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PERMANENT ADDRESS:

MATH DEPT.
U. DE SHERBROOKE
SHERBROOKE, QUÉBEC

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ON $\mathcal{S}_p$ SPACES

BY

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled On L_p Spaces.

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Supervisor

External Examiner

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ABSTRACT

We wish to present some results concerning a collection of B-spaces which, on the one hand, have a nice characterization in terms of finite-dimensional spaces and on the other hand are (in the separable case) isomorphic to complemented subspaces of $L_p[0,1]$.

In chapter 0 we collect together known results concerning these $\mathcal{L}_p$-spaces and introduce some necessary notation. We sometimes will give short heuristic indications of proofs.

Our first chapter concerns the process of direct summing of $\mathcal{L}_p$-spaces in order to form (possibly new) $\mathcal{L}_p$-spaces. The results obtained are of a negative nature and indicate that there is probably only one way to norm an infinite direct sum in order to get a $\mathcal{L}_p$-space.

Chapter 2 deals with tensoring $\mathcal{L}_p$ and $\mathcal{L}_2$ spaces and completion with respect to an appropriate norm.

In the final chapter we determine the isomorphism types of the closed linear spans of subsequences of the Haar system in $L_p[0,1]$ for $1 < p < \infty$; it is shown that l_p and $L_p[0,1]$ are the only possibilities.

There are many open problems in this area of analysis and we will ask questions, when appropriate, which we feel would be important and useful if solved. There is still very much to do; an indication of this comes from the fact that there are at this writing only 9 known isomorphic types of separable infinite-dimensional $\mathcal{L}_p$-spaces for $1 < p < \infty$, $p \neq 2$. It is not even known if the number is finite or not. Showing that two spaces are isomorphic or not is even more difficult than construction of new spaces.

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I would also like to sincerely thank all the mathematicians who have helped me; in particular John Gamlen for all the lively discussions in the coffee room. It is my fervent wish that more people do the same; I also wish on everyone such a pleasant, both mathematically and personally, supervisor-student relationship.

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CHAPTER 0

PRELIMINARIES

We gather here known results concerning $\mathcal{L}_\infty$-spaces. These were introduced in 1968 by J. Lindenstrauss and A.Pelczyński; the lion's share of results since then have been obtained by them and by H.P. Rosenthal.

We start with basic notation and definitions.

<u>Definition 1</u> Unless otherwise specified, all subspaces are assumed to be closed.

<u>Definition 2</u> We say two B-spaces X and Y are isomorphic and write $X \sim Y$ if there is a linear 1 - 1 mapping T of X onto Y with T and T^{-1} continuous. In such a case we get:

$$\frac{1}{\|T^{-1}\|}\|x\| \leq \|Tx\| \leq \|T\|\,\|x\| \quad \text{for all } x \in X .$$

The constants $\|T^{-1}\|$ and $\|T\|$ above are an indication of how close the mapping T is to being an isometry.

<u>Definition 3</u> We define the distance $d(X,Y)$ between two B-spaces X and Y as follows:

If X and Y are not even isomorphic put $d(X,Y) = \infty$.

If $X \sim Y$, we let

$$d(X,Y) = \inf \{\|T\| \|T^{-1}\| \mid T \text{ is an isomorphism of } X \text{ onto } Y.\}.$$

Of course this does not define a metric but these properties are easily seen to hold:

$$1 \le d(X,Y) \le \infty$$

$d(X,Y) = 1$ if X and Y are isometric

$$d(X,Z) \le d(X,Y)d(Y,Z).$$

Hence $d\log d(X,Y)$ is a pseudo-metric. However $d(X,Y)$ is easier to work with so we do so.

<u>Definition 4</u> Let $1 \le p \le \infty$ and $\lambda \ge 1$. A B-space X is called a $\mathcal{L}p,\lambda$-space if for every finite-dimensional subspace E of X there is a finite-dimensional subspace F of X with $E \subset F$ and $d(F,l_p^n) \le \lambda$ where $n = \dim F$.

We say X is a $\mathcal{L}p$-space if it is a $\mathcal{L}p,\lambda$-space for some finite λ.

Heuristically, X is a $\mathcal{L}p$-space if it has subspaces uniformly close to l_p^n spaces "embedded everywhere".

This is a property enjoyed by $Lp(\mu)$ spaces so the concept of $\mathcal{L}p$-spaces generalizes $Lp(\mu)$ spaces. One of the early known results shows how deep the link is:

Definition 5 If Y is a subspace of a B-space X we say Y is complemented in X if there is a continuous linear mapping P of X onto Y with $P^2 = P$. Such a mapping is called a projection of X onto Y.

It is well-known (e.g. by closed graph theorem) that Y is complemented in X if and only if there exists another subspace Y' of X with $Y \cap Y' = \{0\}$ and $X = Y + Y'$. We call such a subspace Y' a complement for Y in X and usually write $X = Y \oplus Y'$.

Theorem 1 (Lindenstrauss & Pelczyński [11].) Let X be a B-space and let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. Assume that for every finite-dimensional subspace E of X there is a subspace $\tilde{E}$ of l_p with $d(E,\tilde{E}) \leq \lambda$. Then there is a measure μ and a subspace Y of $Lp(\mu)$ with $d(X,Y) \leq \lambda$. (We outline the proof: Using the isomorphisms $E \leftrightarrow \tilde{E}$, one defines an extended real-valued mapping on the bounded functions on the unit ball of X^*. We may apply a theorem of Bohnenblust to the subspace of those functions for which this is finite, deducing that the latter is $Lp(\mu)$ for some measure μ. But X can be embedded in bounded functions on the unit ball of its dual (evaluation) and in fact its image lies inside this newly-found space $Lp(\mu)$. The space $\hat{Y}$ is this image of Y; the needed norm inequalities easily hold.)

<u>Theorem 2</u> Let X be a $\mathcal{L}_{2,\lambda}$ space. Then there is a Hilbert space H with $d(X,H) \leq \lambda$. [11]

<u>Theorem 3</u> Let $1 < p < \infty$ and let X be a $\mathcal{L}p$-space. Then X is isomorphic to a complemented subspace of an $Lp(\mu)$ space. [11]

<u>Theorem 4</u> Let X be a $\mathcal{L}_1(\mathcal{L}_\infty)$ space. Then X is isomorphic to a complemented subspace of an $L_1(\mu)$ $(L_\infty(\mu))$ space if and only if X is complemented in X^{**}. [11]

The above are again due to Lindenstrauss and Pełczyński; one needs only to be more careful in the proof of theorem 1.

<u>Theorem 5</u> Let X be a separable $\mathcal{L}p$-space. Then X is isomorphic to a subspace of $Lp[0,1]$. If $1 < p < \infty$ this subspace can be taken to be complemented. [11]

This holds because a separable subspace of an $Lp(\mu)$

space is isomorphic to a subspace of a separable $Lp(\mu)$ space which in turn is isometric to a subspace of $Lp[0,1]$.

<u>Theorem 6</u> Let $1 < p < \infty$. A separable B-space X is isometric to an $Lp(\mu)$ space for some measure μ if and only if X is a $\mathcal{L}_{p,1+\varepsilon}$-space for every $\varepsilon > 0$. [11]

<u>Theorem 7</u> Let X be an infinite-dimensional $\mathcal{L}p$-space with $1 \leq p < \infty$. Then X has a complemented subspace isomorphic to l_p. [11]

<u>Theorem 8</u> Let $1 \leq p \leq r \leq 2$. Then $L_r[0,1]$ is isometric to a subspace of $Lp[0,1]$. [11]

This is good justification for not attempting to find all subspaces of $Lp[0,1]$. On one hand there are too many different isomorphic types, at least for $p < 2$, and on the other hand, the uncomplemented subspaces can possess properties which are vastly different from those enjoyed by the whole space.

<u>Theorem 9</u> Let X be a B-space and $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. Then there is a measure μ and a subspace Y of $Lp(\mu)$ with $d(X,Y) \leq \lambda$ if and only if whenever

we have $\{U_i\}_{i=1}^{n}$, $\{V_j\}_{j=1}^{m} \subset X$ with

$$\sum_{i=1}^{n} |x^*(U_i)|^p \geq \sum_{j=1}^{m} |x^*(V_j)|^p \text{ for all } x^* \in X^*,$$

then we also have $\lambda^p \sum_{i=1}^{n} \|U_i\|^p \geq \sum_{j=1}^{m} \|V_j\|^p$.

(The alteration for $p = \infty$ is obvious.) [11]

This can rule out possible candidates for $\mathcal{L}p$-spaces: if a space is not isomorphic to a subspace of an $Lp(\mu)$ space then it cannot possibly be a $\mathcal{L}p$-space.

<u>Theorem 10</u> (Lindenstrauss & Rosenthal [12].)

Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then a B-space X is a $\mathcal{L}p$-space if and only if X^* is a $\mathcal{L}_q$-space.

<u>Theorem 11</u> (Lindenstrauss & Rosenthal [12].)

Let $1 < p < \infty$. If X is a complemented subspace of a $\mathcal{L}p$-space then either X is a $\mathcal{L}p$-space or X is isomorphic to Hilbert space. If X is a complemented subspace of a $\mathcal{L}_1(\mathcal{L}_\infty)$ space then X is a $\mathcal{L}_1(\mathcal{L}_\infty)$ space.

Combining theorem 5 with theorem 11 we see that the problem of finding isomorphic types of separable $\mathcal{L}p$-spaces $(1 < p < \infty)$ is the same as classifying isomorphic types of the complemented subspaces of $Lp[0,1]$ other than l_2.

