Structures Report No. 69

NUMERICAL ANALYSIS OF GENERAL SHELLS OF REVOLUTION SUBJECTED TO ARBITRARY LOADING

by
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September, 1978

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THE UNIVERSITY OF ALBERTA

NUMERICAL ANALYSIS OF GENERAL SHELLS OF REVOLUTION SUBJECTED TO ARBITRARY LOADING

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF CIVIL ENGINEERING

EDMONTON, ALBERTA

FALL, 1978

ABSTRACT

The governing partial differential equations of a classical shell theory are reduced to a set of eight first order ordinary differential equations. A forward numerical integration process is used to obtain influence coefficients and particular solutions for shells of revolution of general geometric configuration subjected to arbitrary types of loading.

Standard stiffness methods of structural analysis are employed to obtain displacements and stress resultants everywhere within a complex shell structure. A computer program is developed to perform the analysis based on the theory presented. Example problems are selected to test the accuracy of the method. Excellent results, when compared with known solutions, are achieved.

ACKNOWLEDGEMENTS

This study was performed in the Department of Civil Engineering at the University of Alberta. Financial assistance from the National Research Council of Canada is acknowledged.

TABLE OF CONTENTS

CHAPTER PAGE

LIST OF TABLES

LIST OF FIGURES

Figure	Title	Page

NOMENCLATURE

a = constant and equal to the throat radius of the hyperboloid shell of revolution (Eq. 4.9).

$[A]$ = flexibility matrix in Eq. 3.2.

$[A_1],[A_2]\ \{A_3\}$
= coefficient matrices defined in Table 2.1

$[A_{(s)}]$ = coefficient matrix relating the shell fundamental variables and their derivatives at the location s (Eq. 3.5).

b = constant

$[B_1],[B_2],\{B_3\}$
= coefficient matrices defined in Tables 2.2.1 and 2.2.2.

$\{B_{(s)}\}$ = column vector of the loading terms in the basic set of equations (Eq. 3.5)

$\{c\}$ = column vector of the eight arbitrary constant of integration defined in Eq. 3.9.

$[C_1],[C_2],\{C_3\}$
= coefficient matrices defined in Tables 2.3.1 and 2.3.2.

$\dot{D}$ = extensional rigidity of the shell defined in Eq. A.18.1

$\{D\}$ = column vector of the displacements defined in Eq. 2.19.1

$\{D^{\bullet}\}$ = derivative of $\{D\}$ with respect to s (Eq. 2.19.2)

E = modulus of elasticity.

$[Ec]$ = eccentricity transformation matrix defined in Eq. 3.31.

$\{F_s\}$ = column vector of primary stress resultants defined in Eq. 2.15.1.

$\{F^{\bullet}_s\}$ = derivative of $\{F_s\}$ with respect to s (Eq. 2.15.2)

$\{F_\theta\}$ = column vector of secondary stress resultants defined in Eq. 2.15.3.

$\{F^o\}$ = column vector of the fixed end forces in Eq. 3.3.

G = shear modulus defined in Eq. A.13.

$\{h_{(s)}\}$ = column vector of the homogeneous solution of the eight fundamental equations (Eq. 3.6).

$[H_{(s)}]$ = transfer matrix arises from integrating the inhomogeneous terms in the eight fundamental equations (Eq. 3.11).

K = flexural rigidity of the shell defined in Eq. A.18.2.

[K] = stiffness matrix

[L] = transformation matrix from local to global coordinates defined in Eq. 3.29.

M_s, M_θ = meridional and circumferential moments per unit length, respectively.

$M_{s\theta}$,$M_{\theta s}$ = circumferential and meridional twisting moments per unit length, respectively.

n = harmonic number

N_s,N_θ = normal in-plane forces per meridional and circumferential unit length, respectively.

$N_{s\theta}$,$N_{\theta s}$ = in-plane shear forces per meridional and circumferential unit length, respectively.

P_s,P_z,P_θ = intensity of load components in the direction s, z and θ respectively.

Q_s, Q_θ = transverse shear forces per meridional and circumferential unit length, respectively.

$\{Q_{(s)}\}$ = column vector arises from integrating the inhomogeneous terms in the eight fundamental equations (Eq. 3.14).

r = radius of curvature of parallel circles.

r_1 = radius of curvature of meridian.

r_2 = length of the normal between any point on the middle surface and the axis of revolution.

$\dot{r}_1$ = derivative of r_1 with respect to the coordinate s.

R = curvature of parallel circles.

R_1 = first principle curvature = $1/r_1$.

R_2 = second principle curvature = $1/r_2$.

s = coordinate measures the distance along the meridian of the shell.

S_s = effective transverse shearing force per circumferential unit length defined in Eq. 2.6.1.

t = thickness of the shell.

T = change in temperature from arbitrary level.

T_s = effective tangential shearing force per circumferential unit length defined in Eq. 2.6.2.

T_{o1}, T_{o2} = temperature terms defined in Eq. A.19.1.

T_{11}, T_{12} = temperature terms defined in Eq. A.19.2.

T^o, T^i = change in temperature from arbitrary level x measured at the exterior and interior face of the shell, respectively.

U = displacement component in the circumferential direction.

V = displacement component in the meridional direction.

W = displacement component in the rotation direction.

U_z, V_z, W_z = displacement components of a point at distance z from the middle surface.

X = coordinate measures the distance along the axis of revolution.

$\{y_{(s)}\}$ = column vector of the eight fundamental variables (Eq. 3.5).

$\{y^{\bullet}_{(s)}\}$ = derivative of $\{y_{(s)}\}$ with respect to the coordinate s.

α = coefficient of thermal expansion.

β = rotation of the meridian due to deformation.

γ = specific weight of the shell material.

$\gamma_{s\theta}$, $\gamma_{\theta s}$ = shear strain.

ε_s = meridional strain.

ε_θ = hoop strain.

θ = coordinate measures the angle in the circumferential direction.

ν = Poisson's ratio.

σ_s, σ_θ = normal stresses in the meridional and circumferential direction, respectively.

$\tau_{s\theta}$, $\tau_{\theta s}$ = shearing stresses.

ϕ = coordinate measures the angle between any point on the middle surface and the axis of revolution.

$(\ )^{\bullet} = \frac{\partial(\ )}{\partial s}$

$(\ )' = \frac{\partial(\ )}{\partial \theta}$

CHAPTER 1

INTRODUCTION

1.1 Introductory Remarks

Shells of revolution are important structural elements. Many structures such as storage tanks, pressure vessels, silos, chimneys and towers are composed of either a single shell unit or an assemblage consisting of different types of shells. Their behaviour allows the shell thickness to be reduced to a minimum and their advantageous shapes permit more modern architectural concepts.

Although the governing field equations have been known for many years, cases where analytical solutions can be obtained are relatively scarce and are restricted to simple forms of geometry, boundaries and loads. The determination of the forces and deformations in shells constitutes a difficult problem in the theory of elasticity, owing to the complexity of the mathematical equations involved. For conditions in which the analytical solution is complex, or is unknown, the application of numerical methods with the aid of a digital computer has proven to be useful and efficient. This approach allows the solution for generalized geometric configurations and loadings of shells of revolution.

1.2 Purpose of the Study

The objectives of this thesis are:

1) to develop a technique for evaluating the stiffness influence coefficients of any arbitrary element of a shell of revolution of general geometric configuration by means of a direct numerical integration method.
2) to employ the standard stiffness method of structural analysis to analyze assemblages of such elements.
3) to demonstrate the capability of the method to treat arbitrary surface loadings and thermal gradients, including line and concentrated loads.

1.3 Types of Shells of Revolution

The position of a point in a shell of revolution can be given by three orthogonal coordinates s, θ and z (See Appendix A, Sect. A.1, for definitions). The shape of the shell is determined by specifying the two principal radii of curvature r_1 and r_2 of the middle surface and the thickness of the shell (Fig. A.1). Instead of r_2 it is sometimes convenient to use the distance r from a point on the middle surface to the axis of revolution (Fig. A.1) where

$$r = r_2 \sin\phi \qquad 1.1$$

in which ϕ is the angular distance of the point under consideration from the axis of revolution. The generating curve of the middle surface is defined by the equation

$$r = r(x) \qquad 1.2$$

where r(x) represent the radius r as a function of the distance measured along the axis of revolution, x. Therefore the principal radii of curvature can be determined by the following two equations

$$r_1 = \left[1 + \left(\frac{dr}{dx}\right)^2\right]^{3/2} / \frac{d^2r}{dx^2} \qquad 1.3.1$$

$$r_2 = r\left[1 + \left(\frac{dr}{dx}\right)^2\right]^{1/2} \qquad 1.3.2$$

The general shape of any type of shell of revolution (Fig. 1.1 to 1.8) is characterized by particular forms of Eqns. 1.3, as follows:

a) for plates (Fig. 1.1)

$$r_1 = \infty \qquad 1.4.1$$

$$r_2 = \infty \qquad 1.4.2$$

$$\phi = 0 \qquad 1.4.3$$

b) for spheres (Fig. 1.2)

$$r_1 = a \qquad 1.4.4$$

$$r_2 = a \qquad 1.4.5$$

c) for cylinders (Fig. 1.3)

$$r_1 = \infty \qquad 1.4.6$$

$$r_2 = a \qquad 1.4.7$$

$$\phi = \pi/2 \qquad 1.4.8$$

d) for cones (Fig. 1.4)

$$r_1 = \infty \qquad 1.4.9$$

$$r_2 = \frac{r}{\sin\phi} \qquad 1.4.10$$

$$\phi = \text{constant} \qquad 1.4.11$$

e) for toroids (Fig. 1.5) $r_1 = a$ 1.4.12

$$r_2 = \frac{r}{\sin\phi} \qquad 1.4.13$$

f) for ellipsoids (minor axis coincides with the axis of revolution) (Fig. 1.6)

$$r_1 = \frac{a^2b^2}{(a^2\sin^2\phi + b^2\cos^2\phi)^{3/2}} \qquad 1.4.14$$

$$r_2 = \frac{a^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{1/2}} \qquad 1.4.15$$

g) for hyperboloids (Fig. 1.7)

$$r_1 = \frac{-a^2b^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{3/2}} \qquad 1.4.16$$

$$r_1 = \frac{a^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{1/2}} \qquad 1.4.17$$

h) for arbitrary shell elements for which the form of the middle surface cannot be expressed as a closed form function (Fig. 1.8), one can describe r_1, r_2 and r at discrete points along the meridian and interpolate numerically.

1.4 Loadings

Applied surface loads at any point of the shell can be resolved into three components in the three orthogonal directions s, θ and z. This load may vary in the direction along the meridian as well as in the circumferential direction of the shell. Therefore a load component may be written as a function of the coordinates s and θ in the following form

$$P_i = F_{1_i}(s,\theta) \qquad 1.5.1$$

where P_i is the magnitude of the load at the point under consideration in the direction $i(i = s,\theta,z)$ and F_{1_i} is the function representing the applied load. For the special case of axisymmetric loading P_i is independent of θ, thus

$$P_i = F_{2_i}(s) \qquad 1.5.2$$

For non-axisymmetrical loading, the classical method of analysis is to expand this load into a Fourier series, analyze for each harmonic separately and superimpose the effects [10,27]. The number of terms considered in this series must be sufficient to give the desired degree of convergence.

Thermal loading can be considered in either case by algebraically adding the strain due to thermal expansion to the strain due to the surface loading in the stress-strain equations.

1.5 Shell Theory

Shell theories of various degrees of complexity may be derived, depending upon the degree to which the theory of linear elasticity is simplified.

In all cases one begins by reducing the three-dimensional shell problem to a two-dimensional problem expressed in terms of the deformation of the middle surface of the shell. Further simplifications establish various shell theories which may be classified into different categories [3]. Such categories are based on the terms that are retained in the strain and stress-displacement equations with respect to the thickness coordinate.

The second order approximation theory for shells of revolution was presented in 1932 by Flugge [10] and based upon the following assumptions:

1) Consistent with the formulation of the classical theory of elasticity, strains and displacements under loads are small enough so that changes in geometry of the shell will not alter the equations of static equilibrium of the shell (i.e., equations of equilibrium are written in the undeformed configuration).
2) The components of stress normal to the middle surface are small compared to the other components of stress and may be neglected in the stress-strain relationships (i.e., the material may be considered to be in a plane stress condition).
3) Points on lines normal to the middle surface before deformation remain on straight lines normal to the middle surface after deformations (i.e., deformations of the shell due to the radial shears are neglected).

Contrary to other theories, Flugge's theory did not entirely neglect the ratio of the thickness to the radius of curvature (except for an occasional dropping of the fifth and higher order terms) in the stress resultant equations and in the strain-displacement relationships [10, pg. 320].

Applications of this theory have generally been restricted to circular cylindrical shapes, for which some solutions have been obtained [10,16,19] and are considered as standards against which other simplified theories may be compared [16,19].

The fundamental equations of Flugge's theory, upon which this study is based, are presented in Appendix A.

1.6 Methods of Analysis

A structure usually consists of an assemblage of many parts and tends to be complex in nature. Generally, the true structure must be replaced by an idealized approximation, or model, suitable for mathematical analysis. Structural analysis for shells may conveniently be carried out by matrix methods using influence coefficients.

In the literature, the analysis of symmetrically loaded shells of revolution is classically performed using flexibility influence coefficients [3,4,21]. These have been given, in an explicit manner, for very limited number of types of shells of revolution, such as, cylinders, spheres and cones, of uniform thickness.

For symmetrically loaded shells, the membrane solution, which represents the momentless state of stress in the shell, may approximate the particular solution which satisfies the general differential equation of the shell. Using the membrane solution, a flexibility method of analysis can be performed by satisfying the continuity requirements at the joints at which the elements are connected. When the forces and displacements at these joints are known, the conditions within each element may be determined from the solution of the differential equation of the element.

For the case of arbitrary shells of revolutions, under arbitrary systems of loadings, the corresponding expressions for influence coefficients are unknowns. The analytical method of obtaining these expressions would involve the solution of eighth-order differential equations expressed in terms of the geometry of the shell surface and the physical constants. This method is difficult and complicated, even for a simple geometry such as a cylinder, and it is generally impossible for the case of an arbitrary shell under arbitrary loads. However, the latter case is the rule rather than the exception in modern architecture.

As a consequence of the availability of electronic computers, and the increasing familiarity of engineers with this computational tool, the application of numerical methods of analysis to shell problems has become more attractive.

Two numerical methods for the analysis of shells of revolution with arbitrary configurations have received extensive treatment in the technical literature. The first method is the finite difference method, which consists essentially of the direct replacement of the derivatives which appear in the governing differential equations by finite difference approximations. This method transforms the differential equations into a system of algebraic equations which may be solved by an iterative procedure [26] or by means of matrix methods [6]. The method is quite general in application. Replacement of the derivatives by finite difference approximations may be undertaken at any level in the basic formulation of the shell problem. However, it is difficult to introduce boundary conditions into the problem. It also becomes cumbersome when attempting to satisfy equations involving high order derivatives [9,12].

The second method is the finite element method. In this method the displacement of each element into which a shell is subdivided are represented by an approximation [9,11,13,20,24]. The most common practice is to represent each shell by a series of short conical shell elements of uniform thickness. The variation in thickness along the generator of the shell, can be accounted for by considering the average thickness of each element [20]. Once these short segments have been defined, the problem becomes one of analyzing a shell that is an assembly of many short conical shells. Stiffness influence coefficients are evaluated using energy methods. Conditions of continuity are then applied at the boundaries of each segment to evaluate the forces and deformations.

An alternative method of numerical analysis based on direct integration of the shell equations, as proposed by

Goldberg, et al [12] for spherical domes and by Iyer and Simmonds [17] for conical shells, is presented for general shells of revolution in this thesis.

In the application of this method to the analysis of shells of revolution under arbitrary loadings, the governing partial differential equations of a consistent shell theory are expanded by means of Fourier series. Differentiation with respect to the colatitude coordinate θ may be performed to transform the governing equations to ordinary differential equations. These equations are reduced to a set of eight first-order ordinary differential equations involving eight intrinsic dependent variables and their derivatives. The intrinsic variables are the three components of displacement at the middle surface, the rotation of the tangent to the meridian, the membrane normal force in the meridian direction, the moment acting on circumferential sections and two modified shear terms in the directions perpendicular and tangent to the meridian.

This system of equations can be integrated in a stepwise fashion across a given element. By performing matrix operations, which will be described in Chapter 3, the stiffness influence coefficients can be calculated. Fixed end forces can be obtained and stiffness analysis may then be performed to determine the conditions at the element boundaries. These conditions can be used to determine the forces and deformation everywhere within the shell element.

1.7 Outline of Contents

The governing equations of the classical theory of shells of revolution are reduced to a set of eight basic equations in Chapter 2. In Chapter 3, the solution technique for the basic equations is described. Standard stiffness methods for segmented shell structures are outlined. Two example problems are handled in Chapter 4. General types of loadings are considered. In Chapter 5, limitations of the technique are discussed. Conclusions are drawn and possible future developments are outlined in Chapter 6.

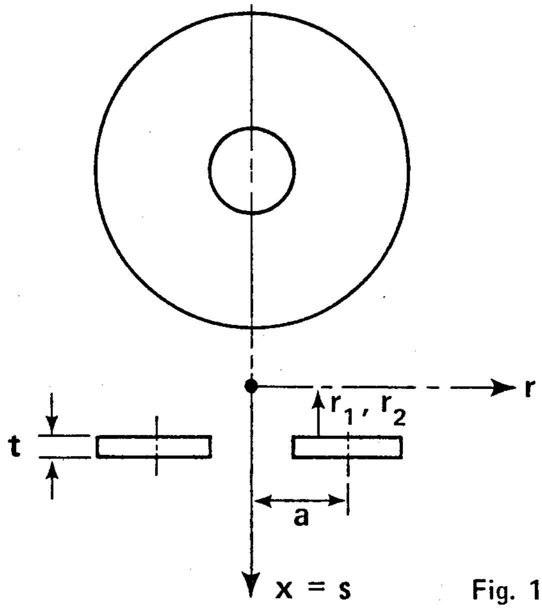


Fig. 1.1 Plate

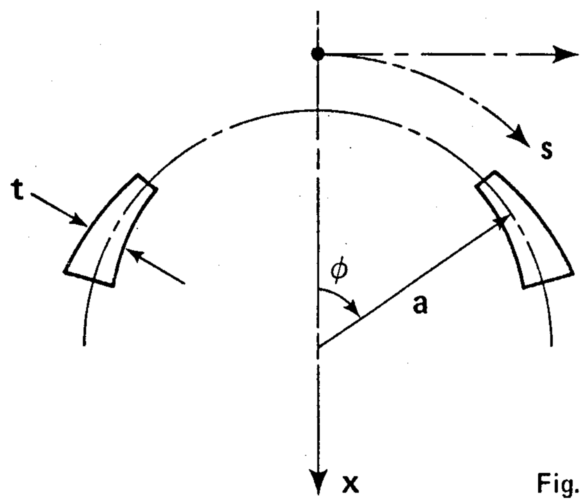


Fig. 1.2 Sphere

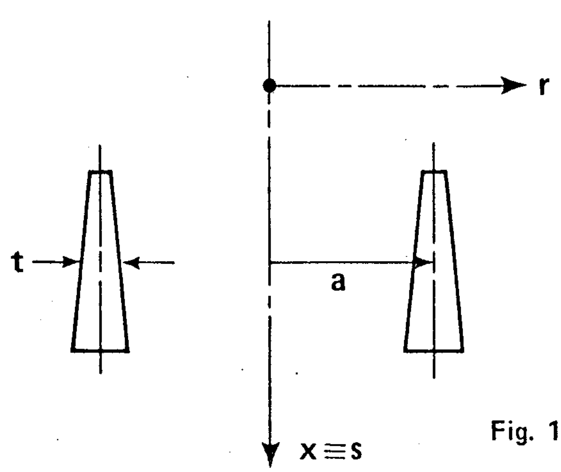


Fig. 1.3 Cylinder

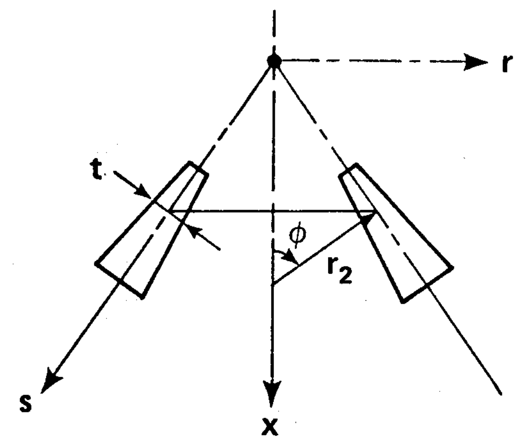


Fig. 1.4 Cone

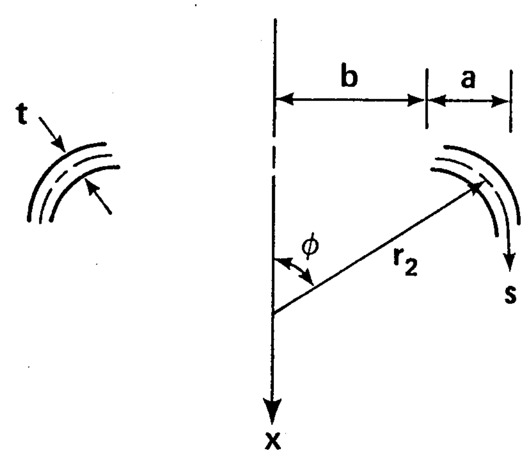


Fig. 1.5 Toroid

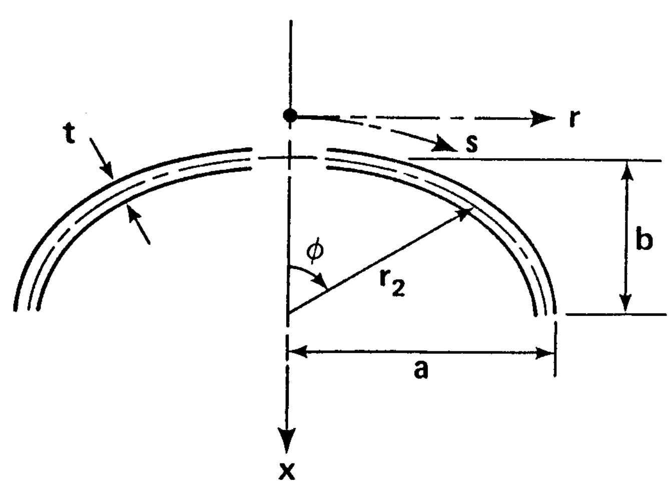


Fig. 1.6 Ellipsoid

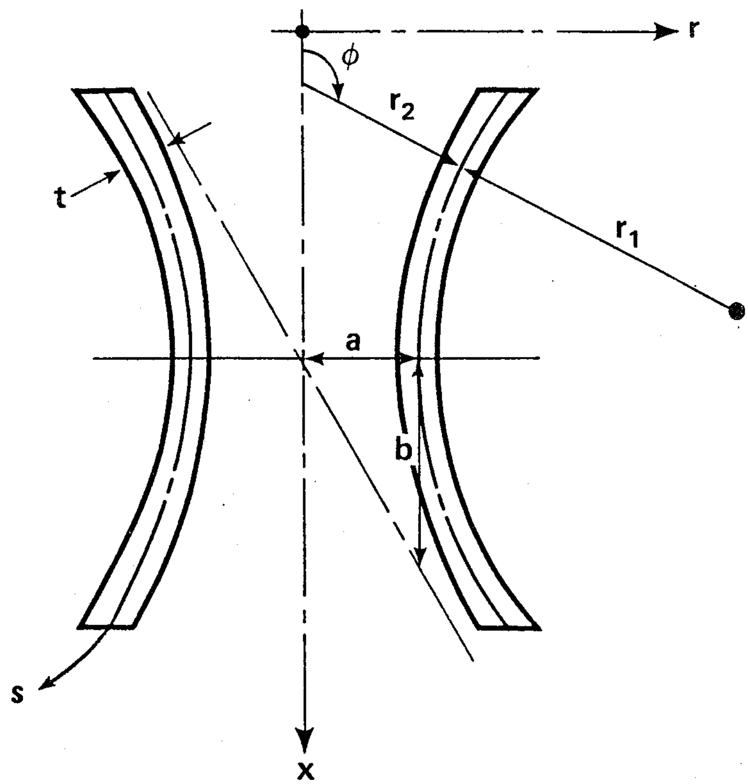


Fig. 1.7 Hyperboloid of Revolution

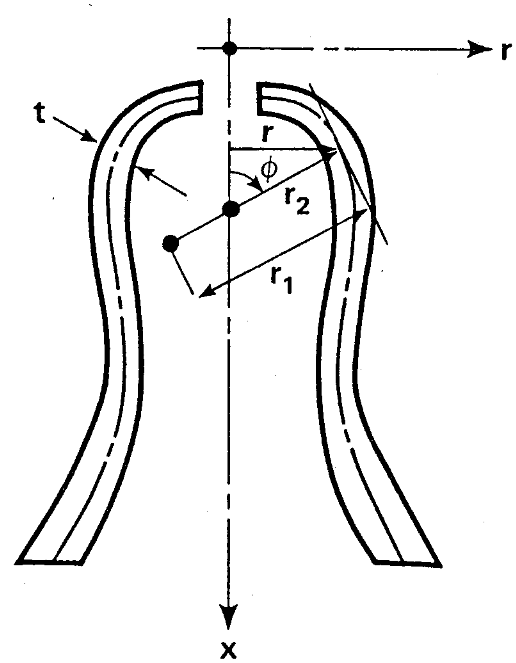


Fig. 1.8 Shell of Revolution of Arbitrary Shape

CHAPTER 2

BASIC EQUATIONS

2.1 Introduction

In this chapter the governing field equations of a shell of revolution, as derived in Appendix A, are expanded using Fourier series. Modified shear terms are introduced into the governing equations to eliminate the inplane shearing force in the circumferential direction and the meridional twisting moment, which appear at the boundaries s = constant. By performing matrix operations to eliminate the stress resultants in the circumferential direction (secondary stress resultants) the governing field equations are reduced to a set of eight first order equations involving four stress resultants (primary stress resultants) and four displacements and their derivatives.

2.2 Fourier Series

The load components and temperature, being arbitrary functions of s and θ, may always be represented in the form

$$P_s = \sum_{n=0}^{\infty} P_{sn}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{P}_{sn}(s) \sin n\theta \qquad 2.1.1$$

$$P_\theta = \sum_{n=0}^{\infty} \overline{P}_{\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} P_{\theta n}(s) \sin n\theta \qquad 2.1.2$$

$$P_z = \sum_{n=0}^{\infty} P_{zn}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{P}_{zn}(s) \sin n\theta \qquad 2.1.3$$

$$T = \sum_{n=0} T_n(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{T}_n(s) \sin n\theta \qquad 2.1.4$$

where P_s, P_θ and P_z are defined in Appendix A, T represents temperature, and the functions of s on the right hand side are Fourier coefficients [10,27]. The corresponding stress resultants and displacements may be expressed as

$$N_s = \sum_{n=0}^{\infty} N_{sn}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{N}_{sn}(s) \sin n\theta \qquad 2.2.1$$

$$N_\theta = \sum_{n=0}^{\infty} N_{\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{N}_{\theta n}(s) \sin n\theta \qquad 2.2.2$$

$$N_{s\theta} = \sum_{n=0}^{\infty} \overline{N}_{s\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} N_{s\theta n}(s) \sin n\theta \qquad 2.2.3$$

$$N_{\theta s} = \sum_{n=0}^{\infty} \overline{N}_{\theta sn}(s) \cos n\theta + \sum_{n=1}^{\infty} N_{\theta sn}(s) \sin n\theta \qquad 2.2.4$$

$$Q_s = \sum_{n=0}^{\infty} Q_{sn}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{Q}_{sn}(s) \sin n\theta \qquad 2.2.5$$

$$Q_\theta = \sum_{n=0}^{\infty} \overline{Q}_{\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} Q_{\theta n}(s) \sin n\theta \qquad 2.2.6$$

$$M_s = \sum_{n=0}^{\infty} M_{sn}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{M}_{sn}(s) \sin n\theta \qquad 2.2.7$$

$$M_\theta = \sum_{n=0}^{\infty} M_{\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} \overline{M}_{\theta n}(s) \sin n\theta \qquad 2.2.8$$

$$M_{s\theta} = \sum_{n=0}^{\infty} \overline{M}_{s\theta n}(s) \cos n\theta + \sum_{n=1}^{\infty} M_{s\theta n}(s) \sin n\theta \qquad 2.2.9$$

$$M_{\theta s} = \sum_{n=0}^{\infty} \overline{M}_{\theta sn}(s) \cos n\theta + \sum_{n=1}^{\infty} M_{\theta sn}(s) \sin n\theta \qquad 2.2.10$$

$$U = \sum_{n=0}^{\infty} U_n (s) \cos n\theta + \sum_{n=1}^{\infty} \bar{U}_n (s) \sin n\theta \qquad 2.2.11$$

$$V = \sum_{n=0}^{\infty} V_n (s) \cos n\theta + \sum_{n=1}^{\infty} \bar{V}_n (s) \sin n\theta \qquad 2.2.12$$

$$W = \sum_{n=0}^{\infty} W_n (s) \cos n\theta + \sum_{n=1}^{\infty} \bar{W}_n (s) \sin n\theta \qquad 2.2.13$$

where the variables on the left hand side are defined in Appendix A, Sects. A.2 and A.3, and where the term (s) indicates that the variable coefficients with subscript n are functions of the coordinate s only and n is the harmonic number. The first and the second series in each expression represent the portions of the variables which are, respectively, symmetric and anti-symmetric with respect to the meridian passing through the line $\theta = 0$.

For an arbitrary applied load expressed as a Fourier series of the order N (Eqs. 2.1.1 to 2.1.3), there are 2N+1 terms that represent each component of load; (n = 0, 1, 2,...,N) for the symmetric series and (n = 1, 2, ..., N) for the anti-symmetric series. For each value of n the (s dependent) variables with the subscript n from each series can be entered in the governing equations of shells of revolution of Appendix A, because the sequences $\cos n\theta$ and $\sin n\theta$ are linearly independent. Differentiations with respect to θ can be performed and the terms grouped according to the common factors, $\cos n\theta$ or $\sin n\theta$. Since the coefficient of each of these factors must be zero, each factor produces a separate equation. For example, Eqn. A.5.1 is

$$rN^{\bullet}_{s} + \cos\phi N_{s} + N'_{\theta s} - \cos\phi N_{\theta} - \frac{r}{r_1} Q_{s} + rP_{s} = 0 \qquad 2.3.1$$

for which, for any n, the cosine terms become

$$rN^{\bullet}_{sn} \cos n\theta + \cos\phi N_{sn} \cos n\theta + nN_{\theta sn} \cos n\theta$$

$$- \cos\phi N_{\theta n} \cos n\theta - \frac{r}{r_1} Q_{sn} \cos n\theta + rP_{sn} \cos n\theta = 0 \qquad 2.3.2$$

which, upon factoring out the common term, yields

$$rN^{\bullet}_{sn} + \cos\phi N_{sn} + n N_{\theta sn} - \cos\phi N_{\theta n} - \frac{r}{r_1} Q_{sn}$$

$$+ rP_{sn} = 0 \qquad 2.3.3$$

If the Fourier expansions of Eqs. 2.1 and 2.2 are truncated such that $0 \leq n \leq N$, the governing system of equations is replaced by (2N+1) ordinary differential systems of equations, each equation of the type illustrated by Eq. 2.3.3, and an analysis is carried out to obtain the solutions for the s dependent coefficients corresponding to each value of n. In the general case there are twenty-six variable coefficients associated with each $n > 0$ in Eqns. 2.2, thirteen associated with each of the factors.

Finally, since linear stress-displacement relations were assumed, the principal of superposition is valid and one can superimpose the (2N+1) solutions so obtained.

The n^{th} set of equations can be written as follows. The six equilibrium equations are obtained from Eqs. A.5 as

$$N^{\bullet}_{sn} + R\cos\phi\, N_{sn} \pm R\,n\,N_{\theta sn} - R\cos\phi\, N_{\theta n} - R_1 Q_{sn} + P_{sn} = 0 \qquad 2.4.1$$

$$N^{\bullet}_{s\theta n} + R\cos\phi\, N_{s\theta n} \mp R\,n\,N_{\theta n} + R\cos\phi\, N_{\theta sn} - R_2 Q_{\theta n} + P_{\theta n} = 0 \qquad 2.4.2$$

$$R_2 N_{\theta n} + R_1 N_{sn} \pm R\,n\,Q_{\theta n} + Q^{\bullet}_{sn} + R\cos\phi\, Q_{sn} - P_{zn} = 0 \qquad 2.4.3$$

$$M^{\bullet}_{sn} + R\cos\phi\, M_{sn} \pm R\,n\,M_{\theta sn} - R\cos\phi\, M_{\theta n} - Q_{sn} = 0 \qquad 2.4.4$$

$$M^{\bullet}_{s\theta n} + R\cos\phi\, M_{s\theta n} \mp R_n M_{\theta n} + R\cos\phi\, M_{\theta sn} - Q_{\theta n} = 0 \qquad 2.4.5$$

$$N_{\theta sn} - N_{s\theta n} - R_2 M_{\theta sn} + R_1 M_{s\theta n} = 0 \qquad 2.4.6$$

where R_1 and R_2 are the principal curvatures of the shell, defined as the reciprocals of the radii of curvature r_1 and r_2, respectively, and R is the curvature of the parallel circle.

The eight stress-displacement equations are obtained from Eqs. A17 as

$$N_{sn} = [K R_1 (R_1 - R_2) r^{\bullet}_1]\, \beta_n - [K(R_1 - R_2)]\beta^{\bullet}_n + [D(R_1 + \nu R_2) + K R^2{}_1 (R_1 - R_2)]W_n$$

$$+ [\nu D R \cos\phi - K R^2{}_1 (R_1 - R_2) r^{\bullet}{}_1] V_n$$

$$+ [D + K R_1 (R_1 - R_2)] V^{\bullet}_n \pm [\nu D R n] U_n$$

$$- [(1 + \nu)\alpha D] T_{o2n} \qquad 2.5.1$$

$$N_{\theta n} = [K R (R_1 - R_2) \cos\phi] \beta_n$$

$$+ [D(R_2 + \nu R_1) + K (R_1 - R_2)(R^2 n^2 - R^2{}_2)] W_n$$

$$+ [D R \cos\phi - K R R_2 \cos\phi (R_1 - R_2)] V_n$$

$$+ [\nu D] V^{\bullet}_n \pm [D R n] U_n$$

$$- [(1 + \nu)\alpha D] T_{o1n} \qquad 2.5.2$$

$$N_{s\theta n} = \frac{1 - \nu}{2} \{ \pm [K R (R_1 - R_2) n] \beta_n$$

$$\pm [K R^2 \cos\phi (R_1 - R_2) n] W_n$$

$$\mp [D R n + K R R_1 (R_1 - R_2) n] V_n$$

$$- [D R \cos\phi - K R \cos\phi (R_1 - R_2)^2] U_n$$

$$+ [D + K (R_1 - R_2)^2] U^{\bullet}_n \} \qquad 2.5.3$$

$$N_{\theta s n} = \frac{1 - \nu}{2} \{ \pm [K R (R_1 - R_2) n] \beta_n$$

$$\mp [K R^2 \cos\phi (R_1 - R_2) n] W_n$$

$$\mp [D R n - K R R_2 (R_1 - R_2) n] V_n$$

$$- [D R \cos\phi] U_n + [D] U^{\bullet}_n \} \qquad 2.5.4$$

$$M_{sn} = [K R_1 r^{\bullet}{}_1 - \nu K R \cos\phi] \beta_n - [K] \beta^{\bullet}_n$$

$$+ [K R_1 (R_1 - R_2) - \nu K R^2 n^2] W_n$$

$$- [K R^2{}_1 r^{\bullet}{}_1] V_n + [K(R_1 - R_2)] V^{\bullet}{}_n$$

$$\mp [\nu K R R_2 n] U_n + [(1 + \nu)\alpha K] T_{12n} \qquad 2.5.5$$

$$M_{\theta n} = [\nu K R_1 r^{\bullet}{}_1 - K R \cos\phi] \beta_n - [\nu K] \beta^{\bullet}{}_n$$

$$- [K R^2 n^2 + K R_2 (R_1 - R_2)] W_n$$

$$- [\nu K R_1 r^{\bullet}{}_1 + K R \cos\phi \, (R_1 - R_2)] V_n$$

$$\mp [K R R_1 n] U_n + [(1 + \nu)\alpha K] T_{11n} \qquad 2.5.6$$

$$M_{s\theta n} = \frac{1 - \nu}{2} \{ \pm [2 K R n] \beta_n \pm [2 K R^2 n \cos\phi] W_n$$

$$\mp [K R R_1 n] V_n - [K R \cos\phi \, (R_1 - 2R_2)] U_n$$

$$+ [K(R_1 - 2R_2)] U^{\bullet}{}_n \} \qquad 2.5.7$$

$$M_{\theta s n} = \frac{1 - \nu}{2} \{ \pm [2 K R n] \beta_n \pm [2 K R^2 n \cos\phi] W_n$$

$$\mp [K R R_2 n] V_n + [K R R_2 \cos\phi] U_n$$

$$- [K R_2] U^{\bullet}{}_n \} \qquad 2.5.8$$

where Eqs. A.7 have been used to eliminate the first and second derivatives of the displacement component W.

Eqs. 2.4 and 2.5 are fourteen equations in terms of the thirteen unknown functions (the variable Fourier series coefficients). As such the system of equations is overspecified. In the theory developed herein, Eq. 2.4.6 is discarded and the remaining thirteen equations allow the solution for the thirteen functions associated with each trignometric functions of the Fourier series expansion.

2.3 Natural Boundary Conditions

According to the classical theory of shells, the quantities which appear in the natural boundary conditions on an edge s = constant of a shell of revolution are the four displacements β, W, V, U and the corresponding four forces M_s, S_s, N_s, T_s.

The forces S_s, T_s are the transverse and tangential effective shears which are commonly known as Kirchhoff's shears [27]. The Kirchhoff shears are work-equivalent forces associated with the displacements W and U, and these effective shears replace the stress resultants Q_s, $N_{s\theta}$ and $M_{s\theta}$. Such replacement is essential to describe simple boundary conditions [10,27]. Although the expressions for these forces may be derived rigorously from variational principles, the following physical interpretation, due to Kelvin and Tait [27, pg.45], is more instructive. If the twisting moment $M_{s\theta}$ acting on an infinitesimally small element of the shell is replaced by a statically equivalent force as shown in Fig. 2.1, one can write from statics the expression for the effective transverse shearing force as [27]

$$S_s = Q_s + \frac{M'_{s\theta}}{r} \qquad 2.6.1$$

and the effective tangential shearing force as

$$T_s = N_{s\theta} - \frac{M_{s\theta}}{r_2} \qquad 2.6.2$$

Writing Eqs. 2.6.1 and 2.6.2 in Fourier series terms, and separating the linearly independent terms, we may define

$$S_{sn} = Q_{sn} \pm \frac{nM_{s\theta n}}{r} \qquad 2.7.1$$

$$T_{sn} = N_{s\theta n} - \frac{M_{s\theta n}}{r_2} \qquad 2.7.2$$

Using the geometrical relations in Eqs. A.2, the derivatives of these forces with respect to the coordinate s may be written as

$$S^{\bullet}_{sn} = Q^{\bullet}_{sn} \pm Rn\,M^{\bullet}_{s\theta n} \mp R^2 n \cos\phi\, M_{s\theta n} \qquad 2.8.1$$

$$T^{\bullet}_{sn} = N^{\bullet}_{s\theta n} - \frac{r_2 M^{\bullet}_{s\theta n} - M_{s\theta n} r^{\bullet}_2}{r^2_2}$$

$$= M^{\bullet}_{s\theta n} - R_2 M^{\bullet}_{s\theta n} - R_2(R_1 - R_2) \cot\phi\, M_{s\theta n} \qquad 2.8.2$$

2.8.2

2.4 Reduction of the Governing Equations

It can be seen that the stress resultants Q_θ, N_θ, M_θ, $M_{\theta s}$ and $N_{\theta s}$ do not enter into any boundary conditions on an edge for which s = constant. Therefore, it is preferable to

$$M_s = F_{11}(\beta, \beta^{\bullet}, W, V, V^{\bullet}, U, T_{12}) \qquad 2.10.5$$

$$M_\theta = F_{12}(\beta, \beta^{\bullet}, W, V, U, T_{11}) \qquad 2.10.6$$

$$M_{s\theta} = F_{13}(\beta, W, V, U, U^{\bullet}) \qquad 2.10.7$$

$$M_{\theta s} = F_{14}(\beta, W, V, U, U^{\bullet}) \qquad 2.10.8$$

where the form of the functions F_1 to F_{14} is obtained from Eqs. 2.4 and 2.5. In addition, Eqs. A.7.1, 2.7 and 2.8, which will be called auxiliary equations, may be written in the following symbolic form,

$$\beta = F_{15} (W^{\bullet}, V) \qquad 2.11.1$$

$$S_s = F_{16} (Q_s, M_{s\theta}) \qquad 2.11.2$$

$$T_s = F_{17} (N_{s\theta}, M_{s\theta}) \qquad 2.11.3$$

$$S^{\bullet}_s = F_{18} (Q^{\bullet}_s, M^{\bullet}_{s\theta}, M_{s\theta}) \qquad 2.11.4$$

$$T^{\bullet}_s = F_{19} (N^{\bullet}_{s\theta}, M_{s\theta}^{\bullet}, M_{s\theta}) \qquad 2.11.5$$

It should be noted that for shells with equal radii of curvature (i.e., $r_1 = r_2$), the sixth equation of equilibrium Eq. 2.4.6 will be an identity as

$$N_{s\theta} = N_{\theta s}$$

$$M_{s\theta} = M_{\theta s}$$

Therefore, this equation will be discarded, as mentioned in eliminate them in terms of the other stress resultants and evaluate them after the solution of the governing equations.

For convenience, the subscript n in Fourier coefficients will be omitted and the governing equations (Eqs. 2.4 and 2.5) can be written, for each set of equations, in the following symbolic form. The equilibrium equations (Eqs. 2.4) may be written symbolically as:

$$F_1\ (N^{\bullet}_s, N_s, N_{\theta s}, N_\theta, Q_s, P_s) = 0 \tag{2.9.1}$$

$$F_2\ (N^{\bullet}_{s\theta}, N_{s\theta}, N_\theta, N_{\theta s}, Q_\theta, P_\theta) = 0 \tag{2.9.2}$$

$$F_3\ (N_\theta, N_s, Q_\theta, Q^{\bullet}_s, Q_s, P_z) = 0 \tag{2.9.3}$$

$$F_4\ (M^{\bullet}_s, M_s, M_{\theta s}, M_\theta, Q_s) = 0 \tag{2.9.4}$$

$$F_5\ (M^{\bullet}_{s\theta}, M_{s\theta}, M_\theta, M_{\theta s}, Q_\theta) = 0 \tag{2.9.5}$$

$$F_6\ (N_{\theta s}, N_{s\theta}, M_{\theta s}, M_{s\theta}) = 0 \tag{2.9.6}$$

The stress resultant-displacement equations (Eqs. 2.5) may be written symbolically as:

$$N_s = F_7(\beta, \beta^{\bullet}, W, V, V^{\bullet}, U, T_{o_2}) \tag{2.10.1}$$

$$N_\theta = F_8(\beta, W, V, V^{\bullet}, U, T_{o_1}) \tag{2.10.2}$$

$$N_{s\theta} = F_9(\beta, W, V, U, U^{\bullet}) \tag{2.10.3}$$

$$M_{\theta s} = F_{10}(\beta, W, V, U, U^{\bullet}) \tag{2.10.4}$$

Appendix A, Sect. A.5 and as noted in Sect. 2.3.

In solving the above equations, the displacements which may be physically imposed on the boundary of a shell are β, W, V and U. The external forces associated with these displacements (in a work-equivalent sense) are M_s, S_s, N_s and T_s. The first set of variables are specified in the case of displacement boundary conditions while the latter set is specified in the case of mechanical or force boundary conditions. In general a combination of four of these eight quantities must be specified to properly define a boundary condition, provided that if one of the displacements is specified the associated force should not be specified and vice-versa. Since the remaining five of the thirteen variables cannot be determined at the boundary, it is desirable to eliminate them from the governing equations. The objective in the following is, therefore, to reduce the thirteen governing equations (Eqs. 2.9.1 to 2.9.5 and Eqs. 2.10) to a set of eight governing equations in terms of the eight variables which may arise in the specification of the boundary conditions. It is also desirable to keep the order of these differential equations to a minimum in order to facilitate the numerical solution. The final set of equations will be of first order.

