 \leq i \leq n-1$,

$$\begin{aligned}&E\Big[\frac{\partial H^n}{\partial t_i}(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big] \\ &= E\Big[E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big] \int_0^{T_i} u_s ds\Big] \\ &= E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) \int_0^{T_i} u_s ds I_{T_i \leq 1}\Big].\end{aligned} \tag{3.4.18}$$

Letting $n \to \infty$ in (3.4.16), because of (3.4.17) and (3.4.18), we obtain (3.4.14). □

We conclude with the following theorem:

THEOREM 3.4.7. *Suppose M is the right continuous martingale*

$$M_t = E[G(T_1 \wedge 1, \dots T_n \wedge 1, \dots) \mid \mathcal{F}_t].$$

Then

$$M_t = E[G] + \int_0^t g_s dQ_s,$$

where

$$g_s = -E\Big[\sum_{i=1}^{\infty} I_{s \leq T_i \leq 1} \frac{\partial G}{\partial t_i}(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \qquad (3.4.19)$$

Proof. The argument is similar to that sketched above: Write

$$\mu(dt) = \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) I_{T_i \leq 1} \delta_{T_i}(dt).$$

Then the right hand side of (3.4.14) can be written as:

$$\begin{aligned} & -E\Big[\int_0^\infty \int_0^t u_s ds \mu(dt)\Big] \\ = & -E\Big[\int_0^\infty \int_0^\infty I_{0 \leq s \leq t < \infty} u_s ds \mu(dt)\Big] \\ = & -E\Big[\int_0^\infty \mu[s, \infty) u_s ds\Big] \\ = & -E\Big[\int_0^\infty \sum_{i=1}^{\infty} I_{s \leq T_i \leq 1} \frac{\partial G}{\partial t_i}(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) u_s ds\Big]. \end{aligned} \qquad (3.4.20)$$

Recall the representation (3.2.10)

$$\begin{aligned} G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) &= E[G] + \sum_{i=1}^{\infty} \int_{T_{i-1} \wedge 1}^{T_i \wedge 1} g^{i-1}(s) dQ_s \\ &= E[G] + \int_0^\infty g_s dQ_s, \end{aligned} \qquad (3.4.21)$$

where

$$g_s = g^{i-1}(s) \qquad \text{if} \quad T_{i-1} \wedge 1 \le s < T_i \wedge 1$$

$$= 0 \qquad \text{if} \quad s \ge 1.$$

Substituting (3.4.21) into the left hand side of (3.4.14) gives

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big)\Big(E[G] + \int_0^\infty g_s dQ_s\Big)\Big] = E\Big[\int_0^\infty u_s g_s ds\Big]. \tag{3.4.22}$$

Let

$$B_s = \sum_{i=1}^{\infty} I_{s \le T_i \le 1} \frac{\partial G}{\partial t_i}(T_1 \wedge 1, \dots, T_n \wedge 1, \dots).$$

Then there is a predictable projection B^* of B such that for each s,

$$B_s^* = E[B_s \mid \mathcal{F}_{s-}] \qquad \text{a.s.}$$

Therefore, for any predictable process $\{u_s,\ s \ge 0\}$,

$$E[u_s B_s] = E[u_s B_s^*].$$

An argument similar to that leading to (3.4.10) then shows that

$$g_s = -E\Big[\sum_{i=1}^{\infty} I_{s \le T_i \le 1} \frac{\partial G}{\partial t_i}(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.}$$

□

3.5. Appendix.

We now give a direct proof that the integrands g obtained in sections 3.2 and 3.4 are equal.

First recall that $g_s = g^{i-1}(s)$ for $T_{i-1} \wedge 1 \le s < T_i \wedge 1$, and that $g^{i-1}(s)$ can be written

$$g^{i-1}(s) = E[H(T_1 \wedge 1, \dots T_n \wedge 1) I_{s<1} \mid T_1, \dots, T_{i-1},\ T_i = s]$$
$$- e^{s-T_{i-1}} E[I_{s<1} I_{T_i > s} H(T_1 \wedge 1, \dots, T_n \wedge 1) \mid T_1, \dots, T_{i-1}]. \tag{3.5.1}$$

Write

$$\begin{aligned}
&E\Big[\sum_{i=1}^{n} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big] \\
&= E\Big[I_{s \le T_1 \le 1} \frac{\partial H}{\partial t_1} \mid \mathcal{F}_{s-}\Big] + \Big\{E\Big[I_{s \le T_1} I_{s \le T_2 \le 1} \frac{\partial H}{\partial t_2} \mid \mathcal{F}_{s-}\Big] \\
&\quad + E\Big[I_{T_1 < s \le T_2} I_{s \le T_2 \le 1} \frac{\partial H}{\partial t_2} \mid \mathcal{F}_{s-}\Big]\Big\} + \dots \\
&\quad + \Big\{E\Big[I_{s \le T_1} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big] + E\Big[I_{T_1 < s \le T_2} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big] \\
&\quad + \dots + E\Big[I_{T_{i-1} < s \le T_i} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big]\Big\} + \dots \\
&= \Big\{E\Big[I_{s \le T_1 \le 1} \frac{\partial H}{\partial t_1} \mid \mathcal{F}_{s-}\Big] + E\Big[I_{s \le T_1} I_{s \le T_2 \le 1} \frac{\partial H}{\partial t_2} \mid \mathcal{F}_{s-}\Big] + \dots \\
&\quad + E\Big[I_{s \le T_1} I_{s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big] + \dots\Big\} + \dots \\
&\quad + \Big\{E\Big[I_{T_{i-1} < s \le T_i \le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\Big] + E\Big[I_{T_{i-1} < s \le T_i} I_{s \le T_{i+1} \le 1} \frac{\partial H}{\partial t_{i+1}} \mid \mathcal{F}_{s-}\Big] \\
&\quad + \dots + E\Big[I_{T_{i-1} < s \le T_i} I_{s \le T_n \le 1} \frac{\partial H}{\partial t_n} \mid \mathcal{F}_{s-}\Big]\Big\} + \dots
\end{aligned} \tag{3.5.2}$$

It suffices to show that the general collection of terms in (3.5.2):

$$E\Big[I_{T_{i-1}<s\le T_i\le 1}\frac{\partial H}{\partial t_i}\mid \mathcal{F}_{s-}\Big]+E\Big[I_{T_{i-1}<s\le T_i}I_{s\le T_{i+1}\le 1}\frac{\partial H}{\partial t_{i+1}}\mid \mathcal{F}_{s-}\Big]$$

$$+\cdots+E\Big[I_{T_{i-1}<s\le T_i}I_{s\le T_n\le 1}\frac{\partial H}{\partial t_n}\mid \mathcal{F}_{s-}\Big]$$

is equal to $g^{i-1}(s)$. To do this, for $i+1\le j\le n-1$, we further write

$$E\Big[I_{T_{i-1}<s\le T_i}I_{s\le T_j\le 1}\frac{\partial H}{\partial t_j}\mid \mathcal{F}_{s-}\Big]$$

$$=E\Big[I_{T_{i-1}<s\le T_i}I_{s\le T_j\le 1}\frac{\partial H}{\partial t_j}\Big(I_{T_{j+1}>1}+I_{T_{j+1}\le 1}I_{T_{j+2}>1}+\dots$$

$$+I_{T_{j+m}\le 1}I_{T_{j+m+1}>1}+\cdots+I_{T_n\le 1}\Big)\mid \mathcal{F}_{s-}\Big].$$

For $g^{i-1}(s)$ in (3.5.1), we decompose:

$$E[I_{s<1}H\mid T_1,\dots,T_{i-1},\ T_i=s]$$

$$=E[I_{s<1}H(I_{T_{i+1}>1}+I_{T_{i+1}\le 1}I_{T_{i+2}>1}+\cdots+I_{T_{i+m}\le 1}I_{T_{i+m+1}>1}+\dots$$

$$+I_{T_n\le 1})\mid T_1,\dots,T_{i-1},\ T_i=s].$$

$$e^{s-T_{i-1}}I_{s<1}E[I_{T_i>s}H\mid T_1,\dots,T_{i-1}]$$

$$=e^{s-T_{i-1}}I_{s<1}E[(I_{s<T_i\le 1}+I_{T_i>1})H\mid T_1,\dots,T_{i-1}].$$

$$e^{s-T_{i-1}}I_{s<1}E[I_{s<T_i\le 1}H\mid T_1,\dots T_{i-1}]$$

$$=e^{s-T_{i-1}}I_{s<1}E[I_{s<T_i\le 1}(I_{T_{i+1}>1}+I_{T_{i+1}\le 1}I_{T_{i+2}>1}$$

$$+\cdots+I_{T_{j-1}\le 1}I_{T_j>1}+\dots)H\mid T_1,\dots,T_{i-1}].$$

By a direct calculation with the aid of conditional densities and integration by parts, it is straightforward to check that:

(i) For $i+1 \le j \le n-1,\ 1 \le m \le n-j-1$,

$$E\left[I_{T_{i-1}<s\le T_i} I_{s\le T_j\le 1} I_{T_{j+m}\le 1} I_{T_{j+m+1}>1} \frac{\partial H}{\partial t_j} \mid \mathcal{F}_{s-}\right] = 0$$

$$E\left[I_{T_{i-1}<s\le T_i} I_{s\le T_j\le 1} I_{T_n\le 1} \frac{\partial H}{\partial t_j} \mid \mathcal{F}_{s-}\right] = 0.$$

(ii) $E\left[I_{T_{i-1}<s\le T_i} I_{s\le T_j\le 1} I_{T_{j+1}>1} \frac{\partial H}{\partial t_j} \mid \mathcal{F}_{s-}\right]$ corresponds to

$$e^{s-T_{i-1}} E[I_{s<T_i\le 1} I_{T_{j-1}\le 1} I_{T_j>1} H \mid T_1, \ldots, T_{i-1}].$$

(iii) $E[I_{T_{i-1}<s\le T_i} I_{T_n\le 1} \frac{\partial H}{\partial t_n} \mid \mathcal{F}_{s-}]$ corresponds to the terms

$$e^{s-T_{i-1}} I_{s<1} E[I_{s<T_i\le 1} I_{T_n\le 1} H \mid T_1, \ldots, T_{i-1}] \qquad \text{and}$$

$$e^{s-T_{i-1}} I_{s<1} E[I_{s<T_i\le 1} I_{T_{n-1}\le 1} I_{T_n>1} H \mid T_1, \ldots, T_{i-1}].$$

(iv) For $0 < m \le n-i-1$,

$$E\left[I_{T_{i-1}<s\le T_i\le 1} I_{T_{i+m}\le 1} I_{T_{i+m+1}>1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\right]$$

corresponds to $E[I_{s<1} H I_{T_{i+m}\le 1} I_{T_{i+m+1}>1} \mid T_1, \ldots, T_{i-1},\ T_i = s]$.

