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INTEGRAL EQUATION METHODS IN ANTI-PLANE ELASTICITY

BY

XIAOFENG SHEN ©

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This is dedicated to my wife and parents.

Abstract

The boundary integral equation method is used to solve the boundary value problem corresponding to anti-plane shear deformation of a homogeneous and anisotropic linearly elastic solid whose cross-section is bounded by an arbitrary (smooth) closed curve. Although both direct and indirect versions of this method are introduced, due to mathematical convenience, the indirect method is used predominantly to solve the problem. The solution is found in the form of a single layer potential based on the principal fundamental solution of anti-plane shear. Uniqueness and existence results are established in the appropriate function spaces. An example of an elastic solid with elliptic cross-section is used to illustrate the theory.

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Table of Contents

List of Tables

List of Figures

Introduction

Within the context of solid mechanics, anti-plane shear deformations are considered as one of the simplest classes of deformations that solids can undergo. With just a single scalar axial displacement, anti-plane shear can be modeled by a single second-order linear or quasi-linear partial differential equation, and is thus viewed as complementary to the plane strain deformation, with its two in-plane displacements.

Surprisingly, despite the fact that the model of anti-plane shear deformations has the virtue of relative mathematical simplicity without loss of essential physical relevance, anti-plane shear deformations have received comparatively little attention in the linear elasticity literature. Undoubtedly, the reason for this is that, in the case of a homogeneous and isotropic, linearly elastic solid, the governing traction boundary value problem describing anti-plane shear deformations is simply the interior Neumann problem for Laplace's equation. In the case of linear anisotropic elastostatics, however, the governing boundary value problem [1] is one of oblique-derivative type involving a second-order linear elliptic partial differential equation. The solution of this boundary value problem is of interest since it can be used to demonstrate the influence of anisotropy on the anti-plane deformation .

The boundary integral equation method is a very powerful tool for solving linear boundary value problems in solid mechanics. There are several known versions of this technique, but all of them can ultimately be placed into two essential classes: direct and indirect methods. The direct method is based on the so-called Somigliana representation formula, where the displacement at any point in the domain is expressed in terms of its value and the value of the stress vector on the boundary. The indirect method provides a particular clear and simple physical illustration of the basic solution procedure by postulating a certain form of the soltion in terms of an unknown abstract function that is chosen purely on the grounds of mathematical convenience. Therefore, the indirect method was widely used for solving the fundamental boundary value problems of elasticity (see, for example, Kupradze [2] and Constanda [3,4]). A similar treatment of the corresponding problems of anti-plane shear deformations, in the case of an anisotropic linearly elastic solid, remains absent from the literature. This can be attributed to the difficulties involved in finding the necessary fundamental solution on which the integral equation method is based.

The purpose of this thesis is to provide a complete theory for the homogeneous and anisotropic linear anti-plane shear problem. In chapter 1, we first introduce some basic kinematics related to the linearly elastic solid and subsequently formulate the mathematical model in terms of a boundary value problem for the anti-plane shear deformation. In chapter 2, we apply the indirect integral equation method to the boundary value problem, including the construction of a suitable fundamental solution for the governing partial differential equation. In doing so, we seek the solution of the boundary value problem in the form of a single layer potential and establish uniqueness and existence results by reducing the problem to a boundary integral equation. In chapter 3, we introduce the direct boundary integral equation method. Finally, an example with elliptic boundary curve is used to illustrate the theory. Numerical results obtained by both indirect and direct methods are compared.

Chapter 1: Foundations of the problem

1.1. Kinematics

We consider the equilibrium of a deformable solid, which, in its unstressed undeformed state, occupies a cylindrical region whose generators are parallel to the X_3-axis of a rectangular Cartesian coordinate system [Figure 1]. The lateral boundary of the cylinder is subjected to a prescribed surface traction t^*, whose only nonzero component is axial and does not vary in the axial direction. Thus we have

$$t_1^* = 0, \quad t_2^* = 0, \quad t_3^* = t_3^*(X_1, X_2), \tag{1.1}$$

on the lateral surface. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. We use X_i ($i = 1,2,3$) for material coordinates and x_i ($i = 1,2,3$) for spatial coordinates. The surface tractions(1.1) would be expected to give rise to a deformation of the form

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u_3(X_1, X_2), \tag{1.2}$$

so that the displacement field is given by

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = u_3(X_1, X_2), \tag{1.3}$$

where the out-of-plane displacement is a function on the cross-section of the cylinder. Such a deformation is called an ***anti-plane shear***. This displacement field may be regarded as a natural complement to that of ***plane strain***, where

$$u_\alpha = u_\alpha(X_1, X_2), \quad (\alpha = 1,2)$$
$$u_3 = 0$$

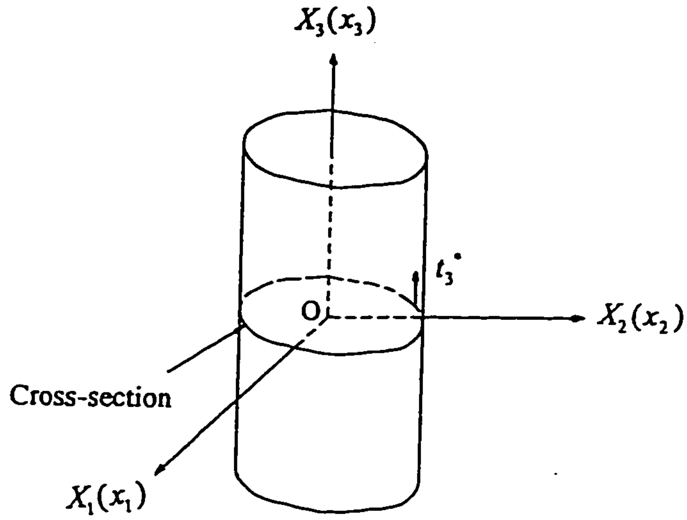

Figure 1 A linearly elastic cylinder in anti-plane shear

1.2. Anti-plane shear deformation of a homogeneous anisotropic linearly elastic cylindrical body

In this section, we assume that the cylinder is composed of a homogeneous anisotropic linearly elastic solid and consider infinitesimal deformations. Thus, we need not distinguish between the material and spatial coordinates [10]. In what follows, $x = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$ are generic points referred to orthogonal Cartesian coordinates in R^2 and R^3 respectively.

The Generalized Hooke's law for an arbitrary homogeneous anisotropic linearly elastic solid in three dimensions is given by

$$T_{ij} = C_{ijkl} E_{kl} \tag{1.4}$$

where T_{ij} denote the components of the second-rank stress tensor, $E_{kl} = (u_{k,l} + u_{l,k})/2$ are the components of the second rank infinitesimal strain tensor, represented, respectively, as $\boldsymbol{T}$ and $\boldsymbol{E}$; C_{ijkl} denote the components of the fourth-rank elastic constant tensor, represented as $\boldsymbol{C}$. The system (1.4) consists of 9 equations. Thus, there are 81 components in $\boldsymbol{C}$. Taking the usual symmetry conditions of $\boldsymbol{C}$ into consideration [9]:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}, \quad C_{ijkl} = C_{klij}, \tag{1.5}$$

the 9 equations in (1.4) are reduced to 6 independent equations and the number of independent components of $\boldsymbol{C}$ is reduced to 21. On using the second relation of (1.5) and the strain-displacement relations we write (1.4) as

$$T_{ij} = C_{ijkl} u_{k,l} \tag{1.6}$$

which, in view of (1.3), reduces to

$$T_{ij} = C_{ij3\alpha} u_{3,\alpha} \tag{1.7}$$

where Greek subscripts range over the values 1, 2. The equilibrium equations, in the absence of body forces, are

$$T_{ij,j} = 0 \tag{1.8}$$

By virtue of (1.7) and (1.3) we see that $T_{ij} = T_{ij}(x_1, x_2)$, so that (1.8) reduce to

$$T_{i\beta,\beta} = 0 \tag{1.9}$$

On substitution of (1.7) into (1.9), we find that the axial displacement u_3 must satisfy

$$C_{i\beta 3\alpha} u_{3,\alpha\beta} = 0 \qquad in\ S \tag{1.10}$$

where S denotes the plane cross-section of the cylinder. The system (1.10) consists of three differential equations for the single unknown u_3, and is thus overdetermined in general. Hence an arbitrary homogeneous anisotropic linearly elastic solid will not, in general, sustain a nontrivial anti-plane shear deformation [1].

We now turn to the boundary conditions (1.1). On employing the relation $t_i = T_{ij} n_j$ where $n_j (j = 1,2,3)$ are the components of the unit outward normal n on the lateral boundary, we find that (1.1) holds provided

$$C_{\alpha\beta 3\gamma} u_{3,\gamma} n_\beta = 0 \qquad on\ \partial S \tag{1.11}$$

$$C_{3\beta 3\gamma} u_{3,\gamma} n_\beta = t_3^{*} \qquad on\ \partial S \tag{1.12}$$

where ∂S is a simple closed curve of length $|\partial S|$ denoting the boundary of S. Its equation in terms of its arc coordinate is

$$x = \psi(s), \quad s \in [0, |\partial S|], \quad \psi(0) = \psi(|\partial S|),$$

with the inverse relationship written as $s = s(x)$, $x \in \partial S$.

