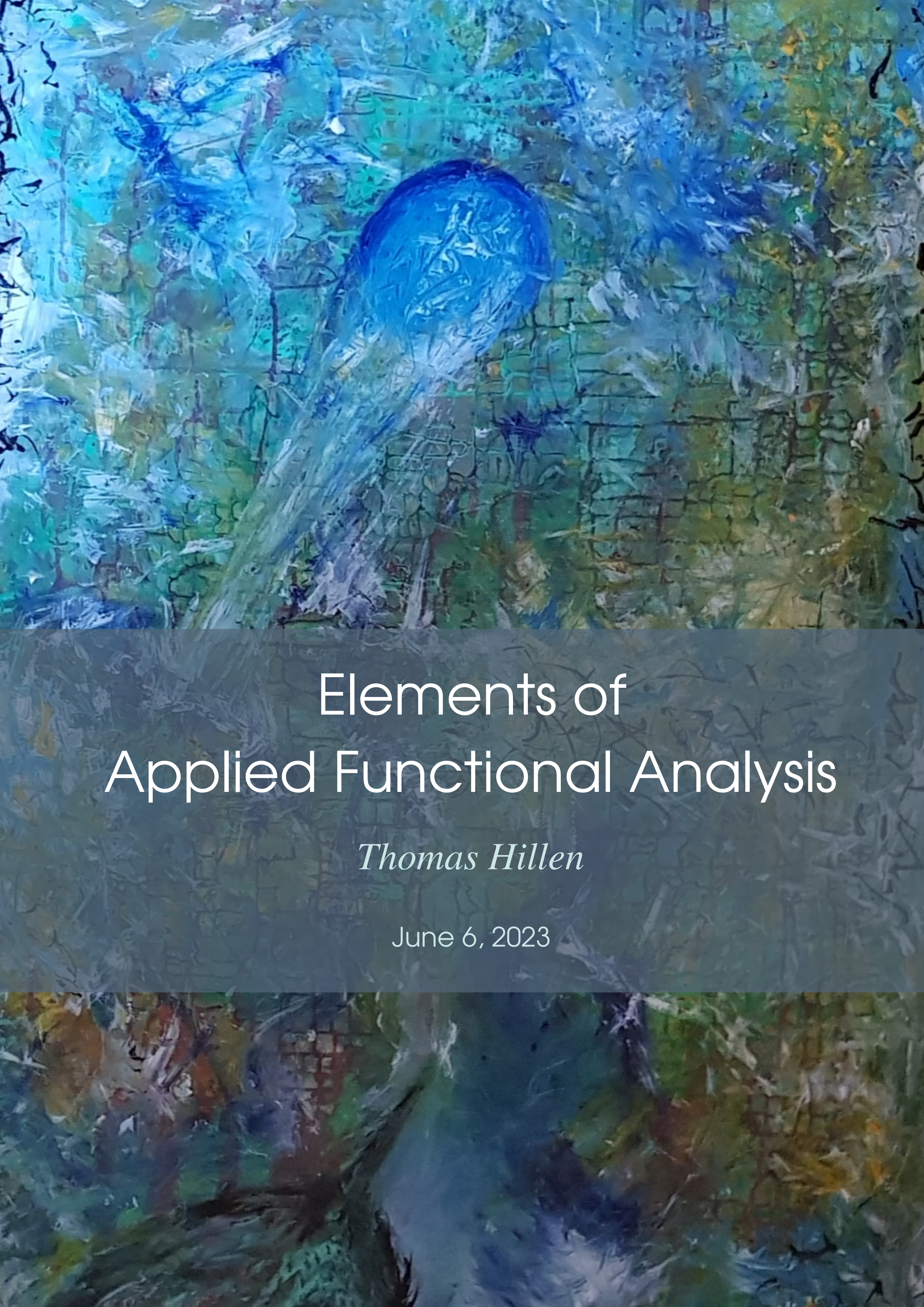

Elements of Applied Functional Analysis

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Preface

Applied Functional Analysis is a branch of mathematics that lies at the intersection of pure mathematics and practical applications to real world problems. It explores the deep an intricate structures of function spaces, often infinite dimensional, and the mappings between them. This text provides a comprehensive introduction to powerful theories such as Banach and Hilbert spaces, linear operators, spectral theory, fixed point methods, variational calculus, and semigroup theory. Each of these topics is developed with full proofs of the main results and many examples for specific applications. I also include several illustrations and pedagogical tools that help learners to better understand the underlying concepts and memorize them easier, such as the *rainbow of function spaces* and the *semigroup triangle*.

The mathematical concepts developed here provide a rigorous framework to model and analyze real world problems from engineering, mechanics, physics, and natural sciences. We will witness how Applied Functional Analysis empowers us to unravel the complexities of the natural world.

I am grateful for support from many of our graduate students, who took this course at the University of Alberta. They found mistakes and asked hard questions, which all helped to improve the presentation. I am very grateful to Pablo Venegas Garcia, who typed up the first version of the linear operator chapter. My thanks also go to Alexandra Shyntar and Ryan Thiessen for careful proof reading of the manuscript. Nevertheless, all remaining errors are entirely my own fault.

> "*Let us now embark on this journey into the realm of Applied Functional Analysis, where the beauty of pure mathematics intertwines with the practicality of real-world applications, offering a powerful toolkit for understanding, analyzing, and transforming our world.*"[1]

Thomas Hillen
Edmonton, May 2023

[1] This last sentence is taken from `chatGPT`, I couldn't say this better.

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1. Introduction

Contents

1.1 Applied Mathematics

Concerned with the mathematical formulation of technical systems, physical problems, and natural phenomena, the field of applied mathematics has undergone drastic changes over the last decades. Traditional applied mathematics was often motivated by theoretical thought experiments. A. Einstein is famous for using thought experiments to formulate plausible theoretical concepts about nature, which can then be formulated as mathematical models and be analysed. Deep mathematical methods were developed for the analysis of these models to gain insight into the engineering, physical, or natural system at hand. A beautiful theory of Applied Functional Analysis developed, which we present here.

Traditionally, data did not play a big role in this process, and data analysis would be left in the capable hands of Statisticians. If quantitative mathematical predictions were needed, applied mathematicians would focus on numerical methods. In fact, numerical analysis and scientific computing became the driving force of applied mathematics for many years. Experimental data would be included, somehow, but a full statistical analysis was not at the centre of interest.

This has changed. The fast development of data collection methods in all areas of sciences in recent years has modified the demands on applied mathematics. Some data sets are so vast that they can be seen as continua of data. Modern applied mathematicians engage in direct collaboration with the sciences, and science requires that their data are included in the modelling from the very beginning. Science expects us to provide an explanation of the available data and possibly make testable predictions. The traditional theoretical approaches are still possible, but applied mathematics has expanded to include inference, statistical learning, data analysis, and AI as applied math tools.

In my view, an applied mathematician in the 21st century needs to gain skills in (i) development of new mathematical models (modelling), (ii) theoretical analysis of mathematical models, (iii) numerical solutions of these models, and (iv) data inferences and statistical learning.

This textbooks focuses on the theoretical aspects of applied mathematics (i.e. item (ii)). This does not mean that the material is old. On the contrary, some parts of this book include quite modern approaches, for example the *rainbow of function spaces*, and the chapter of *semigroup theory*. Other chapters cover very traditional material, such as the chapters on operators and function spaces, and the chapter on variational methods.

1.2 Partial Differential Equations

A main part of applied mathematics are differential equations. Ordinary differential equations (ODEs) include a finite number of differential equations of a single independent variable, and the analysis of those is covered in courses on ODEs, boundary value problems, and dynamical systems. The defining feature of ODEs is the fact that they can be formulated as a finite dimensional systems, often in $\mathbb{R}^n$ for appropriate dimension n.

Partial differential equations (PDEs), however, can be seen as infinite dimensional dynamical systems, a concept that I will make precise in this book. PDEs are used for problems in physics, mechanics, engineering, chemistry, biology and many other disciplines. They are the work-horses of applied mathematics and they form a topic of central importance in our field. Compared to ODEs, PDEs require a new language. The state space of a PDE is a Banach space, the PDE itself can be seen as a combination of operators between Banach spaces, and solutions often arise as weak or weak* limits in those Banach spaces. All these concepts need to be learned, and that is what we do in this book.

As an example, consider a standard reaction-diffusion equation

$$u_t = d\Delta u + f(u)$$

for an unknown function $u(x,t)$ that depends on space x and time t. The constant $d > 0$ denotes the diffusion coefficient, $u_t = \frac{\partial}{\partial t}u(x,t)$ denotes the partial time derivative, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ denotes the *Laplace operator*, and $f(u)$ is a given function that describes growth or decay.

If we introduce a linear operator

$$Au = d\Delta u,$$

we write the reaction diffusion equation as

$$u_t = Au + f(u), \tag{1.1}$$

which now looks like an ODE (ordinary differential equation).

In fact, if it were an ODE, we could use matrix exponentials and the variation of constant formula like

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds, \tag{1.2}$$

where e^{At} is a matrix exponential. But now, $A = d\Delta$ is a differential operator, and not a matrix. The question arises whether we can define a matrix exponential for unbounded operators. The answer is YES, using semigroup theory, which we cover in the final Chapter 9.

If $u(x,t)$ is a solution of the "ODE" (1.1), then for each t, $u(x,t)$ is a function of x. Hence it lies in a function space. But in which function space? We will discuss appropriate choices for a large variety of function spaces, such as Banach spaces, Hilbert spaces, Sobolev spaces, and their dual spaces later in Chapters 2, 4, and 5. In this scenario A becomes a linear operator between those spaces, hence we need to consider operator theory, as we do in Chapter 3.

■ **Example 1.1** As a specific example we consider the one dimensional heat equation on $[0,L]$ with homogeneous Dirichlet boundary conditions.

$$\begin{aligned} u_t &= du_{xx}, & (1.3)\\ u(t,0) &= u(t,L) = 0,\\ u(0,x) &= u_0(x), \end{aligned}$$

with a given initial condition $u_0(x)$. Using separation of constants and the superposition principle [14], we find a solution as Fourier-sine series

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n dt} \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients a_n are the Fourier-sine coefficients of the initial condition

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

The family of functions

$$\mathscr{S} := \left\{\sin\left(\frac{n\pi x}{L}\right), n \in \mathbb{N}\right\}$$

forms an orthogonal set with inner product

$$\langle \phi_n, \phi_m \rangle = \int_0^L \phi_n(x)\phi_m(x)dx.$$

$\mathscr{S}$ can be seen as a basis of the function space $L^2(0,L)$ of square integrable functions on $[0,L]$. Since $\mathscr{S}$ is infinite, we say that $L^2(0,L)$ is infinite dimensional. In this sense we understand (1.3) as an infinite dimensional ODE.

Also note that the Dirichlet boundary conditions $u(t,0) = u(t,L) = 0$ became part of the basis functions $\mathscr{S}$, since the sine functions satisfy these boundary conditions. This suggest that other boundary conditions, such as Neumann boundary conditions, for example, might lead to a different basis functions. In fact, much of PDE theory is concerned about the identification of the "right" basis and the "right" function space. We come back to this question when we talk about domains of definition of unbounded operators in Chapter 3. ■

1.3 Spectrum

If A is a matrix, then the eigenvalues of A tell us something about the stability of the ODE $\dot{u} = Au$. Now, if A is an unbounded operator, (e.g. $A = d\Delta$), what information can we get from the spectrum of A? What actually is the spectrum of A in that case? We will see in the chapter on spectral theory (Chapter 8) that the spectrum of A, denoted as $\sigma(A)$ might contain much more than just eigenvalues.

1.4 Compactness

One of my favorite PhD exam questions is: “What is your favorite compactness result?”

In $\mathbb{R}^n$ we have the theorem of Bolzano-Weierstrass, i.e. if $U \subset \mathbb{R}^n$ is bounded and closed, then it is compact. This, unfortunately, is no longer true in infinite dimensions, since, as we will show that even unit balls in general Banach spaces do not need to be compact.

Another well known compactness result is the theorem of Arzela-Ascoli:

Theorem 1.4.1 — Arzela-Ascoli. Consider a sequence of real-valued continuous functions $\{f_n(x)\}_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a,b]$ of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a uniformly convergent subsequence.

Compactness results are rather essential in functional analysis as they allow us to find limits in function spaces. In the chapters on dual spaces (Chapter 4) and on Sobolev spaces (Chapter 5) we will add a few new compactness results to our menu, such as the Rellich-Kondrachov compactness, weak*-compactness, reflexive weak compactness, and compact Sobolev embeddings.

1.5 Optimization

Another interesting generalization from Calculus in $\mathbb{R}^n$ is the principle of optimization. To optimize a real-valued, twice differentiable function in $\mathbb{R}^n$, we simply look at the gradient ∇f and find its zeroes. Then we study the Hessian matrix $Hess f(x)$ and decide if the critical points are local maxima or minima. The analogy for functions on Banach spaces is called the Calculus of Variations, which we cover in Chapter 7.

1.6 Fixed Point Methods

Now we come back to the reaction-diffusion equation (1.1). A solution is called a *mild* solution, if it satisfies the variation of constant formula (1.2). To show the existence of such a solution, we construct a Picard iteration and use a fixed point argument. Again, the question of the right function space arises. Let us assume X is the Banach space of interest. We take a given function $v \in X$ and we define a nonlinear operator B as

$$B(v) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(v)ds.$$

If we can show that $B : X \to X$, then there is hope to apply a fixed point theorem. If B has a fixed point, $u = B(u)$ in X, then this u is a mild solution of (1.1). We will learn in Chapter 6 a variety of fixed point theorems that apply to this situation.

1.7 Outline

Chapters 2, 3, and 4 contain classical introductory material from functional analysis. We introduce Banach spaces and Hilbert spaces, talk about different norms, and define operators between these spaces. We learn when an operator is bounded, continuous, closed, compact, or symmetric, and we prove the Uniform Boundedness Principle. The introduction of dual spaces, and the important Hahn-Banach theorem, bring additional structure to the function spaces, which allow us to distinguish between weak and weak* convergence. An important consequence is the Alaoglu weak* compactness result, which we prove in Chapter 4. Chapters 2-4 contain essential background that should be studied by readers that are new to this area. These chapters can be skipped by experienced functional analysts.

Chapter 5 on Sobolev spaces makes a distinct jump towards PDEs. General solutions of PDEs often only allow weak derivatives, which we define from a distributional derivative. Sobolev spaces collect functions with weak derivatives into a ranking of increased regularity. The Sobolev embeddings make explicit relations between Sobolev spaces, spaces of integrable functions and spaces of continuous and differentiable functions. We present these relationships in the *Rainbow of function spaces* where we show the relations between smooth functions, Hölder continuous functions, continuous and differentiable functions, integrable functions, measures, Sobolev spaces and their duals. The Rainbow of function spaces is a real highlight of this text, as it gives the reader a visual tool to better understand the relationships between all these spaces.

Chapter 6 on Fixed Point Theorems steers us into very traditional mathematics. Based on topological arguments (e.g. Brouwers fixed point theorem), we develop standard results such as the Banach fixed point theorem and the Schauder fixed point theorem. The Leray-Schauder principle again makes a connection to PDE theory. It shows that *a-priori* estimates can lead to fixed points. Finally, the Lax-Milgram theorem, is not really a fixed point theorem, but it fits into this chapter as it ensures the existence of solutions to bilinear operator equations, as they appear in solution theory of PDEs.

Variational Calculus in Chapter 7, has developed from optimization of mechanical systems, specifically for minimizing the underlying energies. Consequently, we begin Chapter 7 with two classical mechanical problems, the hanging chain and the rolling ball. We find that the language of variational calculus is functional analysis, hence the tools developed so far allow us to formulate optimization problems in a systematic way. We derive the first variation, the Euler-Lagrange equations and the second variation. We discuss Hamiltons principle and we derive conditions such that a minimizer exists.

In Chapter 8 on Spectral Theory we come back to the analysis of operators. Where matrices have eigenvalues, linear operators in general have spectral values which include eigenvalues (point spectrum) plus elements of the continuous and residual spectra. We introduce a new best friend, which is the resolvent $R_\lambda(A)$. The resolvent is needed to identify the different parts of the spectrum. Also, we relate the spectrum of an operator to the spectrum of the adjoint and we formulate some spectral theorems, including the Fredholm alternative.

With Spectral Theory under our belt, we develop in Chapter 9 the Semigroup Theory. Semigroup theory is the framework in which an operator exponential $T(t) = e^{tA}$ will be defined. It can be understood as a solution of an abstract differential equation $u_t = Au$ in a Banach space. The two main theorems in this context are the Hille-Yosida theorem and the Lumer-Phillips theorem, which we prove in Chapter 9. Based on the spectrum of so called "sectorial" operators, we will define analytic semigroups. These are important in many applications since the Laplacian Δ generates analytic semigroups in suitable domains. The semigroup theory is build on intricate connections between the generator A, the semigroup $T(t)$ and the generator $R_\lambda(A)$. To illustrate these relations, we use the *Semigroup Triangle* as a visual tool.

1.8 Recommended Literature

There are excellent textbooks available for each of the topics considered here. As these texts are more specialized they contain much more material than covered here and I recommend those for further reading:

- **Functional analysis:** Robinson [23, 24], Lax [16], Haase [10], Zeidler volume I, II, II [29], Evans [7], Halmos and Sunder [12], Krasnoselsii [15],
- **Sobolev spaces:** Adams [1], Robinson [23], Evans [7], Gilbarg and Trudinger [8], Renardy and Rogers [22]
- **Calculus of Variations:** van Brunt [3], Wan [28], Evans [7],
- **Spectral theory:** Conway [4], Lax [16], Zeidler volume I, II, II [29], Edmunds and Evans [5],
- **Reaction diffusion equations:** Robinson [23], Smoller [26], Taylor [27], Evans [7], Renardy and Rogers [22], Britton [2], Lorenzi and Rhandi [17]
- **Semigroup theory:** Pazy [19], Taylor [27], Engel and Nagel [6], Goldstein [9], Lunardy [18], Lorenzi and Rhandi [17]

2. Basic Functional Analysis

Contents

In this chapter we discuss some basic functional analysis topics such as Banach spaces, Hilbert spaces, their norms and inner products. We will introduce spaces of continuous functions, Hölder continuous functions, and integrable functions. Some essential tools in functional analysis are inequalities, and we will cover the Young, Hölder, and Minkowski inequalities, as well as some integral inequalities and the famous Gronwall Lemma.

2.1 Banach Spaces

A Banach space $(X, ||.||_X)$ is a complete normed space.

■ **Example 2.1** $\mathbb{R}^n$ with each norm is a Banach space. Let $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, then we have many different norms. Some of them are:

$$\begin{aligned}
\text{sum-norm:} \qquad \|x\|_1 &= \sum_{j=1}^{n} |x_j|, \\
\text{Euclidean norm:} \qquad \|x\|_2 &= \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}, \\
\text{p-norm:} \qquad \|x\|_p &= \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}, \qquad p \geq 1, \\
\text{max-norm:} \qquad \|x\|_\infty &= \max_{j=1,\ldots,n} |x_j|.
\end{aligned}$$

■

We list a few basic properties of Banach spaces

1. A subset $Y \subset X$ is *dense* in X, if $\bar{Y} = X$. A Banach space that contains a *dense* and *countable* subset is called *separable*. For example, $\mathbb{R}^n$ is separable, since $\mathbb{Q}^n$ is a dense countable subset.
2. A subset $E \subset X$ is called *compact*, if either (i) each open cover contains a finite subcover, or, equivalently, if (ii) each sequence $\{x_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence.
   For example the closed unit ball in $\mathbb{R}^n$
   $$\bar{B}_1(0) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$$
   is compact by the Theorem of Heine-Borel.
3. Two norms $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ are equivalent, if there exist $a, b > 0$ such that
   $$a\|x\|^{(1)} \leq \|x\|^{(2)} \leq b\|x\|^{(1)}, \qquad \text{for all} \qquad x \in X.$$

2.1.1 Continuous and Differentiable Functions

We use the following notations for spaces of continuous functions. Let $\Omega \subset \mathbb{R}^n$ be a given set. If Ω is bounded, we introduce the domain boundary as $\partial\Omega$, and its closure as $\bar{\Omega}$.

$$\begin{aligned}
C^0(\Omega) &= \{f : \Omega \to \mathbb{R} : \textit{continuous}\}, \\
C^0(\bar{\Omega}) &= \{f : \bar{\Omega} \to \mathbb{R} : \textit{continuous}\}, \\
C^0_b(\Omega) &= \{f \in C^0(\Omega) : \textit{bounded}\}.
\end{aligned}$$

The norm on C^0 is the supremum norm

$$\|u\|_\infty = \sup_{x\in\Omega} |u(x)|.$$

If Ω is bounded, then $C^0_b(\Omega) = C^0(\Omega)$.
By Weierstrass' Theorem we have that

$$\{\text{Polyn. with rational coefficients}\} \quad \underset{\text{dense}}{\subset} \quad \{\text{Polyn.}\} \quad \underset{\text{dense}}{\subset} \quad C^0(\bar{\Omega}),$$

hence $C^0(\bar{\Omega})$ is a separable Banach space.

For spatial derivatives we use a number of notations:

$$\frac{\partial}{\partial x_i} = D_i = \partial_i,$$

as is common practice in analysis.

For a *multiindex* $\alpha = (\alpha_1, \ldots, \alpha_n)$ we denote a combined derivative as

$$D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} f.$$

The *order* of this derivative is written as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Then the spaces of higher derivatives for $r \in \mathbb{N}$ are written as

$$C^r(\Omega) = \{f : \Omega \to \mathbb{R} : D^\alpha f \in C^0(\Omega), \quad \text{for all} \quad |\alpha| \le r\}$$

with norm

$$\|f\|_{C^r} = \sum_{|\alpha| \le r} \sup_{x \in \Omega} |D^\alpha f(x)|.$$

As $r \to \infty$ we then define *smooth functions* as

$$C^\infty(\Omega) = \bigcap_{r=1}^{\infty} C^r(\bar{\Omega}).$$

The support of a function f is defined as

$$\mathrm{supp} f = \overline{\{x : f(x) \neq 0\}},$$

and continuous functions with compact support are denoted as $C_c^r(\Omega)$ and $C_c^\infty(\Omega)$, where we use the double inclusion symbol to indicate compactness:

$$\mathrm{supp} f \Subset \Omega.$$

Note that $C_c^\infty(\Omega)$ is generally not a Banach space. The spaces of *Hölder continuous* functions, for $0 \le \gamma \le 1$ are defined as

$$C^{r,\gamma}(\bar{\Omega}) = \{f \in C^r(\bar{\Omega}) : \exists c > 0, |D^\alpha f(x) - D^\alpha f(y)| \le c|x-y|^\gamma \quad \text{for all} \quad x, y, \in X, \, |\alpha| = r\}$$

with *Hölder norm*

$$\|f\|_{r,\gamma} = \|f\|_{C^r} + \sup_{|\alpha|=r,\, x,y,\in\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\gamma}.$$

If Ω is bounded, we have the inclusions

$$C^{r+1}(\bar{\Omega}) \subset C^{r,\gamma}(\bar{\Omega}) \subset C^r(\bar{\Omega}), \qquad 0 < \gamma \le 1.$$

2.1.2 Integrable Functions

For $1 \leq p < \infty$ we define spaces of integrable functions as

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \|f\|_p := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

and the spaces of locally integrable functions

$$L^p_{loc}(\Omega) := \{ f : f \in L^p(K) \quad \text{for every} \quad K \Subset \Omega \}.$$

For example, consider the function $f(x) = 1$ for $x \in \mathbb{R}$. We have $\int_\mathbb{R} f(x)dx$ is unbounded, hence $f \notin L^1(\mathbb{R})$, but for each compact subset $K \Subset \mathbb{R}$ we have

$$\int_K f(x)dx = |K| < \infty,$$

hence $f \in L^1_{loc}(\mathbb{R})$.