Combining Eq. 2.9.1 and Eq. 2.11.2 to eliminate Q_s, one can get

$$N^{\bullet}_s = R_1 S_s - R\cos\phi\, N_s \mp RR_1 M_{s\theta} + R\cos\phi\, N\theta \mp R_n N_{\theta s} - P_s \qquad 2.12.1$$

Using Eqs. 2.9.2 and 2.9.5 to eliminate Q_θ and then substituting for $M^{\bullet}_{s\theta}$, $N_{s\theta}$, $N^{\bullet}_{s\theta}$ by means of Eqs. 2.11.3 and 2.11.5, one can write

$$T^{\bullet}_{s} = -R\cos\phi\, T_s \mp RR_2 n\, M_\theta + RR_2\cos\phi\, M_{\theta s}$$

$$- R\cos\phi\,(R_1 - R_2)\, M_{s\theta} \pm Rn\, N_\theta$$

$$- R\cos\phi\, N_{\theta s} - P_\theta \qquad 2.12.2$$

Eqs. 2.9.4 and 2.11.2, after eliminating Q_s are in the form

$$M^{\bullet}_{s} = -R\cos\phi\, M_s + S_s + R\cos\phi\, M_\theta$$

$$\mp Rn\, M_{\theta s} \mp Rn\, M_{s\theta} \qquad 2.12.3$$

Using Eq. 2.9.3, after eliminating Q_θ by means of Eq. 2.9.5, together with Eqs. 2.11.2 and 2.11.4 results in the equation

$$S^{\bullet}_{s} = -R\cos\phi\, S_s - R_1 N_s + R^2 n^2 M_\theta \mp R^2 n\cos\phi\, M_{\theta s}$$

$$\mp R^2 n\cos\phi\, M_{s\theta} - R_2 N_\theta + P_z \qquad 2.12.4$$

Eqs. 2.12, which represent the equilibrium equations, may now be written symbolically as

$$M^{\bullet}_{s} = F_{20}\,(M_s, S_s, M_\theta, M_{\theta s}, M_{s\theta}) \qquad 2.13.1$$

$$S^{\bullet}_{s} = F_{21}\ (S_s,\ N_s,\ M_\theta,\ M_{\theta s},\ M_{s\theta},\ N_\theta,\ P_z) \qquad 2.13.2$$

$$N^{\bullet}_{s} = F_{22}\ (S_s,\ N_s,\ M_{s\theta},\ N_\theta,\ N_{\theta s},\ P_s) \qquad 2.13.3$$

$$T^{\bullet}_{s} = F_{23}\ (T_s,\ M_\theta,\ M_{\theta s},\ M_{s\theta},\ N_\theta,\ N_{\theta s},\ P_\theta) \qquad 2.13.4$$

or in matrix notation

$$\{F^{\bullet}_{s}\} = [\ B1 \quad B2\ ] \begin{Bmatrix} F_s \\ F_\theta \end{Bmatrix} + \{B3\} \qquad 2.14$$

where the loading terms in these equations are separated in the vector {B3}, and

$$\langle F_s \rangle = \langle\ M_s \quad S_s \quad N_s \quad T_s\ \rangle \qquad 2.15.1$$

$$\langle F^{\bullet}_{s} \rangle = \langle\ M^{\bullet}_{s} \quad S^{\bullet}_{s} \quad N^{\bullet}_{s} \quad T^{\bullet}_{s}\ \rangle \qquad 2.15.2$$

$$\langle F_\theta \rangle = \langle\ M_\theta \quad M_{\theta s} \quad M_{s\theta} \quad N_\theta \quad N_{\theta s} \rangle \qquad 2.15.3$$

The coefficients of the matrices [B1], [B2] and {B3} are defined in Table 2.1

In Eq. 2.14, the stress resultants have been separated into the vector $\langle F_s \rangle$, whose components are desired in the final formulation, and the vector $\langle F_\theta \rangle$, which remains to be eliminated from the formulation.

Let us now turn our attention to the displacement variables. Eqs. 2.10.1 and 2.10.5, which are two equations in

$\beta^{\bullet}$, $V^{\bullet}$, can be converted to the following two equations

$$\beta^{\bullet} = \frac{1}{CA_2}\{-[\frac{CA_1}{K}]\, M_s + [R_1 - R_2]N_s$$

$$+ [R_1 r^{\bullet}_1 \; CA_2 - \nu R \cos\phi \; CA_1]\,\beta$$

$$- [\nu D R_2(R_1 - R_2) + \nu R^2 n^2 CA_1]W$$

$$- [R^2_1 \, r^{\bullet}_1 \, CA_2 + \nu D R \cos\phi \,(R_1 - R_2)]V$$

$$\mp [\nu R R_1 \, n \, CA_2]\, U + (1 + \nu)\alpha \; [D(R_1 - R_2)\; T_{o2} + CA_1 \, T_{12}]\} \qquad 2.16.1$$

in which

$$CA_1 = D + K R_1 \; (R_1 - R_2) \qquad 2.16.2$$

$$CA_2 = D + K R_2 \; (R_1 - R_2) \qquad 2.16.3$$

and

$$V^{\bullet} = \frac{1}{CA_2}\{-[R_1 - R_2]\; M_s + N_s$$

$$- [\nu K R \cos\phi \; (R_1 - R_2)]\beta$$

$$- [CA_2 \, R_1 + \nu D R_2 + \nu K R^2 n^2 (R_1 - R_2)]W$$

$$- [\nu D R \cos\phi]V \mp [\nu R n \;\; CA_2]U$$

$$+ (1 + \nu)\alpha \;\; [DT_{o2} + K(R_1 - R_2)\; T_{12})\} \qquad 2.16.4$$

Eq. 2.11.1 is in the form

$$W^{\bullet} = -\beta + R_1 V \qquad 2.16.5$$

Finally, Eq. 2.10.3, after eliminating $N_{s\theta}$ by means of Eq. 2.11.3 can be written as

$$U^{\bullet} = \frac{1}{CA_3}\{[\frac{2}{1-\nu}]\ T_s \mp [K\,Rn\,(R_1 - 3R_2)]\beta$$

$$\mp [K\,R^2 n \cos\phi\,(R_1 - 3R_2)]W \pm [Rn\ CA_1 - K\ RR_1R_2\ n]V$$

$$+ [R\cos\phi\ CA_3]U\} \qquad 2.16.6$$

in which

$$CA_3 = D + K(R^2{}_1 - 3R_1R_2 + 3R^2{}_2) \qquad 2.16.7$$

Eqs. 2.16.1, 2.16.4, 2.16.5 and 2.16.6 can be written in a symbolic form as

$$\beta^{\bullet} = F_{24}(\beta,\ W,\ V,\ U,\ M_s,\ N_s,\ T_{o2}, T_{12}) \qquad 2.17.1$$

$$W^{\bullet} = F_{25}(\beta,\ V) \qquad 2.17.2$$

$$V^{\bullet} = F_{26}(\beta,\ W,\ V,\ U,\ M_s,\ N_s,\ T_{o2}, T_{12}) \qquad 2.17.3$$

$$U^{\bullet} = F_{27}(\beta,\ W,\ V,\ U,\ T_s) \qquad 2.17.4$$

or can be written in matrix notation as

$$\{D^{\bullet}\} = [A1 \quad A2] \begin{Bmatrix} D \\ F_s \end{Bmatrix} + \{A3\} \qquad 2.18$$

where

$$\langle D \rangle = \langle \ \beta \quad W \quad V \quad U \ \rangle \qquad 2.19.1$$

$$\langle D^{\bullet} \rangle = \langle \ \beta^{\bullet} \quad W^{\bullet} \quad V^{\bullet} \quad U^{\bullet} \ \rangle \qquad 2.19.2$$

The coefficients of the matrices [A1], [A2] and the column vector {A3} are shown in Tables 2.2.

The stress resultants appearing in the vector $\langle F_\theta \rangle$ of Eq. 2.15.3 can be expressed in terms of the fundamental displacements by selecting Eqs. 2.10.2, 2.10.4, 2.10.6, 2.10.7 and 2.10.8 and writing them in matrix notation

$$\{F_\theta\} = [\ C1 \quad C2\ ] \begin{Bmatrix} D^{\bullet} \\ D \end{Bmatrix} + \{C3\} \qquad 2.20$$

where $\{F_\theta\}$, {D} and $\{D^{\bullet}\}$ are defined in Eqs. 2.15.3, 2.19.1 and 2.19.2 respectively, and the coefficients of [C1], [C2] and {C3} are defined in Tables 2.3.

The three sets of equations (Eqs. 2.14, 2.18 and 2.20) can now be used to form a set of eight first order differential equations relating the eight fundamental variables; four displacements β, W, V, U, and four corresponding forces M_s, S_s, N_s, T_s and their derivatives. By substituting Eq. 2.20 into Eq. 2.14, to replace $\{F_\theta\}$

$$\{F^{\bullet}_s\} = [B1]\{F_s\} + [B2]\left\{[C1]\{D^{\bullet}\} + [C1]\{D\} + \{C3\}\right\} + \{B3\} \qquad 2.21.1$$

and by substituting for $\{D^{\bullet}\}$, from Eq. 2.18,

$$\{F^{\bullet}_s\} = [B1]\{F_s\} + [E1]\left\{[A1]\{D\} + [A2]\{F_s\} + \{A3\}\right\} + [E2]\{D\} + \{E3\} \qquad 2.21.2$$

or

$$\{F^{\bullet}_s\} = [G1]\{D\} + [G2]\{F_s\} + \{G3\} \qquad 2.22$$

where

$$[E1] = [B2][C1] \qquad 2.23.1$$

$$[E2] = [B2][C2] \qquad 2.23.2$$

$$\{E3\} = [B2]\{C3\} + \{B3\} \qquad 2.23.3$$

$$[G1] = [E1][A1] + [E2] \qquad 2.23.4$$

$$[G2] = [E1][A2] + [B1] \qquad 2.23.5$$

$$\{G3\} = [E1]\{A3\} + \{E3\} \qquad 2.23.6$$

Eq. 2.18 and Eq. 2.22, can be combined into one matrix equation as

$$\begin{Bmatrix} D^{\bullet} \\ F^{\bullet}_s \end{Bmatrix} = \begin{bmatrix} A1 & A2 \\ G1 & G2 \end{bmatrix} \begin{Bmatrix} D \\ F_s \end{Bmatrix} + \begin{Bmatrix} A3 \\ G3 \end{Bmatrix}$$

$$= [GA] \begin{Bmatrix} D \\ F_s \end{Bmatrix} + \{GB\} \qquad 2.24$$

Therefore, for the applied loads, stresses and displacements being periodic functions of the coordinate θ, the general governing partial differential equations, Eqs. A.5 and Eqs. A.17, have been transformed into a system of eight first order ordinary differential equations for each harmonic member n. These equations relate, at any point, the eight fundamental dependent variables, that appear in the natural boundary conditions of shells of revolution, and their derivatives with respect to the independent variable s.

As was seen, the reduction of the shell equation into this form involves only straight forward algebraic manipulations. The equations have been put in this form to facilitate numerical integration. They can be integrated with the aid of an appropriate integration technique, such as the Runge-Kutta process. Since a digital computer is needed to perform this integration, the reduction of the equations to the final form, as in Eq. 2.24, is not necessary and can be performed by feeding the three sets of equations (Eqs. 2.14, 2.18 and 2.20) into the computer.

It can be seen that the elements of the matrix GA in Eq. 2.24 are dependent only on the shell thickness, the physical constants and the coordinate s. It should be noted that in the case of shell of revolution under symmetrical loading conditions (i.e., $n = 0$) the system of the eight equations reduces naturally to a system of six first order equations as the displacement component U and the corresponding force T_s vanish due to the symmetry of the applied load.

Matrix B1

$-R\cos\phi$			
	$-R\cos\phi$	$-R_1$	
	R_1	$-R\cos\phi$	
			$-R\cos\phi$

Matrix B2

$R\cos\phi$	$\mp Rn$	$\mp Rn$		
R^2n^2	$\mp R^2n\cos\phi$	$\mp R^2n\cos\phi$	$-R_2$	
		$\mp RR_1n$	$R\cos\phi$	$\mp Rn$
$\mp RR_2n$	$RR_2\cos\phi$	$-R\cos\phi\,(R_1-R_2)$	$\pm Rn$	$-R\cos\phi$

Vector B3

P_r
$-Ps$
$-P_\theta$

TABLE 2.1 Coefficients of Matrices B1, B2 and Load Vector B3 in Eq. 2.14

$R_1 r^{\bullet}_1 - \nu R \cos\phi \frac{CA_1}{CA_2}$	$-\nu \frac{DR_2 (R_1 - R_2)}{CA_2}$ $-\nu R^2 n^2 \frac{CA_1}{CA_2}$	$-R^2{}_1 r^{\bullet}_1$ $-\nu D \frac{R \cos\phi}{CA_2} (R_1 - R_2)$	$\mp \nu R R_1 n$
-1		R_1	
$-\nu K \frac{R \cos\phi}{CA_2} (R_1 - R_2)$	$-R_1 - \frac{\nu D R_2}{CA_2}$ $-\nu K \frac{R^2 n^2}{CA_2} (R_1 - R_2)$	$-\nu \frac{D R}{CA_2} \cos\phi$	$\mp \nu R n$
$\mp K \frac{R n}{CA_3} (R_1 - 3R_2)$	$\mp K \frac{R^2 n \cos\phi}{CA_3} (R_1 - 3R_2)$	$\pm Rn \frac{CA_1}{CA_3}$ $-K \frac{R R_1 R_2}{CA_3} n$	$R \cos\phi$

TABLE 2.2.1 Coefficients of Matrix A1 in Eq. 2.18

$-\frac{CA_1}{K.CA_2}$		$\frac{R_1 - R_2}{CA_2}$	
$-\frac{(R_1 - R_2)}{CA_2}$		$\frac{1}{CA_2}$	
			$(\frac{2}{1 - \nu})\ \frac{1}{CA_3}$

Matrix A2

$\frac{(1 + \nu)\alpha}{CA_2}\ \{D(R_1 - R_2)\ T_{o2} + CA_1T_{12}\}$
$\frac{(1 + \nu)\alpha}{CA_2}\ \{K(R_1 - R_2)\ T_{12} + DT_{o2}\}$

Vector A3

TABLE 2.2.2 Coefficient of Matrix A2 and Load Vector A3 in Eq. 2.18

$-\nu K$			
			$-(\frac{1-\nu}{2})\ K R_2$
			$(\frac{1-\nu}{2})\{K(R_1 - R_2)\}$
		νD	
			$(\frac{1-\nu}{2})D$

Matrix C1

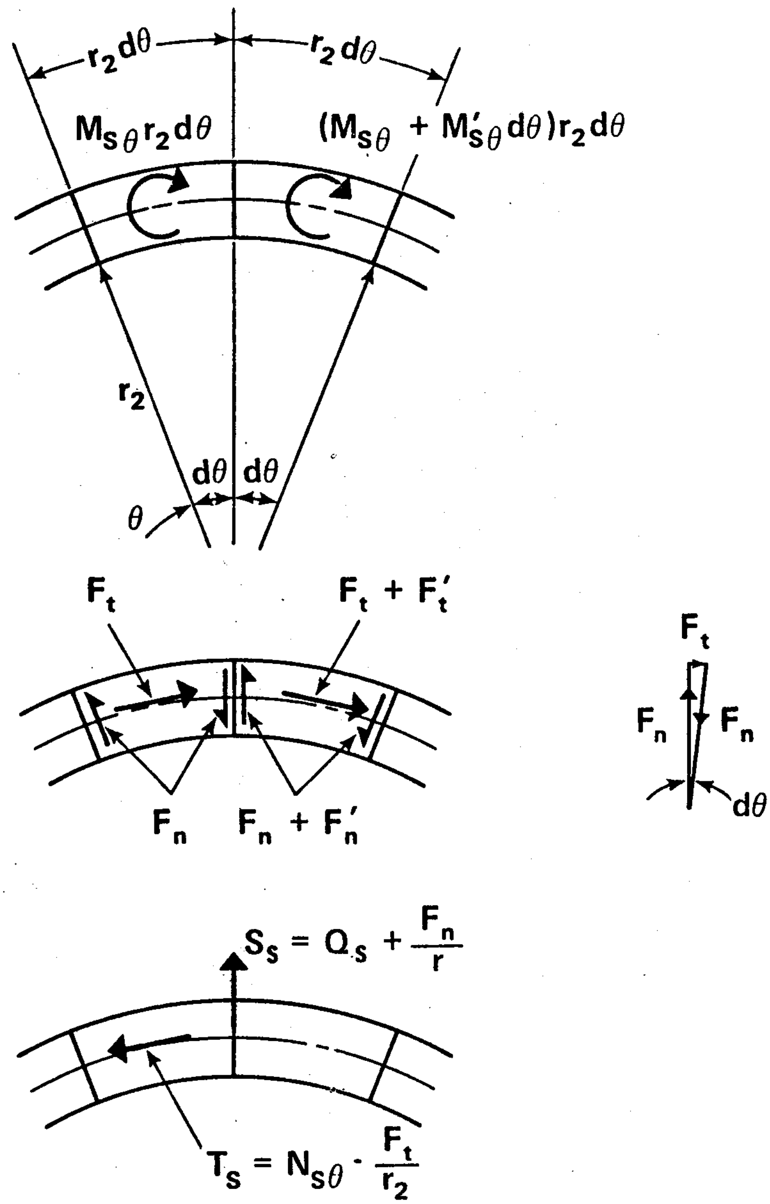
$(1 + \nu)\ \alpha\ K\ T_{11}$
$-\ (1 + \nu)\ \alpha\ D\ T_{o1}$

Vector C3

TABLE 2.3.1 Coefficients of Matrix C1 and Load Vector C3 in Eq. 2.20

$\nu K R_1 r^{\bullet}_1 - K R \cos\phi$	$- K R^2 n^2 - K R_2 \ (R_1 - R_2)$	$- \gamma K R^2_1 r^{\bullet}_1$ $- K R \cos\phi \ (R_1 - R_2)$	$\mp K R R_1 n$
$\pm(\frac{1-\nu}{2}) \ (2K R n)$	$\mp(\frac{1-\nu}{2}) \ (2K R^2 n \cos \ \phi)$	$\mp (\frac{1-\nu}{2}) (K R R_2 n)$	$(\frac{1-\nu}{2}) (K R R_2 \cos\phi)$
$\pm(\frac{1-\nu}{2}) \ (2K R n)$	$(\frac{1-\nu}{2}) \ (2K R^2 n \cos\phi)$	$\mp (\frac{1-\nu}{2}) (K R R_1 n)$	$- (\frac{1-\nu}{2})\{K R \cos\phi \ (R_1 - 2R_2)\}$
$K R \cos\phi \ (R_1 - R_2)$	$D(R_2 + \nu R_1)$ $+ K(R_1 - R_2)(R^2 n^2 - R^2_2)$	$R \cos\phi\{D - K R_2 \ (R_1 - R_2)\}$	$\pm D R n$
$\mp (\frac{1-\nu}{2})\{K R n \ (R_1 - R_2)\}$	$\mp (\frac{1-\nu}{2})\{K R^2 n \cos\phi \ (R_1 - R_2)\}$	$\mp (\frac{1-\nu}{2})\{R n[D - K(R_1 - R_2)^2]\}$	$- (\frac{1-\nu}{2}) D R \cos\phi$

TABLE 2.3.2 Coefficients of Matrix C2 in Eq. 2.20



* F_n, F_t are the Equivalent Static Forces of $M_{s\theta}$

Fig. 2.1 Effective Shearing Forces

CHAPTER 3

STIFFNESS ANALYSIS

3.1 Introduction

This chapter describes how the standard methods of structural analysis can be employed to obtain a complete solution in terms of the stresses and the displacements at any point within a complex structure composed of many elements of any type of shell of revolution, using the eight fundamental equations derived in Chapter 2.

3.2 Influence Coefficients

If a perfectly elastic structural component, supported against rigid body motion, is acted upon by a set of forces F_1, F_2, F_3, ..., F_n at points 1, 2, 3, ..., n and if in addition to this set of forces, intermediate loads are applied simultaneously to the structural member, the induced deformation d_i at the point i due to the forces $F_j (j = 1, 2, 3, ..., n)$ can be expressed in the form

$$d_i = \sum_{j=1}^{n} a_{ij} F_j + d^o_i \qquad 3.1$$

where d^o_i is the additional deformation at point i due to the intermediate applied loads and a_{ij} represents the deflection d_i due to a unit value of F_j. The elastic constants a_{ij} are independent of the magnitude of the applied forces F_j. Eq. 3.1, which represent the linear relation of deformations and forces, can be written in matrix form as

$$\{d\} = [A]\{F\} + \{d^o\} \qquad 3.2$$

where the square matrix [A] is known as the flexibility matrix.

For the same problem the relation given in Eq. 3.2 can also be written in a form which represents the equilibrium conditions for the structure as

$$\{F\} = [K]\{d - d^o\}$$

$$= [K]\{d\} + \{F^o\} \qquad 3.3$$

where the square matrix [K] is defined the element stiffness matrix. The element K_{ij} is defined as the force F_j due to unit displacement at d_j. $\{F^o\}$ is the column vector $\{F^o{}_1, F^o{}_2, ..., F^o{}_n\}$ which are the equivalent forces that replace the intermediate applied loads (equivalent in the sense that the work done during any incremental deformation approximates the work done by the actual applied load). These forces are numerically equal to the so-called fixed end forces.

As stated above, while establishing the flexibility matrix the structure is assumed to be supported against rigid body motion, a condition not necessary for the stiffness matrix. For the latter case the structure can be free to move as a rigid body when a set of nodal displacements is applied. The stiffness matrix, thus obtained, is called "the direct stiffness matrix" which is singular. The elements in any column represent a force system in equilibrium.

The well known advantage of the direct stiffness method is that for an assembly of structural elements, the total structural stiffness matrix can be easily formed by superposition of the individual stiffness matrices irrespective of the boundary conditions. The boundary conditions are considered only after assembly in the actual solution of the system of equations. This permits the consideration of different boundary conditions while the total structure stiffness matrix remains unaltered.

3.3 Shell Element Influence Coefficients

By a procedure similar to that used for a beam element, the stiffness matrix for an element of shell of revolution can be obtained. Since the shell element is a rotationally symmetric element, the nodal points are replaced with nodal circles (Fig. 3.1). The forces and displacements can be expressed as the amplitude of a harmonic number along the nodal circles.

Consider the shell element shown in Fig. 3.1. The eight fundamental stress resultants per unit length at the two edges of the element are shown acting in the positive sense according to the shell theory sign convention. Should the edge $s = a$ of the element be subjected to a unit value of displacement in the direction of W while preventing any other displacement of the two ends of the element, the forces per unit length that would be required to maintain equilibrium of the element are the influence coefficients for a unit displacement W_a. If one writes the degrees of freedom of the element in the order β_a W_a V_a U_a β_b W_b V_b U_b, the element stiffness equation (Eq. 3.3) can be written as

$$\begin{Bmatrix} M_{sa} \\ S_{sa} \\ N_{sa} \\ T_{sa} \\ M_{sb} \\ S_{sb} \\ N_{sb} \\ T_{sb} \end{Bmatrix} = [K] \begin{Bmatrix} \beta_a \\ W_a \\ V_a \\ U_a \\ \beta_b \\ W_b \\ V_b \\ U_b \end{Bmatrix} + \begin{Bmatrix} M^o_{sa} \\ S^o_{sa} \\ N^o_{sa} \\ T^o_{sa} \\ M^o_{sb} \\ S^o_{sb} \\ N^o_{sb} \\ T^o_{sb} \end{Bmatrix} \qquad 3.4$$

where M^o_s, S^o_s, N^o_s, T^o_s are the fixed end stress resultants, at the ends a or b as subscripted, due to the applied loads or thermal gradients.

In order to establish the stiffness matrix and the fixed end stresses, the predetermination of the relationship between the edge displacements and the edge forces is necessary. The stiffness coefficients and fixed end forces for a beam element can be evaluated by the well known method, such as consistent deformation, or solving the pertinent differential equations.

If an analytical solution of the relevant differential equations for a specific geometry of shells is known, both influence coefficients and fixed end forces due to the applied loads can be found. Such a solution is not available for an arbitrary element of shells of revolution with variable thickness and subjected to arbitrary loadings. However, the forces can be evaluated by solving the basic differential equations numerically.

3.4 Solution of the Governing System of Equations

For any given shell element with given geometry and applied load, the governing system of equations (Eq. 2.24) is in the form

$$\{y^{\bullet}_{(s)}\} = [A_{(s)}]\{y_{(s)}\} + \{B_{(s)}\} \qquad 3.5$$

where $\{y_{(s)}\}$ is a vector of the eight dependent variables, four displacements and four corresponding forces, at a particular location s = constant; $\{y^{\bullet}_{(s)}\}$ is a vector of the derivatives of the eight variables with respect to the coordinate s, i.e., $d/ds\{y_{(s)}\}$; $[A_{(s)}]$ is the coefficient matrix relating the variables and their derivatives and consists only of functions of the shell thickness and the geometrical parameters of the location s; and, $\{B_{(s)}\}$ is a vector of the inhomogeneous terms in the equations and is a function of the applied loads and the location s.

The general solution of Eq. 3.5 consists of two parts:

1) The solution of the homogeneous part of the equations, i.e., the differential equations when all loading terms, $\{B_{(s)}\}$, are set equal to zero. This homogeneous solution involves the evaluation of eight constants of integration as the result of eight boundary conditions at the two discontinuous edges of the shell.

2) The particular solution of the equations in which all loading terms are considered. This solution does not depend upon the boundary conditions.

Therefore, the general solution that satisfies the governing system of equations together with the appropriate boundary conditions at the two edges of the shell element can be written as

$$\{y_{(s)}\} = \{h_{(s)}\} + \{P_{(s)}\} \quad 3.6$$

where $\{P_{(s)}\}$ represents the particular solution that satisfies the equation.

$$\{P^{\bullet}_{(s)}\} = [A_{(s)}]\{P_{(s)}\} + \{B_{(s)}\} \quad 3.7$$

and, $\{h_{(s)}\}$ is the homogeneous solution that satisfies the equation.

$$\{h^{\bullet}_{(s)}\} = [A_{(s)}]\{h_{(s)}\} \qquad 3.8$$

Consider now the solution of Eq. 3.8 for an element which spans the region $b \geq s \geq a$. Since Eq. 3.8 represents eight first order linear differential equations, the solution should contain eight arbitrary constants of integration. Let these arbitrary constants of integration be the eight (arbitrary) boundary values which can be imposed at edge "a", and denote these values by $\{c\}$. Then at edge "a"

$$\{h_{(a)}\} = \{c\} \qquad 3.9$$

Substituting into Eq. 3.8 yields, for $s = a$

$$\{h^{\bullet}_{(s)}\}_{s=a} = [A_{(s)}]_{s=a}\{c\} \qquad 3.10$$

Integrating this numerically, as an initial value problem, allows the value of $h_{(s)}$ at any point $s > a$ to be determined as

$$\{h_{(s)}\} = [H_{(s)}]\{c\} \qquad 3.11$$

where $[H_{(s)}]$ represents the matrix arising from the integration of the $[A_{(s)}]$ matrix along the length of the element.

Since the matrix $[H_{(s)}]$ is independent of the initial conditions ($\{c\}$), it is a property of the element and may be interpreted as follows. As $\{c\}$ is arbitrary, assign a unit value to one component, say component c_j, and set the others equal to zero. Then the j^{th} column of $[H_{(s)}]$ represents the values of $\{h_{(s)}\}$ for a unit value of the j^{th} boundary condition at "a". It is apparent that when s = a, Eq. 3.11 must reduce to Eq. 3.9 in order to match the arbitrary boundary conditions, and hence $[H_{(a)}]$ must be the identity matrix, i.e.

$$[H_{(a)}] = [I] \qquad 3.12$$

Eq. 3.12 may be considered to be a "boundary condition" on the numerical integration of the matrix [H].

Turning now to the solution of Eq. 3.7, the equation at s = a may be written as

$$\{P^{\bullet}{}_{(s)}\}_{s=a} = [A_{(s)}]_{s=a}\{c*\} + \{B_{(s)}\} \qquad 3.13$$

where $\{c*\}$ represents an arbitrary set of initial values of $\{P_{(a)}\}$. Numerical integration of this equation yields

$$\{P_{(s)}\} = [H_{(s)}]\{c*\} + \{Q_{(s)}\} \qquad 3.14$$

where $[H_{(s)}]$ is the matrix that arises in Eq. 3.11 and $\{Q_{(s)}\}$ is a vector arising from the integration of the inhomogeneous terms. Since the particular solution is any solution which satisfies the inhomogeneous equations it is adequate to select

$$\{c^*\} = 0 \qquad 3.15$$

in which case Eq. 3.14 reduces to

$$\{P_{(s)}\} = \{Q_{(s)}\} \qquad 3.16$$

and consequently the general solution, Eq. 3.6, becomes

$$\{y_{(s)}\} = [H_{(s)}]\{c\} + \{Q_{(s)}\} \qquad 3.17$$

For s = b, Eq. 3.17 becomes

$$\{y_{(b)}\} = [H_{(b)}]\{y_{(a)}\} + \{Q_{(b)}\} \qquad 3.18$$

in which each column vector of [H] represents the variables at "b" corresponding to each unit variable applied at "a" in the absence of any applied loads. The vector {Q} represents the variables at "b" corresponding to zero displacements and stresses resultants at "a" in the presence of the applied loads.

3.5 Shell Element Stiffness Matrix

The column vector $\{y_{(s)}\}$ represents the vector of Eq. 2.24 consisting of the four displacements as defined by Eq. 2.15.1 and the four stress resultants as defined by Eq. 2.19.1.

Let

$$\{y_{(a)}\} = \begin{Bmatrix} D_a \\ F_a \end{Bmatrix} \qquad 3.19$$

and

$$\{y_{(b)}\} = \begin{Bmatrix} D_b \\ F_b \end{Bmatrix} \qquad 3.20$$

Eq. 3.18 can be written in partitioned form as

$$\begin{Bmatrix} D_b \\ F_b \end{Bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{Bmatrix} D_a \\ F_a \end{Bmatrix} + \begin{Bmatrix} Q_d \\ Q_f \end{Bmatrix} \qquad 3.21$$

where

D represents the four displacement variables,

F represents the four stress resultant variables, and

Q_d, Q_f are the displacement and the stress resultant parts of the particular solution respectively.

The total matrix appearing in Eq. 3.21 is usually referred to as a "transfer matrix" [23]. Eq. 3.21 can be expanded to form two equations as follow

$$\begin{Bmatrix} D_a \\ D_b \end{Bmatrix} = \begin{bmatrix} I & 0 \\ H_1 & H_2 \end{bmatrix} \begin{Bmatrix} D_a \\ F_a \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_d \end{Bmatrix}$$

$$= [Y_1]\begin{Bmatrix} D_a \\ F_a \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_d \end{Bmatrix} \qquad 3.22$$

and

$$\begin{Bmatrix} F_a \\ F_b \end{Bmatrix} = \begin{bmatrix} 0 & I \\ H_3 & H_4 \end{bmatrix} \begin{Bmatrix} D_a \\ F_a \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_f \end{Bmatrix}$$

$$= [Y_2]\begin{Bmatrix} D_a \\ F_a \end{Bmatrix} + \begin{Bmatrix} 0 \\ Q_f \end{Bmatrix} \qquad 3.23$$

Solving Eq. 3.22 for the vector $< D_a \ F_a >^T$ and substituting into Eq. 3.23, yields

$$\begin{Bmatrix} F_a \\ F_b \end{Bmatrix} = [Y_2]\ [Y_1]^{-1} \begin{bmatrix} D_a \\ D_b - Q_d \end{bmatrix} + \begin{Bmatrix} 0 \\ Q_f \end{Bmatrix} \qquad 3.24$$

or

$$\begin{Bmatrix} F_a \\ F_b \end{Bmatrix} = [K]\begin{Bmatrix} D_a \\ D_b \end{Bmatrix} + \begin{Bmatrix} F^o{}_a \\ F^o{}_b \end{Bmatrix} \qquad 3.25$$

It can be seen that each column of [K] represents the stress resultants at each end for a unit displacement applied at one end while the other displacements are restrained and $\{F^o\}$ represents the stress resultants corresponding to the totally restrained boundaries.

3.6 Stiffness Matrix Sign Convention

In the derivation of the element stiffness matrix and the fixed end stresses, the sign convention used corresponds to that generally used in shell theory as given in Fig. 3.1. As a result, the stiffness matrix will have some negative elements on the main diagonal. This can be corrected by adapting the so called "stiffness matrix sign convention". This sign convention is shown in Fig. 3.2. It can be seen that the positive direction of the top normal in plane force N_s, the top tangential shearing force T_s, the bottom moment M_s and the bottom transverse shear S_s have been changed to the opposite direction. Therefore, the stiffness matrix and fixed end stresses in Eq. 3.25 are to be premultiplied by the diagonal matrix

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

3.7 Stress Resultants and Displacements at Intermediate Points

For a shell structure composed of a number of elements, the element stiffness matrices and fixed end forces are evaluated and assembled in the master stiffness equations of the structure. Boundary conditions are imposed, and the displacements at the boundaries of each element are obtained from the solution of the master equations. By substituting the final known boundary displacements of each element into the corresponding element stiffness equation (Eq. 3.25), the primary stress resultants at the element boundaries can be obtained. Thus the correct boundary conditions, displacements and primary stress resultants, for each element are known. In order to evaluate the displacements and the stress resultants at any desired number of intermediate points within the element, the correct boundary conditions at one end of the element, say the end at s = a, are used as initial conditions in integrating the governing set of equations (Eq. 3.5). At each intermediate point, the secondary stress resultants (which were eliminated from the governing equations) can be evaluated, first by evaluating the derivatives of the displacements using Eq. 2.18.

$$\{D^{\bullet}\} = [A1]\{D\} + [A2]\{F_s\} + \{A3\} \qquad 3.26$$

and then substituting into Eq. 2.20

$$\{F_\theta\} = [C1]\{D^{\bullet}\} + [C2]\{D\} + \{C3\} \qquad 3.27$$

A simple check on the results of the integration is that the displacements and the primary stress resultants at the termination end of the element should agree exactly with the known boundary conditions at this end.

3.8 Transformation from Local to Global Coordinates

Displacements and stress resultants, at any point along the generator, are presented in the direction tangent to the meridian at this point and the direction perpendicular to it. Due to possible discontinuity of the meridian curve at a junction between two elements, it is necessary to transform the influence coefficients and the fixed end forces at this junction, to a new coordinates system. It is simplest to adopt the direction of the structure's axis of revolution, x, and the direction perpendicular to it, r. According to the stiffness matrix sign convention, the transformation equations may be written as follow

$$\{D_L\} = [L]\{D_G\} \qquad 3.28.1$$

$$\{F_G\} = [L]^T\{F_L\} \qquad 3.28.2$$

where

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_i & -\cos\phi_i \\ 0 & \cos\phi_i & \sin\phi_i \\ 0 & 0 & 0 \end{bmatrix} \qquad 3.29$$

{D} and {F} represent the displacement and forces respectively and the subscripts L, G represent local and global coordinates. ϕ_i is the angle measured from the global axis of rotation to the meridian at the point s = i (Fig. 3.3).

3.9 Modification of Stiffness Coefficients to Account for Eccentricity

In many cases the middle surfaces of two elements which meet at a node do not coincide at the same point (Fig. 3.4). A transformation of the stiffness coefficients and fixed end stresses to a common reference point is then necessary before assembling the element stiffness matrices into the master stiffness matrix. Eqs. A.10 relate the displacement components of a point at a distance a from the middle surface to the displacement components of a point on the middle surface lying in the same plane. By expanding Eqs. A.10, by means of Fourier series, one can write the following equation

$$\begin{Bmatrix} \beta_z \\ W_z \\ V_z \\ U_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ Z & 0 & 1 & 0 \\ 0 & \frac{nz}{r} & 0 & \frac{r_2+z}{r_2} \end{bmatrix} \begin{Bmatrix} \beta \\ W \\ V \\ U \end{Bmatrix} \qquad 3.30$$

where Eq. A.7.1 has been used to eliminate the derivative of W with respect to s and the subscript z refers to the point at a distance z from the middle surface. Inverting Eq. 3.30 results in

$$\begin{Bmatrix} \beta \\ W \\ V \\ U \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -z & 0 & 1 & 0 \\ 0 & \frac{-nr_2 z}{r(r_2+z)} & 0 & \frac{r_2}{r_2+z} \end{bmatrix} \begin{Bmatrix} \beta_z \\ W_z \\ V_z \\ U_z \end{Bmatrix} \qquad 3.31$$

or

$$\{D\} = [EC]\{D_z\} \qquad 3.32$$

From the work equivalence requirements, the relation between the stresses at the two points can be written as

$$\{F_z\} = [EC]^T\{F\} \qquad 3.33$$

Eq. 3.32 and 3.33 are used to transform the displacements and stresses of a point on the middle surface to a point at a distance z from the middle surface.

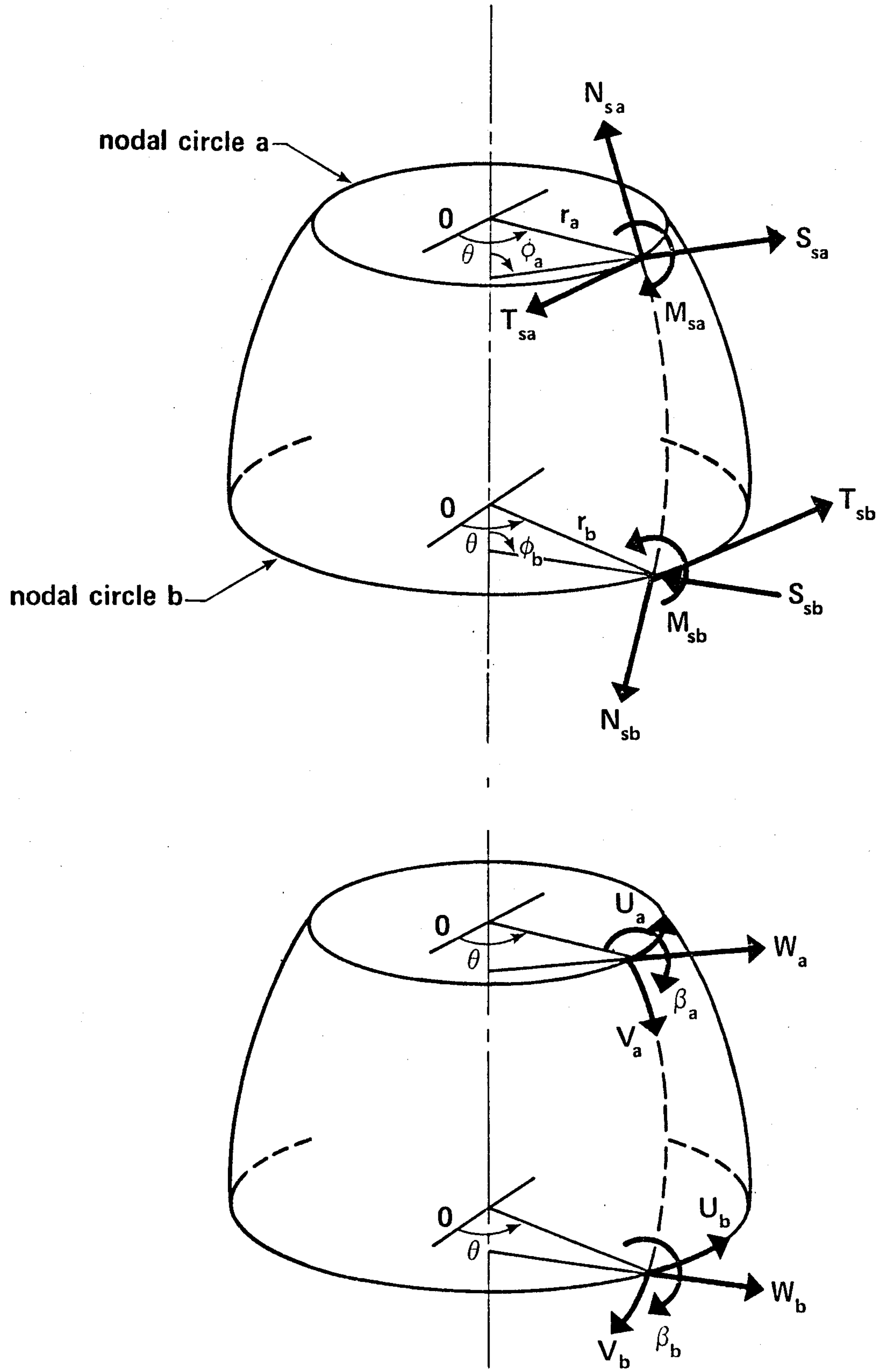


Fig. 3.1 Shell Theory Sign Convention for Strrss Resultants and Displacements

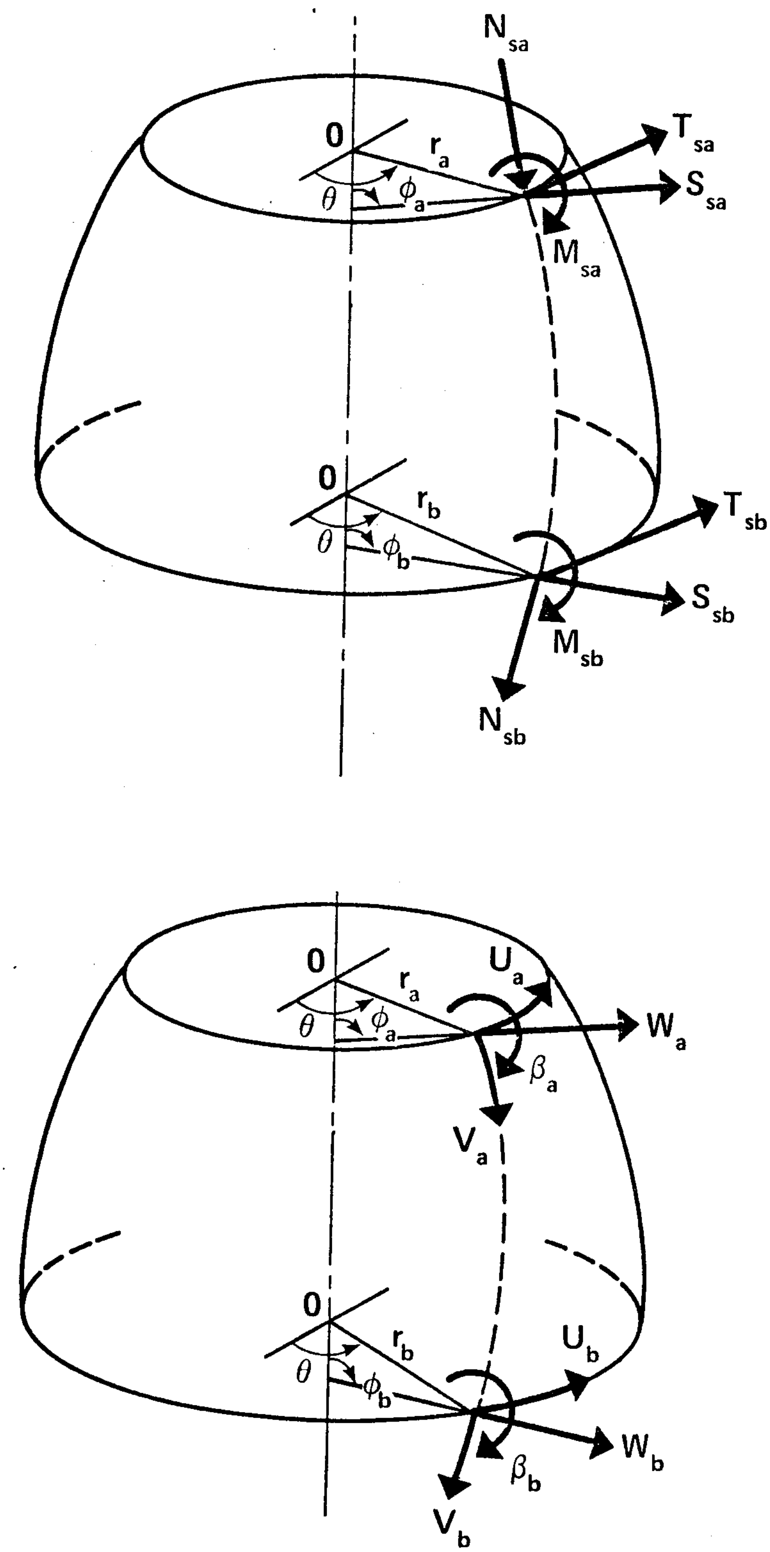


Fig. 3.2 Stiffness Matrix Sign Convention for Stress Resultants and Displacements.

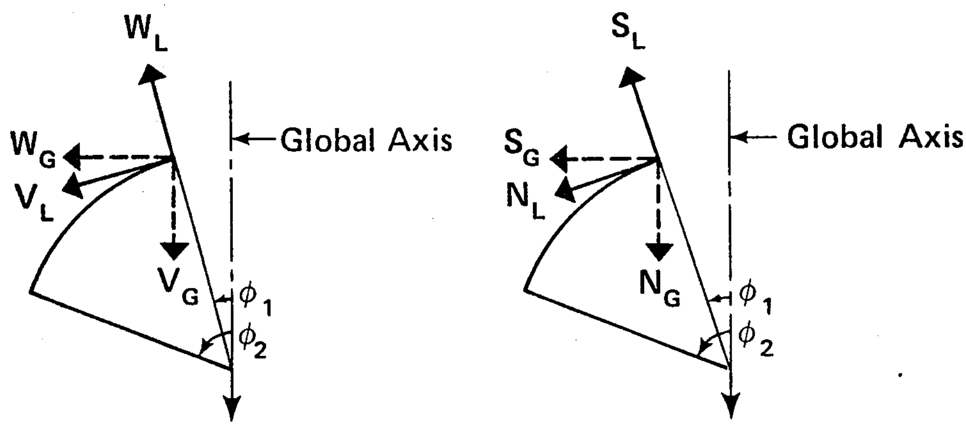


Fig. 3.3 Notation for Transformation of Stress Resultants and Displacements

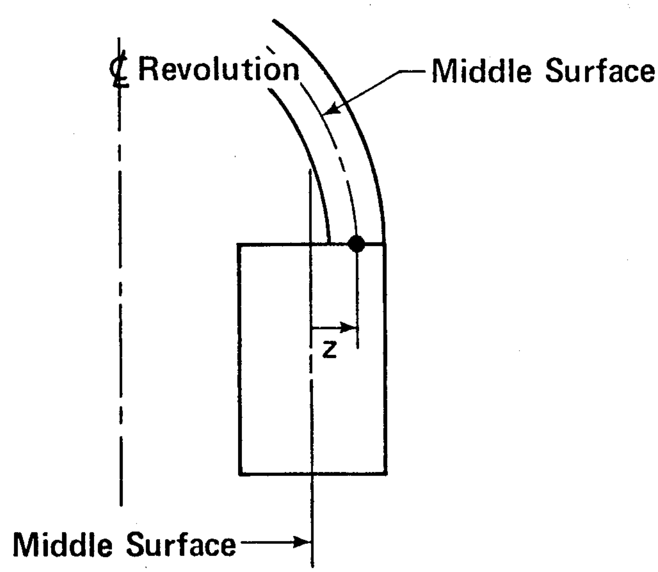


Fig. 3.4 Eccentricity at a Node

CHAPTER 4

EXAMPLE APPLICATIONS

4.1 Introduction

A computer program, named SASHELL, has been developed to perform the stiffness analysis of segmented shell structures based on the theory presented in the preceding chapters. The logic flow of SASHELL is outlined in Sect. 4.2 and listing of the program is included in Appendix C. The results of the analysis of two example applications, using SASHELL, are presented in this chapter.