Furthermore, if X is a $\mathcal{L}_p$-space for $1 < p < \infty$ then X is reflexive and its dual X^* is a $\mathcal{L}_q$ space. Hence if we have two non-isomorphic $\mathcal{L}_p$-spaces, then their duals are non-isomorphic $\mathcal{L}_q$-spaces. Since the case $p = 2$ is done, we see that what we want to do is find the $\mathcal{L}_p$-spaces for $p > 2$.

Thus most of what will now be done will be with $2 < p < \infty$. (It is easier to work with $2 < p < \infty$ than with $1 < p < 2$.)

We will now give a brief summary and construction of the known separable $\mathcal{L}_p$-spaces. $(2 < p < \infty)$ We show how these spaces can be considered as complemented subspaces of $L_p[0,1]$.

1) l_2

Though not a $\mathcal{L}_p$-space for $p \neq 2$, l_2 is isomorphic to a complemented subspace of $L_p[0,1]$ for $1 < p < \infty$; the span of the Rademacher functions in $L_p[0,1]$ gives the desired space. (Khintchine.)

2) l_p

This is easily obtained by considering the closed linear span of a sequence of characteristic functions of disjoint measurable sets of positive measure.

3) $l_p \oplus l_2$

We consider a subspace of $Lp[0,\frac{1}{2}]$ isomorphic to l_p and complemented there. (This is easily done since $Lp[0,\frac{1}{2}]$ is isometric to $Lp[0,1]$ by a natural map.) We also find a subspace of $Lp[\frac{1}{2},1]$ isomorphic to l_2 and complemented. The direct sum of these two spaces is the desired one. This is an example of the following theorem which is part of folklore.

Theorem 12 If X and Y are isomorphic to complemented subspaces of $Lp[0,1]$ then so is $X \oplus Y$.

4) $Z_p = (l_2 \oplus l_2 \oplus \dots)_p$

This is the B-space of all sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in l_2$ for all n for which the norm $\|\{x_n\}\| = (\sum_{n=1}^{\infty} \|x_n\|_2^p)^{\frac{1}{p}}$ is finite.

This is easily obtainable by constructing a Rademacher system in $Lp[n,n+1]$ for each n; this gives spaces $X_n \subset Lp[n,n+1]$ which are uniformly complemented and at a uniformly small distance from l_2.

From the fact that $L_p[0,1]$, $L_p[n,n+1]$ and $L_p[0,\infty)$ are all isometric, it follows that Z_p is isomorphic to $(X_1 \oplus X_2 \oplus \ldots)_p$ which is complemented in $L_p[0,\infty)$. Hence we get, via the isometry, the required space in $L_p[0,1]$.

This again seems to be part of folklore and has been stated without proof by Lindenstrauss and Pelczyński. [11] The proof is quite easy.

Theorem 13. Let X be isomorphic to a complemented subspace of $L_p[0,1]$. Then so is $(\sum X)_p = (X \oplus X \oplus \ldots)_p$.

5) H.P. Rosenthal's space X_p.

Let $2 < p < \infty$ and $\{f_n\}$ be a sequence of independent (probability sense), symmetric ($\mu\{x \mid f_n(x)=\alpha\} = \mu\{x \mid f_n(x) = -\alpha\}$ for every n and α), 3-valued functions in $L_p[0,1]$. If W_p denotes the closed linear span of $\{f_n\}$ in $L_p[0,1]$ then W_p is complemented.

The isomorphic type of W_p is determined by the behaviour of the quotients $W_n = \|f_n\|_2 / \|f_n\|_p$.

If $X_{p,w}(w = \{w_n\})$ denotes the space of all sequences $\{\lambda_n\}$ for which the norm $\|\{\lambda_n\}\| = \max\{(\sum |\lambda_n|^p)^{1/p}, (\sum \lambda_n^2 w_n^2)^{1/2}\}$ is finite then $X_{p,w}$ and W_p are isomorphic.

Conversely any space of sequences $X_{p,w}$ defined as above by means of an arbitrary sequence of positive real numbers $w = \{w_n\}$ (less than 1) is isomorphic to a space W_p obtained as above from some functions in $Lp[0,1]$.

The following hold:

A) If $\sum w_n^{2p/(p-2)} < \infty$ then $X_{p,w} \sim l_p$. (Use Hölder's inequality on $\sum \lambda_n^2 w_n^2$ and apply open mapping theorem.)

B) If $\inf w_n > 0$ then $X_{p,w} \sim l_2$. (Again by open mapping theorem.)

C) If $\{w_n\}$ splits into 2 disjoint subsequences one satisfying A) and the other B) we get $X_{p,w} \sim l_p \oplus l_2$.

D) If none of the above happens, that is $\lim_{n\to\infty} w_n = 0$ and $\sum_{w_n < \varepsilon} w_n^{2p/(p-2)} = \infty$ for all $\varepsilon > 0$, we have a fourth isomorphic type.

We call X_p the space defined by a sequence $\{w_n\}$ with D) holding. It is known that any two spaces $X_{p,w}$ and $X_{p,w'}$ are isomorphic if w and w' both satisfy D). Hence X_p (we are searching for isomorphic types) is well-defined.

6) B_p

For each n define $B_{p,n}$ to be the space X_{p,w^n} for the sequence $w^n = \{w_j^n\}_{j=1}^{\infty}$: $w_j^n = n^{-(p-2)/2p}$ for all j. Each $B_{p,n}$ is isomorphic to l_2 but the (Banach-Mazur) distance increases to ∞. Let $B_p = (B_{p,1} \oplus B_{p,2} \oplus \ldots)_p$.

7) $B_p \oplus X_p$

8) $Z_p \oplus X_p$

9) $Y_p = (X_p \oplus X_p \oplus \ldots)_p$

10) $Lp[0,1]$.

The construction of the spaces X_p, B_p and Y_p and their properties is due to H.P. Rosenthal. [15]. He has shown that the nine known separable $\mathcal{L}p$-spaces listed above are non-isômorphic; part of this is published [15] and the rest is unpublished.

CHAPTER 1

SUMMING OF $\mathcal{L}_p$ SPACES

We saw in Chapter 0 that the direct sum (or product if you prefer) of two $\mathcal{L}p$-spaces or of a $\mathcal{L}p$ and a $\mathcal{L}_2$ space was again a $\mathcal{L}p$-space and that an infinite direct p-sum of either a $\mathcal{L}p$ or a $\mathcal{L}_2$ space was a $\mathcal{L}p$-space.

We investigate here in what ways the infinite sum of a $\mathcal{L}p$ or $\mathcal{L}_2$ space can be normed so as to produce a $\mathcal{L}p$ space. We show that we must use an unconditional basis for some $\mathcal{L}p$-space which contains no copy of l_2.

It is illuminating to realize that all the known separable $\mathcal{L}p$-spaces are obtained by a p-summing process applied to the spaces: $l_p, l_2, X_p, Lp[0,1]$ (either 2 or an infinite number at a time).

We start with some basic definitions:

<u>Definition 1</u> A sequence $\{x_i\}$ in a (necessarily) separable B-space is called a basis for X if every $x \in X$ has a unique representation $x = \sum \lambda_i x_i$ where the convergence is in the norm of X.

We say $\{x_i\}$ is an unconditional basis if every x

can be written uniquely as $x = \sum \lambda_i x_i$ and where the series converges unconditionally in norm to x. (i.e. the series $\sum \lambda_{\sigma(i)} x_{\sigma(i)}$ converges for every permutation σ of the positive integers.).

A sequence $\{x_i\}$ is called a basic sequence (unconditional basic sequence) if it is a basis (unconditional basis) for its closed linear span.

We will write $[x_i]$ for the closed linear span of a sequence $\{x_i\}$. We will sometimes subscript this with p to remind us that we are dealing with the closed linear span in $L_p[0,1]$.

It is well-known that if $\{x_i\}$ is a basic sequence then it is an unconditional basic sequence if and only if whenever $\sum \lambda_i x_i$ converges and $|\mu_i| \le |\lambda_i|$ for all i, then $\sum \mu_i x_i$ also converges; if and only if whenever $\sum \lambda_i x_i$ converges then $\sum \varepsilon_i \lambda_i x_i$ also converges for every choice of $\varepsilon_i = \pm 1$; if and only if whenever $\sum \lambda_i x_i$ converges then $\sum \delta_i \lambda_i x_i$ also converges for every choice of $\delta_i = 0,1$.

In case $\{x_i\}$ is an unconditional basic sequence then there is a new norm on $[x_i]$, equivalent to the original norm, such that if $\sum \lambda_i x_i$ converges and $|\mu_i| \le |\lambda_i|$ for all i, then new norm $(\sum \mu_i x_i) \le$ new norm $(\sum \lambda_i x_i)$.

Definition 2 Let Y be a B-space and $\{x_i\}$ a basic sequence. We define $(\sum Y)_{\{x_i\}} = (Y \oplus Y \oplus \dots)_{\{x_i\}}$ to be the space of all sequences $\{y_i\}$ of elements of Y for which $\|\{y_i\}\| = \|\sum \|y_i\| x_i\|_{[x_i]}$ is finite.

It is easy to see and seems part of folklore that in order for $(\sum Y)_{\{x_i\}}$ to be a linear space, for the above to define a norm and for this linear space to be complete with respect to this norm, what we need is exactly that $\{x_i\}$ be an unconditional basic sequence with $[x_i]$ renormed as mentioned above. That is:

Theorem 1 Let Y be a B-space and $\{x_i\}$ an unconditional basic sequence; suppose $[x_i]$ is renormed as before definition 2. Then $(\sum Y)_{\{x_i\}}$ is a B-space under the norm in definition 2.

