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A model-based approach to nonlinear networked control systems

by

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To my parents

Without their selfless support, this would not have been possible.

ABSTRACT

This thesis is concerned with the analysis of the control design to the nonlinear networked control systems (NCSs).

Ignoring the network connection and cascading actuators, the plant and sensors together, a sampled-data system is obtained. The stabilization problem of nonlinear sampled-data systems is considered under the low measurement rate constraint. Dual-rate control schemes based on the emulation design and discrete-time design approaches respectively are proposed that utilize a numerical integration model to approximately predict the current state of the plant. It is shown that using the dual-rate control schemes, input-to-state stability property will be preserved for the closed loop sampled-data system in a practical sense.

On the other hand, the networked realization of nonlinear control systems is studied and a model-based control scheme is addressed as a solution to reduce the network traffic and resultantly, to attain a higher performance. The NCSs are modeled as continuous-time systems and sampled-data systems, respectively. Under the proposed control scheme, a tradeoff between satisfactory control performance and reduction of network traffic can be achieved. It is shown that by using the estimated values, generated by the plant model, instead of true values of the plant, a significant saving in the required bandwidth is achieved and this makes possible stabilization of the plant even under slow network conditions.

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NOMENCLATURE

NCS - Networked Control Systems
CSMA - Carrier Sense Multiple Access
CAN - Control Area Network
MIMO - Multiple Inputs and Multiple Outputs
LQR - Linear Quadratic Regulator
LTI - Linear Time Invariant
MATI - Maximum Allowable Transfer Internal
TOD - Try-Once-Discard
UGES - Uniformly Globally Exponentially Stable
ZOH - Zero-order Hold
LMI - Linear Matrix Inequality
CTD - Continuous-time Design
DTD - Discrete-time Design
ISS - Input-to-state Stability
A/D - Analog-Discrete
D/A - Discrete-analog
BIBO - Bounded-Input Bounded-Output

Chapter 1

Introduction

The prevalence of network-based control systems is a consequence of the developments in the microelectronic and telecommunication areas. Compared to traditional control systems, this network structure introduces several advantages, such as lower cost, higher flexibility and easier maintenance. Thanks to these advantages, networked control systems have received more and more attention and have been applied widely to spacecrafts, process and automotive industries. However, this type of connection initiates new challenges: the insertion of a network into a previously-designed system affects the system performance and can even destabilize the system. This brings the need to redesign the system to recover the stability of the closed loop. This chapter briefly introduces the background of networked control systems, reviews existing literature and presents an outline of the thesis.

1.1 Background

Traditional control systems composed of interconnected controllers, sensors and actuators have been successfully implemented using a point-to-point architecture. The typical configuration is depicted in Figure 1.1. In this type of architecture, system components are connected directly by dedicated wiring. A point-to-point architecture is somewhat stagnant from a reconfigurability point of view since it is difficult to add, remove, or reconfigure components. As an alternative, the network connection structure offers great flexibility and decreases installation and maintenance costs due to

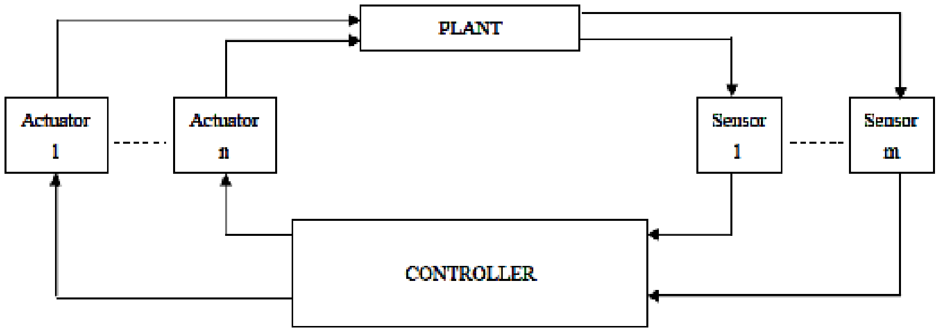

Figure 1.1: The typical setup of a traditional control system

wire reduction. In many complicated control systems, such as manufacturing plants, automobiles and aircrafts, communication networks are being adopted to exchange information and control signals. When a control loop is closed via a communication channel, the interconnection is referred to as a networked control system (NCS) [16, 18, 27, 31, 71]. Figure 1.2 illustrates a standard setup of an NCS, where the information is exchanged using a network among the system components, resulting in a significant decrease of wirings.

The insertion of a network into the control system, however, makes the analysis and design of an NCS complex. Unlike data network, which delivers relatively big data packets with a sporadic nature, control network requires relatively small packets in frequent and timely transmission. Several issues in NCSs may be summarized as follows.

1. Network-induced delays

Due to the feature of time-criticalness, transmitting information over the network induces delays that degrade the system performance or even cause instability. The network-induced delays may be deterministic or random, constant or time varying, depending on the protocol of the control network. The network protocols fall into two general categories: random access and scheduling [71].

Carrier sense multiple access (CSMA) is most often used in random access networks. Popular networks using CSMA protocols include Controller Area Network (CAN) and Ethernet. Before sending its information, each node on the CSMA network verifies network to be idle. As soon as the network is

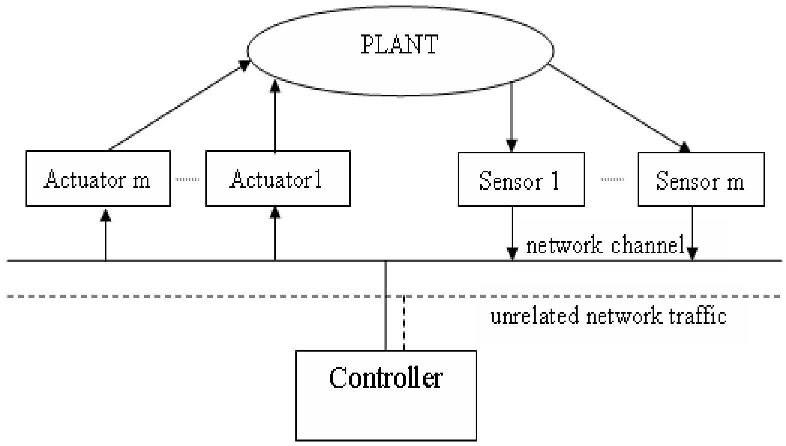

Figure 1.2: The standard setup of an NCS

idle, it begins transmission immediately. However, if several nodes transmit their information simultaneously, a collision occurs. The solution to resolve the collision depends on the protocol used. For CAN bus, the node with the highest priority wins the competition and transmits its information. For Ethernet, each node waits a random time and retransmits its information. Packets on these type of networks are affected by random delays.

Token-passing is often used in scheduling networks, where a node with token is guaranteed to transmit its information freely, without any congestion. Network protocols IEEE 802.4 and IEEE 802.5 have been built on this concept and familiar scheduling networks include PROFIBUS, Foundation Fieldbus, and so on. Packets on these type of networks wait for the token to transmit the information. The induced delays can be constant and bounded.

2. Packet loss

Another issue in NCSs is packet loss. While a data packet is transmitted over the network, it might be lost because of the noise interference, collision with the other packets or other uncontrolled random events. For example, Ethernet employs a CSMA with collision detection protocol. When there is a collision, each node waits for the access to the network. If a node is unable to transmit its information after 16 trials, the packets are dropped.

Usually, feedback systems can tolerate a certain amount of packet loss, but it is useful to determine the packet transmission rate at which the closed loop is stable.

3. Multiple-packet transmission

Due to packet size constraint of the network, large-size data may be transmitted by multiple packets, so-called multiple-packet transmission.

Traditional sampled-data systems assume that the measurements of the plant output and control inputs are delivered at the same time. However, in NCSs with multiple-packet transmission, the controller may not be able to receive all data packets at the time of the control calculation. Moreover, because the data is split into several packets, they may be routed through different paths. Therefore, the recipient may not receive the packets at the order that they have been sent.

Different networks suit different types of transmissions. For instance, Ethernet, created originally for data networks, can hold a maximum of 1500 bytes of data in a single packet. It is convenient and effective to employ single-packet transmission. On the other hand, CAN bus with the character of frequent transmission of small-size packet, suits multiple-packet transmission.

1.2 Literature Review

Due to its advantages, NCS receives more and more attention in recent years. This section briefly reviews the previous work on networked control systems.

1. Discrete-time approach

In [16], Halevi and Ray (1988) consider a continuous-time plant and discrete-time controller and analyze the integrated communication and control systems via a discrete-time approach. The NCS is reduced to a finite dimensional discrete-time model by augmenting the system model to include past values of plant input and output as additional states and the stability is analyzed for the special case of periodic delays.

This approach has been extended recently by [31] in 2003, where a discrete-time model of NCSs with multiple inputs and multiple outputs (MIMO) and multiple distributed communication delays is derived. In addition, the asynchronous sampling mechanisms of distributed sensors are characterized to obtain the actual time delays between sensors and the controller. Based on the proposed NCS model, the stability and performance of a closed-loop system are analyzed, and a linear quadratic regulator (LQR) optimal control is formulated.

Nilsson (1998) models an NCS as a stochastic discrete-time system in [56]. Assume that the discrete process with the sampling period h gives the following form

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma_0 u(k) + \Gamma_1 u(k-1) + v(k), \\ y(k) &= Cx(k) + w(k), \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \Phi &= e^{Ah}, \\ \Gamma_0 &= \int_0^{h-\tau_k^{sc}-\tau_k^{ca}} e^{At} dt B, \\ \Gamma_1 &= \int_{h-\tau_k^{sc}-\tau_k^{ca}}^{h} e^{At} dt B, \end{aligned}$$

x and u are the process state and the control input, τ_k^{sc} and τ_k^{ca} denote the sensor-to-controller and the controller-to-actuator time delays at sampling instant kh respectively, and $v(k)$ and $w(k)$ are uncorrelated white noise with zero mean and covariance matrices R_1 and R_2. An LQG-optimal controller is found for this setup, However, assuming the knowledge of the probability distributions of τ_k^{sc} and τ_k^{ca} makes the result restrictive.

In [18], the network is modeled as an ideal sample and hold device. Stability analysis is performed after lifting the system into a linear time-invariant (LTI) setting and sufficient conditions were obtained on the maximum sampling period.

2. Jump linear systems

In [27], it is assumed that time delays behave according to Markov chains by Krtolica (1994). The plant and the controller are represented by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} p(k+1) &= Fp(k) + Ew(k), \\ v(k) &= Hp(k) + Ew(k), \end{aligned} \tag{1.3}$$

where x, y are the state and output of the plant, and p, v are the state and output of the controller, and w, u are the most recently reported version of y and v. Matrices A, B, C, E, F and H are all constant with compatible dimensions. Assume that

$$\begin{aligned} u(k) &= \textstyle\sum_{i=0}^{D_1} \alpha_i(k) v(k-i), \\ w(k) &= \textstyle\sum_{i=0}^{D_2} \beta_i(k) y(k-i), \end{aligned} \tag{1.4}$$

where $\sum_{i=0}^{D_1} \alpha_i(k) = 1$ and $\sum_{i=0}^{D_2} \beta_i(k) = 1$ with $\alpha_i(k)$ and $\beta_i(k) \in \{0, 1\}$. D_1 and D_2 are two given positive integers which represent upper bounds of sensor-to-controller and controller-to-actuator time delays respectively. Regarding $\alpha(k)$ and $\beta(k)$ as Markovian chains with given probability transition matrices, the NCS is therefore framed to be a discrete-time jump linear system. Necessary and sufficient conditions for mean-square exponential stability are thus derived via the stochastic Lyapunov method.

Similar approach is adopted by Zhang et al. (2005) in [69], where the NCS is modeled as a jump linear system. Using linear matrix inequality (LMI) techniques, a sufficient condition is derived to guarantee that the closed-loop system is stochastically stable.

3. Scheduling

Shin (1991) proposes a dynamic scheduling method which guarantees to minimize the probability of the messages missing their deadlines in [60]. In this method, a bus access mechanism using a poll number is employed. Each node computes a poll number based on the deadlines and the user-defined priorities. The poll number is designed such that the task with the earliest message deadline will have the largest poll number. The node with the largest poll number will access the bus to transmit its information. The simulations in [60] show that poll number method achieves a better performance, in term of meeting the deadlines, compared to that of the token ring. However, it is not flexible to change the message priorities, since the priorities are fixed. Moreover, adding the poll number to the data packet overhead makes this method impossible to use on commercial and comparably less expensive protocols, such as CAN network.

An analytic analysis and design approach is suggested by Walsh (2002) in [66], where the the network effect is modeled as a perturbation on the original system. Walsh et al. consider a continuous-time plant and a continuous controller. The control network is only inserted between the sensor and the controller. They introduce the notion of maximum allowable transfer interval (MATI), denoted by τ, which assumes that successive sensor messages are separated by at most τ seconds. Their goal is to find the value of τ for which the stability of the NCS is guaranteed to be preserved. Walsh et al. analyze the impact of two different scheduling algorithms, token-ring static scheduling and try-once-discard (TOD) on the maximum allowable transfer interval. TOD is a dynamic scheduling protocol where the node with the largest difference between its current value and its last transmitted value accesses the network. After one node accessed the network bus, other nodes discard their current values and are replaced by new data. For each of these scheduling, sufficient conditions on the upper bound of the MATI to preserve stability of the NCS are obtained. Although the upper bound is calculated analytically, the resulting answer is conservative.

The idea in [66] is further generalized by Nesic and Teel (2004) to derive a set of Lyapunov UGES (Uniformly Globally Exponentially Stable) protocols in the $\mathcal{L}_p$ framework [51].

Branicky et al. (2002) apply the rate monotonic scheduling algorithm to schedule a set of NCSs in [5]. The optimal scheduling problem is also formulated under rate-monotonic-schedulability constraints.

4. Continuous-time model

Continuous-time NCS models are considered by several researchers. Göktas et al. (1997) use a modified Padé approximation and consider the delays as an uncertainty in [15]. They use robust control theory, μ-synthesis, to handle worst-case time delays. The tradeoffs between the worst case delay bounds and robust performance parameters are searched through real-time experiments over different communication protocols and media.

Montestruque et al. (2003) address an explicit model of the linear plant to reduce the network traffic in [40]. Sufficient and necessary conditions for stability are derived in terms of the constant update time and the parameters of the plant and those of its model. Furthermore, stochastic stability results with independent identically distributed transmission times are obtained in [41].

Delay-differential equations, on the other hand, are used to characterize

NCSs with delays in the state, input or both. Based on standard algorithms from functional analysis, solutions of linear delay-differential equations can be constructed, and their stability can be analyzed using Lyapunov second methods such as Lyapunov-Krasovskii and/or Lyapunov-Razumikhin stability theorems [29]. In this spirit, new results on stability and $\mathcal{H}_\infty$ performance are proposed in [14] for networked control systems with simultaneous consideration of time delays, data packet dropouts and measurement quantization, by exploiting a new Lyapunov-Krasovskii functional.

Gao et al. (2008) further study the problem of $\mathcal{H}_\infty$ output tracking for network-based control systems [13]. Suppose the physical plant is given by the following linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.5}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the output, and $w(t) \in \mathbb{R}^l$ is the disturbance input that satisfies $w = \{w(t)\} \in L_2[0, \infty)$. A, B, C, D and E are system matrices with appropriate dimensions. Suppose the reference signal $y_r(t)$ is generated by

$$\begin{aligned} y_r(t) &= Hx_r(t), \\ \dot{x}_r(t) &= Gx_r(t) + r(t), \end{aligned} \tag{1.6}$$

where $y_r(t)$ has the same dimension as $y(t)$; $x_r(t)$ and $r(t) \in \mathbb{R}^r$ are, respectively, the reference state and the energy bounded reference input; and G and H are appropriately dimensioned constant matrices with G Hurwitz. Suppose a state feedback controller takes the following form

$$u(t) = K_1 x(t_k - \eta_k) + K_2 x_r(t_k - \eta_k),\ t_k \le t < t_{k+1}, \tag{1.7}$$

where K_1 and K_2 are the state-feedback control gains, t_k is updating instant of the zero order hold (ZOH), and η_k is the signal transmission delays at the instant t_k. A LMI-based procedure is proposed for designing state-feedback controllers, which guarantee that the output of the closed-loop networked control system tracks the output of a given reference model well in the $\mathcal{H}_\infty$ sense. Moreover, the controller design method is further extended to more general cases, where the system matrices of the physical plant contain parameter uncertainties, represented in either polytopic or norm-bounded frameworks.

1.3 Thesis Outline

Most existing literature considers the stabilization of linear NCSs whereas nonlinear NCSs have received little attention. This thesis is concerned with the analysis of the control design to the nonlinear networked control context. This study is important for application since control systems of interest are often nonlinear. Compared to linear systems where the exact solution can be found, one important intrinsic difficulty for nonlinear NCSs is: the explicit solution of a nonlinear differential equation is often non-existent in analytical form.

There are two main topics in this thesis. 1) Ignoring the network connection (communication media and other traffic) in Figure 1.2 and cascading actuators, the plant and sensors together, we obtain a sampled-data system. In practice, hardware restrictions on input and measurement sampling rate can be essentially different. The measurement sampling rate is often made slower than that of the input. In such cases, the single rate results may lead to unstable closed-up sampled data system and it makes sense to configure the control system so that several sample rates co-exist to achieve better performance. We consider the important practical case where hardware restrictions are imposed on the "measurement-A/D conversion" process. Specifically, the stabilization problem of sampled-data nonlinear systems is considered under the low measurement rate constraint. 2) Among several existing methods to ameliorate the NCS performance, the most effective way is to reduce network traffic. The networked realization of nonlinear control systems is studied and an estimation method is presented as a solution to decrease the network traffic and resultantly, to attain a higher performance. It is worth noting that the present study deals with random time-varying updating time. Also, the networked control system is considered in a nonlinear setting. The nonlinear control problem is difficult because the explicit analytic solution of continuous time processes are typically impossible to compute. The model formulated here is essentially different from linear NCSs and is more general. To the best of the authors' knowledge, the stabilization problem of model-based nonlinear networked control systems with random time-varying updating time has not been investigated and still remains challenging, which motivates the present study. A chapter-by-chapter review is as follows.

In Chapter 2 the stabilization problem of sampled-data nonlinear systems is considered under the low measurement rate constraint. A dual-rate control scheme based on the emulation design is proposed that utilizes a numerical integration scheme to approximately predict the current state of the plant. Chapter 2 shows that if one designs a continuous-time controller for a continuous-time plant so that the closed-loop continuous-time system is input-to-state stable and then discretize the controller and implement it using sample and zero order hold devices, then input-to-state stability property will be preserved for the dual-rate system in a practical sense.

Chapter 3, alternatively, considers the problem addressed in Chapter 2 via the discrete-time design approach. Given an approximate discrete-time model of a sampled nonlinear plant and given a family of controllers that stabilizes the plant model in input-to-state sense, Chapter 3 shows that using the dual-rate control scheme based on discrete-time design method, the closed loop sampled data system is input-to-state stable in the semiglobal practical sense.

In Chapter 4, the networked realization of a nonlinear control system is studied and a model-based control scheme is addressed to estimate the missing states due to the network access limitation. By using the estimated values instead of true values of the plant, a significant saving in the required network bandwidth is achieved and this makes possible stabilization of the plant even under slow network conditions.

In Chapter 5, the problem of the analysis and design for NCSs with event-driven digital controller and event-driven holder is considered. The physical plant and the controller are in continuous time and discrete time, respectively, and the NCS is modeled as a sampled-data system with time delays. For such a sampled-data NCS, an estimator to compensate the network-induced delays and reconstruct approximately the undelayed plant state is proposed and the stability result via linearization approach is obtained. A tradeoff between satisfactory control performance and reduction of network traffic can be achieved.

Chapter 2

Input-to-state Stability under Sampling and Emulation

In this chapter the stabilization problem of sampled-data nonlinear systems is considered under the low measurement rate constraint. A dual-rate control scheme based on the emulation design is proposed that utilizes a numerical integration scheme to approximately predict the current state of the plant. The chapter shows that if one designs a continuous-time controller for a continuous-time plant so that the closed-loop continuous-time system is input-to-state stable and then discretizes the controller and implements it using sample and zero order hold devices, then input-to-state stability property will be preserved for the sampled-data dual-rate closed loop system in a practical sense.

The prevalence of digitally implemented controllers and the fact that most systems of interest in control systems are often nonlinear motivate the area of nonlinear sampled-data control systems. Significant progress has been made in this area in recent years, for instance [1, 4, 6, 7, 11, 28, 30, 47, 49, 50, 52–54, 59, 72, 73]. In particular, the sampling results within an input/output setting can be found in [4, 6, 7, 30] and the sampled-data systems with state space framework are considered in [1, 7, 28, 47, 49, 50, 53, 54, 72, 73].

This chapter concentrates on the state space framework. The outline is given as follows. In section 2.1, some definitions, relevant notations and preliminary design approaches for digital controllers used in the sampled-data control literature are introduced. The modeling and problem formulation are presented in section 2.2. In section 2.3 the main results of stability and performance recovery are stated and proved. The results are illustrated by simulations in section 2.4. Finally, the chapter is closed with some concluding remarks.

2.1 Preliminaries

Before proceeding, some notations and terminology as well as some useful properties are introduced.

Recall the basic terminology from stability theory. Denote $\mathbb{R}_{\geq 0} := [0, \infty)$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. It is said to belong to class $\mathcal{K}_\infty$ if it is class $\mathcal{K}$ and is unbounded. Also, a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ belongs to $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to zero as $t \to \infty$. Denote $\mathbb{Z}^+ = \{0, 1, 2 \ldots\}$ and $\mathbb{N}$ as the set of natural numbers. Everywhere, $|\cdot|$ denotes the usual Euclidean norm and $\mathbf{B}(r) := \{x| \; |x| \leq r\}$.

2.1.1 Input-to-State Stability

In practice, control systems are very often affected by noise, expressed for instance as perturbations on controls and errors on observations. Thus, it is desirable for a system not only to be stable, but also to display the so-called input-to-state stability. Intuitively, this means that the behavior of the system should remain bounded when its inputs are bounded, and should tend to equilibrium point when inputs tend to zero. The notion of input-to-state stability (ISS) for nonlinear control systems was proposed in [61] and has been used in stability analysis and control synthesis as illustrated by its numerous applications (see, [22–25, 38, 48, 61–63] and reference therein). ISS provides a natural framework to deal with the perturbation systems. Basically, ISS gives a quantitative bound of the state trajectories in terms of the magnitude of the control input and their initial conditions.

Consider the nonlinear system

$$\dot{x} = f(x, u) \tag{2.1}$$

where $f : D \times D_u$ is locally Lipschitz in x and u, D and D_u are defined respectively by $D = \{x \in \mathbb{R}^n : |x| < a\}$ and $D_u = \{u \in \mathbb{R}^m : |u| < b\}$ for some positive numbers a and b. These assumptions guarantee the existence and uniqueness of the solutions of system (2.1). Assume that the unforced system

$$\dot{x} = f(x, 0) \tag{2.2}$$

has an equilibrium point at the origin $x = 0$.

Definition 2.1. The system (2.1) is said to be *locally input-to-state stable* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $x(t_0) \in D$ and any $u \in D_u$, the solution $x(t)$ starting at t_0 satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|_\infty), \quad \forall t \geq t_0 \geq 0. \tag{2.3}$$

It is said to be (globally) input-to-state stable if $D = \mathbb{R}^n, D_u = \mathbb{R}^m$ and inequality (2.3) is satisfied for any initial state and any bounded input u.

Definition 2.1 implies for a bounded input $u(t)$, the state $x(t)$ will be bounded. Furthermore, as t increases, $\beta(|x(t_0)|, t - t_0) \to 0$ as $t \to \infty$, and the state $x(t)$ will be ultimately bounded. Additionally, consider the unforced system (2.2). Since, with $u(t) \equiv 0$, (2.3) reduces to

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0),$$

local input-to-state stability implies that the origin of the unforced system (2.2) is uniformly asymptotically stable.

The following property, introduced in [38], gives a characterization for input-to-state stability by using Lyapunov-like functions.

