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THE VACUUM IN EXTERNAL FIELD PROBLEMS

BY

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The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "The Vacuum in External Field Problems" submitted by David Lamb in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics.

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ABSTRACT

After a review of general field theory including the construction of various Green's functions and the quantisation of the scalar field, we move on to the external field problem. An overview of some of the work done on the external field problem is then presented with an emphasis given to the problems associated with defining the vacuum relevant to the quantised field. In the third chapter we demonstrate how one can define a vacuum for a scalar field coupled to an external source which vanishes rapidly in any direction. This vacuum is then shown to be a more appropriate vacuum for the interacting field than the vacuum for the in-coming free field, as the energy of the interacting field is lower in this vacuum. Chapter 4 is a review of the gravitational external field problem which is the area in which one can hope to understand things such as Hawking radiation. It is this discovery by Hawking in 1975 which has caused many people to try and develop a more thorough understanding of the external field problem in general.

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CHAPTER ONE

INTRODUCTION TO FIELD THEORY

1.1 Introduction

In this chapter we introduce all the basic tools required to understand Chapters 2 and 3. We start with an introduction to the classical real scalar field and with this field develop most of the analytic tools required, including the Green's functions associated with the Klein Gordon field and how these can be used to find various c-number solutions to the field equations. We also discuss how these functions can be used to construct the Klein Gordon propagator which describes how the field evolves.

We then deal with the quantised scalar field and introduce some operators, which allow us to shift both the operators that make up the field and the field itself by a c-number . These operators will be of use to us in chapters two and three.

The last section of this chapter is an explanation as to why one wants to study the external field problem, and explains some of the unanswered questions involved with the external field problem.

1.2 Classical Scalar Field

What follows is a basic introduction to the classical Klein Gordon field and the associated Green's functions. This introduction is basically the same as that found in any standard introductory field theory text such as Roman [1] or Bjorken and Drell [2]. The real Klein Gordon field is characterised by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) + \rho(x)\phi(x). \tag{1.1}$$

Here, $\phi(x)$ is the field we will be investigating and $\rho(x)\phi(x)$ is the interaction part of the Lagrangian. Therefore if $\rho(x) = 0$ the field is said to be a free field, i.e. free of any interaction. Applying the Euler-Lagrange equation we find the equation of motion for this field to be

$$(\Box + m^2)\phi(x) = \rho(x), \tag{1.2}$$

where $\Box$ is the d'Alambertian operator defined as,

$$\Box \equiv \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2. \tag{1.3}$$

By investigating the Green's functions of (1.2) we can find solutions to the classical field equation (1.2). These solutions will be of use later when we are dealing with a quantised field interacting with a c-number source.

To look at the Green's functions $\mathcal{G}(x-y)$ of (1.2) we want to look at solutions of

$$(\Box_x + m^2)\mathcal{G}(x-y) = \delta(x-y). \tag{1.4}$$

For now we will not concern ourselves with the solution to the homogeneous part of this equation but will just note that this part of the solution contains terms like e^{ikx} and e^{-ikx}, where $k_0 = \omega_k \equiv \sqrt{m^2 + \mathbf{k}^2}$, so that these are what the free fields contain. We can immediately see that the particular solution to this equation can be written as

$$\mathcal{G}(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{m^2 - k^2} d^4k. \tag{1.5}$$

We note that the integrand has two poles involved with the k_0 integration so we must specify how our contour will go around these poles. We can do this in one of two ways. When we are integrating we can alter our contour either above or below each pole or we can move our poles by an infinitesimal amount up or down in the complex plane. We choose the second approach here as it allows us to write different Green's functions explicitly and we don't have to carry around any labels with the different Green's functions about how to deform the contour of integration for each one. The two simplest and most useful Green's functions are the retarded Green's function, $\Delta_r(x)$, and the advanced Green's function $\Delta_a(x)$. These Green's functions have both their poles moved in the same direction, moved up, as in the case of the retarded Green's function, or down, as in the case of the advanced Green's function.

Now having these Green's functions at our disposal we can construct explicit solutions of the particular part of (1.2) i.e.

$$\phi(x) = \int d^4y \mathcal{G}(x-y)\rho(y), \tag{1.6}$$

where we must ensure that the proper Green's function is used so the correct boundary conditions are recovered for the field $\phi(x)$.

We now write out these different functions explicitly and then investigate some properties of these Green's functions and find out why they are appropriately called the advanced and retarded Green's functions. The retarded Green's function as mentioned earlier has both of its poles displaced in the positive i direction. We therefore want the poles of the retarded Green's function to be at

$$k_0 = \omega_k + i\epsilon \qquad \text{and} \qquad k_0 = -\omega_k + i\epsilon. \tag{1.7}$$

We can write the retarded Green's function now as

$$\Delta_r(x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ikx}}{m^2 - k^2 + i\epsilon k_0}. \tag{1.8}$$

We can now evaluate (1.8) to see some of the properties of the retarded Green's function. If we evaluate (1.8) for $x_0 < 0$ we realise that the integral is calculated most easily by closing the contour in the lower half plane when doing the k_0 integration so that the infinite semicircle doesn't contribute to the integral. In the retarded Green's function the poles have both been displaced in positive imaginary direction. This means that there are no poles enclosed in the contour so the integral is zero; therefore

$$\Delta_r(x) = 0 \qquad \text{when} \qquad x_0 < 0. \tag{1.9}$$

We now note that as (1.8) is obtained from (1.5) simply through a deformation of the contour, and (1.5) is a covariant statement then (1.9) is not only true for $x_0 < 0$ but is true in general for $x^2 < 0$, i.e. any spacelike argument. This means that the retarded Green's function is zero for any space like vector x i.e. outside the light cone. We also note that that in doing the integral in (1.8) we didn't impose any conditions on $\mathbf{x}$ so the integral would also be zero if $x^2 > 0, x_0 < 0$. Therefore the retarded Green's function is only nonzero in the future light cone,

$$\Delta_r(x) \neq 0 \qquad \text{only if} \qquad x^2 > 0 \quad \text{and} \quad x_0 > 0. \tag{1.10}$$

Similarly for the advanced Green's function we can deform the contour of integration by displacing both poles below the real axis in the complex plane by just changing the sign of the "shift" of the poles, thus

$$\Delta_a(x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{ikx}}{m^2 - k^2 - i\epsilon k_0}. \tag{1.11}$$

We can also integrate this expression as we did for the retarded Green's function and find that the advanced Green's function only has support in the past light cone,

$$\Delta_a(x) \neq 0 \qquad \text{only if} \qquad x^2 > 0 \text{ and } x_0 < 0. \tag{1.12}$$

We can therefore now write a causal solution to (1.2) using the retarded Green's function $\Delta_r(x)$,

$$\phi(x) = \int d^4y \Delta_r(x-y)\rho(y), \tag{1.13}$$

As of yet we haven't mentioned the homogeneous part of the solution to (1.2), except to say that it contains terms like e^{ikx} and e^{-ikx} where $k_0 = \omega_k$. Although we leave writing these fields out explicitly until we talk about the quantised version of these fields we can now introduce two free fields $\phi_{in}(x)$ and $\phi_{out}(x)$, which satisfy

$$(\Box + m^2)\phi_{in,out}(x) = 0. \tag{1.14}$$

If we now impose a restriction on $\rho(x)$ so that the problem at hand can be regarded as a scattering problem we can then make physical sense of the incoming and outgoing free fields as being the fields before there is any interaction and after there is any interaction respectively. A sufficient condition on $\rho(x)$ is that it vanishes rapidly in any space-time direction. Recalling the properties of the retarded and advanced Green's functions we can now write the complete solution of $\phi(x)$ as

$$\phi(x) = \phi_{in}(x) + \int d^4y \Delta_r(x-y)\rho(y) \tag{1.15}$$

or

$$\phi(x) = \phi_{out}(x) + \int d^4y \Delta_a(x-y)\rho(y). \tag{1.16}$$

We can immediately see that these solutions have the correct boundary conditions in the remote past and future as the retarded Green's function has no support in the remote past and conversely the advanced Green's function has no support in the remote future, i.e.

$$\lim_{x_0 \to \pm\infty} \phi(x) = \phi_{\substack{out \\ in}}(x) \tag{1.17}$$

At this time we can also introduce another useful function, the Schwinger function which is denoted simply by $\Delta(x)$ and is defined as

$$\Delta(x) = \Delta_a(x) - \Delta_r(x). \tag{1.18}$$

We can see that this function also allows us to write a simple expression relating the in and out-fields just introduced. With the help of (1.15) and (1.16) we can now write

$$\phi_{out}(x) = \phi_{in}(x) - \int d^4y \Delta(x-y)\rho(y). \tag{1.19}$$

Recalling our definitions of the retarded and advanced Green's functions we can see that the Schwinger function is just

$$\Delta(x) = \frac{1}{(2\pi)^4} \oint d^4k \frac{e^{ikx}}{m^2 - k^2}, \tag{1.20}$$

where the integral is taken on a closed path which run's clockwise and encloses both poles.

The Schwinger function will be very useful to us later so we now discuss this function in more detail. So far we haven't written out the Schwinger function explicitly so that it would be useful for calculations. It can be shown that with the use of the function

$$\epsilon(k_0) = \theta(k_0) - \theta(-k_0), \tag{1.21}$$

where $\theta(k_0)$ is the standard step function, that we can write the Schwinger function as

$$\Delta(x) = \frac{-i}{(2\pi)^4} \int d^4k \epsilon(k_0) \delta(k^2 - m^2) e^{ikx}, \tag{1.22}$$

where we have used the identity

$$\delta(k^2 - m^2) = \frac{1}{2\omega_k} \{\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\}. \tag{1.23}$$

At this point we can also take note of some properties of the Schwinger function. We can see directly from how we defined the Schwinger function that it only has support in the light cone and also that it is a solution to the homogeneous Klein Gordon equation,

$$\begin{aligned} (\Box + m^2)\Delta(x) &= 0 \\ \Delta(x) &= 0 \quad \text{for} \quad x^2 < 0. \end{aligned} \tag{1.24}$$

We can also show, by calculating the Schwinger function from (1.22) that

$$\partial_0 \Delta(x)\big|_{x_0=0} = \delta(\mathbf{x}). \tag{1.25}$$

This last property of the Schwinger function can be written in proper covariant form as

$$\int_\sigma d\sigma^\mu(x) \partial_\mu \Delta(x - y) = 1 \quad \text{if} \quad y \in \sigma \tag{1.26}$$

where σ is some spacelike surface. We can now demonstrate the most important use of the Schwinger function. It can be used as the Klein Gordon propagator for the free Klein Gordon field. This can be understood best as the famous Cauchy initial value problem. In this problem we would like to be able to determine the value of a field at a later time, or more formally, at a later space-time point, if we know the value of the field and its derivatives on an initial spacelike surface. The following holds true basically by construction,

$$\phi(y) = \int_\sigma d\sigma^\mu(x)\Delta(y-x)\overleftrightarrow{\partial_\mu}\phi(x), \tag{1.27}$$

where

$$a(x)\overleftrightarrow{\partial_\mu}b(x) = a(x)\partial_\mu b(x) - a(x)\overleftarrow{\partial_\mu}b(x) \tag{1.28}$$

We can immediately see that this is the proper solution. Firstly we can see that $\phi(y)$ is a solution to the free Klein Gordon equation because when we act on the right side of (1.27) the only y variable is in the Schwinger function which is a solution to the free Klein Gordon equation. Secondly we can see that due to (1.26) both the value of the field $\phi(x)$ and its derivative $\partial_\mu\phi(x)$ recover their proper values on the original spacelike surface. We also know that this solution is causal when we consider how the Schwinger function was constructed originally from the advanced and retarded Green's functions.