Our question is then the following: What requirements on $\{x_i\}$ do we need so that $(\sum Y)_{\{x_i\}}$ will be a

$\mathcal{S}p$-space whenever Y is?

Theorem 2 If there exists an infinite-dimensional $\mathcal{S}p$-space Y for which $(\sum Y)_{\{x_i\}}$ is a $\mathcal{S}p$-space then $(\sum l_p)_{\{x_i\}}$ is a $\mathcal{S}p$-space. $(1 \leq p < \infty)$.

Proof: By theorem 7 of Chapter 0, Y has a subspace Z which is complemented in Y (by a projection P, say) and isomorphic to l_p (by an isomorphism $T : Z \to l_p$ say.)

Then it is easy to see that $(\sum Z)_{\{x_i\}}$ is a closed subspace of $(\sum Y)_{\{x_i\}}$. It is complemented there by the projection $Q : Q(\{y_i\}) = \{P y_i\}$. This projection Q is clearly continuous since

$$\|Q(\{y_i\})\| = \|\{Py_i\}\| = \| \sum \|Py_i\| x_i \|_{[x_i]}$$

$$\leq \|P\| \, \| \sum \|y_i\| x_i \|_{[x_i]} = \|P\| \, \|\{y_i\}\|$$

by the properties of the unconditional norm introduced on $[x_i]$ and the continuity of P.

Furthermore, $(\sum Z)_{\{x_i\}}$ is isomorphic to $(\sum l_p)_{\{x_i\}}$ by the isomorphism $S : S(\{z_i\}) = \{Tz_i\}$. The continuity properties of S and S^{-1} are inherited from those of

T and T^{-1} by an argument similar to the above.

Thus $(\sum l_p)_{\{x_i\}}$ is isomorphic to a complemented subspace of the $\mathcal{L}p$-space $(\sum Z)_{\{x_i\}}$. But then $(\sum l_p)_{\{x_i\}}$ is clearly not a $\mathcal{L}_2$-space; it is not isomorphic to Hilbert space since it clearly contains a subspace isomorphic to l_p.

Hence by theorem 11 of Chapter 0, $(\sum Z)_{\{x_i\}}$ is a $\mathcal{L}p$-space. But an isomorphic copy of a $\mathcal{L}p$-space is again a $\mathcal{L}p$-space.

Thus $(\sum l_p)_{\{x_i\}}$ is a $\mathcal{L}p$-space.

#

Therefore we want to investigate when $(\sum l_p)_{\{x_i\}}$ is a $\mathcal{L}p$-space. (We have just seen it cannot ever be a $\mathcal{L}_2$-space.)

<u>Theorem 3</u> If $(\sum l_p)_{\{x_i\}}$ is a $\mathcal{L}p$-space then $[x_i]$ is either a $\mathcal{L}p$-space or a $\mathcal{L}_2$-space.

Proof: Take a fixed element $x \neq 0$ of l_p. It is easy to see that $(\sum Rx)_{\{x_i\}}$ is isomorphic to $[x_i]$ and complemented in the $\mathcal{L}p$-space $(\sum l_p)_{\{x_i\}}$. By theorem 11 of Chapter 0, $[x_i]$ is either a $\mathcal{L}p$ or $\mathcal{L}_2$-space.

#

Now it is easy to see that we may as well have assumed, in the definition of infinite sum, that the unconditional basic sequence was normalized.

We quote the following known result:

Theorem 4 In l_2 all normalized unconditional bases are equivalent. (See Singer [18].)

By equivalent we mean the usual

Definition 3 Two bases $\{x_i\}$ and $\{y_i\}$ are called equivalent if $\sum \lambda_i x_i$ oonverges if and only if $\sum \lambda_i y_i$ converges.

If we have equivalent unconditional bases $\{x_i\}$ and $\{y_i\}$ then the correspondence $x_i \leftrightarrow y_i$ defines an isomorphism between $[x_i]$ and $[y_i]$.

We have thus reduced our problem to the following:

A) For what $\mathcal{L}_p$-spaces with unconditional basis $\{x_i\}$ is $(\sum l_p)_{\{x_i\}}$ a $\mathcal{L}_p$-space?

B) Is $(\sum l_p)_{l_2}$ a $\mathcal{L}_p$-space? (Where we mean l_2 using the unit vectors for an unconditional basis.)

The answer to B) is already known and is stated without proof by Lindenstrauss and Pelczyński [11].

<u>Theorem 5</u> If $p \neq 2$ then $(\sum l_p)_{l_2}$ is not a $\mathcal{L}_p$-space. If $2 < p < \infty$, $(\sum l_p)_{l_2}$ is not even isomorphic to a subspace of an $L_p(\mu)$ space.

We will now give a partial answer to question A). We start by quoting some known results which we will need:

<u>Theorem 6</u> (Kadec and Pelczyński [7].) If X is a subspace of $L_p[0,1]$ $(2 < p < \infty)$ isomorphic to l_2, then X is complemented in $Lp[0,1]$.

<u>Definition 4</u> Let $\{x_i\}$ be a basis for a B-space X. Let $\{p_n\}$ be an increasing sequence of positive integers and $\{\lambda_n\}$ a sequence of real numbers. Then a sequence

$\{y_i\}$ in X is called a block basic sequence with respect to $\{x_i\}$ if $y_i \neq 0$ for all i and the $\{y_i\}$ have the form: $y_i = \sum_{j=p_i+1}^{p_{i+1}} \lambda_j x_j$.

Theorem 7 Let Y be an infinite-dimensional subspace of a B-space X with basis $\{x_i\}$. Then there is in Y a basic sequence $\{y_i\}$ which is equivalent to a block basic sequence with respect to $\{x_i\}$. [2]

Since it is easy to see that a block basic sequence with respect to an unconditional basis is itself unconditional, the following holds:

Theorem 8 Let Y be an infinite-dimensional subspace of a B-space X with unconditional basis $\{x_i\}$. Then there is in Y an unconditional basic sequence $\{y_i\}$ which is equivalent to an unconditional block basic sequence with respect to $\{x_i\}$. [2]

But we know what the normalized unconditional basic sequences in l_2 are. We then get:

Theorem 9 Let X be a B-space with unconditional basis

$\{z_i\}$. Suppose Z contains a subspace isomorphic to l_2. Then there is an unconditional block basic sequence with respect to $\{z_i\}$ which is equivalent to the usual basis for l_2.

We will now obtain results which severely restrict possibilities for direct summing.

<u>Theorem 10</u> Let $(\sum_1^n l_p)_2$ denote the direct sum of n copies of l_p, normed with the usual l_2 norm. $(1 \leq n \leq \infty)$. Of course, if $n < \infty$, then $(\sum_1^n l_p)_2$ is isomorphic to l_p so must be a $\mathcal{L}_p$-space; but there does not exist a $\lambda < \infty$ for which $(\sum_1^n l_p)_2$ is an $\mathcal{L}_{p,\lambda}$ space for every n.

<u>Proof:</u> Consider the linear space Y consisting of those sequences $\{x_n\}$ in $(\sum_1^\infty l_p)_2$ with only a finite number of x_n being nonzero. It is clear that Y is dense and is the union of the spaces $(\sum_1^n l_p)_2$. If there were a $\lambda < \infty$ for which $(\sum_1^n l_p)_2$ was an $\mathcal{L}_{p,\lambda}$ space for all $1 \leq n < \infty$ then we could modify the proof of theorem 1 of Chapter 0 (Lindenstrauss and Pelczyński) by considering only the finite dimensional spaces E of Y with

$d(E, l_p^{\dim E}) < \lambda$. This would yield that $(\sum_1^\infty l_p)_2$ is also a $\mathcal{L}_p$-space since it would show that it is isomorphic to a complemented subspace of $L_p[0,1]$. But this is not true. #

Definition 5. Let $1 < p < \infty$ and define F_p to be the B-space of all sequences $\{\lambda_n\}$ for which the norm

$$\|\{\lambda_n\}\| = \Big(\sum_{i=1}^{\infty} \Big(\sum_{n=\frac{i(i-1)}{2}+1}^{i(i+1)/2} \lambda_n^2 \Big)^{p/2}\Big)^{1/p}$$

is finite. It is well-known (Singer [18] p. 544) that F_p is isomorphic to l_p; the proof uses known properties of the Rademacher functions.

Let $(\sum_1^\infty l_p)_{F_p}$ denote the infinite direct sum $(\sum l_p)_{\{e_n\}}$ where e_n are the unit vectors in the norm defining F_p. It is easy to see that

$$(\sum_1^\infty l_p)_{F_p} = (\sum_{n=1}^\infty (\sum_1^n l_p)_2)_p .$$

We will show that for $p \neq 2$, $(\sum_{n=1}^\infty (\sum_1^n l_p)_2)_p$ is not a $\mathcal{L}p$-space.

We first state a slightly stronger version of theorem 11 of Chapter 0.

<u>Theorem 11</u> Let $1 \leq p < \infty$ and $\lambda < \infty$. Suppose X is a $\mathcal{L}_{p,\lambda}$-space and let Y be a subspace of X complemented by a projection P. Suppose Y contains a subspace Z isomorphic to l_p. Then Y is a $\mathcal{L}_{p,\mu}$ space where μ depends only on λ, $\|P\|$ and $d(Z, l_p)$.