Here and throughout what follows we assume that ∂S is a C^2- curve, that is, ψ is twice continuously differentiable on $[0, |\partial S|]$ and

$$\frac{d\psi}{ds}(0+) = \frac{d\psi}{ds}(|\partial S| -),$$

$$\frac{d^2\psi}{ds^2}(0+) = \frac{d^2\psi}{ds^2}(|\partial S| -).$$

A necessary and sufficient condition for a nontrivial (i.e. grad $u \neq$ constant) state of anti-plane shear to be possible subject to the boundary conditions (1.1) now follows from (1.10)-(1.12). Thus if **C** is such that

$$C_{\alpha\beta 3\gamma} \equiv 0 \quad (\alpha, \beta, \gamma = 1,2) \tag{1.13}$$

the first two differential equations in (1.10) are satisfied identically, as are the two boundary conditions (1.11). The axial displacement u_3, henceforth denoted by u, must then satisfy the third of (1.10) and the boundary condition (1.12), so that

$$C_{3\beta 3\alpha} u_{,\alpha\beta} = 0 \qquad in\, S \tag{1.14}$$

$$C_{3\beta 3\alpha} u_{,\alpha} n_\beta = t_3^* \qquad on\, \partial S \tag{1.15}$$

where $t_3^* = t_3^*(x_1, x_2)$ is a prescribed function on ∂S. The boundary value problem (1.14), (1.15) for u is one of oblique-derivative type for a second-order linear partial differential equation with constant coefficients. On employing the divergence theorem and (1.14), it is readily shown that the prescribed traction t_3^* must be such that

$$\int_{\partial S} t_3^* ds = 0 \tag{1.16}$$

for a solution of (1.14), (1.15) to exist. Of course, (1.16) would also follow from the restriction on the data (1.1) necessary for overall force equilibrium. Thus, as shown in [7], the condition (1.13) is *sufficient* to ensure that any solution $u_3 = u$ of (1.14), (1.15) (with $\text{grad}\, u_3 \neq$ constant on S) automatically satisfies (1.10)-(1.12) [1]. Conversely, suppose that u_3 is a solution of (1.10)-(1.12) such that $\text{grad}\, u \neq$ constant on *S*. It was shown [7] that the boundary condition (1.12), with $t_3^* \neq 0$, cannot be satisfied unless condition(1.13) holds. Thus the condition (1.13) is also *necessary* for a nontrivial state of anti-plane shear to be possible subject to the boundary condition (1.1).

1.3 Boundary value problem of anti-plane shear elasticity

In order to facilitate further investigation of elastic symmetries of C_{ijkl}, it is convenient to adopt the contracted notation of linear anisotropic elasticity with single index notation for stress and strain [9], so that (1.4) can be written as

$$T_p = c_{pq} E_q \qquad (p,q = 1,2,\ldots,6) \tag{1.17}$$

where c_{pq} are the components of a 6×6 symmetric matrix c representing the components of C_{ijkl}.

In homogeneous and anisotropic elasticity where c_{pq} are constants, condition (1.13) can be written in terms of the c_{pq} as

$$c_{14} = 0, \quad c_{15} = 0, \quad c_{24} = 0, \quad c_{25} = 0, \quad c_{46} = 0, \quad c_{56} = 0 \tag{1.18}$$

so that there are at most 15 independent nonzero constants in c

Therefore, (1.14) can be written as

$$L(\partial x) u(x) = 0 \tag{1.19}$$

in which $L(\partial x)$ is the partial differential operator defined by

$$L(\partial x) = c_{55} \frac{\partial^2}{\partial x_1^{\,2}} + 2c_{45} \frac{\partial^2}{\partial x_1 \partial x_2} + c_{44} \frac{\partial^2}{\partial x_2^{\,2}} \tag{1.20}$$

Here the elastic constants c_{44}, c_{55} relate a shear strain to a shear stress in the same plane and c_{45} relates a shear strain to a shear stress in a perpendicular plane(see Appendix).

Let $C(S)$, $C^1(S)$ and $C^2(S)$ represent, respectively, the spaces of (real) continuous, continuously differentiable and twice continuously differentiable functions in S. We consider all functions in $C(S)$ ($C^1(S)$ and $C^2(S)$) that are continuously extendible (continuously extendible together with their first and second order derivatives) to $\bar{S} = S \cup \partial S$, and denote by $C(\bar{S})$ ($C^1(\bar{S})$ and $C^2(\bar{S})$) the space of the corresponding extensions.

Hence, the resulting boundary value problem describing the anti-plane shear deformation of a homogeneous and anisotropic linearly elastic solid occupying a cylindrical region whose generators are parallel to the x_3 - axis is defined as follows:

Find $u \in C^2(S) \cap C^1(\bar{S})$ satisfying (1.19) in S and

$$T(\partial x)u(x) = t_3^*(x), \qquad x \in \partial S \qquad (N^+)$$

Here, S represents the plane cross-section of the cylinder, ∂S its boundary and t_3^* is a prescribed function on ∂S characterizing (axial) stress on the lateral surface of the solid. T represents the boundary stress operator defined by

$$T(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, n) = c_{55} n_1 \frac{\partial}{\partial x_1} + c_{45}(n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}) + c_{44} n_2 \frac{\partial}{\partial x_2}$$

where $n = (n_1, n_2)$ is the unit outward normal to ∂S.

Clearly, (1.19) is an elliptic second-order partial differential equation whenever:

$$c_{55} > 0, \qquad c_{55}c_{44} - c_{45}^2 = A^2 > 0 \qquad (1.21)$$

Henceforth, (1.21) will be assumed always to be true.

The stresses arising due to the anti-plane shear in a homogeneous, anisotropic elastic solid follow from (1.7), (1.13) as

$$T_{\alpha\beta} = 0,$$

$$T_{32} = c_{45}\frac{\partial u}{\partial x_1} + c_{44}\frac{\partial u}{\partial x_2}, \tag{1.22}$$

$$T_{31} = c_{55}\frac{\partial u}{\partial x_1} + c_{45}\frac{\partial u}{\partial x_2} \tag{1.23}$$

$$T_{33} = c_{35}\frac{\partial u}{\partial x_1} + c_{34}\frac{\partial u}{\partial x_2} \tag{1.24}$$

respectively. It is of interest to note that the shear stress components $T_{3\alpha}$ ($\alpha = 1, 2$) depend only on the three constants c_{44}, c_{55}, c_{45} (and thus the boundary value problem (N^+) involves only these three constants), while the axial normal stress component T_{33} depends also on two additional different constants, namely c_{34} and c_{35}. Thus, of the 15 constants present in *c*, only 5 of these play a role in the anti-plane shear problem.

It is convenient to express *c*, with (1.18) taken into account, in matrix form as

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & \boxed{c_{34}} & \boxed{c_{35}} & c_{36} \\ 0 & 0 & \boxed{c_{43}} & c_{44} & c_{45} & 0 \\ 0 & 0 & \boxed{c_{53}} & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{pmatrix} \tag{1.25}$$

The three constants arising in (1.20) appear in the 2×2 submatrix indicated by the dashed rectangle in (1.25), while the other two constants arising in the normal stress (1.24) appear in ovals in (1.25). The normal stress component T_{33} arising here is clearly due to anisotropy, in contrast to the situation in finite anti-plane shear for an isotropic material where such a stress arises as a result of nonlinearity. Alternatively, one could interpret the T_{33} as playing a role somewhat analogous to that of axial normal stress arising in plane strain for linear isotropic materials. Thus, from (1.22)-(1.24), T_{33} may be expressed directly in terms of the shear stresses as

$$T_{33} = A^{-2}[(c_{35}c_{44} - c_{34}c_{45})T_{31} + (c_{34}c_{55} - c_{35}c_{45})T_{32}], \qquad (1.26)$$

where $A > 0$ is defined in (1.21).

1.4 Other related boundary value problems

Previously, we noted that for the anti-plane shear deformation of a homogeneous and anisotropic linearly elastic solid, the corresponding boundary value problem turns out to be an interior Neumann problem (N^+) with the boundary subjected to the prescribed traction. It is not difficult to deduce that if the lateral boundary of the elastic cylindrical body is subjected to prescribed displacements whose only nonzero component is axial and which does not vary in the axial direction, i.e.,

$$u_1 = 0 \quad u_2 = 0 \quad u_3 = f(x_1, x_2)$$

(where f is a prescribed function on ∂S), then the corresponding boundary value problem will be the interior Dirichlet problem which is defined as follows:

Find $u \in C^2(S) \cap C^1(\bar{S})$ satisfying (1.19) in S and

$$u(x) = f(x) \qquad x \in \partial S \qquad (D^+)$$

In what follows, it will be necessary to consider the open unbounded complement domain of S, which is denoted by $S^- = R^2 \setminus (S \cup \partial S)$, and the corresponding boundary value problems posed in this domain i.e. the exterior Neumann and Dirichlet problems, defined below.

Let Λ be the class of functions u satisfying the far-field conditions

$$u = O(|x|^{-1}), \qquad \frac{\partial u}{\partial x_\alpha} = O(|x|^{-2}), \qquad \alpha = 1, 2 \tag{1.27}$$

The above far-field conditions restrict the asymptotic behavior of the function u and are shown in Chapter 2 to be sufficient conditions for the uniqueness of solution in the exterior problems.

The exterior Neumann problem (N^-) is defined as follows:

Find $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \Lambda$ satisfying (1.19) in S^- and

$$T(\partial x)u(x) = g(x) \qquad x \in \partial S \tag{N^-}$$

Here, g is a prescribed traction on ∂S.

The exterior Dirichlet problem (D^-) is defined as follows:

Find $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \Lambda$ satisfying (1.19) in S^- and

$$u(x) = h(x) \qquad x \in \partial S \tag{D^-}$$

Here, h is a prescribed displacement on ∂S.