The first example is the pinched cylinder. The exact analytical solution for a long cylinder pinched by a symmetrical circumferential line load (Fig. 4.1) is known [27, pp. 471; 10, pp. 280]. Finite element solutions, using (48 x 48) element stiffness matrices [11] and (24 x 24) element stiffness matrices [5,7], were obtained for the case of a cylindrical shell loaded by diametrically opposed concentrated loads (Fig. 4.2). This solution was compared [2] against an analytical solution based on the inextensional deformation theory (i.e. neglecting entirely the strain in the middle surface of the shell) [27, pp. 501-506].

The second example is the analysis of hyperboloid natural draft cooling tower under the action of wind load and its own weight. Finite element analysis, using the computer program SORIII [14] and using conical shell elements to approximate the geometry [24] are known.

4.2 Logic Flow of SASHELL

In this section, the organization of the computer program SASHELL, which can serve as a summary for the solution technique of segmented shell structure as described in the preceding chapters, is outlined. Details of the required input are given in Appendix B.

1) The structure is divided into elements, each of which is a simple type of shell of revolution, and which are connected along nodal circles or "nodes". A concentrated load applied at a point along the meridional coordinate must be treated as a load acting on a node connecting two elements.

2) Nodal coordinates, system connectivity information and element types, properties and loading conditions are determined. The problem control parameters are established and input (Subroutine READIN and LOADIN).

The program now performs the following operations:

1) Each element is examined. If a coefficient which depends on the element geometric parameters (see the limitations in Sect. 5.3) exceeds a certain limit, the element is divided into subelements (segments) each of which satisfies this limit (subroutine SEGMNT).

2) The connectivity of the system is altered to include the new intermediate nodal points. Geometric parameters and loadings values at each segment boundary are calculated (subroutine SEGEOM).

3) The number of the structure's degrees of freedom are established.

4) If the external applied load on the structure is symmetric, the program discards step 5 and goes to step 6.

5) If the load is non-axisymmetric, the total number of points in the meridional and circumferential direction in the structure is determined and the "results" array is initialized.

6) The harmonic number, n, is set equal to zero.

7) The structure stiffness matrix and load vector are initialized. The nodal load coefficients, if any, which correspond to the harmonic n are added to the load vector.

8) The stiffness analysis starts by calculating the stiffness matrix and fixed end forces for each segment in the structure (subroutine STIFAN).

9) The geometric parameters and external applied loadings at the desired number of integration points in the segment under consideration are calculated. Dead weight of the segment, if required and if n = 0, is superimposed (subroutine PLSEG and DLSEG).

10) The initial conditions at the starting edge of the segment, as stated in Sect. 3.4 (Eqs. 3.12 and 3.15), are set. The governing equations (subroutine FLUGGE) are integrated, using a fourth order Runge-Kutta method, over the desired number of points to obtain the transfer matrix of the segment (subroutine RNGKT).

11) The segment stiffness matrix and fixed end forces are evaluated from the transfer matrix obtained in step 10, as described in Sect. 3.5 (subroutine STIFIX).

12) The results of step 11 are saved (subroutine STORE1).

13) The segment stiffness matrix and fixed end forces are modified to correspond to the stiffness matrix sign convention, as stated in Sect. 3.6.

14) The stiffness influence coefficients and fixed end forces are modified to account for nodal eccentricity, if any, as mentioned in Sect. 3.9. (subroutine ECCNTR).

15) If required, the segment stiffness influence coefficients and fixed end forces are transformed, due to discontinuity of the meridian at the node, to the structural global coordinates, as stated in Sect. 3.8 (subroutine GLTRAN).

16) The segment stiffness influence coefficients are assembled, with respect to the structure's degrees of freedom, into the master stiffness matrix. The segment fixed end forces are subtracted from the

corresponding values in the load vector (subroutine STORE).

17) Steps 9 to 16 are repeated for each segment in the structure.

18) The boundary conditions are imposed on the master stiffness equation (subroutine BOUNDC).

19) Segments edge displacements is obtained, using Gaussian elimination algorithm to solve the master stiffness equation (subroutine SOLVER).

20) The stiffness analysis, for harmonic number n, is completed. The displacements and stress resultants at the desired number of intermediate points in each segment are to be evaluated (subroutine SRADSP).

21) The segment edge displacements, obtained in step 19, are transformed, if necessary, to the segment local coordinates and to account for nodal eccentricity (subroutine GLTRAN and ECCNTR).

22) The known segment edge displacements are substituted in the corresponding segment stiffness equation, saved in step 12. The primary stress resultants at the segment boundaries are evaluated, as described in Sect. 3.7 (subroutine STORE1).

23) The shell equations are integrated, using the initial conditions calculated in step 22, in order to determine the displacements and primary and secondary stress resultants at the intermediate points within the segment, as described in Sect. 3.7 (subroutine RESULT).

24) If the load is symmetric (i.e., the required number of harmonics is zero), the results of step 23 are printed out and the program goes to step 26.

25) If the load is non-axisymmetric (i.e., the required number of harmonic is greater than zero), the displacements and stress resultants are calculated at the desired number of points in the circumferential direction. The results are superimposed in the "results" array (subroutine STORE2).

26) Steps 21 to 25 are repeated for each segment in the structure.

27) The harmonic number, n, is increased by one. Steps 7 to 26 are repeated until n is equal to the required number of harmonics.

28) The results saved in step 25 are printed out and the program stops.

4.3 Pinched Cylinder

The governing differential equation for a circular cylindrical element is

$$\frac{d^2W}{dx^2}\left(K\frac{d^2W}{dx^2}\right) + \frac{Et}{r^2}W = P_z \qquad 4.1$$

where x measures the distance along the axis of the cylinder with the origin (x = 0) at midlength of the element. When the thickness of the shell is constant, this equation reduces to

$$\frac{d^4W}{dx^2} + 4\lambda^4 W = \frac{P_z}{K} \qquad 4.2$$

in which the flexural rigidity, K, is

$$K = \frac{Et^3}{12(1 - \nu^2)} \qquad 4.3$$

and the coefficient λ is defined as

$$\lambda = \sqrt[4]{\frac{3(1 - \nu^2)}{r^2t^2}} \qquad 4.4$$

For the special case of a long cylindrical shell pinched by a live load P uniformly distributed along a circular section (Fig. 4.1), the solution of Eq. 4.2 takes the form [27]

$$W = \frac{-P}{8\lambda^3} e^{-\lambda x} (\cos \lambda x + \sin \lambda x) \qquad 4.5$$

The stress resultants may be expressed in terms of derivatives of the displacement W in the form

$$M_x = K \frac{d^2W}{dx^2} \qquad 4.6.1$$

$$Q_x = K \frac{d^3W}{dx^3} \qquad 4.6.2$$

$$M_\theta = \nu M_x \qquad 4.6.3$$

$$N_\theta = \frac{Et}{r} W \qquad 4.6.4$$

Numerical values of these expressions, at several points along a cylindrical element 20 feet long, 8 feet in diameter and with constant wall thickness equals to 1.24 inches, are shown in Table 4.1 for a live load P = 1 kip/ft. Output of the analysis of the same element using SASHELL is included in Appendix D. The results are presented graphically in Fig. 4.3. The results of the two solutions are identical up to the number of significant figures contained in the output.

For the case of a cylindrical shell pinched by two concentrated loads as shown in Fig. 4.2 the concentrated load is approximated as a live load (see Sect. 5.4) that has a value of 1.0 kip/ft at the loaded points and zero at a short distance from the load points (Fig. 4.4.1). The loading function is expanded in Fourier coefficients and the results of the analysis of each harmonic are superimposed.

The results of this loading case are affected by the number of harmonics considered in the analysis, which will be discussed in Sect. 5.4.

If the circumference of the cylinder shown in Fig. 4.4 is divided to 36 intervals and the load is approximated as shown, the equivalent concentrated load is

$$\overline{P} = \frac{2\pi r}{36} P \qquad 4.7$$

$$= 0.698 \text{ kips}$$

$L = 20'.0$ $\quad$ $E = 30 \times 10^3$ ksi $\quad$ $K = 436$ K.ft

$r = 4'$ $\quad$ $\nu = 0.3$ $\quad$ $\lambda = 2.0$

$t = 1.24''$ $\quad$ $P = 1.0$ K/ft

X (ft)	λ_x	W (10^{-4}ft)	M_x	Q_x	M_θ	N_θ
0	0	-.3583	..1250	-.5000	+.0375	-3.997
0.25	0.5	-.2949	.0302	-.2661	+.0091	-3.290
0.5	1.0	-.1822	-.0138	-.0994	-.0041	-2.032
0.75	1.5	-.0854	-.0258	-.0079	-.0077	-0.953
1.0	2.0	-.0239	-.0224	+.0282	-.0067	-0.266
1.25	2.5	+.0059	-.0144	+.0329	-.0043	+0.066
1.5	3.0	+.0152	-.0070	+.0246	-.0021	+0.169
1.75	3.5	+.0139	-.0022	+.0142	-.0006	+0.155
2.0	4.0	+.0092	+.0002	+.0060	+.0001	+0.102
2.25	4.5	+.0047	+.0011	+.0012	+.0003	+0.052
2.5	5.0	+.0016	+.0011	-.0009	+.0003	+0.018
2.75	5.5	0	+.0007	-.0015	+.0002	0
3.0	6.0	-.0006	+.0004	-.0012	+.0001	-0.066

TABLE 4.1 Stresses and Radial Displacement in a Pinched Cylinder as per Eqs. 4.6

The analytical solution for a concentrated load, based on inextensional deformation theory [27, pg. 506], is of the form

$$W_\phi = \frac{Pr^3}{\pi KL} \sum_{n=2,4,6,\ldots} \frac{1}{(n^2-1)^2} \cos n\theta \qquad 4.8$$

where L is one-half the length of the cylinder. Eq. 4.8 yields

$$(W)_{\theta=0} = 0.382 \times 10^{-3} \text{ ft}$$

$$(W)_{\theta=\pi/2} = -0.351 \times 10^{-3} \text{ ft}$$

A comparison of these values with SASHELL values shown in Fig. 4.4.1 indicates good agreement for $\theta = \pi/2$ but considerable discrepancy for $\theta = 0$. The SASHELL solution predicts a displacement 17% larger than the inextensional solution at $\theta = 0$.

Now referring to Fig. 4.4.2, the cylindrical shell example as choosen by Cantin and Clough [7], is shown. For the same number of intervals (36), the equivalent concentrated load is

$$\overline{P} = 0.0721 \text{ kips}$$

The deflections for this case, from Eq. 4.8, are

$$(W)_{\theta=0} = 0.614 \times 10^{-2} \text{ ft}$$

$$(W)_{\theta=\pi/2} = -0.565 \times 10^{-2} \text{ ft}$$

The ratios of the SASHELL displacements are 1.10 and 1.14 to those of the inextensional theory for $\theta = \pi/2$ and $\theta = 0$, respectively.

4.4 Hyperboloid Natural Draft Cooling Tower

Large capacity power plants generate a substantial amount of operational heat that requires dissipation. One of the major structures in these power plants is the natural draft cooling tower in the form of the shell of revolution. The tower utilizes its height to create the necessary air flow in order to cool a large volume of water in a minimum land area. Hyperboloid cooling towers are the most preferable shape [14] when compared to conical or cylindrical shapes from the aerodynamic point of view.

4.4.1 Geometry of the Tower

The middle surface of a hyperboloid shell is shown in Fig. 1.7. The surface of a hyperboloid of revolution may be classified as a non-developable surface, which means that the surface will not tend to flatten out under load. This surface is generated by the rotation of a hyperbola about a vertical axis. The geometrical equation can be written as

$$\frac{r^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad 4.9$$

in which r is the horizontal radius, x is the vertical coordinate measured from the origin at the throat of the shell, a is the throat radius at x = 0, and b is a constant in which the ratio b/a equals the slope of the asymptotes to the hyperbola.

The principal radii of curvature are given by

$$r_1 = -\frac{a^2b^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{3/2}} \qquad 4.10.1$$

$$r_2 = \frac{a^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{1/2}} \qquad 4.10.2$$

where $\tan\phi = \frac{dx}{dr}$

$$= \frac{b}{a}\sqrt{\frac{r^2}{r^2 - a^2}} \qquad 4.10.3$$

The expression for the derivative of r_1 with respect to the coordinate s (see Eq. 2.5.1) can be obtained by differentiating Eq. 4.10.1 to yield

$$r^{\bullet}_1 = \frac{-3}{a^2b^2}(b^4r\cos\phi + a^4x\sin\phi)\left(\frac{r^2}{a^4} + \frac{x^2}{b^4}\right)^{1/2} \qquad 4.10.4$$

The middle surface of the hyperboloid tower considered in the following analysis is shown in Fig. 4.5. The shell is 355 ft high and is supported by columns evenly spaced on a circular base of 290 ft diameter. The throat of the tower is 165 ft in diameter and is located 60 ft below the top of the shell. The thickness varies from 30 inches at the bottom level of the shell to 6 inches at 25 feet elevation from the bottom. In the top 10 feet of the shell, the thickness also varies from 6 inches to 24 inches. Other than the top and the bottom regions, the shell thickness remains constant at 6 inches. The increased thickness at the top provides a stiffening effect that reduces radial deformation under wind load [24]. The bottom ring at the base acts as an equivalent deep beam bridging between columns.

The constant b of Eqs. 4.10 can be calculated by substituting the values for $r = 145$ at $x = 295$ and $a = 82.5$ into Eq. 4.9, which yields $b = 204.1$ ft. For the purpose of comparing the results with References 14 and 24, the following concrete properties are used

Young's Modulus	E	$= 4 \times 10^6$ psi
Poisson's Ratio	ν	$= 0.15$
Specific Weight	γ	$= 150$ lb/ft^3

4.4.2 <u>Load Description</u>

The tower is analyzed for gravity load and wind pressure load.

The gravity load, which is symmetric with respect to the coordinate θ, is determined from the following equations

$$q = t \times \gamma \qquad 4.11.1$$

$$P_z = - q \cos \phi \qquad 4.11.2$$

$$P_s = q \sin \phi \qquad 4.11.3$$

where q is the intensity of weight per square area of the surface.

The wind load, which is the governing factor in the design of the cooling towers [14], is based on the ACI-ASCE Committee 334 recommendations [1]. The following equivalent static pressure distribution is used

$$q(H,\theta) = G\, C_\theta\, K_H\, q_{30} \qquad 4.12$$

where the factors in this equation are described as follows:

a) $q(H,\theta)$ is the equivalent static normal pressure on the surface of the tower at a location defined by coordinates H,θ. H is the vertical distance measured from the ground level, θ is the circumferential angle measured from the windward meridian.

b) G is the dynamic gust factor which accounts for the overstress due to the tower response to various time variations in the wind pressure.

c) C_θ is the coefficient of wind pressure distribution in the circumferential direction.

d) K_H is the exposure factor which establishes the vertical profile of wind pressure, which in turn, depends on wind speed and roughness of terrain. i.e., K_H can be evaluated from the equation

$$K_H = 2.64 \left(\frac{H}{Hg}\right)^{2\alpha} \quad 4.13$$

where H is the height measured from the ground level; Hg is the gradient height above which the wind velocity is assumed constant and ranges from 900 ft for open country to 1500 ft for center of large cities; α is a constant, depending on the terrain roughness, and ranges from 1/7 for open country to 1/3 for center of large cities.

e) q_{30} is the basic wind pressure (psf) and is equal to the dynamic pressure of the free stream of wind at 30 ft above the ground level at a given site. It may be computed from

$$q_{30} = 0.00256 \, V_{30}^2 \quad 4.14$$

where V_{30} = wind velocity (m.p.h.) at 30 ft above the ground level.

The ACI-ASCE Committee 334 has suggested the normalized wind pressure distribution in the circumferential direction as shown in Fig. 4.6.1. As in Reference 24, assuming Hg = 900 ft, α = 1/7 and V_{30} = 100 m.p.h., the variation of the wind pressure with the height can be taken as shown in Fig. 4.6.2.

4.4.3 Analysis Conclusions

The results of the analysis of the hyperboloid tower, loaded as described above, are represented graphically for dead load in Figs. 4.7.1 to 4.7.3, and for wind load in Figs. 4.8.1 to 4.8.10. Since the actual condition at the bottom of the shell is only partially fixed [24], the results of the analysis for two boundary conditions, fixed and hinged at the bottom, are shown in these figures. Excellent agreement is observed between the displacements and stress resultants shown and the results of References 14 and 24.

The approximation of the geometry with a series of cones [24] does not affect the results for this particular geometry of the tower. However, the meridian curvature R_1 varies only from $-0.002\ ft^{-1}$ at the throat to $-0.0003\ ft^{-1}$ at the bottom of the hyperboloid shell, and is equal to zero for a conical element. For shells with larger R_1 curvature the conical segment approximation may not be as accurate.

4.5 Comments on Results

The examples of this chapter have been selected to test the ability of SASHELL to analyze different types of shells. The pinched cylinder solutions are common test problems because of the difficulty of achieving solutions for concentrated loads. The hyperboloid cooling tower is an illustration of a shell with negative Gaussian curvature under complex loading conditions. It may be concluded that SASHELL is capable of yielding good results on a wide variety of shell problems.

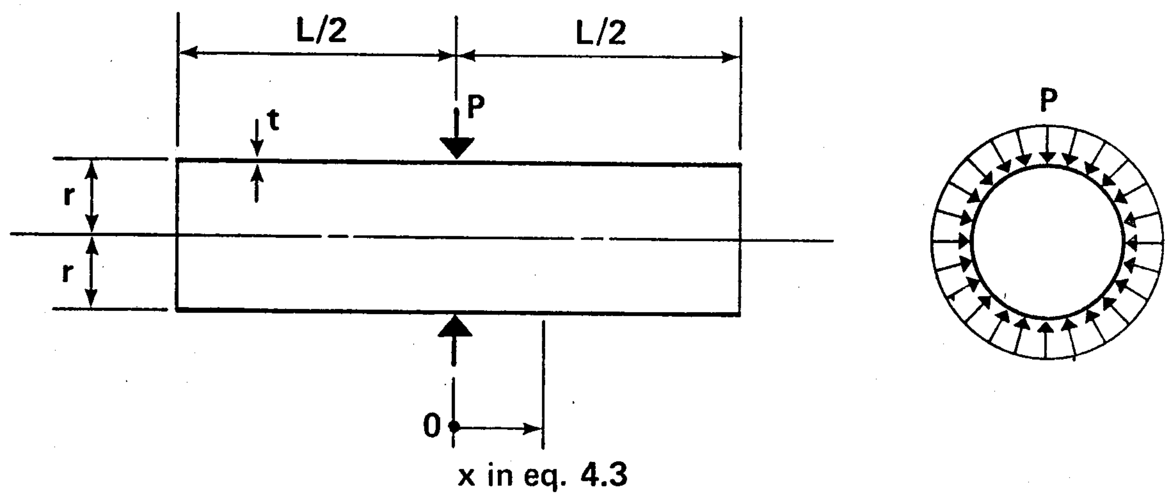


Fig. 4.1 "Pinched Cylinder" Circumferential Line Load

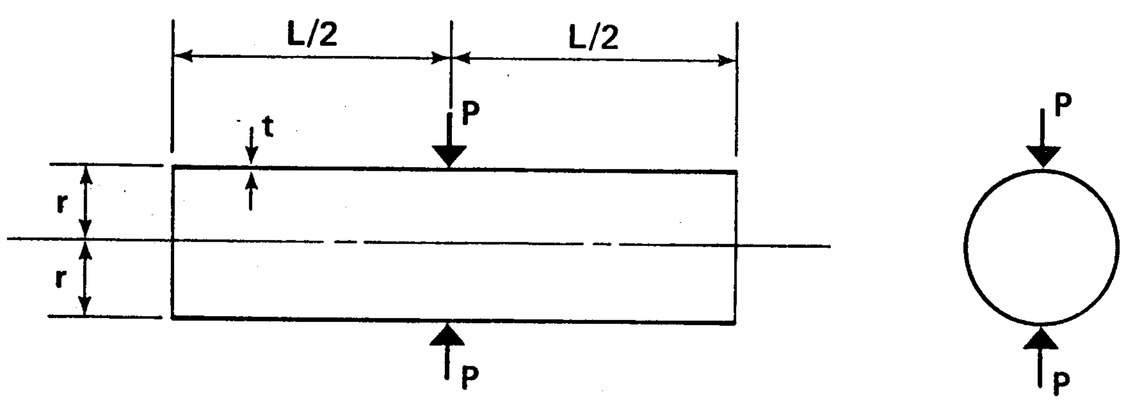


Fig. 4.2 "Pinched Cylinder" Two Concentrated Loads

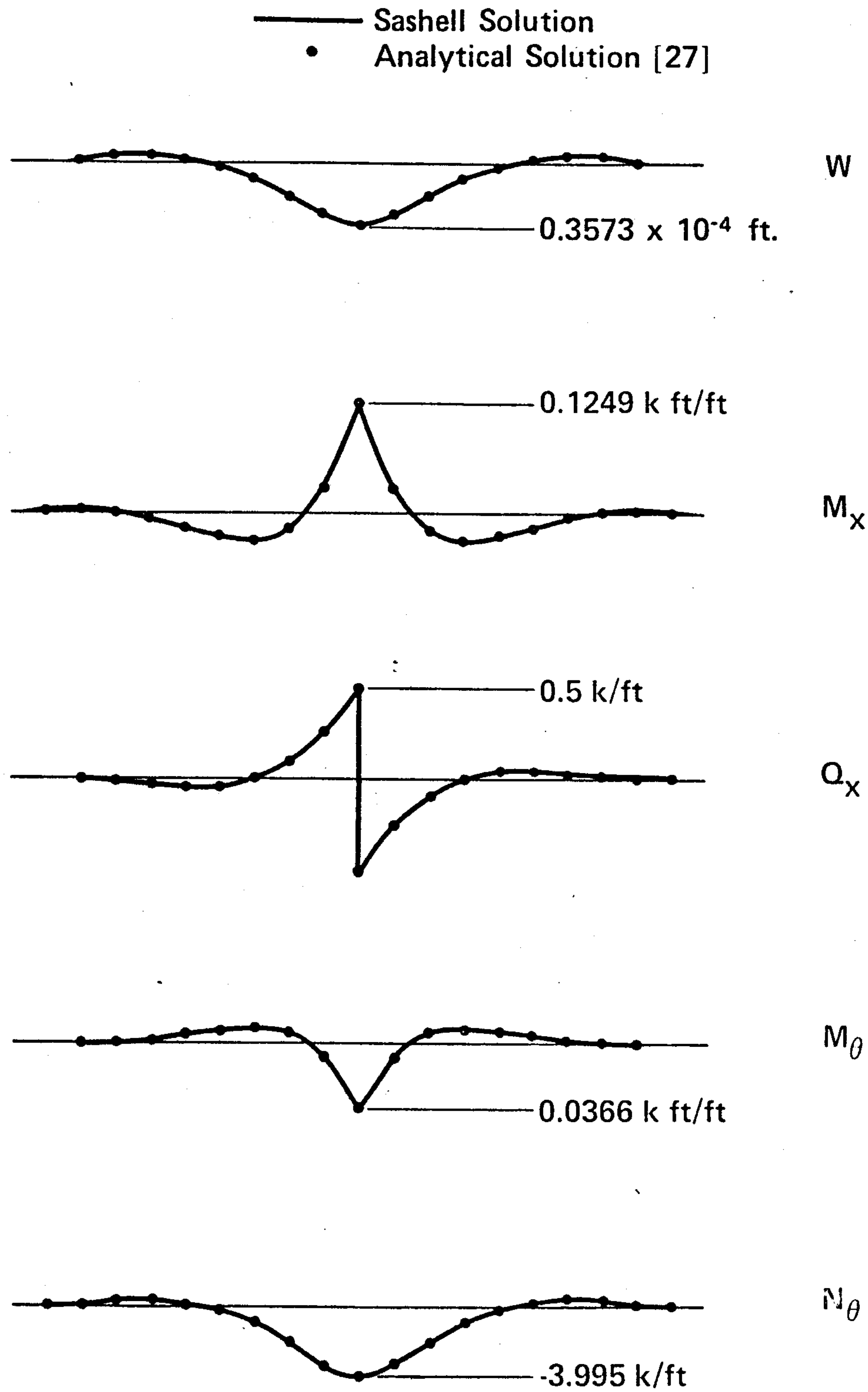
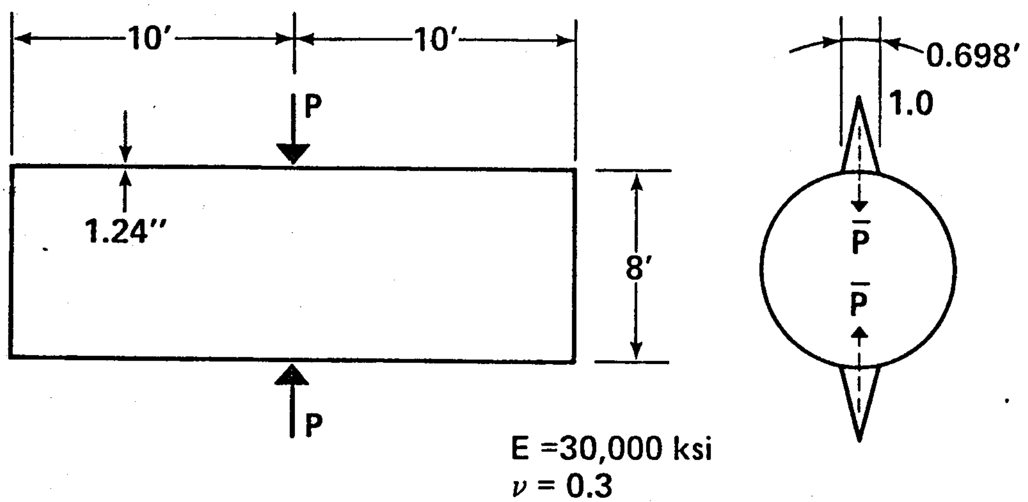


Fig. 4.3

"Pinched Cylinder" Circumferential Line Load Stress Resultants and Radial Displacement



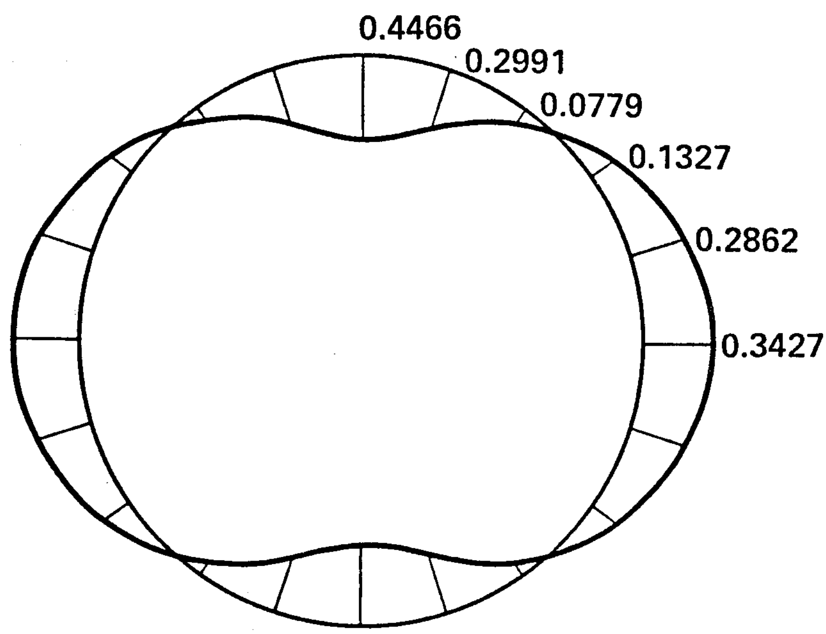


Fig. 4.4.1"Pinched Cylinder" Two Concentrated Loads Radial Displacement (r = 4.0 ft.)

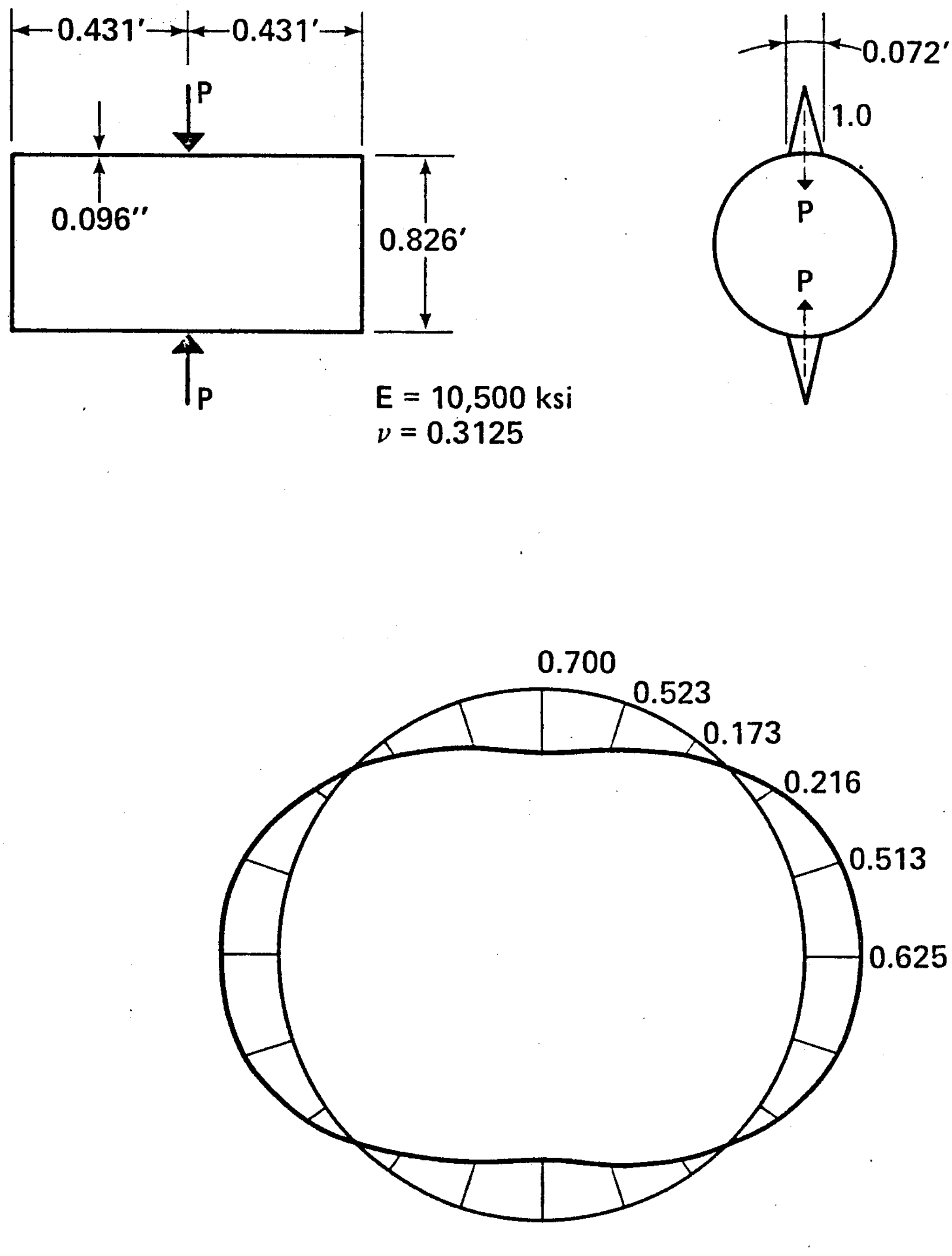


Fig. 4.4.2

"Pinched Cylinder" Two Concentrated Loads Radial Displacement (r = 0.413 ft.)

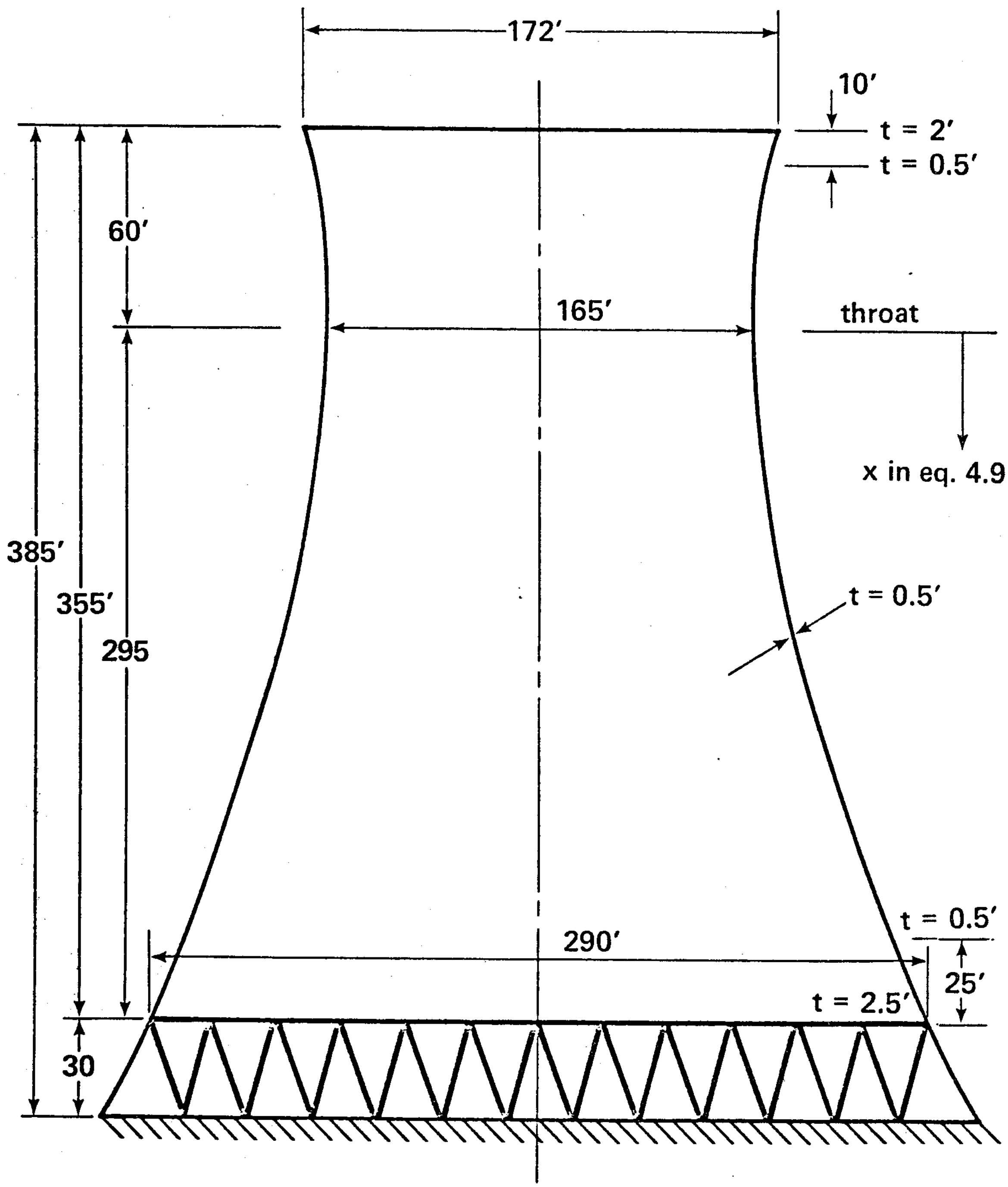


Fig. 4.5 Typical Hyperboloid Natrual Draft Cooling Tower

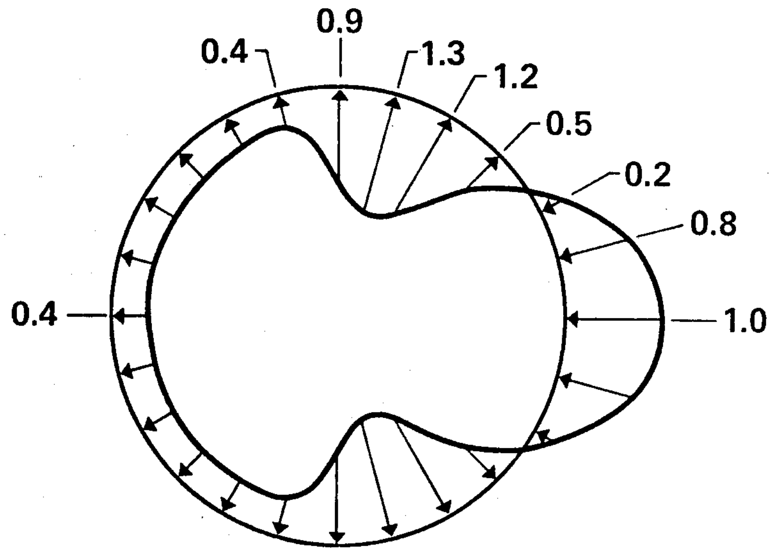


Fig. 4.6.1 Wind Pressure Coefficients C_θ

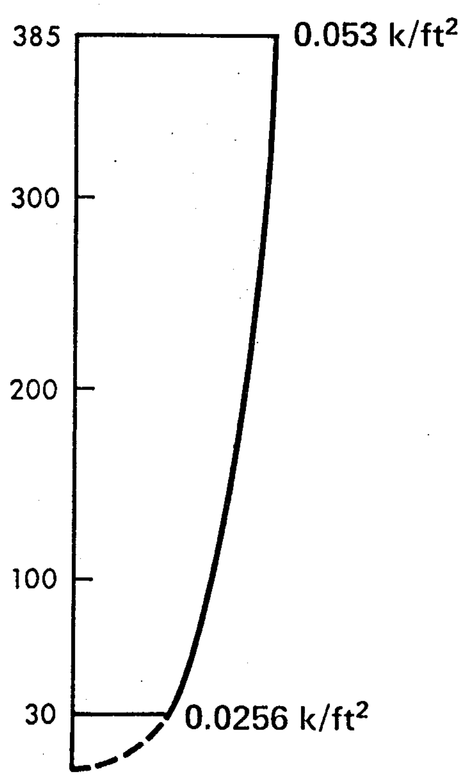


Fig. 4.6.2 Wind Pressure Profile ($\theta = 0$)