(v) $E\left[I_{T_{i-1}<s\le T_i\le 1} \frac{\partial H}{\partial t_i} I_{T_{i+1}\ge 1} \mid \mathcal{F}_{s-}\right]$ corresponds to

$$e^{s-T_i-1} I_{s<1} E[I_{T_i\ge 1} H \mid T_1, \ldots, T_{i-1}] \quad \text{and}$$

$$E[I_{s<1} H I_{T_{i+1}>1} \mid T_1, \ldots, T_{i-1},\ T_i = s].$$

(vi) $E\left[I_{T_{i-1}<s\le T_i\le 1} I_{T_n\le 1} \frac{\partial H}{\partial t_i} \mid \mathcal{F}_{s-}\right]$ corresponds to

$$I_{s<1} E[H I_{T_n\le 1} \mid T_1, \ldots, T_{i-1},\ T_i = s].$$

The proof is, therefore, complete. □

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Chapter 4

INTEGRATION BY PARTS FOR THE SINGLE JUMP PROCESS

4.1. Introduction.

In this chapter the concept of integration by parts is investigated in the fundamental situation of a stochastic process with a single random jump. When the state space of the process is Euclidean space (or, possibly, an open, non-empty subset of a Euclidean space) the techniques of Norris [2] can be specialized to the single jump situation. This method, described in Section 4.2, considers a small ε-perturbation in the state space of the process. The effect of the perturbation can be removed by a Girsanov change of measure, and the integration by parts formula is obtained by differentiating in ε.

However, for a process whose state space is a general measure space, the perturbation of the kind considered by Norris may not make sense. Such processes include those with discrete state spaces, and, in particular, the process which observes a single random instant at a time T. In the latter case the process $p_t = I_{t \geq T}$ takes only the values 0 or 1.

For general jump processes, therefore, an alternative ε-perturbation in the time direction is introduced. By differentiating a new integration by parts formula, which involves a time derivative, is obtained. In the case of the fundamental process p_t

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an alternative expression for the integrand in a martingale representation result is derived.

4.2. Integration by Parts for $\mathbb{R}^d$-Valued Single Jump Processes.

Consider a single jump process with state space $\mathbb{R}^d$ for some $d \geq 1$, which remains at its initial position z_0 until a random time T, when it jumps to a new random position Z. The underlying probability space is taken as $\big([0,\infty] \times \mathbb{R}^d,\ \mathcal{B}([0,\infty]) \times \mathcal{B}(\mathbb{R}^d), \mu\big)$. For $t \geq 0$, let $\mathcal{F}_t$ be the completed σ-field generated by the process up to time t. Suppose (λ, Λ) is the Lévy system for the process (see Elliott [1]). For $A \in \mathcal{B}(\mathbb{R}^d)$, let

$$p(t,A) = I_{t \geq T} I_{Z \in A} \tag{4.2.1}$$

$$\tilde{p}(t,A) = -\int_{]0,t\wedge T]} \lambda(s,A) \frac{dF_s}{F_{s-}} \tag{4.2.2}$$

where $F_t = \mu(]t,\infty] \times \mathbb{R}^d)$. Then $q(t,A) = p(t,A) - \tilde{p}(t,A)$ is an $\mathcal{F}_t$-martingale.

We assume that F_t and λ are absolutely continuous, so that there exist functions f_s and $g(y) > 0$ such that

$$dF_s = f_s ds$$

$$\lambda(s,dy) = g(y)dy.$$

Consequently,

$$\tilde{p}(ds,dy) = \begin{cases} -g(y)\ \frac{f_s}{F_s}\ dyds & \text{if } s \leq T \\ 0 & \text{if } s > T. \end{cases} \tag{4.2.3}$$

Let $v(t,y)$ be an $\mathbb{R}^d$-valued function which satisfies:

(i) $v(t,\cdot)$ is C^1 for each $t \geq 0$; v and $\frac{\partial}{\partial y}\, v(t,y)$ are uniformly bounded.

(ii) supp $v(\cdot,\cdot) \subseteq [0,\infty) \times K$ for some compact $K \subseteq \mathbb{R}^d$.

For small $\varepsilon \in \mathbb{R}$ and $\phi \in L^1(\mu)$, define p^ε by:

$$\int_0^t \int_E \phi(s,y) p^\varepsilon(ds,dy) = \int_0^t \int_E \phi(s,\theta^\varepsilon(s,y)) p(ds,dy), \tag{4.2.4}$$

where

$$\theta^\varepsilon(t,y) = y + \varepsilon v(t,y).$$

Set

$$\lambda^\varepsilon(t,y) = \frac{\partial \theta^\varepsilon(t,y)}{\partial y} \, \frac{g(\theta^\varepsilon(t,y))}{g(y)} \tag{4.2.5}$$

and

$$X_t = \int_0^t \int_E (\lambda^\varepsilon(s,y) - 1) q(ds,dy). \tag{4.2.6}$$

Define the family $\{Z_t^\varepsilon,\ t \geq 0\}$ of exponentials by:

$$\begin{aligned} Z_t^\varepsilon &= \exp(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \\ &= \exp\Big(\int_0^t \int_E \log \lambda^\varepsilon(s,y) dp - \int_0^t \int_E (\lambda^\varepsilon(s,y) - 1) d\tilde{p}\Big). \end{aligned} \tag{4.2.7}$$

Then Z_t^ε satisfies:

$$Z_t^\varepsilon = \int_0^t \int_E Z_{s-}^\varepsilon (\lambda^\varepsilon(s,y) - 1) q(ds,dy) \tag{4.2.8}$$

and $\{Z_t^\varepsilon,\ t \geq 0\}$ is a martingale with $E[Z_t^\varepsilon] = 1$.

Define a new probability measure μ^ε by:

$$\frac{d\mu^\varepsilon}{d\mu} = Z_t^\varepsilon \quad \text{on} \quad \mathcal{F}_t.$$

LEMMA 4.2.1. *Under μ^ε, p^ε has the original law of p.*

Proof. It suffices to check for test functions $\phi \in L^1(\mu)$ and for

$$U_t^\varepsilon = \exp\left\{\int_0^t \int_E \phi(s,y)p^\varepsilon(ds,dy)\right\}Z_t^\varepsilon$$
$$= \exp\left\{\int_0^t \int_E \phi(s,\theta^\varepsilon(s,y))p(ds,dy)\right\}Z_t^\varepsilon$$

that $E[U_t^\varepsilon]$ does not depend on ε. Let

$$Y_t = \exp\left\{\int_0^t \int_E \phi(s,\theta^\varepsilon(s,y))p(ds,dy)\right\}.$$

By the differentiation rule,

$$U_t^\varepsilon = 1 + \int_0^t \int_E Y_{s-}dZ_s^\varepsilon + \int_0^t \int_E Z_{s-}^\varepsilon dY_s + [Y, Z^\varepsilon]_t.$$

But

$$\int_0^t \int_E Z_{s-}^\varepsilon dY_s = \int_0^t \int_E U_{s-}^\varepsilon[\exp(\phi(s,\theta^\varepsilon(s,y))) - 1]p(ds,dy)$$

$$\Delta Y_s = Y_{T-}[\exp\{\phi(T,\theta^\varepsilon(T,Z))\} - 1]I_{s=T}$$

$$\Delta Z_s^\varepsilon = Z_{T-}^\varepsilon[\lambda^\varepsilon(T,Z) - 1]I_{s=T}.$$

Hence,

$$[Y, Z^\varepsilon]_t = \Delta Y_T \Delta Z_T^\varepsilon I_{t\geq T}$$

$$= U_{T-}^\varepsilon[\exp\{\phi(T,\theta^\varepsilon(T,Z))\} - 1][\lambda^\varepsilon(T,Z) - 1]I_{t\geq T}$$

$$= \int_0^t \int_E U_{s-}^\varepsilon[\exp\{\phi(s,\theta^\varepsilon(s,y))\} - 1][\lambda^\varepsilon(s,y) - 1]p(ds,dy).$$

