THE UNIVERSITY OF ALBERTA

ON SOME NONPARAMETRIC METHODS IN REILIABILITY

BY

MICHAEL GUOQING LU

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

EDMONTON, ALBERTA

FALL, 1987

ISBN 0-315-41185-6

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: MICHAEL GUO[illegible]G LU

TITLE OF THESIS: ON SOME NONPARAMETRIC METHODS IN REILIABILITY

DEGREE: MASTER OF SCIENCE

YEAR THIS DEGREE GRANTED: 1987

Permission is hereby granted to THE UNIVERSITY OF ALBERTA LIBRARY to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

(Signed) Lu Guoqing

PERMANENT ADDRESS:

30# HUAIHAI WEST ST

XUZHOU, JIANGSU, P R CHINA

DATE: Sept. 15, 1987

THE UNIVERSITY OF ALBERTA

THE FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled ON SOME NONPARAMETRIC METHODS IN REILIABILITY submitted by MICHAEL GUOQING LU in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE in STATISTICS AND APPLIED PROBABILITY.

Supervisor

External Examiner

DATE:

Dedicated to

my parents

LU BAO-HONG and WANG CHANG-ER

ABSTRACT

We present a unified approach for the problem of testing for the NBU-t_o and NBU classes. In addition, we propose some new tests and introduce some graphical techniques for the same problem.

We also established some weak convergence results for kernel-type quantile processes and percentile residual lifetime processes in both the censored and noncensored cases.

ACKNOWLEDGEMENT

I would like to express my sincere gratitude to Professor E-E. A.A. Aly for proposing the problems considered in this thesis and for his advice, guidance and involvement at various stages of this work. I would also like to thank him for his constant encouragement which has been invaluable to me.

I wish to thank Professor J.R. McGregor and Professor A.N. Al-Hussaini for their encourgement and kind advice during my graduate study at the University of Alberta.

I also wish to thank Mrs. Myrna Janzen for her kindness and help.

Lastly, but not least, I wish to thank Mrs. Laura Heiland and Mrs. Christine Fischer for typing this manuscript efficiently in a relatively short time.

TABLE OF CONTENTS

CHAPTER PAGE

ON SOME NONPARAMETRIC METHODS IN RELIABILITY

FORWORD

In reliability theory, a number of classes of life distributions are defined based on different notions of aging. A large number of these classes is surveyed in Hollander and Proschan (1984). Recently, Hollander et al. (1986) introduced the class NBU-t_o of new better than used of age t_o distributions. The NBU-t_o property states that a new item has stochastically greater life than a used item of age t_o. Hollander et. al. (1986) showed that the NBU-t_o class properly contains the NBU class, where the NBU property states that a new item has stochastically greater life than a used item.

In the statistical theory of reliability and life testing, nonparametric tests for several classes of life distributions are available. Goodness-of-fit tests for the NBU and NBU-t_o classes are in the focus of this thesis.

As to the problem of testing for the NBU class, we have a number of tests. The J-test of Hollander and Proschan (1972), the test of Koul (1977) and a class of tests of Koul (1978). Commenting on the first two tests we note that the relation between them and the real nature of each of them are very implicit.

In Chapter 4, we present a unified approach to the problem of testing for the NBU class. Both the tests of Hollander and Proschan (1972) and Koul (1977) are special cases of our results. In addition, we present some graphical exploratory data analysis techniques for the same problem.

In Chapter 3, we consider the problem of testing for the NBU-t_o class. We present a unified approach which contains that of Hollander et al. (1986) as a special case. In addition, we propose some new tests and introduce a graphical technique for the same problem.

In Chapter 2, we present some preliminaries and in Chapter 1 we give some important definitions.

In order to be able to briefly introduce the material of Chapter 5, we need to introduce some additional notions.

Let X be a random variable with a continuous distribution function F. The corresponding quantile function F^{-1} of F is defined by

$$F^{-1}(y) := \inf\{t : F(t) \geq y\} , \quad 0 < y < 1 .$$

The p-percentile residual life (PRL) function, $R_F(x,p)$, of a life distribution F is the value of r such that $P(X \leq r + x | X \geq x) = p$, i.e.,

$$R_F(x,p) = F^{-1}(1-\bar{p}\,\bar{F}(x)) - x , \quad x > 0 , \quad 0 < p < 1 ,$$

where $\bar{p} = 1-p$ and $\bar{F} \equiv 1-F$.

Yang (1984) studied kernel-type estimators for the quantile function F^{-1} and proved their pointwise consistency and asymptotic normality. The results of Yang can be extended to produce kernel-type estimators for R(x,p). Instead of taking this route, in Chapter 5, we first extend Yang's results to obtain kernel-type quantile processes and prove their weak convergence to a Kiefer process or a sequence of Brownian bridges. This result is also an extension of Theorem 6 of Csörgő, M. and Révész (1978). Next we consider kernel-type PRL processes and prove their weak convergence to a Gaussian process thus extending a result of Csörgő, M. and Csörgő, S. (1986).

I. INTRODUCTION

In this Chapter we present some definitions which will be needed in the sequel.

__Definition 1.1.__ A life distribution F is a distribution satisfying $F(x) = 0$ for $x < 0$. The corresponding survival function is defined as $\overline{F} = 1-F$.

__Definition 1.2.__ A life distribution F is new better than used (NBU) if

(1.1) $$\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y) \quad \text{for all} \quad x,y \geq 0 .$$

The corresponding concept of a new worse than used (NWU) distribution is defined by reversing the inequality in (1.1).

Hollander and Proschan (1972) introduced the concept of NBU distribution and explained that a used NBU device of any fixed age has stochastically small residual life length than that of a new device.

Consider the following classes of life distributions,

(1.2) $$C_0 = \{F:\overline{F}(x+y) = \overline{F}(x)\overline{F}(y) , \quad \text{for all} \quad x,y \geq 0\}$$

and

$$(1.3)\qquad C_A = \{F: \overline{F}(x+y) \le \overline{F}(x)\overline{F}(y), \quad \text{for all } x,y \ge 0, \text{ and the inequality holds for some } x,y > 0\}.$$

We observe that C_0 consists of the boundary members of C_A and is the class of the exponential distributions, i.e.

$$C_0 = \{F: F(x) = 1-\exp(-\lambda x),\ x \ge 0,\ \lambda > 0\}.$$

The hazard function R of a life distribution function F is defined as $R(x) = -\ln \overline{F}(x)$. It is easy to see that F is NBU if and only if its hazard function R is superadditive, i.e.,

$$R(x+y) \ge R(x) + R(y), \quad \text{for all } x,y \ge 0 .$$

<u>Definition 1.3.</u> A life distribution F is an increasing failure rate average (IFRA) distribution if $\frac{1}{x}R(x) = -\frac{1}{x}\ln \overline{F}(x)$ is non-decreasing for $0 < x < \infty$.

Deshpande (1983) indicated that F is an IFRA distribution if and only if

$$(1.4)\qquad \overline{F}(\lambda x) \ge \overline{F}^{\lambda}(x), \quad \text{for all } x \ge 0 \text{ and } 0 \le \lambda \le 1 .$$

This is equivalent to

(1.5) $\overline{F}(bx) \leq \overline{F}^b(x)$, for all $x \geq 0$ and $b \geq 1$.

Taking $x = y$ in (1.1), we obtain

(1.6) $\overline{F}(2x) \leq \overline{F}^2(x)$, for all $x \geq 0$.

We can easily show that if F is NBU, then for any integer $k \geq 2$ we have

(1.7) $\overline{F}(kx) \leq \overline{F}^k(x)$, for all $x \geq 0$.

Comparing (1.5) with (1.7), we observe that (1.7) defines a class larger than that defined by (1.5). Hollander and Proschan (1984) proved that the class of IFRA distributions is a subclass of the class of NBU distributions.

Hollander, et al. (1986) introduced the class of new better than used of age t_o distributions which is defined below.

<u>Definition 1.4.</u> Let $t_o > 0$. A life distribution F is new better than used of age t_o (NBU-t_o), if

(1.8) $\overline{F}(x+t_o) \leq \overline{F}(x)\overline{F}(t_o)$ for all $x \geq 0$.

F is said to be new worse than used of age t_o (NWB-t_o) if we reverse the inequality in (1.8).

The NBU-t_o property states that the chance $\overline{F}(x)$ that a new item will survive to age x is greater than the chance $\overline{F}(x+t_o)/\overline{F}(t_o)$ that an unfailed item of age t_o will survive an additional time x, i.e., a new item has stochastically greater life than a used item of age t_o. Hollander et al. (1986) indicated that the NBU-t_o class contains the NBU class.

Define

(1.9) $$\mathcal{D}_0 = \{F: \overline{F}(x+t_o) = \overline{F}(x)\overline{F}(t_o) \text{ , for all } x \geq 0\}$$

and

(1.10) $$\mathcal{D}_A = \{F: \overline{F}(x+t_o) \leq \overline{F}(x)\overline{F}(t_o), \text{ for all } x \geq 0 \text{ , and the inequality holds for some } x \geq 0\} .$$

It is apparent that $\mathcal{D}_0$ consists of the boundary members of $\mathcal{D}_A$, the NBU-t_o class. In addition, $\mathcal{C}_0$ and $\mathcal{C}_A$ of (1.2) and (1.3) are subclasses of $\mathcal{D}_0$ and $\mathcal{D}_A$, respectively. Hollander et al. (1986) indicated that the distributions in $\mathcal{D}_0$ are only of the three forms F_1, F_2 and F_3 defined as

1. $F_1(x)$ is an exponential distribution.

2. $F_2(x)$ is a life distribution for which $\overline{F}_2(t_o) = 0$.

3. $\bar{F}_3(x) = \bar{G}(x)$ for $0 \le x < t_o$, $= \bar{G}^j(t_o)\bar{G}(x-jt_o)$, for $jt_o \le x \le (j+1)t_o$, $j = 0,1,2,\ldots,$ where G is a life distribution.

In Chapter 3 and Chapter 4, we treat the NBU-t_o and NBU classes. Results similar to those of Chapter 3 and 4 hold true for the NWU-t_o and NWU classes. The new results are obtained by properly changing the signs, reversing the inequalities and/or replacing suprimum by infimum.

II. PRELIMINARIES

Let $X_1, X_2, \ldots, X_n$ $(n \geq 1)$ be a random sample from the continuous distribution function F, and let F^{-1} be its corresponding quantile function which is defined as

$$(2.1) \quad F^{-1}(y) = \inf\{x \in R: F(x) \geq y\}, \; 0 < y \leq 1, \quad F^{-1}(0) = F^{-1}(0+) .$$

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics and F_n be the empirical distribution function of the sample, i.e.,

$$(2.2) \quad F_n(x) = \begin{cases} 0 , & \text{if } x < X_{1:n} , \\ \frac{k}{n} , & \text{if } X_{k:n} \leq x < X_{k+1:n}, \; k=1,2,\ldots,n-1, \\ 1 , & \text{if } x \geq X_{n:n} . \end{cases}$$

Let F_n^{-1} be the corresponding sample quantile function, i.e.,

$$\begin{aligned} F_n^{-1}(y) &= \inf\{x \in R: F_n(x) \geq y\} \\ &= \begin{cases} X_{1:n} & \text{if } y = 0 , \\ X_{k:n} & \text{if } \frac{k-1}{n} < y \leq \frac{k}{n} , \; k = 1,\ldots,n . \end{cases} \end{aligned}$$

Define the uniform empirical process, $\alpha_n(\cdot)$, and the uniform quantile process, $u_n(\cdot)$, by

(2.4) $$\alpha_n(y) = n^{1/2}(E_n(y)-y), \quad 0 \le y \le 1,$$

(2.5) $$u_n(y) = n^{1/2}(y - E_n^{-1}(y)), \quad 0 \le y \le 1,$$

where $E_n(y) = F_nF^{-1}(y)$.

A Brownian bridge $\{B(t) : 0 \le t \le 1\}$ is a separable Gaussian process with mean zero and covariance function

$$E[B(t)B(s)] = s \wedge t - st, \quad \text{for} \quad 0 \le s, t \le 1.$$

where $s \wedge t$ is the minimum of s and t.

A Kiefer process $\{K(s,t) ; 0 \le s \le 1, \ t > 0\}$ is a 2-parameter separable Gaussian process with

$$EK(s,t) = 0$$

and

$$EK(s,t)K(s',t') = (t \wedge t')(s \wedge s' - ss')$$

Let $q(t)$ be any positive function on $(0,1)$ which is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one. Such a function q is called a Chibisov-O'Reilly function if

$$(2.6)\qquad I(q;c) = \int_0^1 [t(1-t)]^{-1}\exp\{-c[t(1-t)]^{-1}q^2(t)\}dt < \infty\ ,\quad \text{for all}\ \ c > 0\ .$$

Condition (2.6) holds for all $c > 0$ if and only if

$$(2.7)\qquad \lim_{\varepsilon\downarrow 0} P\{\sup_{0\le t\le\varepsilon} |B(t)|/q(t) \ge \delta\} = 0\ ,\quad \text{for all}\ \ \delta > 0\ ,$$

where $\{B(t) : 0 \le t \le 1\}$ is a Brownian bridge.

Theorem 2.1. The underlying probability space of the X-sample can be extended in such a way that with a sequence of Brownian bridges $\{B_n(y) : 0 \le y \le 1\}$, we have as $n \to \infty$,

$$(2.8)\qquad \sup_{0\le y\le 1} |\alpha_n(y) - B_n(y)|/q(y) = o_p(1)$$

$$(2.9)\qquad \sup_{1/(n+1)\le y\le n/(n+1)} |u_n(y) - B_n(y)|/q(y) = o_p(1)$$

with any Chibisov–O'Reilly weight function q.

This theorem was first proved by Chibisov (1964) and O'Reilly (1974), under the additional conditions of symmetry and continuity of q. The present form of Theorem 2.1 is due to Csörgő et al. (1986).

The following result of Wellner (1978) will be needed in the sequel.

Theorem 2.2. We have as $n \to \infty$

(2.10) $$\sup_{1/(n+1)\leq y\leq n/(n+1)} FF_n^{-1}(y)/y = 0_p(1)$$

and

(2.11) $$\sup_{1/(n+1)\leq y\leq n/(n+1)} (1-FF_n^{-1}(y))/(1-y) = 0_p(1).$$

Let $Y_1, Y_2, \ldots, Y_m (m \geq 1)$ be a random sample from a continuous distribution G. Let G^{-1}, G_m and G_m^{-1} be the corresponding quantile function, empirical distribution function and sample quantile function, respectively. When two samples are compared, frequently Quantile-Quantile (Q-Q), Procentile-Procentile (P-P) plots are used for statistical data analysis. Recently, Aly (1985) introduced the R-R plot technique to compare the p-percentiles of the remaining life given survival up to time x of two populations.

P-P plot processes as defined in (2.12) will play an important role in the results of this thesis. The empirical P-P plot process, $\ell_N(\cdot)$, is defined as

(2.12) $$\ell_N(y) = N^{1/2}(G_m F_n^{-1}(y) - GF^{-1}(y)) , \quad 0 \leq y \leq 1 ,$$

where $N = m+n$. Aly et al. (1987) proved the following theorem.

<u>Theorem 2.3.</u> Assume that F and G satisfy the conditions: (c.1) f and g are continuous positive density functions on the open support of the distribution functions F and G, respectively, (c.2) With

some Chibisov–O'Reilly function q, we have

(2.13) $$\sup_{0\le y\le 1} \frac{gF^{-1}(y)}{fF^{-1}(y)} q(y) < \infty ,$$

(c.3) there exist a λ, $0 \le \lambda \le \frac{1}{2}$ such that $\lambda \le \lambda_N \leqq 1-\lambda$, where $\lambda_N = n/(n+m)$. Then, on the probability space of Theorem 2.1, we have as $m \wedge n \to \infty$,

(2.14) $$\sup_{0\le y\le 1} |\ell_N(y) - \Gamma_N(y)| = o_p(1) ,$$

where

(2.15) $$\Gamma_N(y) = (1-\lambda_N)^{-\frac{1}{2}} \left(B_m^{(2)}(GF^{-1}(y)) - \lambda_N^{-\frac{1}{2}} \frac{gF^{-1}(y)}{fF^{-1}(y)} B_n^{(1)}(y) \right), \quad 0\le y\le 1,$$

is a Gaussian process and $\{B_m^{(2)}(\cdot)\}$, $\{B_n^{(1)}(\cdot)\}$ are two independent sequences of Brownian bridges.

The p-percentile residual life function, $R_F(x,p)$, of a life distribution F is the p-percentile of the remaining life given survival up to time x, i.e.,

(2.16) $$R_F(x,p) = \begin{cases} F^{-1}(1-\bar{p}\,\bar{F}(x)) - x & \text{if } 0 \le x < F^{-1}(1) \\ 0 & \text{if } F^{-1}(1) < \infty ,\ x \ge F^{-1}(1) . \end{cases}$$

Consider the problem of testing $H_0^*: F = G$ against $H_1^*: R_F(\cdot,p) \geq R_G(\cdot,p)$. Expressing the inequality $R_F(\cdot,p) \geq R_G(\cdot,p)$ in terms of the P-P plot function $GF^{-1}(\cdot)$, Aly (1985) defined the R-R plot function $\Delta(y,p)$ as

$$\Delta(y,p) = GF^{-1}(p+\bar{p}y) - \bar{p}GF^{-1}(y) - p \quad , \quad 0 \leq y \leq 1 \; , \quad 0 \leq p \leq 1 \; .$$

In addition, the empirical R-R plot $\Delta_N(y,p)$ is defined as

$$\Delta_N(y,p) = G_m F_n^{-1}(p+\bar{p}y) - \bar{p}G_m F_n^{-1}(y) - p \; , \quad 0 \leq y \leq 1 \; , \quad 0 \leq p \leq 1 \; .$$

Consequently, the empirical R-R plot process is defined as

$$\text{(2.17)} \qquad \delta_N(y,p) = N^{\frac{1}{2}} \left(\Delta_N(y,p) - \Delta(y,p)\right) \; , \quad 0 \leq y \leq 1 \; , \; 0 \leq p \leq 1 \; .$$

Aly (1985) explained that the problem of testing H_0^* against H_1^* is now reduced to (a) proving a weak convergence result for the process $\delta_N(\cdot,\cdot)$, (b) selecting an appropriate functional, φ, of $\Delta(\cdot,\cdot)$ as a measure of the deviation from H_0^* and (c) using the weak convergence result of (a) above to obtain the asymptotic distribution of the corresponding test statistic, $\varphi(\Delta_N(\cdot,\cdot))$. Another important feature of the results of Aly (1985) is that it introduces a nonparametric graphical techique to deal with the above explained testing problem.

The main aim of the results of Chapter 3 and 4 is to provide a unified treatment of the problems of testing for NBU and for NBU-t_o distributions. Our main approach is guided by the methodology and the techniques of Aly (1985).

III. ON TESTING FOR THE NBU-t_o CLASS

3.1. Introduction

In this chapter, we are interested in the problem of testing

$$(3.1) \qquad H_o: F \in \mathcal{D}_0 \text{ versus } H_1: F \in \mathcal{D}_A$$

where $\mathcal{D}_0$ and $\mathcal{D}_A$ are defined in (1.9) and (1.10) repectively.

Hollander et al. (1986) proposed a test for H_o versus H_1 based on a U-statistic. They have derived the asymptotic distribution of their test statistic and proved its consistency. We propose a general treatment of the above testing problem. The test of Hollander et. al. (1986) is a special case of our results.

3.2. NBU-t_o Processes

According to Definition 1.4, $F \in \mathcal{D}_A$ of (1.9) if and only if $\bar{F}(x+t_o) \le \bar{F}(x)\bar{F}(t_o)$, $0 \le x < \infty$. This is equivalent to

$$(3.2) \qquad \Lambda(y) := F(F^{-1}(y)+t_o) - (1-y)F(t_o) - y \ge 0 , \quad 0 \le y \le 1 .$$

The function $\Lambda(y)$ of (3.2) will be called the NBU-t_o plot function of F. We observe that $F \in \mathcal{D}_0$ of (1.7) if and only if $\Lambda(y) = 0$, for all $0 \le y \le 1$. A natural estimator of $\Lambda(\cdot)$ is its empirical counterpart $\Lambda_n(\cdot)$, which is defined as

(3.3) $$\Lambda_n(y) := F_n(F_n^{-1}(y)+t_o) - (1-y)F_n(t_o) - y , \quad 0 \le y \le 1 .$$

The NBU-t_o plot process, $\lambda_n(\cdot)$, is defined as

(3.4) $$\lambda_n(y) := n^{1/2}(\Lambda_n(y)-\Lambda(y)) , \quad 0 \le y \le 1.$$

Aiming at representing $\lambda_n(\cdot)$ in terms of a certain P-P plot process, we let

(3.5) $$G(x) := F(x+t_o) , \quad \text{for} \quad x \ge -t_o$$

and observe that $G(x)$ is the common distribution function of $Y_i = X_i - t_o$, $1 \le i \le n$. Let G_n be the empirical distribution function of $Y_i = X_i - t_o$, $i = 1,2,\ldots,n$. It is clear that

(3.6) $$G_n(x) = F_n(x+t_o) , \quad x \ge - t_o .$$

By (3.5) and (3.6), we define the P-P plot process, $\ell_n(y)$, as

(3.7) $$\begin{aligned} \ell_n(y) &= n^{1/2}\left(F_n(F_n^{-1}(y)+t_o) - F(F^{-1}(y)+t_o)\right) \\ &= n^{1/2}\left(G_n(F_n^{-1}(y)) - G(F^{-1}(y))\right) , \quad 0 \le y \le 1 . \end{aligned}$$

For a continuous distribution F, (3.4) can be written as

$$\lambda_n(y) = \ell_n(y) - n^{1/2}(1-y)(F_n(t_o)-F(t_o))$$

(3.8)

$$= \ell_n(y) - (1-y)\alpha_n(F(t_o)) ,$$

where $\alpha_n(\cdot)$ is defined in (2.4) and $\ell_n(\cdot)$ is in (3.7).

3.3. On the Asymptotic Theory of NBU-t_o Processes

The following two theorems give some asymptotic results concerning the NBU-t_o plot process.