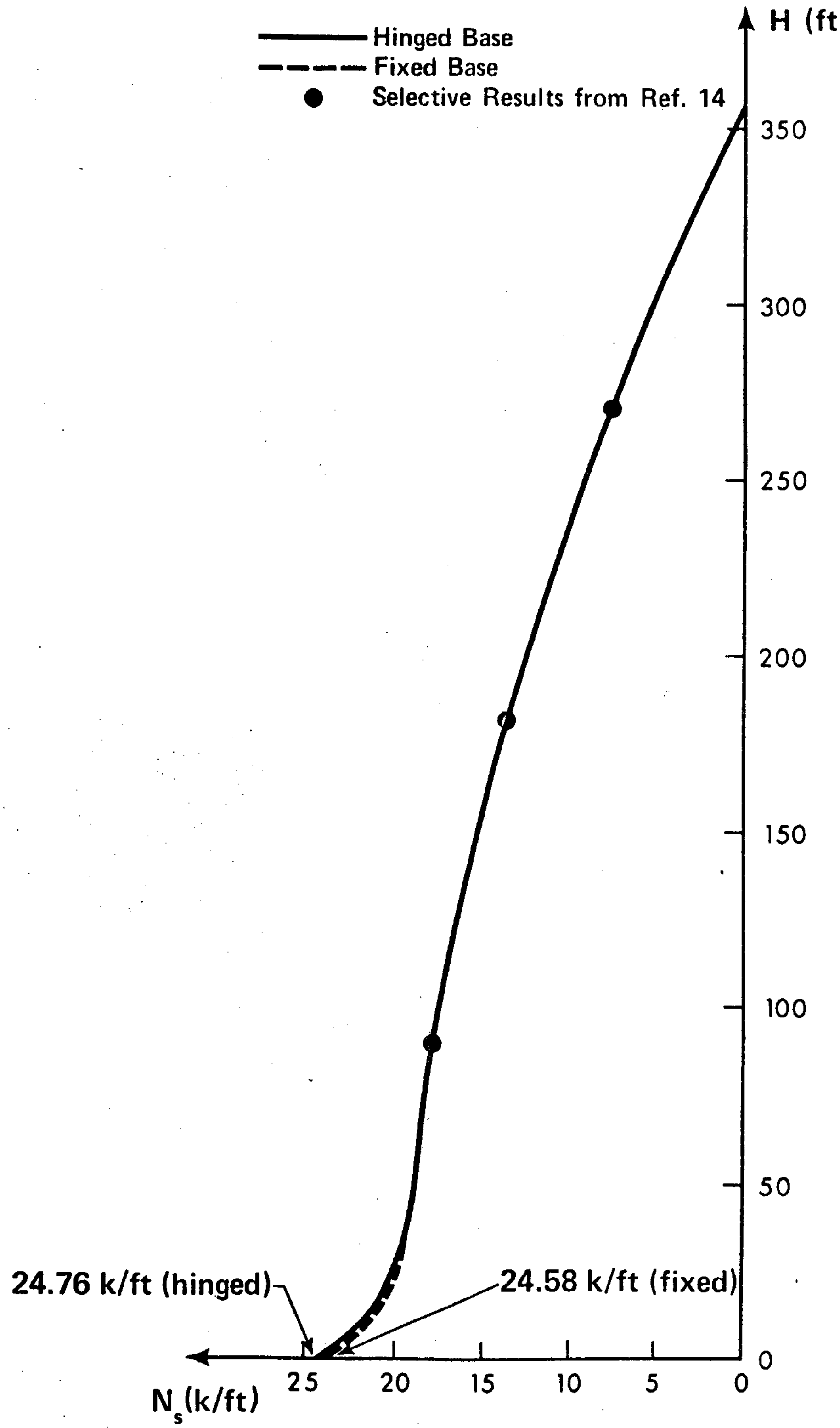


Fig. 4.7.1 Hyperboloid Tower Dead Load Membrane Force N_s

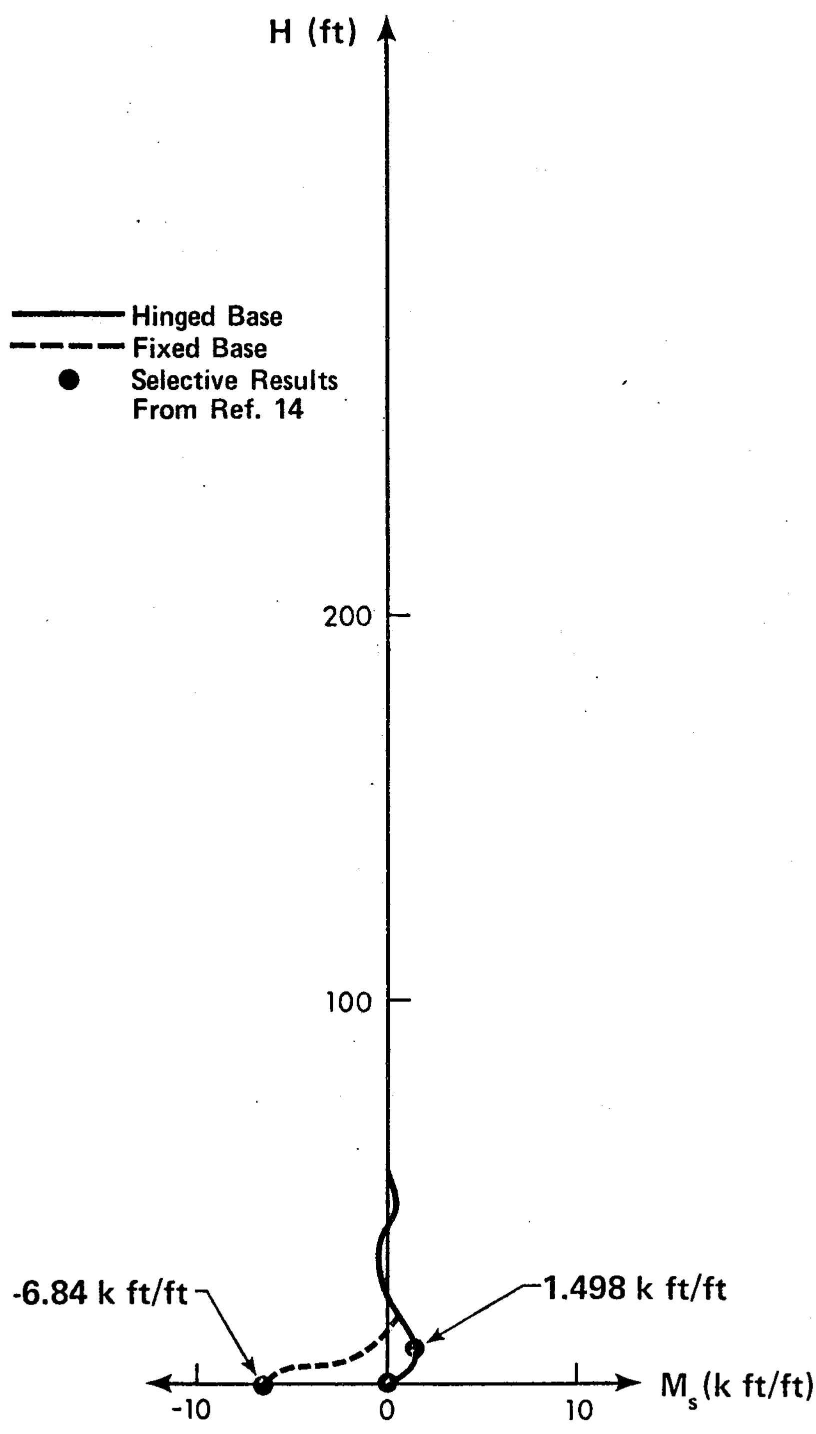


Fig. 4.7.2 Hyperboloid Tower Dead Load Meridianal Moment M_s

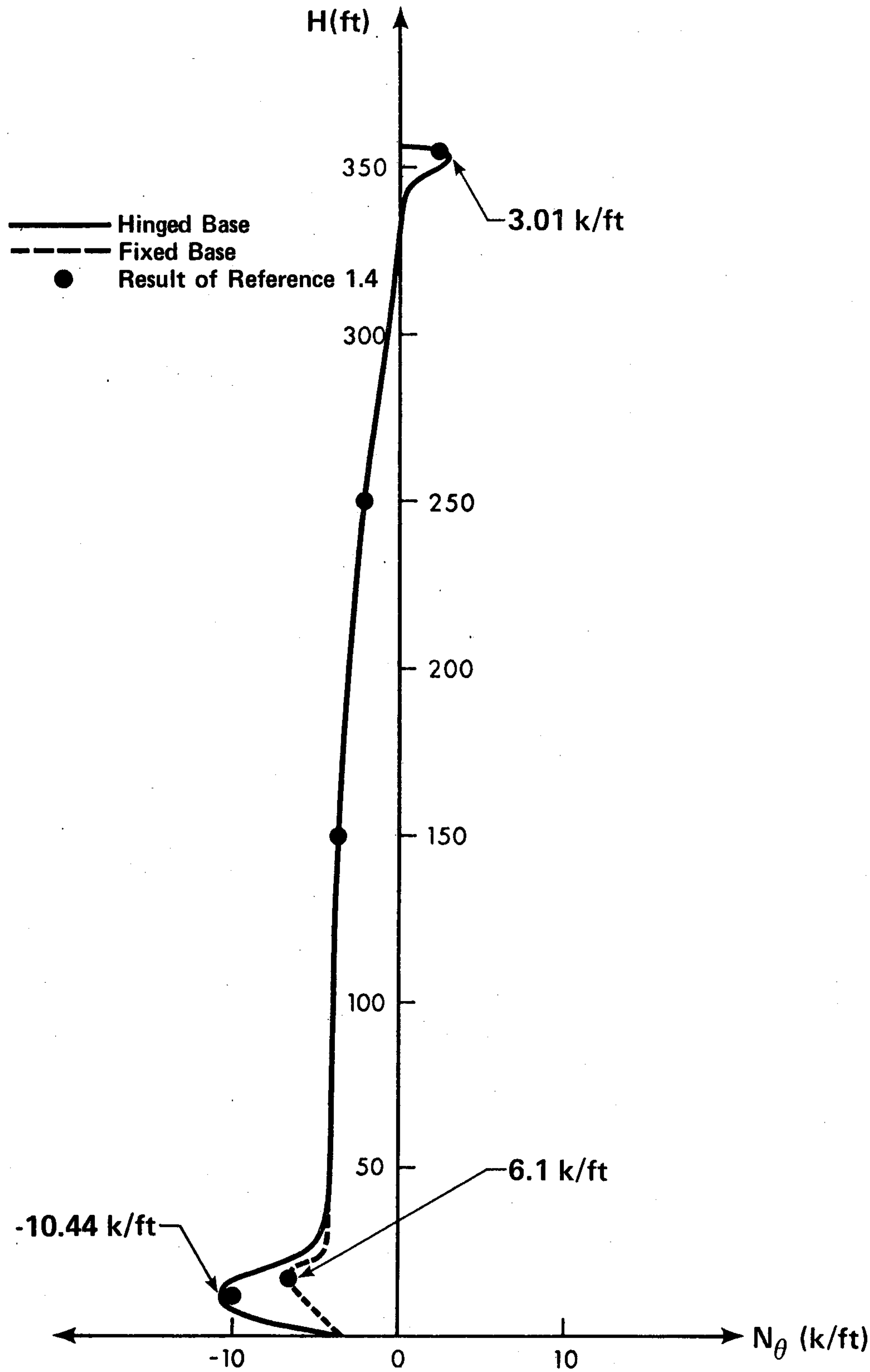


Fig. 4.7.3 Hyperboloid Tower Dead Load Membrane Force N_θ

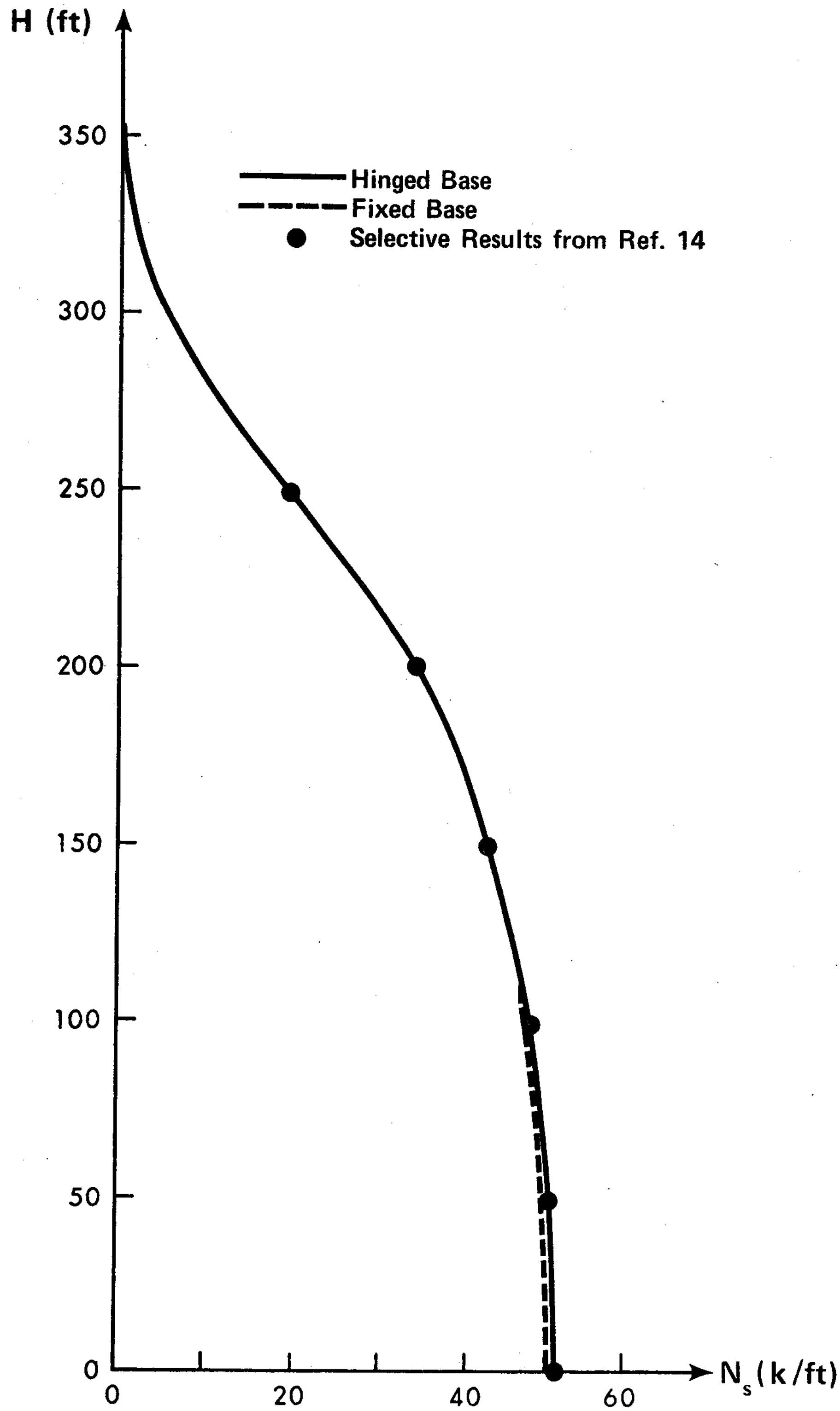


Fig. 4.8.1 Hyperboloid Tower Wind Load Membrane Force N_s ($\theta = 0$)

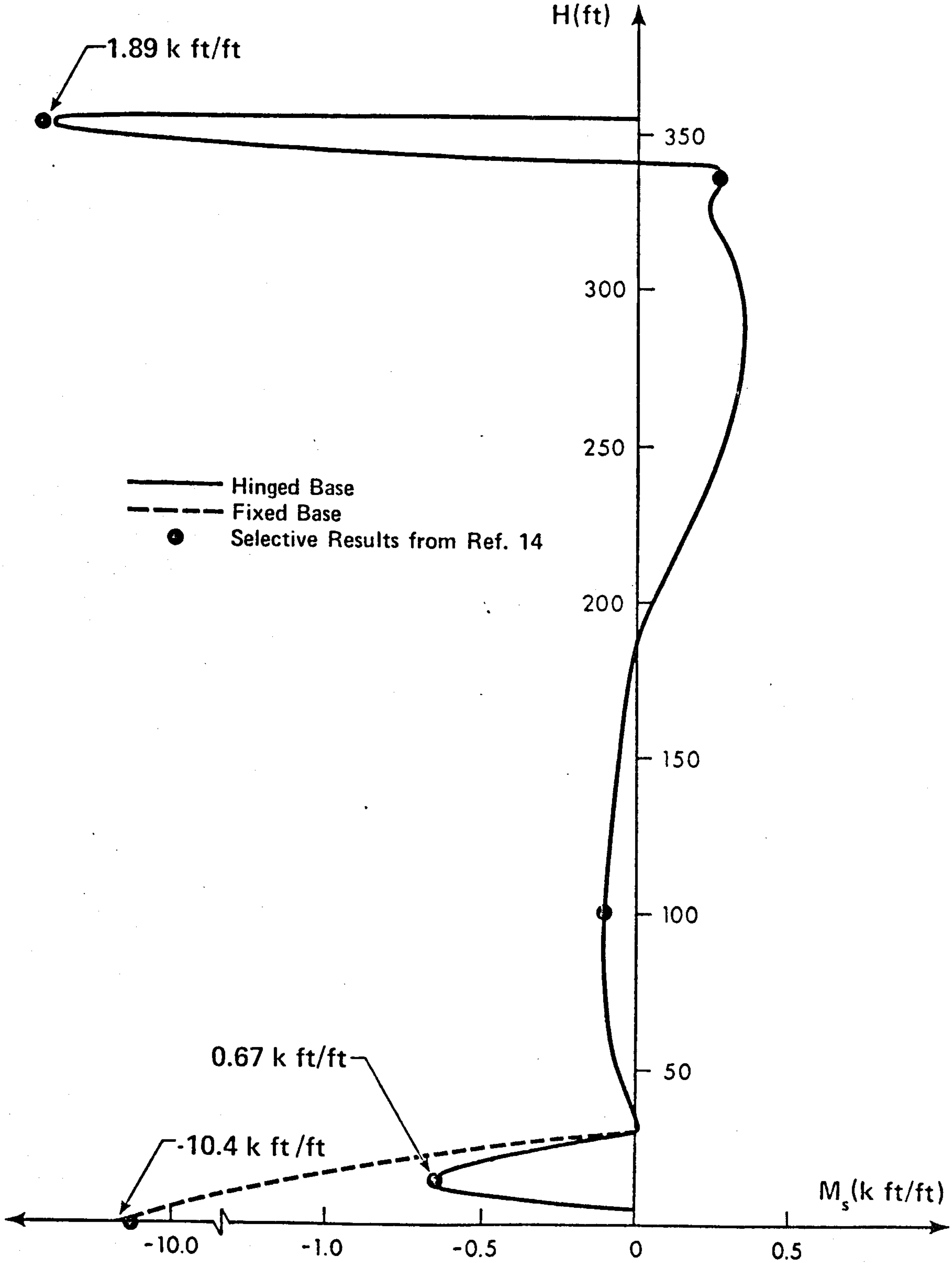


Fig. 4.8.2 Hyperboloid Tower Wind Load Meridional Moment M_s ($\theta = 0$)

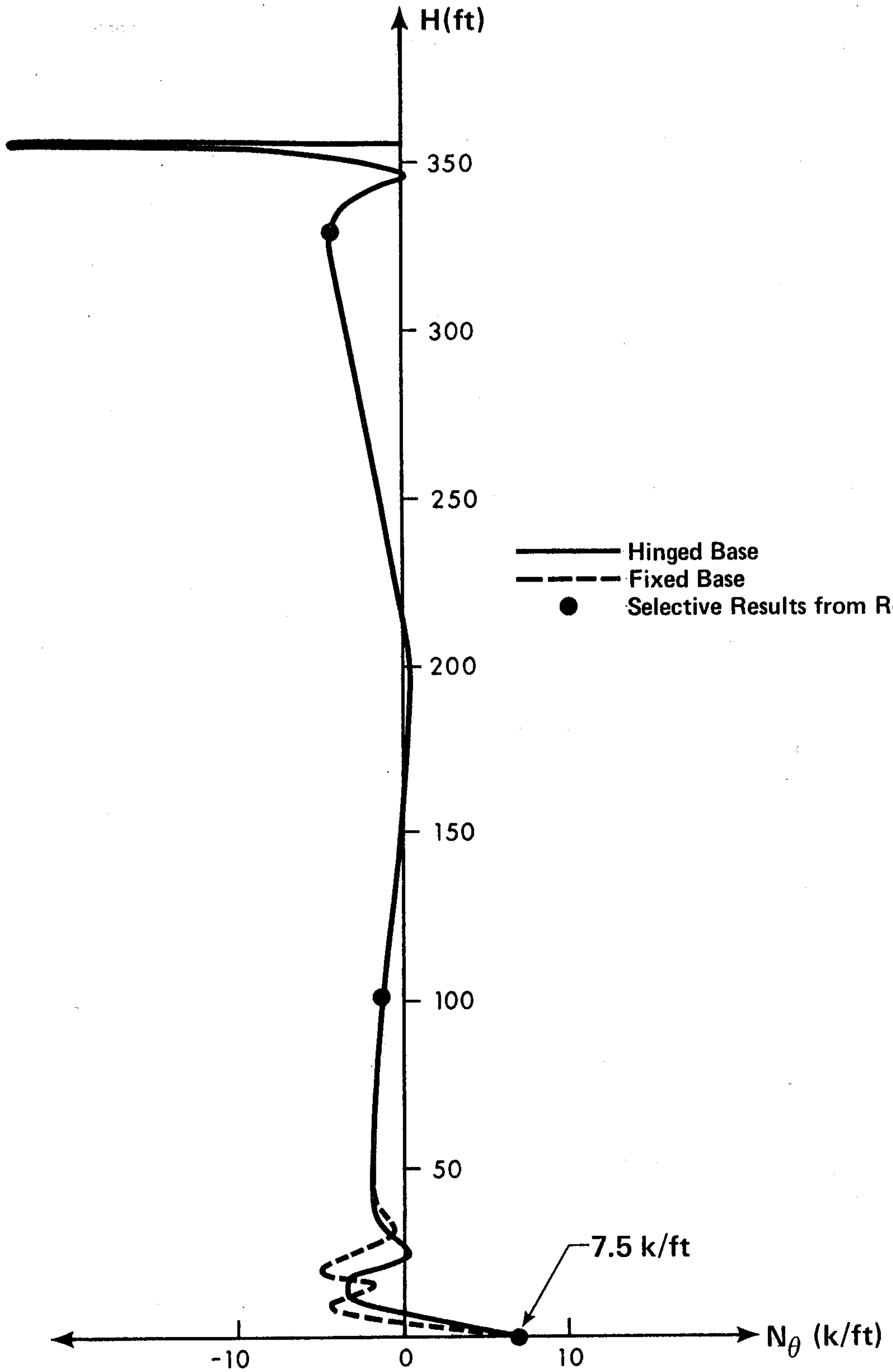


Fig. 4.8.3 Hyperboloid Tower Wind Load Membrane Force N_θ (θ = 0)

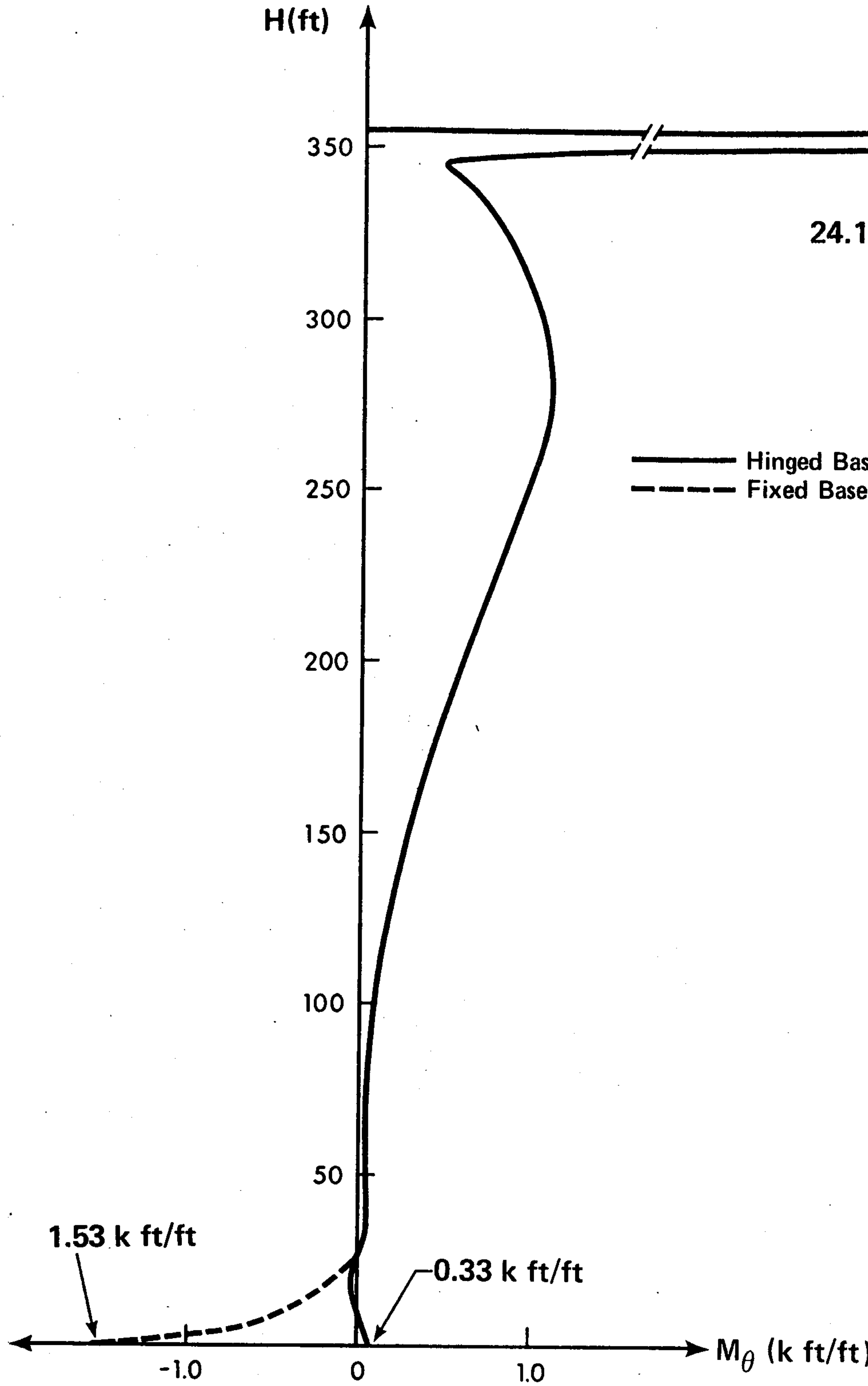


Fig. 4.8.4 Hyperboloid Tower Wind Load Circumferential Moment M_θ ($\theta = 0$)

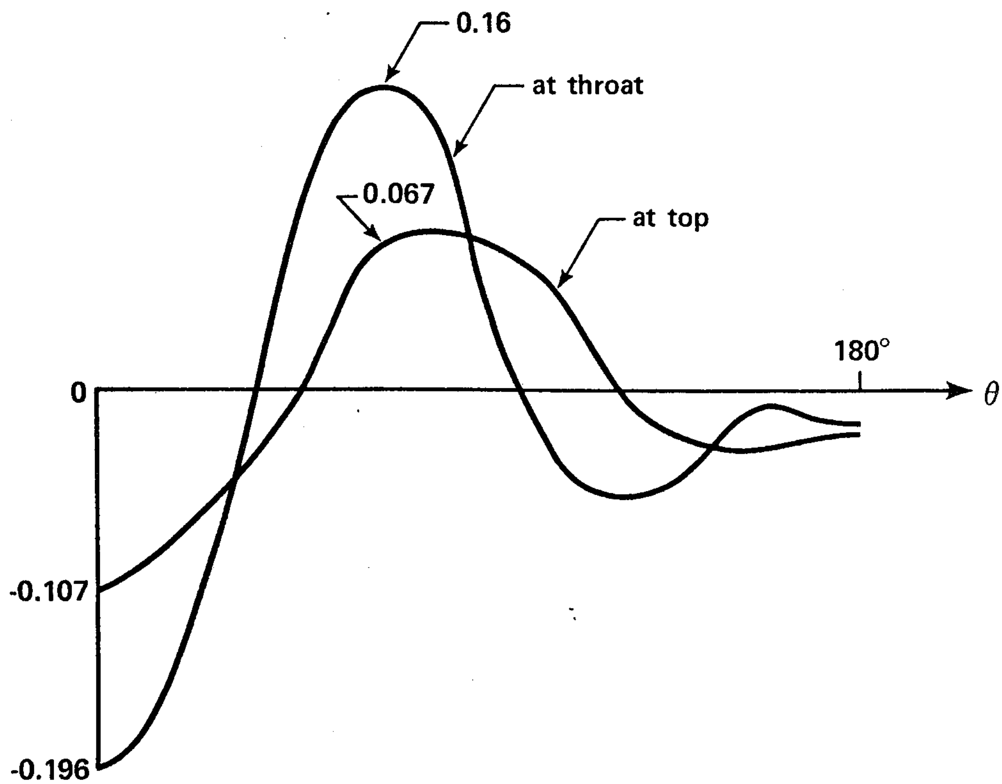


Fig. 4.8.9 Hyperboloid Tower Wind Load Circumferential Variation of W

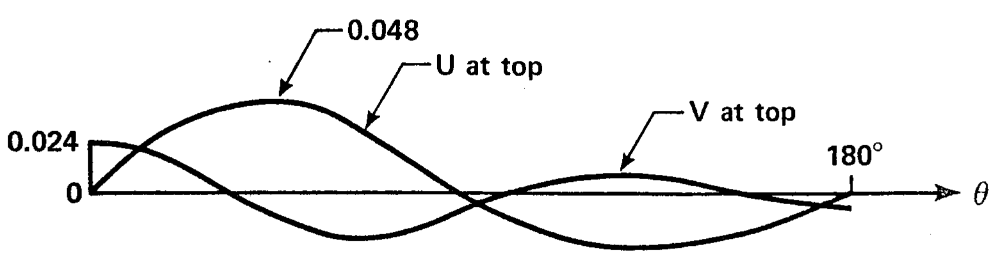


Fig. 4.8.10 Hyperboloid Tower Wind Load Circumferential Variation of V and U

CHAPTER 5

LIMITATIONS

5.1 Introduction

Three factors affect the solution technique presented in this thesis. These are summarized as follows.

1) Singularity of the governing equations at the apex.

2) Stability of the numerical integration process.

3) Convergence of Fourier expansions.

Each factor is discussed separately in the following sections.

5.2 Singularity of the Governing Equations at the Apex

If the shell has a pole (i.e., $r = 0$), coefficients in the governing sets of equations (Eqs. 2.14, 2.18 and 2.20) become singular. This is consistent with classical shell theory. A simple way to handle this situation is to choose the boundary, $s = 0$, not at the pole, but a very short distance away and then impose the boundary conditions at $s = 0$ as follows [6].

1) For harmonic number $n = 0$

$$\beta = V = U = S_s = 0$$

2) For harmonic number $n = 1$

$$W = M_s = N_s = T_s = 0$$

3) For harmonic number $n > 1$

$$W = V = U = M_s = 0$$

5.3 Stability of the Numerical Integration Process

Runge-Kutta fourth order integration method is used, in SASHELL, to integrate the basic set of equations (Eq. 2.24). This method is very well known and has the advantage of being self starting i.e., it needs only the information available at the preceding point. The method can be interpreted as follows:

1) The derivative is evaluated at the starting point of the interval.

2) The above derivative is used to obtain an approximate ordinate to determine an approximate derivative for the midpoint of the interval.

3) The above derivative is used to obtain a second approximation of the derivative at the midpoint of the interval.

4) The above derivative is used to obtain an approximate ordinate to determine an approximate derivative for the end point of the interval.

5) A weighted average of the above four derivatives is taken to determine a total increment in the function for the whole interval.

Analytically, this can be defined as follows:

$$Y_{i+1} = Y_i + \frac{\Delta h}{6} \{ f(Y_i, h_i) + 2f\left(Y^*_{i+1/2}, h_{i+1/2}\right) + 2f\left(Y^{**}_{i+1/2}, h_{i+1/2}\right) + f\left(Y^*_{i+1}, h_{i+1}\right) \} \qquad 5.1$$

where

$$Y^*_{i+1/2} = Y_i + \frac{\Delta h}{2} f(Y_i, h_i) \quad 5.2.1$$

$$Y^{**}_{i+1/2} = Y_i + \frac{\Delta h}{2} f\left(Y^*_{i+1/2}, h_{i+1/2}\right) \quad 5.2.2$$

$$Y^*_{i+1} = Y_i + \Delta h \, f\left(Y^*_{i+1/2}, h_{i+1/2}\right) \quad 5.2.3$$

$$h_{i+1/2} = h_i + \frac{\Delta h}{2} \quad 5.2.4$$

$$h_{i+1} = h_i \, \Delta h \quad 5.2.5$$

in which $f(Y_i, h_i)$ is the value of the function at point i, and Δh is the step size of the interval.

It has been found [25] that the direct integration methods, when applied to a shell problem, suffer a complete loss of accuracy when the generator of the shell exceeds a critical length. The reason for this phenomenon is explained clearly in Reference 18. The general solution of the governing equations, Eq. 3.18, is of the form

$$\{y_{(b)}\} = [H_{(b)}]\{y_{(a)}\} + \{Q_{(b)}\} \quad 5.3$$

Because of the exponentially decaying behavior of the stresses and displacements, it is observed that the coefficients of [H] increase in magnitude in such a way that if the length of the shell element is increased by any factor n, then these coefficients increase in magnitude, approximately, exponentially with n. For example, consider an element spans the region $a \leq s \leq b$. For some prescribed edge conditions at "a", we expect the corresponding solution at "b" to become smaller and smaller when the element ab is increased in length. (i.e., $y_{(b)}$ small, $H_{(b)}$ large and $y_{(a)}$ has a prescribed value). The longer the element, the larger $[H_{(b)}]$ and the smaller $\{y_{(b)}\}$. The only way to get a small value for $\{y_{(b)}\}$ is for the elements of $[H_{(b)}]$ to subtract out and that is the reason that at some critical length of the element all significant digits of $[H_{(b)}]$, in Eq. 5.3, are lost and so is the accuracy.

The loss of accuracy cannot be avoided by choosing a fine mesh for the integration. By taking more steps "partial instability" arises [8]. This means that the numerical solution deviates from the actual solution as we take more steps.

In Reference 18, the critical meridian length is limited with a length factor $\lambda L \leq 3 - 5$, where

$$\lambda L = L \sqrt[4]{\frac{3(1 - \gamma^2}{r^2 h^2}} \qquad 5.4$$

in which

L is the length of the meridian of the shell, R is the minimum radius of curvature, and h is the thickness of the shell.

In the author's opinion, this is very conservative limitation. The loss of accuracy, using the computer program SASHELL, does not arise until the length factor λL exceeds 25.

To demonstrate this, Table 5.1 shows some results of the analysis of clamped pressurized cylindrical shells, as obtained by the computer program SASHELL, with various lengths and different number of steps of integration. From the symmetry of the problem, the end moments and the absolute values of the end shears should be equal. It is obvious, from Table 5.1, that the solution is not affected much by the number of steps adopted for the integration and it begins to break when

1) λL exceeds 20 and the number of steps is less than 20.

2) λL approaches 29 for any number of steps.

As a conservative limitation, the upper bound of λL shall not exceed 25 and the number of points of integration (NP), needed for convergence, is limited as follows:

$$NP \nless 21 \quad \text{for} \quad \lambda L < 20$$

$$NP \nless 31 \quad \text{for} \quad 20 < \lambda L < 25$$

This limitation does not affect the efficiency of this technique. It can be observed from Table 5.1, that the computation time (CPU) required for the analysis is proportional to the number of steps adopted.

5.4 Convergence of Fourier Expansions

The load, when varying with the circumferential coordinates is expanded in a Fourier series. The analysis is carried out to determine the stresses and displacements everywhere within the structure. When the load is represented "exactly", the solution converges.

Theoretically, the number of harmonics required for an arbitrary periodic function to be represented exactly, by means of Fourier series, is infinity, i.e.,

$$f_{(\theta)} = \sum_{n=0}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \qquad 5.5$$

However, the load can always be described at a sufficient number of points to satisfy our engineering judgement of representing the actual loading conditions. Therefore, our concern is to examine the convergence of a function known only at a set of discrete points. If these points are equally spaced, say 2N points, taken over the interval $0 \leq \theta \leq 2\pi$, the spacing is,

$$\theta_i = \frac{2\pi i}{2N} \qquad \text{for } i = 0, 1, 2, \ldots, 2N-1 \qquad 5.6$$

If now an approximation is assumed in the form

$$f_{(\theta)} \approx \sum_{n=0}^{M} A_n \cos n\theta + \sum_{n=1}^{M} B_n \sin n\theta \qquad 5.7$$

where the coefficients are to be determined in such a way that the integrated squared error over the interval of length 2π is least, then the requirement is

$$\int_0^{2\pi} \left[f_{(\theta)} - \sum_{n=0}^{M} A_n \cos n\theta - \sum_{n=1}^{M} B_n \sin n\theta \right]^2 = \min \tag{5.8}$$

Since the function $f_{(\theta)}$ is described only at the points defined by θ_i (Eq. 5.6), we have 2N independent data points which are sufficient to determine the coefficients of 2N terms of an approximation in the form of Eq. 5.7. When $M \leq N$, Eq. 5.8 can be rewritten as follows

$$\sum_{n=0}^{2N-1} \left[f_{(\theta_i)} - \sum_{n=0}^{M} A_n \cos n\theta_i - \sum_{n=1}^{M} \sin n\theta_i \right]^2 = \min \tag{5.9}$$

where $f_{(\theta_i)}$ is the value of the function $f_{(\theta_i)}$ at the point i. The solution is obtained when the partial derivatives of the left hand side of Eq. 5.9 with respect to A_n and B_n are equated to zero [15, pg. 446-457]. The coefficients are in the form

$$A_o = \frac{1}{2N} \sum_{n=0}^{2N-1} f_{(\theta_i)} \tag{5.10.1}$$

$$A_n = \frac{1}{N} \sum_{n=0}^{2N-1} f_{(\theta_i)} \cos n\theta_i \quad (n \neq 0, N) \tag{5.10.2}$$

$$B_n = \frac{1}{N} \sum_{n=0}^{2N-1} f_{(\theta_i)} \sin n\theta_i \tag{5.10.3}$$

The calculation of each coefficient is independent of the calculation of the others and is independent of M as long as $M \leq N$. When M = N, the least-squares criterion (Eq. 5.9) becomes equivalent to the requirement that the "best" approximation of the function (Eq. 5.7) is obtained [15].

The loading cases of the examples discussed in Chapter 4 are shown in Fig. 5.1 and 5.2. The effect of the number of harmonics on the loading associated with these problems is considered in the following. Fourier coefficients of the wind pressure load on the hyperboloid tower are included in Table 5.2.1. Since the load is symmetric with respect to $\theta = 0$, the sine coefficients vanish. The results of superimposing the 12 coefficients, using Eq. 5.7, are shown in Table 5.2.2. It can be seen that these values approximate the load function to a very close agreement. Now examining the coefficients shown in Table 5.2.1. The 9^{th} and the subsequent coefficients are small in comparison with the other coefficients. The magnitude of the 9^{th} coefficient is in the same order of the 10^{th} with opposite sign and so for the 11^{th} and 12^{th}. Therefore we expect that the approximation will be reasonably accurate if the series is terminated at the 8^{th} harmonic. The results of an 8 harmonic approximation are shown in Table 5.2.2.

Fourier coefficients of the two concentrated loads of the pinched cylinder problem discussed in Sect. 4.3 are shown in Table 5.3.1. It can be seen that the odd cosine coefficients as well as the sine coefficients vanish for this case of symmetry with respect to $\theta = 0$ and $\theta = \pi/2$. The approximation with 18 harmonics, for the loading function described at 36 points, represent the load more accurately when compared with the 12 harmonic approximation (Fig. 5.2 and Table 5.3.2). Therefore one can approximate this load by superimposing the results of $n = 0, 2, ..., 16$.

As a general conclusion, one can consider N harmonics in the expansion of a load described at 2N points. A termination of the higher harmonics or exemption of a harmonic number in the series can be decided upon by examining the coefficient of each loading function as a special case.

r = 10 ft   E = 5000 ksi ν = .15   h = 0.5 ft λ = 0.5852

	L (ft)	Number of Elements	Number of Points	CPU Time (sec)	λL	$M_{s(0)}$ (K.ft/ft)	$M_{s(L)}$ (K.ft/ft)	$Q_{s(0)}$ (K/ft)	$Q_{s(L)}$ (K/ft)
1	10	1	11	0.34	15.85	1.442	1.438	-1.676	1.675
2	10	1	15	0.44	5.85	1.442	1.442	-1.676	1.676
3	20	1	21	0.59	11.71	1.430	1.430	-1.675	1.675
4	40	1	21	0.59	23.41	1.451	45.05	-1.697	28.65
5	40	1	31	0.83	23.41	1.427	1.427	-1.672	1.671
6	50	1	21	0.59	29.26	-0.139	-2446.0	0.167	-2173.0
7	50	1	31	0.82	29.26	1.426	1.424	-1.671	1.670
8	50	1	41	1.06	29.26	1.426	1.427	-1.671	1.671
9	50	1	51	1.32	29.26	1.426	1.428	-1.671	1.671
10	50	2	21	1.13	14.63	1.426	1.426	-1.671	1.671
11	60	1	31	0.84	35.11	1.388	121.8	-1.631	-62.74
12	60	1	41	1.08	35.11	1.426	0.876	-1.671	1.795
13	60	1	51	1.31	35.11	1.426	2.367	-1.670	2.271
14	60	1	81	2.10	35.11	1.426	0.511	-1.671	1.235
15	60	2	21	1.12	17.55	1.426	1.426	-1.670	1.670
16	60	2	31	1.60	17.55	1.426	1.426	-1.670	1.670
17	100	1	101	2.50	58.52	- .075	-2.84×10^{8}	-0.068	5.33×10^{9}
18	100	2	31	1.61	29.41	1.425	1.422	-1.670	1.668
19	100	3	21	1.68	19.50	1.425	1.425	-1.670	1.670

Table 5.1 Effect of Length and Number of Integration Points on the Accuracy of the Solution

Harmonic Number	Cosine Coefficient
0	0.3833
1	-0.2792
2	-0.6198
3	-0.5093
4	-0.0917
5	0.1179
6	0.0333
7	-0.0447
8	-0.0083
9	0.0093
10	-0.0136
11	0.0060

TABLE 5.2.1

Fourier Coefficients of Wind Pressure Load

θ	Load Value for 12 Harmonics	Load Value for 8 Harmonics
0	-1.016	-1.010
7.5	-0.955	- .952
15	-0.783	-0.787
22.5	-0.528	-0.535
30	-0.216	-0.219
37.5	0.136	0.141
45	0.516	0.522
52.5	0.888	0.887
60	1.183	1.178
67.5	1.335	1.335
75	1.316	1.319
82.5	1.146	1.144
90	0.883	0.878
97.5	0.610	0.613
105	0.416	0.429
112.5	0.352	0.355
120	0.383	0.366
127.5	0.423	0.406
135	0.416	0.427
142.5	0.387	0.416
150	0.383	0.391
157.5	0.407	0.378
165	0.416	0.388
172.5	0.398	0.409
180	0.383	0.420

TABLE 5.2.2 Effect of Number of Harmonics on Representing the Wind Pressure Load

Harmonic Number	Cosine Coefficient
0	0.0555
1	0
2	0.1111
3	0
4	0.1111
5	0
6	0.1111
7	0
8	0.1111
9	0
10	0.1111
11	0
12	0.1111
13	0
14	0.1111
15	0
16	0.1111
17	0

TABLE 5.3.1

Fourier Coefficients of Two Diametrically Opposed Concentrated Loads

θ	Load Value for 18 Harmonics	Load Value for 12 Harmonics
0	0.994	0.611
5	0.635	0.522
10	0.055	0.301
15	-0.207	0.055
20	-0.055	-0.104
25	0.119	-0.131
30	0.055	-0.055
35	-0.079	0.041
40	-0.055	0.085
45	0.055	0.055
50	0.055	-0.012
55	-0.039	-0.061
60	-0.055	-0.055
65	0.026	-0.005
70	0.055	0.045
75	0.015	0.055
80	-0.055	0.019
85	0.005	-0.032
90	0.055	-0.055

TABLE 5.3.2 Effect of Number of Harmonics on Representing the Two Concentrated Loads

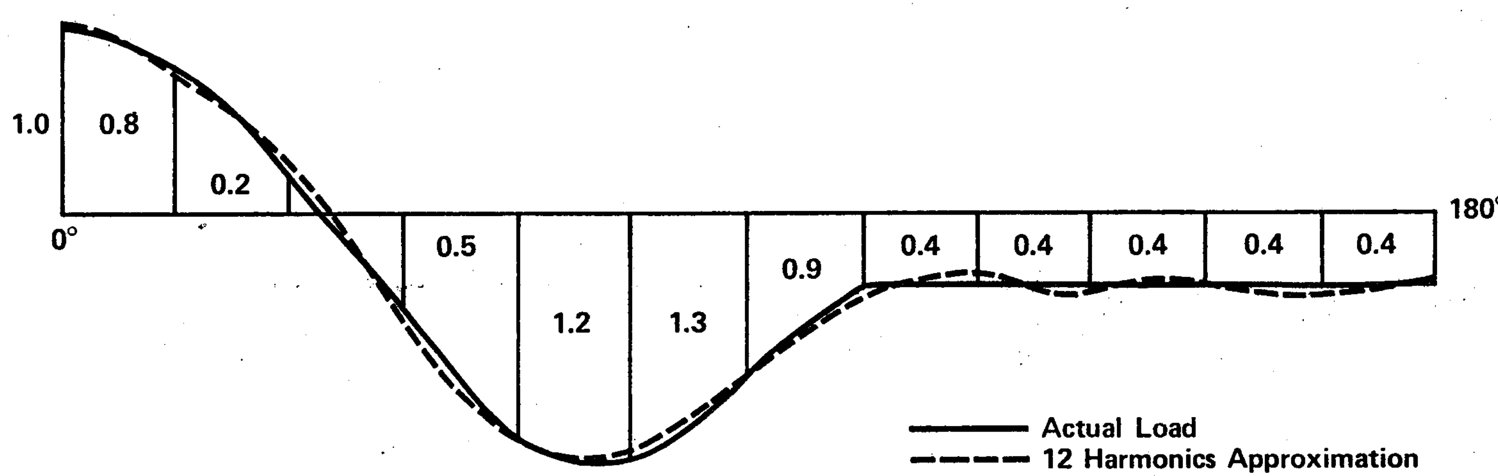


Fig. 5.1 Fourier Approximation for Wind Load

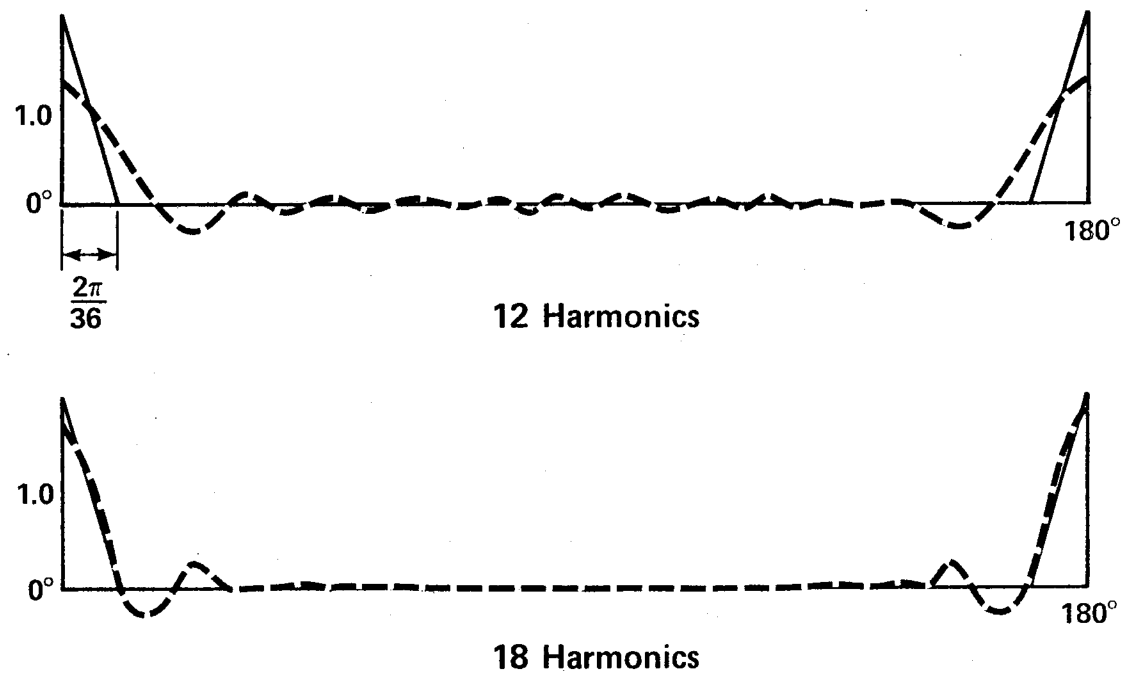


Fig. 5.2 Fourier Approximation for Two Diametrically Opposed Concentrated Loads

CHAPTER 6

SUMMARY AND CONCLUSIONS

In this study, a theory has been generalized. A computer program has been developed for the elastic analysis of axisymmetric segmented shell structure of general geometric configuration. General arbitrary loadings have been considered. Applications for a number of loading cases and elements have been presented.

The reduction of the governing partial differential equations of the classical shell theory to a set of eight first order ordinary differential equations involves only straight forward algebraic manipulations. The classical theory chosen as a foundation for the theory presented herein, Flügge's theory, is considered one of the most accurate theories available in the literature. The approximation in formulating the basic governing equations of this theory are such that the theory may be considered to be exact. The adaptation of the general form of these basic equations allows the geometry of any type of shell element to be considered, without any theoretical approximations which may be suitable for specific dimensions and not for others. Variation of the shell thickness along the meridian can be accounted for with accuracy comparable to that for constant thickness.

The loads, when approximated by means of Fourier series, may be represented with sufficient accuracy by including a number of harmonics, within practical limits. This approximation allows the consideration of general arbitrary types of loadings in a simple manner.

The excellent agreement between the results of the problems presented in this thesis and the results of other known solutions, analytical or numerical, demonstrate the accuracy of the solution technique and the reliability of the numerical integration process used.

It may be concluded that the solution technique presented in this study is simple and general for the elastic analysis of shells of revolution. The accuracy is consistent with a good shell theory. The geometric limitations are of no importance from the practical point of view.

Further development, by applying this method to the study of free vibration and elastic buckling criteria of shells of revolution is possible.

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APPENDIX A

SHELL THEORY

This appendix presents the analytical development of the basic equations, used in this study, as given by Flügge [10].

A.1 Geometry of Shells

The geometry of a shell is defined by specifying the form of the middle surface and the thickness of the shell at each point. The surface of a shell of revolution is generated by the rotation of a plane curve about an axis in its plane. This generating curve is called the "meridian". The intersection of the surface with planes perpendicular to the axis of revolution are "parallel circles". Two coordinates s, θ are required to describe any point on the middle surface of the shell:

a) s measures the distance to the point along the meridian from the intersection of the middle surface with the axis of rotation, or from a datum parallel circle.

b) θ is the angular "distance" of the point from a datum generator.