According to the Fredholm Alternative, when we consider the problem (N^+), it will be necessary to consider also the 'adjoint' Dirichlet problem(D^-).

Chapter 2 Indirect boundary integral equation method applied to the solution of the anti-plane shear problem

The indirect integral equation method is a very powerful tool for solving boundary value problems in solid mechanics , particularly in establishing results on the existence of solutions. In this chapter, we use this method to establish an existence theorem by seeking the solution of the corresponding boundary value problem in the form of a single layer potential, and then reducing the problem to the Fredholm integral equation on the boundary.

2.1 Uniqueness of solution

Theorem 2.1 . If $u \in C^2(S) \cap C^1(\bar{S})$ is a solution of (1.19) in S, then

$$\int_S E(u,u)d\sigma = \int_{\partial S} uTuds \tag{2.1}$$

where $E(u, u)$ is the internal energy density defined on S.

Proof: Let v, $w \in C^2(S) \cap C^1(\bar{S})$ be two arbitrary functions. According to the 2-D Green's formula

$$\int_S \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right)dx_1 dx_2 = \int_{\partial S}(P\cos(n_x, x_1) + Q\cos(n_x, x_2))ds(x)$$

we let

$$Q = v\left(c_{55}\frac{\partial w}{\partial x_1} + c_{45}\frac{\partial w}{\partial x_2}\right), \qquad P = -v\left(c_{45}\frac{\partial w}{\partial x_1} + c_{44}\frac{\partial w}{\partial x_2}\right)$$

and obtain

$$\int_S vLwd\sigma + \int_S E(w,v)d\sigma = \int_{\partial S} vTwds \tag{2.1A}$$

where

$$E(w,v) = (c_{55}\frac{\partial w}{\partial x_1} + c_{45}\frac{\partial w}{\partial x_2})\frac{\partial v}{\partial x_1} + (c_{45}\frac{\partial w}{\partial x_1} + c_{44}\frac{\partial w}{\partial x_2})\frac{\partial v}{\partial x_2}$$

If v and w are two solutions of the anti-plane shear problem (N^+), letting $w = v = u$, we then have

$$\int_S E(u,u)d\sigma = \int_{\partial S} uTuds$$

where $E(u, u)$ is the internal energy density defined on the cross-section S by

$$E(u,u) = c_{55}(\frac{\partial u}{\partial x_1})^2 + 2c_{45}(\frac{\partial u}{\partial x_1})(\frac{\partial u}{\partial x_2}) + c_{44}(\frac{\partial u}{\partial x_2})^2$$

$$= c_{55}[\frac{\partial u}{\partial x_1} + \frac{c_{45}}{c_{55}}(\frac{\partial u}{\partial x_2})]^2 + \frac{A^2}{c_{55}}(\frac{\partial u}{\partial x_2})^2$$

which is a positive quadratic form under the assumption (1.21).

Let $C(S^-), C^1(S^-)$ and $C^2(S^-)$ be respectively the spaces of (real) continuous, continuously differentiable and twice continuously differentiable functions in S^-. Let $C(\bar{S}^-), C^1(\bar{S}^-)$ and $C^2(\bar{S}^-)$ be the space of the corresponding extensions.

Theorem 2.2. If $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \Lambda$ is a solution of (1.19) in S^-, then

$$\int_{S^-} E(u,u)d\sigma = -\int_{\partial S} uTuds \tag{2.2}$$

where, again, $E(u, u)$ represents the internal energy density in S^-.

Proof: Consider a circle K_R with the center at x and radius R sufficiently large so that ∂S lies inside K_R. Then for the bounded region $K_R \cap S^-$, (2.1) can be written as

$$\int_{K_R \cap S^-} E(u,u)d\sigma = \int_{\partial S + \partial K_R} uTuds$$

Let outward normals of both interior and exterior boundaries of bounded region be positive, we have

$$\int_{\partial S + \partial K_R} uTuds = -\int_{\partial S} uTuds + \int_{\partial K_R} uTuds$$

Since $u \in \Lambda$, (2.2) is obtained by the fact that

$$\int_{\partial K_R} uTuds \to 0 \quad as\ R \to \infty$$

Theorem 2.3. (i) Any two solutions of (N^+) differ by an arbitrary constant.

(ii) (D^-) has at most one solution.

(iii) (N^-) has at most one solution.

Proof: (i) Let $u^{(1)}, u^{(2)}$ be any 2 solutions of the problem (N^+), then the difference $u = u^{(1)} - u^{(2)}$ satisfies the governing equation of problem (N^+) and the corresponding boundary condition with $t_3^* = 0$. Therefore, by Theorem 2.1 and the fact that $E(u, u)$ is a positive quadratic form, we have

$$E(u, u) = 0 \qquad \text{in } S.$$

Using the method presented in [4], we can prove that $E(u, u) = 0$ *if and only if* $u = c$, where c is an arbitrary constant.

Hence, $u = c$ in S as required.

(ii) & (iii) are proved similarly using Theorem 2.2, except that the asymptotic conditions imposed on u, in each case, require that $c = 0$, so that $u = 0$ in S^- for (D^-) and (N^-).

2.2 Fundamental solution

The fundamental solution plays a very important role in the boundary integral equation method since the method is possible only when an appropriate fundamental solution can be constructed.

We usually define the fundamental solution of a two-dimensional boundary value problem as a function $u(x, y)$ which, without any consideration of boundary conditions, satisfies the governing differential equation at $x \neq y$, i.e.,

$$L(\partial x)u(x,y) = \delta(x-y).$$

where $x(x_1, x_2)$ and $y(y_1, y_2)$ represent generic points in R^2; δ denotes the Dirac distribution, which tends to infinity at the point $x = y$, and is equal to zero anywhere else. The integral of $\delta(x-y)$ however is equal to one. Physically, the fundamental solution represents a point-load solution in elasticity [8].

Using the methods presented in[5], we can show that the fundamental solution for the operator L is given by

$$D(x,y) = D(y,x) =$$

$$= \frac{1}{2\pi|A|}\{\ln|A| - \frac{1}{2}\ln[c_{44}(x_1-y_1)^2 - 2c_{45}(x_1-y_1)(x_2-y_2) + c_{55}(x_2-y_2)^2]\}$$

Along with $D(x, y)$, we consider the singular solution

$$P(x,y) \equiv P(x,y,n) = T(\partial y,n)D(y,x) = T(\partial y,n)D(x,y)$$

It is easily verified that $D(x, y)$ and $P(x, y)$ satisfy (1.19) at all $x \in R^2$, $x \neq y$.

Proceeding as in [4], we can prove that if $u \in C^2(S) \cap C^1(\bar{S})$ is a solution of (1.19) in S, then

$$\chi(x)u(x) = \int_{\partial S}[D(x,y)T(\partial y)u(y) - P(x,y)u(y)]ds(y) \tag{2.3}$$

where

$$\chi(x) = \begin{cases} 1, & x \in S, \\ \frac{1}{2}, & x \in \partial S, \\ 0, & x \in S^- \end{cases}$$

2.3 Elastic potentials

We introduce the generalized single layer potential

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y)$$

and the generalized double layer potential

$$(W\varphi)(x) = \int_{\partial S} P(x,y)\varphi(y)ds(y)$$

Definition: A function f defined on ∂S is said to be *Hölder continuous* (with index $\alpha \in (0,1)$) on ∂S if

$$|f(x)-f(y)| \le c|x-y|^{\alpha} \quad \text{for all } x,y \in \partial S,$$

where $c =$ constant > 0 is independent of x and y.

We denote by $C^{0,\alpha}(\partial S)$ the vector space of (real) Hölder continuous (with index $\alpha \in (0,1)$) functions on ∂S, and by $C^{1,\alpha}(\partial S)$ the subspace of $C^{1}(\partial S)$ of functions whose first order derivatives belong to $C^{0,\alpha}(\partial S)$.

We have the following properties of V and W:

Theorem 2.4. If $\varphi \in C(\partial S)$, then

(i) $V(\varphi)$ and $W(\varphi)$ are analytic and satisfy (1.19) in $S^{+} \cup S^{-}$;

Proof: Clearly, $V(\varphi)$ and $W(\varphi)$ are twice continuously differentiable at any $x \notin \partial S$ and, by the fact that $D(x, y)$ and $P(x, y)$ satisfy (1.19) at all $x \in R^{2}, x \neq y$, are also solutions of (1.19). Their analyticity follows in the usual way (see, for example [17].)

(ii) $W(\varphi) \in \Lambda$;

Proof: The double layer potential $(W\varphi)(x)$ can be written as

$$(W\varphi)(x) = \int_{\partial S}(c_{55}\frac{\partial D}{\partial y_1}n_1 + c_{45}\frac{\partial D}{\partial y_1}n_2 + c_{45}\frac{\partial D}{\partial y_2}n_1 + c_{44}\frac{\partial D}{\partial y_2}n_2)\varphi(y)ds(y)$$

$$= \frac{A}{2\pi}\int_{\partial S}\frac{1}{r}(\frac{1}{c_{44}\cos^2\theta - 2c_{45}\sin\theta\cos\theta - c_{55}\sin^2\theta})\varphi(y)ds(y)$$

where $r = |x - y|$ is the distance between x and y. As $|x| \to \infty$, since $y \in \partial S$, we have $|x| \cong |x - y|$. Therefore, we can replace r by $|x|$ in the above equation and take it out of the integral , and it is not difficult to conclude that $(W\varphi)(x)$ belongs to Λ.

