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USING A CONSTRUCTIVIST PERSPECTIVE TO INVESTIGATE THE LEARNING AND TEACHING OF MATHEMATICS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled Using a Constructivist Perspective to Investigate the Learning and Teaching of Mathematics submitted by Arthur Craig Loewen in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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This work is dedicated to four very special people:

Rena Ruth Loewen

Arthur Loewen

Sharon Joyce Georgina Loewen

and

Arthur Samuel Cranfield Loewen

ABSTRACT

The purpose of this study was to employ a constructivist model of learning in the investigation of the Direct, Meaning, and Problem Process Teaching approaches. The Direct Teaching approach emphasizes the algorithmic structure of mathematics content. The Meaning Teaching approach extends the Direct approach by emphasizing the representations of, and relationships between mathematical ideas. The Problem Process Teaching approach further extends the Meaning approach by adding a problem solving component at the beginning of each class in which generalized process skills are addressed.

A constructivist model of learning was built around three major components, including propositional knowledge, procedural knowledge, and cognitional knowledge. Propositional knowledge is defined as the meaningful relationships and connections drawn between mathematical concepts. Procedural knowledge is defined as the collection of computational procedures related to a mathematics topic, and cognitional knowledge is defined as a general knowledge form which enables and facilitates the development of both propositional and procedural knowledge.

The study was conducted under a pretest–post test–retention test design. A total of nine teachers and 240 students participated in this study. The teachers were asked to deliver the grade eight percent unit according to an assigned teaching approach: three teachers were assigned to each of the teaching approaches. Development of propositional knowledge was assessed using the structured tree recall task, a card sorting and memory task in which students place together words

that "go together." Development of procedural knowledge was assessed using a diagnostic-performance test built around the objectives in the Alberta grade eight percent unit. Fifteen clinical interviews were also conducted to assess the levels of cognitional knowledge expressed by those students who had shown gain in both propositional and procedural knowledge structures during instruction.

It was found that: (a) the Meaning Teaching approach facilitates the development of propositional knowledge, (b) the Direct and Problem Process Teaching approaches facilitate the development of procedural knowledge, (c) propositional and procedural knowledge grow independently, and (d) cognitional knowledge (particularly identification and synthesis) is related to a students' ability to construct propositional and procedural knowledge.

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CHAPTER ONE

Introduction of the Study

Davis (1983) argues that teachers historically have not been teaching mathematics in a meaningful way, but instead are teaching through rote instruction. He contends that students construct knowledge even when active meaningful construction of knowledge is not supported by the teacher; he argues that this may be an important source of students' misconceptions. Davis also claims that the intuitive understandings children hold prior to instruction are the foundation for the building of powerful mathematical ideas:

> This view holds an important implication both for curriculum and for diagnosis and evaluation: the job of an instructional program is to make solid contact with the mental representations that a student already possesses and to provide those experiences and interpretations that will help the student develop his or her representational capability further, hence becoming able to represent more complex mathematical situations and mathematical knowledge (pg. 108).

Davis concludes his argument by stating that students could learn significantly more mathematics if it were taught from a constructivist perspective.

Impositionism is the opposite to constructivism. The primary goal of impositionism is the achievement of a given student performance ability. The primary goal of constructivism is individually constructed knowledge systems which empower students to complete generalized tasks. Impositionism might best be described as predominantly rote learning or learning through drill and practice. According to Cobb (1988), under impositionism students are forced to adhere to prescribed methods of task completion, and thus develop these perceptions: (a) mathematics is basically arithmetic procedures which are to be memorized, (b) specific facts and skills are really isolated goals, (c) teaching occurs only by direct explanation and the completion of large amounts of homework, and (d) instructional failures are met with the repetition of familiar instructional cycles.

The issue that these researchers address is the relationship between instruction and student learning, a theme adopted by the present study. The general problem this study pursues is to employ a cognitive science framework to investigate the teaching and learning of mathematics.

A HISTORICAL PERSPECTIVE

Romberg and Carpenter (1986) have provided a good historical review of the projects which have culminated in recent research on mathematics students' cognitive constructions and the teaching processes which affect these constructions.

In the 1920s, research on instruction was primarily conducted by behaviorists such as Thorndike who believed that through reinforcement students could learn to perform mathematical processes more effectively. Attempts to implement behaviorist principles into models of instruction failed, and were later replaced with an emphasis on providing students with more meaningful learning experiences.

Authors and researchers such as Brownell in the 1930s, 1940s and 1950s advocated rooting fundamental mathematical ideas in student experiences and providing more meaningful experiences by enabling students to generalize abilities from one content area to another. The goal under this format of instruction was for the student to achieve a level of "meaningful habituation," which means to achieve a high level of understanding with the ability to recall and use information in an almost automatic manner. Brownell (1987, a reprint of much earlier work) advocated the use of drill to achieve this automatic recall ability. He recommended that these drills be implemented after a high level of student understanding was achieved.

The research and work of Piaget became quite prevalent in the 1950s. Piaget was not intending to directly investigate the learning processes in mathematics through his research, but many of the principles and constructs he described did seem to directly relate to this subject discipline. It was difficult to derive implications for classrooms from Piaget's work as he was primarily concerned with describing general stages and processes of student learning, not in relating these stages to teaching processes.

In the 1960s, Gagné provided more specific guidelines on how mathematics may be taught in order to enhance student learning. He employed a task analysis model which enabled the mathematics curriculum to be broken down into a hierarchy of discrete concepts and processes which could then be reassembled into a curriculum. It was believed that if the curriculum was presented in proper sequence and with appropriate pacing, then it would be easy for students to learn and digest. The difficulty with this model was that it did not account for the individuality of students and presumed that the ability of students to learn mathematics was limited primarily to the pacing and sequencing of instruction.

The 1970s and 1980s were characterized by a shift toward an information processing model of instruction which gave greater recognition to the individuality and unique learning styles and abilities of different students. Under the information processing model, the human mind was attributed three major memory components: sensory buffer, short term memory, and long term memory (Shavelson, 1974 and Frederiksen, 1984). The sensory buffer retains information only for a very short period of time, just long enough for it to be classified, coded, stored, or ignored by the short term memory. Long term memory stores information virtually permanently. During instances of learning, information in the long term memory interacts with new knowledge. Information is stored within the long term memory as the collection of nodes and interconnections between nodes. When knowledge is retrieved from long term memory it is recalled in the form of chunks, or groups of nodes which have been stored together by virtue of their interconnectedness. The short term memory contains the information that is currently being used. It is also responsible for monitoring the flow of information from the senses to the long term memory and vice versa to allow processing. In the average adult the short term memory may only contain five to seven items at one time.

Romberg and Carpenter (1986) state that research conducted under the precepts of the information processing model can be divided into two groups: research striving to better understand the components within the information processing system, and research striving to better understand the functioning of the system during the learning of specific content. In various forms, the information processing model is still accorded much discussion today.

The current models of learning and teaching appear to be direct descendants of the 70 year history briefly described above. Individuals store information in long term memory in chunks and networks. The manner in which information is stored is critical to how effectively it may be recalled and used. Thus, the limiting variable of long term memory is the accessibility to information. One variable influencing recall is metacognition, which is one's awareness of one's own mental processes and the ability to control these processes. Students "are not passive learners who simply absorb knowledge. Children come to school with rich informal systems of mathematics. They actively structure incoming information and attempt to fit it into their established cognitive framework" (Romberg and Carpenter, 1986, pg. 858). Current models of learning place a great deal of emphasis on the individual's ability to construct knowledge or assemble ideas in meaningful ways. Current models of teaching attribute to the teacher the role of pointing students toward new and important experiences through which ideas may be accommodated and assimilated by students into existing cognitive structures.

These beliefs have led to concrete guidelines pertaining to how classroom instruction should be delivered, how curriculum may be interpreted, and how research may be conducted. Romberg and Carpenter (1986) state:

> In general, it appears that it is important to stress relationships between concepts, especially higher-order relationships that are related to ways the concepts may be used to solve problems. The analysis of conceptual maps constructed by experts in a field provides a framework for organizing instruction to emphasize important correspondences between related concepts, and the analysis of students' conceptual maps provides a means to evaluate their level of understanding of a topic (pg. 857).

Teachers should: attempt to link and relate concepts; strive to help students develop processes which enable them to provide meaning to their learning through the interaction of new ideas with those stored in long term memory; not overwhelm the limitations of the short term memory (Romberg and Carpenter, 1986); and, recognize that misconceptions are the direct product of incorrectly structured schemata, and not a consequence of absent schemata (Davis, 1983). Curriculum should be prepared for instruction through proper sequencing which enables concepts to be built up or constructed by individual students rather than imposed by teachers (Cobb, 1988). Research should be conducted that will link learning and teaching theories. Despite what is now known about how and what students learn, Romberg and Carpenter argue that research has not employed this knowledge in investigating teaching processes, and that the issues of student learning have not been directly addressed. They state that "We currently know a great deal more about how children learn mathematics than we know about how to apply this knowledge to mathematics instruction" (pg 859), and that "current theories have very little to say about classroom organization or interaction with students" (pg. 859).

Romberg and Carpenter (1986) conclude their article with several research recommendations. One recommendation states that "Research is needed that blends the strengths of current cognitive science research with a concern for the realities of the classroom and focuses on students' learning from instruction over extended periods of time" (pg. 868). A second recommendation states that "The kind of teaching study that needs to be done would bring together both notions about the classroom, the teacher, and the student's role in that environment, and how individuals construct knowledge" (pg 868). The present study has attempted to implement these recommendations.

The present study is an investigation of the relationship between the teaching and learning of mathematics, specifically the effect of different teaching approaches on the construction of knowledge by students. In this study we will derive and present a constructivist model of learning and employ components of this learning model in the investigation of three teaching approaches. This application of a constructivist perspective of learning will also speak to the effectiveness of such a perspective as a means to make classroom decisions. We have adopted Romberg and Carpenter's (1986) recommendations in applying recent developments in the field of cognitive science in real mathematics classrooms for the general purpose of investigating the relationship between teaching and learning.

PARENT STUDIES

Two major works have served as the parent studies to the present research: the Missouri Mathematics Effectiveness Project (MMEP) conducted by Good, Grouws and Ebmeier (1983), and the Meaning in Mathematics Teaching (MMT) Project conducted by Sigurdson and Olson (1988, 1989a, 1989b). The Missouri Mathematics Effectiveness Project was designed to investigate and describe the nature of effective mathematics teaching. The secondary purpose of this study was to determine if teachers could be taught how to deliver effective mathematics instruction given the results of the first phase of the project. The Meaning in Mathematics Teaching Project was designed to investigate the effect of three different teaching approaches on the general achievement, attitude and problem solving abilities of grade eight Alberta students. The MMT employed the findings of Good, Grouws and Ebmeier in their design of the teaching approaches.

Missouri Mathematics Effectiveness Project

The Good, Grouws, and Ebmeier (1983) project was conducted in the mid 1970s and early 1980s. It was characterized by a series of smaller research projects imbedded within a larger structure intended to study and identify the qualities of effective mathematics instruction. A variety of naturalistic and experimental study designs were employed. Their project evolved from a concern that though many studies had been conducted to identify the qualities and attributes of effective instruction, there appeared to be little consistency among the findings.

The first phase of the Missouri Mathematics Effectiveness Project (MMEP) involved identifying the qualities of effective teachers. This phase of the project was conducted in approximately 100 grade three and grade four classrooms. The researchers began by identifying nine teachers who had proven themselves to be effective over at least two consecutive years, and nine teachers who had proven themselves to be ineffective over at least two consecutive years. Teacher effectiveness was defined as: "student performance (residual gain) on a standardized achievement test" (pg. 6), and was measured by tabulating student scores on the Iowa Test of Basic Skills. Teachers whose students showed high residual gain scores were considered effective. The researchers then randomly selected another 23 classrooms (to protect the identities of the subjects) and with the help of research assistants visited all 41 classrooms approximately seven times. On each visit the following data was recorded: time spent on instruction, time spent on development, time spent on practice, teacher-student interaction patterns, teacher behavior and managerial style, materials employed, and homework assignments. The researchers discovered

> high residual mean scores appeared to be strongly associated with the following teacher behaviors: (1) large-group instruction; (2) generally clear instruction and availability of information to students as needed (process feedback, in particular); (3) a nonevaluative and relaxed learning environment which was task focused; (4) higher achievement expectations (more homework, faster pace); and (5) classrooms which were relatively free of major behavior disorders (pg. 8).

From these generalizations the authors were able to develop a program of instruction which could be prepared and implemented by other groups of teachers.

The second phase of the MMEP was designed to investigate whether teachers could alter existing instructional approaches in order to implement the qualities of effective instruction listed above. This phase involved a sample of 40 fourth grade teachers who were randomly assigned to treatment and control groups.

The treatment group was provided with two workshops to introduce the specific model of instruction which was a product of the first phase of the study (see Figure 1). These teachers were then asked to implement this model. The researchers made six visits to each classroom to evaluate the degree to which the model was being implemented, and tested students' achievements and changes in attitude. The researchers state: "At this point the most reasonable interpretation is that the total instructional treatment program, when implemented, had a positive impact upon mean student achievement" (pg. 91). The second phase of the MMEP proved that teachers could change instructional patterns, and that these changes could result in improved student performance on standardized tests and on verbal problem solving tasks.

In the third and final phase of the MMEP, the researchers attempted to determine if the same results could be achieved when working with junior high rather than elementary school teachers. A sample of 19 volunteers was found, and these grade eight teachers were randomly assigned to control and treatment groups. Each teacher was observed a total of twelve times. Again the researchers considered the teacher's abilities to implement the instructional model and the achievement levels of his or her students. The researchers report that the teachers could implement the instructional model, and that this instructional model had desirable outcome effects on student achievement, attitude, and problem solving ability.

Meaning in Mathematics Teaching Project

The second major work which served as a parent study to the present research was conducted by Sigurdson and Olson (1988, 1989a, 1989b).

Figure 1: Good, Grouws, and Ebmeier (1983) model of instruction: key instructional behaviors

Daily Review (first 8 minutes except Mondays):

1. Review the concepts and skills associated with the homework
2. Collect and deal with homework assignments
3. Ask several mental computation exercises

Development (about 20 minutes):

1. Briefly focus on prerequisite skills and concepts
2. Focus on meaning and promoting student understanding by using lively explanations, demonstrations, process explanations, illustrations, etc.
3. Assess student comprehension
   a. Using process-product questions (active interaction)
   b. Using controlled practice
4. Repeat and elaborate on the meaning portion as necessary

Seatwork (about 15 minutes):

1. Provide uninterrupted successful practice
2. Momentum - keep the ball rolling - get everyone involved, then sustain involvement
3. Alerting - let students know their work will be checked at end of period
4. Accountability - check students' work

Homework Assignment:

1. Assigned on a regular basis at the end of each math class except Fridays
2. Should involve about 15 minutes of work to be done at home
3. Should include one or two review problems

Special Reviews:

1. Weekly review/maintenance
   a. Conduct during the first 20 minutes each Monday
   b. Focus on skills and concepts covered during the previous week
2. Monthly review/maintenance
   a. Conduct every fourth Monday
   b. Focus on skills and concepts covered since the last monthly review

According to these researchers "The major objective of this study is to compare these three instructional approaches, using as a criterion for this comparison, mathematical achievement and attitude" (Sigurdson & Olson, 1989, pg. 1). The general areas of interest in the MMT study included: (a) the mathematical achievement of students under the three models, (b) the students' attitude toward mathematics under the three models, (c) the performance of high and low achievers under the three models, and (d) the ability of teachers to implement the treatments (Sigurdson & Olson, 1988).

In the MMT, three different instructional models and one control group were designed by the researchers and implemented in 54 grade eight mathematics classrooms: conventional textbook instruction (control group - CTI), the direct model (DI), the meaning model (MI), and the problem process model (PPI). Thirteen teachers were assigned to the CTI group, 13 to the DI group, 14 to the MI group, and 14 to the PPI group. Approximately 1200 students were enrolled in these 54 classes.

The Direct Teaching Approach. The Direct Teaching approach is a lesson delivery format which incorporates the instructional pattern described by Good, Grouws, and Ebmeier (1983, summarized in Figure 1). Teachers who implement this model of instruction will: avoid any explicit attempts to link new ideas to past learning, avoid the use of manipulatives, address problem solving only in an attempt to unveil correct solution processes (but will not attempt to describe or summarize these processes), and concentrate mainly on enabling students to correctly employ and carry out algorithms. Development of concepts under this model will typically entail several instances of teacher demonstration of algorithms followed by teacher guided episodes of student practice. Under the Direct Teaching approach, it is assumed that students will independently generate relationships between mathematical concepts even though the teacher will make no explicit attempts to address such relatio[illegible] Given that students are active interpreters of their own environments, and are actively and independently involved in the construction of mathematical knowledge, this teaching approach is a viable classroom approach.

The Meaning Teaching Approach. The Meaning Teaching approach is also a lesson delivery format which incorporates the Active Mathematics Teaching lesson format described in Figure 1. Under the Meaning Teaching approach, the teacher tries to: enable students to make links between present learning and past ideas, provide applications of mathematical ideas thus relating these ideas to the lives of the students or to the "real world," provide manipulative experiences which enhanced a sense of 'acceptance,' generalize processes to enable transfer to novel situations, engage in a great deal of discussion with much student input, and ask many process questions which force students to articulate relationships between mathematical concepts.

Teaching methods under this approach involve attending to prerequisites, attending to relationships, attending to representations, and attending to perceptions (Good and Grouws, 1987). Attending to prerequisites means that the teachers will ensure that students have mastered important form elements prior to developing the interconnections or understanding elements which constitute a new concept. Attending to relationships means that ideas must always be linked to similar or related ideas rather than being presented in a vacuum. Attending to representation implies that teachers must make clear what the symbols used in instruction represent. Attending to perceptions means that teachers will help students "become aware of the relevant aspects of the mathematical concept under study" (pg. 29). Students should also become aware of other aspects that may be present but are not relevant. Such awareness enables students to develop a mental image of the mathematical concepts under study, and use this image to relate to previous learning and to generalize to further contexts.

Under the Meaning Teaching approach, the teacher is responsible for explaining why algorithms work, not just introducing them and having the students practice them. Teachers are also responsible for showing how skills and ideas are interrelated as well as showing how concepts can be distinguished one from another. The teacher also provides labels for concepts and provides extensions and applications of mathematical ideas in order to facilitate transfer (Good and Grouws, 1987). Other specific behaviors the teacher may engage in include the employment of concrete manipulative materials, and the linking of syntax, language and mathematical principles to these manipulatives.

Rathmell (1986) provides a list of some specific teacher behaviors which should be avoided when striving to teach for meaning: fail to check for student understanding of prerequisites, employ a rule-example-practice approach without supplying any explanation, fail to use models and provide only a symbolic explanation, use models without clear explanations, use models without correct thinking, use models without connecting them to the symbolic work, use inappropriate numbers for examples, ask questions but fail to answer or explain, fail to clearly explain how to write the algorithm, develop an idea but fail to relate it to the topic, fail to prepare students for transition to seatwork, and fail to ask questions to check for understanding.

<u>The Problem Process Teaching Approach</u>. The Problem Process Teaching approach is an extension of the Meaning Teaching approach described above except time is provided (approximately 8 to 10 minutes usually at the beginning of a

period) to solve given problems and to articulate the processes required to solve these problems. The Problem Process Teaching approach involves both the construction of meaning and the development of problem solving processes through specifically designed exercises. In the MMT project, the actual selection of problems was left to the teacher; however, workshops were conducted by the researchers with the teachers to discuss the major principles of this teaching approach, and some problems were constructed and organized during the workshop sessions for instructional purposes. Some of the major principles of problem solving instruction discussed with the teachers at the workshop sessions are described by Charles and Lester (1984) and Good and Grouws (1987).

Effective methods of problem solving instruction are not clearly understood (Charles and Lester, 1984), but three principles are generally accepted: (a) students must solve problems in order to become good problem solvers. Students do not develop problem solving skills solely from experience with heuristics or the development of specific skills such as translation. These skills and heuristics must be drawn together through more generalized and complete first-hand experiences. (b) ability in problem solving develops over a prolonged period of time. Students do not become good problem solvers during one or two week courses, the specific skills involved take longer than that to develop, and (c) the program in which problem solving is taught must be systematically planned. The skills and concepts must be consciously and regularly introduced and practiced in order for efficient learning to occur.

Good and Grouws (1987) suggest that the teacher should ask students how they received their answers, should provide more exercises in mental computation and estimation, and should place more attention on the multiple methods of problem solving. Asking students how they received their answers encourages them to focus more clearly on the process than on the product, while the mental computation and estimation skills provide students with a way of checking, evaluating and monitoring their work. Attention to the multiple methods of problem solving helps students develop an awareness of the variety of available problem solving strategies and heuristics.

Each of the three treatment group teaching approaches were asked to use the Missouri Mathematics lesson format (see Figure 1). This lesson format was implemented in an attempt to standardize the teaching models. Sigurdson and Olson (1989a) state:

> In this way, the major differences between the instructional models [were] the approach to the mathematics content in the lessons: algorithmic-practice in the DI, meaning in the MI, meaning and problem processes in the PPI. By using the Missouri Mathematics Project lesson format, many instructional variables, such as student-teacher interaction, amount of homework, use of review and the like, would be held constant. An additional element of control was a common text across all classrooms in the study (pg. 3).

Several workshop sessions were held in order to assist the teachers in implementing the lesson format and their randomly assigned instructional model. The workshop session delivered to the DI teachers focused on the Missouri Mathematics Project lesson format, as well as means to teach algorithms and procedures clearly. The workshop sessions delivered to the MI teachers also addressed the lesson format, but ways to implement a meaning focus were also discussed. The teachers spent some time in developing and sharing appropriate classroom activities. The workshop sessions delivered to the PPI teachers were similar to that received by the MI teachers, except time was also spent on developing appropriate problem process activities (Sigurdson & Olson, 1989a).

Data was collected in the Meaning in Mathematics Teaching Project using the following research tools: (a) trained observers visited each classroom a minimum of five times and recorded the sequence of events during class periods to assess the teachers' abilities to implement the models as instructed; (b) pre, post and retention tests were developed and administered to students to test a "range of knowledge, comprehension and problem solving items" (Sigurdson & Olson, 1989a, pg. 5); (c) an attitude scale was developed and delivered to students both pre and post treatment; and (d) a questionnaire was administered and several teachers were interviewed to assess teachers' perceptions of the models they implemented.

Many important conclusions were drawn and organized around six fundamental questions. The first question involved comparison of student performance in the group of control teachers with the students taught by teachers in the innovative models. The researchers found that all three models (DI, MI, and PPI) produced statistically significantly higher student achievement scores, but no statistically significant difference between the three innovative instruction groups. The second research question involved comparison of the performance of students by ability level. The researchers found that high achieving students benefitted equally from the direct and meaning teaching models, but did not seem to benefit from the problem process treatment. The researchers also found that:

> Low achieving students are the only group who do not lose from the problem process approach (when compared to the meaning group). One thing is clear, something different is going on in the Problem Process approach than in the other two, and this "something" is of benefit to low achieving students (Sigurdson & Olson, 1989b, pg. 43).

The researchers later reanalyzed their data eliminating low implementers from each group (i.e., those teachers who had shown the least tendency to implement their innovative teaching model effectively). The researchers were then able to report that only medium and high ability students benefit under the meaning model, while all students benefit under the problem process model. This observation was a strong endorsement of the problem process model.

In their third research question, the researchers attempted to compare the effectiveness of the instructional models (DI, MI, and PPI) in above and below-average classes. They found that the below-average classes, in general, did not benefit from the meaning or problem process models, and thus concluded that the extra effort required to implement these models could not be justified in classes of general low ability. The fourth research question attempted to investigate the effect of the four teaching approaches on six attitude factors, including: the ease of doing math, independence from the teacher, the importance of studying math, the fun of doing math, the meaning of math, and the relationship between math and problem solving. No significant results were found. The same comparison was employed in the fifth research question involving only students from general high ability classes, and it was found that students in the problem process model found mathematics to be more enjoyable and fun.

The sixth research question intended to determine if teachers could implement their assigned model of instruction. The researchers found that the models were not of equal ease to implement, that is, teachers had less difficulty implementing the direct model than the meaning model. Teachers also had less trouble implementing the meaning model than the problem process model. As a consequence of this discovery, the researchers were forced to conclude that (a) though the meaning model was adequately implemented in low achieving classrooms, it seemed to have little positive effect on the students, and (b) the low implementation level of the problem process model may have prevented the demonstration of the full potential or effectiveness of this model.

In their entirely, the results of the Meaning in Mathematics Teaching project present a strong endorsement of the problem process model. As stated in the final report (Sigurdson & Olson, 1989b):

> The above conclusions taken together support the problem process approach. When implemented to a high degree, it does not lag behind the meaning approach; in above-average classes although lagging behind meaning in achievement, it benefits the low students as well as the high and medium; and finally it transmits a more positive attitude to learning mathematics. While it does seem to have the greatest potential, it also offers the greatest difficulty for implementation (pg. 67).

These results also provide a strong endorsement of the Missouri Mathematics Project lesson format (Sigurdson & Olson, 1989a).

In general, the Meaning in Mathematics Teaching Project conducted by Sigurdson and Olson may be characterized as an attempt to look at achievement at a gross level. There exists a deeper level at which achievement or learning may be investigated: the level at which knowledge is constructed. The purpose of the present study is to investigate students' construction of knowledge, and to investigate the affects of the three defined teaching approaches on the knowledge constructed. This investigation attempts to draw links between approaches to teaching and learning processes, necessitating the adoption of a particular process or description of learning. The constructivist perspective has been adopted as a current, widely-accepted description of learning (Davis, Maher & Noddings, 1990).

The results of the MMT show clearly that the three different teaching models did have different affects in terms of gross measures of student learning, but given the tools employed, this finding says little about the manner in which the mathematics was learned, or the differential affects of the teaching models on the knowledge that was constructed. The MMT was ultimately a curriculum implementation study, whereas the present study deliberately adopts a constructivist perspective on learning and employs this perspective to investigate the teaching approaches. In turn, this application of a constructivist learning model will act as a means to test its viability, i.e., its potential to describe the processes involved in the learning of mathematics in a classroom environment.

The present study and the MMT are similar in that both are teaching experiments, but they do differ in a number of dimensions. First, the present study employs only nine of the 54 teachers involved in the MMT. Second, because this project is build around a particular model of student learning and mathematical understanding, it has adopted instrumentation appropriate to this model. In the present study the purpose is to investigate the construction of knowledge, thus specific tests have been adapted and employed to measure forms of knowledge constructed. The major purpose of the MMT was to investigate students' achievement and attitudes, thus different tests appropriate to this purpose were employed. Finally, the present study was conducted over the course of a single teaching unit, while the MMT was conducted during the better part of a full school year. Because the purpose of this study was to investigate students' knowledge construction, it was necessary to limit the investigation to a particular context and content area (as recommended by Hiebert and Wearne, 1988).

LIMITATIONS AND DELIMITATIONS

Limitations are defined as the variables over which a researcher has no control. The primary limitation of this study is that there currently exists no way of definitively and directly measuring students' cognitive structures. The development of more reliable and valid measures is a current research interest. The fact that no such measures exist is primarily a consequence of two variables: (a) any cognitive structure is affected by emotions, and (b) all cognitive structures are undergoing constant change as new information is assimilated and accommodated. Because of the dynamic nature of cognitive constructions and the unpredictable affect of emotions on these constructions, any attempt to investigate cognitive constructions must be recognized as somewhat limited. Tests do exist for providing momentary descriptions (called cognitive maps) of students' constructions, but these maps must be considered simply snapshots (or even reflections) of students' cognitive networks.

An unfortunate second limitation to this study exists. In the design of this study fifteen interviews were planned. All fifteen interviews were conducted, but only eleven of the interviews were fully transcribable: three interviews were lost, and one was only partially transcribable. These interviews were lost due to technical failure of the recording apparatus. The loss of this data was unfortunate but unavoidable.

Delimitations are the variables or dimensions which the researcher chooses not to address or include within a study. Three delimitations exist: (a) this study does not directly attempt to examine learning in any other content domain than the grade eight percent unit. This decision was made based upon the limited resources of the researcher, the time at which the unit was taught, the time commitment of the teacher participants, and the need to provide a specific content domain. (b) this study attempts only to investigate student knowledge construction despite the fact that many other outcomes of learning exist, including: students' attitudes, students' sense of efficacy, and students' social development. These other outcomes were considered in the Meaning in Mathematics Teaching Project and thus are not considered here. (c) this study does not attempt to investigate the degree to which the teachers implemented their respective instructional models. The teachers who participated in this study were selected according to their level of implementation as determined by their participation in the MMT study (i.e., they were all high implementers). Furthermore, because the question of the ability of teachers to implement a given model was a research interest of the MMT, it was not pursued here, and an assumption was made that teachers were in fact following the instructional format to which they had been assigned.

The percent teaching unit was investigated for three reasons: (a) the time of year in which it was taught, and the fact that this time coincided with the researcher's opportunity to undertake the research, (b) the percent unit is a general unit in that it may be easily related to other math topics and units such as data management, fractions and decimals, and ratio and proportion. Its close relationship with many other topics permits many interconnections to be developed. Furthermore, its close relationship to these topics implies that if students perform well in this unit they are likely to perform well in the related units, and (c) the study attempted to compare the influence of three instructional approaches on student learning, and in so far as all three are viable approaches to instruction, it is irrelevant which unit is chosen; thus, the arguments of accessibility and generalizability prevail.

The actual topics which were to be addressed during the instruction in this phase of the study were entirely determined by the parameters of the Sigurdson and Olson (1988) study, but even within that study, topics were primarily determined by the demands of the Alberta curriculum and the resources made available through the adopted textbook. In the Journeys in Math 8 text, the percent teaching unit contains the following topics in order: meaning of percent, expressing fractions and decimals as percents, percents as decimals and fractions, finding a percent of a number, percent and circle graphs, finding a number when a percent of it is known, discount, sales tax, simple interest, percent gain, and percent loss.

CHAPTER TWO

Review of the Literature

The purpose of this study is to investigate three teaching approaches through the application of a theoretical model of learning derived from constructivist principles. The purpose of this chapter is to summarize the literature which describes and summarizes the constructivist perspective. Several teaching experiments attempting to relate teaching method to learning outcomes will be reviewed. This chapter concludes with a description of the theoretical constructivist learning model which will be used to analyze the Direct, Meaning, and Problem Process Teaching approaches.

CONSTRUCTIVISM AND INSTRUCTION

Plunkett (1981) has argued that knowledge must be constructed in the learning of mathematics. To try to separate 'mathematics' from 'self' and demand that math exists objectively in the world and therefore must be learned in some objective form, makes mathematics unlearnable. Plunkett asks:

> What is mathematics? ... I think that the question is the wrong one. It assumes that there is a thing called mathematics which has some sort of objective existence, and thus a nature which can be defined. But there is no such thing: we are deceived into thinking that there is by our inveterate habit of using nouns ... treating mathematics as a human activity rather than an ontological problem has the distinct advantage that we can feel we are dealing with an answerable question (pg 47).

Plunkett goes on to argue that all that can be known about mathematics is what people do when they do mathematics. If this reasoning is accepted, then it must be concluded that individuals construct their own mathematical realities as a result of their own experiences. This is the fundamental precept of the constructivist perspective.

Cobb (1988) contends that there are two forms of instruction: constructivism and impositionism. He defines the constructivist position:

> From this perspective, mathematical structures are not perceived, intuited, or taken in but are constructed by reflectively abstracting from and reorganizing sensorimotor and conceptual activity. They are inventions of the mind. Consequently, the teacher who points to mathematical structures is consciously reflecting on mathematical objects that he or she has previously constructed. Because teachers and students each construct their own meanings for words and events in the context of the ongoing interaction, it is readily apparent why communication often breaks down, why teachers and students frequently talk past each other. The constructivist's problem is to account for successful communication. ...Similarly, teachers and students who might be said to share mathematical meanings are each making imperceptible accommodations in their ways of knowing. From this perspective, the process of successfully sharing or exchanging mathematical thoughts and ideas is not viewed as one of transmission. Instead, it is characterized as a dynamic continually changing fit between the meaning-making of active interpreters of language and action (pg. 89).