Theorem 3.1. Assume that $f = F'$ is continuous and

(3.9) $$\sup_{0\leq y\leq 1} \frac{f(F^{-1}(y)+t_o)}{f(F^{-1}(y))} q(y) < \infty ,$$

for some Chibisov-O'Reilly weight function q. Then on the probability space of Theorem 2.1 and with the same sequence $\{B_n(\cdot)\}$ of Brownian bridges, we have as $n \to \infty$,

(3.10) $$\sup_{0\leq y\leq 1} |\lambda_n(y) - \Gamma_n(y)| = o_p(1) ,$$

where

$$(3.11)\quad \Gamma_n(y) = B_n(F(F^{-1}(y)+t_o)) - \frac{f(F^{-1}(y)+t_o)}{f(F^{-1}(y))} B_n(y) - (1-y)B_n(F(t_o)) , \quad 0 \le y \le 1 .$$

It is easy to see that, if $F \in \mathcal{D}_0$, $\frac{f(F^{-1}(y)+t_o)}{f(f^{-1}(y))}$ is a constant and hence condition (3.9) is satisfied, thus (3.10) holds true. In the following theorem, we show that (3.10) holds true for all continuous $F \in \mathcal{D}_0$ without requiring the existence of $f = F'$.

Theorem 3.2. Assume that $F \in \mathcal{D}_0$ is a continuous distribution function. Then, on the probability space of Theorem 2.1, we have as $n \to \infty$,

$$(3.12)\quad \sup_{0 \le y \le 1} |\lambda_n(y) - \Gamma_n^0(y)| = o_p(1) ,$$

where

$$(3.13)\quad \Gamma_n^0(y) = B_n(y+(1-y)F(t_o)) - (1-F(t_o))B_n(y) - (1-y)B_n(F(t_o)) , \quad 0 \le y \le 1 .$$

We observe that $\Gamma_n^0(\cdot)$ is a zero mean Gaussian process with convariance function

$$(3.14)\quad E\Gamma_n^0(x)\Gamma_n^0(y) = \bar{F}(t_o)[(1+\bar{F}(t_o))(x\wedge y) + F(t_o)xy + (1-y)(x\wedge F(t_o))$$
$$+ (1-x)(y\wedge F(t_o)) - y\wedge(x+(1-x)F(t_o)) - x\wedge(y+(1-y)F(t_o))]$$
$$0 \le x\ ,\ y \le 1\ .$$

Let $\Gamma^0(y)$ be a zero mean Gaussian process with the convariance function of (3.14). Then

$$\{\Gamma_n^0(y):\ 0 \le y \le 1\} \overset{D}{=} \{\Gamma^0(y):\quad 0 \le y \le 1\},\ \text{for each } n \ge 1\ ,$$

and by Theorem 3.2, we have

$$(3.15)\qquad \lambda_n(y) \xrightarrow{D} \Gamma^0(y) \quad \text{as} \quad n \longrightarrow \infty.$$

The proof of Theorem 3.1 is similar to and simpler than that of Theorem 4.1 of the next chapter. For this reason we will only prove Theorem 3.2 here.

<u>Proof of Theorem 3.2</u>: Recall that $F \in \mathcal{D}_0$ if and only if $\Lambda(y) = 0$. This is equivalent to

$$(3.16)\qquad F(F^{-1}(y)+t_o) = y + (1-y)F(t_o)$$
$$= F(t_o) + (1-F(t_o))y\ ,\quad 0 \le y \le 1\ .$$

By Theorem 2.1, we have

$$(3.17)\quad \sup_{0\le y\le 1} |(1-y)\alpha_n(F(t_o)) - (1-y)B_n(F(t_o))|$$

$$= |\alpha_n(F(t_o))-B_n(F(t_o))| = o_p(1)\ , \quad \text{as}\quad n\to\infty\ .$$

By (3.5) and (3.6), we have $G_nF_n^{-1}(y) = F_n(F_n^{-1}(y)+t_o)$ $= F_nF^{-1}(F(F_n^{-1}(y)+t_o)) = F_nF^{-1}(GF_n^{-1}(y))$. Hence,

$$(3.18)\quad \ell_n(y) = n^{1/2}(G_nF_n^{-1}(y) - GF_n^{-1}(y)) - n^{1/2}(GF^{-1}(y)-GF_n^{-1}(y))$$

$$= n^{1/2}(F_nF^{-1}(GF_n^{-1}(y))-GF_n^{-1}(y)) - \gamma_n(y)$$

$$= \alpha_n(GF_n^{-1}(y)) - \gamma_n(y)\ ,$$

where $\gamma_n(y) := n^{1/2}(GF^{-1}(y)-GF_n^{-1}(y))$. By Theorem 2.1

$$(3.19)\quad \sup_{0\le y\le 1} |\alpha_n(GF_n^{-1}(y)) - B_n(GF_n^{-1}(y))| = o_p(1)\ .$$

and

$$(3.20)\quad \sup_{0\le y\le 1} |FF_n^{-1}(y) - y| = o_p(1)$$

Since $B_n(GF^{-1}(y))$ is almost surely continuous for each n, by (3.20), we obtain,

$$(3.21)\qquad \sup_{0\le y\le 1} |B_n(GF_n^{-1}(y)) - B_n(GF^{-1}(y))|$$

$$= \sup_{0\le y\le 1} |B_n(GF^{-1}(FF_n^{-1}(y))) - B_n(GF^{-1}(y))| = o_p(1) .$$

Thus (3.19) and (3.21) yield

$$\sup_{0\le y\le 1} |\alpha_n(GF_n^{-1}(y)) - B_n(GF^{-1}(y))| = o_p(1) .$$

Hence, by (3.16) we obtain

$$(3.22)\qquad \sup_{0\le y\le 1} |\alpha_n(GF_n^{-1}(y)) - B_n(y+(1-y)F(t_o))| = o_p(1) .$$

Next, we consider $\gamma_n(\cdot)$ of (3.18). By (3.16), we have

$$(3.23)\qquad \gamma_n(y) = n^{1/2}\left(GF^{-1}(y) - GF_n^{-1}(y)\right)$$

$$= n^{1/2}\left(GF^{-1}(y) - GF^{-1}(FF_n^{-1}(y))\right)$$

$$= n^{1/2}(1-F(t_o))(y-FF_n^{-1}(y))$$

$$= (1-F(t_o))u_n(y) .$$

By Theorem 2.1

$$(3.24)\qquad \sup_{1/(n+1)\le y\le n/(n+1)} |\gamma_n(y) - (1-F(t_o))B_n(y)|$$

$$= (1-F(t_o)) \sup_{1/(n+1)\le y\le n/(n+1)} |u_n(y) - B_n(y)| = o_p(1)$$

Now, we prove that as $n \to \infty$,

$$(3.25)\qquad \sup_{0\le y\le 1/(n+1)} |\gamma_n(y)| = o_p(1) ,$$

$$(3.26)\qquad \sup_{n/(n+1)\le y\le 1} |\gamma_n(y)| = o_p(1) ,$$

$$(3.27)\qquad \sup_{0\le y\le 1/(n+1)} |B_n(y)| = o_p(1) ,$$

and

$$(3.28)\qquad \sup_{n/(n+1)\le y\le 1} |B_n(y)| = o_p(1) .$$

We will only prove (3.25), since (3.26) follows by a similar argument. In addition, (3.27) and (3.28) follow by P. Lévy modulus of continuity of a Brownian motion. (cf. for example, Theorem 1.6.1 of Csörgő and Révész (1981)). Since

$$\sup_{0\le y\le 1/(n+1)} |\gamma_n(y)| = (1-F(t_o)) \sup_{0\le y\le 1/(n+1)} |u_n(y)|$$

$$\leq \sup_{0 \leq y \leq 1/(n+1)} n^{1/2}|y-U_{1:n}| \leq \left(\frac{n^{1/2}}{n+1} - n^{1/2}U_{1:n}\right) \vee (n^{1/2}U_{1:n}) ,$$

where $U_{1:n}$, $U_{2:n}$, ..., $U_{n:n}$ are the order statistics of a random sample from the uniform distribution on (0,1), and $s \vee t = \max(s,t)$. To prove (3.25), is suffices to show that

$$n^{1/2}U_{1:n} = o_p(1) , \quad \text{as} \quad n \to \infty . \tag{3.29}$$

For any $\varepsilon > 0$, we have $P(|n^{1/2}U_{1:n}| > \varepsilon) = P(U_{1:n} > \varepsilon n^{-1/2}) = (1-\varepsilon n^{-1/2})^n \leq (e^{-\varepsilon n^{-1/2}})^n = e^{-n^{1/2}\varepsilon} \to 0$, as $n \to \infty$. Thus (3.29) is proved.

By (3.24)-(3.28), we have

$$\sup_{0 \leq y \leq 1} |\gamma_n(y) - (1-F(t_o))B_n(y)| = o_p(1) . \tag{3.30}$$

From (3.8), (3.17), (3.18), (3.22) and (3.30), we get (3.11) and (3.12).

3.4 On Testing For the NBU-t_o Class.

Consider the problem of testing for H_o against H_1 of (3.1). Measures of the deviation from H_o in favour of H_1 can be constructed as appropriate functionals of $\Lambda(\cdot)$ of (3.2) which are zero under H_o and positive (or negative) under H_1 . For example, the measure

$$\varphi_1 = \varphi_1(\Lambda(y)) = \int_0^1 \Lambda(y)dy$$

is proposed by Hollander et al (1986). Their proposed test statistic is given by

(3.31) $$\varphi_{1n} = \varphi_1(\Lambda_n(y)) = \int_0^1 \Lambda_n(y)dy = \frac{1}{n^2}\sum_{k=1}^{n} R_{k,t_o} - \frac{1}{2n} R_{t_o} - \frac{1}{2},$$

where R_{k,t_o} is the rank of $X_{k:n} + t_o$ in the combined X, X + t_o sample, R_{t_o} is the number of X_i which are less than or equal to t_o.

By Theorem 3.2, under H_o, we have

(3.32) $$n^{1/2}\varphi_{1n} = \int_0^1 n^{1/2} \Lambda_n(y)dy = \int_0^1 \lambda_n(y)dy \xrightarrow{D} \int_0^1 \Gamma^0(y)dy,$$

where $\Gamma^0(y)$ is the Gaussian process of (3.15). The random variable $\int_0^1 \Gamma^0(y)dy$ is a normal random variable with mean zero and the variance σ_o^2 given by

(3.33) $$\sigma_o^2 = \frac{1}{12}\bar{F}(t_o) + \frac{1}{12}\bar{F}^2(t_o) - \frac{1}{6}\bar{F}^3(t_o).$$

This agrees with the results of Hollander et al. (1986).

In addition to the above test, one may develop other test statistics based on different measures of the derivation from H_o: $\bar{F}(x+t_o) = \bar{F}(x)\bar{F}(t_o)$, $0 \leq x < \infty$. For example, we may consider any of the following measures:

$$\varphi_2 = \varphi_2(\Lambda(y)) := \sup_{0\leq y\leq 1} \Lambda(y)$$

and

$$\varphi_3 = \varphi_3(\Lambda(y)) := \int_0^1 \Lambda^2(y)dy .$$

The corresponding test statistics are $\varphi_{in} = \varphi_i(n^{1/2}\Lambda_n(y))$, $i = 2,3$.

Now, if H_o: $F \in \mathcal{D}_0$ holds true, by Theorem 3.1, we have as $n \to \infty$,

$$\varphi_{in} = \varphi_i(n^{1/2}(\Lambda_n(y))) \xrightarrow{D} \varphi_i(\Gamma^0(y)) , \; i = 2,3,$$

where $\Gamma^0(y)$ is the Gaussian process of (3.15). More precisely

$$(3.34) \quad \varphi_{2n} = n^{-1/2} \max_{1\leq k\leq n} \left[R_{k,t_o} - \left(1-\frac{k-1}{n}\right)R_{t_o} - k+1\right] \xrightarrow{D} \sup_{0\leq y\leq 1} \Gamma^0(y),$$

(3.35)
$$\varphi_{3n} = n^{-1} \sum_{k=1}^{n} [n^{-1} R_{k,t_o}^2 - 2R_{k,t_o} (R_{t_o} + \frac{2k-1}{2n^2} (1+R_{t_o}))]$$
$$+ R_{t_o} + \frac{1}{3n} (1-R_{t_o})^2 \xrightarrow{D} \int_0^1 \Gamma^{0^2}(y)dy.$$

3.5. A Graphical Approach

An important feature of employing NBU-t_o plot processes is their immediate use as possible graphical tools. An essential step in this direction is the following theorem, in which we prove that $\Lambda_n(\cdot)$ is a uniformly consisent estimator of $\Lambda(\cdot)$.

Theorem 3.3. Suppose that F is a continuous distribution function. We have, as $n \to \infty$,

(3.36)
$$\sup_{0 \le y \le 1} |\Lambda_n(y) - \Lambda(y)| = o_p(1).$$

We leave the proof of this theorem to the next chapter. In fact, (3.36) is a direct consequence of (4.64) of Theorem 4.3.

A graphical appraoch to test H_o is to plot $\Lambda_n(y)$ as a function of y, $0 \le y \le 1$, if H_o is true, then, the NBU-t_o plot $(y, \Lambda_n(y))$ should closely evolve along the horizontal axis. A plot which falls "above" the horizontal axis would imply that H_1 is true, i.e., the distribution is NBU-t_o. In addition, a plot which falls "under" the horizontal axis would imply that the true distribution is NWU-t_o.

<u>Example 3.1.</u> Bryson and Siddiqui (1969, P. 1483) gave a data corresponding to the survival period (in days) of 43 patients from the date of diagnosis of chronic granulocytic leukemia. To apply the NBU-t_o graphical technique, we choose $t_o = 1825$ ($\approx$ 5 years) for illustrative purpose. We obtain the NBU-t_o plot in Figure 3.1. From the figure, we can see that the plot is "above" the horizontal axis on most of the [0,1]. This suggests that we should reject H_0 and conclude that H_1 is true, i.e., a newly diagnosed patient has stochastically greater residual life than does a patient after five years. This conclusion agrees with that of Hollander et. al. (1986).

Insert Figure 3.1 about here.

A formal graphical approach to the above problem is to construct a confidence band for the unknown NBU-t_o plot function $\Lambda(\cdot)$ of (3.2). From Theorem 3.1 we observe that the convariance function of limiting Gaussian process of $\lambda_n(\cdot)$ depends on the unknown distribution function F in a complicated way. To avoid this difficulty, we may construct bootstrapped confidence bands for $\Lambda(\cdot)$. This is explained in what follows.

Let $n_1 \le n$. Given $X_1, X_2, \ldots, X_n$, let $X_1', \ldots, X_n'$ be conditionally independent r.v.'s with common distribution function

F_n. The empirical distribution function of $X'_1, \ldots, X'_n$,

$$(3.37) \qquad F_{n_1,n}(t) := \frac{1}{n_1} \#\{1 \le i \le n_1 : X'_i \le t\}$$

is called the bootstrapped empirical distribution function of F_n. The bootstrapped empirical quantile function is $F^{-1}_{n_1,n}$ with $F^{-1}_{n_1,n}(0) = 0$. Using these bootstrapped functions, we can define the bootstrapped empirical NBU-t_o plot as

$$(3.38) \qquad \Lambda_{n_1,n}(y) := F_{n_1,n}(F^{-1}_{n_1,n}(y)+t_o)-(1-y)F_{n_1,n}(t_o)-y \ , \ 0 \le y \le 1,$$

and the bootstrapped NBU-t_o plot process as

$$(3.39) \qquad \lambda_{n_1,n}(y) := n_1^{1/2}(\Lambda_{n_1,n}(y)-\Lambda_n(y)), \quad 0 \le y \le 1 .$$

Now, generating the bootstrapped processes $\lambda^{(i)}_{n_1,n}(y)$, $1 \le i \le M$, M times. Let

$$(3.40) \qquad G_{M,n_1,n}(x) = \frac{1}{M} \#\{1 \le i \le M : \sup_{0\le y\le 1} |\lambda^{(i)}_{n_1,n}(y)| \le x\} ,$$

and

(3.41) $$c_{M,n_1,n}(\alpha) = \inf\{x \geq 0: G_{M,n_1,n}(x) \geq 1-\alpha\} = G^{-1}_{M,n_1,n}(1-\alpha) ,$$

where $\alpha \in (0,1)$ is fixed.

An asymptotically $(1-\alpha)100\%$ confidence band for $\Lambda(\cdot)$ is given by

(3.42) $$\{\Lambda_n(y) - c_{M,n_1,n}(\alpha)n^{-\frac{1}{2}} \leq \Lambda(y) \leq \Lambda_n(y) + c_{M,n_1,n}(\alpha)n^{-\frac{1}{2}} , \quad 0 \leq y \leq 1\}.$$

In order to test H_0: $F \in \mathcal{D}_0$ against H_1: $F \in \mathcal{D}_A$, we plot the $(1-\alpha)100\%$ confidence band of (3.42), if the band contains the horizontal axis, we conclude that H_0 is true; if the lower limit of the confidence band falls above the horizontl axis, then we conclude that H_1 is true.

For the data of Example 3.1 and with $t_0 = 1825$, we plot the 95% bootstrapped confidence band for $\Lambda(y)$ in Figure 3.2. The lower limit of the band is essentially above the horizontal axis.

Insert Figure 3.2 about here.

<u>Remark 3.1.</u> The construction of the bootstrapped distribtuion $G_{M,n_1,n}(\cdot)$ of (3.40) requires the existence of a limiting distribution for $\lambda_n(\cdot)$. Theorem 3.1 can be used in this regard provided that .

condition (3.9) is satisfied. However, if the latter condition is not satisfied, we may use the weaker result that for any fixed $\varepsilon \in (0, \frac{1}{2})$

$$\sup_{\varepsilon \le y \le 1-\varepsilon} |\lambda_n(y) - \Gamma_n(y)| = o_p(1) \tag{3.43}$$

The above result is sufficient for all practical applications of the methodology.

We mention here that the consistence of the bootstrapped process can be proved by a routine application of the results of Chapter 17 of Csörgő et. al. (1986).

IV ON TESTING FOR THE NBU CLASS

4.1 Introduction

In this chapter, we consider the problem of testing for the NBU class, i.e.,

$$(4.1) \qquad H_0' = F \in \overline{C}_0 \quad \text{versus} \quad H_1' = F \in C_A$$

Hollander and Prochan (1972) proposed a test based on a U-statistic and proved that their test is unbiased and consistent. They also showed that their test with $\alpha = \binom{2n-1}{n}^{-1}$, $n \geq 3$, has power 1 against the subclass $F_{a,b}$ of C_A, where

$$(4.2) \qquad F_{a,b} = \{F : F \text{ is a continuous distribution function on the interval } [a,b]\}, \quad 0 < a < b < 2a .$$

Koul (1977) gave a Kolmogorov-Smirov type test for testing H_0' against H_1'. He also proved that his test is unbiased, consistent and having power 1 against $F_{a,b}$, $0 < a < b < 2a$.

4.2. NBU Process

By Definition 1.2, $F \in C_A$, if and only if

$$\overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t), \quad 0 \leq x, t < \infty .$$

This is true if and only if $\Lambda^*(\cdot,\cdot) \geq 0$ or $\Lambda(\cdot,\cdot) \geq 0$, where

(4.3) $\Lambda^*(y,t) := F(F^{-1}(y)+t)-(1-y)F(t)-y$, $0 \leq y \leq 1$, $0 \leq t < \infty$,

and

(4.4) $\Lambda(y,z) := F(F^{-1}(y) + F^{-1}(z))-(1-y)z-y$, $0 \leq y,z \leq 1$.

The function $\Lambda^*(\cdot,\cdot)$ of (4.3) will be called the NBU plot function and $\Lambda(\cdot,\cdot)$ of (4.4) will be called the symmetric NBU (SNBU) plot function. The empirical NBU plot function, $\Lambda_n^*(\cdot,\cdot)$, and the empirical SNBU plot function, $\Lambda_n(\cdot,\cdot)$ are defined respectively by

(4.5) $\Lambda_n^*(y,t) := F_n(F_n^{-1}(y)+t)-(1-y)F_n(t)-y$, $0 \leq y \leq 1$, $0 \leq t < \infty$.

and

(4.6) $\Lambda_n(y,z) := F_n(F_n^{-1}(y) + F_n^{-1}(z))-(1-y)z-y$, $0 \leq y$, $z \leq 1$.

Now, we define the NBU plot process, $\lambda_n^*(\cdot,\cdot)$, and the SNBU plot process, $\lambda_n(\cdot,\cdot)$ by

(4.7) $\lambda_n^*(y,t) := n^{1/2}(\Lambda_n^*(y,t) - \Lambda^*(y,t))$, $0 \leq y \leq 1$, $0 \leq t < \infty$,

and

(4.8) $\lambda_n(y,z) := n^{1/2}(\Lambda_n(y,z) - \Lambda(y,z))$, $0 \leq y$, $z \leq 1$,

respectively.

Taking $z = y$ in (4.4), (4.6) and (4.8) , we have

(4.9)
$$\tilde{\lambda}_n(y) = n^{1/2}(\Lambda_n(y,y) - \Lambda(y,y))$$
$$= n^{1/2}(F_n(2F_n^{-1}(y)) - F(2F^{-1}(y))) \ , \quad 0 \leq y \leq 1 \ .$$

which will be called the diagonal new better than used (DNBU) plot process.

4.3 On the Asymptotic Theory for NBU Processes.

The asymptotic theory of $\lambda_n^*(\cdot,\cdot)$, $\lambda_n(\cdot,\cdot)$, and $\tilde{\lambda}_n(\cdot)$ is similar to that of $\lambda_n(\cdot)$ of the last section. The proof of Theorem 3.1 is contained in that of the following Theorem.

Theorem 4.1. Assume that $f = F'$ is continuous, and

(4.10)
$$\sup_{0\leq y\leq 1} \ \sup_{0\leq t<\infty} \frac{f(F^{-1}(y)+t)}{f(F^{-1}(y))} q(y) < \infty$$

for some Chibsov-O'Reilly weight function q. Then on the probability space of Theorem 2.1, we have as $n \to \infty$,

(4.11) $$\sup_{0\le y\le 1} \sup_{0\le t<\infty} |\lambda_n^*(y,t) - \Gamma_n^*(y,t)| = o_p(1) ,$$

and

(4.12) $$\sup_{0\le y\le 1} \sup_{0\le z\le 1} |\lambda_n(y,z) - \Gamma_n(y,z)| = o_p(1) ,$$

where

(4.13) $$\Gamma_n^*(y,t) := B_n(F(F^{-1}(y)+t)) - \frac{f(F^{-1}(y)+t)}{f(F^{-1}(y))} B_n(y) - (1-y)B_n(F(t)),$$

and

(4.14) $$\Gamma_n(y,z) = B_n(F(F^{-1}(y)+F^{-1}(z))) - \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y) - \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(z))} B_n(z) .$$

<u>Theorem 4.2.</u> Assume that $f = F'$ is continuous, and

(4.15) $$\sup_{0\le y\le 1-} \frac{f(2F^{-1}(y))}{f(F^{-1}(y))} q(y) < \infty,$$

for some Chibisov–O'Reilly weight function q. Then on the probability space of Theorem 2.1, we have as $n \to \infty$,

(4.16) $$\sup_{0 \le y \le 1} |\tilde{\lambda}_n(y) - \tilde{\Gamma}_n(y)| = o_p(1),$$

where

(4.17) $$\tilde{\Gamma}_n(y) = B_n(F(2F^{-1}(y))) - 2\,\frac{f(2F^{-1}(y))}{f(F^{-1}(y))}\,B_n(y).$$

Theorem 4.2 follows from Theorem 2.3 by taking $G(x) = F(2x)$, $x \ge 0$.