A third coordinate z is required to measure the distance along a normal to the middle surface.

The radii of curvature of a shell of revolution are:

a) r is the radius of curvature of parallel circles

b) r_1 is the radius of curvature of meridian

c) r_2 is the length of the normal between any point on the middle surface and the axis of revolution.

The following fundamental geometrical relations can clearly be seen in Fig. A.1

$$r = r_2 \sin\phi \qquad \text{A.1.1}$$

$$ds = r_1 \, d\phi \qquad \text{A.1.2}$$

$$dr = ds \cos\phi \qquad \text{A.1.3}$$

$$dx = ds \sin\phi \qquad \text{A.1.4}$$

where ϕ represents the angular distance of the point under consideration from the axis of rotation.

From Eqs. A.1 one can write

$$\frac{dr}{ds} = \cos\phi \qquad \text{A.2.1}$$

$$\frac{dr_2}{ds} = \frac{r_1 - r_2}{r_1} \cot\phi \qquad \text{A.2.2}$$

For simplicity the derivatives with respect to s and θ will be indicated by dots and primes respectively, i.e.,

$$\frac{\partial}{\partial s}(\;) \equiv (\;)^{\bullet} \qquad \text{A.3.1}$$

$$\frac{\partial}{\partial \theta}(\;) \equiv (\;)' \qquad \text{A.3.2}$$

A.2 Equations of Equilibrium

If a shell element is cut out by two meridians and two parallel circles, each pair infinitely close as seen in Fig. A.2, the element is stressed by ten stress resultant components which must be in equilibrium with the external applied load. These stress resultants are:

N_s, N_θ = normal in-plane forces per meridional and circumferential unit length, respectively.

$N_{s\theta}$, $N_{\theta s}$ = in-plane shear forces per meridional and circumferential unit length, respectively.

Q_s, Q_θ = transverse shear forces per meridional and circumferential unit lengths, respectively.

M_s, M_θ = meridional and circumferential moments per unit length, respectively.

$M_{s\theta}$, $M_{\theta s}$ = circumferential and meridional twisting moments per unit length, respectively.

The sign convention for the forces is shown in Fig. A.2

Referring to the three orthogonal axis s, θ, z, one can obtain six equations of equilibrium

$$\Sigma N_i = 0 \qquad \text{A.4.1}$$

$$\Sigma M_i = 0 \qquad \text{A.4.2}$$

where ΣN_i is the sum of the forces in the i direction ($i = s, \theta, z$).

ΣM_i is the sum of the moments about the i axis

$(i = s, \theta, z)$

The six equilibrium equations in terms of the ten stress resultants are

$$r_1(rN_s)^{\bullet} + r_1N'_{\theta s} - r_1N_\theta \cos\phi - r\,Q_s + rr_1\,P_s = 0 \qquad \text{A.5.1}$$

$$r_1(rN_{s\theta})^{\bullet} + r_1N'_\theta + r_1\,N_{\theta s}\cos\phi - r_1Q_\theta \sin\phi + rr_1P_\theta = 0 \qquad \text{A.5.2}$$

$$r_1\,N_\theta \sin\phi + rN_s + r_1\,Q'_\theta + r_1(rQ_s)^{\bullet} - rr_1\,P_z = 0 \qquad \text{A.5.3}$$

$$r_1(rM_s)^{\bullet} + r_1M'_{\theta s} - r_1\,M_\theta \cos\phi - rr_1\,Q_s = 0 \qquad \text{A.5.4}$$

$$r_1(rM_{s\theta})^{\bullet} + r_1M'_\theta + r_1\,M_{\theta s}\cos\phi - rr_1\,Q_\theta = 0 \qquad \text{A.5.5}$$

$$rr_1\,N_{\theta s} - rr_1N_{s\theta} - r_1\,M_{\theta s}\sin\phi + r\,M_{s\theta} = 0 \qquad \text{A.5.6}$$

where P_s, P_θ, P_z are the resolved components of the external applied load in the s, θ, z, respectively. If the subscript s, in the above equations, is replaced by ϕ, and the relations

$$\frac{1}{r_1}\frac{\partial}{\partial\phi}(\;) = (\;)^{\bullet} \qquad \text{A.6}$$

is observed. The above set of equations (Eqs. A.5) can be reduced to the equilibrium equations in Reference 10, pg. 318. Six equations of equilibrium are not enough to determine the ten stress resultants. The shell element is four times internally statically indeterminate and therefore, an analysis of the deformation of the shell is required in order to obtain the solution.

A.3 Strain-Displacement Relations

The displacement vector of a point lying on the middle surface of an element may be described by its three orthogonal components, defined as:

W = the displacement component in the radial direction, positive when it points away from the centre of curvature.

V = the displacement component in the meridional direction, positive in the direction of increasing the coordinate s.

U = the displacement component in the direction of the tangent to the parallel circle, positive in the direction of increasing the coordinate θ.

In addition, the auxiliary variable β, which represents the angle by which an element of the meridian rotates during deformation may be expressed in terms of the displacement components, Fig. A.3, as [10].

$$\beta = -W^{\bullet} + \frac{V}{r_1} \qquad \text{A.7.1}$$

By differentiation, the change in slope of the meridian due to deformation, is

$$\beta^{\bullet} = -W^{\bullet\bullet} + \frac{r_1 V^{\bullet} - r_1^{\bullet} V}{r_1^2} \qquad \text{A.7.2}$$

Referring to Fig. A.4, one can obtain the relationship between the strains and the displacement components of a point on the middle surface.

The meridional strain is

$$\varepsilon_s = \frac{\text{Elongation of the line element ds}}{\text{ds}}$$

$$= \frac{W}{r_1} + V^{\bullet} \qquad \text{A.8.1}$$

The hoop strain is

$$\varepsilon_\theta = \frac{\text{Elongation of the line element rd}\theta}{\text{rd}\theta}$$

$$= \frac{U' + V\cos\phi + W\sin\phi}{r} \qquad \text{A.8.2}$$

The shear strain, which is the change of the right angle between the two line elements ds and rdθ (Fig. A.4.3), is equivalent to

$$\gamma_{s\theta} = \gamma_1 + \gamma_2 =$$

$$= \frac{V'}{r} + U^{\bullet} + \frac{U}{r}\cos\phi \qquad \text{A.8.3}$$

Since the displacement is assumed to be very small in comparison with the principal radii of curvature, all products of two displacement components have been dropped in deriving the foregoing equations.

One may use the preceding equations to express the relationship between the strains and the displacement components of an arbitrary point at a distance z from the middle surface by simply replacing W, V, U with the displacement components of this point W_z, V_z, U_z, and replacing the radii r_1, r_2 with $r_1 + z$ and $r_2 + z$, respectively.

Then

$$\varepsilon_s = \frac{W_z}{r_1 + z} + V^{\bullet}_z \qquad \text{A.9.1}$$

$$\varepsilon_\theta = U'_z + \frac{V_z \cos\phi + W_z \sin\phi}{(r_2 + z)\sin\phi} \qquad \text{A.9.2}$$

$$\gamma_{s\theta} = U^{\bullet}_z - \frac{U_z \cos\phi - V'_z}{(r_2 + z)\sin\phi} \qquad \text{A.9.3}$$

By introducing the assumption that lines normal to the middle surface before deformation remain normal after deformation, it can clearly be seen from Fig. A.5 that

$$W_z = W \qquad \text{A.10.1}$$

$$V_z = V\frac{r_1 + z}{r_1} + W^{\bullet}z \qquad \text{A.10.2}$$

$$U_z = U\frac{r_2 + z}{r_2} + W'\frac{z}{r} \qquad \text{A.10.3}$$

Therefore, the strains at a distance z from the middle surface in terms of the displacements W, V, U at the middle surface can be obtained by substituting Eqs. A.10 into Eqs. A.9 to obtain

$$\varepsilon_s = \frac{1}{r_1 + z} W + \frac{r^{\bullet}_1 z}{r_1 + z} W^{\bullet} - \frac{r_1 z}{r_1 + z} W^{\bullet\bullet}$$

$$+ V^{\bullet} - \frac{r^{\bullet}_1 z}{r_1 (r_1 + z)} V \qquad \text{A.11.1}$$

$$\varepsilon_\theta = \frac{1}{r_2 + z} W - \frac{z \cot\phi}{r_2 + z} W^{\bullet} - \frac{z}{r \sin\phi \, (r_2 + z)} W''$$

$$+ \frac{\cot\phi \, (r_1 + z)}{r_1 (r_2 + z)} V + \frac{1}{r} U' \qquad \text{A.11.2}$$

$$\gamma_{s\theta} = \frac{r_1 (r_2 + z)}{r_2 (r_1 + z)} U^{\bullet} - \frac{r_1 (r_2 + z)}{r_2^2 (r_1 + z)} \cot\phi \, U$$

$$+ \frac{(r_2 + z)}{r_1 \sin\phi (r_2 + z)} V' - \frac{z}{\sin\phi} \left(\frac{1}{r_2 + z} + \frac{r_1}{r_2 (r_1 + z)}\right) W'^{\bullet}$$

$$+ \frac{\cot\phi}{r_2 \sin\phi} \left(\frac{z}{r_2 + z} + \frac{r_1 z}{r_2 (r_1 + z)}\right) W' \qquad \text{A.11.3}$$

A.4 Stress-Strain Relations

Hooke's law relates the strains to the corresponding stresses in linearized form, as long as the stresses remain within the elastic limit. If T is the change in temperature measured from arbitrary level. Hooke's law may be written, in index notation, as

$$E\varepsilon_i = \sigma_i - \nu(\sigma_j + \sigma_k) + E\alpha T \qquad \text{A.12.1}$$

$$G\gamma_{ij} = \tau_{ij} \qquad \text{A.12.2}$$

where

i, j, t	take in turns the direction s, θ, z
σ_i	is the normal stress in the i direction
τ_{ij}	is the shearing stress in the i plane and j direction

$$G = \frac{E}{2(1 + \nu)} \qquad A.13$$

The modulus of elasticity, E, Poisson's ratio, ν, the coefficient of thermal expansion, α, and thus the shear modulus, G, are the material constants.

As in the theory of plates, except in the immediate vicinity of concentrated forces, the stresses in z direction are small in comparison with the stresses in s, θ directions and their influence in Hooke's law may be neglected, i.e., it is assumed that

$$\sigma_z = \gamma_{sz} = \gamma_{\theta z} = 0 \qquad A.14$$

Therefore, the stress-strain relations can be written as

$$\sigma_s = \frac{E}{1-\nu^2}\left[\varepsilon_s + \nu\varepsilon_\theta - (1 + \nu)\ \alpha T\right] \qquad A.15.1$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\varepsilon_\theta + \nu\varepsilon_s - (1 + \nu)\ \alpha T\right] \qquad A.15.2$$

$$\tau_{s\theta} = \frac{E}{2(1+\nu)} \gamma_{s\theta} \qquad \text{A.15.3}$$

A.5 Elastic Law

The internal stress resultants can be determined by integrating the stresses through the shell thickness (Fig. A.6). They are defined as

$$N_s = \int_{-t/2}^{t/2} \sigma_s \frac{r_2 + z}{r_2} dz \qquad \text{A.16.1}$$

$$N_\theta = \int_{-t/2}^{t/2} \sigma_\theta \frac{r_1 + z}{r_1} dz \qquad \text{A.16.2}$$

$$N_{s\theta} = \int_{-t/2}^{t/2} \tau_{s\theta} \frac{r_1 + z}{r_2} dz \qquad \text{A.16.3}$$

$$N_{\theta s} = \int_{-t/2}^{t/2} \tau_{\theta s} \frac{r_1 + z}{r_1} dz \qquad \text{A.16.4}$$

$$M_s = \int_{-t/2}^{t/2} \sigma_s \frac{r_2 + z}{r_2} zdz \qquad \text{A.16.5}$$

$$M_\theta = - \int_{-t/2}^{t/2} \sigma_\theta \frac{r_1 + z}{r_1} zdz \qquad \text{A.16.6}$$

$$M_{s\theta} = \int_{-t/2}^{t/2} \tau_{s\theta} \frac{r_2 + z}{r_2} zdz \qquad A.16.7$$

$$M_{\theta s} = \int_{-t/2}^{t/2} \tau_{\theta s} \frac{r_1 + z}{r_1} zdz \qquad A.16.8$$

It should be noted that the expressions for Q_s, Q_θ have been omitted as they are equal to the integral of the shearing stresses in z direction. Also the minus sign in these expressions correspond to the positive directions assumed for the stress resultants as shown in Fig. A.6.

The expressions for σ_s, σ_θ, $\tau_{s\theta}$ (Eqs. A.15) can be entered into Eqs. A.16. Then Eqs. A.11 are substituted for the displacements and the integration with respect to z is performed. The results in the stress-resultant-displacement relationships.

$$N_s = D\left[V^{\bullet} + \frac{W}{r_1} + \nu \frac{U' + V\cos\phi + W\sin\phi}{r}\right]$$

$$+ \frac{K}{r_1^2}\frac{r_2 - r_1}{r_2}\left[(\frac{V}{r_1} - W^{\bullet})\, r^{\bullet}_1 + r_1 W^{\bullet} + \frac{W}{r_1}\right]$$

$$- (1 + \nu)\,\alpha\, D\, T_{o2} \qquad A.17.1$$

$$N_\theta = D\left[\frac{U' + V\cos\phi + W\sin\phi}{r} + \nu(V^{\bullet} + \frac{W}{r_1})\right]$$

$$- \frac{K}{rr_1}\frac{r_2 - r_1}{r_2}\left[\frac{V}{r_1}(\frac{r_1 - r_2}{r_2})\cos\phi + \frac{W\sin\phi}{r_2}\right.$$

$$\left. + \frac{W''}{r} + W^{\bullet}\cos\phi\right] - (1 + \nu)\,\alpha\, D\, T_{o1} \qquad A.17.2$$

$$N_{s\theta} = D\left(\frac{1-\nu}{2}\right)\left[U^{\bullet} + \frac{V' - U\cos\phi}{r}\right]$$

$$+ \frac{K}{r_1^{\,2}}\left(\frac{1-\nu}{2}\right)\frac{r_2 - r_1}{r_2}\left[U^{\bullet}\left(\frac{r_2 - r_1}{r_2}\right)\right.$$

$$\left. + U\,\frac{r_1 - r_2}{r_2}\,\frac{\cot\phi}{r_2} + W'^{\bullet}\,\frac{r_1}{r} - W'\,\frac{r_1\cos\phi}{r^2}\right] \qquad \text{A.17.3}$$

$$N_{\theta s} = D\left(\frac{1-\nu}{2}\right)\left[U^{\bullet} + \frac{V' - U\cos\phi}{r}\right]$$

$$+ \frac{K}{rr_1}\left(\frac{1-\nu}{2}\right)\frac{r_2 - r_1}{r_2}\left[V'\,\frac{r_2 - r_1}{r_1 r_2} - W'^{\bullet} + \frac{W'\cos\phi}{r}\right]$$

A.17.4

$$M_s = K\left[W^{\bullet\bullet} - W^{\bullet}\,\frac{r^{\bullet}_1}{r_1} - W\,\frac{r_1 - r_2}{r_2}\,\frac{1}{r_1^{\,2}} - \frac{V^{\bullet}}{r_2} + V\,\frac{r^{\bullet}_1}{r_1^{\,2}}\right.$$

$$\left. + \nu\,\frac{W''}{r^2} + \nu\,\frac{W^{\bullet}\cos\phi}{r} - \nu\,\frac{U'}{rr_2} - \nu\,\frac{V\cos\phi}{rr_1}\right]$$

$$+ (1+\nu)\,\alpha\,K\,T_{12} \qquad \text{A.17.5}$$

$$M_\theta = K\left[\frac{W''}{r^2} + \frac{W^{\bullet}\cos\phi}{r} - \frac{W}{r_2^{\,2}}\,\frac{r_2 - r_1}{r_1} - \frac{U'}{rr_1}\right.$$

$$- \frac{V\cos\phi}{rr_1}\,\frac{2r_2 - r_1}{r_2} + \nu W^{\bullet\bullet} - \nu\,W^{\bullet}\,\frac{r_1^{\bullet}}{r_1}$$

$$\left. - \nu\,\frac{V^{\bullet}}{r_1} + \nu\,\frac{V\,r_1^{\bullet}}{r_1^{\,2}}\right] + (1+\nu)\,\alpha\,K\,T_{11} \qquad \text{A.17.6}$$

$$M_{s\theta} = K \frac{1 - \nu}{2} \left[\frac{2W'^{\bullet}}{r} - \frac{2W' \cos\phi}{r^2} - \frac{U^{\bullet}}{r_2} \frac{2r_1 - r_2}{r_2} \right.$$

$$\left. + \frac{U}{r_2{}^2} \frac{2r_1 - r_2}{r_1} \cot\phi - \frac{V'}{rr_1} \right] \qquad \text{A.17.7}$$

$$M_{\theta s} = K \frac{1 - \nu}{2} \left[\frac{2W'^{\bullet}}{r} - \frac{2W'}{r^2} \cos\phi - \frac{U^{\bullet}}{r_2} \right.$$

$$\left. + \frac{U \cot\phi}{r_2{}^2} - \frac{V'}{rr_1} \frac{2r_2 - r_1}{r_2} \right] \qquad \text{A.17.8}$$

Where the extensional rigidity, D, is defined as

$$D = \frac{Et}{1 - \nu^2} \qquad \text{A.18.1}$$

and the flexural rigidity, K, is defined as

$$K = \frac{Et^3}{12(1 - \nu^2)} \qquad \text{A.18.2}$$

The temperature terms T_{ok} and $T1_k$ $(k = 1, 2)$ are defined as follows:

$$T_{ok} = \frac{1}{t} \int_{-t/2}^{t/2} Tdz + \frac{1}{tr_k} \int_{-t/2}^{t/2} Tzdz \qquad \text{A.19.1}$$

$$T1_k = \frac{12}{t^3} \int_{-t/2}^{t/2} Tzdz + \frac{12}{t^3 r_k} \int_{-t/2}^{t/2} Tz^2dz \qquad \text{A.19.2}$$

If a linear variation of the temperature T through the thickness is assumed, Eqs. A.19 can be integrated by parts to yield

$$T_{ok} = \{\frac{T^o + T^i}{2} + (T^o - T^i)\ \frac{t}{12r_k}\} \qquad \text{A.20.1}$$

$$T_{1_k} = \{\frac{T^o - T^i}{t} + \frac{T^o + T^i}{2r_k}\} \qquad \text{A.20.2}$$

in which T^o and T^i are the temperature measured at the outer and inner face of the shell respectively. If the term t/r_k is neglected when compared with unity, the subscript k disappears from Eqs. A.20 and T_o, T_1 can be defined as the average temperature measured on the middle surface and the temperature gradient respectively.

By substituting Eq. A.6 into Eqs. A.17 and changing the subscript s to ϕ, Eqs. A.17 reduce to the elastic law in Reference 10, pg. 322.

Eqs. A.5 and A.17 are the governing equations for a shell of revolution. The sixth equation of equilibrium (Eq. A.5.6) is identically satisfied if $r_1 = r_2$. Therefore, the five remaining equations of equilibrium (Eqs. A.5) and the eight equations of the elastic law (Eqs. A.17) are 13 equations in 13 unknowns (the three displacement components and ten stress resultants). Theoretically, one can solve for the stresses and displacements at any point in the shell using these equations.

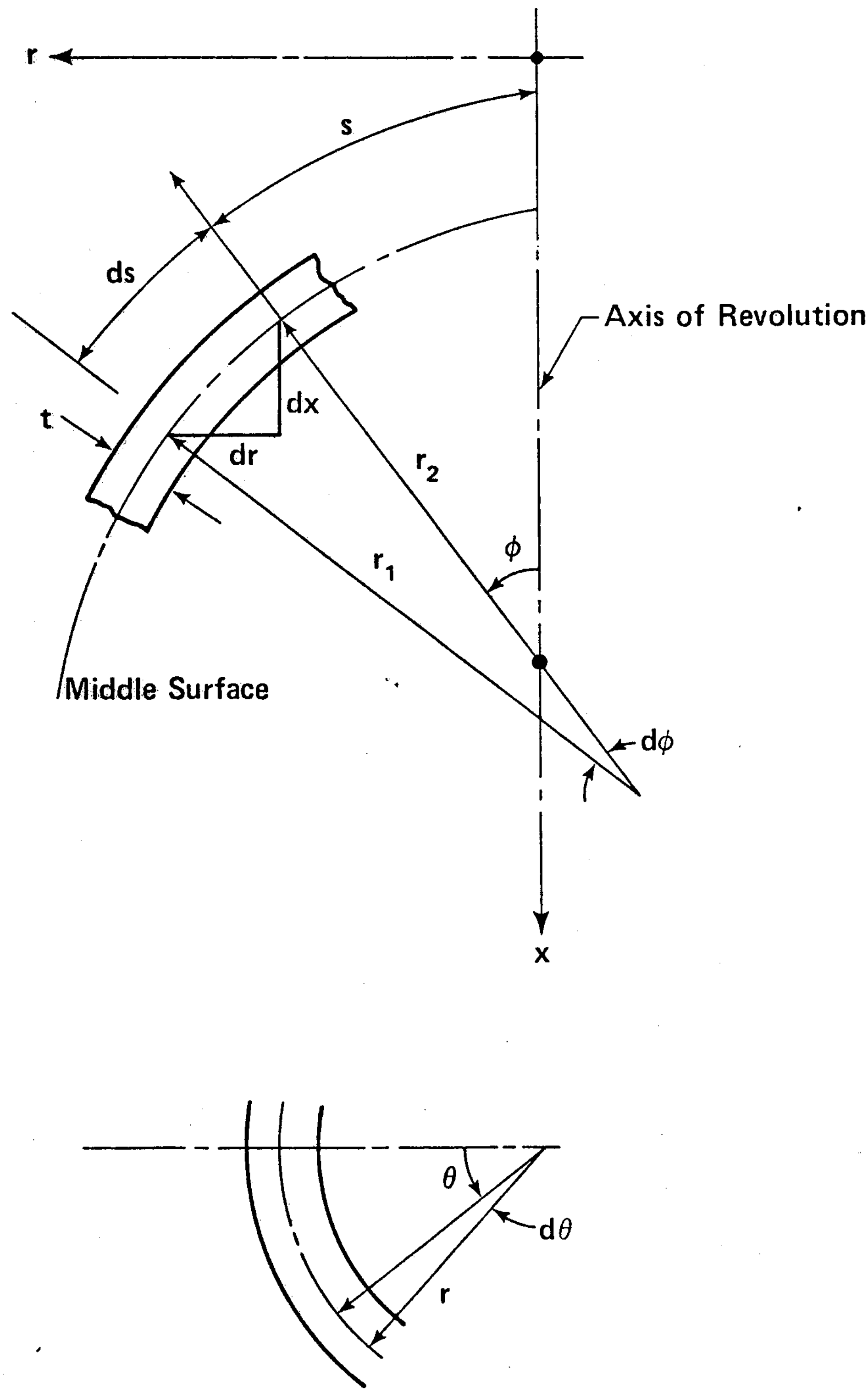


Fig. A.1 Meridian and Parallel Circle of a Shell of Revolution

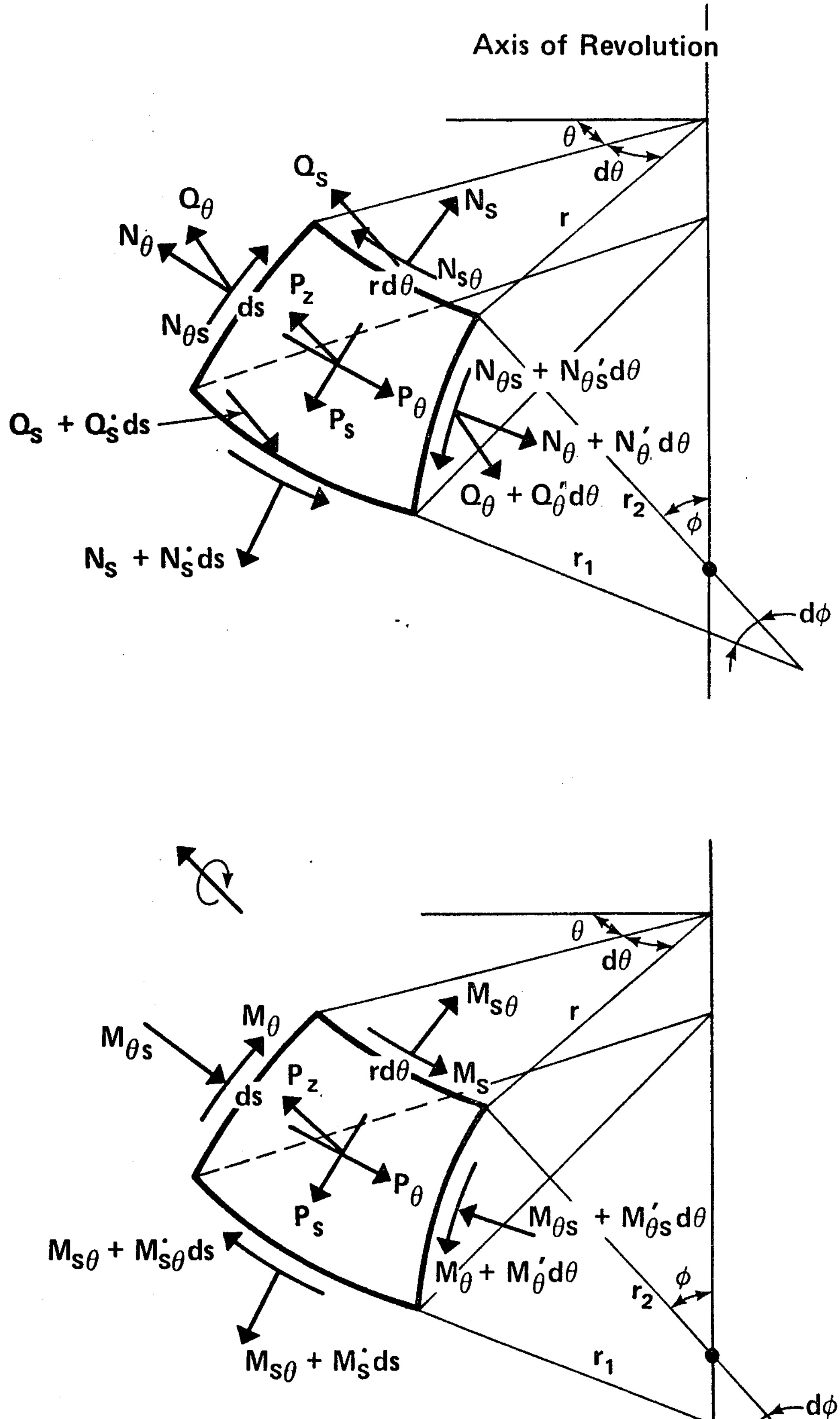


Fig. A.2 Shell Element Stress Resultants and Load Components

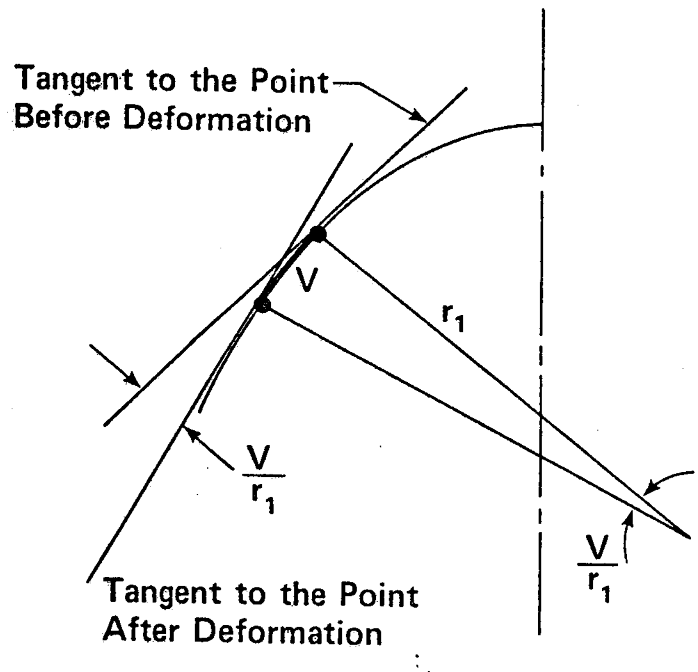


Fig. A.3.1 Meridional Rotation Due to Displacement V

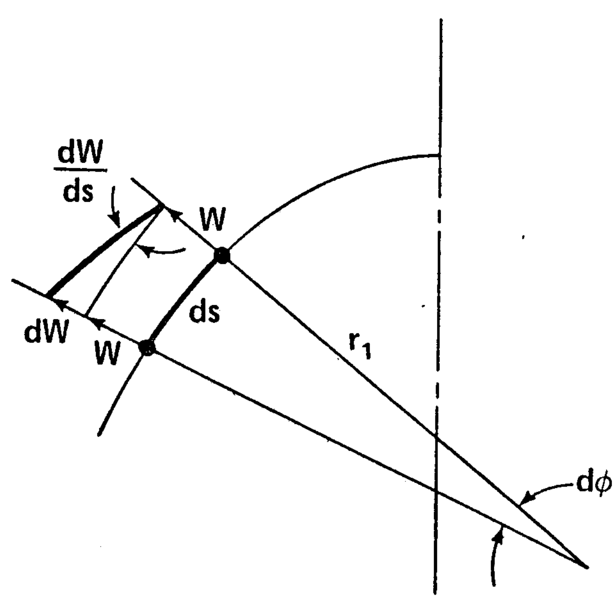


Fig. A.3.2 Meridional Rotation Due to Displacement W

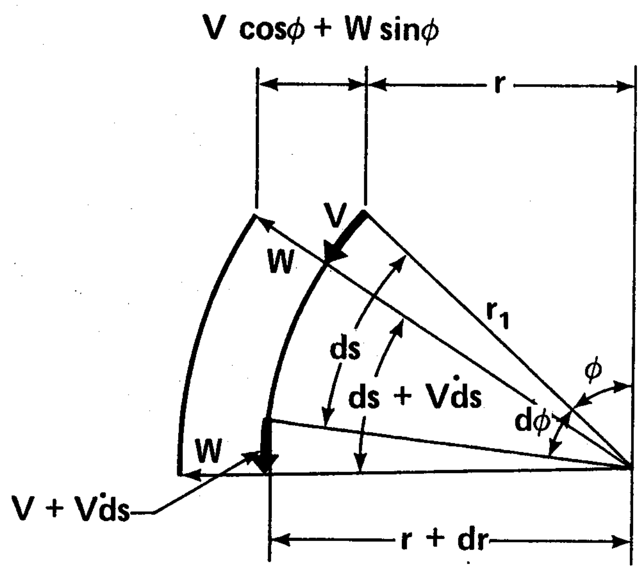


Fig. A.4.I Meridian of a Shell Before and After Deformation

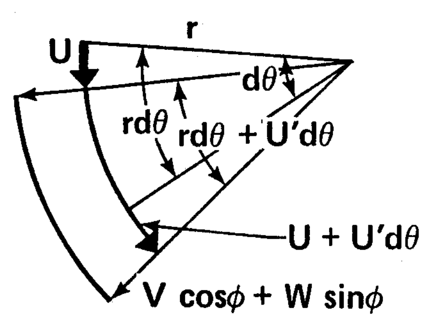


Fig. A.4.2 Parallel Circle Before and After Deformation

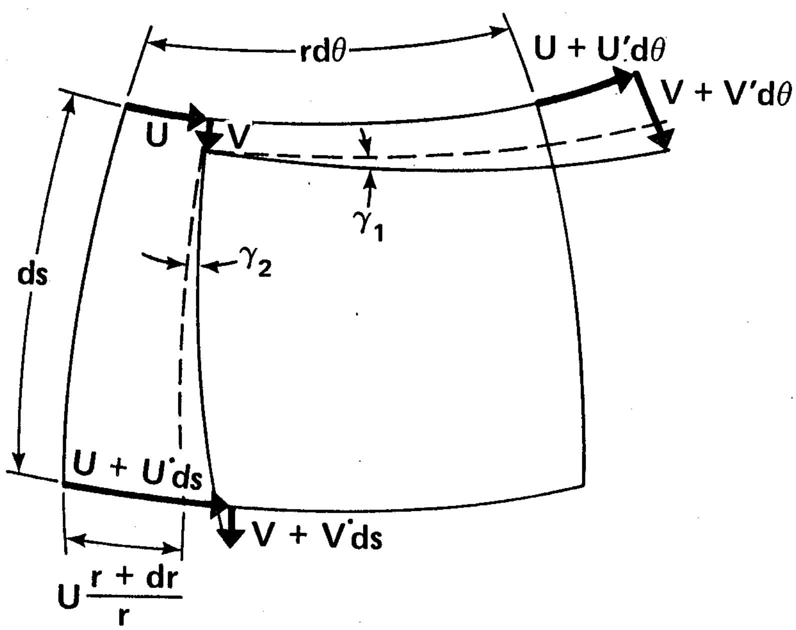
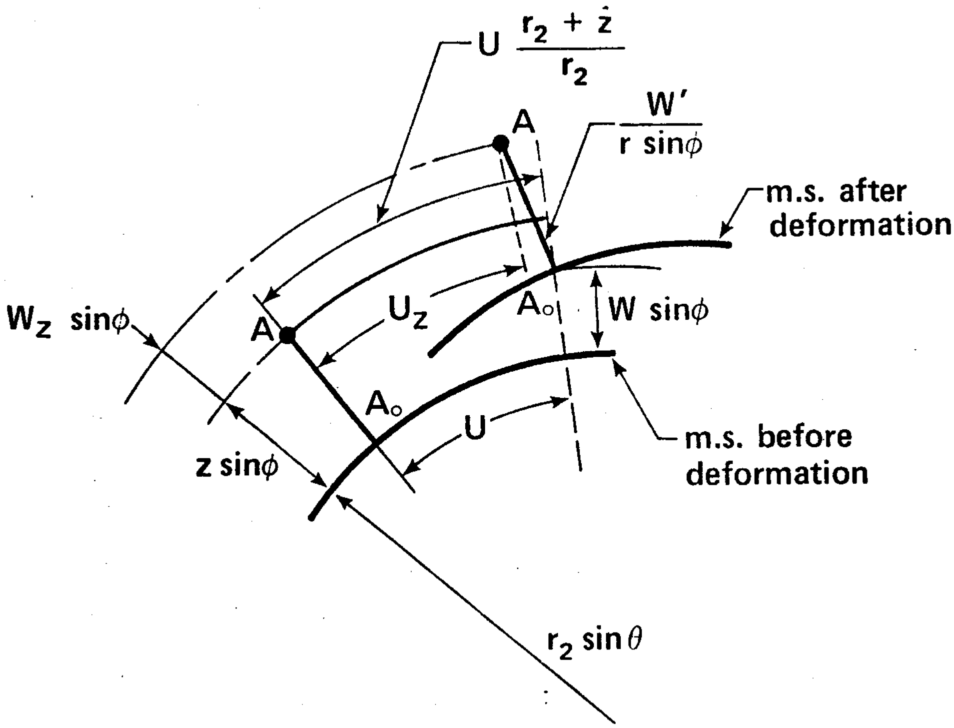
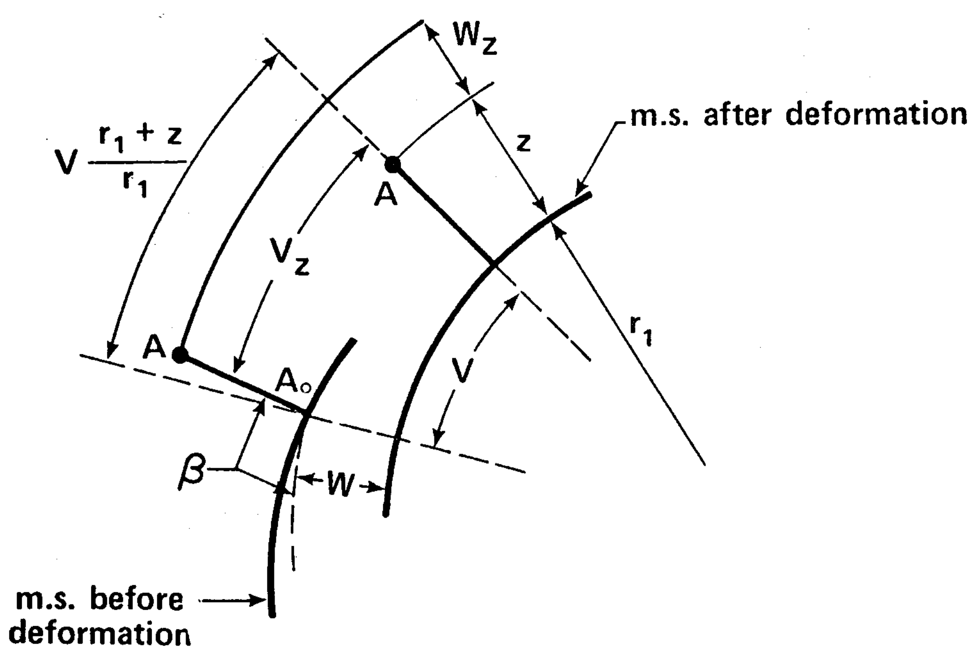


Fig. A.4.3 Change of the Right Angle Between Line Elements After Deformation



(a) Plane of Parallel Circle



(b) Meridian

Fig. A.5 Relation Between Displacement of Two Points on Line Normal to the Middle Surface

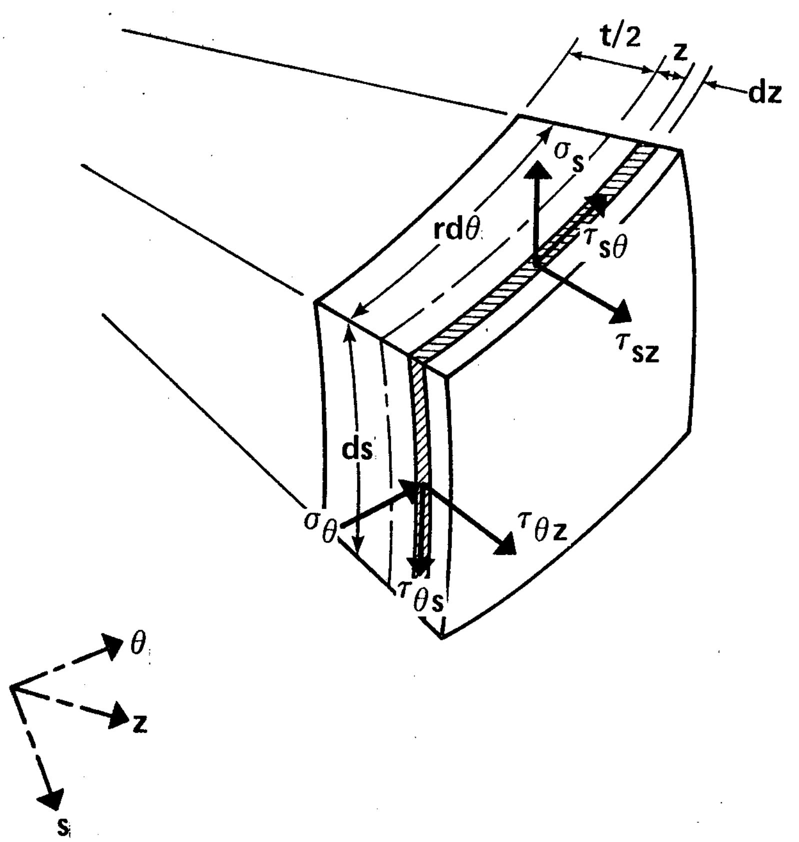


Fig. A.6 Stresses Acting on a Shell Element

APPENDIX B

USER'S MANUAL FOR PROGRAM SASHELL

Program SASHELL computes stress resultants and displacements for axisymmetric branched segmented shell structures due to their own weight, external applied loads and differential temperature variation (along the meridian or circumference).

In the present stage of development, the program is capable of analysing five types of shells of revolution of variable thickness. These are cylinders, circular plates, spheres, cones and hyperboloids of revolution. Loadings may be symmetric or non-axisymmetric with respect to the collatitude coordinate and may vary along the meridian.

The analysis procedure is based on the theory presented in this thesis and the program logic flow outlined in Sect. 4.2. A complete listing of the program is given in Appendix C. Input of the problems discussed in Chapter 4 and output of the pinched cylinder for a concentrated line load are given in Appendix D.

The input to SASHELL consists of several types of input cards. Certain card types may be repeated as required.

A typical explanation of a card type consists of the card type, a descriptive name indicating the nature of the data being entered and the format for the data on that card. This is followed by a symbolic line of input which, in turn, is followed by definitions of the input variables.

Limitations on SASHELL, due to the dimensions of the arrays in the program, are outlined following the explanation of the input cards.

TYPE 1: TITLE CARD (Format 10A8)

80
AN IDENTIFIER STRING

One card which contains any title for the problem

TYPE 2: ANALYSIS CONTROL CARD (Format 4I4,F7.0)

4	8	12	16	23
IPRINT	NP	NPCR	LDC	BETA

IPRINT : Print control parameter

If IPRINT = 0, the output will contain an echo check of the completed data. The loadings will be expanded in Fourier series, if required for the analysis, and the coefficients will be printed out.

If IPRINT = 1, the output will contain the echo check of the input data and the final results.

If IPRINT = 2, the output will contain the echo check of the input data, the results of the analysis for each harmonic and the superimposed final results.

If IPRINT = 3, full output including intermediate values will be printed out. (Used for checking purposes only.)

NP : Number of points along the element meridian for which the Runge-Kutta integration process is used. NP should not be specified less than 21 (see Limitations).

NPCR : Number of the circumferential points at which the final results are required.
If the loads are symmetric or antisymmetric with respect to the meridian passing through $\theta = 0$, NPCR is the number of points along half the circumference $(0, \pi)$. If the loads vary randomly in the circumferential direction, NPCR is the number of points along the full circumference of the element $(0, 2\pi)$.
If the loads are constants in the circumferential direction (symmetric), NPCR is equal to one.

LDC : Dead load control parameter.
If LDC = 0, dead load is excluded from the analysis.
If LDC = 1, dead load is evaluated and superimposed on the external applied loadings.

BETA : Maximum element length coefficient. BETA should not exceed 25 (see Limitations).

<u>TYPE 3: STRUCTURE DATA CARD</u> (Format 2I3,4F12.0)

3	6	18	30	42	54
NE	NJ	EG	PUG	GAMG	TKG

NE : Number of elements

NJ : Number of junctions between elements (nodes).

EG : Global modulus of elasticity.

PUG : Global Poisson's ratio.

GAMG : Global specific weight.

TKG : Global coefficient of thermal expansion

NOTE: If the structure consists of elements of different materials, the global properties are to be omitted and the structural data card specifies the number of elements and nodes only.

TYPE 4: NODAL DATA CARDS (format I4,2F10.0,4I4)

One card is required for each node.

4	14	24	28	32	36	40
I	XCOOR(I)	RCOOR(I)	IDF(I,1)	IDF(I,2)	IDF(I,3)	IDF(I,4)

I : Node number

XCOOR(I) : Global X coordinate of node I along the axis of revolution directed downward from the top of the structure.

RCOOR(I) : Radius of the parallel circle passing through node I. RCOOR(I) should not be specified as zero (see Limitations).

IDF(I,J) : Identification of the jth degree of freedom at node I. J = 1,4 for the rotation of the meridian (β), the radial displacement component (W), the meridional displacement component (V) and the circumferential displacement component (U), respectively. When

IDF(I,J) = 0, the corresponding degree of freedom is not restrained.

When IDF(I,J) = 1, the corresponding degree of freedom is restrained.

TYPE 5: ELEMENT DATA CARDS

Two cards are required for each element.

First card   Format (5I6,4F10.0)

6	12	18	24	30	40	50	60	70
I	IT(I)	NC(I,1)	NC(I,2)	NIP(I)	TH(I,1)	TH(I,2)	EC(I,1)	EC(I,2)

I : Element number

IT(I) : Element type

If IT(I) = 1, element I is a cylinder

If IT(I) = 2, element I is a cone or a circular plate.

If IT(I) = 3, element I is a sphere of which $r_1 = r_2$.

If IT(I) = 4, element I is a sphere of which $r_1 \neq r_2$.

If IT(I) = 5, element I is a hyperboloid of revolution.

NC(I,1) : Node number at the top of element I.

NC(I,2) : Node number at the bottom of element I.

NIP(I) : Integer to indicate the number of intermediate points in the element at which the final results are not required. The number of equally spaced points at which the final results will be printed out are

$\frac{NP-1}{NIP(I)} + 1$. A value of NP-1 will be assigned to NIP(I) when input is zero.

TH(I,1) : Element thickness at the top.

TH(I,2) : Element thickness at the bottom.

EC(I,1) : Eccentricity of the top node from the middle surface of the element at the top.

EC(I,2) : Eccentricity of the bottom node from the middle surface of the element at the bottom.

NOTE: The eccentricity is defined such that the radius of the parallel circle passes through the middle surface of the element is

$$r_{midsurface} = r_{node} - EC(I,J)$$

Thus, EC(I,J) is positive when directed inward from the node to the middle surface of the element.

Second card (Format 7F10.0)

10	20	30	40	50	60	70
HPCN(1,I)	HPCN(2,I)	HPCN(3,I)	GAMA(I)	E(I)	PU(I)	TCOEF(I)

HPCN(1,I) : Radius of curvature of the meridian for spherical element of type 4 (i.e., $r_1 \neq r_2$), or throat radius of a hyperboloid element (type 5). For elements other than of type 4 and type 5, HPCN(1,I) is 0.0.