Proposition 2.1. *The system (2.1) is locally input-to-state stable if and only if there exist a continuously differentiable function $V : D \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$, and γ such that for all $x \in D, u \in D_u$ the following holds*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2.4}$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(|x|) + \gamma(\|u\|_\infty). \tag{2.5}$$

Moreover, if $D = \mathbb{R}^n, D_u = \mathbb{R}^m$ and $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$, then the system (2.1) is input-to-state stable.

2.1.2 Emulation

Due to the complexity of the problem, results on the digital controller design for sampled-data nonlinear systems are rare. Unlike linear systems, $\dot{x}(t) = Ax+Bu$, where the exact discrete-time model of the plant can be given by $x(k+1) = e^{AT}x(k) + \int_0^T e^{As}dsBu(k)$, one important intrinsic difficulty for nonlinear sampled-data control systems is: the exact discrete model of the plant cannot be found. The absence of a good model of a sampled-data nonlinear plant for a digital controller design has leaded to several methods that use different type of approximate models. In the sampled-data control literature, two such methods attracted lots of attention, namely (see [7]): continuous-time design (CTD method) and discrete-time design (DTD method). The first approach involves digital implementation of a continuous-time stabilizing controller and the second is to design a digital controller based on a discrete-time equivalent of the plant.

CTD method is often referred to as the controller emulation design. It is a well-established method to design digital controllers for continuous-time plants. The design procedure is depicted in Table 2.1. In this method, one first designs a continuous-time controller based on the continuous-time plant. At this step the sampling is completely ignored. Then, at the second step, the obtained continuous-time controller is discretized and implemented using a sample and hold device.

continuous − time plant model
⇓
continuous − time controller
⇓
discretize controller
⇓
implement the controller

Table 2.1: The emulation Method

In the case of the emulation method, controller design is the topic of the area of continuous-time nonlinear control. Many results for nonlinear sampled-data systems use this method because traditional continuous-time controller design can be used directly for the design of digital controllers. See

[1, 7, 28, 49, 59, 72, 73].

2.1.3 Multi-rate Sampling

The sampling rate is a critical design parameter in the design of digital controllers. Usually, as the sampling rate is faster, the performance of a digital control system improves; however, computer costs also increase because less time is available to process the controller equations. Additionally, faster sampling rates require faster A/D conversion speed which may also increase system costs. Aside from costs, the selection of sampling rates depends on many factors, such as hardware restriction.

In single-rate sampled-data systems, the analog-discrete (A/D) and discrete-analog (D/A) conversions are made at the same rate. However, hardware restrictions on input and measurement sampling rate can be essentially different. For example, the D/A converters are generally faster than the A/D converters, so the measurement sampling rate is often made slower than that of the input. The situations where the systems have several samplers operating at different rates is called multi-rate sampling (see, for instance, [3]). Such cases may be necessary for systems with special data transmission links or special sensors and actuators and are useful for improving the performance of the system. Use of Multi-rate sampling is also natural in multiprocessor systems.

2.1.4 One-step consistency

Consider the nonlinear continuous-time control system (2.1). Let $F_T^e(x,u)$ and $F_T^a(x,u)$ denote the exact discrete-time and approximate discrete-time model of the system with the sampling period T. Basically, one-step consistency guarantees that the error of solutions between $F_T^e(x,u)$ and $F_T^a(x,u)$ starting from the same initial condition is small over one step, relative to the size of the step.

Definition 2.2. F_T^a is said to be one-step consistent with F_T^e if for any strictly positive real numbers (Δ_1, Δ_2), there exists a $\mathcal{K}$-class function $\rho(\cdot)$ and $T^* > 0$ such that $|F_T^e(x,u) - F_T^a(x,u)| \leq T\rho(T)$ for each fixed $T \in (0, T^*]$ and $x \in \mathbf{B}(\Delta_1), u \in \mathbf{B}(\Delta_2)$.

A sufficient condition for one-step consistency is the following ([53]).

Lemma 2.2. *F_T^a is one-step consistency with F_T^e if*

- *F_T^a is one-step consistent with F_T^{Euler} where $F_T^{Euler}(x,u) := x+Tf(x,u)$,*
- *for each compact set $\chi \subset \mathbb{R}^n$ there exist $\rho \in \mathcal{K}_\infty, M > 0$ and $T^* > 0$ such that for all $T \in (0, T^*]$ and all $x, y \in \chi$, we have $|f(y,u(x))| \leq M$ and $|f(y,u(x)) - f(x,u(x))| \leq \rho(|y-x|)$.*

Remark 2.1. Consistency property specifies how the model should be discretized for the emulation procedure to yield desired results. The checkable conditions for one-step consistency property is presented in Lemma 2.2. Notice that if the exact discrete-time model of the plant can be obtained, then it is not necessary to use an approximate discrete-time model and consistency become superfluous. It holds automatically. One such case is the emulation design for linear systems in [12].

2.2 Modeling and Problem Formulation

In this section the problem of sampled-data stabilization of nonlinear systems with disturbances under the "low measurement rate" constraint is considered and the design of dual-rate controllers based on emulation as a solution to this problem is addressed.

2.2.1 Motivation

As mentioned in [49], the main question in the emulation design is whether the desired properties of the continuous-time closed loop system that the designed controller yielded will be preserved and if so, in what sense for the closed-loop sampled-data system. This question has been addressed and answered for the emulation method for single-rate nonlinear systems in [28], where the authors showed that if a continuous-time controller stabilizes the approximate discrete-time plant model in input-to-state sense, then sampling the controller fast enough, the resulting sampled-data closed-loop system has the same property. However, this result requires fast sampling which means it may not be implementable in practice in cases when the required sampling period is too small to be realized with the available hardware. Hardware restrictions on input and measurement sampling rate can be essentially different. The D/A converters are generally faster than the A/D converters, so the measurement sampling rate is often made slower than that of the input. In such cases, it makes sense to configure the control system so that several sample rates co-exist to achieve better performance.

In this chapter a different scenario is considered, namely, the use of a dual-rate control scheme, where the output is measured at a relatively slow rate, whereas the control signal is adjusted faster. The performance and stability robustness analysis of such a dual-rate control system under the emulation design is studied.

2.2.2 Modeling

Consider a continuous-time nonlinear plant with initialization at t_0

$$\dot{x} = f(x, u, w), \quad t \geq t_0 \geq 0, \tag{2.6}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ are respectively the state, control input and exogenous disturbance and f is continuous, locally Lipschitz and $f(0,0,0) = 0$. Consider a dynamic feedback controller

$$\begin{aligned} \dot{z} &= g(x, z), \\ u &= u(x, z), \end{aligned} \tag{2.7}$$

where $z \in \mathbb{R}^{n_z}$ is the state of the controller and g, u are continuous, locally Lipschitz and $g(0,0) = 0, u(0,0) = 0$. A starting point in the emulation design is to assume that the closed-loop continuous-time system (2.6)-(2.7) possess input-to-state stability. Then as a second step the controller is discretized and implemented. The discretization of the controller can be written as

$$\begin{aligned} z(i+1) &= z(i) + \int_{t_0+iT_i}^{t_0+(i+1)T_i} g(x(i), z(s))ds & (2.8)\\ &=: G^e_{T_i}(x(i), z(i)), \\ u(i) &= u(x(i), z(i)), & (2.9) \end{aligned}$$

where the state measurement $x(i)$ is assumed to be given and T_i is the input sampling period. It is emphasized that the discretization (2.8)-(2.9) can not be computed exactly in most cases. Hence assume that an approximate discrete-time model of the controller

$$z(i+1) = G^a_{T_i,h}(x(i), z(i)), \tag{2.10}$$

$$u(i) = u(x(i), z(i)) \tag{2.11}$$

corresponding to the input sampling period T_i is available, parameterized by the modeling parameter $h > 0$. Here, h, which is the integration period of the numerical integration used to generate the approximate model of the controller, may be different from the sampling period. Similar set-up was also used in current literature, for example, in [50, 59]. Let

$$\begin{aligned} x(i+1) &= x(i) + \int_{t_0+iT}^{t_0+(i+1)T} f(x(s), u(i), w(s))ds \\ &=: F^e_T(x(i), u(i), w[i]) \end{aligned} \tag{2.12}$$

be the exact discrete-time model of the continuous-time plant (2.6) with the sampling period $T > 0$. Let

$$x(i+1) = F^a_{T,h}(x(i), u(i), w[i]) \tag{2.13}$$

be a family of approximate discrete-time models of the plant corresponding to the sampling period T, parameterized by the modeling parameter h. The approximate models can be obtained using many different numerical integration methods. For example, the simplest such model has the following form: $x(i+1) = x(i) + \int_{t_0+ih}^{t_0+(i+1)h} f(x(i), u(i), w(t))dt$ with the sampling period T equal to the integration period h.

Remark 2.2. A number of studies have shown that the classical Euler approximation may not yield satisfactory performance and is not always appropriate to use. For instance, the Euler model is not recommended in [44] for singularly perturbed systems that exhibit two-time scale behavior. In [8] the authors showed that bilinear approximation is superior to Euler for a particular application. Motivated by this fact, this chapter considers the numerical integration scheme when the integration period is different from the sampling period. This setup appears to be much more reasonable and in this case the approximate model can predict much better the states of the system.

The following notation will be used. For a given function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, denote $w[i] := \{w(t) : t \in [t_0 + iT, t_0 + (i+1)T), i \in \mathbb{Z}^+\}$ with the norm $\|w[i]\|_\infty = esssup_{\tau \in [t_0+iT, t_0+(i+1)T]} |w(\tau)|$, where $t_0 \geq 0$ is initial time. To shorten notations, denote $w_f = w[i]$ and $\widetilde{x} := (x^T, z^T)^T$.

The following definitions are needed to guarantee that the mismatch of the approximation scheme is small in some sense.

Definition 2.3. The approximate model of the plant $F^a_{T,h}$ is said to be one-step consistent with F^e_T if for any strictly positive real numbers $(\Delta_1, \Delta_2, \Delta_3)$ there exist a $\mathcal{K}$-class function $\rho(\cdot)$ and $T^* > 0$ such that for each fixed $T \in (0, T^*]$, there exists $h^* \in (0, T]$ such that $|F^e_T(x, u, w_f) - F^a_{T,h}(x, u, w_f)| \leq T\rho(h)$ for all $x \in \mathbf{B}(\Delta_1), u \in \mathbf{B}(\Delta_2), \|w\|_\infty \leq \Delta_3$ and $h \in (0, h^*)$.

Definition 2.4. The approximation scheme of the controller $G^a_{T_i,h}$ is said to be one-step consistent with $G^e_{T_i}$ if for any strictly positive real numbers (Δ_1, Δ_2) there exist a $\mathcal{K}$-class function $\rho(\cdot)$ and $T^* > 0$ such that for each fixed $T \in (0, T^*]$, there exists $h^* \in (0, T]$ such that $|G^e_{T_i}(x, z) - G^e_{T_i,h}(x, z)| \leq T\rho(h)$ for all $x \in \mathbf{B}(\Delta_1), z \in \mathbf{B}(\Delta_2)$ and $h \in (0, h^*)$.

2.2.3 Problem setting

Assume that the input sampling period is equal to the sampling period of system (2.12), that is $T_i = T$. Suppose that due to physical constraints, one cannot sample the measurement as fast as he wishes. Let the measurement sampling T_m be a multiple of T, i.e. $T_m = lT$ for some integer $l \geq 1$.

For this setting, this chapter considers a dual-rate inferential control scheme and for such a scheme to work, a discrete-time model with fast sampling rate for the nonlinear plant (2.6) is needed. Note that in general the exact discrete-time model F^e_T in (2.12) is unknown. Instead, the approximate discrete-time model (2.13) is used. In most applications, it is too strong to assume that disturbance $w[i]$ is known for all i. One way to fix this problem is to assume that the approximate model is run with zero disturbance. Assume that the approximate discrete-time plant model with zero disturbance

$$x(i+1) = F^a_{T,h}(x(i), u(i), 0)$$

is available. The proposed control scheme is the following: in order to feed back the digital controller a fast rate signal, the scheme uses the slow sampled measurement every lT period, giving $x(0), x(lT), x(2lT)$, etc., and uses the zero disturbance model $F^a_{T,h}(x, u, 0)$ to get the estimated state to fill in the missing samples. Such a process is depicted in Figure 2.1 where f, F_h and K represent the continuous-time plant, the approximate model with zero disturbance and the fast rate controller, respectively. The two systems f and K are interfaced by the A/D and D/A converters, modeled by a slow sampler S_s operating with the sampling period T_m and the zero-order hold H_f with fast sampling period T. A periodic switch is introduced which connects to

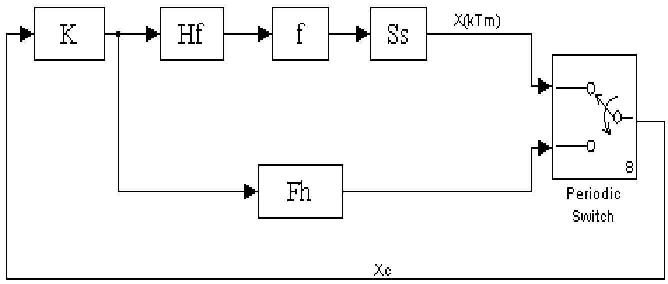

Figure 2.1: The sampled-data inferential control system

the actual state x at time $t_0 + klT$ and connects to the approximate estimate of the state at $t = t_0 + klT + jT,\ j \in \{1, 2, \ldots, l-1\}$, which is reconstructed by $F^a_{T,h}(x, u, 0)$ with periodically updated initialization at sampling instant $i = t_0 + klT$ by the actual state. $x(kT_m)$ and x_c are the measured slow-sampling output and the output of periodic switch, respectively. Thus the output of the switch is a fast rate signal given by

$$x_c(i{+}1) = \begin{cases} x(i+1), \quad i+1 = kl, k \in \mathbb{Z}^+, \\ F^a_{T,h}(x_c(i), u(i), 0)\ \textit{with initial value}\ x_c(kl) = x(kl), \textit{otherwise}. \end{cases} \tag{2.14}$$

The controller depends on the output of the switch $x_c(i)$ and is implemented digitally by H_f. To summarize, the dual-rate control scheme uses a fast-rate approximate model, a fast-rate controller and a periodic switch. The sampled-data closed loop system consists of the continuous-time plant and the digital controller

$$\begin{aligned} \dot{x} &= f(x,u,w), &(2.15)\\ z(i+1) &= G^{a}_{T,h}(x_c(i),z(i)), &(2.16)\\ u(i) &= u(x_c(i),z(i)). &(2.17)\end{aligned}$$

The discrete-time model of this sampled-data system consists of the exact discrete-time model (2.12) and the controller (2.16)-(2.17).

Remark 2.3. A similar approach is used in [30] to study linear sampled-data systems and in [59] to study practical asymptotic stability of the inferential control system. Unlike [30], in this chapter we investigate the stabilization of the nonlinear sampled-data systems which may generalize the results presented in [30]. Different from [59], this chapter focuses on input-to-state stability (ISS) for the nonlinear context with disturbance. The difference is important given an intrinsic robustness problem associated with the inferential control approach. Indeed, the presence of disturbances (in addition to the model mismatch originated by the approximate plant model) implies that model estimates will deviate from the true plant output measured by a sensor.

This chapter considers the important practical case where hardware restrictions are imposed on the "measurement-A/D conversion" process. More precisely, this chapter addresses the problem of dual-rate sampled-data stabilization of system (2.6) under the "low measurement rate" constraint. In this case, the single rate design method presented in [28] may lead to unstable closed-loop sampled data system (see example in Section 2.4). This chapter endeavors to solve the following: for any given low measurement sampling period T_m, find a controller which makes (possibly choosing T_i appropriately small) the closed-loop system input-to-state stable in the semiglobal practical sense under the emulation design. It is emphasized that the results are prescriptive since they can be used as a guide when one designs digital controllers based on the emulation approach.

Remark 2.4. Here, *semiglobal* means that the region of attraction can be rendered as large as desired by reducing the sampling period. *Practical* means that the state of the system converges to a neighborhood of zero.

2.3 Main results

Assumption 2.1. The closed-loop continuous time system (2.6) and (2.7) is globally input-to-state stable, which implies there exists a smooth function $V : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$ such that for all $(x, z) \in \mathbb{R}^{n_{\tilde{x}}}$ and $w \in \mathbb{R}^p$, the following holds

$$\alpha_1(|(x,z)|) \leq V(x,z) \leq \alpha_2(|(x,z)|) \tag{2.18}$$
$$\frac{\partial V}{\partial x} f(x, u(x,z), w) + \frac{\partial V}{\partial z} g(x,z) \leq -\alpha_3(|(x,z)|) + \gamma(\|w\|_\infty). \tag{2.19}$$

Assumption 2.2. $F^a_{T,h}(x, u, w_f)$ is one-step consistent with $F^e_T(x, u, w_f)$.

Assumption 2.3. $G^a_{T,h}(x, z)$ is one-step consistent with $G^e_T(x, z)$.

In the sequel, denote the discrete-time state $x(i)$ to be the sampled continuous-time state $x(t_0 + iT)$. Motivated by [1], we introduce the state of the sampled-data systems: $\bar{x}(t) = (x^T(t), z^T(i))^T$ for $t \in [t_0 + iT, t_0 + (i+1)T)$.

Theorem 2.3. *Under assumptions 2.1- 2.3, there exist $\bar{\beta} \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ such that the following holds. Given any strictly positive real numbers $(\Delta_x, \Delta_w, \mu)$, there exists $T^* > 0$, such that for each $T \in (0, T^*]$ there exists $h^* \in (0, T]$ such that for all $|\bar{x}(t_0)| \leq \Delta_x$, $\|w\|_\infty \leq \Delta_w$ and all $h \in (0, h^*)$, the sampled-data systems (2.15)-(2.17) satisfy:*

$$|\bar{x}(t)| \leq \bar{\beta}(|\bar{x}(t_0)|, t - t_0) + \gamma_2(\|w\|_\infty) + \mu, \quad \forall t \geq t_0 \geq 0. \tag{2.20}$$

Remark 2.5. It is useful when the disturbance w is only assumed to be a measurable function of time. Indeed, if Theorem 2.3 imposes an additional condition on disturbances, say, $\dot{w}$ is uniformly bounded, which is mentioned, for example, in [28], then the result may be restricted to hold for a small set of disturbance and this setting in most reality may not be available.

Let us begin with the following lemmas to establish the proof of the main results.

Lemma 2.4. *Given any strictly positive real numbers (D_1, D_3, ε), there exists $T_1 > 0$ such that for any fixed $T \in (0, T_1]$ there exists $h_1 \in (0, T]$ such that for each $h \in (0, h_1), |\tilde{x}(0)| \leq D_1$ and $\|w\|_\infty \leq D_3$, the following holds: if $max_{i \in \{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1$ for some $k \in \{0, 1, \ldots\}$ then $|x(k) - x_c(k)| \leq \varepsilon$.*

Proof. Let (D_1, D_3, ε) be given. Define $\Delta_1 = D_1 + \varepsilon$. By the local Lipschitz property of u and the fact that u is zero at zero, there exists $D_2 > 0$ such that $|u(\tilde{x})| \leq D_2$ for all $\tilde{x} \in \mathbf{B}(\Delta_1)$. Let $T_{11} > 0$ and $h_{11} > 0$ be as in Assumption 2.2 corresponding to $\Delta_1 = D_1 + \varepsilon, \Delta_2 = D_2$ and $\Delta_3 = D_3$, and let $\rho(\cdot) \in \mathcal{K}_\infty$ be a function from Assumption 2.2. Also, let $a > 0$ be a number that satisfies $|f(x, u, w)| \leq a$ for all $|\tilde{x}| \leq \Delta_1, |u| \leq D_2$ and $\|w\|_\infty \leq D_3$. Let $T_{12} > 0, h_{12} > 0$ be such that $T_{12}(2(l-1)a + \rho(h_{12})) \leq \varepsilon$. Finally define $T_1 = \min\{T_{11}, T_{12}, \varepsilon/a\}$ and $h_1 = \min\{h_{11}, h_{12}\}$.

Suppose $T \in (0, T_1], h \in (0, h_1)$ and $max_{i \in \{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1$ for some $k \in \{0, 1, \ldots\}$. Consider k by the following three cases. If $k = jl$ for some $j \in \{0, 1, \ldots\}$, then it is obvious that $|x(k) - x_c(k)| = 0$. If $k = jl + 1$, then it follows from Assumption 2.2 and triangle inequalities that

$$\begin{aligned} |x(k) - x_c(k)| &= |F^e_T(x(jl), u(jl), w[jl]) - F^a_{T,h}(x(jl), u(jl), 0)| \\ &\leq |F^e_T(x(jl), u(jl), w[jl]) - F^e_T(x(jl), u(jl), 0)| \\ &\quad + |F^e_T(x(jl), u(jl), 0) - F^a_{T,h}(x(jl), u(jl), 0)| \\ &\leq 2aT + T\rho(h). \end{aligned} \tag{2.21}$$

Otherwise,

$$\begin{aligned} &|x(k) - x_c(k)| \\ &= |F^e_T(x(k-1), u(k-1), w[k-1]) - F^a_{T,h}(x_c(k-1), u(k-1), 0)| \\ &\leq |F^e_T(x(k-1), u(k-1), w[k-1]) - F^e_T(x_c(k-1), u(k-1), 0)| \\ &\quad + |F^e_T(x_c(k-1), u(k-1), 0) - F^a_{T,h}(x_c(k-1), u(k-1), 0)| \\ &\leq |x(k-1) - x_c(k-1)| + 2aT + T\rho(h) \\ &\leq 2(k - jl)aT + T\rho(h) \end{aligned} \tag{2.22}$$

holds for all $k \in \{jl + 2, \ldots, (j+1)l - 1\}$. From the choice of T and h, it follows that $T(2ai + \rho(h)) \leq \varepsilon$ for all $i \in \{1, 2, \ldots, l-1\}$. This completes the proof of Lemma 2.4. ■

The following lemma will show that the Lyapunov function associated with the continuous-time system is decreasing along the trajectories of the exact discrete-time system, at least when the size of x is large relative to the size of w and both are bounded.

Lemma 2.5. *The exact discrete-time closed-loop model (2.12) and (2.16)-(2.17) of the sampled-data system (2.15)-(2.17) has the following property: under assumptions 2.1- 2.3, given any strictly positive numbers* (D_1', D_2', δ) *there exists* $T_2 > 0$ *such that for any* $T \in (0, T_2]$ *there exists* $h_2 \in (0, T]$ *such that for each* $|\tilde{x}(0)| \leq D_1'$, $\|w\|_\infty \leq D_2'$ *and* $h \in (0, h_2]$, *the following holds: if* $max_{i\in\{0,1,\ldots,k\}}|\tilde{x}(i)| \leq D_1'$ *for some* $k \in \mathbb{Z}^+$, *then*

$$\frac{V(x(k+1), z(k+1)) - V(x(k), z(k))}{T} \leq -\alpha_3(|(\tilde{x}(k))|) + \gamma_1(\|w\|_\infty) + \delta. \tag{2.23}$$

Proof. Let the strictly positive numbers (D_1', D_2', δ) be given. Define $\Delta := D_1' + 1$. Let $T_{21} > 0$ and $h_{21} > 0$ be as in Assumption 2.3 corresponding to $\Delta_1 = \Delta$ and $\Delta_2 = \Delta$, and let $\tilde{\rho}(\cdot) \in \mathcal{K}_\infty$ be a function from Assumption 2.3. Let $L_1, L_2 > 0$ be the Lipschitz constants of $f(x, u, w)$ and $g(x, z)$ respectively, on the sets where $|x| \leq \Delta, |z| \leq \Delta$ and $\|w\|_\infty \leq \Delta_w$. Also, let $b > 0$ be a number that satisfies $max\{|\frac{\partial V}{\partial x}|, |\frac{\partial V}{\partial z}|, |f(x, u, w)|, |g(x, z)|\} \leq b$ for all $|x| \leq \Delta, |z| \leq \Delta$ and $\|w\|_\infty \leq D_2'$. It is easy to see that such $b > 0$ always exists because of the continuous differentiability of V, the local Lipschitz properties of f and g, and the fact that they are in a closed set. Take any $\varepsilon_1 \in (0, min\{1, \frac{\delta}{8bL_2}\})$. From Lemma 2.4, let $T_{22} > 0$ and $h_{22} > 0$ be generated by $(D_1', D_2', \varepsilon_1)$. Let strictly positive numbers T_{23}, h_{23} and h_{24} be such that $T_{23}b^2(L_1 + L_2) \leq \frac{\delta}{4}, T_{21}\tilde{\rho}(h_{23}) \leq \frac{1}{2}$ and $b\tilde{\rho}(h_{24}) \leq \frac{\delta}{4}$. By the continuity of $\frac{\partial V}{\partial x}$, it implies that $\frac{\partial V}{\partial x}$ is uniformly continuous on compact sets. Hence, it follows that given any $\epsilon > 0$, there exists $T_\epsilon > 0$ such that $\left|\frac{\partial V}{\partial x}\Big|_{(x_1,z_1)} - \frac{\partial V}{\partial x}\Big|_{(x,z)}\right| \leq \epsilon$ for any $T \in (0, T_\epsilon), |x_1 - x| \leq Tb$ and $|z_1 - z| \leq Tb$. Take $\epsilon = \frac{\delta}{4b}$ and denote $T_{24} := T_\epsilon$ such that for all $T \in (0, T_{24}), |\tilde{x}| \leq D_1'$ and $\|w\|_\infty \leq D_2'$, the following holds

$$\left|\frac{\partial V}{\partial x}\bigg|_{(x_1,z_1)} - \frac{\partial V}{\partial x}\bigg|_{(x,z)}\right| \leq \frac{\delta}{4b}. \tag{2.24}$$

Likewise, let $T_{25} > 0$ be such that for all $T \in (0, T_{25}), |\tilde{x}| \leq D_1'$ and $\|w\|_\infty \leq D_2'$,

$$\left|\frac{\partial V}{\partial z}\bigg|_{(x_2,z_2)} - \frac{\partial V}{\partial z}\bigg|_{(x,z)}\right| \leq \frac{\delta}{8b}. \tag{2.25}$$

Finally, define $T_2 = min\{T_{21}, T_{22}, T_{23}, T_{24}, T_{25}, \frac{1}{2b}\}$ and $h_{25} = min\{h_{21}, h_{22}, h_{23}, h_{24}\}$. For any fixed $T \in (0, T_2]$, define $h_2 = min\{h_{25}, T\}$. Suppose $max_{i\in\{0,1,\ldots,k\}}|\tilde{x}(i)| \le D_1'$ for some $k \in \mathbb{Z}^+$, $\|w\|_\infty \le D_2'$, $h \in (0, h_2]$ and $T \in (0, T_2]$.