According to Cobb's description, the major principles of constructivism are: (a) individuals create their own mathematical understandings through the processes of abstraction and reorganization from sensorimotor and conceptual data, (b) the teacher serves the function of pointing or directing the students to manifestations of new concepts, (c) communication is an essential and problematic element in helping students construct knowledge, and (d) learning is characterized as a dynamic activity, embodied in language and action, in which students and teachers develop a fit between constructed meanings of their collective and individual experiences.

Noddings (1990) has argued that constructivism has many strengths as a pedagogical view. She reiterates Cobb's (1988) assertion that knowledge is a constructed entity, but goes on to provide a summarized list of the major principles of constructivism, including:

1. All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.

2. There exist cognitive structures that are activated in the processes of construction. These structures account for the construction; that is, they explain the result of cognitive activity in roughly the way a computer program accounts for the output of a computer.

3. Cognitive structures are under continual development. Purposive activity induces transformation of existing structures. The environment presses the organism to adapt.

4. Acknowledgement of constructivism as a cognitive position leads to the adoption of methodological constructivism (pg. 10).

It could be argued that Noddings' second point is somewhat deterministic and thus inappropriate as it relates to learning situations. The element of predictability associated with programming cannot be generalized to learning contexts. Her intent (that what students bring in the sense of conceptual networks to the learning context affects the process of knowledge construction) remains a critical precept of the constructivist perspective. Noddings does however support the notion of the dynamic nature of cognitive structures and the importance of the environment in the learning context. She concludes by stating that adoption of constructivist principles necessitates adoption of particular teaching methods commesurate with those learning principles.

A common theme to the works of Cobb (1988) and Noddings (1990) is the importance of the environment or context in which construction occurs. Sigurdson (1988) argues that context is the means whereby a teacher can talk about mathematical ideas with students. He contends that mathematics is difficult to discuss once devoid of context. The learner will retain only that information which is linked within a knowledge network, and linking may occur through demonstration models, manipulative models, pictures, diagrams, and application stories (word problems). Sigurdson's discussion contributes a means whereby students may construct knowledge.

Doyle (1988) argues that the tasks that teachers assign their students constitute the context in which instruction (and thus knowledge construction) occurs. He defines a task as having four components: (a) a goal state or end product to be achieved, (b) a problem space or set of conditions and resources available to accomplish the task, (c) the operations involved in assembling and using resources to reach the goal state or generate the product, and (d) the importance of the task in the overall work system of the class.

Plunkett (1981), Cobb (1988), and Noddings (1990) have provided an overview and description of constructivist principles as they apply to instructional situations. We have seen that the learner acts as a mediator in his or her own learning, and that the learner is an active participant in the learning situation, reflectively abstracting from and generalizing to his or her environment. We have also seen that the environment constitutes the context in which construction takes place, that the teacher (also an active interpreter and constructor of knowledge) acts within this frame as an important component of the context.

EVIDENCE FOR CONSTRUCTIVISM

The constructivist perspective claims that learners act as constructors or builders of their own knowledge. In order to substantiate this claim it must therefore be shown that learners do act as "active interpreters of language and action" (Cobb, 1988, pg. 89), and as mediators in their own learning activities. Several researchers have attempted to validate this constructivist principle.

Winne and Marx (1982) completed a study using the stimulus-recall technique. The study involved five teachers and 113 pupils in grades four through seven. Teachers had ten lessons videotaped in a variety of content areas. After each taped lesson, the teachers were interviewed to determine the times during the lesson at which they intended the students to think in particular ways, and to identify several incidents about which students should be interviewed. Following the teacher interview, students were interviewed in groups (as small as two students and as large as six) using a standardized process. The researchers were able to summarize a list of cognitive behaviors that teachers intended for their students. Whenever either the student or the teacher reported that a particular way of thinking was necessary or implied by the teacher behavior, the researchers looked for an identifiable instructional stimuli for that thought process. It was found that many instructional stimuli could be used to cue the same student thought process.

The research methodology which Winne and Marx (1980) employed allowed them to draw several interesting conclusions, namely: (a) there was no direct correspondence between the instructional stimuli which teachers intended and the cognitive responses of the students, (b) there was an inverse relationship between the amount of information students needed to process and the effectiveness of the teacher stimulus, (c) students could respond more easily to a teacher stimulus when it demanded a well-developed cognitive response, (d) mastery over content would limit the effectiveness of the teacher stimulus to evoke desired cognitive processes, i.e., student content mastery was a prerequisite to activation of cognitive processes by a teacher stimuli, and (e) students played a role in determining whether specific cognitive process would be evoked. This final conclusion is vital as it implies that "students will construct meaning for classroom activities regardless of whether the teacher (or an instructional theory) does" (Winne and Marx, 1982, pg. 515). The authors draw the general conclusion that this study supports the major principle: students' cognitive processes mediate instructional outcomes.

Peterson, Swing, Braverman and Buss (1982) and Peterson and Swing (1982) also used the stimulated recall methodology to investigate students' thought processes during instruction. In this study 72 fifth and sixth grade students of medium ability were taught a two day course on simple probability. Lessons were delivered in the first hour of 2 three-hour sessions each morning. The classes were observed and each students' behavior (whether he or she was on or off task) was recorded every 20 seconds. Students were interviewed after each lesson using the stimulated recall format. The results of the study were: (a) students who indicated a higher degree of attention during class time scored higher on achievement measures. Students' claim to having paid attention was a better predictor of achievement than was the observational data, (b) students' reports of understanding were significantly and positively related to performance measures, (c) students who could articulate the specific strategies that they employed during instruction performed better than those who could only describe general learning strategies. The specific learning strategies that were identified include:

> repeating and reviewing information to oneself, relating information to prior knowledge, anticipating an answer to a teacher's question, trying to understand the teacher or figure out a problem, checking one's answer with a teacher or a student, reworking a problem in one's head or on paper if the answer was incorrect, reading or rereading directions or problems, and motivating oneself with self-thoughts (pg. 487).

(d) two specific learning strategies were most advantageous if employed: relating information to past knowledge, and trying to understand the teacher or problem, and (e) students who reported motivational self-thoughts tended to have more positive attitudes toward mathematics at the end of the session regardless of initial motivation levels.

In another study intended to replicate the one described above, Peterson, Swing, Stark and Waas (1984) found that the same effects held true, but they also concluded that attitude was an important variable in student learning:

> independent of students' mathematics ability, students' reports of negative evaluative self-thoughts were negatively related to students' seatwork scores and to students' achievement scores and attitude posttest scores ... motivation may affect academic outcomes because of the student's willingness to engage in a task or to persist in engagement or because of the degree of processing the student engages in (pg. 511).

The authors state nicely that simply attending to the lesson is not the most important mediating process, but that the "actual cognitive processes involved in processing the mathematics information presented during classroom instruction . . . may be as important or possibly even more important than the quantity of that time" (pg. 512). Again, these studies point out that students are mediators in their own learning.

Leinhardt (1988) conducted two studies within the topics of subtraction and fractions. The first study was conducted with two high, four medium, and two low ability grade two students. The students were interviewed once before, once after, and twice during instruction. The researcher also conducted pre and post tests. In the fractions study, the researcher used three high, five medium, and three low ability grade four students. These students were interviewed twice before instruction and once afterward. An in-class think aloud session was conducted once with each student, and each also received a stimulated-recall interview. The researcher found that the students were able to maintain a high level of performance, but upon probing, misconceptions in students' cognitive networks appeared. Leinhardt states:

> We have started a description of how and what children learn during instruction. We have seen that in some cases at least, they enter into

> the process of instruction with some fairly powerful intuitive concepts. During instruction, however, neither the texts nor the teachers seemed to capitalize on this knowledge as a basis for either instruction or explanation of new material. We have also seen that two children in different classes learned the computational portions of their math to a high level of skill, and it is important to not underestimate that accomplishment in itself. However, when we probed in a somewhat elaborate way the concrete representations and computational skills, we found that gaps still existed in both the computational and concrete systems of knowledge (pg. 140).

In the fractions study it was found that students' intuitive knowledge was suppressed by the memorization of algorithms. Three important conclusions may be drawn from this study: (a) the student is seen as one who constructs knowledge, (b) knowledge may be linked to intuitive understandings of the world, and (c) in general, teachers make no attempt to link mathematical knowledge to intuitive knowledge.

In a project conducted by Steffe (1983) six children, all seven years of age, were interviewed and two (Scenetra and James) were selected for detailed analysis due to their different (operative vs. formative) counting schemes. Operative schemes involve internal mental operations, while formative schemes do not. During the interviews the students were given addition questions. For example, seven blocks would be placed on the table and the student would verify the number. The blocks would be covered with a handkerchief, and then four more blocks would be added. The student would be asked to tell how many blocks there were altogether, and the child's method of determining the solution would be noted. Scenetra, possessing an operative counting scheme, simply counted on the remaining four to achieve a total of eleven. She knew that by counting to seven by imagining the blocks she would reach seven again, therefore she needed only to start at eight. James, on the other hand, possessing a formative counting scheme, had to reconstruct the original seven by starting over at one and counting up to eleven. Scenetra had, in fact, created an algorithm for counting which related the words for the numbers to the quantity itself. James had not. Steffe states: "child generated algorithms should be viewed as comprising a substantive part of the child's arithmetical knowledge. They should be nurtured and allowed to grow into increasingly powerful and sophisticated schemes" (pg. 119). Children do create their own schemes as they build complex knowledge systems from intuitive knowledge. From this perspective, the student in the mathematics classroom can be seen as one who actively constructs knowledge through learning experiences.

In summary, there appears to be a good deal of theoretical and research evidence to support the claim that as a consequence of being mediators in their own learning, students must perceive and interpret stimuli from teachers, draw relationships between present knowledge and intuitive knowledge or past learning, and build meaningful representations of concepts from varied contexts. Children are active constructors of knowledge.

TEACHING EXPERIMENTS

The studies summarized and reviewed below are all teaching experiments in that they attempt to investigate the changes evoked in students' learning which are a consequence of teaching methods. The studies can be divided into three groups: (a) those that deal only with analyzing or investigating representations of students' cognitive structures, (b) those that attempt to analyze or investigate the differential affects of alternate teaching approaches, and (c) those that attempt to describe the cognitive tools that empower knowledge construction.

Investigations of Cognitive Structures

Fensham, Garrard and West (1981) provide an overview of studies completed prior to 1981. They state that traditional cognitive mapping teaching experiments have involved three stages. The first stage is identified by data gathering. Historically many different forms for data collection have been employed including: word association, sentence writing, defining, card sorting, essay writing, writing descriptions, interviewing, and solution articulating. The purpose of this stage is to gain data from learners regarding concept associations. The second stage is characterized by data analysis. The authors state: "Here the researchers organize these data into a variety of structural forms using various procedures that include coding, qualitative categorizing, quantitative scoring, and dimensional scaling" (pg 122). The final stage is depicted by data feedback. In this stage the researcher will report student concept organization to the cooperating teacher for purposes of remediation and planning for instruction. The authors note that the third stage is often omitted.

Fensham et al. (1981) describe a teaching experiment conducted with grade eleven chemistry students. Although few details of their study are reported, the authors identify a list of seven key instructional concepts for each week of the three week study. They asked the students to identify the degree of relationship between pairs of the instructional concepts. The students were also asked to list how each concept was associated to classroom activities. The information was represented to the students by means of a two dimensional cognitive map. The students were asked to write descriptive words on linkage lines which showed where associations had been identified. The authors present this experiment as an example of how cognitive mapping may be used as an effective instructional process in traditional classrooms.

Shavelson (1972) investigated changes in students' cognitive structures through a study in which 40 high school students (28 experimental subjects and 12 control subjects) accepted instruction in select topics in physics. The key question of the study was "To what extent does the structure in the student's memory after learning, correspond to the structure in the instructional material" (pg. 225). The study was conducted under a pre and post test design in which only the experimental group subjects received instruction. Analysis of students' cognitive structures was completed through a word association task and an achievement test. Analysis of the programmed materials was completed through digraph analysis to chart structure and relationships between key concepts and words. The researcher found that: (a) achievement of the experimental group increased significantly, (b) cognitive structures of the experimental group were significantly affected by instruction and these structures changed to resemble more closely the structures inherent within the instructional materials, and (c) subjects in the experimental group reported significantly more associations between key concepts after instruction.

Geeslin and Shavelson (1975) repeated the Shavelson (1972) study but used grade eight mathematics students as their sample. As in the earlier study, their intent was to compare students' cognitive structures with content structure after instruction. The authors summarize their study:

> The study investigated learning of mathematical structure. Eighth grade students (N=87) were assigned randomly to read either a programmed text on probability (experimental group) or one on prime numbers (control group). The subject matter structure of the probability text was mapped with the method of directed graphs. Structure in students' memories, cognitive structures, was

> investigated using a word association technique. Cognitive structure and achievement data were gathered at pretest, posttest, and retention test. The directed graphs provided an interpretable map of subject matter structure (pg. 21).

The results of the Geeslin and Shavelson study replicated those of Shavelson's earlier study in that (a) students in the experimental group learned to solve significantly more problems than those in the control group, and (b) cognitive structure changed as a consequence of instruction. The cognitive structure in students' memories changed to more closely approximate the content structure.

Hewson and Hewson (1981) investigated the role student prior knowledge played in the development of scientific conceptualizations. Their study was conducted using a pre and posttest design with a sample of 90 grade nine students. The sample was divided in half to form a treatment and a control group as described by Hewson (1981):

> The research involved the development of experimental and control materials used with two similar student groups ... A test designed to assess whether students possessed scientific or alternative concepts for mass, volume and density served as both pre- and post-test. The alternative concepts were those previously identified in an equivalent group of students (pg. 3).

Instruction in the treatment group entailed: extrapolation, instantiation, and elaboration to integrate new concepts with existing concepts; introduction of conceptual conflicts to encourage exchange of existing concepts with new concepts; and, provision of examples to clarify concepts. Instruction in the control group was described as "traditional." Both groups were taught the same four instructional units in four class hours. The researchers found that: (a) some concepts held a strong resistance to change, and (b) where students' prior knowledge was considered in instruction, these students achieved greater gains in scientific conceptions. Hewson and Hewson explain this finding by claiming that for conceptual change to occur, a new concept must be seen as intelligible, plausible and fruitful. They argue that attention to students' prior knowledge provides concepts with a sense of plausibility thus increasing the potential for conceptual change to occur.

Champagne, Gunstone and Klopfer (1983) conducted two studies to investigate conceptual changes during instruction in mechanics. The first study was conducted in Pittsburgh, while the second study was conducted in Victoria, Australia. The Pittsburgh sample consisted of 23 academically gifted middle school students (thirteen students in the experimental group and ten students in the control group). The study employed a pre and post test design in which changes in students' cognitive structures were measured. Instruction in the experimental group was delivered via class discussions in which students were given opportunity to argue their perceptions of specific scientific problems and events both before and after demonstrations. Instruction was delivered one day per week over the course of eight weeks.

Champagne et al. (1983) employed five research measures including a free sort task, a tree construction task, and a word association task. The subjects were also asked to complete a conSAT task and a real event task. The conSAT task consisted of a taped interview in which subjects were given a series of words placed on index cards. The subjects were asked to sort the cards and explain the formed groups to the interviewer. In the real event task a series of events was described, and the subject was asked to predict the outcome. The event was then enacted, and the subject was asked to describe the finding and any differences between his or her prediction and the outcome. The real event task was also tape recorded.

Champagne et al. (1983) reported that the cognitive structures of the experimental and control groups were similar both prior to instruction and after instruction, and that little change in cognitive structure had occurred in either group. The researchers were forced to conclude that these cognitive structures were highly resistant to change. More promising findings were reported in the Victorian phase of the study.

The students employed in the Victorian sample included six undergraduate science majors enrolled in a teacher training program. These students volunteered to participate in the study based upon a feeling that their knowledge of mechanics was limited. These students were given five full days of instruction, and completed the same five pre and post test tasks as did the Pittsburgh sample. The Victorian group was given the added task of reflecting upon their learning experiences in writing by keeping a journal. Champagne et al. (1983) state that the greatest evidence to support the claim of conceptual change came through the real event task where after instruction students gave much more precise descriptions and employed a greater number of physical principles to support predictions. The researchers reported a 29% increase in word associations as well as an increase in accuracy on this task.

Champagne et al. (1983) caution that three major differences exist between the Pittsburgh and Victorian sample groups: (a) age, (b) motivation levels - the Victoria group felt their knowledge in mechanics was limited and that this knowledge was necessary for their future teaching careers, and (c) prior knowledge level - the Victoria group had a higher level of general science knowledge prior to instruction. The authors conclude that

> ... we have then changed cognitive structures in a content area for which we have found this process to be very difficult. However we have achieved this only with mature and motivated students with a reasonable store of existing propositional knowledge, and not for all

students (pg. 24).

Although the researchers recognized the limitations of their findings, they claimed success based upon the fact that their modest results were obtained over the short period of five days, optimistically implying that much more could be done over a longer time span.

Investigations of the Differential Effects of Alternate Teaching Forms

Dunn (1983) identified six different instructional approaches (see Figure 2) and used these approaches to teach a contrived concept to 230 university chemistry students who were divided into six groups. The contrived concept was a 'mib' and was defined as "a right triangle with an external segment or line perpendicular to the center of the shortest side" (pg. 648). Two tests were used to evaluate the students' understanding of the contrived concept: a 20 item multiple choice test, and the construction of a written definition of a mib. Dunn found that the students taught through the prototype and combination instructional approaches were: less distracted by nonrelevant attributes, more likely to include more critical attributes of a mib in their definitions, and more consistent in applying their definitions to correctly identify mibs. Dunn also found that students taught through the discovery approach had the lowest performance level. The author concludes that the most effective models of instruction are the prototype and combination instructional approaches.

Stiff (1989) investigated the effects of relevant knowledge, teaching strategy, and strategy length on the learning of a contrived concept. He defines relevant knowledge as "known information about subordinate concepts and relationships that should be useful to the learner" (pg. 228). The teaching strategies

Figure 2: Dunn models of instruction.

Method	Treatment Components
Discovery	Identifying mibs through Trial and Error + Feedback on Correctness
Expository (A)	Definition + Pictorial Example + Drawing Task
Expository (B)	Definition + Selection Task
Prototype Development	Definition + Explanation of Examples and Non-Examples
Interrogatory	Definition + Questions related to Illustration
Combination	Combination of Prototype Development and Interrogatory without Selection Task

were defined as a series of E (exemplification) and C (characterization) moves. An E-move entails the giving of examples and non-examples. A C-move entails the giving of definitions and analogies. An example teaching strategy is an ECE strategy which would be comprised of a series of E-moves followed by a series of C-moves followed by a second series of E-moves. Stiff contends that high levels of relevant knowledge are necessary for C-moves, but not so for E-moves.

Stiff (1989) used a 'mat' as the contrived concept. A mat was defined as an ordered pair of positive integers where the sum of the coordinates is even, and at least one coordinate is divisible by three. An example mat is (6,8). The researcher also defined a del, tag and terse: "A mat is a del whose tags are terse. A tag is the first or second coordinate of an ordered pair, a del is an ordered pair of positive integers in which a coordinate is divisible by three, and tags are terse if they have the same parity" (pg. 230). The mat, del, tag and terse represent the concepts which were taught during the instructional component of the project.

Stiff's (1989) project involved 326 senior high students enrolled in advanced mathematics courses. These students were divided into 18 treatment groups based upon two different teaching strategies, three levels of relevant knowledge, and three degrees of strategy length. Instruction occurred through programmed materials over a single class period. Three different criterion tests were administered immediately after instruction to evaluate student achievement: true and false questions to identify mats (Test A), questions where examples of mats had to be provided by the subjects (Test B), and questions where subjects identified paraphrased definitions of mats (Test C). Stiff found that: (a) students with medium and high relevant knowledge performed significantly better on Tests A and C; (b) students with high relevant knowledge performed better on Test B than did students with medium relevant knowledge who in turn performed better than students with low relevant knowledge; (c) C strategy instruction produced higher means on Tests A and B, while E strategy instruction produced higher means on Test C; and (d) there was no significant difference among groups on the strategy length. Stiff concludes that an increase in relevant knowledge leads to increased learning, and teachers should select the best teaching strategy depending upon the relevant knowledge levels of the students.

Mayer and Greeno (1972) investigated the learning outcomes produced from two different instructional approaches in the learning of binomial distributions. Their research was motivated by the idea that "different instructional procedures may result in learning outcomes that are qualitatively or structurally different" (pg. 165). The aspect of the instructional approach varied was the sequencing of activities. One group of students was taught the elements of the general binomial distribution formula prior to the joining of these elements into a cohesive formula, while the second group was introduced to the general formula first followed by a detailed discussion of each of the elements. The researchers completed three studies, but only two provided useful results. In the first study 20 paid female volunteers were divided into two groups, Group G and Group F. The instruction for the two groups differed in that "the material given Group G included more discussion of concepts, while the booklet for Group F had the character of a set of instructions and could be likened to a computer program for finding a numerical answer" (pg. 167). The performance of the two groups was measured on four different types of problems. Type F (familiar) problems were identical to those used in instruction, while Type T (transformed) problems were identical in deep structure but involved a novel context. Type U (unanswerable) problems could not be solved, while Type Q (question) problems were those which involved the discussion of general principles or properties. Mayer and Greeno found that Group F performed significantly better on Type F and Type T problems, while Group G performed significantly better on Type U and Type Q problems.

The second study completed by Mayer and Greeno (1972) was intended to extend the results found in the first study. In the second study the researchers included gender as a variable (32 female and 32 male subjects), and introduced two new instructional approaches. Instruction occurred through programmed materials. Along with Group G and Group F, there was a Group G/F and a Group F/G. Group G/F worked through both booklets completing the Group G book first followed by the Group F book. Group F/G also worked through both booklets but in reverse order. The researchers found that there was a significant interaction between performance on the four question types and the instructional model, but also found that Group G females did not perform as well as females in the other groups.

One major contribution of the study completed by Mayer and Greeno (1972) was that different instructional approaches can produce qualitatively different learning outcomes. They summarize their results:

> One explanation of the difference that seems straightforward is that subjects in the different instructional treatments encoded the information presented about the binomial formula in different ways. One reasonable hypothesis is that the booklets emphasizing general concepts tended to activate structures in the subjects' previous knowledge involving concepts familiar to them in general experience, while the booklets emphasizing the formula tended to activate structures involving the ideas and techniques associated with arithmetic and mathematical calculations ... For subjects who received the formula emphasis, the new ideas would be assimilated to schemas involving calculational techniques, while for subjects receiving emphasis on general concepts, the new material would be assimilated to ideas of a more general kind, involving the subjects' experience with random events (pg. 171).

A second major contribution of this study is evidence of two different forms of learning outcomes: a performance ability, which is apparently easily addressed and investigated under many instructional approaches; and a recognition of relationships, which enables the learner to employ and recognize the underlying properties and relationships of given concepts in a general way.

Mayer (1977) completed a series of three studies to investigate the claim that different instructional methods will result in different learning outcomes. In each study Mayer taught the subjects to count through the first nineteen numbers in base three. In the first study Mayer divided 24 university students into two groups. The first group (Group Letter) was taught base three where each digit 0, 1, and 2 had been replaced by a single letter (i.e., 0=w, 1=d, and 2=r). The second group (Group Number) was taught base three using the digits 0, 1, and 2. The achievement variables Mayer investigated included speed of learning as well as the number of errors made while learning and the ability to generalize to addition and subtraction problems. Group Letter found the learning process to be more time consuming, made more mistakes while learning, and was not as successful with the transfer problems. It was found that Group Letter tried to learn by rote memory while Group Number generalized from a knowledge of base ten.

The second study completed by Mayer (1977) also involved instruction in base three. In this study 24 university students were divided into two equal groups, but both groups were taught base three in the letter form (i.e., where letters were substituted for the numerals). The first group (Group Before) was given a conversion table prior to instruction, while the second group (Group After) was not given the table until after instruction. Mayer found that prior exposure to the table made Group Before more efficient learners, and enabled them to relate their learning to previously held constructs.

The third study completed by Mayer (1977) essentially attempted to replicate the findings of the second study except further limitations were introduced: the sample was entirely comprised of females, a shorter counting sequence was employed, and a time limit was imposed on responses. Mayer found that the group given the conversion table after instruction made significantly more errors in the learning process, while the group given the conversion table before instruction performed significantly better in the transfer and counting tasks.

Mayer's (1977) three studies may be summarized to a few general conclusions. First, Mayer has shown that different forms of instruction produce qualitatively different learning outcomes indicated by subjects' differential abilities to perform on transfer tasks. Second, Mayer provides evidence of two different learning sets. The rote learning set "involves stimulus learning, response production, and formation of associations among stimuli" (pg. 544). The meaningful learning set "involves relating the presented materials to an integrated set of existing knowledge" (pg. 544). Finally, Mayer has shown that the context in which the learning occurs establishes the rote or meaningful learning set. He concludes: "...instructional objectives should be sensitive not only to what behavior is learned but also to how it is learned and structured in memory" (pg. 545).

Investigations of the Cognitive Tools of Learning

Peterson has provided a description of the knowledge forms for classroom learning and the knowledge forms for classroom teaching and how they interact within the classroom as student's thinking and cognitions mediate between teacher behavior and student achievement:

> Cognitive science researchers have shown that children develop informal systems of mathematics outside of the classroom, and they do not simply absorb what they are taught. They structure and interpret the presented mathematics curriculum and instruction in light of their existing knowledge (pg. 11).

Peterson identified eight different forms of knowledge (summarized in Figure 3). In a general discussion of the forms of knowledge students employ in their learning, Peterson states:

> To learn effectively in a classroom, a student needs to have both general knowledge of strategies for learning and acquiring information during classroom instruction, and content-specific knowledge of strategies that enable him or her to learn the specific subject matter content. More sophisticated and effective learners may have an additional level of knowledge which consists of a self-awareness of both the general and content-specific cognitive processes and strategies for learning and acquiring information in a classroom (pg. 7).

Peterson describes four forms of knowledge for classroom learning. General cognitional knowledge is evident when the student possesses such processes as

Figure 3: Peterson model of cognitional and meta-cognitional knowledge.

	Cognitional Knowledge		Metacognitional Knowledge	
For classroom learning	General	Content Specific	General	Content Specific
For classroom teaching	General	Content Specific	General	Content Specific

summarizing, organizing, and relating. Content-specific cognitional knowledge would entail the prerequisite principles and mathematical concepts necessary for learning a given concept. General meta-cognitional knowledge is evident in the student's ability to selectively call upon and use the cognitional knowledge described above. Students who employ this form of knowledge will (publicly or privately) consciously make such statements as: "it is time to look back and summarize what I have done so far." Content-specific metacognitional knowledge is employed when students selectively call upon and use the mathematical principles related to immediate content. An example is provided by the student who consciously decides that the correct operation to use in a given problem is multiplication rather than addition.

In a study reported by Peterson (1988), involving 30 classes of grade four students, half of the teachers were given inservice training on teaching specific cognitional knowledge to their students. This knowledge included: defining and describing, comparing, thinking of reasons, and summarizing. The other half of the teachers were given workshops on improving engage time and academic learning time. It was found that the treatment group showed "significant ability-by-treatment interactions for udents' high level mathematics achievement, conceptual mathematics achievement, and for achievement on story problems" (pg. 9). The lower ability students benefitted greatly from the instruction, while the higher ability students benefitted only marginally. The study implies that general cognitional knowledge and general metacognitional knowledge are valuable assets in learning, and that these knowledge forms may distinguish between low and high ability students. Direct instruction in these skills may provide a remedial effect for certain learning skills.

In contrast to Peterson's list of cognitional knowledge forms developed and listed through research conducted in classrooms, Sierpinska (1990) provides a generalized theoretical analysis of the notion of understanding. It is important to note that Sierpinska's notion of understanding is consistent with the notion of meaningful learning as found in constructivist literature. Sierpinska argues that understanding may be conceptualized both as a process and as a set of acts. Through a theoretical analysis of the works of such scholars as Skemp, Dewey, Lakatos, Ricour, Herscovics and Bergeron, Locke, and Hoyles she provides a categorization of the acts of understanding. Sierpinska asserts that there are four acts of understanding: identification, discrimination, generalization, and synthesis. Identification has been defined as the "identification of objects that belong to the denotation of the concept (related to the concept in question), or: identification of a term as having a scientific status" (pg. 39). Discrimination involves the separation of two concepts which the individual had previously confused. Generalization is described as "becoming aware of the non-essentiality of some assumption or of the possibility to extend the range of application" (pg. 39). Finally, synthesis is defined as the recognition of "relations between two or more properties, facts, objects and arranging them into a consistent whole" (pg. 39). Sierpinska contends that these four acts both constitute understanding, and serve as the mechanism by which knowledge is constructed.

Confrey (1981, 1982 and Confrey and Lanier, 1980) has adopted a list of problem solving behaviors from the works of Krutetskii (1976) and Erlwanger (1974), and has argued that these problem solving behaviors relate to students' abilities to understand mathematics (Confrey and Lanier, 1980). The researchers conducted a study which was designed to investigate "the abilities and strategies of general mathematics students and their resulting conceptions of mathematics" (pg. 549). They conducted a series of clinical interviews with grade nine students in which the students' responses to the given problems were analyzed according to the categories established by Krutetskii: information gathering (the collection of data germane to a particular problem), generalization (the ability to transfer from one context to another or recognize the similarity between given contexts), reversibility (the ability to mentally reverse operations - e.g., know that the difference between 50 and 34 is 16 given that the sum of 34 and 16 is 50), curtailment (the ability to regenerate skills and knowledge prerequisite to a given concept on demand), and flexibility (the ability to switch between problem solving strategies when necessary and not be limited by an initial mind set). Several implications for the teaching of general mathematics classes were derived, including: (a) investigation of these cognitive strategies will provide for more informed teaching, i.e., teachers will become aware of the thought processes these students employ in learning mathematics, and this should enhance the learning opportunities of these students; (b) teachers will be able to reconsider the assumptions they make about the learning capabilities of their students; and (c) the teacher may address the specific abilities these students lack.

There exists a remarkable similarity between the forms of cognitional knowledge described by Peterson (1988) and the acts of understanding as described by Sierpinska (1990). Peterson's notion of defining and describing corresponds with Sierpinska's notion of identification. Similarly, comparing corresponds with discrimination and summarizing corresponds with generalization or synthesis depending upon the nature of the summary. Peterson's fourth form of cognitional knowledge, thinking of reasons, is less specific than the first three, and may contain elements of identification, discrimination, synthesis or generalization. Because of the similarity between Peterson's cognitional knowledge forms and Sierpinska's acts of understanding only one needs to be pursued, and due to the more robust descriptions provided by Sierpinska, the terms identification, discrimination, synthesis and generalization for forms of cognitional knowledge are adopted in this study.

Confrey and Lanier's (1980) four problem solving behaviors do not correspond well with Sierpinska's acts of understanding. Generally, Sierpinska's acts might be described as a learner's attempt to classify and cluster concepts, whereas Confrey and Lanier's behaviors might be described as the learner's manipulations of these concepts (via mental processes). Upon a close inspection, the reader will find that some relationships do exist between Sierpinska's acts and Confrey's problem solving behaviors. For example, flexibility (the ability to switch from one strategy to another within a given problem solving context) must surely involve aspects of identification (e.g., recognition of a problem as being of a certain type thus appropriately solved via a given strategy) and discrimination (e.g., recognition of the difference between two possible strategies one being more or less viable than another). Collectively, Peterson (1988), Sierpinska (1990), and Confrey and Lanier (1980) have supplied a list of the cognitive tools associated with the learning of mathematics.

Summary

Thus far research has been able to show that students do act as mediators in their own learning experiences, and furthermore are active participants and interpreters of their environments. We have also noted that the knowledge and attitudes with which students enter into learning sequences have an influence on that which is learned. We also know that teachers constitute an integral part of the learning context. Perhaps most importantly, research has been able to show that learning is a dynamic activity, that cognitive structures are constantly undergoing change (Noddings, 1990) as new concepts are added to existing structures or as new concepts replace those determined faulty. Finally, we have seen that there exists a "toolkit" (Davis, Maher & Noddings, 1990) of cognitional knowledge that monitors and facilitates the construction of knowledge.