<u>Proof of Theorem 4.1.</u> Let

$$G_t(x) = F(x+t), \quad G_{t,n}(x) = F_n(x+t), \quad t > 0, \quad x > -t,$$

and define

(4.18) $$\ell_n(y,t) = n^{1/2}(G_{t,n}(F_n^{-1}(y)) - G_t(F^{-1}(y))).$$

Now, similar to (3.8) and (3.18), we have

(4.19) $$\lambda_n^*(y,t) = \ell_n(y,t) - (1-y)\alpha_n(F(t)),$$

and

(4.20) $$\ell_n(y,t) = \alpha_n(G_tF_n^{-1}(y)) - \gamma_n(y,t) ,$$

where

(4.21) $$\gamma_n(y,t) = n^{1/2}(G_tF^{-1}(y) - G_tF_n^{-1}(y)) .$$

Next, since $0 \leq G_tF_n^{-1}(y) \leq 1$ for all $0 \leq y \leq 1$, $0 \leq t < \infty$, so by Theorem 2.1, we have, as $n \to \infty$,

(4.22) $$\sup_{0 \leq t < \infty} \sup_{0 \leq y \leq 1} |\alpha_n(G_tF_n^{-1}(y)) - B_n(G_tF_n^{-1}(y))| \leq \sup_{0 \leq z \leq 1} |\alpha_n(z) - B_n(z)| = o_p(1) .$$

Since the continuous distribution function F is also uniformly continuous, it is easy to verify that the family of functions $\{G_tF^{-1}(y), 0 \leq t < \infty\}$ is equicontinuous, consequently, the family of stochastic processes $\{B_n(G_tF^{-1}(y)), 0 \leq t < \infty\}$ is almost surely equicontinuous for each n . This combined with $G_tF_n^{-1}(y) = G_tF^{-1}(FF_n^{-1}(y))$, and (3.20) yield

(4.23) $$\sup_{0 \leq t < \infty} \sup_{0 \leq y \leq 1} |B_n(G_tF_n^{-1}(y)) - B_n(G_tF^{-1}(y))| = o_p(1) .$$

By (4.22) and (4.23), we obtain

$$(4.24)\qquad \sup_{0\le t<1}\ \sup_{0\le y\le 1} |\alpha_n(G_tF_n^{-1}(y)) - B_n(G_tF^{-1}(y))| = o_p(1) .$$

Now, we consider $\gamma_n(y,t)$ of (4.21). First, we show that

$$(4.25)\qquad \sup_{0\le t<\infty}\ \sup_{0\le y<1/(n+1)} |\gamma_n(y,t)| = o_p(1) .$$

Since

$$(4.26)\qquad \sup_{0\le y\le 1/(n+1)} |\gamma_n(y,t)| \le n^{\frac{1}{2}} G_tF_n^{-1}(\tfrac{1}{n+1}) + n^{\frac{1}{2}} G_tF^{-1}(\tfrac{1}{n+1})$$

$$= n^{\frac{1}{2}} G_tF^{-1}(FF_n^{-1}(\tfrac{1}{n+1})) + n^{\frac{1}{2}} G_tF^{-1}(\tfrac{1}{n+1}) ,$$

by Theorem 2.2, it is sufficient to show that

$$(4.27)\qquad \sup_{0\le t<\infty} n^{\frac{1}{2}} G_tF^{-1}(\tfrac{c}{n+1}) = o(1) \quad , \quad \text{for all fixed } c > 0 .$$

By (4.10), we have

$$(4.28)\qquad \sup_{0\le t<\infty} G_tF^{-1}(y) \le K \int_0^y (1/q(s))ds \quad \text{with some } K > 0 .$$

Since for any Chibisov-O'Reilly function q, we have $s^{-\frac{1}{2}} q(s) \to \infty$, as $s \to 0$ by (4.28), we get (4.27).

By the one-term Taylor expansion,

$$\gamma_n(y,t) = u_n(y) \frac{g_t F^{-1}(\xi_n(y))}{fF^{-1}(\xi_n(y))}$$

where $g_t(x) = f(x+t)$ and

(4.30) $$FF_n^{-1}(y) \wedge y \leq \xi_n(y) \leq FF_n^{-1}(y) \vee y$$

thus, for any $0 < \varepsilon \leq \frac{1}{2}$,

(4.31) $$\sup_{1/(n+1)\leq y\leq\varepsilon} |\gamma_n(y,t)| = \sup_{1/(n+1)\leq y\leq\varepsilon} |u_n(y)| \frac{g_t F^{-1}(\xi_n(y))}{fF^{-1}(\xi_n(y))}.$$

Since $q(\lambda y)$ is a Chibisov-O'Reilly function for all $\lambda > 0$, whenever $q(y)$ is, by Theorem 2.1 and (2.7), we get

(4.32) $$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P\{\sup_{1/(n+1)\leq y\leq\varepsilon} |u_n(y)|/q(\lambda y) > \delta\} = 0,$$
for all $\delta>0$.

By (4.30), (3.20), (4.10) and (4.32), we obtain

$$(4.33)\quad \lim_{\varepsilon\downarrow 0}\ \limsup_{n\to\infty} P\{ \sup_{0\le t<\infty}\ \sup_{1/(n+1)\le y\le \varepsilon} |\gamma_n(y,t)| > \delta\} = 0 , \quad \text{for all } \delta > 0 .$$

Similar to (4.26) and (4.33) , we also have

$$(4.34)\quad \sup_{0\le t<\infty}\ \sup_{n/(n+1)\le y\le 1} |\gamma_n(y,t)| = o_p(1) ,$$

and

$$(4.35)\quad \lim_{\varepsilon\downarrow 0}\ \limsup_{n\to\infty} P\{ \sup_{0\le t<\infty}\ \sup_{1-\varepsilon\le y\le n/(n+1)} |\gamma_n(y,t)| > \delta\} = 0 , \quad \text{for all } \delta > 0 .$$

By (4.10) and (2.7) , we have

$$(4.36)\quad \lim_{\varepsilon\downarrow 0} P\{ \sup_{0\le t<\infty}\ \sup_{0\le y\le \varepsilon} \left|\frac{g_t F^{-1}(y)}{fF^{-1}(y)} B_n(y)\right| > \delta\} = 0 , \quad \text{for all } \delta > 0 .$$

By the symmetric behaviour of $q(t)$ in (2.6)

$$(4.37)\quad \lim_{\varepsilon\downarrow 0} P\{ \sup_{0\le t<\infty}\ \sup_{1-\varepsilon\le y\le 1} \left|\frac{g_t F^{-1}(y)}{fF^{-1}(y)} B_n(y)\right| > \delta\} = 0 , \quad \text{for all } \delta > 0 ,$$

By assumption, $f(x)$ is continuous, hence it is uniformly continuous on $[F^{-1}(\varepsilon) , \infty)$ for any $\varepsilon > 0$, so the family of functions

$$\{g_t F^{-1}(y)/fF^{-1}(y),\quad 0 \le t < \infty\} = \{f(F^{-1}(y)+t)/f(F^{-1}(y)),\ 0 \le t < \infty\}$$

is equicontinuous on $[\varepsilon, 1-\varepsilon]$. Note that (3.20) and (4.30) imply that $\sup_{0\le y\le 1} |\xi_n(y) - y| = o_p(1)$. Then, by Theorem 2.1,

$$\text{(4.38)}\qquad \sup_{0\le t<\infty}\ \sup_{\varepsilon\le y\le 1-\varepsilon} \left|\gamma_n(y,t) - \frac{g_t(F^{-1}(y))}{f(F^{-1}(y))} B_n(y)\right| = o_p(1),$$

for all $\varepsilon \in \left(0, \frac{1}{2}\right)$.

Combining (4.26), (4.33) - (4.38), we get

$$\text{(4.39)}\qquad \sup_{0\le t<\infty}\ \sup_{0\le y\le 1} \left|\gamma_n(y,t) - \frac{g_t(F^{-1}(y))}{f(F^{-1}(y))} B_n(y)\right| = o_p(1).$$

By (4.24) and (4.39), we have

$$\text{(4.40)}\qquad \sup_{0\le t<\infty}\ \sup_{0\le y\le 1} \Big|\ell_n(y,t) - B_n(F(F^{-1}(y)+t)) + \frac{f(F^{-1}(y)+t)}{f(F^{-1}(y))} B_n(y)\Big| = o_p(1).$$

Now, by Theorem 2.1, we have

$$(4.41)\qquad \sup_{0\le t<\infty}\ \sup_{0\le y\le 1} |(1-y)\alpha_n(F(t)) - (1-y)B_n(F(t))|$$

$$= \sup_{0\le t<\infty} |\alpha_n(F(t))-B_n(F(t))|$$

$$= \sup_{0\le s\le 1} |\alpha_n(s)-B_n(s)|$$

$$= o_p(1) \ , \qquad \text{as } n \to \infty \ .$$

Consequently, (4.40) and (4.41) yield (4.11).

Next, we prove (4.12). By the definition of $\lambda_n(y,z)$, we have

$$(4.42)\qquad \lambda_n(y,z) = n^{1/2}(F_n(F_n^{-1}(y) + F_n^{-1}(z)) - F(F^{-1}(y) + F^{-1}(z)))$$

$$= n^{1/2}(F_n(F_n^{-1}(y) + F_n^{-1}(z)) - F(F^{-1}(y) + F_n^{-1}(z)))$$

$$- n^{1/2}(F(F^{-1}(y) + F^{-1}(z)) - F(F^{-1}(y) + F_n^{-1}(z)))$$

$$= \ell_n(y,F_n^{-1}(z)) - \gamma_n(z,F^{-1}(y)) \ ,$$

where $\ell_n(\cdot,\cdot)$ is as in (4.18) and $\gamma_n(\cdot,\cdot)$ is as in (4.21).

By (4.40), as $n \to \infty$,

$$(4.43)\qquad \sup_{0\le z\le 1}\ \sup_{0\le y\le 1} |\ell_n(y,F_n^{-1}(z)) - B_n(F(F^{-1}(y) + F_n^{-1}(z))) + \frac{f(F^{-1}(y)+F_n^{-1}(z))}{f(F^{-1}(y))} B_n(y)|$$

$$\leq \sup_{0\leq t<\infty} \sup_{0\leq y\leq 1} \Big| \ell_n(y,t) - B_n(F(F^{-1}(y)+t)) + \frac{f(F^{-1}(y)+t)}{f(F^{-1}(y))} B_n(y) \Big|$$

$$= o_p(1) .$$

Similarly, (4.36) implies

$$(4.44) \quad \lim_{\varepsilon\downarrow 0} P\Big\{ \sup_{0\leq z\leq 1} \sup_{0\leq y\leq \varepsilon} \Big| \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y) \Big| \geq \delta \Big\} = 0 ,$$

and

$$(4.45) \quad \lim_{\varepsilon\downarrow 0} P\Big\{ \sup_{0\leq z\leq 1} \sup_{0\leq y\leq \varepsilon} \Big| \frac{f(F^{-1}(y)+F_n^{-1}(z))}{f(F^{-1}(y))} B_n(y) \Big| \geq \delta \Big\} = 0 .$$

On the other hand, (4.37) implies

$$(4.46) \quad \lim_{\varepsilon\downarrow 0} P\Big\{ \sup_{0\leq z\leq 1} \sup_{1-\varepsilon\leq y\leq 1} \Big| \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y) \Big| \geq \delta \Big\} = 0 ,$$

and

$$(4.47) \quad \lim_{\varepsilon\downarrow 0} P\Big\{ \sup_{0\leq z\leq 1} \sup_{1-\varepsilon\leq y\leq 1} \Big| \frac{f(F^{-1}(y)+F_n^{-1}(z))}{f(F^{-1}(y))} B_n(y) \Big| \geq \delta \Big\} = 0 .$$

For any $0 < \varepsilon < \frac{1}{2}$, since $1/f(F^{-1}(y))$ is bounded on $[\varepsilon, 1-\varepsilon]$, and the family of functions $\{H_y(t) = f(F^{-1}(y)+t), \ \varepsilon \le y \le 1-\varepsilon\}$ is equicontinuous on $[0,\infty)$, then the family of functions $\{\tilde{H}_y(z) = f(F^{-1}(y) + F^{-1}(z))/f(F^{-1}(y)), \ \varepsilon \le y \le 1-\varepsilon\}$ is equicontinuous on $[0,1]$. By (3.20), as $n \to \infty$

$$\sup_{0\le z\le 1} \sup_{\varepsilon\le y\le 1-\varepsilon} \left| \frac{f(F^{-1}(y)+F_n^{-1}(z))}{f(F^{-1}(y))} B_n(y) - \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y) \right| = o_p(1). \tag{4.48}$$

Combining (4.44) - (4.48), we have

$$\sup_{0\le z\le 1} \sup_{0\le y\le 1} \left| \frac{f(F^{-1}(y)+F_n^{-1}(z))}{f(F^{-1}(y))} B_n(y) - \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y) \right| = o_p(1). \tag{4.49}$$

Next, we observe that $\{F(F^{-1}(y) + F^{-1}(z)), \ 0 \le y \le 1\}$ is equicontinuous on $[0,1]$ and hence $\{B_n(F(F^{-1}(y) + F^{-1}(z))), \ 0 \le y \le 1\}$ is equicontinuous on $[0,1]$, almost surely. Then, by (3.20),

$$(4.50)\quad \sup_{0\le z\le 1}\ \sup_{0\le y\le 1} \left|B_n(F(F^{-1}(y) + F_n^{-1}(z))) - B_n(F(F^{-1}(y) + F^{-1}(z)))\right| = o_p(1) .$$

By (4.43), (4.49) and (4.50), we have

$$(4.51)\quad \sup_{0\le z\le 1}\ \sup_{0\le y\le 1} \left|\ell_n(y,F_n^{-1}(z)) - B_n(F(F^{-1}(y)+F^{-1}(z))) + \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(y))} B_n(y)\right| = o_p(1).$$

Replacing y by z in (4.39), we have

$$(4.52)\quad \sup_{0\le z\le 1}\ \sup_{0\le y\le 1} \left|\gamma_n(z,F^{-1}(y)) - \frac{f(F^{-1}(y)+F^{-1}(z))}{f(F^{-1}(z))} B_n(z)\right| = o_p(1) .$$

Finally, by (4.51) and (4.52), we get (4.12).

Remark 4.1. When $F \in {}_0$, i.e. F is an exponential distribution, the conditons of Theorem 4.1 and Theorem 4.2 are satisfied, and hence (4.11) and (4.12) hold true. In this case, the limit processes have simpler structures. For example $\Gamma_n(y,z)$ of (4.14) becomes

$$(4.53)\quad \Gamma_n^0(y,z) = B_n(y+(1-z)y) - (1-z)B_n(y) - (1-y)B_n(z),$$

which is a Gaussian process with $E\Gamma_n^0(y,z) = 0$, $0 \le y,z \le 1$, and the covariance function:

(4.54) $$E\Gamma_n^0(y_1,z_1)\Gamma_n^0(y_2,z_2) = y_1(1-z_2) + z_2(1-y_1) - y_1z_1(1-y_2)(1-z_2)$$

$$+ (1-y_1)(1-z_2)(y_2 \wedge z_1) + (1-y_2)(1-z_1)(y_1 \wedge z_2)$$

$$- (1-y_2)([(1-z_1)y_1 + z_1] \wedge z_2) - (1-z_2)([(1-y_1)z_1 + y_1] \wedge y_2),$$

$$0 \le y_1 \le y_2 \le 1 \quad , \quad 0 \le z_1 \le z_2 \le 1 .$$

Let $\Gamma^0(y,z)$ be a Gaussian process on $[0,1]\times[0,1]$, with zero mean and the covariance function of (4.54). Then, Theorem 4.1 implies that for all $F \in C_0$,

(4.55) $$\lambda_n(y,z) \xrightarrow{D} \Gamma^0(y,z) .$$

Similarly, when $F \in C_0$, $\tilde{\Gamma}_n(y)$ of (4.1) becomes

(4.56) $$\tilde{\Gamma}_n^0(y) = B_n(2y-y^2) - 2(1-y)B_n(y),$$

which is a Guassian process with $E\tilde{\Gamma}_n^0(y) = 0$, $0 \le y \le 1$, and the convariance function:

$$(4.57)\quad E\tilde{\Gamma}_n^0(x)\tilde{\Gamma}_n^0(y) = \begin{cases} (1-y)^2(2-x^2) - 2(1-y)(1-x)^2, & \text{if } x \le y \le 2x-y^2, \\ -x^2(1-y)^2, & \text{if } x \le 2x-y^2 < y. \end{cases}$$

Let $\tilde{\Gamma}^0(y) \overset{D}{=} \tilde{\Gamma}_n^0(y)$, $0 \le y \le 1$, then Theorem 4.2 implies that, for $F \in C_0$

$$(4.58)\qquad \tilde{\lambda}_n(y) \xrightarrow{D} \tilde{\Gamma}^0(y) .$$

4.4 On testing for the NBU Class

Consider the problem of testing H_0': $F \in C_0$, i.e., $F(x) = 1-e^{-\lambda x}$, $x \in R, \lambda > 0$ against H_1': $F \in C_A$, i.e., F is a NBU distribution. The deviation from H_0' in favor of H_1' may be measured by any appropriate functional of $\Lambda(\cdot,\cdot)$. Some possible functionals are given by

$$\psi_1 = \psi_1(\Lambda(\cdot,\cdot)) = \int_0^1\int_0^1 \Lambda(y,z)dydz,$$

$$\psi_2 = \psi_2(\Lambda(\cdot,\cdot)) = \sup_{0\le y\le 1}\ \sup_{0\le z\le 1} \Lambda(y,z) ,$$

and

$$\psi_3 = \psi_3(\Lambda(\cdot,\cdot)) = \sup_{0\le y\le 1} \sup_{0\le z\le 1} |\Lambda(y,z)|,$$

the corresponding test statistics are $\psi_i(\Lambda_n(\cdot,\cdot))$, $i = 1,2,3$, respectively. By Theorem 4.1 and Remark 4.1, we have

$$(4.59) \qquad n^{1/2}\psi_{in} = n^{1/2}\psi_i(\Lambda_n(\cdot,\cdot)) \xrightarrow{D} \psi_i(\Gamma^0(\cdot,\cdot))$$

where $\Gamma^0(\cdot,\cdot)$ is the Gaussian process of (4.55).

Now, using the notation of Koul (1977), $\Lambda_n(\cdot,\cdot)$ can be written as

$$(4.60) \qquad \Lambda_n(y,z) = n^{-1}\bar{S}_{ij}-(1-y)z-y \;, \quad \frac{i-1}{n} < y \le \frac{i}{n}\;,\; \frac{j-1}{n} < z \le \frac{j}{n}\;, \quad 1 \le i,j \le n\;,$$

where

$$(4.61) \qquad \bar{S}_{ij} = \sum_{k=1}^{n} I(X_k \le X_{i:n} + X_{j:n})\;, \quad 1 \le i,j \le n\;.$$

In particular, we have

$$(4.62) \qquad \psi_{1n} = \{n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{S}_{ij} - \tfrac{3}{4}\}\;,$$

$$(4.63) \qquad \psi_{2n} = n^{-2} \max_{1\le i\le j\le n} [n\bar{S}_{ij} - (n-i-1)(j-1) - n(i-1)]\;,$$

and

$$\psi_{3n} = n^{-2} \max_{1 \leq i \leq j \leq n} \{|\rho_{ij}|, |\rho^*_{ij}|\},$$

where

$$\rho_{ij} = n\bar{S}_{ij} - (n-i-1)(j-1) - n(i-1), \quad \rho^*_{ij} = n\bar{S}_{ij} - (n-i)j - ni.$$

We observe that $n^2[(n-1)(n-2)]^{-1}(\psi_{1n} + \frac{3}{4}) = -J_n$ where $J_n = -[n(n-1)(n-2)]^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{S}_{ij}$ is the U-statistic of Hollander and Proschan (1972). In addition, $n^2\psi_{2n} = -T_n$, which is the test statistic proposed by Koul (1977). We mention here that the consistency of J_n and T_n follow from that of $\Lambda_n(\cdot,\cdot)$.

We may also consider appropriate functionals of $\tilde{\Lambda}(y) := \Lambda(y,y)$, $0 \leq y \leq 1$, as measures of the deviation from H_0' in favour of H_1'. For exmaple, we may use any of the statistics

$$(4.65) \qquad \varphi_{1n} = \int_0^1 \tilde{\Lambda}_n(y)dy = n^{-2} \sum_{i=1}^{n} R_i^{(2)} - \frac{2}{3},$$

and

$$\varphi_{2n} = \sup_{0\le y\le 1} \tilde{\Lambda}_n(y) = n^{-2} \max_{1\le i\le n} [nR_i^{(2)} - 2n(i-1) + (i-1)^2]$$

where $R_i^{(2)}$ is the rank of $X_{i:n}$ in the combined X and 2X sample, $1 \le i \le n$, and $\tilde{\Lambda}_n(y) := \Lambda_n(y,y)$, $0 \le y \le 1$. Theorem 4.2 and Remark 4.1 implies that under H_0', as $n \to \infty$,

(4.67) $$n^{1/2}\varphi_{1n} \xrightarrow{D} \int_0^1 \tilde{\Gamma}^0(y)dy,$$

(4.68) $$n^{1/2}\varphi_{2n} \xrightarrow{D} \sup_{0\le y\le 1} \tilde{\Gamma}^0(y)dy,$$

where $\tilde{\Gamma}^0(y)$ is the Gaussian process of (4.58).

Observe that $\int_0^1 \tilde{\Gamma}^0(y)dy$ is a normal r.v. with mean zero and variance $\sigma_{\varphi_1} = \int_0^1\int_0^1 \operatorname{cov}(x,y)dxdy = \frac{11}{630}$. But $\sup_{0\le y\le 1} \tilde{\Gamma}^0(y)$, is not a normal r.v. To make the test staistic φ_{2n} applicable, we present the percentage points of $n^{1/2}\varphi_{2n}$ for $1 \le n \le 50$ in Table 4.1 which is established by using Monte Carlo method. The percentage points of the limiting distribution of $n^{1/2}\varphi_{2n}$ is given in the last row of Table 4.1.