To shorten the notations, denote $x := x(k), z := z(k), x_c := x_c(k), u := u(k), f := f(x(k), u(k), w(k))$ and $g := g(x(k), z(k))$ in the sequel. Consider

$$
\begin{aligned}
&\frac{V(x(k+1), z(k+1)) - V(x(k), z(k))}{T} \\
&= \underbrace{\frac{\partial V}{\partial x}\bigg|_{\tilde{x}} f + \frac{\partial V}{\partial z}\bigg|_{\tilde{x}} g}_{J_1} \\
&+ \underbrace{\frac{1}{T}\{V(x(k+1), z(k+1)) - V(x + Tf, z + Tg(x_c, z))\}}_{J_2} \\
&+ \underbrace{\frac{1}{T}\{V(x + Tf, z + Tg(x_c, z)) - V(x, z) - \frac{\partial V}{\partial x}_{|\tilde{x}} f - \frac{\partial V}{\partial z}_{|\tilde{x}} g\}}_{J_3}.
\end{aligned} \tag{2.26}
$$

J_1: By Assumption 2.1, it follows that

$$J_1 \le -\alpha_3(|(\tilde{x}(k)|) + \gamma(\|w\|_\infty). \tag{2.27}$$

J_2: By the Mean Value Theorem, it can be obtained that

$$J_2 \le J_{21} + J_{22}, \tag{2.28}$$

where

$$
\begin{aligned}
J_{21} &= \tfrac{1}{T}\,\tfrac{\partial V}{\partial x}\Big|_{(x_3, z(k+1))} |x(k+1) - x - Tf| \\
J_{22} &= \tfrac{1}{T}\,\tfrac{\partial V}{\partial z}\Big|_{(x+Tf, z_3)} |z(k+1) - z - Tg(x_c, z)|
\end{aligned} \tag{2.29}
$$

with $x_3 = x + Tf + \theta_1\{x(k+1) - x - Tf\}$, $z_3 = z + Tg(x_c, z) + \theta_2\{z(k+1) - z - Tg(x_c, z)\}$ and $\theta_1, \theta_2 \in (0, 1)$.

J_{21}: If $\{x(k+1) - x - Tf\} \le 0$, then it follows from the choice of T_2, in particular $T_2 \le \frac{1}{2b}$, that $|x_3| \le |x + Tf| \le D_1' + 1 = \Delta$, else, $|x_3| \le |x(k+1)|$ and $|x_3| \le |x(k+1)| \le D_1' + \frac{1}{2} \le \Delta$ hold because here $x(t)$ is the solution of initial value problem: $\dot{x}(t) = f(x(t), u, w(t)), \forall t \in [t_0 + kT, t_0 + (k+1)T]$ with the initial value $x(k)$. Since by Lemma 2.4 $|x_c| \le D_1' + \varepsilon_1 \le \Delta$, it follows from Assumption 2.3, triangular inequality as well as the choice of h_{23}, that

$$\begin{aligned}
&|z(k+1)| = |G^a_{T,h}(x_c, z)| \\
&\le |G^e_T(x_c, z)| + |G^e_T(x_c, z) - G^a_{T,h}(x_c, z)| \\
&\le D'_1 + \tfrac{1}{2} + T\tilde{\rho}(h) \\
&\le D'_1 + \tfrac{1}{2} + \tfrac{1}{2} \\
&\le \Delta.
\end{aligned} \tag{2.30}$$

Since $|x_3| \le \Delta$ and $|z(k+1)| \le \Delta$, it yields that $\left|\frac{\partial V}{\partial x}\Big|_{(x_3, z(k+1))}\right| \le b$. Note that

$$\begin{aligned}
&|x(k+1) - x - Tf| \\
&\le T|f(x(kT+\theta_3 T), u, w(kT+\theta_3 T)) - f(x, u, w)| \\
&\le T|f(x(kT+\theta_3 T), u, w(kT+\theta_3 T)) - f(x, u, w(kT+\theta_3 T))| \\
&\quad + T|f(x, u, w(kT+\theta_3 T)) - f(x, u, w)| \\
&\le TL_1\{|x(kT+\theta_3 T) - x| + |w(kT+\theta_3 T) - w|\} \\
&\le \theta_3 L_1 T^2 |f(x(kT+\alpha\theta_3 T), u, w(kT+\alpha\theta_3 T))| + 2L_1 T\|w\|_\infty \\
&\le L_1 T^2 b + 2L_1 T\|w\|_\infty
\end{aligned} \tag{2.31}$$

where $\theta_3, \alpha \in (0,1)$. Hence,

$$J_{21} \le Tb^2 L_1 + 2L_1 b\|w\|_\infty. \tag{2.32}$$

J_{22}: In exactly the same way, it implies from $max\{|z+Tg(x_c,z)|, |z(k+1)|\} \le \Delta$ that $|z_3| \le \Delta$. Moreover, $|x+Tf| \le \Delta$, which yields $\left|\frac{\partial V}{\partial z}\Big|_{(x+Tf, z_3)}\right| \le b$. It follows from triangular inequality and Assumption 2.3 that

$$\begin{aligned}
&|z(k+1) - z - Tg(x_c, z)| \\
&\le |G^a_{T,h}(x_c, z) - G^e_T(x_c, z)| + |G^e_T(x_c, z) - z - Tg(x_c, z)| \\
&\le T\tilde{\rho}(h) + |G^e_T(x_c, z) - z - Tg(x_c, z)|.
\end{aligned} \tag{2.33}$$

Applying the Mean Value Theorem to the last term of (2.33), gives

$$\begin{aligned}
|G^e_T(x_c, z) - z - Tg(x_c, z)| &\le T|g(x_c, z(kT+\theta_4 T)) - g(x_c, z)| \\
&\le TL_2|z(kT+\theta_4 T) - z| \\
&\le \beta L_2 T^2 |g(x_c, z(kT+\beta\theta_4 T))| \\
&\le bL_2 T^2
\end{aligned} \tag{2.34}$$

where $\theta_4, \beta \in (0,1)$. Hence,

$$J_{22} \leq b\tilde{\rho}(h) + Tb^2 L_2. \tag{2.35}$$

By the choice of T_{23} and h_{24}, it follows that

$$J_2 \leq 2L_1 b\|w\|_\infty + \frac{\delta}{2}. \tag{2.36}$$

J_3: Let $x_4 := x + \theta_5 Tf$ and $z_4 := z + \theta_6 Tg(x_c, z)$ where $\theta_5, \theta_6 \in (0,1)$. Since $|x_4 - x| \leq Tb$ and $|z_4 - z| \leq Tb$, from the choice of T_{24} and T_{25}, it follows that

$$\left| \frac{\partial V}{\partial x}\bigg|_{(x_4, z+Tg(x_c,z))} - \frac{\partial V}{\partial x}\bigg|_{(x,z)} \right| \leq \frac{\delta}{4b}, \tag{2.37}$$

$$\left| \frac{\partial V}{\partial z}\bigg|_{(x,z_4)} - \frac{\partial V}{\partial z}\bigg|_{(x,z)} \right| \leq \frac{\delta}{8b}. \tag{2.38}$$

Hence,

$$\begin{aligned} J_3 \leq\ & \tfrac{\partial V}{\partial x}\big|_{(x_4,z+Tg(x_c,z))} f + \tfrac{\partial V}{\partial z}\big|_{(x,z_4)} g(x_c,z) - \tfrac{\partial V}{\partial x}\big|_{(x,z)} f - \tfrac{\partial V}{\partial z}\big|_{(x,z)} g \\ \leq\ & b \left| \tfrac{\partial V}{\partial x}\big|_{(x_4,z+Tg(x_c,z))} - \tfrac{\partial V}{\partial x}\big|_{(x,z)} \right| + \tfrac{\partial V}{\partial z}\big|_{(x,z_4)} g(x_c,z) - \tfrac{\partial V}{\partial z}\big|_{(x,z)} g \\ \leq\ & \tfrac{\delta}{4} + J_{31} + J_{32}, \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} J_{31} &= \tfrac{\partial V}{\partial z}\big|_{(x,z_4)} g(x_c,z) - \tfrac{\partial V}{\partial z}\big|_{(x,z)} g(x_c,z), \\ J_{32} &= \tfrac{\partial V}{\partial z}\big|_{(x,z)} g(x_c,z) - \tfrac{\partial V}{\partial z}\big|_{(x,z)} g(x,z). \end{aligned} \tag{2.40}$$

In (2.39), first we applied the Mean Value Theorem by adding and subtracting $V(x, z + Tg(x_c, z))$, then used the bounds of f and (2.37). It implies from (2.38) that J_{31} is bounded by $\frac{\delta}{8}$. By Lemma 2.4 and the local Lipschitz property of g, it is easy to show that

$$J_{31} \leq bL_2\varepsilon_1 \leq \frac{\delta}{8}.$$

Thus,

$$J_3 \leq \tfrac{\delta}{4} + \tfrac{\delta}{8} + \tfrac{\delta}{8} = \tfrac{\delta}{2}. \tag{2.41}$$

It follows from the bounds of J_1 to J_3 that

$$\begin{aligned}&\frac{V(x(k+1),z(k+1))-V(x(k),z(k))}{T}\\&\leq -\alpha_3(|(\tilde{x}(k)|)+\gamma(\|w\|_\infty)+2L_1b\|w\|_\infty+\tfrac{\delta}{2}+\tfrac{\delta}{2}\\&\leq -\alpha_3(|(\tilde{x}(k)|)+\gamma_1(\|w\|_\infty)+\delta,\end{aligned} \tag{2.42}$$

where $\gamma_1(s) := \gamma(s) + 2L_1bs$. The proof of Lemma 2.5 is complete. ■

Remark 2.6. It follows from the proof of Lemma 2.4 and 2.5 that in the case when the states of the plant are sampled with the different rates l_iT ($l_i \geq 1$), Lemma 2.4 and 2.5 still hold.

Lemma 2.6. *Given any strictly positive real numbers* (d_1, d_2, ν_1, ν_2), *there exists* $T_3 > 0$ *such that for any* $T \in (0, T_3]$ *the following holds: if* $|\tilde{x}(k)| \leq d_1$ *and* $\|w\|_\infty \leq d_2$, *then the solution* $x(t)$ *of the system* $\dot{x}(t) = f(x(t), u, w(t))$ *exists for all* $t \in [t_0 + kT, t_0 + (k+1)T)$ *and satisfies* $|x(t) - x(k)| \leq \nu_1$. *Moreover, for any fixed* $T \in (0, T_3]$ *there exists* $h_3 \in (0, T]$ *such that if* $max_{i\in\{0,1,\ldots,k\}}|\tilde{x}(i)| \leq d_1$ *for some* $k \in \mathbb{Z}^+$, *then* $V(x(k+1), z(k+1)) \leq V(x(k), z(k)) + \nu_2$.

Lemma 2.6 is a standard inter-sample growth result. This result is trivial to prove and the proof is omitted. In fact, the proof of the first part of Lemma 2.6 is standard (see, for example, [26] (the proof of Theorem 2.2)) and the proof of the second part comes directly from the continuity of V and Lemma 2.4.

Now the proof of Theorem 2.3 can be finalized as follows.

Proof of Theorem 2.3. Given numbers $(\Delta_x, \Delta_w, \mu)$, let μ_1 and ν_1 be strictly positive real numbers such that $\mu_1 + \nu_1 = \mu$. Define $D_2' := \Delta_w$. Let δ, ν_2 and ν_3 be strictly positive real numbers such that $\nu_3 := \alpha_2 \circ \alpha_3^{-1}(4\delta)$ and $\max_{s\in[0,\alpha_2\circ\alpha_3^{-1}(4\gamma_1(\Delta_w))]}\{\alpha_1^{-1}(s+\nu_2+\nu_3) - \alpha_1^{-1}(s)\} \leq \mu_1$. Define $D_1' = max\{\alpha_1^{-1}\circ\alpha_2(\Delta_x), \alpha_1^{-1}(\alpha_2\circ\alpha_3^{-1}(2\gamma_1(\Delta_w)+2\delta)+\nu_2)\}$. From Lemma 2.5, let $T_1^* > 0, h_1^* > 0$ be generated by (D_1', D_2', δ). From Lemma 2.6, let $T_2^* > 0, h_2^* > 0$ be generated by $(D_1', D_2', \nu_1, \nu_2)$. Take $T^* = min\{T_1^*, T_2^*\}$ and $h^* = min\{h_1^*, h_2^*\}$. Consider arbitrary $T \in (0, T^*), h \in (0, h^*)$ and arbitrary $\bar{x}(t_0), w(t)$ with $|\bar{x}(t_0)| \leq \Delta_x, \|w\|_\infty \leq \Delta_w$. First of all, let us establish the following claim.

Claim 2.7. *If* $V(\tilde{x}(0)) \leq \alpha_1(D_1')$ *then* $V(\tilde{x}(i)) \leq \alpha_1(D_1'), \forall i \in \mathbb{N}$.

Proof of Claim 2.7. Let $V(\tilde{x}(i)) \leq \alpha_1(D_1')$ holds for all $i \leq k$, which implies $|\tilde{x}(i)| \leq D_1'$. Either

$$V(\tilde{x}(k)) \geq \alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\Delta_w) + 2\delta), \tag{2.43}$$

in which case it follows from (2.18) and Lemma 2.5 that $V(\tilde{x}(k+1) - V(\tilde{x}(k) \leq -\frac{T}{2}\alpha_3(|\tilde{x}(k)|)$, which implies $V(\tilde{x}(k+1) \leq V(\tilde{x}(k)) \leq \alpha_1(D_1')$. Or

$$V(\tilde{x}(k)) \leq \alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\Delta_w) + 2\delta), \tag{2.44}$$

which implies, from Lemma 2.6 and the definition of D_1', $V(\tilde{x}(k+1)) \leq V(\tilde{x}(k)) + \nu_2 \leq \alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\Delta_w) + 2\delta) + \nu_2 \leq \alpha_1(D_1')$. By induction, the proof of Claim 2.7 is complete.

Now we continue the proof of Theorem 2.3. From the choice of D_1', it is obvious that $V(\tilde{x}(0)) \leq \alpha_2(|\tilde{x}(0)|) \leq \alpha_2(\Delta_x) \leq \alpha_1(D_1')$. It follows from Claim 2.7 that $V(\tilde{x}(i)) \leq \alpha_1(D_1'), \forall i \in \mathbb{Z}^+$. Hence, $|\tilde{x}(i)| \leq D_1', \forall i \in \mathbb{Z}^+$, which implies by Lemma 2.5 that (2.23) holds for all $k \geq 0$. Denote $V_i := V(\tilde{x}(i))$ and introduce the variable

$$p(s) := V_i + (\tfrac{s}{T} - i)(V_{i+1} - V_i), s \in [iT, (i+1)T), \forall i \geq 0. \tag{2.45}$$

Note that $p(s)$ is continuous, piecewise linear and $0 \leq p(s) \leq \max\{V_i, V_{i+1}\}$. It follows from the second part of Lemma 2.6 that

$$p(s) \leq V_i + \nu_2, \quad \forall s \in [iT, (i+1)T), i \geq 0,$$

which, together with (2.23), implies that

$$\begin{aligned} & p(s) \geq \alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\|w\|_\infty) + 2\delta) + \nu_2 \\ \Rightarrow\ & V_i \geq \alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\|w\|_\infty) + 2\delta) \\ \Rightarrow\ & \dot{p}(s) \leq -\frac{1}{2}\alpha_3(\alpha_2^{-1}(p(s))). \end{aligned} \tag{2.46}$$

It is a well-known fact (see [26]) that (2.46) implies the existence of $\beta_p \in \mathcal{KL}$ such that $p(s) \leq \beta_p(p(0), s)$. Since $p(0) = V_0 \leq \alpha_2(|\tilde{x}(0)|)$ and $p(iT) = V_i \geq \alpha_1(|\tilde{x}(i)|)$, the following holds:

$$|\tilde{x}(i)| \leq \max\{\beta(|\tilde{x}(0)|, iT), \alpha_1^{-1}(\alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\|w\|_\infty) + 2\delta) + \nu_2)\} \tag{2.47}$$

where $\beta(s,t) := \alpha_1^{-1} \circ \beta_p(\alpha_2(s), t))$. It follows from the first part of Lemma 2.6 that $|\bar{x}(t) - \tilde{x}(i)| \leq \nu_1, \forall t \in [t_0 + iT, t_0 + (i+1)T)$. Thus,

$$|\bar{x}(t)| \leq \max\{\beta(|\bar{x}(t_0)|, iT), \alpha_1^{-1}(\alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\|w\|_\infty) + 2\delta) + \nu_2)\} + \nu_1,$$

where the initial value of the state of the sampled-data system $\bar{x}(t_0)$ is exactly the same with that of the discrete model $\tilde{x}(0)$. It was shown in [1] (Lemma 1 and corollary 1) that there exists $\bar{\beta} \in \mathcal{KL}$ such that $\beta(s, iT) \leq \bar{\beta}(s, (i+1)T)$. Moreover, note that $t - t_0 \leq (i+1)T, \ \forall t \in [t_0 + iT, t_0 + (i+1)T)$. Hence,

$$\begin{aligned}
|\bar{x}(t)| &\leq \max\{\bar{\beta}(|\bar{x}(t_0)|, t - t_0), \alpha_1^{-1}(\alpha_2 \circ \alpha_3^{-1}(2\gamma_1(\|w\|_\infty) + 2\delta) + \nu_2)\} + \nu_1 \\
&\leq \max\{\bar{\beta}(|\bar{x}(t_0)|, t - t_0), \alpha_1^{-1}(\alpha_2 \circ \alpha_3^{-1}(4\gamma_1(\|w\|_\infty)) + \nu_2 + \nu_3)\} + \nu_1 \\
&\leq \bar{\beta}(|\bar{x}(t_0)|, t - t_0) + \gamma_2(\|w\|_\infty) + \mu_1 + \nu_1 \\
&\leq \bar{\beta}(|\bar{x}(t_0)|, t - t_0) + \gamma_2(\|w\|_\infty) + \mu, \qquad (2.48)
\end{aligned}$$

where $\nu_3 := \alpha_2 \circ \alpha_3^{-1}(4\delta)$ and $\gamma_2(s) := \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(4\gamma_1(s))$. Here a weak form of the triangle inequality, which holds for any function ψ of class $\mathcal{K}$ (in particular, take $\psi := \alpha_2 \circ \alpha_3^{-1}$) and any $a, b \in \mathbb{R}_{\geq 0}$,

$$\psi(a+b) \leq \psi(2a) + \psi(2b), \qquad (2.49)$$

and the definition of μ_1 and μ, were used. The proof of Theorem 2.3 is complete. ■

Remark 2.7. This study uses the zero disturbance model $F^a_{T,h}(x, u, 0)$ to estimate the states of the plant between samples. If in practice, $F^a_{T,h}(x, u, w_f)$ is available, then the results of input-to-state stability may be obtained by modifying lightly the assumptions and following the similar reasoning as Lemma 2.4-2.6 and Theorem 2.3. (See [35].)

2.4 Numerical Examples

Consider the continuous-time plant with a continuous-time controller

$$\begin{aligned}
\dot{x}(t) &= x^3(t) + u(t) + w(t), \\
u(t) &= -x(t) - 3x^3(t).
\end{aligned} \qquad (2.50)$$

It is obvious that the closed loop continuous-time system is ISS. Let $f_h(x,u,w)$ represent one step of the numerical integration routine on the sampling interval $[iT,(i+1)T)$ defined by

$$\begin{aligned} f_h(x,u,w) &= x + h(x^3+u) + \int_{iT}^{iT+h} w(s)ds \\ &:= f_h^1(x,u,w). \end{aligned} \tag{2.51}$$

The approximate discrete-time model of the plant $F^a_{T,h}(\cdot,\cdot,\cdot)$ is generated by

$$\begin{aligned} &f_h(k,x,u,w) := x + h(x^3+u) + \textstyle\int_{iT+kh}^{iT+(k+1)h} w(s)ds, \\ &f_h^{k+1}(x,u,w) := f_h(k+1, f_h^k, u, w), \\ &F^a_{T,h}(x,u,w_f) := f_h^N(x,u,w), \quad k = 1,2,\cdots, \end{aligned} \tag{2.52}$$

where h represents the integration period, T is the sampling period and

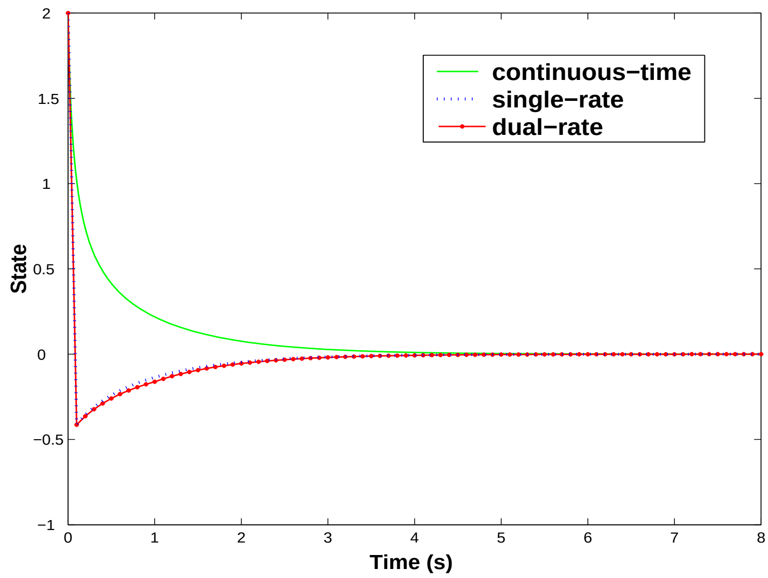

Figure 2.2: The performance of the closed-loop sampled-data system with $T_m = 1s$ and $T_i = 0.1s$

$N = \frac{T}{h}$. Since the controller is a static feedback controller, we only need to check the assumption 2.2. By Lemma II.2 in [47], f_h is one-step consistent with F^e_h which is the exact discrete-time model with the sampling period h. Also, the multi-step consistency is guaranteed by the one-step consistency plus the uniform Lipschitz condition on f_h (see Remark 13 in [50]). Then it follows closely from the conclusions of Corollary 4 and Remark 14 in [50] that $F^a_{T,h}(x, u, w_f)$ is one-step consistent with $F^e_T(x, u, w_f)$. It is concluded from Theorem 2.3 that the exact closed loop system is semiglobally practically input-to-state stable.