2.1.3 Mollifiers

Mollifiers are approximations by smooth functions.

Theorem 2.1.1 Given $f \in C^0_c(\Omega)$. For each $\varepsilon > 0$ there exists a $\phi \in C^\infty_c(\Omega)$ such that

$$\|f - \phi\|_\infty \leq \varepsilon.$$

Proof. By direct construction. We define the *standard mollifier* as

$$\rho(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right) & , \quad |x| \leq 1, \\ 0 & , \quad |x| > 1, \end{cases}$$

where c is chosen such that

$$\int_{\mathbb{R}^n} \rho(x)dx = 1.$$

As the exponential function is smooth, and converges to 0 as $|x| \to 1$, it follows that $\rho(x) \in C^\infty_c(\mathbb{R}^n)$. We define the *standard mollifier of size* $h > 0$ as

$$\rho_h(x) = \frac{1}{h^n} \rho\left(\frac{x}{h}\right),$$

where we have

$$\int_{\mathbb{R}^n} \rho_h(x)dx = 1. \tag{2.1}$$

The standard mollifier and the rescaled version is shown in Figure 2.1. Given $f \in C^0_c(\Omega)$ then the *mollification* f_h of f for $h < \operatorname{dist}(\operatorname{supp} f, \partial\Omega)$ is

$$f_h(x) = f * \rho_h(x) = \frac{1}{h^n} \int_\Omega \rho\left(\frac{x-y}{h}\right) f(y)dy.$$

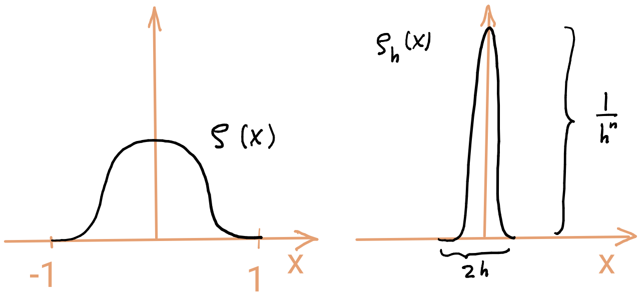


Figure 2.1: Sketch of the standard mollifier and its rescaled version.

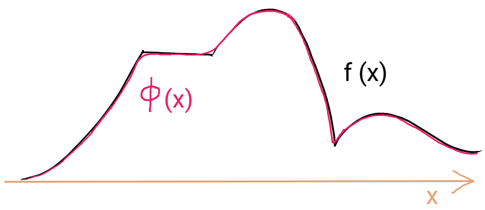


Figure 2.2: Sketch of an approximation of f by a smooth function ϕ.

Since $\rho \in C^\infty$, we immediately have that

$$f_h(x) = \frac{1}{h^n}\int_\Omega \rho\left(\frac{x-y}{h}\right) f(y)dy \in C^\infty(\Omega).$$

Furthermore, if f has compact support, so does f_h. For the convergence of $f_h \to f$ we use the fact that ρ_h integrates to one, (2.1), and write

$$\begin{aligned} |f_h(x) - f(x)| &= \left|\frac{1}{h^n}\int_\Omega \rho\left(\frac{x-y}{h}\right)(f(y)-f(x))dy\right| \\ &\le \sup_{|x-y|\le h} |f(y)-f(x)| \left|\frac{1}{h^n}\int \rho\left(\frac{x-y}{h}\right)dy\right|. \end{aligned}$$

Since f is continuous, we can make h small such that $|f(y)-f(x)| < \varepsilon$ for all $|x-y| < h$, and then let h go to zero. ■

2.2 Inequalities

Here we collect a number of inequalities which we will use in later chapters. The first three are cited from measure theory without proof. They give us conditions such that taking limits and integration can be interchanged.

2.2.1 Integral Inequalities

Theorem 2.2.1 — Monotone Convergence. Consider a sequence of measurable functions $0 \leq f_1 \leq f_2 \leq \dots$ a.e. $x \in \Omega$. Then

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

Theorem 2.2.2 — Fatou's Lemma. Assume $f_j \geq 0$ are measurable functions, then

$$\int_\Omega \left(\liminf_{n\to\infty} f_n(x)\right) dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx.$$

Theorem 2.2.3 — Dominated Convergence. Assume $f_j(x) \to f(x)$ as $j \to \infty$ for all $x \in \Omega$, and $|f_j(x)| \leq g(x)$ with $g \in L^1(\Omega)$. Then

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

Proof. The proofs of these Theorems can be found in standard textbooks on measure theory, or Lebesgue integration [11, 12]. See also [23] p 22 Theorem 1.7. ■

2.2.2 Young, Hölder, Minkowski

The Young, Hölder and Minkowski inequalities belong to the standard toolbox of each mathematician who works in analysis. You should never leave your home without these!

Theorem 2.2.4 — Young's inequality. Consider $a, b \geq 0, p, q > 1$ and p and q are conjugate $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

If $\varepsilon > 0$ we write

$$ab \leq \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q.$$

The most common version of Young's inequality is the version with $p = q = 2$:

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2. \tag{2.2}$$

This last inequality is also known as "Peter-Paul" inequality. You take money from Peter and pay Paul, i.e. you make the a-term small through a small ε, but you have to pay the price by making the b-term large with ε^{-1}.

Proof. We rewrite the inequality by multiplying with b^{-q}. Then $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$ is equivalent with

$$\frac{a^p b^{-q}}{p} + \frac{1}{q} - ab^{1-q} \geq 0.$$

Since p and q are conjugate indices, we have $1-q=-\frac{q}{p}$ and we write the above inequality as

$$\frac{\left(ab^{-\frac{q}{p}}\right)^p}{p}+\frac{1}{q}-ab^{-\frac{q}{p}}\geq 0.$$

With the function

$$f(t)=\frac{t^p}{p}+\frac{1}{q}-t$$

the left hand side is exactly $f(ab^{\frac{-q}{p}})$. Now it is easy to show that $f(t)$ has a global minimum of 0 at $f(1)=0$. Indeed, $f'(t)=t^{p-1}-1$ and $t=1$ is a critical point. As we have $f''(1)=p-1>0$ we find a global minimum at 1.

Finally, to obtain the Peter-Paul estimate (2.2) we write $ab=\sqrt{\varepsilon}a\,\frac{b}{\sqrt{\varepsilon}}$ and use Youngs inequality with $p=q=2$. ■

Theorem 2.2.5 — Hölder inequality. Let (p,q) be conjugate indices with $1<p\leq\infty$ and suppose $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$. Then $fg\in L^1(\Omega)$ and

$$\|fg\|_1\leq\|f\|_p\|g\|_q.$$

Proof. We first assume $1<p<\infty$ and use Youngs inequality

$$\begin{aligned}\int_\Omega\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}dx &\leq \int_\Omega\frac{1}{p}\frac{|f|^p}{\|f\|_p^p}+\frac{1}{q}\frac{|g|^q}{\|g\|_q^q}\,dx\\ &= \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

In the case of $p=\infty$ we simply estimate with the supremum of f:

$$\int|fg|dx\leq\|f\|_\infty\int_\Omega|g|dx=\|f\|_\infty\|g\|_1.$$

Hence in both cases $\int|fg|dx$ is bounded and $fg\in L^1(\Omega)$. ■

Theorem 2.2.6 — Minkowski inequality. Assume $1\leq p<\infty$ and consider $f,g\in L^p(\Omega)$. Then $f+g\in L^p(\Omega)$ and

$$\|f+g\|_p\leq\|f\|_p+\|g\|_p.$$

Proof. Notice that for a constant c large enough we have

$$|f(x)+g(x)|^p\leq(|f(x)|+|g(x)|)^p\leq c(|f(x)|^p+|g(x)|^p),$$

which implies $f+g\in L^p(\Omega)$.

If (p,q) are conjugate, then $(p-1)q=p$ and $|f+g|^{p-1}\in L^q(\Omega)$. Now

$$\begin{aligned}|f+g|^p &= |f+g|^{p-1}|f+g|\\ &\leq |f+g|^{p-1}(|f|+|g|)\\ &= |f+g|^{p-1}|f|+|f+g|^{p-1}|g|.\end{aligned}$$

Then, using Hölders inequality on each of these terms, we find

$$\begin{aligned}\|f+g\|_p^p &\leq \||f+g|^{p-1}\|_q(\|f\|_p+\|g\|_p) \\ &= \left(\int_\Omega |f+g|^p dx\right)^{\frac{1}{q}}(\|f\|_p+\|g\|_p) \\ &= \|f+g\|_p^{\frac{p}{q}}(\|f\|_p+\|g\|_p),\end{aligned}$$

leading to

$$\|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p+\|g\|_p.$$

The exponent of the last term on the right hand side is

$$p-\frac{p}{q} = p\left(1-\frac{1}{q}\right) = \frac{p}{p} = 1.$$

■

Theorem 2.2.7 $L^p(\Omega)$ for $1 \leq p < \infty$ is a Banach space.

Proof. To show completeness we assume $\{f_n\} \subset L^p(\Omega)$ converges pointwise $f_n(x) \to f(x)$ a.e. $x \in \Omega$ and that $\|f_n\|_p < C$ for all n. Then by Fatou's Lemma (Theorem 2.2.2) we have

$$\begin{aligned}\|f\|_p^p &= \int \liminf_{n\to\infty} |f_n(x)|^p dx \leq \liminf_{n\to\infty} \int |f_n|^p dx \\ &\leq \lim_{n\to\infty} \int |f_n|^p dx \leq C^p.\end{aligned}$$

■

Theorem 2.2.8 Let Ω be bounded, then $C^0(\Omega)$ is dense in $L^p(\Omega)$,

$$L^p(\Omega) = \overline{C_c^0(\Omega)}^{\|\cdot\|_p},$$

and $L^p(\Omega)$ is separable.

Proof. (Sketch of proof): As indicated in Figure 2.3 we approximate L^p functions with simple functions of the form

$$S_n = \sum_{j=1}^n c_j \left(\chi_{I_j}(x)\right)_h,$$

where χ_{I_j} are indicator functions of sub-intervals I_j, and the index h denotes their mollification. If we choose rational coefficients, we also conclude that L^p is separable. ■

With the same argument we show that

$$L^p(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_p}.$$

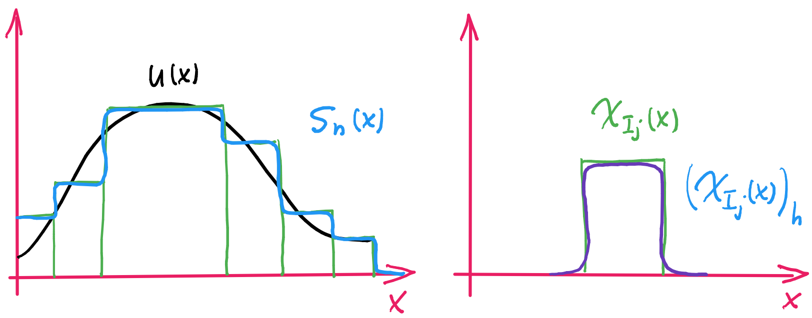


Figure 2.3: Sketch of an approximation of u with mollified indicator functions.

Lemma 2.2.1 Let Ω be bounded and $p > 1$. Then $f \in L^p(\Omega)$ implies $f \in L^{p'}(\Omega)$ for each $1 < p' < p$.

Proof. We use Hölder's inequality with index p/p' and q such that

$$\frac{1}{p/p'} + \frac{1}{q} = 1.$$

Hence

$$q = \frac{p}{p - p'}.$$

Then with Hölders inequality we get

$$\begin{aligned}
\int_\Omega |f(x)|^{p'} dx &\leq \| |f|^{p'} \|_{\frac{p}{p'}} \|1\|_q \\
&= |\Omega| \left(\int_\Omega \left(|f|^{p'} \right)^{\frac{p}{p'}} dx \right)^{\frac{p'}{p}} \\
&= |\Omega| \|f\|_p^{p'}.
\end{aligned}$$

■

2.3 L^∞–spaces

The *essential supremum* of a function $f(x)$ is defined as

$$\|f(x)\|_\infty := \text{ess sup}_\Omega |f(x)| = \inf\left\{ \sup_{x \in S} |f(x)| : S \subset \bar{\Omega}, \text{ and } \Omega \backslash S \text{ has measure zero } \right\}.$$

■ **Example 2.2** The function

$$f(x) = \begin{cases} 2 & \text{for } x = 0 \\ 1 & \text{else} \end{cases}$$

has

$$\text{ess sup}_{[-1,1]}|f(x)| = 1 \neq \sup_{[-1,1]} |f(x)| = 2.$$

■

■ **Example 2.3** There is a natural connection between L^∞ and C^0 in case when functions are continuous. If $f \in C^0(\Omega)$ then

$$\text{ess sup}_\Omega |f(x)| = \sup_\Omega |f(x)|.$$

In this sense we have

$$\|f\|_{L^\infty} = \|f\|_{C^0} = \|f\|_\infty.$$

■

Theorem 2.3.1 Let Ω have finite volume, then

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$

and if $\|f\|_p \leq K$ for all p, then $\|f\|_\infty \leq K$.

Proof. We simply compute

$$\|f\|_p = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} \leq \|f\|_\infty \left(\int_\Omega 1^p dx\right)^{\frac{1}{p}} = |\Omega|^{\frac{1}{p}} \|f\|_\infty,$$

which implies that

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

Moreover, for each $\varepsilon > 0$ there exists a set A of non-zero measure such that

$$|f(x)| \geq \|f\|_\infty - \varepsilon, \qquad \text{for all} \quad x \in A.$$

Therefore

$$\int_\Omega |f(x)|^p dx \geq \int_A |f(x)|^p dx \geq |A| \left(\|f\|_\infty - \varepsilon\right)^p,$$

which implies

$$\|f\|_p \geq |A|^{\frac{1}{p}} \left(\|f\|_\infty - \varepsilon\right).$$

We obtain in the limit that

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty,$$

proving our claim. ■

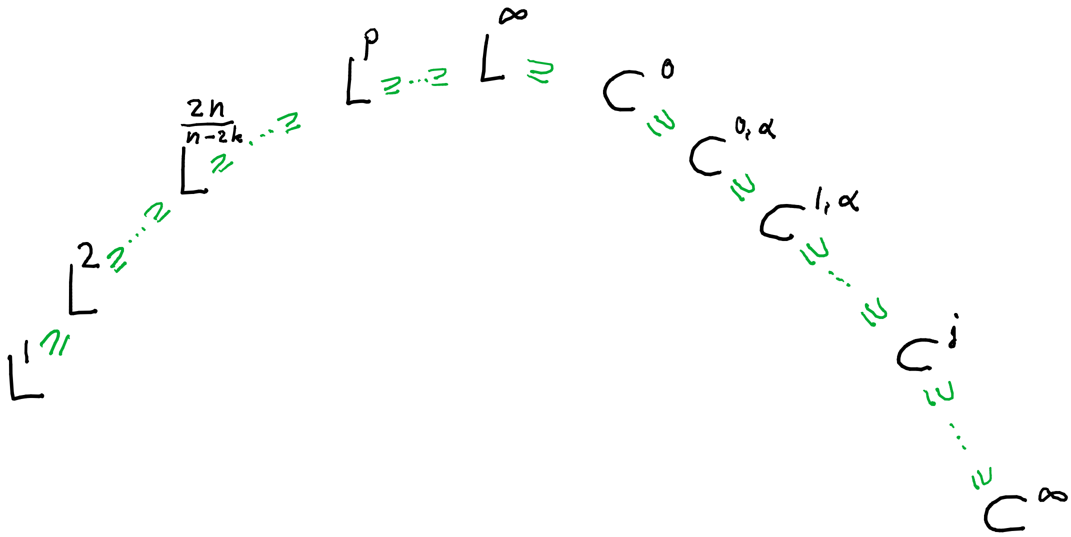


Figure 2.4: A part of the *Rainbow of Function Spaces* for L^p and C^j spaces. Ω is a bounded domain, and the argument (Ω) is suppressed to reduce cluttering the image.

Theorem 2.3.2 $L^\infty(\Omega)$ is complete, i.e. a Banach space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$. Then

$$|f_n(x)| \leq \|f_n\|_\infty \qquad \text{a.e.}$$

and

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \qquad \text{a.e.}$$

Hence for a.e. $x \in \Omega$ the set $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, which has a limit

$$f_n(x) \to f(x) \qquad \text{a.e. in} \quad \Omega.$$

Hence we find a well defined function $f(x)$, which based on the above estimates, is in $L^\infty(\Omega)$. ∎

In Figure 2.4 we show the relationships of some of the function spaces that we have discussed so far. Note that for these inclusions we assume Ω to be bounded. This *Rainbow of Function Spaces* will be extended with dual spaces and Sobolev spaces as we carry on through this course. See Figure 5.3 for the full Rainbow of Function Spaces.

2.4 Differential Inequalities

The arguably most important tool in PDE analysis is the famous *Gronwall's lemma*, which allows to integrate a differential inequality. Here we derive a rather general differential inequality result, which then as a Corollary, includes the Gronwall Lemma and several of its cousin results.

We define a one-sided derivative for a real function $x:\mathbb{R}\to\mathbb{R}$

$$\frac{d}{dt^+}x(t) = \lim_{h\to 0^+}\frac{x(t+h)-x(t)}{h}.$$

Of course, if $x\in C^1$ then $\frac{d}{dt^+}x(t) = \frac{d}{dt}x(t)$.

Theorem 2.4.1 Assume $f(x,t)$ is Lipschitz continuous and consider the differential inequality for a real function $x:\mathbb{R}\to\mathbb{R}$

$$\frac{d}{dt^+}x(t) \le f(x,t), \qquad x(0)=x_0, \tag{2.3}$$

and the corresponding differential equation

$$\frac{d}{dt}y(t) = f(y,t), \qquad y(0)=y_0.$$

If initially $x_0\le y_0$, then

$$x(t)\le y(t)$$

for as long as the solution $y(t)$ exists.

Proof. Let $\{y_n(t)\}_{n\in\mathbb{N}}$ solve the modified equation

$$\frac{d}{dt}y_n(t) = f(y_n,t)+\frac{1}{n}, \qquad y_n(0)=y_0.$$

The solution exists and is unique, since F is Lipschitz. We claim that for any given $n>0$ we have $x(t)\le y_n(t)$.

Assume this is not the case. Then there exists a time t_1 where x is bigger: $x(t_1)>y_n(t_1)$. This means there must be an interval, containing t_1 such that $x(t_1)$ is bigger, i.e. we find a $t_2\in[0,t_1)$ such that $x(t_2)=y_n(t_2)$ and $x(t)>y(t)$ for $t\in(t_2,t_1]$. Then the one sided derivative of $x(t)$ at t_2 satisfies for h small enough

$$\frac{x(t_2+h)-x(t_2)}{h} > \frac{y(t_2+h)-y(t_2)}{h}$$

and

$$\frac{d}{dt^+}x(t_2) \ge f(y_n(t_2),t_2)+\frac{1}{n} = f(x(t_2),t_2)+\frac{1}{n},$$

which is a contradiction to the differential inequality (2.3). This means that $x(t)\le y_n(t)$ for all t and all $n\in\mathbb{N}_+$. By the result on the continuous dependence of the solution on model parameters, we pass to the limit for $n\to\infty$ and get $\lim_{n\to\infty}y_n(t)=y(t)$. Hence we keep the inequality:

$$x(t)\le y(t)$$

for all t such that $y(t)$ exists. ■

The application of this result to linear differential inequalities is Gronwalls Lemma.

Corollary 2.4.2 — Gronwall's Lemma. Consider a real function $x : \mathbb{R} \to \mathbb{R}$ and two point-wise defined and integrable functions $g(t), h(t)$ and assume

$$\frac{d}{dt^+} x(t) \leq g(t)x(t) + h(t).$$

Then

$$x(t) \leq x(0) \exp\left(\int_0^t g(s)ds\right) + \int_0^t \exp\left(\int_s^t g(\tau)d\tau\right) h(s)ds. \tag{2.4}$$

Corollary 2.4.3 — Special Case. If

$$\frac{d}{dt^+} x(t) \leq ax(t) + b, \qquad x(0) = x_0,$$

then

$$x(t) \leq \left(x_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}.$$

Proof. (of Corollary 2.4.2). The right hand side of (2.4) is the unique solution of

$$\dot{y}(t) = g(t)y(t) + h(t), \qquad y(0) = x(0).$$

Hence we apply Theorem 2.4.1. ■

2.5 Hilbert Spaces

Besides Banach spaces, Hilbert spaces are the work horses for Applied Functional Analysis. These are complete normed vector spaces with an inner product.

Definition 2.5.1 An *inner product* $(\cdot,\cdot) : X \times X \to \mathbb{R}$ on a vector space X satisfies

$$\begin{array}{ll} (i) & (\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z) \quad \text{for all} \quad x,y,z \in X, \quad \lambda,\mu \in \mathbb{R}, \\ (ii) & (y,x) = (x,y), \quad \text{for all} \quad x,y \in X, \\ (iii) & (x,x) \geq 0, \quad \text{for all} \quad x \in X \quad \text{and} \quad (x,x) = 0 \quad \text{only for} \quad x = 0. \end{array}$$

The associated norm is

$$\|x\| = \sqrt{(x,x)}.$$

Proposition 2.5.1 — Cauchy-Schwartz Inequality. For all $x, y \in X$ we have

$$|(x,y)| \leq \|x\| \, \|y\|.$$

Proof. If either x or y equals zero then the inequality is trivial. Hence we consider non-zero $x, y \in X$ and a $\lambda \in \mathbb{R}, \lambda \neq 0$. We have

$$0 \leq \|x - \lambda y\|^2 = (x - \lambda y, x - \lambda y) = \|x\|^2 - 2\lambda(x,y) + \lambda^2\|y\|^2,$$

which implies for $\lambda > 0$ that

$$|(x,y)| \leq \frac{1}{2\lambda}\|x\|^2 + \frac{\lambda}{2}\|y\|^2.$$

Now we chose a very special $\lambda = \frac{\|x\|}{\|y\|}$ to obtain

$$|(x,y)| \leq \frac{1}{2}\|x\|\|y\| + \frac{1}{2}\|x\|\|y\|.$$

■

Definition 2.5.2 A complete inner product space is called a *Hilbert Space*.