The proof is identical to the proof of theorem 11 of Chapter 0 but uses in addition the fact that a $\mathcal{L}_{p,\lambda}$ space is distance $\leq \lambda$ to a subspace of an $L_p(\mu)$ space complemented by a projection with norm $\leq \lambda$. (See Lindenstrauss and Pelczyński [11].)

<u>Theorem 12</u> $(\sum_{n=1}^{\infty} (\sum_1^n l_p)_2)_p$ is not a $\mathcal{L}p$-space for $p \neq 2$.

<u>Proof:</u> By contradiction.

Suppose $(\sum_{n=1}^{\infty} (\sum_1^n l_p)_2)_p$ is a $\mathcal{L}_{p,\lambda}$ space for some $\lambda < \infty$. Now every $(\sum_1^n l_p)_2$ contains a subspace isometric to l_p. Furthermore, each such subspace is complemented by a projection of norm 1. We then apply theorem 11 and get that there is a μ, depending only on λ, for which $(\sum_1^n l_p)_2$ is a $\mathcal{L}_{p,\mu}$ space for every n. This contradicts Theorem 11.   //

Theorem 13   Let $\{x_i\}$ be an unconditional basis for a $\mathcal{L}p$-space X. Suppose there exists $\lambda < \infty$ and for each n an n-dimensional subspace X_n of X with X_n spanned by elements $y_1^n \cdots y_n^n$ with $x_i'(y_j^n) \cdot x_i'(y_k^n) = 0$ for every i and $j \neq k$; suppose all these subspaces X_n are complemented by projections with norm $\leq \lambda$, and that for each n, if T_n maps y_i^n to the i^{th} unit vector in l_2^n, then $\|T_n\| \|T_n^{-1}\| \leq \lambda$. Then $(\sum l_p)_{\{x_i\}}$ is not a $\mathcal{L}p$-space.

Proof: Suppose $(\sum l_p)_{\{x_i\}}$ is a $\mathcal{L}p$-space. Then $(\sum(\sum l_p)_{\{x_i\}})_p$ is also a $\mathcal{L}p$-space. We will show this cannot happen: we will show this latter space contains a complemented subspace isomorphic to $(\sum_{n=1}^{\infty}(\sum_1^n l_p)_2)_p$, which is neither a $\mathcal{L}p$ nor a $\mathcal{L}_2$ space.

The space is constructed as follows: for each n consider $y_1^n \cdots y_n^n$ in the n^{th} copy of $(\sum l_p)_{\{x_i\}}$ in the direct sum $(\sum(\sum l_p)_{\{x_i\}})_p$. The assumption on

y_i^n says that no two can share a co-ordinate. That is there are constants λ_j and sets of integers (disjoint) $A_1^n \ldots A_n^n$ with $y_i^n = \sum_{j \varepsilon A_i} \lambda_j^n x_j^n$. (We superscript x_i to denote that we are dealing with the x_i found in the n^{th} copy of $(\sum l_p)_{\{x_i\}}$.) Consider, in the n^{th} co-ordinate, the space Y_n generated by the elements of the form

$\{\lambda_j^n\{\mu_k\}\}_{j\varepsilon A_i^n}$ for some $\{\mu_k\}\varepsilon\ l_p$.

In this way we construct the space

$$(\sum_n Y_n)_p \quad \text{in} \quad (\sum_n (\sum l_p)_{\{x_i\}})_p .$$

We first show this space is isomorphic to $(\sum(\sum_1^n l_p)_2)_p$. It is clear that it is enough to show that Y_n are uniformly isomorphic to $(\sum_1^n l_p)_2$. A typical element of Y_n is $\{\lambda_j^n\{\mu_k^i\}\}_{j\varepsilon A_i^n}$, $i=1\ldots n$

for some n sequences $\{\mu_k^i\}$ in l_p. We wish to correspond to this the element $\{\{\mu_k^i\}\}_i$ in $(\sum_1^n l_p)_2$.

The norm of the former is

$$\|\sum_i \sum_{j\varepsilon A_i^n} |\lambda_j^n| \, \|\{\mu_k^i\}\|_p \, x_j^n\|_{[x_i]} \quad \text{which equals}$$

$$\|\sum_i \sum_{j\varepsilon A_i^n} \lambda_j^n \, \|\{\mu_k^i\}\|_p \, x_j^n\|_{[x_i]} \quad \text{since we have normed}$$

$[x_i]$ to obtain a triangle inequality and obtain this as a corrollary.

But then this equals

$\|\sum_i \|\{\mu_k^i\}\|_p\, y_i^n\|_{[x_i]}$ which is λ-equivalent to $(\sum_i \|\{\mu_k^i\}\|_p^2)^{1/2}$ by our known isomorphism. This equals the norm of $\{\{\mu_k^i\}\}_i$ in $(\sum_1^\mu l_p)_2$.

Thus Y_n and $(\sum_1^n l_p)_2$ are distance less than or equal to λ by a naturally obtained isomorphism.

We now show that for each n, Y_n is complemented by a projection of norm smaller than or equal to λ.

First we may have assumed with no loss of generality that $\lambda_j^n > 0$ for all j. Suppose P_n is the given projection of X onto X_n with $\|P_n\| \leq \lambda$. It is clear that we can assume (because of unconditionality) that $P_n x_j = 0$ if $j \notin A_i$ for some $i \in 1 \ldots n$. Then write P_n: $P_n(\lambda_j x_j^n) = \mu_j \sum_{k \in A_i} \lambda_k x_k^n$ if $j \in A_i$. Consider now the map $\tilde{P}_n$ given by $\tilde{P}_n(\lambda_j x_j^n) = \frac{|\mu_j|}{\sum_{k \in A_i} |\mu_k|} \sum_{k \in A_i} \lambda_k x_k^n$.

It is easy to check that $\tilde{P}_n$ is a linear idempotent onto

X_n. We now wish to show continuity.

A typical element of X is $\sum \nu_j x_j^n$. Then

$$\tilde{P}_n\left(\sum \nu_j x_j^n\right) = \sum_{i=1}^{n} \sum_{j\varepsilon A_i} \frac{|\mu_j| \nu_j}{\lambda_j} \frac{\sum_{k\varepsilon A_i} \lambda_k x_k^n}{\sum_{k\varepsilon A_i} |\mu_k|}$$

But this equals $P_n(x)$ where

$$x = \sum_{i=1}^{n} \sum_{j\varepsilon A_i} \nu_j \frac{\operatorname{sgn} \mu_j x_j^n}{\sum_{k\varepsilon A_i} |\mu_k|} .$$

Hence $\left\| \tilde{P}_n\left(\sum \nu_j x_j^n\right) \right\| \le \lambda \left\| \sum_{i=1}^{n} \sum_{j\varepsilon A_i} \frac{\nu_j \operatorname{sgn} \mu_j x_j^n}{\sum_{k\varepsilon A_i} |\mu_k|} \right\|$

$$= \lambda \left\| \sum_{i=1}^{n} \sum_{j\varepsilon A_i} \nu_i x_j^n \Big/ \sum_{k\varepsilon A_i} |\mu_k| \right\|$$

$$\le \lambda \left\| \sum_{i=1}^{n} \sum_{j\varepsilon A_i} \nu_j x_j^n \right\| = \lambda \left\| \sum \nu_j x_j^n \right\|$$

and so $\|\tilde{P}_n\| \le \lambda$. The first equality holds since $|\nu_j| = |\nu_j \operatorname{sgn} \mu_j|$ for all j and the second inequality since $\sum_{k\varepsilon A_i} |\mu_k| \ge |\sum_{k\varepsilon A_i} \mu_k| \ge 1$ for all i.

What we have just succeeded in showing is that there is no loss in generality in assuming that $\mu_k > 0$ for every j in the definition of P_n. This will be crucial in showing that the induced projections on $Y(\sum l_p)_{\{x_i\}}$ are uniformly bounded.

We now define P'_n from $(\sum l_p)_{\{x_i\}}$ onto Y_n by mapping the element $\{\{\mu_k^j\}\}_j$ of $(\sum l_p)_{\{x_i\}}$ which is zero except in the j^{th} co-ordinate to the sequence in Y_n whose i^{th} co-ordinate is the sequence

$$\{i^{th}\text{co-ord of } P_n \mu_k^j x_j^n\}_k .$$

By the triangle inequality for the norm on l_p we can see that

$\|P'_n\{\{\mu_k^j\}_k\}_j\| \le \|P_n\{\|\{\mu_k^j\}_k\|_p\}_j\|$ which in turn is less than or equal to

$\lambda\|\sum\|\mu_k^j\|_p x_j\| = \lambda\|\{\{\mu_k^i\}\}\|$. The fact that $\mu_j > 0$ for all j is also used.

We have thus constructed Y_n in the n^{th} copy of $(\sum l_p)_{\{x_i\}}$ with $d(Y_n, (\sum_1^n l_p)_2) \le \lambda$ and Y_n complemented with a projection of norm less than or equal to λ. Thus we have a complemented subspace of

$(\sum(\sum l_p)_{\{x_i\}})_p$ isomorphic to $(\sum l_p)_{l_p}$ which means that $(\sum(\sum l_p)_{\{x_i\}})_p$ is not a $\mathcal{L}p$-space and so $(\sum l_p)_{\{x_i\}}$ is not either.

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We now prove a theorem of similar style. The proof is not as messy.