The only thing that we haven't done that would be of use to us later is to write out the solution to (1.2) explicitly, this we will save until the next section when we will deal with the field constructed with creation and annihilation operators. To recover the classical field from the one that is going to be introduced in the next section one just has to interpret the expansion coefficients as c-numbers, instead of as q-numbers as they will be introduced.

1.3 The quantised Klein Gordon Field

We now want to consider the Klein Gordon field as a q-number instead of a c-number as we have done so far. For simplicity we will illustrate the quantisation procedure for the free Klein Gordon field. We first note that with our Lagrangian (1.1), with $\rho(x) = 0$, we can construct a Hamiltonian,

$$\mathcal{H}(x) = \pi(x)\partial_0\phi - \mathcal{L}(x) \tag{1.29}$$

where

$$\pi(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_0\phi(x))} = \dot{\phi}(x). \tag{1.30}$$

This Hamiltonian written in terms of the field $\phi(x)$ and its derivatives is

$$\mathcal{H}(x) = \frac{1}{2}\left\{m^2\phi(x)\phi(x) + \nabla\phi(x)\cdot\nabla\phi(x) + \dot{\phi}(x)\dot{\phi}(x)\right\}. \tag{1.31}$$

As we now have a Hamiltonian and operators, $\phi(x)$ and $\pi(x)$, we now must ensure that the Heisenberg equations of motion are satisfied and are consistent with the field equation for free fields (1.14). The Heisenberg equations for the operators are

$$\partial_0\pi(x) = -i\left[\pi(x), \mathcal{H}(x)\right] \tag{1.32}$$

and

$$\partial_0\phi(x) = -i\left[\phi(x), \mathcal{H}(x)\right]. \tag{1.33}$$

We find that we can construct a consistent algebra by assuming the equal time commutation relations,

$$\begin{aligned} [\phi(x), \pi(y)]|_{x_0=y_0} &= i\delta(\mathbf{x}-\mathbf{y}) \\ [\phi(x), \phi(y)]|_{x_0=y_0} &= [\pi(x), \pi(y)]|_{x_0=y_0} = 0. \end{aligned} \tag{1.34}$$

These commutation relations can be seen to agree in appearance with the normal quantum mechanical commutation relation between position and momentum, they also allow us to write the total Hamiltonian in a way that allows us to introduce an obvious particle interpretation. This will be done shortly; first we will write out the field explicitly and then we will rewrite the Hamiltonian to illustrate this point. By performing a Fourier transform of the field and applying equation (1.14) we can see that the field must look like

$$\phi_{in}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3p}{\omega_p} \left\{ a_{in}(\mathbf{p}) e^{-ipx} + a^{\dagger}_{in}(\mathbf{p}) e^{ipx} \right\}. \tag{1.35}$$

with

$$p_0 = \omega_p. \tag{1.36}$$

As the only parts of this field that are not c-numbers are the $a_{in}(\mathbf{p})$ and $a^{\dagger}_{in}(\mathbf{p})$, we can calculate the commutation relations of these operators

$$\begin{aligned} [\, a(\mathbf{p}), a^{\dagger}(\mathbf{q})\,] &= \omega_p \delta(\mathbf{p} - \mathbf{q}) \\ [a(\mathbf{p}), a(\mathbf{q})] &= [\, a^{\dagger}(\mathbf{p}), a^{\dagger}(\mathbf{q})\,] = 0. \end{aligned} \tag{1.37}$$

It is now instructive to rewrite our Hamiltonian in terms of these operators $a(\mathbf{p})$ and $a^{\dagger}(\mathbf{p})$, the total Hamiltonian H is,

$$H = \int d^3x \mathcal{H}(x) \tag{1.38}$$

writing this in terms of the $a(\mathbf{p})$ and $a^{\dagger}(\mathbf{p})$ we find,

$$\begin{aligned} H &= \int \frac{d^3p}{\omega_p} \omega_p \frac{1}{2} \left(a(\mathbf{p}) a^{\dagger}(\mathbf{p}) + a^{\dagger}(\mathbf{p}) a(\mathbf{p}) \right) \\ &= \int \frac{d^3p}{\omega_p} \omega_p \left(a^{\dagger}(\mathbf{p}) a(\mathbf{p}) + \frac{\omega_p}{2} \right). \end{aligned} \tag{1.39}$$

It is at this point that we are led to a particle interpretation of the field in terms of the operators $a(\mathbf{p})$ and $a^{\dagger}(\mathbf{p})$. We first note that the total Hamiltonian H commutes with what we will call the number operator $a^{\dagger}(\mathbf{p})a(\mathbf{p})$. We can show that this operator has eigenstates of the usual Fock space built on the cyclic vacuum $|0\rangle$ defined by

$$a(\mathbf{p})\,|0\rangle = 0 \quad \forall\, \mathbf{p}. \tag{1.40}$$

We now notice that our total Hamiltonian diverges even if we look at its expectation value in the vacuum due to the factor of $\frac{1}{2}$ multiplying the frequency

ω_p. As this is unacceptable we must somehow eliminate this divergence. This is normally done in one of two ways. We can either subtract out the divergence so that the vacuum expectation value is now zero or we can do something called normal order the Hamiltonian. What we mean by normal ordering is that any product of annihilation and creation operators, $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ respectively, are ordered with the annihilation operator to the right of the creation operator, we see that this also eliminates the divergence problem with the vacuum. We can now write the normal ordered Hamiltonian as

$$H = \int \frac{d^3p}{\omega_p} \omega_p \; a^\dagger(\mathbf{p}) a(\mathbf{p}). \tag{1.41}$$

We now introduce two operators. One of these operators shifts the creation and annihilation operators by a given c-number function and its complex conjugate respectively and the other shifts the value of the field by a time independent term. To illustrate how these operators work we first introduce some relationships which have operators as part of the exponential. The following relationships can be proven by expanding the exponentials as a series.

$$\text{If} \quad [A, B] = c \quad \text{a c} - \text{number}, \tag{1.42}$$

then

$$e^A e^B = e^B e^A e^c \tag{1.43}$$

$$\left[A, e^B\right] = c e^B \tag{1.44}$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}c}. \tag{1.45}$$

We can now construct the first operator we mentioned earlier, the unitary operator S,

$$S = exp\left(-\int \frac{d^3p}{\omega_p} \left\{h(\mathbf{p}) a^\dagger_{in}(\mathbf{p}) - h^*(\mathbf{p}) a_{in}(\mathbf{p})\right\}\right) \tag{1.46}$$

using the relationships just introduced it is straightforward to show that

$$S^{-1} a_{in}(\mathbf{p}) S = a_{in}(\mathbf{p}) + h(\mathbf{p}) \tag{1.47}$$

and

$$S^{-1}a_{in}^{\dagger}(\mathbf{p})S = a_{in}^{\dagger}(\mathbf{p}) + h^{*}(\mathbf{p}). \tag{1.48}$$

Using this S operator we can now easily express the out-field introduced earlier in terms of the in-field using (1.19),

$$S^{-1}\phi_{in}(x)S = \phi_{out}(x) \tag{1.49}$$

with

$$h(\mathbf{p}) = \frac{-\sqrt{2}i}{\omega_p}\int dy_0 \tilde{\rho}(y_0, \mathbf{p})e^{iy_0\omega_p}, \tag{1.50}$$

where $\tilde{\rho}(y_0, \mathbf{p})$ is the three dimensional Fourier transform of the source $\rho(y)$. We also see that this operator can be used to get the out-vacuum from the in-vacuum, these vacuums are defined as,

$$a_{in}(\mathbf{p})\,|0_{in}\rangle = 0 \quad \forall\ \mathbf{p} \quad \text{and} \quad a_{out}(\mathbf{p})\,|0_{out}\rangle = 0 \quad \forall\ \mathbf{p}. \tag{1.51}$$

With these definitions it follows that

$$S^{-1}\,|0_{in}\rangle = |0_{out}\rangle\,. \tag{1.52}$$

We also introduce a second operator $U(x_0)$ which shifts the field by a time independent term. With

$$U(x_0) = e^{-i\int_{y_0=x_0} d^3y C(\mathbf{y})\dot{\phi}_{in}(y)} \tag{1.53}$$

$$U^{-1}(x_0)\phi_{in}(x)U(x_0) = \phi_{in}(x) + C(\mathbf{x}). \tag{1.54}$$

For this relationship we have used the commutation relations (1.35).

1.4 The external field problem

The problem of a quantum field coupled to an externally prescribed source has been studied almost since the dawn of quantum field theory. Although originally most of the work in this area was done involving the Dirac field coupled to an external electromagnetic field, recently people have started looking into this problem in the context of quantum fields in curved spacetime. In this area there are still many unanswered questions, as can been seen in Fulling's recent book "Aspects of quantum field theory in curved spacetime" [3].

As more work is done with the more complicated problem of quantum fields in curved space it has become clear that even the problem of quantum fields coupled to simpler external sources is not well understood. As was pointed out as recently as 1978 [4] "the present status of quantum field theory in external fields must be regarded as unsatisfactory". H. Rumpf et al. [4] mention that the main difficulty of the external field theory approach is describing what the physical states of the quantised field are. For us to approach this problem the most natural thing to try and do is to try and define a relevant vacuum for these states to be built upon, however this problem was not really addressed until the 70's [5] and there is still not overwhelming agreement as to how this should be done. Even in what many people refer to as the most up to date textbook on quantum field theory [6] where there is an entire chapter dealing with quantum fields interacting with external fields there is no mention of this vacuum which we would like to have, and very little written on anything but the asymptotic fields, before and after the interaction has taken place. Rumpf et al. in the paper mentioned earlier [4] go on to mention that the problems inherent in the external field problem may not attract a lot of attention as the external field problem is an approximation to the fully quantised theory which at least in the case of electromagnetism, already exists to a considerable extent. However as was mentioned in Ruijsenaars Ph.D. thesis [7] for the most part the theory for interacting quantised fields are for various reasons purely formal from a mathematical point of view. The main reason for this mathematical problem is that the field equations are nonlinear. Even for the case of quantum electrodynamics, the fully quantised theory is not as yet engraved in stone, let alone for the case of coupling to a general quantised field. We can hope and expect, however, to learn a lot about what might happen in fully quantised theories by looking at what happens in the case of fields coupled to externally prescribed sources. In a first approximation we can also treat these external sources as classical. We are forced to treat the gravitational field as classical since we don't as yet have a quantum mechanical description of gravitation.