The research evidence before us necessitates the conclusion that the constructivist perspective is a viable perspective from which to interpret classroom learning events. We know that learner's cognitive constructions can be affected by instruction, and that these constructions can be differentially affected by different forms of teaching. However, we have not yet begun to describe the nature of that which is constructed during these learning sequences.

A CONSTRUCTIVIST VIEW OF LEARNING

The present study represents an attempt to employ a constructivist model of learning as a tool to investigate three different instructional models. In order to complete this task it is necessary to derive such a model of learning. The model described below represents a concatenation of current constructivist literature and current ideas in the field of cognitive psychology.

Cognitive psychologists have been able to show that the learner is an active participant in the learning process, that is, he or she is not a passive recipient of knowledge (Davis, Maher & Noddings, 1990). We know that students act as interpreters of language and action (Cobb, 1988), and we know that students construct their own knowledge regardless of the teacher's attempts to facilitate or thwart these constructions (Winne and Marx, 1982; Davis, 1983). Significantly less is known about the actual processes involved in knowledge construction (Cobb, 1988), but many attempts have been made to describe and classify the many products of construction.

Skemp (1971, 1987) and Frederiksen (1984) have both provided descriptions of the processes involved in cognitive constructions. Skemp has attempted to describe meaningful learning (where relationships are formed through the association of common objects) in a way that would inform instructional processes. Frederiksen employs a more technical approach consistent with psychological research.

Skemp (1971, 1987) describes learning through the processes of abstracting and classifying. The process of abstraction requires identifying the major properties of an object or class of objects, while the process of classifying entails placing or associating an object within its class. Thus a concept is interpreted as the awareness of the relationship between an object and its class. Primary concepts are those which are derived from our sensory and motor experiences of the world, while secondary concepts are those that have been built up from the primary concepts. Skemp (1971) states:

> A concept is a way of processing data which enables the user to bring past experience usefully to bear on the present situation. Without language each individual has to form his own concepts direct from the environment. Without language, these primary concepts cannot be brought together to form concepts of higher order. By language, however, the first process can be speeded up, and the second is made possible (pg. 28).

Language plays an important role in this model, for it is through language that we devise symbols and thus classify objects. This process is completed by 'delta 1' (see Figure 4) which is the mechanism of cognitive activity. Delta 1 is monitored and controlled by 'delta 2' which might be considered the mechanism of metacognition. The networks that are built up from the primary concepts are called schemas, and understanding involves relating information to an appropriate schema.

According to Skemp (1971, 1987) many variables may inhibit the construction process. Students who possess a poor attitude are unwilling to participate in this process. This poor attitude may be a product of boring rote learning (memorization). Rote learning does not provide the student with the necessary precursor concepts to construct higher order understanding. In summary, Skemp's model of the intellect is one in which perceptions are drawn from the environment and used to build concepts through the processes of abstraction and classification. These concepts are used to build up schemas.

Frederiksen (1984) describes three kinds of memory: sensory buffer, short term, and long term (see Figure 5). The sensory buffer retains information only for

Figure 4: Skemp model of cognitive activity.

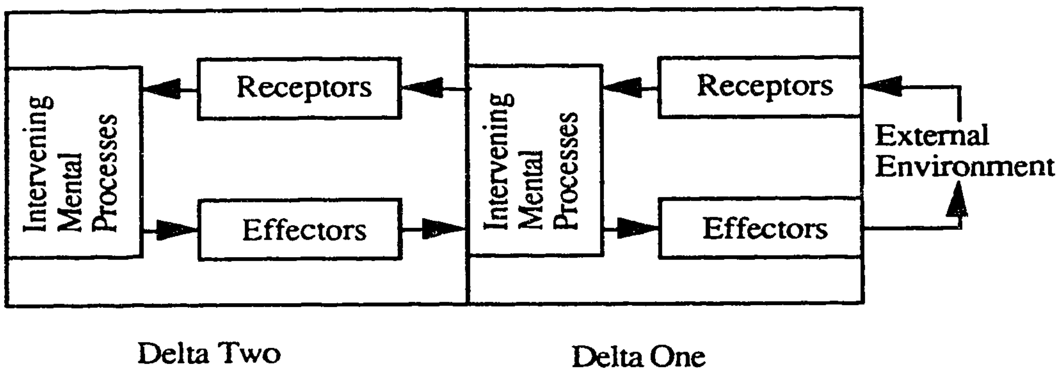
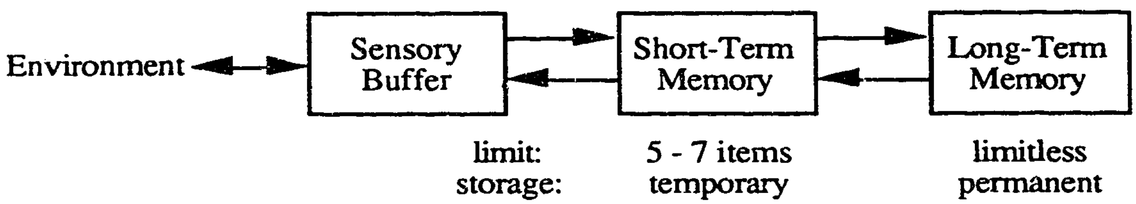


Figure 5: Frederiksen model of cognitive information processing.



a very short period of time, just long enough for it to be classified, coded, stored, or ignored by the short term memory. Long term memory stores information virtually permanently. During instances of learning, information in this memory interacts with new knowledge.

Information is stored within the long term memory as nodes and the interconnections or relationships between these nodes. When knowledge is retrieved from the long term memory it is recalled in the form of chunks, or groups of nodes which have been stored together by virtue of their interconnectedness. The short term memory (STM) contains the information that is currently being used. It is also responsible for monitoring the flow of information from the senses to the long term memory and vice versa to allow processing. In the average adult the STM may only contain five to seven items at one time.

Frederiksen (1984) identifies two forms of information processing: controlled and automatic. Controlled processing requires the attention of the individual, and this rapidly overwhelms the short term memory, thus limiting the number of operations which can be carried out. Automatic processing does not require the attention of the individual, enabling many more operations to be carried out simultaneously. Those operations which require control can become automatic through practice. Repetitive practice allows the learner to become proficient at a single operation, while varied practice enables the learner to transfer skills. In this model, the learner is seen as one who receives information from the environment, links the information in stored networks, and then recalls information in chunks when required.

Smock (1976) has drawn from Piaget's work in an attempt to describe this linking of chunks as being achieved through the processes of accommodation and assimilation. In accommodation past cognitive constructions must be adapted, deconstructed, or replaced to permit the linkage of new learnings. Accommodation occurs when new learnings are found to be incompatible with (i.e., they contradict) past learnings. Assimilation is employed when new learnings are compatible with past learnings, but the new learnings are of a more complex level. In this case the new cognitive constructions are simply added to past cognitive constructions to create more complex cognitive networks.

These conceptualizations of the processes involved in the construction of knowledge have much in common. First, cognitive networks are formed through the recognition of similarities and differences between objects and through the recognition of the representations of these objects in the environment. Second, both researchers present their theories so as to imply that these cognitive processes which enable the construction of cognitive schemas and networks are innate. That is, in the works of Skemp (1971, 1987), Frederiksen (1984) and Smock (1976) it is assumed that these acts of knowledge construction are natural and unschooled acts in which all learners engage.

Cognitive psychology has not addressed the linkage between the construction of cognitive networks, and the actual creation of knowledge. That is, they have not articulated how the processes of accommodation, assimilation, abstraction and classification create particular knowledge forms. However, many authors have attempted to provide descriptions of the knowledge forms which are constructed (see for example Kieren, 1988 and Lienhart, 1988).

What does it mean when one claims to understand mathematics or to understand a particular mathematical concept? Or more directly, when one understands a mathematical concept, what knowledge forms have been constructed? According to Romberg and Carpenter (1986),

> Understanding involves fitting information into the learner's existing cognitive framework. This means taking into account the knowledge of the mathematics under consideration that the learner brings to the situation, connecting semantic knowledge and procedural skills, and encouraging integration of related concepts (pg. 859).

In their discussion, Romberg and Carpenter recognize two components of understanding: semantic knowledge and procedural skills. These components correspond closely to Shavelson's (1981) notion of propositional and procedural structures:

> The propositional structure of a subject matter refers to the meaning

> of mathematical concepts and operations. More accurately, it refers to the verbal and visual representation of meaning ... Procedural structure refers to a set of rules and heuristics that specify, at least partially, the step-by-step procedures leading from the specification of a particular task to a goal state (pg. 25-27).

Therefore, when we refer to understanding (and thus the development of understanding or the learning of a new concept), we recognize that it contains the organization and sequencing of individual concepts along with the structures relating the concepts and the processes and procedures to which they may be applied. These two forms of constructed knowledge have been recognized as early as 1977 by Mayer who employed two different measures of learning in his teaching experiments: the development of performance criteria and associations between concepts.

More recent authors have begun to argue for a form of cognitional knowledge which facilitates the meaningful learning of mathematics (and thus the construction of propositional and procedural knowledge). According to Noddings (1990; Davis, Maher & Noddings, 1990):

> It is assumed that learners have to construct their own knowledge - individually and collectively. Each learner has a tool kit of conceptions and skills with which he or she must construct knowledge to solve problems presented by the environment (pg. 3).

Noddings does not articulate the nature of this tool kit of conceptions and skills, but other authors have provided more complete descriptions. Peterson (1988, whose research confirmed that these conceptions and skills are teachable and are related to students' achievement levels) lists the following: defining and describing, comparing, thinking of reasons, and summarizing. Confrey (1981, 1982) and Confrey and Lanier (1980) have also argued that there exists a set of cognitive skills which discriminate between capable and less capable learners. Their list includes: information gathering, reversibility, generalization, curtailment, and flexibility.

And finally, Sierpinska (1990) in her attempts to describe understanding argues for the following acts: identification, discrimination, generalization, and synthesis. Peterson is the only author who specifically names these conceptual processes, and thus her term (cognitional knowledge) will be adopted in this study along with the forms of cognitional knowledge as described by Sierpinska and Confrey and Lanier. Beyond these forms of cognitional knowledge Peterson has presented a description of metacognitional knowledge which is the ability to control and engage in cognitional knowledge.

Both Confrey (1980, 1981) and Peterson (1988) set about to deliberately address cognitional knowledge through instruction in their studies. Thus we know only that cognitional knowledge is not innate (as evidenced by the fact that not all learners possess it), it is teachable, and it is related to learners' successes. We do not know whether cognitional knowledge is constructed only through instruction explicitly intended to address it (i.e., the teacher sets out to address cognitive skills in identification, synthesis, flexibility and the like), or whether cognitional knowledge develops concurrently with instruction centred around mathematical topics (i.e., if cognitional knowledge develops concurrently with instruction in percents, fractions, regrouping or other mathematical topics).

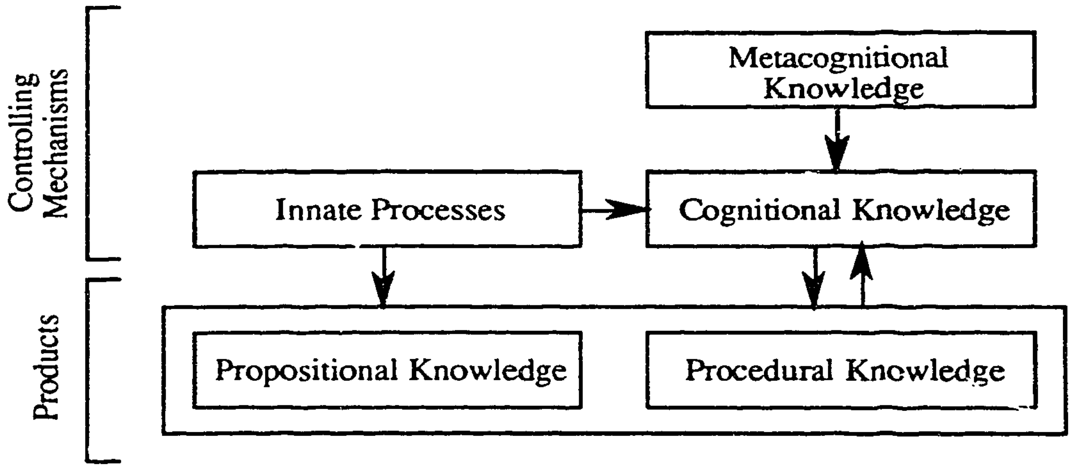
It is reasonable to ask what relationship exists between cognitional knowledge and the cognitive behaviors as described by Skemp (1971, 1987) and Smock (1976), as it remains a possibility that they are simply the same thing. Peterson (1988) and Confrey (1981, 1982) specifically argue that cognitional knowledge discerns between effective learners and less effective learners, and that cognitional knowledge can be developed through instruction. That cognitional knowledge can be taught was confirmed by both researchers in their work with schoolchildren. No claim has been made that abstraction, classification, accommodation, or assimilation (as they have been defined by Skemp and Smock) are teachable, or that the ability to engage in these differentiates the capability of learners. Abstraction and classification are descriptions of processes in which learners manipulate sensory and perceptual data. Accommodation and assimilation are descriptions of mental processes through which learners reorganize conceptual networks in long term memory.

A Constructivist Model of Learning

In the preceding literature we have argued for four basic elements of a constructivist model of learning, including: (a) the act of constructing knowledge involves the innate mental behaviors of abstraction and classification (of perceptual and sensory data taken from the environment), and assimilation and accommodation (occurring between existing conceptual structures and new concepts). By innate we mean that all learners engage in these processes, (b) the knowledge forms constructed include propositional knowledge, procedural knowledge and cognitional knowledge, and that these cannot be considered innate as teaching experiments have shown that different individuals construct these knowledge forms differently, (c) cognitional knowledge acts as a facilitating and controlling mechanism whereby propositional and procedural knowledge are constructed, and (d) metacognitional knowledge is an awareness of cognitional knowledge and enables the conscious engagement of cognitional knowledge in a learning situation or problem solving task. These elements are organized into the diagram shown in Figure 6.

In this study, this learning model will be used to investigate the Direct,

Figure 6: A constructivist model of learning.



Meaning, and Problem Process Teaching approaches. Furthermore, the learning model itself can be investigated by studying relationships between propositional knowledge and procedural knowledge, as well as relationships between cognitional knowledge and propositional and procedural knowledge.

CHAPTER THREE

Research Methodology

In this study we investigate the Direct, Meaning, and Problem Process Teaching approaches through a constructivist model of learning. This model of learning (presented in the preceding chapter) is essentially comprised of three components: propositional knowledge, procedural knowledge, and cognitional knowledge. The learning model is built upon the argument that propositional and procedural knowledge are constructed during learning events, and that cognitional knowledge facilitates and enables such construction. In this study we investigate the change in students' propositional and procedural knowledge using the teaching approach under which the percent unit is delivered as the basis of comparison. It is necessary therefore to develop and present techniques of measuring the propositional and procedural constructions expressed by the students under the three teaching models. We present here the structured tree recall task as a measure of students' propositional constructions, and the diagnostic-performance test as a measure of procedural constructions.

In applying this constructivist model of learning to the investigation of teaching approaches, much may be learned about the viability and usefulness of the learning model itself. It has been argued that cognitional knowledge facilitates and monitors the construction of propositional and procedural knowledge. If this assertion is valid, we would expect that students who have shown gains in both propositional and procedural knowledge would also possess higher levels of cognitional knowledge. In order to investigate the relationships between cognitional knowledge and the other knowledge forms, a series of problem solving tasks were administered. We present here the clinical interview as a means whereby students' cognitional knowledge may be investigated.

THE STRUCTURED TREE RECALL TASK

This research methodology is founded upon a theory of organization of cognitive structures. The purpose of the structured tree recall technique within the present study was to provide an image of students' cognitive structures as measured by associations students draw between major concepts taught within a given curriculum unit. According to Naveh-Benjamin et al. (1986), the structured tree recall task

> ... is based on a theory of mental organization which assumes that single concepts or sets of concepts are mentally organized into a hierarchy whose lowest level terminal nodes represent the single concepts and non-terminal nodes represent a mental code that stands for its constituents. The technique capitalizes on the fact that people have a tendency to recall all items of one chunk of information before moving on to the next chunk. Chunks, then, are inferred by inspecting all trials for groups of items that appear together from a set of cued and uncued trials, an algorithm efficiently finds the set of all chunks for each subject and represents this set as an "ordered tree." The obtained ordered tree may be considered to be a representation of a subject's knowledge structure (pg. 131).

The purpose of this methodology is to provide an image of a subject's organization of concepts based upon the assumption that individuals recall information in related units called chunks. This image constitutes a measure of students' propositional knowledge in that (by definition) chunks are collections of concepts that are linked and stored together in long term memory due to the relationships a subject perceives between them. For example, it is reasonable to expect that the general concepts 'fraction' and 'decimal' may be found together, linked in a chunk, due to their common ability to represent 'a part of a whole.' The linking of concepts together based upon similarity or common attributes is the construction of propositional knowledge, therefore a test measure which attempts to provide a model or image of this linking effectively also provides an image of a subjects' propositional knowledge. The structured tree recall task is such a test measure.

The structured tree recall technique involves the sorting, memorization, and recall of a list of given words. The number of words which were given to the subjects was determined by the pilot study and reports of related research. Reitman and Rueter (1980) completed two studies using this technique. In the first study subjects were given 24 words to memorize; in the second study, subjects were given 16 words. In an experiment completed by McKeithen et al. (1981), subjects were asked to memorize 21 words. Naveh-Benjamin et al. (1986) argues that such long lists of words creates a performance effect in that some subjects do not possess the necessary memory capabilities. In the pilot study students were given a list of only 10 words, and it was found that all students, regardless of general ability level could easily memorize the list. In the present study it was decided to use 16 words in order to increase the challenge for the highly talented students without making the task impossible for the less talented students. Increasing the number of words had the added effect of increasing all subjects' dependence upon the meaningful sorting or clustering of words.

To begin the structured tree recall task, subjects were asked to remove a set of 16 cards from the given envelopes (see Appendix A for copies of structured tree recall task materials). On each card was a different key word taken from the grade eight percent unit (distractor words are marked with an asterisk in the following list). The words were: *adjacent, cost, decimal, denominator, discount, fraction, hundredths, interest, markup, part, percent, ratio, sale price, sales tax, *square, *zero. The subjects were asked to spread the 16 cards out in front of them and sort the cards by placing together words that naturally go together. The phrase "go together" was heavily stressed in the instructions and repeated several times thus encouraging students to look for words that were somehow related. In order to encourage sorting of the words based upon associations, four different examples were provided to each class (adapted from Shavelson, 1974):

A HORSE is like a PONY.
A RAKE and a FLOWER may both be found in a GARDEN.
A POODLE is a type of DOG.
The AREA of a rectangle equals its LENGTH multiplied by its WIDTH.

Students were informed that if they had the words HORSE and PONY they may want to put them in a group together because the two words describe things that are 'alike.' Similarly RAKE, FLOWER, and GARDEN may make a good group because they describe objects that are often 'found together.' The students were told that they could sort the words any way they wished, and were told that they could have any number of groups of any size.

After sorting, the subjects were asked to memorize the words in the sorted groups. Ten minutes was allowed for this task. The subjects were told that they could not work together, and that a good way to memorize the words was to practice repeating them silently so as not to disturb the others sitting around them.

Finally, the students were asked to return the cards to the envelope, take out the recall sheets, and to record the memorized words in their sorted groups. This task was completed eight times on eight separate pages. After completing one page, the subject was to turn it over and not look back at it. Of the eight recall sheets, six represented cued trials while two represented uncued trials. In a cued trial, one of the 16 words was given and the subjects were to respond by giving the remainder of the group in which the word had been placed when the cards were sorted. After listing the entirety of that group, the students were to list the remaining groups until all 16 words had been recalled. The purpose of the cued trials was to encourage variety and force the recall of words in clusters. In an uncued trial the subject could begin with any word he or she wished. Uncued trials were necessary to enable the analysis of all structures including the root or terminal node.

The decision to employ eight trials was made through pilot testing and through analysis of the technique as described in the literature. In their first study, Reitman and Rueter (1980) used each of 24 words as a cue word thus asking subjects to repeat the recall task 24 times. In their second study they asked their subjects to recall the word list (uncued) once on each of ten successive days, thus illustrating a second way to reduce stereotypy. McKeithen et al. (1981) asked their subjects to repeat the recall task 25 times: each of 21 words was used once to cue one trial, with four interspersed uncued trials. Naveh-Benjamin et al. (1986) argued that little more could be learned by having the subjects repeat the task many times than could be learned by having the subjects repeat the task four times. Naveh-Benjamin et al. administered only two cued and two uncued trials. In the pilot study it was found that one ambiguous structure resulted even after five trials, thus Naveh-Benjamin's assertion was rejected and more repetitions were added. Reitman and Rueter used approximately half as many task repetitions as the number of words in the word list, thus eight cued repetitions was deemed ample for the present study.

Once data for all trials was collected, the word sequences were entered into a computer program (see Appendix B) to identify chunks according to the algorithm specified by Reitman and Rueter (1980). The computer program functioned by accepting the recalled words in a matrix in sequence. The program then searched for combinations of words which occurred together in any order in all trials. These clusters of words were output as chunks. This process is presented as a flowchart in Appendix B. Chunks were defined as groups of words which occurred together on all cued and uncued trials, except in cases where a member of that chunk occurred as the cue word. Once chunks were identified, structured trees were drawn and assigned a PRO (possible recall order) score according to the complexity of the tree. PRO scores could range from zero to 30.67, with zero denoting a highly structured tree. Similarity scores were used to compare students' structured trees determined prior to instruction with their structured trees found after instruction. Similarity scores range from 0 to 1 with 0 showing no correspondence and 1 showing perfect correspondence. The algorithms used to calculate PRO and Sim scores are defined and demonstrated below.

Naveh-Benjamin, Lin, McKeachie, and Tucker (1986) provided a succinct definition of the PRO score:

> Amount of organization was measured by the possible recall order (PRO), which is the natural logarithm of the number of different written orders that can be obtained by traversal of a given structure, or alternatively, of the number of written orders that contain its chunks. For example, if words were listed randomly each time, the number of possible recall orders would be great; on the other hand, a subject's structure that listed all of the concepts in the same order on every trial could be created by only one possible order...In general, the smaller the PRO, the more organization in the structure (pg. 133).

The similarity score was described as a measure of the similarity between two structures, and is defined as "the natural logarithm of the total number of chunks the two trees have in common plus one, divided by the natural logarithm of the total number of chunks in both trees plus one" (pg. 133). A high value (near 1.00) indicates high similarity between the content structure of two trees.

As an example, consider the following data obtained from one student during the pilot study. For purposes of this discussion, each word was replaced by a letter. The student recalled the ten given words in the following order (underlined letters show cued trials):

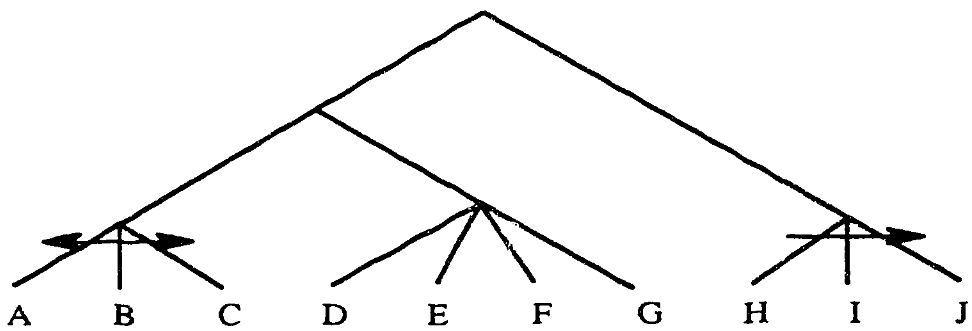
1: A B C D E F G H I J
2: J H I G E F D A B C
3: G D E F A B C H I J
4: H I J F D G E C B A
5: F G E D A B C H I J
6: H I J F D G E A B C
7: I H J G F E D C B A
8: C B A F D G E H I J

The above data was entered into a computer program, and the following word clusters were identified: ABC, ABCDEFG, ABCDEFGHIJ, BC, DEFG, HI, and HIJ. From these clusters, the structured tree shown in Figure 7 was drawn.

In a structured tree, each cluster can be either uni-directional, bi-directional, or non-directional. Cluster ABC is bi-directional, meaning it is always recalled by the subject either in the order ABC or the order CBA. Cluster HIJ is uni-directional, meaning the subject always begins with the word specified by the letter H and follows through with I and J in sequence. Cluster DEFG is non-directional, meaning the subject begins on different trials with different words and recalls the remaining words in different orders. A uni-directional cluster is indicative of the highest degree of structure, while a non-directional cluster is indicative of the lowest degree of structure.

The PRO score of a given tree is calculated as the product of the number of possible recall orders of its substituent parts (as shown in Figure 7). Similarity score is a measure of the similarity between comparable trees. Assume that the tree shown in Figure 8 was provided by the same student prior to instruction. In comparing the trees shown in Figure 7 and Figure 8, it is found that the trees have

Figure 7: Example structured tree from given data on pilot test.



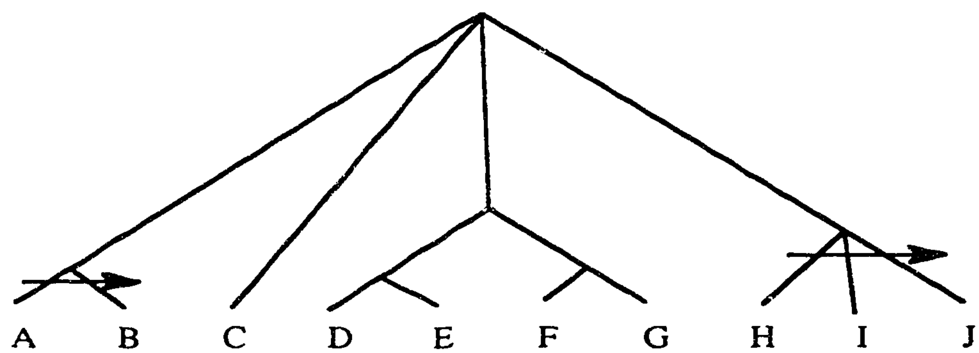
$$PRO_{tree} = \ln(PRO_{A\text{-}C} \times PRO_{A\text{-}G} \times PRO_{D\text{-}G} \times PRO_{A\text{-}J} \times PRO_{H\text{-}J})$$

$$= \ln[(2) \times (2) \times (4!) \times (2) \times (1)]$$

$$= \ln(192)$$

$$= 5.26$$

Figure 8: Example tree to demonstrate similarity score.



$$\text{Similarity Score} = \ln(2 + 1) / \ln(9 + 1)$$

$$= \ln(3) / \ln(10)$$

$$= 0.48$$

chunks HIJ, and DEFG in common. The two trees have a total of two chunks in common, with a total of nine different chunks in both trees (the root node is not counted). The similarity score of the trees is shown in Figure 8. In the present study the PRO score was used to describe the amount of structure in students' trees. The similarity score was used to compare the content and structure of students' trees before and after instruction.

The purpose of the Naveh-Benjamin et al. (1986) and the Reitman and Rueter (1980) studies was to investigate the feasibility and usefulness of the structured tree technique as a research methodology. Naveh-Benjamin et al. report that subjects found the act of organizing words to be a meaningful task as it made them think about the meaning of the words. These authors found a significant interaction between subjects' grade point average and tree complexity. In their study, Naveh-Benjamin et al. report (while working with 154 university psychology students) that the median PRO score decreases during instruction, indicating an increase in tree complexity and knowledge structure. With respect to their experimentation, Reitman and Rueter conclude: "We have shown here that our technique provides reliable, interesting descriptions of some of the regularities in recall and represents them as an ordered tree" (pg 578). Collectively these studies show that the structured tree recall technique is a viable means to investigate student learning.

THE DIAGNOSTIC-PERFORMANCE TEST

The diagnostic-performance test is a paper and pencil survey test (Underhill, Uprichard and Heddens, 1980) comprised of questions drawn from topics in the Alberta junior high mathematics curriculum and the Journeys in Math text series. The purpose of the diagnostic-performance test within this study was to serve as an indicator of procedural knowledge. Acquisition of procedural knowledge has been described as the acquisition of the set of rules and routines which enable a subject to complete a specific task. In the diagnostic-performance test emphasis is placed on subject ability to recall algorithms not on subject ability to transfer between contexts, thus reducing the possibility that confusion resulting from a new context would mask ability to recall rules and procedures. As a result the questions on the diagnostic-performance test are stated in simple, clear language, often in chart form and virtually devoid of context (as opposed to the format typically found in familiar achievement tests). The diagnostic-performance test measured students' simple knowledge of calculation algorithms.

The diagnostic-performance test was developed by the researcher according to the following process: (a) a list of all grade seven, eight, and nine objectives in the percent unit was obtained from the Alberta Junior High Mathematics Curriculum Guide (1988), (b) objectives addressed in the Journeys in Math 8 (1987) text chapter were added to this list in sequence to get a complete objective list, (c) for each of these objectives three test questions were constructed and used as a pilot study test, (d) the questions were sequenced: in increasing order according to grade level, to reflect the probable order of instruction, and in increasing order of complexity, (e) three test forms were piloted with 18 grade nine pupils, (f) from the performance of the students on the pilot study tests the appropriate question difficulty level was determined, (g) the final version of the diagnostic-performance test was constructed by selecting questions from the piloted versions and by devising new comparable questions which employed comparable number values (typically drawn from the set of whole numbers), (h) in the final version each objective was represented by two questions, and (i) alternate test forms were created by changing only the given numbers in each question; the format was held constant as was the magnitude of the number values given in each question. A sample of diagnostic-performance test form A is found in Appendix C. Underhill et al. (1980) recommend the use of three questions for each objective, but it was found during the pilot study that such a test was too long to be administered in a 40 minute period. A test of this length would have imposed a time factor, and by definition, the test was not to be timed (Underhill et al., 1980).

In order to demonstrate mastery of a given objective, the student had to correctly answer both of the given questions. The student was assigned a score on the diagnostic-performance test (called an objective score) according to the number of objectives over which he or she had shown mastery. Scores could range from a low of 0 to a high of 27, although no student scored below 2 on any of the test forms, and no student scored higher that 26 on any test form.

ANALYSIS OF THE STRUCTURED TREE RECALL TASK AND DIAGNOSTIC-PERFORMANCE TEST

After completing the first structured tree recall (STR) task, students were sorted from highest to lowest according to their possible recall order (PRO) scores. The top fifth were denoted as high ability students, the middle fifth as medium ability students, and the bottom fifth as low ability students. The same process was used to assign students to high, medium, and low groups according to their performance on the first diagnostic-performance (D-P) test. Student identification numbers, model under which they were taught, scores from the tests, and student ability level were entered into a large matrix in the Statview™ and SPSS programs for analysis. A multivariate analysis of covariance to compare means between groups was run in which student ability and teaching approach were used as independent variables and post test scores were used as dependent variables. Pretest scores were used as covariates. Correlations between test scores were also calculated using the Statview software package.

THE CLINICAL INTERVIEW

In the constructivist learning model it was claimed that cognitional knowledge acts as a mechanism that monitors and facilitates the construction of propositional and procedural knowledge. If this is true, then we should expect that those students who have shown changes in propositional and/or increases in procedural knowledge, should also demonstrate more frequent incidents of cognitional knowledge. Thus, a tool was necessary for the investigation of the relationship between the three knowledge forms. The clinical interview was instituted for this purpose.

The clinical interview was designed by Piaget to investigate the creative and erroneous answers given by students on standardized tests. According to Piaget (discussed in Ginsburg, 1981), the clinical interview may have three purposes: discovery of cognitive activities, the identification of cognitive activities, and the evaluation of competence levels. The first purpose enables the researcher to elucidate students' error patterns and ways of thinking during instances of problem solving. The second purpose enables the researcher to investigate more completely patterns of behavior in a variety of problem solving situations. The final purpose enables the researcher to elucidate the highest mathematical level at which the subject is able to perform. The clinical interview may focus on any one or more of these purposes in a given interview or series of related interviews. Within the present study the clinical interview was used to identify students' cognitive activities, specifically their demonstrated forms of cognitional knowledge.