(iii) $V(\varphi) \in C^{0,\alpha}(R^2)$ for any index $\alpha \in (0,1)$, where $C^{0,\alpha}(R^2)$ denotes the vector space of (real) Hölder continuous (with index $\alpha \in (0,1)$) functions in R^2.

Proof. This property can be proved using classical procedures (see, for example [4]).

Theorem 2.5. If $\varphi \in C^{0,\alpha}(\partial S), \alpha \in (0,1)$, then

(i) $W(\varphi)$ has $C^{0,\alpha}$-extensions W^+ and W^- to $\bar{S}^+$ and $\bar{S}^-$, respectively. These extensions are given by

$$W^+(x) = \begin{cases} W(x) & x \in S^+ \\ -\frac{1}{2}\varphi(x) + W_0(x), & x \in \partial S \end{cases} \tag{2.4}$$

$$W^-(x) = \begin{cases} W(x) & x \in S^- \\ \frac{1}{2}\varphi(x) + W_0(x), & x \in \partial S \end{cases} \tag{2.5}$$

where

$$W_0(x) = \int_{\partial S} P(x,y)\varphi(y)ds(y), \qquad x \in \partial S,$$

the integral being understood as principal value;

(ii) The first order derivatives of $V(\varphi)$ in S^+ and S^- have $C^{0,\alpha}$-extensions to $\bar{S}^+$ and $\bar{S}^-$, respectively. In addition, we have

$$(TV)^+(x) = \begin{cases} (TV)(x) & x \in S^+ \\ +\frac{1}{2}\varphi(x) + (TV)_0(x), & x \in \partial S \end{cases} \tag{2.6}$$

$$(TV)^-(x) = \begin{cases} (TV)(x) & x \in S^- \\ -\frac{1}{2}\varphi(x) + (TV)_0(x), & x \in \partial S \end{cases} \tag{2.7}$$

where

$$(TV)_0(x) = \int_{\partial S} T(\partial x) D(x,y)\varphi(y) ds(y), \qquad x \in \partial S,$$

the integral being understood as principal value.

Theorem 2.6. If $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, then the first order derivatives of $W(\varphi)$ in S^+ and S^- have $C^{1,\alpha}$ - extensions to $\bar{S}^+$ and $\bar{S}^-$, respectively. In addition, these extensions satisfy

$$(TW)^+ = (TW)^- \qquad on \quad \partial S \tag{2.8}$$

Theorem 2.5, Theorem 2.6 can also be proved using classical procedures (see, for example, [4]).

2.4 Complex variable treatment

In the analysis of two-dimensional boundary value problems by boundary integral equation methods, it is often convenient to express certain properties of functions in terms of complex variables . For an arbitrary function f given on ∂S we write

$$f(z) \equiv f(x)$$

where $z = x_1 + ix_2$.

Suppose now that $C(\partial S)$ and $C^1(\partial S)$ are complex spaces, and construct the complex spaces $C^{0,\alpha}(\partial S)$ and $C^{1,\alpha}(\partial S)$ by defining Hölder continuity in terms of the inequality

$$|f(z) - f(\zeta)| \leq c|z-\zeta|^{\alpha} \qquad \text{for all } z,\zeta \in \partial S$$

and the derivative as

$$f'(z) = \frac{d}{dz} f(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}, \qquad z,\zeta \in \partial S,$$

if this limit exists.

Since $|z-\zeta| = |x-y|$, where $\zeta = y_1 + iy_2$, it is obvious that Hölder continuity with respect to z and Hölder continuity with respect to x (or s) are equivalent . The same can also be said about Hölder continuous differentiability on ∂S [4].

For mathematical convenience, the fundamental solution of (N^+) can also be written in the complex form

$$D(x,y) = \frac{1}{\pi} \mathrm{Re}\left(\frac{1}{|A|} \log(\frac{1}{\sigma}) \right)$$

$$= -\frac{1}{\pi} \mathrm{Re}\left(\frac{1}{|A|} \log(\sigma) \right)$$

Here, $\sigma = (x_1 - y_1) + \alpha(x_2 - y_2)$

$\alpha = a + ib$

$a = -\frac{c_{45}}{c_{44}}, \; b = \frac{|A|}{c_{44}}$

and log(...) denotes the complex logarithm.

2.5 Existence Theorem

2.5.1 Boundary integral equation from a single layer potential

In order to apply the Fredholm Alternative to establish the existence theorem, we seek the solution of the anti-plane shear problem (N^+) in the form of a single layer potential

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y) \tag{2.9}$$

Where the unknown density $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$. In view of Theorem 2.4(i), (iii) and Theorem 2.5 (ii), the boundary value problem (N^+) reduces to the boundary integral equation

$$\frac{1}{2}\varphi(x) + (TV)_0(x) = t_3^*(x) \qquad x \in \partial S$$

Here,

$$(TV)_0(x) = \int_{\partial S} T(\partial x)D(x,y)\varphi(y)ds(y),$$

the integral being understood as principal value.

Lemma 2.1. The boundary integral equation

$$\frac{1}{2}\varphi(x) + (TV)_0(x) = t_3^*(x) \qquad x \in \partial S \tag{η^+}$$

is a Cauchy singular integral equation.

Proof. For convenience, we investigate the boundary integral equation in terms of complex variables. Therefore, we seek the solution of the anti-plane shear problem (N^+)

in the complex form of single layer potential:

$$u(x) = (V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y)$$
$$= -\frac{1}{\pi}\int_{\partial S}[\operatorname{Re}(\frac{1}{|A|}\log\sigma)]\varphi(y)ds(y)$$

The integral equation now becomes

$$-\varphi(x) + \frac{1}{\pi}\int_{\partial S}[\operatorname{Re}(\frac{1}{|A|}T(\partial x)\log\sigma]\varphi(y)ds(y) = -2t_3^*(x), \qquad x \in \partial S \qquad (2.10)$$

A straightforward calculation shows that

$$\operatorname{Re}\left(\frac{1}{|A|}T(\partial x)\log\sigma\right) = \operatorname{Re}\left(\frac{1}{|A|}\left[\frac{(c_{55}+c_{45}\alpha)\cos(n(x),x_1)+(c_{45}+c_{44}\alpha)\cos(n(x),x_2)}{\sigma}\right]\right)$$

$$= \operatorname{Im}\left((1-iB)\left[\frac{\cos(n(x),x_2)-\alpha\cos(n(x),x_1)}{\sigma}\right]\right)$$

$$= \operatorname{Im}\left((1-iB)\frac{\partial}{\partial s(x)}\log\sigma\right)$$

where $B = \dfrac{c_{55}-c_{45}}{|A|}$

Hence, (2.10) becomes

$$-\varphi(x) + \frac{1}{\pi}\operatorname{Im}\left((1-iB)\int_{\partial S}\left(\frac{\partial}{\partial s(x)}\ln\sigma\right).\varphi(y)ds(y)\right) = -2t_3^*(x) \qquad x \in \partial S \qquad (2.11)$$

The kernel $\frac{\partial}{\partial s(x)}\log\sigma$ can be written as

$$\frac{\partial}{\partial s(x)}\log\sigma = (\frac{\partial}{\partial s(x)}\log\sigma + \frac{\partial}{\partial s(y)}\log\sigma) - \frac{\partial}{\partial s(y)}\log\sigma$$

and

$$\frac{\partial}{\partial s(y)}\log\sigma = \frac{\partial}{\partial s(y)}\log r + \frac{\partial}{\partial s(y)}\log\frac{\sigma}{r}$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

It is easy to show [2] that

$$\frac{\partial}{\partial s(y)}\log r ds(y) = \frac{d\xi}{\xi - z} - id\theta$$

Here, ξ and z are the arc-coordinates of the points y and x on the boundary.

Therefore, the integral equation becomes

$$\frac{1}{2}\varphi(x) + \frac{B}{2\pi}\int_{\partial s}\frac{\varphi(\xi)}{\xi - z}d\xi - \mathrm{K}\varphi + \frac{1}{2\pi}\mathrm{Im}(1 - iB)\int_{\partial s}(\frac{\partial}{\partial s(x)}\log\sigma + \frac{\partial}{\partial s(y)}\log\sigma)\varphi(y)ds(y) = t_3^{*}(x) \quad (2.12)$$

where

$$\mathrm{K}\varphi = \frac{1}{2\pi}\mathrm{Im}\int_{\partial s}\varphi(t)d[(1 - iB)(\log\frac{\sigma}{r} - i\theta)]$$

and K is a completely continuous operator [2]. The expression

$$\frac{\partial}{\partial s(x)}\log\sigma + \frac{\partial}{\partial s(y)}\log\sigma$$

is also continuous as the point x moves along the contour ∂S [2] . Thus, the integral equation

$$\frac{1}{2}\varphi(x) + (TV)_0(x) = t_3^*(x) \qquad x \in \partial S$$

can be rewritten as

$$\frac{1}{2}\varphi(z) + \frac{B}{2\pi}\int_{\partial S}\frac{\varphi(\xi)}{\xi - z}d\xi + (TV)_1(z) = t_3^*(z) \qquad z \in \partial S$$

where $(TV)_1$ is a completely continuous operator. Therefore the equation is a singular integral equation with Cauchy kernel [6].