We should expect many reasonable results from the treatment of external sources, both classical and quantised, in the same way that the Schrödinger equation gave many reasonable results for situations where it was a good approximation.

CHAPTER TWO

A REVIEW OF THE EXTERNAL FIELD PROBLEM

2.1 Introduction

Work done in the past on the external field problem can be grouped with relative ease into one of three groups. The earliest work in this area was done trying to couple a quantised field to a classical time independent source. This area of research goes back to 1940 [8] and for this reason we will first give an overview of some of the work done in this area.

The other two areas of research share one common aspect in that they both involve time dependence of some sort. The first area of research in this second group involves looking at only the fields and states before and after the interaction has taken place. Therefore in this area the object of most interest is the S matrix or S operator which connects the two asymptotically free regions, normally referred to as the in and out regions.

The third area of interest is work done with a time dependent interaction where the fields of interest are not the asymptotically free fields but the fields that are interacting with the time dependent source. For these fields the division of the field into positive and negative frequency parts is not easily done in a causal manner, and thus the definition of the vacuum in terms of the positive frequency part of the field is also not easily obtained. This third area of research is the one that has received the least amount of attention and also is the area of interest in chapter 3.

In this chapter we only concern ourselves with spin 0 and $\frac{1}{2}$ fields there are enough problems with these fields that are not well understood without going to higher spin-fields. For some of the problems associated with higher spin-fields one can refer to [10] where problems such as noncausal propagation are investigated.

2.2 Time independent external fields

As was mentioned earlier the external field problem for time independent external fields was studied as early as 1940 [8],[9]. There is one major advantage and one disadvantage in looking at the time independent problem.

The major advantage as mentioned by Fulling [11] is that in the time independent theory the division of the fields into positive and negative frequency parts is done with relative ease, this does not however mean that this division is unique. In the case of time dependent problems there is largely still no agreement as to how this division should be done. Unfortunately even in Fulling's case the notion of positive frequency is still not unique as it can be different for different coordinate systems. This disadvantage for the time dependent model will be discussed later in this chapter and is also the topic of interest in chapter 3.

The disadvantage to this time independent model is on some levels much more fundamental. Because in this model our interaction is time independent there are no asymptotically free fields and therefore no natural Fock spaces to build the theory on. As was pointed out by Labonté [12] there exists the possibility of different quantisations leading to different Fock-Hilbert spaces. These quantisations were only shown lead to the same Fock-Hilbert space for certain special potentials. In particular in the case of a Dirac field minimally coupled to an electromagnetic potential Moses and Friedrichs [21] [22] were only able to show that this problem didn't arise for certain scalar potentials without any magnetic field present. This problem with the magnetic field unfortunately appears in many places when people try to couple the Dirac field to an electromagnetic potential whether time independent or dependent.

As was mentioned earlier the consensus in the literature is that the first work on the external field problem was done in 1940 in two papers by H. Snyder, J. Weinberg and L.I. Schiff, [8] , [9]. In these two papers the authors discuss some problems that they encounter studying a charged scalar field minimally coupled to an electrostatic field. In these papers the authors study the fields associated with the Lagrangian density

$$\mathcal{L}(x) = \{\partial_t - ieV(\mathbf{x})\}\,\Psi^*(x)\,\{\partial_t + ieV(\mathbf{x})\}\,\Psi(x) - \nabla\Psi^*(x)\cdot\nabla\Psi(x) - \Psi^*(x)\Psi(x) \tag{2.1}$$

where we have set the mass $m = 1$.

When one applies the usual quantisation procedure to this Lagrangian one demands that the Heisenberg equations are satisfied so the time dependence for any function made up of the field, it's complex conjugate, and their respective conjugate momenta is determined from

$$i\dot{f}(\Psi, \Psi^*, \pi, \pi^*)(x) = [f(\Psi, \Psi^*, \pi, \pi^*)(x), H]. \tag{2.2}$$

When we then construct the charge density,

$$\rho(x) = ie(\pi^*(x)\Psi^*(x) - \pi(x)\Psi(x)) \tag{2.3}$$

we find that the total charge q is a constant of the motion i.e.

$$i\dot{q} = [q, H] = 0. \tag{2.4}$$

Because of this we should be able to diagonalize both q and H simultaneously and thus have states of definite charge and energy. Unfortunately because of our electrostatic term V in the Lagrangian one also finds this "extra" term in our conjugate momenta expressions. Because of this the normal quantisation procedure is quite difficult as one normally imposes commutation relations between the fields and their conjugate momenta. For this reason the authors perform the quantisation in a different manner and then later show that the two procedures are equivalent. To perform this different quantisation the authors first calculate the field equations from the Lagrangian density (2.1),

$$(\partial_t + ieV(\mathbf{x}))^2 \Psi(x) = \nabla^2\Psi(x) - \Psi(x). \tag{2.5}$$

They now perform a time Fourier transform on this field equation and find that the time Fourier transformed fields satisfy,

$$(E_k - eV(\mathbf{x}))^2 \Psi_k(\mathbf{x}) = \Psi_k(\mathbf{x}) - \nabla^2\Psi_k(\mathbf{x}) \tag{2.6}$$

where

$$\Psi(\mathbf{x}, t) = \int dE_k e^{-iE_k t}\Psi_k(\mathbf{x}) \tag{2.7}$$

If we now use Green's theorem and assume that the fields go to zero at infinity we find that the Ψ_k's are not orthogonal to one another but satisfy a modified orthogonality relation of the form

$$\int d^3x \Psi_k^*(\mathbf{x}) (E_k + E_l - 2eV) \Psi_l(\mathbf{x}) = \epsilon_k \delta_{kl} \tag{2.8}$$

where $\epsilon_k = \pm 1$. The meaning of ϵ_k can be seen if we calculate the total charge and Hamiltonian, for these we find

$$q = \Sigma_k a_k^\dagger a_k \epsilon_k \tag{2.9}$$

and

$$H = \Sigma_k a_k^\dagger a_k E_k \epsilon_k. \tag{2.10}$$

It is now evident that the division of the Ψ_k into states of positive and negative ϵ_k correspond to states with positive and negative charge.

To calculate the commutation relations for the a_k and $a_k^\dagger$ we calculate the time derivative of these operators in two different ways and require that both methods give the same result. The first way one can calculate this time derivative is to notice that one can write out any $\Psi(\mathbf{x},t)$ as a series expansion over the Ψ_k,

$$\Psi(\mathbf{x},t) = \Sigma_k \, a_k \Psi_k(\mathbf{x}). \tag{2.11}$$

The only time dependence on the right hand side of this equation must come from the a_k therefore we thus must have

$$\dot{a}_k = -iE_k a_k. \tag{2.12}$$

If one now also calculates the time derivative using the Hamiltonian and the Heisenberg equations (2.4) one finds that one must have,

$$[a_k, a_l^*] = \frac{\delta_{kl}}{\epsilon_k}. \tag{2.13}$$

If we use these commutation relations to calculate the commutation relations between the fields and their conjugate momenta we find that we must impose conditions involving the ϵ_k's as well as the fields if these commutation relations are to agree with the usual equal time commutation relations imposed on the fields. This quantisation procedure is then shown to be equivalent to the usual one by constructing the evolution of the fields in terms of Green's functions. It is then found that for these evolution equations to reproduce the required boundary conditions on the field and it's derivative the same conditions must be imposed on the fields and the ϵ_k's. We can see that from (2.13) that if ϵ_k is $+1$ then a_k is the usual annihilation operator and $a_k^\dagger$ is the usual creation operator. If however ϵ_k is -1 then their roles are reversed. At this point the authors then find a canonical transformation to get from the old field quantities to some new field quantities such that the charge operator and Hamiltonian will be in diagonal form. In the second paper on basically the same subject the authors point out that it is not possible to perform these canonical transformations if any of the frequencies E_k go complex. In this situation our Hamiltonian is no longer self-adjoint and one must find a self-adjoint extension before finding the appropriate transformations. Unfortunately it is shown that this can occur even for the simple model of a sufficiently deep well.

This same problem involving strong interactions is later discussed by Schroer et al. [18] where they show that for a time independent interaction, if the interaction is strong enough, one must either introduce an indefinite metric or do without a vacuum state. The type of interaction that the authors look at involves the interaction of the charged scalar field with an external scalar interaction $V(\mathbf{x})$ such that the relevant field equation is

$$\left(\partial_t^2 - \nabla^2 + m(m - V(\mathbf{x}))\right) A(x) = 0. \tag{2.14}$$

The authors then look for the stationary solutions to this field equation in the usual way by expressing the field as a Fourier transform. Unfortunately they find that the only way in which they can construct a Fock space consistently is to introduce an indefinite metric. What one finds is that for the imaginary energy modes the commutation relations are

$$[a, b^\dagger] = i \tag{2.15}$$

and all other commutators are zero. From this point the only way of then constructing a Fock space consistently is to introduce an indefinite metric such that,

$$\begin{aligned} a\,|0\rangle = b\,|0\rangle &= 0 \\ a^\dagger\,|0\rangle = |a\rangle & \qquad b^\dagger\,|0\rangle = |b\rangle \\ \langle a|\,a\,\rangle = \langle b|\,b\,\rangle &= 0 \\ \langle a|\,b\,\rangle &= i \end{aligned} \tag{2.16}$$

With this construction it is then possible to show that both the energy and charge are conserved although the number operators for the imaginary energy modes must be different from the positive energy ones which are of the usual form, the number operators for the imaginary energy states are,

$$\begin{aligned} N_{a_i} &= i a_i^\dagger b_i \\ N_{b_i} &= -i b_i^\dagger a_i. \end{aligned} \tag{2.17}$$

For the positive energy operators the usual Fock representation is the only possibility. However, for the imaginary energy operators there is an alternative to the representation requiring an indefinite metric as briefly outlined above. Schroer et al. show that if one introduces different canonical variables one can have a Fock-Space representation for the imaginary energy modes with a positive definite metric, howevei part of the Hamiltonian describes repulsive oscillators. For this reason it is therefore impossible to have a particle like interpretation for these modes as they are without a vacuum state, i.e. no lowest energy state exists. Schroer et al. also mention that these are the only two possible quantisations for the Hamiltonian at hand because the continuous part of the Hamiltonian has the form

$$\int d^3k\ \omega_k\,(N_a(\mathbf{k}) + N_b(\mathbf{k}))\,, \tag{2.18}$$

and therefore the Hamiltonian uniquely selects the Fock representation of the $a(\mathbf{k})$ and $b(\mathbf{k})$ so that the Hamiltonian makes sense as a self adjoint operator in the representation space. We are therefore left with one of two choices, abandon the usual structure of quantum theory and introduce an indefinite metric, or introduce a positive definite metric and do without a vacuum state.