Ginsburg (1981) described the clinical interview as "an unstructured and open-ended method intended to give the child the opportunity to display his 'natural inclination'" (pg. 6). Confrey (1981) states: "by clinical interviewing, I am referring to task-oriented, flexible interviews between a student and interviewer wherein the interviewer is expected to follow and pursue the student's thinking, asking questions until the student's reasons for response are understandable to the interviewer" (pg. 6). The main qualities of the clinical interview include: (a) it is unstructured, that is, the direction the interview takes is largely contingent upon the responses given by the subject, (b) it is based upon some given task through which the students' cognitions and the origins of such cognitions are made known to the interviewer.

The interviewer's role in this methodology is to present problems which are challenging but not impossible for students to solve, and to ask questions to identify students' knowledge structures, processes in problem solving, and confidence. Confrey (1981) states:

> A clinical interview aims to ... ascertain what a student believes, why s/he believes, how s/he came to believe it and what predictions s/he might make as a result of those beliefs. Both the interviewee and the interviewer assume active roles in the process, with the student for the most part guiding the inquiry. At times, the interviewer strives to clarify the meaning of the interviewee's statements, while at other times, s/he is more interactive, actively hypothesizing about the implications of the students' responses, posing new questions to test those hypotheses (pg. 15).

In order to achieve these purposes the questions must be selected carefully. Confrey states that the initial questions must be good, disturbing, and compelling, but should consider the appropriate difficulty level for the students interviewed, their familiarity with the topic area, and the intent of the study.

In the present study, each interview consisted of twelve problems as shown in Appendix D. The interview problems were derived using the structure defined by Mayer and Greeno (1972), who were attempting to show that different cognitive constructions result from alternate teaching approaches. They adopted four types of problems which they believed would induce variety in the testing situation and would test the variety of different concepts students may have constructed.

> Learning was evaluated using four types of test problems: (a) familiar problems (Type F) which were stated in the same way as example problems given during training; (b) problems requiring a transformation (Type T), usually of an algebraic nature, to be put into the familiar form; (c) unanswerable problems (Type U) which looked like familiar problems but actually set up inconsistent or otherwise impossible conditions, and (d) questions (Type Q) where the subject was required to give a property of the formula or a constraint on situations in which the formula can be applied, rather than a computational answer (pg. 166).

To ensure that the Type F problems were indeed familiar to the students, these problems were taken from the Journeys in Math text series being used by all the teachers in the study. These problems were typically identical in intent, structure and even context to those found in the text, but the number values were changed. The type F problems in the clinical interviews included: conversion problems, balloon problem, growth problem, and sales tax problem (see Appendix D).

Type T problems are best described as those which require some transfer in order for students to recognize that they can be reduced to a familiar structure. Two forms of Type T problems were included: those that related directly to the grade eight percent unit, and those that related to sister units such as ratio and proportion, or fractions and decimals. Problems chosen from related units were selected because they provided students with the opportunity to generalize knowledge structures from their studies of ratios and rational numbers. Repetitive application of a known formula was required in the first form of the Type T problem. Type T problems that required repetitive application of a known formula included: bouncing ball problem, and interest problem. In the second form of the Type T problems, subjects needed to translate from fractional or proportional representations of values to percent representations. There were five Type T problems used in the interview, including: smarties problem I, smarties problem II, smarties problem III, pizza problem, and photocopier problem.

In the clinical interviews conducted in this study, one unanswerable problem was given. An unanswerable problem involves the introduction of a set of conditions that can not exist. The unanswerable problem was the circlegraph problem which read: "Jane spent 50% of her allowance on a movie, 30% on a new pencil case, and 30% on flowers for her mother. Draw a circlegraph and explain your drawing." Although this problem looks like a traditional problem we might find in a textbook, the percentages have a sum which exceeds 100%. The reader should note that there are two ways of interpreting this problem. In the first interpretation, Jane has a fixed amount of money which is her allowance, and she partitions it as described and thus spends more than she has. This interpretation introduces an inconsistency (to spend 110% of a fixed amount). In the second interpretation, Jane's allowance is simply used as a marker to describe the total amount spent. In this sense it seems quite reasonable for Jane to spend more than her allowance through the addition of external funds. In this second interpretation, 110% of Jane's allowance becomes the fixed cost used to construct the circlegraph. Because of the multiple interpretations possible for this problem, it also served as a Type Q problem where students were expected to discuss properties rather than make computations.

In addition to the criterion described by Mayer and Greeno (1972), three other conditions were imposed to ensure that the interviews remained manageable and interesting: (a) the interview should not exceed two hours in length, (b) if possible, questions should enable the use of a physical object so that the object may be made available to the student, and (c) all problems should be pilot tested to ensure that they were of appropriate difficulty and interest to students of a grade eight level.

In the present study not all 241 students were interviewed. The purpose of the clinical interview was to determine if there was a relationship between procedural and propositional knowledge. Therefore students were selected according to their changes in propositional and procedural knowledge during the study using the steps described below.

Step 1. Students whose scores showed large improvement on the diagnostic-performance test and large increases in cognitive structure on the structured tree recall task during instruction were identified. See Figure 9.

Step 2. These students were divided into four categories as defined in Figure 10 according to their performance on the initial structured tree recall task and their initial diagnostic-performance test score. The definition for 'beginning low' was a score in the bottom 33 percentile. The definition for 'beginning high' was a score in the top 33 percentile.

Step 3. In each of the four categories shown in Figure 10, the three students who best typified the descriptors of the category were selected for interviewing, however, due to mechanical failure one interview from each of groups A, C and D was lost. One interview from group B was only partially transcribable.

Figure 9: First step in the selection process for clinical interviews.

Change from pre to post test on D-P Test		Change from pre to post test on STR task: Improvement	Change from pre to post test on STR task: No Improvement
	Improvement	Subject Group (12 Students)	
	No Improvement		Control Group (3 Students)

Figure 10: Classifications of students showing improvement on both test forms.

Performance on initial D-P Test		Performance on initial STR task: Began Low	Performance on initial STR task: Began High
	Began Low	A	B
	Began High	C	D

Step 4. For comparison purposes, three students were selected who did not improve on either the diagnostic-performance test or the structured tree recall task (the group marked as the control group in Figure 9). A total of 11 viable interviews were collected.

In the selection of the students, no consideration was given to: the gender of the student, the teaching approach under which the student was being taught, or the location or socio-economic status of the school. Teachers were consulted regarding the student selections to ensure that students were willing to be interviewed, and to ensure that students would cooperate to the best of their abilities during the interview.

On the day of the interview, each student was paired with one classmate of the same sex. This pairing helped to eliminate stress on the students, as well as provide someone with whom the student could discuss and share ideas. According to Noddings (1982), clinical interviews performed in small groups have the following advantages: (a) they remove the interviewer from a position of authority, (b) they reduce the amount of necessary interrogation by the interviewer, (c) they allow the individual to talk aloud and talk to him or herself without appearing rude, (d) they permit more reflexive talk that may reveal odd and interesting heuristics not normally seen in the presence of an authority, and (e) they allow students to learn from each other. The partners were selected for each student by that student's teacher according to the following criterion: (a) the students should get along although not necessarily be good friends, and (b) the student should be of a general lesser mathematical ability. The second criterion was stipulated to ensure that the majority of problem solving would actually be done by the student selected according to the steps listed above.

Interview times were pre-arranged with the teacher, and interviews were typically conducted during regularly scheduled math periods. Interviews ranged in time from a length of one hour to one and one-half hours. The interviewer went to the students' classroom and walked with them to the location of the interview, which in each case was either a school counsellor's office or a school sick room. These locations were chosen to maximize privacy and to minimize external distractions. If students became particularly irritated or frustrated by a problem, the problem was set aside with the students' consent, and was left to a later time in the interview.

Each interview was begun by asking students to provide their full names. The students were informed that the researcher was very interested in their results from one or more of the tests recently completed, and thus was interested in how they would solve some varied word problems. The students were informed that the interviews would be taped using a small recorder so that the researcher could remember exactly what the students had done while solving the problems. The problems were randomly sequenced for the first interview, but this sequence was then maintained for the remaining interviews. Each question was presented to the students one at a time on laminated cards, and each question was read aloud by the interviewer. Students were told that they could work together, and were given several pieces of scrap paper as well as calculators and pencils. Where possible, manipulative activities were made available to help illustrate problems. Generally, students seemed disinterested in the manipulatives and chose not to use them. The interviews were not timed.

ANALYSIS OF THE CLINICAL INTERVIEWS

One of the initial concerns a researcher faces (when employing verbal data such as that found in a clinical interview) is a concern for the reliability and validity of the emerging data. Swanson et al. (1981) argue that: (a) verbal data do have a place in cognitive research, (b) there are important limits and constraints on their use: the level of questioning imposed upon subjects must be restricted to that at which answers can reasonably be provided; for example, subjects cannot reasonably be asked to comment on their neural functioning, (c) effective use of verbal data requires paying careful attention to these limits and constraints, (d) provided this is done, any of the remaining problems with using verbal reflections are the same as those which apply to traditional research methods. From this perspective the authors argue that the clinical interview is a viable research methodology.

After the completion of all fifteen interviews, the tapes were transcribed and analyzed according to the procedure documented by Confrey (1982): (a) randomly select 2/3 of the interviews, (b) construct summary sheets of the students' cognitive processes and instances of demonstrated cognitional knowledge for each problem in the interview, (c) construct a description of the instances of displayed cognitional knowledge of these first 2/3 of the students interviewed (including identification, discrimination, generalization, synthesis, reversibility, curtailment and flexibility), (d) read the remaining transcripts and construct summary sheets as in the third step above, (e) decide if the lists of cognitional knowledge adequately describe the remaining transcripts, (f) revise the list if necessary, and (g) re-read all transcripts to ensure that the revised summary list of student cognitional knowledge conforms to all transcript data.

RESEARCH QUESTIONS

The purpose of this study is to employ a constructivist view of learning to the investigation of three different teaching approaches. The three teaching approaches include the Direct Teaching approach, the Meaning Teaching approach, and the Problem Process Teaching approach. A model to describe learning has been derived employing constructivist principles. This learning model contains three basic components including propositional knowledge, procedural knowledge and cognitional knowledge. It has been argued that as learners construct knowledge they construct propositional and procedural knowledge, and that cognitional knowledge performs a monitoring and facilitating role in this learning process. We can then use the development of, or change in both propositional and procedural knowledge as a measure of the differential outcomes of the three learning models. Furthermore, knowing that cognitional knowledge is described as providing a monitoring and facilitating role in the construction of propositional and procedural knowledge, we can therefore use this theoretical relationship as one measure of the viability of the learning model.

The questions that are posed below function as the specific guiding questions of this study. They can be classed into two groups. The first set of questions pertain to the investigation of the three teaching models through the evaluation of the growth and change in propositional and procedural knowledge.

Question 1: Under which teaching approach do students make the greatest changes in propositional knowledge?

Question 2: Under which teaching approach do students of different ability levels make the greatest changes in propositional knowledge?

Question 3: Under which teaching approach do students make the greatest gains in procedural knowledge?

Question 4: Under which teaching approach do students of different ability levels make the greatest gains in procedural knowledge?

Question 5: Under which teaching approach do students best retain their changes in propositional knowledge and/or gains in procedural knowledge over a ten week time period?

In an attempt to expand and investigate the derived constructivist model of learning, a second set of two questions is established that pertain to the viability of the learning model.

Question 6: Do changes in propositional knowledge correlate with gains in procedural knowledge?

Question 7: Do students who have shown changes in propositional knowledge and gains in procedural knowledge also demonstrate cognitional knowledge as described by Sierpinska and Confrey?

STUDY FLOW

To answer the questions listed above, this study was conducted under a pretest, post test, retention test design and thus was completed in three phases: the Pre-Instructional Phase, the Instructional Phase, and the Post-Instructional Phase (see Figure 11).

The Pre-Instructional Phase. In this phase of the study nine teachers were selected from among those participating in the Sigurdson and Olson (1988) Meaning in Mathematics Teaching Project. These teachers were selected according to the following criterion: (a) they were willing to participate (volunteers), (b) they were identified by the observers in the Sigurdson and Olson study as teachers who were effectively and accurately implementing their assigned instructional model, (c) they had more than one class of grade eight mathematics students, and (d) exactly three teachers in each model were selected. In selecting the teachers, no consideration was given to: the socio-economic status of the public served by the school in which they taught, the gender of the teacher, or the size of his or her classes.

One class for each teacher was identified as the group that would participate

Figure 11: Study timeline.

PRE-INSTRUCTIONAL PHASE (two weeks):

- Select nine teachers as participants, three from each model: direct, meaning, and problem process.
- Administer: Structured Tree Recall Task (STA)
  Diagnostic-Performance Test Form "A" (DPA)

INSTRUCTIONAL PHASE (four weeks):

- Percent unit is taught (three weeks).
- At conclusion of unit, administer: Structured Tree Recall Task (STB)
  D-P Test Form "B" (DPB)
- Identify 15 students for participation in clinical interviews.
- Administer clinical interviews (one week).

POST-INSTRUCTIONAL PHASE (eight to ten weeks):

- Classroom teaching in other units resumes.
- After about ten weeks, administer: Structured Tree Recall Task (STC)
  D-P Test Form "C" (DPC)
- Data Analysis

in this project. Two stipulations affected the selection of this class: the class could not be the same class as the one participating in the Meaning in Mathematics Teaching Project, and the class must not have been identified as a unique class (such as special remedial or special gifted class). A total of 245 students were enrolled in the nine chosen classes. The smallest class had an enrollment of 26 students while the largest class had an enrollment of 32 students. The average class size was approximately 27 pupils per class. Students could choose not to participate in the study thus there were a total of 241 pupils who participated in at least one of the three phases of the study.

Within a two week time span, the first Structured Tree Recall Task (STA) was administered to each of the nine classes. On the next consecutive school day, the Diagnostic-Performance Test Form "A" (DPA) was administered to the same classes. In each case these tests were administered within one week prior to the start of the percent unit. All tests were administered by the researcher. Each student was assigned a number code for purposes of identification, and the scores of each student were stored in the form of a computer spreadsheet for later analysis.

Instructional Phase. In this phase of the study the teachers taught their planned percent units to their respective classes employing their specific instructional format. The length of time spent teaching the percent unit was generally related to the length of time allotted to this topic under the Sigurdson and Olson (1988) project: approximately three weeks. Teachers were instructed to notify the researcher as soon as they knew the date they would be finished their unit. At that time a second round of testing comprised of both the structured tree recall task (STB) and the diagnostic-performance test (DPB) was completed. The purpose of this second set of tests was to assess the changes in students' propositional and procedural knowledge which were a result of instruction.

These tests (STB and DPB) were administered to each class on two consecutive days. All nine classes were tested within a two week time span. All tests were administered by the researcher. Tests were scored immediately after their administration in order that students could be selected to participate in the clinical interviews.

Post-Instructional Phase. Approximately ten weeks after the conclusion of the unit the researcher returned to each of the nine classes to administer one final structured tree recall task (STC) and the diagnostic-performance test (DPC). This time lapse was dictated by the pacing of the individual teacher, so it was not possible to give each class the exact same number of school days between the second and third test sets. The purpose of the post-instructional phase was to assess student retention of constructed knowledge.

CHAPTER FOUR

Results: The Teaching Approaches

The purpose of this study is to investigate three different teaching approaches through the application of a constructivist model of learning. One of the three teaching approaches is the Direct Teaching approach in which the mathematics content is portrayed to students with no specific attempt made to connect new ideas to previously learned ideas. Emphasis in the Direct Teaching approach is placed on showing students how to complete specific, isolated tasks through the application of algorithms. In the Meaning Teaching approach much emphasis is placed on connecting concepts by showing relationships between them. In the Meaning Teaching approach representations for concepts are developed and used to define a context in which the concepts may be added to existing cognitive networks. In the Problem Process Teaching approach students receive the same form of teaching as is found in the Meaning Teaching approach except a portion of class time (eight to ten minutes at the beginning of each period) is set aside to solve teacher selected problems. During this time of problem solving teachers carefully describe, model, and lead students through the processes which are involved in solving the problem.

The constructivist model of learning derived for the analysis of these teaching approaches contains three major components: propositional knowledge, procedural knowledge, and cognitional knowledge. Propositional knowledge is the collection of relationships that students construct between concepts. Procedural knowledge is the collection of rules, steps and algorithms associated with the completion of a defined task. These constructions are stored in memory as chunks and are recalled in chunks. It has been argued that when students learn mathematics they build up or construct both propositional and procedural knowledge (regardless of the teaching approach). Cognitional knowledge is the collection of cognitive skills which enables and facilitates the construction of propositional and procedural knowledge. The construction of propositional and procedural knowledge has been used in this study as a means to investigate the differential effects of the Direct, Meaning, and Problem Process Teaching approaches.

QUESTION ONE

Under which teaching approach do students make the greatest changes in propositional knowledge?

In this question we deal with change rather than improvement or gain. In the constructivist perspective, the manner in which students draw relationships is a matter over which they alone have control, thus it would be inappropriate to label their constructions as either correct or incorrect. However, student constructions can be compared to traditionally accepted mathematical relationships (e.g., fractions are like decimals) where such relationships are well defined, and individual cognitive networks can also be compared to the cognitive networks of other individuals such as teachers or other students (as was done by Naveh-Benjamin, et. al, 1986). In this study we are primarily interested in how individual student constructions are affected by varied teaching approaches thus comparisons are made between each student's cognitive networks before and after instruction in percents.

We have adopted two different measures of propositional knowledge within the structured tree recall task. The first measure is the Possible Recall Order (or PRO) score. This score provides a measure of the structure in a subject's cognitive network. The second measure is the called the Similarity (or Sim) score. This measure is a measure of change, and reflects the amount of change in both structure and content of a subject's cognitive network from one task event to the next. Both measures are reported here because they measure different elements of the structured tree. The PRO score is determined only by the complexity and degree of organization of the structured tree, while the Sim score is determined by both the complexity of the tree structure and the manner in which specific concepts are arranged.

In order to measure change in structure and complexity of students' structured trees, a multivariate analysis of covariance was calculated using teaching approach and student ability as independent variables, post test PRO as the dependent variable, and pretest PRO as covariate. The mean adjusted PRO score by teaching approach and student ability level is shown as a chart in Table 1 and as a bar graph in Figure 12. No statistically significant differences were found by teaching approach (F=1.97, df=2/79, p>.05). This result shows that the teaching approaches did not have a differential affect on the degree of structure found in students' structured trees.

The PRO score is a measure of the complexity of the structured tree only, whereas the similarity score is a measure of both the complexity of the tree and the content of the tree (i.e., the actual relationships drawn between concepts). The mean similarity scores under the Direct, Meaning and Problem Process Teaching approaches were 0.35, 0.17, and 0.34 respectively. In this case, a significant difference was found between the three teaching approaches (F=5.56, df=2/78, p<.01): the students in the Meaning Teaching approach made greater changes in their tree structure and content than did the students in the other teaching.

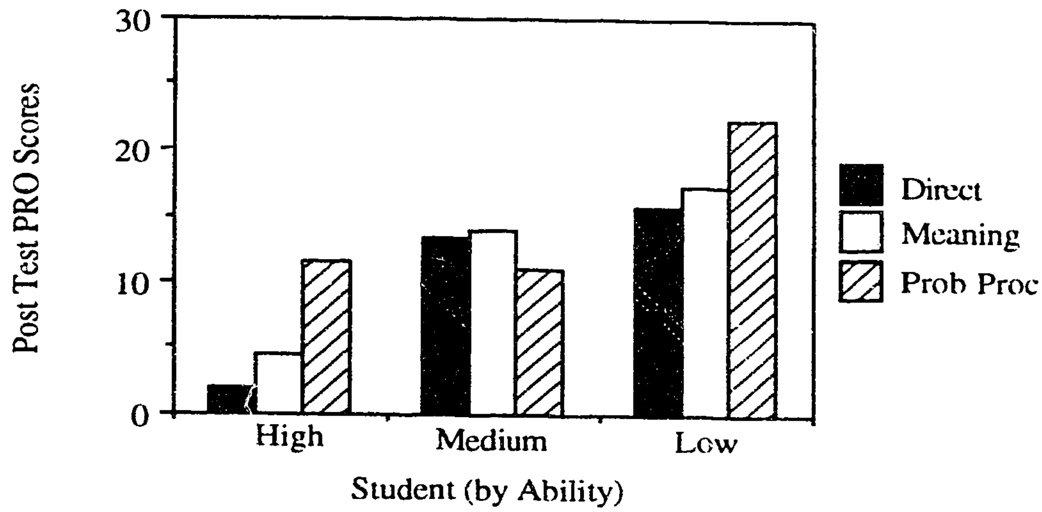
Table 1: Mean pretest and adjusted post test PRO scores by teaching approach and student ability.

	N	Pretest	Adjusted Post Test
DIRECT	26	18.41	11.85
High	6	2.33	1.89
Medium	8	13.96	13.43
Low	12	29.41	15.78
MEANING	33	12.46	10.58
High	15	2.39	4.51
Medium	9	13.14	13.97
Low	9	28.58	17.28
PROB PROC	30	16.92	15.26
High	7	3.76	11.61
Medium	12	13.35	10.90
Low	11	29.17	22.33
HIGH	28	2.72	5.72
MEDIUM	29	13.45	12.55
LOW	32	29.09	18.45
TOTAL SAMPLE	89	15.70	12.52

approaches.

These results may at first appear contradictory, but it is important to remember the difference between the PRO score and the Sim score. The PRO score measures only organizational complexity in the structured tree whereas the Sim

Figure 12: Bar graph of mean adjusted PRO scores on post tests by teaching approach and student ability.



score measures both complexity and content. Taken together, the results above imply that the Meaning Teaching approach may result in greater changes in student propositional knowledge than do the Direct and Problem Process Teaching approaches. Furthermore, because we were able to identify change in content, but not in complexity, this implies that the nature of the changes includes the re-sorting and reorganization of concepts, not an increase in the rigidity or structure of these concepts in memory. In short, it appears that the Meaning Teaching approach encourages students to exchange one mathematical idea for another resulting in a new arrangement of mathematical ideas, but not necessarily a more or less complicated arrangement.

Perhaps we should not be particularly surprised by the effect of the Meaning Teaching approach on students' construction of propositional knowledge, after all, this approach was specifically designed to address representations of, and connections between concepts. However, we should therefore expect similar results from the students in the Problem Process Teaching approach given the similarity between these two approaches; such results did not materialize. We are left to conclude that either the Problem Process teachers did not follow the model as it was intended, or that the problem solving component included at the beginning of the lesson in some manner interfered with the construction of propositional knowledge and the reorganization of concepts within cognitive structures. It may be that the time required to deliver the problem solving component at the beginning of the class introduced a time restriction on teachers thus pressuring them to revert to a rather direct mode of instruction. A second possibility is that the problem solving component became a virtual "how to" session and was thus perceived by the students in a manner similar to Direct instruction.

We are left to conclude that the Meaning Teaching approach has the greatest impact on students' construction and reorganization of propositional knowledge. Furthermore, we know that the nature of these constructions entails the reorganization or restructuring of concepts, not the institution of new, firmer structures. We also know that the addition of the problem solving component in the Problem Process Teaching approach probably reduces the tendency of students to construct propositional knowledge as was evidenced in the Meaning Teaching approach.

QUESTION TWO

Under which teaching approach do students of different ability levels make the greatest changes in propositional knowledge?

In this question, ability level refers to the students' ability or tendency to construct propositional knowledge or to relate concepts and form clusters. In this study, this form of student ability was determined from the initial structured tree task. To determine student ability level, students were sorted from lowest to highest according to their PRO scores on the pretest. A low PRO score represents a high degree of structure in an individual's cognitive network, while a high PRO score indicates very little structure in an individual's cognitive network. Students with PRO scores in the first 20th percentile were defined as the high group. Students with PRO scores in the middle 20th percentile were defined as the medium group, and students with PRO scores in the last 20th percentile were defined as the low group.

We used two different measures of change in propositional knowledge: the PRO scores and the similarity scores. A multivariate analysis of covariance was calculated on post test PRO scores using teaching approach and student ability as independent variables and pretest PRO as covariate. We failed to achieve a significant student ability by teaching approach effect ($F=1.46$, $df=4/79$, $p>.05$), thus we can not report that a particular teaching approach had a differential effect on students of a particular ability level (results are reported in Table 1 and Figure 12 shown above).

However, some interesting patterns result from an inspection of Table 1. In this table it can be seen that while medium ability students made relatively few changes in the complexity of their cognitive structures (a change from a mean PRO score of 13.45 to an adjusted mean of 12.55), the low ability students demonstrated a large increase in the structure of their cognitive networks (a change from a mean of 29.09 to an adjusted mean of 18.45), and the high ability students showed a decrease in structure (the mean PRO score rose from 2.72 to 5.72). In fact, the high ability students in the Problem Process Teaching approach ended with cognitive structures comparable in degree of structure with the Problem Process medium ability students. This result implies that low ability students are those students who have made very few links between concepts, i.e., they have very loose organizational structures in which concepts are stored, and the construction of more rigid cognitive structures is an important process they undergo in learning situations. It is not clear why high ability students show a decrease in structure, but this may be a product of the structure tree task itself, that is, a ceiling effect.

The second measure of change in propositional knowledge is the similarity score. An analysis of variance was conducted on Sim scores in which student ability and teaching approach were used as independent variables. Mean similarity scores by student ability level and teaching approach are shown in Table 2. We failed to achieve a statistically significant difference between similarity scores by student ability and teaching approach ($F=2.09$, $df=4/78$, $p>.05$). Thus we cannot report that the teaching approaches had a differential effect on the construction of propositional knowledge according to student ability level. More simply phrased, we cannot say (for example) that the construction f propositional knowledge was facilitated for low ability students by the Meaning Teaching approach. Likewise, we cannot say that any teaching approach facilitated the construction of propositional knowledge for students of any particular ability level.

It is interesting to note however that a statistically significant difference between similarity scores was found with respect to student ability level ($F=11.18$, $df=2/78$, $p<.001$). Remembering that a low score indicates change, from Table 2 we can see that low ability students made the greatest changes in cognitive network

Table 2: Mean similarity scores denoting change from pre to post test.

	Student Ability Level			
	High	Medium	Low	Mean
Direct	0.50	0.60	0.08	0.35
Meaning	0.13	0.34	0.05	0.17
Prob Proc	0.39	0.38	0.25	0.34
Mean	0.28	0.42	0.13	0.27

structure and content. The high ability group made more changes than did the medium ability group. This result is somewhat surprising as we would probably expect that the high ability group would make the greatest number of changes in their propositional knowledge as they would appear to have the greater general ability to construct propositional knowledge.

A possible ceiling effect in this data would enable low ability students to make large changes while restricting high ability students to small changes. However, if we accept this explanation, then the observation that the high ability students made more changes in propositional knowledge than did the medium students seems all the more surprising. The possible evidence of a ceiling effect does not adequately describe this data. This result implies that one attribute which truly separates high ability students from all others is their ability to construct and reorganize propositional knowledge, regardless of the teaching approach. This observation can be extended to conclude that all students regardless of ability level or teaching approach made changes to their existing cognitive networks. This observation is consistent with the constructivist perspective which claims that students will construct knowledge regardless of the teachers' attempts to facilitate or thwart this construction (Winne and Marx, 1982).

The learning model employed in this study specifies two forms of mathematical knowledge which are constructed during learning events. The structured tree recall task was used as a tool to investigate propositional knowledge, the first of the two knowledge forms. The diagnostic-performance test was used to investigate the second knowledge form, procedural knowledge. The diagnostic -performance test was scored by counting the number of objectives over which a student had proven mastery. This score was called the objective (Obj) score. Two variables were considered in the analysis of Obj scores including teaching approach (Direct, Meaning or Problem Process Teaching approach), and student ability.

QUESTION THREE

Under which teaching approach do students make the greatest gains in procedural knowledge?

A multivariate analysis of covariance was calculated on post test objective scores using teaching approach and student ability as independent variables and pretest objective scores as the covariate. The mean objective score by teaching approach and ability level is reported as a chart in Table 3 and as a bar graph in Figure 13. Although the Meaning approach students did not show as great an increase in procedural knowledge as did the students in the other teaching models, an analysis of these means shows that the differences do not reach significance ($F=2.35$, $df=2/79$, $p>.05$). This statistic implies that though a trend favoring the

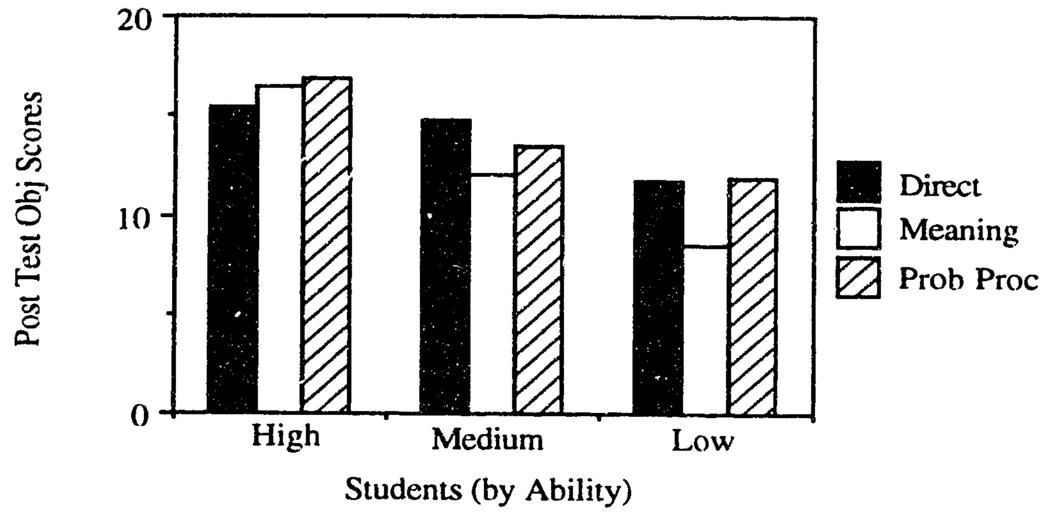
Table 3: Mean pretest and adjusted post test objective scores by teaching approach and student ability.

	N	Pretest	Adjusted Post Test
DIRECT	24	9.92	14.15
High	6	15.83	15.39
Medium	12	9.42	14.73
Low	6	5.00	11.74
MEANING	30	10.93	12.80
High	12	16.50	16.53
Medium	9	9.78	12.08
Low	9	4.67	8.53
PROB PROC	35	9.17	13.67
High	9	16.89	16.90
Medium	11	9.36	13.43
Low	15	4.40	11.92
HIGH	27	16.48	16.40
MEDIUM	32	9.50	13.54
LOW	30	4.60	10.86
TOTAL SAMPLE	89	9.97	13.51

Direct and Problem Process Teaching approaches exists, we cannot statistically claim that one teaching approach has a greater effect on student construction of procedural knowledge.

In one sense, this result is somewhat surprising, as one would expect that

Figure 13: Bar graph of mean adjusted Obj scores on post tests by teaching approach and student ability.



the students taught under the Direct Teaching approach would show a substantial increase in procedural knowledge compared to students in the other teaching models. It should be realized however that all of the teachers in each of the models would be addressing some form of procedural knowledge. The very nature of what happens in classrooms is that teachers want students to demonstrate their learning through applications on given tasks. Hence, the teachers would not have been satisfied under any of the teaching approaches if the students had not shown some increase in procedural knowledge. Increase in procedural knowledge is a widely recognized measure of learning, thus it is natural for teachers to ensure that students have shown increases in procedural knowledge before proceeding further in the unit or course. The emphasis teachers are likely to place on procedural knowledge would reduce the likelihood of reaching a statistical difference on such a test measure.