HPCN(2,I) : Angle in degrees measured from the axis of revolution to the top edge of the spherical element of type 4, or hyperboloid constant in which the ratio $\frac{HPCN(2,I)}{HPCN(1,I)}$

equals to the slope of the asymptotes of the hyperbola. For elements other than of type 4 and 5, HPCN(2,I) is 0.0.

HPCH(3,I) : Angle in degrees measured from the axis of revolution to the top edge of the spherical element of type 4, or global X coordinate of the throat of the hyperboloid element. HPCH(3,I) is equal to 0.0 for elements other than of type 4 and 5.

GAMA(I) : Element specific weight. Specified if different from the global specific weight, otherwise GAMA(I) = 0.0 or blank.

E(I) : Element modulus of elasticity. Specified if different from the global modulus of elasticity, otherwise E(I) = 0.0 or blank.

PU(I) : Element Poisson's ratio. Specified if different from the global Poisson's ratio, otherwise PU(I) = 0.0 or blank.

TCOEF(I) : Element coefficient of thermal expansion. Specified if different from the global coefficient otherwise TCOEF(I) = 0.0 or blank.

<u>NOTE</u>: For a segmented structure which does not include spherical or hyperboloid elements and for which the other elements of which the structure consists have the same material properties, this card is a blank card.

TYPE 6: LOADING SPECIFICATION CARD (Format (6I5))

5	10	15	20	25	30
NEL	NJL	NHL	NTL	NHPL	NHIN

NEL : Number of externally loaded elements in the structure.

NJL : Number of loaded nodes in the structure.

NHL : Maximum number of harmonics required for the analysis including the zero harmonic (i.e., NHL = maximum harmonic number +1).

NTL : Loading type character.

If NTL = 0, loading is symmetric and the analysis is required for the zero harmonic only.

If NTL = 1, loading is non-axisymmetric in the circumferential direction, input is provided at a number of discrete points along the circumference of the shell, and the analysis is required for the cosine coefficients of Fourier series only (i.e., loading is symmetric with respect to a meridian passes through θ = constant).

If NTL = 2, loading is non-axisymmetric in the circumferential direction, input is provided at a number of discrete points along the circumference of the shell, and the analysis is required for both are cosine and sine coefficients of Fourier series.

If NTL = 3, loading is nonaxisymmetric along the along the circumferential direction and input directly as cosine coefficients of Fourier expansion only.

NHPL : Number of points along the circumference of the shell at which the loads values are described. These points are defined such that the circumferential coordinate is $\frac{2\pi i}{\text{NHPL}}$ where i = 0, 1, 2, ..., NHPL - 1.

NHIN : Integer which defines the increment in the harmonics, starting from the zero harmonic, to be specified when the analysis is required for a harmonic number (0, NHIN, 2 NHIN, ..., NHL - 1). The program sets NHIN = 1 when it is specified as zero or blank.

<u>NOTE</u>: If the load is described at NHPL points, NHL can be specified as $\frac{\text{NHPL}}{2}$ and NHIN can be set equal to one. When IPRINT = 0, the Fourier coefficients of the input load are obtained with the echo check of the input data. Then, the user may decide, upon examining these coefficients, on the final values of NHL and NHIN (see Chapter 5, Sect. 5.2.3).

<u>TYPE 7: ELEMENT LOADING CONTROL INPUT CARDS</u> (Format 6I5)

One card for each loaded element

5	10 15 20 25 30
LL(I)	ILOAD (LL(I), K), K = 1,5

I : Integer takes the value of 1 to NEL

LL(I) : Number of the loaded element.

ILOAD(LL(I),K)

: Identifier for loading type on the element LL(I) in the order:

ILOAD(*,1) for loading in the direction tangent to the meridian (s).

ILOAD(*,2) for loading in the direction tangent to the parallel circle (θ).

ILOAD(*,3) for loading in the direction perpendicular to the tangent to the meridian (z).

ILOAD(*,4) for temperature at the shell exterior face.

ILOAD(*,5) for temperature at the shell interior face.

If ILOAD(*,K) = 0, no load of type K is applied.

If ILOAD(*,K) = 1, the applied load of type K is constant along the meridian and to be specified at one end of the element only.

If ILOAD(*,K) = 2, the applied load of type K varies linearly along the meridian and is to be specified at the two ends of the element.

If ILOAD(*,K) = 3, the applied load of type K varies as a second degree function along the meridian and is to be specified at the two ends of the element.

NOTE: Element loads are input in the element local coordinates.

TYPE 8: ELEMENT LOADING CARDS

This type of card depends upon NTL and is classified as TYPE 8-A, TYPE 8-B and TYPE 8-C for NTL equal to 0, 1 or 2, and 3 respectively.

TYPE 8-A: ELEMENT SYMMETRICAL LOADING CARDS (Format 2F10.0)

This type is required if NTL is equal to zero. Number of cards required, for each loaded element, is equal to, and input in the order consistent with, each non-zero term in the corresponding row in ILOAD array.

10	20
ACEL(K1)	ACEL(K2)

ACEL(K1) : Magnitude of the load at the element top.

ACEL(K2) : Magnitude of the load at the element bottom.

TYPE 8-B ELEMENT ASYMMETRIC TABULATED LOADING CARDS (Format 8F10.0)

This type is required if NTL is equal to 1 or 2. Number of cards required, for each loaded element, is equal to $\frac{NHPL}{8}$ for, and input in the order consistent with, each non-zero term in the corresponding row in ILOAD array.

10	20	30	40	50	60	70	80
W(I), I = 1, NHPL							

W(I) : Magnitude of the load, at the points defined by $\frac{2\pi i}{NHPL}$ (i = 0, 1, 2, ..., NHPL-1) along the circumference of the element, and for which Fourier expansion is required.

TYPE 8-C ELEMENT LOADING FOURIER COEFFICIENTS CARDS

(Format 8F10.0)

This type is required if NTL is equal to 4. Number of cards required, for each loaded element, is equal to $\frac{NHL}{8}$ for, and input in the order consistent with, each non-zero term in the corresponding row in ILOAD array.

10	20	30	40	50	60	70	80
AL(K)	, K = 1, NHL						

AL(K) : Cosine coefficients of Fourier expansion for Harmonics 0, 1, 2, ..., NHL-1.

TYPE 9: NODAL LOADING CONTROL INPUT CARDS (Format I5)

One card for each loaded node (NJL cards)

5
J

J : Number of loaded node.

TYPE 10: NODAL LOADING CARDS

This type of card depends upon NTL and is classified as TYPE 10-A, TYPE 10-B and TYPE 10-C for NTL equal to 0, 1 or 2, and 3, respectively.

TYPE 10-A: NODAL SYMMETRICAL LOADING CARDS (Format (4F10.0))

This type is required if NTL is equal to zero. One card is required for each loaded node.

10	20	30	40
ANJL (II,1)	,	II = 1,4	

ANJL(II,1): Magnitude of the nodal load, in the global structure coordinates, in the following order.

II = 1, meridional couple, M_s, positive when rotating in the clockwise direction.

II = 2, force normal to the tangent to meridian at the corresponding node, S_s, positive in the outward direction from the axis of revolution.

II = 3, force in the direction tangent to the meridian, N_s, positive when directed downward parallel to the axis of revolution.

II = 4, force in the direction tangent to the parallel circle, T_s, positive when directed in the anticlockwise rotating direction around the structure.

NOTE: The sign convention of the nodal loads as defined above is equivalent to the stiffness matrix sign convention as described in Sect. 3.7.

TYPE 10-B: NODAL ASYMMETRICAL TABULATED LOADING CARDS

This type is required if NTL is equal to 1 or 2 and consists of the following cards, for each loaded node.

(1) Asymmetric nodal load control input card. Required to identify which to the nodal loads, classified as in Type 10-A cards (M_s, S_s, N_s, T_s), is applied an to be input. One card (Format 4I4) is required

4	8	16	20
KLD(K) , K = 1,4			

K : Integer which takes a value of 1 to 4 and represents the nodal forces, as defined in TYPE 10-A cards, in the order M_s, S_s, N_s and T_s, respectively.

KLD(K) : If equal to zero, nodal force of the type K is not applied.
If equal to 1, nodal force of the type K is applied.

(2) Nodal loading magnitude card (Format 8F10.0)
Number of cards required is equal to $\frac{NHPL}{8}$ for, and input in the order consistent with, each non-zero term in KLD.

10	20	30	40	50	60	70	80
W(I) , I = 1, NHPL							

W(I) : Magnitude of the nodal load, at the points defined by $\frac{2\pi i}{NHPL}$ (i = 0, 1, 2, ..., NHPL - 1) along the circumference of the nodal circle, and for which Fourier expansion is required.

TYPE 10-C: NODAL LOADING FOURIER COEFFICIENTS CARDS

This type is required if NTL is equal to 3 and NJL is greater than 0. It consists of the following cards, for each loaded node:

(1) Asymmetric nodal load control input card of the same type described in TYPE 10-B.

4	8	12	20
KLD(K)	, K = 1, 4		

(2) Nodal loading magnitude card (Format 8F10.0)
Number of cards required is equal to $\frac{NHL}{8}$ for, and input in the order consistent with, each non-zero term in KLD.

10	20	30	40	50	60	70	80
AL (N),	N = 1,	NHL					

AL(N) : Cosine coefficients of FOURIER expansion for harmonics 0, 1, 2, ... NHL - 1.

LIMITATIONS

Number of elements NE	$\not> 20$
Number of nodes NJ	$\not> 21$
Number of integration points NP	$\not> 51$
Number of circumferential points NPCR	$\not> 11$
Number of harmonics NHL	$\not> 20$
Number of circumferential points at which loadings is specified NHPL	$\not> 40$
Full band width NHB	$\not> 80$
Total number of Fourier coefficients in the problem for elements loading, associated with either cosine or sine factor and defined by NHL x NEL	$\not> 200$
Total number of Fourier coefficients in the problem for nodal loadings, associated with both cosine and sine factor and defined by 8 x NHL x NJL	$\not> 200$

Number of points in a non-axisymmetrically loaded structure at which the final results are required, and which can be calculated as

$$\sum_{i=1}^{NE} NS_i \times \left\{ \frac{(NP-1)}{NIP_i} + 1 \right\} \not> 200$$

where NS_i = number of segments into which element i is subdivided and can be obtained by $\frac{L}{BETA} \sqrt[4]{3(1 - \nu^2)/r^2 t^2}$ in which, L, is the element length, r, is the element radius, t, is the element thickness and ν is Poisson's ratio.

The above limitations are due to the dimension statements in the program SASHELL as listed in Appendix C. The dimension of the corresponding arrays may be modified to change these limitations.

Limitations on the theory employed in SASHELL are as follows (see Chapter 5).

1) RCOOR should not be specified as zero.

2) The length factor coefficient BETA should not be specified greater than 25.

3) The number of points of integration NP should not be less than 21 when BETA is specified less than 20, and not less than 31 when BETA is greater than 20 and less than 25.