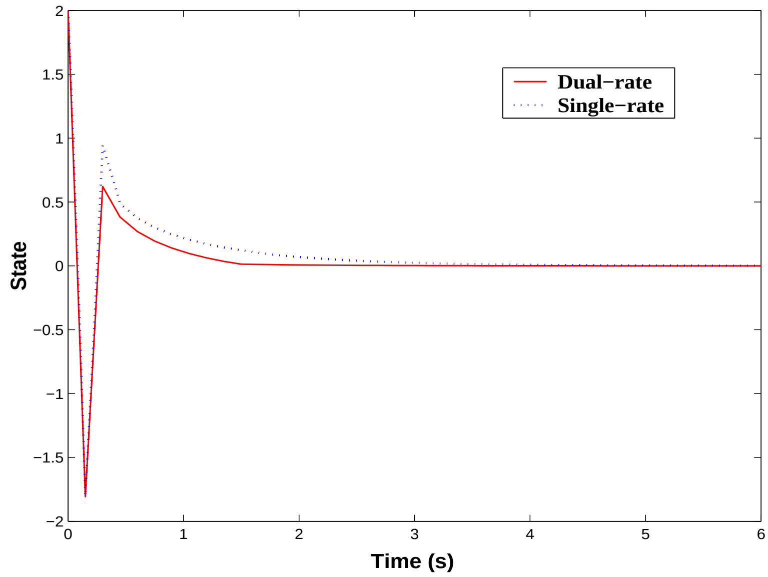

Figure 2.3: The performance of the closed-loop sampled-data system with $T_m = 1.5s$ and $T_i = 0.15s$

First we study a single-rate sampled data implementation of this controller by the emulation method. Assume the initial value is $x_0 = 2.0$ and the disturbances $w(t) \equiv 0$ for the purpose of simplicity. The simulation shows that the single-rate sampled-data controller stabilizes the system with disturbance free only when the sampling period $T \leq 0.15$ seconds. On the other hand, we study the system with the low measurement sampling rate $T_m = 1$ seconds. Setting the input sampling period $T_i = 0.1$ seconds and the integration step $h = 0.005$ seconds, the closed-loop computer simulation shows that stability can be recovered (Figure 2.2). Suppose the measurement rate is lower, say, $T_m = 1.5$ seconds. Figure 2.3 shows that even though the same input sampling period with that of the single-rate scheme is taken, that is, $T_i = 0.15$ seconds, the stability is maintained and the closed loop performance is better.

2.5 Conclusions

This chapter investigated the problem of sampled-data input-to-state stabilization of nonlinear systems under the low measurement constraint based on the emulation design. The results are very important for applications since in the nonlinear system the controller design is usually carried out in continuous time and the digital implementation of a nonlinear controller usually requires some form of approximate discretization. Moreover, the fact that the sampled-data control systems are of the semi-global practical stability property motivates implementation of such algorithms.

Since the fast sampling results may not be implemented due to hardware limitations, this chapter addressed the dual-rate control scheme. The main idea is to introduce a controller that contains a "fast" numerical integration model that reconstructs approximately the missing states between samples. The control action depends on the state of this model and the state of this model is corrected from time to time using the low rate measurements of the actual state of the plant. It was shown that if a continuous-time controller input-to-state stabilizes a continuous-time plant, then under some standard assumptions the proposed dual-rate scheme makes the closed loop sampled data system input-to-state stable in the semiglobal practical sense.

Notes. A version of this chapter has been published.
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Chapter 3

Input-to-state stability via discrete-time design method

So far, the design of stabilizing controllers for sampled-data nonlinear systems under the low measurement rate using the emulation method has been presented. This chapter considers this problem via the discrete-time design approach. Given an approximate discrete-time model of a sampled nonlinear plant and given a family of controllers that stabilizes the plant model in input-to-state sense, it is showed that the closed loop sampled-data control system is input-to-state stable under the dual-rate control scheme based on the DTD method.

This chapter is organized as follows. Section 3.1 reviews the discrete-time design method and introduces some preliminary notions. A brief description of problem statement is presented in section 3.3. The main results are presented in section 3.4 and illustrated via simulations in section 3.5. Finally, this chapter is closed with conclusions in the last section.

3.1 Discrete-time design method

As indicated in chapter 2 the discrete-time design method (DTD method) is one of fundamental approaches to design discrete-time controllers for continuous-time plants. This approach parallels the classical approach to analog controller design. The design procedure is depicted in Table 3.1. In this method, one first derives a discrete-time equivalent of the plant and then directly designs a discrete-time controller to control the discretized plant and finally implements the discrete-time controller using a sampler and hold device.

In principle, DTD method is more straightforward for linear systems than for nonlinear systems. Indeed, for linear systems an explicit discrete-time model can be written down while typically for nonlinear systems this is not the case. Moreover, the exact discrete-time model of a linear system is linear while the exact discrete-time model for a nonlinear system does not usually preserve important structures of the underlying continuous-time nonlinear system ([49]). Hence, it is unusual for nonlinear systems to assume the knowledge of the exact discrete-time model of the plant while this assumption is made for most of linear systems. In the nonlinear literature, more often than not, one assumes that the approximate discrete-time model of the nonlinear system is available and such a discrete-time design is referred to as approximate DTD method.

$$\boxed{\begin{array}{c} continuous-time\ plant\ model \\ \Downarrow \\ discretize\ plant\ model \\ \Downarrow \\ discrete-time\ controller \\ \Downarrow \\ implement\ the\ controller \end{array}}$$

Table 3.1: The DTD Method

Despite the difficulties one faced in the approximate DTD method for nonlinear systems, there is a strong motivation for pursuing this approach since it deals with the issue of sampling naturally and effectively. As mentioned in [12, 53], the approximate DTD method may yield better results than the emulation design. Many results for nonlinear sampled-data systems based on the DTD method have been investigated, for instance [1, 7, 30, 37, 47, 49, 50, 53, 59].

One may take it for granted that if a digital controller is designed for an approximate discrete-time model of the plant with a sufficiently small sampling period then the same controller will also stabilize the exact discrete-time model. It is important that if this is true, then one could directly design digital controllers for the approximate model. However, this reasoning is wrong since no matter how small the sampling period is, it is possible to find a controller that stabilizes the approximate model for that sampling period but destabilizes the exact model, as illustrated by the following example. Consider the continuous-time system ([49])

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u. \end{aligned} \tag{3.1}$$

Its Euler approximate model is

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= x_2(k) + Tx_3(k), \\ x_3(k+1) &= x_3(k) + Tu(k). \end{aligned} \tag{3.2}$$

Consider a dead-beat controller for the Euler model given by

$$u(k) = -\tfrac{x_1(k)}{T^3} - \tfrac{3x_2(k)}{T^2} - \tfrac{3x_3(k)}{T}. \tag{3.3}$$

The system matrix of the closed-loop system consisting of (3.2) and (3.3) is

$$\mathbf{A} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -\frac{1}{T^2} & -\frac{3}{T} & -2 \end{bmatrix}.$$

By calculating the eigenvalues of $\mathbf{A}$, one knows that the closed loop system has all poles equal to zero for all $T > 0$. Hence this Euler-based closed loop system is asymptotically stable for all $T > 0$. On the other hand, the exact discrete-time closed loop system consisting of (3.1) and (3.3) has a pole at around -2.64 for all $T > 0$. Hence the closed-loop sampled-data control system is unstable for all $T > 0$. It follows that, to design a stable controller using the DTD method, it is not sufficient to only design a stable controller for an approximate discrete-time model of the plant for sufficiently small T. Extra conditions are needed.

The main question in the DTD method is whether or not the closed-loop system consisting of the exact discrete-time plant model and the discrete-time controller will have similar properties as the closed-loop system consisting of the approximate discrete-time plant model and the discrete-time controller. The chapter will present some answers to this question for this method. It is emphasized that the results do not contain algorithms for digital controller design. In the case of the DTD method, controller design algorithms are under development and the results provide a guide for doing this.

3.2 Multi-step consistency

Let $F^e_T(x,u)$ and $F^a_T(x,u)$ denote the exact discrete-time and approximate models of the nonlinear continuous-time control system

$$\dot{x} = f(x,u). \tag{3.4}$$

with the sampling period T. The exact discrete-time model $F^e_T(x,u)$ is obtained by integrating the initial value problem

$$\dot{x} = f(x,u(k)), \tag{3.5}$$

with given $u(k)$ and initial value x_0 over the sampling interval $[kT,(k+1)T]$. Let $x(t)$ denotes the solution of the initial value problem. Then the exact discrete-time model for the system (3.4) can be written as

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(s),u(k))ds, \\ &:= F^e_T(x(k),u(k)). \end{aligned} \tag{3.6}$$

The approximate model is obtained from (3.5) using one of the numerical integration methods, for example, Euler, Runge-Kutta, and so on.

Definition 3.1. The family (u,F^a_T) is said to be *multi-step consistent with* (u,F^e_T) if for each $L>0,\eta>0$ and each compact set $\chi\in\mathbb{R}^n$, there exist a function $\alpha:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}\cup\{\infty\}$ and $T^*>0$ such that for all $T\in[0,T^*)$ we have that $x,z\in\chi, |x-z|\leq\delta$ implies

$$|F^e_T(x,u(x)) - F^a_T(z,u(z))| \leq \alpha(\delta,T) \tag{3.7}$$

and

$$k \leq L/T \Rightarrow \alpha^k(0,T) := \overbrace{\alpha(\cdots\alpha(\alpha}^{k}(0,T),T)\cdots,T) \leq \eta. \tag{3.8}$$

Multi-step consistency guarantees that the error of solutions between $F^e_T(x,u)$ and $F^a_T(x,u)$ is small over multiple steps corresponding continuous-time intervals with length of order one. A sufficient condition for multi-step consistency is given in the following([49]).

Lemma 3.1. *If for each compact set $\chi \subset \mathbb{R}^n$ there exist $K > 0, \rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $T \in [T^*, 0)$ and all $x, z \in \chi$ we have*

$$|F_T^e(x, u(x)) - F_T^a(z, u(z))| \leq (1 + KT)|x - z| + T\rho(T) \tag{3.9}$$

then (u, F_T^a) is multi-step consistent with (u, F_T^e).

Observe that relative to one-step consistency condition, the condition of Lemma 3.1 is guaranteed by one-step consistency plus the following Lipschitz condition on either (u, F_T^e) or (u, F_T^a): for each compact set $\chi \subset \mathbb{R}^n$ there exist $K > 0$ and $T^* > 0$ such that for all $T \in [T^*, 0)$ and all $x, z \in \chi$,

$$|F_T^e(x, u(x)) - F_T^a(z, u(z))| \leq (1 + KT)|x - z|. \tag{3.10}$$

This condition is guaranteed for F_T^e when $f(x, u)$ and $u(x)$ are locally Lipschitz, uniformly in small T.

3.3 Problem statement

Consider the nonlinear continuous-time plant

$$\dot{x}(t) = f(x(t), u(t), w(t)), \tag{3.11}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ are respectively the state, control input and exogenous disturbance and f is locally Lipschitz and $f(0, 0, 0) = 0$. Let

$$x(i + 1) = F_T^e(x(i), u(i), w[i]) \tag{3.12}$$

be the exact discrete-time model of (3.11) with the sampling period $T > 0$. Assume that the input sampling period T_i is equal to T. If the measurements of all states of the plant are available at the sampling instants $kT, k \in \mathbb{Z}^+$, then it is a single-rate sampled-data system. Moreover, Nesic and Laila showed in [47] that if a digital controller input-to-state stabilizes the approximate discrete-time model of the plant, then it would also input-to-state stabilize the exact discrete-time model.

Suppose now that the states of the plant are sampled with period $T_m = lT$ for some integer $l \geq 1$. For such a dual-rate control system, a model-based inferential controller using the DTD method is proposed. Since the explicit solution of a nonlinear differential equation is often non-existent in analytical form, the approximate discrete-time model of (3.12) $F^a_{T,h}(x(i), u(i), w[i])$ is used instead. Assume that the approximate model with zero disturbance

$$x(i+1) = F^a_{T,h}(x(i), u(i), 0) \tag{3.13}$$

is available. The main idea is the following: since the plant input is sampled "l" times faster than the output, the fast-rate model (3.13) is used to estimate the inter-samples of the measurements and then supply them to the digitally implemented controller. For each "l" input of the controller, one is the true measurement of the plant and the remaining "$l-1$" are the estimations given by the zero disturbance model (3.13). This scheme is updated periodically using the actual states of the plant. Hence, the input of the controller is given by

$$x_c(i+1) = \begin{cases} x(i+1), \quad i+1 = kl, k \in \mathbb{Z}^+, \\ F^a_{T,h}(x_c(i), u(i), 0) \; with\ initial\ value\ x_c(kl) = x(kl), otherwise. \end{cases} \tag{3.14}$$

Consider the dynamic feedback controller

$$z(i+1) \;= G_{T,h}(x_c(i), z(i)), \tag{3.15}$$

$$u(i) \;= U_{T,h}(x_c(i), z(i)), \tag{3.16}$$

where $z \in \mathbb{R}^{n_z}$ and $G_{T,h}(0,0) = 0, U_{T,h}(0,0) = 0$. Notice that the control law, in turn, depends on the states of the model rather than the actual states of the plant.

This chapter presents the framework of controller design for sampled-data nonlinear systems under the low measurement rate using the DTD method. It is the purpose of this chapter to show that under some mild assumptions the closed loop dual-rate system is input-to-state stable in the semiglobal practical sense. The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. The main results can be used to determine if the sampling period is sufficiently small.

Introduce the following definitions. To shorten the notation, denote $\tilde{F}^a_{T,h}(\widetilde{x}, w_f) := \begin{bmatrix} F^a_{T,h}(x, U_{T,h}(x,z), w_f) \\ G_{T,h}(x,z) \end{bmatrix}$.

Definition 3.2. The system $\tilde{x}(i+1) = \tilde{F}^a_{T,h}(\tilde{x}(i), w[i])$ is equi-Lipschitz Lyapunov-ISS if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty, \tilde{\gamma} \in \mathcal{K}$ and for any positive real numbers (Δ_1, Δ_2) there exist $T^* > 0$ such that for each fixed $T \in (0, T^*]$ there exists $h^* \in (0, T]$ such that for all $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$, $\|w\|_\infty \leq \Delta_2$ and $h \in (0, h^*)$ there exists a function $V_{T,h} : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}_{\geq 0}$ with the following properties:

$$\alpha_1(|\tilde{x}|) \leq V_{T,h}(\tilde{x}) \leq \alpha_2(|\tilde{x}|), \tag{3.17}$$

$$V_{T,h}(\tilde{F}^a_{T,h}(\tilde{x}, w_f)) - V_{T,h}(\tilde{x}) \leq -T\alpha_3(|\tilde{x}|) + T\tilde{\gamma}(\|w\|_\infty) \tag{3.18}$$

and, for all $\tilde{x}_1, \tilde{x}_2 \in \mathbf{B}(\Delta_1)$, there exists $M > 0$ such that $|V_{T,h}(\tilde{x}_1) - V_{T,h}(\tilde{x}_2)| \leq M|\tilde{x}_1 - \tilde{x}_2|$.

Remark 3.1. Note that Definition 3.2 is closely related to Definition 2.1 (ISS). In particular, it was shown in [24] that the non-parameterized discrete-time system is ISS if and only if (3.17)-(3.18) holds. It would be important to provide conditions under which one can construct a Lyapunov function from Definition 3.2. Illustration how such Lyapunov function can be constructed and used in controller design based on approximate discrete-time model can be found in [50].

Definition 3.3. The control law $(G_{T,h}, U_{T,h})$ is said to be uniformly locally Lipschitz if for any $\Delta_1 > 0$ there exist $L_1, L_2 > 0$ and $T^* > 0$ such that for each fixed $T \in (0, T^*]$ there exists $h^* \in (0, T]$ such that for all $\xi_1, \xi_2 \in \mathbf{B}(\Delta_1)$ and $h \in (0, h^*]$, we have $|G_{T,h}(\xi_1) - G_{T,h}(\xi_2)| \leq L_1|\xi_1 - \xi_2|, |U_{T,h}(\xi_1) - U_{T,h}(\xi_2)| \leq L_2|\xi_1 - \xi_2|$, where $\xi := (x_c^T, z^T)^T$.

3.4 Main results

In this section, the main results are stated and proved. The results specify conditions which guarantee that the dual-rate controller input-to-state stabilizes the closed-loop sampled-data system. More precisely, the stabilization problem under the following assumptions is addressed.

Assumption 3.1. $\tilde{F}^a_{T,h}(\tilde{x}, w_f)$ is equi-Lipschitz Lyapunov-ISS.

Assumption 3.2. $F^a_{T,h}(x, u, w_f)$ is one-step consistent with the exact discrete-time model $F^e_T(x, u, w_f)$.

Assumption 3.3. The controller (3.15)-(3.16) is uniformly locally Lipschitz.

Remark 3.2. From assumption 3.3 and $U_{T,h}(0,0) = 0$, it follows that given positive numbers (Δ_1, Δ_2) there exist $T^*, h^* > 0$ such that for all $\xi := (x_c^T, z^T)^T \in \mathbf{B}(\Delta_1)$ and $h \in (0, h^*)$, $|U_{T,h}(\xi)| \leq \Delta_2$ holds. That is, the output of the controller is locally uniformly bounded (see [26]).

Theorem 3.2. *Under assumptions 3.1-3.3, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the following holds. Given any positive real numbers $(\Delta_{\tilde{x}}, \Delta_w, \delta)$, there exists $T^* > 0$ such that for each $T \in (0, T^*]$ there exists $h^* \in (0, T]$ such that for all $|\tilde{x}(0)| \leq \Delta_{\tilde{x}}, \|w\|_\infty \leq \Delta_w$ and all $h \in (0, h^*]$, the exact closed loop discrete-time model (3.12) and (3.14)-(3.16) satisfies $|\tilde{x}(i)| \leq \beta(|\tilde{x}(0)|, iT) + \gamma(\|w\|_\infty) + \delta$.*

Let us begin with the following lemmas.

Lemma 3.3. *Consider the exact closed loop discrete-time model (3.12) and (3.14)-(3.16). Given any strictly positive real numbers (D_1, D_3, ε), there exists $T_1 > 0$ such that for any fixed $T \in (0, T_1]$ there exists $h_1 \in (0, T]$ such that for all $h \in (0, h_1], |\tilde{x}(0)| \leq D_1$ and $\|w\|_\infty \leq D_3$, the following holds: if $\max_{i\in\{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1$ for some $k \in \{0, 1, \ldots\}$ then the exact discrete-time state of the plant satisfies: $|x(k) - x_c(k)| \leq T\varepsilon + T\lambda\|w\|_\infty$, for some $\lambda > 0$.*

Proof. Let (D_1, D_3, ε) be given. Define $\Delta_1 = D_1 + \varepsilon + 1$. By Remark 3.2, for given $D_2 > 0$ there exist $T_{11} > 0$ and $h_{11} > 0$ such that $|U_{T,h}(x_c, z)| \leq D_2$ for all $(x_c^T, z^T)^T \in \mathbf{B}(\Delta_1)$. Let $L > 0$ be the Lipschitz constant of function f. Also, let $\lambda > 0$ be a number such that $e^{L(l-1)T} - 1 \leq \lambda T$ for any $T \in (0, T_{11}]$. Let $T_{12} > 0$ and $h_{12} > 0$ be as in Assumption 3.2 corresponding to $\Delta_1 = D_1 + \varepsilon + 1, \Delta_2 = D_2$ and $\Delta_3 = D_3$, and let $\rho(\cdot) \in \mathcal{K}_\infty$ be a function from Assumption 3.2. Let $T_{13} > 0, h_{13} > 0$ be such that $\rho(h_{13})(e^{L(l-1)T_{13}} - 1)/(e^{LT_{13}} - 1) \leq \varepsilon$. Finally define $T_1 = \min\{T_{11}, T_{12}, T_{13}, 1/\lambda D_3, 1\}$ and $h_1 = \min\{h_{11}, h_{12}, h_{13}\}$.

Suppose $T \in (0, T_1], h \in (0, h_1]$ and $\max_{i\in\{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1$ for some $k \in \{0, 1, \ldots\}$. First claim that $|(x_c^T(k), z^T(k))^T| \leq \Delta_1$ for some $k \in \{0, 1, \ldots\}$ follows by induction. Consider k in the following three cases. If $k = jl$ for some $j \in \{0, 1, \ldots\}$, then it is obvious that $|x(k) - x_c(k)| = 0$. If $k = jl + 1$, then it follows from Assumption 3.2 and triangle inequalities that

$$\begin{aligned}
&|x(k) - x_c(k)| \\
&= |F_T^e(x(jl), U_{T,h}(x(jl), z(jl)), w[jl]) - F_{T,h}^a(x(jl), U_{T,h}(x(jl), z(jl)), 0)| \\
&\leq |F_T^e(x(jl), U_{T,h}(x(jl), z(jl)), w[jl]) - F_T^e(x(jl), U_{T,h}(x(jl), z(jl)), 0)| \\
&\quad + T\rho(h) \\
&\leq (e^{LT} - 1)\|w\|_\infty + T\rho(h).
\end{aligned} \tag{3.19}$$

Otherwise, the following holds for all $k \in \{jl + 2, \ldots, (j + 1)l - 1\}$:

$$\begin{aligned}
|x(k) - x_c(k)| &\leq T\rho(h) + e^{LT}|x(k-1) - x_c(k-1)| + (e^{LT} - 1)\|w\|_\infty \\
&\leq T\rho(h)\frac{e^{(k-jl)LT} - 1}{e^{LT} - 1} + (e^{(k-jl)LT} - 1)\|w\|_\infty.
\end{aligned} \tag{3.20}$$

It implies from the choice of T and h that $|x(k) - x_c(k)| \leq T\varepsilon + T\lambda\|w\|_\infty$. This completes the proof of Lemma 3.3. ■

Lemma 3.4. *Consider the exact closed loop discrete-time model (3.12) and (3.14)-(3.16). For any strictly positive real numbers (D_1', D_3') there exists $T_2 > 0$ such that for any fixed $T \in (0, T_2]$ there exists $h_2 \in (0, T], \Delta > 0$ such that for all $h \in (0, h_2], |\tilde{x}(0)| \leq D_1'$ and $\|w\|_\infty \leq D_3'$, the following holds: if $\max_{i\in\{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1'$ for some $k \in \{0, 1, \ldots\}$ then the exact state of closed loop system $\tilde{x}(k + 1)$ and the approximation satisfy $\tilde{F}_{T,h}^a(\tilde{x}(k), w[k]) \in \mathbf{B}(\Delta)$.*

Proof. Let (D_1', D_3') be given. Take $\varepsilon_1 > 0$. From Lemma 3.3, let $(D_1', D_3', \varepsilon_1)$ generate $T_{21} > 0, h_{21} > 0$ and let $\lambda > 0$ be from in Lemma 3.3. From Assumption 3.1, let $T_{22} > 0, h_{22} > 0$ be generated by $\Delta_1 = D_1' + \varepsilon_1 + 1, \Delta_2 = D_3'$ and let $T_{23}, h_{23}, L_1, L_2 > 0$ be as in Assumption 3.3. Let $T_{24} > 0, h_{24} > 0$ and $\varepsilon_2 > 0$ be such that $T_{24}(\rho(h_{24}) + L_1(\varepsilon_1 + \lambda\|w\|_\infty) + L_2(e^{LT_{24}} - 1)(\varepsilon_1 + \lambda\|w\|_\infty)) \leq \varepsilon_2$. Define $\Delta = \alpha_1^{-1}(\alpha_2(D_1') + \tilde{\gamma}(D_3')) + \varepsilon_2$. Let $T_2 = \min\{T_{21}, T_{22}, T_{23}, T_{24}\}$ and $h_2 = \min\{h_{21}, h_{22}, h_{23}, h_{24}\}$.