As mentioned earlier much attention has been given to the areas of research involving external fields due to the interest in quantum field theory in curved spacetime. Many of the problems one encounters when trying to interpret the physics of this problem were discussed or at least alluded to in a paper by S.A. Fulling in 1973 [11]. In this paper Fulling discusses some of the ambiguities which arise when one tries to construct a quantum field theory in a background metric which is not Minkowskian. It is Fulling's goal to follow the most common strategy and to choose, if possible, a coordinate system where the field equations are separable and to then quantise the resulting normal mode structure in close analogy to what is normally done in the standard quantisation of a free field in Minkowski spacetime. Fulling also proposes a test for any quantisation prescription which supposedly is applicable to static metrics. The quantisation prescription must yield physically reasonable results if it is applied to a flat spacetime equipped with an unusual co-ordinate system. It is hoped that this test may be used to eliminate some of the ambiguities by eliminating incorrect theories on physical grounds.

To demonstrate this quantisation procedure Fulling quantises a scalar field on a Riemannian manifold of dimension n which is equipped with a metric tensor $g_{\mu\nu}$. It is assumed that there is a coordinate system in which the metric is static i.e.

$$\partial_{x_0} g_{\mu\nu} = 0 \tag{2.19}$$

and

$$g_{oj} = 0 \quad \text{for} \quad j \neq 0 \tag{2.20}$$

The generalisation of the Klein Gordon equation in this situation is

$$(\Box_c + m^2)\phi(x) = 0 \tag{2.21}$$

where

$$\Box_c = |g|^{-\frac{1}{2}}\partial_\mu(|g|^{\frac{1}{2}} g^{\mu\nu}\partial_\nu) \quad \text{and} \quad g = det\,\{g_{\mu\nu}\}\,. \tag{2.22}$$

When the metric is static the field equation (2.21) can be solved by separation of the time variable,

$$\phi(t, \mathbf{x}) = \Psi_j(\mathbf{x})e^{\pm iE_j t}. \tag{2.23}$$

With this substitution one obtains the eigenvalue equation,

$$|g|^{-\frac{1}{2}} g_{00}\partial_l(|g|^{\frac{1}{2}} g^{lk}\partial_k\Psi_j(\mathbf{x})) + g_{00}m^2\Psi_j(\mathbf{x}) \equiv K\Psi_j(\mathbf{x}) = E_j^2\Psi_j(\mathbf{x}). \tag{2.24}$$

Fulling then shows that K is hermitian and its expectation values are positive which implies that all the E_j^2 are non-negative. For convenience it is assumed that the numbers in the spectrum can be classified as either being part of a point spectrum σ_p or a continuous spectrum σ_c and that a complete set of eigenfunctions exists such that the Hilbert space can be expanded as

$$F(x) = \int d\mu(j)\overline{f}(j)\Psi_j(x) \tag{2.25}$$

where $\int d\mu(j)$ means to sum over the point spectrum and to integrate over the continuous spectrum. We can now write the general solution to the field equation (2.21) as

$$\phi(t, \mathbf{x}) = \int \frac{d\mu(j)}{(2E_j)^{\frac{1}{2}}} \left[a_j \Psi_j(\mathbf{x}) e^{-iE_j t} + a_j^\dagger \Psi_j(\mathbf{x}) e^{iE_j t}\right]. \tag{2.26}$$

At this point one can normal order the operators and in analogy with normal free field theory in flat spacetime this eliminates the problem of an infinite c-number contribution to the vacuum energy.

As this procedure should be applicable to any static metric Fulling then applies this to the two dimensional Rindler space wedge. With this procedure Fulling is able to find the eigenfunctions of the appropriate field equation and writes these eigenfunctions in terms of the Rindler coordinates v and z,

$$\Psi_j(z) = \pi^{-1}\left[2j sinh(\pi j)\right]^{\frac{1}{2}} K_{ij}(mz), \tag{2.27}$$

where the K_{ij} are the modified Bessel functions of imaginary order. One is now able to write out an expansion of the field in terms of creation and annihilation operators

$$\phi(v, z) = \int dj \frac{1}{(2j)^{\frac{1}{2}}} \Psi_j(z)(e^{-ijv} a_j + e^{ijv} a_j^\dagger). \tag{2.28}$$

Problems now arise if one writes out the positive frequency part of this field, in terms of Rindler coordinates, and compares this quantity to the positive frequency part of the free field quantised in terms of cartesian coordinates in Minkowski space. If one relates these two different fields one finds that the positive frequency part of one of these fields has to be expressed in terms of both the positive frequency and negative frequency parts of the other. Because of this fact the two quantisations are different. The subtleties of the problem at hand can partially be illustrated by studying the problem of generalising to a Riemannian manifold the Green's functions associated with the Klein Gordon equation. Although some of these functions are determined solely by the manifold and restrictions on their supports, some are not. The positive frequency part of the Schwinger function can be defined as

$$\Delta^{+}(x,y) = \langle 0|\, \phi(x)\phi(y)\, |0\rangle\,, \tag{2.29}$$

however this definition obviously requires a unique vacuum or a notion of positive frequency. Unfortunately we have seen that this is not a trivial object to find even in the time independent problem. Of course the problem is even more difficult with the time dependent interaction as has been mentioned before. Fulling goes on to point out that the two quantisations giving different vacuums is reminiscent of a Casimir type effect, as the quantisation in Rindler space can be interpreted as that appropriate to the physical situation of an impenetrable wall located on the light cone. He then reminds us of this point by showing that the vacuum state and the energy density of a free field in a box with periodic boundary conditions differ from those of a region of the same size in infinite space.

Perhaps the most comprehensive discussion on the external field theory problem was given in 1975 by G. Labonté [12], although most of the paper deals with the time dependent model he does mention some interesting results for the time independent problem. Labonté finds that "exactly as in free field theory: i) the field is always a well defined operator valued distribution on the same Fock-Hilbert space as determined at one time. ii) There exists a self adjoint, non-negative definite, energy operator, the eigenvalues of which are simply the sums of the energies of all the individual energy quanta contained in the system. iii) At all times, the system can be completely described in terms of these and none of these are ever created nor annihilated ". Labonté also investigates the case where the potential is a good c-number scattering potential. The scattering matrix is obtained with the LSZ formalism [20] and it is found that any (anti)particle bound in the potential remains bound and any (anti)particle with enough energy scatters individually in the potential. As this is for the time independent potential there is no creation or annihilation of (anti)particles only a distortion of the individual wave functions. Labonté illustrates that for the time independent model the problem is completely analogous to the quantised free field. The physical interpretation of this problem is just that implicit in the Furry representation [19] . Landau and Lifshitz [19] refer to the Furry representation as being intermediate between the Heisenberg and interaction representation. It should be noted that in this treatment by Labonté there is a restriction on the potential such that the energy operator remains self adjoint.

2.3 The S operator

With regards to the external field problem there has been a lot of successful work done in finding the S operator and its properties. We can recall from chapter 1 (1.49) that the S operator connects the incoming and outgoing fields and also the incoming and outgoing states,

$$\phi_{out}(x) = (S^{-1}\phi_{in}S)(x) \tag{2.30}$$

$$S\,|a_{out}\rangle = |a_{in}\rangle\,. \tag{2.31}$$

As can been seen from above the S operator can be understood as inducing a transformation between the in creation and annihilation operators and the out creation and annihilation operators. Therefore,

$$\begin{aligned} a_{out}(\mathbf{k}) = S^{-1}a_{in}(\mathbf{k})S &= \int d^3q\alpha(\mathbf{k},\mathbf{q})a_{in}(\mathbf{q}) + \beta(\mathbf{k},\mathbf{q})a^{\dagger}_{in}(\mathbf{q}) \\ a^{\dagger}_{out}(\mathbf{k}) = S^{-1}a^{\dagger}_{in}(\mathbf{k})S &= \int d^3q\alpha^*(\mathbf{k},\mathbf{q})a^{\dagger}_{in}(\mathbf{q}) + \beta^*(\mathbf{k},\mathbf{q})a_{in}(\mathbf{q}) \end{aligned} \tag{2.32}$$

In this way it can be seen how the S matrix can be used to calculate the number of particles which have been created during the interaction. To calculate the number of particles created one just has to calculate the expectation value of the number operator. Before the interaction,

$$\langle 0_{in}|\, a^\dagger_{in}(\mathbf{k})a_{in}(\mathbf{k})\, |0_{in}\rangle = 0 \tag{2.33}$$

however after the interaction the relevant operators are a_{out} and $a^\dagger_{out}$ so when one finds the expectation value of the number operator one finds, using (2.33)

$$\langle 0_{in}|\, a^\dagger_{out}(\mathbf{k})a_{out}(\mathbf{k})\, |0_{in}\rangle = \int d^3k'\, |\beta(\mathbf{k},\mathbf{k}')|^2 \tag{2.34}$$

so that number of particles created is related to the amount of mixing there is between positive and negative frequencies of the in and out-fields. This is one of the physically interesting things one can do with the S operator and is therefore prevalent in the literature. Of course all of the physics comes into the problem as to how one arrives at these transformations (2.33). As one can see at this point the problems are not with interpreting one's results after one has found the S operator or S matrix but actually showing that the operator or matrix exists and that it is unitary so that the two Fock spaces {in and out} are unitarily equivalent.

In this respect R. Seiler [15] has shown the existence and uniqueness of the S matrix for two different models. He has shown these properties for the S matrix for the scalar field coupled to an external scalar source and for the Dirac field coupled to an external electromagnetic source. Capri [16] has shown similarly the existence and unitarity of the S matrix for the Dirac field coupled to an external electromagnetic field and in this way has shown that the in-fields and out-fields are unitarily equivalent. As both the in-fields and out-fields satisfy the same commutation relations it was sufficient for Capri to show that there exists a vacuum for the out-field in the Hilbert space of the in-field i.e. some combination of the in-states yields the out-vacuum.

An interesting point was raised by Labonté in his 1975 paper [12] about the uniqueness of the asymptotic vacua. As one would expect there are problems in defining unique asymptotic vacua if there are any bound states. This can be best understood by looking at the scattering particles which are the only thing we have to define the asymptotic vacua in terms of. The bound state particles create problems for us here because they do not interact with the scattering particles as we only regard the particles as interacting with the external potential. In this sense our asymptotic vacua could contain any number of bound particles or antiparticles and this would not affect our definition of the free vacua. We avoid any problems which would be associated with bound states as we only consider potentials which vanish rapidly in any spacetime direction.

More recently (1975) R.M. Wald [13] has also shown that the S matrix exists for a gravitational field of compact support and has calculated the number of particles created. In a later paper [14] Wald shows that a sufficient condition for an S matrix to exist, and for it to be unique is that a certain operator B, which connects certain in and out one particle Hilbert spaces, is Hilbert-Schmidt ($\mathrm{tr}B^{\dagger}B < \infty$). This condition is related to there only being a finite number of particles created during the interaction.

2.4 The interpolating field

Although this is not the approach that we will be using in chapter 3 in our investigation of the interpolating field there has been much work done in trying to use the S matrix to construct the interpolating fields, unfortunately this procedure is not unique and not very straightforward. In fact in the second Lehmann, Symanzik, and Zimmerman (L.S.Z.) paper on the formulation of quantised field theories [20] the authors show that for any given S matrix there exist many invariant fields $A(x)$ which tend asymptotically to $A_{out}(x)$ and $A_{in}(x)$ for $t \rightarrow \pm\infty$ respectively. The reason that different fields can tend asymptotically to the correct limit is shown in this paper to be related to extrapolating various functions $\tilde{h}(\mathbf{k})$ off the mass shell.