In considering the trends evident within the data, there is one major surprise. We would probably expect that the students in the Meaning Teaching approach would perform at a level comparable to the students in the Problem Process approach. Instead we find that the students in the Problem Process classes performed almost equally as well as the Direct group students on measures of procedural knowledge. Assuming the teaching approaches were delivered as intended, this result implies that the problem solving component included at the beginning of each class in the Problem Process Teaching approach had an important effect in the development of procedural knowledge, allowing these students to perform on measures of procedural knowledge as ably as the students in the Direct Teaching approach.

QUESTION FOUR

Under which teaching approach do students of different ability levels make the greatest gains in procedural knowledge?

Student ability was determined by student performance on the first diagnostic-performance test. To determine student ability level, students were sorted from highest to lowest, and those students whose objective scores were in the top 20th percentile were assigned to the high group. The students who fell in the middle 20th percentile were assigned to the medium group, while the students who scored in the lowest 20th percentile were assigned to the low group.

A multivariate analysis of covariance was conducted on post test objective scores using teaching approach and student ability as independent variables, and pretest objective scores as covariate. No significant difference between students of varying ability levels was found between teaching approaches ($F=1.18$, $df=4/79$, $p>.05$). Thus, we are not able to conclude that students of a given ability level are more likely to show an increase in procedural knowledge under a given teaching approach. Stated as an example, we are unable to claim with confidence that low ability students show greater increases in procedural knowledge under the Direct (or any other) Teaching approach.

Some trends do exist within the data (see Table 3 and Figure 13). Two interesting trends emerge: (a) medium and low ability students in the Direct and Problem Process Teaching approaches seem to show the greatest increase in procedural knowledge, (b) high ability students in general show little or no gain across the three teaching approaches.

The first trend would imply that the Direct and Problem Process Teaching approaches seem to facilitate procedural knowledge construction for students of a general lower ability level. Consider first the nature of the three instructional approaches. In the Direct approach, algorithms are directly and simply taught as steps and rules. This form of instruction would require little interpretation of classroom events on the part of the learner. The ability to interpret teacher presentations is probably not a skill most lower ability learners possess (see Winne and Marx, 1980), thus this skill may be circumvented in the Direct approach allowing the low ability learner to be successful. The Meaning and Problem Process Teaching approaches both require a greater degree of interpretation of the mathematical environment. However, unlike the Meaning approach, the Problem Process Teaching approach attempts to address this interpretational ability through the inclusion of a problem solving component. Thus, we could expect that lower ability students would perform least effectively under the Meaning Teaching approach where no attempt is made to address the cognitive skills they lack. We could also expect that lower ability students would perform relatively better under the Direct and Problem Process Teaching approaches where the need for these skills are circumvented (as in the case of the Direct approach), or specifically addressed and developed (as in the case of the Problem Process approach).

The second trend found in the data (that high ability students show little or no gain across instructional approaches) should also be explained. The existence of a possible ceiling effect may account for this trend. The diagnostic-performance test was constructed using objectives from the grade seven, eight, and nine curricula. We should expect that after instruction students would demonstrate mastery over the grade seven and eight objectives, but it is unreasonable to expect them to demonstrate mastery over the grade nine objectives: these objectives were not taught by any of the teachers. Therefore, those higher ability students who could answer all of the grade seven and many of the grade eight level questions on the pretest did not have the same opportunity for improvement as the lower ability students who could answer relatively few questions on the pretest. Alternatively, the static performance across teaching approaches by the high ability students may be accounted for by the sheer ability of these students. That is, high ability students are likely to develop procedures for task completion regardless of the teaching approach. In discussing these results with one of the teachers in the study, she commented that high ability students are likely to learn "in spite of instruction, not because of it." In other words, one characteristic of these students is their ability to learn, that is, construct procedural knowledge under any teaching approach.

QUESTION FIVE

Under which teaching approach do students best retain their changes in propositional knowledge and/or gains in procedural knowledge over a ten week time period?

In the design of this study, three test sessions were conducted with each class: a pretest, a post test, and a retention test. The change noted in propositional knowledge and procedural knowledge between the pre and post tests was used to answer questions one through four. In this question, change between the post and retention tests as well as the change between pre and retention tests (net change) are considered. The change from post to retention test provides an indication of change resulting from memory decay. The change from pre to retention test provides an overview of total change which is a product both of instruction and decay.

To answer question five several different analyses were conducted. To investigate cognitive structure change due to memory decay, three tests were conducted: (a) a multivariate analysis of covariance was calculated on retention test PRO scores using teaching approach and student ability as independent variables and post test PRO scores as covariate, (b) an analysis of variance was calculated on similarity scores (found by comparing retention test and post test structured trees) using teaching approach and student ability as independent variables, and (c) a multivariate analysis of covariance was calculated on retention test objective scores using teaching approach and student ability as independent variables and post test objective scores as covariate. To investigate cognitive structure change due to both instruction and memory decay, tests (a) and (c) above were repeated, substituting pretest scores as covariates. An analysis of variance was also calculated on similarity scores (found by comparing retention test and pretest structured trees) using teaching approach and student ability as independent variables. The results of these statistical tests are reported in Table 4.

The only score with which we were able to reach significance is the net change objective score which was calculated by conducting a multivariate analysis of covariance on retention objective scores using pretest objective scores as covariate ($F=5.69$, $df=2/79$, $p<.01$). This result indicates that there was a significant difference in net change between teaching approaches. As can be inferred from Table 5, the students under the Direct Teaching approach made an average adjusted gain of 3.81 objectives, while the students in the Problem Process Teaching approach made an average adjusted gain of 3.33 objectives. This may be held in contrast to the students in the Meaning Teaching approach who showed an average adjusted net gain of 0.08 objectives. This statistic indicates that the Direct and Problem Process Teaching approaches were most effective with respect to net gain and retention of procedural knowledge.

However, on this same score there was also a significant interaction between teaching approach and student ability level ($F=2.51$, $df=4/79$, $p<.05$). That we reached significance on this statistic implies that the three teaching approaches did produce statistically significant differences in procedural knowledge, but that this is only true for students of particular ability levels. From Table 5 we can determine that it was only the medium and low ability students in the Direct and Problem Process Teaching approaches that made large positive changes in procedural knowledge. Table 5 shows that high ability students showed decreases under each of the teaching approaches. This result may indicate a ceiling effect as described earlier. Of interest is the fact that none of the ability groups

Table 4: Effect of teaching approach on retention and net change scores.

Retention Scores			
PRO	F=0.02	df=2/79	p>.05
Sim	F=1.46	df=2/77	p>.05
Obj	F=2.88	df=2/79	p>.05
Net Change Scores			
PRO	F=1.04	df=2/79	p>.05
Sim	F=1.69	df=2/76	p>.05
Obj	F=5.69	df=2/79	p<.01

under the Meaning Teaching approach showed large positive increases. This result implies that specific attempts to address representations and connections between concepts does not result in a more rigorous or better retained system of procedural knowledge. Most importantly, this statistic affirms that these teaching approaches do have a differential effect on the construction of procedural knowledge for students of medium and low ability.

CONCLUSIONS AND IMPLICATIONS FOR TEACHERS

In this study we are investigating three different teaching approaches using a constructivist framework for learning. Our purpose is not to determine the 'best' teaching approach, but to determine if the teaching approaches have a differential effect on the knowledge forms students construct. Some differences are evident.

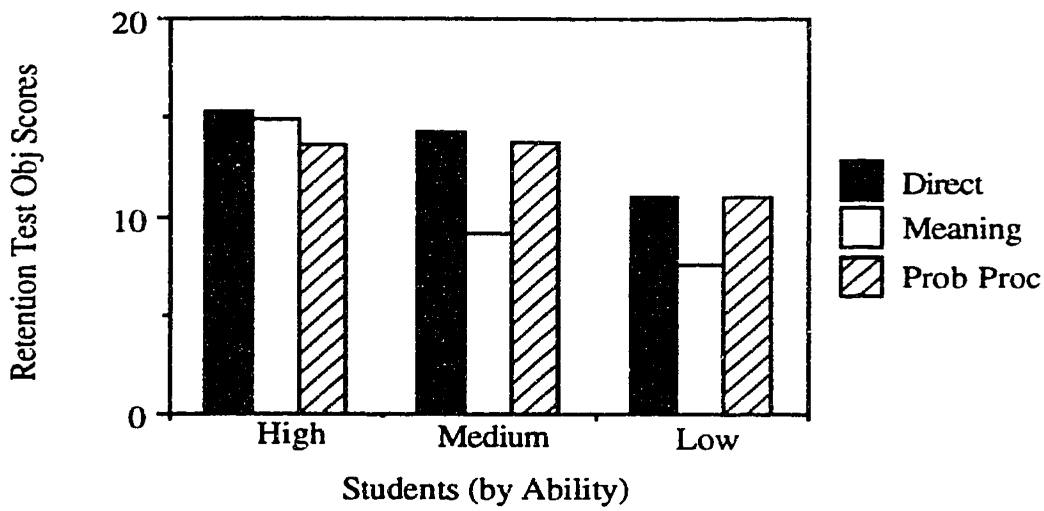
First, the Meaning Teaching approach has the greatest effect on

Table 5: Mean objective scores on pre and retention tests by teaching approach and student ability.

	N	Pretest	Adjusted Retention Test
DIRECT	24	9.92	13.73
High	6	15.83	15.29
Medium	12	9.42	14.30
Low	6	5.00	11.02
MEANING	30	10.93	11.01
High	12	16.50	14.92
Medium	9	9.78	9.19
Low	9	4.67	7.60
PROB PROC	35	9.17	12.50
High	9	16.89	13.63
Medium	11	9.36	13.71
Low	15	4.40	10.94
HIGH	27	16.48	14.57
MEDIUM	32	9.50	12.66
LOW	30	4.60	9.95
TOTAL SAMPLE	89	9.97	12.33

propositional knowledge. In this study we notice that on average the students in the Meaning approach have much lower similarity scores (from pre to post test) than do the students in the other two instructional approaches. Given that we do not see a change in PRO scores, we note that the change in propositional knowledge is a reorganization of concepts within cognitive structures as opposed to the

Figure 14: Bar graph of mean adjusted Obj scores on retention tests by teaching approach and student ability.



construction of new structures. To the teacher of mathematics these observations should imply that attention to representations of, and connections between concepts does result in differential learning outcomes. The students exposed to this form of teaching will apparently undergo greater instances of conceptual exchange (Hewson & Hewson, 1981).

Second, a comparison between the three levels of student ability shows that the greatest changes in propositional knowledge occur within the low ability group. For these students we notice that the change involves both content and structure. It is clear then that low ability students undergo a process of conceptual capture (Hewson & Hewson, 1981) where new concepts are assimilated in the creation of new conceptual structures. To the mathematics teacher this result will demonstrate a difference between the low and high ability student. Whereas the low ability student is undergoing a process of conceptual capture, the high ability student is undergoing a process of conceptual exchange. This difference highlights the need to begin with first principles when teaching any new concept to lower ability students and to develop these principles carefully linking into students past understandings and representations of concepts.

Third, it has been determined that teaching approach does differentially effect construction of procedural knowledge. Students of low and medium ability developed and retained significantly more procedural knowledge under the Direct and Problem Process Teaching approaches. This significant difference is not due to instruction alone, but due to both instruction and retention. To the mathematics teacher this result should imply that particular forms of instruction can facilitate the development of procedural knowledge.

On the surface, it is somewhat surprising that the Direct and Problem Process Teaching approaches would both serve this purpose. Both Peterson (1988) and Confrey (1981, 1982) have determined that there exists a form of cognitional knowledge which is related to student ability. The Problem Process Teaching approach may address this knowledge through generalized problem solving contexts. That is, in the Problem Process Teaching approach the teacher employs and addresses skills such as synthesis, flexibility, and generalization, and if low ability students are in turn developing these skills, then it is reasonable to expect that their performance on measures of procedural knowledge will be enhanced. In the Direct approach, no attempt is made to address these skills, but given the nature of the Direct approach, these skills are also not required: the Direct Teaching approach effectively circumvents the necessity for these cognitive skills thus allowing low ability students (who do not have these skills) to be successful. The teacher should therefore realize that it is possible to teach both the collection of rules and algorithms required for completing mathematical tasks, and the set of cognitive skills which are characteristic of high ability learners. Toward this end, these results serve as a strong endorsement of the Problem Process approach.

Finally, we have seen that once propositional knowledge structures are formed after instruction, there appears to be comparable change and/or decay in these structures across teaching approaches. We found significant differences between the similarity scores of the three teaching model groups immediately after instruction, but this significance did not generalize to the retention period. This observation is a testimonial to the ever-changing nature of these cognitive structures. Propositional knowledge structures may be held in contrast to procedural knowledge structures. We did not find significant differences between the objective scores of the three teaching approaches immediately after instruction, but we did find significant differences after the retention period. Perhaps this result should not be surprising. All of our teachers (regardless of teaching approach) would have addressed some method of solving specific mathematical tasks, therefore the real test of procedural knowledge becomes its resistance to decay some time after instruction (as confirmed by Cobb, 1988). To the teacher of mathematics these observations should imply that it is necessary to be continually addressing representations of, and connections between mathematical concepts, as conceptual structures should not be considered static but constantly changing. Teachers should also be aware that the true test of the development of procedural knowledge structures is the ability of students to employ these structures after a significant time lapse.

In summary, we have shown that teaching approach does differentially effect students' construction of both propositional and procedural knowledge. We have shown that where attention is paid to the representation of, and relationships between concepts (as in the Meaning Teaching approach) then propositional knowledge is effected. We have also shown that where attention is paid to algorithms and generalized problem solving skills, procedural knowledge is effected. Finally, we have seen that none of the Direct, Meaning, or Problem Process Teaching approaches seems to simultaneously address both propositional and procedural knowledge.

CHAPTER FIVE

Results: The Constructivist Learning Model

This project has applied a constructivist model of learning to the investigation of three different teaching approaches. The three different teaching approaches have been found to have different effects on students' learning outcomes. The application of the constructivist learning model to this investigation allows some observations to be drawn with respect to the learning model itself.

The constructivist learning model contains three major components: propositional knowledge, procedural knowledge, and cognitional knowledge. Propositional knowledge was defined as the collection of representations and relationships which students construct between given concepts during learning sequences. Procedural knowledge was defined as the collection of rules and algorithms which enable the completion of a given mathematical task. Cognitional knowledge was defined as the cognitive mechanisms which facilitate and enable the development of both propositional and procedural knowledge. Sierpinska (1990) has argued that there exists a variety of acts of understanding, and these acts both constitute mathematical knowledge and the mechanism whereby mathematical knowledge is constructed. Both Confrey (1981, 1982) and Peterson (1988) have argued that there are teachable forms of cognitional knowledge. Sierpinska's acts of understanding correspond closely to the cognitional forms described by Peterson and include: identification, discrimination, generalization, and synthesis. Like Sierpinska, Confrey provides a list of four forms of cognitional knowledge: reversibility, generalization, curtailment, and flexibility.

The purpose of this chapter is to reflect upon the derived constructivist learning model using the results obtained through the structured tree recall task and the diagnostic-performance test. However, because we are primarily interested in the mental processes in which students engage (i.e., their cognitional knowledge), it is necessary to employ a further research tool to investigate these skills. In this study we have employed the clinical interview in which students are given a variety of word problems and asked to verbalize their thinking processes as they undertake a search for the solution. Through the clinical interview the cognitional knowledge forms demonstrated by selected students can be reported.

In this chapter we will look primarily at two elements of our constructivist learning model. Romberg and Carpenter have stated that "Understanding involves ... connecting semantic knowledge and procedural skills, and encouraging integration of related concepts" (pg. 859). If semantic knowledge and procedural skills are outcomes of learning, and are to be connected by a student during learning events, then it seems reasonable that change in propositional knowledge should correlate with gain in procedural knowledge. In this chapter the relationship between propositional and procedural knowledge will be investigated. It has also been claimed that there exists a relationship between cognitional knowledge and propositional and procedural knowledge (as evident in the work of Confrey 1981, 1982 and Peterson, 1988). To validate this claim, it is necessary to show that those students who have shown both a change in propositional knowledge and a gain in procedural knowledge also possess certain forms of cognitional knowledge. These issues are pursued in the two questions that follow.

QUESTION SIX

Do changes in propositional knowledge correlate with gains in procedural knowledge?

In the design of this study, students completed both a pre and post test separated by a two-week period of instruction in percents. Measures of change in propositional knowledge included the similarity score and possible recall order (or PRO) score obtained by completion of the structured tree recall task. The measure of gain in procedural knowledge was an objective score obtained through completion of the diagnostic-performance test. To answer this question change in similarity score was correlated to change in objective score from pre to post test. Likewise change in PRO score was correlated to change in objective score. These correlations are listed in Table 6.

The correlations in Table 6 are small and negative, and neither is statistically significant. The negative correlation is easily understood as a low similarity score and a low PRO score represent an increase in propositional knowledge, while a high objective score indicates an increase in procedural knowledge. The small correlation is somewhat disappointing, but perhaps not surprising as Geeslin and Shavelson (1975) were also unable to show a similar relationship. They state: "A comparison of word association, achievement, and attitude data indicated that learning of structure may differ from learning measured by achievement tests" (pg. 21). In their study, word association tasks were used to measure conceptual relationships drawn by subjects, and achievement tests were used to measure general problem solving abilities. These researchers were unable to verify that changes in conceptual relationships correlated with changes in general problem solving abilities.

The inability to draw a relationship between change in propositional

Table 6: Correlation of change in propositional knowledge and gain in procedural knowledge.

Variables		Correlation	N	* t	p
Sim	Obj	-0.03	144	-0.36	p>.05
PRO	Obj	-0.13	146	-1.57	p>.05

$$* \, t = r\sqrt{\frac{N - 2}{1 - r^2}}$$

knowledge and gain in procedural knowledge is an important result as it implies that teachers cannot, indeed must not, assume that students have developed more sophisticated representations and relationships between mathematical concepts based solely on their ability to perform more sophisticated or complex computations. Likewise, it cannot be assumed that students who have developed more sophisticated representations of mathematical concepts will necessarily be able to perform given computations or follow given algorithms. This result implies that teachers must therefore address both propositional knowledge and procedural knowledge in learning sequences.

The data presented in Table 6 represents the scores obtained from all students under three teaching approaches. However, it is possible that greater correlations can be found under one teaching approach than under the others. Such a correlation would provide a strong endorsement of that approach as it would imply that both propositional and procedural knowledge can be simultaneously addressed under that approach. To test this hypothesis the correlations were recalculated taking the Direct, Meaning, and Problem Process teaching approaches separately. No statistically significant results were found (see Table 7) thus supporting and extending Mayer and Greeno's (1972) conclusion: changes in propositional knowledge do not necessarily translate into gains in procedural knowledge regardless of teaching approach (Direct, Meaning, or Problem Process), and changes in procedural knowledge can not necessarily be attributed to changes in propositional knowledge. To the classroom teacher this should imply that: (a) none of the three approaches to instruction considered here simultaneously enable development of both propositional and procedural knowledge, and (b) where the teacher wishes to address the erroneous conceptions of students, specific attempts must be made to do so; it cannot be assumed that misconceptions will be addressed through drill and practice in correct procedures, i.e., propositional knowledge will not necessarily be affected by the explicit attempts of a teacher to address procedural knowledge; the reverse is also true.

QUESTION SEVEN

Do students who have shown changes in propositional knowledge and gains in procedural knowledge also demonstrate cognitional knowledge as described by Sierpinska and Confrey?

In order to answer this question, students who had shown an increase in both propositional and procedural knowledge were selected to undergo clinical interviews. Because we were unable to establish a correlation between propositional knowledge change and procedural knowledge gain, it was necessary to select a relatively wide variety of students for the subject group. We selected students from four categories: those who had started high in both propositional and

Table 7: Correlation of change in propositional knowledge and gain in procedural knowledge by teaching approach.

	Variables		Correlation	N	* t	p
Direct	Sim	Obj	-0.03	41	-0.19	p>.05
	PRO	Obj	0.10	42	0.64	p>.05
Meaning	Sim	Obj	-0.11	51	-0.77	p>.05
	PRO	Obj	-0.20	51	-1.43	p>.05
Prob Proc	Sim	Obj	-0.07	52	-0.50	p>.05
	PRO	Obj	-0.25	53	-1.84	p>.05

$$* \, t = r\sqrt{\frac{N-2}{1-r^2}}$$

procedural knowledge (Jason and Derek, Kristine and Erin, and Cec and Sheriden), those who had started low in both propositional and procedural knowledge (Linda and Lyanne, Connie and Megan, and Teresa and Jennifer), those who had started high in propositional knowledge but low in procedural knowledge (Kenya and Paula, Patricia and Marcie, and William and J.J.), and those who had started low in propositional knowledge but high in procedural knowledge (Steve and Matthew, Brad and Chris, and Carl and Kevin).

The purpose of this question within the present study is to verify that the forms of cognitional knowledge described by Sierpinska (1990), Confrey (1981, 1982) and Confrey and Lanier (1980) are in fact demonstrated, and found in the cognitive activity of students. A total of 175 instances of cognitional knowledge as described by Sierpinska were found in the interviews with the students who had demonstrated change in both propositional and procedural knowledge. A total of 67 instances of cognitional knowledge as described by Confrey were found in the interviews of the same group. Where possible, several examples of each form of cognitional knowledge are provided below.

Identification. Identification has been given a two-fold description as the "identification of objects that belong to the denotation of the concept (related to the concept in question), or: identification of a term as having a scientific status" (Sierpinska, pg. 14). Identification is readily found in students' problem solving actions, and appears to be one of the most common of Sierpinska's acts of understanding. A total of 54 instances of identification were found in the cognitive behaviors of the interviewed students.

The first form of identification occurs when a student provides an alternate representation for a concept, object or idea. This process most often occurs when students make statements of equality or similarity. Some examples are listed below. The first set of examples shows where students have substituted a single value or number for simple computation, these substitutions typically all follow the format: A is {equal to} B.

Matthew: ... 10% of one hundred is 10 ...

Kristine: ... 30% of one hundred is 30 ...

Jason: ... She spends 50% of her allowance on a movie, and that's half ...

Connie: That would make it three out of three, so that would be one ... a hundred [percent]...

Because of the ease of the computation, students find it easier to communicate in terms of the number the phrase represents rather than in terms of the phrase itself, such as 10% of 100. In one example above, a student has substituted the word 'half' for 50%. Certain percentages (such as 100%, 50% and 25%) seem so common to students that they will substitute the fractional equivalents for these percentages.

The more complicated forms of identification involve a substitutive process in which students replace or define a concept with a more familiar concept or definition. Some examples are given below:

> Jason: ... 300 cm equals 250% of the inflated balloon ... and the original size was 100%.
>
> Jason: ... Because if it's just her allowance, she has 100%, that's all of it and if you add them all up it's 110% in all, and she can't have 110% of her allowance.
>
> Kevin: One whole [pizza] is 100% ...
>
> Jennifer: [At] 180 cm tall he's 100% fully grown, like at two years he was 45% and I think he was 180 cm tall he was full grown so that he's 100% fully grown...

In each of these examples the students are attempting to define a percentage in terms of a physical object. This fixes the percentage in terms of a concrete representation which is easier to conceptualize. Often the student is looking to establish an equivalent for 100%. In the first example the student has identified 100% as being a deflated balloon, while in the second example 100% is the whole amount of the allowance, and therefore 110% of the whole, fixed amount is impossible. In the third example the student represents 100% as one whole pizza, while in the fourth example the height of 180 cm is established as 100%. The ability to establish a fixed representation of one hundred percent is an important skill as it enables the construction of ratios as one route to the solution of a problem.

The second form of identification occurs when students use a known word to represent a new idea or object. For example, a student might state "20% of the smarties in set one are red. That is its color. We must get the same color in set two." In this case the word 'color' is actually used as a way of defining a particular ratio or proportion between different colored smarties in a set. Such definitions often expedite discussion and communication when shared between cooperating partners. There are very few examples of this form of identification in the clinical interview data. One example appears when students begin to use the word "drop" as a noun to define each stage of the bouncing ball problem.

> Jason: I just times it by ... 100 by 60 and the percent equal. And then I got 60% of 60 in the same way. [Eventually] it came to 21.4 and you take 1 down and that would be 4 times ... like it would be 4 drops [to get to] 20 cm.

> Jennifer: ... When you bounce it and then it comes back up and then it bounces again ... it needs as much space between the 60 cm and the third drop as it did between the 100 and the first and second bounce.

In the first example the student is saying that you must calculate 60% of a number four times, and each calculation represents one drop. In both examples the students use the word 'drop' to define the successive bounces of the ball. Instances of this second form of identification are difficult to report as students rarely preface comments with statements such as "let's define this as..." More commonly, students simply introduce and begin to use words without fully defining or sharing their constructed representations.

In the examples listed above, the students have attempted to simplify their communication through the substitution of a single word or phrase to represent a more complex idea or process. The ability to complete such a substitution indicates an ability to recognize and categorize objects and ideas. In this sense, identification becomes a form of cognitional knowledge.

Discrimination. Sierpinska's second act of understanding is discrimination. She defines it as the "discrimination between two objects, properties, ideas that were confused before" (pg. 39). This definition implies that an act of discrimination involves an identification (through behavior or verbal comment) of objects, properties or ideas in which confusion plays a role, followed by a second act of identification in which the two entities are separated, re-established, and clarified. A total of 5 instances of discrimination are found in the interviews of the students who had demonstrated a change in both propositional and procedural knowledge.

In the first example of discrimination, Paula confuses the words deflated and inflated in the balloon problem. Paula is uncertain as to whether the respective volumes associated with those words should be larger or smaller than the given volume. Paula knows that she has been given the inflated volume (however she does not understand 'cubic centimeters' and so converts 300 cubic centimeters to 900 cm by multiplying 300 by 3) but believes that the deflated volume of the balloon will be larger than the inflated volume; she multiplies by 2.5 (having converted 250% to 2.5).

Paula: I got ... 2,250.

How did you get that?

Paula: What I did first was 300 x 3 and multiplied by 2.5 ... equals 2,250.

Does that answer sound reasonable to you?

Paula: Yes, because I also tried dividing and 900 divided by 2.5 is 360.

OK. (Pause)

Paula: That doesn't sound very good since ...

Since what?

Paula: No, that's totally wrong. I just feel like ...

OK. Tell me about it.

Paula: I realize like you want the deflated volume, if the volume is 900 when it's inflated then it's got to be 360 at its smallest point.

Paula's initial calculations show that she is looking for the largest value possible by multiplying the numbers available in the problem. She changes her mind however when she realizes that a deflated volume must be less than the initial volume of 900, and thus reverts to an earlier calculation where she divided 900 by 2.5. Initially Paula has confused the respective magnitudes of the volumes associated with the words inflated and deflated, but has resolved her confusion by comparing the meanings of the words inflated and deflated with the values she has calculated.

A second example of discrimination is provided by Connie and Megan. They have been presented with two sets of colored smarties. In the first set there is one red and two green smarties. In the second set there are four red smarties and six green smarties. The students have been asked to add or remove only green smarties from the second set in order to get the percent of red equal in the two sets. One student asks: "Well, if you have four reds, how can that equal the other [set]?" Here we can see that the students are confusing the empirical count of the red smarties with the percentage of red in the entire set. Connie and Megan do resolve their confusion by shifting their focus away from the number of reds in the two sets to a comparison of the ratios between colors in the sets.

A third example of discrimination is provided by Chris and Brad who are working together on the same problem as described above with Connie and Megan. Brad has decided that he should remove four green smarties from set two to establish a 2:1 ratio in both sets.

Brad: I'd take away 4 from there, from set two.

Why?

Brad: Cause it's half of the red ... it'd be 2 greens and 4 reds, so that'd be half ... and there's 1 red and 2 green which is also half [in the other set].

At this point in the problem solving process, Brad has confused getting a representation of a 1:2 ratio in both sets with getting an equal percent of a particular color in each set. The resolution of Brad's confusion is shown later after he pairs each red smartie with two green smarties. Brad concludes that rather than taking away 4 green smarties, two should be added to complete the grouping process. In a later statement Brad demonstrates his understanding that the proportion of colors must be comparable:

> Brad: ... there's half as many smarties in each set cause there's 8 greens in set two and 4 reds in set two, so ... the same with that one [set one] ... there's 2 greens and 1 red.

In this example Brad resolves his confusion over looking for comparable ratios in each set to looking for comparable ratios between the two sets.

The three examples of discrimination given above show how students either individually or in cooperation with their interview partners come to recognize that they have two ideas or objects confused and are thus able to separate the ideas again. This separation of ideas often enables the students to correctly complete their given problems. It is interesting to note that there are other instances which could not be labeled as acts of discrimination because the students were not successful in distinguishing between ideas or objects that had been confused. For example, one student confused the calculation of a ratio with the calculation of an arithmetic mean. In each of these cases the students either stated their confusion or were visibly frustrated. These students were unable to follow through to a correct problem solution, thus implying that the inability to discriminate between confused ideas and objects acts as significant barrier to the effective solution of given problems. Teachers may therefore wish to encourage their students to describe their processes, describe what they believe is expected from them in a given problem, and to describe what they believe to be the root of their confusion. The ability to engage in such descriptions may constitute a first step leading toward the ability to effectively discriminate.

Generalization. Sierpinska's third act of understanding is generalization. She describes it as "becoming aware of the non-essentiality of some assumption or of the possibility to extend the range of applications" (pg. 39). Like identification, generalization may also be expressed in one of two ways. The first form of generalization requires the student to make a verbal or non-verbal assumption (non-verbal assumptions are evidenced in student action), and eventually come to a point in the problem solution where it becomes obvious that the assumption is not valid or not necessary. When this realization takes place, the student makes some statement or takes some action demonstrating the rejection of the assumption. A total of 14 instances of generalization were found in the interviews of the students who had shown a change in both propositional and procedural knowledge.

The first example of generalization is provided by J.J. who was trying to solve the pizza problem which reads:

> $5^1/4$ pizzas are to be split between three people. What percent of a pizza does each person get?

J.J.'s first idea was to take all of the pizzas and divide them up into quarters and then begin to distribute quarters to each of the three people. J.J. immediately however makes another suggestion which provides a shorter route to the solution:

J.J.: There's five pizzas ... one to each ... that leaves $2^1/4$ left.

J.J. goes on to solve the problem by dividing the remaining pizza into nine quarters and thus distributing three quarters to each person. The realization that not all of the pizzas need to be split into quarters, that in fact an easier solution is available when a larger equal portion could be given to each person, constitutes an example of Sierpinska's notion of generalization.

Brad provides two examples of generalization. The first example arises during the solution of the bouncing ball problem, while the second example arises during the solution of the sales tax problem. The bouncing ball problem reads:

> A rubber ball bounces back up to 60% of its original height when it is dropped. If the ball is dropped from a height of 1 m how many bounces before its height is less than 20 cm?

Brad's initial assumption is that the ball drops 40 cm from bounce to bounce. After a short discussion, Brad concludes that the ball loses 40% of its decreasing height after each bounce.

> Brad: Okay so it bounces and it stops at 60% so each bounce it would really drop about 40% I think ... and so after 60 it would go down to 20, I think.
>
> Chris: Does it lose 60% of one hundred each time, or 60% of the bounce it took to hit the floor?
>
> I'll let you discuss that with Brad.
>
> Brad: Yeah, because like you'd bounce from there and go up and then take it from there right from where it's the highest point so it'd go down.
>
> So what's your idea Brad? How does this work?
>
> Brad: I ... like you drop the ball from the original height and it hits the ground and goes back up to it's highest point which is 60% of the original, and then you just take from there ... 60% ... so it will hit the ground and just go up the 60% of the 60.

It is interesting to note that Brad makes a shift from talking about the amount of decrease between bounces to talking about the height the ball returns to after each bounce. In so doing Brad correctly conceptualizes the problem and goes on to correctly solve the problem. In this problem Brad makes an incorrect assumption (that the ball will lose 40% of its height, or 40 cm, after each bounce), but comes to realize that this assumption is incorrect and moves to a correct conceptualization of the problem (that the ball returns to 60% of the height of its previous bounce). Brad's awareness of the non-essentiality of his assumption constitutes an example of Sierpinska's notion of generalization.