2.5.2 Boundary integral equation from a double layer potential

If the solution of the boundary value problem (D^-) is sought in the form of a double layer potential

$$(W\varphi)(x) = \int_{\partial S} P(x,y)\varphi(y)ds(y) \tag{2.13}$$

where the unknown density $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, in view of Theorem 2.4(i), (ii), Theorem 2.5(i) and Theorem 2.6, the boundary value problem (D^-) reduces to the singular integral equation

$$\frac{1}{2}\varphi(x)+W_0(x)=h(x) \qquad x\in\partial S \qquad (\omega^-)$$

where,

$$W_0(x)=\int_{\partial S}P(x,y)\varphi(y)ds(y),$$

the integral being understood as principal value.

The corresponding homogeneous equations are denoted by (η_0^+) and (ω_0^-) respectively as follows:

$$\frac{1}{2}\varphi(x)+(TV)_0(x)=0 \qquad x\in\partial S \qquad (\eta_0^+)$$

$$\frac{1}{2}\varphi(x)+W_0(x)=0 \qquad x\in\partial S \qquad (\omega_0^-)$$

The following result can be proved as in [3].

Theorem 2.7 If $\alpha\in(0,1)$, any $C^{0,\alpha}$- solution φ, of the integral equation (ω_0^-) is of the class $C^{1,\alpha}(\partial S)$.

Theorem 2.8 The equation (ω_0^-) has only the constant solution in $C^{0,\alpha}(\partial S)$,.

Proof: In view of Theorem 2.7, it suffices to prove the result in $C^{1,\alpha}(\partial S), \alpha\in(0,1)$. It is clear that the constant function $u(x)=c$ is a solution of the homogeneous boundary value problem (N_0^+). Further, since $Tu=0$, replacing u by c in (2.3), we have

$$\frac{1}{2}c+\int_{\partial S}P(x,y)c\,ds(y)=0 \qquad x\in\partial S$$

This means that $u = c$ is a solution of (ω_0^-) for arbitrary c.

Let φ^0 be an arbitrary solution of (ω_o^-), then

$$f = \varphi^0 - c \tag{2.14}$$

is also a $C^{1\alpha}$- solution of (ω_0^-) for any constant c. Consequently, we have

$$W^-(f) = 0 \quad \text{on } \partial S\ ,$$

and from Theorem 2.4-2.6, $W^-(f)$ is a solution of homogeneous boundary value problem (D_0^-). From Theorem 2.3(ii), we have

$$W^-(f) = 0 \quad \text{in } S^-.$$

This means

$$(TW)^-(f) = 0 \quad on \partial S,$$

which, in turn, by Theorem 2.8, implies that

$$(TW)^+(f) = 0 \quad on \partial S.$$

Hence, $W^+(f)$ is a solution of the homogeneous boundary value problem (N_0^+) and, by Theorem 2.3(i),

$$W^+(f) = W^+(\varphi^0) - W^+(c) = K \qquad in\, S \tag{2.15}$$

where K is an arbitrary constant.

Without loss of generality, suppose that the origin of the coordinates lies in S, we choose the c, so that $K = 0$, by, for example, requiring that

$$(W^+(f))(0) = (W^+(\varphi^0))(0) - (W^+(c))(0) = 0$$

or

$$c = \frac{(W^+(\varphi^0))(0)}{(W^+(1))(0)} \tag{2.16}$$

The fact that such c exists is proved as follows. Let $c = c^*$ be a solution of the homogeneous equation

$$c^*(W^+(1))(0) = (W^+(c^*))(0) = 0 \tag{2.17}$$

We shall show that the only solution of (2.17) is $c^* = 0$, so that

$$(W^+(1)(0)) \neq 0 .$$

Taking $\varphi^0 = 0$ and $c = c^*$ in (2.14), we have, as above, that $W^+(c^*)$ solves the homogeneous problem $(N_0^{\ +})$. Hence, from Theorem 2.3(i), $W^+(c^*)$ is an arbitrary constant in S. In view of (2.17), we conclude that

$$W^+(c^*) = 0 \quad \text{in } S$$

so that

$$(TW)^+(c^*) = 0 \quad on\,\partial S .$$

From (2.8), we have

$$(TW^-)(c^*) = 0 \quad on\,\partial S .$$

Thus $W^-(c^*)$ solves the homogeneous exterior Neumann problem $(N_0^{\ -})$ which, by Theorem 2.3(iii) implies that

$$W^-(c^*) = 0 \quad in\, S^- .$$

Since

$$W^+(c^*) = 0 \quad in\, S,$$

from Theorem 2.5(i), it follows that

$$c^* = W^-(c^*) - W^+(c^*) = 0 \quad on\,\partial S .$$

Hence (2.17) has only the trivial solution $c^* = 0$ and c given by (2.16) exists uniquely. With this c, we have, from(2.15),

$$W^{+}(f) = 0 \quad in\ S.$$

But, from above,

$$W^{-}(f) = 0 \quad in\ S^{-}.$$

Applying Theorem 2.5(i) once more, we obtain

$$f = W^{-}(f) - W^{+}(f) = 0 \quad on\ \partial S$$

Finally, from (2.14), the only $C^{1,\alpha}$ - solution of (ω_0^-) is the constant solution.

Theorem 2.9 The Fredholm Alternative holds for (η^+) and (ω^-) in the real dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$ $\alpha \in (0,1)$, with the bilinear from

$$(\varphi,\psi) = \int_{\partial S} \varphi\psi ds$$

Proof: Denoting by η and ω and the integral operators occurring in (η^+), (ω^-), respectively, it is clear that for any $\varphi,\psi \in C^{0,\alpha}(\partial S)$

$$(\eta\varphi,\psi) = (\varphi,\omega\psi) \text{ and } (\omega\varphi,\psi) = (\varphi,\eta\psi)$$

which means that η and ω are mutually adjoint in the given dual system. Proceeding as in [4], it remains to show that the index [6] of each of the corresponding singular integral operators is zero. In fact, from (2.12), the index of the operator

$$\eta = (TV)_0 + \frac{1}{2}$$

is given by the formula

$$\aleph = \frac{1}{2\pi}\left[\arg\left(\frac{1-Bi}{1+Bi}\right)\right]_{\partial S}$$

where $[\ldots]_{\partial S}$ denotes the change in $[\ldots]$ as z traverses ∂S once anti-clockwise.

Since $B = \frac{c_{55} - c_{45}}{|A|}$ and $1 + Bi \neq 0$, the expression in (...) is a constant and therefore there is no change in $[\ldots]$ along closed curve ∂S, i.e., index $\aleph = 0$.

The same is true for the operator

$$\omega = W_0 + \frac{1}{2}.$$

Theorem 2.10. The condition

$$\int_{\partial S} t_3^* ds = 0$$

is necessary and sufficient for the solvability of the anti-plane shear problem for any $t_3^* \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$. The solution is unique up to a constant and can be represented as a single layer potential $V(\varphi)$ with density $\varphi \in C^{0,\alpha}(\partial S)$ obtained from the singular integral equation (η^+)

Proof: (1) Necessity:

Let u be a solution of the anti-plane shear problem (N^+). According to (2.1A), let $v \equiv 1$, we have

$$\int_S Lu d\sigma = \int_{\partial S} Tu ds,$$

therefore

$$\int_{\partial S} t_3^{*} ds = \int_{\partial S} Tu ds = \int_{S} Lu d\sigma = 0$$

(2) Sufficiency

We see that u solves the interior Neumann problem (N^{+}) if φ solves the integral equation

$$\frac{1}{2}\varphi(x) + \int_{\partial S} T(\partial x) D(x,y)\varphi(y) ds(y) = t_3^{*}(x)$$

In the proof of Theorem 2.8, we saw that the only solutions of the adjoint homogeneous equation

$$\frac{1}{2}\psi(x) + \int_{\partial S} P(x,y)\psi(y) ds(y) = 0$$

are the constant functions ψ =constant, therefore

$$\int_{\partial S} t_3^{*} ds = 0$$

coincides with the condition

$$\int_{\partial S} t_3^{*}\psi ds = 0$$

which by Fredholm's alternative guarantees the existence of a density $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, for which $V(\varphi)$ solves the anti-plane shear problem (N^{+}).

Chapter 3 Direct boundary integral equation method

The direct boundary integral equation method is also an efficient method for solving boundary value problems. In this method, the direct boundary integral equation can be derived using "integral identities" [e.g., Green's second identity, Betti's reciprocal theorem, equation (2.1A)]. Nevertheless, both indirect and direct methods utilize the same infinite region unit solutions (e.g., fundamental solutions) to generate the components of the kernels in the final boundary integral equations.

In this chapter, the direct boundary integral equation is obtained from the equation (2.1A) through the weighted residual technique. To do this, we have to establish the weak statement for the residuals of anti-plane shear problem (N^+). The residual functions in the field and on the boundary are:

$$R = Lu \qquad in\ S \tag{3.1}$$

$$R_{\partial S} = Tu - t_3^* \qquad on\ \partial S \tag{3.2}$$

respectively. Then the weak statement of problem (N^+) is

$$\int_S w_i R d\sigma + \int_{\partial S} w_i R_{\partial S} ds = 0 \tag{3.3}$$

Here w_i is a weighted function. From (3.1) and (3.2), we have

$$\int_S w_i R d\sigma = \int_{\partial S} w_i Tu ds - \int_{\partial S} w_i t_3^* ds \tag{3.4}$$

From equation (2.1A), we obtain the relations

$$\int_S w_i Lud\sigma = \int_{\partial S} w_i Tuds - \int_S E(u,w_i)d\sigma \qquad (3.5)$$

$$\int_S uLw_i d\sigma = \int_{\partial S} uTw_i ds - \int_S E(u,w_i)d\sigma \qquad (3.6)$$

By applying (3.5) and (3.6) in turn to (3.4), we obtain

$$\int_S uLw_i d\sigma = \int_{\partial S} uTw_i ds - \int_{\partial S} (w_i)t_3^* ds \qquad (3.7)$$

This is an important equation as it is usually the starting point for the application of numerical techniques, such as the boundary element method. Our aim is now to reduce formula (3.7) to a boundary integral equation. This is done by using a special type of weighting function w_i, i.e. the fundamental solution. We choose the same fundamental solution used in the indirect method - $D(x,y)$ as the weighting function, so that, as in Chapter 2, it satisfies the equation

$$L(\partial x)D(x,y) = \delta^i$$

Where δ^i represents a Dirac Delta function which tend to infinity at the point at a point 'i', i.e., $y = x^i$ and is equal to zero anywhere else. The integral of δ^i however is equal to one.