The fields constructed from these functions reproduce the in and out-fields as these fields only depend on the functions evaluated on the mass shell. This ambiguity arises because there isn't a unique prescription for extrapolating these functions off the mass shell. To attempt to eliminate some or all of this ambiguity the authors introduce the concept of a causal scattering matrix by requiring that the interpolating field be causal, the authors call an operator causal if,

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{2.35}$$

In terms of these functions mentioned earlier the $\tilde{h}(\mathbf{k})$, this means that a particular scattering matrix is causal if there exists at least one continuous extrapolation of these functions off the mass shell such that the commutator (2.35) is satisfied. Unfortunately the authors then leave the question open as to whether any such scattering matrix exists which is causal in this sense or whether this is too stringent a demand. It should be recalled at this time that when we talk of the S matrix or S operator we are talking about the operator which connects the asymptotically free fields and that the problems mentioned above are not problems with the S matrix or operator but with trying to use it to find the interpolating fields.

Instead of this approach we will use the Green's functions mentioned in chapter 1 to investigate the interpolating fields. Capri [16] also used this approach in dealing with a Dirac field minimally coupled to an electromagnetic field and was able to show that the anticommutator of the field evaluated at two different points yields a Schwinger type function with the same properties as the Schwinger function introduced in chapter 1. Capri actually expressed the fields in terms of the field operators smeared with test functions and therefore the relationship that Capri actually proved was,

$$\{\Psi(f), \Psi(g)\} = iS_R^B(f,g) - iS_A^B(f,g) \tag{2.36}$$

where f and g are the test functions and $S_R^B(f,g)$ and $S_A^B(f,g)$ are the retarded and advanced Green's functions respectively.

Of course in the simple model we introduced in Chapter 1 showing that the interpolating fields are local is easy as the interpolating fields satisfy the same algebra as the free fields. This is because of the simple way in which we have coupled the scalar field to the c-number source.

Problems with the interpolating scalar field have also been looked at by R. Seiler and others in two papers [15] [17] where the potential is coupled to the scalar field in a slightly different way than we have looked at thus far, they look at the scalar field satisfying the field equation,

$$(\Box + m^2)\phi(x) = V(x)\phi(x) \tag{2.37}$$

In these two papers the authors discuss problems related to the time evolution operator which takes the field from a time t_1 to some other time t_2 as long as $t_1 < t_2$. They manage to construct a unitary operator in the Fock space generated by the in-field by showing that a certain Hilbert-Schmidt condition of the classical time-evolution kernel is satisfied. The problem of this operator being in the Fock space generated by the in-fields is then equivalent to showing the existence of a vacuum for the creation and annihilation operators $a(t), a^{\dagger}(t)$ at any time t in the Fock space of the in-fields.

Although some of our discussion would lead one believe that the only problem at hand is finding a time evolution operator to induce a Bogoliubov transformation to connect the creation and annihilation operators from one time to another, Labonté and Capri pointed out that for the situation of the interpolating field interacting with a time dependent source this is not the case. For the interpolating field in this case it is not even clear how one is supposed to causally identify the positive and negative frequency parts of the field and in this way identify the annihilation and creation operators of the field respectively. There are two straightforward ways of identifying the positive frequency part of the interpolating field,

$$\phi^{+}(t,\mathbf{x}) = \frac{1}{2\pi}\int dk_0 \theta(k_0) e^{ik_0 t}\int dt' e^{-ik_0 t'}\phi(t',\mathbf{x}) \tag{2.38}$$

or

$$\phi^{+}(x) = \phi^{+}_{\text{in}}(x) + \int dy (\Delta_R)^{(+)}(x-y)\rho(y). \tag{2.39}$$

Both of these definitions are however acausal. The first one is acausal as the t' integral runs from $-\infty$ to $+\infty$ so one has to know the field at all times before one can define the positive frequency at one time. In the second definition the positive frequency portion of the retarded Green's has support in both the future and past light cones and is therefore also acausal. To get around this problem Labonté and Capri introduce an auxiliary field which is coupled to a time independent source which matches the time dependent source at a particular time. One can then use information about this auxiliary field to deduce information about the interpolating field at that particular time. This is the procedure that we will now use in chapter 3 to define the vacuum relevant to the interpolating scalar field interacting with a time dependent source introduced in the first chapter.

CHAPTER THREE

VACUUM FOR AN INTERACTING FIELD

3.1 Introduction

From our discussion in Chapter 2 we saw that only in the last twenty years has the question of instantaneous vacuum definition for the time dependent external field problem been addressed. The most straightforward way of defining a vacuum for a quantised field is to somehow extract the positive frequency part of the field and thus define the vacuum as the state which is annihilated by this part of the field. Unfortunately if the interaction we wish to study is time dependent the positive frequency part of the field is not easy to come by in a causal way.

We show that by introducing an auxiliary field, similar to that introduced by G. Labonté and A.Z. Capri [5], we can define a vacuum at any time during the interaction. This auxiliary field is coupled to a time independent source which matches the time dependent source at a particular time. Thus the equations for the auxiliary field are time-translation invariant and the field may be decomposed in a causal manner into positive and negative frequency parts. Futhermore the auxiliary field is matched to the Heisenberg field (coupled to the time dependent source) at that particular time. We then define the vacuum for the auxiliary field and this is then also the vacuum for the Heisenberg field at that particular time. We find this vacuum and show that it is a more appropriate vacuum for the interacting field than the vacuum for the incoming free Heisenberg field in the sense that the energy of the field is lower in the vacuum we propose as compared to the energy of the field in the vacuum for the incoming Heisenberg field.

3.2 Auxiliary Field

We consider the simple model consisting of a real scalar field $\Phi(x)$ coupled to a time dependent c-number source. An auxiliary field $\Phi_\tau(x)$ is introduced to match this field at a particular time τ and since the Heisenberg field at time τ coincides with the auxiliary field Φ_τ at time τ we can deduce things about the Heisenberg field at time τ. The source is assumed to vanish rapidly in the remote past and future and therefore the Heisenberg field is asymptotically a free field. This is also what was assumed by G. Labonté and A.Z. Capri [5] for the Dirac field coupled to a time dependent electromagnetic field.

The equations of motion of the Heisenberg and auxiliary fields are respectively

$$(\Box + m^2)\Phi(x) = \rho(x) \tag{3.1}$$

$$(\Box + m^2)\Phi_\tau(x) = \rho(\tau, \mathbf{x}) \tag{3.2}$$

with the boundary conditions

$$\lim_{x_0 \to -\infty} \Phi(x) = \Phi_{in}(x) \tag{3.3}$$

$$\Phi_\tau(x)\,|_{x0=\tau} = \Phi(x)|_{x0=\tau} \tag{3.4}$$

$$(\partial_0 \Phi_\tau(x))\,|_{x0=\tau} = (\partial_0 \Phi(x))\,|_{x0=\tau} \tag{3.5}$$

We can immediately write the field $\Phi_\tau(x)$ in terms of the Heisenberg field $\Phi(x)$ by using the boundary conditions imposed on $\Phi_\tau(x)$ with the usual Klein Gordon propagator and a c-number term due to the source. Thus,

$$\Phi_\tau(x) = \int_{y0=\tau} d^3y \Delta(x-y) \overleftrightarrow{\partial_0} \Phi(y) + S(x) \tag{3.6}$$

where the c-number function S(x) satisfies the inhomogeneous part of the equation of motion and the following boundary conditions

$$S(x)\,|_{x0=\tau} = 0 \tag{3.7}$$

$$\partial_{x_0} S(x)\,|_{x0=\tau} = 0. \tag{3.8}$$

We can find S(x) explicitly by expressing the field $\Phi_\tau(x)$ as a three dimensional Fourier decomposition and imposing the boundary conditions (3.7) and (3.8). The three dimensional Fourier transform of $S(x)$, $\tilde{S}(x_0, \mathbf{k})$, can then be written as

$$\tilde{S}(x_0, \mathbf{k}) = \frac{\tilde{\rho}(\tau, \mathbf{k})}{\omega_k^2}\left\{1 - \frac{e^{i\omega_k(x_0-\tau)}}{2} - \frac{e^{-i\omega_k(x_0-\tau)}}{2}\right\} \tag{3.9}$$

where $\tilde{\rho}(\tau, \mathbf{k})$ is the three dimensional Fourier transform of the time independent source. Thus, the auxiliary field can be written as

$$\Phi_\tau(x) = \int_{y_0=\tau} d^3y \Delta(x-y) \overleftrightarrow{\partial_0} \Phi(y) + \int \frac{d^3k}{(2\pi)^3}\tilde{S}(x_0, \mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{3.10}$$

If we now write the Heisenberg field $\Phi(y)$ as

$$\Phi(y) = \Phi_{in}(y) + \int d^4z \Delta_R(y-z)\rho(z) \tag{3.11}$$

and

$$\Phi_{in}(x) = \frac{1}{\sqrt{2(2\pi)^3}}\int \frac{d^3p}{\omega_p}\left\{a_{in}(\mathbf{p})e^{-ip\cdot x} + a_{in}^\dagger(\mathbf{p})^{ip\cdot x}\right\} \tag{3.12}$$

we can then write the auxiliary field as

$$\Phi_\tau(x) = \int \frac{d^3p}{\omega_p\sqrt{2(2\pi)^3}}\left\{(\, a_{in}(\mathbf{p}) + h(\mathbf{p})\,)\, e^{-ip\cdot x} + \left(a_{in}^\dagger(\mathbf{p}) + h^*(\mathbf{p})\right) e^{ip\cdot x}\right\} + C(\mathbf{x}). \tag{3.13}$$

Here

$$h^*(\mathbf{p}) = \frac{1}{\sqrt{2(2\pi)^3}}\left(\int \frac{dk_0}{(2\pi)}\frac{\tilde{\rho}(k_0, \mathbf{p})(\omega_p + k_0)}{(\omega_p^2 - k_0^2 + i\epsilon k_0)}e^{-i(\omega_p - k_0)\tau} - \frac{\tilde{\rho}(\tau, \mathbf{p})}{\omega_p}e^{-i\omega_p\tau}\right) \tag{3.14}$$

and

$$C(\mathbf{x}) = \int \frac{d^3p}{\omega_p^2(2\pi)^3}\tilde{\rho}(\tau, \mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}. \tag{3.15}$$

3.3 Vacuum Definition

The Heisenberg field is defined on the Fock space $\mathcal{H}_{in}$ of incoming states with the cyclic vacuum $|0_{in}\rangle$. To define the vacuum for the auxiliary field $\Phi_\tau(x)$, (which is to be the vacuum for the Heisenberg field $\Phi(x)$ at time $x_0 = \tau$), in terms of the states in $\mathcal{H}_{in}$ we first express the field, $\Phi_\tau(x)$, in terms of the incoming field, $\Phi_{in}(x)$. This is accomplished by using two operators S and $U(x_0)$ such that

$$\Phi_\tau(x) = U^{-1}(x_0) S^{-1} \Phi_{in}(x) S U(x_0) \tag{3.16}$$

where S and $U(x_0)$ are given by

$$S = exp\left[\int \frac{d^3p}{\omega_p}\{h(\mathbf{p})a^\dagger_{in}(\mathbf{p}) - h^*(\mathbf{p})a_{in}(\mathbf{p})\}\right] \tag{3.17}$$

and

$$U(x_0) = exp\left[-i\int_{y_0=x_0} d^3y C(\mathbf{y})\partial_0\Phi_{in}(y)\right] \tag{3.18}$$