The second example of generalization which Brad provides is found in his solution to the sales tax problem, which reads:

> The Canadian government is about to impose a 9% sales tax. If you bought a $20 T-shirt, a $5 pair of socks, and a $50 pair of jeans, how much sales tax would you have to pay?

Brad's method of solving this problem is to take each item separately, add 9% to it, and then add them together:

> <u>Brad</u>: You'd have to do for each thing you buy ... you'd have to do it separately so you go 20 plus 9%, you get your answer ... then you do the next one 5 plus 9%, then 50 plus 9% and then you add them all together.

His interview partner believes that you must add all of the items together first, and then take 9% of the total in order to calculate the sales tax. The boys work separately and upon completing their calculations and comparing their answers, find they have reached the same solution. This is a surprise to Brad as he believed that calculating the sales tax for each item separately would give a smaller tax:

> <u>Brad</u>: If you add all the costs together the percent would be increased I thought, but if you did it separately I thought it'd be slightly lower ... if you work it out from putting them altogether.

Brad's shift from his first erroneous assumption (that you can achieve a decreased total tax by calculating tax for each item separately) to a correct understanding (that you pay the same amount of tax whether you calculate the tax based on the total price of the purchases or based on the price of each individual item) represents a third example of generalization.

Jason provides a further example of generalization as he solves the photocopier problem. In this problem, the student is shown a line 10 cm in length.

The student is told that when run through the photocopier, this line shrinks in length from 10 cm to 7 cm. The student is then told that a square 10 cm on a side is created and run through the same photocopier. The student is asked to express the area of the square produced by the photocopier as a percent of the area of the original square. Jason solves the problem, but seems uncertain of his solution.

> Jason: I'm not sure how much it would be decreased, like I think every side would decrease by 3 cm ... and each line would come out to 7 cm ... and then you just ... 49 cm square. Like I just ... the square would be 7 cm instead of 10. I'm not sure what it would work out to.

Jason goes on to take his answer of 49 square cm and subtract it from 100 to receive a final answer of 51 square centimeters. When asked why, he responds

> Jason: Well it seems so small ... like compared to the first one. It seemed like so much less.

Jason apparently feels that the answer 49 simply seems too small, and so he subtracts it from 100 in order to get a slightly larger answer. He does eventually overcome the deception of the size of the solution and does conclude that the correct answer is 49 square centimeters.

> Jason: It is 49, cause I was taking 49 out of 100, but it should be 49 because if each side is 7 cm then 7 x 7 is 49 and not 51. 51 is just what's ... what was left after taking away, I think.

When asked if his answer surprises him, Jason states:

> Jason: Sort of. It does a bit.
>
> Why?
>
> Jason: Because I thought it would be larger 'cause I thought it wouldn't be like ... I thought the percent would be more than that left ... I didn't think that much would be left after taking away just a percentage from each side.

The original assumption under which Jason is operating is that removing 30% from each side of a square would not significantly reduce the area of the square. Probably Jason senses that the final answer should be about 70% of the original area of the square (this is a common solution given by many of the other interviewed students, and incidentally also given by Jason's interview partner Derek). As Jason becomes more confident in the picture he has drawn, he becomes convinced that his answer is correct and that his assumption (that the reduction of the square would reduce the area of the square only marginally) is incorrect. This shift in perception constitutes an example of generalization.

Synthesis. The final of Sierpinska's four acts of understanding is synthesis. According to Sierpinska, synthesis is "grasping relations between two or more properties, facts, objects and organizing them into a consistent whole" (pg. 39). To constitute an act of synthesis, the student must show (either verbally or through action) an awareness of at least two properties, facts or objects, and must show (either verbally or through action) that the two are employed together in a consistent manner. It is not uncommon for the joining of these entities to create a new idea or a new process which may lead to a correct solution. A total of 102 instances of synthesis were evident in the interviews of the students who had shown a change in both propositional and procedural knowledge.

Sierpinska's notion of synthesis is apparently an extremely common form of cognitional knowledge as there are many examples available in the clinical interviews. The pizza problem proved to provide an excellent opportunity for students to link together their notions of dividing and distribution with their knowledge of fractions and parts of a whole. Paula chose to distribute one whole pizza to each person, leaving her with $2^1/4$ pizzas left to share. She then divided the two whole pizzas into halves and distributed one half to each person. This effectively left her with one half of one pizza and one quarter of one pizza. The half pizza was further split into quarters so that each individual could be dealt one quarter. In this manner Paula was able to determine that each person received $1^3/_4$ or 175% of a pizza. Her solution process is diagrammed in Figure 15. Four other students independently employed a limited successive division strategy to solve the pizza problem. In this solution the students distributed one whole pizza to each individual, leaving a remainder of $2^1/_4$ pizzas. The two whole pizzas were split into quarters to obtain a total of nine quarters. These quarters could then be distributed evenly to the three people named in the problem. This solution process is also diagrammed in Figure 15. Both of these solutions illustrate how the students have drawn together the notions of 'division into successively smaller parts' and 'distribution' in order to solve the problems. The students also had to link visual representations of parts of a whole with the fractions and percents that they represent in order to complete the problem. The linking of these concepts constitutes synthesis under Sierpinska's definition.

Several examples of synthesis were also evidenced in the solutions employed in the bouncing ball problem. Jennifer and Matthew describe their solutions to the problem:

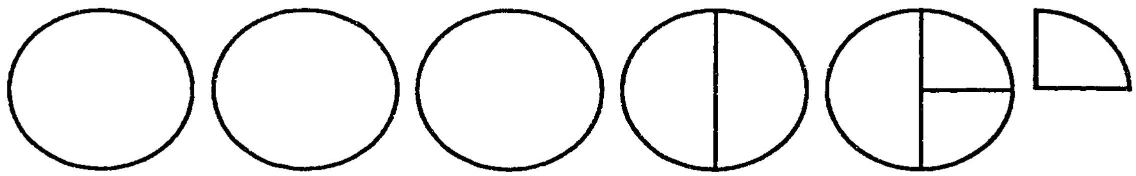
> Jennifer: So if it bounced and it was 60% of the metre stick then it ... then when it bounced again it would be 60% of what the number it bounced up to before. So you have to find out what, how many cm is 60% of the metre stick ...So it would be 60 on the metre stick where it bounced and it would go up to 60 on the metre stick ... like 60% on the metre stick, 60 cm ... 60 over 100. Then it bounces again from that height and ... 60% of 60.

> Matthew: That is equal to 100 cm ... so 60% of 100 cm is 60 ... 60% of 60 is 36 and 60% of 36 is 21 ... it would take 4 [bounces].

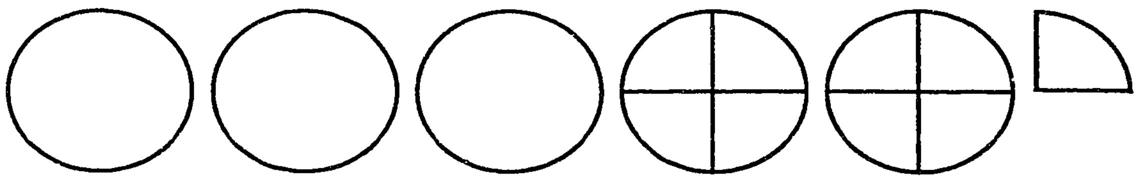
In her solution Jennifer has drawn together several concepts (percent of a number, measurement on a metre stick, relationship between metres and centimeters, and repetition of a process) in order to create a new and appropriate strategy for the

Figure 15: Solutions to the pizza problem.

(a) Successive division shown by Paula:



(b) Limited successive division:



calculation of the number of bounces required. The same relationships can be seen in the comments made by Matthew.

The balloon problem also served as an environment to elicit some responses indicative of synthesis. The balloon problem states: When inflated, a balloon is 250% of its deflated volume. If its inflated volume is 300 cubic centimeters, what is its deflated volume? Brad describes his solution process:

> <u>Brad</u>: You could divide that ... 300 cm cubed by the 250% and you'd get the centimeters cubed of the deflated volume of the balloon.
>
> How do you know? How do you know to divide?
>
> <u>Brad</u>: Cause if you multiply it would be greater ... it would be a greater number than it is inflated so you'd have to do the opposite of multiplying ... just divide.

In Brad's response to the question 'how do you know to divide?' he draws together his knowledge of the effect of multiplication with his knowledge of the expected volume (based on his understanding of the concepts inflated and deflated). The resultant strategy leads him through to a correct solution to the problem.

During the solving of the growth problem Kristine also provides an example of synthesis. Kristine has calculated that Bob must have been 81 cm tall when he was 2 years old. Kristine is asked to comment on whether her answer seems to be correct. She states:

> Kristine: [the answer would] have to be less than half of 180.
>
> Why do you say that?
>
> Kristine: ... 45% is less than 50%, so it'd have to be less than half of 180 [cm]. Half of 180 is 90.

In this example, Kristine has drawn together her understanding of the concept of half (half of 180 is 90) and the relationship of 45% to 50% (45% is less than 50%) to express confidence in her answer. Her answer of 81 cm seems correct as it is less than half of 180 cm.

In each of the cases described above, the students have drawn together one or more facts, concepts or representations to create a new, consistent whole. In this manner the examples above show that students do in fact engage in this form of cognitional knowledge as described by Sierpinska.

Reversibility. Confrey and Lanier (1980) have adopted this definition of reversibility: "the ability to restructure the 'direction' of a mental process, to change from a direct to reverse train of thought" (pg. 551). To constitute an example of reversibility then, the student must clearly identify or employ a specific process, and thereafter clearly identify or employ a reverse process. Confrey and Lanier provide an example of students who do not possess reversibility: "when given repeated problems of the variety, 17 x 13 = 221 what is 221 ÷ 13, in which their attention is called to the first statement, they persisted on undertaking the entire calculation rather than simply reversing" (pg. 552). In this sense, reversibility is the ability to recognize and employ related subtraction and division equations to addition and multiplication situations. The students that were interviewed provided 15 examples of reversibility, some of which are presented below. The first example is provided by Chris and Brad while solving the first smartie problem.

> Chris: We're going to find out how many different colors of smarties there are and divide it into the total.

Chris and Brad go on to determine that there are eight different colors of smarties and so divide eight into 100 to determine the number of reds in the box of 100 smarties. Later Chris is asked how he could check his answer, he responds

> Chris: 12.5 times the total number of [colors of] smarties.

In this instance Chris has recognized and employed the two equations 100 divided by 8 equals 12.5 and 12.5 multiplied by 8 equals 100. Another example is provided by Matthew and Steve who have just calculated the 9% sales tax on $75 to be $6.75. When asked if they could check their answer the following exchange occurs:

> Matthew: What'd we say the sales tax was? ... $6.75 ... So, if we put 6.75 divided by the sales tax ... by 0.09 ...
>
> Steve: [We would end] up with your beginning price of $75.

In this example Matthew and Steve have recognized the reverse relationships of the two statements 75 multiplied by 0.09 is 6.75, and 6.75 divided by 0.09 is 75.

Confrey and Lanier's interpretation of reversibility could be considered quite narrow as it includes only computational processes and not mental processes. More general cognitive processes may also be indicative of reversibility. For example in a problem such as "There are three green smarties and 1 brown smartie in a set. What percent of the set is green?" the student may recognize that the percent of green smarties and percent of brown smarties have a special relationship (they must total 100%). Thus, the student may prefer to calculate the percent of brown smarties (because 1 of 4 is easily recognized as 25%) and find the percent of green smarties by subtracting 25 from 100. In this sense, the student has recognized a reverse relationship between properties within a given context. This form of reversibility was evidenced by three groups of students who chose to deal with the 40% drop rather than the 60% recovery between bounces in the bouncing ball problem.

> Carl: Well, whenever it bounces it loses 40% of its bounce, and the ball comes up only 60% of its height, which is the same ...

> Paula: I thought ... it would only drop 40% because it bounces back 60% of its original height.

> Teresa: Maybe like you go 60 out of 100 cm ... it's 40 in between so you just go down 40.

To reverse the calculation as these students have done creates a more difficult solution process because it requires a greater number of steps, i.e., to calculate 60% of a number requires only one multiplication (x 0.6), whereas to reduce a number by 40% requires one multiplication (x 0.4) and a subtraction (the result must be subtracted from the original number). Drawing an analogy from photography, reversibility in this context occurs when students seem more readily cognizant of the negative image of an object rather than the object itself.

A second example of this form of reversibility is provided by Carl as he completes the photocopier problem. In his solution Carl calculates the area that is lost in the reduction and subtracts that from 100, rather than making the simpler calculation of finding the area of the reduced square.

> Carl: So if it would become 3 shorter going both ways and in lengthwise that would 51 shorter that'd be about ... yeah ... it'll be 51.

> How did you come to 51?

Carl: Well 3 lines of 10 up here is 30 plus 7 lines of 3 that's 21, so ... okay ... that's 51.

The question asks you to express the area of the final square as a percent of the original square.

Carl: 49%.

In his solution Carl has imagined that the original square (10 cm on a side) is made up of 100 tiny squares. To complete the problem Carl mentally removes 3 lines of 10 squares each from each side and then calculates the total number of tiny squares removed (see Figure 16). This number is subtracted from 100 to find the correct final answer. In this example of reversibility Carl has chosen to deal with the area lost by the reduction rather than the final area resulting from the reduction, but his ability to employ this strategy is dependent upon his recognition of the complementarity of the two areas.

There is a third type of reversibility which these students employed in their problem solving attempts. In this type of reversibility students reverse a complex mental process to calculate or verify an answer. After solving the bouncing ball problem Chris and Brad were asked how they could check their answer.

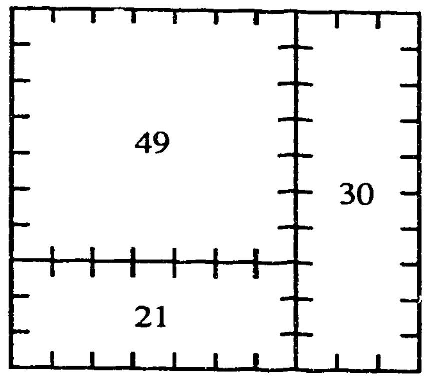
Chris: Just redo the question. Sort of double check it making sure of your calculations.

Any other ideas?

Brad: Just do it from the opposite ... start from the bottom.

In this case the students were willing to accept the end condition (20 cm) as 60% of some number and then repetitively apply this calculation to find how many bounces would be necessary to reach the given condition (100 cm). In essence, the students were starting at the end point of the problem and working backward. This recognition of a backward process constitutes a third form of reversibility. Two groups of students recognized this method as a means to solve the problem.

Figure 16: Carl's solution to the photocopier problem.



Because of the extremely complex nature of these types of solutions (they require the reversal of many calculations or many stages of the problem) it was rarely employed.

In the examples listed above, the students have readily shown their ability to reverse not only simple computations, but complex mental processes. The students have demonstrated three forms of reversibility: reversibility of simple computations, reversibility based upon recognition of shared properties, and reversibility of complex sequences.

Generalization. Confrey and Lanier's description of generalization differs from that of Sierpinska. In Sierpinska's definition of generalization, the individual recognizes that some assumption under which he or she is working is not essential. In this sense generalization becomes a looking back process involving analysis and evaluation of earlier thinking in light of later conclusions. Sierpinska also recognizes a second form of generalization in the ability to see how a concept may be applied in a different context. To Confrey, generalization is a grouping process involving the recognition of similarities and likenesses: the individual may recognize the general case from a collection of single cases, or recognize the single case exemplifying the general case. Confrey and Lanier define generalization as "a) the ability to subsume a particular case under a known general concept; b) the ability to see something general from particular cases, to form a concept" (pg. 551). In the first instance of generalization, the individual refers to a general known case while trying to solve a specific problem. A total of 21 instances of generalization are found in the interview data. Some examples are discussed below.

In the first example, Connie has been asked to give the equivalent percent to the fraction 3/10. She correctly answers "30%." When asked how she found that answer she replies

> Connie: Like over ten. Anything over ten ... like say it was four, it would be forty. That's how I do it.

In this instance Connie recognizes that "anything over ten" may be easily converted to a percent simply by adding a zero to the digit in the numerator. She has applied a known rule to a specific case in order to explain her answer. A similar example is provided by Paula. She has been asked to convert a series of fractions to their equivalent percents. After completing three conversions correctly she states

> Paula: No, we all do it the same. Just divide by the bottom and do long division if you can't use the calculator.

To Paula, all fractions can be converted to equivalent percents simply by dividing and moving the decimal (a process she never fully describes, but repeatedly employs). Paula's repeated use of this division algorithm exemplifies how a generalized process may be employed in the resolution of a specific task.

Confrey and Lanier's first form of generalization may be expressed by students in another way. In this form of generalization the individual will create an analogy between a given and a known context. The known context is more familiar to the individual and so properties from that context may be borrowed and applied in the given context. Carl provides an example while solving the growth problem:

> Carl: Okay ... I usually do it by tests because it seems easier.

Carl goes on to use the analogy of tests, that is, he pretends he is solving the problem by working with test scores rather than with height measurements. He thinks of 50% as being a bad score and then uses 50% of 180 as a starting point to find 45% of 180. Paula also uses an analogy to create a simple solution to a given problem. She has been asked to give a percent equivalent to 1/4. After correctly answering 25% she is asked to explain her answer:

> Paula: If you think about coins, you have four of them make a dollar, so if you take one of them, or a fourth you have 25 cents, so 25%.

In this example Paula has borrowed the context of money in order to help her solve a problem. In neither of the examples listed above do the students actually state a general case, but rather they draw an equivalence between a given (but relatively unfamiliar) context and a known context. The ability to draw this analogy requires the ability to recognize (although not necessarily verbalize) a general case. In this sense, the students have provided examples of generalization.

No students provided spontaneous examples of Confrey and Lanier's second form of generalization. In this form of generalization, students would have to link together several specific cases in order to generate a new concept. In the clinical interviews conducted during this study, students were only given one of each type of problem, with one exception. In this case the student did create a new concept. The example is provided by Brad and Chris.

> This is the sales tax problem. The Canadian Government is about to impose a 9% sales tax. If you bought a $20 T-shirt, $5 pair of socks, and a $50 pair of jeans, how much sales tax would you have to pay? Brad what's your idea?

Brad: You'd have to do for each thing you buy ... you'd have to do it separately so you go 20 plus 9% you get your answer then you do the next one 5 plus 9% then 50 plus 9% and then you add them all together.

Alright, you go ahead and start on that and I'll ask you now Chris. What's your idea?

Chris: I'm not sure but I'm pretty sure that when they add sales tax it's on to the total of the bill so I think you'd add them together and then add the sales tax.

So you'd do it differently then.

Chris: Yes.

(Some time passes while the students calculate.)

So you did get the same answer. Does that surprise you?

Chris: A little bit ... I thought it'd be different.

Brad: If you add all the costs together the percent would be increased I thought, but if you did it separately I thought it'd be slightly lower if you work it out from putting them altogether.

Okay. Let's say its 10% sales tax in Saskatchewan and let's say you but a $50 shirt and a $50 pair of jeans. Now would it be different if you took them separately than if you took them together?

Chris: It would be the same.

How do you know?

Chris: Well, it's the same in this one ... so that gave me a clue ... 10% of 100 which is 50 plus 50 is $10 so then 10% of 50 would be 5 and 10% of 50 would be 5 and 5 plus 5 is $10 again.

Initially both Brad and Chris believe that the total tax calculated by taking the cost of each item separately will be less than the total tax calculated by taking the total cost of the items. After solving a second problem similar in nature to the first, Chris states that the same sales tax will be calculated either way. A more powerful concept is created through the analysis of two similar problems. Due to an attempt to standardize the clinical interviews, there are very few instances where spontaneous problems were presented to students, thus few examples of Confrey's second form of generalization are available.

Curtailment. Confrey and Lanier define curtailment as "the shortening of mathematical processes which can be recalled and explained in detail upon request" (pg. 552). Few instances of curtailment were reported in the clinical interviews. This is probably a product of the restrictive definition of curtailment. This definition requires the following three events to occur: a identifiable process must be evidenced, a request must be made for an explanation of the process, and the student must provide a clear explanation of his or her process. In rigidly applying these criterion, many possible instances of curtailment cannot be reported, i.e., students will not be able to provide a clear explanation of the processes they employed. Of the 15 recorded examples, a few are described below.

The most common examples of curtailment occur while students are trying to calculate percents equivalent to given fractions. For example, Kristine has been asked to describe her process for knowing that $^1/_4$ is equal to 25%. She states

> Kristine: I've always known, but divide 1 by 4 and you'd get 0.25 ... and then multiply by 100 and get 25.

In this example Kristine does not initially regenerate the relationship between $^1/_4$, its decimal equivalent, and its percent equivalent until specifically asked to do so. She has curtailed this process and recalls it only when needed. Jennifer provides a second example when she is asked to express $^4/_{25}$ as a percent.

> Jennifer: 4 goes into 100 25 times, right? ... Yes it does. It's ... 16%.
>
> How did you come to that answer?
>
> Jennifer: Because 25 goes into 100 four times so you multiply 4 times 25 to get over 100 and if you do the same to the top as what you did to the bottom ...

A third example is given by Carl. He is working with a set of 2 red and 3 green smarties and has just been asked to describe what would happen to the percent of reds if one more green smartie is added.

Carl: The reds would go down to 50% of the greens.

And how do you know?

Carl: Well 2 is half of 4, and just times it by 5 ... 8 ... no, 25 I guess, and you get 4 is a hundred and then 2 becomes 50.

In each of the examples listed above, the students have curtailed the process of translating a simple fraction into its percent equivalent.

Curtailment can also occur on a more complex level, where students rely upon a known principle, and thus do not feel the need to complete a calculation. An example is provided by J.J. and William working together on the first smartie problem. In this problem the boys have been given two sets. In the first set there is one red and two green smarties. In the second set there are four red and six green smarties. The boys have been asked to add or remove green smarties from the second set to get the percentage of red in each set equal.

William: We're not supposed to touch this set, but we're supposed to make that set's red even to the percentage of this one. But it also says ...

J.J.: But we can add them though, we can just add 2 more green ones. It would be the same as 2 to 1, it would be 8 to 4.

William: Yeah, it would too.

And would that make the percentages the same?

William: Yup.

How do you know?

William: It would be like 2 to 1, here it would be 8 to 4. All you have to do is to round it down, or whatever you say.

In this example of curtailment, the boys do not feel the need to actually calculate the percent of red in both sets in order to prove that the percents are equal. They are convinced of the equality because they know they can 'round it down' (i.e., reduce $4/8$ to $1/2$). They know the fractions must represent equivalent percents because they recognize $4/8$ and $1/2$ to be equivalent fractions. In this case the computation process has been curtailed by the introduction of the notion of equivalent fractions.

Flexibility. Confrey and Lanier define flexibility as the ability to "accept a variety of methods, to remember each one distinctly and to develop ease and efficiency when given a variety of methods" (pg. 552). Confrey and Lanier report that less able problem solvers commonly confuse various problem solving methods, and in fact prefer to restrict themselves to one method. The students interviewed in this study seemed quite ready to employ more than one problem solving strategy and to switch between them when necessary, especially at a point of confusion. A total of 16 examples of flexibility were found in the interview data.

The pizza problem served to provide several examples of flexibility. Carl first calculates the percent of a pizza that each person would receive by dividing the pizza using a pictorial representation, he then goes back and verifies his solution through numerical computations.

> Carl: Times 5 by 4 equals 20 then you'd add on another one which becomes 21. You divide by 3 ... equals 7.

In this solution Carl calculates that there are a total of 21 quarters in $5\frac{1}{4}$ pizzas. By accepting a basic unit of quarters, he is free to concentrate on 21 pieces which .e knows can be divided evenly three ways. In his first conceptualization of the problem Carl employs a pictorial distribution algorithm, while in his second conceptualization he employs a known math fact ($21 \div 3 = 7$). Two students, Jason and Jennifer, did the opposite of Carl in that they switched to the pictorial representation after completing computations. Jason's example is given below.

Jason: I'll find the ratio here ... and divide 5.25 by 3 and figure the percent each person gets.

As illustrated above, Jason's first solution is entirely computational. Later Jason is asked it he can think of another way to solve the problem. He begins to work with the diagram:

Jason: ... first get a pizza for each person since there's only 3 people and then with 2 1/4 left divide those between 3 people ...

How would you do that?

Jason: Like, okay ... Maybe draw the two extra circles into quarters, no just wait ... into 4 quarters and then just draw up another quarter and that would be 9 quarters altogether.

Jason actually goes on to provide a third solution, which entails dividing the 5 1/4 pizzas into quarters right from the start. This solution is similar to Carl's above.

Diagrams were also commonly used in the bouncing ball problem. Kristine solved the ball problem initially by multiplying 100 by 0.6, and taking that answer and multiplying by 0.6, etc. until she reached 20 cm. This gave her a total of 4 bounces. As a way of keeping track of the number of bounces (i.e., the number of times she had to multiply by 0.6) she created a chart like the one shown in Figure 17. In this chart she was able to ensure that her successive answers were smaller than previous answers, and it was also easy to go back and count the number of answers that had been computed. Jason also provided examples of how pictures could be used to help solve the bouncing ball problem. In his first attempt he draws a series of sticks to represent the heights of each bounce (see Figure 17). The sticks are never employed in any further problem solving approach, but are used simply as a means to better understand the intent of the problem. Jason later suggests another diagram which could be used more effectively to calculate the number of bounces:

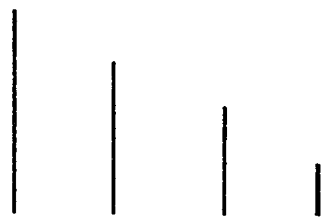
Can you think of a different way?

Figure 17: Solutions to the bouncing ball problem.

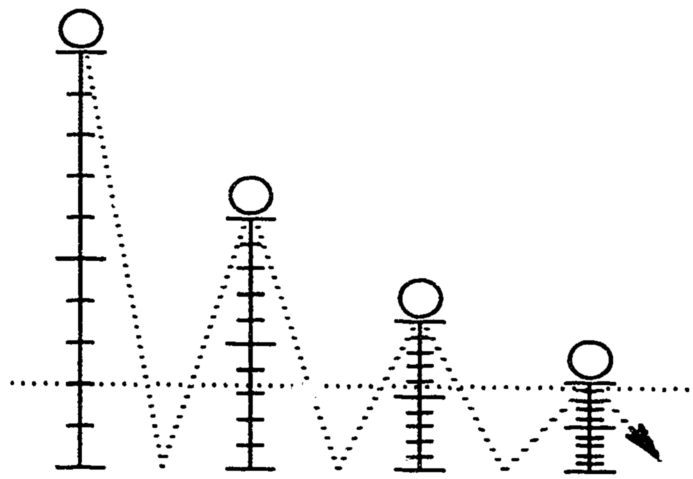
(a) Chart used by Kristine to keep track of her answers.

height	100	60	36	21.6	12.96
bounces		1	2	3	4

(b) Stick diagrams drawn by Jason to help him visualize the problem.



(c) Diagrammatic solution devised by William based upon line lengths.



Jason: Like a chart. Like a picture. First do 1 m and calculate ... and then draw 60% of that and 60% of the next one, 60% of the next one until you get below 20 cm.

Jason does not actually complete this drawing, but William does. The exploded diagram William creates is also found in Figure 17. In this diagram, William draws a line approximately 10 cm in length. This line is then divided in half, and then each half is divided into 5 equal sections. William then draws a new line which is the same height as six sections of the previous line. He repeats this process, finding the correct height of each bounce by pictorially taking 60% of the previous line. William knows he has drawn enough lines when his last line is less than two sections of the very first line (where each section represents 10 cm). This strategy can lead to a correct solution of the problem without any computations ever being performed.

Chris and Brad also provide an example of flexibility while solving the second smartie problem. They have already concluded that they need to add two smarties to set two to get an equal percent of red in both sets when they are asked how they could check their answer. The boys decide that they could pair the smarties to verify their solution. They find that they are missing two green smarties and therefore conclude that the addition of two green smarties would produce equal percent of reds in both sets.

The guess and test strategy was commonly employed or at least suggested by most of the interviewed students. It was regularly employed by students who had forgotten or were confused by the routine to calculate the percent of a number. These students would guess a value, calculate its percent, and then adjust their guess. In this example Carl is solving the bouncing ball problem. He has just calculated that 60% of 100 is 60, and 60% of 60 is 36. He is unsure as to how to find 60% of 36. Carl implements a guess and check strategy:

> Carl: When it drops [to] 60 cm, when it bounces again it goes to 36 cm so we ... you divide ...keep your number and you divide by 36 and when you get your 60% you know its right. I think.
>
> Show me what numbers you are going to punch into the calculator now.
>
> Carl: Oh, I'm trying 21 at the moment ... [divide by] 36 ... 58, so [try] 22 ... two zeroes ... divided by 36 is 61. So you'd have to go to 20.5 ...

Several students employ guess and test as an alternate method when they get stuck in a problem, but rarely does it seem to lead to a solution. In Carl's case he became quickly confused and was not able to complete the question.

As has been shown, the students did in fact employ a broad variety of problem solving strategies, and regularly suggested alternate methods of solving a problem or checking an answer when asked. These examples verify that the students interviewed in the clinical interviews had mastery over a variety of techniques and were able to employ them at will. This ability is consistent with Confrey's notion of flexibility.

While the preceding evidence (that cognitional knowledge is present in students' cognitive activities) does provide some face validity to these knowledge forms, it does not demonstrate that there exists a relationship between cognitional knowledge and propositional and procedural knowledge. To address this relationship a comparison must be made between those students who have shown a change in propositional and procedural knowledge (subject group) and those who have not. In order to complete this comparison an additional three sets of clinical interviews were conducted with students who did not show changes or improvements in propositional and procedural knowledge (see Figure 9). These interviews were conducted with Kyle and Trevor, Claudia and Corinne, and Shawna and Karen. These interviews are used in this study to constitute a control group. The frequency with which students engage in each type of cognitional knowledge is summarized in Table 8.

From this table a few observations may be made. First, all cognitional knowledge forms appear more frequently in the problem solutions of the subject

Table 8: Frequencies of instances demonstrating cognitional knowledge.

	Student	Sierpinska IDE	DIS	GEN	SYN	Confrey REV	GEN	CUR	FLE
Subject Group	Connie	9	1	4	8	1	0	0	0
	Jennifer	11	0	1	9	1	2	1	1
	Carl	8	0	2	14	4	4	2	2
	Brad	3	2	2	10	3	3	2	0
	Matthew*	2	0	0	5	2	1	2	2
	Paula	8	1	0	16	2	3	1	2
	J.J.	4	1	2	8	1	1	3	2
	Kristine	5	0	0	16	1	3	2	3
	Jason	9	0	3	16	0	4	2	4
	AVE.	6.6	0.6	1.6	11.3	1.7	2.3	1.7	1.8
Control Group	Kyle	1	0	0	2	0	1	1	0
	Claudia	4	0	0	0	0	0	1	0
	Shawna	6	0	0	4	0	2	0	0
	AVE.	3.7	0.0	0.0	2.0	0.0	1.0	0.7	0.0

* This interview was only partially transcribable.

group than in the control group. Due to the sample size, it would be inappropriate to conduct a statistical test of these differences, but it seems quite reasonable to claim that the differences imply a relationship between cognitional knowledge and propositional and procedural knowledge: those students who have shown changes in both propositional and procedural knowledge do seem to demonstrate a greater number of instances of cognitional knowledge. Note that we have not determined causality in this investigation. We have simply observed that where propositional and procedural knowledge were developed, cognitional knowledge was also present in those learners. Peterson (1988) and Confrey (1981, 1982) have argued that these cognitional knowledge forms can be taught. Therefore, given that these knowledge forms relate to the development of propositional and procedural knowledge, it stands to reason that this cognitional knowledge should be specifically addressed in the mathematics curriculum.

A second observation is that within the subject group the average number of occurrences of synthesis is approximately twice that of identification; within the control group the reverse is true. This observation implies that though the control group is apparently capable of formulating representations of mathematical properties they seem less disposed toward joining those representations thus forming more complex structures. The inability to create these structures may be an important factor in the inability of the control group students to engage in propositional and procedural knowledge construction.