Hence,

$$U_t^\varepsilon = 1 + \text{Martingale} + \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1]\lambda^\varepsilon(s,y)p(ds,dy)$$

$$= 1 + \text{Martingale} + \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1]\lambda^\varepsilon(s,y)\tilde{p}(ds,dy)$$

$$= 1 + \text{Martingale} - \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1]\lambda^\varepsilon(s,y)g(y)\ \frac{f_s}{F_s}\ dyds.$$

Thus

$$E[U_t^\varepsilon] = 1 - \int_0^t \int_E E[U_s^\varepsilon][\exp\{\phi(s,\theta^\varepsilon(s,y))\} - 1]g(\theta^\varepsilon(s,y))\ \frac{\partial\theta^\varepsilon(s,y)}{\partial y}\ \frac{f_s}{F_s}\ dyds$$

$$= 1 - \int_0^t \int_E E[U_s^\varepsilon][\exp\{\phi(s,y)\} - 1]g(y)\ \frac{f_s}{F_s}\ dyds$$

by the Jacobian formula. Thus $E[U_t^\varepsilon]$ is independent of ε. □

As a consequence of Lemma 4.2.1, we have

$$E[Z_T^\varepsilon \exp\{\phi(T, Z + \varepsilon V(T,Z))\}] = E[\exp\{\phi(T,Z)\}] \tag{4.2.9}$$

which leads us to the following theorem:

THEOREM 4.2.2. *Suppose $G : [0,\infty] \times \mathbb{R}^d \to \mathbb{R}$ is positive, bounded and that its partial derivative $\frac{\partial G(t,z)}{\partial z}$ exists and is bounded. Then*

$$E\Big[\Big(\int_0^T \int_E \Big(\frac{\partial}{\partial y}\ V(t,y) + \frac{g'(y)}{g(y)}\ V(t,y)\Big) q(ds,dy)\Big) G(T,Z)\Big] = -E\Big[\frac{\partial G(T,Z)}{\partial z}\ V(T,Z)\Big]. \tag{4.2.10}$$

Proof. Differentiate (4.2.9) with respect to ε, then set $\varepsilon = 0$ to obtain

$$E\Big[\frac{d}{d\varepsilon} Z_T^\varepsilon\Big|_{\varepsilon=0} \exp\{\phi(T, Z + \varepsilon V(T,Z))\}\Big|_{\varepsilon=0}\Big]$$

$$+ E\Big[Z_T^\varepsilon\Big|_{\varepsilon=0}\ \frac{d}{d\varepsilon}\ \exp\{\phi(T, Z + \varepsilon V(T,Z))\}\Big|_{\varepsilon=0}\Big] = 0. \tag{4.2.11}$$

From (4.2.8),

$$\frac{d}{d\varepsilon}\, Z_T^\varepsilon = \int_0^T \int_E \frac{dZ_{t-}^\varepsilon}{d\varepsilon}(\lambda^\varepsilon(t,y)-1)q(dt,dy) + \int_0^T \int_E Z_{t-}^\varepsilon\, \frac{d}{d\varepsilon}\, \lambda^\varepsilon(t,y)q(dt,dy).$$

From the definition of $\lambda^\varepsilon(t,y)$,

$$\lambda^\varepsilon(t,y)\Big|_{\varepsilon=0} = 1$$

and from (4.2.7),

$$Z_{t-}^\varepsilon\Big|_{\varepsilon=0} = 1.$$

Also,

$$\frac{d\lambda^\varepsilon(t,y)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y).$$

Hence

$$\frac{d}{d\varepsilon}\, Z_T^\varepsilon\Big|_{\varepsilon=0} = \int_0^T \int_E \Big(\frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y)\Big) q(dt,dy).$$

Thus (4.2.11) becomes

$$E\Big[\Big(\int_0^T \int_E \Big(\frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y)\Big) q(dt,dy)\Big) \exp(\phi(T,Z))\Big] = -E\Big[\exp\{\phi(T,Z)\}\Big(\frac{\partial}{\partial z}\phi(T,Z)\Big)V(T,Z)\Big]. \tag{4.2.12}$$

Let $\phi(T,Z) = \log G(T,Z)$. Then (4.2.12) becomes (4.2.10) and the proof is complete. □

4.3. Integration by Parts for a General Jump Process.

Consider a single jump process with values in a Lusin space $(E, \mathcal{E})$. The underlying probability space is $([0,\infty] \times E,\ \mathcal{B}([0,\infty]) \times \mathcal{E},\ \mu)$. In this section we suppose that for every $t \geq 0$, $F_t > 0$, and both F_t and Λ_t are continuous in t. Furthermore, we assume that there exists a function $\alpha(s)$, with $\alpha(s) > 0$ for all $s \geq 0$, such that

$$\Lambda_t = \int_0^t \alpha(s) ds.$$

Let $u : [0,\infty] \times E \to \mathbb{R}$ be a bounded, positive, deterministic function such that

$$u_s(y) = 0 \quad \text{if} \quad s \notin [0, b]$$

for some fixed $b \in \mathbb{R}$. For $\varepsilon > 0$, define

$$\Lambda_t^\varepsilon = \int_0^t \int_E (1 + \varepsilon u_s(y)) \lambda(s, dy) d\Lambda_s. \tag{4.3.1}$$

Consider the new measure μ^ε which has a Lévy system $(\lambda, \Lambda^\varepsilon)$. Then (see Elliott [1]) $\mu^\varepsilon \ll \mu$, and if

$$L^\varepsilon = \frac{d\mu^\varepsilon}{d\mu},$$

we have

$$L^\varepsilon(t) = \int_E (1 + \varepsilon u_t(y)) \lambda(t, dy) \exp\Big\{ - \int_0^t \int_E \varepsilon u_s(y)) \lambda(s, dy) d\Lambda_s \Big\}. \tag{4.3.2}$$

Furthermore, if $L_t^\varepsilon = E[L^\varepsilon(t) \mid \mathcal{F}_t]$ then $\{L_t^\varepsilon,\ t \geq 0\}$ satisfies

$$\begin{aligned} L_t^\varepsilon &= 1 + \int_0^t L_{s-}^\varepsilon dM_s \\ &= 1 + \int_0^t L_{s-}^\varepsilon \int_E \varepsilon u_s(y) \lambda(s, dy) q(ds, E), \end{aligned} \tag{4.3.3}$$

where

$$M_t = \int_0^t \int_E \varepsilon u_s(y)\lambda(s,dy)q(ds,E).$$

If $F_t^\varepsilon = \mu^\varepsilon(]t,\infty] \times E)$, then

$$F_t^\varepsilon = F_t \exp\Big\{ - \int_0^t \int_E \varepsilon u_s(y))\lambda(s,dy)d\Lambda_s\Big\}. \tag{4.3.4}$$

Define

$$\psi_\varepsilon(t) = \sup\{s : F_s^\varepsilon \geq F_t\}.$$

Then $\psi_\varepsilon(t)$ is an increasing function of t, and $F^\varepsilon_{\psi_\varepsilon(t)} = F_t$, i.e.,

$$\mu^\varepsilon(]\psi_\varepsilon(t),\infty] \times E) = \mu(]t,\infty] \times E).$$

Hence if we let $\phi_\varepsilon(t) = \psi_\varepsilon^{-1}(t)$, then under μ^ε, $\phi_\varepsilon(T)$ has the same distribution as T under μ. This observation leads us to the following theorem:

THEOREM 4.3.1. *Let $G(t,z)$ be a real-valued function defined on $[0,\infty] \times E$, which is bounded and has bounded partial derivative $\frac{\partial}{\partial t}G(t,z)$. Then*

$$E\Big[\Big(\int_0^T \int_E u_t(y)\lambda(t,dy)q(dt,E)\Big)G(T,Z)\Big]$$
$$= -E\Big[\frac{\partial G(T,Z)}{\partial t}\ \frac{1}{\alpha(T)}\int_0^T \int_E u_t(y)\lambda(t,dy)\alpha_t dt\Big]. \tag{4.3.5}$$

Proof. From the above discussion we have

$$E[G(T,Z)] = E^\varepsilon[G(\phi_\varepsilon(T),Z)]$$

$$= E[L_T^\varepsilon G(\phi_\varepsilon(T),Z)] \tag{4.3.6}$$

where E^ε denotes that expectation is taken with respect to μ^ε. Differentiate (4.3.6) with respect to ε, then set $\varepsilon = 0$ to obtain

$$E\left[\frac{dL_T^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} G(\phi_\varepsilon(T), Z)\Big|_{\varepsilon=0}\right] + E\left[L_T^\varepsilon\Big|_{\varepsilon=0} \frac{d}{d\varepsilon}\, G(\phi_\varepsilon(T), Z)\Big|_{\varepsilon=0}\right] = 0. \qquad (4.3.7)$$

From (4.3.2) and (4.3.3),

$$\frac{dL_T^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^T \int_E u_t(y)\lambda(t, dy)q(dt, E). \qquad (4.3.8)$$

Also,

$$\frac{d}{d\varepsilon}\, G(\phi_\varepsilon(T), Z)\Big|_{\varepsilon=0} = \frac{\partial}{\partial t}\, G(T, Z)\, \frac{\partial}{\partial \varepsilon}\, \phi_\varepsilon(T)\Big|_{\varepsilon=0}.$$

To evaluate $\frac{\partial}{\partial \varepsilon}\, \phi_\varepsilon(T)\Big|_{\varepsilon=0}$, note that $F^\varepsilon_{\psi_\varepsilon(t)} = F_t$. Hence

$$F_t^\varepsilon = F_{\phi_\varepsilon(t)} = F_t \exp\Big\{ - \int_0^t \int_E \varepsilon u_s(y))\lambda(s, dy)d\Lambda_s\Big\}$$