Here S shifts the operators $a_{in}(p)$ and $a^\dagger_{in}(p)$ by $h(p)$ and $h^*(p)$ respectively while $U(x_0)$ shifts the entire field by the time independent term $C(\mathbf{x})$. Although the operator $U(x_0)$ shifts the field by a time independent term it is operating on the time dependent field $\Phi_{in}(x)$ and for this reason must have time dependence for the net result not to have any time dependence. We can now define a vacuum for the field $\Phi_\tau(x)$ by looking at the positive frequency part of the field given in equation (3.13). For now we ignore the time independent term $C(\mathbf{x})$. This vacuum is thus defined by:

$$a_\tau(\mathbf{k})\,|0_\tau\rangle = 0 \qquad \forall \mathbf{k} \tag{3.19}$$

where

$$a_\tau(\mathbf{k}) = a_{in}(\mathbf{k}) + h(\mathbf{k}) \tag{3.20}$$

This vacuum can be obtained from the in-vacuum using the S operator introduced earlier as,

$$\Big|0_\tau\Big\rangle = S^{-1}\Big|0_{in}\Big\rangle \tag{3.21}$$

To show that this vacuum is a more appropriate vacuum for the field $\Phi_\tau(x)$, than the vacuum for the incoming field, and thus a more appropriate vacuum for the Heisenberg field $\Phi(x)$ at time $x_0 = \tau$, we calculate the energy of the field in both vacuums and show that the field has a lower energy in the vacuum we have proposed. We thus calculate the expectation value of the total hamiltonian in both vacua and compare. The easiest way to do this is to just calculate

$$E = \int d^3x \left\langle 0_{in} \left| H_\tau(x) - SH_\tau(x)S^{-1} \right| 0_{in} \right\rangle |_{x_0=\tau} \tag{3.22}$$

where $H_\tau(x)$ is the Hamiltonian density of the field $\Phi_\tau(x)$ given by

$$H_\tau(x) = \frac{1}{2}\left(m^2(\Phi_\tau(x))^2 + (\dot{\Phi}_\tau(x))^2 + (\nabla\Phi_\tau(x))^2\right) - \rho(\tau, \mathbf{x})\Phi_\tau(x). \tag{3.23}$$

Note that at $x_0 = \tau$ this coincides with $H(x)$, the hamiltonian density of the Heisenberg field $\Phi(x)$. To calculate E it is convenient to first rewrite $h^*(p)$ as

$$h^*(p) = \frac{1}{\sqrt{2(2\pi)^3}}\left(d(\mathbf{p}) - \frac{\tilde{\rho}(\tau, \mathbf{p})}{\omega_p} e^{-i\omega_p \tau}\right) \tag{3.24}$$

where

$$\begin{aligned} d(\mathbf{p}) &= \int \frac{dk_0}{(2\pi)} \frac{\tilde{\rho}(k_0, \mathbf{p})}{(\omega_p^2 - k_0^2 + i\epsilon k_0)} (\omega_p + k_0) e^{-i(\omega_p - k_0)\tau} \\ &= -i \int dx_0 \theta(\tau - x_0) \tilde{\rho}(x_0, \mathbf{p}) e^{-i\omega_p x_0}. \end{aligned} \tag{3.25}$$

Now most of the terms can be made to look like

$$\int \frac{d^3p}{2(2\pi)^3} \frac{\tilde{\rho}^*(\tau, \mathbf{p})}{\omega_p} \left(d(\mathbf{p}) - \frac{\tilde{\rho}(\tau, \mathbf{p})}{\omega_p} e^{-i\omega_p \tau)}\right) e^{i\omega_p x_0} \tag{3.26}$$

or its complex conjugate. If we now rewrite $\rho(\tau, \mathbf{x})$ as

$$\rho(\tau, \mathbf{x}) = \int \frac{d^3k}{2(2\pi)^3} \left(\tilde{\rho}(\tau, \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + \tilde{\rho}^*(\tau, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\right) \tag{3.27}$$

we find that when the energy difference E is evaluated at $x_0 = \tau$ almost all the terms cancel leaving,

$$E = \int \frac{d^3p}{2(2\pi)^3} \left\{ \left| i \int dx_0 \theta(\tau - x_0) \tilde{\rho}(x_0, \mathbf{p}) e^{-i\omega_p x_0} + \frac{\tilde{\rho}(\tau, \mathbf{p})}{\omega_p} e^{-i\omega_p \tau} \right|^2 \right\}, \tag{3.28}$$

and therefore as this difference is positive definite the energy of the field is lower in the vacuum $|0_\tau\rangle$ than in the in-vacuum $|0_{in}\rangle$.

So far we have ignored the time independent term in the auxiliary field $\Phi_\tau(x)$. If we try and include this term in hopes that a vacuum that incorporates this term will somehow be better, a possible thing to do is to break it up so that part contributes to our new annihilation operator $a'_\tau(\mathbf{k})$ and the rest to the creation operator $a'^\dagger_\tau(\mathbf{k})$.

An obvious way of breaking up $C(\mathbf{x})$ is to write it as

$$C(\mathbf{x}) = \frac{1}{2(2\pi)^3}\int \frac{d^3p}{\omega_p^2}\left\{\tilde{\rho}(\tau,\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + \tilde{\rho}^*(\tau,\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\right\}, \tag{3.29}$$

in this way the two parts of $C(\mathbf{x})$ are each other's complex conjugate and thus the new annihilation and creation operators are still the hermitian conjugates of each other. We can now define a new vacuum $|0'_\tau\rangle$ as

$$a'_\tau(\mathbf{k})\,|0'_\tau\rangle = 0 \qquad \forall\mathbf{k} \tag{3.30}$$

where

$$a'_\tau(\mathbf{k}) = a_{in}(\mathbf{k}) + h(\mathbf{k}) + \frac{1}{\sqrt{2(2\pi)^3}}\frac{\tilde{\rho}^*(\tau,\mathbf{p})}{\omega_p}. \tag{3.31}$$

If we again calculate the difference of the energies in the different vacua

$$E' = \int d^3x\left(\langle 0_{in}|H_\tau(x)|\,0_{in}\rangle - \langle 0'_\tau|H_\tau(x)|\,0'_\tau\rangle\right)|_{x_0=\tau} \tag{3.32}$$

we find that

$$E' = \int \frac{d^3p}{2(2\pi)^3}\left\{\left|i\int dx_0\theta(\tau-x_0)\tilde{\rho}(x_0,\mathbf{p})e^{-i\omega_p x_0} + \frac{\tilde{\rho}(\tau,\mathbf{p})}{\omega_p}e^{-i\omega_p\tau}\right|^2 - \left|\frac{\tilde{\rho}(\tau,\mathbf{p})}{\omega_p}\right|^2\right\} \tag{3.33}$$

and is thus less than the energy difference (E) calculated earlier. This means that the energy of the interacting Heisenberg field has a lower energy in the first vacuum $|0_\tau\rangle$, defined by equation (3.19), than in the second vacuum $|0'_\tau\rangle$, defined by equation (3.30).

3.4 Conclusion

We have therefore shown that since the energy difference defined by equation (3.28) is positive definite that indeed the vacuum we have proposed $|0_\tau\rangle$, as defined in equation (3.19), is a more appropriate vacuum (in the sense that the energy of the field is lower in this vacuum) for the auxiliary field $\Phi_\tau(x)$, and thus a more appropriate vacuum (in the same sense) for the Heisenberg field $\Phi(x)$ at $x_0 = \tau$, than the in-vacuum for the incoming Heisenberg field $\Phi_{in}(x)$. We have also shown that the obvious choice of vacuum for the field is better than the vacuum defined by trying to incorporate the time independent term of the field into a new vacuum.

CHAPTER FOUR

QUANTUM FIELD THEORY IN CURVED SPACETIME

4.1 Introduction

Quantum field theory in curved spacetime has received much interest since 1975 following Hawking's discovery of particle creation by black holes [23]. Unfortunately this theory is still not well understood and many people are still working to try and comprehend some of the details of the theory.

When studying quantum field theory in curved spacetime people's work can normally be grouped into one of three categories, with some papers dealing with topics of interest involving more than one group. The main interest of this thesis has been in the category that one could label as formalism and this will be the first topic dealt with in this chapter. Here one wishes to find a procedure to build a quantum field theory in curved space in a manner very similar to what one does in Minkowski space. We would like to construct a Fock space with the use of creation and annihilation operators which would then lead one to a particle concept similar to what we know and understand in Minkowski space. With this particle concept in place one could then hope to better understand exactly how a strong gravitational field can create particles and how we may be able to mimic this type of effect in experiments we could perform ourselves. It is also hoped that if one had a better understanding of the physics of quantum fields in curved space one could also have a more physical understanding as to which techniques make physical sense for the regularisation and renormalisation of the stress tensor and other observables . The second and third category therefore contain those papers dealing with particle creation and those dealing with the regularisation and renormalisation of the stress tensor.

4.2 Formalism

The first thing that one must do before dealing with some of the interesting questions which arise when talking about quantum field theory in curved spacetime is to generalise our Lagrangian and thus our field equations from Minkowski space to a more general spacetime. After this of course we will have to deal with the question of how to quantise the solutions to these field equations.

The simplest Lagrangian we could generalise to curved space is the free version of the Lagrangian we introduced at the beginning of chapter 1 (1.1),

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x). \tag{4.1}$$

We can rewrite the first term in this expression as

$$\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x), \tag{4.2}$$

which can be generalised to

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x). \tag{4.3}$$

If we now generalise the partial derivatives to covariant derivatives ($\partial_\mu \rightarrow \nabla_\mu$) we can see that the expression

$$\mathcal{L}(x) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \xi R(x)\phi^2(x) \tag{4.4}$$

reduces to the Lagrangian given earlier (4.1) in Minkowski space. This is the most general Lagrangian quadratic in ϕ that is invariant under general coordinate transformations and not coupled to $R_{\mu\nu}$ but only to R. Here ξ is a dimensionless constant and $R(x)$ is the Ricci scalar curvature. One now has to question why we have included the scalar curvature in this way and only in this way. There are actually at least two reasons [24]. The first reason is that when $m = 0$ and ξ has a particular numerical value (1/6 for 4 Dim.) the action and equation of motion exhibit conformal invariance. This means that a solution in a spacetime with metric $g_{\mu\nu}(x)$ is also, after a simple rescaling, a solution in the metric $\Omega(x)g_{\mu\nu}(x)$ where Ω is any given positive function [25]. The second reason is that it is known that the renormalisation of an interacting field in curved spacetime will involve a counter term proportional to $R\phi^2$. In this way ξ can be regarded as a coupling constant which will then be renormalised.