Insert Table 4.1 about here.

Remark 4.2. Comparing ψ_{1n} of (4.62) and φ_{1n} of (4.65), we note that φ_{1n} has simpler expression. Both ψ_{1n} and φ_{1n} have normal limit distributions with zero means and distribution-free variances $\sigma^2_{\varphi_1}$ and $\sigma^2_{\varphi_1}$ respectively, where $\sigma^2_{\psi_1} = \frac{5}{432} \approx 0.01157$ as given by Hollander et al. (1972) and $\sigma^2_{\psi_1} = \frac{11}{630} \approx 0.01746$. Consequently, ψ_{1n} is more efficient than φ_{1n}. A measure of their relative efficiency is given by

$$R = e(\psi_{1n}, \varphi_{1n}) = \sigma^2_{\varphi_1} / \sigma^2_{\psi_1} = 0.66287. \tag{4.69}$$

4.5 A Graphical Approach

The graphical methods stated in Chapter 3 can also be used for testing $H_0' : F \in C_0$ versus $H_1' : F \in C_A$. First, we prove the consistency of $\Lambda^*_n(\cdot,\cdot)$, $\Lambda_n(\cdot,\cdot)$ and $\tilde{\Lambda}_n(\cdot)$.

Theorem 4.3. For any continuous distriubtion function F, we have, as $n \to \infty$,

$$\sup_{0 \le y \le 1} \sup_{0 \le t < \infty} |\Lambda^*_n(y,t) - \Lambda^*(y,t)| = o_p(1) \tag{4.70}$$

(4.71) $$\sup_{0\le y\le 1}\ \sup_{0\le z\le 1} |\Lambda_n(y,z) - \Lambda(y,z)| = o_p(1)$$

and

(4.72) $$\sup_{0\le y\le 1} |\tilde{\Lambda}_n(y) - \tilde{\Lambda}(y)| = o_p(1)$$

Proof. By (4.18) and (4.19), we have

$$|\Lambda_n^*(y,t) - \tilde{\Lambda}(y,t)| = |G_{t,n}F_n^{-1}(y) - G_tF^{-1}(y) - (1-y)(F_n(t) - F(t))|$$

$$\le |F_n(F_n^{-1}(y)+t) - F(F_n^{-1}(y)+t)|$$

$$+ |G_t(F_n^{-1}(y)) - G_t(F^{-1}(y))| + (1-y)|F_n(t)-F(t)|,$$

hence

(4.73) $$\sup_{0\le t<\infty}\ \sup_{0\le y\le 1} |\Lambda_n^*(y,t) - \Lambda^*(y,t)|$$

$$\le \sup_{0\le x<\infty} |F_n(x)-F(x)| + \sup_{0\le t<\infty}\ \sup_{0\le y\le 1} |G_tF^{-1}(FF_n(y))-G_tF^{-1}(y)|$$

$$+ \sup_{0\le t<\infty} |F_n(t)-F(t)|$$

By Glivenko-Cantelli Theorem, the first and third terms on the right side of (4.73) tends to zero almost surely, and the middle term tends to zero in probability by the equicontinuity of $\{G_t F^{-1}(y), 0 \le t < \infty\}$ and (3.20). This proves (4.70).

Next, since

$$|\Lambda_n(y,z) - \Lambda(y,z)| = |F_n(F_n^{-1}(y) + F_n^{-1}(z)) - F(F^{-1}(y) - F^{-1}(z))|$$

$$\le |F_n(F_n^{-1}(y) + F_n^{-1}(z)) - F(F^{-1}(y) + F_n^{-1}(z))|$$

$$+ |F(F^{-1}(y) + F_n^{-1}(z)) - F(F^{-1}(y) + F^{-1}(z))|$$

then, by (4.70), (3.20) and the equicontinuity of $\{G_s F^{-1}(y),$ $0 \le s < \infty\}$,

$$\sup_{0\le y\le 1} \sup_{0\le z\le 1} |\Lambda_n(y,z) - \Lambda(y,z)|$$

$$\le \sup_{0\le y\le 1} \sup_{0\le t<\infty} |G_{t,n}F_n^{-1}(\text{[illegible]}(y)| + \sup_{0\le z\le 1} \sup_{0\le s<\infty} |G_s F_n^{-1}(z) - G_s F^{-1}(z)|$$

$$= o_p(1)$$

This gives (4.71).

Taking $z = y$ in (4.71), we obtain (4.72). The proof is completed.

In order to test H_0' against H_1', we may plot $\Lambda_n(y,z)$ as a function of (y,z), $0 \leq y,z \leq 1$. If H_0 is ture, then the SNBU plot $(y,z,\Lambda_n(y,z))$ should closely evolve along the Y,Z-plane. A plot which falls "above" the Y,Z-plane would imply that H_1 is true. A plot which falls "under" the Y,Z-plane would imply that the data comes from a NWU distribution. A plot of $\Lambda_n(y,z)$ for the data of Example 3.1 is presented in Figure 4.1.

Insert Figure 4.1 about here.

If the alternative is the subclass $F_{a,b}$, $0 < a < b < 2a$, of (4.2) of C_A, then it is easy to verify that $\bar{S}_{ij} \equiv n$, where $\bar{S}_{ij}$ is defined by (4.61). By (4.60), we have that for all $F \in F_{a,b}$, $\Lambda_n(y,z) = (1-y)(1-z) > 0$ for all $0 \leq y,z < 1$. Hence, the above explained graphical test based on $\Lambda_n(y,z)$ has power 1 against $F_{a,b}$. This implies that both the test of Hollander and Proschan (1972), based on J_n, and that of Koul (1977), based on T_n, have power 1 against F_{ab}. This fact was also proved by Hollander and Proschan (1972) and Koul (1977).

The above explained graphical approach requires plotting in a three-dimensional space. We may avoid this difficulty, by plotting $\tilde{\Lambda}_n(\cdot)$ instead of $\Lambda_n(\cdot,\cdot)$. If H_0' is ture, the DNBU plot $(y,\tilde{\Lambda}_n(y))$

should closely evolve along the horizontal axis. A plot which falls "above" (resp. "under") the horizontal axis would imply that H_1' is true, i.e. the underlying distribution is NBU (resp. NWU).

Bootstrapped confidence bands for $\Lambda(\cdot,\cdot)$ and $\tilde{\Lambda}(\cdot)$ can be constructed in a fashion similar to that of Chapter 3. In the following we only show how the method works.

Let $X_1', X_2', \ldots, X_{n_1}'$ be conditionally independent r.v.'s with common distribution function F_n. The bootstrapped empirical distriubtion function $F_{n_1,n}$ is defined by (3.37). We define the bootstrapped empirical SNBU plot $\Lambda_{n_1,n}(\cdot,\cdot)$ and the bootstrapped empirical DNBU plot $\tilde{\Lambda}_{n_1,n}(\cdot)$ by,

(4.74) $$\Lambda_{n_1,n}(y,z) := F_{n_1,n}(F^{-1}_{n_1,n}(y)+F^{-1}_{n_1,n}(z)) - (1-y)z-y, \quad 0 \le y,z \le 1,$$

and

(4.75) $$\tilde{\Lambda}_{n_1,n}(y) := F_{n_1,n}(2F^{-1}_{n_1,n}(y)) - 2y + y^2, \quad 0 \le y \le 1.$$

The bootstrapped SNBU plot process and the bootstrapped DNBU plot process are defined by

(4.76) $$\lambda_{n_1,n}(y,z) := n_1^{1/2}(\Lambda_{n_1,n}(y,z)-\Lambda_n(y,z)), \quad 0 \le y,z \le 1 ,$$

and

(4.77) $\tilde{\lambda}_{n_1,n}(y) := n_1^{1/2}(\tilde{\Lambda}_{n_1,n}(y) - \tilde{\Lambda}_n(y)), \quad 0 \leq y \leq 1$.

Now generating bootstrapped processes $\lambda^{(i)}_{n_1,n}(y,z)$, $\tilde{\lambda}^{(i)}_{n_1,n}(y)$, $1 \leq i \leq N$, N times, we let

(4.78) $$G^*_{N,n_1,n}(x) = \frac{1}{N} \# \{ 1 \leq i \leq N : \sup_{0 \leq y \leq 1} \sup_{0 \leq z \leq 1} |\lambda^{(i)}_{n_1,n}(y,z)| \leq x \},$$

(4.79) $$\tilde{G}^*_{N,n_1,n}(x) = \frac{1}{N} \# \{ 1 \leq i \leq N : \sup_{0 \leq y \leq 1} |\tilde{\lambda}^{(i)}_{n_1,n}(y)| \leq x \},$$

(4.80) $$C^*_{N,n_1,n}(\alpha) = G^{*-1}_{N,n_1,n}(1-\alpha),$$

and

(4.81) $$\tilde{C}_{N,n_1,n}(\alpha) = \tilde{G}^{-1}_{N,n_1,n}(1-\alpha),$$

where $\alpha \in (0,1)$ is fixed.

An asymptotically $(1-\alpha)100\%$ confidence band for $\Lambda(\cdot,\cdot)$ is given by

(4.82) $$\{\Lambda_n(y,z) - C^*_{N,n_1,n}(\alpha)n^{-\frac{1}{2}} \leq \Lambda(y,z) \leq \Lambda_n(y,z) + C^*_{N,n_1,n}(\alpha)n^{-\frac{1}{2}}, \quad 0 \leq y,z \leq 1\}$$

and an asymptotically $(1-\alpha)100\%$ confidence band for $\tilde{\Lambda}(\cdot)$ is given by

$$(4.83) \quad \{\tilde{\Lambda}_n(y) - \tilde{C}_{N,n_1,n}(\alpha)n^{-\frac{1}{2}} \le \tilde{\Lambda}(y) \le \tilde{\Lambda}_n(y) + \tilde{C}_{N,n_1,n}(\alpha)n^{-\frac{1}{2}},$$

$$0 \le y \le 1\}.$$

For the problem of testing H_0' : F is an exponential distribution against H_1' : F is NBU but not exponential, we may construct the bootstrapped confidence band for $\Lambda(y,z)$ or for $\tilde{\Lambda}(y)$. If the (1-α)100% confidence band in (4.75) contains the Y,Z-plane, then we conclude that H_0' is true. If, on the other hand, the lower limit of the confidence band falls above the Y,Z-plane, then we conclude that H_1' is true. Similarly, we can use the DNBU bootstrapped confidence band for the test.

For the data of Example 3.1, we present a 95% bootstrapped confidence band for the DNBU plot function $\tilde{\Lambda}(\cdot)$ in Figure 4.2. We see that both the upper and the lower limits have crossed the horizontal axis. This suggests that the distribution of the survival period of the patients is neither exponential nor NBU. Recall that Figure 3.2 implies that the distribution is new better than used of age 1825.

Insert Figure 3.1 about here.

V. SMOOTHED QUANTILE AND PERCENTILE RESIDUAL LIFETIME PROCESSES

5.1. Introduction

In this chapter we prove a number of results concerning the weak convergence of kernel-type quantile and percentile residual lifetime processes. We will consider both the censored and noncensored cases.

In the following, we list a number of conditions of which certain subsets will be needed in different places in the sequel.

In the following group of conditions, F is a distribution function

(F.1) F is twice differentiable on (a,b), where $a = \sup\{x: F(x) = 0\}$, $b = \inf\{x: F(x) = 1\}$, $-\infty \leq a < b \leq \infty$.

(F.2) $F'(x) = f(x) > 0$ on (a,b).

(F.3.1) There exists a $\gamma > 0$ such that

$$\sup_{a<x<b} F(x)(1-F(x)) \frac{|f'(x)|}{f^2(x)} < \gamma . \tag{5.1}$$

(F.3.2) For a fixed $t > 0$, there exists a $\gamma > 0$, such that

$$\sup_{t<x<b} F(x)(1-F(x)) \frac{|f'(x)|}{f^2(x)} < \gamma \tag{5.2}$$

(F.3.3) There exists a $\gamma > 0$ and a $p^* \in (0,1)$ such that

(5.3) $$\sup_{0<t<p^*} t \frac{|f'Q(t)|}{(fQ(t))^2} < \gamma$$

where $Q(t) = F^{-1}(t)$ is the quantile function of F.

(F.4) $\lim_{x\to\infty} x^{\alpha}(F(-x)+1-F(x)) = 0$, for some $\alpha > 0$.

In the following group of conditions K is a probability density function.

(K.1) K has finite support, i.e., $K(x) = 0$ for $|x| > c$ for some constant $c > 0$.

(K.1) K satisfies a Lipschitz condition, i.e., there exists a constant Γ such that for all x,y, $|K(x) - K(y)| \leq \Gamma|x-y|$.

In the following two conditions, $\{h_n\}$ is a sequence of positive numbers.

(H.1) $n^{1/2}h_n \to 0$, as $n \to \infty$.

(H.2) $h_n \to 0$, as $n \to \infty$ and $\sum_{n=2}^{\infty} (\log n)^{1/2}/(h_n n^{(\delta_0-4)/2}) < \infty$, for some $\delta_0 > 4$.

5.2 On Kernel-type Quantile Processes

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with an absolutely continuous distributi function F. Let $X_{1:n} \leq X_{2:n} \leq , \ldots, \leq X_{n:n}$ be the order statistic of the X_i's. A natural estimator of the quantile function $F^{-1}(\lambda)$ of F is the λth quantile $F_n^{-1}(\lambda)$ given by

$$(5.4) \quad F_n^{-1}(\lambda) = X_{k:n}, \text{ if } (k-1)/n < \lambda \leq k/n, \quad i = 1, \ldots, n,$$

and $F_n^{-1}(0) = X_{1:n}$.

Csorgo and Révész (1978) defined the quantile process $\rho_n(y)$ as

$$(5.5) \quad \rho_n(y) = n^{1/2} fF^{-1}(y)(F_n^{-1}(y) - F^{-1}(y)), \quad 0 < y < 1.$$

They have proved a number of strong approximation results for $\rho_n(y)$.

Yang (1985) considered a smoothed alternative to $F_n^{-1}(\cdot)$, which i given by

$$(5.6) \quad \tilde{Q}_n(\lambda) := n^{-1} h_n^{-1} \sum_{i=1}^{n} X_{i:n} K\left(\frac{(i/n)-\lambda}{h_n}\right).$$

It is motivated from the following kernel-type estimator

$$(5.7) \qquad Q_n(\lambda) := \int_0^1 F_n^{-1}(t) h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$= \sum_{i=1}^{n} X_{i:n} h_n^{-1} \int_{(i-1)/n}^{i/n} K\left(\frac{t-\lambda}{h_n}\right) dt,$$

where K is some density function and $h_n \to 0$ as $n \to \infty$. Yang (1985) proved that, for any given $\lambda \in (0,1)$, $n^{1/2}[Q_n(\lambda) - F^{-1}(\lambda)]$ is asymptotically normal. He has also proved the asymptotic mean square equivalence of $Q_n(\lambda)$ and $\tilde{Q}_n(\lambda)$.

In this section, we will prove the weak convergence of a kernel-type quantile process. This result is an extension of the results of Csörgö and Révész (1978), Csörgö et. al. (1982) and Yang (1985). First, we state the following theorem.

<u>Theorem A</u> (Csörgö et al. (1982)). Assume that conditions (F.1), (F.2) and (F.3.1) are satisfied. On an appropriate probability space, there exists a sequence of Brownian bridges $\{B_n(y) := n^{1/2}K(y,n),\ 0 \le y \le 1\}_{n=1}^{\infty}$, where $K(\cdot,\cdot)$ is a Kiefer process, such that

$$(5.8) \qquad \sup_{1/(n+1) \le y \le n/(n+1)} |\rho_n(y) - B_n(y)|$$

$$\overset{a.s.}{=} \begin{cases} O(n^{-1/2}(\log\log n)^{1+\gamma}), & \gamma \le 1, \\ O(n^{-1/2}(\log\log n)^{\gamma}(\log n)^{(1+\varepsilon)(\gamma-1)}), & \gamma > 1, \end{cases}$$

where $\varepsilon > 0$ is arbitrary and γ is as in condition (F.3.1).

Following Yang (1985), we define the kernel-type quantile process $\rho_n^*(\cdot)$ as

$$\text{(5.9)} \qquad \rho_n^*(\lambda) := n^{1/2} fF^{-1}(\lambda)(Q_n(\lambda) - F^{-1}(\lambda)),$$

where $Q_n(\cdot)$ is as in (5.7).

The main result of this secion is the following theorem.

Theorem 5.1. Assume that conditions (F.1), (F.2), (F.3.1), (K.1) and (H.1) are satisfied. On the probability space of Theorem A, and with the same sequence of Borwnian bridges, we have

$$\text{(5.10)} \qquad \sup_{1/\sqrt{n} \leq \lambda \leq (1-1/\sqrt{n})} |\rho_n^*(\lambda) - B_n(\lambda)| = o_p(1), \quad \text{as} \quad n \to \infty.$$

In order to be able to prove Theorem 5.1, we need the following three Lemmas.

Lemma 5.1. Suppose that conditions (F.1),(F.2) and (F.3.1) are satisfied. Then for any $0 < r,s < 1$ we have

$$\text{(5.11)} \qquad \frac{fF^{-1}(r)}{fF^{-1}(s)} \leq \left\{\frac{r \vee s}{r \wedge s} \cdot \frac{1-(r\wedge s)}{1-(r \vee s)}\right\}^{\gamma} .$$

Lemma 5.1 was proved by Csörgö and Révész (1978).

<u>Lemma 5.2</u> Suppose that conditions (F.1), (F.2), (F.3.1), (K.1) and (H.2) are satisfied. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$, $\varepsilon_n^{-1} h_n \to 0$, as $n \to \infty$, then we have

$$(5.12) \qquad \sup_{\varepsilon_n \le \lambda \le 1-\varepsilon_n} \int_{-\infty}^{\infty} \left|\frac{fF^{-1}(\lambda)}{fF^{-1}(t)} - 1\right| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt = o(1), \quad \text{as } n \to \infty,$$

and

$$(5.13) \qquad \sup_{\varepsilon_n \le \lambda \le 1-\varepsilon_n} \int_{-\infty}^{\infty} \left|\frac{fF^{-1}(x)}{fF^{-1}(t)} h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right)\right| dt \le M < \infty,$$

where M is a constant.

<u>Proof</u>. We note tht (5.13) follows directly from (5.12). Hence, we will prove (5.12) only. By Lemma 5.1,

$$\left\{\frac{\lambda \wedge t}{\lambda \vee t} \cdot \frac{1-(\lambda \vee t)}{1-(\lambda \wedge t)}\right\}^{\gamma} \le \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} \le \left\{\frac{\lambda \vee t}{\lambda \wedge t} \cdot \frac{1-(\lambda \wedge t)}{1-(\lambda \vee t)}\right\}^{\gamma}$$

Now, by the above inequality and the fact that $a-1 \ge 1 - \frac{1}{a}$, for all $a \ge 1$, we have

$$(5.14)\quad \left|\frac{fF^{-1}(\lambda)}{fF^{-1}(t)}-1\right| \le \left(\left\{\frac{\lambda\vee t}{\lambda\wedge t}\cdot\frac{1-(\lambda\vee t)}{1-(\lambda\wedge t)}\right\}^{\gamma}-1\right)\vee\left(1-\left\{\frac{\lambda\wedge t}{\lambda\vee t}\cdot\frac{1-(\lambda\vee t)}{1-(\lambda\wedge t)}\right\}^{\gamma}\right)$$

$$=\left\{\frac{\lambda\vee t}{\lambda\wedge t}\cdot\frac{1-(\lambda\wedge t)}{1-(\lambda\vee t)}\right\}^{\gamma}-1.$$

This combined with condition (K.1) give

$$\sup_{\varepsilon_n\le\lambda\le 1-\varepsilon_n}\int_{-\infty}^{\infty}\left|\frac{fF^{-1}(\lambda)}{fF^{-1}(t)}-1\right|h_n^{-1}K\left(\frac{t-\lambda}{h_n}\right)dt$$

$$=\sup_{\varepsilon_n\le\lambda\le 1-\varepsilon_n}\int_{\lambda-ch_n}^{\lambda+ch_n}\left|\frac{fF^{-1}(\lambda)}{fF^{-1}(t)}-1\right|h_n^{-1}K\left(\frac{t-\lambda}{h_n}\right)dt$$

$$\le\sup_{\varepsilon_n\le\lambda\le 1-\varepsilon_n}\ \sup_{|\lambda-t|\le ch_n}\left|\frac{fF^{-1}(\lambda)}{fF^{-1}(t)}-1\right|\int_{-\infty}^{\infty}K(y)dy$$

$$\le\sup_{\varepsilon_n\le\lambda\le 1-\varepsilon_n}\ \sup_{|\lambda-t|\le ch_n}\left\{\frac{\lambda\vee t}{\lambda\wedge t}\cdot\frac{1-(\lambda\wedge t)}{1-(\lambda\vee t)}\right\}^{\gamma}-1$$

$$\le\sup_{\varepsilon_n\le\lambda\le 1-\varepsilon_n}\left\{\left(\frac{\lambda}{\lambda-ch_n}\cdot\frac{1-\lambda+ch_n}{1-\lambda}\right)\vee\left(\frac{\lambda+ch_n}{\lambda}\cdot\frac{1-\lambda}{1-\lambda-ch_n}\right)\right\}^{\gamma}-1$$

$$\le\left(\frac{\varepsilon_n}{\varepsilon_n-ch_n}\cdot\frac{\varepsilon_n+ch_n}{\varepsilon_n}\right)^{\gamma}-1$$

$$= \left(\frac{1+c\varepsilon_n^{-1}h_n}{1-c\varepsilon_n^{-1}h_n}\right)^{\gamma} - 1$$

$$= o(1), \quad \text{as} \quad n \to \infty.$$

This completes the proof of (5.12).