Suppose $T \in (0, T_2], h \in (0, h_2]$ and $\max_{i\in\{0,1,\ldots,k\}} |\tilde{x}(i)| \leq D_1'$ for some $k \in \{0, 1, \ldots\}$. It follows from Assumption 3.1 that

$$\begin{aligned}
\tilde{F}_{T,h}^a(\tilde{x}(k), w[k]) &\leq \alpha_1^{-1} \circ V_{T,h}(\tilde{F}_{T,h}^a(\tilde{x}(k), w[k])) \\
&\leq \alpha_1^{-1}(V_{T,h}(\tilde{x}(k)) + \tilde{\gamma}(D_3')) \\
&\leq \Delta.
\end{aligned} \tag{3.21}$$

Applying Assumption 3.2 and 3.3 as well as triangle inequalities to $|\tilde{x}(k+1) - \tilde{F}^a_{T,h}(\tilde{x}(k), w[k])|$ gives

$$|\tilde{x}(k+1) - \tilde{F}^a_{T,h}(\tilde{x}(k), w[k])| \leq T(\rho(h) + L_1|x(k) - x_c(k)| + L_2(e^{LT} - 1)|x(k) - x_c(k)|.$$

Thus, from Lemma 3.3 and the choice of T_{24}, h_{24} and Δ,

$$|\tilde{x}(k+1)| \leq |\tilde{F}^a_{T,h}(\tilde{x}(k), w[k])| + \varepsilon_2 = \Delta. \tag{3.22}$$

This completes the proof of Lemma 3.4. ∎

Lemma 3.5. *Let* $\mathcal{W} = \{w \in \mathcal{L}_\infty | \; \|w\|_\infty \leq C_w, \forall\, C_w > 0\}$ *and let* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$*. Let strictly positive real numbers* (d, D) *be such that* $\alpha_1(D) \geq d$ *and let* $T_3 > 0$ *be such that for each fixed* $T \in (0, T_3]$ *there exists* $h_3 \in (0, T]$ *such that for any* $h \in (0, h_3]$ *there exists a function* $V_{T,h} : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}_{\geq 0}$ *such that for all* $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}, \alpha_1(|\tilde{x}|) \leq V_{T,h}(\tilde{x}) \leq \alpha_2(|\tilde{x}|)$ *and for all* $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$ *with* $|\tilde{x}| \leq D$*, all* $w \in \mathcal{W}$ *and* $\max\{V_{T,h}(\tilde{x}(i+1)), V_{T,h}(\tilde{x}(i))\} \geq d$*, the following holds:* $V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \leq -\frac{T}{4}\alpha_3(|\tilde{x}(i)|)$*. Then,* $|\tilde{x}(i)| \leq D$ *for all* $|\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(D), w \in \mathcal{W}$ *and all* $i \in \mathbb{Z}^+$*, and the solution of the exact closed loop discrete-time model exists and satisfies*

$$|\tilde{x}(i)| \leq \beta(|\tilde{x}(0)|, iT) + \alpha_1^{-1}(d). \tag{3.23}$$

Proof. The definitions of d and D imply that

$$|\tilde{x}(0)| \leq \max\{\alpha_1^{-1} \circ V_{T,h}(\tilde{x}(0)), \alpha_1^{-1}(d)\} \leq D. \tag{3.24}$$

So either $V_{T,h}(\tilde{x}(1)) \geq d$ which, from the condition of Lemma 3.5, implies $V_{T,h}(\tilde{x}(1)) \leq V_{T,h}(\tilde{x}(0))$, or else $V_{T,h}(\tilde{x}(1)) \leq d$. Then, in either case, $V_{T,h}(\tilde{x}(1)) \leq \max\{V_{T,h}(\tilde{x}(0)), d\}$. Hence $V_{T,h}(\tilde{x}(i)) \leq \max\{V_{T,h}(\tilde{x}(0)), d\}$ follows by induction and $|\tilde{x}(i)| \leq D$ holds as well. Consequently (3.23) follows, using an argument similar to the proof of Theorem 2 in [53]. ∎

Lemma 3.6. *Consider the exact closed loop discrete-time model (3.12) and (3.14)-(3.16). There exists* $\hat{\gamma} \in \mathcal{K}_\infty$ *such that the following holds. For any strictly positive real numbers* $(C_{\tilde{x}}, C_w, \nu)$ *with* $C_{\tilde{x}} \geq \alpha_1^{-1}(\hat{\gamma}(C_w) + \nu)$*, there* *exists $T_4 > 0$ such that for each $T \in (0, T_4]$ there exists $h_4 \in (0, T]$ such that for all $h \in (0, h_4]$, $|\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(C_{\tilde{x}})$, $\|w\|_\infty \leq C_w$ and all $i \in \mathbb{Z}^+$,*

$$\begin{aligned} &\max\{V_{T,h}(\tilde{x}(i+1)), V_{T,h}(\tilde{x}(i))\} \geq \hat{\gamma}(\|w\|_\infty) + \nu \\ &\Rightarrow V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \leq -\tfrac{T}{4}\alpha_3(|\tilde{x}(i)|). \end{aligned} \tag{3.25}$$

Proof. Let positive real numbers $(C_{\tilde{x}}, C_w, \nu)$ be given. Define $\varepsilon_2 = \frac{1}{2}\alpha_2^{-1}(\frac{\nu}{2}), \varepsilon_3 = \alpha_2^{-1}(\frac{1}{2}\alpha_1(\varepsilon_2))$, and $\Delta = \alpha_1^{-1}(\alpha_2(C_{\tilde{x}}) + \tilde{\gamma}(C_w)) + \varepsilon_2$. Take any $\varepsilon_1 > 0$ which satisfies the inequality: $ML_1\varepsilon_1 \leq \frac{1}{4}\alpha_3(\varepsilon_3)$. From Lemma 3.3, let $(C_{\tilde{x}}, C_w, \varepsilon_1)$ generate T_{41}, h_{41} and let $\lambda > 0$ be as in Lemma 3.3. Let T_{42} and h_{42} come from Lemma 3.4 corresponding to $(C_{\tilde{x}}, C_w)$ and also the following holds

$$T_{42}(\rho(h_{42}) + L_1(\varepsilon_1 + \lambda\|w\|_\infty) + L_2(e^{LT_{42}} - 1)(\varepsilon_1 + \lambda\|w\|_\infty)) \leq \varepsilon_2.$$

Let positive real numbers $T_{43}, h_{43}, T_{44}, h_{44}$ and T_{45} be such that

$$\begin{aligned} &T_{43}(M(\rho(h_{43}) + L_1(\varepsilon_1 + \lambda\|w\|_\infty) + L_2(e^{LT_{43}} - 1)(\varepsilon_1 + \lambda\|w\|_\infty)) + \\ &\tilde{\gamma}(\|w\|_\infty) + \tfrac{1}{4}\alpha_3(C_{\tilde{x}})) \leq \tfrac{\nu}{2}, \\ &M\rho(h_{44}) + ML_2(e^{LT_{44}} - 1)\varepsilon_1 \leq \tfrac{1}{4}\alpha_3(\varepsilon_3), \end{aligned}$$

and $T_{45}\tilde{\gamma}(C_w) \leq \frac{1}{2}\alpha_1(\varepsilon_2)$. Let $\hat{\gamma}(s) = \alpha_2 \circ \alpha_3^{-1}(4(\tilde{\gamma}(s) + \lambda M(L_1 s + L_2(e^L - 1)s)))$. Take $T_4 = \min\{T_{41}, T_{42}, T_{43}, T_{44}, T_{45}\}$ and $h_4 = \min\{h_{41}, h_{42}, h_{43}, h_{44}\}$. Consider any $T \in (0, T_4], h \in (0, h_4], |\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(C_{\tilde{x}})$ and $\|w\|_\infty \leq C_w$.

First of all, claim that $|\tilde{x}(i)| \leq C_{\tilde{x}}$ for any $i \in \mathbb{Z}^+$. From now suppose this to be true. It follows from Lemma 3.4 that $|\tilde{F}^a_{T,h}(\tilde{x}(i), w[i])| \leq \Delta$ and $|\tilde{x}(i+1)| \leq \Delta$.

Now suppose that

$$V_{T,h}(\tilde{x}(i+1)) \geq \hat{\gamma}(\|w\|_\infty) + \tfrac{\nu}{2}. \tag{3.26}$$

Using Assumption 3.1 and triangle inequalities, the following holds

$$\begin{aligned} &V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \leq -T\alpha_3(|\tilde{x}(i)|) + T\tilde{\gamma}(\|w\|_\infty) + \\ &M|\tilde{x}(i+1) - \tilde{F}^a_{T,h}(\tilde{x}(i), w[i])|. \end{aligned} \tag{3.27}$$

Applying Assumption 3.2-3.3 and triangle inequalities and from the choice of T_{41} and h_{41}, it is easy to see that

$$\begin{aligned} &V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \leq -T\alpha_3(|\tilde{x}(i)|) + T\tilde{\gamma}(\|w\|_\infty) + \\ &TM(\rho(h) + L_1(\varepsilon_1 + \lambda\|w\|_\infty) + L_2(e^{LT} - 1)(\varepsilon_1 + \lambda\|w\|_\infty)). \end{aligned} \tag{3.28}$$

Denote $\mu_1 := ML_1\varepsilon_1, \mu_2 := M(\rho(h) + L_2(e^{LT} - 1)\varepsilon_1)$ and $\kappa(s) := \tilde{\gamma}(s) + M\lambda(L_1 + L_2(e^{LT} - 1))s$. Then

$$\begin{aligned}
& V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \\
& \leq -\tfrac{T}{4}\alpha_3(|\tilde{x}(i)|) \underbrace{-\frac{T}{4}\alpha_3(\alpha_2^{-1}(V_{T,h}(\tilde{x}(i)))) + T\kappa(\|w\|_\infty)}_{J_1} \\
& \underbrace{-\frac{T}{4}\alpha_3(|\tilde{x}(i)|) + T\mu_1}_{J_2} \underbrace{-\frac{T}{4}\alpha_3(|\tilde{x}(i)|) + T\mu_2}_{J_3}.
\end{aligned} \tag{3.29}$$

Supposition (3.26) implies that

$$\begin{aligned}
\hat{\gamma}(\|w\|_\infty) + \frac{\nu}{2} \quad \leq \quad & |V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{F}^a_{T,h}(\tilde{x}(i), w[i]))| + \\
& V_{T,h}(\tilde{F}^a_{T,h}(\tilde{x}(i), w[i])) - V_{T,h}(\tilde{x}(i)) + \\
& V_{T,h}(\tilde{x}(i)) \\
\leq \quad & MT\rho(h) + MTL_1(\varepsilon_1 + \lambda\|w\|_\infty) + \\
& MTL_2(e^{LT} - 1)(\varepsilon_1 + \lambda\|w\|_\infty) \\
& + T\tilde{\gamma}(\|w\|_\infty) + \\
& V_{T,h}(\tilde{x}(i)).
\end{aligned}$$

It follows from the choice of T_{43} and h_{43} that

$$\tilde{\gamma}(\|w\|_\infty) + \tfrac{\nu}{2} \leq \tfrac{\nu}{2} + V_h(\tilde{x}(i)). \tag{3.30}$$

Thus

$$V_{T,h}(\tilde{x}(i+1)) \geq \hat{\gamma}(\|w\|_\infty) + \tfrac{\nu}{2} \Rightarrow V_{T,h}(\tilde{x}(i)) \geq \hat{\gamma}(\|w\|_\infty),$$

and $J_1 \leq 0$ holds by the definition of $\hat{\gamma}(\cdot)$. Supposition (3.26) gives that

$$\tilde{x}(i+1) \geq \alpha_2^{-1}(\tfrac{\nu}{2}) = 2\varepsilon_2. \tag{3.31}$$

Then from the choice of T_{42} and h_{42}, it is obtained that

$$\begin{aligned}
|\tilde{F}^a_{T,h}(\tilde{x}(i), w[i])| \quad \geq \quad & |\tilde{x}(i+1)| - |\tilde{x}(i+1) - \tilde{F}^a_{T,h}(\tilde{x}(i), w[i])| \\
\geq \quad & 2\varepsilon_2 - \varepsilon_2 = \varepsilon_2.
\end{aligned} \tag{3.32}$$

Using the choice of T_{45}, it follows that

$$\begin{aligned}\alpha_2(|\tilde{x}(i)|) &\geq V_{T,h}(\tilde{F}^a_{T,h}(\tilde{x}(i), w[i])) - T\tilde{\gamma}(C_w)\\ &\geq \alpha_1(|\tilde{F}^a_{T,h}(\tilde{x}(i), w[i])|) - T\tilde{\gamma}(C_w)\\ &\geq \alpha_1(\varepsilon_2) - \frac{1}{2}\alpha_1(\varepsilon_2) = \frac{1}{2}\alpha_1(\varepsilon_2),\end{aligned}$$

which implies

$$\begin{aligned}|\tilde{x}(i)| &\geq \alpha_2^{-1}(\frac{1}{2}\alpha_1(\varepsilon_2)) = \varepsilon_3\\ &\geq \alpha_3^{-1}(4\mu_1) \qquad (3.33)\end{aligned}$$

and then $J_2 \leq 0$ holds. Moreover, it follows from the choice of T_{44} and h_{44} that

$$|\tilde{x}(i)| \geq \varepsilon_3 \Rightarrow -\tfrac{T}{4}\alpha_3(|\tilde{x}(i)|) + T\mu_2 \leq 0. \qquad (3.34)$$

That is, $J_3 \leq 0$ holds. Hence, supposition (3.26) implies

$$V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i)) \leq -\tfrac{T}{4}\alpha_3(|\tilde{x}(i)|). \qquad (3.35)$$

Suppose

$$V_{T,h}(\tilde{x}(i+1)) \leq \hat{\gamma}(\|w\|_\infty) + \tfrac{\nu}{2} \qquad (3.36)$$

and

$$V_{T,h}(\tilde{x}(i)) \geq \hat{\gamma}(\|w\|_\infty) + \nu. \qquad (3.37)$$

From our choice of T_{43}, it follows that

$$\begin{aligned}&V_{T,h}(\tilde{x}(i+1)) - V_{T,h}(\tilde{x}(i))\\ &\leq \hat{\gamma}(\|w\|_\infty) + \tfrac{\nu}{2} - V_{T,h}(\tilde{x}(i)) + \tfrac{\nu}{2} - \tfrac{\nu}{2}\\ &\leq \hat{\gamma}(\|w\|_\infty) + \nu - V_{T,h}(\tilde{x}(i)) - \tfrac{\nu}{2}\\ &\leq -\tfrac{\nu}{2} \leq -\tfrac{T}{4}\alpha_3(|\tilde{x}(i)|).\end{aligned} \qquad (3.38)$$

It remains to establish the initial claim: $|\tilde{x}(i)| \leq C_{\tilde{x}}$ for any $i \in \mathbb{Z}^+$. This claim follows by induction. Indeed, it clearly holds for $i = 0$, since the definition of $\tilde{x}(0)$ implies

$$|\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(C_{\tilde{x}}) \leq C_{\tilde{x}}. \qquad (3.39)$$

Then (3.25) holds for $i = 0$ from the deduction above. Lemma 3.5 (Take $D = C_{\tilde{x}}$ and $d = \hat{\gamma}(\|w\|_\infty) + \nu$.) gives that $|\tilde{x}(1)| \leq C_{\tilde{x}}$. That is, this claim holds for $i = 1$ as well. Then

$$|\tilde{x}(i)| \leq C_{\tilde{x}}, \quad i \in \mathbb{Z}^+ \tag{3.40}$$

follows by induction. The proof of Lemma 3.6 is complete. ■

Now we complete the proof of Theorem 3.2 as follows.

Proof of Theorem 3.2. Let $(\Delta_{\tilde{x}}, \Delta_w, \delta)$ be given and let all conditions in Theorem 3.2 hold. Let $\hat{\gamma} \in \mathcal{K}_\infty$ come from Lemma 3.6. Define $(C_{\tilde{x}}, C_w, \nu)$ as $C_w := \Delta_w$, $\nu > 0$ is such that

$$\sup_{s\in[0,\Delta_w]}[\alpha_1^{-1}(\hat{\gamma}(s) + \nu) - \alpha_1^{-1}(\hat{\gamma}(s))] \leq \delta,$$

and

$$C_{\tilde{x}} := \max\{\alpha_1^{-1}(\hat{\gamma}(\Delta_w) + \nu), \alpha_1^{-1} \circ \alpha_2(\Delta_{\tilde{x}})\}.$$

It follows from the choice of $(C_{\tilde{x}}, C_w, \nu)$ that

$$C_{\tilde{x}} \geq \alpha_1^{-1}(\hat{\gamma}(C_w) + \nu) \tag{3.41}$$

and

$$|\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(C_{\tilde{x}}). \tag{3.42}$$

From Lemma 3.6, let $(C_{\tilde{x}}, C_w, \nu)$ generate $T^* > 0, h^* > 0$ such that (3.25) holds. Let $D = C_{\tilde{x}}$ and $d = \hat{\gamma}(\|w\|_\infty) + \nu$, which imply that $\alpha_1(D) \geq d$. It follows from the definition of (D, d) that all conditions of Lemma 3.5 are satisfied. Therefore for all $h \in (0, h^*)$, $|\tilde{x}(0)| \leq \Delta_{\tilde{x}}$ and $\|w\|_\infty \leq \Delta_w$,

$$\begin{aligned} |\tilde{x}(i)| &\leq \beta(|\tilde{x}(0)|, iT) + \alpha_1^{-1}(d) \\ &\leq \beta(|\tilde{x}(0)|, iT) + \alpha_1^{-1}(\hat{\gamma}(\|w\|_\infty) + \nu) \\ &\leq \beta(|\tilde{x}(0)|, iT) + \gamma(\|w\|_\infty)) + \delta, \end{aligned} \tag{3.43}$$

where $\gamma(s) := \alpha_1^{-1} \circ \hat{\gamma}(s)$. This completes the proof of Theorem 3.2. ■

Remark 3.3. Following the proof of Theorem 3.2, it is easy to see that if Assumption 3.1 is relaxed slightly to the assumption of practical Lyapunov-ISS, that is, $V_{T,h}(\tilde{F}^a_{T,h}(\tilde{x}, w_f)) - V_{T,h}(\tilde{x}) \leq -T\alpha_3(|\tilde{x}|) + T\tilde{\gamma}(\|w\|_\infty) + T\delta_1$, then Theorem 3.2 still holds.

3.5 Numerical examples

Consider the continuous-time plant

$$\dot{x}(t) = x^3(t) + u(t) + w(t). \tag{3.44}$$

Let $x(i+1) = F_T^e(x(i), u(i), w[i])$ be the exact discrete-time model of the continuous-time plant with the sampling period T. Let $f_h(x, u, w)$ represent one step of the numerical integration routine on the sampling interval $[iT, (i+1)T)$ defined by $f_h(x, u, w) = x + h(x^3 + u) + \int_{iT}^{iT+h} w(s)ds := f_h^1(x, u, w)$. The numerically integrated approximate model $F_{T,h}^a(\cdot, \cdot, \cdot)$ can be generated by

$$\begin{aligned} & f_h(k, x, u, w) := x + h(x^3 + u) + \int_{iT+kh}^{iT+(k+1)h} w(s)ds, \\ & f_h^{k+1}(x, u, w) := f_h(k+1, f_h^k, u, w), \\ & F_{T,h}^a(x, u, w_f) := f_h^N(x, u, w),\ k = 1, 2, \ldots \end{aligned} \tag{3.45}$$

where h and T represent the integration period and the sampling period respectively, and $N = \frac{T}{h}$. Moreover, the approximate model with disturbance free, which reconstructs the missing plant states between samples, is generated by $F_{T,h}^a(x, u, 0)$. Consider a digital controller

$$u(i) = -x(i) - x^3(i). \tag{3.46}$$

First check the consistency of the approximation scheme. By Lemma II.2 in [47], f_h is one-step consistent with F_h^e which is the exact discrete-time model with the sampling period h. Also, the multi-step consistency is guaranteed by the one-step consistency plus the uniform Lipschitz condition on f_h (see Remark 13 in [50]). Then it follows closely from the conclusions of Corollary 4 and Remark 14 in [50] that $F_{T,h}^a(x, u, w_f)$ is one-step consistent with $F_T^e(x, u, w_f)$. Take $V_{T,h}(x) = |x|$. It follows that the approximate closed loop discrete-time model

$$\begin{aligned} x(i+1) &= F_{T,h}^a(x(i), u(i), w[i]), \\ u(i) &= -x(i) - x^3(i) \end{aligned} \tag{3.47}$$

is practically Lyapunov-ISS with $\alpha_3(|x|) = |x|$ and $\tilde{\gamma}(\|w\|_\infty) = \|w\|_\infty$. Moreover it is easy to see that assumption 3.3 is also satisfied. It is concluded from Theorem 3.2 that the exact closed loop system is semiglobally practically input-to-state stable.

Assume the initial state $x(0) = 3.4$. The simulation shows that the single-rate method stabilizes the system without disturbance only when $T < 0.205$ seconds. On the other hand, consider the dual-rate method with the low measurement rate $T_m = 1.5$ seconds. Setting $T_i = 0.15$ seconds and $h = 0.0075$ seconds ($N = 20$), the simulation shows that the dual-rate controller stabilizes the system successfully (Figure 3.1). Compared with

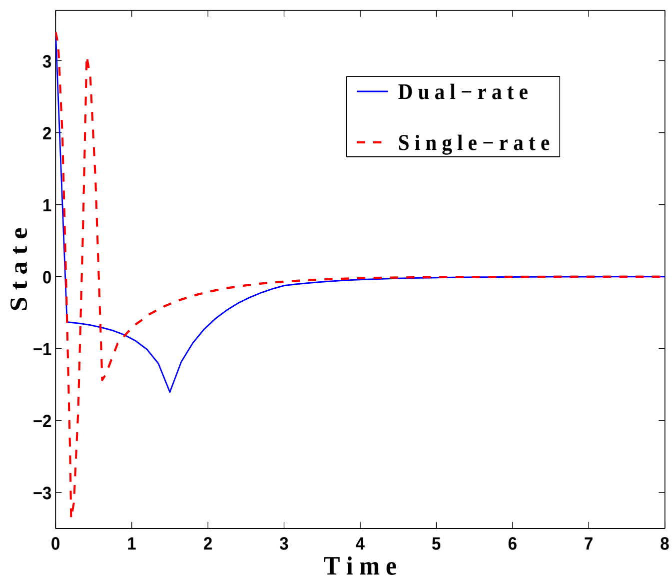

Figure 3.1: The performance of the closed-loop system without disturbance under two control schemes

the single rate scheme, the dual-rate scheme is more effective, which can render a stable closed loop using much lower sampling rate. In Figure 3.2 a sinusoidal disturbance of amplitude 0.8 and frequency 2 rad/s is considered and the simulation shows that the closed-loop system is practically ISS. This example shows that the dual-rate inferential system is indeed more robust than the corresponding fast single-rate system.

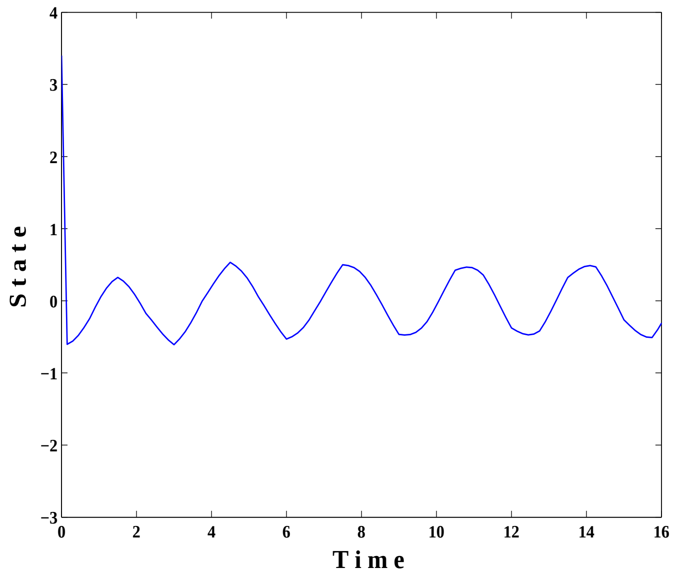

Figure 3.2: The closed-loop performance for dual-rate control system with disturbance

3.6 Conclusions

This chapter is a detailed version of the work in [36]. It concentrates on the problem of stabilization of nonlinear systems under the low measurement constraint. The approach to the solution of this problem employs a dual-rate scheme based on approximate DTD method. The main idea is to introduce a controller that includes an approximate discrete-time model of the plant. In the setting of this chapter, the exact and approximate discrete-time models were considered as functions of the sampling period T and the integration period h. Indeed, it is important to note that the role of T and h are different when pursuing the approximate DTD method. Typically, the size of the domain of attraction may be controlled by reducing T whereas the accuracy of the closed loop behavior may be controlled by tuning design parameters.