Example 2.4 The function space $L^2(\Omega)$ is a Hilbert space with inner product

$$(x,y) = \int_\Omega x(t)y(t)dt,$$

or, if we consider complex valued functions

$$(x,y) = \int_\Omega \bar{x}(t)y(t)dt.$$

When working in L^2 we will often use the single line for the L^2-norm:

$$|f| = \|f\|_2 = \|f\|_{L^2} = \sqrt{(f,f)}.$$

■

If $M \subset H$ is a subset, then we define the *orthogonal complement*

$$M^\perp := \{u \in H : (u,v) = 0, \quad \text{for all} \quad v \in M\}.$$

Proposition 2.5.2 If $M \subset H$ is a closed linear subspace, then each $x \in H$ has a unique decomposition as

$$x = u + v, \qquad u \in M, \quad v \in M^\perp.$$

Proof. Suppose $x \in H$. If $x \in M$ then we chose $u = x$ and $v = 0$ and we are done. So we assume $x \notin M$. Then, since M is closed, we can find a point $u \in M$ of minimal distance to x (see Figure 2.5):

$$\|x-u\| = \delta := \inf_{y\in M}\|x-y\|.$$

Then we define $v = x - u$. Now we want an explicit parametrization of this minimum. For that we take a $y \in M$ and consider

$$D(t) := \|x-(u-ty)\|^2 = \|v+ty\|^2 = \|v\|^2 + 2t(v,y) + t^2\|y\|^2.$$

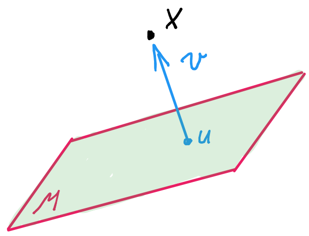


Figure 2.5: Sketch of the shortest distance from x to a point $u \in M$.

We know that $D(t)$ is minimal for $t = 0$, hence

$$D'(t) = 2(v,y) = 0, \qquad \text{for all} \qquad y \in M.$$

This means $v \in M^{\perp}$.

To show uniqueness of this decomposition, we assume that there are two decompositions

$$x = u_1 + v_1 = u_2 + v_2, \qquad u_1, u_2 \in M, \quad v_1, v_2 \in M^{\perp}.$$

Then $v_1 - v_2 = u_2 - u_1$ and

$$\|v_1 - v_2\|^2 = (v_1 - v_2, v_1 - v_2) = (v_1 - v_2, u_2 - u_1) = 0.$$

■

The above proposition allows us to define *orthogonal projections* of x onto M as

$$P_M : H \to M, \quad P_M x = P_M(u+v) = u,$$

where $x = u + v$ is the unique decomposition of Proposition 2.5.2. The projection has the property

$$P_M^2 = P_M, \qquad \text{and} \qquad \|P_M\|_{op} \leq 1,$$

where we will introduce the operator norm in the next chapter.

We can take this orthogonal decomposition to the extreme, by decomposing with respect to one-dimensional subspaces. Doing this we quite naturally arrive at a Hilbert basis.

Definition 2.5.3 An orthogonal set $\{e_i\}_{i\in\mathbb{N}}$ is called a *Hilbert basis*, if

$$x = \sum_{j=1}^{\infty} (x, e_j) e_j, \qquad \text{for all} \quad x \in H.$$

Proposition 2.5.3 An orthonormal set $\{e_i\}_{i\in\mathbb{N}}$ is a Hilbert basis if and only if

$$\|x\|^2 = \sum_{j=1}^{\infty} (x, e_j)^2, \qquad \text{for all} \quad x \in H. \tag{2.5}$$

Proof. We prove this equivalence in two parts:

"$\Longrightarrow$:" Let $\{e_i\}$ be a Hilbert basis. Then for each finite n we have

$$\left\| \sum_{j=1}^{n} (x,e_j)e_j \right\|^2 = \sum_{j=1}^{n} (x,e_j)^2$$

and the claim follows for $n \to \infty$.
"$\Longleftarrow$:" Assume (2.5). We define a set

$$Y := \left\{ x : x = \sum_{j=1}^{\infty} (x,e_j)e_j \right\}$$

and we aim to show that $Y = H$. For that we show that Y is closed in H and also that Y is dense in H. Together this implies $Y = H$.
Claim: Y is closed. Consider a Cauchy sequence $\{y_n\} \subset Y$. Each member of this sequence can be identified with its coefficients, which are elements of the sequence space l^2:

$$y_n \sim ((y_n,e_1),(y_n,e_2),\ldots,(y_n,e_n),\ldots) \subset l^2.$$

Since l^2 is complete, the sequence has a limit in l^2, lets call it

$$y^* \sim (a_1,a_2,\ldots,a_n,\ldots).$$

Then

$$y^* = \sum_{j=1}^{\infty} a_j e_j \qquad \text{and} \qquad a_j = (y^*,e_j),$$

i.e. $y^* \in Y$. Hence Y is closed.
Claim: Y is dense. Assume it is not dense. Then $H \backslash Y$ contains a non-zero vector and since Y is closed, this means that $Y^\perp$ contains a non-trivial element $x \in Y^\perp$. But by definition, x is perpendicular to all basis vectors, i.e.

$$\|x\|^2 = \sum_i (x,e_i)^2 = 0.$$

Then $Y^\perp = \{0\}$, which is a contradiction to the assumption that Y is not dense. Hence Y must be dense. ■

Proposition 2.5.4 *H is separable if and only if it has a countable basis.*

Proof. Again, we consider both directions.
"$\Longleftarrow$:" If we have a countable basis, we construct a countable dense subset by linear combinations of basis functions with rational coefficients.
"$\Longrightarrow$:" If H is separable, then it has a countable dense subset $\{x_n\}_{n\in\mathbb{N}}$, where we assume that $0 \notin \{x_n\}$. We turn this into a countable basis as follows.

$$\begin{aligned} e_1 &= \frac{x_1}{\|x_1\|}, \\ e_n &= \frac{y_n}{\|y_n\|}, \qquad y_n = x_n - \sum_{i=1}^{n-1} (x_n,e_i)e_i. \end{aligned}$$

■

Proposition 2.5.5 The unit ball in an infinite dimensional Hilbert space is not compact.

Proof. Let $\{e_i\}$ be an orthonormal basis of the Hilbert space. Then each $e_i \in \overline{B}_1(0)$ for all $i \in \mathbb{N}$. But

$$||e_n - e_m||^2 = (e_n - e_m, e_n - e_m) = ||e_n||^2 + ||e_m||^2 = 2$$

for all $n \neq m$. Hence, $\{e_j\}$ is a bounded sequence, but not a Cauchy sequence, and it does not converge in $\overline{B}_1(0)$. ■

2.6 Exercises

As a guideline about the difficulty of these exercises, I add a level to the problems. Level 1 indicates elementary and straight forward questions, level 2 are of intermetiate difficulty and level 3 problems are hard.

Exercise 2.1 (Equivalent norms) (level 1)
Let $||.||_p$ denote the p-norm in $\mathbb{R}^2$.

1. Show that $||.||_1, ||.||_2, ||.||_\infty$ are equivalent norms.
2. Let B_p denote the closed unit ball in the norm $p = 1, 2, \infty$. Show that

$$B_1 \subsetneq B_2 \subsetneq B_\infty.$$

3. Plot the unit balls B_1, B_2, B_∞ .

■

Exercise 2.2 (Spectral theorem for matrices) (level 1)
Prove the following theorem for a real matrix A:
(a) If μ is an eigenvalue of a real matrix A, then $\lambda = e^\mu$ is an eigenvalue of e^A.
(b) $Re\mu < 0$ if and only if $|\lambda| < 1$. ■

Exercise 2.3 (Example of convergence) (level 2)
Consider the family of functions $\{f_n\}_{n\in\mathbb{N}}$ that are piecewise defined as

$$f_n(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}\left(t - \frac{1}{2}\right) & \frac{1}{2} - \frac{1}{n} < t \leq \frac{1}{2} + \frac{1}{n}, \\ 1 & \frac{1}{2} + \frac{1}{n} < t \leq 1. \end{cases}$$

Plot $f_n(t)$ for general n, and show that this family $\{f_n\}_n$ converges in $L^2([0,1])$ but not in $L^\infty([0,1])$. ■

Exercise 2.4 (All those functions) (level 2)

1. Find a function $f \in C_c^\infty(\mathbb{R})$ with supp$f \subset [a,b]$, where $a < b \in \mathbb{R}$.
2. Let $\Omega \subset \mathbb{R}^n$ be bounded. Show that if $f \in L^2(\Omega)$, then it follows that $f \in L^1(\Omega)$.
3. If Ω is unbounded the above statement is not true. Show that $\rho(x) = \frac{1}{1+x}$ is contained in $L^2([0,\infty))$ but not in $L^1([0,\infty))$.

4. Show that $\rho(x) = e^{-x}x^{-\frac{2}{3}}$ is contained in $L^1([0,\infty))$ but not in $L^2([0,\infty))$.
5. Find a value $\gamma^* \in [0,1]$ such that the function $f(x) = x^{\frac{3}{2}}$ is element of the Hölder space $C^{1,\gamma}([0,1])$ for $\gamma \leq \gamma^*$ and $f(x)$ is not contained in $C^{1,\gamma}([0,1])$ for $\gamma > \gamma^*$.

■

Exercise 2.5 (Mollifier) (level 1)

1. The mollification of a function can be written in two ways. Show that

$$\frac{1}{h^n}\int_{\mathbb{R}^n} \rho\left(\frac{x-z}{h}\right) u(z)dz = \frac{1}{h^n}\int_{\mathbb{R}^n} \rho\left(\frac{z}{h}\right) u(x-z)dz.$$

2. Assume a Lipschitz continuous function $u \in C^{0,1}(\mathbb{R}^n)$ is uniformly Lipschitz continuous with constant K:

$$|u(x) - u(y)| \leq K|x-y|.$$

Show that each mollification $u_h = \rho_h * u$ is uniformly Lipschitz continuous with the same constant K.

■

Exercise 2.6 (Fourier Transform) (level 1)

The Fourier transform

$$\hat{f}(\omega) = C\int_{-\infty}^{\infty} e^{i\omega x} f(x)dx$$

is used by different authors with all kind of constants, like $C = 1, C = (2\pi)^{-1}, C = \sqrt{(2\pi)}^{-1}$ etc; very much confusing beginning students. The Fourier transform can be seen as a map of $L^2(-\infty,\infty)$ into itself, where $L^2(-\infty,\infty)$ is a complex Hilbert space, with inner product $\langle f,g\rangle = \int f(x)\bar{g}(x)dx$. For which constant C is the Fourier transform an isometry? Justify your answer.

Hint: You might use the identity

$$\int_{-\infty}^{\infty} e^{i\omega x} d\omega = 2\pi\delta_0(x).$$

■

Exercise 2.7 (Interpolation Inequality) (level 2)

Use Hölder's inequality to show the *interpolation inequality*: Assume $1 \leq p \leq q \leq r < \infty$ and consider $\lambda \in (0,1)$ such that $\frac{1}{q} = \lambda\frac{1}{p} + (1-\lambda)\frac{1}{r}$. Show

$$\|u\|_{L^q} \leq \|u\|_{L^p}^{\lambda}\|u\|_{L^r}^{(1-\lambda)}.$$

■

3. Linear Operators

Contents

3.1 Introduction

Operators, in particular linear operators, are the key focus of functional analysis. Operators connect function spaces with each other, describe properties of physical systems, describe cost functions, and solve partial differential equations (PDEs). This text is mostly concerned with linear operators on Banach spaces with one exception in the chapter on Fixed-Point theorems (Chapter 6), where we also consider fixed points of non-linear operators.

During this chapter[1], and then continuing through the entire text, we will see four

[1]I am very grateful to *Pablo Venegas Garcia* for his help in writing this chapter.

types of operators as standard examples: (i) a matrix A on $\mathbb{R}^n$, which is used to relate the abstract theory back to what we know from linear algebra; (ii) the Laplacian operator Δ, is of central importance for the analysis of PDEs, and it is and unbounded linear operator with its own challenges; (iii) an integral operator, which is compact. Compactness will, of course, make many arguments much easier. Finally, (iv) the first derivative $\frac{\partial}{\partial x}$ arises as an unbounded generator of the shift semigroup (see Chapter 9).

3.2 Linear Operators

Definition 3.2.1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and $A : X \to Y$ a map. A is *linear* if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for each $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. It is *bounded* if $\|Ax\|_Y \leq M\|x\|_X$ for all $x \in X$ and some $M > 0$. We denote the set of all linear and bounded operators as

$$\mathscr{L}(X,Y) := \{A : X \to Y, A \text{ is linear and bounded } \}.$$

This space can be equipped with the *operator norm*

$$\begin{aligned} \|A\|_{op} &:= \inf\{M : \|Ax\|_Y \leq M\|x\|_Y \quad \text{for all} \quad x \in X\} \\ &= \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} \\ &= \sup_{\|x\|=1} \|Ax\|_Y. \end{aligned}$$

Proposition 3.2.1 $\mathscr{L}(X,Y)$ is a Banach space.

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\mathscr{L}(X,Y)$. Given $\varepsilon > 0$ there exists an N such that

$$\|A_m - A_n\|_{op} \leq \varepsilon \qquad \text{for all} \quad m,n \geq N.$$

Now for fixed $x \in X$ we have

$$\|A_m x - A_n x\| \leq \|(A_m - A_n)x\| \leq \|A_m - A_n\|_{op}\|x\| \leq \varepsilon\|x\|.$$

Hence $\{A_n x\}$ is a Cauchy sequence in Y and since Y is complete, we have a limit

$$A_n x \to y \text{ in } Y.$$

As we do this for each $x \in X$ we define a mapping $A : X \to Y : Ax = y$. A is the candidate for the limit of $\{A_n\}$. We see that A is linear and bounded. Indeed,

$$\lim_{n\to\infty} A_n(\alpha x_1 + \beta x_2) = \alpha \lim_{n\to\infty} A_n x_1 + \beta \lim_{n\to\infty} a_n x_2 = \alpha y_1 + \beta y_2$$

and

$$\|A_n - A\|_{op} \leq \varepsilon \quad \text{for all} \quad n > N.$$

Hence, $A_n \to A \in \mathscr{L}(X,Y)$. ∎

The next proposition records a central fact for linear operators.

Proposition 3.2.2 Let $A : X \to Y$ be linear, then

$$A \text{ is bounded} \iff A \text{ is continuous.}$$

Proof. "$\Rightarrow$" If A is bounded, then for all $x_n, x \in X$

$$\|A(x_n - x)\|_Y \leq \|A\|_{op}\|x_n - x\|_X,$$

hence A is continuous.

"$\Leftarrow$" Now let A be continuous and assume A is not bounded. Then for each $n \in \mathbb{N}$ there exists $y_n \in X$ with

$$\|Ay_n\|_Y \geq n^2\|y_n\|_X.$$

Define $x_n := \frac{y_n}{n\|y_n\|}$. Then $\|x_n\|_X \to 0$ for $n \to \infty$. But $\|Ax_n\| \geq n \to \infty$ so A is not continuous, which is a contradiction.

■

Here we introduce two of our standard examples, an integral operator K, and the Laplacian Δ:

■ **Example 3.1** Let Ω be a bounded domain. The integral operator $K : L^2(\Omega) \to L^2(\Omega)$ given by

$$Ku(x) = \int_\Omega k(x,y)u(y)dy \quad \text{with} \quad \int_\Omega \int_\Omega |k(x,y)|^2 dxdy = c < \infty$$

is bounded on $L^2(\Omega)$ ■

Proof. We consider the L^2-norm of K and use the Cauchy Schwarz inequality:

$$\begin{aligned}
\|Ku\|_2^2 &= \int_\Omega \left| \int_\Omega k(x,y)u(y)dy \right|^2 dx \\
&\leq \int_\Omega \left(\int_\Omega |k(x,y)|^2 dy \right) \left(\int_\Omega |u(y)|^2 dy \right) dx \\
&= c\|u\|_2^2.
\end{aligned}$$

Hence $\|K\|_{op} \leq \sqrt{c}$. ■

■ **Example 3.2** We consider the second derivative $A := -\frac{d^2}{dx^2}$ on $L^2[0,1]$. Since $C^2[0,1]$ is dense on $L^2[0,1]$ we define A only on its *domain*

$$D(A) = \{f \in C^2[0,1] : Af \in L^2[0,1]\}.$$

Then $A : D(A) \to L^2[0,1]$ is *unbounded*. ■

Proof. We apply A to a decaying exponential:

$$\|Ae^{-kx}\|_2 = \left\| -\frac{d^2}{dx^2} e^{-kx} \right\|_2 = k^2 \|e^{-kx}\|_2 = k^{\frac{3}{2}} \frac{1-e^{-2k}}{2},$$

which is certainly unbounded at $k \to \infty$. ■

An important lesson learned from these two examples is that for an unbounded operator the *domain belongs to the operator*. We write $(A, D(A))$. Here are some specific examples.

$$\left(A = -\frac{d^2}{dx^2}, \quad D(A) \text{ from above} \right),$$

$$\text{or } \left(A = -\frac{d^2}{dx^2}, \quad D(A) = \{f \in C^2[0,1], Af \in L^2[0,1], f(0) = 0, f(1) = 0\} \right),$$

$$\text{or } \left(A = -\frac{d^2}{dx^2}, \quad D(A) = \left\{ f \in C^2[0,1], Af \in L^2(0,1), \frac{\partial f}{\partial x}(0) = 0, \frac{\partial f}{\partial x}(1) = 0 \right\} \right).$$

Although all three operators have the same A as negative second derivative, they need to be considered as different operators simply because the domains are different.

The *range* of an operator $A : D(A) \to Y$ is defined as

$$R(A) = \{g \in Y : g = Af, f \in D(A)\}$$

and the *kernel* or *nullspace* is

$$\mathrm{Ker}(A) = \{f \in D(A) : Af = 0\}.$$

Proposition 3.2.3 $A : D(A) \to R(A)$ is invertible if and only if $\mathrm{Ker}(A) = \{0\}$.

Proof. For each $y \in R(A)$ find a unique solution $x \in X$ with $Ax = y$.

If $\mathrm{Ker}(A) \neq \{0\}$ then there exists a $\varphi \neq 0$, $\varphi \in \mathrm{Ker}(A)$ and $A\varphi = 0$. But also $A(0) = 0$, which means A is not invertible.

Now assume $\mathrm{Ker}(A) = \{0\}$. A is surjective by assumption. To show injectivity assume $Ax_1 = y$, $Ax_2 = y$. Then $A(x_1 - x_2) = y - y = 0$, which implies $x_1 - x_2 \in \mathrm{Ker}(A)$, i.e. $x_1 = x_2$. ■

3.3 The Baire Category Theroem

We borrow the Baire Category Theorem from topology to show the uniform boundedness principle for linear operators.

Theorem 3.3.1 — The Baire Category Theorem. If G_i is a countable family of dense open sets of a Banach space X, then,

$$G = \bigcap_{n=1}^{\infty} G_n$$

is dense in X.

Proof. We need to show that for any $x \in X$ and any $r > 0$, $B_r(x) \bigcap G \neq \emptyset$, where $B_r(x)$ denotes an open ball and $\overline{B_r}(x)$ denotes the closed ball. Since all G_n are dense and open we have a $y \in G_n$ and a smaller radius $s < r$ with

$$B_r(x) \cap G_n \supseteq B_{2s}(y) \supseteq \overline{B_s}(y)$$

We form a sequence of nested sets:

$$\begin{aligned}
x_1 \in X, \quad r_1 < \tfrac{1}{2}: &\quad \overline{B_{r_1}}(x_1) \subseteq G_1 \cap B_r(x) \\
x_2 \in X, \quad r_2 < \tfrac{1}{2^2}: &\quad \overline{B_{r_2}}(x_2) \subseteq G_2 \cap B_{r_1}(x_1) \\
&\vdots \\
x_n \in X, \quad r_n < \tfrac{1}{2^n}: &\quad \overline{B_{r_n}}(x_n) \subseteq G_n \cap B_{r_{n-1}}(x_{n-1}).
\end{aligned}$$

Then, as illustrated in Figure 3.1

$$\overline{B_{r_1}}(x_1) \supseteq \overline{B_{r_2}}(x_2) \supseteq \cdots \overline{B_{r_n}}(x_n) \supseteq \cdots$$

and

$$\bigcap_{n=1}^{\infty} \overline{B_{r_n}}(x_n) \text{ converges to a point } \{x_0\}.$$

Hence $x_0 \in \overline{B_{r_1}}(x_1) \subseteq B_r(x)$ and $x_0 \in G_n$ for all n.

$$\Rightarrow x_0 \in B_r(x) \cap G \text{ and } B_r(x) \cap G \neq \emptyset.$$

■

In other words, a countable collection of open dense sets has many *common points*.

■ **Example 3.3** An example for Baire's category theorem. Consider $X = \mathbb{R} \bmod 2\pi$, i.e. the interval $[0, 2\pi)$ with its periodic extension. Then X is a Banach space. Now define the family of dense sets as

$$G_n = [0, 2\pi) \setminus \left\{ \frac{1}{n} \right\}.$$

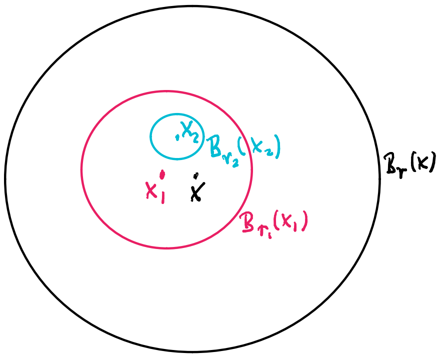


Figure 3.1: Illustration of the nested balls in the proof of Baire's Category Theorem.

Then

$$\bigcap_n G_n = [0, 2\pi) \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

which is still dense in X. ■

The original Baire's Theorem was actually formulated for nowhere dense sets.

Corollary 3.3.2 — Original Baire Category Theorem. F is called nowhere dense in X (of the first category) if $\overline{F}$ contains no nonempty open sets. Let F_j be a countable sequences of nowhere dense sets. Then

$$\bigcup_{j=1}^{\infty} F_j \neq X.$$

Proof. Define $G_j = X \setminus \overline{F}_j$, then G_j is open and dense. Indeed consider $x \in X$. If $x \notin \overline{F}_j$ then $x \in G_j$. If $x \in \overline{F}_j$ then there exists points in X that are arbitrary close. Hence G_j is dense in X. Then, by the Baire Category Theorem 3.3.1, $\bigcap G_j$ is dense in X and

$$G = \bigcap G_j = \bigcap X \setminus \overline{F}_j = X \setminus \bigcup \overline{F}_j.$$

Hence $\bigcup \overline{F}_j \neq X$. ■

After our excursion to topology, we come back to linear operators.