The integral of a Dirac delta function multiplied by any other function is equal to the value of the latter at the point x^i [20]. Hence,

$$\int_S u(LD)d\sigma = \int_S u(-\delta^i)d\sigma = -u^i$$

Therefore, we obtain

$$u^i + \int_{\partial S} u(TD(x,y))ds = \int_{\partial S} D(x,y)t_3^* ds \tag{3.8}$$

We have now deduced an equation (3.8) which is valid for any point within the domain S. We apply (3.8) on the boundary to find out what happens when the point x^i is on ∂S. A simple way to do this is to consider the point i that is on the boundary but the domain itself is augmented by semicircle of radius ε as shown in Figure 2. The point x^i is considered to be at the center and the radius ε is taken to zero.

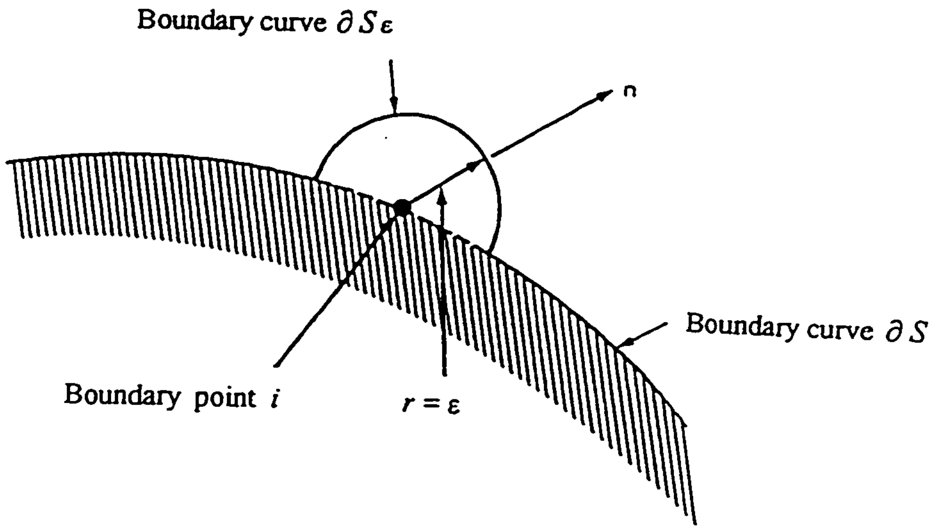

Figure 2 Boundary point for cross-section of the cylinder in anti-plane shear, augmented by a small semicircle

Using the similar procedure in [20], we can obtain the direct boundary integral equation for anti-plane problem (N^+) as:

$$\frac{1}{2}u^{i} + \int_{\partial S} u(TD(x,y))ds = \int_{\partial S} D(x,y)t_3^{*}ds\,, \tag{3.9}$$

the integrals are understood as principal values. In this boundary integral equation, the only unknown is the displacement of the anti-plane shear, rather than the 'fictitious' density in the indirect boundary integral equation.

Usually, equation (3.9) is used as a starting point for boundary element method by discretizing the equation.

Chapter 4. Examples

4.1 Explicit solution of anisotropic anti-plane shear problem

In chapter 2, we obtained the boundary integral equation by seeking the solution in the form of single layer potential. If we can find the explicit solution of density φ in the integral equation, and substitute it into the single layer potential , we can obtain the analytical solution of the anti-plane shear deformation in terms of displacement. However, only those boundary integral equations with some regular boundary contours may be solved explicitly due to their special geometric properties.

In this chapter, we find the explicit solution for the anti-plane shear problem (N^+) in the case when the curve ∂S represents an ellipse. To this end, suppose that the parametric equations of the ellipse are

$$\begin{cases} y_1 = a_1 \cos\theta, \\ y_2 = a_2 \sin\theta \end{cases}$$

where $a_1 \geq a_2$, $0 \leq \theta \leq 2\pi$

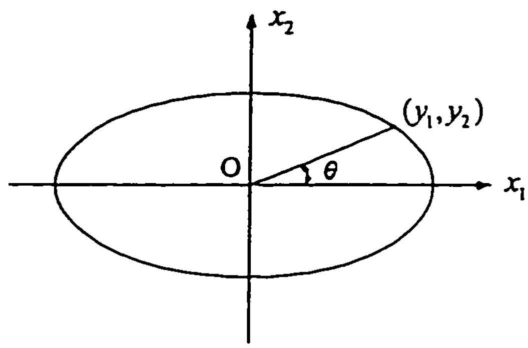

Figure 2 Elliptic contour of the cylinder

For mathematical convenience, the fundamental solution of anti-plane shear problem (N^+) can be written in the complex form as in (2.4.1)

$$D(x,y) = -\frac{1}{\pi}\mathrm{Re}\left(\frac{1}{|A|}\log(\sigma)\right)$$

and the boundary integral equation can be written as (2.11).

On the ellipse ∂S , if θ_0 is the value of the parameter θ corresponding to $x \in \partial S$, we have

$$\frac{\partial}{\partial s(x)}\log\sigma =$$

$$= \frac{2[\alpha\cos(n(x),x_1) - \cos(n(x),x_2)]}{(a_1 - i\alpha_2)(e^{i\theta_0} - \lambda e^{-i\theta_0})}\left(\frac{\lambda e^{-i\theta_0}}{e^{i\theta_0} - \lambda e^{-i\theta_0}} - \frac{e^{i\theta_0}}{e^{i\theta} - e^{i\theta_0}}\right)$$

and

$$\frac{\lambda e^{-i\theta_0}}{e^{i\theta_0} - \lambda e^{-i\theta_0}} = \sum_{n=1}^{\infty}\lambda^n e^{-in(\theta+\theta_0)}$$

$$-\frac{e^{i\theta_0}}{e^{i\theta} - e^{i\theta_0}} = \frac{1}{2} + \frac{i}{2}\cot\frac{\theta - \theta_0}{2}$$

Here,

$$\lambda = \frac{a_1 + ia_2\alpha}{a_1 - ia_2\alpha}\ .$$

Next, as in [2], define

$$X_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)d\theta$$

$$X_n = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)e^{in\theta}d\theta \qquad n \geq 1$$

$$X_{-n} = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)e^{-in\theta}d\theta \qquad n \geq 1$$

Substitute the above into (2.11) to obtain

$$
\begin{aligned}
&(K\psi)(\theta_0) = \\
&\quad = -\psi(\theta_0) + \frac{B}{2\pi}\int_0^{2\pi} \cot(\frac{\theta - \theta_0}{2})\psi(\theta)d\theta + X_0 + 2\mathrm{Re}(1 - \mathrm{iB})\sum_{n=1}^{\infty} \lambda^n e^{-in\theta_0} X_{-n} \\
&\quad = f(\theta_0) \qquad (4.1)
\end{aligned}
$$

where,

$$
\begin{aligned}
\psi(\theta) &= \varphi(\theta)\sqrt{a_1^2 \sin^2\theta + a_2^2 \cos^2\theta} \\
f(\theta) &= 2t_3^*(\theta)\sqrt{a_1^2 \sin^2\theta + a_2^2 \cos^2\theta}
\end{aligned} \qquad (4.2)
$$

Equation (4.1) is a singular integral equation. In order to find its explicit solution, we should simplify the equation further by applying the regularizing operator[6].

We choose the regularizing operator as

$$
M\chi(\theta_0) = -\chi(\theta_0) - \frac{\theta}{2\pi}\int_0^{2\pi} (\cot\frac{\theta - \theta_0}{2})\chi(\theta)d\theta
$$

and applying the well known formulas

$$
\int_0^{2\pi} \cot\frac{\theta - \theta_0}{2} d\theta = 0
$$

$$
-\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \cot(\frac{\theta - \theta_0}{2}) \cot(\frac{w - \theta_0}{2})\psi(w)dwd\theta = \psi(\theta_0) - X_0
$$

$$
\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} \cot(\frac{\theta - \theta_0}{2})d\theta = ie^{in\theta_0} \quad (n = \pm 1, \pm 2, \ldots)
$$

In fact, we obtain,

$$(MK)(\psi)(\theta_0) = (1 + B^2)[\psi(\theta_0) - X_0] - 2(1 - iB)^2 \sum_{n=1}^{\infty} \lambda^n e^{-in\theta_0} X_{-n} = (Mf)(\theta_0) \tag{4.3}$$

Finally, defining

$$f_{\pm n} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{in\theta} d\theta \qquad n = 0, 1, 2\cdots,$$

multiplying equation (4.3) in turn, by $\frac{1}{2\pi} e^{in\theta_0} d\theta_0$ and $\frac{1}{2\pi} e^{-in\theta_0} d\theta_0$, and integrating over the interval $(0, 2\pi)$ we obtain

$$(1 + B^2)(X_0 - X_0) = \frac{1}{2\pi} \int_0^{2\pi} (Mf)(\theta) d\theta = \frac{1}{\pi} \int_{\partial S} \overset{\bullet}{t}_3(y) ds(y) \tag{4.4}$$

$$-(1 + iB) X_n + (1 - iB) \lambda^n X_{-n} = f_n \tag{4.5}$$

$$-(1 - iB) X_{-n} + (1 + iB) \overline{\lambda^n} X_n = f_{-n} \tag{4.6}$$

($\overline{\lambda}$ denotes the complex conjugate of λ).