With this Lagrangian at our disposal we can now find the generalisation of the free Klein Gordon equation for a general spacetime,

$$\Box_c \phi(x) + (m^2 + \xi R(x))\phi(x) = 0 \tag{4.5}$$

where

$$\Box_c \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu [g^{\mu\nu} \sqrt{g} \partial_\nu \phi(x)], \tag{4.6}$$

which is an explicit expression for $g^{\mu\nu}\nabla_\nu\nabla_\mu\phi$.

Once we have our field equation we now wish to construct a Fock representation of the Hilbert space in which the solutions to these field equations will be said to act when quantised and viewed as operators. One wishes to construct this Fock space with creation and annihilation operators in analogy with what one does in normal quantum field theory in Minkowski space. To identify these operators the most natural thing to attempt to do is to try and decompose the field into two parts and identify one part with some generalised notion of positive frequency and the other with some generalised notion of negative frequency. Unfortunately as was illustrated in chapter 2, even in the case of a static metric there isn't an agreed upon unique prescription for performing this decomposition.

The problem with performing this decomposition of the field into "positive and negative" frequencies is that in general spacetimes unlike Minkowski space we do not have the Poincaré group as a symmetry group of the spacetime [26]. In some special classes of spacetime (such as de Sitter space) there exist many "natural" coordinates associated with the Killing vectors but unfortunately they do not enjoy the same central physical status in curved space as their counterparts in Minkowski space. In these special spacetimes with many symmetries and thus many Killing vectors there may be many local, or global timelike Killing vectors to associate a "natural" time with and thus no "natural" means of performing the positive and negative frequency decomposition. If the metric is not static but time dependent in some way then the question becomes even more confusing as to how one could go about defining an appropriate time.

There have been attempts to in some way define a time in curved space which could then be used to perform the decomposition into positive and negative frequency parts of a free field propagating in a curved background. In 1975 Ashtekar et al. [26] proposed a means of decomposing fields by using the global timelike Killing field and an energy condition which then picks out the complex structure of the field and thus the decomposition. In the case of a static metric where the timelike Killing fields remain timelike the procedure picks out the "correct" complex structure by requiring that the classical energy of the field agrees with the energy calculated for the quantised Hamiltonian in the one particle Hilbert space. The authors thus require "that our complex structure be so chosen that the energy of each one particle state equals that of the corresponding classical field." The decomposition is characterised by a selection of the complex structure J, such that $J\phi = i\phi^+ - i\phi^-$ where $\phi^\pm$ are the positive/negative frequency parts of the field. It can be seen by writing J this way that for models involving a time dependent external interaction the J could also become time dependent and this is where frequency mixing would come from causing particle creation. Of course this could only happen in the case where the underlying spacetime is not stationary and there is not a global timelike Killing field. The procedure outlined above must however be modified if the timelike Killing field does not remain timelike everywhere and thus the spacetime is not stationary. In this case a timelike hypersurface orthogonal unit vector field ξ^a is introduced on the given spacetime. Once again the energy condition picks out our now time dependent complex structure $J(t)$. Unfortunately there does not seem to be a unique means of picking out our vector field ξ^a even though different choices will lead to different decompositions and thus different physics. One could interpret the elements of a particular Fock space which is chosen as being the quantum states of particles as seen by the observers following these Killing trajectories. Ashtekar et al. [26] thus conclude that the definition of a particle in this situation is natural only in so far as the motions along Killing trajectories are natural. Even in the static case one could have selected a different vector field and have obtained a completely different notion of particles.

This complex structure approach to decomposing a field into its positive and negative frequency parts can be shown to yield similar results to methods which seek this decomposition by using the Feynman propagator. As was mentioned by Fulling [35] some Green's functions can be characterised by their support properties in spacetime, namely G_{adv}, G_{ret} and different combinations of these. However other Green's functions such as the positive and negative frequency parts of the commutator and combinations of these (such as the Feynman propagator) "have definitions which hinge on the decomposition of solutions into positive and negative frequency parts." There have been attempts to perform a decomposition of the field into positive and negative frequency parts by imposing physically reasonable boundary conditions on the Feynman propagator. Of course some sort of boundary conditions are required to define this propagator uniquely as it is the solution to a differential equation. The problem is generalising these boundary conditions to curved space. For a free field in Minkowski space where we have a positive and negative frequency decomposition we can impose "causal" boundary conditions; positive frequencies are propagated into the future while negative frequencies are propagated into the past. Panangaden [31] shows that if a spacetime has asymptotically static regimes and one can define complex structures in the past and future, J_p and J_f respectively, one can then construct Feynman propagator in terms of these complex structures where the Feynman propagator is given by the Schwinger representation,

$$G_F(x,x') = i\frac{\langle 0_{out}|\, T[\phi(x),\phi(x')]\, |0_{in}\rangle}{\langle 0_{out}|\, 0_{in}\,\rangle}. \tag{4.7}$$

We are then able to calculate the S matrix in a much more straightforward manner as compared the how we would have calculated in the original formalism of the complex structure J which was done with a * algebra approach to quantum fields in curved space.

One can also find approximations for the Feynman propagator. Birrell and Davies [36] illustrate the procedure for obtaining an adiabatic expansion of G_F. In some applications only the high frequency behaviour of the field are of interest as the high frequency components only probe the immediate vicinity of the spacetime point where the metric only changes a small amount. In this case one is then interested in $G_F(x,x')$ in the limit where $x \to x'$. Unfortunately for questions of a more fundamental nature such as frequency decomposition this does not solve the problem. Because we have not imposed global boundary conditions on the differential equation defining the Feynman propagator the adiabatic expansion of the propagator does not determine the particular vacuum states in (4.7). This procedure of imposing boundary conditions in natural regions of spacetime is an approach which is usually used by people trying to demonstrate particle creation in a certain physical situation. In this approach "natural" boundary conditions are imposed upon the states or operators in certain regions (i.e. horizons, past infinity etc.) and then it is shown that in one of the regions (i.e. future infinity) there are particles and therefore particle creation. There are many papers dealing with this topic and thus we will leave this topic until the next section when we deal with particle creation.

There have also been a number of attempts to decompose the field based on the diagonalization of the Hamiltonian or energy operator at each instant of time. In this way the particle definition is implicit in the quantisation procedure as the Hamiltonian can now be regarded as the sum of the individual energies of each particle at each instant. Although this procedure sounds reasonable at first there are many ambiguities which arise in performing calculations. In fact the Fock representations at different times can be unitarily inequivalent, even though the asymptotic representations of the in and out fields are equivalent. Fulling in 1979 [27] illustrates that if one uses this procedure the prescription is loose enough to be made consistent with any ansatz you wish including unphysical ones.

In the mid 70's P. Hájíček [28] [29] proposed another way of defining positive and negative frequency for fields in curved space. In these papers Hájíček concludes that our notion of particles must be slightly "fuzzy" in curved space.. In this way P. Hájíček gives some credit to those people that believe that there is little or no hope of us ever having a complete notion of a particle in curved space. For example P.C.W. Davies [37] believes the particle concept becomes useless in curved space and therefore one should not be looking to define particles. He suggests that perhaps the energy momentum tensor should be playing the role of the central observable. Hájíček proposes that because any point in curved space has a neighborhood which is almost flat one can then distinguish the positive and negative frequencies in the usual way if one requires that the states of the field be localized to such an extent that only these neighborhoods are sufficient for their definition. In this way one can see that the definitions reached through this means can only be approximate as these are only neighborhoods we are dealing with and not exactly pieces of Minkowski space. Of course one could also question how different our notion of particle is in this case because we are limiting the space in which we are defining our states. In this way we could be introducing boundary type effects which are not really there in the physics we are trying to understand.

More recently Capri and Roy [30] have proposed a means of defining a unique time and vacuum for a given observer in a curved spacetime. The goal of this paper is to define "the" instantaneous vacuum as well as "the" direction of time in a coordinate independent manner. The geometry of the spacetime and the observers position and velocity are the only things that are used to define a unique vacuum for a given observer. The direction of time is chosen in the following way. If there is a single globally timelike Killing vector in the spacetime, this vector defines the direction of time. "If there is more than one such globally timelike Killing vector then the time direction is chosen to be that Killing vector which defines a symmetry generator which commutes with at least one other generator." In this way one can pick out the generator of time translations which commutes with the generator of space translations and is thus distinguished from the Lorentz boost generator.

Unfortunately, in general one does have not spacetime symmetries which will provide us with global timelike Killing vectors. In this case one then constructs a spacelike hypersurface from the spacelike geodesics which pass through the observers position. The direction of time is then given by the normal to this surface. If one now constructs a subspace using this normal and any one of the spacelike geodesics one has a surface with Poincaré symmetry. The Killing vector associated with time translation on this surface then gives us the parametrisation along this timelike normal vector. The vacuum for an observer at this point is then given by extracting the annihilation operator by defining positive frequency in terms of this time variable.

4.3 Particle creation

Particle creation by a time dependent gravitational field is of course the most interesting effect one can demonstrate when studying quantum fields in curved space. The easiest way one can see this particle creation is a model of a quantised field coupled to a classical gravitational field of compact support. For example Wald in 1979 [14] showed the existence of the S matrix for the model of a field coupled to an external gravitational field of compact support. Another model in which one can demonstrate particle creation which is perhaps a little more reasonable as a physical model, as one doesn't really expect a gravitational field to have compact support, is that of a two dimensional Robertson-Walker universe with the line element,

$$ds^2 = dt^2 - a^2(t)dx^2. \tag{4.8}$$

Introducing η as a new time parameter where $d\eta = dt/a$ and

$$t = \int_0^\eta a(\eta')d\eta' \tag{4.9}$$

we can now write our line element as

$$ds^2 = a^2(\eta)(d^2\eta - d^2x). \tag{4.10}$$

If we now choose $a(\eta)$ such that

$$a^2(\eta) = A + Btanh(\rho\eta) \tag{4.11}$$

where A, B, and ρ are constants then in the remote past and future the spacetime becomes Minkowskian as,

$$lim_{\eta\to\pm\infty}a^2(\eta) = A \pm B. \tag{4.12}$$

It can be shown [38] that because in the asymptotic past and future the positive frequency modes over which one expands the field are different this necessarily means that there is going to be frequency mixing in the Bogoliubov transformations connecting the operators associated with these modes and thus there will be particle creation. It should be mentioned here that there is an interesting subtlety in this model which involves particle creation. "There is only particle creation when the conformal symmetry is broken by the presence of a mass which provides a length scale for the theory" [39].