Theorem 3.3.3 — The Uniform Boundedness Principle. Let X be a Banach-space and Y a normed space. Consider $S \subset \mathscr{L}(X,Y)$ where

$$\sup_{T \in S} \|Tx\|_Y < \infty \quad \text{for all} \quad x \in X.$$

Then

$$\sup_{T \in S} \|T\|_{op} < \infty.$$

Proof. We define closed sets $F_j = \{x \in X : \|Tx\|_Y \leq j \text{ for all } T \in S\}$. Then by the above assumption, we have $\bigcup_j F_j = X$. By Corollary 3.3.2 at least one of the F_j is not nowhere dense, i.e., it has non empty interior. We call this set F_n with

$$B_r(y) \subset F_n.$$

This means that $y + x \in F_n$ with $\|x\|_X \leq r$ and

$$\|Tx\|_Y = \|T(y+x) + T(-y)\| \leq n + \|Ty\|_Y \leq R$$

for some $R > 0$. In particular for $\|x\| = r$, we have

$$\|Tx\|_Y \leq \frac{R}{r}\|x\|_X \quad \text{for all} \quad T \in S.$$

Since for arbitrary $x \in X$ we can consider $\tilde{x} = r\frac{x}{\|x\|}$ with $\|\tilde{x}\| = r$, and have

$$\|Tx\| = \frac{\|x\|}{r}\|T\tilde{x}\| \leq \frac{\|x\|}{r}\frac{R}{r}\|\tilde{x}\| = \frac{R}{r}\|x\|.$$

Hence we conclude that $\|T\|_{op} \leq \frac{R}{r}$. ■

3.4 Compact Operators

Definition 3.4.1 An operator $K : X \to Y$ between normed spaces is compact if for each bounded $U \subset X$ the set $\overline{K(U)}$ is compact in Y.

$$\begin{aligned} &\iff \quad K(U) \text{ is relative compact in } Y. \\ &\iff \quad \{u_n\} \subseteq X \text{ bounded sequence} \quad \Rightarrow \{Ku_n\}_{n\in\mathbb{N}} \text{ has a convergent subsequence in } Y. \end{aligned}$$

Proposition 3.4.1 A compact operator is bounded.

Proof. The set $B_1(0) \subset X$ is bounded, hence by compactness of the operator K the set $K(B_1(0))$ is compact and hence bounded in Y. This implies that

$$\sup_{\|x\| \leq 1} \|Kx\|_X \leq M$$

for some $M > 0$. Then $\|K\|_{op} \leq M$ and K is bounded. ■

Next we find that the limit of compact operators is again a compact operator.

Theorem 3.4.2 Let X be a normed space and Y a Banach space and consider a sequence $\{K_n\}$ of compact operators in $\mathscr{L}(X,Y)$. Assume $K_n \longrightarrow K$ for $n \to \infty$ in the $\|\cdot\|_{op}$-norm. Then K is also compact.

Proof. We want to show that for each bounded sequence $\{x_n\} \in X$, the limit operator K maps the bounded sequence into a pre-compact sequences. For this we use a diagonal argument. Let $\{x_n\} \subseteq X$ be a bounded sequence. Then

$$\begin{array}{rcl} \{K_1(x_n)\} & \text{has a convergent subsequence} & \{K_1(x_{n_{1j}})\}, \\ \{K_2(x_{n_{1j}})\} & \text{has a convergent subsequence} & \{K_2(x_{n_{2j}})\}, \\ \vdots & \vdots & \vdots \\ \{K_l(x_{n_{(l-1)j}})\} & \text{has a convergent subsequence} & \{K_l(x_{n_{lj}})\}, \\ \vdots & \vdots & \vdots \end{array}$$

Consider the diagonal sequence $y_j = x_{n_{jj}}$. Then

$$\begin{aligned} \|K(y_i) - K(y_j)\|_Y &\leq \|K(y_i) - K_n(y_i)\| + \|K_n(y_i) - K_n(y_j)\| + \|K_n(y_j) - K(y_j)\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \end{aligned}$$

for n and i,j large enough. So $\{K(y_n)\}$ is a Cauchy sequence in Y and it converges, i.e., K is compact. ■

As a consequence:

Corollary 3.4.3 Let $\mathscr{K}(X,Y)$ denote the set of linear compact operators from a normed space into a Banach space, then $\mathscr{K}(X,Y)$ is a Banach space.

■ **Example 3.4** Our standard example of the second derivative operator

$$A = -\frac{d^2}{dx^2} : \quad D(A) \longrightarrow L^2(0,1)$$

is unbounded, hence not compact. ■

Proposition 3.4.4 If a bounded operator $A : X \longrightarrow Y$ has finite dimensional range, then A is compact.

Proof. $\{Ax_n\}$is a bounded sequence in a finite dimensional vector space $R(A)$. By Bolzano-Weierstrass it has a convergent subsequence. ■

■ **Example 3.5** Our second standard example

$$Kf(x) := \int_\Omega k(x,y)f(y)dy \quad K : L^2(\Omega) \longrightarrow L^2(\Omega),$$

with $k \in L^2(\Omega \times \Omega)$ is compact. ■

Proof. We choose $\{\phi_j\}$ as an orthonormal base (ONB) of $L^2(\Omega)$. Then $\{\phi_j\phi_i\}$ is ONB of $L^2(\Omega\times\Omega)$. Write

$$k(x,y) = \sum_{i,j=1}^{\infty} k_{ij}\phi_i(x)\phi_j(y) \qquad \text{with} \qquad \|k\|^2_{L^2(\Omega\times\Omega)} = \sum_{i,j=1}^{\infty} \mid k_{ij} \mid^2 .$$

We consider finite truncations

$$k_n(x,y) := \sum_{i,j=1}^{n} k_{ij}\phi_i(x)\phi_j(y),$$

with corresponding intergal operators

$$K_n u := \int_\Omega k_n(x,y)u(y)dy.$$

If $u = \sum_{l=1}^{\infty} c_l\phi_l(x)$, then

$$\begin{aligned} K_n u &= \int_\Omega \sum_{i,j,l=1}^{n,n,\infty} k_{ij}\, c_l\, \phi_i(x)\, \phi_j(y)\, \phi_l(y)dy \\ &= \sum_{i,j=1}^{n} k_{ij}\, c_j\, \phi_i(x). \end{aligned}$$

Hence K_n has finite dimensional range (rank$= n$), which means all operators K_n are compact. These compact operators approximate K:

$$\|K-K_n\|^2 \le \int_\Omega\int_\Omega \mid k(x,y)-k_n(x,y) \mid^2 dxdy = \sum_{i,j=n+1}^{\infty} \mid k_{ij} \mid^2 \quad \to 0$$

for $n\to\infty$. Hence $K_n \longrightarrow K$ for $n\to\infty$ and hence K is compact as well. ■

▪ **Example 3.6 — Solution of Poisson's equation.** We consider the Poisson equation with homogeneous Neumann boundary conditions on the interval $[0,L]$.

$$-u'' = f, \qquad u'(0) = u'(L) = 0,$$

where $f(x)$ is a given function in $L^2(0,L)$. We see immediately that if $u(x)$ is a solution, then $u(x)+c$ is also a solution for any constant c. Hence, to obtain unique solutions, we specify the total mass of the solution

$$\int_0^L u(x)dx = M.$$

The corresponding differential operator is

$$(A,D(A)) = \left(-\frac{d^2}{dx^2}; D(A) = \{u\in C^2 : u'(0) = u'(L) = 0,\ \int_0^L u(x)dx = M\}\right).$$

Then the solution to the Poisson equation with mass M is written as

$$u = A^{-1}f,$$

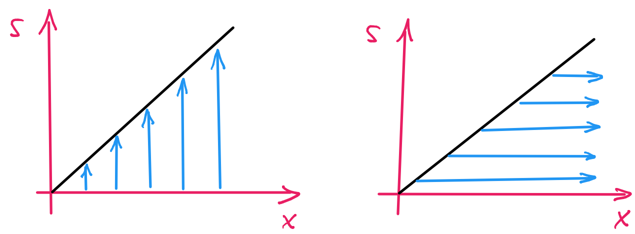


Figure 3.2: Application of Fubini's theorem.

and A^{-1} is compact.

To show this claim we explicitly solve the Poisson equation by integrating twice

$$-u'(y)+u'(0)=\int_0^y f(s)ds,$$

where $u'(0)=0$. We integrate once more

$$-u(x)+u(0)=\int_0^x\int_0^y f(s)ds\,dy.$$

We do not know $u(0)$, and we compute it later from the mass condition. By Fubini (see Figure 3.2) we have

$$\begin{aligned} u(x) &= -\int_0^x\int_0^y f(s)ds\,dy+u(0)\\ &= -\int_0^x\int_s^x f(s)dy\,ds+u(0)\\ &= -\int_0^x (x-s)f(s)ds+u(0)\\ &= \int_0^x (s-x)f(s)ds+u(0). \end{aligned} \tag{3.1}$$

Integrating this further, we get the total mass

$$\begin{aligned} M=\int_0^L u(x)dx &= \int_0^L\int_0^x (s-x)f(s)ds\,dx+u(0)L\\ &= \int_0^L\int_s^L (s-x)f(s)dx\,ds+u(0)L\\ &= \int_0^L\left(sL-\frac{L^2}{2}-s^2+\frac{s^2}{2}\right)f(s)ds+u(0)L\\ &= -\int_0^L\frac{(L-s)^2}{2}f(s)ds+u(0)L. \end{aligned}$$

Hence,

$$u(0)=\frac{M}{L}+\frac{1}{2L}\int_0^L (L-s)^2 f(s)ds.$$

Then from (3.1) we write $u(x)$ as an integral operator:

$$\begin{aligned} u(x) &= \frac{M}{L} + \int_0^x (s-x)f(s)ds + \frac{1}{2L}\int_0^L (L-s)^2 f(s)ds \\ &= \frac{M}{L} + \int_0^L k(x,s)f(s)ds, \end{aligned}$$

with integral kernel

$$k(x,s) = \chi_{[0,x]}(s)(s-x) + \frac{(L-s)^2}{2L}.$$

This integral kernel satisfies $k \in L^2((0,L)\times(0,L))$, hence the integral operator is compact. Then

$$A^{-1} : f \mapsto \frac{M}{L} + \int_0^L k(x,s)f(s)ds,$$

is compact. ■

3.5 Symmetry and Positivity

Another important property for operators on Hilbert spaces is symmetry.

Definition 3.5.1 Let $A : D(A) \longrightarrow H$ on a Hilbert space H. A is *symmetric*, if

$$(Ax,y) = (x,Ay) \quad \text{for all} \quad x,y \in D(A).$$

■ **Example 3.7** if $k(x,y) = k(y,x)$, then K is symmetric on $L^2(\Omega)$

$$\begin{aligned} (Kf,g) &= \int\int k(x,y)f(y)g(x)dydx \\ &= \int\int f(y)k(y,x)g(x)dxdy \\ &= (f,Kg). \end{aligned}$$

■

■ **Example 3.8** Consider

$$A = -\frac{d^2}{dx^2}, \quad D(A) = \left\{ f \in L^2[0,1],\ Af \in L^2[0,1],\ \frac{df}{dx}(0) = 0,\ \frac{df}{dx}(1) = 0 \right\}$$

Then

$$\begin{aligned} (Au,v) &= \int_0^1 -\frac{d^2}{dx^2}u(x)v(x)dx \\ &= -\frac{d}{dx}uv\Big|_0^1 + \int_0^1 \frac{d}{dx}u\frac{d}{dx}vdx \\ &= u\frac{d}{dx}v\Big|_0^1 - \int_0^1 u\frac{d^2}{dx^2}vdx \\ &= (u,Av), \end{aligned}$$

hence $(A,D(A))$ is symmetric. ■

Definition 3.5.2 An operator A on a Hilbert space is *positive* if there is a constant $k > 0$ such that

$$(Au,u) \geq k\|u\|^2, \quad u \in D(A)$$

In this case, all its eigenvalues are positive (see Spectral Theory in Chapter 8).

Example 3.9 If $k(x,y) \geq \delta > 0$, then K is positive on L^2_+.

$$\begin{aligned}
(Ku,u) &= \int\int k(x,y)u(x)u(y)dxdy \\
&\geq \delta \int\int u(x)u(y)dxdy \\
&\geq \delta \int\int_{\{y=x\}} u(x)u(y)dydx \\
&= \delta\|u\|_2^2.
\end{aligned}$$

■

Example 3.10 $A = -\frac{d^2}{dx^2}$, $\quad D(A)$ as above is positive.

$$\begin{aligned}
(Au,u) &= \int_0^1 -\frac{d^2}{dx^2}u(x)u(x)dx \\
&= \int_0^1 \frac{d}{dx}u\frac{d}{dx}udx \\
&= \int_0^1 \left|\frac{d}{dx}u\right|^2 dx \\
\text{(Poincaré Inequality)} \quad &\geq \int_0^1 |u|^2 dx \\
&= \|u\|^2.
\end{aligned}$$

■

3.6 Closed Operators

The fact that an unbounded operator is only defined once its domain of definition is given, leads to some interesting relationship between operators. First of all there is the question of what is the *natural* domain for a given operator. For example the second derivative:

Example 3.11 We define $A = -\frac{d^2}{dx^2}$ on $L^2(0,1)$ with

$$D_1(A) = \left\{f \in C^2[0,1] + \text{boundary conditions } \right\}.$$

Is this the best choice of $D(A)$? What about

$$\begin{aligned}
D_2(A) &= \{f \text{ has two weak derivatives + boundary conditions }\} \\
&= H_0^2(0,1) \text{ (see Sobolev spaces in chapter 5).}
\end{aligned}$$

How are $(A, D_1(A))$ and $(A, D_2(A))$ related? The answer lies in the following definition. ■

Definition 3.6.1 An operator $(\hat{A}, D(\hat{A}))$ is an extension of $(A, D(A))$ if

$$D(\hat{A}) \supset D(A) \text{ and } \hat{A}\Big|_{D(A)} = A.$$

The closed extension arises by including limit points:

$$\text{If} \quad \underbrace{x_n}_{\in D(A)} \to \underbrace{x}_{\in X} \quad \text{and} \quad \underbrace{Ax_n}_{\in X} \to \underbrace{y}_{\in X} \quad \text{then we define} \quad y = \hat{A}x$$

and $x \in D(\hat{A})$.

Definition 3.6.2 A is closed if $\{x_n\} \in D(A)$ $x_n \longrightarrow x$ in X implies

$$x \in D(A) \text{ and } Ax = y = \lim_{n \to \infty} Ax_n.$$

Proposition 3.6.1 If A is a symmetric operator on a Hilbert space with dense domain $D(A)$, then it has a unique closed extension $(\hat{A}, D(\hat{A}))$ which is also symmetric. The extension is usually also called $(A, D(A))$

Proof. Consider $x_n \longrightarrow x$ with $x_n \in D(A)$ and $Ax_n \longrightarrow y$ in Y. We define the extension as follows

$$D(\hat{A}) := \{x \in X : \text{ there exists a sequence } \{x_n\} \subset D(A) : x_n \longrightarrow x\}.$$

Define $\hat{A}$ by $y = \hat{A}x$. $\hat{A}$ is well-defined from the above limits. If there are two sequences $x_n \to x$ and $x_n^* \to x$ with $Ax_n \to y$, $Ax_n^* \to y^*$ then for each $u \in D(A)$ we have

$$\begin{aligned}
(y^* - y, u) &= \lim_{n\to\infty} (Ax_n^* - Ax_n, u) \\
&= \lim_{n\to\infty} (x_n^* - x_n, Au) \\
&= 0 \quad \text{for all} \quad u \in D(A).
\end{aligned}$$

Since $D(A)$ is dense it follows that $y^* = y$ and the extension $\hat{A}$ is well defined and $x \in D(\hat{A})$ and $\hat{A}$ is closed. ■

3.7 A Glance Ahead to Spectral Theory

In Chapter 8, we will discuss the spectral theory for linear operators in detail. Here we present some of the main results on eigenvalues and spectral bounds. As we discussed the two standard examples of the second derivative operator and the integral operator as examples of unbounded versus compact operators, respectively, a discussion of their spectra fits very naturally in this chapter. The proofs of the spectral results are given later in Chapter 8.

The integral operator K is our example of a compact operator. It satisfies the Hilbert-Schmidt theorem:

Theorem 3.7.1 — Hilbert-Schmidt. Let $A : H \to H$ be a linear, symmetric, compact operator on a Hilbert space H. Then

1. All eigenvalues of A are real and there is at most one accumulation point at 0.
2. The eigenvectors $\{w_j\}$ can be chosen to form an orthonormal basis and A has a *spectral representation*

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j) w_j.$$

The second derivative is an unbounded operator. Depending on the boundary conditions we often are in a situation where the solution operator of a Poisson equation is compact (see our previous example 3.6). Hence the next spectral theorem is a typical situation for a Laplace operator problem.

Theorem 3.7.2 — Spectral Theorem. Let $A : D(A) \to H, R(A) = H$ be symmetric, linear, and unbounded, and let A^{-1} exists and be compact. Then

1. There exists an infinite set $\{\lambda_n\}$ of real eigenvalues with

$$\lim_{n\to\infty} |\lambda_n| = +\infty.$$

2. The eigenvectors $\{w_j\}$ can be chosen to form an orthonormal basis and

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j) w_j.$$

■ **Example 3.12** An example for the Hilbert-Schmidt spectral theorem. Consider the simple integral operator $K : L^2(0,L) \to L^2(0,L)$ given by

$$K(f) = \int_0^1 f(s)ds.$$

Then the eigenvalue problem for K reads

$$\int_0^1 f(s)ds = \lambda f(x), \qquad \text{for all} \qquad x \in [0,1],$$

which is only true for constant functions $f(x) = c$, and the eigenvalue is $\lambda_1 = 1$. The normalized eigenfunction is $w_1 = 1$, and these are all the eigenfunctions. Writing out the spectral representation of K, we have

$$K(f) = \lambda_1 (f, w_1) w_1 = \int_0^1 f(s)ds,$$

as it should be. ■

3.7.1 Fractional Powers

The above spectral representations provide a natural way to define fractional operators such as $\sqrt{-\Delta}$ for example. Fractional operators are feared by some and loved by others. They are popular in the analysis of non-standard random walks such as Levy flights [21]. The analysis of fractional power operators is tricky, and the spectral representation below gives us a powerful tool for their analysis.

Definition 3.7.1 If a positive operator A on a Hilbert space has a representation as

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, w_j) w_j$$

then we define the *fractional powers* of A as

$$A^{\alpha} u = \sum_{j=1}^{\infty} \lambda_j^{\alpha} (u, w_j) w_j$$

for $\alpha \in \mathbb{R},\ \alpha \geq 0$ with domain

$$D(A^{\alpha}) = \{u : \|A^{\alpha} u\| < \infty\} = \left\{ u : u = \sum_{j=1}^{\infty} c_j w_j, \quad \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^{2\alpha} < \infty \right\}.$$

It can be shown that $D(A^{\alpha})$ is a Hilbert space with inner product.

$$(u, v)_{D(A^{\alpha})} := (A^{\alpha} u, A^{\alpha} v)$$

with the norm

$$\|u\|_{D(A^{\alpha})} = \|A^{\alpha} u\|.$$

Example 3.13 Consider $A = -\Delta$. Then we define $A^{\frac{1}{2}}$ as operator on L^2 with domain

$$D(A^{\frac{1}{2}}) = \left\{ u = \sum c_j w_j, \quad \sum_{j=1}^{\infty} |\, c_j \,| \lambda_j^2 < \infty \right\} \quad \text{and norm} \quad \|u\|_{\frac{1}{2}} = \sum_{j=1}^{\infty} |\, c_j \,| \lambda_j^2.$$

3.8 Exercises

Exercise 3.1 (Closed nullspace) (level 1)
Let X be a Banach space and $A : X \to X$ a bounded linear map. Show that the null-space $\ker(A)$ is closed.

Exercise 3.2 (Closed range) (level 1)
Let X be a Banach space and $A : X \to X$ a bounded linear map. Assume there is a $K > 0$ such that

$$\|x\| \leq K \|Ax\|, \qquad \text{for all} \qquad x \in X.$$

Show that the range of A is closed.

Exercise 3.3 (Spectral representation) (level 2)
Let A be a symmetric linear operator on a Hilbert space H with $R(A) = H$ and with compact inverse A^{-1}. The natural domain of definition is

$$\mathscr{D}(A) = \{u \in H; Au \in H\}.$$

Show that there exists an orthonormal basis $\{w_j\}$ of H such that

$$\mathscr{D}(A) = \left\{u; u = \sum c_j w_j, \qquad \sum |c_j|^2 \lambda_j^2 < \infty\right\}.$$

■

Exercise 3.4 (Normal Integral Operators) (level 2)
For $k \in L^2(\Omega \times \Omega)$ consider the integral operator

$$u \mapsto Ku(x) := \int_\Omega k(x,y)u(y)dy.$$

1. Find a condition on the kernel k such that the integral operator K is normal.
2. Find an example of a normal integral operator that is not symmetric. Make sure to chose k and Ω so that $k \in L^2(\Omega \times \Omega)$.

■

Exercise 3.5 (Root of Laplacian) (level 1)
For $A = -\Delta$ show that on a bounded domain the norm on $\mathscr{D}(A^{\frac{1}{2}})$ is equivalent to the norm on H_0^1. (Hint: If A is symmetric, $(Au, v) = (u, Av)$, then also $A^{\frac{1}{2}}$). ■

4. Dual Spaces

Contents

A key tool in functional analysis is the concept of a dual space. Instead of looking at elements of a Banach space directly, we look at all linear maps from the Banach space to the real numbers. Imagine that the elements of a Banach space represent something real, for example cars. The dual space would then contain all measurements that we can make on a car, i.e. size, make, horsepower, color, engine etc. This example is a bit flawed, since most of these measurements would not be linear maps, for example, what is 2*blue + 3*red, but you get my idea. We plan to associate an object (car) with all its properties (color, horsepower, make, ...). Once we know all its specifications, we can uniquely identify the car. This is, in essence, the Hahn-Banach theorem.

4.1 Dual Spaces

Definition 4.1.1 Let X be a Banach space. The *dual space* X^* is the space of all linear

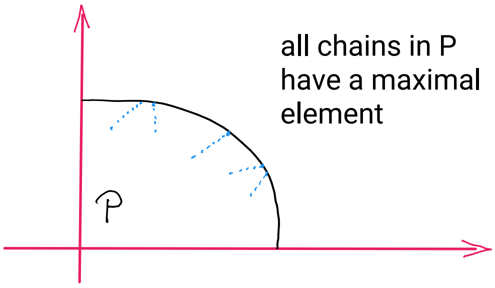


Figure 4.1: Illustration of Zorn's Lemma.

functionals from $X \to \mathbb{R}$. This space has the operator norm and it is a Banach space

$$(X^*, \|.\|_*), \qquad \|A\|_* = \|A\|_{op}.$$

4.1.1 Zorn's Lemma

In any advanced theory comes the moment, where a new concept needs to be based on very deep classical results from set theory or topology. This moment is here as we prepare for the famous Hahn-Banach Theorem of dual spaces. We will use Zorn's Lemma on partially ordered sets.

Let P be a set with a partial order $\leq$. A partial order satisfies

- $a \leq a$ for all $a \in P$,
- $a \leq b$ and $b \leq c$ implies $a \leq c$,
- if $a \leq b$ and $b \leq a$ then $a = b$.

A set $C \subset P$ is a *chain* if any two objects can be compared. An *upper bound* M satisfies $a \leq M$ for all $a \in P$ and a maximal element $m \in P$ satisfies that $m \leq a$ implies $a = m$.

Proposition 4.1.1 — Zorn's Lemma. Let P be a partially ordered set such that all chains in P have an upper bound. Then P has a maximal element.

To illustrate Zorn's lemma we consider a domain P as indicated in Figure 4.1, where each chain has a maximal element.