In (4.4), the condition(1.16) (vanishing of the net external force) imply that $X_0 = C$, where C is an arbitrary constant.

To find X_{-n}, multiply equation (4.5) on the left by $\overline{\lambda^n}$, and add the resulting equation to (4.6) , we obtain

$$X_{-n} = \frac{f_n \overline{\lambda^n} + f_{-n}}{(1 - iB)(\overline{\lambda^n} \lambda^n - 1)}$$

Using these expressions for X_0, X_{-n} in (4.3) , we obtain $\psi(w_0)$ on the boundary ∂S:

$$\psi(w_0) = 2\,\mathrm{Re}\frac{(1-iB)^2}{1+B^2}\sum_{n=1}^{\infty}[\lambda^n e^{-in\theta_0}\frac{f_n\overline{\lambda^n}+f_{-n}}{(1-iB)(\overline{\lambda^n}\lambda^n-1)}]+\frac{(Mf)(\theta_0)}{1+B^2}+C \tag{4.7}$$

Once $\psi(\theta_0)$ is found for $\theta_0 \in \partial S$ from (4.7), φ on ∂S can be found from (4.2) and the displacement

$$u(x) = (V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y) \tag{4.8}$$

4.2. Numerical examples

Using the above method, the density φ in the boundary integral equation of homogeneous anisotropic anti-plane shear problem can be solved explicitly and the displacement can be obtained easily.

In this section, we first give a numerical example for the anti-plane shear problem (N^+) of elastic cylindrical body composed of homogeneous and anisotropic material with elliptic contour.

Secondly, we reduce the problem (N^+) to an interior Neumann problem of Laplace's equation by choosing appropriate values of elastic constants c_{44}, c_{55} and c_{45}.

4.2.1 Example 1

Choose the three elastic constants in the (N^+) as following values:

$c_{55} = 2.56 \times 10^{11}$ $dynes / cm^2$

$c_{44} = 4.13 \times 10^{11} \quad dynes / cm^2$

$c_{45} = 1.58 \times 10^{11} \quad dynes / cm^2$

(Conversion factors to other units are:

$10^{10} dynes / cm^2 = 0.145 \times 10^6 lb / in.^2 = 1.02 \times 10^4 kg / cm^2$).

We let the two axes of ellipse be

$a_1 = 2cm$

$a_2 = 1cm$

and let $t_3^* = \sin 2\theta$.

For convenience, choose the arbitrary constant C in (4.3) be zero.

Since it is difficult (if not impossible) to determine the integral (4.8) in closed form, we apply a numerical integration technique to solve this problem. A computer code was written using FORTRAN based on the well-known Simpson Rule. The results and the related data are given in Table 1.

4.2.2 Example 2

In this example, we reduce the homogeneous and anisotropic problem to the homogeneous and isotropic problem by letting $c_{55} = c_{44}$, $c_{45} = 0$. The problem then becomes the interior Neumann problem of Laplace's equation. The two methods are applied to obtained the results for this example. The first method applied is the analytical method, i.e., the method used in Example 1; The second method is a numerical method - boundary element method. For mathematical convenience , we can further simplify the problem by letting the two axes of the ellipse be $a_1 = a_2 = 1cm$, so that the boundary curve of the elastic cylindrical body becomes a circle.

Since this example is a special case of Example 1, it is still sufficient to illustrate the theory developed in this thesis .

Method 1

Choose the values of the elastic constants as follow:

$$c_{55} = c_{44} = 4.13 \times 10^{11} \quad dynes/cm^2$$

$$c_{45} = 0 \ .$$

Using the same method and code in Example 1, we can obtain the displacements of internal points for the homogeneous isotropic elastic circular bar. The results and related data are given in the Table 2

Method 2

The boundary element method is applied to this example to compare the results obtained by Method 1. The boundary integral equation used in boundary element method in this example is derived according to the formulations in Chapter 3.

By discretizing the boundary curve into ten constant elements, a BEM code was written to solve the problem. The results obtained by BEM are given in the Table 3

Although the surface traction $t_3^* = \sin 2\theta$ is symmetric with respect to the x_3-axis in both Example 1 and Example 2, we note that the displacements indicated in Table 1 are not symmetric with respect to the x_3-axis. However, the displacements indicated in Table 2 are symmetric with respect to the x_3-axis. Obviously, the asymmetry of displacements in Example 1 is caused by anisotropy of the material. We also note that the results obtained by Method 2 are very close to the results obtained by Method 1. Any difference is attributed to the insufficient number of discretized elements in Method 2.

internal point	X (cm)	Y (cm)	Displacement (10^{-12} cm)
1	.100E+01	.000E+00	.182E+00
2	.809E+00	.294E+00	.384E+00
3	.309E+00	.476E+00	.378E+00
4	-.309E+00	.476E+00	.160E+00
5	-.809E+00	.294E+00	.461E-01
6	-.100E+01	.000E+00	.182E+00
7	-.809E+00	-294E+00	.384E+00
8	-.309E+00	-476E+00	.378E+00
9	.309E+00	-.476E+00	.160E+00
10	.809E+00	-.294E+00	.461E-01
11	.160E+01	.000E+00	-.332E-02
12	.129E+01	.470E+00	.471E+00
13	.494E+01	.762E+00	.465E+00
14	-.494E+00	.762E+00	-.917E-01
15	-.129E+00	.470E+00	-.334E+00
16	-.160E+01	.000E+00	-.332E-02
17	-.129E+01	-.470E+00	.471E+00
18	-.494E+01	-.762E+00	.465E+00
19	.494E+00	-.762E+00	-.917E-01
20	.129E+00	-.470E+00	-.334E+00

Table 1 Results of Example 1

internal point	X (cm)	Y (cm)	Displacement (10^{-12}) cm)
1	.500E+00	.000E+00	.146E-00
2	.405E+00	.293E+00	-.146E-00
3	.155E+00	.476E+00	-.239E-00
4	-.155E+00	.476E+00	-.222E-05
5	-.405E+00	.293E+00	.239E-00
6	-.500E+00	.000E+00	.146E-00
7	-.405E+00	-.293E+00	-.146E-00
8	-.155E+00	-.476E+00	-.239E-00
9	.155E+00	-.476E+00	-.222E-05
10	.405E+00	-.293E+00	.239E-00
11	.800E+00	.000E+00	.378E-00
12	.648E+00	.469E+00	-.378E-00
13	.248E+00	.762E+00	-.608E-00
14	-.248E+00	.762E+00	.386E-00
15	-.648E+00	.469E+00	.608E-00
16	.800E+00	.000E+00	.378E-00
17	.648E+00	.469E+00	-.378E-00
18	.248E+00	.762E+00	-.608E-00
19	-.248E+00	.762E+00	.386E-05
20	-.648E+00	.469E+00	.608E-00

Table 2 Results of Example 2 (Method 1)

internal point	X (cm)	Y (cm)	Displacement (10^{-12} cm)
1	.500E+00	.000E+00	.169E-00
2	.405E+00	.293E+00	-.170E-00
3	.155E+00	.476E+00	-.278E-00
4	-.155E+00	.476E+00	-.821E-03
5	-.405E+00	.293E+00	.275E-00
6	-.500E+00	.000E+00	.169E-00
7	-.405E+00	-.293E+00	-.170E-00
8	-.155E+00	-.476E+00	-.278E-00
9	.155E+00	-.476E+00	-.821E-03
10	.405E+00	-.293E+00	.275E-00
11	.800E+00	.000E+00	.436E-00
12	.648E+00	.469E+00	-.435E-00
13	.248E+00	.762E+00	-.711E-00
14	-.248E+00	.762E+00	-.676E-03
15	-.648E+00	.469E+00	.707E-00
16	.800E+00	.000E+00	.436E-00
17	.648E+00	.469E+00	-.435E-00
18	.248E+00	.762E+00	-.711E-00
19	-.248E+00	.762E+00	-.676E-03
20	-.648E+00	.469E+00	.707E-00

Table 3 Results of Example 2 (Method 2)

Chapter 5 Conclusion

The displacement relating to anti-plane shear of a homogeneous and anisotropic linearly elastic solid with surface traction (1.1) was obtained using the indirect boundary integral equation method in the form of single layer potential. The corresponding stresses can be obtained from (1.22)-(1.24). Although the most irregular boundary geometries preclude any analytical solution of the problem, the example presented in chapter 4 indicate that the explicit solution can be obtained when the boundary curve is an ellipse. It is reasonable to expect that the problem with the Pascal limacon, the epitrochoid, the hypotrochoid and other boundary curves can also be solved explicitly.