Hawking's discovery of particle creation by black holes [23] was revolutionary because the presence of a black hole can cause the locally negligible effects of particle creation to add up and actually cause particles to be emitted to infinity. In this way the particles emitted to infinity avoid the problems of particle definition in curved space as at infinity the spacetime becomes asymptotically flat and one can use one's knowledge of Minkowski space to define the particles unambiguously. In this paper Hawking studies a massless scalar field propagating in the spacetime exterior to a collapsing star. The metric describing the space where the field is then propagating in is the Schwarzchild metric,

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (4.13)$$

where there is a coordinate singularity at $r = 2M$ due to a bad choice of coordinates. The solutions to the field equations involving this metric must satisfy physically reasonable boundary conditions and have a definite positive and negative frequency decomposition in the regions where we wish to define particles if we are to predict particle creation. To determine the field completely we must specify the field on a complete Cauchy surface so we have a well defined Cauchy problem. Because for a massless field past null infinity ($\mathcal{I}^-$) is one of these surfaces at this surface we can decompose the field just as we do in Minkowski space,

$$\phi = \Sigma_i(f_i\mathbf{a}_i + \overline{f}_i\mathbf{a}_i^\dagger). \qquad (4.14)$$

Here the $\{f_i\}$ form a complete orthonormal set of positive frequency modes at $\mathcal{I}^-$. In this way the operators $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ have the usual interpretation of being the annihilation and creation operators for ingoing particles (particles at past null infinity). Because massless fields are completely determined by their data on $\mathcal{I}^-$ we can express the field ϕ in the form (4.14) everywhere. If we wish to also describe outgoing particles (particles at future null infinity) we must also describe our field completely on some complete Cauchy surface involving future null infinity. However because of the presence of the horizon (loosely speaking a sort of hole) our complete Cauchy surface includes both the event horizon and future null infinity. As the field in the region outside the event horizon is determined completely from this data we can also express ϕ in the form,

$$\phi = \Sigma_i\{p_i\mathbf{b}_i + \overline{p}_i\mathbf{b}_i^\dagger + q_i\mathbf{c}_i + \overline{q}_i\mathbf{c}_i^\dagger\}. \tag{4.15}$$

Here the $\{p_i\}$ are a complete set of positive frequency solutions on future null infinity ($\mathcal{I}^+$), they also have zero Cauchy data on the event horizon. In this way the operators $\mathbf{b}_i$ and $\mathbf{b}_i^\dagger$ have the usual interpretation of being the annihilation and creation operators for outgoing particles. The $\{q_i\}$ are a complete set of solutions on the event horizon and have zero Cauchy data on future null infinity. The choice $\{q_i\}$ does not affect the final results so we do not complicate things by trying to also impose some sort of positive frequency condition on them to try and limit this choice. Because both of our expansions for the field are in terms of complete sets we can express the $\{q_i\}$ and $\{p_i\}$ as linear combinations of the $\{f_i\}$ and $\{\overline{f}_i\}$. This then allows us to compute the Bogoliubov transformation relating the $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ and the $\mathbf{b}_i$ and $\mathbf{b}_i^\dagger$. One can then define the incoming vacuum in terms of the $\mathbf{a}_i$ and compute the number of outgoing particles just as was done in earlier examples using the S matrix. In this way Hawking shows that a black hole emits particles as if it were a hot body with temperature $\hbar\kappa/2\pi k_B$ where κ is the surface gravity of the black hole.

Although the calculations done by Hawking indicate that the effects on the particles emitted by the details of the collapse process decay exponentially, some people wish to try and do away with the collapse process. This led people like Boulware [32] Unruh [33] and Hartle and Hawking [40] to try and study particle creation by primordial black holes. Unfortunately this means one can no longer avoid the problem of defining positive frequency in regions which are not Minkowskian, a problem which Hawking didn't have to deal with in his original paper. These three papers immediately led to the definition of three different "vacuum" states, the Boulware vacuum $|0_B\rangle$, the Unruh vacuum $|0_U\rangle$, and the Hartle Hawking vacuum $|0_H\rangle$. These different states are defined by choosing a coordinate system which leads to a static metric and then defining positive and negative frequency with respect to the time coordinate of this system. The confusion which arose as to which of these "vacuum" states are reasonable or not lead Fulling to write a paper entitled "Alternative vacuum states in static spacetimes with horizons" [34]. These different vacua are introduced to try and satisfy different boundary conditions which the authors believe to be physically reasonable.

The first vacuum state introduced, the Boulware vacuum was constructed in terms of the coordinates (t, r, θ, ϕ), the coordinates in terms of which we originally wrote the Schwarzchild metric(4.13). Thus Boulware is able to describe particles at infinity where the spacetime is basically Minkowskian. Unfortunately as we mentioned when we introduced this metric, in the form we wrote it, there was a coordinate singularity at the event horizon. This singular behaviour also plagues calculations such as expectation values of operators calculated in this state. This singular nature can be easily traced to the fact that the Killing vector with respect to which positive frequency was defined becomes null on the horizon.

The other two vacua do not suffer from any singular nature but do describe similar physical situations. Both describe a thermal flux of particles emitted by the black hole and observable at future infinity. The difference between these two states is at past infinity. Here the Unruh vacuum predicts a void of particles even though the black hole is forever present. The Hartle Hawking vacuum however has a thermal bath of particles at past infinity. In this way the Hartle Hawking state really describes a black hole in equilibrium with a thermal bath of particles. Although it is interesting to calculate different expectation values in these states and in this way eliminate some states on physical grounds and say a particular state is physically reasonable or not there are still questions which must be answered. The question of whether there is a unique vacuum with which some physical predictions can be made is not addressed by these papers. Unfortunately these states appear to be constructed using a coordinate system which gives them the answer they believe to be true.

We have mentioned earlier that particles may not be defined precisely enough to be useful observables with which physical insight can be gained. The expectation value of the stress tensor however is defined locally and therefore does not suffer the same global problems that plague particles. Calculating the renormalised expectation value of the stress tensor is a topic in itself which we address in the next section. There are many people who believe that putting a particle detector into one's physical system is a reasonable way of avoiding the problems of particle definition or the regularisation and renormalisation of calculations involving the stress tensor. It could however be argued that the very presence of the detector may interfere with the physics of the system and cause the system to appear differently than it would without the detector. This sort of thinking lead Hájíček to ask the question "is it possible to distinguish particles created by the gravitational field from particles "created" by the detection process?" [28]. The particle detector does however give one a means of making predictions, one can predict how often a particle detector will "click" and as long as one is not overly concerned as to where the click came from there is no real problem with using particle detectors to describe some physical situations. Some authors have made interesting predictions involving their detectors. For example Unruh [33] predicts a detector will click when accelerated through Minkowski space in the same way as if it were stationary and exposed to a thermal bath of particles. The calculation of the renormalised stress tensor however in this situation is zero just as it is in Minkowski space (the acceleration in this case is mimicked by a coordinate transformation). Unfortunately the particle detector approach in curved space has some of the same problems as our original approach of trying to decompose the field into positive and negative frequency and then construct a Fock space. When one calculates the detector response function one finds that although it is independent of the details of the detector it is dependent on the positive frequency Wightman function [36] something one needs an appropriate notion of vacuum to define.

4.4 The stress tensor

The stress tensor $T_{\mu\nu}(x)$ appears to be an entity which avoids many of the problems associated with other possible observables one might be tempted to evaluate. The "fuzziness" of the particle definition and the fact that time dependent external fields can create particles implies that the particle number operator may not be a well defined observable, the particle detector on the other hand by its very presence and the fact that it is explicitly coupled to the field suggests that there may be a problem with using a detector to describe the physics of the system without the detector there. The stress tensor evaluated at a point $T_{\mu\nu}(x)$ seems to be an object which might escape these problems. This object when calculated in terms of a quantised field is itself an operator and therefore one wants to look at expectation values of this operator. There are two problems with this. The first problem is that the the operator when formally calculated contains objects such as, $\langle\phi(x)\phi(x)\rangle$ being the multiplication of two operator valued distributions evaluated at the same point. The second point unfortunately is the same problem that we have been seeing all along. We still must decide which expectation value should be taken (i.e. which state), a problem that will once again cause problems because of our inability to understand what global boundary conditions are relevant in curved space.

The first problem is of course not specific to quantum field theory in curved space but is also a problem in Minkowski space as well. What can be done to give some meaning to this entity is to look at the regularised quantity $\langle\phi(x)\phi(x')\rangle$

after the subtraction of the vacuum contributions look at the limit as $x' \to x$. The second problem can not be dealt with in the same way as we did in Minkowski space. When we calculated the vacuum expectation value of the energy in chapter 1 we found that we could discard the vacuum contribution using a process where we normal order our creation and annihilation operators. The reasons we can't do this in curved space is obvious as we don't have any natural operators to use. In fact we can't even remove the vacuum contributions using Minkowski type terms, it can be shown that even in a simple Robertson-Walker type universe the energy cannot be renormalised using Minkowski type terms [41].

Of course in the gravitational context one must be very careful about how one renormalises the stress tensor. Although in nongravitationl physics all that matters are energy differences, in the gravitational problem energy, just like mass, is also responsible for gravitational effects. In this case a more elaborate scheme involving the dynamics of the gravitational field is required.

Regularisation involves redefining the expectation value of the stress tensor in terms of a parameter ϵ, the value is finite for $\epsilon \neq 0$ and is infinite or ill defined when $\epsilon = 0$. The expression is then modified so that in the end the limit $\epsilon \to 0$ can be taken without any problems such as divergences. There are three different regularisation, renormalisation schemes which are common in the literature with regards to curved space, dimensional regularisation, split point regularisation, and adiabatic regularisation.

Bunch in 1979 [42] showed that by using a dimensional regularisation scheme one can regularise and renormalise the expectation value of the stress tensor in curved space. He also illustrated in this paper that one can understand the renormalisation process as moving the divergent part of the expectation value from the right side of Einstein's field equation to the left side by renormalising the coupling constants in the equation. The dimensional regularisation scheme then involves calculating the expectation value of the stress tensor in n dimensions. Therefore in terms of the general regularisation prescription we mentioned earlier $\epsilon = n - 4$ if one wishes to calculate the renormalised expectation value in four dimensions. Bunch also mentions that the expression for the divergences are equivalent to the divergences calculated by Christensen [43] using the point splitting technique.

Split point regularisation is the technique which we mentioned earlier where one looks at expressions like $\langle\phi(x)\phi(x')\rangle$ in the limit $x' \rightarrow x$. In this prescription ϵ represents the distance between these two points. There is an ambiguity which arises here because the second point x' can be taken any direction if we don't limit it in some way. Davies and Fulling [44] specify that these points can be connected by any non-null geodesic. The detailed behaviour of $\langle T_{\mu\nu}(\epsilon)\rangle$ depends on the direction of the approach of x' to x, to eliminate this dependence and also renormalise the expectation value Davies and Fulling discard all of the terms in the expectation value which depend on ϵ or t^{ρ}, the tangent vector at x along the geodesic connecting x and x'.

Adiabatic regularisation is actually a subtraction scheme which in some respects sounds similar to the subtraction process induced through normal ordering in Minkowski space. Once again one tries to identify the vacuum contributions which are then subtracted. One major advantage of this procedure is that the subtraction can be done mode by mode before the sum over modes is performed. In this way only finite integrals are performed. Bunch has shown [45] that calculations of the renormalised stress tensor using adiabatic regularisation agree with those done using point splitting but the adiabatic approach is "much simpler and more elegant".

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