Theorem 4.1.2 — Hahn-Banach. Let X be a Banach space and $M \subset X$ a linear subspace. Consider $f \in \mathscr{L}(M, \mathbb{R})$ with $|f(x)| \leq k\|x\|$ for all $x \in M$. Then there exists an extension F of f on X and

$$F \in \mathscr{L}(X, \mathbb{R}), \qquad |F(x)| \leq k\|x\|, \quad \text{for all} \quad x \in X.$$

Proof. We define a set

$$E := \{(g, G) : g \text{ is an extension of } f \text{ on } G \text{ and } |g(x)| \leq k\|x\|, \text{ for all } x \in G, M \subset G\}.$$

We have that $E \neq \emptyset$ since $(f, M) \in E$. The set E can be partially ordered as $(h, H) \leq (g, G)$, if g is an extension of h with $H \subset G$ and $g\Big|_H = h$.

We take a chain $C = (g_i, G_i)$ in E and find an upper bound (u, U) with

$$u(x) = g_i(x), \qquad \text{for } x \in G_i \backslash G_{i-1}, \qquad \text{and } U = \cup_i G_i.$$

Hence by Zorn's Lemma there exists a maximal element $(F,W) \in E$.

Now we need to show that $W = X$. Assume not, then there exists a non-trivial element $z \in X \backslash W$, which we use to define another extension. We define $Z := W + \mathrm{span}(z)$ and each $x \in Z$ can be written as $x = w + \alpha z$ with $w \in W$, and $\alpha \in \mathbb{R}$. We define an extension of F as

$$g(w+\alpha z) = F(w) + \alpha c,$$

for an appropriately chosen constant $c > 0$. We need to specify c such that $|g(x)| \leq k\|x\|$ in Z. This means

$$|F(w)+\alpha c| \leq k\|w+\alpha z\|, \qquad \text{for all } w \in W, \alpha \in \mathbb{R}.$$

If $F(w)+\alpha c > 0$ then $F(w)+\alpha c \leq k\|w+\alpha z\|$ and

$$\alpha c \leq k\|w+\alpha z\| - F(w), \qquad \text{for all } w \in W. \tag{4.1}$$

In the other case, when $F(w)+\alpha c < 0$, then $-F(w)-\alpha c \leq k\|w+\alpha z\|$ which leads to $F(-w)-\alpha c \leq k\|-w-\alpha z\|$. As this needs to hold for all $w \in W$, we simply name $v = -w$ and write the above as condition for v: If $-F(v)+\alpha c \leq 0$ then

$$-\alpha c \leq k\|v-\alpha z\| - F(v), \qquad \text{for all } v \in W. \tag{4.2}$$

If $\alpha < 0$ then (4.2) becomes (4.1) with α replaced by $-\alpha$, and vice versa. Hence we only need (4.1) and (4.2) for $\alpha > 0$. Then the condition becomes

$$-\frac{k}{\alpha_1}\|v-\alpha_1 z\| + \frac{F(v)}{\alpha_1} \leq c \leq \frac{k}{\alpha_2}\|w+\alpha_2 z\| - \frac{F(w)}{\alpha_2} \qquad \text{for all } v,w \in W, \alpha_1, \alpha_2 > 0.$$

The left hand side must be less than the right hand side, which we write as

$$\begin{aligned} \alpha_2 F(v) + \alpha_1 F(w) &\leq \alpha_1 k\|w+\alpha_2 z\| + \alpha_2 k\|v-\alpha_1 z\| \\ F(\alpha_2 v + \alpha_1 w) &\leq k\left(\|\alpha_2 v - \alpha_1\alpha_2 z\| + \|\alpha_1 w + \alpha_1\alpha_2 z\|\right). \end{aligned}$$

This last inequality is true, since

$$|F(\alpha_2 v+\alpha_1 w) \leq k\|\alpha_2 v+\alpha_1 w\| \leq k\left(\|\alpha_2 v-\alpha_1\alpha_2 z\| + \|\alpha_1 w+\alpha_1\alpha_2 z\|\right).$$

We can indeed further extend (F,W), which contradicts (F,W) being the maximal element. Consequently, $W = X$ and we are done. ■

R Note that M could be one-dimensional and we can still extend it to all of X.

The following corollary is often called The Hahn-Banach Theorem:

Corollary 4.1.3 — Hahn-Banach version II. Let $x,y \in X$. If $f(x) = f(y)$ for all $f \in X^*$ then $x = y$.

Proof. Assume $f(x) = f(y)$ for all $f \in X^*$, but $x \neq y$. Then on $\mathrm{span}(x,y)$ we define a functional $\phi(\alpha x + \beta y) = \alpha|x|$. Note, ϕ is a linear map and $\phi(x) = |x| \neq \phi(y) = 0$. By the Hahn-Banach Theorem (Theorem 4.1.2) we can extend ϕ to a linear form $f \in X^*$ giving $f(x) \neq f(y)$, contradicting the assumption. Hence $x = y$. ■

■ **Example 4.1 — Car example.** Let us come back to the illustrative example of car characterizations. We view the set of all cars to be the original space, and the set of all car characteristics (horse power, color, make, size, engine size, fuel consumption etc.,) as elements of the dual space. Now, Hahn-Banach in this context says that if you have two cars x and y and you find that ALL characteristics are the same, i.e. same horse power, same color, same age, same key, same mileage, etc. Then x and y must be the same car. ■

The following version of the Hahn-Banach theorem is also very useful.

Corollary 4.1.4 — Hahn-Banach version III. For each $x \in X$ there exists an $f \in X^*$ with

$$f(x) = |x| \qquad \text{and} \qquad \|f\|_{op} = 1.$$

Proof. We define $f(x) = |x|$ on $\mathrm{span}(x)$ and extend it to X^*. ■

■ **Example 4.2** On $\mathbb{R}^n$ we have $(\mathbb{R}^n)^* = \mathbb{R}^n$. Given a vector $x \in \mathbb{R}^n$ we find the dual element $f_x \in \mathbb{R}^n$ by $f_x(y) = x \cdot y$. This means to the row-vector x we assign the column vector $f_x = x^T$. Then, naturally, $\|f_x\|_{op} = \|x^T\| = \|x\|$. ■

■ **Example 4.3** For a bounded smooth set Ω and for $1 < p < \infty$ we use Hahn-Banach to show that $(L^p(\Omega))^* = L^q(\Omega)$, where p and q are conjugate. For $f \in L^q(\Omega)$ we define a linear functional through integration

$$L_f(g) = \int_\Omega f(x)g(x)dx.$$

By Hölders inequality we have

$$|L_f(g)| \le \|f\|_q \|g\|_p.$$

This shows that $L_f \in (L^p(\Omega))^*$ and that $\|L_f\|_{op} \le \|f\|_q$. But we can get more. We can show that we have an isometry between L^q and $(L^p)^*$. To show this we choose $g(x) = |f(x)|^{q-2} f(x)$. Then

$$\|g\|_p^p = \int_\Omega \left| |f(x)^{q-2} f(x) \right|^p dx = \int_\Omega |f(x)|^q dx = \|f\|_q^q.$$

Hence $\|g\|_p = \|f\|_q^{\frac{q}{p}}$. Now

$$|L_f(g)| = \int_\Omega f(x)|f(x)|^{q-2} f(x)dx = \int_\Omega |f(x)|^q dx = \|f\|_q^q = \|f\|_q \|f\|_q^{\frac{q}{p}} = \|f\|_q \|g\|_p.$$

Hence

$$\|L_f\|_{(L^p)^*} = \|L_f\|_{op} = \|f\|_q,$$

and the map $L^q(\Omega) \to (L^p(\Omega))^*$ is an isometry.

What is missing is to show that each element in $(L^p)^*$ can be written as an integral operator. This is done via measure theory, and not covered in this book, but can be found in [11]. However, in case of $p = 2$ we can prove this result using the Riesz Representation Theorem 4.2.1. In any case, we have

$$(L^p(\Omega))^* \simeq L^q(\Omega)$$

and we simply write

$$(L^p(\Omega))^* = L^q(\Omega).$$

■

■ **Example 4.4** L^∞ is the dual of L^1 but not vice versa. For $f \in L^\infty(\Omega)$ we again define an integral operator as

$$L_f(g) = \int_\Omega f(x)g(x)dx,$$

with

$$|L_f(g)| \leq \|f\|_\infty \|g\|_1.$$

Similar as before we show that

$$\|L_f\|_{(L^1)^*} = \|L_f\|_{op} = \|f\|_\infty$$

and we get an isometry

$$(L^1(\Omega))^* = L^\infty(\Omega).$$

However, as we will not explicitly show here, we only have

$$L^1(\Omega) \subsetneq (L^\infty)^*.$$

The dual space $M(\Omega) = (L^\infty(\Omega))^*$ is a measure space equipped with the total variation norm [11]). ■

4.2 Dual of Hilbert Spaces

In a Hilbert space the dual can be identified with the original space. This is the key result of the Riesz Representation Theorem.

Theorem 4.2.1 — Riesz Representation Theorem. Let H be a Hilbert space then

$$H^* \simeq H.$$

1. Each $x \in H$ has an associated dual element $l_x \in H^*$ defined by $l_x(y) = (x, y)$. Then $\|l_x\|_* = \|x\|$.
2. For every $l \in H^*$ there exists a unique $x_l \in H$ such that $l(y) = (x, y)$ for all $y \in H$ and $\|l\|_* = \|x_l\|$. The map $l \mapsto x_l$ is continuous.

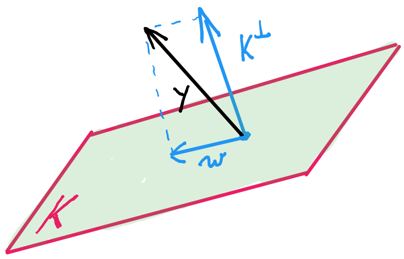


Figure 4.2: Orthogonal decomposition on K.

Proof. 1. By definition l_x is a linear functional on H and from the Cauchy Schwartz inequality we find

$$|l_x(y)| = |(x,y)| \leq \|x\|\|y\|.$$

Hence $\|l_x\|_{op} \leq \|x\|$. If we chose $y = x$ then we get $l_x(x) = \|x\|^2$, and $\|l_x\|_{op} = \|x\|$.

2. Suppose $l \in H^*$ is given. The linear space $K := \ker(l)$ is a closed subspace of H and we claim that $K^{\perp}$ is one dimensional. Indeed, if we have two elements $u, v \in K^{\perp}$, then the linear combination is also in $K^{\perp}$: $l(v)u - l(u)v \in K^{\perp}$. However,

$$l(l(u)v - l(v)u) = l(v)l(u) - l(u)l(v) = 0.$$

Hence $l(v)u - l(u)v \in K$. This means $l(v)u - l(u)v = 0$ and u and v are linear dependent.
Now we chose a unit vector $z \in K^{\perp}$, and decompose $H = K \oplus K^{\perp}$ as (see Figure 4.2)

$$y \in H, \qquad y = (z,y)z + w, \qquad w \in K.$$

Then $l(y) = (z,y)l(z)$. So we define $x_l := l(z)z$, which gives

$$(x_l, y) = (l(z)z, y) = l(z)(z,y) = l(y).$$

■

4.3 Reflexive Spaces

Let us consider some examples first.

■ **Example 4.5** Let H be a Hilbert space with inner product $(\cdot,\cdot)$. For $x, y \in H$ we interpret the inner product in two ways: Firstly

$$(\underbrace{x}_{\text{acts on } y}, y) \qquad x \in H^*, y \in H$$

and secondly

$$(\underbrace{x}_{\in H^*}, \underbrace{y}_{\text{acts on } x}), \qquad x \in H^*, y \in (H^*)^*.$$

Hence in a natural way we identify the bi-dual H^{**} with the original space H. ■

■ **Example 4.6** For $1 < p < \infty$ we consider the duality of L^p and L^q, where p and q are conjugate. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$.

$$\int_\Omega \underbrace{f(x)}_{\in L^p} \underbrace{g(x)}_{\in (L^p)^*} dx \qquad \text{or} \qquad \int_\Omega \underbrace{f(x)}_{\in (L^p)^{**}} \underbrace{g(x)}_{\in (L^p)^*} dx.$$

■

Definition 4.3.1 A Banach space X is *reflexive* if

$$X^{**} \simeq X.$$

■ **Example 4.7** Some well known examples of reflexive spaces:

- $\mathbb{R}^n$ is reflexive.
- Hilbert spaces are reflexive.
- $L^p(\Omega)$ is reflexive for $1 < p < \infty$ and Ω bounded and smooth.
- L^1 and L^∞ are not reflexive.

■

4.4 Weak Convergence

In a Banach space, in particular in spaces of infinite dimension, we find a large variety of notions of convergence. The default notion of convergence would be using the norm in a Banach space, i.e. a sequence $\{x_n\} \subset X$ converges *strongly* to $x \in X$, if and only if

$$\|x_n - x\| \to 0, \qquad \text{as} \quad n \to \infty.$$

A weaker form of convergence is the following definition, which uses the dual space as the judge of convergence.

Definition 4.4.1 — Weak convergence. Let X be a Banach space. A sequence $\{x_n\} \subset X$ *converges weakly* to x, which we denote as

$$x_n \rightharpoonup x,$$

if $f(x_n) \to f(x)$ for all $f \in X^*$ as $n \to \infty$.

We see immediately that weak convergence is weaker than strong convergence in the next Lemma:

Proposition 4.4.1 Strong convergence $x_n \to x$ implies weak convergence $x_n \rightharpoonup x$.

Proof. Each $f \in X^*$ is continuous by definition, hence as $x_n \to x$ we automatically have $f(x_n) \to f(x)$, which implies weak convergence. ■

On the other hand, a weak convergent sequence does not necessarily have to be strongly convergent as the following example shows.

■ **Example 4.8 — weak does not imply strong.** Let H be a separable Hilbert space with countable orthonormal basis $\{e_i\}$. We will show that $e_i \rightharpoonup 0$ as $i \to \infty$. We test with elements from H^*. By the Riesz Representation Theorem 4.2.1 each $f \in H^*$ has a unique representation $x_f \in H$ such that $f(x) = (x_f, x)$ for all $x \in H$ and $\|f\| = \|x_f\|$. In particular $f(e_i) = (x_f, e_i)$. The norm of x_f can be expressed as

$$\|f\|^2 = \|x_f\|^2 = \sum_{i=1}^{\infty} |(x_f, e_i)|^2,$$

which is a convergent series. Hence $|(x_f, e_i)| \to 0$ for $i \to \infty$ and we have weak convergence to 0.

On the other hand $\|e_i - 0\| = \|e_i\| = 1$ for all i, hence e_i does not converge strongly to 0. ∎

Proposition 4.4.2 Weak convergent sequences are bounded and weak limits are unique.

Proof. Assume $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$. Then $f(x) = f(y)$ for all $f \in H^*$ and by the Hahn-Banach Theorem 4.1.3 we find $x = y$, i.e. uniqueness.

To show boundedness we consider a test function $f \in X^*$. Then $\{f(x_n)\}$ is a convergent sequence in $\mathbb{R}$, hence it is bounded $|f(x_n)| \leq C_f$ for all n. Now we associate x_n with an element $G_n \in X^{**}$ as $G_n(f) = f(x_n)$ for all $f \in X^*$. Then $|G_n(f)| \leq C_f$ for all n. By the Uniform Boundedness Principle 3.3.3 it follows that $\|G_n\|_{**}$ is bounded. But $\|G_n\|_{**} = \|x_n\|$ by definition, hence $\{x_n\}$ is bounded. ∎

Theorem 4.4.3 — weak + compact implies strong . Suppose $A : X \to Y$ is compact and $x_n \rightharpoonup x$. Then $Ax_n \to Ax$ in Y.

Proof. For each of $f \in X^*$ we have $f(x_n) \to f(x)$. Now consider $g \in Y^*$. Then $gA \in X^*$ and

$$g(Ax_n) = (gA)(x_n) \to (gA)(x) = g(Ax), \quad \text{for} \quad n \to \infty,$$

hence we have already weak convergence $Ax_n \rightharpoonup Ax$. Now $\{x_n\}$ is bounded and A is compact. Hence $\{Ax_n\}$ has a convergent subsequence which converges strongly in Y. But such a strong convergent sequence is also weakly convergent, and weak limits are unique. Hence we must have that $Ax_n \to Ax$ in Y. ∎

4.5 Weak* Convergence

Weak* convergence is often perceived of something very strange. I hope to convince you that this is not true. It is a close cousin to weak convergence with the only difference that test functions are not chosen from the dual but from the pre-dual. Let us look at this in detail.

Definition 4.5.1 Let X be a Banach space. A sequence $\{f_n\} \subset X^*$ is weak* convergent, denoted by

$$f_n \overset{*}{\rightharpoonup} f,$$

if $f_n(x) \to f(x)$ for all $x \in X$.

Proposition 4.5.1 Weak* convergent sequences are bounded and weak* limits are unique.

Proof. The proof is essentially the same as the proof of Proposition 4.4.2 ■

Compared to weak convergence in Definition 4.4.1, it seems we simply switched the roles of x and f. That's a good way to see it. In applications, you need to understand if you are currently working in the function space X or in its dual space X^*. In X, you would use weak convergence and in X^* you would by default, use weak* convergence.

However, as X^* is again a Banach space, we can use test functions in X^{**} and still talk about weak convergence in X^*. If the Banach space is reflexive $X^{**} = X$, then chosing test functions in X or in X^{**} is the same, hence then weak convergence = weak* convergence.

Weak* convergence has one special gem, which does not hold in general for weak convergence, and that is a compactness result.

Theorem 4.5.2 — Alaoglu weak*-compactness. Let X be a separable Banach space and $\{f_n\} \subset X^*$ a bounded sequence in the dual space. Then $\{f_n\}$ has a weak* convergent subsequence.

Proof. The proof uses a very classical diagonal sequence argument. Let $\{x_k\} \subset X$ denote a countable dense subset of X. For the first element x_1 we consider the sequence $\{f_n(x_1)\}$.

- $\{f_n(x_1)\}$ is a bounded sequence in $\mathbb{R}$ and it has a convergent subsequence $\{f_{n_{1j}}(x_1)\}_j$.

We take this subsequence and evaluate it at x_2 and so forth.

- $\{f_{n_{1j}}(x_2)\}$ is a bounded sequence in $\mathbb{R}$ and it has a convergent subsequence $\{f_{n_{2j}}(x_2)\}_j$.
- $\vdots$
- $\{f_{n_{kj}}(x_{k+1})\}$ is a bounded sequence in $\mathbb{R}$ and it has a convergent subsequence $\{f_{n_{(k+1)j}}(x_{k+1})\}_j$.

We obtain nested sequences

$$\{f_n\} \supseteq \underbrace{\underbrace{\underbrace{\{f_{n_{1j}}\}}_{\text{convergent at } x_1} \supseteq \{f_{n_{2j}}\}}_{\text{convergent at } x_1, x_2} \supseteq \cdots \supseteq \{f_{n_{kj}}\}}_{\text{convergent at } x_1,\ldots,x_k} \supseteq \cdots$$

Hence we define a diagonal sequence $g_j := f_{n_{jj}}$ such that $\{g_j(x_k)\}$ converges for each x_k. In $\varepsilon - \delta$ notation this means that for each $\varepsilon > 0$ we can find an index $J > 0$ such that

$$|g_j(x_n) - g_l(x_n)| < \varepsilon, \quad \text{for all } j,l > J.$$

Now for $x \in X$ we can always find an $x_k \in X$ with $\|x - x_k\| < \varepsilon$. By assumption all g_j are bounded and continuous, hence

$$|g_j(x_k) - g_j(x)| \leq \|g_j\|_{op}\|x_k - x\| \leq M\varepsilon,$$

where M denotes the bound on the f_n. Taking all these estimates together we find

$$\begin{aligned}|g_j(x)-g_l(x)| &\leq |g_j(x)-g_j(x_k)|+|g_j(x_k)-g_l(x_k)|+|g_l(x_k)-g_l(x)|\\ &\leq M\varepsilon+\varepsilon+M\varepsilon=(2M+1)\varepsilon.\end{aligned}$$

Then $\{g_j(x)\}_j$ is a Cauchy sequence in $\mathbb{R}$ for each $x\in X$. Hence, by definition, $\{g_j\}$ is weak* convergent $g_j \overset{*}{\rightharpoonup} g$. It is now easy to show that the limit g is linear and bounded, hence $g\in X^*$.

■

A direct consequence is the reflexive-weak compactness result.

Corollary 4.5.3 Let X be a reflexive Banach space and $\{x_n\}\subset X$ a bounded sequence. Then $\{x_n\}$ has a weak convergent subsequence.

Proof. We know that $(X^*)^*=X$, hence by Alaoglu a bounded sequence in X is a bounded sequence in the dual of the dual $(X^*)^*$, and has a weak* convergent subsequence with test functions in X^*. But this is weak convergence in X. ■

■ **Example 4.9** Let $\{f_k\}\subset L^p(\Omega)$ be bounded for a smooth bounded domain Ω and $1<p<\infty$. Then

$$(L^p(\Omega))^*=L^q(\Omega),\quad \frac{1}{p}+\frac{1}{q}=1,\quad (L^p(\Omega))^{**}=L^p(\Omega).$$

Hence $\{f_k\}$ has a weak convergent subsequence $\{f_{n_k}\}$. This means there exists an $f\in L^p(\Omega)$ such that

$$\int_\Omega g f_{n_k}dx\to\int_\Omega g f dx \quad \text{as } k\to\infty,\quad \text{for all } g\in L^q(\Omega).$$

■

■ **Example 4.10** The spaces L^1 and L^∞ are not reflexive, hence there is a real difference between weak and weak* convergence. If we consider a bounded sequence $\{f_n\}\subset L^\infty(\Omega)$, then we understand $L^\infty(\Omega)=(L^1(\Omega))^*$ as dual space of L^1. Hence $\{f_n\}$ is a bounded sequence in $(L^1(\Omega))^*$ and from Alaoglu's Theorem 4.5.2 we get a weak* convergent subsequence $\{f_{n_k}\}$ such that

$$\int_\Omega f_{n_k} g dx\to\int_\Omega f g dx \quad \text{for all } g\in L^1(\Omega).$$

■

4.6 Exercises

Exercise 4.1 (Orthogonal Complements) (level 2)
Let M be a subspace of a Hilbert space. Show that $(M^{\perp})^{\perp} = M$ if and only if M is closed.

Exercise 4.2 (Projections) (level 2)
Let H be a Hilbert space with orthonormal basis (ONB) $\{\psi_i\}_{i\geq 1}$. For each $n \geq 1$ define the projection operator P_n as

$$P_n u = \sum_{i=1}^{n} (\psi_i, u)\psi_i.$$

1. Show that P_n is a projection, i.e. $P_n^2 = P_n$.
2. Show that P_n is self-adjoint.
3. The orthogonal complement of P_n is $Q_n = \mathbb{I} - P_n$. Show that for each u:

$$\lim_{n\to\infty} \|Q_n u\| = 0.$$

Exercise 4.3 (Dense Subspaces) (level 3)
Let V be a linear subspace of a Banach space X. Show that V is not dense in X if and only if there exists $l \in X^*$, $l \neq 0$ such that $l(x) = 0$ for all $x \in V$.

Exercise 4.4 (weak convergence) (level 1)
Let $T \in \mathscr{L}(X,Y)$ and X,Y are Banach spaces. Let x_n be a weak convergent sequence in X. Show that Tx_n converges weakly in Y.