Notes. A version of this chapter has been published.
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Chapter 4

Model-based Nonlinear NCSs

So far, the stabilization problem of sampled-data nonlinear systems has been investigated via continuous-time design and discrete-time design methods. This chapter studies the networked realization of nonlinear control systems and presents an estimation method as a solution to reduce network traffic to achieve satisfactory performance.

In this chapter the network effects are treated actively. That is, we concentrate on the interwinding effects between the control system and the communication network. System performance is not only dependent on the performance on its components, but also on their network interaction. In general as the number of messages on the network increases, system nodes experience longer waiting time to access the network, which may degrade the system performance. One of the effective ways to improve the performance of a networked control system is to reduce network traffic. Estimation methods in networked control systems can reduce network traffic and thus ameliorate control performance.

This chapter addresses the stabilization problem of nonlinear networked control systems and proposes a model-based control scheme to estimate the missing states due to the network access limitation. By adding an estimator, a tradeoff between satisfactory control performance and reduction of network traffic can be achieved.

The outline of this chapter is as follows. After the description of problem statement and relevant definitions and notations, the main results for stability analysis of networked control systems in the nonlinear context are derived. The effectiveness of the proposed strategy is verified via simulations. Finally, this chapter is closed with conclusions in the last section.

4.1 Related approaches

In the literature, several approaches have been proposed to reduce the network traffic and improve the system performance, which may be summarized as follows.

In [67] Wong et al. introduced the concept of a recursive coder-estimator sequence to vary the coding scheme from transmission to transmission and investigated a state estimation problem involving finite communication capacity. Stability results were established for mean coder-estimator and equal-partition coder-estimator schemes. Other coding and control schemes for deterministic LTI systems were subsequently proposed and analyzed. Nair studied the exponential stability of finite-dimensional LTI plants with limited feedback data rate in [45] and obtained the infimum data rate by using coder-controller and quantisation theory. Analogous results for stochastic linear systems were reported in [46]. In a close spirit, Liberzon et al. considered the stabilization of linear and nonlinear systems with encoded measurements of the state in [32, 33].

Hristu investigated the stability of systems which were governed by linear dynamics under limited communication [19]. Each system and its feedback controller were viewed as users on a shared network which granted access only to a few nodes. The author proposed a network admission policy for the use of communication sequences which specified the amount of time available for each system to complete its feedback loop, thus reducing network usage. Although the results were conservative, they represented a significant improvement over previous estimates for the amount of communication required to guarantee stability.

In [34] the authors used power spectral density to determine a dropout compensator that minimized the regulator's output power. Data packets may be excessively delayed due to network congestion. As a result, it is often desirable to purposely drop measurements that are delayed excessively. This analysis derived a closed form expression for a control system's output power spectral density as a function of the data dropouts and used this result to synthesize an optimal dropout compensator that minimized the regulator's output power. Stochastic asymptotic stability was obtained in the mean square sense.

The work in [40], upon which this chapter is based, addressed an explicit model of the linear plant to reduce the network traffic. Sufficient and necessary conditions for stability were derived in terms of the constant update time and the parameters of the plant and those of its model. These results were extended in [42] to a particular class of nonlinear control systems. Stochastic stability results with independent identically distributed and Markov-chain driven transmission times were obtained in [41]. Sufficient stability conditions for two types of static and dynamic quantization schemes were derived in [43]. The work in [70] followed the same line to handle the constraints of the network realization of an LTI control system. It was shown that the problem of regulation can be reduced to that of an asymptotically stable observer design for linear systems with missing measurements. The ideas behind the extension with intermittent feedback were summarized in [9, 10].

Nilsson addressed a timeout control scheme based on predictions in [55] such that the measurements that arrive after the timeout can be used to compute the next control signal instead of been neglected. Adjustable deadband was introduced to reduce network traffic in [57]. With a deadband defined on a node, the node will not broadcast a new message if the node signal is within the deadband. Using a similar idea, Yook et al. [68] described a new framework for distributed control systems in which estimators were used at each node to estimate the values of the outputs of the system. When the estimated value deviated from the true value by more than a tolerance, the actual value was broadcast to the rest of the system. The communication only occurred when the difference between the measured actual output and the estimate output of the system was above the threshold value and Bounded-Input Bounded-Output (BIBO) stability was obtained.

4.2 Problem statement

Consider the nonlinear networked control system whose plant dynamics is described by

$$\dot{x}(t) = f_p(t, x, u) + r_p(t, x, u) \tag{4.1}$$

with the initial time zero, where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are the plant state and the control vectors respectively. $f_p : [0, \infty) \times D \times D_u \to \mathbb{R}^n$ and $r_p : [0, \infty) \times D \times D_u \to \mathbb{R}^n$ are continuous in t and locally Lipschitz in x and u on $[0, \infty) \times D \times D_u$, $D \subset \mathbb{R}^n$ is a domain that contains $x = 0$, $D_u \subset \mathbb{R}^m$ is a domain that contains $u = 0$, and $f_p(t, 0, 0) = 0$. $r_p(t, x, u)$ represents a perturbation term which could result from model errors, aging, or uncertainties and disturbances in realistic problems. Consider the state feedback controller

$$u(t) = g_c(t, x) \tag{4.2}$$

where $g_c : [0, \infty) \times D \to \mathbb{R}^m$ is continuous in t and locally Lipschitz in x on $[0, \infty) \times D$ and $g_c(t, 0) = 0$.

The communication access constraint imposed by the network is the main focus. Like [40, 65, 66], the controller dynamics are considered continuous and sampling delay is ignored, because the access interval to the network is much larger than the processing period of the controller and smart sensors. Once the sensor has the access, the data is assumed to transmit instantly, because most of the control network is a local area network with very high data rate and a physical range less than 100 meters.

Denote the updating time of the controller as t_k and the network access interval at t_k as $t_{k+1} - t_k = \tau_k$. A natural assumption on τ_k can be made as follows

$$0 \leq \tau_k \leq \tau_m \tag{4.3}$$

where τ_m denotes the access deadline. The model of the networked control system formulated here deserves the following remarks.

Remark 4.1. It is worth noting that [40] considers a constant updating time, while this chapter deals with random time-varying access updating time. Also, in the literature, several references consider networked control systems as stochastic models (see, for instance, [27]). These models usually assume a probabilistic structure on the delays, while the approach proposed here places only a bound on the network access interval.

Remark 4.2. The networked control system is considered in a nonlinear time-varying setting, i.e. the physical plant and the model estimator are both nonlinear. The nonlinear control problem is difficult because the explicit analytic solution of continuous time processes are typically impossible to compute. The model formulated here is essentially different from that in [40], and is more general. It is important to point out that recent results in [42] (an extended version can be found in Chapter 6, [39]), presented a comprehensive study on networked control systems and in particular, investigated the local stability of a particular class of nonlinear control system described by $\dot{x} = f(x) + g(u)$ and $u = h(x)$, with constant updating time. To the best of the authors' knowledge, the stabilization problem of model-based nonlinear networked control systems with random time-varying updating time has not been investigated and still remains challenging, which motivates the present study.

In this chapter, we first present conditions under which stability is preserved when the communication channel is inserted into the control loop. Because of the network access limitation, the controller receives the signal only at t_k and keeps the value for the whole access interval. In this manner, the controller behaves in the exact same way as if having a zero order hold at its input.

In the remaining of this chapter, we address an alternative approach based on knowledge of the plant dynamics to improve the system performance. It is important to note that one of the central issues of networked control systems is the large amount of bandwidth required in the transmission of data. This can be addressed in two ways: by reducing the number of packets transmitted, or by reducing the size of the data. It has been shown that reducing the number of packets transmitted brings better benefits than data compression [41], which motivates our idea of reducing the network traffic. In this manner, more bandwidth will be available to allocate resources without sacrificing stability and ultimate performance of the overall system.

Specifically, the idea is the following: a model-based controller is used to reconstruct approximately the missing states between transmission times.

The model state is updated from time to time using the measurement of the actual state of the plant and the control law, in turn, depends on the states of the model rather than the actual states of the plant.

Such a process is depicted in Figure 4.1. When there is no estimator, the controller is piecewise constant and the input to the controller $\hat{x}(t) = x(t_k)$ for $t \in [t_k, t_{k+1})$. Alternatively, a plant model is used at the controller side to reconstruct the plant behavior when the sensor data x is not available due to the network access limitation, and to generate the estimate $\hat{x}$ to feed to the controller block. This setting is to perform the feedback by updating the state of the model using the actual state of the plant provided by the sensor. The rest of the time the control action is based on the plant model and is running in open loop for a period of τ_k seconds. Indeed, this idea can be interpreted as a switch between open loop and closed loop control. Having knowledge of the plant model at the controller side enables us to run it in open loop, while the update of the model state provides the close loop information needed to overcome model mismatch. The use of plant models in

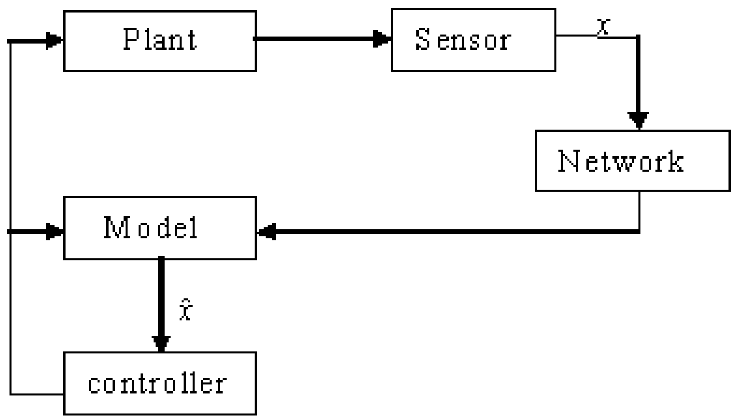

Figure 4.1: The setup of the model-based NCS

the controller is not new. It has been used in previous work on sampled data systems, for example [1, 35, 36]. The works in [35, 36] utilized a numerical integration model to design a dual-rate controller to guarantee the input-to-state stability of the closed loop. Arcak et al. proposed a design framework for single rate sampled-data systems via a discrete-time model in [1].

Remark 4.3. Note that the controller and the actuator are combined together into a single node. That is, the network is between the sensor and the controller/actuator nodes. Assuming that the controller and the actuator physically coexist is reasonable since embedded microprocessors are usually incorporated into the actuator to process the data and execute the commands.

In this chapter, the primary objective is to efficiently use the finite bus capacity while maintaining good closed-loop control system performance. This chapter concentrates on characterizing the network access interval to achieve a tradeoff between better performance and lower network traffic. It is shown that if a controller exponentially stabilize the non-networked system, then the proposed control scheme would guarantee the nonlinear networked control system to preserve desired performance (stability and ultimate boundedness).

The following notation and definitions will be used throughout this chapter. Denote $\mathbb{Z}^+$ as the set of nonnegative integers, $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbf{B}(r) := \{x | \ |x| \leq r\}$. The Euclidean norm of a vector is denoted as $|\cdot|$.

Definition 4.1. The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be exponentially stable if there exist α, β and $a > 0$, such that $|x(t)| \leq \beta |x(t_0)| e^{-\alpha(t-t_0)}$ whenever $|x(t_0)| \leq a$.

Definition 4.2. $\dot{x} = f(t, x)$ is said to be uniformly ultimately bounded if there exist positive constants a and λ, such that for all $|x(t_0)| \leq a$, there exists a positive constant $T > 0$ such that $|x(t)| \leq \lambda, \forall t \geq t_0 + T$.

4.3 Main results

In this section the main results are stated and proved. The effect of the network on the system performance is considered and an estimation method to reduce network traffic and improve system performance is presented.

A starting point is to assume that the control law is designed in advance without considering the effect of the network. In this way traditional controller design techniques can be applied. Precisely, the problem formulated in the previous section is addressed under the following assumption.

Assumption 4.1. Consider the nominal system of (4.1) and (4.2). There exist positive constants c_1, c_2, c_3, c_4 and for any $\Delta > 0$ there exist a continuous function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^n$ with $|x| \leq \Delta$, the following holds

$$c_1|x|^2 \leq V(t,x) \leq c_2|x|^2 \tag{4.4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f_p(t,x,g_c(t,x)) \leq -c_3|x|^2 \tag{4.5}$$

$$\left|\frac{\partial V}{\partial x}\right| \leq c_4|x|. \tag{4.6}$$

Remark 4.4. Note that assumption 4.1 is closely related to Definition 4.1. In particular, it was shown in [26] that the continuous-time system is exponentially stable if (4.4) and (4.5) holds. Moreover, a converse theorem for exponential stability would confirm that if the origin is exponentially stable, then there exists a Lyapunov function satisfying (4.4)-(4.6) (see for example, Section 3.6 in [26]). It would be important to provide conditions under which one can construct a Lyapunov function from this assumption.

4.3.1 Stability analysis without estimator

In a typical situation, one does not know the additive uncertainty $r_p(t,x,u)$, but have some information about it, such as a linear growth bound on it. Consider the case of a vanishing perturbation: $r_p(t,0,0) = 0$. Suppose for any $\Delta > 0$ the perturbation term $r_p(t,x,g_c(t,x)) =: r(t,x)$ satisfies $|r(t,x)| \leq \gamma|x|$ whenever $|x| \leq \Delta$, where γ is a nonnegative constant. Consider the networked control system in Figure 4.1 with the plant dynamics (4.1) and the controller

$$u = g_c(t,\hat{x}), \tag{4.7}$$

where $\hat{x}$ is the state of the controller. When no plant model is used at the controller side, the controller receives the signal only at t_k because of the network access limitation. The resulting controller is piecewise constant, i.e. $\hat{x}(t) = x(t_k)$ for $t \in [t_k, t_{k+1})$.

Begin with the following lemma [66], which can be viewed as a reverse time extension of the Gronwall Inequality.

Lemma 4.1. *Let $\lambda(t)$ be continuous and differentiable, and let $k(t)$ be continuous and nonnegative. If the function $y(t)$ satisfies*

$$y(t) \le \lambda(t) + \int_t^{t_f} k(s)y(s)ds, \ \forall\ t_f \ge t \ge 0, \tag{4.8}$$

then

$$y(t) \le \lambda(t_f)e^{\int_t^{t_f} k(s)ds} - \int_t^{t_f} \dot{\lambda}(s)e^{\int_t^s k(\tau)d\tau}ds. \tag{4.9}$$

Theorem 4.2. *Consider the networked control system (4.1) and (4.7). Suppose there exists a Lyapunov function $V(t,x)$ that satisfies (4.4)-(4.6) and suppose $|r(t,x)| \le \gamma|x|$ with $\gamma < \frac{c_3}{c_4}$. Then the following holds. Given $\zeta_1 > 0$, there exists $\tau_m > 0$ with the following properties:*

$$(i)\ L_2(e^{(L_1+L_3)\tau_m} - 1) < 1,$$
$$(ii)\ \frac{L_2(\gamma+L_1+L_1L_2)(e^{(L_1+L_3)\tau_m}-1)e^{(L_1+L_3)\tau_m}}{1-L_2(e^{(L_1+L_3)\tau_m}-1)} < \frac{c_3}{c_4} - \gamma,$$

where L_i $(i = 1,2,3)$ are locally Lipschitz constants of $f_p(t,x,u), g_c(t,x)$ and $r_p(t,x,u)$, such that for all $|x(t_0)| \le \zeta_1$ and $\tau_k \in [0,\tau_m]$, the origin of the NCS is exponentially stable.

Proof. Let $\zeta_1 > 0$ be given. Define $\Delta_x := \sqrt{\frac{c_2}{c_1}}\zeta_1$ and $\Delta_1 := \Delta_x + 1$. By the local Lipschitz property of u and the fact that u is zero at zero, there exists $\Delta_2 \ge 0$ such that $|u(t)| \le \Delta_2$ for all $x \in \mathbf{B}(\Delta_x) \subset D$. Let $L_1, L_2, L_3 > 0$ be the Lipschitz constants of $f_p(t,x,u), g_c(t,x)$ and $r_p(t,x,u)$ for all $|x| \le \Delta_1, |u| \le \Delta_2$ with $\mathbf{B}(\Delta_1) \subset D$ and $\mathbf{B}(\Delta_2) \subset D_u$. Let b be a number that satisfies $\max\{|f_p(t,x,u)|, |r_p(t,x,u)|\} \le b$ for all $|x| \le \Delta_1, |u| \le \Delta_2$.

First of all, claim that $|x(t_k)| \le \Delta_x$ for $k \in \mathbb{Z}^+$. Assume from now on that this claim is true. Since f_p and r_p are bounded by b, the solution $x(t)$ of the initial value problem: $\dot{x}(t) = f_p(t,x,u)+r_p(t,x,u)$ with the initial value $x(t_k)$ exists (in particular, we can take $\tau_k = \frac{1}{2b}$), and $|x(t)| \le \Delta_1, \forall t \in [t_k, t_{k+1})$. Define the network induced error signal as

$$e(t) = x(t) - \hat{x}(t). \tag{4.10}$$

At each transmission time t_k, it follows that $e(t_k) = 0$, for all $k \in \mathbb{Z}^+$. The closed-loop dynamics in between two successive transmission times can be written as

$$\begin{aligned} \dot{x}(t) &= f_p(t, x, g_c(t, \hat{x})) + r_p(t, x, g_c(t, \hat{x})), \\ \hat{x}(t) &= x(t_k) \end{aligned} \tag{4.11}$$

$\forall t \in [t_k, t_{k+1})$. Then it follows that for any $t \in [t_k, t_{k+1})$,

$$\dot{e}(t) = f_p(t, x, g_c(t, \hat{x})) + r_p((t, x, g_c(t, \hat{x}))), \tag{4.12}$$

which implies from the Lipshitz property of f_p, g_c and $f_p(t, 0, 0) = 0$ and $g_c(t, 0) = 0$ that

$$|\dot{e}(t)| \leq (L_1 + L_3)|e| + \gamma + L_1 + L_1 L_2 |\hat{x}|. \tag{4.13}$$

Application of comparison lemma results in

$$|e(t)| \leq \tfrac{(\gamma + L_1 + L_1 L_2)}{L_1 + L_3}(e^{(L_1 + L_3)\tau_k} - 1)|\hat{x}|, \tag{4.14}$$

where we have used the fact that $\frac{d|e(t)|}{dt} \leq |\frac{de(t)}{dt}|$. Fix the final time t_f and let the initial time t be interchangeable, that is, $t_k \leq t \leq t_f < t_{k+1}$. It follows from

$$|x(t)| \leq |x(t_f)| + L_2(L_1 + L_3)|\hat{x}|(t_f - t) + \textstyle\int_{t_f}^{t} (L_1 + L_3)|x(s)|ds$$

and Lemma 4.1 that

$$|x(t)| \leq |x(t_f)|e^{(L_1 + L_3)(t_f - t)} + L_2(L_1 + L_3)|\hat{x}|(e^{(L_1 + L_3)(t_f - t)} - 1). \tag{4.15}$$

Let $t = t_k$, and $t_f = t$. Then

$$|\hat{x}(t)| \leq \tfrac{e^{(L_1 + L_3)\tau_k}}{1 - L_2(e^{(L_1 + L_3)\tau_k} - 1)}|x(t)| \tag{4.16}$$

and thus

$$|e(t)| \leq \tfrac{(\gamma + L_1 + L_1 L_2)(e^{(L_1 + L_3)\tau_k} - 1)e^{(L_1 + L_3)\tau_k}}{(L_1 + L_3)(1 - L_2(e^{(L_1 + L_3)\tau_k} - 1))}|x(t)|. \tag{4.17}$$

It follows from assumption 4.1 with $\Delta = \Delta_1$ that the derivative of $V(t, x)$ along the trajectories of (4.11) is given by

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f_p(t, x, g_c(t, \hat{x})) + r_p(t, x, g_c(t, \hat{x}))) \\ &\leq -(c_3 - c_4\gamma)|x|^2 + c_4 L_2(L_1 + L_3)|x||e| \\ &\leq -(c_3 - c_4\gamma - \mu)|x|^2 \end{aligned} \tag{4.18}$$

where $\mu = \frac{c_4 L_2(\gamma+L_1+L_1L_2)(e^{(L_1+L_3)\tau_m}-1)}{1-L_2(e^{(L_1+L_3)\tau_m}-1)} e^{(L_1+L_3)\tau_m}$. Under the conditions (i) and (ii), we have $\dot{V} \leq -c|x|^2$, with $c = c_3 - c_4\gamma - \mu > 0, \forall t \in [t_k, t_{k+1})$. Hence, the origin is exponentially stable.

It remains to establish the initial claim. This claim follows by induction. Indeed, it clearly holds for $k = 0$, since $c_1|x(t_0)|^2 \leq V(t_0) \leq c_2\zeta_1^2 \leq c_1\Delta_x^2$ by the definition of Δ_x. Then from the deduction above, $\dot{V} \leq -c|x|^2 \leq -\frac{c}{c_2}V$ holds for the transmission interval between t_0 and t_1. By the comparison lemma, it follows that $V(t_1) \leq V(t_0) \leq c_1\Delta_x^2$ and thus $|x(t_1)| \leq \Delta_x$, which implies that this claim holds for $k = 1$ as well. Then $|x(t_k)| \leq \Delta_x, k \in \mathbb{Z}^+$ follows by induction. This establishes the claim and the proof is complete. ■

Remark 4.5. The convergence rate of the closed-loop system is easy to estimate in Theorem 4.2. It follows from $\dot{V} \leq -c|x^2|$ and (4.4) that $|x(t)| \leq \sqrt{\frac{c_2}{c_1}}|x(t_0)|e^{-\frac{c_3-c_4\gamma-\mu}{2c_2}(t-t_0)}$. Hence, we conclude that the convergence rate should not be greater than $-\frac{c_3-c_4\gamma-\mu}{2c_2}$. Here, μ depending on the access deadline τ_m, is the main factor that affects heavily the convergence rate of the NCS. One can see that if τ_m is sufficiently large, then the larger access interval leads to the slower convergence rate.

4.3.2 Stability analysis with estimator

Since the lack of awareness of the inner plant behavior in the access interval may result in large bandwidth and fast update time of the network, in the following, we seek an alternative approach, which utilizes the knowledge of the plant dynamics, to lower network usage and ameliorate the system performance.

Consider the performance of such a model-based scheme depicted in Figure 4.1. Let the plant model in between two successive transmission times $[t_k, t_{k+1})$ be

$$\dot{\hat{x}}(t) = f_p(t, \hat{x}(t), u(t)) \tag{4.19}$$

with initial value $\hat{x}(t_k) = x(t_k)$, where $\hat{x}(t) \in \mathbb{R}^n$ is the model state to feed the controller. The closed-loop dynamics of the plant model with the controller (4.7) can be written as

$$\dot{\hat{x}}(t) = f_p(t, \hat{x}, g_c(t, \hat{x})). \tag{4.20}$$

Theorem 4.3. *Consider the NCS whose plant dynamics are described by (4.1) with the model-based controller (4.7) and (4.19). Suppose there exists a Lyapunov function $V(t,x)$ that satisfies (4.4)-(4.6) and suppose $|r(t,x)| \leq \gamma|x|$ with $\gamma < \frac{c_3}{c_4}$. Then the following holds. Given $\zeta_2 > 0$, there exists $\tau_m > 0$ with the following properties:*

(i) $\frac{\alpha(\gamma+L_1L_2+L_3)}{L_1+L_3-\beta}(e^{(L_1+L_3-\beta)\tau_m}-1) < 1$

(ii) $\frac{\alpha\gamma L_2(L_1+L_3)(e^{2(L_1+L_3)\tau_k}-e^{(L_1+L_3-\beta)\tau_m})}{1-\frac{\alpha(\gamma+L_1L_2+L_3)}{L_1+L_3-\beta}(e^{(L_1+L_3-\beta)\tau_m}-1)} < (\frac{c_3}{c_4}-\gamma)(L_1+L_3+\beta)$

where $\alpha = \sqrt{\frac{c_2}{c_1}}$ and $\beta = \frac{c_3}{2c_2}$, such that for all $|x(t_0)| \leq \zeta_2$ and $\tau_k \in [0,\tau_m]$, the model-based NCS is exponentially stable.