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon}\Big|_{\varepsilon=0} = F_t\Big(- \int_0^t \int_E u_s(y)\lambda(s, dy)d\Lambda_s\Big). \qquad (4.3.9)$$

On the other hand (see Elliott [1]),

$$F_t = \exp\Big(- \int_0^t \alpha(s)ds\Big)$$

so

$$F_{\phi_\varepsilon(t)} = \exp\Big(- \int_0^{\phi_\varepsilon(t)} \alpha(s)ds\Big).$$

Thus

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon} = -\alpha(\phi_\varepsilon(t))\, \frac{d\phi_\varepsilon(t)}{d\varepsilon}\, \exp\Big(- \int_0^{\phi_\varepsilon(t)} \alpha(s)ds\Big)$$

and

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon}\Big|_{\varepsilon=0} = -\alpha(t)\,\frac{d\phi_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0} F_t. \tag{4.3.10}$$

From (4.3.9) and (4.3.10), we obtain

$$\frac{d\phi_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{1}{\alpha(t)}\int_0^t \int_E u_s(y)\lambda(s,dy)\alpha(s)ds. \tag{4.3.11}$$

Now from (4.3.8) and (4.3.11), we have (4.3.5). □

4.4. Integration by Parts and Martingale Representation.

In Section 4.3, we considered a single jump process with values in a Lusin space. Now suppose that at its random jump time T, the process jumps to a fixed position $z_1 \in E$. If we define Λ_t^ε simply by

$$\Lambda_t^\varepsilon = \int_0^t (1+\varepsilon u_s)d\Lambda_s$$

where u is just a function of the time, which is positive, bounded and vanishes outside a bounded interval, then the method described in Section 4.3 would give us the simpler integration by parts formula:

$$E\Big[\Big(\int_0^T u_s dq_s\Big)G(T)\Big] = -E\Big[\frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\int_0^T u_s\alpha_s ds\Big] \tag{4.4.1}$$

where G is a bounded function defined on $[0,\infty]$ with bounded derivative. On the other hand, if we assume $E[G(T)]=0$, then $G(T)$ has the martingale representation (see Elliott [1]):

$$G(T) = \int_0^T \gamma_s dq_s \tag{4.4.2}$$

where

$$\gamma_s = G(s) - F_s^{-1}\int_0^s G(v)dF_v.$$

If we substitute (4.4.2) into the left side of (4.4.1), we have

$$\begin{aligned} E\Big[\Big(\int_0^T u_s dq_s\Big)\Big(\int_0^T \gamma_s dq_s\Big)\Big] &= E\Big[\int_0^T u_s\gamma_s d\langle q,q\rangle_s\Big] \\ &= -E\Big[\int_0^T u_s\gamma_s \frac{dF_s}{F_s}\Big] \\ &= E\Big[\int_0^T u_s\gamma_s\alpha_s ds\Big] \\ &= E\Big[\int_0^\infty I_{s\le T}\ u_s\gamma_s\alpha_s ds\Big]. \qquad (4.4.3) \end{aligned}$$

Now, if we define the measure π by:

$$\pi(dt) = \frac{dG(T)}{dt}\ \frac{1}{\alpha(T)}\ \delta_T(dt),$$

then the right side of (4.4.1) is

$$\begin{aligned} -E\Big[\int_0^\infty\int_0^t u_s\alpha_s ds\mu(dt)\Big] &= -E\Big[\int_0^\infty\int_0^\infty I_{0\le s\le t<\infty}\ u_s\alpha_s ds\mu(dt)\Big] \\ &= -E\Big[\int_0^\infty \pi[s,\infty)u_s\alpha_s ds\Big] \\ &= -E\Big[\int_0^\infty \frac{dG(T)}{dt}\ \frac{1}{\alpha(T)}\ I_{s\le T<\infty}\ u_s\alpha_s ds\Big]. \qquad (4.4.4) \end{aligned}$$

A comparison between (4.4.3) and (4.4.4) leads us to the following expression for γ:

THEOREM 4.4.1. *The integrand γ that appears in the martingale representation (4.4.2) is given by:*

$$\gamma_s = -E\Big[\frac{dG(T)}{dt}\ \frac{1}{\alpha(T)}\ I_{s\le T<\infty} \mid \mathcal{F}_{s-}\Big]. \qquad (4.4.5)$$

Proof.

$$
\begin{aligned}
E\Big[\frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\,I_{s\le T<\infty}\mid \mathcal{F}_{s-}\Big] &= -F_s^{-1}\int_s^\infty \frac{dG(t)}{dt}\,\frac{1}{\alpha(t)}\,dF_t \\
&= F_s^{-1}\int_s^\infty \frac{dG(t)}{dt}\,\frac{1}{\alpha(t)}\,F_t\alpha(t)dt \\
&= F_s^{-1}\int_s^\infty F_t dG(t) \\
&= F_s^{-1}\Big(-F_sG(s)-\int_s^\infty G(t)dF_t\Big) \\
&= -G(s)+F_s^{-1}\int_0^s G(r)dF_r \\
&= -\gamma_s
\end{aligned}
$$

□

References

[1] R.J. Elliott, Stochastic Calculus and Applications. *Applications of Mathematics*, Vol. 18, Springer–Verlag, Berlin-Heidelberg-New York, 1982.

[2] J.R. Norris, Integration by Parts for Jump Processes. *Séminaire de Probabilités XXII*, Lecture Notes in Math 1321, (271–315), Springer–Verlag, Berlin, 1988.

Chapter 5

TIME REVERSAL OF NON-MARKOV POINT PROCESSES

5.0. Introduction.

The time reversal of stochastic processes has been investigated for some years. One motivation comes from quantum theory, and this is discussed in the book of Nelson [8]. The time reversal of Markov diffusions is treated in, for example, the papers of Elliott and Anderson [3], and Haussman and Pardoux [6]. However, the first discussion of time reversal for a non-Markov process on Wiener space appears in the paper by Föllmer [5], in which he uses an integration-by-parts formula related to the Malliavin calculus.

In the present chapter an analog of the Fréchet derivative is introduced for functionals of a Poisson process. The integration-by-parts formula on Poisson space, see Chapter 3, is formulated in terms of this derivative and counterparts of Föllmer's formulae are obtained.

In Section 5.1 we will discuss the time reversal of a Brownian motion introduced by Föllmer. In Section 5.2 the time reversed form of the standard Poisson process is derived. Section 5.3 considers a point (counting) process N with Markov intensity $h(N_t)$, so that $Q_t = N_t - \int_0^t h(N_s)ds$ is a martingale, and obtains the reverse time decomposition of Q for $t \in (0,1]$. Finally, in Section 5.4, the situation when h

1. A version of this chapter has been submitted for publication. Robert J. Elliott and Allanus H. Tsoi.

is predictable is considered using the "Fréchet" derivative and integration-by-parts techniques mentioned above.

5.1. Time Reversal on Wiener Space.

We will now consider the time reversal of a Brownian motion in the non-Markovian case following Föllmer [5]. Let (X_t) be the coordinate process defined on the filtered probability space $(C[0,1], \mathcal{F}, (\mathcal{F}_t), P^*)$, where P^* is the Brownian measure, and $\mathcal{F}_t$ is the complete σ-field generated by $\sigma\{X_s,\ s \le t\}$. We also use $\widehat{\mathcal{F}}_t$ to denote the complete σ-field generated by $\sigma\{X_s : t \le s \le 1\}$.

Let (b_t) be an adapted process with $\int_0^1 b_t^2 dt < \infty$, P^* a.s. Then by the Girsanov transform ([1]), the process

$$W_t^b = X_t - X_0 - \int_0^t b_s ds$$

is a Brownian motion under P, where P is the measure defined by

$$\frac{dP}{dP^*}\Big|_{\mathcal{F}_t} = G_t = \exp\Big(\int_0^t b_s dX_s - \frac{1}{2}\int_0^t b_s^2 ds\Big).$$

We say that P has finite entropy with respect to the Brownian measure P^* if

$$H(P \mid P^*) = E\Big[\log \frac{dP}{dP^*}\Big] < \infty.$$

Let H_t be the σ-field defined by

$$H_t = \widehat{\mathcal{F}}_t \vee \sigma(X_0).$$

It is known (see [7]) that the process W_t^1 defined by

$$W_t^1 = X_t - X_1 + \int_t^1 \frac{X_s - X_0}{s}\, ds$$

is a reverse time H_t-Brownian motion under P^*. If P has finite entropy with respect to P^*, it then follows that there exists an $\widehat{\mathcal{F}}_t$-adapted process $(\hat{b}_t)$ such that

$$\widehat{W}_t = X_t - X_1 - \int_t^1 \hat{b}_s ds \tag{5.1.1}$$

is a reverse time $(\widehat{\mathcal{F}}_t)$-Brownian motion under P. Notice that from (5.1.1), we have

$$X_{t-h} - X_t = \widehat{W}_{t-h} - \widehat{W}_t + \int_{t-h}^t \hat{b}_s ds$$

so that

$$\hat{b}_t = \lim_{h \downarrow 0} \frac{1}{h} E[X_{t-h} - X_t \mid \widehat{\mathcal{F}}_t]. \tag{5.1.2}$$

We now recall what an L^2-differentiable function is. For a bounded predictable process (u_t), put

$$U_t = \int_0^t u_s ds.$$

Write

$$X_t^{\varepsilon,U} = X_t + \varepsilon U_t, \qquad 0 \leq t \leq 1.$$

DEFINITION. *A function $F \in L^2(P^*)$ is called L^2-differentiable if there is a measurable process (φ_t) such that, for any bounded predictable process (u_t),*

$$DF(\cdot, U) = \lim_{\varepsilon \downarrow 0} \frac{F(X^{\varepsilon,U}) - F(X)}{\varepsilon} = \int_0^1 u_s \varphi_s ds \text{ in } L^2(P^*). \tag{5.1.3}$$

If F is Fréchet-differentiable on $C[0,1]$ with bounded derivative $DF(w, dt)$,then (5.1.3) holds with $\varphi_s(w) = DF(w, [s,1])$. Also, if F is L^2-differentiable, then Bismut's integration by parts formula (2.3) holds for F.