Third, the most common forms of cognitional knowledge in which students engage are identification and synthesis. Given that identification and synthesis are the two forms of cognitional knowledge most frequently associated with knowledge construction, teachers should employ these knowledge forms in classroom activities to facilitate their development. Teachers could employ these knowledge forms by clearly labeling and discussing any objects, pictures or symbols used in instruction, and by clearly articulating each property, principle or idea used in the development of a concept. The teacher should explain where each concept was last found, and why the concept or property is important, necessary, or helpful in the present context. These discussions may help students build representations for new concepts (identification), and may assist students in drawing relationships between new and previously learned concepts (synthesis). The teaching behaviors which are described are those employed within the Meaning and Problem Process Teaching approaches. In this sense, this observation serves as an endorsement of the Meaning and Problem Process approaches to teaching.

Fourth, we notice that there is a much greater tendency to engage in the cognitional knowledge forms as described by Sierpinska than in those described by Confrey. However, it is important to note that the higher frequency for the Sierpinska forms is a product primarily of the large number of instances of identification and synthesis. The infrequency of discrimination and generalization may be a product of their definitions.

Sierpinska describes discrimination as the "discrimination between two objects, properties, ideas that were confused before" (pg. 39). In this definition the student must clearly identify two objects, properties and ideas, as well as show that these ideas have been confused. The student must also shown that he or she has resolved this confusion by clearly separating them again. Discrimination contains three major components: identification, confusion, and resolution. In holding to this definition rigidly, it was difficult to find instances of discrimination in the clinical interviews as students would frequently omit one or more of the components. The most commonly omitted component was resolution, as students tended not to be able to separate concepts once confused. This may have been a product of the limited time of the interview; students had little time to leisurely reflect even though they were never rushed. Typically students would become frustrated when confused and opt to try a different problem.

Like discrimination, the definition of generalization limited the frequency of its use: generalization is "becoming aware of the non-essentiality of some assumption or of the possibility to extend the range of applications" (Sierpinska, 1990, pg. 39). In the first form of generalization, students had to name or identify an assumption, employ that assumption in some train of thought, and recognize that the assumption was not helpful, fruitful, or was simply incorrect. Generalization entailed three components: identification, use, and rejection. Like discrimination, generalization was infrequent as students tended to omit one element: rejection. Students would occasionally make non-essential assumptions, but tended to stick with an assumption once made. The second form of generalization (that of extending the range of applications) was also limited in that the interviews were comprised of specific questions placed on laminated cards. No student ever made a comment regarding the application of a concept to a different context than that described or implied on the card (such spontaneous transfer is probably more likely to occur when new concepts are formed rather than when previously formed concepts are employed as was the case in these interviews). It is interesting to note that the students occasionally employed a reverse form of transfer in that they would borrow from other contexts to help explain a present context (for example, using percentage scores on a test to help understand the magnitude of a given percent in a problem not involving test scores as a context).

In contrast to the large disparity in frequencies among Sierpinska's forms of cognitional knowledge, the Confrey forms of cognitional knowledge are much more uniform: all four of Confrey's forms of cognitional knowledge have approximately equal frequencies. The Sierpinska forms of cognitional knowledge were spontaneously expressed by students in their problem solving activities (during the clinical interviews), but the Confrey forms were not: the interviewer had to specifically ask questions such as "Can you think of another way?" to ascertain flexibility, and "How do you know?" to ascertain curtailment. The uniformity among the Confrey forms is probably due to the questioning role the researcher had to take in the interview. Given that the Sierpinska forms were spontaneously provided while the Confrey forms were not makes the Sierpinska forms more effective as a research tool: they do not require interference on the part of the researcher to encourage engagement in cognitional knowledge. The Sierpinska forms of cognitional knowledge are a more natural expression of students' cognitive processes.

The large disparity among the Sierpinska cognitional knowledge forms, and the large difference between the Sierpinska forms and the Confrey forms implies that only two of the eight cognitional knowledge forms investigated in this study do discriminate well between students who do and students who do not readily construct knowledge. These two cognitional knowledge forms are identification and synthesis. In surveying the eight cognitional knowledge forms, it is not surprising that identification and synthesis are the most frequent as these two forms seem to correspond well to the act of construction: identification may be conceptualized as the identification of knowledge building blocks, while synthesis is the assembly of these blocks into larger structures.

In conclusion, the results of this study imply that the cognitional knowledge forms as described by Peterson (1988), Sierpinska (1990), and Confrey (1981, 1982) can be found in the cognitive behaviors of students. Furthermore, our results show that those students who demonstrate a change in propositional knowledge and a gain in procedural knowledge, may demonstrate more frequent instances of identification and synthesis. These observations provide some insight into our derived constructivist model of learning. First, we have found that propositional and procedural knowledge may be independent knowledge forms, and that they may develop independently. Second, we have seen a relationship between cognitional knowledge and the other two knowledge forms. We have also found that the most frequently displayed and employed forms of cognitional knowledge are identification and synthesis. This observation would imply that the most important cognitional knowledge students bring to their learning experiences is that which enables them to code (acts of identification) and relate the information that they perceive (acts of synthesis).

CHAPTER SIX

Reflections and Directions for Future Research

This study employed a constructivist model of learning in the investigation of three different teaching approaches. Having completed the investigation, it is now appropriate to reflect back on the work done and summarize the findings of our investigation and discuss the implications of these findings. Given that this study was conducted within the confines of a parent study (the Meaning in Mathematics Teaching Project), we will compare our findings with those of the MMT. We will also reflect on the constructivist learning model which provided a framework for this study.

REFLECTIONS ON THE TEACHING APPROACHES

The three teaching approaches which were investigated included the Direct, Meaning, and Problem Process Teaching approaches. In the Direct approach an emphasis was placed on the teaching of algorithms and rules which would enable students to perform mathematical tasks. The Meaning Teaching approach built upon the Direct approach with emphasis on the representations of, and relationships between mathematical concepts. In the Meaning Teaching approach teachers tried to relate new concepts in a unified network, as well as link new learning to past learnings. The Problem Process Teaching approach was a further extension of the Meaning approach in which a problem solving component was added at the beginning of each math period. During this problem solving component teachers would engage students in an activity in which mathematical processes could be discovered, modeled, and developed. The general problem this study pursued was to determine if these instructional approaches had differential effects on students' learning as described by a constructivist view.

The three teaching approaches were investigated by testing their effect on students' propositional and procedural knowledge. By propositional knowledge we mean the representations that students create for mathematical concepts, and the relationships that they draw between them. By procedural knowledge we mean the series of steps and procedures which enable the completion of a defined mathematical task. The constructivist learning model posited that change in propositional knowledge and gain in procedural knowledge were outcomes of learning. It was found that change in propositional knowledge was facilitated by one teaching approach, while gain in procedural knowledge was facilitated by the other teaching approaches. Thus, the results of this study imply that students' cognitive constructions can be influenced by an instructional approach. Perhaps more importantly, this effect has been found in an actual classroom environment; earlier studies were conducted either in laboratory settings or with contrived concepts or both (see for example Shavelson, 1972; Geeslin and Shavelson, 1972; Mayer and Greeno, 1972; Mayer, 1977; Dunn, 1983; and Stiff, 1989).

Whereas the present study attempted to draw comparisons between the three approaches to teaching, the Meaning in Mathematics Teaching Project compared the three teaching approaches to a conventional classroom setting (where emphasis was simply placed on following textbook activities). In the Meaning in Mathematics Teaching Project, Sigurdson and Olson (1989) found that there was a significant difference between the conventional classroom teaching approach and the three treatment teaching approaches; however, they were unable to show a statistically significant difference between the three treatment models on measures o. achievement. That is, the conventional classroom approach was significantly different (and less effective than) the three treatment approaches, but there was no difference between the Direct, Meaning, and Problem Process approaches. The present study found that there were differences between the three treatment approaches: the Meaning Teaching approach apparently influenced propositional knowledge more so than did the Direct and Problem Process approaches, while the Direct and Problem Process Teaching approaches influenced procedural knowledge more so than did the Meaning approach. It is important to remember however, that the Meaning in Mathematics Teaching Project used general achievement as a means to compare the teaching models, and this is quite different from measuring specific knowledge forms. Measures of general achievement encompass both measures of propositional and procedural knowledge. In essence, the MMT showed that something different was happening in the treatment groups than was found in the conventional teaching group. The results reported here support that finding by suggesting that in the Meaning Teaching approach students made significant changes in propositional knowledge, while in the Direct and Problem Process approaches, students made significant gains in procedural knowledge.

Both the Meaning in Mathematics Teaching project and the present study investigated whether the treatment teaching approaches had an impact on student learning for students of varied ability level. The two studies showed some similarities and some differences in these investigations. The present study finds that: (a) the development of propositional knowledge may be facilitated for high and medium ability students in the Meaning Teaching approach, but not facilitated by the Direct or Problem Process Teaching approaches, (b) the development of propositional knowledge may be facilitated for low ability students by all three teaching approaches, (c) the development of procedural knowledge may be facilitated for medium and low ability students in the Direct and Problem Process Teaching approaches, but not facilitated by the Meaning Teaching approach, and (d) the development of procedural knowledge may not be facilitated for high ability students in the Problem Process Teaching approach, but is marginally facilitated by the Direct and Meaning Teaching approaches. These observations are summarized in Figure 18.

The MMT project included two different forms of data analysis. First, Sigurdson and Olson (1989a, 1989b) labeled their classes as relatively high ability or low ability classes and looked for differences in measures of achievement between these classes according to teaching approach. They were unable to find any statistically significant differences, but did find that the Meaning Teaching approach seemed to consistently provide the highest means for high ability classes. Sigurdson and Olson (1989b) also found that "High achievement students do not benefit much more from a meaning than a direct approach but they do seem not to gain from the Problem Process treatment" (pg. 43). Note that classification at this level of analysis was based on class means and not individual student performance. The second level of analysis in the MMT project involved identifying which students within these classes contributed to the performance of the whole class. In this analysis individual students were labeled as high, medium and low ability. Sigurdson and Olson found that "medium and high students benefit from a meaning approach. However the Problem Process approach allows all students to benefit" (pg. 47).

Some of the findings of the MMT project are similar to the findings of this

Figure 18: Facilitation of knowledge forms by teaching approach for students of varying ability level.

Propositional Knowledge

	Direct	Meaning	Prob Proc
High	no	yes	no
Medium	no	marginal	no
Low	yes	yes	marginal

Procedural Knowledge

	Direct	Meaning	Prob Proc
High	marginal	marginal	no
Medium	yes	no	yes
Low	yes	no	yes

study: (a) high ability students do seem to perform well under the Meaning Teaching approach, and (b) low ability students perform quite poorly under the Meaning Teaching approach. There are some differences however: (a) in the MMT project low ability students performed quite poorly under the Direct Teaching approach, while these same students performed very well on measures of both propositional and procedural knowledge in this study, (b) in the MMT project medium ability students performed well under the Meaning Teaching approach, while in this study medium ability students performed poorly on measures of procedural knowledge, and (c) in the MMT project students of all ability levels performed well under the Problem Process Teaching approach, while it was found that high ability students did not perform well under the Problem Process approach in this study. The MMT project concurred with our results only on measures of high class performance not measures of individual student performance. To a large extent these differences can be explained by the differences in that which was measured. Measures of general achievement will contain elements of both propositional knowledge and procedural knowledge, and will employ elements of cognitional knowledge through the application of concepts to new contexts. This study attempted to separate measures of propositional and procedural knowledge as well as eliminate elements of transfer.

Assuming that the mathematics teacher wishes to address both propositional and procedural knowledge, a few general observations may be made from Figure 18. When a teacher is working with relatively high ability students, then the Meaning approach to teaching may be a viable method and may provide the greatest opportunity to address both propositional and procedural knowledge. The results of this study imply that the Problem Process approach to teaching with high ability students may offer the least potential as we observed little growth in procedural knowledge. The Direct approach with high ability students may be viable from the perspective of development in procedural knowledge, but this approach shows small change scores on measures of propositional knowledge. When working with medium ability students, either the Direct or Problem Process methods of teaching seem to offer approximately equal potential. It is worth noting that while both teaching models were found to have similar influence on procedural knowledge, the Problem Process approach may hold greater potential in its ability to influence propositional knowledge. Medium ability students do not seem to gain in procedural knowledge under the Meaning Teaching approach indicating this approach (in general) may have little potential in the instruction of these students. The results of this study also imply that when working with lower ability students, the Direct approach to teaching may be most appropriate. We found that low ability students in the Direct Teaching approach showed a great deal of change in both propositional and procedural knowledge. The general poor performance of low and medium ability students in the Meaning Teaching approach on our measures of procedural knowledge may show that the effort required to deliver the Meaning approach to these students might not be justified. In short, this study found that where the teacher is attempting to maximize effect on both propositional and procedural knowledge, the Meaning Teaching approach may be most appropriate with high ability students, the Problem Process approach may be most appropriate with medium ability students, and the Direct Teaching approach may be most appropriate with low ability students.

Not withstanding the implications listed above, it is important to note that the purpose of this study was not to describe or identify particular teaching approaches for students of varying ability levels. We have not shown, or attempted to show that particular methods of instruction are categorically more effective with students of defined ability levels. To investigate the effectiveness of particular teaching approaches on particular students requires a different study design than the one employed here, including the constitution of highly controlled, statistically comparable subject groups. In this study we employed somewhat atypical notions of ability (i.e., we defined ability as the tendency to make representations of concepts, to draw connections between concepts, to recall algorithms, and to employ algorithms) and applied them to students who were selected for this study because they were enrolled in a particular teacher's class, not because of any individual or personal attribute. That is, we did not select students based upon more typical, externally derived notions of ability, and did not attempt to select comparable classes of comparable ability levels. Therefore, in interpreting the specific conclusions drawn above, the reader should merely observe that different teaching approaches may have different learning outcomes for students of differing ability levels. More importantly, the reader should observe that meaning structures can be addressed for all students of all ability levels, and that certain classroom activities may help in facilitating the development of such structures.

The results of this study imply that there are different learning outcomes which may be attributed to the Meaning Teaching approach than to the other two models. No evidence is available however to show that differences exist between the Direct and Problem Process Teaching approaches. This observation is somewhat disappointing given the nature of the three teaching approaches. We would expect the Direct and Problem Process results to be the least similar with the results from the Meaning approach somewhere inbetween. Sigurdson and Olson (1989b) found that the Problem Process Teaching approach was the most difficult of the approaches to deliver, followed by the Meaning and finally the Direct approach. Given this finding and the similarity between the Direct and Problem Process Teaching approaches, a shadow of suspicion is cast over the Problem Process approach. Either it is not a viable approach in itself (due to teachers' general inability to implement it well), or the problem solving component it contains acts as a detriment to the development of propositional knowledge as evidenced under the Meaning approach. In either case, some reservations must be expressed with regard to the Problem Process Teaching approach. Of interest, the MMT project came to the opposite conclusion and found this approach to be the most effective for all students across ability levels once poor implementers had been removed from the subject group.

REFLECTIONS ON THE LEARNING MODEL

In the constructivist learning model presented in Chapter Two, we adopted four basic principles synthesized from earlier research, including: (a) the act of constructing knowledge is an innate act involving abstraction and classification of perceptual and sensory data taken from the environment, (b) the knowledge forms constructed include propositional, procedural, and cognitional knowledge, (c) cognitional knowledge acts as a facilitating and controlling mechanism whereby propositional and procedural knowledge are constructed, and (d) metacognitional knowledge is an awareness of cognitional knowledge and enables the conscious engagement of cognitional knowledge. Our study specifically employed the second and third postulates only.

In our application of the learning model we observed that: (a) change in propositional knowledge and gain in procedural knowledge do not correlate implying that these changes occur independently, (b) those students who had shown both a change in propositional knowledge and an increase in procedural knowledge were also the same students who demonstrated particular forms of cognitional knowledge, and (c) synthesis and identification were the two most important or common forms of cognitional knowledge employed by these students.

The first observation is strangely unsatisfying. We have long recognized that students can be trained to complete algorithmic functions through simple drill and practice techniques. But, we have more recently come to believe that students must develop representations of math concepts and draw relationships between these concepts if routines are to be learned and retained in a meaningful way.

This result is of critical import to classroom teachers both in terms of instruction and evaluation. We have historically emphasized the mastery of computational algorithms at all grade levels, and this emphasis has been reflected in our testing instruments. We have left the development of representations of ideas and relationships between concepts to the student, assuming that propositional knowledge will develop in accordance with gain in procedural knowledge. The results in this study would imply that we can no longer make this assumption. To the classroom teacher this result implies that specific attempts should be made to address both propositional and procedural knowledge.

The Meaning Teaching approach may provide one avenue for addressing propositional knowledge. The type of activities that these teachers employed included: using manipulatives, reviewing to recall past learnings, having students engage in discussions encorporating new concepts, and having students discuss applications of new concepts. Within discussions teachers would help students talk about how ideas relate, how they are similar or different from other ideas or processes, and how routines can generalize from one context to another. In working with manipulatives, teachers would carefully describe how physical objects represent abstract concepts, carefully labeling manipulative pieces while encouraging questioning, language development, and verbal skills.

While the Meaning Teaching approach works well to address propositional knowledge, the Direct or Problem Process Teaching approaches can be employed to address procedural knowledge. There is an issue of practicality which becomes important however. The Problem Process Teaching approach and the Meaning Teaching approach are more alike than are the Direct Teaching approach and the Meaning Teaching approach. Given the time restrictions placed on teachers by the demands of the curriculum content, teachers may want to opt for the Problem Process Teaching approach over the Direct Teaching approach. In the Problem Process Teaching approach, teachers spent the first 8 to 10 minutes in problem solving situations with their students. The problems should be carefully selected such that they relate to the content being addressed (and thus also function as review), and encourage discussion from students. The problem solving session could be allowed to end in a formal summary as one would find under the Direct Teaching approach. In this way we have borrowed the best from each teaching approach to address both propositional and procedural knowledge. One cautionary note: in this study we found that the Problem Process Teaching approach functioned much more like the Direct Teaching approach than the Meaning Teaching approach thus implying that the problem solving component added in the Problem Process Teaching approach negated the effects of the Meaning Teaching approach. Teachers must therefore be careful to ensure that full time and effort is accorded to the lesson development component where the Meaning Teaching approach exerts its greatest influence.

Our second observation with respect to the learning model is that those students who showed a change in propositional knowledge and a gain in procedural knowledge also demonstrated particular forms of cognitional knowledge. Peterson (1988) has argued that these cognitional knowledge forms relate to the general mathematical ability of students. Confrey (1981, 1982) and Sierpinska (1990) have gone so far as to argue that these cognitive behaviors are those through which understanding is developed. The relationship between propositional and procedural knowledge and cognitional knowledge is an important finding of this study: it shows that where students are active constructors of knowledge there may exist a body of cognitional knowledge related to the ability to construct knowledge. Peterson and Confrey have determined that this cognitional knowledge is teachable. Their conclusions together with the findings in this study constitute an important implication for classroom teachers: during instruction, teachers may want to address this cognitional knowledge, for by addressing this knowledge lower ability students may be assisted in becoming more effective builders of mathematical concepts. The question is, how can this be done? This study did not set about to test changes in cognitional knowledge as a consequence of instruction, rather our purpose was to investigate a possible relationship between cognitional knowledge and propositional and procedural knowledge. However, Confrey (1981) has provided a very specific model:

1. Identify relevant concepts to be taught.
2. Determine students' alternative, private conceptions of the concepts, perhaps through their responses to a problem.
3. Identify terminology and symbols attached to those public concepts, and to those private conceptions.
4. Propose possible routes from private to public concepts through a series of development stages – these should be both conceptual and linguistical stages.
5. Apply a theory of conceptual change. One possible method is to construct or search out problems that unite the concepts to be learned and to pose these as challenges. To be effective problems for the students it's likely that these should conflict with the students' privately held conceptions.

6. Devote attention to the processes necessary to form the concepts, such as generalization, prediction, abstraction, curtailment, etc.

7. Assess the students on problems that involve flexible and original instances of the concepts, and that require problem solving strategies as well as recall of previous instances (pg. 12).

Within this model it is clear that reversibility, generalization, curtailment, and flexibility have been addressed. Likewise identification, discrimination, and synthesis (Sierpinska's forms of cognitional knowledge) may also be addressed. The Problem Process Teaching approach as described in this study holds the greatest potential to encorporate the suggestions made by Confrey in that its problem solving component may serve as an effective and efficient time to present the problems mentioned in stages two, five and seven above.

The third major observation with regard to the learning model employed in this study, is that two forms of cognitional knowledge were most prevalent in the cognitive behaviors of students showing changes in propositional knowledge and gain in procedural knowledge: identification and synthesis. This observation is important in its implications for teachers. During instruction students may benefit when teachers specifically address representations and relationships between mathematical concepts. Identification is the ability to develop representations of mathematical concepts. If we address these representations, we may assist those students who are slow to develop (or unable to independently develop) representations thus enabling them to link new concepts to past conceptual networks. Furthermore, by the consistent employment of identification we may facilitate the development of this behavior in all students. Synthesis is the ability to develop relationships or connections between mathematical concepts, that is, to link concepts together into a consistent whole. The teacher may find it easiest to specifically address identification and synthesis in the Meaning and Problem

Process Teaching approaches where representations of concepts and relationships between concepts are stressed.

Within this study we did not assess the effect of the teaching approaches on the development of (or change in) cognitional knowledge. Confrey (1981, 1982), Confrey and Lanier (1980) and Peterson (1988) have all shown that cognitional knowledge can be addressed and developed through classroom instruction, but it is still unclear whether cognitional knowledge is only affected when deliberate attempts are made to address it, or whether its development is a by-product of instruction in other content. In short, Confrey, Confrey and Lanier, and Peterson have verified that cognitional knowledge is a constructed knowledge form, but the manner in which it is constructed and the teaching approaches which best influence this construction have not been pursued.

In our constructivist learning model we also argued that cognitional knowledge acts as a facilitating and controlling mechanism whereby propositional and procedural knowledge are constructed. This postulate has been argued by Sierpinska (1990) more generally in her assertion that identification, generalization, discrimination and synthesis act as the means whereby knowledge and understanding are demonstrated. We chose not to pursue a causal link in this study, but to pursue instead the more fundamental and necessary first step of reaffirming the relationship Confrey (1981, 1982) and Peterson (1988) found between the presence of cognitional knowledge and the development of propositional and procedural knowledge. We have observed such a relationship, and our findings imply that further research in this area may be a fruitful endeavor.

We did not find inconsistencies within the learning model that would challenge its viability. However, further research is needed to verify certain relationships within the model, such as to determine if there exists a causal link between propositional and procedural knowledge and cognitional knowledge, and to determine if there does exist a correlation or other verifiable relationship between propositional knowledge and procedural knowledge. Beyond research into the model itself, it is difficult to recommend this learning model as a tool for further teaching experiments: it is probably more useful as a means to understand and develop curriculum appropriate for the learning of individual children.

The constructivist perspective is founded upon the notion that children build their own mathematical knowledge through their interactions with their environments. Therefore, regardless of the teaching approach, we would expect to see changes in both propositional and procedural knowledge as children undertake these constructions. No research methodology currently available lets us into the child's mind to determine whether these constructions are productive or valuable constructions. Until such time as such a tool exists we are unable to determine the effectiveness of a teaching approach with respect to the fruitfulness of a child's constructions. The learning model is helpful however in providing insight into the knowledge forms which are being constructed and employed by mathematics students. We know that from a curriculum and classroom perspective we must address all of propositional, procedural and cognitional knowledge. In this sense, the constructivist learning model provides more guidance in terms of understanding the learning of individual students than it provides a means of evaluating the relative effectiveness of instruction.

REFLECTIONS ON THE STRUCTURED TREE RECALL TECHNIQUE

Measurement of propositional knowledge, or more generally, the measurement of students' internal cognitive constructions is a continuing problem for the cognitive scientist. Many different methods have been tried (see Shavelson, 1974; Fensham, Garrard and West, 1981; Champagne, Gunstone and Klopfer, 1983), but the method chosen for this study was that which had been subjected to the highest degree of scrutiny (see Reitman and Rueter, 1980; McKeithen, et al., 1981; Naveh-Benjamin, et al., 1986). However, several problems were encountered in implementing this research tool.

In completing the structured tree recall task, the students were asked to recall the word list eight separate times. Naveh-Benjamin, et al. (1986) have argued that not much more can be learned from having the students recall the words four times than can be learned from having them recall the words many times. We strongly disagree. We found that even after eight recall events, there were still some students whose structured trees could not be determined (these students were subsequently dropped from the subject pool). This difficulty not only led to the unfortunate loss of some data, but it casts doubt on the effectiveness of the structured tree to provide an accurate mapping of students' cognitive structures.

A second major difficulty was encountered with the interpretation of the data from the structured tree recall task. Similarity scores are a measure of change in a subject's structured tree, but change can be either positive or negative. The similarity score does not differentiate between the student who begins with a very unstructured tree and moves to a highly structured tree and the student who begins with a highly structured tree and moves to a highly unstructured tree. Surely the first scenario is that which is intended in instruction, yet the second scenario was observed with some students in this study. Both the students exemplified above would show low similarity scores, and thus the similar coding of students with highly different learning outcomes confounds the calculation of a correlation. It was not possible to delete those students who showed massive degeneration of conceptual structures, as this would simply be an arbitrary decision on the part of the researcher (for the degeneration of cognitive structures may be one possible outcome of a given teaching approach). We were therefore limited to looking at change in general as opposed to improvement in cognitive networks. The similarity score associated with the structured tree recall task assumes students will make productive changes in cognitive networks, but this study has found that this assumption is questionable. An alternative to the method employed in this study would entail comparing students' structured trees to an external tree rather than to their own previous trees. For example, the students' structured trees could be compared to one provided by their teacher (as was done by Naveh-Benjamin, et al., 1986), or to a content map constructed using an analysis of textual materials (as was done by Shavelson, 1972). Either of these methods would allow the researcher to describe how the subjects' cognitive networks were changing toward a likeness of a defined structure.

A third difficulty was found with respect to the structured tree recall technique: some students were found to employ mnemonic devices in memorizing the key words. For example, some students chose to memorize the words in alphabetical order. Though the inability to employ a more productive means of sorting and memorizing the key words does say something about students' cognitive networks, it should imply that these students recognize few meaningful associations between major concepts. Instead, those students who memorized and recalled the words in alphabetical order performed in a very consistent manner and were therefore shown to have highly complex and highly structured cognitive trees.

Those students who were found to have recalled the words in alphabetical order were eliminated from the data, but it is impossible to know how many other students employed a variety of other different mnemonic devices. These students apparently believed that the task was really one of memory, and not one of demonstrating relationships between known concepts. It cannot be known how many students, if any, employed mnemonic devices other than alphabetization, but the suspicion remains that not all trees provide a representation of the relationships students perceive between concepts. The easiest way to alleviate this difficulty in future research is to forego the memory element of the structured tree recall technique. One could simply have students paperclip together the word cards in groups that go together. These groups could then be compared to previous groupings or to other external structures as described above.

In reflection on the structured tree recall task, we believe that the nature of this task does not facilitate the correlation of propositional and procedural knowledge. There were three groups of key words chosen for use in this task: concept words, application words, and distractor words. Procedural knowledge relates to the ability to employ rules and algorithms to achieve a mathematical task, but there were no words available that could be assembled to represent a formula or algorithm (for example, words like interest, principal, rate and time when placed together resemble a formula). There may be subtle ways to demonstrate procedural knowledge in this task, for example a student may place together 'sales tax' and 'interest' because they both require the computation of the percent of a number. However, the students may also have put them together merely because the words reminded them of a store or bank. A different task to illustrate the relationship between propositional and procedural knowledge was needed. In future research, one possibility includes asking the students to describe or justify the sorting of words thus forcing and elaborating on the meaningful association of concepts (see Champagne, Gunstone and Klopfer, 1983 on the conSAT task; see Fensham, Garrard and West, 1981 on the use of cognitive maps in instruction).

We have listed four important difficulties with respect to the structured tree recall task, and each of these difficulties would confound the computation of a correlation between propositional and procedural knowledge. We maintain that the structured tree recall task was appropriate for comparison between teaching models as each of the difficulties listed above had an approximately equal probability of occurring under each of the teaching models. Assuming our sample size was sufficient to support this claim, the comparison between teaching approaches maintains some validity. In short, the structured tree recall task is probably a more effective tool for comparing between teaching approaches than it is for comparing between knowledge forms within the learning model.

The concerns listed above describe specific implementational and interpretational difficulties of the structured tree recall task. A further concern exists with respect to the ability (or viability) of the structured tree recall task to adequately and completely describe propositional knowledge. This study has adopted the definition for propositional knowledge as given by Shavelson (1981):

> The propositional structure of a subject matter refers to the meaning of mathematical concepts and operations. More accurately, it refers to the verbal and visual representation of meaning (pg. 25).

In this definition we notice that propositional structures entail both verbal and visual representations of meaning. The structured tree recall task could be conceptualized as a task which primarily measures representation of meaning in a verbal form. At least, the task itself is built around the recall of definitions and relationships as stored in memory under given words or simple phrases. In this sense, the structured tree recall task predominantly measures one aspect of propositional knowledge: verbal representation. But, what do we mean by the visual representation of meaning? An example has been given for us in William's solution to the bouncing ball problem (see Figure 16 in Chapter Five). In his solution to the problem, William constructs a detailed pictorial representation of the problem, accurately drawing lines the lengths of which represent the height of the ball on each consecutive bounce. William's solution leads him through to the correct answer without making a single computation. This solution is both highly visual in nature, and demonstrative of William's ability to construct meaning for the problem conditions. Is it reasonable to expect that the student who is able to construct such complex meaningful representations of concepts would also demonstrate this ability through the structured tree task? It is important to observe that though this task may provide one measure (a verbal measure) of propositional knowledge, the visual dimension of propositional knowledge is not addressed.

Knowing that the structured tree task predominantly measures the verbal representation of meaning, it is fair to ask what implications may be drawn for the results of this study. First, we found that no correlation existed between propositional and procedural knowledge, but it may be that there exists a relationship between procedural knowledge and the visual dimension of propositional knowledge. Is it possible that a link may be found between visual representations of meaning and procedural knowledge, especially insofar as visual representations often imply a solution route within problem solving situations? Second, we failed to identify a teaching approach which clearly addresses both propositional and procedural knowledge, but this conclusion may have to be further investigated accounting for the visual representation of meaning. Future research may wish to reconsider the design and intent of the Meaning and Problem Process approaches to clearly account for both the verbal and visual representation of meaning. Some questions remain: Would we have achieved the same findings if we replaced or supplemented the structured tree task with a visual representation of meaning? and How do verbal and visual representations of meaning correspond or differ, especially as applied within a problem solving context?

CONSTRUCTIVISM IN THE CLASSROOM

Noddings (1990) has presented four major principles of the constructivist perspective, including:

> 1. All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.
>
> 2. There exist cognitive structures that are activated in the processes of construction ...
>
> 3. Cognitive structures are under continual development. Purposive activity induces transformation of existing structures. The environment presses the organism to adapt.
>
> 4. Acknowledgement of constructivism as a cognitive position leads to the adoption of methodological constructivism (pg. 10).

These four principles have many implications for teachers of mathematics.

First, knowing that all mathematical knowledge is constructed shifts the collective responsibilities of teacher and student. Under impositionism, where the teacher is perceived as the one who owns and imparts all mathematical knowledge, correct transmission of knowledge is the responsibility of the teacher. Where the student is accepted as one who constructs knowledge, the student shares the responsibility of effective communication which facilitates knowledge construction.

Under constructivism the teacher is responsible for manipulation of the environment in order to challenge students' prior conceptions and thus encourage conceptual exchange. It is also the teacher's responsibility to manipulate the environment so that students will have maximal opportunity to link new concepts to prior conceptions and thus facilitate conceptual capture. The constructivist perspective places responsibility for knowledge construction on the shoulder of the student while placing responsibility for manipulation of the environment on the shoulder of the teacher. This constitutes a shift in roles where the teacher is no longer perceived as being directly responsible for the transmission of knowledge. We believe that this shift in responsibilities aids in maintaining the integrity of the teacher, the student, and the learning environment.