APPENDIX C

PROGRAM LISTING

```
C     ************************************
C     *         PROGRAM SASHELL          *
C     ************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CO1/IPRINT,NP,BETA,NDOF,NF,NP1,NDIM,NDIM1
      COMMON/CO2/NE,NJ,NT,LT,LDC,NHL,NEL,NJL,NHIN,NPCR
      COMMON/CO3/XCR,RCR,TH,EC,E,PU,TCOEF,GAMA,DSP,HPCN
      COMMON/CO4/IT,NC,NDIV,IDF,ILOAD,IDCO
      COMMON/CO5/ACEL,BCEL,APL,BPL,PS,DCN,DSN
      DIMENSION XCR(21),RCR(21),DSP(21,4),EC(20,2),TH(20,2),
     * GAMA(20),TCOEF(20),E(20),PU(20),SC(20,6),GEO(80,6),
     * PE(6,2),SS(200,80),RHS(200,2),ANJL(100,50),INJL(21),
     * ACEL(200),APL(200,20),BPL(200,20),BCEL(200),DCN(11),
     * DSN(11),IDF(21,4),NC(20,2),NCON(20,2),NS(20),NDIV(20),
     * II(20),IDCO(20,10,2),MM(6),ILOAD(20,5),IDL(5),
     * PS(10,2),SM(6,6),DSRI1(10,2),HPCN(3,20),
     * DSPT(4,11,200),PSRT(4,11,200),SSRT(6,11,200)
      PI=3.141592654
      CALL READIN(IPRINT,NP,NPCR,BETA,NE,NJ,NT,LDC,NHL,NHIN,
     *  NEL,NJL,XCR,RCR,TH,EC,IT,NC,NDIV,E,PU,TCOEF,
     * GAMA,IDF,ILOAD,ACEL,BCEL,INJL,ANJL,HPCN)
      CALL SEGMNT(MDIF,NJT,NS,GEO,NCON)
C     -------------------------------------------------------
C     IDENTIFY TYPE OF LOADING
C     ---------------------------------------------------------
C     NDOF IS THE NODAL DEGREES OF FREDOM
      NDOF=4
      NF=8
      LT=1
      IF(NT.EQ.0.OR.IPRINT.GT.2) GO TO 37
      MAXE=0
      NP1=NP-1
      DO 35 I=1,NE
   35 MAXE=MAXE+NS(I)*(NP1/NDIV(I)+1)
      DO 36 J=1,MAXE
      DO 36 K=1,NPCR
      DO 36 L=1,6
      IF(L.GT.4) GO TO 38
      DSPT(L,K,J)=0.0
      PSRT(L,K,J)=0.0
   38 SSRT(L,K,J)=0.0
   36 CONTINUE
   37 CONTINUE
      NPCR1=NPCR-1
C
C     ANALYSIS FOR ALL HARMONICS
      N=1
      NH=0
  500 CONTINUE
      GO TO 502
  501 CONTINUE
      N=N+NHIN
      NH=N-1
  502 CONTINUE
      IF(NT.EQ.2.AND.N.GT.1) LT=2
      NP1=NP+LT
      IF(IPRINT.GT.2.OR.NHL.EQ.1) GO TO 43
      IF(NH.EQ.0) GO TO 42
      AT=LT
      THETA=AT*PI/NPCR1
      DO 40 J=1,NPCR
      THETAN=THETA*NH*(J-1)
      DCN(J)=DCOS(THETAN)
   40 DSN(J)=DSIN(THETAN)
      GO TO 43
   42 DO 41 J=1,NPCR
      DCN(J)=1.0
   41 DSN(J)=0.0
   43 CONTINUE
      NHB=NDOF*(MDIF+1)
      NFB=2*NHB-1
      NEQ=NDOF*NJT
      WRITE(6,1020)NH,NEQ,NFB
 1020 FORMAT(//'***********************************'/
     *       '*     HARMONIC NUMBER      =',I5,'     */
     *       '*     NUMBER OF EQUATIONS  =',I5,'     */
     *       '*     FULL BAND WIDTH      =',I5,'     */
     *       '***********************************')
      DO 90 J=1,NFB
      DO 90 I=1,NEQ
   90 SS(I,J)=0.0
      DO 191 L=1,LT
      DO 191 J=1,NEQ
  191 RHS(J,L)=0.0
      DO 80 L=1,LT
      DO 80 J=1,NF1
   80 DSRI1(J,L)=0.0
      DO 81 J=1,LT
      JJ=NF+J
   81 DSRI1(JJ,J)=1.0
      DO 83 L=1,LT
      DO 83 J=1,10
   83 PS(J,L)=0.0
C     ------------------------------------------------
C     ADD NODAL LOADS CONTRIBUTION TO RHS
C     -------------------------------------------------
      IF(NJL.EQ.0) GO TO 201
      LJT=4
      IF(NT.EQ.1) LJT=8
      DO 200 I=1,NE
      IK=NC(I,1)
      IF(INJL(IK).EQ.0) GO TO 200
      L1=IDCO(I,1,1)
      LA=(L1-1)*NDOF
      DO 62 J=1,LT
      JT=(J-1)*4
      DO 62 L=1,NDOF
      LB=LA+L
      LJ1=(IK-1)*LJT+L+JT
   62 RHS(LB,J)=ANJL(LJ1,N)
  200 CONTINUE
      DO 65 L=1,LT
      WRITE(6,2001) L
   65 WRITE(6,2000) (RHS(J,L),J=1,NEQ)
 2001 FORMAT(/'NODAL LOADS ASSEMBELED IN RHS    LT=',I10/)
 2000 FORMAT(4E15.6)
  201 CONTINUE
C
      CALL STIFAN(NH,NHB,NS,GEO,SM,PE,SS,RHS,DSRI1)
C
```

```
C     -------------------------------------------------------
C     MODIFY STIFFNESS MATRIX AND RIGHT HAND SIDE VECTOR
C     FOR PRESPECIFIED BOUNDARY CONDITIONS
C     ----------------------------------------------------------
      CALL BOUNDC(NE,NJ,NDOF,LT,NHB,NEQ,NC,NCON,IDF,DSP,SS,RHS)
C
      CALL SOLVER(SS,RHS,LT,NEQ,NFB)
      WRITE(6,1400)
      IF(LT.EQ.2) WRITE(6,1500)
      DO 82 L=1,LT
      IF(L.EQ.2) WRITE(6,1600)
   82 WRITE(6,1700) (RHS(KI,L),KI=1,NEQ)
 1400 FORMAT(//'**** DISPLACEMENT AT NODES'//)
 1500 FORMAT('****SYMMETRIC PART'/)
 1600 FORMAT(/'ASSYMMETRIC PART'/)
 1700 FORMAT(5X,'R',14X,'W',15X,'V',15X,'U'/(4E15.6))
C     ------------------------------------------------------
C     EVALUATE DISPLACEMENT AND STRESS RESULTANTS AT
C     INTERMEDIATE POINTS AND SUPERIMPOSE THE RESULTS
C     ------------------------------------------------------
      LIMIT=0
      CALL SRADC2(LIMIT,NH,NS,GEO,DS,SM,PE,SS,RHS,DSRI1,DSPT,
     *  PSPT,SSPT)
      IF(LIMIT.NHL) GO TO 501
      IF(IPRINT.GT.2.OR.NHL.EQ.1) GO TO 71
C
      K0=-1
      L1=0
      L2=0
      DO 70 I=1,NZ
      NXI=NDIV(I)
      NSEG=NS(I)
      NXT=NP1/NXI+1
      K0=K0+1
      DO 70 K=1,NSEG
      K0=K0+1
      K1=K0+1
      X0=GEO(K0,1)
      X1=GEO(K1,1)
      XJ=(X1-X0)*NXI/NP1
      WRITE(6,2220) I,K
      L1=L2+1
      L2=L1+NXT-1
      LL=0
      DO 70 L=L1,L2
      XX=X0+LL*XJ
      LL=LL+1
      WRITE(6,2221) L,XX
      WRITE(6,2201) ((DSPT(J,M,L),J=1,4),M=1,NPCR)
      WRITE(6,2202) ((PSPT(J,M,L),J=1,4),M=1,NPCR)
      WRITE(6,2203) ((SSPT(J,M,L),J=1,6),M=1,NPCR)
   70 CONTINUE
   71 CONTINUE
 2220 FORMAT(//'**********************************************'/
     *  'ELEMENT',I10,10X,'SEGMENT',I10/
     *  '*********************************************'/)
 2221 FORMAT(/'POINT NUMBER',I10,5X,'AT X =',F10.4)
 2201 FORMAT(/'DISPLACEMENT'/(4E12.4))
 2202 FORMAT(/'PRIMARY STRESS RESULTANTS'/(4E12.4))
 2203 FORMAT('SECONDARY STRESS RESULTANTS'/(6E12.4))
      STOP
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE SEGMNT(NDIP,NJT,NS,GEO,NCON)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CO1/IPRINT,NP,BETA,NDOF,NZ,NP1,NDIM,NTIM1
      COMMON/CO2/NE,NJ,NT,LT,LDC,NHL,NEL,NJL,NHIN,NPCR
      COMMON/CO3/XCR,RCR,TH,EC,E,PU,TCOEF,GAMA,DSP,HPCN
      COMMON/CO4/IT,NC,NDIV,IDF,ILOAD,IDCO
      COMMON/CO5/ACEL,BCEL,APL,BPL,PS
      COMMON/HYP/AH,BH,XH
      DIMENSION XCR(21),RCR(21),DSP(21,4),EC(20,2),TH(20,2),
     *  GAMA(20),TCOEF(20),E(20),PU(20),SC(20,6),GEO(60,6),
     *  PS(10,2),SM(8,8),HPCN(3,20),
     *  ACEL(200),APL(200,20),BPL(200,20),BCEL(200),
     *  IDF(21,4),NC(20,2),NCON(20,2),NS(20),NDIV(20),IT(20),
     *  ITA(20),IDCO(20,10,2),MM(8),ILOAD(20,5),IDL(5)
      PI=3.141592654
C
      K0=0
      DO 10 I=1,NE
      K1=NC(I,1)
      K2=NC(I,2)
      RT=RCR(K1)-EC(I,1)
      RB=RCR(K2)-EC(I,2)
      THI=TH(I,1)
      THF=TH(I,2)
      AH=HPCN(1,I)
      BH=HPCN(2,I)
      XH=HPCN(3,I)
C
      CALL SEGEOM   (IT(I),NSEG,RT,RB,XCR(K1),XCR(K2),THI,
     1          THF,BETA,PU(I),SC)
C
      NS(I)=NSEG
      NS1=NSEG+1
      DO 10 K=1,NS1
      K0=K0+1
      DO 10 J=1,6
      GEO(K0,J)=SC(K,J)
   10 CONTINUE
C     ----------------------------------------------------------
C     MODIFY NODAL NUMBERING SYSTEM TO INCLUD THE NEW NODES
C     ----------------------------------------------------------
      DO 19 I=1,NE
      DO 19 J=1,2
   19 NCON(I,J)=0.0
      MAX=0
      NEE=NE
      DO 20 I=1,NE
      I1=I+1
      NSEG=NS(I)
      NSEG1=NSEG-1
      IF(NCON(I,1).GT.0) GO TO 55
      MAX=MAX+1
      NCON(I,1)=MAX
      IF(I.EQ.NEE) GO TO 55
```

```
      DO 15 IN=I1,NZ
      DO 15 J=1,2
      IF(NC(IN,J).NE.NC(I,1)) GO TO 15
      NCON(IN,J)=NCON(I,J)
   15 CONTINUE
   55 CONTINUE
      IDCO(I,1,1)=NCON(I,1)
      IF(NSEG.EQ.1) GO TO 16
      DO 16 K=1,NSEG1
      MAX=MAX+1
      K1=K+1
      IDCO(I,K,2)=MAX
      IDCO(I,K1,1)=MAX
   16 CONTINUE
      IF(NCON(I,2).GT.0) GO TO 57
      MAX=MAX+1
      NCON(I,2)=MAX
      IF(I.EQ.NEE) GO TO 57
      DO 17 IN=I1,NZ
      DO 17 J=1,2
      IF(NC(IN,J).NE.NC(I,2)) GO TO 17
      NCON(IN,J)=NCON(I,2)
   17 CONTINUE
   57 CONTINUE
      IDCO(I,NSEG,2)=NCON(I,2)
   20 CONTINUE
C
      WRITE(6,1000)
      DO 11 I=1,NE
      NSEG=NS(I)
      DO 11 K=1,NSEG
      WRITE(6,1001) I,K,(IDCO(I,K,J),J=1,2)
   11 CONTINUE
 1000 FORMAT('****    CONNECTIVITY ARRAY    ***'//
     1 '    ELEMENT    SEGMENT    TOP NODE    BOT NODE')
 1001 FORMAT(4I10)
      WRITE(6,1002)
 1002 FORMAT(//'****    COORDINATE SYSTEM    ***'/
     * 'ELEMENT   NUMB OF SEG',
     *    17X,'Z',11X,'R',10X,'TH',9X,'PHI',10X,'S',10X,'R2'/)
      K0=0
      DO 12 I=1,NE
      NSEG=NS(I)
      WRITE(6,1003)I,NSEG
      NS1=NS(I)+1
      DO 12 K=1,NS1
      K0=K0+1
      WRITE(6,1004) (GEO(K0,J),J=1,6)
   12 CONTINUE
 1003 FORMAT(I5,6X,I5)
 1004 FORMAT(30X,6F12.3)
C     -----------------------------------------------------------
C     DETERMINE TOTAL NUMBER OF SEG. AND MAX DIFFERENCE
C     IN NODAL NUMBERS
C     -----------------------------------------------------------
      M=0
      DO 30 I=1,NZ
      NSEG=NS(I)
      DO 30 K=1,NSEG
      L=IABS(IDCO(I,K,1)-IDCO(I,K,2))
      IF(L.GT.M) M=L
   30 CONTINUE
C
      MDIF=M
      NJT=MAX
      IF(LDC.EQ.1.OR.NEL.EQ.0) GO TO 82
      NLD=(MAX+NZ)*5
      DO 80 N=1,NHL
      DO 80 LS=1,NLD
   80 APL(LS,N)=0.0
      DO 81 N=1,NHL
      DO 81 LS=1,NLD
   81 BPL(LS,N)=0.0
      CALL PLOAD(IPRINT,NE,NT,NHL,NS,ILOAD,ACEL,BCEL,
     *  APL,BPL)
   82 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE STIFAN(NH,NHS,NS,GEO,SM,FE,SS,RHS,DSPI1)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CO1/IPRINT,NP,BETA,NDOF,NF,PI,NDIM,NDIM1
      COMMON/CO2/NE,NJ,NT,LT,LDC,NHL,NEL,NCL,NHIN,NPCH
      COMMON/CO3/XCR,RCR,TH,EC,E,PU,TCOEF,GAMA,DSP,HPCN
      COMMON/CO4/IT,NC,NDIV,IDF,ILOAD,IDCO
      COMMON/CO5/ACEL,BCEL,APL,BPL,PS
      COMMON/HYP/AH,BH,XH
      DIMENSION XCR(21),RCR(21),DSP(21,4),EC(20,2),TH(20,2),
     *  GAMA(20),TCOEF(20),E(20),PU(20),SC(20,6),GEO(80,6),
     *  RK(2),THK(2),PHK(2),SK(2),RR(2),SM(8,8),YS(8,10),
     *  FE(8,2),SS(200,80),RHS(200,2),DS(8,2),DSPI1(10,2),
     *  ACEL(200),APL(200,20),BPL(200,20),BCEL(200),HPCN(3,20),
     *  IDF(21,4),NC(20,2),NCON(20,2),NS(20),NDIV(20),IT(20),
     *  PS(10,2),IDCO(20,10,2),NN(8),ILOAD(20,5),IDL(5)
     PI=3.141592654
     N=NH+1
     LOC=0
     LMS1=0
     K0=-1
     DO 200 I=1,NE
     I1=I-1
     IF(I1.EQ.0) GO TO 91
     LOC=LOC+(NS(I1)+1)*5
  91 CONTINUE
     LOC2=LOC
     DO 98 J=1,5
  98 IDL(J)=ILOAD(I,J)
     ITYP=IT(I)
     NSEG=NS(I)
     AH=HPCN(1,I)
     BH=HPCN(2,I)
     XH=HPCN(3,I)
     K0=K0+1
     DO 100 K=1,NSEG
     K0=K0+1
     K1=K0+1
     RK(1)=GEO(K0,2)
```

```
      RK(2)=GEO(K1,2)
      THK(1)=GEO(K0,3)
      THK(2)=GEO(K1,3)
      PHK(1)=GEO(K0,4)
      PHK(2)=GEO(K1,4)
      SK(1)=GEO(K0,5)
      SK(2)=GEO(K1,5)
      RP(1)=GEO(K0,6)
      RP(2)=GEO(K1,6)
      IF(NEL.EQ.0) GO TO 93
      LOC1=LOC2+1
      LOC2=LOC1+4
      LOC3=LOC2+5
      KL=0
      DO 95 L=LOC1,LOC3
      KL=KL+1
   95 PS(KL,1)=APL(L,N)
   92 CONTINUE
      IF(LT.EQ.1) GO TO 93
      KL=0
      DO 96 L=LOC1,LOC3
      KL=KL+1
   96 PS(KL,2)=BPL(L,N)
   93 CONTINUE
C
      CALL RNGKT(IPRINT,ITYP,LT,LDC,NEL,NH,NP,GAMA(I),E(I),
     *  PO(I),TCOEF(I),RK,THK,PHK,SK,IDL,PS,YS)
      CALL STIFIX(IPRINT,LT,YS,SM,FE)
C
      CALL STORE1(1,LMS1,IPRINT,LT,NF,SM,FE,DS,DSRI1)
C     ----------------------------------------------------
C     MODIFICATION FOR STIFFNESS MATRIX SIGN CONVENTION
C     ----------------------------------------------------
      LL=NDOF+2
      DO 52 J=1,NF
      DO 52 L=3,LL
   52 SM(L,J)=-SM(L,J)
      DO 53 J=1,LT
      DO 53 L=3,LL
   53 FE(L,J)=-FE(L,J)
C     ----------------------------------------------------
C     MODIFICATION FOR ECCENTRICITY
C     ----------------------------------------------------
      DO 51 L=1,2
      IF(L.EQ.1) L1=1
      IF(L.EQ.2) L1=NSEG
      IF(K.NE.L1) GO TO 51
      Z=EC(I,L)
      IF(Z.EQ.0) GO TO 51
      R1=RK(L)
      R2=RP(L)
      CALL ECCNTF(IPRINT,L,LT,NH,NP,Z,R1,R2,SM,FE,DS)
   51 CONTINUE
      IF(ITYP.EQ.1) GO TO 58
C     ----------------------------------------------------
C     TRANSFORMATION OF STIFFNESS COEF TO GLOBAL COORDINATES
C     ----------------------------------------------------
      DO 56 L=1,2
      IF(L.EQ.1) L1=1
      IF(L.EQ.2) L1=NSEG
      IF(K.NE.L1) GO TO 56
      PHIK=PHK(L)
      CALL GLTRAN(IPRINT,L,LT,NH,NP,PHIK,SM,FE,DS)
   56 CONTINUE
   58 CONTINUE
C     ----------------------------------------------------
C     ASSEMBELE SEG. STIF. MATRIX INTO MASTER STIF.MATRIX
C     ----------------------------------------------------
      CALL STORE(I,K,NDOF,NF,LT,NHB,IDCO,SM,FE,SS,RHS)
  100 CONTINUE
  200 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE STORE(I,K,NDOF,NF,LT,NHB,IDCO,SM,FE,SS,RHS)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION IDCO(20,10,2),SM(8,8),FE(8,2),SS(200,80),
     * RHS(200,2),MM(8)
      L1=IDCO(I,K,1)
      L2=IDCO(I,K,2)
      J1=(L1-1)*NDOF
      J2=(L2-1)*NDOF
      DO 60 JJ=1,NDOF
      JL=JJ+NDOF
      MM(JJ)=J1+JJ
      MM(JL)=J2+JJ
   60 CONTINUE
      DO 61 J=1,NF
      JJ=MM(J)
      DO 61 L=1,NF

      LL=MM(L)-JJ+NHB
      SS(JJ,LL)=SS(JJ,LL)+SM(J,L)
   61 CONTINUE
C     ----------------------------------------------------
C     ASSEMBLE RIGHT HAND SIDE VECTOR
C     ----------------------------------------------------
      DO 62 L=1,LT
      DO 62 J=1,NDOF
      JA=J1+J
      JB=J2+J
      JN=J+NDOF
      RHS(JA,L)=RHS(JA,L)-FE(J,L)
      RHS(JB,L)=RHS(JB,L)-FE(JN,L)
   62 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE BOUNDC(NE,NJ,NDOF,LT,NHB,NEQ,NC,NCON,IDF,
     * DSP,SS,RHS)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION NC(20,2),NCON(20,2),IDF(21,4),DSP(21,4),
     *  SS(200,80),RHS(200,2)
      NB=NHB+1
```

```
      NFB=2*NHB-1
      DO 300 I=1,NJ
      DO 320 L=1,NE
      DO 320 J=1,2
      LL=L
      JJ=J
      L1=NC(L,J)
      IF(L1.EQ.I) GO TO 302
  320 CONTINUE
  302 CONTINUE
      K1=LCON(LL,JJ)
      DO 307 K=1,NDOF
      IK=IDF(I,K)
      IF(IK-1)   307,305,305
  305 CONTINUE
      D=0.0
      KK=(K1-1)*NDOF+K
      DO 309 II=NB,NFB
      KL=KK+II-NHB
      KU=KK-II+NHB
      IF(KU.LE.0) GO TO 306
      DO 10 M=1,LT
   10 RHS(KU,M)=RHS(KU,M)-SS(KU,II)*D
      SS(KU,II)=0.0
  306 IF(KL.GT.NEQ) GO TO 309
      KI=NFB+1-II
      DO 20 M=1,LT
   20 RHS(KL,M)=RHS(KL,M)-SS(KL,KI)*D
      SS(KK,KI)=0.0
  309 CONTINUE
      DO 308 KJ=1,NFB
  308 SS(KK,KJ)=0.0
      SS(KK,NHB)=1.0
      DO 30 M=1,LT
   30 RHS(KK,M)=D
  307 CONTINUE
  300 CONTINUE
C
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE SRADSP(LIMIT,NH,NS,GEO,DS,SM,FE,SS,RHS,DSRI1,
     * DSPT,PSPT,SSRT)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CO1/IPRINT,NP,BETA,NDOF,NF,NF1,NDIM,NDIM1
      COMMON/CO2/NE,NC,NT,LT,LDC,NHL,NEL,NJL,NHIN,NPCR
      COMMON/CO3/XCR,RCR,TH,EC,E,PU,TCOEF,GAMA,DSP,HPCN
      COMMON/CO4/IT,NC,NDIV,IDF,ILOAD,IDCO
      COMMON/CO5/ACEL,BCEL,APL,BPL,PS,DCN,DSN
      COMMON/CO6/DSPL,PSP,SSR
      COMMON/HYP/AH,BH,XH
      DIMENSION XCR(21),RCR(21),DSP(21,4),EC(20,2),TH(20,2),
     * GAMA(20),TCOEF(20),E(20),PU(20),SC(20,6),GEO(80,6),
     * RK(4),THK(2),PHK(2),SK(2),RR(2),SM(8,8),FE(8,2),
     * ACEL(200),APL(200,20),BPL(200,20),BCEL(200),SS(200,80),
     * RHS(200,2),IDF(21,4),NC(20,2),NCON(20,2),NS(20),NDIV(20),
     * IT(20),IDCO(20,10,2),MM(8),ILOAD(20,5),IDL(5),DS(8,2),
     * DSPL(8,120),PSR(8,120),SSR(12,120),DSRI1(10,2),
     * HPCN(3,20),DCN(11),DSN(11),DSPT(4,11,200),PSRT(4,11,200),
     * SSRT(6,11,200),PS(10,2)
      N=NL
      K0=-
      LOC=0
      LMS1=0
      DO 500 I=1,NE
      I1=I-1
      IF(I1.EQ.0) GO TO 491
      LOC=LOC+(NS(I1)+1)*5
  491 CONTINUE
      LOC2=LOC
      DO 498 J=1,5
  498 IDL(J)=ILOAD(I,J)
      ITYP=IT(I)
      NSEG=NS(I)
      NXI=NDIV(I)
      NXT=(NP-1)/NXI+1
      AH=HPCN(1,I)
      BH=HPCN(2,I)
      XH=HPCN(3,I)
      K0=K0+1
      DO 400 K=1,NSEG
      K0=K0+1
      K1=K0+1
      RK(1)=GEO(K0,2)
      RK(2)=GEO(K1,2)
      THK(1)=GEO(K0,3)
      THK(2)=GEO(K1,3)
      PHK(1)=GEO(K0,4)
      PHK(2)=GEO(K1,4)
      SK(1)=GEO(K0,5)
      SK(2)=GEO(K1,5)
      RR(1)=GEO(K0,6)
      RR(2)=GEO(K1,6)
      IF(NEL.EQ.0) GO TO 493
      LOC1=LOC2+1
      LOC2=LOC1+4
      LOC3=LOC2+5
      KL=0
      DO 495 L=LOC1,LOC3
      KL=KL+1
  495 PS(KL,1)=APL(L,N)
  492 CONTINUE
      IF(LT.EQ.1) GO TO 493
      KL=0
      DO 496 L=LOC1,LOC3
      KL=KL+1
  496 PS(KL,2)=BPL(L,N)
  493 CONTINUE
C
      IK1=IDCO(I,K,1)
      IK2=IDCO(I,K,2)
      JK1=(IK1-1)*NDOF
      JK2=(IK2-1)*NDOF
      DO 410 L=1,LT
      DO 410 J=1,NDOF
      JJ=J+NDOF
      JJ1=JK1+J
```

```
      JJ2=JK2+J
      DS(J,L)=RHS(JJ1,L)
  410 DS(JJ,L)=RHS(JJ2,L)
C
      DO 430 L=3,4
      IF(L.EQ.3) L1=1
      IF(L.EQ.4) L1=NSEG
      IF(K.NE.L1) GO TO 430
      L2=L-2
      Z=ZC(I,L2)
      IF(Z.EQ.0) GO TO 430
      R1=RK(L2)
      R2=RP(L2)
      CALL ECCNTR(IPRINT,L,LT,NH,NF,Z,R1,R2,SM,FE,DS)
  430 CONTINUE
      IF(ITYP.EQ.1) GO TO 480
      DO 440 L=3,4
      IF(L.EQ.3) L1=1
      IF(L.EQ.4) L1=NSEG
      IF(K.NE.L1) GO TO 440
      L2=L-2
      PHIK=PHK(L2)
      CALL GLTRAN(IPRINT,L,LT,NH,NF,PHIK,SM,FE,DS)
  440 CONTINUE
  480 CONTINUE
      IF(IPRINT.GT.1) WRITE(6,900) I,K
      CALL STORE1(2,LMS1,IPRINT,LT,NF,SM,FE,DS,DSRI1)
      CALL RESULT(IPRINT,ITYP,LT,LDC,NEL,NH,NF,NP,GAMA(I),E(I),
     *  PO(I),TCOEF(I),RK,THK,PHK,SK,IDL,PS,DSRI1,DSPL,PSR,SSR)
      IF(IPRINT.LT.2) GO TO 56
      IF(LT.EQ.1) WRITE(6,1100)
      IF(LT.EQ.2) WRITE(6,1101)
      LN1=LT*NDOF
      LN2=LT*6
      DO 50 J=1,NP
   50 WRITE(6,1200) (J,(DSPL(L,J),L=1,LN1))
      IF(LT.EQ.1) WRITE(6,1140)
      IF(LT.EQ.2) WRITE(6,1141)
      DO 52 J=1,NP
   52 WRITE(6,1200) (J,(PSR(L,J),L=1,LN1))
      WRITE(6,1150)
      DO 55 J=1,NP
      IF(LT.EQ.1) WRITE(6,1251) (J,(SSR(L,J),L=1,LN2))
   55 IF(LT.EQ.2) WRITE(6,1250) (J,(SSR(L,J),L=1,LN2))
   56 IF(NHL.EQ.1) GO TO 400
      CALL STORE2(LIMIT,NH,NDOF,NP,NXI,LT,NPCR,DCN,DSN,DSPL,PSR,
     *  SSR,DSRT,PSRT,SSRT)
      LIMIT=LIMIT+NXT
C
  900 FORMAT(//'*********************************************'/
     *  'ELEMENT',I10,10X,'SEGMENT',I10/
     *  '*********************************************'/)
 1100 FORMAT(/'DISPLACEMENT '/10X,'R',12X,'W',12X,'V',12X,'U'/)
 1101 FORMAT(/'DISPLACEMENT'/10X,'R',12X,'W',12X'V',12X,'U',12X,
     *  'P',12X,'W',12X,'V',12X,'U'/)
 1140 FORMAT(//'PRIMARY STRESS RESULTANTS'/
     *  10X,'MS',11X,'SS',10X,'NS',10X,'TS'/)
 1141 FORMAT(//'PRIMARY STRESS RESULTANTS'/10X,'MS',11X,'SS',10X,
     *  'NS',10X,'TS',10X,'MS',10X,'SS',10X,'NS',10X,'TS'/)
 1150 FORMAT(//'SECONDARY STRESS RESULTANTS'/
     *  10X,'MT',9X,'MTS',9X,'MST',9X,'NT',9X,'NTS',9X,'NST'/)
 1250 FORMAT(I3,3X,6E12.4/(6X,6E12.4))
 1251 FORMAT(I3,3X,6E12.4)
 1200 FORMAT(I3,3X,8E12.4)
C
  400 CONTINUE
  500 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE STORE1(LM,LM1,IPT,LT,NF,SM,FE,DS,DSRI1)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION SM(8,8),FE(8,2),DS(8,2),DSRI1(10,2),FIX(2,200),
     *  SMS(8,200),SR(8,2)
      GO TO (10,20),LM
C
   10 DO 1 L=1,NF
      LL=LM1+L
      DO 1 J=1,NF
    1 SMS(J,LL)=SM(J,L)
      DO 2 L=1,NF
      LL=LM1+L
      DO 2 J=1,LT
    2 FIX(J,LL)=FE(L,J)
      LM1=LM1+NF
      RETURN
C
   20 CONTINUE
      DO 5 L=1,NF
      LL=LM1+L
      DO 5 J=1,NF
    5 SM(J,L)=SMS(J,LL)
      DO 7 L=1,NF
      LL=LM1+L
      DO 7 J=1,LT
    7 FE(L,J)=FIX(J,LL)
      CALL MPROD(SM,DS,SR,8,8,NF,NF,LT)
      IF(IPT.LT.2) GO TO 15
      WRITE(6,500) ((SM(I,K),K=1,NF),I=1,NF)
      WRITE(6,501) ((DS(I,L),I=1,NF),L=1,LT)
      WRITE(6,505) ((SR(I,L),I=1,NF),L=1,LT)
      WRITE(6,506) ((FE(I,L),I=1,NF),L=1,LT)
   15 CONTINUE
      DO 8 L=1,LT
      DO 8 J=1,NF
    8 SR(J,L)=SR(J,L)+FE(J,L)
      NDOF=NF/2
      DO 9 L=1,LT
      DO 9 J=1,NDOF
      JJ=J+NDOF
      DSRI1(J,L)=DS(J,L)
    9 DSRI1(JJ,L)=SR(J,L)
      IF(IPT.LT.2) GO TO 16
      WRITE(6,507) ((SR(I,L),I=1,NF),L=1,LT)
      WRITE(6,510) ((DSRI1(I,L),I=1,NF),L=1,LT)
  500 FORMAT(//'STIFFNESS MATRIX BEFORE MODF.'/(8E14.6))
  501 FORMAT(/'DISPLACEMENT'/(8E14.6))
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  505 FORMAT(/'STRESS RESULTANTS DUE TO DISPL'/(8E14.6))
  506 FORMAT(/'FIXED END STRESSES DUE TO LOAD'/(8E14.6))
  507 FORMAT(/'STRESSES AT ENDS'/(6E14.6))
  510 FORMAT(/'INITIAL CONDITIONS'/(8E14.6))
   16 LM1=LM1+NF
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE STORE2(LM,NH,ND,NP,NXI,LT,NPCR,DC,DS,DSPL,
     *  PSR,SSR,DSPT,PSRT,SSRT)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION DSPL(8,1),PSR(8,1),SSR(12,1),DC(11),DS(11),
     *  DSPT(4,11,1),PSRT(4,11,1),SSRT(6,11,1)
C
      KO=0
      DO 200 K=1,NP,NXI
      KO=KO+1
      KK=LM+KO
      DO 100 J=1,NPCR
      D1=DC(J)
      D2=DS(J)
      DO 50 I=1,3
      DSPT(I,J,KK)=DSPT(I,J,KK)+D1*DSPL(I,K)
      IF(LT.EQ.1) GO TO 50
      II=I+4
      DSPT(I,J,KK)=DSPT(I,J,KK)+D2*DSPL(II,K)
   50 CONTINUE
      DSPT(4,J,KK)=DSPT(4,J,KK)+D2*DSPL(4,K)
      IF(LT.EQ.1) GO TO 51
      DSPT(4,J,KK)=DSPT(4,J,KK)+D1*DSPL(8,K)
   51 CONTINUE
      DO 60 I=1,3
      PSRT(I,J,KK)=PSRT(I,J,KK)+D1*PSR(I,K)
      IF(LT.EQ.1) GO TO 60
      II=I+4
      PSRT(I,J,KK)=PSRT(I,J,KK)+D2*PSR(II,K)
   60 CONTINUE
      PSRT(4,J,KK)=PSRT(4,J,KK)+D2*PSR(4,K)
      IF(LT.EQ.1) GO TO 61
      PSRT(4,J,KK)=PSRT(4,J,KK)+D1*PSR(8,K)
   61 CONTINUE
      DO 70 I=1,4,3
      SSRT(I,J,KK)=SSRT(I,J,KK)+D1*SSR(I,K)
      IF(LT.EQ.1) GO TO 70
      II=I+6
      SSRT(I,J,KK)=SSRT(I,J,KK)+D2*SSR(II,K)
   70 CONTINUE
      DO 71 I=2,6
      IF(I.EQ.4) GO TO 71
      SSRT(I,J,KK)=SSRT(I,J,KK)+D2*SSR(I,K)
      IF(LT.EQ.1) GO TO 71
      II=I+6
      SSRT(I,J,KK)=SSRT(I,J,KK)+D1*SSR(II,K)
   71 CONTINUE
  100 CONTINUE
  200 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE SOLVER(A,B,LT,NEQ,NFB)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION A(200,80),B(200,2)
C
      NHB=(NFB+1)/2
      NL=NEQ-1
      NB=NHB+1
      NM=NEQ-NHB+1
      NR=NFB
C
      DO 250 N=1,NL
      IF(A(N,NHB).LE.0.) GO TO 700
      N1=N+1
      N2=N+NHB-1
      IF(N.LE.NM) GO TO 10
      NR=NEQ+NHB-N
      N2=NEQ
   10 CONTINUE
C
      DO 100 K=NB,NR
  100 A(N,K)=A(N,K)/A(N,NHB)
      DO 110 L=1,LT
  110 B(N,L)=B(N,L)/A(N,NHB)
C
      NN=NHB
      DO 250 I=N1,N2
      NN=NN-1
      IF(A(I,NN).EQ.0.) GO TO 250
      C=A(I,NN)
      JJ=NN
      DO 200 J=NB,NR
      JJ=JJ+1
  200 A(I,JJ)=A(I,JJ)-C*A(N,J)
      DO 210 L=1,LT
  210 B(I,L)=B(I,L)-C*B(N,L)
  250 CONTINUE
C     BACK SUBSTITUTION
C
      I=NEQ
      DO 300 L=1,LT
  300 B(NEQ,L)=B(NEQ,L)/A(NEQ,NHB)
C
      DO 400 N=1,NL
      I=I-1
      IF(N.LT.NHB) NR=NHB+N
      DO 400 L=1,LT
      DO 400 J=NB,NR
      K=I+J-NHB
  400 B(I,L)=B(I,L)-A(I,J)*B(K,L)
C
C
      RETURN
  700 WRITE(6,3000) N,A(N,NHB)
      STOP
 3000 FORMAT('NEGATIVE OR ZERO ELEMENT ON MAIN DIAGONAL',I5,
```

```
     *  E12.4)
        END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE READIN(IPRINT,NP,NPCR,BETA,NE,NJ,NTL,LDC,NHL,
     * NHIN,NEL,NJL,XCOOR,RCOOR,TH,EC,IT,NC,NPI,E,PU,TCOEF,
     *  GAMA,IDF,ILOAD,ACEL,BCEL,INJL,ANJL,HPCN)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XCOOR(21),RCOOR(21),IDF(21,4),DSP(21,4),IT(20),
     C  NPI(20),NC(20,2),EC(20,2),TH(20,2),GAMA(20),E(20),
     C   PU(20),TCOEF(20),ACEL(200),BCEL(200),HPCN(3,20),
     C INJL(21),ANJL(200,20),ILOAD(20,5),TITLE(10)
C
      READ (5,100) TITLE
      WRITE(6,200) TITLE
C     ANALYSIS DATA
      READ (5,101) IPRINT,NP,NPCR,LDC,BETA
      IF(NPCR.EQ.0) NPCR=1
      WRITE(6,201) IPRINT,NP,NPCR,LDC,BETA
C     STRUCTURE DATA
      READ (5,110) NE,NJ,EG,PUG,GAMG,TKG
      WRITE(6,210) NE,NJ,EG,PUG,GAMG,TKG
C     NODAL DATA
      READ (5,120) (I,XCOOR(I),RCOOR(I),(IDF(I,J),J=1,4),I=1,NJ)
      WRITE(6,220) (I,XCOOR(I),RCOOR(I),(IDF(I,J),J=1,4),I=1,NJ)
C     ELEMENT DATA
      WRITE(6,240)
      DO 10 I=1,NE
      READ (5,140) I,IT(I),NC(I,1),NC(I,2),NPI(I),TH(I,1),
     1      TH(I,2),EC(I,1),EC(I,2)
      READ(5,145) (HPCN(J,I),J=1,3),GAMA(I),PU(I),E(I),TCOEF(I)
      IF(TH(I,2).EQ.0) TH(I,2)=TH(I,1)
      IF(TCOEF(I).EQ.0)    TCOEF(I)=TKG
      IF(E(I).EQ.0)        E(I)=EG
      IF(PU(I).EQ.0)      PU(I)=PUG
      IF(GAMA(I).EQ.0)   GAMA(I)=GAMG
      IF(NPI(I).EQ.0)NPI(I)=NP-1
      WRITE(6,241) I,IT(I),NC(I,1),NC(I,2),NPI(I),TH(I,1),
     1  TH(I,2),EC(I,1),EC(I,2),(HPCN(J,I),J=1,3),GAMA(I),
     2  PU(I),E(I),TCOEF(I)
   10 CONTINUE
C     LOADING SPECIFICATION
      READ (5,150) NEL,NJL,NHL,NTL,NHPL,NHIN
      IF(NHL.EQ.0) NHL=1
      IF(NHIN.EQ.0) NHIN=1
      NSPL=2
      WRITE(6,250) NEL,NJL,NHL,NTL,NHPL,NHIN
      IF(NEL.EQ.0.AND.NJL.EQ.0) GO TO 40
      CALL LOADIN(NE,NJ,NEL,NJL,NTL,NHL,NSPL,NHPL,ILOAD,
     *  ACEL,BCEL,INJL,ANJL)
   40 CONTINUE
      IF(IPRINT.EQ.0) GO TO 50
      RETURN
C*****************************
C     FORMAT STATEMENTS
C*****************************
  100 FORMAT(10A8)
  200 FORMAT('1',10A8)
  101 FORMAT(4I4,F7.0)
  201 FORMAT(/'***    ANALYSIS DATA    ***'//
     1        'IPRINT        =',I4/
     1        'NP            =',I4/
     2        'NPCR          =',I4/
     3        'LDC           =',I4/
     4        'BETA          =',F7.3//)
  110 FORMAT(2I3,4F12.0)
  210 FORMAT(/'***    STRUCTURAL DATA    ***'//
     1  'NUMBER OF ELEMENTS              =',I3/
     2  'NUMBER OF NODES                 =',I3/
     4  'GLOBAL YOUNG S MODULUS          =',E12.4/
     5  'GLOBAL POISSON S RATIO          =',E12.4/
     6  'GLOBAL UNIT WEIGHT              =',E12.4/
     7  'GLOBAL THERMAL COEFFICIENT      =',E12.4//)
  120 FORMAT(I4,2F10.0,4I4)
  220 FORMAT('***     NODAL DATA      ***'//
     *  'NODE        XCOOR         RCOOR',17X,'R   W   V   U'//
     *  (I4,2F12.4,15X,4I3))
  140 FORMAT(5I6,4F10.0)
  145 FORMAT(7F10.0)
  240 FORMAT(//'***    ELEMENT DATA    ***'//
     * 'EL IT TN BN NPI    TH1      TH2       EC1',
     * 6X,'EC2',6X,'HP1',6X,'HP2',6X,'HP3     UNW    PR',
     * 8X,'E',9X,'TCOEF'/)
  241 FORMAT(I2,4I3,7F9.4,2F6.4,2E12.4)
  150 FORMAT(6I5)
  250 FORMAT(//'***    LOADING SPECIFICATIONS    ***'//
     1  'NUMBER OF LOADED ELEMENTS     =',I5/
     2  'NUMBER OF LOADED NODES        =',I5/
     3  'NUMBER OF HARMONICS TO FIT    =',I5/
     4  'LOADING TYPE CODE             =',I5/
     5  'NUMBER OF CIRCUMFR POINTS     =',I5/
     6  'HARMONIC INCREMENTAL          =',I5//)
   50 STOP
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE LOADIN(NE,NJ,NEL,NJL,NTL,NHL,NSPL,NHPL,ILOAD,
     *  ACEL,BCEL,INJL,ANJL)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ILOAD(20,5),ACEL(200),BCEL(200),AL(20),BL(20),
     *  INJL(21),ANJL(200,20),KLD(4),W(40),LL(20)
C
      IF(NEL.EQ.0) GO TO 200
      DO 1 I=1,NE
    1 LL(I)=0
      READ(5,1001)  (LL(I),(ILOAD(LL(I),K),K=1,5),I=1,NEL)
      K1=0
      K2=0
      WRITE(6,3000)
      IF(NTL.NE.0) GO TO 10
      DO 20 I=1,NEL
      L=LL(I)
      DO 20 K=1,5
      KL=ILOAD(L,K)
      IF(KL.EQ.0) GO TO 20
      K1=K2+1
```

```
      K2=K1+1
      READ(5,1100) (ACEL(KK),KK=K1,K2)
      J=0
      WRITE(6,2101)L,K,KL
      DO 15 KK=K1,K2
      J=J+1
      WRITE(6,2101) L,K,KL
      WRITE(6,2102) J,ACEL(KK)
   15 CONTINUE
   20 CONTINUE
C
      GO TO 200
   10 CONTINUE
      NL=NHPL/2
      ML=NHL-1
      DO 30 I=1,NEL
      L=LL(I)
      DO 30 K=1,5
      KL=ILOAD(L,K)
      IF(KL.EQ.0) GO TO 30
      DO 40 J=1,NSPL
      IF(KL.EQ.1.AND.J.GT.1) GO TO 28
      IF(NTL.EQ.3) GO TO 23
      READ(5,1400) (W(JJ),JJ=1,NHPL)
      CALL FORIT(W,NL,ML,AL,BL)
   23 IF(NTL.EQ.3) READ(6,1400) (AL(JJ),JJ=1,NHL)
   28 CONTINUE
      K1=K2+1
      K2=K1+NHL-1
      K3=0
      IF(K.EQ.2) GO TO 24
      DO 35 KK=K1,K2
      K3=K3+1
   35 ACEL(KK)=AL(K3)
      IF(NTL.GT.1) GO TO 27
      K3=0
      DO 36 KK=K1,K2
      K3=K3+1
   36 BCEL(KK)=BL(K3)
      GO TO 27
   24 DO 25 KK=K1,K2
      K3=K3+1
   25 BCEL(KK)=AL(K3)
      IF(NTL.GT.1) GO TO 27
      K3=0
      DO 26 KK=K1,K2
      K3=K3+1
   26 ACEL(KK)=BL(K3)
   27 CONTINUE
      WRITE(6,2202) J,(ACEL(KK),KK=K1,K2)
      IF(NTL.GT.1) GO TO 40
      WRITE(6,2202) J,(BCEL(KK),KK=K1,K2)
   40 CONTINUE
   30 CONTINUE
  200 CONTINUE
C
      IF(NJL.EQ.0) RETURN
      WRITE(6,2710)
      DO 150 I=1,NJ
  150 INJL(I)=0
      DO 155 I=1,NJL
      READ(5,2700) J
      INJL(J)=1
  155 CONTINUE
      IF(NTL.NE.0) GO TO 171
      NJ4=NJ*4
      DO 151 J=1,NJ4
  151 ANJL(J,1)=0.0
      DO 170 I=1,NJ
      IF(INJL(I).EQ.0) GO TO 170
      IN4=I*4
      IN=IN4-3
      N=1
      NH=0
      READ(5,1600) (ANJL(II,N),II=IN,IN4)
      WRITE(6,2700) I
  172 WRITE(6,2720) NH,(ANJL(II,N),II=IN,IN4)
  170 CONTINUE
      RETURN
  171 NL=NHPL/2
      ML=NHL-1
      NJ8=NJ*8
      DO 210 N=1,NHL
      DO 210 J=1,NJ8
  210 ANJL(J,N)=0.0
      DO 250 I=1,NJ
      IF(INJL(I).EQ.0) GO TO 250
      READ(5,1650) (KLD(K),K=1,4)
      IN8=I*8
      IN=IN8-7
      DO 180 K=1,4
      IF(KLD(K).EQ.0) GO TO 180
      IF(NTL.EQ.3) GO TO 160
      READ(5,1400) (W(KK),KK=1,NHPL)
      CALL FORIT(W,NL,ML,AL,BL)
  160 IF(NTL.EQ.3) READ(5,1400) (AL(KK),KK=1,NHL)
      DO 180 N=1,NHL
      IF(K.EQ.4) GO TO 181
      K1=IN+K-1
      K2=K1+4
      GO TO 182
  181 K2=IN+K-1
      K1=K2+4
  182 ANJL(K1,N)=AL(N)
      IF(NTL.EQ.3) GO TO 180
      ANJL(K2,N)=BL(N)
  180 CONTINUE
      WRITE(6,2700) I
      DO 190 N=1,NHL
      NH=N-1
      WRITE(6,2720) NH,(ANJL(K,N),K=IN,IN8)
  190 CONTINUE
  250 CONTINUE
  251 RETURN
C
C     FORMAT STATMENTS
C
 1001 FORMAT(6I5)
 1100 FORMAT(2F10.0)
 1400 FORMAT(12F10.0)
```

```
 2101 FORMAT(I4,2I9)
 2102 FORMAT(22X,I9,24X,E12.4/(22X,I9,24X,E12.4))
 2202 FORMAT(22X,I9,10X,5E12.4/(41X,5E12.4))
 3060 FORMAT(//'***    ELEMENT LOAD    ***'/
     *  'ELEM     L.TYPE     L.ID        1=TOP,2=BOT',15X,
     *  ' LOAD VALUES OR FOURIER COEFFICIENTS'/)
 1600 FORMAT(4F10.0)
 1650 FORMAT(4I5)
 2700 FORMAT(I4)
 2710 FORMAT(//'*** NODAL LOADS ****'//
     *  'NODE',10X,'H. NUMB',20X,'MS',10X,'SS',10X,'NS',10X,'TS'/)
 2720 FORMAT(15X,I3,20X,4E12.4/(38X,4E12.4))
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE FORIT(FNT,N,M,A,B)
C      FOURIER ANALYSIS FOR A PERIODICALLY TABULATED FUNCTION.
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(1),B(1),FNT(1)
      PI=3.141592654
      AN=N
      COEF=1./AN
      CONST=PI*COEF
      S1=DSIN(CONST)
      C1=DCOS(CONST)
      C=1.0
      S=0.0
      J=1
      FNTZ=FNT(1)
   10 U2=0.0
      U1=0.0
      I=2*N
C
   20 U0=FNT(I)+2.0*C*U1-U2
      U2=U1
      U1=U0
      I=I-1
      IF(I-1) 30,30,20
   30 A(J)=COEF*(FNTZ+C*U1-U2)
      B(J)=COEF*S*U1
      IF(J-(M+1)) 40,50,50
   40 Q=C1*C-S1*S
      S=C1*S+S1*C
      C=Q
      J=J+1
      GO TO 10
   50 A(1)=A(1)*0.5
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE SEGEOM(ITYP,NSEG,RI,RF,XI,XF,THI,THF,BL,U,SC)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HYP/AH,BH,XH
      DIMENSION SC(20,6),HPCN(3,20)
      PI=3.141592654
      S1=RF-RI
      S2=XF-XI
      THIK=(THF+THI)/2.
      GO TO(1,2,3,4,5),ITYP
C     CYLINDER
    1 SI=XI
      SF=XF
      AL=PI/2.
      RAD=RI
      GO TO 50
C     CONE
    2 AL=DATAN2(S2,S1)
      DC=DCOS(AL)
      DS=DSIN(AL)
      SI=RI/DC
      SF=RF/DC
      RAD=(RI+RF)/2.
      IF(DS.NE.0.0) RAD=RAD/DS
      GO TO 50
C     SPHERE R1 EQ. R2
    3 X=(RF*RF-RI*RI-S2*S2)/2./S2
      ZX=S2+X
      RX=(RF*RF+X*X)**.5
      ALI=DATAN2(RI,ZX)
      ALF=DATAN2(RF,X)
      SI=RX*ALI
      SF=RX*ALF
      RAD=RX
      GO TO 50
C     SPERE R1 NE. R2
    4 CONTINUE
      ALI=BH*PI/180.
      ALF=XH*PI/180.
      SI=AH*ALI
      SF=AH*ALF
      AL=(ALI+ALF)/2.
      DS=DSIN(AL)
      RX=AH*DSIN(ALI)-RI
      RAD=(RI+RF)/2./DS
      GO TO 50
C
C     HYPERBOLOID
    5 AB=(BH/AH)**2
      AH2=AH*AH
      PHI=DATAN(AB)
      DS=DSIN(PHI)
      TL=S2/DS
      RAD=AH
      GO TO 55
   50 CONTINUE
      TL=SF-SI
      TH=(THF-THI)/TL
   55 B=(3.*(1-U**2)/(RAD*THIK)**2)**.25
      SEG=B*TL/BL
      NSEG=SEG
      DSEG=SEG-NSEG
      IF(DSEG.GT.0) NSEG=NSEG+1
      NSEG1=NSEG+1
      IF(ITYP.EQ.5) GO TO 60
      TLS=TL/NSEG
      THS=TH*TLS
```

```
C
C
C
      GO TO(21,22,23,23),ITYP
   21 DO 20 K=1,NSEG1
      K1=K-1
      SC(K,1)=XI+K1*TLS
      SC(K,2)=RI
      SC(K,3)=THI+K1*THS
      SC(K,4)=AL
      SC(K,5)=SC(K,1)
      SC(K,6)=RI
   20 CONTINUE
      RETURN
   22 DO 30 K=1,NSEG1
      K1=K-1
      SC(K,1)=XI+K1*TLS*DS
      SC(K,2)=RI+K1*TLS*DC
      SC(K,3)=THI+K1*THS
      SC(K,4)=AL
      SC(K,5)=SI+K1*TLS
      SC(K,6)=0.0
      IF(DS.NE.0.0)  SC(K,6)=SC(K,2)/DS
   30 CONTINUE
      RETURN
   23 PHS=(ALF-ALI)/NSEG
      DO 40 K=1,NSEG1
      K1=K-1
      SC(K,1)=XI+K1*S2/NSEG
      AL=ALI+K1*PHS
      DS=DSIN(AL)
      IF(ITYP.EQ.3)  SC(K,2)=RX*DS
      IF(ITYP.EQ.4)  SC(K,2)=AH*DS-RX
      SC(K,3)=THI+K1*THS
      SC(K,4)=AL
      SC(K,5)=SI+K1*TLS
      SC(K,6)=SC(K,2)/DS
   40 CONTINUE
      RETURN
   60 CONTINUE
      TLS=(XP-XI)/NSEG
      THS=(THF-THI)/(XP-XI)
      DO 61 K=1,NSEG1
      K1=K-1
      SC(K,1)=XI+K1*TLS
      XX=XI-XH+K1*TLS
      X2=XX*XX
      XB=X2/AE
      RR2=XB+AH2
      SC(K,2)=(RR2)**.5
      SC(K,3)=THI+K1*THS*TLS
      IF(RR2.LE.AH2)  GO TO 62
      DT=(RR2/XB)**.5
      DT=DT*BH/AH
      PHI=DATAN(DT)
      IF(XX.LE.0.0)  PHI=PI-PHI
      GO TO 63
   62 PHI=PI/2.
   63 SC(K,4)=PHI
      SC(K,5)=SC(K,1)
      DS=DSIN(PHI)
      SC(K,6)=SC(K,2)/DS
   61 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE JINVER(A,B,NM,NQ)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C     INVERTS THE MATRIX A BY THE JACOBI METHOD
C     AND STORE THE RESULT IN MATRIX B
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(10,10),B(10,10)
C     INITIALIZE THE B MATRIX
      DO 10 J=1,NM
      DO 10 I=1,NM
   10 B(I,J)=0.0
      DO 20 J=1,NQ
   20 B(J,J)=1.0
C
C     BEGIN JACOBI REDUCTION OF MATRIX A AND ALSO OPERATE ON B
      DO 100 N=1,NQ
      IF(DABS(A(N,N)).LT.1.0D-09) GO TO 999
      C=1./A(N,N)
      N1=N+1
      IF(N.EQ.NQ)  GO TO 50
      DO 40 J=N1,NQ
      AJ=A(N,J)*C
      BJ=B(N,J)*C
      DO 30 I=1,NQ
      A(I,J)=A(I,J)-AJ*A(I,N)
   30 B(I,J)=B(I,J)-BJ*A(I,N)
      A(N,J)=AJ
   40 B(N,J)=BJ
   50 DO 90 J=1,N
      BJ=B(N,J)*C
      DO 80 I=1,NQ
   80 B(I,J)=B(I,J)-BJ*A(I,N)
   90 B(N,J)=BJ
  100 CONTINUE
      RETURN
  999 WRITE(6,1000)N
 1000 FORMAT('**ZERO ELEMENT ON MAIN DIAGONAL FOR EQUATION',I4)
      STOP
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE RNGKT(IPT,IT,LT,LDC,NEL,NH,NP,UNW,E,U,TK,RK,
     *      THK,PHK,SK,IDL,PS,YS)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HYP/AH,BH,XH
      DIMENSION  TPP(10,10),TDPP(8,10),AK(8,10),
     *   BK(8,10),CK(8,10),DK(8,10),SA(8,10),RK(2),THK(2),
     *   PX(10),PHK(2),SK(2),RX(51),TX(51),PX(51),SX(51),
     *   YS(8,10),PS(10,2),IDL(5),PLD(10,51),PIX(10)
      NP=6
      NP1=NP+LT
```

```
      LM=1
      IF(NH.GT.0) LM=0
      NRK=NP
      NM1=NRK-1
      IF(IT.EQ.5) GO TO 26
      RJ=(RK(2)-RK(1))/NM1
      TJ=(THK(2)-THK(1))/NM1
      FJ=(PHK(2)-PHK(1))/NM1
      SJ=(SK(2)-SK(1))/NM1
      DO 20 I=1,NRK
      I1=I-1
      TX(I)=THK(1)+I1*TJ
      FX(I)=PHK(1)+I1*FJ
   20 CONTINUE
      GO TO (21,22,23,23),IT
   21 DO 15 I=1,NRK
   15 RX(I)=RK(1)
      GO TO 24
   22 DO 16 I=1,NRK
      I1=I-1
   16 RX(I)=RK(1)+I1*RJ
      GO TO 24
   23 PHI=PHK(1)
      DSN=DSIN(PHI)
      R2=RK(1)/DSN
      RX(1)=RK(1)
      XR=AH*DSN-RK(1)
      DO 17 I=2,NRK
      PHI=FX(I)
      DSN=DSIN(PHI)
      IF(IT.EQ.3) RX(I)=R2/DSN
      IF(IT.EQ.4) RX(I)=AH*DSN-XR
   17 CONTINUE
      GO TO 24
   26 CALL HYPRB(LM,NRK,RK,THK,SK,RX,FX,TX,XJ,TJ)
   24 CONTINUE
      II=5*LT
      DO 25 K=1,NP
      DO 25 I=1,II
   25 PLD(I,K)=0.0
      DO 255 I=1,II
  255 PIX(I)=0.0
   57 IF(LDC.EQ.1.OR.NEL.EQ.0) GO TO 55
      CALL PLSEG(IPT,LT,NP,IDL,PS,PLD,PIX)
   55 CONTINUE
      IF(LDC.EQ.0.OR.NH.GT.0) GO TO 56
      CALL DLSEG(IT,ITA,NP,UNW,FX,TX,PLD)
   56 CONTINUE
C
C     INITIAL VALUES
      DO 2 K=1,NP1
      DO 2 I=1,NP
    2 YS(I,K)=0.0
      DO 4 I=1,NP
    4 YS(I,I)=1.0
      DO 1 K=1,NP1
      DO 1 I=1,NP1
    1 TFP(I,K)=0.0
      DO 11 K=1,LT
      KK=NP+K
   11 TFP(KK,KK)=1.0
C     START INTEGRATION BY RUNGE KUTTA
C
      DO 100 J=1,NM1
      J1=J+1
      IF(IT.NE.5) GO TO 27
      XJ2=XJ*XJ
      RJ=RX(J1)-RX(J)
      RJ2=RJ*RJ
      SJ=(XJ2+RJ2)**.5
      FJ=FX(J1)-FX(J)
   27 CONTINUE
      DO 3 K=1,NP1
      DO 3 I=1,NP
    3 TFP(I,K)=YS(I,K)
      IF(J.GT.1) GO TO 30
      X1=RX(J)
      X2=TX(J)
      X3=FX(J)
      DO 40 I=1,II
   40 PX(I)=PLD(I,J)
      CALL PLUGGE(IT,LT,NH,NP,X1,X2,X3,PX,E,U,TK,R2,SA)
      CALL MPROD(SA,TFP,TDPP,8,10,NP,NP1,NP1)
   30 CONTINUE
      DO 5 K=1,NP1
      DO 5 I=1,NP
      AK(I,K)=SJ*TDPP(I,K)
    5 TFP(I,K)=YS(I,K)+AK(I,K)/2.
C
      X15=RX(J)+RJ/2.
      X25=TX(J)+TJ/2.
      X35=FX(J)+FJ/2.
      DO 41 I=1,II
   41 PX(I)=PLD(I,J)+PIX(I)
      CALL PLUGGE(IT,LT,NH,NP,X15,X25,X35,PX,E,U,TK,R2,SA)
      CALL MPROD(SA,TFP,TDPP,8,10,NP,NP1,NP1)
      DO 6 K=1,NP1
      DO 6 I=1,NP
      BK(I,K)=SJ*TDPP(I,K)
    6 TFP(I,K)=YS(I,K)+BK(I,K)/2.
C
      CALL MPROD(SA,TFP,TDPP,8,10,NP,NP1,NP1)
      DO 7 K=1,NP1
      DO 7 I=1,NP
      CK(I,K)=SJ*TDPP(I,K)
    7 TFP(I,K)=YS(I,K)+CK(I,K)
C
      J1=J+1
      X10=RX(J1)
      X20=TX(J1)
      X30=FX(J1)
      DO 42 I=1,II
   42 PX(I)=PLD(I,J1)
      CALL PLUGGE(IT,LT,NH,NP,X10,X20,X30,PX,E,U,TK,R2,SA)
      CALL MPROD(SA,TFP,TDPP,8,10,NP,NP1,NP1)
      DO 9 K=1,NP1
      DO 9 I=1,NP
    9 DK(I,K)=SJ*TDPP(I,K)
C
      DO 10 K=1,NP1
```

```
      DO 10 I=1,NP
   10 YS(I,K)=YS(I,K)+(1/6.)*(AK(I,K)+2.*BK(I,K)+
     C          2.*CK(I,K)+DK(I,K))
      IF(IPT.LT.8) GO TO 100
      WRITE(6,2000) J1,SJ
       WRITE(6,1100) ((YS(I,K),K=1,NP1),I=1,NP)
 2000 FORMAT(//'YS AT POINT',I10,10X,'SJ=',E15.4//)
  100 CONTINUE
  101 CONTINUE
      IF(IPT.LT.6) GO TO 110
      WRITE(6,1000)
      WRITE(6,1100)  ((YS(I,K),K=1,NF1),I=1,NP)
  110 CONTINUE
      RETURN
 1000 FORMAT(//'*** TRANSFER MATRIX ***'//)
 1100 FORMAT(9E12.4)
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE STIFIX (IPT,LT,YS,SM,FE)
C     EVALUATES SEG. STIF. MATRIX AND FIXED END FORCES
C
C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION YD(10,10),ZP(8,10),YDI(10,10),SM(8,8),
     *    FE(8,2),WW(8,10),YS(8,10)
      NP=6
      NP1=NP+LT
      NP2=NP/2
C     NP=NUMBER OF SEGMENT D O F
C     NP2=NUMBER OF NODAL D O F
      DO 2 I=1,NP1
      DO 2 K=1,NP1
    2 YD(I,K)=0.0
      DO 3 K=1,NP1
      DO 3 I=1,NP
    3 ZP(I,K)=0.0
      DO 5 K=1,NP1
      DO 5 I=1,NP2
      II=I+NP2
      YD(II,K)=YS(I,K)
      ZP(II,K)=YS(II,K)
    5 CONTINUE
      DO 6 I=1,NP2
      II=I+NP2
      YD(I,I)=1.0
      ZP(I,II)=1.0
    6 CONTINUE
      DO 7 K=1,LT
      KK=NP+K
    7 YD(KK,KK)=1.0
      CALL JINVER(YD,YDI,NP1,NP1)
      CALL MPROD(ZP,YDI,WW,8,10,NP,NP1,NP1)
      DO 22 I=1,NP
      DO 22 K=1,NP
   22 SM(I,K)=WW(I,K)
      DO 23 K=1,LT
      KK=NP+K
      DO 23 I=1,NP
   23 FE(I,K)=WW(I,KK)
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE GLTRAN(IPT,L,LT,N,NF,PHIK,S,F,D)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION S(8,8),F(8,2),D(8,2)
      DS=DSIN(PHIK)
      DC=DCOS(PHIK)
      GO TO (10,20,30,40),L
   10 DO 1 I=1,NF
      T2= S(2,I)*DS+S(3,I)*DC
      T3=-S(2,I)*DC+S(3,I)*DS
      S(2,I)=T2
      S(3,I)=T3
    1 CONTINUE
      DO 2 I=1,NF
      T2= S(I,2)*DS+S(I,3)*DC
      T3=-S(I,2)*DC+S(I,3)*DS
      S(I,2)=T2
      S(I,3)=T3
    2 CONTINUE
      DO 8 K=1,LT
      T2= F(2,K)*DS+F(3,K)*DC
      T3=-F(2,K)*DC+F(3,K)*DS
      F(2,K)=T2
    8 F(3,K)=T3
      RETURN
   20 CONTINUE
      IF(NF.EQ.6) J=5
      IF(NF.EQ.8) J=6
      J1=J+1
      DO 3 I=1,NF
      TJ = S(J,I)*DS+S(J1,I)*DC
      TJ1=-S(J,I)*DC+S(J1,I)*DS
      S(J,I)=TJ
      S(J1,I)=TJ1
    3 CONTINUE
      DO 4 I=1,NF
      TJ = S(I,J)*DS+S(I,J1)*DC
      TJ1=-S(I,J)*DC+S(I,J1)*DS
      S(I,J)=TJ
      S(I,J1)=TJ1
    4 CONTINUE
      DO 9 K=1,LT
      TJ = F(J,K)*DS+F(J1,K)*DC
      TJ1=-F(J,K)*DC+F(J1,K)*DS
      F(J,K)=TJ
    9 F(J1,K)=TJ1
      RETURN
   30 CONTINUE
      DO 15 K=1,LT
      DJ2=D(2,K)*DS-D(3,K)*DC
      DJ3=D(2,K)*DC+D(3,K)*DS
      D(2,K)=DJ2
   15 D(3,K)=DJ3
      RETURN
   40 CONTINUE
```

```
      IF(NP.EQ.6) J=5
      IF(NP.EQ.8) J=6
      J1=J+1
      DO 16 K=1,LT
      DJ1=D(J,K)*DS-D(J1,K)*DC
      DJ2=D(J,K)*DC+D(J1,K)*DS
      D(J,K)=DJ1
   16 D(J1,K)=DJ2
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE ECCNTR(IPT,L,LT,N,NP,Z,R,R2,S,F,D)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION S(8,8),F(8,2),D(8,2)
      A=-Z
      B=-N*N*R2/R/(R2+Z)
      C=R2/(R2+Z)
      GO TO (10,20,30,40),L
   10 DO 1 I=1,NP
      S(1,I)=S(1,I)+A*S(3,I)
      IF(NP.EQ.6) GO TO 1
      S(2,I)=S(2,I)+B*S(4,I)
      S(4,I)=S(4,I)*C
    1 CONTINUE
      DO 2 I=1,NP
      S(I,1)=S(I,1)+A*S(I,3)
      IF(NP.EQ.6) GO TO 2
      S(I,2)=S(I,2)+B*S(I,4)
      S(I,4)=S(I,4)*C
    2 CONTINUE
      DO 5 K=1,LT
      F(1,K)=F(1,K)+A*F(3,K)
      IF(NP.EQ.6) GO TO 5
      F(2,K)=F(2,K)+B*F(4,K)
      F(4,K)=F(4,K)*C
    5 CONTINUE
      RETURN
   20 IF(NP.EQ.6) J=4
      IF(NP.EQ.8) J=5
      J1=J+1
      J2=J+2
      J3=J+3
      DO 3 I=1,NP
      S(I,J)=S(I,J)+A*S(I,J2)
      IF(NP.EQ.6) GO TO 3
      S(I,J1)=S(I,J1)+B*S(I,J3)
      S(I,J3)=S(I,J3)*C
    3 CONTINUE
      DO 4 I=1,NP
      S(J,I)=S(J,I)+A*S(J2,I)
      IF(NP.EQ.6) GO TO 4
      S(J1,I)=S(J1,I)+B*S(J3,I)
      S(J3,I)=S(J3,I)*C
    4 CONTINUE
      DO 15 K=1,LT
      F(J,K)=F(J,K)+A*F(J2,K)
      IF(NP.EQ.6) GO TO 15
      F(J1,K)=F(J1,K)+B*F(J3,K)
      F(J3,K)=F(J3,K)*C
   15 CONTINUE
      RETURN
C
   30 CONTINUE
      DO 16 K=I,LT
      D(3,K)=D(1,K)*A+D(3,K)
      IF(NP.EQ.6) GO TO 16
      D(4,K)=D(2,K)*B+D(4,K)*C
   16 CONTINUE
      RETURN
   40 CONTINUE
      DO 17 K=1,LT
      D(7,K)=D(5,K)*A+D(7,K)
      IF(NP.EQ.6) GO TO 17
      D(8,K)=D(6,K)*B+D(8,K)*C
   17 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE RESULT(IPT,IT,LT,LDC,NEL,NE,NF,NP,UNE,E,J,
     *       TK,RK,THK,PHK,SK,IDL,PS,YI,DSP,PSR,SSR)
C
C     EVALUATE THE DISPLACEMENTS AND THE STRESSES AT
C     INTERMIDIATE POINTS BY RUNGE KUTTA PROCESS USING THE
C     INITIAL CONDITIONS YI AT THE BOUNDARY.
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HYP/AH,BH,XH,XW
      COMMON/EM/E1,E2,E3
      DIMENSION PSR(8,1),DSP(8,1),SSR(12,1),YI(10,2),TFP(10,2),
     *   AK(8,2),BK(8,2),CK(8,2),DK(8,2),SA(8,10),RK(2),THK(2),PZ(10)
     *   SK(2),RX(51),TX(51),FX(51),SX(51),PS(10,2),IDL(5),
     *   PIX(10),E1(5,4),E2(5,4),E3(5,2),ES1(5,2),ES2(5,2),TDS(4,2),
     *   TDFP(8,2),PHK(2),PLD(10,51),DS(4,2)
      LM=0
      NF1=NF+LT
      NDOF=NF/2
      NRK=NP
      NM1=NRK-1
      IF(IT.EQ.5) GO TO 26
      RJ=(RK(2)-RK(1))/NM1
      TJ=(THK(2)-THK(1))/NM1
      FJ=(PHK(2)-PHK(1))/NM1
      SJ=(SK(2)-SK(1))/NM1
      DO 20 I=1,NRK
      I1=I-1
      TX(I)=THK(1)+I1*TJ
      FX(I)=PHK(1)+I1*FJ
   20 CONTINUE
      GO TO (21,22,23,23),IT
   21 DO 15 I=1,NRK
   15 RX(I)=RK(1)
      GO TO 24
   22 DO 16 I=1,NRK
      I1=I-1
   16 RX(I)=RK(1)+I1*RJ
```

```
      GO TO 24
   23 PHI=PHK(1)
      DSN=DSIN(PHI)
      R2=RK(1)/DSN
      RX(1)=RK(1)
      XR=AH*DSN-RK(1)
      DO 17 I=2,NRK
      PHI=PK(I)
      DSN=DSIN(PHI)
      IF(IT.EQ.3) RX(I)=R2/DSN
      IF(IT.EQ.4) RX(I)=AH*DSN-XR
   17 CONTINUE
      GO TO 24
   26 CALL HYPRB(LM,NRK,RK,THK,SK,RX,FX,TX,XJ,TJ)
   24 CONTINUE
      II=5*LT
      DO 25 K=1,NP
      DO 25 I=1,II
   25 PLD(I,K)=0.0
      DO 255 I=1,II
  255 PIX(I)=0.0
   57 IF(LDC.EQ.1.OR.NEL.EQ.0) GO TO 55
      CALL PLSEG(IPT,LT,NP,IDL,PS,PLD,PIX)
   55 CONTINUE
      IF(LDC.EQ.0.OR.NH.GT.0) GO TO 56
      CALL DLSEG(IT,ITA,NP,UNW,FX,TX,PLD)
   56 CONTINUE
C
      IF(IPT.LT.3) GO TO 60
      WRITE(6,1210) NP
      DO 59 K=1,NP
   59 WRITE(6,1200) K,(PLD(I,K),I=1,II)
 1210 FORMAT(//'*** LOADS ON SEGMENT AT ',I5,'   POINTS'/
     *  '  POINT NUMBER',44X,'PX',12X,'PY',12X,'PZ',10X,
     *  'TEMP',9X,'T.GRAD.'//)
 1200 FORMAT(I5,5X,10E10.3)
   60 CONTINUE
      DO 11 K=1,LT
      DO 11 I=1,NF1
   11 TPP(I,K)=0.0
      DO 12 K=1,LT
      KK=NP+K
   12 TPP(KK,K)=1.0
C     START INTEGRATION BY RUNGE KUTTA
C
      DO 100 J=1,NRK
      DO 1 I=1,NDOF
    1 DSP(I,J)=YI(I,1)
      IF(LT.EQ.1) GO TO 72
      DO 71 I=1,NDOF
      KK=I+NDOF
   71 DSP(KK,J)=YI(I,2)
   72 CONTINUE
      DO 2 I=1,NDOF
      KK=I+NDOF
    2 PSR(I,J)=YI(KK,1)
      IF(LT.EQ.1) GO TO 74
      DO 73 I=1,NDOF
      KK=I+NDOF
   73 PSR(KK,J)=YI(KK,2)
   74 CONTINUE
      IF(J.EQ.NRK) GO TO 30
      J1=J+1
      IF(IT.NE.5) GO TO 27
      XJ2=XJ*XJ
      RJ=RX(J1)-RX(J)
      RJ2=RJ*RJ
      SJ=(XJ2+RJ2)**.5
      FJ=FX(J1)-FX(J)
   27 CONTINUE
      DO 3 K=1,LT
      DO 3 I=1,NF
    3 TPP(I,K)=YI(I,K)
      IF(J.GT.1) GO TO 50
      X1=RX(J)
      X2=TX(J)
      X3=FX(J)
      DO 40 I=1,II
   40 PX(I)=PLD(I,J)
      CALL FLUGGE(IT,LT,NH,NF,X1,X2,X3,PX,E,G,TK,R2,SA)
      CALL MPROD(SA,TPP,TDFP,8,10,NF,NF1,LT)
C
   30 CONTINUE
      DO 80 K=1,LT
      DO 80 I=1,NDOF
   80 DDS(I,K)=TDFP(I,K)
      DO 81 I=1,NDOF
      KK=I+NDOF
      DS(I,1)=DSP(I,J)
   81 DS(I,2)=DSP(KK,J)
      CALL MPROD(E1,DDS,ES1,5,4,5,NDOF,LT)
      CALL MPROD(E2,DS,ES2,5,4,5,NDOF,LT)
      DO 82 I=1,5
   82 SSR(I,J)=ES1(I,1)+ES2(I,1)+E3(I,1)
      IF(NF.NE.8) GO TO 84
      SSR(6,J)=PSR(4,J)+R2*SSR(3,J)
      GO TO 86
   84 CONTINUE
      SSR(6,J)=R2*SSR(3,J)
   86 IF(LT.EQ.1) GO TO 87
      DO 85 I=1,5
      KK=6+I
   85 SSR(KK,J)=ES1(I,2)+ES2(I,2)+E3(I,2)
      IF(NF.NE.8) GO TO 88
      SSR(12,J)=PSR(8,J)+R2*SSR(9,J)
      GO TO 87
   88 CONTINUE
      SSR(12,J)=R2*SSR(9,J)
   87 CONTINUE
      IF(J.EQ.NRK) GO TO 200
      DO 4 K=1,LT
      DO 4 I=1,NF
    4 AK(I,K)=SJ*TDFP(I,K)
      DO 5 K=1,LT
      DO 5 I=1,NF
    5 TPP(I,K)=YI(I,K)+AK(I,K)/2.
C
      X15=RX(J)+RJ/2.
      X25=TX(J)+TJ/2.
      X35=FX(J)+FJ/2.
```

```
      DO 41 I=1,II
   41 PX(I)=PLD(I,J)+PIX(I)
      CALL PLUGGE(IT,LT,NH,NP,X15,X25,X35,PX,E,U,TK,R2,SA)
      CALL MPROD(SA,TFP,TDFP,8,10,NP,NP1,LT)
      DO 6 K=1,LT
      DO 6 I=1,NP
    6 BK(I,K)=SJ*TDFP(I,K)
      DO 7 K=1,LT
      DO 7 I=1,NP
    7 TFP(I,K)=YI(I,K)+BK(I,K)/2.
C
      CALL MPROD(SA,TFP,TDFP,8,10,NP,NP1,LT)
      DO 8 K=1,LT
      DO 8 I=1,NP
    8 CK(I,K)=SJ*TDFP(I,K)
      DO 9 K=1,LT
      DO 9 I=1,NP
    9 TFP(I,K)=YI(I,K)+CK(I,K)
C
      J1=J+1
      X10=RX(J1)
      X20=TX(J1)
      X30=PX(J1)
      DO 42 I=1,II
   42 PX(I)=PLD(I,J1)
      CALL PLUGGE(IT,LT,NH,NP,X10,X20,X30,PX,E,U,TK,R2,SA)
      CALL MPROD(SA,TFP,TDFP,8,10,NP,NP1,LT)
      DO 10 K=1,LT
      DO 10 I=1,NP
   10 DK(I,K)=SJ*TDFP(I,K)
C
      DO 18 K=1,LT
      DO 18 I=1,NP
   18 YI(I,K)=YI(I,K)+(1/6.)*(AK(I,K)+2.*BK(I,K)+2.*CK(I,K)+
     *  DK(I,K))
  100 CONTINUE
  200 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE PLUGGE(IT,LT,N,NF,R,T,PHI,PS,E,PNU,TK,R2,SA)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/EK/E1,E2,E3
      COMMON/HYP/AH,BH,XH
      DIMENSION SA(8,10),
     C    A1(4,4),A2(4,4),A3(4,2),B1(4,4),B2(4,5),B3(4,2),
     C    E1(5,4),E2(5,4),E3(5,2),G1(4,4),G2(4,4),G3(4,2),
     C    BE21(4,4),BE22(4,4),BE23(4,2),BEA1(4,4),BEA2(4,4),
     C    BEA3(4,2),PS(10),TEMP(4,2)
      PI=3.141592654
      ED=E*T/(1-PNU*PNU)
      EK=E*(T**3)/12./(1-PNU*PNU)
      DC=DCOS(PHI)
      DS=DSIN(PHI)
      NF1=NF+LT
      NF2=NF/2
      IF(IT.EQ.5) GO TO 25
      R=1./R
      R2=R*DS
      R1=0.0
      DR1=0.0
      IF(IT.EQ.3) R1=R2
      IF(IT.EQ.4) R1=1./AH
C
      GO TO 30
   25 CONTINUE
      AH2=AH*AH
      AH4=AH2*AH2
      BH2=BH*BH
      BH4=BH2*BH2
      AB=AH2*BH2
      AR=(AH2*DS*DS-BH2*DC*DC)**.5
      Z=BH2*DC/AR
      RH=(R*R/AH4+Z*Z/BH4)**.5
      R1=-AB*(RH)**3
      DR1=-3*AB*RH*(R*DC/AH4+Z*DS/BH4)
      R2=R/DS
C     WRITE(6,500) R,R1,R2,DR1,PHI,T,DC,DS
  500 FORMAT(/8E12.4)
      R1=1./R1
      R=1./R
      R2=R*DS
   30 CONTINUE
C
      DO 1 K=1,4
      DO 1 I=1,4
    1 A1(I,K)=0.0
      DO 2 K=1,4
      DO 2 I=1,4
    2 A2(I,K)=0.0
      DO 3 K=1,LT
      DO 3 I=1,4
    3 A3(I,K)=0.0
      DO 4 K=1,4
      DO 4 I=1,4
    4 B1(I,K)=0.0
      DO 5 K=1,5
      DO 5 I=1,4
    5 B2(I,K)=0.0
      DO 6 K=1,LT
      DO 6 I=1,4
    6 B3(I,K)=0.0
C
      DO 7 K=1,4
      DO 7 I=1,5
    7 E1(I,K)=0.0
      DO 8 K=1,4
      DO 8 I=1,5
    8 E2(I,K)=0.0
      DO 9 K=1,LT
      DO 9 I=1,5
    9 E3(I,K)=0.0
C
      R12=R1-R2
      R1R=R1*R12
      R2R=R2*R12
      RC=R*DC
```

```
      PS=P*S
      URC=PNU*RC
      URN=PNU*RN
      UDRC=ED*URC
      UKRC=EK*URC
      EKRC=EK*RC
      EKRN=EK*RN
      EDRN=ED*RN
      EDRC=ED*RC
      PU2=(1.-PNU)/2.
      TKU=TK*(1+PNU)
      U2EK=PU2*EK
      U2ED=PU2*ED
      UEK=PNU*EK
      UED=PNU*ED
      U2KRN=PU2*EKRN
      CA1=ED+EK*R1R
      CA2=ED+EK*R2R
      CA3=ED+EK*(R1*R1-3.*R2R)
      CA12=CA1/CA2
C
      A1(1,1)= DR1*R1-URC*CA12
      A1(1,2)=-UED*R2R/CA2-URN*RN*CA12
      A1(1,3)=-DR1*R1*R1-UDRC*R12/CA2
      A1(1,4)=-URN*R1
      A1(2,1)=-1.0
      A1(2,3)= R1
      A1(3,1)=-UKRC*R12/CA2
      A1(3,2)=-R1-UED*R2/CA2-URN*EKRN*R12/CA2
      A1(3,3)=-UDRC/CA2
      A1(3,4)=-URN
      A1(4,1)=-EKRN*(R1-3.*R2)/CA3
      A1(4,2)=-EKRN*RC*(R1-3.*R2)/CA3
      A1(4,3)= RN*CA1/CA3-EKRN*R1*R2/CA3
      A1(4,4)= RC
C
      A2(1,1)=-CA12/EK
      A2(1,3)= R12/CA2
      A2(3,1)=-R12/CA2
      A2(3,3)= 1./CA2
      A2(4,4)= 1./PU2/CA3
C
      DO 15 K=1,LT
      KK=4+(K-1)*5
      K1=KK+1
      TP1=(PS(KK)+PS(K1))/2.
      TP2=PS(KK)-PS(K1)
      T01=TP1+TP2*T*R1/12.
      T02=TP1+TP2*T*R2/12.
      T11=TP2/T+TP1*R1
      T12=TP2/T+TP1*R2
      A3(1,K)=TKU*(ED*R12*T02/CA2+T12*CA12)
      A3(3,K)=TKU*(ED*T02+EK*R12*T12)/CA2
      E3(1,K)=TKU*EK*T11
      E3(4,K)=-TKU*ED*T01
   15 CONTINUE
C
      B1(1,1)=-RC
      B1(1,2)= 1.0
      B1(2,2)=-RC
      B1(2,3)=-R1
      B1(3,2)= R1
      B1(3,3)=-RC
      B1(4,4)=-RC
C
      B2(1,1)= RC
      B2(1,2)=-RN
      B2(1,3)=-RN
      B2(2,1)= RN*RN
      B2(2,2)=-RN*RC
      B2(2,3)=-RN*RC
      B2(2,4)=-R2
      B2(3,3)=-RN*R1
      B2(3,4)= RC
      B2(3,5)=-RN
      B2(4,1)=-RN*R2
      B2(4,2)= RC*R2
      B2(4,3)=-RC*R12
      B2(4,4)= RN
      B2(4,5)=-RC
C
      DO 16 K=1,LT
      KK=1+(K-1)*5
      K1=KK+1
      K2=KK+2
      B3(2,K)= PS(K2)
      B3(3,K)=-PS(KK)
   16 B3(4,K)=-PS(K1)
C
      E1(1,1)=-UEK
      E1(2,4)=-U2EK*R2
      E1(3,4)= U2EK*(R1-2.*R2)
      E1(4,3)= UED
      E1(5,4)= U2ED
C
      E2(1,1)=-EKRC+UEK*R1*DR1
      E2(1,2)=-EKRN*RN-EK*R2R
      E2(1,3)=-EKRC*R12-UEK*R1*R1*DR1
      E2(1,4)=-EKRN*R1
      E2(2,1)= 2.*U2KRN
      E2(2,2)= 2.*U2KRN*RC
      E2(2,3)=-U2KRN*R2
      E2(2,4)= PU2*EKRC*R2
      E2(3,1)= 2.*U2KRN
      E2(3,2)= 2.*U2KRN*RC
      E2(3,3)=-2.*U2KRN*R1
      E2(3,4)=-PU2*EKRC*(R1-2.*R2)
      E2(4,1)= EKRC*R12
      E2(4,2)= ED*(R2+PNU*R1)+EK*R12*(RN*RN-R2*R2)
      E2(4,3)= EDRC-EKRC*R2R
      E2(4,4)= EDRN
      E2(5,1)=-U2KRN*R12
      E2(5,2)=-U2KRN*RC*R12
      E2(5,3)=-PU2*EDRN+U2KRN*R12*R12
      E2(5,4)=-PU2*EDRC
C
      CALL MPROD(B2,E1,BE21,4,5,4,5,4)
      CALL MPROD(B2,E2,BE22,4,5,4,5,4)
      CALL MPROD(B2,E3,BE23,4,5,4,5,LT)
      CALL MPROD(BE21,A1,BEA1,4,4,4,4,4)
```

```
      CALL MPROD(BE21,A2,BEA2,4,4,4,4,4)
      CALL MPROD(BE21,A3,BEA3,4,4,4,4,LT)
C
      CALL DGMADD(BE22,BEA1,G1,4,4)
      CALL DGMADD(B1,BEA2,G2,4,4)
      CALL DGMADD(B3,BE23,TEMP,4,LT)
      CALL DGMADD(TEMP,BEA3,G3,4,LT)
C
      DO 50 K=1,NF1
      DO 50 I=1,NP
   50 SA(I,K)=0.0
      DO 60 L=1,NP2
      LN=L+NF2
      DO 60 I=1,NF2
      IN=I+NF2
      SA(I,L)=A1(I,L)
      SA(I,LN)=A2(I,L)
      SA(IN,L)=G1(I,L)
      SA(IN,LN)=G2(I,L)
   60 CONTINUE
      DO 70 K=1,LT
      IK=NF+K
      DO 70 I=1,NF2
      IN=I+NF2
      SA(I,IK)=A3(I,K)
      SA(IN,IK)=G3(I,K)
   70 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE HYPRB(LM,NP,RK,THK,SK,RX,PX,TX,SJ,TJ)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HYP/AH,BH,XH
      DIMENSION RK(1),SK(1),THK(1),RX(1),PX(1),TX(1)
C
C
      IF(LM.EQ.0) GO TO 15
      WRITE(6,800)
  800 FORMAT(//' HYPERBOLOID CONSTANTS IN THE ORDER J X R R1 ',
     + 'R2 DR1  PHI  TH'/)
   15 PI=3.141592654
      NM1=N2-1
      SJ=(SK(2)-SK(1))/NM1
      TJ=(THK(2)-THK(1))/NM1
      S1=SK(1)
      AB=AH/BH
      AH2=AH*AH
      DO 100 J=1,NP
      J1=J-1
      XX=S1-XH+J1*SJ
      X2=(XX*AB)**2
      RR2=X2+AH2
      RX(J)=(RR2)**.5
      IF(RR2.LE.AH2) GO TO 10
      DT=(RR2/X2)**.5
      DT=DT/AB
      PHI=DATAN(DT)
      IF(XX.LT.0.0) PHI=PI-PHI
      PX(J)=PHI
      GO TO 11
   10 PX(J)=PI/2.
   11 CONTINUE
      TX(J)=THK(1)+J1*TJ
C
      IF(LM.EQ.0) GO TO 100
C
      DC=DCOS(PHI)
      DS=DSIN(PHI)
      R=RX(J)
      R2=R/DS
      AH2=AH*AH
      BH2=BH*BH
      AH4=AH2*AH2
      BH4=BH2*BH2
      A1=(AH2*DS*DS-BH2*DC*DC)**.5
      Z=BH2*DC/A1
      RH=(R*R/AH4+Z*Z/BH4)**.5
      R1=-AH2*BH2*(RH)**3
      DR1=-3*AH2*BH2*RH*(R*DC/AH4+Z*DS/BH4)
C
      WRITE(6,1000) J,XX,R,R1,R2,DR1,PX(J),TX(J)
 1000 FORMAT(I3,7E12.4)
  100 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE PLOAD(IPT,NE,NTL,NHL,NS,ILOAD,ACEL,BCEL,
     *   APL,BPL)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION NS(20),ILOAD(20,5),ACEL(200),BCEL(200),
     *  APL(200,20),BPL(200,20)
C
      K1=0
      K2=0
      MAX=0
      DO 500 I=1,NE
      NSEG=NS(I)
      NSEG1=NSEG+1
      I1=I-1
      IF(I.EQ.1) GO TO 10
      MAX=MAX+(NS(I1)+1)*5
   10 CONTINUE
C
      DO 400 K=1,5
      KIL=ILOAD(I,K)
      IF(KIL.EQ.0) GO TO 400
      IF(NTL.NE.0) GO TO 100
      K1=K2+1
      K2=K1+1
      ATLD=ACEL(K1)
      ABLD=ACEL(K2)
      GO TO(5,6,7,8),KIL
    5 CONTINUE
      DO 15 KN=1,NSEG1
```

```
      KN1=KN-1
      K5=MAX+KN1*5+K
   15 APL(K5,1)=ATLD
      GO TO 400
    6 CONTINUE
      AINC=(ALLD-ATLD)/NSEG
      DO 20 KN=1,NSEG1
      KN1=KN-1
      K5=MAX+KN1*5+K
   20 APL(K5,1)=ATLD+KN1*AINC
      GO TO 400
    7 AINC=ABLD-ATLD
      NSEG2=NSEG*NSEG
      DO 30 KN=1,NSEG1
      KN1=KN-1
      KN2=KN1*KN1
      K5=MAX+KN1*5+K
   30 APL(K5,1)=ATLD+AINC*KN2/NSEG2
      GO TO 400
C
    8 CONTINUE
      GO TO 400
  100 CONTINUE
      K1=K2
      K2=K1+NHL
      DO 200 N=1,NHL
      K1=K1+1
      K2=K2+1
      IF(NTL.EQ.3) GO TO 40
      ATLD=ACEL(K1)
      ABLD=ACEL(K2)
      AINC=(ABLD-ATLD)/NSEG
   40 CONTINUE
      IF(NTL.EQ.2) GO TO 45
      BTLD=BCEL(K1)
      BBLD=BCEL(K2)
      BINC=(BBLD-BTLD)/NSEG
   45 CONTINUE
C
      GO TO(50,50,60,80),KIL
   50 IF(NTL.EQ.3) GO TO 51
      DO 55 KN=1,NSEG1
      KN1=KN-1
      K5=MAX+KN1*5+K
   55 APL(K5,N)=ATLD+KN1*AINC
   51 CONTINUE
      IF(NTL.EQ.2) GO TO 200
      DO 56 KN=1,NSEG1
      KN1=KN-1
      K5=MAX+KN1*5+K
   56 BPL(K5,N)=BTLD+KN1*BINC
      GO TO 200
   60 CONTINUE
      IF(NTL.EQ.3) GO TO 61
      DO 65 KN=1,NSEG1
      KN1=KN-1
      KN2=KN1*KN1
      K5=MAX+KN1*5+K
   65 APL(K5,N)=ATLD+AINC*KN2/NSEG
   61 CONTINUE
      IF(NTL.EQ.2) GO TO 200
      DO 66 KN=1,NSEG1
      KN1=KN-1
      KN2=KN1*KN1
      K5=MAX+KN1*5+K
   66 BPL(K5,N)=BTLD+BINC*KN2/NSEG
      GO TO 200
   80 CONTINUE
  200 CONTINUE
  400 CONTINUE
  500 CONTINUE
C
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE PLSEG(IPRINT,LT,NP,IDL,PS,PLD,PIX)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION IDL(5),PS(10,2),PLD(10,51),PIX(10),PINC(2)
      II=2*LT
      DO 2 I=1,II
    2 PIX(I)=0.0
C
      NM1=NP-1
      DO 100 I=1,5
      I5=I+5
      ILOAD=IDL(I)
      IF(ILOAD.EQ.0) GO TO 100
      GO TO (10,20,30),ILOAD
   10 CONTINUE
      DO 11 K=1,NP
      PLD(I,K)=PS(I,1)
      IF(LT.EQ.1) GO TO 11
      PLD(I5,K)=PS(I,2)
   11 CONTINUE
      GO TO 100
   20 CONTINUE
      DO 12 L=1,LT
   12 PINC(L)=(PS(I5,L)-PS(I,L))/NM1
      PIX(I)=PINC(1)/2
      IF(LT.EQ.1) GO TO 14
      PIX(I5)=PINC(2)/2
   14 CONTINUE
      DO 21 K=1,NP
      KM1=K-1
      PLD(I,K)=PS(I,1)+KM1*PINC(1)
      IF(LT.EQ.1) GO TO 21
      PLD(I5,K)=PS(I,2)+KM1*PINC(2)
   21 CONTINUE
      GO TO 100
C
   30 CONTINUE
      NM2=NM1*NM1
      DO 15 L=1,LT
   15 PINC(L)=(PS(I5,L)-PS(I,L))/NM2
      PIX(I)=PINC(1)/4
      IF(LT.EQ.1) GO TO 16
      PIX(I5)=PINC(2)/4.
```

```
   16 CONTINUE
      DO 32 K=1,NP
      KM1=K-1
      KM2=KM1*KM1
      PLD(I,K)=PS(I,1)+KM2*PINC(1)
      IF(LT.EQ.1) GO TO 32
      PLD(I5,K)=PS(I5,2)+KM2*PINC(2)
   32 CONTINUE
  100 CONTINUE
  200 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE DLSEG(IT,ITA,NP,UNW,FX,TX,PLD)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION FX(120),TX(120),PLD(10,1)
C
      IF(IT.NE.1) GO TO 20
C     GRAVITY LOAD FOR CYLINDER
   10 DO 1 I=1,NP
      PX=UNW*TX(I)
    1 PLD(1,I)=PLD(1,I)+PX
      GO TO 30
C     GRAVITY LOAD FOR CONE AND SPHERE
   20 DO 2 I=1,NP
      PHI=FX(I)
      Q=UNW*TX(I)
      DC=DCOS(PHI)
      DS=DSIN(PHI)
      PX=Q*DS
      PZ=-Q*DC
      PLD(1,I)=PLD(1,I)+PX
      PLD(3,I)=PLD(3,I)+PZ
    2 CONTINUE
   30 CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE MPROD(A,B,C,N1,N2,N,M,L)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N1,1),B(N2,1),C(N1,1)
      DO 20 I=1,N
      DO 20 J=1,L
      T=0.0
      DO 10 K=1,M
   10 T=T+A(I,K)*B(K,J)
   20 C(I,J)=T
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      SUBROUTINE DGMADD(A,B,C,N,M)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,M),B(N,M),C(N,M)
      DO 10 I=1,N
      DO 10 J=1,M
   10 C(I,J)=A(I,J)+B(I,J)
      RETURN
      END
END OF FILE
```