If the drift (b_t) is a bounded smooth function on $C[0,1] \times [0,1]$ with bounded Fréchet derivative $Db_s(\cdot, ds)$, then it follows from the definition of G_1 that G_1 is L^2-differentiable; i.e.., there is a measurable process (γ_t) such that

$$DG_1(\cdot, U) = \int_0^1 u_t \gamma_t dt. \tag{5.1.4}$$

We now want to show that

$$\gamma_t G_1^{-1} = b_t + \int_t^1 Db_r(\cdot, [t, r]) dW_r^b. \tag{5.1.5}$$

For $\varepsilon > 0$, put

$$G_1^{\varepsilon,U} = G_1(X + \varepsilon U).$$

Then $G_1^{\varepsilon,U}$ satisfies

$$G_1^{\varepsilon,U} = 1 + \int_0^1 G_s^{\varepsilon,U} b_s(X + \varepsilon U) d(X_s + \varepsilon U_s).$$

Thus

$$DG_1(\cdot, U) = \int_0^1 G_s b_s u_s ds + \int_0^1 (DG_s(\cdot, U) b_s + G_s Db_s(\cdot, U)) dX_s. \tag{5.1.6}$$

By the method of variation of constants, we obtain

$$DG_1(\cdot, U) = G_1 \Big\{ \int_0^1 b_s u_s ds + \int_0^1 Db_s(\cdot, U) dX_s - \int_0^1 Db_s(\cdot, U) b_s ds \Big\}.$$

Thus

$$\begin{aligned} DG_1(\cdot, U) G_1^{-1} &= \int_0^1 b_s u_s ds + \int_0^1 Db_s(\cdot, U) dW_s^b \\ &= \int_0^1 b_s u_s ds + \int_0^1 \int_s^1 Db_r(\cdot, [s, r]) dW_r^b u_s ds. \end{aligned} \tag{5.1.7}$$

From (5.1.4), we have

$$DG_1(\cdot, U)G_1^{-1} = \int_0^1 u_t \gamma_t G_1^{-1} dt. \tag{5.1.8}$$

By equating (5.1.7) and (5.1.8), we obtain

$$\int_0^1 u_s \gamma_s G_1^{-1} ds = \int_0^1 \Big(b_s + \int_s^1 Db_r(\cdot, [s, r]) dW_r^b \Big) u_s ds. \tag{5.1.9}$$

Let $u_s = I_{[t-\varepsilon, t]}(s)$. Then (5.1.9) gives

$$\int_{t-\varepsilon}^t \gamma_s G_1^{-1} ds = \int_{t-\varepsilon}^t \Big(b_s + \int_s^1 Db_r(\cdot, [s, r]) dW_r^b \Big) ds. \tag{5.1.10}$$

If we divide both sides of (5.1.10) by ε, and then let $\varepsilon \downarrow 0$, we obtain (5.1.5).

By using the identities (5.1.2), (5.1.5) and Bismut's integration by parts formula (2.3), Föllmer obtains the following explicit expression for the drift $\hat{b}_t$:

$$\hat{b}_t = -E[b_t + a_t \mid \widehat{\mathcal{F}}_t]$$

where

$$a_t = \frac{1}{t} \Big(W_t^b - \int_0^t \int_s^1 Db_r(\cdot, [s, r]) dW_r^b ds \Big) + \int_t^1 Db_r(\cdot, [t, r]) dW_r^b. \tag{5.1.11}$$

5.2. Time Reversal Under the Original Measure.

Consider a standard Poisson process $N = \{N_t : 0 \leq t \leq 1\}$ on $(\Omega, \mathcal{F}, P)$. We take $N_0 = 0$. Let $\{\mathcal{F}_t\}$ be the right-continuous, complete filtration generated by N. Let $G_t^0 = \sigma\{N_s : t \leq s \leq 1\}$ and $\{G_t\}$ be the left-continuous, completion of $\{G_t^0\}$.

THEOREM 5.2.1. *Under P, N is a reverse time G_t-quasimartingale, and it has the decomposition:*

$$N_t = N_1 + M_t - \int_t^1 \frac{N_s}{s}\, ds,$$

where M is a reverse time G_t-martingale.

Proof. Since N is Markov, we have, for $\varepsilon > 0$,

$$E[N_{t-\varepsilon} - N_t \mid G_t] = E[N_{t-\varepsilon} - N_t \mid N_t]$$

$$= -\frac{\varepsilon}{t}\, N_t \qquad (5.2.1)$$

(see [4] and [7]). Thus

$$\int_0^t E|E[N_{s-\varepsilon} - N_s \mid G_s]|ds = O(\varepsilon).$$

By Stricker's theorem [9], N_t is a reverse time G_t-quasimartingale. Considering approximate Laplacians we see it has the decomposition

$$N_t = N_1 + M_t + \int_t^1 \alpha_s ds \qquad (5.2.2)$$

where from (5.2.1) and (5.2.2),

$$\alpha_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t E[\alpha_s \mid G_t]ds$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E[N_{t-\varepsilon} - N_t \mid G_t]$$

$$= -\frac{N_t}{t}.$$

5.3. Time Reversal After A Change of Measure: The Markov Case.

Consider a process $h_t = h(N_t)$ which satisfies: There exist positive constants A, $K > 0$ such that $0 < A < h(N_t) \leq K$ for all t, a.s.

Define the family $\{\Lambda_t,\ 0 \leq t \leq 1\}$ of exponentials:

$$\Lambda_t = \prod_{0 \leq u \leq t} \Big(1 + \big(h(N_{u-}) - 1\big)\Delta N_u\Big) \exp\Big(\int_0^t (1 - h(N_{u-}))du\Big).$$

Then Λ is an $(\mathcal{F}_t)$-martingale under P, and is the unique solution of the equation

$$\Lambda_t = 1 + \int_0^t \Lambda_{u-}(h(N_{u-}) - 1)(dN_u - du).$$

Define a new probability measure P^h by

$$\frac{dP^h}{dP} = \Lambda_1.$$

Then under P^h, the process $H_t = N_t - \int_0^t h(N_{u-})du$ is an $(\mathcal{F}_t)$-martingale (see [2]). Let $\beta(t) = \int_0^t h(N_{u-})du$ so that β is positive and increasing in t. Write

$$\psi(t) = \beta^{-1}(t)$$

$$N'_t = N_{\psi(t)}$$

$$\mathcal{F}'_t = \mathcal{F}_{\psi(t)}.$$

LEMMA 5.3.1. *(N'_t) is a Poisson process under $(\Omega, \mathcal{F}, (\mathcal{F}'_t), P^h)$.*

Proof. Since $H_t = N_t - \beta(t)$ is an $(\mathcal{F}_t)$-martingale under P^h, $H'_t = H_{\psi(t)} = N_{\psi(t)} - t$ is an $(\mathcal{F}'_t)$-martingale under P^h. By Itô's rule,

$$H'^2_t = 2\int_0^t H'_{s-}dH'_s + \sum_{s \leq t}(\Delta N_{\psi(s)})^2$$

$$= 2\int_0^t H'_{s-}dH'_s + N_{\psi(t)}.$$

Hence $H^2_{\psi(t)} - t$ is also an $(\mathcal{F}'_t)$-martingale under P^h. Therefore, $\{N'_t\}$ is Poisson by Lévy's characterization (Theorem 12.31 in [1]). □

LEMMA 5.3.2. *N is Markov under P^h.*

Proof. Consider any $\phi \in C_0^\infty(\mathbb{R})$. For $t \geq s$, by Baye's formula,

$$E^h[\phi(N_t) \mid \mathcal{F}_s] = \frac{E[\Lambda_t \phi(N_t) \mid \mathcal{F}_s]}{E[\Lambda_t \mid \mathcal{F}_s]}$$

$$= E[\Lambda_s^t \phi(N_t) \mid \mathcal{F}_s]$$

$$= E[\Lambda_s^t \phi(N_t) \mid N_s],$$

because N is Markov under P, where

$$\Lambda_s^t = \prod_{s<u\leq t} \Big(1 + \big(h(N_u) - 1\big)\Delta N_u\Big) \exp\Big(\int_s^t (1 - h(N_u))du\Big).$$

Hence

$$E^h[\phi(N_t) \mid \mathcal{F}_s] = E^h[\phi(N_t) \mid N_s]$$

and N is Markov under P^h. □

Note that

$$H_t = H_1 + N_t - N_1 + \int_t^1 h(N_s)ds. \tag{5.3.1}$$

Thus H_t is a reverse time G_t-quasimartingale under P^h if and only if N_t is. To determine the reverse time decomposition we again investigate the approximate Laplacians, as in [3].