Regarding the corresponding boundary value problems relating to the *inhomogeneous* anti-plane shear deformations, we can apply exactly the same theory and basic procedures developed in Chapter 2-4 if the fundamental solutions of these problems can be constructed. Unfortunately, the fundamental solutions of second-order linear elliptic partial differential equations with variable coefficients are very difficult to obtain in spite of the availability of general forms given by Miranda in [17]. However, some investigations on the fundamental solutions have been done so far by the author, which were largely motivated by the fact that there are various applications in inhomogeneous linear elasticity (see, for example [23], [24]).

Reasonably, we should also extend our theory to the other two types of basic problems in classical elasticity - oscillation and dynamic problems. The governing boundary value problems relating to the steady oscillation problems of anti-plane shear deformations may be proved to be still elliptic [8]. These problems can be treated similarly by the same theory used in the static problem without any consideration of initial conditions. In studying dynamic problems of anti-plane shear deformations, however, both boundary and initial conditions have to be considered. In addition, loss of ellipticity of the governing equation can occur. In the homogeneous anisotropic case, the governing equation is hyperbolic under the condition (1.21) [1]. Thus, new mathematical difficulties arise, reflecting the more complicated nature of the dynamic state as compared to the static or oscillation case. However, it can be readily verified that (1.13) is still sufficient to ensure that a dynamic nontrivial state of anti-plane shear can exist [1]. This gives us the possibility to apply our theory to the dynamic problems with some modifications.

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Appendix : Physical significance of the elastic constants in anisotropic materials

In 1678 Robert Hooke proposed the law for the stress-strain relation in arbitrary anisotropic linear elastic solid, which is now named after him. The law was actually discovered in 1660 and was first published in 1676 as an anagram. It was proposed independently by Mariotte in 1680.

In more specific terms, Hooke's law can be expressed as follows : Each stress component is directly proportional to each strain component; or, in symbols:

$$T_{ij} = c_{ijkl} E_{kl} \qquad \text{(A.1)}$$

$$E_{ij} = s_{ijkl} T_{kl} \qquad \text{(A.2)}$$

where i, j, k, l = 1,2 or 3; the c_{ijkl} are termed the elastic stiffnesses, and the s_{ijkl} the elastic compliances. As the equations stand , there are 81 stiffnesses and compliances, but owning to the usual symmetry relations,

$$c_{ijkl} = c_{jikl} \qquad c_{ijkl} = c_{ijlk} \qquad c_{ijkl} = c_{klij} \qquad \text{(A.3)}$$

$$s_{ijkl} = s_{jikl} \qquad s_{ijkl} = s_{ijlk} \qquad s_{ijkl} = s_{klij} \qquad \text{(A.4)}$$

the number of independent stiffnesses and compliances is 21 in the most general case.

For convenience, the full tensor suffixes of the stresses and strains are frequently contracted with single index notation:

$$T_{11} = T_1, \quad T_{22} = T_2, \quad T_{33} = T_3 \quad T_{23} = T_4, \quad T_{13} = T_5, \quad T_{12} = T_6 \qquad \text{(A.5)}$$

$$E_{11} = E_1, \quad E_{22} = E_2, \quad E_{33} = E_3 \quad 2E_{23} = E_4, \quad 2E_{13} = E_5, \quad 2E_{12} = E_6 \qquad \text{(A.6)}$$

The occurrence of the factor 2 in the equations relating to the shear strains in (A.6) should be particularly noted and the 'tensor' shear strains E_{ij} $(i,j = 1,2,3,\ i \neq j)$ carefully distinguished from the 'engineering ' shear strains E_q $(q = 4,5,6)$ [9]. The contracted notation (A5) and (A6) is almost invariably used in experimental work on elasticity. The generalized Hooke's law then becomes:

$$T_q = c_{qr} E_r \qquad \text{(A.7)}$$

$$E_q = s_{qr} T_r \quad (q,r = 1,2,3,4,5 \text{ or } 6) \qquad \text{(A.8)}$$

If we expand (A.1) for $i = j = 1$, we get

$$\begin{aligned} T_{11} &= c_{1111}E_{11} + c_{1112}E_{12} + c_{1113}E_{13} + c_{1121}E_{21} + c_{1122}E_{22} + c_{1123}E_{23} + c_{1131}E_{31} + c_{1132}E_{32} + c_{1133}E_{33} \\ &= c_{1111}E_{11} + 2c_{1112}E_{12} + 2c_{1113}E_{13} + c_{1122}E_{22} + 2c_{1123}E_{23} + c_{1133}E_{33} \end{aligned}$$

Introducing the contracted stresses and strains from (A.5) and (A.6), we have

$$T_1 = c_{1111}E_1 + 2c_{1112}\frac{1}{2}E_6 + 2c_{1113}\frac{1}{2}E_5 + c_{1122}E_2 + 2c_{1123}\frac{1}{2}E_4 + c_{1133}E_3$$

and comparing with the expanded form of (A.7),

$$T_1 = c_{11}E_1 + c_{12}E_2 + c_{13}E_3 + c_{14}E_4 + c_{15}E_5 + c_{16}E_6$$

we find $c_{1111} = c_{11}, \quad c_{1122} = c_{12}, \ldots, c_{1112} = c_{16}$,

and by similar expansion and comparison for other values of i, j, it can be shown that

$$c_{ijkl} = c_{qr} \tag{A.9}$$

On the other hand, expanding (A.2) for $i = j = 1$ and contracting by (A.5) and (A.6), we get

$$E_1 = s_{1111}T_1 + 2s_{1112}T_6 + 2s_{1113}T_5 + s_{1122}T_2 + 2s_{1123}T_4 + s_{1133}T_3$$

Comparing this with the expanded form of (A.8), we find

$$s_{1111} = s_{11}, \quad s_{1122} = s_{12}, \ldots, 2s_{1112} = s_{16}$$

and by similar expansion and comparison for other values of i, j, it can be shown that:

$$\left.\begin{aligned} s_{ijkl} &= s_{qr} \quad for \quad q,r = 1,2,3 \\ 2s_{ijkl} &= s_{qr} \quad for \quad q = 1,2,3,\ r = 4,5,6 \\ 4s_{ijkl} &= s_{qr} \quad for \quad q,r = 4,5,6 \end{aligned}\right\} \tag{A.10}$$

The use of the symbols s for compliance and c for stiffness is now almost invariably followed.

If we expand equation (A.8) we get

$$E_1 = s_{11}T_1 + s_{12}T_2 + s_{13}T_3 + s_{14}T_4 + s_{15}T_5 + s_{16}T_6 \tag{A.11}$$

. . . .

$$E_4 = s_{41}T_1 + s_{42}T_2 + s_{43}T_3 + s_{44}T_4 + s_{45}T_5 + s_{46}T_6 \tag{A.12}$$

. . . .

Now if the only stress acting is T_1 then all six strains $E_1, \ldots, E_4, \ldots$ exist. In the particular case of E_1, (A.11) gives

$$E_1 = s_{11}T_1$$

and therefore

$$s_{11} = 1 / K_1 \tag{A.13}$$

where K_1 is the Young's modules.

If the only stress is T_2, then

$$E_1 = s_{12} T_2$$

But we have

$$E_1 / E_2 = -\nu_{21},$$

where ν_{21} is the Poisson's ratio and $E_2 = s_{22} T_2$ [9]. Therefore

$$s_{12} = -\nu_{21} / K_2$$

and since $s_{12} = s_{21}$ [9]

$$s_{12} = -\nu_{21} / K_2 = -\nu_{12} / K_1 \tag{A.14}$$

Similarly, if T_4 is the only stress different from zero, we find that s_{14} relates a shear stress in the $x_2 x_3$ plane to an extensional strain in the x_1 direction. From equation (A.12) and the reciprocal relations we find that $s_{14} (= s_{41})$ also connects an extensional stress in the x_1 direction to shear strain in the $x_2 x_3$ plane [9].

In the same way, s_{24} relates a tensile stress in the x_2 direction to a shear stain in the $x_2 x_3$ plane, or alternatively, a tensile strain in the x_2 direction to a shear stress in the $x_2 x_3$ plane; and s_{44} relates a shear stress to shear strain, both in the same plane. We thus have

$$s_{44} = 1 / G_{23}$$

where G_{23} is the shear modules in the $x_2 x_3$ plane.

The last typical compliance to consider is s_{45}, and we find that this relates a shear stress in the x_1x_3 plane to a shear strain in the x_2x_3 plane and vice versa.

The above findings can be generalized as follows:

(i) $s_{qq}(\equiv s_{iiii})(q = 1,2,3)$ relates an extensional stress to an extensional strain both in the same direction, and $s_{qq} = 1 / K_q$;

(ii) $s_{qr}(\equiv s_{iijj})(q \neq r,\ q,r = 1,2,3)$ relates an extensional strain to a perpendicular extensional stress, and $s_{qr} = -\nu_{rq} / K_r = -\nu_{qr} / K_q$;

(iii) (a) $s_{qr}(\equiv 2s_{iiij})(\ q = 1,2,3\ or\ r = 4,5,6)$ relates an extensional strain to a shear stress and vice versa, both in the same plane;

(iii) (b) $s_{qr}(\equiv 2s_{iijk})(\ q = 1,2,3\ or\ r = 4,5,6)$ relates an extensional strain to a shear stress in a perpendicular plane, and vice versa;

(iv) $s_{qq}(\equiv 4s_{ijij})(q = 4,5,6)$ relates a shear strain to a shear stress both in the same plane, and $s_{qq} = 1 / G_q$.

(v) $s_{qr}(\equiv 4s_{ijik})(q \neq r,\ q,r = 4,5,6)$ relates a shear strain to a shear stress in a perpendicular plane.

The physical significance of the stiffnesses c can be investigated similarly.