<u>Theorem 14</u> Let $\{x_i\}$ be an unconditional basis for a $\mathcal{L}p$-space for $2 < p < \infty$. Suppose X contains a subspace isomorphic to l_2. Then $(\sum l_p)_{\{x_i\}}$ is not a $\mathcal{L}p$-space.

<u>Proof:</u> We will construct in $(\sum l_p)_{\{x_i\}}$ a subspace isomorphic to $(\sum l_p)_2$. If the space $(\sum l_p)_{\{x_i\}}$ were a $\mathcal{L}p$-space then it would be isomorphic to a subspace of $L_p[0,1]$ and hence so would $(\sum l_p)_2$. But this contradicts theorem 5.

By theorem 9 there is a block basic sequence with respect to $\{x_i\}$ which is equivalent to the usual basis of l_2. Suppose $y_j = \sum_{i=p_j+1}^{p_{j+1}} \lambda_i x_i$ be such a block basic sequence. $(p_1 < p_2 < \ldots)$. (wlog $\lambda_i > 0$). Consider then the space Y in $(\sum l_p)_{\{x_i\}}$ generated by elements of the form $\{\{\lambda_i \mu_k^j\}_k\}_{i=p_j+1}^{p_{j+1}}$ for some $\{\mu_k^j\}_j$ in l_p. The norm

of a typical element $\{\{\lambda_i \mu_k^j\}_k\}_{p_j < \lambda_i \leq p_{j+1}}$

for sequences $\{\mu_k^j\}_k$ ($j = 1\ldots$) in l_p, is

$$\left\| \sum_{j=1}^{\infty} \sum_{i=p_j+1}^{p_{j+1}} |\lambda_i| \, \| \{\mu_k^j\} \| x_i \right\|$$

$$= \left\| \sum_{j=1}^{\infty} \| \{\mu_k^j\} \| \sum_{i=p_j+1}^{p_{j+1}} |x_i| x_i \right\|$$

$$= \left\| \sum_{j=1}^{\infty} \| \{\mu_k^j\} \| y_j \right\| \sim \left(\sum_{j=1}^{\infty} \| \{\mu_k^j\} \|^2 \right)^{1/2}$$

by the isomorphism on the blocks with respect to x_i. We have thus showed that Y is closed in $(\sum l_p)_{\{x_i\}}$ and isomorphic to $(\sum l_p)_2$, as desired.

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The theorem 14 says that only bases for $\mathcal{L}p$-spaces with no subspace isomorphic to Hilbert space have any chance of being usable for norming the infinite sum of

l_p (and hence also the infinite sum of any $\mathcal{L}p$-space.)

It would appear that l_p is the only such $\mathcal{L}p$-space; credence is given to this conjecture by the following two facts: 1) this is clearly the case among the known $\mathcal{L}p$-spaces 2) a result of Kadec & Pełczyński [7]: Let X be a subspace of $L_p[0,1]$. Then X contains no subspace isomorphic to l_2 if and only if every subspace of X contains a subspace isomorphic to l_p and complemented in $L_p[0,1]$ (hence also in X). Thus follows easily from their theorem 3.

Open problem: Let $p > 2$. Is l_p the only separable $\mathcal{L}p$-space which contains no isomorphic copy of l_2? (The only one with unconditional basis?)

With respect to the second question we would like to mention a result of Johnson, Rosenthal and Zippin [6] which states that every separable $\mathcal{L}p$ space has a basis. It does not seem unreasonable to suspect that they all possess unconditional bases. $(p \neq 1, p \neq \infty)$.

A positive answer to the above problem would restrict attention to the unconditional bases for l_p. In this respect, it is our feeling that the following problem has a positive answer:

Open problem: Is every unconditional basis for l_p equivalent to a block basic sequence with respect to the usual basis for l_p?

A negative answer to the first problem would easily imply the existence of more $\mathcal{L}_p$-spaces.

CHAPTER 2

TENSORING OF $\mathcal{L}_p$ AND $\mathcal{L}_2$ SPACES

In this chapter we consider tensor products of $\mathcal{L}_p$-spaces and of a $\mathcal{L}_p$-space and l_2. We show that if the tensor product is appropriately normed and then the completion taken with respect to this norm then the result is always a $\mathcal{L}_p$-space.

The definition of tensor product of linear space is too well-known to state. The famous memoir by Grothendieck gathers together much of what is known about tensor products of B-spaces.

It is easy to see from the definition of tensor products that $L_p[0,1] \otimes L_p[0,1]$ can be naturally identified with a linear subspace of the L_p space on the unit square with Lebesque measure. Hence $L_p[0,1] \otimes L_p[0,1]$ inherits a natural norm. It is not easy to see how to compute the norm of an arbitrary element of this tensor product; it is, however, easy to see that the completion under the inherited norm of the image of $L_p[0,1] \otimes L_p[0,1]$ is the whole L_p space on the square.

S. Chevet [4] has introduced a norm on arbitrary tensor products of B-spaces; she has shown that this norm is equivalent to the norm on $L_p[0,1] \otimes L_p[0,1]$ inherited from the norm of L_p of the square. This norm has the benefit that certain computations are easier to make. In addition, it generalizes certain cross-norms considered by Grothendieck and Saphar.

Definition 1 Let X and Y be B-spaces, and $1 < p < \infty$ (with $1/p + 1/q = 1$). Define a norm $\| \|_p$ on $X \otimes Y$ as follows:

$$\|u\|_p = \inf \{ (\sum_{i=1}^{n} \|y_i\|^p)^{1/p} \sup_{\substack{x' \varepsilon X' \\ \|x'\| \leq 1}} (\sum_{i=1}^{n} |x'(x_i)|^q)^{1/q} \}$$

where the infinium is taken over all possible representations $u = \sum_{i=1}^{n} x_i \otimes y_i$ of $u \varepsilon X \otimes Y$. Denote by $X \hat{\otimes}_p Y$ the completion of $X \otimes Y$ with respect to the norm $\| \|_p$. [4]

As already mentioned the following is known:

Theorem 1 On $L_p[0,1] \otimes L_p[0,1]$ the norm $\| \|_p$ is equal to the norm inherited from $L_p([0,1] \times [0,1])$; $L_p[0,1] \hat{\otimes}_p L_p[0,1]$ is isometric to $L_p([0,1] \times [0,1])$. [4]

We wish to do computations involving $X \hat{\otimes}_p Y$ where X and Y are either $\mathcal{L}p$ or $\mathcal{L}_2$ spaces.

Hence consider X and Y, B-spaces which are complemented subspaces of $L_p[0,1]$. (Results of an isomorphic nature will follow easily later on.) We wish to show that $X \hat{\otimes}_p Y$ is a complemented subspace of $L_p[0,1] \hat{\otimes}_p L_p[0,1]$, hence a complemented subspace of $L_p([0,1] \times [0,1])$ by theorem 1, hence a complemented subspace of $L_p[0,1]$.

<u>Theorem 2</u> Let X and Y be complemented subspaces of $L_p[0,1]$. Then $X \otimes Y$ is a linear subspace of $L_p[0,1] \otimes L_p[0,1]$. There are immediately two norms available for $X \otimes Y$: the norm $\| \|_p$ of definition 1 and the norm, call it $\| \|^p$, inherited from the $\| \|_p$ norm of definition 1 on $L_p[0,1] \otimes L_p[0,1]$. These two norms are equivalent, so that $X \hat{\otimes}_p Y$ is a closed subspace of $L_p[0,1] \hat{\otimes}_p L_p[0,1]$.

<u>Proof</u>: Let P be a projection of $L_p[0,1]$ onto X and Q a projection onto Y. Let $u \in X \otimes Y$.

Then $\|u\|^p = \inf \left((\sum \|y_i\|^p)^{1/p} \sup_{x' \in L_q, \|x'\| \leq 1} (\sum |x'(x_i)|^q)^{1/q} \right)$
(inf over $u = \sum x_i \otimes y_i$, $x_i, y_i \in L_p^*$)
$\leq \inf \left((\sum \|y_i\|^p)^{1/p} \sup_{x' \in L_q, \|x'\| \leq 1} (\sum |x'(x_i)|^q)^{1/q} \right)$
(inf over $u = \sum x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$). But every $x' \in L_q$ induces a continuous functional on X with norm no larger; hence $\|u\|^p \leq \|u\|_p$.

On the other hand, let $u = \sum x_i \otimes y_i$, $x_i, y_i \in L_p$ be an arbitrary representation used in computing $\|u\|^p$. Then $u = \sum Px_i \otimes Qy_i$ is a typical representation used in computing $\|u\|_p$. But

$$\left((\sum \|Qy_i\|^p)^{1/p} \sup_{x' \in X', \|x'\| \leq 1} (\sum |x'(Px_i)|^q)^{1/q} \right)$$
$$\leq \|P\| \ \|Q\| \left((\sum \|y_i\|^p)^{1/p} \sup_{x' \in X', \|x'\| \leq 1} (\sum |x'P/\|P\|(x_i)|^q)^{1/q} \right)$$
$$\leq \|P\| \ \|Q\| \left((\sum \|y_i\|^p)^{1/p} \sup_{x' \in L_q, \|x'\| \leq 1} (\sum |x'(x_i)|^q)^{1/q} \right)$$

since every $x'P/\|P\|$ for $x' \in X'$ with norm at most 1 defines an $x' \in L_q$ with norm at most 1.

If we take inf first on the left side then on the right side we get that $\|u\|_p \leq \|P\| \ \|Q\| \ \|u\|^p$ as wanted.