THEOREM 5.3.3.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[N_{t-\varepsilon} - N_t \mid G_t] = -E^h\Big[h(N_t - 1)\frac{N_t}{\int_0^t h(N_u)du} \;\Big|\; N_t\Big]. \tag{5.3.2}$$

Proof. By Lemma 5.3.2,

$$E^h[N_t - N_{t-\varepsilon} \mid G_t] = E^h[N_t - N_{t-\varepsilon} \mid N_t].$$

Consider a bounded, differentiable function ϕ on $\mathbb{R}$ and its restriction to $\mathbf{Z}$ (the range of N). By Itô's rule,

$$\phi(N_t) = \phi(N_{t-\varepsilon}) + \int_{t-\varepsilon}^{t} \phi'(N_{s-})dN_s + \sum_{t-\varepsilon<s\le t} (\phi(N_s) - \phi(N_{s-}) - \phi'(N_{s-})\Delta N_s)$$

$$= \phi(N_{t-\varepsilon}) + \int_{t-\varepsilon}^{t} (\phi(N_{s-}+1) - \phi(N_{s-}))dN_s.$$

So

$$\phi(N_t)(N_t - N_{t-\varepsilon}) = \int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\phi(N_{s-}+1) - \phi(N_{s-}))dN_s$$

$$+ \int_{t-\varepsilon}^{t} \phi(N_{s-})dN_s + \sum_{t-\varepsilon<s\le t} \Delta\phi(N_s)\Delta N_s$$

$$= \int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\phi(N_{s-}+1) - \phi(N_{s-}))dN_s + \int_{t-\varepsilon}^{t} \phi(N_{s-}+1)dN_s.$$

Since

$$H_t = N_t - \int_0^t h(N_s)ds$$

$$= N_t - \int_0^t h(N_{s-})ds$$

is a martingale under P^h,

$$E^h[\phi(N_t)(N_t - N_{t-\varepsilon})]$$

$$= E^h\Big[\int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\phi(N_{s-}+1) - \phi(N_{s-}))h(N_{s-})ds\Big]$$

$$+ E^h\Big[\int_{t-\varepsilon}^{t} \phi(N_{s-}+1)h(N_{s-})ds\Big]. \quad (5.3.3)$$

Now, if $|\phi| \leq C$,

$$\left|E^h\Big[\int_{t-\varepsilon}^{t}(N_{s-}-N_{t-\varepsilon})(\phi(N_{s-}+1)-\phi(N_{s-}))h(N_{s-})ds\Big]\right|$$

$$\leq 2KC\int_{t-\varepsilon}^{t}E^h[|N_{s-}-N_{t-\varepsilon}|]ds$$

$$\leq 2KC\int_{t-\varepsilon}^{t}E^h\Big[\Big|N_{s-}-N_{t-\varepsilon}-\int_{t-\varepsilon}^{s-}h(N_{u-})du\Big|\Big]+E^h\Big[\Big|\int_{t-\varepsilon}^{s-}h(N_{u-})du\Big|\Big]ds$$

$$\leq 2KC\int_{t-\varepsilon}^{t}\Big\{\Big[E^h\Big|N_{s-}-N_{t-\varepsilon}-\int_{t-\varepsilon}^{s-}h(N_{u-})du\Big|^2\Big]^{1/2}+K\varepsilon\Big\}ds$$

$$\leq 2KC\int_{t-\varepsilon}^{t}\Big\{E^h\Big[\int_{t-\varepsilon}^{t}h(N_{u-})du\Big]^{1/2}+K\varepsilon\Big\}ds$$

$$\leq 2KC\int_{t-\varepsilon}^{t}((K\varepsilon)^{1/2}+K\varepsilon)ds$$

$$\leq K'\varepsilon^{3/2}+K''\varepsilon^2.$$

Thus from (5.3.3),

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\,E^h[\phi(N_t)(N_t-N_{t-\varepsilon})]=E^h[\phi(N_{t-}+1)h(N_{t-})]$$

$$=E^h[\phi(N_t+1)h(N_t)]. \qquad (5.3.4)$$

However,

$$E^h[\phi(N_t+1)h(N_t)]=E^h[\phi(N_{\psi(\beta(t))}+1)h(N_{\psi(\beta(t))})]$$

$$=E^h[\phi(N'_{\beta(t)}+1)h(N'_{\beta(t)})]$$

$$=E^h\big[E^h[\phi(N'_{\beta(t)}+1)h(N'_{\beta(t)})\mid\beta(t)]\big].$$

And

$$E^h[\phi(N'_{\beta(t)}+1)h(N'_{\beta(t)}) \mid \beta(t)] = \sum_{k=0}^{\infty} \phi(k+1)h(k)\frac{\beta(t)^k e^{-\beta(t)}}{k!}$$

$$= \sum_{\ell=0}^{\infty} \phi(\ell)h(\ell-1)\frac{\beta(t)^\ell e^{-\beta(t)}}{\ell!}\ \frac{\ell}{\beta(t)}$$

$$= E^h\left[\phi(N'_{\beta(t)})h(N'_{\beta(t)}-1)\frac{N'_{\beta(t)}}{\beta(t)} \mid \beta(t)\right]$$

$$= E^h\left[\phi(N_t)h(N_t-1)\frac{N_t}{\beta(t)} \mid \beta(t)\right].$$

Hence,

$$E^h[\phi(N_t+1)h(N_t)] = E^h\left[\phi(N_t)h(N_t-1)\frac{N_t}{\int_0^t h(N_u)du}\right]. \tag{5.3.5}$$

Thus from (5.3.4) and (5.3.5),

$$\lim_{\varepsilon\downarrow 0} E^h\left[\phi(N_t)\frac{(N_t - N_{t-\varepsilon})}{\varepsilon}\right] = E^h\left[\phi(N_t)h(N_t-1)\frac{N_t}{\int_0^t h(N_u)du}\right],$$

or

$$\lim_{\varepsilon\downarrow 0} E^h\left[\frac{N_{t-\varepsilon} - N_t}{\varepsilon} \mid G_t\right] = -E^h\left[h(N_t-1)\frac{N_t}{\int_0^t h(N_u)du} \mid N_t\right].$$

□

By Theorem 5.3.3 and an argument similar to that in [3], we see that N, and hence H, is a reverse time G_t-quasimartingale under P^h, and it has the decomposition

$$H_t = H_1 + M_t + \int_t^1 \alpha_t d_t. \tag{5.3.6}$$

Moreover, we have the following expression for α_t:

THEOREM 5.3.4. *The integrand α_t that appears in (5.3.6) is given by*

$$\alpha_t = h(N_t) - E^h\Big[h(N_t - 1)\frac{N_t}{\int_0^t h(N_u)du} \mid N_t\Big].$$

Proof. From (5.3.1) and (5.3.6),

$$\begin{aligned} E^h[H_{t-\varepsilon} - H_t \mid G_t] &= E^h\Big[\int_{t-\varepsilon}^t \alpha_s ds \mid G_t\Big] \\ &= E^h\Big[N_{t-\varepsilon} - N_t \mid G_t\Big] + E^h\Big[\int_{t-\varepsilon}^t h(N_s)ds \mid G_t\Big]. \end{aligned}$$

Thus

$$\begin{aligned} \alpha_t &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\ E^h\Big[\int_{t-\varepsilon}^t \alpha_s ds \mid G_t\Big] \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\ E^h[N_{t-\varepsilon} - N_t \mid G_t] + h(N_t). \end{aligned}$$

From Theorem 5.3.3, α_t has the stated form. □

5.4. Time Reversal After a Change of Measure: The Non-Markov Case.

Suppose $\{N_t : 0 \le t \le 1\}$ is a Poisson process with jump times $T_1 \wedge 1, \dots, T_n \wedge 1, \dots$. Let $\{u_t\}$ be a real predictable process satisfying:

(i) $\{u_t\}$ is positive and bounded a.s.

(ii) There exists a bounded interval, say, $[0, b]$, such that $u_s(w) = 0$ a.s. if $s \notin [0, b]$.

For $\varepsilon > 0$, consider the family of exponentials:

$$\Lambda_t^\varepsilon = \prod_{0\le s\le t} (1 + \varepsilon u_s \Delta N_s)\exp\Big(-\int_0^t \varepsilon u_s ds\Big).$$

Then $\{\Lambda_t^\varepsilon\}$ is an $\{\mathcal{F}_t\}$-martingale with $E[\Lambda_t^\varepsilon] = 1$. See Chapter 3. Define a probability measure P^ε on $\mathcal{F}_1$ by

$$\frac{dP^\varepsilon}{dP} = \Lambda_1^\varepsilon.$$

Set

$$\phi_\varepsilon(t) = \int_0^t (1 + \varepsilon u_s) ds$$

and write

$$\psi_\varepsilon(t) = \varphi_\varepsilon^{-1}(t)$$

$$= \int_0^t \frac{1}{1 + \varepsilon u_{\psi_\varepsilon(s)}}\, ds$$

$$\mathcal{F}_t^\varepsilon = \mathcal{F}_{\psi_\varepsilon(t)}.$$

Then the process $N_t^\varepsilon = N_{\psi_\varepsilon(t)}$ is Poisson on $(\Omega, \mathcal{F}, (\mathcal{F}_t^\varepsilon), P^\varepsilon)$ with jump times $\phi_\varepsilon(T_1) \wedge 1, \ldots \phi_\varepsilon(T_n) \wedge 1, \ldots$. See Chapter 3.