Second, certain teaching behaviors and activities will be adopted by the teacher who recognizes that cognitive structures exist and that these structures must be activated in learning sequences. Teachers will need to attend to past learning. The teacher will not want to merely present a new concept in isolation without discussing or having students discover how new concepts are like or different from past concepts. Given that these conceptual structures must be activated, the teacher will not want to merely present ideas to students. Students must be challenged, and encouraged to encounter and reflect upon mathematical properties, and this can only be accomplished when students are involved by way of discussions and questioning. Other ways of activating structures include challenging problem solving activities where students must apply known concepts within new contexts, thus encouraging both transfer and application. Adoption of the constructivist precepts implies a very active form of instruction. Traditionally an effective teacher is defined as one who possesses good teaching skills, primarily good communication skills. Constructivism implies that good teaching involves the ability to manipulate classroom environments such that students may challenge past structures, link new concepts to existing structures, and thus build more powerful cognitive networks.

Third, the constructivist perspective is useful to the classroom teacher as it provides an alternate means to characterize the curriculum. The constructivist perspective specifies particular forms of knowledge which are constructed. This study has argued for three such forms including propositional knowledge, procedural knowledge, and cognitional knowledge. The results of this study imply that different teaching approaches have differential affects on the construction of propositional and procedural knowledge, and imply that a relationship exists between propositional and procedural knowledge and cognitional knowledge. Therefore, the teacher who adopts the constructivist perspective may wish to attend to the propositional, procedural, and more general cognitional knowledge structures of any given topic or unit. By propositional structures we mean that teachers should address the representations of concepts as well as the relationships between concepts, specifically how students are 'seeing' mathematical ideas and how students are drawing connections between them. By procedural structures we mean the collection of algorithms and processes which enable computations. Cognitional knowledge structures are more general cognitive processes which enable students to classify new concepts, discriminate between concepts, link concepts, think flexibility, as well as reverse and curtail mental operations.

Historically, mathematics teachers have focused almost exclusively on the procedural knowledge dimension of the curriculum to the exclusion of the propositional and cognitional knowledge components: we have taught half the curriculum, and probably not the most important half. Consider again the solution William provided to the bouncing ball problem (see Figure 16). Through William's powerful understanding of the problem (that is, his ability to represent the event and draw relationships between the concepts inherent within the problem) he was able to effectively solve the problem without making a single calculation: William's solution provides evidence of the power and importance of propositional knowledge structures. Propositional knowledge may be the cornerstone of constructivism as it constitutes the meaningful association of concepts to form powerful ideas.

Finally, adoption of the constructivist perspective represents a challenge to the classroom teacher: a challenge to identify the 'tool kit' (Davis, Maher and Noddings, 1990) of cognitional knowledge which facilitates concept construction, and a challenge to identify those teaching approaches which facilitate the construction of all three knowledge forms. In this study we have found that none of the Direct, Meaning, or Problem Process Teaching approaches in isolation are able to address both propositional and procedural knowledge for students of all ability levels. The teacher's challenge is to find a combination of teaching approaches which enable the simultaneous construction of propositional and procedural knowledge while enabling the development of cognitional knowledge structures. Constructivism challenges the teacher to focus not only on what knowledge is built, but on how it is built, and who is building it.

DIRECTIONS FOR FUTURE RESEARCH

Our previous summaries have implied a research agenda. We have already noted that there remains the challenge of verifying a link between propositional and procedural knowledge, and the challenge of determining the nature of the relationship of cognitional knowledge to the other two knowledge forms. We have also already noted the need for continued research into the visual nature of propositional knowledge, and development of a valid methodology to investigate students' cognitive constructions. These research directions are a direct result of the findings and frustrations within this study.

(a) In this study we found that propositional knowledge could be more highly influenced by the Meaning Teaching approach than the Direct and Problem Process approaches, and that procedural knowledge could be more highly influenced by the Direct or Problem Process Teaching approaches than the Meaning approach. Can a single teaching approach be developed that influences all of propositional, procedural, and cognitional knowledge?

(b) This project did not find a relationship between change in propositional knowledge and growth in procedural knowledge. The investigation of the relationship between propositional and procedural knowledge may entail a study of error patterns, as the consistent errors students make while employing algorithms may be indicative of inconsistent or flawed propositional knowledge structures. Can change in propositional and procedural knowledge structures be accounted for through the study of student error patterns?

(c) In this study we attempted to determine if a relationship exists between the three knowledge forms, specifically if cognitional knowledge is related to the other two knowledge forms. Using only descriptive statistics, we found such a relationship. Confrey (1981 1982) and Peterson (1988) have both verified that cognitional knowledge is a teachable knowledge form. However, both researchers set out to deliberately teach cognitional knowledge. Is cognitional knowledge simply constructed as one product of instruction, or must it be directly addressed in instructional sequences?

(d) The observation that cognitional knowledge may be associated with students' ability to construct propositional and procedural knowledge provides a strong impetus for further research into cognitional knowledge. The scope or variety of cognitional knowledge forms should also be pursued. This study considered only those previously described by Confrey and Lanier (1980, and Confrey 1981, 1982) and Sierpinska (1990). The question remains, do other forms of cognitional knowledge exist?

CONCLUSION

The purpose of this study was to employ a constructivist model of learning as a means to investigate the Direct, Meaning, and Problem Process Teaching approaches. The constructivist model of learning was comprised of three major components: propositional knowledge, procedural knowledge, and cognitional knowledge. Propositional knowledge and procedural knowledge were presented as outcomes of learning. Cognitional knowledge was presented as a mechanism that controls and facilitates the development of propositional and procedural knowledge. Measures of propositional knowledge and procedural knowledge were used to compare the three teaching approaches. We found that the Meaning Teaching approach had the most influence over the construction of propositional knowledge. We also found that the Direct and Problem Process Teaching approaches had the most influence over the construction of procedural knowledge. In short, these results imply that alternate teaching approaches when employed in actual classroom situations do have differential affects on students' cognitive constructions.

A secondary purpose of this study was to reflect upon the constructivist learning model in light of its application to the investigation of the teaching approaches. In this study we did not find a correlation between the development of propositional and procedural knowledge; however, we did find that there may exist a relationship between cognitional knowledge and the other two knowledge forms. We found that those students who showed a change in propositional knowledge and a gain in procedural knowledge demonstrated a greater number of instances of cognitional knowledge during problem solving events. This study did not attempt to validate this synthesized constructivist model of learning.

In the evolution of cognitive psychology, increasing emphasis has been placed on developing plausible descriptions of student learning within teaching environments. This study has attempted to draw a connection between the Direct, the Meaning, and the Problem Process Teaching approaches and the manner in which students construct knowledge, and we have attempted to make this connection within actual teaching environments. Much more needs to be done to develop more effective research methodology, and more needs to be done to further define, refine, and validate the constructivist learning model.

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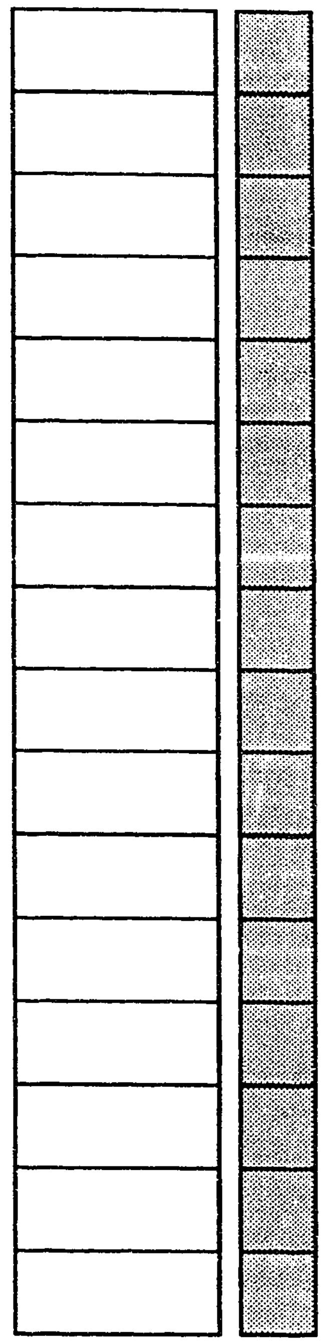
APPENDIX A

Structured Tree Task Materials

Recall Form:



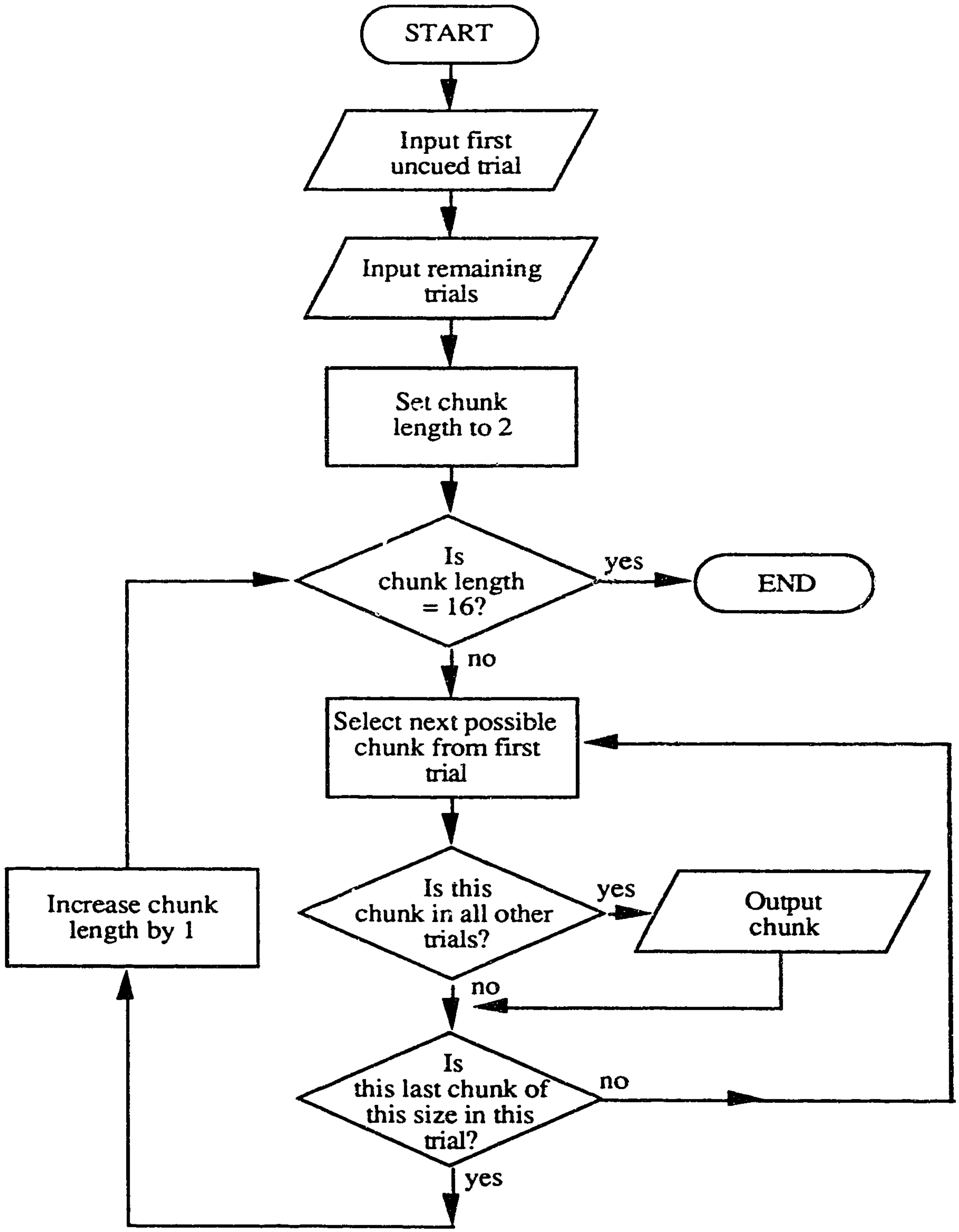
Cards:

adjacent	cost	decimal
denominator	discount	fraction
hundredths	interest	markup
part	percent	ratio
sale price	sales tax	square
	zero	

APPENDIX B

BASIC Program for the Cluster Analysis of Structured Recall Form Data

```
10 REM Written by A. Craig Loewen
20 REM CLUSTER:  Analysis of Structured Ordered Tree Forms
30 REM October 27, 1988
40 INPUT "Array length:  ";al
50 INPUT "Array width:  ";aw
60 DIM a(aw,al)
70 REM Input Data
80 FOR x = 2 TO aw:  Print "row: ";x
90 FOR y = 1 TO al: a(1,y) = y
100 PRINT "(";x;",";y;")";"INPUT a(x,y)
110 NEXT y
120 NEXT x
130 PRINT "***MATRIX*********************************"
140 FOR x= 1 TO aw: PRINT "ROW"x": ";
150 FOR y = 1 TO al
160 PRINT a(x,y)" ";
170 NEXT y: PRINT: NEXT x
180 PRINT "*******************************************"
190 INPUT "Errors? <y/n>: ";a$
200 IF a$="n" THEN 240
210 INPUT "Row, Column: ";r,c
220 INPUT "Entry: ";e
230 a(r,c) = e: PRINT: GOTO 130
240 PRINT "BEGINNING ANALYSIS -----------------"
250 PRINT "Chunk Listing:"
300 FOR x = 1 TO al - 1
310 FOR y = x + 1 TO al
320 low = a(1,x)
330 high = a(1,y)
340 wid = y-x+1
350 flag = 0
360 FOR row = 2 TO aw
370 FOR col = 1 TO al - wid + 1
380 FOR z = col TO col + wid - 1
390 IF a(row,z) < low OR a(row,z) > high THEN 420
400 NEXT z
410 flag = flag + 1: GOTO 430
420 NEXT col: GOTO 450
430 NEXT row
440 IF flag = aw - 1 THEN PRINT "chunk:  ";low;"-";high
450 NEXT y
460 NEXT x
470 PRINT "END ANALYSIS ----------------------------------------"
```


START
Input first uncued trial
Input remaining trials
Set chunk length to 2
Is chunk length = 16?
yes
END
no
Select next possible chunk from first trial
Is this chunk in all other trials?
yes
Output chunk
no
Is this last chunk of this size in this trial?
no
yes
Increase chunk length by 1

APPENDIX C

Diagnostic-Performance Test
Form "A"

Name:
School:
Teacher:
Date:

FOR RESEARCH USE ONLY:

Split-Half Scores - A:	B:

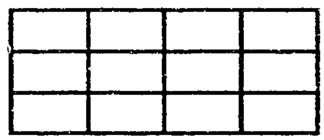
Please leave the test closed until you are told to begin.

Diagnostic-Performance Test - Form "A"

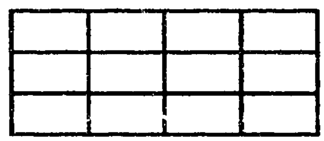
Instructions: Write your answers in this booklet in the spaces provided. Please be neat, otherwise your responses cannot be counted. The questions will get increasingly more difficult as you progress through the booklet. *It is very important that you attempt the questions and problems in the order in which they are given.* If you know how to solve a problem, then solve it and place your answer in the blank provided. If you do not even know how to start a problem, then skip it and go on to the next one. You may find you cannot answer all the questions in this booklet. This is to be expected, so do not be discouraged! There is no time limit, but you should work as quickly and accurately as you can. You may use a calculator if you wish. Please use the margins and the backs of the pages as scrap paper.

(1) Shade in the correct number of boxes:

A. Shaded : Whole = 2 : 3



B. Shaded : Whole = 3 : 4



(2) Place a check (√) in the box if the statement is true:

A. $\frac{12}{10} = \frac{6}{5}$ ☐

B. $\frac{7}{15} = \frac{21}{30}$ ☐

(3) Find the missing number:

A. $\frac{?}{12} = \frac{40}{48}$

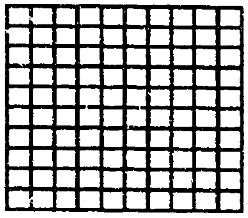
Answer:

B. $\frac{75}{?} = \frac{5}{6}$

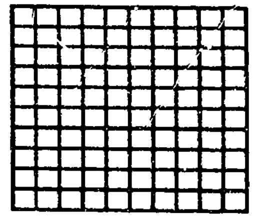
Answer:

(4) Shade in the correct number of boxes:

A. $\frac{25}{100}$



B. $\frac{60}{100}$



---

(5) Complete the chart:

	Fraction	Percent	Decimal
A.	4/5		
B.	9/25		

---

(6) Complete the chart:

	Fraction	Percent	Decimal
A.		65%	0.65
B.		4%	0.04

(7) Calculate:

A. Find 2% of 1300.

Answer:

B. Find 15% of 120.

Answer:

---

(8) Calculate:

A. 7 is what percent of 10?

Answer:

B. 15 is what percent of 25?

Answer:

(9) Give two examples of situations where it is possible to have a percent greater than 100%.

Explain your answers.

A.

B.

(10) Complete the chart:

	Fraction	Percent
A.	1 1/4	
B.	2 1/10	

(11) Complete the chart:

	Fraction	Percent
A.		150%
B.		225%

(12) Calculate:

A. 60 is 125% of what number?

Answer:

B. 15 is 60% of what number?

Answer:

(13) Complete the chart:

	Percent Discount	Regular Price	Sales Price
A.	25%	$120	
B.	20%	$250	

(14) Complete the chart:

	Sales Tax Rate	Regular Price	Total Cost
A.	5%	$120	
B.	8%	$350	

(15) Complete the chart:

	Principal	Annual Rate	Time	Interest
A.	$10	10%	1 year	
B.	$300	12%	9 months	

(16) Calculate:

A. Increase 120 by 40%.

Answer:

B. Increase 200 by 150%.

Answer:

(17) Calculate:

A. Decrease 40 by 20%.

Answer:

B. Decrease 8000 by 75%.

Answer:

(18) Complete the chart:

	Percent Discount	Regular Price	Sales Price
A.	10%		$54
B.	25%		$135

(19) Complete the chart:

	Sales Tax Rate	Regular Price	Total Cost
A.	10%		$33
B.	15%		$161

(20) Calculate:

A. A number was increased by 30% to get 260.
Find the number.

Answer:

B. A number was increased by 70% to get 51.
Find the number.

Answer:

(21) Calculate:

A. A number was decreased by 20% to get 40.
Find the number.

Answer:

B. A number was decreased by 5% to get 285.
Find the number.

Answer:

(22) Complete the chart:

	Percent	Decimal
A.	87 1/2%	
B.	56 3/4%	

(23) Complete the chart:

	Percent	Fraction
A.	12 1/2%	
B.	31 1/4%	

(24) Complete the chart:

	Principal	Annual Rate	Time	Interest
A.	$50		1 years	$6
B.		6%	6 months	$9

(25) Complete the chart:

	Sales	Commission	Pay
A.		20%	$15
B.	$800		$56

(26) Complete the chart:

	Cost Price	Selling Price	Percent Profit
A.	$15	$45	
B.		$200	150%

(27) Complete the chart:

	Cost Price	Selling Price	Percent Loss
A.	$90		50%
B.		$180	10%

END OF TEST

APPENDIX D

Clinical Interview Questions

(1) Students are presented with a single box of smarties candies and are asked to open the box and spread the candies out on the tabletop.

Smarties Problem I: If you had a box of 100 smarties with the same colors in the same ratio as your box, how many red ones would there be?

(2) Students are given two loops. One red and two green smarties placed inside of the first loop (set I). Four red and six green smarties are placed inside of the of the second loop (set II).

Smarties Problem II: How many green smarties should be added or removed from the second set to make the percent of red in each set equal?

(3) Students are given one loop. Two red and three green smarties are placed inside the loop.

Smarties Problem III: What would happen to the percent of red smarties if: (a) one green smartie was added? (b) one red smartie was added? (c) one red and one green smarties was added?

(4) Students were presented with a high-bounce rubber ball and allowed to bounce the ball a number of times and describe the behavior of the ball.

Bouncing Ball Problem: A rubber ball bounces back up to 60% of its original height when it is dropped. If the ball is dropped from a height of 1 m how many bounces before its height is less than 20 cm?

(5) The students are shown a card with a series of five ratios placed in fraction form.

Conversion Problems: Express each of the following ratios as a percent:

$\frac{1}{4}$ $\frac{4}{25}$ $\frac{3}{10}$ $\frac{18}{24}$ $\frac{7}{30}$

(6) The students were presented with a balloon and were asked to blow it up and then to let the air out of it again.

Balloon Problem: When inflated, a balloon is 250% of its deflated volume. If its inflated volume is 300 cm^3, what is its deflated volume?

(7) No physical or manipulative material was presented with this problem.

Growth Problem: At two years of age a child is 45% of the height he will be when full grown. Bob grew up to be 180 cm tall. How tall was he when he was 2 years old?

(8) No physical or manipulative material was presented with this problem.

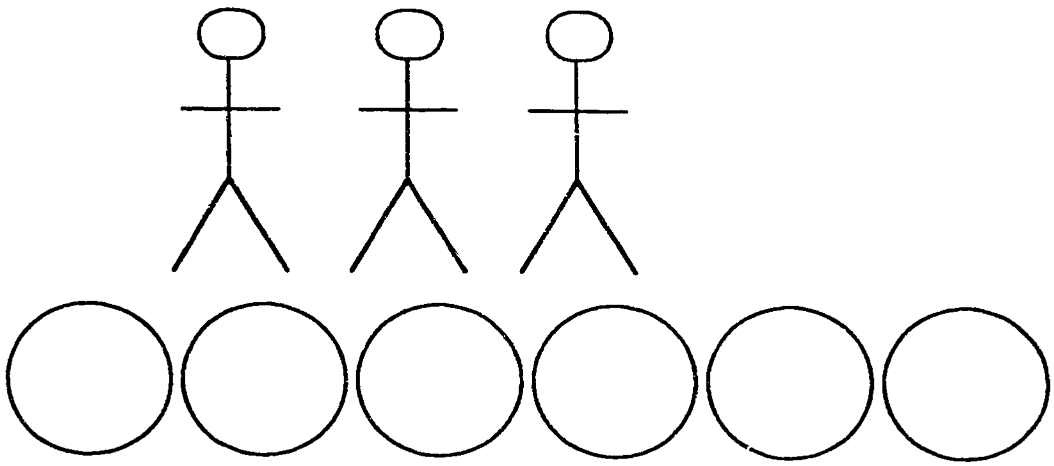
Interest Problem: Kelly puts $100 in the bank. At the end of each year Kelly gets 10% interest which is added directly into the account. After 3 years, how much money is in the account?

(9) No physical or manipulative material was presented with this problem.

Sales Tax Problem: The Canadian government is about to impose a 9% sales tax. If you bought a $20 T-shirt, a $5 pair of socks, and a $50 pair of jeans, how much sales tax would you have to pay?

(10) A diagram like the one shown below was presented to the student along with this problem.

Pizza Problem: 5.1/4 pizzas are to be split among 3 people. What percent of a pizza does each person get?



(11) A diagram consisting of two large circles was presented to the student along with this problem.

Circlegraph Problem: Jane spent 50% of her allowance on a movie, 30% on a new pencil case, and 30% flowers for her mother. Draw a circlegraph and explain your drawing.

(12) Three cards were presented along with this problem. On the first card a line 7 cm in length had been drawn. The second card contained a line 10 cm in length, and the third card contained a square 10 cm on a side.

Photocopier Problem: A 10 cm long line is drawn on a piece of paper and then reduced with a photocopier. The photocopied line comes out 7 cm long. A square (10 cm on a side) is constructed and reduced using the same machine. Express the area of the final square as a percent of the original square.

APPENDIX E

Data Tables

Table 9: ST task means by class and teaching approach.

Sample	PRO-Pre M	PRO-Pre n	PRO-Pre SD	PRO-Post M	PRO-Post n	PRO-Post SD	PRO-Ret M	PRO-Ret n	PRO-Ret SD
Class 11	14.3	21	9.35	13.7	22	9.82	11.7	21	9.86
Class 12	19.2	22	8.49	17.4	23	8.41	18.9	20	10.15
Class 13	20.0	22	10.17	12.2	20	9.44	13.4	18	12.10
Class 21	18.2	22	11.48	12.3	21	9.69	11.6	22	9.82
Class 22	9.8	24	7.27	12.0	26	10.17	9.4	26	8.50
Class 23	13.9	24	9.61	13.6	22	9.65	14.2	21	11.19
Class 31	18.6	22	8.10	15.8	23	10.16	13.9	21	10.51
Class 32	16.9	27	7.47	12.8	22	6.73	13.9	22	9.20
Class 33	15.6	22	11.21	14.7	27	10.72	13.9	24	10.62
Direct	17.9	65	9.55	14.6	65	9.33	14.7	59	10.96
Meaning	13.9	70	10.01	12.6	69	9.74	11.6	69	9.85
Pr Proc	17.0	71	8.92	14.5	72	9.43	13.9	67	9.98
Total	16.2	206	9.61	13.9	206	9.50	13.3	195	10.28

Table 10: D-P test means by class and teaching approach.

Sample	Obj-Pre			Obj-Post			Obj-Ret		
	M	n	SD	M	n	SD	M	n	SD
Class 11	10.8	24	5.20	14.5	25	5.39	13.4	25	5.70
Class 12	10.0	25	3.70	11.7	24	4.19	12.2	25	5.22
Class 13	8.4	25	4.00	13.4	21	5.05	11.0	23	5.30
Class 21	10.3	23	4.04	15.7	21	4.95	14.4	21	5.16
Class 22	10.4	28	4.43	12.8	27	5.58	9.8	28	4.40
Class 23	8.3	27	3.31	10.0	25	4.36	9.5	24	5.19
Class 31	10.3	24	5.09	14.5	24	5.67	13.5	24	4.92
Class 32	9.0	28	3.77	11.8	24	4.44	11.9	23	4.03
Class 33	8.3	25	4.70	12.3	26	6.00	11.1	22	4.62
Direct	9.7	74	4.39	13.2	70	4.98	12.2	73	5.43
Meaning	9.7	78	4.02	12.7	73	5.42	11.0	73	5.29
Pr Proc	9.2	77	4.53	12.9	74	5.49	12.2	69	4.58
Total	9.5	229	4.31	12.9	217	5.29	11.8	215	5.13

Table 11: ST task means by ability groups.

Sample	n	PRO-Pre M	PRO-Pre SD	PRO-Post M	PRO-Post SD	PRO-Ret M	PRO-Ret SD
D - Hi	6	2.3	2.08	1.7	1.41	5.7	9.18
Med	8	14.0	1.34	13.4	7.62	9.9	9.83
Lo	12	29.4	1.67	16.0	12.03	18.5	11.68
M - Hi	15	2.4	2.33	4.3	4.52	6.2	8.94
Med	9	13.1	1.18	13.9	9.42	13.0	9.32
Lo	9	28.6	1.74	17.5	11.27	19.0	10.19
PP - Hi	7	3.8	2.72	11.4	11.10	9.7	8.68
Med	12	13.4	0.97	10.9	6.02	13.3	7.20
Low	11	29.2	1.53	22.5	7.56	23.1	10.57
Direct	26	18.4	11.36	11.9	10.68	12.9	11.58
Meaning	33	12.5	11.15	10.5	9.89	11.6	10.59
Pr Proc	30	16.9	10.33	15.3	9.51	16.1	10.28
High	28	2.7	2.37	5.5	7.18	7.0	8.74
Medium	29	13.5	1.15	12.5	7.49	12.3	8.46
Low	32	29.1	1.63	18.7	10.54	20.2	10.76
Total	89	15.7	11.18	12.5	10.10	13.5	10.83

Table 12: D-P test means by ability groups.

Sample	n	Obj-Pre M	Obj-Pre SD	Obj-Post M	Obj-Post SD	Obj-Ret M	Obj-Ret SD
D - Hi	6	15.8	1.72	17.2	5.85	17.7	5.82
Med	12	9.4	0.52	14.5	1.62	14.1	1.68
Lo	6	5.0	0.89	9.7	3.27	9.0	4.43
M - Hi	12	16.5	3.29	19.3	4.75	17.6	5.33
Med	9	9.8	0.44	12.0	3.97	9.1	3.69
Lo	9	4.7	1.12	6.3	2.35	5.4	2.46
PS - Hi	9	16.9	2.93	19.8	2.59	16.4	4.19
Med	11	9.4	0.51	13.2	3.66	13.5	2.81
Lo	15	4.4	0.99	9.6	4.64	8.7	3.11
Direct	24	9.9	4.06	14.1	4.46	13.7	4.79
Meaning	30	10.9	5.47	13.2	6.68	11.4	6.68
Pr Proc	35	9.2	5.32	13.3	5.62	12.2	4.61
High	27	16.5	2.82	19.1	4.33	17.2	4.93
Medium	32	9.5	0.51	13.3	3.22	12.5	3.42
Low	30	4.6	1.00	8.6	4.01	7.8	3.48
Total	89	10.0	5.07	13.5	5.68	12.3	5.45

Table 13: Similarity score means by ability groups.

Sample	n	Sim Pre/Post M	Sim Pre/Post SD	Sim Post/Ret M	Sim Post/Ret SD	Sim Pre/Ret M	Sim Pre/Ret SD
D - Hi	6	0.50	0.30	0.34	0.27	0.41	0.39
Med	8	0.60	0.15	0.43	0.24	0.39	0.23
Lo	12	0.08*	0.19	0.22*	0.23	0.20**	0.42
M - Hi	15	0.13	0.23	0.19	0.21	0.08	0.14
Med	9	0.34	0.23	0.58*	0.32	0.27	0.21
Lo	9	0.05	0.14	0.15	0.19	0.24	0.35
PP - Hi	7	0.39	0.20	0.49	0.24	0.32	0.16
Med	12	0.38	0.37	0.47	0.19	0.33	0.28
Low	11	0.25*	0.36	0.26*	0.28	0.05*	0.16
Direct	26	0.35*	0.31	0.32*	0.25	0.32**	0.36
Meaning	33	0.17	0.24	0.28*	0.29	0.18	0.24
Pr Proc	30	0.34*	0.33	0.41*	0.25	0.23*	0.25
High	28	0.28	0.28	0.30	0.26	0.21	0.26
Medium	29	0.42	0.29	0.49*	0.24	0.33	0.24
Low	32	0.13**	0.26	0.21**	0.24	0.16***	0.33
Total	89	0.27**	0.30	0.33***	0.27	0.23***	0.28

Note: A similarity score which was undefined resulted in a reduced sample size. Instances where this occured are marked with an asterisk. The number of asterisks represents the number of undefined scores.

Table 14: Proportions of students correctly answering questions on D-P test.

Question	Pre Test n=229		Post Test n=217		Retention Test n=215	
	Raw	%	Raw	%	Raw	%
1	165	72.1	143	65.9	154	71.6
2	216	94.3	197	90.8	200	93.0
3	184	80.3	173	79.7	167	77.7
4	224	97.8	213	98.2	212	98.6
5	176	76.9	187	86.2	175	81.4
6	142	62.0	158	72.8	162	75.3
7	73	31.9	172	79.3	143	66.5
8	159	69.4	150	69.1	137	63.7
9	46	20.0	77	35.5	73	34.0
10	137	59.8	136	62.7	141	65.6
11	135	59.0	139	64.1	128	59.5
12	58	25.3	108	49.8	90	41.9
13	87	38.0	162	74.7	130	60.5
14	73	31.9	167	77.0	140	65.1
15	18	7.9	36	16.6	64	29.8
16	60	26.2	116	53.5	98	45.6
17	67	29.3	120	55.3	106	49.3
18	11	4.8	30	13.8	25	11.6
19	7	3.1	28	12.9	20	9.3
20	11	4.8	31	14.3	22	10.2
21	13	5.7	32	14.7	23	10.7
22	60	26.2	113	52.1	106	49.3
23	14	6.1	32	14.7	14	6.5
24	3	1.3	18	8.3	7	3.3
25	17	7.4	52	24.0	40	18.6
26	4	1.7	6	2.8	2	0.9
27	19	8.2	15	6.9	10	4.7