Exercise 4.5 (strong convergence) (level 1)
Let H be a Hilbert space. Let x_n converge weakly to x and assume in addition that the norms converge as well: $\|x_n\| \to \|x\|$. Show that $x_n \to x$ strongly in H.

5. Sobolev Spaces

Contents

5.1 Distributional and Weak Derivatives

First we define distributions and the distributional derivative. We consider $\Omega \subset \mathbb{R}$ and define the set of test functions as

$$\mathscr{D}(\Omega) := C_c^\infty(\Omega).$$

Note that $\mathscr{D}(\Omega)$ is not a Banach space, since it is not closed with respect to the sup-norm.

Definition 5.1.1

1. A *distribution* (generalized function) $f \in \mathscr{D}'(\Omega)$ is continuous in the following sense: If $\phi_n \to \phi$ in $\mathscr{D}(\Omega)$, then $f(\phi_n) \to f(\phi)$ in $\mathbb{R}$. We often write the linear map induced by f as

$$(f, \phi) = f(\phi).$$

2. Given $f \in \mathscr{D}'(\Omega)$. The *distributional derivative* of f is defined as a distribution

$v \in \mathscr{D}'(\Omega)$ that satisfies

$$(v,\phi) = -\left(f, \frac{d\phi}{dx}\right), \qquad \text{for all} \qquad \phi \in \mathscr{D}(\Omega).$$

We can relax the assumption of compact support on the test functions by making it a bit bigger. Then we can introduce a norm and define the *Schwartz space* of functions that decay quickly enough as $|x| \to \infty$:

$$\mathscr{S}(\mathbb{R}) = \left\{ \phi \in C^\infty(\mathbb{R}) : \left\| |x|^k \left(\frac{d}{dx}\right)^\alpha \phi \right\|_\infty < \infty, \quad \text{for all} \quad k \geq 0, \alpha \geq 0 \quad \text{multiindex.} \right\}.$$

Elements of the dual space $\mathscr{S}(\Omega)$ are then called *tempered distributions*. Note that

$$\mathscr{D}(\Omega) \subset \mathscr{S}(\Omega), \qquad \text{hence} \qquad \mathscr{S}'(\Omega) \subset \mathscr{D}'(\Omega),$$

each tempered distribution is automatically a distribution. We call the distributional derivative a weak derivative, if it is integrable:

Definition 5.1.2 Consider $f \in L^1_{loc}(\Omega)$. If the distributional derivative of f satisfies $f' \in L^1_{loc}(\Omega)$, then we call it the *weak derivative* of f.

We then use integration to express the linear map of the derivative as

$$\int_\Omega v\phi dx = -\int_\Omega f\frac{d\phi}{dx}dx \qquad \text{for all} \quad \phi \in \mathscr{D}(\Omega).$$

We write

$$v = f' = \frac{df}{dx} = Df.$$

If u is differentiable and ϕ has compact support, then from integration by parts we have

$$\int_\Omega \frac{du}{dx}\phi dx = -\int_\Omega u\frac{d\phi}{dx}dx,$$

hence weak and classical derivative coincide for differentiable functions.

If α is a multiindex, then the weak derivative can be generalized as

$$\int_\Omega D^\alpha u\, \phi dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\phi dx, \qquad \text{for all} \quad \phi \in \mathscr{D}(\Omega).$$

■ **Example 5.1** Consider the function on the left of Figure 5.1

$$u(x) = \begin{cases} x^2, & 0 \leq x < 1 \\ 1, & 1 < x \leq 2 \end{cases}.$$

Then, using the definition of the weak derivative, we find for each test function $\phi \in \mathscr{D}([0,2])$ that

$$\begin{aligned}
\int_0^2 \frac{du}{dx}\phi dx &= -\int_0^2 u\phi' dx = -\int_0^1 x^2\phi' dx - \int_1^2 \phi' dx \\
&= -x^2\phi|_0^1 + \int_0^1 2x\phi dx - \phi(2) + \phi(1) \\
&= -\phi(1) + \int_0^1 2x\phi dx + \phi(1) \\
&= \int_0^1 2x\phi dx + \int_1^2 0\cdot\phi dx.
\end{aligned}$$

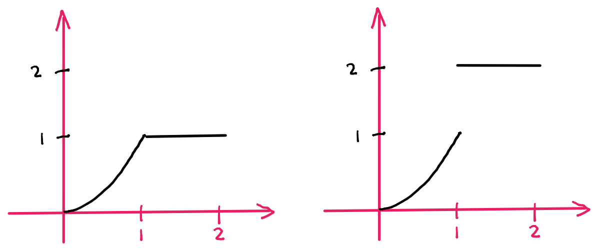


Figure 5.1: Graphs for the examples 5.1 and 5.2.

Hence the distributional derivative is

$$u'(x) = \begin{cases} 2x, & 0 \le x < 1 \\ 0, & 1 < x \le 2 \end{cases}.$$

This is clearly integrable on $[0,2]$, hence it is also the weak derivative of u. ■

■ **Example 5.2** Consider a slightly modified example as shown on the right of Figure 5.1

$$u(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 2, & 1 < x \le 2 \end{cases}$$

Then, for each test function $\phi \in \mathscr{D}([0,2])$ we have

$$\begin{aligned} \int_0^2 \frac{du}{dx}\phi dx &= -\int_0^2 u\phi' dx = -\int_0^1 x^2\phi' dx - \int_1^2 2\phi' dx \\ &= -x^2\phi|_0^1 + \int_0^1 2x\phi dx - 2\phi(2) + 2\phi(1) \\ &= -\phi(1) + \int_0^1 2x\phi dx + 2\phi(1) \\ &= \int_0^1 2x\phi dx + \int_1^2 0\cdot\phi dx + \phi(1). \end{aligned}$$

Here the question arises how to write the evaluation $\phi(1)$ as a linear map $\phi \mapsto \phi(1)$. The *Dirac delta distribution* is the answer. We define $\delta \in \mathscr{D}'(\Omega)$ by its action

$$(\delta_x, \phi) = \phi(x), \qquad \text{for all} \quad \phi \in \mathscr{D}(\Omega).$$

Then the distributional derivative is

$$u'(x) = \delta_1(x) + \begin{cases} 2x, & 0 \le x < 1 \\ 0, & 1 < x \le 2 \end{cases}.$$

This is not integrable on $[0,2]$, since δ_1 it is not a measurable function, it is a distribution. Hence we find a distributional derivative which is not a weak derivative of u. ■

5.2 Sobolev Spaces

Definition 5.2.1 Let $k \in \mathbb{N}$ and $1 \le p \le \infty$. A *Sobolev space* $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) : \ D^\alpha u \in L^p(\Omega), \ 0 \le |\alpha| \le k\},$$

with norm

$$\|u\|_{k,p} = \left(\sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}.$$

For $p = 2$ we define Hilbert spaces $H^k(\Omega) = W^{k,2}(\Omega)$ with inner product

$$(u, v)_{H^k} = \sum_{0 \le |\alpha| \le k} (D^\alpha u, D^\alpha v)$$

and norm

$$\|u\|_{H^k} = \left(\sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_2^2 \right)^{\frac{1}{2}}.$$

Theorem 5.2.1 $W^{k,p}(\Omega)$ is a separable Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^{k,p}(\Omega)$. Then $\{u_n\}$ and $\{D^\alpha u_n\}$ for all $0 \le |\alpha| \le k$ are Cauchy sequences in the Banach space $L^p(\Omega)$. Hence they converge in L^p:

$$u_n \to u, \qquad D^\alpha u_n \to u_\alpha \qquad \text{as} \quad n \to \infty.$$

We take a test function $\phi \in C_c^\infty(\Omega)$ and evaluate the weak dervative

$$\begin{aligned} \int_\Omega D^\alpha u \phi dx &= (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \lim_{n \to \infty} \int_\Omega u_n D^\alpha \phi dx \\ &= \lim_{n \to \infty} \int_\Omega D^\alpha u_n \, \phi dx \\ &= \int_\Omega u_\alpha \phi dx. \end{aligned}$$

Hence $D^\alpha u = u_\alpha$ almost everywhere in Ω. Since in $L^p(\Omega)$ functions are only unique almost everywhere, these two functions are identical in $L^p(\Omega)$: $D^\alpha u = u_\alpha$. This implies $u \in W^{k,p}(\Omega)$.

To prove separability, we consider the map

$$u \mapsto \{u_\alpha\}_{|\alpha| \le k}.$$

This map is an isometry that maps $W^{k,p}(\Omega)$ to the product space $\Pi_{|\alpha| \le k} L^p(\Omega)$, which is a finite product of separable Banach spaces. Hence it is again separable. ■

Another useful space is a Sobolev space for functions that are zero at the boundary.

Definition 5.2.2

$$H_0^k(\Omega) = \overline{C_c^\infty(\Omega)}^{H^k} = \{u \in H^k(\Omega) : u|_{\partial\Omega} = 0\}.$$

For convenience we will often ignore the argument (Ω) on the Sobolev spaces when the domain Ω is clear. For example

$$W^{k,p} = W^{k,p}(\Omega), \qquad H_0^1 = H_0^1(\Omega).$$

Theorem 5.2.2 — Poincaré Inequality. Let Ω be bounded in at least one direction. Without loss of generality this is the x_1-direction and $|x_1| \leq d < \infty$. Then there is a constant $C > 0$ such that for $u \in H_0^1$ we have

$$\|u\|_2 \leq C\|\nabla u\|_{(L^2)^n},$$

Where the index $(L^2)^n$ indicates that each component of the gradient ∇u is an element of $L^2(\Omega)$.

Proof. We consider $u \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}
\|u\|_2^2 &= \int_\Omega |u(x)|^2 dx = -\int_\Omega x_1 \frac{\partial}{\partial x_1}|u(x)|^2 dx \\
&= -\int_\Omega 2x_1 u(x) \frac{\partial}{\partial x_1} u(x) dx \\
&\leq 2d\|u\|_2 \left\| \frac{\partial}{\partial x_1} u \right\|_2 .
\end{aligned}$$

Hence

$$\|u\|_2 \leq 2d \left\| \frac{\partial}{\partial x_1} u \right\|_2 \leq 2d\|\nabla u\|_{(L^2)^n}.$$

Since H_0^k is the closure of C_c^∞ in the H^k norm, the same estimates apply to $u \in H_0^k$. ■

Using the Poincare inequality, we write for a function $u \in H_0^1(\Omega)$

$$\|Du\|_2^2 \leq \|u\|_{H^1}^2 = \|u\|_2^2 + \|Du\|_2^2 \leq (1+C)\|Du\|_2^2.$$

Hence $\|Du\|_2$ and $\|u\|_{H_0^1}$ are equivalent norms on H_0^1. We define

$$\begin{aligned}
\|u\|_{H_0^1} &:= \|Du\|_2, \\
\|u\|_{H_0^k}^2 &:= \sum_{|\alpha|=k} \|D^\alpha u\|_2^2, \\
(u,v)_{H_0^k} &:= \sum_{|\alpha|=k} (D^\alpha u, D^\alpha v).
\end{aligned}$$

Definition 5.2.3 The dual space of H_0^k is defined as the (Hilbert-) dual space

$$H^{-k} := \left(H_0^k(\Omega)\right)^*.$$

We find that functions in H^{-k} are k-times weak derivatives of L^2 functions.

Lemma 5.2.1 Let $f \in H^{-k}$. Then there are functions $g_\alpha \in L^2$ such that

$$f = \sum_{|\alpha|=k} D^\alpha g_\alpha.$$

Proof. Since H^{-k} is the dual of a Hilbert space, it is also a Hilbert space. Then by the Riesz Representation Theorem 4.2.1 there is a representative $u_f \in H_0^k$ such that the action of f can be written as

$$f(v) = (u_f, v)_{H_0^k} \qquad \text{for all} \quad v \in H_0^k.$$

In particular for $\phi \in C_c^\infty$ we have

$$f(\phi) = (u_f, \phi)_{H_0^k} = \sum_{|\alpha|=k} (D^\alpha u_f, D^\alpha \phi) = \sum_{|\alpha|=k} (-1)^{|\alpha|} (D^{2\alpha} u_f, \phi).$$

Hence we have the representation of f as

$$f = \sum_{|\alpha|=k} (-1)^{|\alpha|} D^{2\alpha} u_f = \sum_{|\alpha|=k} D^\alpha \left((-1)^{|\alpha|} D^\alpha u_f\right),$$

where

$$g_\alpha = (-1)^{|\alpha|} D^\alpha u_f \in L^2.$$

■

Lemma 5.2.1 implies that for $u \in H^k(\Omega)$ we have $D^\alpha u \in H^{k-|\alpha|}$. Using the dual spaces, we extend our rainbow of function spaces as shown in Figure 5.2

5.3 Embeddings

Definition 5.3.1 An *embedding* for X into Y is a structure preserving injective map Ψ. In case of Banach spaces, the structure that is preserved is the norm in the sense that $\|\Psi(u)\|_Y \leq \|u\|_X$. We write an embedding as

$$X \hookrightarrow Y.$$

A *compact embedding*

$$X \hookrightarrow\hookrightarrow Y$$

maps bounded sets in X into relatively compact sets in Y.

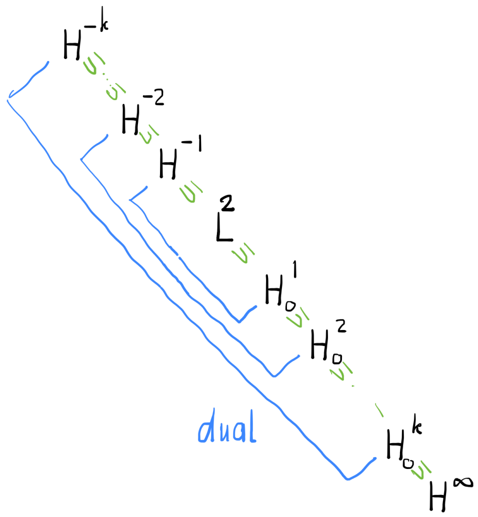


Figure 5.2: A part of the *Rainbow of Function Spaces* for square integrable Sobolev spaces and their duals. Ω is a bounded domain, and the argument (Ω) is suppressed to reduce cluttering the image.

Here we list some inclusion and embedding theorems for Sobolev spaces without giving proofs. The embedding theory is rather involved and technical and can be found in Adams [1] and Robinson [23].

Lemma 5.3.1
1. $C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.
2. $C_c^\infty(\mathbb{R}^n)$ is dense in $H_0^k(\mathbb{R}^n)$.

Theorem 5.3.1 — Sobolev Embedding Theorem. Let $\Omega \subset \mathbb{R}^n$ be bounded with smooth boundary $\partial\Omega \in C^k$. Suppose $u \in H^k$. Then
1. If $k < \frac{n}{2}$, then $u \in L^{\frac{2n}{n-2k}}$ and
$$\|u\|_{\frac{2n}{n-2k}} \leq C\|u\|_{H^k}.$$
The exponent $\frac{2n}{n-2k}$ is called the *Sobolev exponent*.
2. If $k = \frac{n}{2}$, then $u \in L^p$ for all $1 \leq p < \infty$ and
$$\|u\|_p \leq C(p)\|u\|_{H^k}.$$
3. If $k > j + \frac{n}{2}$, $j \in \mathbb{N}$, then $u \in C^j(\bar{\Omega})$ and
$$\|u\|_{C^j} \leq C_j\|u\|_{H^k}.$$

The following Theorem of Rellich Kondrachov is an important compactness result, which we will use often in later chapters. Compactness is a strong property and it has to come from somewhere. In this case it comes from the Arzela-Ascoli theorem, making a strong connection to Real Analysis.

Theorem 5.3.2 — Rellich-Kondrachov Compactness. Let Ω be a bounded C^1 domain, then

$$H^1(\Omega) \hookrightarrow L^2(\Omega).$$

Proof. Take a bounded sequence $\{u_k\} \subset H^1(\Omega)$ and extend it to a bounded sequence $\{v_k\} \subset H^1(\mathbb{R}^n)$ with compact support U. Now we mollify all those functions v_k to $(v_k)_h$, also with compact support, which we call again U. As before, we denote the standard mollifier as $\rho_h(x)$ and we find

$$\begin{aligned}
|v_k(x) - (v_k)_h(x)| &\leq \int_{B_1(0)} \rho_h(z)|v_k(x) - v_k(x-hz)|dz \\
&= \int_{B_1(0)} \rho_h(z) \left| \int_0^{h|z|} \frac{\partial}{\partial r} v_k(x - r\frac{z}{|z|})dr \right| dz \\
&\leq \int_{B_1(0)} \rho_h(z) \int_0^{h|z|} \left| \frac{\partial}{\partial r} v_k(x-rs) \right| dr\, dz,
\end{aligned}$$

where we used a abbreviation of $s = \frac{z}{|z|}$. Integrating this inequality over the compact support U we find

$$\begin{aligned}
\int_U |v_k(x) - (v_k)_h(x)|dx &\leq h \int_U |Dv_k|dx \int_{B_1(0)} |z|\rho(z)dz \\
&\leq h|U|^{1/2}\|Dv_k\|_2 \\
&\leq Ch.
\end{aligned}$$

Thus $(v_k)_h \to v_k$ uniformly on $L^1(\Omega)$. Using the interpolation inequality (Exercise 2.7) we also estimate the L^2 norm as

$$\|v_k - (v_k)_h\|_2 \leq \|v_k - (v_k)_h\|_1^{\frac{r-2}{2(r-1)}} \|v_k - (v_k)_h\|_r^{\frac{r}{2(r-1)}},$$

where the first term converges to zero, and the second term is bounded for $r > 0$ large enough such that $H^1(U) \subset L^r(U)$. Then we obtain that $(v_k)_h \to v_k$ uniformly on L^2.

Finally we employ an Arzela-Ascoli argument and we show that for fixed h the sequence $\{(v_k)_h\}$ is uniformly bounded and equicontinuous. Indeed

$$\begin{aligned}
|(v_k)_h(x)| &\leq \int_{B_1(0)} \rho(z)|v_k(x-zh)|dz \\
&\leq h^{-n}\|\rho\|_\infty \|v_k\|_1 \\
&\leq Ch^{-n}
\end{aligned}$$

and

$$\begin{aligned}
|D(v_k)_h(x)| &\leq \left| \int_{B_1(0)} \rho(z)(D_x v_k)(x-zh)dz \right| \\
&= h^{-1} \left| \int_{B_1(0)} \rho(z)(D_z v_k)(x-zh)dz \right| \\
&= h^{-1} \left| \int_{B_1(0)} D\rho(z) v_k(x-zh)dz \right| \\
&\leq h^{-n-1}\|D\rho\|_\infty \|v_k\|_1.
\end{aligned}$$

Hence by the Arzela-Ascoli Theorem, there exists a convergent subsequence of $\{(v_k)_h\}$ for reach h>0, which uniformly converges in U. Then, by our previous estimates, we build a diagonal sequence . We find subsequences such that

$$\begin{array}{lll} \{(v_{k_{1j}})_1\} & \text{convergent for} & h=1 \\ \{(v_{k_{2j}})_h\} & \text{convergent for} & h=1,\frac{1}{2} \\ \vdots & & \\ \{(v_{k_{lj}})_h\} & \text{convergent for} & h=1,\frac{1}{2},\dots,\frac{1}{l}. \\ \vdots & & \end{array}$$

We set $w_j := v_{k_{jj}}$, where $\{(w_j)_{\frac{1}{j}}\}$ converges for all h in L^2. Then for $n > m$ we have

$$\|w_n - w_m\|_2 \leq \|w_n - (w_n)_{\frac{1}{n}}\|_2 + \|w_m - (w_m)_{\frac{1}{n}}\|_2 + \|(w_n)_{\frac{1}{n}} - (w_m)_{\frac{1}{n}}\|_2,$$

where all terms converge in L^2. ■

R In Figure 5.3 the *Rainbow of Function Spaces*, we summarize all the inclusions and embeddings that we discussed so far. We obtain a scale of spaces from the largest measure space $\mathscr{D}'$ to the smallest function space here, which is C_c^∞. The spaces are related by inclusions, embeddings and as dual spaces and they include spaces of differentiable funcitons, spaces of Hölder continuous functions, spaces of integrable functions and Sobolev spaces. Sobolev embeddings make interesting short-cuts between these spaces.

5.4 Trace Theorem

For integrable functions, or for Sobolev functions on bounded domains Ω, it is not entirely clear how these functions are defined on the domain boundary $\partial\Omega$. In particular, L^p functions are unique only up to sets of measure zero. The boundary $\partial\Omega$ is a set of measure zero, hence an L^p function can have any value on the boundary. Functions in Sobolev spaces benefit from the Sobolev embeddings into continuous functions (see Theorem 5.3.1). In particular if $p > n$ then we have the embedding (see [1, 17, 22]

$$W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

In such cases the functions are continuous as we approach the boundary and they are well defined on $\partial\Omega$. However, when $p \leq n$, we need to define what the boundary (trace) of a function is. This is done via the *trace operator* $\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(\partial\Omega)$.

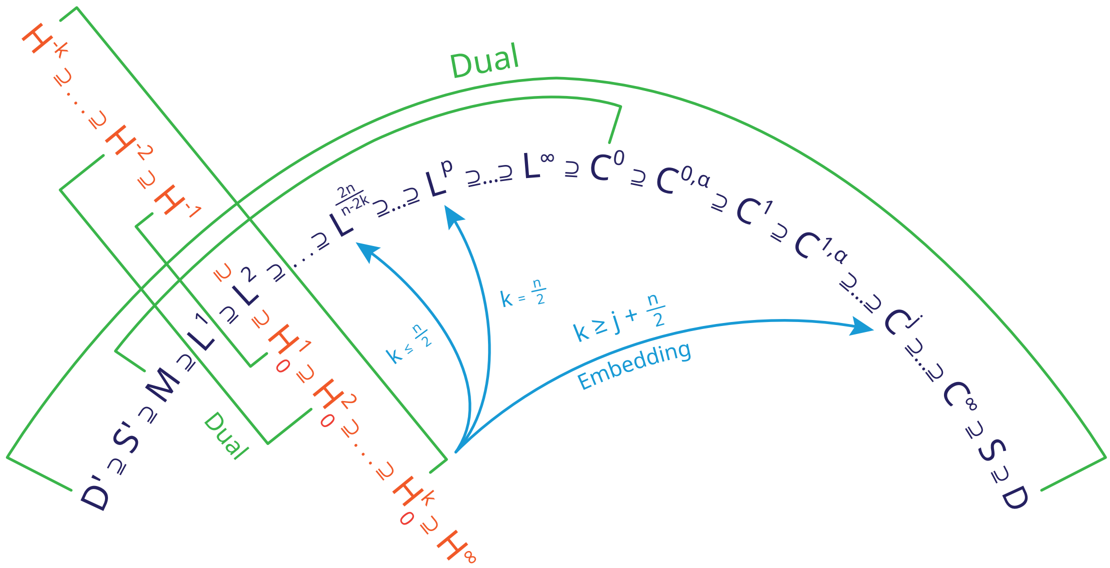


Figure 5.3: The Rainbow of Function Spaces. A visual tool to help to understand the relationships between the function spaces that were mentioned. Note that this is a schematic for pedagogical purposes. Many details are left out. For example, inclusion here is understood as embedding, where the norms are respected. For example $H_0^1 \subset L^2$ means that elements of H_0^1 are also elements of L^2 and a bounded H_0^1 norm implies a bounded L^2 norm. Also the use of the index p as compared to the Sobolev index $\frac{2n}{n-2k}$ is ambiguous, since there are p that are less than the Sobolev index as well, indicated by the dots. All spaces here are Banach spaces, except the space of test functions D. I am very grateful to George Shyntar, who prepared this beautiful rendering of the rainbow.