Lemma 5.3. Assume that conditions (F.1), (F.2), (F.3.1), (K.1) and (H.1) are satisfied. Then, as $n \to \infty$,

$$\text{(5.15)} \qquad \sup_{1/\sqrt{n} \le \lambda \le (1-1/\sqrt{n})} n^{1/2} fF^{-1}(\lambda) \int_0^1 [F^{-1}(t) - F^{-1}(\lambda)] h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt = o(1) .$$

Proof. By the one-term Taylor's expansion on $F^{-1}(\cdot)$, we have

$$n^{1/2} fF^{-1}(\lambda) \int_0^1 [F^{-1}(t) - F^{-1}(\lambda)] h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$= n^{1/2} h_n \int_{\lambda-ch_n}^{\lambda+ch_n} \frac{fF^{-1}(\lambda)}{fF^{-1}(\xi_t)} \left(\frac{t-\lambda}{h_n}\right) h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

where ξ_t is a value between λ and t. By Lemma 1,

$$\sup_{1/\sqrt{n} \le \lambda \le (1-1/\sqrt{n})} n^{1/2} fF^{-1}(\lambda) \int_0^1 [F^{-1}(t) - F^{-1}(\lambda)] h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$\leq n^{1/2} h_n \sup_{1/\sqrt{n}\leq\lambda\leq(1-1/\sqrt{n})} \sup_{|t-\lambda|\leq ch_n} \frac{fF^{-1}(\lambda)}{fF^{-1}(\xi_t)} \int_{-c}^{c} yK(y)d$$

$$\leq n^{1/2} h_n c \sup_{1/\sqrt{n}\leq\lambda\leq(1-1/\sqrt{n})} \sup_{|\xi_t-\lambda|\leq ch_n} \frac{fF^{-1}(\lambda)}{fF^{-1}(\xi_t)}$$

$$\leq n^{1/2} h_n c \sup_{1/\sqrt{n}\leq\lambda\leq(1-1/\sqrt{n})} \sup_{|\xi_t-\lambda|\leq ch_n} \left\{\frac{\xi_t\vee\lambda}{\xi_t\wedge\lambda} \cdot \frac{1-(\xi_t\wedge\lambda)}{1-(\xi_t\vee\lambda)}\right\}^{\gamma}$$

$$\leq n^{1/2} h_n c \left(\frac{1+ch_n\sqrt{n}}{1-ch_n\sqrt{n}}\right)^{\gamma}$$

$$= o(1), \text{ as } n \to \infty.$$

This proves (5.15)

<u>Proof of Theorem 5.1.</u> By the definitions of $\rho_n^*(\cdot)$ of (5.9) and $\rho_n(\cdot)$ of (5.5), we have

$$\rho_n^*(\lambda) = n^{1/2} fF^{-1}(\lambda)(Q_n(\lambda) - F^{-1}(\lambda))$$

$$= n^{1/2} fF^{-1}(\lambda)\left[\int_0^1 F_n^{-1}(t) h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt - F^{-1}(\lambda)\right]$$

$$= \int_0^1 \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} \rho_n(t) h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$+ n^{1/2} fF^{-1}(\lambda) \int_0^1 [F^{-1}(t) - F^{-1}(\lambda)] h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt.$$

Hence

$$(5.17) \qquad |\rho_n^*(\lambda) - B_n(\lambda)| \leq \int_0^1 \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} |\rho_n(t) - B_n(t)| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$+ \int_0^1 \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} |B_n(t) - B_n(\lambda)| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$+ B_n(\lambda) \int_0^1 \left| \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} - 1 \right| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$+ \left| n^{1/2} fF^{-1}(\lambda) \int_0^1 [F^{-1}(t) - F^{-1}(\lambda)] h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt \right|$$

$$:= I_{1n}(\lambda) + I_{2n}(\lambda) + I_{3n}(\lambda) + I_{4n}(\lambda).$$

By Theorem A, condition (H.1) and (5.13) of Lemma 2, we have

$$(5.18) \qquad \sup_{1/\sqrt{n} \leq \lambda \leq (1-1/\sqrt{n})} I_{1n}(\lambda)$$

$$= \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} \int_{\lambda-ch_n}^{\lambda+ch_n} \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} |\rho_n(t) - B_n(t)| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$\le \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} \sup_{|t-\lambda|\le ch_n} |\rho_n(t) - B_n(t)| \int_{-\infty}^{\infty} \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$\le \sup_{1/\sqrt{n}-ch_n\le t\le 1-1/\sqrt{n}+ch_n} |\rho_n(t) - B_n(t)| \cdot$$

$$\sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} \int_{-\infty}^{\infty} \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$\le M \sup_{1/(n+1)\le t\le(1-1/n+1)} |\rho_n(t) - B_n(t)|$$

$$\overset{a.s}{=} o(1) , \quad \text{as } n \to \infty.$$

The last inequality above follows from the observation that $\frac{1}{\sqrt{n}} - ch_n = \frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \ge \frac{1}{n+1}$ for large enough n.

Next, we note that a Brownian bridge $B(\cdot)$ is almost surely uniformly continuous. This combined with (5.13) of Lemma 2 implies that

$$(5.19) \quad \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} I_{2n}(\lambda)$$

$$= \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} \int_{\lambda-ch_n}^{\lambda+ch_n} \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} |B_n(t)-B_n(\lambda)| h_n^{-1} K(\frac{t-\lambda}{h_n})dt$$

$$\le \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} \sup_{|t-\lambda|\le ch_n} |B_n(t) - B_n(\lambda)| \int_{-\infty}^{\infty} \frac{fF^{-1}(\lambda)}{fF^{-1}(t)} h_n^{-1} K(\frac{t-\lambda}{h_n})dt$$

$$\overset{P}{=} o(1), \quad \text{as} \quad n \to \infty.$$

As to I_{3n} of (5.17), we use the fact that $\sup_{0\le t\le 1} B(t)$ is bounded in probability together with (5.12) of Lemma 2 to obtain

(5.20) $$\sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} I_{3n}(\lambda) \overset{P}{=} o(1)$$

Now Lemma 3 implies that

(5.21) $$\sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} I_{4n}(\lambda) = o(1) \quad \text{as} \quad n \to \infty.$$

Combining (5.17) - (5.21), we obtain (5.10).

Yang (1985) proved the mean square asymptotical equivalence of $\tilde{Q}_n(\lambda)$ and $Q_n(\lambda)$ stated in the following theorem.

<u>Theorem B.</u> Suppse the assumptions (F.4), (K.1) and (K.2) are satisfied, then we have, as $n \to \infty$,

(5.22) $$E[(\tilde{Q}_n(\lambda)-Q_n(\lambda))^2] = o[1/(nh_n^2a_n^2)]$$

uniformly in λ, where $a_n \to 0$ and $a_n/(nh_n) \to 0$, as $n \to \infty$.

If we choose $a_n = 1/(n^{1/4}h_n)$, and $1/(n^{5/8}h_n) \to 0$, then Theorem B implies that

$$\text{(5.23)} \qquad \sup_{0<\lambda<1} |\tilde{Q}_n(\lambda) - Q_n(\lambda)| = o_p(1/\sqrt{n})$$

therefore $\tilde{Q}_n(\lambda)$ and $Q_n(\lambda)$ have the same asymptotic distribution. So the result of Theorem 5.1 also holds true for $\tilde{Q}_n(\lambda)$ provided $f(x)$ is bounded. We state this conclusion in the following corollary.

<u>Corollary 5.1</u> Suppose the conditions (F.1), (F.2), (F.3.1), (K.1), (K.2) and (H.1) are satisfied, $f(x)$ is bounded on (a,b) and h_n is choosen such that $1/[n^{5/8}h_n] \to 0$. Then, on the probability space of Theorem A and with the same Brownian bridges $B_n(\cdot)$, we have, as $n \to \infty$

$$\text{(5.24)} \qquad \sup_{1/\sqrt{n}\le\lambda\le(1-1/\sqrt{n})} |n^{1/2} fF^{-1}(\lambda)(\tilde{Q}_n(\lambda) - F^{-1}(\lambda)) - B_n(\lambda)| = o_p(1) .$$

Proof: Parzen (1980) proved that condition (F.3.1) impies that $E|x|^{\alpha} < \infty$, $0 \le \alpha \le \frac{1}{\gamma}$, which implies (F.4). The result follows by (5.10) and (5.23).

5.3. On Kernel-type Percentile Residual Lifetime Processes

Heines and Singpurwalla (1974) introduced the concept of the p-percentile residual lifetime $R(t,p)$ defined in (2.16), A natural estimator of $R(t,p)$ is $\hat{R}_n(t,p)$ defined by

$$\hat{R}_n(t,p) = \begin{cases} F_n^{-1}(1-\bar{p}\,\bar{F}_n(t)) - t\,, & 0 \le t \le X_{n:n}\,,\quad 0 \le p \le 1, \\ 0\,, & t > X_{n:n}\,. \end{cases} \tag{5.25}$$

Csörgö (1983) defined the p-percentile residual lifetime process, $r_n(t,p)$, as

$$\begin{aligned} r_n(t,p) &= n^{1/2} fF^{-1}(1-\bar{p}\,\bar{F}(t))[\hat{R}_n(t,p) - R(t,p)] \\ &= n^{1/2} f(R(t,p) + t)[\hat{R}_n(t,p) - R(t,p)]\,. \end{aligned} \tag{5.26}$$

M. Csörgö and S. Csörgö (1986) proved that for any given $t \ge 0$ and $0 < p < 1$, $r_n(t,p)$ is asymptotically normal. They have also proved some strong approximation results for $r_n(t,p)$ as a stochastic process in either one or both of its parameters p and t. They have used these asymptotic results to construct some confidence bands for $R(\cdot,\cdot)$.

Let $t \ge 0$ be arbitrary but fixed. For $0 \le p \le 1$, we define a kernel-type estimator of $R(t,p)$ as follows:

$$(5.27) \qquad \tilde{R}_n(t,p) = \int_0^1 \hat{R}_n(t,q) h_n^{-1} K\left(\frac{q-p}{h_n}\right) dq \; .$$

where K is some density function and $h_n \to 0$, as $n \to \infty$.

Note that for $0 \leq p \leq 1$,

$$(5.28) \qquad \hat{R}_n(t,p) = \begin{cases} X_{[k+(n-k)p]:n} - t, & \text{if } X_{k:n} \leq t < X_{k+1:n}, \; k = 1,2,\ldots,n-1, \\ 0, & \text{if } t \geq X_{n:n}, \end{cases}$$

where $X_{0:n} \equiv 0$. Using (5.28), we can write $\tilde{R}_n(t,p)$ of (5.27) as

$$(5.29) \qquad \tilde{R}_n(t,p) = \begin{cases} \sum_{i=k+1}^{n} X_{i:n} h_n^{-1} \int_{(i-k-1)/(n-k)}^{(i-k)/(n-k)} K\left(\frac{q-p}{h_n}\right) dq, \\ \qquad \text{if } X_{k:n} \leq t < X_{k+1:n}, \; k=0,1,\ldots,n-1, \\ 0, \qquad \text{if } t \geq X_{n:n}. \end{cases}$$

Motivated by (5.29), we can estimate $R(t,\cdot)$ by

$$
(5.30)\quad \tilde{R}_n(t,p) = \begin{cases} (n-k)^{-1} \sum_{i=k+1}^{n} X_{i:n} h_n^{-1} K\left(\frac{\frac{i-k}{n-k} - p}{h_n}\right), & \\ \qquad X_{k:n} \le t < X_{k+1:n}, \; k=0,1,\ldots, n-1; & \\ 0, \qquad t \ge X_{n:n}. & \end{cases}
$$

Clearly, $\tilde{R}_n(t,p)$ approximately equal to $R_n(t,p)$.

Notice that the estimators $\tilde{R}(t,p)$ and $R_n(t,p)$ are essentially weighted averages of the $X_{i:n}$'s, weighting those $X_{i:n}$ for which $\frac{i-k}{n-k}$ is close to p more heavily than those $X_{i:n}$ for which $\frac{i-k}{n-k}$ is far from p. To show the effect of the smoothness of $\tilde{R}n(t,p)$, plots of $\tilde{R}_n(t,p)$ versus p for fixed t based on a real data set are given in Figures 5.1; 5.2. This data set is taken from Bickel and Doksum (1977) and it consists of the elapsed time spent above a certain high level for a series of 66 wave records taken at San Francisco Bay. In Figure 5.1, we fixed $t = 2.0$ $h_n = 0.30$ and in Figure 5.2 we fixed $t = 3.0$, $h_n = 0.35$. In both figures, the triangular density function $K(u) = (1-|u|)I_{(|u|\le 1)}$ was used as the weighted function for $\tilde{R}_n(t,p)$. When the support of $K(\frac{u-\lambda}{h_n})$ lies before the left end of the range $[\frac{1}{n}, 1]$, i.e., when $\lambda - \frac{1}{n} < h_n$, K is replaced by the truncated density function K_ℓ given by

$$K_{\ell}(u) = C_{\ell} I_{[(\frac{1}{n} - \lambda)/h_n < u \le 0]} + C_{\ell}(1-u) I_{[0<u<1]} ,$$

where $C_{\ell} > 0$ is a constant such that $\int K_{\ell}(u)du = 1$. On the other sider, if the support of $K(\frac{u-\lambda}{h_n})$ lies after the right end of the range $[\frac{1}{n}, 1]$, i.e., when $1-\lambda<h_n$, the K is replaced by the truncated density function K_r give by

$$K_r(u) = (1+u)C_r I_{[-1\le u\le(1-\lambda)/h_u]} ,$$

where C_r is a constant such that $\int_{-\infty}^{\infty} K_r(u)du = 1$.

Insert Figures 5.1, 5.2 about here.

For $t > 0$, $0 < p \le 1$, the smoothed p-PRL process $r_n^*(\cdot,\cdot)$ is defined by

(5.31) $$r_n^*(t,p) = n^{1/2} fF^{-1}(1-p\,\bar{F}(t))[R_n(t,p) - R(t,p)]$$

where $R_n(t,p)$ is given in (5.27).

The main result of this section is an extension of the following theorem of M. Csörgő and S. Csörgő (1986).

<u>Theorem C</u> Assume that conditions (F.1), (F.2) and (F.3) are satisfied. Let $B = \limsup_{x \uparrow b} f(x)$. Assume in addition that either $0 < B < \infty$ or $B = 0$ and $f(x)$ is non-increasing on an interval to the left of b. Then, on an appropriate probability space, we have,

(5.32) $$\sup_{0<p<1} |r_n(t,p) - G_t(p,n)| \overset{a.s.}{=} o(n^{-1/4}(\log\log n)^{1/4}(\log n)^{1/2})$$

where

(5.33) $$G_t(p,n) := n^{-1/2}\{K((1-p\bar{F}(t)),n) - pK(F(t), n)\} .$$

and $\{K(\lambda,s) : 0 \leq \lambda \leq 1 , s \geq 0\}$ is a Kiefer process.

The following theorem is our main result of this section.

<u>Theorem 5.2.</u> Assume that the conditions of Theorem C are satisfied and that conditions (K,1) and (H,1) hold true. Then, on the probability space of Theorem C and with the same Gaussian process $G_t(p,n)$ of (5.33), we have, for the fixed $t > 0$ of condition (F.3.2),

(5.34) $$\sup_{0\leq p\leq(1-1/\sqrt{n})} |r_n^*(t,p) - G_t(p,n)| = o_p(1) \quad , \text{ as } n \to \infty .$$

Proof. The proof of Theorem 5.2 is essentially the same as that of Theorem 5.1. Indeed, if we can prove a result similar to (5.12) of Lemma 5.2, then the rest of the proof follows automatically in the same way as that of Theorem 5.1. Thus, we only need to prove that as $n \to \infty$,

$$(5.35) \qquad \sup_{0 \le p \le 1-\varepsilon_n} \int_{-\infty}^{\infty} \left| \frac{fF^{-1}(1-p\,\bar{F}(t))}{fF^{-1}(1-q\,\bar{F}(t))} - 1 \right| h_n^{-1} K\left(\frac{q-p}{h_n}\right) dq = o(1) .$$

Observe that assumption (F. 3.2) implies that (5.11) of Lemma 5.1 holds true only for $F(t) < r,\ s < 1$. But $F(t) < 1 - p\,\bar{F}(t) < 1$ if and only if $0 < p < 1$. So by Lemma 5.1, for any $0 < p,\ q < 1$, we have

$$(5.36) \qquad \frac{fF^{-1}(1-p\,\bar{F}(t))}{fF^{-1}(1-q\,\bar{F}(t))} \le \left\{ \frac{(1-p\,\bar{F}(t)) \vee (1-q\,\bar{F}(t))}{(1-p\,\bar{F}(t)) \wedge (1-q\,\bar{F}(t))} \cdot \frac{1-[(1-p\,\bar{F}(t)) \wedge (1-q\,\bar{F}(t))]}{1-[(1-p\,\bar{F}(t)) \vee (1-q\,\bar{F}(t))]} \right\}^{\gamma}$$

$$= \left\{ \frac{1-(p \vee q)\bar{F}(t)}{1-(p \wedge q)\bar{F}(t)} \cdot \frac{p \wedge q}{p \vee q} \right\}^{\gamma} .$$

Thus,

$$(5.37) \qquad \left| \frac{fF^{-1}(1-p\,\bar{F}(t))}{fF^{-1}(1-q\,\bar{F}(t))} - 1 \right| \le \left\{ \frac{1-(p \vee q)\bar{F}(t)}{1-(p \wedge q)\bar{F}(t)} \cdot \frac{p \wedge q}{p \vee q} \right\}^{\gamma} - 1 .$$

Consequently,

$$\sup_{0\le p\le 1-\varepsilon_n} \int_{-\infty}^{\infty} \left|\frac{fF^{-1}(1-\bar{p}\,\bar{F}(t))}{fF^{-1}(1-\bar{q}\,\bar{F}(t))} -1\right| h_n^{-1}K\left(\frac{q-p}{h_n}\right)dq$$

$$\le \sup_{0\le p\le 1-\varepsilon_n} \sup_{|q-p|\le ch} \left\{\frac{1-(\overline{p\vee q})\bar{F}(t)}{1-(\overline{p\wedge q})\bar{F}(t)} \cdot \frac{\overline{p\wedge q}}{\overline{p\vee q}}\right\}^{\gamma} -1$$

$$\le \sup_{0\le p\le 1-\varepsilon_n} \left\{\left[\frac{(1-(\overline{p+ch_n})\bar{F}(t))\bar{p}}{(1-\bar{p}\,\bar{F}(t))(\overline{p+ch_n})}\right] \vee \left[\frac{(1-\bar{p}\,\bar{F}(t))(\overline{p-ch_n})}{(1-(\overline{p-ch_n})\bar{F}(t))\bar{p}}\right]\right\}^{\gamma} -1$$

$$= \sup_{0\le p\le 1-\varepsilon_n} \left\{\left[\frac{1-\bar{p}\,\bar{F}(t)-ch_n\bar{F}(t)}{1-\bar{p}\,\bar{F}(t)} \cdot \frac{\bar{p}}{\bar{p}-ch_n}\right] \vee \left[\frac{1-\bar{p}\,\bar{F}(t)}{1-\bar{p}\,\bar{F}(t)+ch_n\bar{F}(t)} \cdot \frac{\bar{p}+ch_n}{\bar{p}}\right]\right\}^{\gamma}-1$$

$$\le \left\{\left[\frac{1-\varepsilon_n\bar{F}(t)-ch_n\bar{F}(t)}{1-\varepsilon_n\bar{F}(t)} \cdot \frac{\varepsilon_n}{\varepsilon_n-ch_n}\right] \vee \left[\frac{1-\varepsilon_n\bar{F}(t)}{1-\varepsilon_n\bar{F}(t)+ch_n\bar{F}(t)} \cdot \frac{\bar{p}+ch_n}{\bar{p}}\right]\right\}^{\gamma} -1$$

$$= \left\{\left[\frac{1-\varepsilon_n\bar{F}(t)-ch_n\bar{F}(t)}{(1-\varepsilon_n\bar{F}(t))(1-ch_n\varepsilon_n^{-1})}\right] \vee \left[\frac{(1-\varepsilon_n\bar{F}(t))(1+ch_n\varepsilon_n^{-1})}{1-\varepsilon_n\bar{F}(t)+ch_n\bar{F}(t)}\right]\right\}^{\gamma} -1$$

$= o(1)$, as $n \to \infty$.

This completes the proof of (5.35).

In the following theorem, we prove the equivalence of $\tilde{R}_n(t,\cdot)$ and $R_n(t,\cdot)$ in probability.

<u>Theorem 5.3</u> Suppose that conditions (F.4), (K.1), (K.2) and (H.2) are satisfied. Let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$, $h_n \varepsilon_n^{-1} \to 0$ as $n \to \infty$. Then, for any $a < t < b$ we have

$$(5.38) \qquad \sup_{0 \le p \le 1-\varepsilon_n} |\tilde{R}_n(t,p) - R_n(t,p)| = o_p(1/(n^{1/2} h_n a_n))$$

where $a_n \to \infty$ and $a_n/(nh_n) \to 0$, as $n \to \infty$.

<u>Proof:</u> Let

$$(5.39) \qquad J_n(u) = a_n\Big[K\Big(\frac{u-p}{h_n}\Big) - (n-k)\int_{u-1/(n-k)}^{u} K\Big(\frac{q-p}{h_n}\Big)dq\Big] ,$$

and

$$(5.40) \qquad J_{n,k}(u) = J_n\Big(\frac{u-(k/n)}{1-(k/n)}\Big)\, I_{[\frac{k}{n} < u \le 1]}(u) ,$$

where k is the random integer such that $X_{k:n} \le t < X_{k+1:n}$, and $I_A(u)$ is the indicator function of A. Thus both $J_n(\cdot)$ and

$J_{n,k}(\cdot)$ are random variables. By (5.39), we can write

$$(5.41)\qquad \tilde{R}_n(t,p) - R_n(t,p) = \sum_{i=1}^{n} J_{n,k}(i/n)X_{i:n}/[(n-k)h_n a_n] \quad \text{a.s.}$$

Let u_n be some interior point of $(u$ [illegible] $u)$ such that

$$(5.42)\qquad (n-k)\int_{u-1/(n-k)}^{u} K\left(\frac{q-p}{h_n}\right)dq = K\left(\frac{u_n-p}{h_n}\right) \quad \text{a.s.}$$

and $u_n' = u_n\left(1-\frac{k}{n}\right) + \frac{k}{n}$.