For $\{u_t\}$ satisfying (i) and (ii) above, set $U_t = \displaystyle\int_0^t u_s ds$. Suppose $g_s(w)$ is an $\{F_t\}$-predictable function on $[0,1]$. Then for $0 \le s \le T_1 \wedge 1$,

$$g_s(w) = g(s),$$

and in general, for $T_{n-1} \wedge 1 < s \le T_n \wedge 1$,

$$g_s(w) = g(s, T_1 \wedge 1, \ldots, T_{n-1} \wedge 1).$$

Note that by setting $g_s(0,0,\ldots) = g(s)$ for $0 \le s \le T_1 \wedge 1$, $g_s((s - T_1) \vee 0, \ldots, (s - T_{n-1}) \vee 0), 0, 0 \ldots)$ for $T_{n-1} \wedge 1 < s \le T_n \wedge 1$, etc., such a g can be written in the form

$$g_s(w) = g_s((s - T_1) \vee 0,\ (s - T_2) \vee 0,\ \ldots), \qquad s \in [0,1]. \tag{5.4.1}$$

Therefore, we shall consider a predictable function g of this form, and further assume that if

$$g = g_s(t_1, t_2, \dots),$$

then all the partial derivatives $\frac{\partial g_s}{\partial t_i}$ exist for all s, and there is a constant $K > 0$ such that

$$\left|\frac{\partial g_s}{\partial t_i}\right| < K \qquad \text{for all } i, \text{ and for all } s. \tag{5.4.2}$$

We now define the analog of the Fréchet derivative for functionals of the Poisson process.

Write

$$g_s^\varepsilon = g_s((s - \phi_\varepsilon(T_1)) \vee 0, \ \dots, \ (s - \phi_\varepsilon(T_u)) \vee 0, \dots).$$

Then

$$\frac{\partial g_s^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} = -\sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} g_s((s - T_1) \vee 0, \ \dots, \ (s - T_n) \vee 0, \dots) \int_0^{T_i} u_r dr I_{T_i < s}. \tag{5.4.3}$$

Define

$$\mu(dt) = -\sum_{i=1}^{\infty} \frac{\partial g_s}{\partial t_i} I_{T_i < s} \delta_{T_i}(dt)$$

where δ_{T_i} is the point mass at T_i. Then

$$\begin{aligned}
\frac{\partial g_s^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} &= \int_0^s \int_0^t u_r dr \mu(dt) \\
&= \int_0^s \int_0^s I_{0 \le r \le t \le s} u_r dr \mu(dt) \\
&= \int_0^s \mu([r, s]) u_r dr \\
&= -\int_0^s \sum_{i=1}^{\infty} I_{r \le T_i < s} \frac{\partial g_s}{\partial t_i} u_r dr \\
&= \int_0^s Dg_s(\cdot, [r, s]) u_r dr,
\end{aligned}$$

where

$$Dg_s(\cdot,[r,s]) = -\sum_{i=1}^{\infty} I_{r\le T_i<s}\frac{\partial g_s}{\partial t_i}.$$

Write

$$Dg_s(\cdot,U) = \int_0^s Dg_s(\cdot,[r,s])u_r dr.$$

Note that

$$Dg_{T_i}(\cdot,U) = -\sum_{j=1}^{i-1}\frac{\partial g_{T_i}}{\partial t_j}\int_0^{T_j} u_r dr. \tag{5.4.4}$$

DEFINITION 5.4.1. *A process $\{g_s\}$ of the form (5.4.1) is said to be differentiable if it satisfies (5.4.2) and (5.4.3) for all u satisfying (i) and (ii) above, and for all s. We call $Dg_s(\cdot,U)$ the derivative of g_s in the direction U. It is of interest to note that this concept of differentiability of a function of a Poisson process is an analog of the Fréchet derivative of a function of a continuous process. See Föllmer [5], where similar formulae arise using the Fréchet derivative.*

Now suppose $\{h_s\}$ is a bounded, $\{F_t\}$-predictable process of the form given by (5.4.1), which satisfies:

(a) h is differentiable in the sense of Definition 5.4.1.

(b) $\frac{\partial h_s}{\partial s}$ exists, and there exists a constant $A > 0$ such that $\left|\frac{\partial h_s}{\partial s}\right| < A$ for all s, a.s.

(c) There are constants $B > 0$, $C > 0$ such that $0 < B < h_s < C$ for all s, a.s.

It is easy to check that $h_s = h_s((s-T_1)\vee 0,\ (s-T_2)\vee 0,\dots)$ is predictable. Consider the family of exponentials:

$$G_t = \prod_{0\le s\le t}(1+(h_s-1)\Delta N_s)\exp\Big(\int_0^t(1-h_s)ds\Big) \tag{5.4.5}$$

$$= \Big(\prod_{0\le T_i\le t} h_{T_i}\Big)\exp\Big(\int_0^t(1-h_s)ds\Big).$$

Then $\{G_t\}$ is a martingale with $E[G_t] = 1$. Since for each fixed w, if $T_{n-1}(w) < t \le T_n(w)$, G_t is a function of $(t, T_1(w), \ldots, T_{n-1}(w))$, we see as above that G_t can be considered to be of the form

$$G_t = G_t((t - T_1) \vee 0, \ \ldots, \ (t - T_n) \vee 0, \ \ldots).$$

THEOREM 5.4.2. *(G_t) defined in (5.4.5) is differentiable in the sense of Definition 5.4.1.*

Moreover,

$$\begin{aligned} DG_1(\cdot, U)G_1^{-1} &= \int_0^1 \gamma_s u_s G_1^{-1} ds \\ &= \int_0^1 \int_r^1 \left[\frac{\partial h_s}{\partial s} + \sum_{j=1}^{\infty} I_{\{T_j < s\}} \frac{\partial h_s}{\partial t_j} + Dh_s(\cdot, [r, s])\right] \frac{1}{h_s} dN_s u_r dr \\ &\quad - \int_0^1 \int_r^1 Dh_s(\cdot, [r, s]) ds u_r dr \qquad \text{a.s.} \end{aligned} \tag{5.4.6}$$

where

$$\gamma_s = -\sum_{i=1}^{\infty} I_{s \le T_i \le 1} \frac{\partial}{\partial t_i} G_1((1 - T_1) \vee 0, \ldots, \ (1 - T_n) \vee 0, \ \ldots).$$

Proof. The first identity follows from the definition and properties of the derivative. To determine $DG_t(\cdot, U)$ we calculate the derivative of G_t^ε at $\varepsilon = 0$. Write

$$h_s^\varepsilon = h_s((s - \phi_\varepsilon(T_1)) \vee 0, \ldots, (s - \phi_\varepsilon(T_n)) \vee 0, \ldots),$$

so

$$G_t^\varepsilon = \prod_{0\le s\le t} \left(1 + (h_s^\varepsilon - 1)\Delta N_{\psi_\varepsilon(s)}\right) \exp\Big(\int_0^t (1 - h_s^\varepsilon) ds\Big)$$
$$= \Big(\prod_{0\le \phi_\varepsilon(T_i)\le t} h^\varepsilon_{\phi_\varepsilon(T_i)}\Big) \exp\Big(\int_0^t (1 - h_s^\varepsilon) ds\Big)$$
$$= \Big(\prod_{0\le T_i\le \psi_\varepsilon(t)} h^\varepsilon_{\phi_\varepsilon(T_i)}\Big) \exp\Big(\int_0^t (1 - h_s^\varepsilon) ds\Big).$$

Then

$$\log G_t^\varepsilon = \sum_{i=1}^{\infty} I_{T_i\le\psi_\varepsilon(t)} \log h^\varepsilon_{\phi_\varepsilon(T_i)} + \int_0^t (1 - h_s^\varepsilon) ds. \tag{5.4.7}$$

Differentiate (5.4.7) with respect to ε, and then set $\varepsilon = 0$, to see

$$DG_t(\cdot, U)\frac{1}{G_t} = \sum_{i=1}^{\infty}\Big\{I_{T_i\le t}\Big[\frac{\partial h_{T_i}}{\partial t}\int_0^{T_i} u_r dr + \sum_{j=1}^{i-1}\frac{\partial h_{T_i}}{\partial t_j}\Big(\int_0^{T_i} u_r dr - \int_0^{T_j} u_r dr\Big)\Big]\frac{1}{h_{T_i}}\Big\}$$
$$- \int_0^t Dh_s(\cdot, U) ds \qquad \text{a.s.}$$

From (5.4.4) this is

$$= \sum_{i=1}^{\infty}\Big\{I_{T_i\le t}\Big[\frac{\partial h_{T_i}}{\partial t}\int_0^{T_i} u_r dr + \sum_{j=1}^{i-1}\frac{\partial h_{T_i}}{\partial t_j}\int_0^{T_i} u_r dr + Dh_{T_i}(\cdot, U)\Big]\frac{1}{h_{T_i}}\Big\}$$
$$- \int_0^t Dh_s(\cdot, U) ds$$
$$= \int_0^t \Big[\frac{\partial h_s}{\partial s}\int_0^s u_r dr + \sum_{j=1}^{\infty} I_{\{T_j<s\}}\frac{\partial h_s}{\partial t_j}\int_0^s u_r dr + Dh_s(\cdot, U)\Big]\frac{1}{h_s} dN_s$$
$$- \int_0^t Dh_s(\cdot, U) ds. \tag{5.4.8}$$