Thus $\|u\|^p \le \|u\|_p \le \|P\| \|Q\| \|u\|^p$ for every $u \in X \otimes Y$.

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Thus, from now on, we will not worry about which of these two norms we are using on $X \otimes Y$.

<u>Theorem 3</u> Let X and Y be complemented subspaces of $L_p[0,1]$. Then $X \hat{\otimes}_p Y$ is a complemented subspace of $L_p[0,1] \hat{\otimes}_p L_p[0,1]$.

<u>Proof:</u> Suppose X is complemented in $L_p[0,1]$ by a projection P and Y by a projection Q. We wish to define a projection of $L_p[0,1] \hat{\otimes}_p L_p[0,1]$ onto $X \hat{\otimes}_p Y$.

From the definition of tensor product, it is easy to see that $P \otimes Q\ (f \otimes g) = Pf \otimes Qg$ defines a linear idempotent of $L_p[0,1] \otimes L_p[0,1]$ onto $X \otimes Y$ $(f,g \in L_p[0,1])$. If $u = \sum f_i \otimes g_i$ is then a typical element of $L_p[0,1] \otimes L_p[0,1]$,

$$\|u\| = \inf \{ (\sum \|g_i\|^p)^{1/p} \sup (\sum |x'(f_i)|^q)^{1/q} \}$$

$$\|P\otimes Q(u)\| = \inf \{ (\sum \|Qg_i\|^p)^{1/p} \sup_{x' \in X', \|x'\| \le 1} (\sum |x'Pf_i|^q)^{1/q} \}$$

Using an argument as in theorem 2 we get

$\|P \otimes Q(u)\| \leq \|P\| \|Q\| \|u\|$, after, of course taking the correct infimums (infimum for $P \otimes Q(u)$ first).

By continuity we extend to a continuous linear map of $L_p[0,1] \hat{\otimes}_p L_p[0,1]$. This map is easily seen to be idempotent and onto $X \hat{\otimes}_p Y$ since $P \otimes Q = I$ on $X \otimes Y$.

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It is not difficult to show that the following is then true:

<u>Theorem 4</u> Let X and Y be $\mathcal{L}p$ or $\mathcal{L}_2$ spaces. Then $X \hat{\otimes}_p Y$ is a $\mathcal{L}p$ or a $\mathcal{L}_2$ space.

We now wish to improve a bit on this and show that if X and Y are $\mathcal{L}_2$ spaces then $X \hat{\otimes}_p Y$ is a $\mathcal{L}_2$ space and that $X \hat{\otimes}_p Y$ is a $\mathcal{L}p$-space in the other cases.

<u>Theorem 5</u> If $\{x_i\}$ is a basis for a reflexive space X and $\{f_i\}$ are the associated coefficient functionals then $\{f_i\}$ is a basis for X'. (Karlin [8].)

<u>Theorem 6</u> If $1 < p < \infty$ then $l_2 \hat{\otimes}_p l_2$ is isomorphic to l_2 hence is a $\mathcal{L}_2$-space.

Proof: The case $p = 2$ is trivial since $l_2 \hat{\otimes}_2 l_2$ is a subspace of L_2.

Suppose now $2 < p < \infty$. Consider the subspace X of $L_p[0,1]$ spanned by the Rademacher functions $\{r_i\}$. It is well-known that X is a complemented subspace of $L_p[0,1]$ which is isomorphic to l_2. Hence we need to show that $X \hat{\otimes}_{p^\sigma} X$ is isomorphic to l_2.

Consider then the double sequence $\{r_i \otimes r_j\}$. It is not difficult to see that this is orthonormal (from the orthonormality of the Rademacher system.)

Since $p > 2$ we immediately get

$$\| \sum \lambda_{ij} \, r_j \otimes r_j \|_p \geq \| \sum \lambda_{ij} \, r_j \otimes r_j \|_2$$

$$= \left(\iint | \sum \lambda_{ij} \, r_i(s) r_j(t) |^2 ds dt \right)^{1/2}$$

$$= \left(\sum \lambda_{ij}^2 \right)^{1/2}$$

by orthonormality.

On the other hand

$$\| \sum \lambda_{ij} \, r_j \otimes r_j \|_p = \left(\iint | \sum \lambda_{ij} \, r_j(s) r_j(t) |^p ds dt \right)^{1/p}$$

$$\leq C \left(\int \left(\sum_i \; | \sum_j \lambda_{ij} r_j(t) |^2 \right)^{p/2} dt \right)^{1/p}$$

$$\le C\left(\sum_i \left\| \left|\sum_j \lambda_{ij} r_j\right|^2 \right\|_{p/2}\right)^{1/2}$$

$$= C\left(\sum_i \left(\int \left|\sum_j \lambda_{ij} r_j(t)\right|^p dt\right)^{2/p}\right)^{1/2}$$

$$\le C^2\left(\sum_{i,j} \lambda_{ij}^2\right)^{1/2} .$$

The first and last inequalities follow from Khintchine's inequality for $\{r_i\}$ and the second is a triangle inequality for $L_{p/2}[0,1]$ $(p > 2)$.

Thus the space of elements of the form $\sum \lambda_{ij}\, r_i \otimes r_j$ is isomorphic to l_2, hence is complete, hence equals $X \hat{\otimes}_p X$ and so $X \hat{\otimes}_p X$ is isomorphic to l_2 if $2 < p < \infty$. It also follows that $\{r_i \otimes r_j\}$ is an unconditional basis for $X \hat{\otimes}_p X$ equivalent to the usual basis for l_2.

If $1 \le p \le 2$ then $\left\| \sum \lambda_{ij}\, r_i \otimes r_j \right\|_p \ge \left\| \sum \lambda_{ij}\, r_i \otimes r_j \right\|_1$

$$= \iint \left|\sum \lambda_{ij} r_i(s) r_j(t)\right| ds\,dt = \iint \left|\sum_i \sum_j \lambda_{ij} r_j(t) r_i(s)\right| ds\,dt$$

$$\ge C \int \left(\sum_i \left|\sum_j \lambda_{ij} r_j(t)\right|^2\right)^{\frac{1}{2}} dt \ge C\left(\sum_i \left(\int \left|\sum_j \lambda_{ij} r_j(t)\right| dt\right)^2\right)^{\frac{1}{2}}$$

$$\ge C^2 \left(\sum_i \left(\sum_j |\lambda_{ij}|^2\right)\right)^{\frac{1}{2}} = C^2 \left(\sum \lambda_{ij}^2\right)^{\frac{1}{2}} .$$

The second inequality from the last comes from writing $f_i = \sum \lambda_{ij} r_j$ and seeing that

$$\left(\sum \left(\int |f_i|\right)^2\right)^{\frac{1}{2}} = \sup_{\sum c_i^2 = 1} \sum c_i \int |f_i| \le \int \sup_{\sum c_i^2 = 1} \sum c_i |f_i|$$

$$\leq \int \sup_{\sum c_i^2 \leq 1} \left(\sum c_i^2\right)^{\frac{1}{2}} \left(\sum |f_i|^2\right)^{\frac{1}{2}} = \int \left(\sum |f_i|^2\right)^{\frac{1}{2}}$$

essentially by Holder's inequality. We also use Khintchine's inequality for $p=1$. The rest is easy.

#

Theorem 7 Suppose X and Y are $\mathcal{L}p$ or $\mathcal{L}_2$ spaces, not both $\mathcal{L}_2$ spaces. Then $X \hat{\otimes}_p Y$ is a $\mathcal{L}p$-space.

Proof: We have already seen that $X \hat{\otimes}_p Y$ is either a $\mathcal{L}_2$ or $\mathcal{L}p$-space. We show it cannot be a $\mathcal{L}_2$-space.

Without loss of generality we may assume X is a $\mathcal{L}p$-space. Then X contains a subspace Z complemented in X and isomorphic to l_p. If y is any nonzero element of Y then it is easy to see that $Z \otimes Ry$ is a closed subspace of $X \hat{\otimes}_p Y$ isomorphic to l_p. But then $X \hat{\otimes}_p Y$ cannot be isomorphic to l_2, so cannot be a $\mathcal{L}_2$ space.

#

Theorem 8 Let X be either a $\mathcal{L}p$ or $\mathcal{L}_2$ space. Then $X \hat{\otimes}_p l_p$ is isometric to $(\sum X)_p$.

Proof: Without loss of generality, X is an actual complemented subspace of $L_p[0,1]$.

Suppose now Y is the closed linear span in $L_p[0,1]$ of a sequence of disjoint measurable sets of positive measure. Then Y is isometric to l_p so $X\hat{\otimes}_p l_p$ is isometric to $X\hat{\otimes}_p Y$.

But the isometry between $X\hat{\otimes}_p Y$ and $(\sum X)_p$ follows easily.

) #

To determine the isomorphism types of the spaces $X\hat{\otimes}_p Y$ where X and Y range over the known complemented subspaces of $L_p[0,1]$ seems difficult. It is usually easy to say that a particular tensor product is a complemented subspace of a known nontrivial $\mathcal{L}p$-space. However, it is not even known, say, whether the complemented subspaces of X_p are only X_p, l_p, l_2 and $l_p \oplus l_2$. Even determining the complemented subspaces of $l_p \oplus l_2$ (they are the obvious ones) was very difficult. (This was done by a student of Pelczyński but it is not yet published and I do not yet have a preprint. The result was communicated in Israel.)