Proof. Let $\zeta_2 > 0$ be given. Define $\Delta_x := \sqrt{\frac{c_2}{c_1}}\zeta_2$ and $\Delta_{\hat{x}} := \sqrt{\frac{c_2}{c_1}}\Delta_x$. By the Lipschitz property of u and the fact that u is zero at zero, there exists $\Delta_u \geq 0$ such that $|u(t)| \leq \Delta_u$ for all $\hat{x} \in \mathbf{B}(\Delta_{\hat{x}})$. Define $\Delta_1 := \Delta_x + 1$ and $\Delta_2 := \max\{\Delta_1, \Delta_{\hat{x}}\}$. Let $L_1, L_2, L_3 > 0$ be the Lipschitz constants of $f_p(t,x,u)$, $g_c(t,x)$ and $r_p(t,x,u)$ for all $|x| \leq \Delta_2$, $|u| \leq \Delta_u$.

By induction, we claim that $|x(t_k)| \leq \Delta_x$ for $k \in \mathbb{Z}^+$. Define $e(t) = x(t) - \hat{x}(t)$. Since the model state $\hat{x}$ is updated at t_k and resets to $x(t_k)$, it follows that $e(t_k) = 0$. The closed-loop plant dynamics in $[t_k, t_{k+1})$ is

$$\dot{x}(t) = f_p(t, x(t), g_c(t, \hat{x}(t))) + r_p(t, x(t), g_c(t, \hat{x}(t))). \tag{4.21}$$

Combining with (4.20), it is easy to obtain that for $t \in [t_k, t_{k+1})$,

$$|\dot{e}(t)| \leq (L_1 + L_3)|e| + \gamma|\hat{x}|, \tag{4.22}$$

which implies by the comparison lemma that

$$|e(t)| \leq \gamma \int_{t_k}^{t} e^{(L_1+L_3)(t-s)}|\hat{x}(s)|ds.$$

Use $V(t,x)$ as the Lyapunov function candidate. The derivative of V along the trajectories of the closed loop plant model (4.20) is given by $\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f_p(t, \hat{x}, g_c(t, \hat{x}))$. It follows from assumption 4.1 that the plant model is exponentially stable. Thus,

$$|\hat{x}(t)| \leq \sqrt{\frac{c_2}{c_1}}|\hat{x}(t_k)|e^{-\frac{c_3}{2c_2}(t-t_k)} \tag{4.23}$$

and

$$|e(t)| \leq \tfrac{\alpha\gamma}{L_1+L_3+\beta}|\hat{x}(t_k)|(e^{(L_1+L_3)\tau_k} - e^{-\beta\tau_k}) \tag{4.24}$$

where $\alpha := \sqrt{\frac{c_2}{c_1}}$ and $\beta := \frac{c_3}{2c_2}$. It follows from Lemma 4.1 and simple calculations that

$$|\hat{x}(t_k)| \leq \tfrac{e^{(L_1+L_3)\tau_k}}{1-\frac{\alpha(\gamma+L_1L_2+L_3)}{L_1+L_3-\beta}(e^{(L_1+L_3-\beta)\tau_k}-1)}|x(t)|. \tag{4.25}$$

Substitution (4.24) gives

$$|e(t)| \leq \underbrace{\frac{\alpha\gamma(e^{2(L_1+L_3)\tau_m} - e^{(L_1+L_3-\beta)\tau_m})}{(L_1+L_3+\beta)(1-\frac{\alpha(\gamma+L_1L_2+L_3)}{L_1+L_3-\beta}(e^{(L_1+L_3-\beta)\tau_m}-1))}}_{\rho}|x(t)|. \tag{4.26}$$

It follows from assumption 4.1 and condition (ii) that the derivative of V along the trajectories of (4.21) is

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f_p(t,x,g_c(t,\hat{x})) + r_p(t,x,g_c(t,\hat{x})))\\ &\leq -(c_3 - c_4\gamma)|x|^2 + c_4L_2(L_1+L_3)|x||e|\\ &\leq -\eta|x|^2 \end{aligned} \tag{4.27}$$

with $\eta = c_3 - C_4\gamma - c_4L_1L_2\rho > 0$. Hence the model-based NCS is exponentially stable with the convergent rate not greater than $-\eta/2c_2$. This completes the proof of Theorem 4.3. ■

In the remaining of this subsection, we turn to consider the case of a nonvanishing perturbation, where we do not know $r_p(t,0,0) = 0$ and the origin $x = 0$ may not be an equilibrium point of the perturbed plant. Suppose that the perturbation is bounded by a sufficient small bound. That is, the plant is of robustness property having exponentially stable equilibrium at the origin such that $|\hat{x}(t)|$ will be bounded for sufficiently large t and arbitrarily small perturbations will not result in large deviations from the origin. We will conclude that under the proposed model-based control scheme the nonlinear networked control system is uniformly ultimately bounded.

Theorem 4.4. *Consider the NCS whose plant dynamics are described by (4.1) with the model-based controller (4.7) and (4.19). Suppose there exists a Lyapunov function $V(t,x)$ that satisfies (4.4)-(4.6) and suppose $|r(t,\hat{x})| \leq \delta$*

for all $t \geq 0$. Then the model-based NCS is uniformly ultimately bounded: the trajectories of the closed loop system satisfiy

$$|x(t)| \leq \sqrt{\tfrac{c_2}{c_1}} \max\{|x(t_0)|e^{-\frac{(1-\theta)c_3}{2c_2}(t-t_0)}, \tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))\}.$$

Proof. Take $V(t,x)$ as a Lyapunov candidate. The derivative of V along the trajectories of the closed loop NCS is given by

$$\dot{V} \leq -c_3|x|^2 + c_4|x|\delta + c_4(L_1L_2 + L_3)|x||e|. \tag{4.28}$$

The states of the plant $x(t)$ and the plant model $\hat{x}(t)$ can be rewritten as

$$\begin{aligned} x(t) &= x(t_k) + \textstyle\int_{t_k}^t f_p(s, x(s), g_c(s,\hat{x}) + r_p(s, x(s), g_c(s,\hat{x})\ ds \\ \hat{x}(t) &= \hat{x}(t_k) + \textstyle\int_{t_k}^t f_p(s, \hat{x}(s), g_c(s,\hat{x}))ds. \end{aligned} \tag{4.29}$$

Subtracting the two equations and taking norms yield

$$|e(t)| \leq \delta(t - t_k) + (L_1 + L_3) \textstyle\int_{t_k}^t |e(s)|ds,\ \forall t \in [t_k, t_{k+1}). \tag{4.30}$$

Application of the Gronwall Inequality to $|e(t)|$ results in

$$|e(t)| \leq \delta(t - t_k) + \textstyle\int_{t_k}^t \delta(L_1 + L_3)(s - t_k)e^{(L_1+L_3)(t-s)}ds.$$

Integrating the right-hand side by parts, we obtain

$$|e(t)| \leq \tfrac{\delta}{L_1+L_3}(e^{(L_1+L_3)\tau_k} - 1),\ \forall t \in [t_k, t_{k+1}). \tag{4.31}$$

Then

$$\dot{V} \leq -c_3|x|^2 + c_4|x|\delta(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_k} - 1)). \tag{4.32}$$

Using (4.4), we get

$$V \geq c_2\{\tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_k} - 1))\}^2 \Rightarrow \dot{V} \leq -\tfrac{(1-\theta)c_3}{c_2}V \tag{4.33}$$

for some positive constant $\theta < 1$, which implies

$$V(t) \leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k)}V(t_k), c_2\{\tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))\}^2\} \tag{4.34}$$

for $\forall t \in [t_k, t_{k+1})$, where denote $V(t) := V(t, x(t))$ and $V(t_k) := V(t_k, x(t_k))$ to shorten notations. Note that $V(t_k) = V(t_k^-)$, where $V(t_k^-)$ denotes the left limit of V at t_k. Using (4.34) to calculate $V(t_k^-)$ on $[t_{k-1}, t_k)$, we obtain that

$$V(t_k) \leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}\tau_{k-1}} V(t_{k-1}), c_2\{\tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))\}^2\} \tag{4.35}$$

and thus

$$V(t) \leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k+\tau_{k-1})} V(t_{k-1}), c_2\{\tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))\}^2\}. \tag{4.36}$$

Denote $N := c_2\{\frac{c_4\delta}{c_3\theta}(1 + \frac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_k} - 1))\}^2$. Iterating this process, we get $\forall t \in [t_k, t_{k+1})$,

$$\begin{aligned} V(t) &\leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k+\tau_{k-1}+\tau_{k-2})} V(t_{k-2}), e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k+\tau_{k-1})} N, N\} \\ &\leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k+\tau_{k-1}+\tau_{k-2})} V(t_{k-2}), N\} \\ &\leq \cdots \cdots \\ &\leq \max\{e^{-\frac{(1-\theta)c_3}{c_2}(t-t_k+\sum_{s=0}^{k-1}\tau_s)} V(t_0), N\}. \end{aligned} \tag{4.37}$$

It follows from $V(t) \geq c_1|x(t)|^2, V(t_0) \leq c_2|x(t_0)|^2$ and $t-t_k+\sum_{s=0}^{k-1}\tau_s = t-t_0$ that

$$|x(t)| \leq \sqrt{\tfrac{c_2}{c_1}} \max\{|x(t_0)|e^{-\frac{(1-\theta)c_3}{2c_2}(t-t_0)}, \tfrac{c_4\delta}{c_3\theta}(1 + \tfrac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))\}. \tag{4.38}$$

This completes the proof of Theorem 4.4. ■

Remark 4.6. Note from (4.38) that the ultimate bound, $\sqrt{\frac{c_2}{c_1}}\frac{c_4\delta}{c_3\theta}(1 + \frac{L_1L_2}{L_1+L_3}(e^{(L_1+L_3)\tau_m} - 1))$, is proportional to δ. This shows that arbitrarily small perturbations will not result in large steady-state deviations. Also it is possible to make $\delta = 0$ by having the exact plant. This agrees with the intuition that without model errors, the networked system will be equivalent to the original system and have the same desired behavior regardless of how large are the access interval. This is the main reason why the proposed control scheme is attractive: in the ideal case, we can expect to recover the performance of the non-networked system.

Remark 4.7. It needs to be pointed out that the derived bounds could be conservative, as the simulations in section 4.4 demonstrate. The conservatism is due to the conservative nature of the Gronwall Inequality and is also a consequence of the worst-case analysis which has been adopted from the beginning.

Consider the more general case. Suppose that$|r(t, \hat{x})| \leq \gamma|\hat{x}| + \delta$ where γ and δ are nonnegative constants, the following corollary is obtained.

Corollary 4.5. *Consider the NCS whose plant dynamics are described by (4.1) with the model-based controller (4.7) and (4.19). Suppose there exists a Lyapunov function $V(t, x)$ that satisfies (4.4)-(4.6). Suppose $|r(t, \hat{x})| \leq \gamma|\hat{x}| + \delta$ with $\gamma < \frac{c_3}{c_4}$ and τ_m satisfies the conditions of Theorem 4.3. Then the closed-loop NCS is uniformly ultimately bounded.*

4.4 Numerical examples

Consider the continuous-time plant

$$\dot{x}(t) = x^3(t) + u(t) + 0.5x\sin(5t) \tag{4.39}$$

with the continuous-time controller

$$u(t) = -x(t) - 3x^3(t). \tag{4.40}$$

Take the Lyapunov function

$$V(t, x) = \tfrac{1}{2}x^2. \tag{4.41}$$

Since

$$\dot{V} = -x^2 - 2x^4 + 0.5x^2\sin(5t) \leq -\tfrac{1}{2}x^2,$$

it follows that the non-networked closed loop system is of robustness having globally exponentially stable equilibrium at the origin and satisfies Assumption 4.1 with $c_1 = c_2 = 0.5$ and $c_3 = c_4 = 1$.

First of all, consider that the networked controller holds the last value received from the sensor until the next time the sensor data has the access to network. Consider the networked control implementation under this piecewise constant control scheme. Assuming the initial value is $x(0) = 1.0$, it follows from Theorem 4.2 with $\gamma = 0.5,\ L_1 = 3, L_2 = 10$ and $L_3 = 0.5$ that the stability of the NCS is preserved for $\tau_k < 0.0005$ seconds. On the

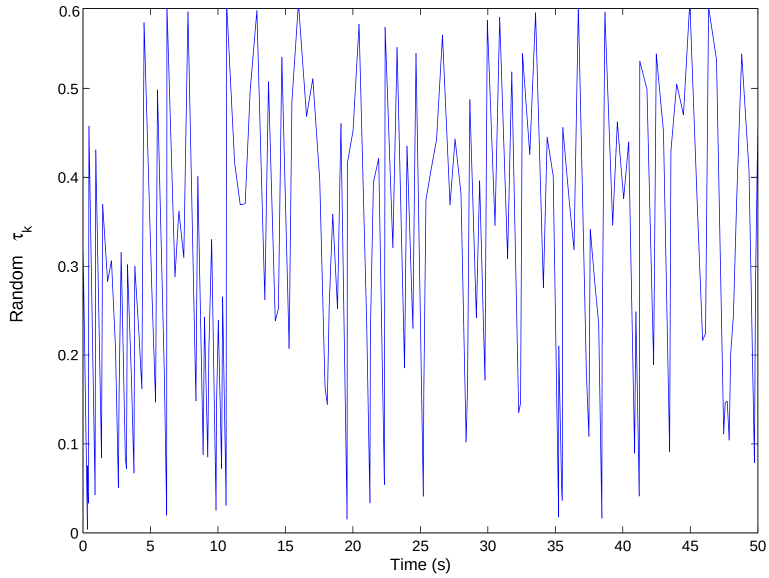

Figure 4.2: Random access intervals with upper bound $\tau_m = 0.6s$.

other hand, study the networked controller implementation based on a plant model given by

$$\dot{\hat{x}}(t) = \hat{x}^3 + u. \tag{4.42}$$

The access deadline obtained from Theorem 4.3 with $\alpha = 1$ and $\beta = 1$ is 0.015 seconds, which shows the advantages and the effectiveness of the introduction of the estimator to improve the system performance.

In the simulation the network access intervals are generated randomly according to the access bound and shown in Figure 4.2. Our simulations show that the NCS using the first control scheme remains stable for τ_m around 0.6 seconds (see Figure 4.3). For the purpose of comparison consider also the NCS response under model-based controller with the same update time and the closed loop response of the non-network system. It is easy to see that the form of response curve under model-based control scheme is almost identical to the one with network free.

On the other hand, our simulations show that using the control scheme based on the plant model (4.42), the access deadline to recover the stability is around 16 seconds (see Figure 4.4a), which is more than twenty times larger than that of the system without the estimator. Moreover, the simulation

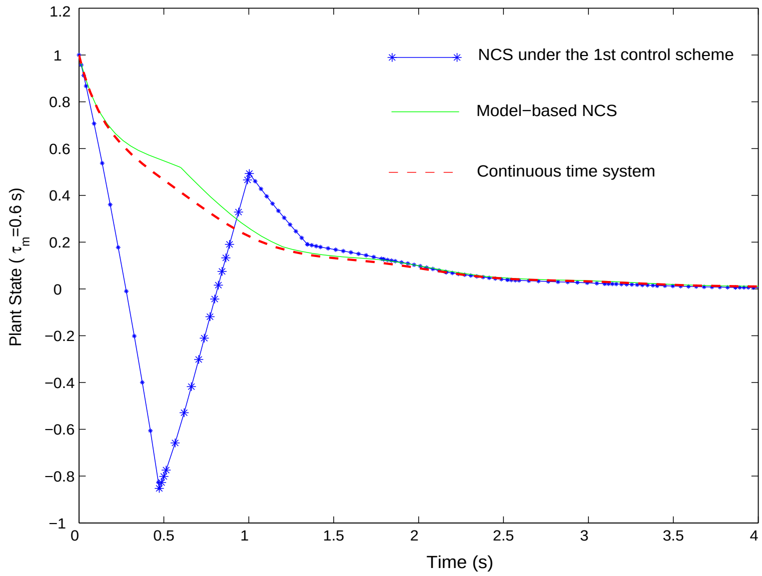

Figure 4.3: The NCS responses under two control schemes with $\tau_m = 0.6s$.

of the networked controller implementation based on the plant model with nonvanishing perturbation given by

$$\dot{x}(t) = x^3 + u + 0.5\sin(x) \tag{4.43}$$

and the bound $\tau_m = 1.5$ seconds is shown in Figure 4.4b. Advantages of model-based approach are clear when compared with the response obtained under the first control scheme (piecewise constant controller), which a much larger access deadline to guarantee the stability is achieved and hence a significant saving in the required bandwidth is obtained and more bandwidth will be available to allocate more resources.

As one can see, for small access interval both the system with and without an estimator are stable, but the system with an estimator is stable for network much slower than that without an estimator and the difference in the amount of bandwidth used is large. This means that when it is possible, it is worth to add an estimator to allow for slower network traffic, or in other words to allow for an increased network load.

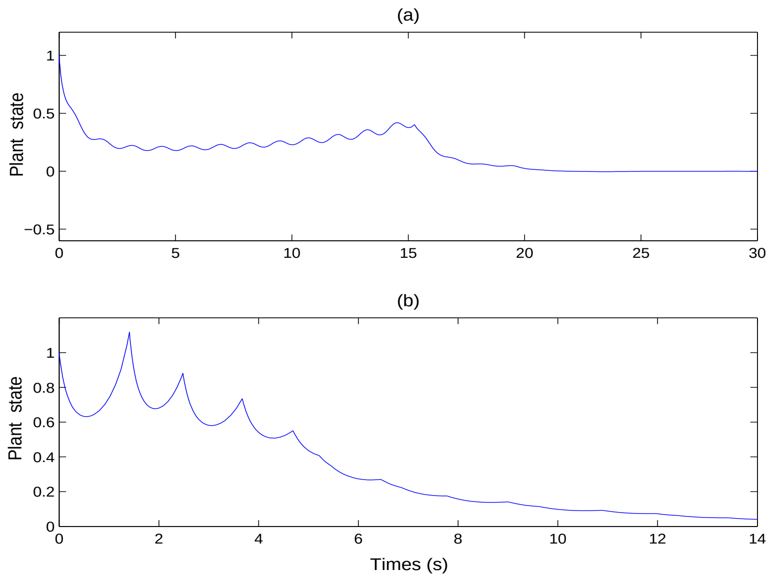

Figure 4.4: The model-based NCS responses with vanishing and non-vanishing perturbations.

4.5 Conclusions

In this chapter we studied the networked realization of a nonlinear control system. We focus on the effect of the network on the closed loop performance as well as present an estimation method as a solution to reduce network traffic. The conditions for the network access interval to preserve desired performance of the networked control system are derived.

Two different approaches to model the networked control system are considered: piecewise constant control scheme and model-based scheme. It is shown that if a controller exponentially stabilizes the non-networked system, then both of the control schemes would guarantee the networked control system to preserve desired performance by employing access deadline.

It needs to be pointed out that there is a tradeoff between the performance of the control system and bandwidth usage. In essence, we trade increased computational demands for decreased communication. By adding the estimator, the computation load is increased due to the additional computation related to the estimator. However, by using the estimated values instead of true values of the plant, a significant saving in the required bandwidth is achieved and this makes possible stabilization of the plant even under slow network conditions.

Notes. A version of this chapter has been submitted for publication.
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Chapter 5

Sampled-data control of nonlinear NCSs

Basically, an NCS is hybrid, which involves a continuous plant, networks and time-driven or event-driven devices, for instance, sampler, controller and holder. Hybrid nature of NCSs makes the synthesis and analysis problems difficult. In this chapter, the problem of analysis for the nonlinear networked control systems with event-driven digital controller and event-driven holder is considered. The configuration shows that this NCS model is quite general and it removes the assumption that the time delay is less than one sampling period used in [55, 71]. The physical plant and the controller are in continuous-time and discrete time respectively. The NCS is modeled as a sampled-data system with time delays. Sampled-data control system formulation has been recognized a modeling method for NCSs for years, see, for example, [13, 21, 71]. In this chapter, for such a sampled-data NCS, a compensation scheme for the network-induced delays is addressed, which employs an estimator to reconstruct approximately the undelayed plant state and feed it back to the digital controller, and the stability results via linearization approach [20] are derived. An illustrative numerical example is presented to show the usefulness and effectiveness of the proposed scheme.

The remainder of this chapter is organized as follows. The problem formulation of the NCS with network-induced delays is presented in section 5.1. The stability and performance issues of the system are examined in section 5.3 and the results are illustrated via simulations in section 5.4. Finally, this chapter is closed with conclusions in the last section.

5.1 Problem Formulation

The NCS model considering network-induced delays is shown in Figure 5.1, in which a continuous plant is controlled by a digital controller. Suppose that the physical plant is given by the following continuous-time nonlinear system

$$\dot{x}(t) = f(x, u) \tag{5.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are the plant state and the control vectors respectively. It is assumed that $x(t)$ is online measurable and the measurements of $x(t)$ are transmitted in a single packet, which means that the sensor data are lumped together into one network packet and transmitted over the communication channel at the same time.

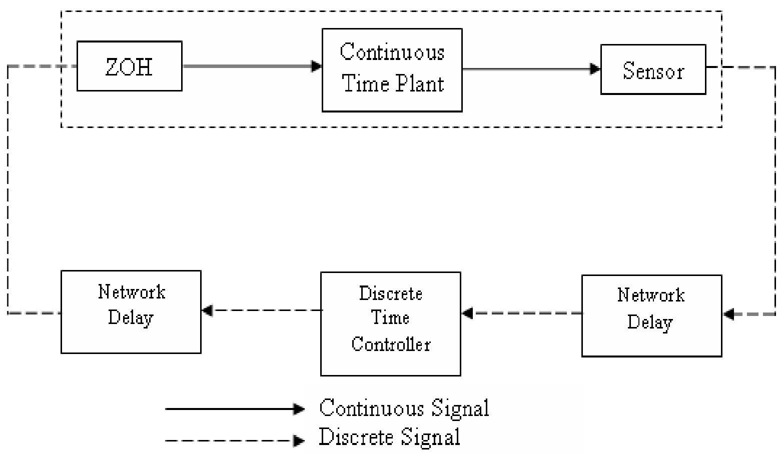

Figure 5.1: The setup of the sampled-data control of NCS

As Figure 5.1 shows, there are two different sources of delays from the network: sensor-to-controller delay τ_{sc} and controller-to-actuator delay τ_{ca}. For time-invariant controllers, the sensor-to-controller delay and controller-to-actuator delay can be lumped together as $\tau = \tau_{sc} + \tau_{ca}$ for the purpose of analysis [71].

Because of the asynchronous nature in NCS, this chapter considers the setup with a time-driven sensor, an event-driven controller and an event-driven ZOH. Event-driven control strategies ([2, 17, 64]) have been proposed to make such a compromise between processor load and control performance. In contrast to a time-driven system, in a event-driven system, time no longer serves the purpose of driving such a system and in between event occurrences, the state variables are unaffected and their values change only at certain points in which events takes place. There is no mechanism to specify how events might interact over time or how their time of occurrence might be determined.

The sampling period is assumed to be a positive real constant, denoted by T. The time-driven sensor sample the plant outputs periodically at the sampling instants. The event-driven controller can be implemented by an external event interrupt mechanism and calculates the control signal as soon as the sensor data arrives. The event-driven actuator means that the plant inputs are changed as soon as the data become available. In an event-driven controller or actuator, it is the occurrence of an event rather than the passing of time, that decides when the next sample should be taken. Moreover, the aim of event-driven control is to create a better balance between the control performance and other aspects of the system, such as communication load [17].