For any fixed $a < t < b$, there exists a $\delta > 0$ such that $0 < F(t) - \delta < F(t) < F(t) + \delta < 1$. Since $\frac{k}{n} = F_n(t) \overset{a.s}{\to} F(t)$, so there exist a set Ω_o with $P(\Omega_o) = 1$ such that $F_n(t,\omega) \to F(t)$ for all $\omega \in \Omega_o$. Hence, for any $\omega \in \Omega_o$, there exists an $n(\omega) \geq 1$ such that

$$(5.43)\qquad F(t) - \delta < k(\omega)/n < F(t) + \delta, \qquad \text{for } n \geq n(\omega).$$

Let $A_n = \{u: F(t) - \delta \leq u \leq 1 - (\varepsilon_n - ch_n)(1 - F(t) - \delta)\}$. We prove that for any $\omega \in \Omega_o$,

$$(5.44)\qquad \sup_{0 \leq p \leq 1-\varepsilon_n} |J_{n,k}(u,\omega)| = \sup_{0 \leq p \leq 1-\varepsilon_n} |J_{n,k}(u,\omega)|\, I_{A_n}(u), \quad \text{for } n \geq n(\omega).$$

Observe that (5.44) holds true, if $\{u: |J_{n,k}(u,\omega)| > 0,\ 0 \le p \le 1-\varepsilon_n\} \subseteq A_n$. In fact, by condition (K.1) and (5.43)

$$(5.45)\quad \{u: |J_{n,k}(u,\omega)| > 0,\ 0 \le p \le 1-\varepsilon_n\}$$

$$= \{u: |J_n(\tfrac{u-k(\omega)/n}{1-k(\omega)/n})| I_{[k(\omega)/n < u \le 1]} > 0,\ 0 \le p \le 1-\varepsilon_n\}$$

$$\subseteq \{u: k(\omega)/n \le u \le 1 \text{ and } K((\tfrac{u-k(\omega)/n}{1-k(\omega)/n} - p)/h_n) > 0 \text{ or } K((\tfrac{u'_n-k(\omega)/n}{1-k(\omega)/n} - p)/h_n) > 0,\ 0 \le p \le 1-\varepsilon_n\}$$

$$= \{u: k(\omega)/n < u \le 1 \text{ and } (p-ch_n)(1-k(\omega)/n) \le u'_n-k(\omega)/n \le u-k(\omega)/n \le (p+ch_n)(1-k(\omega)/n),\ 0 \le p \le 1-\varepsilon_n\}$$

$$\subseteq \{u: k(\omega)/n < u \le 1,\ (p-ch_n)(1-k(\omega)/n) \le u-(k(\omega)+1)/n \le u-k(\omega)/n \le (p+ch_n)(1-k(\omega)/n),\ 0 \le p \le 1-\varepsilon_n\}$$

$$= \{u: [(p-ch_n)(1-k(\omega)/n) + (k(\omega)+1)/n] \vee k(\omega)/n \le u \le (p+ch_n)(1-k(\omega)/n) + k(\omega)/n,\ 0 \le p \le 1-\varepsilon_n\}.$$

$$\subseteq \{u: k(\omega)/n \le u \le (1-\varepsilon_n+ch_n)(1-k(\omega)/n) + k(\omega)/n,\ 0 \le p \le 1-\varepsilon_n\}$$

$$= \{u: k(\omega)/n \le u \le 1-(\varepsilon_n+ch_n)(1-k(\omega)/n),\ 0 \le p \le 1-\varepsilon_n\}$$

$$\subseteq A_n, \quad \text{when } n \ge n(\omega).$$

So, (5.44) follows by (5.45).

By conditions (K.2) and (5.42), we have

$$\sup_{0\le p\le 1-\varepsilon_n} |J_{n,k}(n)| \le \sup_{0\le p\le 1-\varepsilon_n} \left|J_n\left(\frac{u-k/n}{1-k/n}\right)\right|$$

$$\le a_n \Gamma \left|\frac{u-k/n}{1-k/n} - u_n\right| / h_n$$

$$= a_n \Gamma |u-u_n'| / [h_n(1-k/n)]$$

$$\le a_n \Gamma / [n\, h_n(1-k/n)]$$

Denoting $J_n^*(u) = (a_n \Gamma / n\, h_n) I_{A_n}(u)$, then by (5.44) and (5.46) for any $\omega \in \Omega_o$.

(5.47) $$\sup_{0\le p\le 1-\varepsilon_n} (1-k(\omega)/n)|J_{n,k}(u,\omega)| \le J_n^*(u).$$

Observe that $J_n^*(u) = 0$ for $0 \le u \le F(t) - \delta$ and $1-(\varepsilon_n - ch_n)(1-F(t)-\delta) < n \le 1$, (Since $h_n = o(\varepsilon_n)$, so $1-(\varepsilon_n - ch_n)(1-F(t)-\delta) < 1$ for sufficiently large n) and by the assumption on a_n, $J_n^*(u) \to 0$. Let $D_n = n^{-1}\sum_{i=1}^{n} J_n^*(i/n) X_{i:n}$. Note that condition (F.4) implies that Theorem 5 and Remark 2 of Stingler (1974) are applicable to D_n. Hence, we have

$$\lim_{n\to\infty} n^{1/2}E(D_n) = \lim_{n\to\infty} n\,\mathrm{Var}(D_n) = 0 \quad ,$$

Since $E(D_n^2) = \{\mathrm{Var}(D_n) + (ED_n)^2\} = O(n^{-1})$, so

(5.48) $$D_n = o_p(n^{-\frac{1}{2}})$$

By (5.47), for $\omega \in \Omega_o$,

(5.49) $$\sup_{0\le p\le 1-\varepsilon_n} \sum_{i=1}^{n} |J_{n,k}(i/n, \omega)| X_{i:n}(\omega)/[(n-k(\omega))h_n a_n]$$

$$\sup_{0\le p\le 1-\varepsilon_n} \sum_{i=1}^{n} J_n^*(i/n,w) X_{i:n}(\omega)/[(1- k(\omega)/n)^2 h_n a_n]$$

$$= D_n(\omega)/[(1- k(\omega)/n)^2 h_n a_n]$$

Notice that $P(\Omega_o) = 1$, (5.48), (5.49) and the fact that $(1- k(\omega)/n) = \overline{F}_n(t,\omega) \to \overline{F}(t) > 0$ give

(5.50) $$\sup_{0\le p\le 1-\varepsilon_n} \sum_{i=1}^{n} |J_{n,k}(i/n)| X_{i:n}/[(n-k)h_n a_n] = o_p(1/(n^{1/2}h_n a_n)) \; .$$

Now, recall that X is a non-negative random variable. By (5.41) and (5.50), we have

$$(5.51)\qquad \sup_{0\le p\le 1-\varepsilon_n} |\tilde{R}_n(x,p) - R_n(x,p)|$$

$$\overset{a.s}{\le} \sup_{0\le p\le 1-\varepsilon_n} \sum_{i=1}^{n} |J_{n,k}(i/n)| X_{i:n}/[(n-k)h_n a_n]$$

$$= o(1/n^{1/2}h_n a_n))$$

Hence, (5.38) is proved.

Similar to Corollary 5.1, Theorem 5.2 and Theorem 5.3 give the following Corollary.

Corollary 5.2. Assume that conditions (F.1), (F.2), (F.3.2), (K.1), (K.2) and (H1) are satisfied and h_n is chosen such that $1/[n^{5/8}h_n] \to 0$, as $n \to \infty$. Then, on the probability space of Theorem C and with the same Gaussian process $\{G_t(p,n)\}$ of (5.33), for the fixed $a < t < b$, we have, as $n \to \infty$,

$$(5.52)\qquad \sup_{0\le p\le (1-1/\sqrt{n})} |n^{1/2} fF^{-1}(1-p\bar{F}(t))(\tilde{R}_n(t,p)-R_n(t,p))-G_t(p,n)|$$

$$= o_p(1) .$$

5. On Kernel-type Quantile Processes Under Random Censorship

Let $X_1^0, X_2^0, \ldots, X_n^0$ be i.i.d. r.v. with a continuous distribution function F^0. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. r.v. with a continuous distribution function H. Suppose that the two sequences $\{X_i^0\}$ and $\{Y_i\}$ are independent. For any distribution function L denote

$$T_L = \inf\{t: L(t) = 1\} \tag{5.53}$$

In the random censorship model from the right, the X_i^0 may be censore on the right by the Y_i, i.e., one observes only the pairs (X_i, δ_i), $i = 1,2,\ldots,n$, where X_i is the minimum of X_i^0, Y_i and δ_i is the indicator function of the event $X_i^0 \le Y_i$, i.e.,

$$X_i = \begin{cases} X_i^0 & \text{and} \quad \delta_i = 1 \quad , \quad \text{if} \quad X_i^0 \le Y_i \ , \\ Y_i & \text{and} \quad \delta_i = 0 \quad , \quad \text{if} \quad X_i^0 > Y_i \ , \end{cases} \tag{5.54}$$

$i = 1,2,\ldots,n$. Thus, [illegible], $i = 1,2,\ldots,n$, are i.i.d. r.v. with distribution function:

$$\text{[illegible]} = 1-(1-F^0(t))(1-H(t)) \ , \quad -\infty < t < \infty . \tag{5.55}$$

In the random censorship model from the right, a widely used estimator of F^0 is the product-limit (or, Kaplan-Meier (1958)) estimator $\hat{F}_n^0(t)$ defined by

$$(5.56)\quad 1-\hat{F}_n^0(t) = \begin{cases} 1, & 0 \le t \le X_{1:n}, \\ \prod_{i=1}^{k-1} \left(\frac{n-i}{n-i+1}\right)^{\delta_i'}, & X_{k-1:n} < t \le X_{k:n}, \; k = 2,\ldots,n, \\ 0, & t > X_{n:n}, \end{cases}$$

where $(X_{i:n}, \delta_i')$, $i = 1,\ldots,n$, denote the ordered X_i's along with their corresponding δ_i's. Let s_i denote the jump of $\hat{F}_n^0$ at $X_{i:n}$, i.e.,

$$(5.57)\quad s_i = \begin{cases} \hat{F}_n^0(X_{2:n}), & i = 1, \\ \hat{F}_n^0(X_{i+1:n}) - \hat{F}_n^0(X_{i:n}), & i = 2,\ldots,n-1, \\ 1-\hat{F}_n^0(X_{n:n}), & i = n. \end{cases}$$

and denote

$$(5.58)\quad S_i = \begin{cases} 0, & i = 0, \\ \hat{F}_n^0(X_{i+1:n}) = \sum_{j=1}^{i} s_j, & i = 1,2,\ldots,n-1, \\ 1, & i = n. \end{cases}$$

Note that $s_i = 0$ if and only if $\delta_i' = 0$, $i < n$, i.e., when $X_{i:n}$

is a censored observation.

Based on the randomly right-censored data, a natural estimator of the quantile function Q^0 of F^0 is the sample PL-quantile function $\hat{Q}_n^0$ defined by

$$(5.59) \qquad \hat{Q}_n^0(y) = \inf\{t: \hat{F}_n^0(t) \geq y\} .$$

A more direct way of looking at $\hat{Q}_n^0(\cdot)$ is given by

$$(5.60) \qquad \hat{Q}_n^0(y) = X_{i:n} \ , \quad \text{if} \quad S_{i-1} < y \leq S_i \ , \qquad i = 1,2,\ldots,n \ ,$$

where S_i , $i = 0,1,\ldots,n$, are defined in (5.58).

Aly, et al. (1985) defined the PL-quantile process ρ_n^0 as

$$(5.61) \quad \rho_n^0(y) = n^{1/2} f^0 Q^0(y)(Q^0(y) - \hat{Q}_n^0(y)) \quad , \quad 0 < y < 1 \ ,$$

where f^0 is the derivative of F^0. They proved some strong approximation results for $\rho_n^0(y)$. In the following we state their main result.

__Theorem D.__ Assume that F^0 satisfies conditions (F.1), (F.2) and

(F.3.3). Then, on an appropriate probability space, there exist a sequence of Gaussian processes $\{G_n(\cdot),\ n \geq 1\}$ and a positive constant C such that with $\delta_n = Cn^{-1}\log\log n$, we have

$$(5.62)\qquad \sup_{\delta_n \leq y \leq p^0} |\rho_n^0(y) - G_n(y)| \overset{a.s.}{=} O(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}) .$$

where $\{G_n(y) := n^{-1/2}K^*(y,n),\ n \geq 1\}$ and $K^*(\cdot,\cdot)$ is a generalized Kiefer process with $EK^*(t,s) = 0$, and

$$(5.63)\qquad EK^*(t,s)K^*(t',s') \text{ [illegible] } (1-t')d^*(t \wedge t') .$$

with

$$d^*(t) \text{ [illegible] } du ,$$

[illegible] $(Q$ [illegible]$))$

$p^0 < (p^* \wedge T_{F^*})$, $F^*(\cdot) \equiv \tilde{F}Q^0(\cdot)$ and $\tilde{F}(\cdot)$ is as in (5.55).

Padgett (1986) proposed the kernel-type quantile function estimator $Q_n^0(\lambda)$ and its approximation $\tilde{Q}_n^0(\lambda)$ defined respectively by

$$(5.64)\qquad Q_n^0(\lambda) = \int_0^1 \hat{Q}_n^0(t)\, h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$= h_n^{-1} \sum_{i=1}^{n} X_{i:n} \int_{S_{i-1}}^{S_i} K(\frac{t-\lambda}{h_n})dt ,$$

and

$$(5.65) \qquad \tilde{Q}_n^0(\lambda) = h_n^{-1} \sum_{i=1}^{n} X_{i:n} s_i K(\frac{S_i-\lambda}{h_n}) .$$

Padgett (1986) proved that $Q_n^0(\lambda)$ is almost surely consistent for every $0 \leq p \leq 1$. Lio, Padgett and Yu (1986) proved that for every continuity point λ of $f^0Q^0(\cdot)$, $n^{1/2}[Q_n^0(\lambda) - Q(\lambda)]$ is asymptotically normal. Lio and Padgette (1987) proved the asymptotic normality of $n^{1/2}[Q_n^0(\lambda) - Q^0(\lambda,h_n)]$ under a weaker condition for h_n, where

$$(5.66) \qquad Q^0(\lambda,h_n) = h_n^{-1} \int_0^1 Q^0(t)K(\frac{t-\lambda}{h_n})dt , \qquad \text{for } 0 \leq \lambda \leq 1 .$$

They also proved the asymptotic mean and mean square equivalences of Q_n^0 and $\tilde{Q}_n^0$ and the mean square convergence of $\tilde{Q}_n^0(\lambda)$ to $Q^0(\lambda)$.

In this section, we will prove the weak convergence of a kernel-type process under random censorship. This result is an extension of the results of Aly et. al. (1985), Lio et. al. (1986) and Lio and Padgett (1987). First, we define the kernel-type quantile process $\overset{*}{\rho}{}_n^0(\cdot)$ as

(5.67) $\overset{*0}{\rho}_n(\lambda) := n^{1/2} f^0 Q^0(\lambda)(Q_n^0(\lambda) - Q^0(\lambda))$, $0 \leq \lambda \leq 1$,

where Q_n^0 is as in (5.64).

The main result of this section is the following theorem.

Theorem 5.4. Assume the condition (F.1), (F.2), (F.3.3), (K.1), (H.1) and (H.2) are satisfied. Then, on the probability space of Theorem D and with the same sequence of Gaussian processes $\{G_n(\cdot)\}$, we have, as $n \to \infty$,

$$\sup_{1/\sqrt{n} \leq \lambda \leq p^0} |\overset{*0}{\rho}_n(\lambda) - G_n(\lambda)| = o_p(1) , \tag{5.68}$$

where $p^0 < (p^* \wedge T_{F^*})$.

If the process is centered with $Q^0(\lambda, h_n)$ of (5.66) instead of $Q^0(\lambda)$, the result of Theorem 5.4 can be obtained with a slower rate of convergence of h_n to zero as stated in the following theorem.

Theorem 5.4'. Assume that all the conditions of Theorem 5.4 are satisfied except that condition ~~(H.1)~~ is replaced by $h_n = o((\log n)^{-3/2})$. Let ε_n be a sequence of positive numbers such that $\varepsilon_n \to 0$, $h_n = o(\varepsilon_n)$, as $n \to \infty$. Then, on the probability space of Theorem D and with the same sequence of Gaussian processes $\{G_n(\cdot)\}$, we have, as $n \to \infty$,

$$(5.69) \qquad \sup_{\varepsilon_n \le \lambda \le p^0} |n^{1/2} f^0 Q^0(\lambda)[Q_n^0(\lambda) - Q^0(\lambda, h_n)] - G_n(\lambda)| = o_p(1)$$

where $Q^0(\lambda, h_n)$ is as in (5.66).

In order to prove Theorem 5.4, we need the following result of Aly, Csörgő and Horvăth (Theorem 8.2.1 of Csörgő (1983)).

Lemma 5.4. Let $K^*(\cdot,\cdot)$ be the generalized Kiefer process of (5.63) and assume that $h_n \to 0$ satisfies condition (H.2). Then, with $p^0 < T_{F^*}$; we have

$$(5.70) \qquad \sup_{0 \le s \le p^0 - h_n} \sup_{0 \le t \le h_n} |K^*(s+t,n) - K^*(s,n)| \overset{a.s.}{=} O(n^{1/2}(\log n)^{3/4} h_n^{1/2})$$

Proof of Theorem 5.4. Again, we only need to prove that

$$(5.71) \qquad \sup_{\varepsilon_n \le \lambda \le p^0} \int_{-\infty}^{\infty} \left|\frac{f^0 Q^0(\lambda)}{f^0 Q^0(t)} - 1\right| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt = o(1) ,$$

and

(5.72) $$\sup_{\varepsilon_n \le \lambda \le p^0} \sup_{|t-\lambda| \le ch_n} |G_n(t) - G_n(\lambda)| = o_p(1) .$$

Since $G_n(t) = h^{1/2} K^*(t,n)$ and $h_n = o((\log n)^{-3/2})$, (5.72) follows directly from (5.70) of Lemma 5.4.

To prove (5.71), similar to the proof of Lemma 5.1, one can show that condition (F.3.3) implies that for any $0 < r,s < p^*$,

(5.73) $$\frac{f^0 Q^0(s)}{f^0 Q^0(r)} \le \left\{\frac{s \vee r}{s \wedge r}\right\}^{\gamma}$$

By (5.73), we have, for any $0 < \lambda , t < p^*$,

$$\left|\frac{f^0 Q^0(\lambda)}{f^0 Q^0(t)} - 1\right| \le \left\{\frac{\lambda \vee t}{\lambda \wedge t}\right\}^{\gamma} .$$

Notice that $p^0 < p^*$ thus $p^0 + ch_n < p^*$ for sufficiently large n, so we have

$$\sup_{\varepsilon_n \le \lambda \le p^0} \int_{-\infty}^{\infty} \left|\frac{f^0 Q^0(\lambda)}{f^0 Q^0(t)} - 1\right| h_n^{-1} K\left(\frac{t-\lambda}{h_n}\right) dt$$

$$\le \sup_{\varepsilon_n \le \lambda \le p^0} \sup_{|\lambda - t| \le ch_n} \left\{\frac{\lambda \vee t}{\lambda \wedge t}\right\}^{\gamma} - 1$$

$$\leq \sup_{\varepsilon_n \leq \lambda \leq p^0} \left[\left(\frac{\lambda}{\lambda - ch_n} \right) \vee \left(\frac{\lambda + ch_n}{\lambda} \right) \right]^{\gamma} - 1$$

$$\leq \left(\frac{\varepsilon_n}{\varepsilon_n - ch_n} \right)^{\gamma} - 1$$

$$= \left(\frac{\varepsilon_n}{1 - ch_n \varepsilon_n^{-1}} \right)^{\gamma} - 1$$

$$= o(1) .$$

This gives (5.71).

The proof of Theorem 5.4' is included in the proof of Theorem 5.4 in which Lemma 5.3 is not used, so condition (H.1) is not necessary.

Lio and Padgett (1987) proved the equivalence of $\tilde{Q}_n^0$ and Q_n^0 stated in the following.

<u>Theorem E.</u> Assume that conditions (F.1), (F.2), (K.1), and (K.2) hold true and [illegible] $< \infty$, $h_n \to 0$ as $n \to \infty$. Then as $n \to \infty$,

$$\sup_{0 \leq \lambda \leq p_0} [\tilde{Q}_n^0(\lambda) - Q_n(\lambda)]^2 = o_p(n^{-1})$$

provided $h_n^{-4}\sqrt{\log \log n/n} \to 0$, where $p_0 < T_{F^*}$

Since there is no common range for h_n in Theorem 5.4 and Theorem E, we are not able to claim that the result of (5.68) of Theorem 5.4 holds true for $Q_n^0(\cdot)$. However, Theorem 5.4' and Theorem E give the following corollary.

Corollary 5.3. Assume that conditions (F.1), (F.2), (F.3.3) (K.1), and (K.2) are satisfied. In addition assume that $f^0Q^0(\cdot)$ is bounded on $[0,p^*]$, $EX^{0^2} < \infty$, and $\{h_n\}$ is chosen such that $h_n = o((\log n)^{-3/2})$, $h_n^{-4}\sqrt{\log\log n/n} \to 0$. Then, on the probability space of Theorem D and with the same sequence of Gaussian processes $G_n(\cdot)$, we have, as $n \to \infty$,

$$(5.75) \qquad \sup_{0\le\lambda\le p^0} |n^{1/2} f^0Q^0(\lambda)[\tilde{Q}_n^0(\lambda) - Q^0(\lambda,h_n)] - G_n(\lambda)| = o_p(1)$$

where $Q^0(\lambda,h_n)$ is as in (5.66).

5.5 On Kernel-type Percentile Residual Lifetime Processes Under Random Censorship

In the random censorship from the right model, a natural estimator of p-percentile residual lifetime is $\hat{R}_n^0(x,p)$ defined as

$$(5.76)\qquad \hat{R}_n^0(x,p) = \begin{cases} \hat{Q}_n^0(1-p\,\hat{\bar{F}}_n^0(t))-t\ , & 0 \le t < X_{n:n}\ , \\ 0\ , & t \ge X_{n:n}\ , \end{cases}$$

$0 \le p \le 1$. Using the notation of section 5.4, $\hat{R}_n^0(t,p)$ of (5.76) is also given by

$$(5.77)\qquad \hat{R}_n^0(t,p) = \begin{cases} X_{i:n}-t\ , & \text{when } X_{k-1:n} \le t < X_{k:n}\ ,\ \dfrac{S_{i-1}-S_k}{1-S_k} < p \le \dfrac{S_i-S_k}{1-S_k}\ , \\ & k = 1,2,\ldots,n\ ,\quad i = k+1,\ldots,n, \\ 0\ , & \text{when } t \ge X_{n:n}\ , \end{cases}$$

where S_i , $0 \le i \le n$, are as in (5.58).

Chung (1987) defined the PL p-percentile residual process $r_n^0(t,p)$ by

$$(5.78)\qquad r_n^0(t,p) := n^{1/2} f^0(Q^0(1-p\bar{F}^0(t)))[\hat{R}_n^0(t,p)-R(t,p)],\quad 0 \le p \le 1,\ t>0.$$

He has proved the strong convergence of $r_n^0(t,p)$ to a Gaussian process. We state his result in the following theorem.