Theorem 5.4.1 — Trace Theorem. Let $\Omega \subset \mathbb{R}^n$ for $n > 1$ be a smooth bounded domain and $p \in (1,\infty)$. If $p > n$ let $q \in [p,\infty)$ and if $p \le n$ then let $q = \frac{p(n-1)}{n-p}$. Then there exists a bounded linear operator $\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ such that for each $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ we have

$$\mathrm{Tr}[u] = u|_{\partial\Omega}.$$

Proof. The proof of the trace theorem is more involved than is healthy for this text. A challenge arises as we need to define the corresponding measure on the $n-1$-dimensional manifold $\partial\Omega$. We can, for example, use the $n-1$ dimensional Hausdorff measure. But then we have to understand $L^q(\partial\Omega)$ in the right way, which leads to the introduction of Besov spaces, which exceeds the purpose of this text. Detailed proofs of the trace theorem can be found in the [7, 17, 22]. ■

5.5 Banach-Space Valued Functions

Banach-space valued function become important when we solve partial differential equations (PDEs). As an example let us look at a *reaction diffusion equation* for a physical or biological quantity $u(x,t)$:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u). \tag{5.1}$$

If $u(x,t)$ denotes the solution, then for each time $t \ge 0$ we have a function of space x: $u(t) : x \mapsto u(x,t)$, hence this map is in some functions space and we write $u(t) \in X$ for all $t \ge 0$. Typical function spaces in the context of PDEs are given below.

For a given time T, we denote a time interval I as either $(0,T)$ or $[0,T]$. Then

$$C^0(I,X) = \{u : I \to X, \quad \text{continuous in } t \text{ with respect to the norm in } X\}.$$

For example $C^0(0,T;L^p(\Omega))$ has the norm

$$\|u\|_{C^0(0,T;L^p(\Omega))} = \sup_{0\le t\le T} \|u(t)\|_p.$$

The space $L^p(0,T;L^q(\Omega))$ has the norm

$$\|u\|_{L^p(0,T;L^p(\Omega))} = \left(\int_0^T \|u(t)\|_q^p dt\right)^{\frac{1}{p}}.$$

The norm in $L^p(0,T;L^p(\Omega)$ can be written in two ways

$$\|u\|_{L^p(0,T;L^p(\Omega))} = \left(\int_0^T \|u(t)\|_p^p dt\right)^{\frac{1}{p}} = \left(\int_0^T \int_\Omega |u(x,t)|^p dxdt\right)^{\frac{1}{p}} = \|u\|_{L^p([0,T]\times\Omega)}.$$

The space $L^2(0,T;L^2(\Omega))$ is a Hilbert space with inner product

$$(u,v)_2 = \int_0^T \int_\Omega u(x,t)v(x,t)dxdt.$$

We even prove a Fundamental Theorem of Calculus for Banach-space valued functions.

Theorem 5.5.1 — Fundamental Theorem for Sobolev Spaces. Let X be a Banach space and $u \in W^{1,p}(0,T;X)$, $1 \le p \le \infty$. Then

$$u(t) = u(s) + \int_s^t \frac{du}{dt}(\tau)d\tau, \qquad 0 \le s \le t \le T. \tag{5.2}$$

Furthermore $u \in C^0([0,T],X)$ and

$$\sup_{0 \le t \le T} \|u(t)\|_X \le C\|u\|_{W^{1,p}(0,T;X)}.$$

Proof. If $p = \infty$ then $u \in W^{1,\tilde{p}}(0,T;X)$ for some $\tilde{p} < \infty$. Hence we consider $p < \infty$ right away. We mollify u to be equal to zero outside of the interval $[0,T]$ and call the mollification u_h. It is easy to check that

$$\frac{du_h(t)}{dt} = \left(\frac{du(t)}{dt}\right)_h$$

and

$$u_h \to u, \qquad \text{and} \qquad \frac{du_h}{dt} \to \frac{du}{dt} \qquad \text{in} \quad L^p \qquad \text{for } h \to 0.$$

For $h > 0$ the function u_h is continuously differentiable and we use the Fundamental Theorem of Calculus:

$$u_h(t) = u_h(s) + \int_s^t \frac{du_h}{dt}(\tau)d\tau.$$

Taking the limit as $h \to 0$ gives equation (5.2).

For the sup-norm estimate we chose $t = 0$ and $s = t$ and find

$$\|u(0)\|_X \le \|u(t)\|_X + \int_0^t \|u'(\tau)\|_X d\tau,$$

which integrated from 0 to T becomes

$$\begin{aligned} T\|u(0)\|_X &\le \int_0^T \|u(t)\|_X dt + \int_0^T \int_0^t \|u'(\tau)\|_X d\tau dt \\ &\le \|u\|_{L^1(0,T;X)} + T\|u'\|_{L^1(0,T;X)} \\ &\le T^{\frac{1}{q}}\|u\|_{L^p(0,T;X)} + T^{\frac{1+q}{q}}\|u'\|_{L^p(0,T;X)}, \end{aligned}$$

where we used Hölders inequality and the fact that $\frac{1+q}{q} = \frac{1}{q} + 1$ in the last step. We use (5.2) again and the previous estimates to get

$$\begin{aligned} \|u(t)\|_X &\le \|u(0)\|_X + \int_0^t \|u'(\tau)\|_X d\tau \\ &\le \|u(0)\|_X + T^{\frac{1}{q}}\|u'\|_{L^p(0,T;X)} \\ &\le T^{\frac{1-q}{q}}\|u\|_{L^p} + 2T^{\frac{1}{q}}\|u'\|_{L^p} \\ &\le C\|u\|_{W^{1,p}(0,T;X)}, \end{aligned}$$

which proves the last estimate of the Theorem. ■

■ **Example 5.3** If we apply the Fundamental Theorem for Sobolev Spaces 5.5.1 to reaction diffusion equations

$$u_t = D\Delta u + f(u)$$

we often have that

$$u \in L^2(0,T;H_0^1(\Omega)) \quad \text{and} \quad u_t \in L^2(0,T;H^{-1}).$$

Since $H^1 \subset H^{-1}$ it follows that

$$u \in H^1(0,T;H^{-1}).$$

Then, by the above Theorem 5.5.1 we find

$$u \in C^0([0,T];H^{-1}).$$

In this context, the reaction diffusion equation presents itself as an equality in H^{-1}:

$$\underbrace{u_t}_{H^{-1}} = \underbrace{D\Delta \underbrace{u}_{H^1}}_{H^{-1}} + \underbrace{f(u)}_{H^{-1}}.$$

Of course, we made no assumptions here on the growth term $f(u)$. But it is clear what the assumptions of f should be. One possibility is to assume that the map $f : H^1 \to H^{-1}$ is globally Lipschitz continuous. ■

We cite one more result in this context without giving a proof. The proof can be found in Robinson page 214 ff. [23].

Theorem 5.5.2 Let $X \subset\subset H \subset Y$ be Banach spaces, and H reflexive. Suppose $\{u_n\} \subset L^2(0,T;X)$ is uniformly bounded and the time derivatives $\{u_{nt}\} \subset L^p(0,T;Y)$ are uniformly bounded for some $p > 1$. Then there exists a subsequence that converges strongly in $L^2(0,T;H)$.

■ **Example 5.4** The previous theorem opens the door for numerical approximations of partial differential equations such as the *Galerkin Method*. For the Galerkin method we consider a finite set of n orthogonal basis functions, for example sine and cosine functions, and project the differential equation onto the subspace spanned by those basis functions. Since this subspace is finite dimensional, the projection gives us an n-dimensional ordinary differential equation (ODE). We call the solution u_n. We are often able to show that

$$u_n \subset L^2(0,T;H_0^1) \qquad u_{n,t} \subset L^2(0,T;H^{-1})$$

is uniformly bounded in those spaces. Then chosing $X = H_0^1, H = L^2, Y = H^{-1}$ the previous Theorem provides us with a convergent subsequence

$$u_{n_k} \to u \qquad \text{in} \qquad L^2((0,T)\times\Omega).$$

This limit is then a candidate for a solution of the original reaction diffusion equation. This is the essential step of the Galerkin method. However, the set of basis functions need to be chosen carefully. Also, more work is needed to show that the time derivative converges, and that the limit function is continuous in the right spaces. Here we refer to other textbooks such as [23], where the entire proof of the Galerkin method occupies about 6 pages. We develop a simpler case in Exercise 5.3.

■

5.6 Exercises

Exercise 5.1 (Conserved integral) (level 2)
Let V be the subspace of $H^1(\Omega)$, consisting of functions with zero integral

$$V := \left\{ u \in H^1(\Omega) : \int_\Omega u(x)dx = 0 \right\}.$$

Arguing by contradiction, use Rellich-Kondrachov compactness to show that there exists a constant $C > 0$ such that we have a Poincaré inequality

$$\|u\|_2 \leq C\|\nabla u\|_2.$$

■

Exercise 5.2 (Distributions) (level 1)
For $\psi \in C_c^\infty(\Omega)$ and $u \in D'(\Omega)$ we define a distribution $u\psi$ by

$$\langle \psi u, \phi \rangle = \langle u, \psi\phi \rangle, \qquad \text{for all} \qquad \phi \in C_c^\infty(\Omega).$$

Show that indeed $\psi u \in D'(\Omega)$ and that

$$D(\psi u) = uD\psi + \psi Du.$$

■

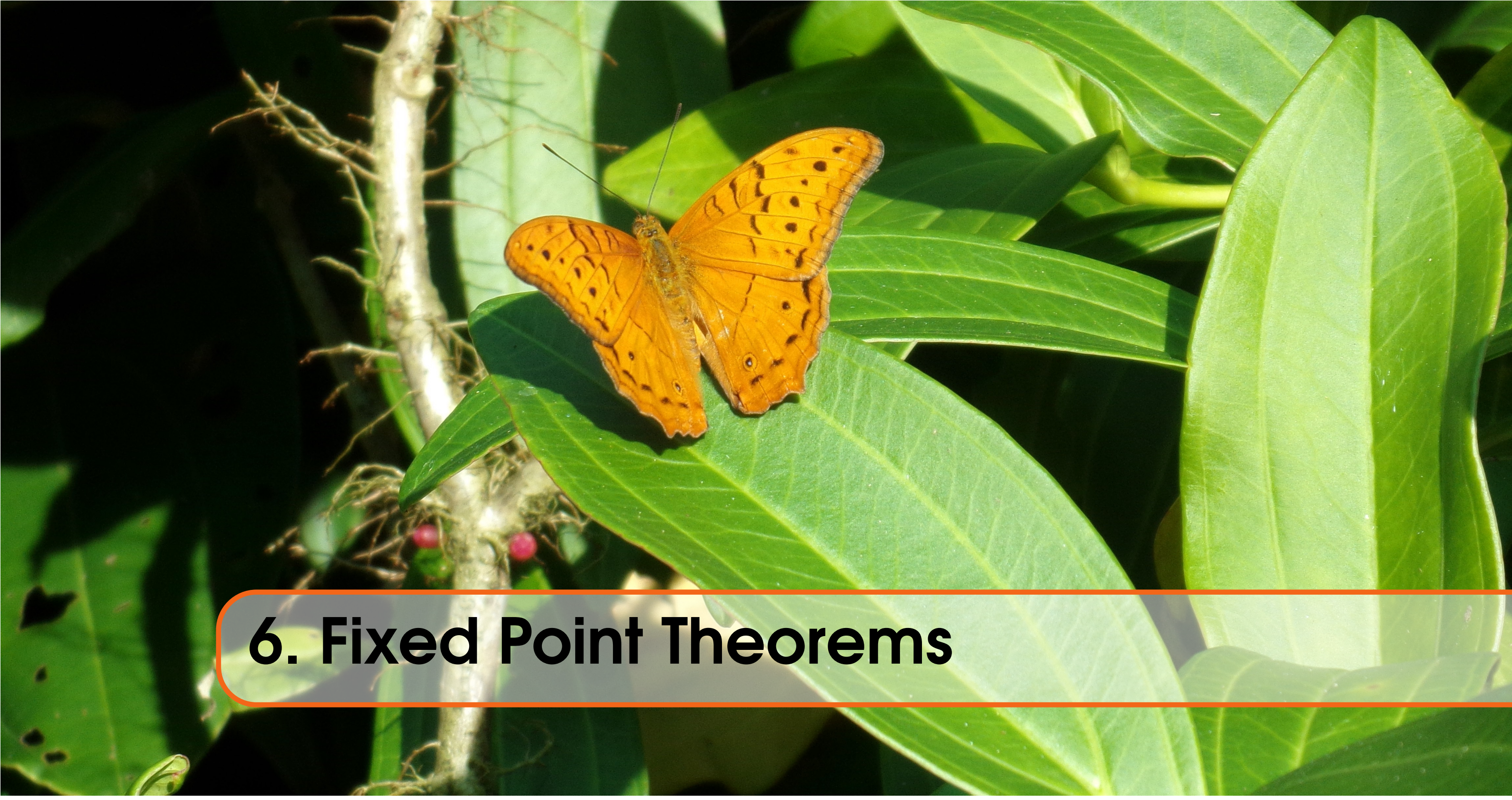
Exercise 5.3 (Galerkin method) (level 3)
Consider the reaction-diffusion equation on Ω:

$$\begin{aligned} u_t &= \Delta u + \alpha u, \\ u|_{\partial\Omega} &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{5.3}$$

with $u_0 \in L^2(\Omega)$. Let $\{\psi_i\}_{i \geq 0}$ denote an ONB of $L^2(\Omega)$ of eigenfunctions of Δ with eigenvalues λ_i. To construct a solution for (9.25), we apply projections P_n to (9.25) and pass to the limit for large n:

1. Apply the projections P_n to (9.25) and argue that the resulting system has a unique solution u_n for each $n \geq 1$.
2. Derive the estimate
$$\frac{d}{dt}\frac{\|u_n\|_2^2}{2} + \|\nabla u_n\|_2^2 \leq \alpha\|u_n\|_2^2. \tag{5.4}$$
3. Use Gronwall's Lemma to show that for each $T > 0$ u_n is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.
4. Integrate (5.4) from 0 to T and show that u_n is uniformly bounded in the space $L^2(0,T;H^1(\Omega))$.

5. Show that u_n has a convergent subsequence with limit u. Identify the type of convergence and the function space of the limit function u.
6. Explain in which sense $P_{n_j} u_{n_j}$ converges to u?
7. Explain in which sense Δu_{n_j} converges to Δu?
8. Use (9.25) and explain in which sense $\frac{\partial u_{n_j}}{\partial t}$ converges to $\frac{\partial u}{\partial t}$?
9. Conclude that the limit u is indeed a weak solution of (9.25).



6. Fixed Point Theorems

Contents

Fixed point theorems are an important tool in applied mathematics. They are used in many different contexts. For example in existence theories, a linear or non-linear map is defined such that a fixed-point corresponds to a solution of the problem. In other cases, these results can be used to compute spectra of operators. In this chapter, we cover some of the most important fixed-point theorems of Banach, Brouwer, Schauder, and the Leray-Schauder Principle.

Definition 6.0.1 Given a map $T : X \to X$ from a Banach space into itself. Each solution of $T(x) = x$ is called a *fixed point*.

6.1 The Banach Fixed-Point Theorem

The main idea is to solve a fixed-point problem $x = Tx$ iteratively, i.e., start at some x_0 and build a sequence

$$x_0 \in X, \qquad x_{n+1} = Tx_n$$

and show that x_n converges to x in some appropriate sense.

Definition 6.1.1 Let (X,d) be a metric space and $M \subset X$. A (nonlinear) map $T : M \to X$ is called

- *k-contractive* if and only if

$$d(T(x),T(y)) \leq k\, d(x,y), \qquad \text{for all} \quad x,y \in M, \qquad 0<k<1,$$

- *contractive* if and only if

$$d(T(x),T(y)) < d(x,y), \qquad \text{for all} \quad x,y \in M, \qquad x \neq y.$$

The main theorem of this section is:

Theorem 6.1.1 — Banach's fixed-point theorem. Let $M \subset X$ be a non-empty, closed subset of a complete metric space X, and let $T : M \to M$ be k-contractive with $k < 1$. Then

1. T has exactly one fixed point in M.
2. The sequence $x_{n+1} = Tx_n$ converges to the fixed point x for each initial point $x_0 \in M$.

Proof. Consider $x_0 \in M$ and $x_{n+1} = Tx_n$. Then

$$d(x_{n+1},x_n) = d(T(x_n),T(x_{n-1})) \leq k\, d(x_n,x_{n-1}) \leq \cdots \leq k^n d(x_1,x_0).$$

and then

$$\begin{aligned}
d(x_{n+m},x_n) &\leq d(x_{n+m},x_{n+m-1}) + d(x_{n+m-1},x_{n+m-2}) + \cdots + d(x_{n+1},x_n) \\
&\leq \left(k^{n+m-1} + k^{n+m-2} + \cdots + k^n\right) d(x_1,x_0) \\
&\leq k^n \sum_{j=0}^{m-1} k^j\, d(x_1,x_0) \\
&\leq \frac{k^n}{1-k} d(x_1,x_0) \\
&\to 0, \qquad \text{for} \qquad n \to \infty.
\end{aligned}$$

Hence $\{x_n\}_n$ is a Cauchy sequence, and since X is complete, it converges to a point $x \in X$.

Moreover, $T : M \to M$ and M is closed, hence the limit $x \in M$. While T is a k-contraction, it is also continuous, hence $T(x) = x$ and x is a fixed point.

To show uniqueness, we assume there is another fixed point $y \in M$. Then

$$d(x,y) = d(T(x),T(y)) \leq k\, d(x,y)$$

and $k < 1$, hence $d(x,y) = 0$, and we find $x = y$. ■

■ **Example 6.1 — Ordinary differential equations.** Arguably, the most known application of the Banach Fixed Point Theorem 6.1.1 is the Theorem of Picard and Lindelöff to solve ordinary differential equations. We invite the readers to consult standard text books on ODEs [20]. ■

6.2 The Brouwer and Schauder fixed-point theorems

The Brouwer fixed point theorem is a central piece in algebraic topology in finite dimensions $\mathbb{R}^n$. It is equivalent to the negative retraction principle, and this is where we start our arguments. The Brouwer fixed point theorem is a rather general result for finite dimensional spaces. We lift it to the infinite dimensional setting in Schauders fixed-point theorem, where we employ some sense of compactness.

Definition 6.2.1 Let X be a topological space and $T : X \to M$ a continuous map with $M \subset X$. T is called a *retraction* on M if $T(x) = x$ for all $x \in M$. In this case M is called a *retract* of X.

■ **Example 6.2** For example $X = \mathbb{R}^n, M = B_R(0)$ and

$$T(x) = \begin{cases} x & \text{if} \quad x \in B_R(0), \\ R\frac{x}{|x|} & \text{if} \quad x \notin B_R(0), \end{cases}$$

is a continuous retraction of X to M. ■

Theorem 6.2.1 — Negative retract principle. In $\mathbb{R}^n$ there is no continuous map $T : \bar{B}_1(0) \to \partial B_1(0)$, which leaves the boundary points fixed, i.e.

$$T(x) = x, \qquad \text{for all} \qquad x \in \partial B_1(0).$$

Proof. See textbooks on topology, for example [25]. ■

Theorem 6.2.2 — Brouwer's fixed point theorem. Each continuous map $T : \bar{B}_1(0) \to \bar{B}_1(0)$ in $\mathbb{R}^n$ has a fixed point.

Proof. Assume otherwise, i.e. $T(x) \neq x$ for all $x \in \bar{B}_1(0)$. Then we construct a continuous map from the ball to the boundary $r : \bar{B}_1(0) \to \partial B_1(0)$ as follows. For each $x \in B_1(0)$ we follow the line segment from $T(x)$ to x to the boundary. Since $T(x) \neq x$ for all x, this line segment is well defined, and it has a unique intersection with the boundary, called $r(x)$. This map is continuous, since T is continuous, and it satisfies $r(x) = x$ for all $x \in \partial B_1(0)$. This is a contradiction to the negative retract principle in Theorem 6.2.1. ■

Corollary 6.2.3 If $M \subset \mathbb{R}^n$ is homeomorphic to $\bar{B}_1(0)$, then each continuous map $T : M \to M$ has a fixed point.

Proof. Let $\phi : M \to \bar{B}_1(0)$ denote the homeomorphism and ϕ^{-1} its continuous inverse. Then we apply Brouwers fixed point theorem to the conjugate map

$$f := \phi \circ T \circ \phi^{-1} : \bar{B}_1(0) \to \bar{B}_1(0).$$

■

■ **Example 6.3** The previous corollary includes nonempty convex compact sets in $\mathbb{R}^n$, star-shaped domains, and p-norms. ■

Before we proceed to Schauder's fixed point theorem, we need a technical Lemma about convex combinations.

Proposition 6.2.4 Let $K \Subset X$ be a compact subset and let $\varepsilon > 0$ be given. Let A denote a finite set $A \subset K$ such that

$$K \subset \bigcup_{a \in A} B_\varepsilon(a)$$

is a finite open covering of K. We define a convex combination of the elements of A as a map $\phi_A : K \to K$ as

$$\phi_A(x) = \frac{\sum_{a \in A} m_a(x)\, a}{\sum_{a \in A} m_a(x)}, \quad \text{where} \quad m_a(x) = \begin{cases} 0 & \text{for } \|x-a\| \geq \varepsilon \\ \varepsilon - \|x-a\| & \text{for } \|x-a\| < \varepsilon \end{cases}.$$

The function $\phi_A(x)$ denotes a weighted average of the anchor points $a_i \in A$, weighted by the distance to x. Then $\phi_A(x)$ is continuous on K and for each $x \in K$ we have

$$\|\phi_A(x) - x\| < \varepsilon.$$

Proof. Note that $m_a(x) \geq 0$ and for each $x \in K$ there is at least one $a \in A$ such that $x \in B_\varepsilon(a)$. Hence $\sum_{a \in A} m_a(x) > 0$ and $\phi_A(x)$ is well defined on K. The map $m_a : K \to [0, \varepsilon]$ is continuous, hence ϕ_A is continuous on K. Now, if $x \in K$ then

$$\phi_A(x) - x = \frac{\sum_{a \in A} m_a(x)(a-x)}{\sum_{a \in A} m_a(x)},$$

and for those m_a with $m_a(x) > 0$ we have $\|x-a\| < \varepsilon$. Hence

$$\|\phi_A(x) - x\| \leq \frac{\sum_{a \in A} m_a(x)\, \varepsilon}{\sum_{a \in A} m_a(x)} = \varepsilon.$$

■

Theorem 6.2.5 — Schauder's fixed point theorems. Let X be a Banach space and $M \subset X$ a nonempty, bounded, and convex subset. Consider a continuous map $T : M \to M$.