CHAPTER 3

SUBSEQUENCES OF THE HAAR SYSTEM IN $L_p[0,1]$

In this chapter we deal with the classification of the isomorphic types of closed linear spans of subsequences of the Haar system in $L_p[0,1]$. $(1 < p < \infty)$.

This is a partial result in the direction of finding all separable $\mathcal{L}p$-spaces $(1 < p < \infty)$ with unconditional basis; in fact, finding all those with unconditional basis and complement with unconditional basis is the same problem as that of determining the isomorphic types of subsequences of arbitrary unconditional bases for $L_p[0,1]$. $(1 < p < \infty)$.

Since the Haar system is in a sense the most meager basis for $L_p[0,1]$, it is a natural starting point. Note that by the results of [6], all $\mathcal{L}p$-spaces have bases.

It is our feeling that all separable $\mathcal{L}p$-spaces for $1 < p < \infty$ will be recovered by looking at complemented blocks with respect to the Haar system. Hence, a result concerning subsequences seems a good beginning.

The Haar functions on the unit interval are usually defined as follows:

$$y_1(t) = 1$$

and
$$y_{2^n+m}(t) = \begin{cases} 1 & \text{if } \frac{2m-2}{2^{n+1}} \le t < \frac{2m-1}{2^{n+1}} \\ -1 & \text{if } \frac{2m-1}{2^{n+1}} \le t < \frac{2m}{2^{n+1}} \\ 0 & \text{otherwise} \end{cases}$$

for $n = 0 \ldots$ and $m = 1 \ldots 2^n$.

It is standard to normalize the Haar functions so that the biorthogonal functional associated with the i^{th} Haar function is simply integration with respect to the i^{th} Haar function. For our purposes, however, it is more convenient to have these functions normalized with supremum norm as above.

It is well-known that the Haar functions form an unconditional basis for $L_p[0,1]$ if $1 < p < \infty$. The proof is due to R.E.A.C. Paley and J. Marcinkiewicz and can be found in I. Singer [18].

We now state some well-known properties of unconditional basic sequences and some properties of l_p and $L_p[0,1]$ which we will use.

Lemma 1 If $\{x_i\}$ is an unconditional basic sequence and $\{x_{i_n}\}$ a subsequence, then the map P :

$$P(x_{i_n}) = x_{i_n}, \quad n = 1 \ldots \text{ and } P(x_i) = 0 \text{ for other}$$

indices, defines a projection of $[x_i]$ onto $[x_{i_n}]$. In fact an upper bound can be given for the norms of such projections from a particular unconditional basis.

Lemma 2 If $\{x_i\}$ is an unconditional basic sequence in a B-space X, if $[x_i]$ is complemented in X by a projection P and $\{y_i\}$ is a sequence in X with $\sum_{i=1}^{\infty} \|P\| \, \|x_i'\| \, \|x_i - y_i\| < 1$ where x_i' is the coefficient functional associated with x_i, then $\{y_i\}$ is an unconditional basic sequence, $[y_i]$ is complemented in X and the correspondence $x_i \leftrightarrow y_1$ defines an isomorphism [2].

Lemma 3 An infinite-dimensional complemented subspace of l_p is isomorphic to l_p. $(1 < p < \infty)$ [14].

Lemma 4 A closed sublattice of $L_p[0,1]$ is complemented by a projection of norm 1. $(1 < p < \infty)$ [1].

We are now ready to state and prove:

Theorem: Let $\{x_i\}$ be a subsequence of the Haar system. Then if $1 < p < \infty$ and X is the closed linear span of $\{x_i\}$ in $L_p[0,1]$ either

X is isomorphic to l_p

or

X is isomorphic to $L_p[0,1]$.

Proof: Consider the set

$A = \{t \in [0,1] \mid t \in$ support x_i for infinitely many indices $i\}$.

The set A is clearly measurable.

We show that:

if $\mu(A) = 0$ then $X \sim l_p$

if $\mu(A) > 0$ then $X \sim L_p[0,1]$,

where $\sim$ means, as usual, isomorphic.

Case 1 $\mu(A) = 0$

We will show that X is a complemented subspace of a space isometric to l_p and appeal to Lemma 3. (X is clearly infinite-dimensional since it arises from a subsequence (infinite) of a basic sequence.)

Firstly it is clear that if $\{S_n\}$ is a sequence of mutually disjoint measurable sets with $\mu([0,1] - \cup S_n) = 0$ and $X \mid_{S_n}$ is the restriction of X to the set S_n then $(\sum X \mid S_n)_p$ is a subspace of $L_p[0,1]$ which contains X. The space X is clearly complemented in any such space by restriction of the projection of $L_p[0,1]$ onto X given in Lemma 1.

We will now choose sets $\{S_n\}$ with the above properties and so that $(\sum X \mid S_n)_p$ will obviously be isometric to l_p. In fact we will pick the sets so that X is constant on each one.

Consider the sets A_n, $n = 0, 1, \ldots,$:

$A_n = \{t \in [0,1] \mid t \in$ support x_i for exactly n indices $i\}$. We mean of course the supports as given by the original definition of the Haar functions and not the support of a function equal ae to a Haar function.

The sets A and A_n are clearly pairwise disjoint and $[0,1] - \cup A_n = A$ so this set has measure zero.

Fix n for a moment and consider only those functions x_i which appear in the definition of A_n. By a maximality argument repeated n times we can split this (possibly finite) sequence into n subsequences $\{x_i\}$ $i \in B_{n,j}$, $j = 1 \ldots n$ with each subsequence consisting of disjoint functions and so that the supports get finer, that is:

if $i \in B_{n,j}$ for $j \geq 2$ then there exists $k \in B_{n,j-1}$ with $\operatorname{supp} x_k \supset \operatorname{supp} x_i$ (properly).

It is then easy to see that $A_n = \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} x_k)$

$$= \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} x_k^+) \cup \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} x_k^-) .$$

The sets in the final union clearly are disjoint, have union A_n and X is constant on each one.

The case $n = 0$ is easy since X is constant 0 on A_0.

Doing this for all n and numbering our sets properly, we arrive at the promised sets S_n ; X is constant on each set S_n so $X \mid S_n$ is either 0 or 1- dimensional for all n.

Case 2 $\mu(A) > 0$

By Lemma 1, X is complemented in $L_p[0,1]$ with complement X' say.

Suppose we can show that X contains a subspace Y isomorphic to $L_p[0,1]$ and with complement, say Y', in X.

We will then get

$$X \sim Y \oplus_p Y' \; ; \; Y \sim L_p[0,1] \, , \; L_p[0,1] \sim X \oplus_p X'$$

so that

$$\begin{aligned} X \oplus_p L_p[0,1] &\sim (Y \oplus_p Y') \oplus_p L_p[0,1] \\ &\sim L_p[0,1] \oplus_p Y' \oplus_p L_p[0,1] \\ &\sim L_p[0,1] \oplus_p Y' \\ &\sim Y \oplus_p Y' \\ &\sim X \end{aligned}$$

and

$$= X \oplus_p L_p[0,1] \cong X \oplus_p \left(\sum_{n=1}^{\infty} L_p[0,1]\right)_p$$

$$\cong X \oplus_p \left(\sum_{n=1}^{\infty} X \oplus_p X'\right)_p$$

$$\cong \left(\sum_{n=1}^{\infty} X \oplus_p X'\right)_p$$

$$\cong \left(\sum_{n=1}^{\infty} L_p[0,1]\right)_p$$

$$\cong L_p[0,1]$$

Hence $X \cong X \oplus_p L_p[0,1] \cong L_p[0,1]$, our desired result.

We now proceed to construct such a subspace Y.

Consider the sequence $\{x_i\}$. By induction and maximality argument we can find a countable numer of subsequences $\{k_i\}$ $i \in N_k$ $(k = 1 \ldots)$, each possibly finite or infinite, with

$$N_k \cap N_j = \phi \text{ if } k \neq j$$

$$\cup N_k = N$$

$i, j \in N_k$ for some k, then $\operatorname{supp} x_i \cap \operatorname{supp} x_j = \phi$

unless $i = j$

$i \in N_k$, for some $k \geq 2$, then there exists

$j \in N_{k-1}$ with $\operatorname{supp} x_i \subset \operatorname{supp} x_j$.

What we have really done is split up the sequence into a countable number of subsequences so that support of functions get finer from one subsequence to the next and so that the functions in any one subsequence are disjoint from one another. This is similar to what we did in the previous case on the A_n. From the fact that $\mu(A) > 0$, it is easy to see from what follows that we actually have an infinite number of subsequences.

It is not difficult to see that the sets $A_k = \bigcup_{i \in N_k} \operatorname{supp} x_i$ are decreasing, contain A and $\bigcap_{k=1}^{\infty} A_k = A$.

For $n = -1, 0, \ldots$ and given positive constants c_n, pick k_n increasingly strictly with n so that

$$\mu(A_{k_n} - A) < c_n .$$

Then define functions b_i, $i = 1, 2, \ldots$ as follows:

$$b_1 = \sum_{i \in N_{k_{-1}}} x_i \quad , \qquad b_2 = \sum_{i \in N_{k_0}} x_i$$

and inductively thereafter by

$$b_{2^n+m} = \sum x_i \ , \quad \text{for } i \in N_{k_n} \ , \ \operatorname{supp} x_i \subset \operatorname{supp} \left(b_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}\right)$$

if m is odd .

and

$$b_{2^n+m} = \sum x_i \ , \quad \text{for} \quad i \in N_{k_n} \ , \ \text{supp}\ x_i \subset \text{supp}\ (b_{2^{n-1}+[\frac{m+1}{2}]})$$

if m is even, for $n = 1 \ldots$ and $m = 1 \ldots 2^n$.