<u>Theorem F.</u> Let $t > 0$ be a fixed number with $t < T_{\tilde{F}}$. Let

$p_1 \in (0,1)$ be fixed such that $Q^0(1-\bar{p}_1\bar{F}^0(t)) < T_{\tilde{F}}$. Suppose that conditions (F.1), (F.2) are satisfied. Then on an appropriate probability space, we have, as $n \to \infty$,

$$(5.79) \quad \sup_{0\le\lambda\le p_1} |r_n^0(t,p) - G_t^0(p,n)| \overset{a.s}{=} o(n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}) ,$$

where $\{G_t^0(p,n) := n^{-\frac{1}{2}}[K^*((1-\bar{p}\,\bar{F}^0(t)), n) - \bar{p}\,K^*(F^0(t),n)] , n \ge 1\}$ is a Gaussian process and $K^*(\cdot,\cdot)$ is the generalized Kiefer process defined in (5.63).

Now we define the kernel-type PL-estimator of the p-percentile residual lifetime by

$$(5.80) \quad R_n^0(t,p) = \int_0^1 \hat{R}_n(t,q)h_n^{-1} K(\frac{q-p}{h_n})dq \quad , \quad 0 \le p \le 1 \; , \quad t \ge 0 \; .$$

where K is some density function and $h_n \to 0$, as $n \to \infty$. By (5.77), $R_n^0(t,p)$ of (5.80) can be written for any $0 \le p \le 1$, as

$$(5.81)\quad \overset{0}{R}_n(t,p) = \begin{cases} h_n^{-1} \sum_{i=k+1}^{n} X_{i:n} \int_{(S_{i-1}-S_k)/(1-S_k)}^{(S_i-S_k)/(1-S_k)} K\left(\frac{q-p}{h_n}\right)dq\ , \\ \qquad X_{k-1:n} < t \le X_{k:n}\ ,\ k = 1,2,\ldots,n. \\ 0 \qquad\qquad t > X_{n:n}\ . \end{cases}$$

We may also estimate $R^0(t,p)$ by $\tilde{R}_n^0(t,p)$, where

$$(5.82)\quad \tilde{R}_n^0(x,p) = \begin{cases} h_n^{-1}(1-S_k)^{-1} \sum_{i=k+1}^{n} X_{i:n} s_i\, K\left(\dfrac{\frac{S_i-S_k}{1-S_k} - p}{h_n}\right)\ , \\ \qquad X_{k-1:n} < t \le X_{n:n},\ k = 1,2,\ldots,n\ , \\ 0 \qquad\qquad t > X_{n:n}\ . \end{cases}$$

A kernel-type p-percentile residual lifetime process $\overset{*0}{r}_n(t,p)$ is defined as

$$(5.83)\quad \overset{*0}{r}_n(t,p) = f^0 Q^0(1-\bar{p}\,\bar{F}^0(t))[\overset{0}{R}_n(t,p) - R^0(t,p)]\ ,\quad 0 \le p \le 1,\quad t > 0\ .$$

where $R^0(t,p) = Q^0(1-\bar{p}\,\bar{F}^0(t)) - t$, is the p-PRL of F^0.

The main result of this section is the following theorem which extends Chung (1987)'s approximation result of p-PRL process, $r_n^0(\cdot,\cdot)$ to that of the kernel-type p-PRL process $\overset{*0}{r}_n(\cdot,\cdot)$.

Theorem 5.5. In addition to the conditions of Theorem F, assume that conditions (K.1), (H.1) and (H.2) are satisfied. Then, on the probability space of Theorem F and with the same Gaussian process $G_t^0(\cdot,n)$, we have, as $n \to \infty$,

$$\sup_{0 \le p \le p_1} |\overset{*0}{r}_n(t,p) - G_t^0(p,n)| = o_p(1) \quad . \tag{5.84}$$

where $p_1 \in (0,1)$ is such that [illegible]$(1-\bar{p}\,\bar{F}^0(t)) < T_{\tilde{F}}$ and $\tilde{F}$ is as in (5.55).

In order to be able to state Theorem 5.5', we define

$$R^0(t,p,h_n) = \int_0^1 R^0(t,p)\, h_n^{-1} K\left(\frac{q-p}{h_n}\right) dq \quad . \tag{5.85}$$

Theorem 5.5'. Assume that all the conditions of Theorem 5.5 are satisfied except condition (H.1) is replaced by $h_n = o((\log n)^{-3/2})$. Then, on the probability space of Theorem F and with the same Gaussian process $G_t^0(\cdot,n)$, we have, as $n \to \infty$,

$$\sup_{0 \le p \le p_1} |n^{1/2} f^0 Q^0 (1-\bar{p}\,\bar{F}^0(t))[R_n^0(t,p) - R^0(t,p,h_n)] - G_t^0(p,n)| = o_p(1) \quad . \tag{5.86}$$

where $R^0(t,p,h_n)$ is as in (5.85) .

Proof of Theorem 5.5 and Theorem 5.5'. Similar to the proof of Theorem 5.4 and Theorem 5.4'. We only need to prove that for the fixed $t > 0$ as $n \to \infty$,

$$(5.87) \qquad \sup_{0 \le p \le p_1} \int_0^1 \left| \frac{f^0 Q^0(1-\bar{p}\,\bar{F}^0(t))}{f^0 Q^0(1-\bar{q}\,\bar{F}^0(t))} - 1 \right| h_n^{-1} K\left(\frac{q-p}{h_n}\right) dq = o(1)$$

and

$$(5.88) \qquad \sup_{0 \le p \le p_1} \; \sup_{|q-p| \le ch_n} |G_t^0(p,n) - G_t^0(q,n)| = o_p(1)$$

By the definition of $G_t^0(\cdot,n)$, Lemma 5.4 and the assumption that $h_n = o((\log n)^{-3/2})$, we have (5.88) .

In order to prove (5.87), we observe that the continuity of $f^0(\cdot)$ implies that $\psi(p) = f^0 Q^0(1-\bar{p}\,\bar{F}^0(t))$ is uniformly continuous on $[0,p_1]$ for any fixed $t > 0$, and condition (F.2) implies that $1/\psi(p)$ is bounded on $[0,p_1]$, i.e., $1/\psi(p) < L$, $0 \le p \le p_1$ for some positive number L . Hence

$$\sup_{0 \le p \le p_1} \int_0^1 \left| \frac{f^0 Q^0(1-\bar{p}\,\bar{F}^0(t))}{f^0 Q^0(1-\bar{q}\,\bar{F}^0(t))} - 1 \right| h_n^{-1} K\left(\frac{q-p}{h_n}\right) dq$$

$$\leq \sup_{0\leq p\leq p_1} \sup_{|q-p|\leq ch_n} \left|\frac{\psi(p)-\psi(q)}{\psi(q)}\right|$$

$$\leq L \sup_{0\leq p\leq p_1} \sup_{|q-p|\leq ch_n} |\psi(p) - \psi(q)|$$

$$= o(1) , \qquad \text{as } n \to \infty .$$

This completes the proof of (5.87).

Now, the equivalence of $\tilde{R}_n^0(t,p)$ and $R_n^0(t,p)$ is given in the following theorem.

Theorem 5.6 Assume that conditions (F.1), (F.2), (K.1) and (K.2) are satisfied, $E(X_1^{0^2}) < \infty$ and $h_n \to 0$ as $n \to \infty$, then for any fixed $0 < t < T_{F^0}$, we have as $n \to \infty$,

$$\text{(5.89)} \qquad \sup_{0\leq p\leq p_0} n[\tilde{R}_n^0(t,p) - R_n^0(t,p)]^2 = o_p(1)$$

provided $h_n^{-4}\sqrt{(\log\log n)/n} \to 0$, where $p_0 < T_{F^*}$ and $F^*(\cdot) = \tilde{F}Q^0(\cdot)$.

Proof. For the fixed $0 < t < T_{F^0}$. Let k be the random integer such that $X_{k-1:n} < t \leq X_{k:n}$. For $0 \leq p \leq p_1$, when $\delta_i > 0$, i.e. when $X_{i:n}$ is uncensored, let S_i^* be a interior point of the interval

(S_{i-1}, S_i) , such that

$$(5.90)\quad s_i(1-S_k)^{-1}K\left(\frac{\frac{S_i^*-S_k}{1-S_k}-p}{h_n}\right) = \int_{(S_{i-1}-S_k)/(1-S_k)}^{(S_i-S_k)/(1-S_k)} K\left(\frac{q-p}{h_n}\right)dq \quad \text{a.s.,}$$

$$k = 1,2,\ldots,n\ ,\ i = k+1,\ldots,n\ .$$

Let i^* be the smallest $i \le n$ such that $S_{i+1}-p_1 > ch_n$, where c is the constant in (K.1). If no such i exists, then let $i^* = n$. By condition (K.2), (F.1), (5.90) and the fact that $X_{i:n} \ge 0$, $1 \le i \le n$,

$$(5.91)\quad n[\tilde{R}_n^0(t,p) - R_n^0(t,p)]^2(1-\hat{F}_n^0(t))^4 I_{[0,p_1]}(p)$$

$$= n\, h_n^{-2}\Big\{ \sum_{i=k+1}^{n} x_{i:n}s_i(1-S_k)^{-1}\Big[K\Big(\frac{\frac{S_i-S_k}{1-S_k}-p}{h_n}\Big) - K\Big(\frac{\frac{S_i^*-S_k}{1-S_n}-p}{h_n}\Big)\Big] \cdot$$

$$I_{[0,p_1]}(p)\Big\}^2(1-\hat{F}_n^0(t))^4$$

$$\le \Gamma^2 nh_n^{-2}\Big\{ \sum_{n=k+1}^{n} X_{n:n}s_i(1-S_k)^{-2}h_n^{-1}|S_i-S_i^*| I_{[0,p_1]}(p)\Big\}^2(1-\hat{F}_n^0(t))^4$$

$$\le \Gamma^2 nh_n^{-4}\Big\{ \sum_{i=1}^{n} X_{i:n}s_i^2(1-S_k)^{-2}\, I_{[0,p_1]}(p)\Big\}^2(1-S_k)^4$$

$$\le \Gamma^2 h_n^{-4} \sum_{i=1}^{n} X_{i:n}^2 s_i^3\, n\, I_{[0,i^*]}(i)$$

$$\leq \Gamma^2 h_n^{-4}\, 2 \sup_{0\leq t\leq T_{F^0}} |\hat{F}_n^0(t)-F^0(t)| \int_0^1 x^2 d\, \hat{F}_n^0(x)\, ns_i\, I_{[0,i^*]}(i)$$

From Saunder (1975), for $i \leq i^*$, $0 \leq s_i \leq [1-H(F^{0^{-1}}(p))]^{-1} + o(1)$.

By the results of Földes and Rejtö (1981),

$$(5.92)\qquad \sup_{0\leq t\leq T_{F^0}} |\hat{F}_n^0(t) - F^0(t)| \overset{a.s}{=} O(\sqrt{\log \log n/n})$$

and by a proof similar to that for Theorem 4.1 of Mauro (1985).

$\int_0^1 x^2 d\, \hat{F}_n^0(x) \to E(X^{0^2})$ in probability. Therefore,

$$(5.93)\qquad \sup_{0\leq p\leq p_1} n[\tilde{R}_n^0(t,p) - R_n^0(t,p)]^2 (1-\hat{F}_n^0(t))^4 = 0_p(h_n^{-4}(\log \log n/n)^{1/2})$$

By (5.92); $(1-\hat{F}_n^0(t)) \overset{a.s}{\to} \bar{F}^0(t) > 0$. Hence we have

$$(5.94)\qquad \sup_{0\leq p\leq p_1} n[\tilde{R}_n^0(t,p) - R_n^0(t,p)]^2 = 0_p(h_n^{-4}(\log \log n/n)^{1/2}) .$$

This complete the proof of Theorem 5.6.

From Theorem 5.5' and Theorem 5.6, we have the following corollary.

<u>Corollary 5.4.</u> Assume that conditions (F.1), (F.2), (K.1), and (K.2) are satisfied. Suppose, in addition, $EX_0^2 < \infty$, and h_n is chosen such that $h_n = o((\log n)^{-3/2})$, $h_n^{-4}\sqrt{\log\log n/n} \to 0$, as $n \to \infty$. Then, on the probability space of Theorem F and with the same Gaussian process $G_t^0(\cdot,n)$, we have, as $n \to \infty$,

$$(5.95)\quad \sup_{0 \le p \le p_0} \left| n^{1/2} f^0 Q^0(1-\bar{p}\,\bar{F}^0(t))[\tilde{R}_n^0(t,p) - R^0(t,p,h_n)] - G_t^0(p,n) \right|$$

$$= o_p(1),$$

where $R^0(t,p,h_n)$ is as in (5.85), and $p_0 < p_1 \wedge T_{F^*}$.

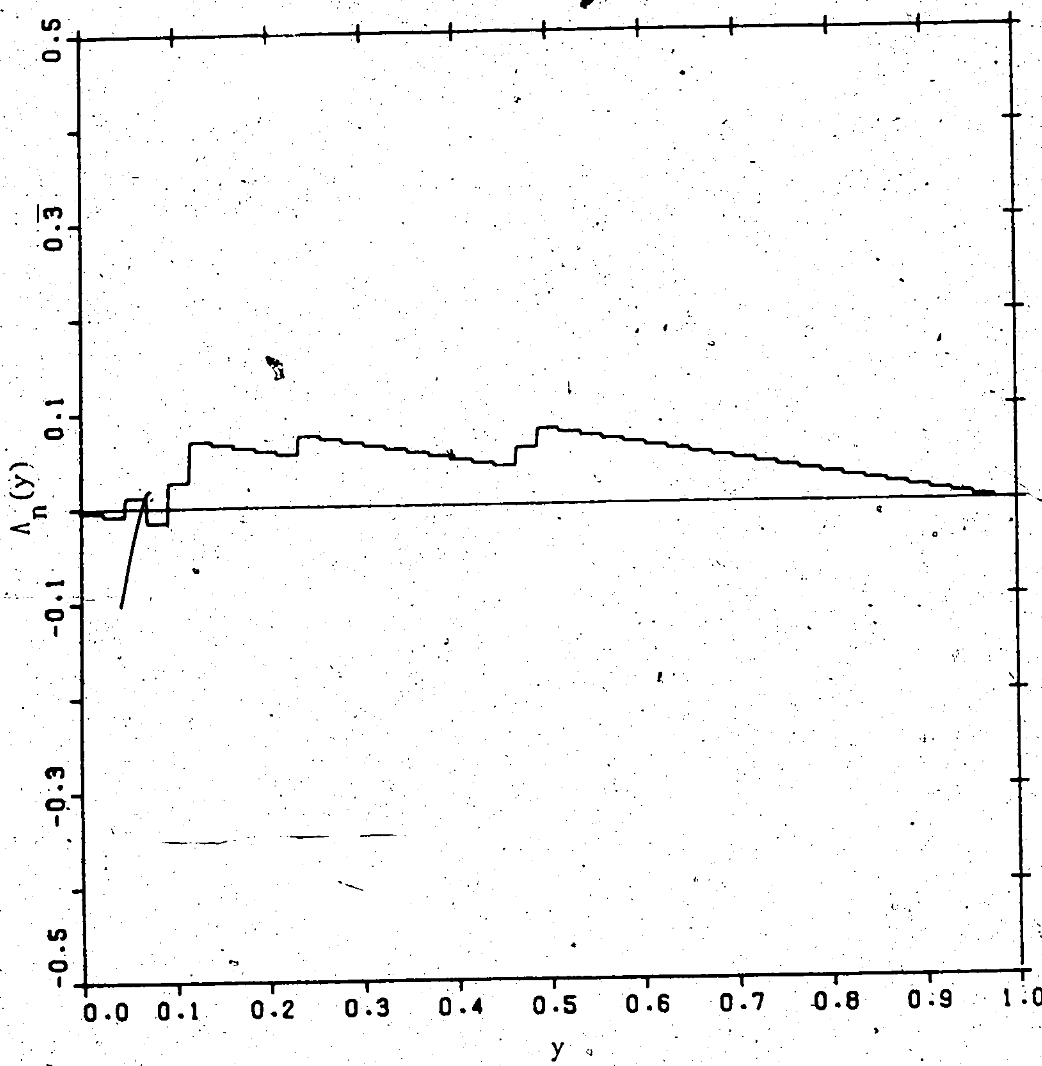

Figure 3.1. Plot of NBU-t_o process $\Lambda_n(\cdot)$ (t_o = 1825)

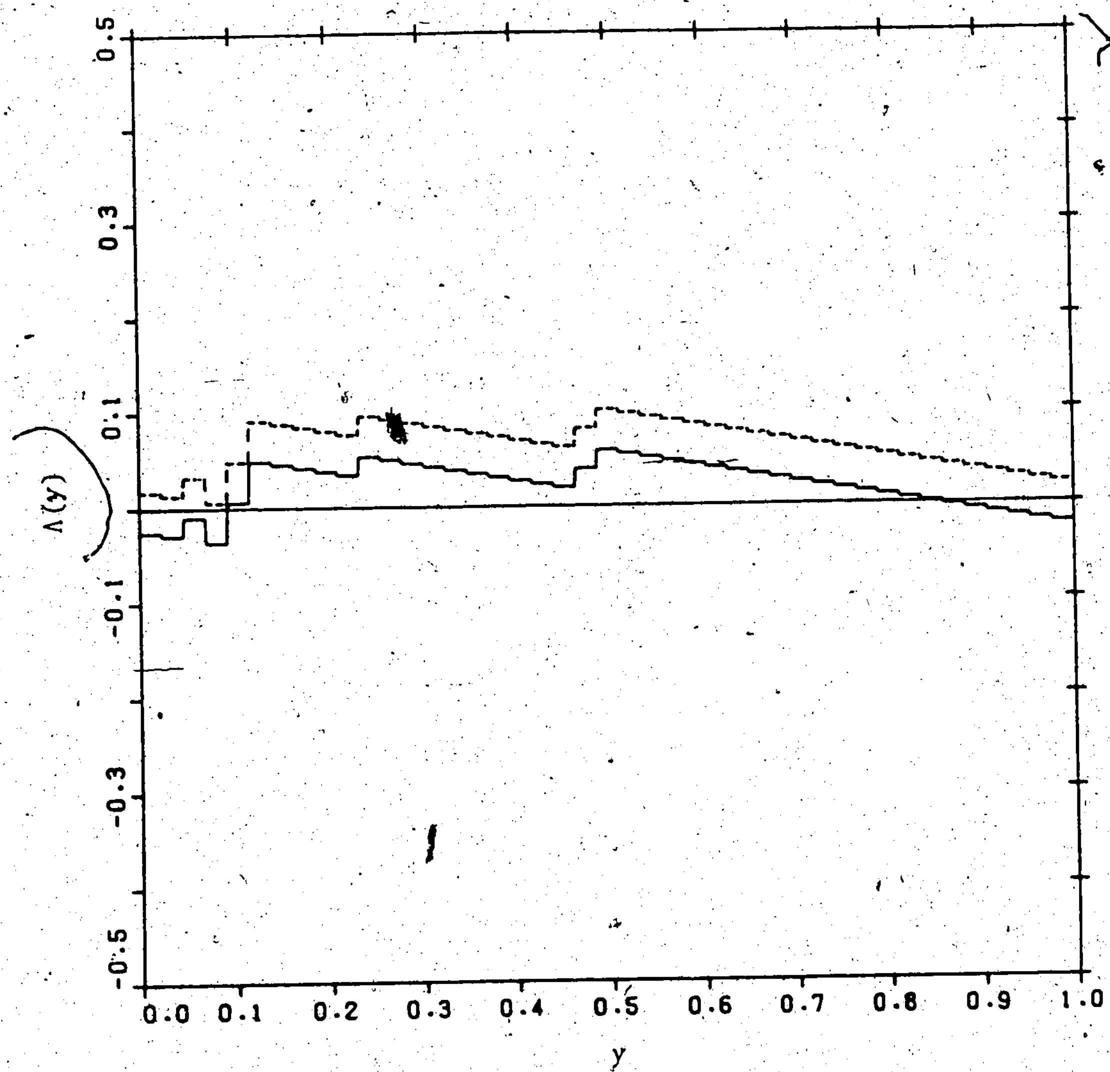

Figure 3.2. Plot of Bootstrapped 95% Confidence Band for the NBU-t_o Plot Function $\Lambda(\cdot)$ (t_o = 1825)