Denote the updating instant of the ZOH as t_k, and suppose that the updating signal (successfully transmitted signal from the sampler to the controller and to the ZOH) at the instant t_k has experienced signal transmission delays τ_k with $\tau_k = \tau_{sc}^k + \tau_{ca}^k$. For this formulation, it is not difficult to know that the controller switches at the updating instants $t_k = kT + \tau_k$ and the control action is held over the interval $[t_k, t_{k+1})$. Therefore, the digital controller takes the following form

$$u(t_k) = g(x(t_k - \tau_k)), \tag{5.2}$$

and

$$u(t) = g(x(t_k - \tau_k)) \tag{5.3}$$

for $t_k \le t < t_{k+1}$, $k \in \mathbb{Z}^+$.

Remark 5.1. it is important to note that in (5.3), t_k refers to the updating instant of the ZOH, not the sampling instant. It should be noted that the controller and actuator are event-driven, which mean we do not need to synchronize the ZOH actuator and the sampler. The ZOH may be updated between sampling instants.

5.2 Compensation for the delays

A NCS is concerned primarily with the quality of real-time service. Queues are not desirable in a NCS, since in real-time control systems, the newest data is the best. This brings us the idea to compensate the delays and make the new data available.

An estimator is used to reconstruct an approximation to the undelayed plant state and make it available for the controller calculation. Assume that the delays of each sample is larger than $(l-1)$ sampling period and less than l sampling period for some integer $l \geq 1$, that is, $(l-1)T < \tau_k < lT$ for all k. From the bound of the network-induced delays, we know that one control sample is received every sample period for $k \geq l-1$. Since $(l-1)T < \tau_k < lT$, the first updating time t_0 is in the interval $((l-1)T, lT)$ and the kth updating time t_k is in the interval $((k+l-1)T, (k+l)T)$ by recursion. The estimation of the plant state at the updating time t_k is based on the sensor measurement of the plant at the sampling instant $(k+l-1)T$ and the network-induced delays τ_k.

Specifically, the main idea behind this approach is the following: a controller containing a model-based scheme is proposed that allows us to reconstruct approximately the plant states $x(t_k)$. This scheme is updated periodically using the measurements of the plant states at the sampling instants and the control law, in turn, depends on the states of the model rather than the actual states of the plant. Here, the model doesn't assume complete knowledge of the plant. In other words, there is model mismatch in the model dynamics. The use of plant model in the controller is not new. The stabilization problem of NCSs based on its plant model has been addressed recently in [40, 41, 58].

5.3 Main results

It is assumed that the functions f and g are continuous differentiable with $f(0,0)=0$ and $g(0)=0$. For functions $f(x,u)$ and $g(x)$, let $A=\frac{\partial f(0,0)}{\partial x}$ and $B=\frac{\partial f(0,0)}{\partial u}$. Linearizing system (5.1) at the origin gives

$$\dot{x}(t)=Ax+Bu+F(x,u), \tag{5.4}$$

where $\lim_{(x,u)\to(0,0)}\frac{F(x,u)}{\sqrt{|x|^2+|u|^2}}=0$.

Consider the continuous-time plant (5.4). Let $\tilde{x}(t_k)$ denote the estimate of the plant state at the time t_k. To compensate for the network delays, introduce the approximate model

$$\dot{\tilde{x}}(t)=A\tilde{x}+Bu(t_{k-1}), \quad t\in[(k+l-1)T,t_k) \tag{5.5}$$

with periodically updated initialization $\tilde{x}((k+l-1)T)=x((k+l-1)T)$ to estimate $x(t_k)$. Here, denote $u(t_{-1})=0$ in (5.5) for $k=0$. It follows from (5.5) that $\tilde{x}(t_k)$ can be calculated by

$$\tilde{x}(t_k)=e^{A(\tau_k-(l-1)T)}x((k+l-1)T)+\int_0^{\tau_k-(l-1)T}e^{As}ds\ Bu(t_{k-1}). \tag{5.6}$$

Thus the control signal is computed by $u(t)=g(\tilde{x}(t_k))$ for $t\in[t_k,t_{k+1})$. Let $C=\frac{dg(0)}{dx}$. Then the controller can be rewritten as

$$u(t)=C\tilde{x}(t_k)+G(\tilde{x}(t_k)), \quad t\in[t_k,t_{k+1}) \tag{5.7}$$

where $\lim_{x\to 0}\frac{G(x)}{|x|}=0$. It follows from this control law (5.7) and (5.4) that

$$\begin{aligned} x(t_{k+1}) \ = \ & e^{A(T+\tau_{k+1}-\tau_k)}x(t_k)+\int_0^{T+\tau_{k+1}-\tau_k}e^{As}ds\ BC\tilde{x}(t_k)+\\ & \int_0^{T+\tau_{k+1}-\tau_k}e^{As}ds\ BG(\tilde{x}(t_k))+\\ & \int_{t_k}^{t_{k+1}}e^{A(t_{k+1}-s)}F(x(s),u(t_k))ds \end{aligned} \tag{5.8}$$

Let $z(k)=(x^T(t_k),\tilde{x}^T(t_k))^T$, where the superscript T means the transpose of a vector. Hence, the augmented closed-loop system takes the following form

$$z(k+1)=\Phi_k z(k)+\Omega_k, \tag{5.9}$$

where

$$\Phi_k = \begin{pmatrix} e^{A(T+\tau_{k+1}-\tau_k)} & \int_0^{T+\tau_{k+1}-\tau_k} e^{As}ds\ BC \\ 0 & \int_0^{\tau_{k+1}-(l-1)T} e^{As}ds\ BC \end{pmatrix}, \tag{5.10}$$

and

$$\Omega_k = \begin{pmatrix} \int_0^{T+\tau_{k+1}-\tau_k} e^{As}ds\ BG(\tilde{x}(t_k)) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)}F(x(s),u(t_k))ds \\ \int_0^{\tau_{k+1}-(l-1)T} e^{As}ds\ BG(\tilde{x}(t_k)) + e^{A(\tau_{k+1}-(l-1)T)}x((k+l)T) \end{pmatrix}. \tag{5.11}$$

It is noted that the augmented closed-loop system in (5.9) is in the form of a sampled-data system depending on the variable and uncertain network-induced delays.

The main results are stated and proved in the following.

Theorem 5.1. *Consider the networked control system with a continuous-time nonlinear plant described by (5.1), a state estimator described by (5.6) and a digital controller described by (5.7). Assume that* $\limsup_{k\to\infty} |\Phi_k| < 1$. *Then the equilibrium point* $x = 0$ *of the closed-loop networked system is exponentially stable.*

The following lemma is needed to establish the proof of Theorem 5.1.

Lemma 5.2. *For any given* $\nu > 0$, *there exists* $\delta_1 = \delta_1(\nu) > 0$, *such that* $|x(t)| \le c|z(k)|$ *(here,* c *is a constant) and* $|\Omega_k| \le \nu|z(k)|$ *for* $t \in [t_k, t_{k+1})$, *whenever* $|z(k)| \le \delta_1$ *for all* $k \in \mathbb{Z}$.

Proof. It follows from (5.4) that for $t \in [t_k, t_{k+1})$,

$$x(t) = x(t_k) + B(t-t_k)u(t_k) + \int_{t_k}^{t} Ax(s) + F(x(s),u(t_k))ds. \tag{5.12}$$

By continuity, there exists $\delta_2 > 0$ such that $|F(x,u)| \le |x| + |u|$ and $|G(x)| \le |x|$, whenever $|x| \le \delta_2$ and $|u| \le \delta_2$. First, claim that there exists $\delta_3 > 0$ ($\delta_3 < \min\{\frac{\delta_2}{c}, \frac{\delta_2}{|C|+1}\}$) such that $|x(t)| < \delta_2$ for all $t \in [t_k, t_{k+1})$ whenever $|z(k)| \le \delta_3$. This claim can be established by contradiction. Since $\delta_3 < \delta_2$, it is assumed that there exists a $\bar{t} \in [t_k, t_{k+1})$, such that $|x(t)| < \delta_2$, for all $t \in [t_k, \bar{t})$ and $|x(\bar{t})| = \delta_2$. It follows from $|z(k)| \le \delta_3$ that $|u(t_k)| \le |C||\tilde{x}(t_k)| + |\tilde{x}(t_k)| \le (|C|+1)\delta_3 \le \delta_2$. Thus,

$$\begin{aligned} |x(t)| &\le |x(t_k)| + |B|(t_{k+1}-t_k)|u(t_k)| + \int_{t_k}^{t} (|A|+1)|x(s)| + |u(t_k)|ds \\ &\le |x(t_k)| + 2(|B|+1)T|u(t_k)| + \int_{t_k}^{t} (|A|+1)|x(s)|ds. \end{aligned}$$

It follows by Gronwall Inequality that

$$\begin{aligned} |x(t)| &\leq (|x(t_k)| + 2(|B|+1)T|u(t_k)|)e^{2(|A|+1)T} \\ &\leq c|z(k)| < c \cdot \frac{\delta_2}{c} < \delta_2 \end{aligned} \tag{5.13}$$

where $c := [1 + 2(|B|+1)(|C|+1)T]e^{2(|A|+1)T}$. Thus $|x(\bar{t})| < \delta_2$, which is a contradiction. Therefore, there exists $\delta_3 > 0$ such that $|x(t)| < \delta_2$ for all $t \in [t_k, t_{k+1})$ whenever $|z(k)| \leq \delta_3$. Moveover, it is known that $x(t) \leq c|z(k)|$ for $t \in [t_k, t_{k+1})$ whenever $|z(k)| \leq \delta_3$.

Now consider Ω_k: for given $\nu > 0$, let $\nu = (c + \varepsilon_1|B|T)e^{|A|T} + 2\varepsilon_1(|B| + \sqrt{c^2 + (|C| + \varepsilon_1)^2})Te^{2|A|T}$. There exists $\delta_4 > 0$ such that $|F(x,u)| \leq \varepsilon_1\sqrt{|x|^2 + |u|^2}$, whenever $\sqrt{|x|^2 + |u|^2} \leq \delta_4$ and $|G(x)| \leq \varepsilon_1|x|$, whenever $|x| \leq \delta_4$. Let $\delta_1 = \min\{\delta_3, \frac{\delta_4}{\sqrt{2}c}, \frac{\delta_4}{\sqrt{2}(|C|+\varepsilon_1)}\}$. It follows from the definition of δ_1 that $|x(t)| \leq c|z(k)| \leq \delta_4$ and $\sqrt{|x(t)|^2 + |u(t_k)|^2} \leq \delta_4$ for $t \in [t_k, t_{k+1})$. Using (5.11) and the definition of ν, we obtain

$$\begin{aligned} |\Omega_k| &\leq \varepsilon_1 \int_0^{T+\tau_{k+1}-\tau_k} e^{|A|(T+\tau_{k+1}-\tau_k)} ds|B||z(k)| + c|z(k)|e^{|A|(\tau_{k+1}-(l-1)T)} \\ &\quad + \varepsilon_1 \int_0^{\tau_{k+1}-(l-1)T} e^{|A|(\tau_{k+1}-(l-1)T)} ds|B||z(k)| \\ &\quad + \varepsilon_1 e^{2|A|T} \int_{t_k}^{t_{k+1}} \sqrt{|x(s)|^2 + |u(t_k)|^2} ds \\ &\leq 2\varepsilon_1(|B| + \sqrt{c^2 + (|C| + \varepsilon_1)^2})Te^{2|A|T}|z(k)| \\ &\quad + (c + \varepsilon_1|B|T)e^{|A|T}|z(k)| \\ &\leq \nu|z(k)|. \end{aligned} \tag{5.14}$$

This completes the proof of Lemma 5.2. ∎

Now the proof of Theorem 5.1 can be finalized as follows.

Proof. Let $V(z(k)) = z(k)^T z(k)$ be a Lyapunov candidate for system (5.9). Then

$$\begin{aligned} V(z(k+1)) &= (\Phi_i z(k) + \Omega_k)^T(\Phi_k z(k) + \Omega_k) \\ &\leq |\Phi_k|^2|z(k)|^2 + 2|\Phi_k||z(k)||\Omega_k| + |\Omega_k|^2. \end{aligned} \tag{5.15}$$

Since $\limsup_{k\to\infty} |\Phi_k| < 1$, there exist q $(0 < q < 1)$ and $k_0 \in \mathbb{Z}$ such that $|\Phi_k| < q$ when $k \geq k_0$. Hence,

$$V(z(k+1)) \leq q^2|z(k)|^2 + 2q|z(k)||\Omega_k| + |\Omega_k|^2, \quad k \geq k_0. \tag{5.16}$$

Choose $0 < \nu < 1$ such that $r := \sqrt{q^2 + 2q\nu + \nu^2} < 1$. By Lemma 5.2, there exists $\delta_1 > 0$ such that $|\Omega_k| \leq \nu|z(k)|$ and $|x(t)| \leq c|z(k)|$ for $t \in [t_k, t_{k+1})$, when $|z(k)| \leq \delta_1$ for all $k \in \mathbb{Z}$. For any $\varepsilon > 0$, define $\delta := \frac{\min\{\delta_1, \varepsilon e^{-\alpha k_0}\}}{\prod_{j=0}^{k_0-1}(|\Phi_j|+1)}$ with $\alpha := -\ln r > 0$. It follows from the part of the proof of Lemma 5.2 that $|z(0)| \leq \delta$ holds by choosing a proper bound for $|x(0)|$. Now for $|z(0)| \leq \delta$, it follows by the definition of δ that $|z(0)| \leq \delta_1$. Using (5.9), we obtain that $|z(k)| \leq \delta_1$ for $0 \leq k \leq k_0$, which implies

$$\begin{aligned}
&V(z(k_0+1)) \leq |(q^2 + 2q\nu + \nu^2)|z(k_0)|^2 \leq r^2|z(k_0)|^2 \\
&\Rightarrow |z(k_0+1)| \leq r|z(k_0)| \leq \delta_1.
\end{aligned}$$

By induction, we have $|z(k)| \leq \delta_1$ for all $k \in \mathbb{Z}$. It follows that

$$|z(k)| \leq r|z(k-1)| \leq r^2|z(k-2)| \leq \ldots \leq r^{k-k_0}|z(k_0)|, \quad k \geq k_0. \tag{5.17}$$

Note that by the definition of δ,

$$\begin{aligned}
|z(k)| &\leq (|\Phi_{k-1}|+1)\cdots(|\Phi_0|+1)|z(0)| \\
&\leq \varepsilon e^{-\alpha k}
\end{aligned} \tag{5.18}$$

for $0 \leq k \leq k_0$. Hence, it follows that

$$|z(k)| \leq \varepsilon r^{k-k_0} e^{-\alpha k_0} \leq \varepsilon e^{-\alpha k} \tag{5.19}$$

for all $k \in \mathbb{Z}$ and $|x(t)| \leq c\varepsilon e^{-\alpha k}$. This completes the proof of Theorem 5.1. ■

5.4 Numerical examples

Consider the continuous-time plant

$$\dot{x}(t) = x - 2\sin(x) + u \tag{5.20}$$

with the digital controller

$$u(kT) = -0.25x(kT) - x(kT)^2. \tag{5.21}$$

It is assumed that $T = 1, l = 2$ and $T < \tau_k < 2T$. Let $\tilde{x}(t_k))$ denote the estimate of the plant state at the time t_k. To compensate for the network delays, introduce the approximate model

$$\dot{\tilde{x}}(t) = -\tilde{x} + u(t_{k-1}), \tag{5.22}$$

for $t \in [(k+1)T, t_k)$ and $k \in \mathbb{Z}$ with periodically updated initialization $\tilde{x}((k+1)T) = x((k+1)T)$. Here $u(t_{-1}) = 0$ for $k = 0$. It follows from (5.22) that $\tilde{x}(t_k))$ can be calculated by

$$\tilde{x}(t_k) = e^{A(\tau_k - T)} x((k+1)T) + \int_0^{\tau_k - T} e^{-s} ds \; u(t_{k-1}). \tag{5.23}$$

Thus the control signal can be computed by

$$u(t) = -0.25\tilde{x}(t_k) - \tilde{x}(t_k)^2 \tag{5.24}$$

for $t \in [t_k, t_{k+1})$. To summarize, the model-based NCS consists of (5.23), (5.24) and (5.20). Denote

$$\Phi_k = \begin{pmatrix} e^{-(T+\tau_{k+1}-\tau_k)} & -\frac{1}{4}\int_0^{T+\tau_{k+1}-\tau_k} e^{-s} ds \\ 0 & -\frac{1}{4}\int_0^{\tau_{k+1}-(l-1)T} e^{-s} ds \end{pmatrix}. \tag{5.25}$$

For the purpose of simplification, consider the constant delays $\tau = 1.5$. It follows from calculations that all eigenvalues of $\Phi^T \Phi$ satisfy $\lambda < 1$, which implies from Theorem 5.1 that the model-based NCS is exponentially stable.

Consider the networked control implementation under the model-based control scheme. Assume the initial value is $x(0) = 1.88$. The simulation shows that the model-based NCS is stable. (see Figure 5.2). For the purpose of comparison consider also the NCS response without compensation with the same initial value. The simulation shows that the closed loop is unstable. On the other hand, the simulations under this two control scheme with initial value $x(0) = 1.2$ is shown in Figure 5.3, which implies the advantages of model-based approach: a larger region of attraction can be obtained with better closed loop response. This means that when it is possible, it is worth to add an estimator to compensate the networked delays and achieve better performance.

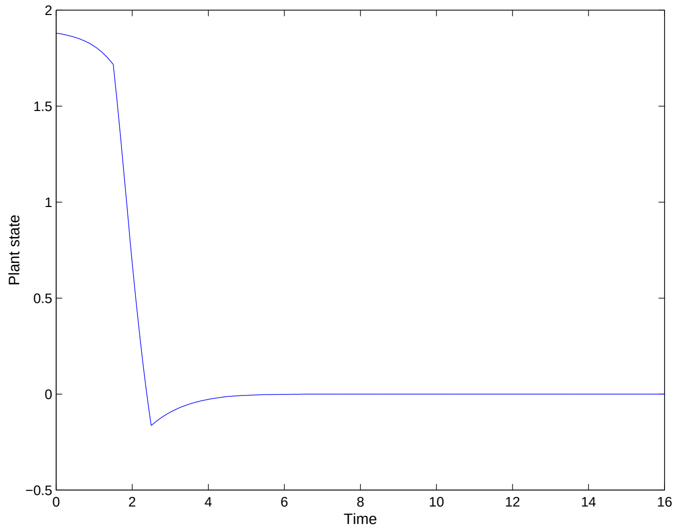

Figure 5.2: The model-based NCS responses with $x(0) = 1.88$.

5.5 Conclusions

This chapter considered the networked control systems with event-driven digital controller and event-driven holder. The physical plant and the controller are in continuous time and discrete time, and the NCS is modeled as a sampled-data system with time delays. For such a sampled-data NCS, an estimator to compensate the network-induced delays and reconstruct approximately the undelayed plant state is proposed and the stability result via linearization approach is obtained. Because of the nonlinearities, local rather than global results are obtained. It needs to be pointed out that in general it is difficult to obtain good estimates of the region of attraction by the present method. Furthermore, this limitation is inherent in most methodologies involving linearizations.

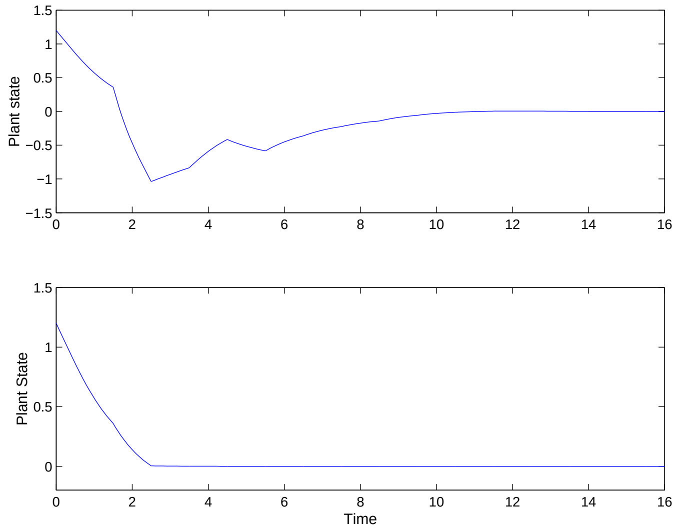

Figure 5.3: The NCS responses under two control scheme with $x(0) = 1.2$.

Chapter 6

Conclusions

As a consequence of lower cost, easy maintenance and higher flexibility, the network-based systems have replaced the traditional counterpart and have been applied to the industry. However, some network-inherited properties, such as time delays, data packet loss, etc., challenge the NCS superiority over the traditional control systems. The insertion of a network into a previously-designed system affects the closed-loop system performance and can have destabilizing effect.

This study is concerned with the analysis of the control design for nonlinear networked control systems. The main contributions are the following.

Ignoring the network connection and cascading actuators, the plant and sensors together, a sampled-data control system is obtained. We considered the important practical case where hardware restrictions are imposed on the "measurement-A/D conversion" process. First of all, the problem of sampled-data stabilization of system under the "low measurement rate" constraint is addressed via the emulation design method. Since the single rate results presented in [28] may lead to unstable closed-up sampled data system, we address the dual-rate control scheme. The main idea is to introduce a controller that contains a "fast" numerical integration model that reconstructs approximately the missing states between samples. It was shown that if a continuous-time controller input-to-state stabilizes a continuous-time plant, then under some standard assumptions the proposed dual-rate scheme makes the closed loop sampled data system input-to-state stable in the semiglobal practical sense. Our results are important in applications since the fast sampling results may not be possible to implement due to hardware limitations. It is emphasized that the results are prescriptive and they can be used as a guide when one designs digital controllers based on the emulation approach.

We also present some answers to the problem of dual-rate sampled-data stabilization of system using the approximate discrete-time design method. Given an approximate discrete-time model of a sampled nonlinear plant and given a family of controllers that stabilizes the plant model in input-to-state sense, it is shown that the closed loop sampled-data control system is input-to-state stable under the dual-rate control scheme based on the DTD method. In our setting the exact and approximate discrete-time models were considered as functions of the sampling period T and the integration period h. Indeed, it is important to note that the role of T and h are different when pursuing the approximate DTD method. Typically, the size of the domain of attraction may be controlled by reducing T whereas the accuracy of the closed loop behavior may be controlled by tuning design parameters.

The networked realization of nonlinear control systems was also studied. Among several existing methods to ameliorate the NCS performance, the most effective method is to decrease the network traffic. We focus on the communication access constraint imposed by the network and consider NCSs as continuous-time models. For such NCS models, we study the effect of the network on the closed loop performance as well as present a model-based control scheme as a solution to reduce network traffic. Specifically, a plant model is used at the controller side to reconstruct approximately the plant behavior between transmission times and feedback the estimate to the controller. The model state is updated from time to time using the measurement of the actual state of the plant and the control law, in turn, depends on the states of the model rather than the actual states of the plant. The primary objective is to efficiently use the finite bus capacity while maintaining good closed-loop control system performance. It is shown that if a controller exponentially stabilizes the non-networked system, then the proposed two control schemes, piecewise constant control and model-based scheme, would guarantee the nonlinear networked control system to preserve desired performance (stability and ultimate boundedness) by employing network access deadline. Moreover, in model-based scheme, by using the estimated values instead of true values of the plant, a significant saving in the required bandwidth is achieved and this makes possible stabilization of the plant even under slow network conditions.

Finally, the stability analysis for NCSs with event-driven digital controller and event-driven holder was considered. It is important to note that the controller and actuator are event-driven, which mean we do not need to synchronize the actuator and the sampler. The physical plant and the controller are in continuous time and discrete time, respectively, and the NCS is modeled as a sampled-data system with time delays. For such a sampled-data NCS, a model-based controller was proposed, which employs an estimator to compensate the network-induced delays because as a real-time control system, queues are not desirable in a NCS and the newest data is the best. The stability results via linearization approach are derived. Because of the nonlinearities, local rather than global results are obtained. It needs to be pointed out that in general it is difficult to obtain good estimates of the region of attraction by the linearization method. Furthermore, this limitation is inherent in most methodologies involving linearizations.

We remark that the central issues for NCSs addressed in this study are the lack of access caused by the inability of the shared medium to accommodate all sensors simultaneously and network-induced delays. These occur independently of any other communication constraints, such as possible packet losses, which are neglected here. Modifying the NCS model to include and compensate them may be interesting issues. As the nodes distributed independently, the multi-rate sampling is natural for NCSs. This topic will be a subject of future research.

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