APPENDIX D

DATA FILES AND SAMPLE OUTPUT

OF EXAMPLE APPLICATIONS

```
NATURAL DRAFT HYPERBOLOID COOLING TOWER; WIND LOAD ;
1,21,8,0,25.0,
5,6,4000.0,.15,.15,0.0,
1,0.0,85.99,
2,10.0,84.94,
3,116.67,85.62,
4,223.33,105.7,
5,330.0,136.8,
6,355.0,145.0,0,1,1,1,
1,5,1,2,2,2.0,0.5,
82.5,204.1,60.0,0.0,
2,5,2,3,2,0.5,0.5,
82.5,204.1,60.0,0.0,
3,5,3,4,2,0.5,0.5,
82.5,204.1,60.0,0.0,
4,5,4,5,2,0.5,0.5,
82.5,204.1,60.0,0.0,
5,5,5,6,2,0.5,2.5,
82.5,204.1,60.0,0.0,
5,0,8,2,24,1,
1,0,0,2,0,0,
2,0,0,2,0,0,
3,0,0,2,0,0,
4,0,0,2,0,0,
5,0,0,2,0,0,
-.0530,-.0424,-.0106,.0265,.0636,.0689,.0477,.0212,
.0212,.0212,.0212,.0212,.0212,.0212,.0212,.0212,
.0212,.0212,.0477,.0689,.0636,.0265,-.0106,-.0424,
-.0526,-.0421,-.0105,.0263,.0632,.0684,.0474,.0210,
.0210,.0210,.0210,.0210,.0210,.0210,.0210,.0210,
.0210,.0210,.0474,.0684,.0632,.0263,-.0105,-.0421,
-.0526,-.0421,-.0105,.0263,.0632,.0684,.0474,.0210,
.0210,.0210,.0210,.0210,.0210,.0210,.0210,.0210,
.0210,.0210,.0474,.0684,.0632,.0263,-.0105,-.0421,
-.0478,-.0383,-.0096,.0239,.0574,.0622,.0430,.0191,
.0191,.0191,.0191,.0191,.0191,.0191,.0191,.0191,
.0191,.0191,.0430,.0622,.0574,.0239,-.0096,-.0383,
-.0478,-.0383,-.0096,.0239,.0574,.0622,.0430,.0191,
.0191,.0191,.0191,.0191,.0191,.0191,.0191,.0191,
.0191,.0191,.0430,.0622,.0574,.0239,-.0096,-.0383,
-.0414,-.0331,-.0083,.0207,.0497,.0538,.0372,.0165,
.0165,.0165,.0165,.0165,.0165,.0165,.0165,.0165,
.0165,.0165,.0372,.0538,.0497,.0207,-.0883,-.0331,
-.0414,-.0331,-.0083,.0207,.0497,.0538,.0372,.0165,
.0165,.0165,.0165,.0165,.0165,.0165,.0165,.0165,
.0165,.0165,.0372,.0538,.0497,.0207,-.0883,-.0331,
-.0304,-.0243,-.0061,.0152,.0365,.0395,.0273,.0122,
.0122,.0122,.0122,.0122,.0122,.0122,.0122,.0122,
.0122,.0122,.0273,.0395,.0365,.0152,-.0061,-.0243,
-.0304,-.0243,-.0061,.0152,.0365,.0395,.0273,.0122,
.0122,.0122,.0122,.0122,.0122,.0122,.0122,.0122,
.0122,.0122,.0273,.0395,.0365,.0152,-.0061,-.0243,
-.0256,-.0205,-.0051,.0127,.0307,.0332,.023,.0102,
.0102,.0102,.0102,.0102,.0102,.0102,.0102,.0102,
.0102,.0102,.0230,.0332,.0307,.0127,-.0051,-.0205,
```

```
NATURAL DRAFT HYPERBOLOID COOLING TOWER; DEAD LOAD
2,21,1,1,25.0,
3,4,576000.0,.15,.15,0.0,
1,0.0,86.0,
2,10.0,84.94,
3,330.0,136.81,
4,355.0,145.0,0,1,1,1,
1,5,1,2,1,2.0,0.5,
82.5,204.1,60.0,0.0,
2,5,2,3,1,0.5,0.5,
82.5,204.1,60.0,0.0,
3,5,3,4,1,0.5,2.5,
82.5,204.1,60.0,0.0,
0,0,0,0,
PINCHED CYLINDER  'LINE CIRCUMFERENTIAL LOAD'
2,41,1,0,25.0,
2,3,4320000.,,.3,.15,
1,0.0,4.0,1,1,1,1,
2,10.0,4.,0,0,0,0,
3,20.0,4.,1,1,1,1,
1,1,1,2,1,0.1033,
0.0,
2,1,2,3,1,0.1033,
0.0,
0,1,0,0,0,0,
2,
0.0,-1.0,0.0,0.0,
PINCHED CYLINDER  '2 CONCENTRATED LOADS'
2,41,1,0,25.0,
2,3,4320000.,,.3,.15,
1,0.0,4.0,1,1,1,1,
2,10.0,4.,0,0,0,0,
3,20.0,4.,1,1,1,1,
1,1,1,2,1,0.1033,
0.0,
2,1,2,3,1,0.1033,
0.0,
0,1,18,2,36,2,
2,
0,1,0,0,
-1.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
0.0,0.0,-1.0,0.0,0.0,0.0,0.0,0.0,
0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
```

```
1PINCHED CYLINDER  'LINE CIRCUMFERENTIAL LOAD'

***   ANALYSIS DATA   ***

IPRINT        =   2
NP            =  41
NPCR          =   1
LDC           =   0
BETA          = 25.000



***   STRUCTURAL DATA   ***

NUMBER OF ELEMENTS          =  2
NUMBER OF NODES             =  3
GLOBAL YOUNG S MODULUS      =  0.4320E+07
GLOBAL POISSON S RATIO      =  0.3000E+00
GLOBAL UNIT WEIGHT          =  0.1500E+00
GLOBAL THERMAL COEFFICIENT  =  0.0


***     NODAL DATA     ***

NODE      XCOOR        RCOOR                   R  W  V  U

   1       0.0        4.0000                   1  1  1  1
   2      10.0000     4.0000                   0  0  0  0
   3      20.0000     4.0000                   1  1  1  1


***   ELEMENT DATA   ***

EL IT TN DN NPI   TH1      TH2      EC1      EC2      HP1      HP2      HP3    UNW   PR         E          TCOEF

 1  1  1  2  1   0.1033   0.1033   0.0      0.0      0.0      0.0      0.0    0.15000.3000  0.4320E+07  0.0
 2  1  2  3  1   0.1033   0.1033   0.0      0.0      0.0      0.0      0.0    0.15000.3000  0.4320E+07  0.0


***   LOADING SPECIFICATIONS   ***

NUMBER OF LOADED ELEMENTS    =   0
NUMBER OF LOADED NODES       =   1
NUMBER OF HARMONICS TO FIT   =   1
LOADING TYPE CODE            =   0
NUMBER OF CIRCUMFR POINTS    =   0
HARMONIC INCREMENTAL         =   1



END OF FILE
```

```
*** NODAL LOADS ****

NODE          H. NUMB                      MS          SS           NS          TS

   2
                 0                         0.0       -0.1000E+01  0.0          0.0

****   CONNECTIVITY ARRAY    ***

   ELEMENT   SEGMENT   TOP NODE   BOT NODE
       1         1         1          2
       2         1         2          3


****   COORDINATE SYSTEM    ***
ELEMENT   NUMB OF SEG                X           R          TH          PHI          S          R2

    1          1
                                    0.0        4.000       0.103       1.571       0.0        4.000
                                  10.000       4.000       0.103       1.571      10.000      4.000
    2          1
                                  10.000       4.000       0.103       1.571      10.000      4.000
                                  20.000       4.000       0.103       1.571      20.000      4.000


***********************************
*   HARMONIC NUMBER       =    0    *
*   NUMBER OF EQUATIONS =   12    *
*   FULL BAND WIDTH       =   15    *
***********************************

NODAL LOADS ASSEMBELED IN RHS   LT=          1

   0.0            0.0              0.0          0.0
   0.0           -0.100000E+01     0.0          0.0
   0.0            0.0              0.0          0.0


**** DISPLACEMENT AT NODES


     R               W                V              U
  -0.0            -0.0             -0.0           -0.0
  -0.142532E-11   -0.357270E-04    0.285508E-11  -0.0
  -0.0            -0.0             -0.0            0.0


END OF FILE
```

```
********************************************
ELEMENT          1            SEGMENT        1
********************************************



STIFFNESS MATRIX BEFORE MODF.
  0.174608E+04 -0.349552E+04  0.432283E+03  0.0           -0.414982E+01 -0.161937E+02 -0.432284E+03  0.0
 -0.349552E+04  0.139950E+05 -0.168689E+04  0.0            0.161937E+02  0.631924E+02  0.168689E+04  0.0
 -0.432283E+03  0.168689E+04 -0.450307E+05  0.0            0.432284E+03  0.168689E+04  0.450307E+05  0.0
  0.0           0.0           0.0          -0.171666E+05  0.0           0.0           0.0           0.171666E+05
  0.414978E+01 -0.161938E+02  0.432284E+03  0.0           -0.174608E+04 -0.349552E+04 -0.432285E+03  0.0
  0.161936E+02 -0.631928E+02  0.168689E+04  0.0           -0.349552E+04 -0.139950E+05 -0.168689E+04  0.0
 -0.432284E+03  0.168689E+04 -0.450307E+05  0.0            0.432285E+03  0.168689E+04  0.450307E+05  0.0
  0.0           0.0           0.0          -0.171666E+05  0.0           0.0           0.0           0.171665E+05

DISPLACEMENT
 -0.0          -0.0          -0.0          -0.0           -0.142532E-11 -0.357270E-04  0.285508E-11 -0.0

STRESS RESULTANTS DUE TO DISPL
  0.578553E-03 -0.225767E-02 -0.602675E-01  0.0            0.124884E+00  0.500000E+00 -0.602676E-01  0.0

FIXED END STRESSES DUE TO LOAD
  0.0           0.0           0.0           0.0            0.0           0.0           0.0           0.0

STRESSES AT ENDS
  0.578553E-03 -0.225767E-02 -0.602675E-01  0.0            0.124884E+00  0.500000E+00 -0.602676E-01  0.0

INITIAL CONDITIONS
 -0.0          -0.0          -0.0          -0.0            0.578553E-03 -0.225767E-02 -0.602675E-01  0.0
END OF FILE
```

DISPLACEMENT

	R	W	V	U
1	-0.0	-0.0	-0.0	-0.0
2	-0.1891E-06	0.2870E-07	-0.3087E-07	0.0
3	-0.2005E-06	0.8049E-07	-0.6259E-07	0.0
4	-0.1423E-06	0.1247E-06	-0.9525E-07	0.0
5	-0.7697E-07	0.1523E-06	-0.1286E-06	0.0
6	-0.2930E-07	0.1654E-06	-0.1623E-06	0.0
7	-0.2902E-08	0.1691E-06	-0.1962E-06	0.0
8	0.7569E-08	0.1682E-06	-0.2301E-06	0.0
9	0.9008E-08	0.1660E-06	-0.2640E-06	0.0
10	0.6731E-08	0.1640E-06	-0.2978E-06	0.0
11	0.3831E-08	0.1626E-06	-0.3316E-06	0.0
12	0.1627E-08	0.1620E-06	-0.3653E-06	0.0
13	0.3816E-09	0.1617E-06	-0.3991E-06	0.0
14	-0.1243E-09	0.1617E-06	-0.4329E-06	0.0
15	-0.2467E-09	0.1618E-06	-0.4666E-06	0.0
16	-0.3246E-09	0.1618E-06	-0.5004E-06	0.0
17	-0.6310E-09	0.1619E-06	-0.5341E-06	0.0
18	-0.1341E-08	0.1622E-06	-0.5679E-06	0.0
19	-0.2451E-08	0.1626E-06	-0.6017E-06	0.0
20	-0.3596E-08	0.1634E-06	-0.6354E-06	0.0
21	-0.3798E-08	0.1644E-06	-0.6692E-06	0.0
22	-0.1217E-08	0.1651E-06	-0.7030E-06	0.0
23	0.6807E-08	0.1645E-06	-0.7369E-06	0.0
24	0.2298E-07	0.1610E-06	-0.7707E-06	0.0
25	0.4792E-07	0.1522E-06	-0.8043E-06	0.0
26	0.7604E-07	0.1367E-06	-0.8378E-06	0.0
27	0.8958E-07	0.1154E-06	-0.8709E-06	0.0
28	0.5236E-07	0.9608E-07	-0.9036E-06	0.0
29	-0.9190E-07	0.9821E-07	-0.9360E-06	0.0
30	-0.4078E-06	0.1567E-06	-0.9689E-06	0.0
31	-0.9278E-06	0.3200E-06	-0.1004E-05	0.0
32	-0.1574E-05	0.6326E-06	-0.1043E-05	0.0
33	-0.2027E-05	0.1092E-05	-0.1090E-05	0.0
34	-0.1584E-05	0.1573E-05	-0.1146E-05	0.0
35	0.9264E-06	0.1711E-05	-0.1209E-05	0.0
36	0.6975E-05	0.8070E-06	-0.1267E-05	0.0
37	0.1763E-04	-0.2178E-05	-0.1291E-05	0.0
38	0.3200E-04	-0.8348E-05	-0.1232E-05	0.0
39	0.4456E-04	-0.1807E-04	-0.1022E-05	0.0
40	0.4183E-04	-0.2939E-04	-0.6055E-06	0.0
41	-0.1391E-11	-0.3573E-04	0.2855E-11	0.0

END OF FILE

PRIMARY STRESS RESULTANTS

	MS	SS	NS	TS
1	0.5786E-03	-0.2258E-02	-0.6027E-01	0.0
2	0.1493E-03	-0.1200E-02	-0.6027E-01	0.0
3	-0.5068E-04	-0.4388E-03	-0.6027E-01	0.0
4	-0.1032E-03	-0.2554E-04	-0.6027E-01	0.0
5	-0.8552E-04	0.1324E-03	-0.6027E-01	0.0
6	-0.4792E-04	0.1484E-03	-0.6027E-01	0.0
7	-0.1478E-04	0.1080E-03	-0.6027E-01	0.0
8	0.6421E-05	0.5971E-04	-0.6027E-01	0.0
9	0.1668E-04	0.2358E-04	-0.6027E-01	0.0
10	0.1977E-04	0.3070E-05	-0.6027E-01	0.0
11	0.1929E-04	-0.5344E-05	-0.6027E-01	0.0
12	0.1766E-04	-0.6745E-05	-0.6027E-01	0.0
13	0.1612E-04	-0.5053E-05	-0.6027E-01	0.0
14	0.1515E-04	-0.2555E-05	-0.6027E-01	0.0
15	0.1480E-04	-0.2266E-06	-0.6027E-01	0.0
16	0.1499E-04	0.1655E-05	-0.6027E-01	0.0
17	0.1558E-04	0.2914E-05	-0.6027E-01	0.0
18	0.1635E-04	0.2991E-05	-0.6027E-01	0.0
19	0.1687E-04	0.6502E-06	-0.6027E-01	0.0
20	0.1631E-04	-0.5786E-05	-0.6027E-01	0.0
21	0.1335E-04	-0.1876E-04	-0.6027E-01	0.0
22	0.6384E-05	-0.3772E-04	-0.6027E-01	0.0
23	-0.5662E-05	-0.5791E-04	-0.6027E-01	0.0
24	-0.2152E-04	-0.6483E-04	-0.6027E-01	0.0
25	-0.3462E-04	-0.3010E-04	-0.6027E-01	0.0
26	-0.2943E-04	0.8902E-04	-0.6027E-01	0.0
27	0.2113E-04	0.3388E-03	-0.6027E-01	0.0
28	0.1531E-03	0.7370E-03	-0.6027E-01	0.0
29	0.3969E-03	0.1209E-02	-0.6027E-01	0.0
30	0.7432E-03	0.1492E-02	-0.6027E-01	0.0
31	0.1081E-02	0.1028E-02	-0.6027E-01	0.0
32	0.1119E-02	-0.1080E-02	-0.6027E-01	0.0
33	0.3083E-03	-0.5910E-02	-0.6027E-01	0.0
34	-0.2134E-02	-0.1414E-01	-0.6027E-01	0.0
35	-0.6987E-02	-0.2480E-01	-0.6027E-01	0.0
36	-0.1438E-01	-0.3325E-01	-0.6027E-01	0.0
37	-0.2256E-01	-0.2874E-01	-0.6027E-01	0.0
38	-0.2611E-01	0.7238E-02	-0.6027E-01	0.0
39	-0.1419E-01	0.9891E-01	-0.6027E-01	0.0
40	0.2990E-01	0.2661E+00	-0.6027E-01	0.0
41	0.1249E+00	0.5000E+00	-0.6027E-01	0.0

END OF FILE

SECONDARY STRESS RESULTANTS

	MT	MTS	MST	NT	NTS	NST
1	0.1696E-03	0.0	0.0	-0.1804E-01	0.0	0.0
2	0.4328E-04	0.0	0.0	-0.1477E-01	0.0	0.0
3	-0.1896E-04	0.0	0.0	-0.9037E-02	0.0	0.0
4	-0.3476E-04	0.0	0.0	-0.4148E-02	0.0	0.0
5	-0.2833E-04	0.0	0.0	-0.1084E-02	0.0	0.0
6	-0.1581E-04	0.0	0.0	0.3640E-03	0.0	0.0
7	-0.4970E-05	0.0	0.0	0.7758E-03	0.0	0.0
8	0.1662E-05	0.0	0.0	0.6837E-03	0.0	0.0
9	0.5153E-05	0.0	0.0	0.4376E-03	0.0	0.0
10	0.6104E-05	0.0	0.0	0.2123E-03	0.0	0.0
11	0.5914E-05	0.0	0.0	0.6431E-04	0.0	0.0
12	0.5366E-05	0.0	0.0	-0.1043E-04	0.0	0.0
13	0.4865E-05	0.0	0.0	-0.3676E-04	0.0	0.0
14	0.4549E-05	0.0	0.0	-0.3914E-04	0.0	0.0
15	0.4434E-05	0.0	0.0	-0.3345E-04	0.0	0.0
16	0.4488E-05	0.0	0.0	-0.2563E-04	0.0	0.0
17	0.4667E-05	0.0	0.0	-0.1297E-04	0.0	0.0
18	0.4904E-05	0.0	0.0	0.1363E-04	0.0	0.0
19	0.5068E-05	0.0	0.0	0.6593E-04	0.0	0.0
20	0.4909E-05	0.0	0.0	0.1511E-03	0.0	0.0
21	0.4031E-05	0.0	0.0	0.2580E-03	0.0	0.0
22	0.1947E-05	0.0	0.0	0.3367E-03	0.0	0.0
23	-0.1676E-05	0.0	0.0	0.2737E-03	0.0	0.0
24	-0.6478E-05	0.0	0.0	-0.1224E-03	0.0	0.0
25	-0.1051E-04	0.0	0.0	-0.1096E-02	0.0	0.0
26	-0.7128E-05	0.0	0.0	-0.2833E-02	0.0	0.0
27	0.5810E-05	0.0	0.0	-0.5209E-02	0.0	0.0
28	0.4521E-04	0.0	0.0	-0.7355E-02	0.0	0.0
29	0.1184E-03	0.0	0.0	-0.7108E-02	0.0	0.0
30	0.2231E-03	0.0	0.0	-0.5682E-03	0.0	0.0
31	0.3265E-03	0.0	0.0	0.1765E-01	0.0	0.0
32	0.3412E-03	0.0	0.0	0.5253E-01	0.0	0.0
33	0.1031E-03	0.0	0.0	0.1038E+00	0.0	0.0
34	-0.6246E-03	0.0	0.0	0.1573E+00	0.0	0.0
35	-0.2080E-02	0.0	0.0	0.1725E+00	0.0	0.0
36	-0.4310E-02	0.0	0.0	0.7145E-01	0.0	0.0
37	-0.6799E-02	0.0	0.0	-0.2619E+00	0.0	0.0
38	-0.7937E-02	0.0	0.0	-0.9503E+00	0.0	0.0
39	-0.4467E-02	0.0	0.0	-0.2034E+01	0.0	0.0
40	0.8643E-02	0.0	0.0	-0.3296E+01	0.0	0.0
41	0.3709E-01	0.0	0.0	-0.3999E+01	0.0	0.0

END OF FILE

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******************************************
ELEMENT          2            SEGMENT          1
******************************************



STIFFNESS MATRIX BEFORE MODF.
  0.174608E+04 -0.349552E+04  0.432283E+03  0.0           -0.414982E+01 -0.161937E+02 -0.432284E+03  0.0
 -0.349552E+04  0.139950E+05 -0.168689E+04  0.0            0.161937E+02  0.631924E+02  0.168689E+04  0.0
 -0.432283E+03  0.168689E+04 -0.450307E+05  0.0            0.432284E+03  0.168689E+04  0.450307E+05  0.0
  0.0           0.0           0.0          -0.171666E+05  0.0           0.0           0.0           0.171666E+05
  0.414978E+01 -0.161938E+02  0.432284E+03  0.0           -0.174608E+04 -0.349552E+04 -0.432285E+03  0.0
  0.161936E+02 -0.631928E+02  0.168689E+04  0.0           -0.349552E+04 -0.139950E+05 -0.168689E+04  0.0
 -0.432284E+03  0.168689E+04 -0.450307E+05  0.0            0.432285E+03  0.168689E+04  0.450307E+05  0.0
  0.0           0.0           0.0          -0.171666E+05  0.0           0.0           0.0           0.171666E+05

DISPLACEMENT
 -0.142532E-11 -0.357270E-04  0.285508E-11 -0.0           -0.0          -0.0          -0.0           0.0

STRESS RESULTANTS DUE TO DISPL
  0.124884E+00 -0.500000E+00 -0.602676E-01  0.0            0.578557E-03  0.225770E-02 -0.602677E-01  0.0

FIXED END STRESSES DUE TO LOAD
  0.0           0.0           0.0           0.0            0.0           0.0           0.0           0.0

STRESSES AT ENDS
  0.124884E+00 -0.500000E+00 -0.602676E-01  0.0            0.578557E-03  0.225770E-02 -0.602677E-01  0.0

INITIAL CONDITIONS
 -0.142532E-11 -0.357270E-04  0.285508E-11 -0.0            0.124884E+00 -0.500000E+00 -0.602676E-01  0.0

END OF FILE
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DISPLACEMENT

	R	W	V	U
1	-0.1425E-11	-0.3573E-04	0.2855E-11	-0.0
2	-0.4188E-04	-0.2937E-04	0.6064E-06	0.0
3	-0.4440E-04	-0.1790E-04	0.1026E-05	0.0
4	-0.3152E-04	-0.8117E-05	0.1236E-05	0.0
5	-0.1705E-04	-0.1991E-05	0.1292E-05	0.0
6	-0.6489E-05	0.9008E-06	0.1266E-05	0.0
7	-0.6414E-06	0.1719E-05	0.1206E-05	0.0
8	0.1678E-05	0.1531E-05	0.1143E-05	0.0
9	0.1996E-05	0.1036E-05	0.1088E-05	0.0
10	0.1490E-05	0.5848E-06	0.1042E-05	0.0
11	0.8424E-06	0.2395E-06	0.1003E-05	0.0
12	0.3454E-06	0.1427E-06	0.9690E-06	0.0
13	0.5763E-07	0.9558E-07	0.9362E-06	0.0
14	-0.6462E-07	0.9918E-07	0.9038E-06	0.0
15	-0.8881E-07	0.1201E-06	0.8710E-06	0.0
16	-0.6992E-07	0.1407E-06	0.8379E-06	0.0
17	-0.4130E-07	0.1548E-06	0.8044E-06	0.0
18	-0.1807E-07	0.1622E-06	0.7707E-06	0.0
19	-0.4027E-08	0.1648E-06	0.7369E-06	0.0
20	0.2311E-08	0.1649E-06	0.7030E-06	0.0
21	0.3885E-08	0.1641E-06	0.6692E-06	0.0
22	0.3252E-08	0.1631E-06	0.6354E-06	0.0
23	0.2040E-08	0.1625E-06	0.6017E-06	0.0
24	0.1035E-08	0.1621E-06	0.5679E-06	0.0
25	0.4668E-09	0.1619E-06	0.5341E-06	0.0
26	0.2690E-09	0.1618E-06	0.5004E-06	0.0
27	0.2356E-09	0.1617E-06	0.4666E-06	0.0
28	0.8581E-10	0.1617E-06	0.4329E-06	0.0
29	-0.5120E-09	0.1617E-06	0.3991E-06	0.0
30	-0.1888E-08	0.1620E-06	0.3653E-06	0.0
31	-0.4204E-08	0.1628E-06	0.3316E-06	0.0
32	-0.7103E-08	0.1642E-06	0.2978E-06	0.0
33	-0.9146E-08	0.1663E-06	0.2640E-06	0.0
34	-0.7146E-08	0.1684E-06	0.2301E-06	0.0
35	0.4187E-08	0.1690E-06	0.1962E-06	0.0
36	0.3150E-07	0.1650E-06	0.1623E-06	0.0
37	0.7961E-07	0.1515E-06	0.1286E-06	0.0
38	0.1445E-06	0.1236E-06	0.9527E-07	0.0
39	0.2012E-06	0.7974E-07	0.6261E-07	0.0
40	0.1891E-06	0.2860E-07	0.3088E-07	0.0
41	0.1285E-10	-0.2959E-11	-0.1048E-12	0.0

END OF FILE

PRIMARY STRESS RESULTANTS

	MS	SS	NS	TS
1	0.1249E+00	-0.5000E+00	-0.6027E-01	0.0
2	0.2981E-01	-0.2657E+00	-0.6027E-01	0.0
3	-0.1447E-01	-0.9719E-01	-0.6027E-01	0.0
4	-0.2609E-01	-0.5656E-02	-0.6027E-01	0.0
5	-0.2219E-01	0.2932E-01	-0.6027E-01	0.0
6	-0.1386E-01	0.3287E-01	-0.6027E-01	0.0
7	-0.6519E-02	0.2392E-01	-0.6027E-01	0.0
8	-0.1824E-02	0.1323E-01	-0.6027E-01	0.0
9	0.4505E-03	0.5233E-02	-0.6027E-01	0.0
10	0.1139E-02	0.7052E-03	-0.6027E-01	0.0
11	0.1037E-02	-0.1156E-02	-0.6027E-01	0.0
12	0.6835E-03	-0.1471E-02	-0.6027E-01	0.0
13	0.3464E-03	-0.1127E-02	-0.6027E-01	0.0
14	0.1209E-03	-0.6511E-03	-0.6027E-01	0.0
15	0.6121E-05	-0.2760E-03	-0.6027E-01	0.0
16	-0.3282E-04	-0.5405E-04	-0.6027E-01	0.0
17	-0.3212E-04	0.4324E-04	-0.6027E-01	0.0
18	-0.1728E-04	0.6504E-04	-0.6027E-01	0.0
19	-0.1930E-05	0.5268E-04	-0.6027E-01	0.0
20	0.8820E-05	0.3177E-04	-0.6027E-01	0.0
21	0.1454E-04	0.1424E-04	-0.6027E-01	0.0
22	0.1665E-04	0.3362E-05	-0.6027E-01	0.0
23	0.1676E-04	-0.1742E-05	-0.6027E-01	0.0
24	0.1608E-04	-0.3130E-05	-0.6027E-01	0.0
25	0.1533E-04	-0.2581E-05	-0.6027E-01	0.0
26	0.1484E-04	-0.1140E-05	-0.6027E-01	0.0
27	0.1479E-04	0.7958E-06	-0.6027E-01	0.0
28	0.1527E-04	0.3115E-05	-0.6027E-01	0.0
29	0.1635E-04	0.5502E-05	-0.6027E-01	0.0
30	0.1794E-04	0.6859E-05	-0.6027E-01	0.0
31	0.1950E-04	0.4755E-05	-0.6027E-01	0.0
32	0.1969E-04	-0.4802E-05	-0.6027E-01	0.0
33	0.1605E-04	-0.2665E-04	-0.6027E-01	0.0
34	0.5026E-05	-0.6383E-04	-0.6027E-01	0.0
35	-0.1689E-04	-0.1120E-03	-0.6027E-01	0.0
36	-0.5027E-04	-0.1502E-03	-0.6027E-01	0.0
37	-0.8719E-04	-0.1298E-03	-0.6027E-01	0.0
38	-0.1032E-03	0.3267E-04	-0.6027E-01	0.0
39	-0.4941E-04	0.4466E-03	-0.6027E-01	0.0
40	0.1497E-03	0.1201E-02	-0.6027E-01	0.0
41	0.5785E-03	0.2258E-02	-0.6027E-01	0.0

END OF FILE

SECONDARY STRESS RESULTANTS

	MT	MTS	MST	NT	NTS	NST
1	0.3658E-01	0.0	0.0	-0.3995E+01	0.0	0.0
2	0.8612E-02	0.0	0.0	-0.3272E+01	0.0	0.0
3	-0.5173E-02	0.0	0.0	-0.2001E+01	0.0	0.0
4	-0.8671E-02	0.0	0.0	-0.9186E+00	0.0	0.0
5	-0.7249E-02	0.0	0.0	-0.2401E+00	0.0	0.0
6	-0.4475E-02	0.0	0.0	0.8061E-01	0.0	0.0
7	-0.2075E-02	0.0	0.0	0.1718E+00	0.0	0.0
8	-0.5571E-03	0.0	0.0	0.1513E+00	0.0	0.0
9	0.1679E-03	0.0	0.0	0.9679E-01	0.0	0.0
10	0.3798E-03	0.0	0.0	0.4688E-01	0.0	0.0
11	0.3397E-03	0.0	0.0	0.1419E-01	0.0	0.0
12	0.2208E-03	0.0	0.0	-0.2080E-02	0.0	0.0
13	0.1103E-03	0.0	0.0	-0.7330E-02	0.0	0.0
14	0.3726E-04	0.0	0.0	-0.6950E-02	0.0	0.0
15	0.5599E-06	0.0	0.0	-0.4648E-02	0.0	0.0
16	-0.1154E-04	0.0	0.0	-0.2366E-02	0.0	0.0
17	-0.1096E-04	0.0	0.0	-0.8022E-03	0.0	0.0
18	-0.5957E-05	0.0	0.0	0.1469E-04	0.0	0.0
19	-0.9127E-06	0.0	0.0	0.3064E-03	0.0	0.0
20	0.2575E-05	0.0	0.0	0.3172E-03	0.0	0.0
21	0.4410E-05	0.0	0.0	0.2228E-03	0.0	0.0
22	0.5069E-05	0.0	0.0	0.1198E-03	0.0	0.0
23	0.5088E-05	0.0	0.0	0.4519E-04	0.0	0.0
24	0.4862E-05	0.0	0.0	0.2779E-05	0.0	0.0
25	0.4615E-05	0.0	0.0	-0.1738E-04	0.0	0.0
26	0.4456E-05	0.0	0.0	-0.2707E-04	0.0	0.0
27	0.4432E-05	0.0	0.0	-0.3402E-04	0.0	0.0
28	0.4574E-05	0.0	0.0	-0.3907E-04	0.0	0.0
29	0.4900E-05	0.0	0.0	-0.3445E-04	0.0	0.0
30	0.5381E-05	0.0	0.0	-0.2917E-05	0.0	0.0
31	0.5858E-05	0.0	0.0	0.8019E-04	0.0	0.0
32	0.5932E-05	0.0	0.0	0.2378E-03	0.0	0.0
33	0.4862E-05	0.0	0.0	0.4692E-03	0.0	0.0
34	0.1577E-05	0.0	0.0	0.7105E-03	0.0	0.0
35	-0.4996E-05	0.0	0.0	0.7792E-03	0.0	0.0
36	-0.1506E-04	0.0	0.0	0.3227E-03	0.0	0.0
37	-0.2630E-04	0.0	0.0	-0.1182E-02	0.0	0.0
38	-0.3144E-04	0.0	0.0	-0.4291E-02	0.0	0.0
39	-0.1577E-04	0.0	0.0	-0.9185E-02	0.0	0.0
40	0.4342E-04	0.0	0.0	-0.1488E-01	0.0	0.0
41	0.1719E-03	0.0	0.0	-0.1806E-01	0.0	0.0

END OF FILE