There is, of course, no problem with convergence since in all cases we are taking disjoint sums of functions with absolute value 1 .

We will show that if the constants c_n are picked "small enough" in some sense yet to be determined, then an appropriate subsequence of $\{b_i\}$ will yield our desired space Y .

Define $d_1 = |b_1| \,|_A$ and $d_i = b_i|_A$ for $i \geq 2$. Write D for the space generated by $\{d_i\}$.

We now search for conditions on the constants c_n so that: 1. the correspondence $d_i \leftrightarrow y_i$ defines an isomorphism between D and $L_p[0,1]$ which preserves lattice structure and 2. $\sum \|d_i'\| \, \|d_i - b_i\|$ is small enough where the summation is over some suitable set of integers.

Now if $n \geq 1$ and $1 \leq m \leq 2^n$ we have

$$\mu(b^{+}_{2^n+m}) \geq \mu(b^{+}_{2^n+m}|A)$$

$$= \mu(b^{+}_{2^n+m}) - \mu(b^{+}_{2^n+m}|A^c)$$

$$\geq \tfrac{1}{2}\,\mu(b_{2^n+m}) - \mu(b^{+}_{2^n+m}|A^c)$$

$$\geq \tfrac{1}{2}\,\mu(b_{2^n+m}) - c_n .$$

On the other hand

$$\mu(b_{2^n+m}|A) = \mu(b^{+}_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}|A)$$

$$= \mu(b^{+}_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}) - \mu(b^{+}_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}|A^c)$$

$$= \tfrac{1}{2}\,\mu(b_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}) - \mu(b^{+}_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}|A^c)$$

$$\geq \tfrac{1}{2}\,\mu(b_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}) - c_{n-1}$$

Combining these two and using induction we get

$$\mu(b^{\pm}_{2^n+m}|A) \geq \frac{1}{2^{n+1}}\mu(b_2) - \sum_{j=0}^{n} c_j\, 2^{j-n} .$$

This can be rewritten

$$\frac{\mu(d_i^{\pm})}{\mu(y_i^{\pm})} \geq \mu(b_2) - \sum_{j=0}^{\infty} c_j\, 2^{j+1} \quad , \quad i \geq 2 .$$

since the supports of Haar functions are easily computed.

But by definitions $\mu(d_1) \geq \mu(b_1)$ and $\mu(y_1) = 1$.

Hence we get the above inequality holding for all i , except for d_1^- , y_1^- .

On the other hand

$$\mu(b^{\pm}_{2^n+m}|_A) \leq \mu(b^{\pm}_{2^n+m}) = \tfrac{1}{2}\,\mu(b_{2^n+m})$$

$$\leq \tfrac{1}{4}\,\mu(b^{\pm}_{2^{n-1}+\lceil\frac{m+1}{2}\rceil}) \leq \cdots$$

so that

$$\mu(b^{\pm}_{2^n+m}|_A) \leq \frac{1}{2^{n+1}}\,\mu(b_2)$$

which implies

$$\frac{\mu(d_i^{\pm})}{\mu(y_i^{\pm})} \leq \mu(b_2) \quad \text{for } i \geq 2 .$$

But then $\mu(d_1) \leq c_{-1} + \mu(A)$.

We finally get

$$c_{-1} + \mu(A) \geq \frac{\mu(d_i^{+})}{\mu(y_i^{+})} \geq \mu(A) - \sum c_j 2^{j+1}$$

since $\mu(A) \leq \mu(b_2) \leq \mu(b_1) \leq c_{-1} + \mu(A)$.

If we now consider constants $\lambda, \lambda_1, \ldots, \lambda_n$ and write

$$D_\lambda = \{t \mid \sum_{i=1}^{n} \lambda_i d_i(t) = \lambda\} \quad , \quad Y_\lambda = \{t \mid \sum_{i=1}^{n} \lambda_i y_i(t) = \lambda\}$$

then the set Y_λ is a union (disjoint) of sets of the form supp y_i^{+}. The set D_λ is the corresponding union of disjoint sets supp d_i^{+}. This is because the functions d_i and y_i attain the same values and the functions behave with respect to one another in the same lattice fashion.

From this observation and the inequalities just obtained we get $c_{-1} + \mu(A) \geq \dfrac{\mu(D_\lambda)}{\mu(Y_\lambda)} \geq \mu(A) - \sum c_j 2^{j+1}$

This implies in turn that if $\lambda_1 \ldots \lambda_n$ are constants then

$$(c_{-1}+\mu(A))^{1/p} \geq \frac{\| \sum \lambda_i d_i \|}{\| \sum \lambda_i y_i \|} \geq (\mu(A) - \sum c_j 2^{j+1})^{1/p} .$$

It is then easy to see that if $\sum c_j 2^{j+1} < \mu(A)$

then the correspondence $T(d_i) = y_i$ defines a lattice isomorphism between D and $L_p[0,1]$.

We in fact have

$$\|T\| \le (\mu(A) - \sum c_j 2^{j+1})^{1/p} \quad \text{and} \quad \|T^{-1}\| \le (c_{-1} + \mu(A))^{1/p} .$$

Thus, with the above requirement, D forms a sublattice of $L_p[0,1]$ so is complemented by a projection of norm 1.

Consider the index set I consisting of 2 and all integers of the form 2^n+m for $n = 1 \ldots$ and $m = 1 \ldots 2^{n-1}$.

We wish to impose additional restrictions on the constants c_n so that the closed linear span of $\{b_i\}_{i\in I}$ will do for our space Y.

Firstly $[y_i]_{i\in I}$ is complemented in $L_p[0,1]$ by a projection with norm less than or equal to a constant K, which depends only on the Haar system and the number p.

We can then see that $[d_i]_{i\in I}$ is complemented in D by a projection with norm less than or equal to $K\|T\|\,\|T^{-1}\|$ and hence also in $L_p[0,1]$ by a projection with no greater norm. (D is complemented, norm 1, in $L_p[0,1]$.)

We wish to apply Lemma 2 to $\{d_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$.

Firstly $y_i'T(d_j) = y_i'(y_j) = \delta_{ij}$

so that $d_i' = y_i'T$ for every i.

But the functional y_i' consists in integrating with respect to y_i, but with a normalizing factor thrown in. It is not difficult to determine that $\|y'_{2^n+m}\| = 2^{n/p}$.

Thus $\|d'_{2^n+m}\| \le \|y'_{2^n+m}\|\,\|T\| \le 2^{n/p}\,(\mu(A) - \sum c_j 2^{j+1})^{1/p}$.

Secondly

$$\|d_{2^n+m} - b_{2^n+m}\| = (\mu(b_{2^n+m}|A^c))^{1/p} \le c_n^{1/p}$$

Thus if

$\sum_{i\in I} \|P\|\,\|d_i'\|\,\|d_i - b_i\| = W$, we have

$$W \le K\left\{\frac{c_{-1}+\mu(A)}{\mu(A) - \sum c_j 2^{j+1}}\right\}^{1/p} \frac{\sum_{i\in I}\|y_i'\|\,\|d_i - b_i\|}{(\mu(A) - \sum c_j 2^{j+1})^{1/p}}$$

$$\le K\left\{\frac{c_{-1}+\mu(A)}{\mu(A) - \sum c_j 2^{j+1}}\right\}^{1/p} \cdot \frac{c_0^{1/p} + \sum c_j^{1/p} 2^{j-1} 2^{j/p}}{(\mu(A) - \sum c_j 2^{j+1})^{1/p}}$$

Hence we can make $W < 1$ by making

$$\left\{\frac{c^{-1} + \mu(A)}{\mu(A) - c_j 2^{j+1}}\right\}^{1/p} < 2 \quad \text{and}$$

$$K\frac{c_0^{1/p} + \sum c_j^{1/p} 2^{j+j/p-1}}{(\mu(A) - \sum c_j 2^{j+1})^{1/p}} < 1/4$$

This can easily be done at the same time as making

$$\sum c_j 2^{j+1} < \mu(A) .$$

We then get that $[b_i]_{i\epsilon I}$ is complemented in $L_p[0,1]$ so also in X and that $b_i \leftrightarrow d_i$ defines an isomorphism.

However, we already know that $d_i \leftrightarrow y_i$ defines an isomorphism. Thus $[b_i]_{i\epsilon I}$ is isomorphic to $[y_i]_{i\epsilon I}$. By our choice of I and unconditionality of the Haar system this space is clearly isomorphic to $L_p[0,\frac{1}{2}]$ hence also to $L_p[0,1]$.

The space $[b_i]_{i\epsilon I}$ satisfies our desired properties.

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We feel that a good, though probably difficult, next step in finding $\mathfrak{L}p$-spaces is to find the isomorphic types of complemented closed linear spans of blocks with respect to the Haar system and/or showing that a large class of $\mathcal{L}_p$-spaces can be obtained in this way.

Note: It has been pointed out to us that there are easier ways to get the sets N_k in the proof. H.P.Rosenthal points out that if an element x of the Haar system is called a predecessor of another element y of the Haar system when support x $\supset$ support y but $x \neq y$, then

$N_k = \{ i \mid x_i$ has exactly k-1 predecessors in $\{x_i\} \}$.

It has also been pointed out by A.Meir that if

$M = \{ i \mid$ the i'th Haar function belongs to the sequence $\{x_i\} \}$

then the statement :"M has positive density in the positive integers" also separates the cases l_p and L_p.

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