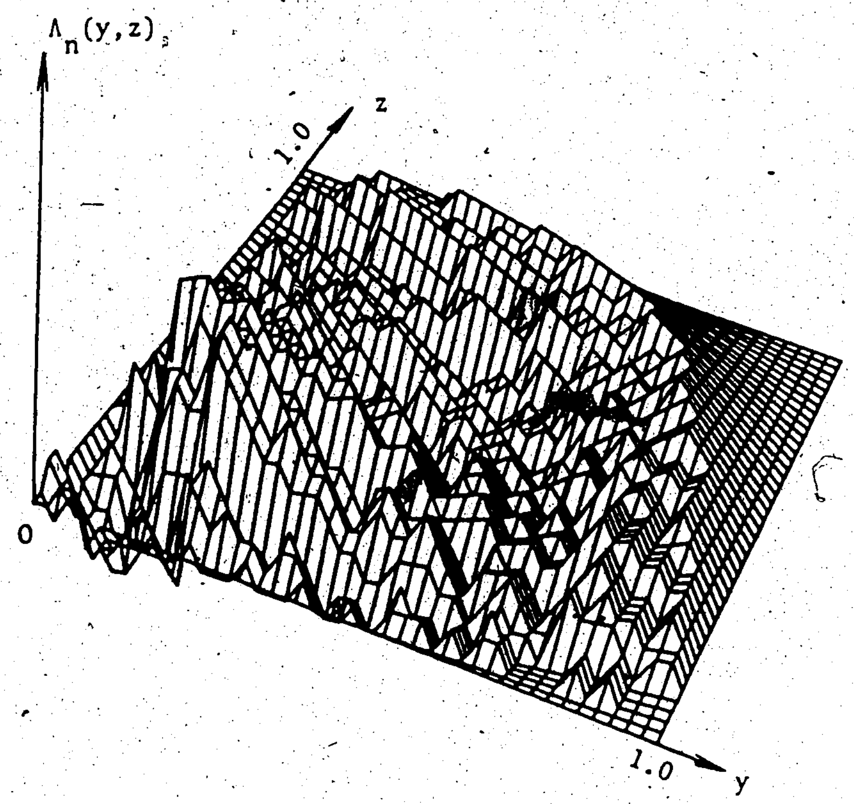

Figure 4.1. Plot of the Empirical SNBU Function $\Lambda_n(\cdot,\cdot)$

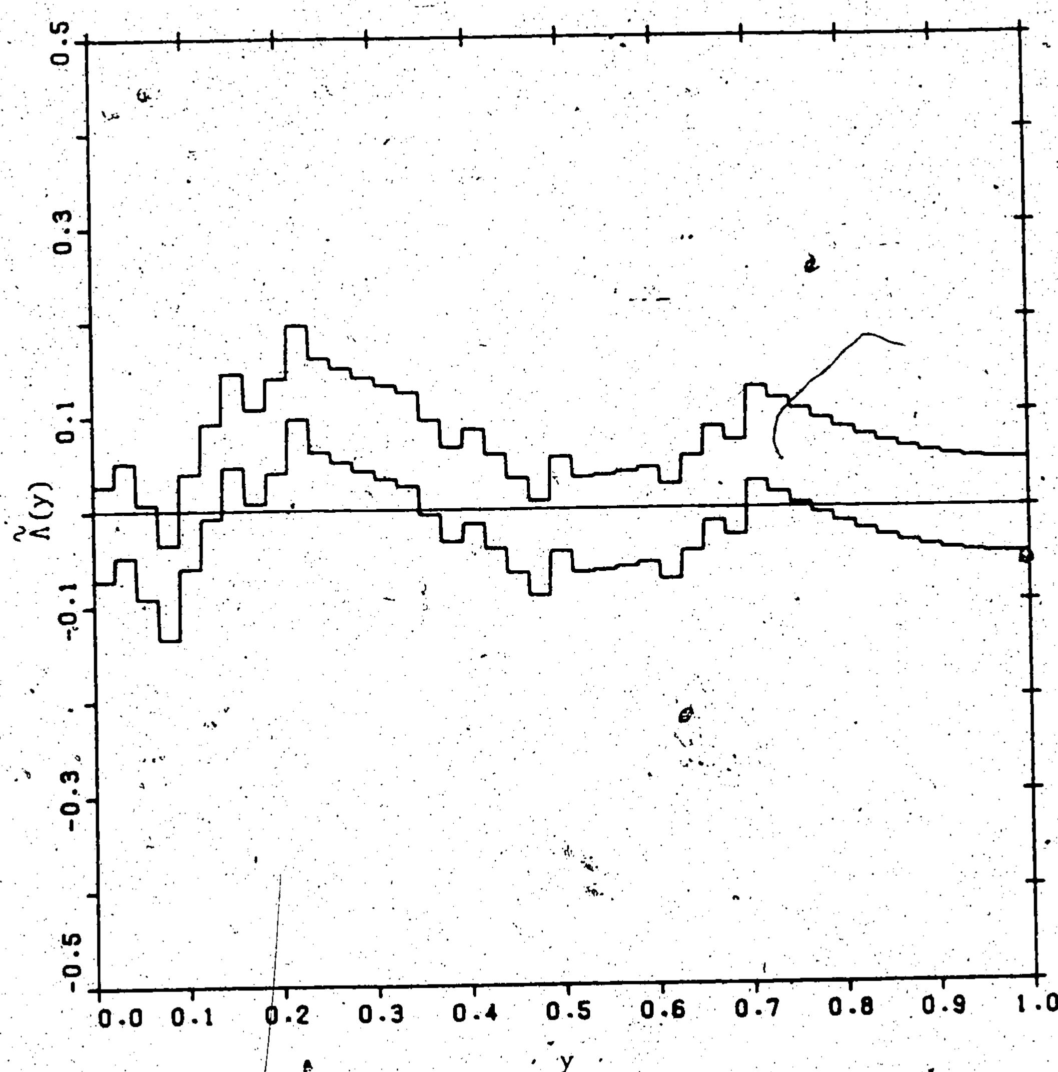

Figure 4.2. Plot of Bootstrapped 95% Confidence Band for the DNBU plot Function $\tilde{\Lambda}(\cdot)$

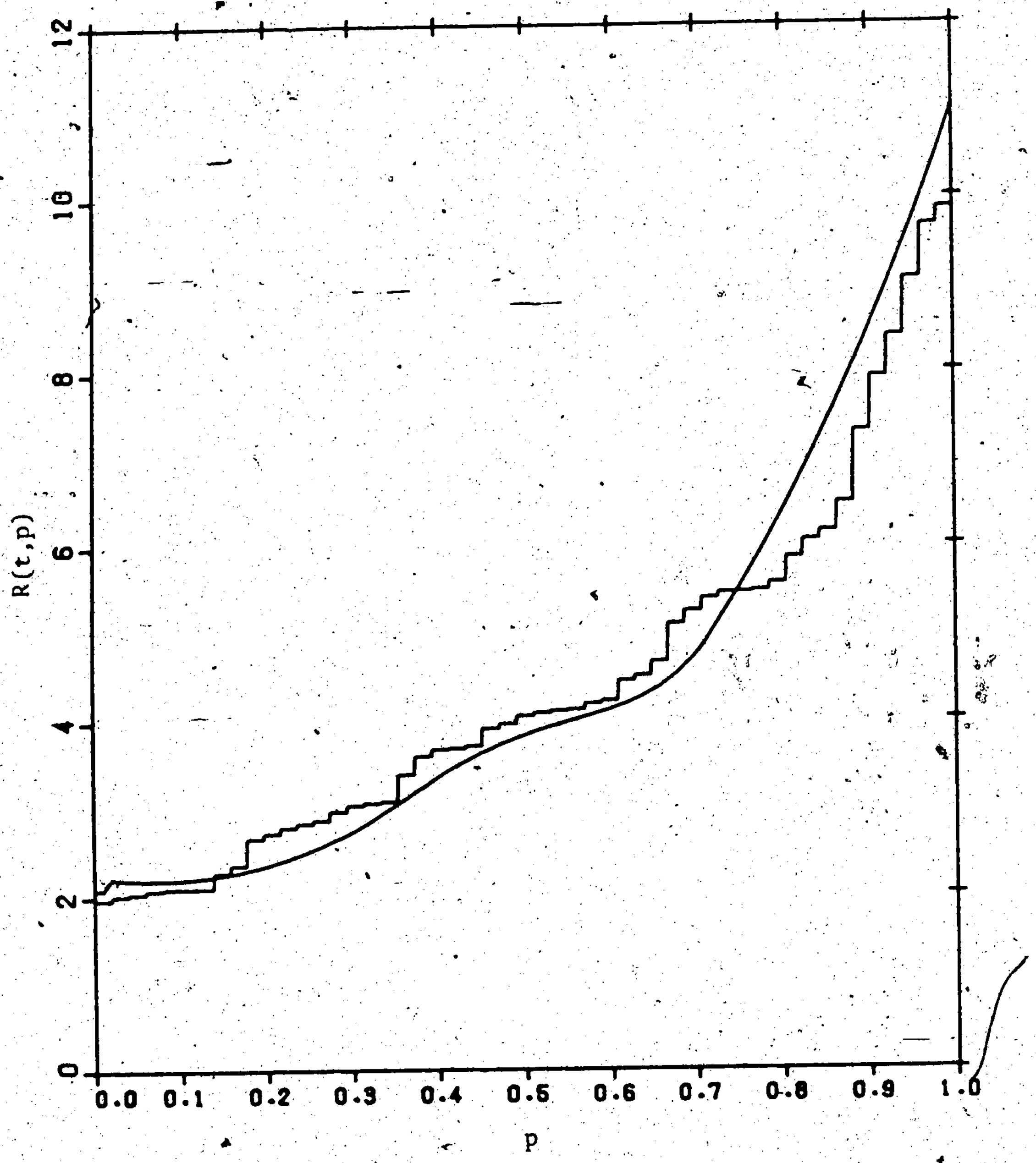

Figure 5.1. Plots of Empirical PRL $\hat{R}_n(\cdot,\cdot)$, and Smooth PRL Estimator $\tilde{R}_n(\cdot,\cdot)$ $(t = 2.0,\ h_n = 0.30)$

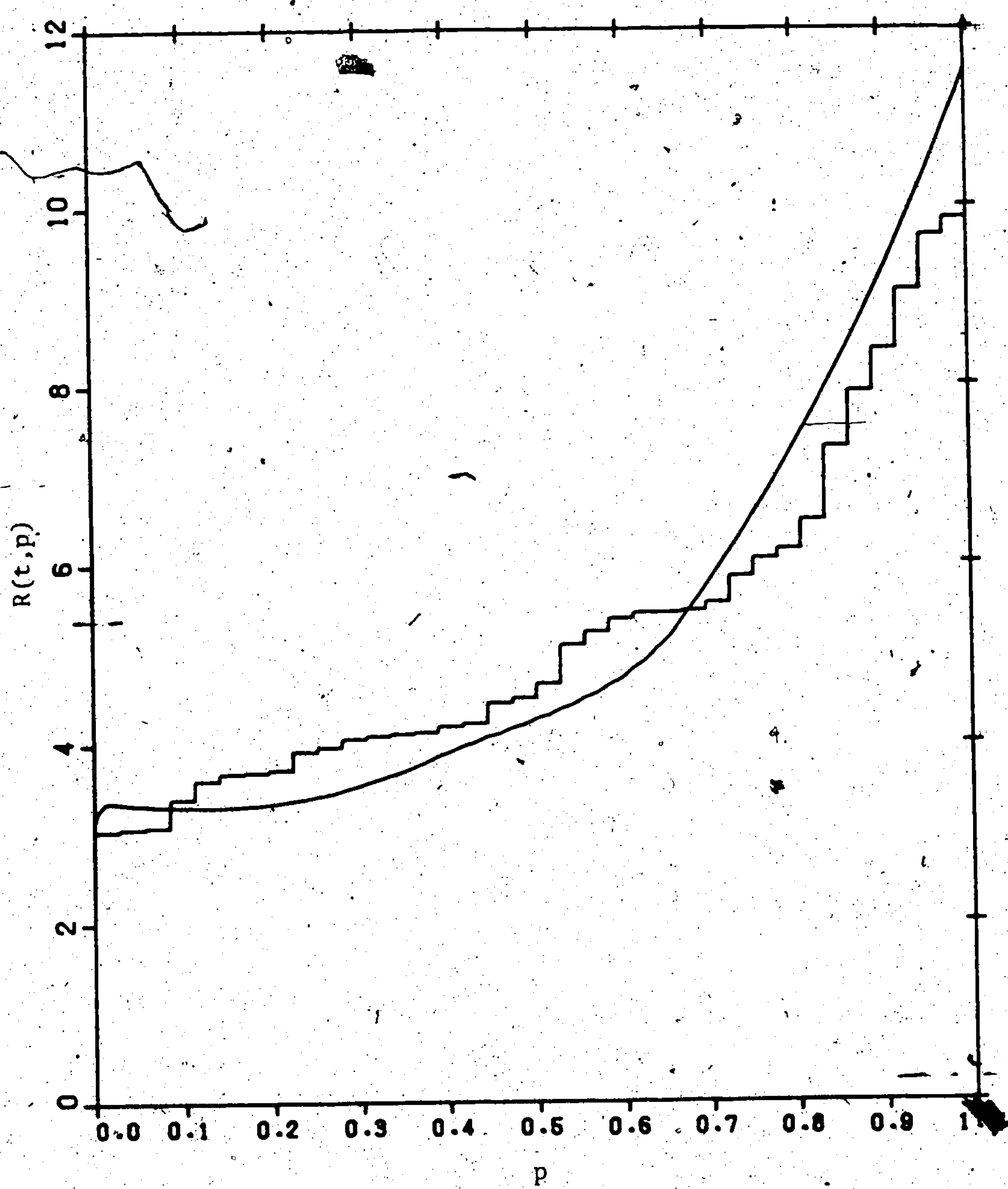

Figure 5.2. Plots of Empirical PRL $\hat{R}_n(\cdot,\cdot)$ and Smooth PRL Estimator $\tilde{R}_n(\cdot,\cdot)$ ($t = 3.0$, $h_n = 0.35$)

TABLE 4.1. PERCENTAGE POINTS OF $n^{1/2}\phi_{2n}$

N	0.75	0.80	0.85	0.90	0.91	0.93	0.95	0.97	0.99
3	1.155	1.155	1.155	1.155	1.732	1.732	1.732	1.732	1.732
4	1.125	1.125	1.125	1.500	1.500	1.500	1.500	1.500	2.000
5	0.984	1.342	1.342	1.342	1.431	1.431	1.431	1.789	1.789
6	1.225	1.225	1.293	1.293	1.293	1.633	1.633	1.701	2.041
7	1.134	1.188	1.188	1.512	1.512	1.512	1.566	1.566	1.944
8	1.105	1.105	1.237	1.414	1.414	1.458	1.458	1.768	1.812
9	1.037	1.148	1.333	1.370	1.370	1.370	1.481	1.667	2.000
10	1.075	1.233	1.265	1.297	1.391	1.391	1.581	1.613	1.929
11	1.151	1.206	1.233	1.343	1.453	1.508	1.535	1.617	1.919
12	1.155	1.179	1.251	1.371	1.443	1.467	1.540	1.660	1.828
13	1.109	1.131	1.195	1.387	1.408	1.451	1.472	1.685	1.963
14	1.088	1.145	1.241	1.355	1.374	1.413	1.604	1.680	1.890
15	1.102	1.188	1.291	1.360	1.360	1.446	1.566	1.618	1.876
16	1.063	1.141	1.250	1.313	1.391	1.500	1.516	1.641	1.891
17	1.084	1.184	1.227	1.327	1.341	1.455	1.512	1.683	1.826
18	1.061	1.179	1.192	1.309	1.388	1.427	1.506	1.650	1.899
19	1.111	1.159	1.220	1.352	1.376	1.425	1.509	1.654	1.884
20	1.118	1.163	1.219	1.353	1.386	1.442	1.521	1.610	1.845
21	1.101	1.133	1.257	1.351	1.351	1.465	1.538	1.621	1.912
22	1.095	1.153	1.221	1.318	1.366	1.[illegible]34	1.521	1.647	1.928
23	1.079	1.160	1.260	1.333	1.369	1.460	1.496	1.677	1.877
24	1.055	1.157	1.233	1.327	1.361	1.437	1.505	1.641	1.846
25	1.072	1.168	1.208	1.328	1.392	1.432	1.488	1.608	1.808
26	1.086	1.169	1.207	1.350	1.365	1.403	1.493	1.599	1.886
27	1.076	1.155	1.219	1.333	1.347	1.411	1.461	1.604	1.803
28	1.114	1.161	1.242	1.350	1.377	1.431	1.492	1.573	1.809
29	1.082	1.148	1.217	1.332	1.345	1.402	1.511	1.633	1.863
30	1.102	1.156	1.248	1.339	1.394	1.467	1.515	1.649	1.850
31	1.089	1.130	1.222	1.309	1.350	1.443	1.489	1.628	1.825
32	1.083	1.155	1.260	1.331	1.376	1.436	1.508	1.613	1.906
33	1.066	1.129	1.224	1.303	1.350	1.408	1.477	1.588	1.825
34	1.069	1.125	1.211	1.327	1.352	1.392	1.498	1.609	1.811
35	1.082	1.154	1.227	1.323	1.357	1.405	1.497	1.599	1.879
36	1.060	1.130	1.208	1.333	1.338	1.394	1.500	1.574	1.875
37	1.097	1.151	1.204	1.320	1.346	1.408	1.484	1.591	1.804
38	1.084	1.153	1.221	1.319	1.336	1.400	1.481	1.588	1.806
39	1.072	1.158	1.224	1.322	1.347	1.429	1.482	1.618	1.852
40	1.091	1.142	1.206	1.300	1.328	1.407	1.459	1.597	1.802
41	1.086	1.127	1.230	1.310	1.329	1.398	1.467	1.577	1.798
42	1.080	1.139	1.223	1.326	1.367	1.422	1.492	1.602	1.833
43	1.082	1.124	1.220	1.309	1.348	1.422	1.507	1.599	1.830
44	1.072	1.124	1.223	1.329	1.360	1.398	1.484	1.593	1.826
45	1.073	1.146	1.222	1.312	1.345	1.375	1.461	1.574	1.822
46	1.074	1.144	1.221	1.337	1.353	1.426	1.500	1.590	1.798
47	1.074	1.133	1.204	1.322	1.338	1.397	1.483	1.611	1.856
48	1.058	1.155	1.230	1.311	1.347	1.398	1.491	1.600	1.831
49	1.073	1.143	1.216	1.329	1.353	1.420	1.475	1.598	1.808
50	1.061	1.131	1.202	1.312	1.344	1.397	1.474	1.581	1.822
∞	1.076	1.125	1.216	1.321	1.336	1.396	1.476	1.596	1.829

REFERENCES

[1] Aly, E.E.(1985) "R-R Plot Processes and Percentile Residual Life Functions". Preprint.

[2] Aly, E.E., Csörgö, M. and Horváth, L.(1985). "Strong Approximation of the Quantile Process of the Product-limit Estimator". Journal of Multivariate Analysis. Vol. 16, p. 185-210.

[3] Aly, E.E., Csörgö, M. and Horváth, L. (1987). "P-P Plots, Rank Processes, and Chernoff-Savage Theorems". New Perspectives in Theoretical and Applied Statistics. Puri, M.L., Vilaplana J.P. and Wertz, W. (eds). John Wiley & Sons, Inc.

[4] Bickel, P.J. and Doksum, K.A.(1977) Mathematical Statistics: Basic Ideas and Selected Topics. San Francisco: Holden-Day Inc.

[5] Brigson, M.C. and Siddiqui, M.M.(1969) "Some Criteria for Aging". Journal of the American Statistical Association. Vol. 64, p. 1472-1483.

[6] Chibisov, D.(1964) "Some Theorems on the Limiting Behaviour of Empirical Distribution Functions" Selected Translations in Mathematical Statistics and Probability Vol. 6, p. 147-156.

[7] Chung, C-J.F.(1986) "Confidence Bands for Quantile Function and Percentile Residual Lifetime under Random Censorship". Ph.D. Thesis, Carleton University, May 1986.

[8] Csörgö, M.(1983) Quantile Processes with Statistical Applications (CBMS-NSF Regional Conference Series in Applied Mathematics, 4.2): SIAM, Philadelphia.

[9] Csörgo, M. and Csörgo, S.(1986) "Estimation of Percentile Residual Life". Technical Report. No. 79. Carleton University. p. 36-53.

[10] Csörgo, M., Csörgo, S. and Horváth, L.(1986) Asymptotic Theory for Empirical Reliability and Concentration Processes, Lecutre Notes in Statistics, 33. Springer-Verlag, Berlin.

[11] Csörgo, M., Csörgo, S., Horváth, L. and Mason, D.M.(1986). "Weighted Empirical and Quantile Processes". Annals of Probability. Vol. 14, p. 31-85.

[12] Csörgo, M., Csörgo, S., Horváth, L. and Révész, P.(1982) "On Weak and Strong Approximations of the Quantile Process" Proc. of the 7th Conference on Probability Thoery. Brasov, Aug. 29-Sept. 4.

[13] Csörgo, M. and Révész, P.(1978) "Strong Approximations of the Quantile Function". Annals of Statistics Vol. 6, P. 882-894.

[14] Csörgo, M. and Révész, P.(1981) Strong Approximations in Probability and Statistics. Academic Press.

[15] Deshpande, J.V.(1983) "A Class of Tests for Exponentiality Against Increasing Failure Rate Average Alternatives". Biometrika. Vol. 70, p. 514-518.

[16] Földes, A. and Réjtö, L.(1981) "A LIL Type Result for the Product Limit Estimator". Zeitschrift fur Wahrscheinlielikeits-theorie and Verwande Gebiete Vol. 56, p. 76-86.

[17] Heines, A.L. and Singpurwalla, N.D.(1974) "Some Contributions to the Stochastic Characterization of Wear". In Reliability and Biometry: Statistical Analysis of Life Length, F. Prochan and R.J. Serfling (eds.), SIAM.

[18] Hollander, M., Park, D.H. and Preschan, F.(1986) "A Class of Life Distributions for Aging". Journal of the American Statistical Associaton. Vol. 81, P. 91-95.

[19] Hollander, M. and Proschan, F.(1972) "Testing Whether New Is Better then Used". Annals of Mathematical Statistics Vol. 43, p. 1136-1146.

[20] Hollander, M. and Proschan, F.(1984) "Nonparametric Concepts and Methods in Reliabiity". Nonparametric Methods. Handbook of Statistics. VOl. 4, P.R. Krishnaiah and P.K. Sen (eds.) North-Holland. P. 613-655.

[21] Kaplan, E.L. and Meier, P.(1958) "Nonparametric Estimation from Incomplete Observations". Journal of the American Statistical Association, Vol. 53, p. 457-481.

[22] Koul, H.L.(1977) "A Test for New Better then Used". Communications in Statistics, Part A - Theory and Methods Vol. 6, p. 563-573.

[23] Koul, H.L.(1978), "A Class of Tests for Testing 'New Is Better then Used'". Canadian Journal of Statistics, Vol. 6, p. 249-271.

[24] Lio, Y.L. and Padgett, W.J.(1987) "Some Convergence Results for Kernel-Type Quantile Estimators Under Censoring". Statistics and Probability Letters, Vol. 5, p. 5-14.

[25] Lio, Y.L., Padgett, W.J. and Yu, K.F.(1986) "On the Asymptotic Properties of a Kernel-Type Quantile Estimator from Censored Samples". to appear in Journal of Statistical Planning and Inference.

[26] Mauro, D.(1985) "A Combinatoric Approach to the Kaplan-Meier Estimator" Annals of Statistics, Vol. 13, p. 142-149.

[27] O'Reilly, N.(1974) "On the Weak Convergence of Empirical Process in Sup-Norm Metries". Annals of Probability Vol. 2, p. 642-651.

[28] Padgett, W.J.(1986) "A Kernel-Type Estimator of a Quantile Function from Right-Censored Data". Journal of the American Statistical Association. Vol. 81, p. 215-222.

[29] Parzen, E.(1980) "Quantile Functions, Convergence in Quantile, and Extrem Value Distribution Theory" Technical Report. B-3, Statistical Institute, Texas A & M University. College Station TX.

[30] Sander, J.(1975) "The Weak Convergence of Quantiles of the Product-Limit Estimator" Technical Report 5. Stanford University, Dept. of Statistics.

[31] Stingler, S.M.(1974) "Linear Function of Order Statistics". The Annals of Statistics Vol. 2, p. 676-693.

[32] Wellner, J.A.(1978) "Limit Theorems for the Ratio of the Empirical Distribution Function to the True Distribution Function". Z. Wahrsch. Verw. Gebiete. Vol. 45, p. 73-88.

[33] Yang, S.S.(1985) "A Smooth Nonparametric Estimator of a Quantile Function". Journal of the American Statistical Association, Vol. 80, p. 1004-1011.