(Formally, the differentiation of the indicator functions $I_{T_i\le\psi_\varepsilon(t)}$ introduces Dirac measures $\delta(t - T_i)$. However, $P(T_i = t) = 0$ and we later will take expectations, so these can be ignored.) From (5.4.8),

$$
\begin{aligned}
DG_1(\cdot,U)G_1^{-1} &= \int_0^1 \Big\{ \frac{\partial h_s}{\partial s}\int_0^s u_r dr + \sum_{j=1}^{\infty} I_{\{T_j<s\}}\frac{\partial h_s}{\partial t_j}\int_0^s u_r dr \\
&\quad + \int_0^s Dh_s(\cdot,[r,s])u_r dr \Big\} \frac{1}{h_s} dN_s - \int_0^1 \int_0^s Dh_s(\cdot,[r,s])u_r dr ds \\
&= \int_0^1 \int_0^1 I_{0\le r\le s\le 1}\Big\{ \frac{\partial h_s}{\partial s} u_r \frac{1}{h_s} + \sum_{j=1}^{\infty} I_{\{T_j<s\}}\frac{\partial h_s}{\partial t_j} u_r \frac{1}{h_s} \\
&\quad + Dh_s(\cdot,[r,s])u_r \frac{1}{h_s}\Big\} dr dN_s - \int_0^1 \int_0^1 I_{0\le r\le s\le 1} Dh_s(\cdot,[r,s])u_r dr ds \\
&= \int_0^1 \int_r^1 \Big[\frac{\partial h_s}{\partial s} + \sum_{j=1}^{\infty} I_{\{T_j<s\}}\frac{\partial h_s}{\partial t_j} + Dh_s(\cdot,[r,s])\Big] \frac{1}{h_s} dN_s u_r dr \\
&\quad - \int_0^1 \int_r^1 Dh_s(\cdot,[r,s]) ds u_r dr,
\end{aligned}
$$

which is (5.4.6). □

Consider the family of exponentials defined by (5.4.5) and define a new probability measure P^h on $\mathcal{F}_1$ by:

$$\frac{dP^h}{dP} = G_1.$$

Then (see [2]) the process

$$
\begin{aligned}
Z_t &= N_t - \int_0^t h_s ds \\
&= Q_t - \int_0^t (h_s - 1) ds, \qquad (5.4.9)
\end{aligned}
$$

where $Q_t = N_t - t$, is an $(\mathcal{F}_t)$-martingale under P^h. We want to show that Z_t is a reverse time G_t-quasimartingale under P^h, having the decomposition

$$Z_t = Z_1 + M_t + \int_t^1 \alpha_s ds. \qquad (5.4.10)$$

From (5.4.9), we can write

$$Z_t = Z_1 + Q_t - Q_1 + \int_t^1 (h_s - 1)ds.$$

Now

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h\Big[\int_{t-\varepsilon}^t (h_s - 1)ds \mid G_t\Big] = E^h[h_t - 1 \mid G_t].$$

Hence, to show that Z_t has the decomposition given by (5.4.10), it again suffices to consider approximate Laplacians as in [3] and show that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[Q_{t-\varepsilon} - Q_t \mid G_t]$$

exists.

THEOREM 5.4.3.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[Q_t - Q_{t-\varepsilon} \mid G_t] = \frac{1}{t} E^h[Q_t + a_t \mid G_t] - E^h[b_t \mid G_t] \tag{5.4.11}$$

where

$$a_t = \int_0^t \int_s^1 \Big[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + Dh_r(\cdot, [s,r])\Big] \frac{1}{h_r} dN_r ds$$
$$- \int_0^t \int_s^1 Dh_r(\cdot, [s,r])$$

and

$$b_t = \int_t^1 \Big[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + Dh_r(\cdot, [t,r])\Big] \frac{1}{h_r} dN_r$$
$$- \int_t^1 Dh_r(\cdot, [t,r])dr.$$

Proof. First we note that if $H((1-T_1)\vee 0,\dots,(1-T_n)\vee 0,\dots)$ is a square integrable functional and its first partial derivatives are all bounded by a constant, then, using a similar argument as in Chapter 3, we have the integration by parts formula

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big)H\Big] = -E[DH(\cdot,U)] \tag{5.4.12}$$

where $DH(\cdot,U)$ is the derivative in direction U of Definition 5.4.1.

A direct consequence is the product rule

$$E\Big[FH\Big(\int_0^1 u_s dQ_s\Big)\Big] = -E[FDH(\cdot,U)] - E[HDF(\cdot,U)]. \tag{5.4.13}$$

Let $H = G_1$ be the Girsanov density, then (5.4.13) becomes

$$E^h\Big[F\int_0^1 u_s dQ_s\Big] = -E^h[DF(\cdot,U)] - E^h[FG_1^{-1}DG_1(\cdot,U)]. \tag{5.4.14}$$

Now fix $t_0 \in (0,1)$. Write $T_k(t_0)$ for the k-th jump time of N_t greater than t_0. Suppose F is a bounded and G_{t_0} measurable function. Furthermore, we suppose that F is a differentiable function (in the sense of Definition 5.4.1) of the form

$$F((1-T_1(t_0))\vee 0,\dots,(1-T_k(t_0))\vee 0,\dots),$$

and that the derivatives of F are bounded. Then the measure $DF(\cdot,dt)$ is concentrated on $[t_0,1]$ and (5.4.14) holds for such an F. Take $u_s = I_{[t_0-\varepsilon,t_0]}(s)$ in (5.4.14). For such an F

$$\begin{aligned} DF(\cdot,U) &= \int_{t_0-\varepsilon}^{t_0} DF(\cdot,[r,1])dr \\ &= \int_{t_0-\varepsilon}^{t_0} DF(\cdot,[t_0,1])dr \\ &= \varepsilon DF(\cdot,[t_0,1]). \end{aligned}$$

Therefore, we have from (5.4.14)

$$E^h[(Q_{t_0} - Q_{t_0-\varepsilon})F] = -\varepsilon E^h[DF(\cdot,[t_0,1])] + E^h\Big[FG_1^{-1}\int_{t_0-\varepsilon}^{t_0}\sum_{i=1}^{\infty} I_{s\le T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\Big]. \tag{5.4.15}$$

From (5.4.15),

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\,[(Q_{t_0} - Q_{t_0-\varepsilon})F] = -E^h[DF(\cdot,[t_0,1])] + E^h\Big[FG_1^{-1}\sum_{i=1}^{\infty} I_{t_0\le T_i<1}\frac{\partial G_1}{\partial t_i}\Big]. \tag{5.4.16}$$

Using (5.4.15) again with $\varepsilon = t_0 = t$, we have

$$-E^h[DF(\cdot,[t,1])] = \frac{1}{t}E^h[Q_tF] - \frac{1}{t}E^h\Big[FG_1^{-1}\int_0^t\sum_{i=1}^{\infty} I_{s\le T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\Big]. \tag{5.4.17}$$

Now let $u_s = I_{[0,t]}(s)$ in Theorem 5.4.2 to obtain

$$-\int_0^t\Big(\sum_{i=1}^{\infty} I_{s\le T_i<1}\frac{\partial G_1}{\partial t_i}\Big)G_1^{-1}ds$$

$$= \int_0^t\int_s^1\Big[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j<r\}}\frac{\partial h_r}{\partial t_j} + Dh_r(\cdot,[s,r])\Big]\frac{1}{h_r}dN_r ds$$

$$-\int_0^t\int_s^1 Dh_r(\cdot,[s,r])dr ds.$$

Hence (5.4.17) becomes

$$-E^h[DF(\cdot,[t,1])] = \frac{1}{t}E^h[Q_tF] + \frac{1}{t}E^h[a_tF]. \tag{5.4.18}$$

Now take $u_s = I_{[t-\varepsilon,t]}(s)$ in Theorem 5.4.2 to obtain

$$-\int_{t-\varepsilon}^t\Big(\sum_{i=1}^{\infty} I_{s\le T_i<1}\frac{\partial G_1}{\partial t_i}\Big)G_1^{-1}ds$$

$$= \int_{t-\varepsilon}^t\int_s^1\Big[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j<r\}}\frac{\partial h_r}{\partial t_j} + Dh_r(\cdot,[s,r])\Big]\frac{1}{h_r}dN_r ds$$

$$-\int_{t-\varepsilon}^t\int_s^1 Dh_r(\cdot,[s,r])dr ds. \tag{5.4.19}$$

Multiply both sides of (5.4.19) by F, and then take expectations

$$-E^h\Big[F\int_{t-\varepsilon}^{t}\Big(\sum_{i=1}^{\infty} I_{s\le T_i<1}\frac{\partial G_1}{\partial t_i}\Big)G_1^{-1}ds\Big] = E^h\Big[F\int_{t-\varepsilon}^{t} b_s ds\Big]. \tag{5.4.20}$$

Divide both sides of (5.4.20) by ε, and then let $\varepsilon \downarrow 0$, to obtain

$$-E^h\Big[F\Big(\sum_{i=1}^{\infty} I_{t\le T_i<1}\frac{\partial G_1}{\partial t_i}\Big)G_1^{-1}\Big] = E^h[b_t F]. \tag{5.4.21}$$

Combining (5.4.16), (5.4.18) and (5.4.21), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\, E^h[(Q_t - Q_{t-\varepsilon})F] = \frac{1}{t}E^h[(a_t + Q_t)F] - E^h[b_t F].$$

Thus we have proved (5.4.11). □

As a consequence of Theorem 5.4.3, Z_t is a reverse time G_t-quasimartingale having the decomposition given by (5.4.10). It follows immediately that the integrand α_t in (5.4.10) is given by

$$\alpha_t = E^h[b_t + h_t - 1 \mid G_t] - \frac{1}{t}E^h[a_t + Q_t \mid G_t].$$

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