1. If M is compact, then T has a fixed point.
2. If T is compact, then T has a fixed point.

Proof. We define $K = \overline{T(M)}$. In both cases the set K is compact and for each $n \in \mathbb{N}$ we can find a finite covering with balls of radius $1/n$:

$$K \subset \bigcup_{a \in A_n} B_{\frac{1}{n}}(a),$$

where A_n is a finite set. We use the above function ϕ_A to define ϕ_{A_n}.

The element $\phi_{A_n}(x) \in X$ is a convex combination of elements $a \in A_n$, hence

$$\phi_n(K) \subset \text{convex hull}(K) \subset M,$$

since $T : M \to M$ and M is convex. Then we define maps $T_n = \phi_{A_n} \circ T : M \to M$ with

$$\|T_n(x) - T(x)\| \leq \frac{1}{n},$$

by Proposition 6.2.4.

We define $M_n := M \cap \operatorname{span}(A_n)$. Since $\operatorname{span}(A_n)$ is a finite dimensional subspace, the sets M_n are a bounded, closed and convex subsets of a finite dimensional subspace, and $T_n : M_n \to M_n$ is continuous. By the Corollary to Brouwers fixed point theorem (Corollary 6.2.3) we have a fixed point x_n for each n: $T_n(x_n) = x_n$.

1. If M is compact, then the bounded sequence $\{x_n\}_n$ has a convergent subsequence

$$x_{n_j} \to x, \qquad \text{for} \quad j \to \infty,$$

and since T is continuous we have $T(x) = x$.
2. If T is compact, then $\{T(x_n)\}_n$ has a convergent subsequence, and the result is the same as under item 1.

■

■ **Example 6.4 — Elliptic equations.** Schauder theory is an important tool to solve elliptic partial differential equations. Please see the comprehensive introduction [8]. ■

6.3 The Leray-Schauder Principle

The Leray-Schauder principle shows how a-priori estimates can be used to find solutions to equations and fixed-points of operators.

Theorem 6.3.1 Let X be a Banach space and $A : X \to X$ a compact linear map. Suppose each solution of $u = \gamma A u$ satisfies an a-priori estimate

$$\|u\| \le c \qquad \text{for all} \quad \gamma \in [0,1].$$

Then $u = Au$ has a solution.

Proof. Set $M := \{u \in X, \|u\| \le 2c\}$ and

$$Lu := \begin{cases} Au & \text{if } \|Au\| \le 2c, \\ 2c\frac{Au}{\|Au\|} & \text{if } \|Au\| > 2c. \end{cases}$$

Then $\|Lu\| \le 2c$ for all $u \in X$ and $L : M \to M$. The map L is continuous and compact, since A is continuous and compact. By Schauder's fixed point theorem, L has a fixed point $Lu = u$, $u \in M$. If $\|Au\| \le 2c$, then $Au = Lu = u$ and we are done. If $\|Au\| > 2c$, then

$$\|u\| = \|Lu\| = 2c\frac{\|Au\|}{\|Au\|} = 2c.$$

On the other hand we find that

$$u = Lu = \frac{2c}{\|Au\|} Au = \gamma A u, \qquad \text{for some } \gamma \in (0,1).$$

Hence the a-priori estimate applies to u and we find $\|u\| \le c$, which is a contradiction. The fixed point of L satisfies $\|Au\| \le 2c$ and it is also a fixed point of A. ■

6.4 The Lax Milgram Lemma

(The presentation is adapted from L.C. Evans' book [7])

The Lax Milgram Lemma is not really a fixed-point theorem, but it deals with solutions to nonlinear equations, in particular quadratic operator equations. It is a generalization of Riesz representation theorem.

Theorem 6.4.1 — Lax Milgram. Let H be a Hilbert space and $B: H \times H \to \mathbb{R}$ a continuous, bilinear mapping. We assume that here are two constants $\alpha, \beta > 0$ such that

$$\begin{array}{rcllr} |B(u,v)| & \leq & \alpha \|u\| \, \|v\| & \text{for all} & u,v \in H, \\ B(u,u) & \geq & \beta \|u\|^2 & \text{for all} & u \in H. \end{array}$$

The second condition is known as the *coercivity condition*. Then for each $f \in H^*$ there exists a solution $u \in H$ of

$$B(u,v) = f(v), \qquad \text{for all} \qquad v \in H.$$

Proof. We proceed in several steps.

1. For each fixed $u \in H$, the mapping $v \mapsto B(u,v)$ is linear and bounded, hence in H^*. By the Riesz representation theorem 4.2.1, there exists a unique representative $w \in H$ such that

   $$(w,v) = B(u,v), \qquad \text{for all} \qquad v \in H.$$

   We write $Au = w$, defining a map $A: H \to H$ such that

   $$B(u,v) = (Au,v).$$

   To prove the Lax-Milgram results, we need to show that A is invertible.
2. We show first that $A: H \to H$ is linear and bounded. By direct computation we get:

   $$(A(a_1u_1 + a_2u_2), v) = B(a_1u_1 + a_2u_2, v) = a_1B(u_1,v) + a_2B(u_2,v) = (a_1Au_1 + a_2Au_2, v).$$

   Furthermore

   $$\|Au\|^2 = (Au,Au) = B(u,Au) \leq \alpha \|u\| \, \|Au\|.$$

   Hence

   $$\|Au\| \leq \alpha \|u\| \qquad \text{and} \qquad \|A\| \leq \alpha.$$

3. From the coercivity condition we get

   $$\beta \|u\|^2 \leq B(u,u) = (Au,u) \leq \|Au\| \, \|u\|.$$

   Hence

   $$\beta \|u\| \leq \|Au\|.$$

   This implies that the mapping A is injective and the range $R(A)$ is closed.

4. We now show that $R(A) = H$. If not, then since the range is closed, we find a nontrivial orthogonal element $z \in R(A)^\perp$. For this z we have

$$\beta \|z\|^2 \leq B(z,z) = (Az,z) = 0,$$

which is a contradiction. Hence $R(A) = H$ and we can invert A.

5. Then for each $f \in H^*$, we find $w \in H$ and $u \in H$ such that

$$f(v) = (w,v) = (Au,v) = B(u,v),$$

which was to be shown.

6. The last step is to prove uniqueness of the solution u. Assume we have two solutions u and $\tilde{u}$. Then $B(u-\tilde{u},v) = 0$ for all $v \in H$. But by coercivity

$$\beta \|u-\tilde{u}\|^2 \leq B(u-\tilde{u},u-\tilde{u}) = 0$$

and $u = \tilde{u}$.

■

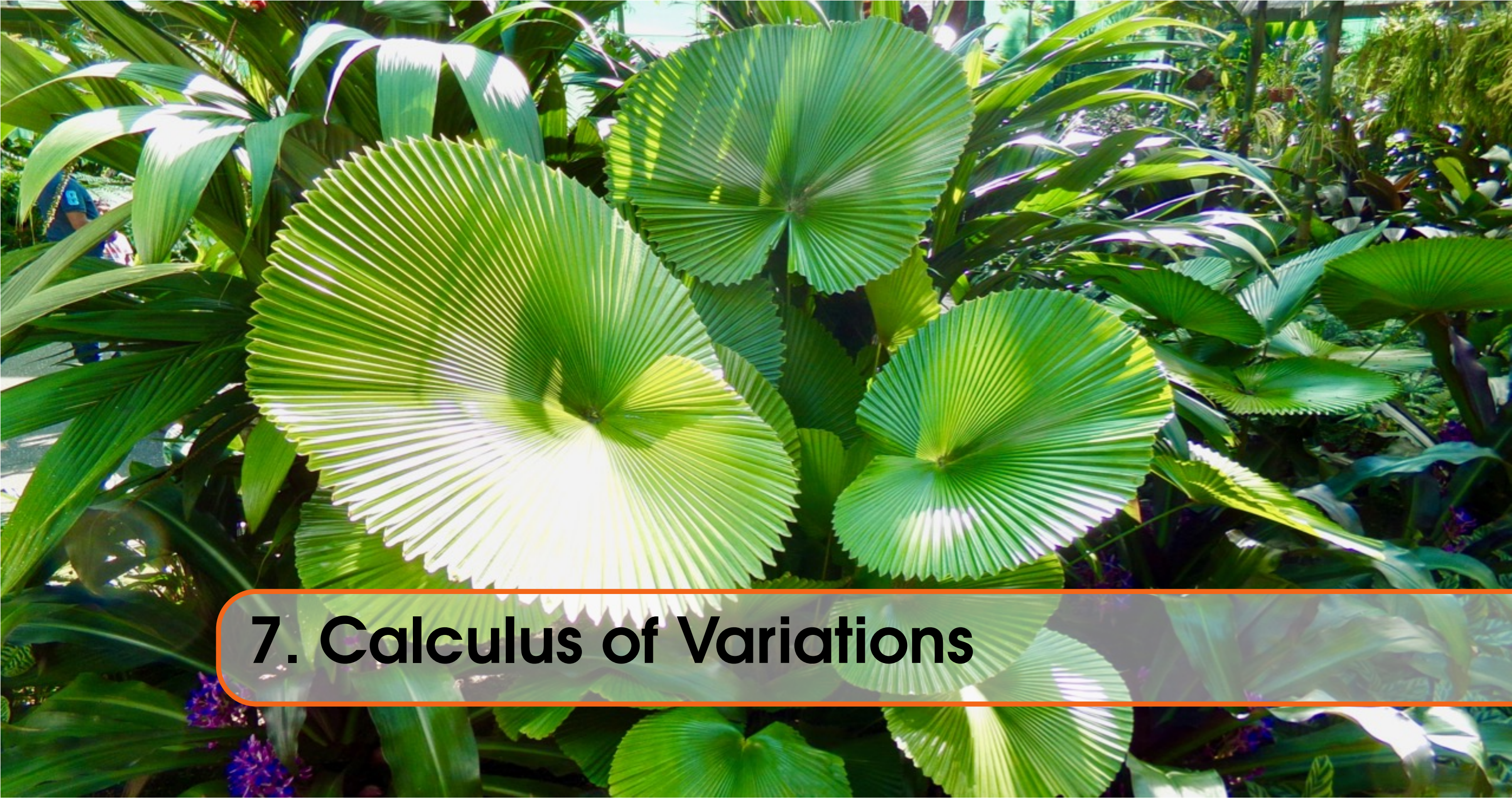
■ **Example 6.5 — Application to PDEs.** Suppose that we want to solve a nonhomogeneous PDE of the form

$$Lu + \mu u = f$$

where L is a general second order differential operator. Using weak formulation with test functions v we write

$$\underbrace{\int Lu\, v\, dx + \mu \int uv dx}_{=B(u,v)} = \underbrace{\int fv dx}_{=(f,v)},$$

and the problem becomes: given $f \in X^*$, find $u \in X$ to solve $B(u,v) = f(v)$, i.e. a Lax-Milgram situation. The details of the estimates in Theorem 6.4.1 need to be worked out, depending on the second order differential operator and possible boundary conditions. ■



7. Calculus of Variations

Contents

See textbooks of L.C. Ewans [7] and B. van Brunt [3].

7.1 Introductory Examples

The Calculus of Variations is a rather traditional area of Analysis. It was developed in the context of mechanics, but it has grown from there to cover most areas of applied mathematics, including PDE analysis, financial markets, math biology, etc. While teaching Calculus of Variations, we want to ensure that the roots in mechanics as well as the use for other areas of applied mathematics are covered. We begin with some classical examples.

7.1.1 The Catenary (Hanging Chain)

We consider a chain hanging between two points (x_0, y_0) and (x_1, y_1). The mass per unit length is denoted by m and the height is parameterized as $y(s)$, $s \in [0, L]$ with $y(0) = y_0$ and $y(L) = y_1$, and L denotes the length of the chain.

The potential energy in a gravitational field is

$$W_p(y) = \int_0^L mgy(s)\, ds.$$

We transform the integral from arc-length s to the x-coordinate by using the arc-length formula

$$ds = \sqrt{1 + y'(x)^2}\, dx,$$

which gives

$$W_p(y) = \int_{x_0}^{x_1} mgy(x)\sqrt{1 + y'(x)^2} dx,$$

where m and g are constants. To find the profile $y(x)$ of the hanging chain, we minimize the potential energy, i.e. we minimize

$$J(y) := \int_{x_0}^{x_1} y\sqrt{1 + y'^2} dx, \qquad y(x_0) = y_0, \qquad y(x_1) = y_1,$$

where we ignore the constant factors m and g. The question arises: How to find the minimum profile $y(x)$? We come back to this question in Example 7.4.

7.1.2 The Brachystochrone (The Fastest Roller)

Here we are interested of the fastest path that a rolling frictionless object can take to get from a point (x_0, y_0) to a lower point (x_1, y_1) with $x_0 < x_1$ and $y_0 > y_1$. Again we use s for the arc-length along the path and we denote the speed of the ball as $v(s)$, $s \in [0, L]$, where L is the length of the path. The differential time element is $dt = \frac{ds}{v(s)}$, such that the total time is

$$T(y) = \int_0^L \frac{ds}{v(s)}.$$

Since there is no friction, we conserve the total energy

$$\frac{1}{2}mv^2(x) + mgy(x) = C, \qquad \text{for all} \qquad x \in [x_0, x_1],$$

where the constant C is given from the initial condition as

$$C = \frac{1}{2}mv^2(x_0) + mgy_0.$$

Then

$$v(x) = \sqrt{\frac{2C}{m} - 2gy(x)}.$$

Using again the transformation into the local coordinate x, as was done for the catenary, we find the total time to be

$$T(y) = \int_{x_0}^{x_1} \frac{\sqrt{1 + y(x)'^2}}{\sqrt{\frac{2C}{m} - 2gy(x)}} dx.$$

We combine some of the constants and y into a new function

$$w(x) = \frac{1}{2g}\left(\frac{2C}{m} - 2gy(x)\right),$$

such that $w' = -y'$ and we find the optimization problem is to minimize

$$J(y) = \int_{x_0}^{x_1} \frac{\sqrt{1 + w'^2}}{\sqrt{w}} dx, \qquad w(x_0) = \frac{1}{2g}\left(\frac{2C}{m} - 2gy_0\right), \qquad w(x_1) = \frac{1}{2g}\left(\frac{2C}{m} - 2gy_1\right).$$

Again we find a complicated minimization problem and we will show how this is solved in Example 7.5.

7.1.3 Motivation from PDEs

An abstract, nonlinear, time independent PDE can be written as

$$A[u] = 0.$$

Here we would like to understand the differential equation as a chracterization of some energy minimizers. In this case we are looking for cases where A is something like a "derivative" of some "energy functional" J with $J' = A$ so the differential equations becomes

$$J'[u] = 0.$$

Solving this equation then corresponds to finding local maxima, minima and saddle points of J. The Calculus of Variations, as we develop here, is exactly the method that connects abstract energy functional J with a functional derivative $J' = A$. Finding minimum and maximum then reduces to solving the *Euler-Lagrange equations*. Once such a relation is established we encounter two cases in the literature. Sometimes it is easier to optimize J than to solve the Euler-Lagrange PDEs, while in other cases it is easier to solve the Euler-Lagrange PDEs rather than to optimize J. We will discuss examples for both of these cases.

If the functional gradient of J is defined, we can also study the *gradient flow*

$$u_t = -J'[u], \qquad \text{or} \qquad u_t = -A[u].$$

For example when $A = -\frac{\partial^2}{\partial x^2}$ we get the heat equation as gradient flow. But what is the corresponding energy J ? To answer this question we need to understand what J' actually means, and then make a formal connection between the PDE and the optimization problem.

7.2 First Variation and the Euler-Lagrange Equations

Suppose that $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary $\partial\Omega$.

7.2.1 Lagrangian

The *Lagrangian* is twice continuously differentiable function

$$L : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

We apply L to a real function $u(x)$ and its gradient $\nabla u(x) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ as $L(\nabla u, u, x)$. In the abstract formulation we often use the symbol p for the gradient and z for u and write $L(p, z, x)$. In this notation $\nabla_p L$ denotes the gradient of L as it depends on the vector component p, and similarly $\frac{\partial}{\partial z} L$ denotes the derivative of L with respect to the second entry z, and for short we write $L \in C^2$.

We define the *action* or *energy* as

$$J(u) = \int_\Omega L(\nabla u, u, x) dx,$$

and we stipulate boundary conditions

$$u\Big|_{\partial\Omega} = g.$$

7.2.2 First Variation

Assume now that $u(x)$ is a minimizer of $J(u)$ and assume that it is smooth $u \in C^2(\bar{\Omega})$. Choose any perturbation function $v \in C_c^\infty(\Omega)$ and consider the perturbed energy $J(u + \varepsilon v)$ for small ε. Now $\varepsilon \mapsto J(u + \varepsilon v)$ is a real function and we can use real calculus to find a minimum. A necessary condition is

$$\frac{d}{d\varepsilon} J(u + \varepsilon v)\Big|_{\varepsilon = 0} = 0.$$

This derivative is simply called the *First Variation* for J and denoted as

$$\delta J(u) = \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0}.$$

Let us compute the first variation:

$$\begin{aligned}
&\frac{d}{d\varepsilon} J(u+\varepsilon v) \\
= &\int_\Omega \nabla_p L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\cdot\nabla v + \frac{\partial}{\partial z} L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\, v\, dx \qquad (7.1) \\
= &-\int_\Omega \nabla\left(\nabla_p L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\right) v dv \\
&+\int_\Omega \frac{\partial}{\partial z} L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\, v\, dx,
\end{aligned}$$

where we used that v has compact support, hence there are no boundary terms through integration by parts. Then

$$\delta J(u) = \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = \int_\Omega \left[-\nabla(\nabla_p L(\nabla u, u, x)) + \frac{\partial}{\partial z} L(\nabla u, u, x)\right] v\, dx,$$

for all test function $v\in C_c^\infty(\Omega)$. Hence, the first variation is zero when the *Euler-Lagrange Equation*

$$-\nabla\big(\nabla_p L(\nabla u, u, x)\big) + \frac{\partial}{\partial z} L(\nabla u, u, x) = 0 \qquad (7.2)$$

is satisfied, in the sense of distributions. Since we assumed that u and L are smooth, the equation holds in the classical sense as well. Note that we use ∇ for the spatial gradient and ∇_p for the partial derivative of L with respect to the vector quantity p.

■ **Example 7.1 — Dirichlet's Principle.** We consider the Lagrangian

$$L(\nabla u, u, x) = \frac{1}{2}|\nabla u|^2.$$

The corresponding energy is called the *Dirichlet energy*, or the *Dirichlet integral*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$$

(and sometimes the factor 1/2 is not used). Let us compute the Euler-Lagrange equation (7.2) of this energy.

$$\frac{\partial}{\partial z} L = 0$$

and

$$\begin{aligned}
\nabla_p L &= \frac{1}{2}\nabla_p |p|^2 \\
&= \frac{1}{2}\left(\frac{\partial}{\partial p_1}(p_1^2+\cdots+p_n^2), \ldots, \frac{\partial}{\partial p_n}(p_1^2+\cdots+p_n^2)\right) \\
&= (p_1, p_2, \ldots, p_n) = p = \nabla u.
\end{aligned}$$

Then the Euler-Lagrange equation (7.2) becomes

$$-\nabla\left(\nabla_p L\right)+\frac{\partial}{\partial z}L=-\nabla\nabla u=-\Delta u=0,$$

i.e. we get the *Poisson equation*

$$\Delta u=0.$$

■

■ **Example 7.2 — Generalized Dirichlet Principle.** We can generalize the above Dirichlet principle for any quadratic Lagrangian. In the general form we have

$$L(\nabla u,u,x)=\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)\underbrace{\frac{\partial u}{\partial x_i}}_{p_i}\underbrace{\frac{\partial u}{\partial x_j}}_{p_j}-f(x)u$$

Then

$$\frac{\partial}{\partial p_i}L=\sum_{j=1}^{n}a^{ij}(x)\frac{\partial u}{\partial x_j},\qquad \text{and}\qquad \frac{\partial}{\partial z}L=f(x).$$

The energy is

$$J(u)=\int_\Omega\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-f(x)u\,dx$$

and the Euler-Lagrange equation becomes

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sum_{j=1}^{n}a^{ij}(x)\frac{\partial}{\partial x_j}u(x)=f(x).$$

Using matrix notation we write this as

$$\nabla\Big(A\nabla u\Big)=f,\qquad A(x)=\left(a^{ij}(x)\right)_{i,j},$$

which is an *anisotropic Helmholtz equation in the Fickian form*.

■

7.3 Special Cases

7.3.1 Special Case I: no u-dependence

Here we assume that the Lagrangian is independent of $u(x)$, where x is a one-dimensional variable. Then

$$L(u',x),\qquad J(u)=\int_{x_0}^{x_1}L(u',x)dx.$$

In this case the Euler-Lagrange equation (7.2) simplifies to

$$\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u'}\right)=0.$$

This is integrated to find

$$\frac{\partial L}{\partial u'} = \text{constant} = c_1.$$

We call this relationship the *First Integral*. Here is an example:

■ **Example 7.3** Consider

$$L(u',x) = e^x\sqrt{1+u(x)'^2}, \quad \text{and} \quad J(u) = \int_{x_0}^{x_1} e^x\sqrt{1+u(x)'^2}dx.$$

The first integral of the Euler-Lagrange equation becomes

$$\frac{\partial L}{\partial u'} = \frac{e^x u'}{\sqrt{1+u'^2}} = c_1,$$

which is transformed to read

$$u' = \frac{c_1}{\sqrt{e^{2x}-c_1^2}}. \tag{7.3}$$

Note here that

$$c_1 = \frac{e^x u'}{\sqrt{1+u'^2}} < e^x,$$

hence there is no concern about the sign of the square root term in (7.3) The equation (7.3) is integrated to find an explicit solution

$$u(x) = \sec^{-1}\left(\frac{e^x}{c_1}\right) + c_2.$$

■

7.3.2 Special Case II: no x-dependence

Again, we consider a one-dimensional spatial variable x, but now, L does not explicitly depend on x:

$$L(u,u'), \qquad J(u) = \int_{x_0}^{x_1} L(u,u')dx. \tag{7.4}$$

In this case we also find a first integral, which turns out to be a Hamiltonian of the system.

Theorem 7.3.1 The function

$$H(u,u') = u'\frac{\partial L(u,u')}{\partial u'} - L(u.u')$$

is a Hamiltonian of system (7.4), i.e., H is conserved along solution curves $u(x)$.

Proof. We compute the derivative of H:

$$\begin{aligned}
\frac{d}{dx}H(u,u') &= \frac{d}{dx}\left(u'\frac{\partial L}{\partial u'} - L\right) \\
&= u''\frac{\partial L}{\partial u'} + u'\frac{d}{dx}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}u' - \frac{\partial L}{\partial u'}u'' \\
&= u'\left(\frac{d}{dx}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}\right) \\
&= 0,
\end{aligned}$$

using the Eurler-Lagrange equation in the last step. ■

With these techniques in place we can come back to the initial examples of the hanging chain (catenary) and the rolling ball (brachystochrone).

■ **Example 7.4 — Hanging chain.** For the hanging chain problem, we found the energy

$$J(u) = \int_{x_0}^{x_1} u\sqrt{1+u'^2}dx.$$

Here the Lagrangian does not explicitly depend on x, hence we define the Hamiltonian as in Theorem 7.3.1:

$$\begin{aligned}
H(u,u') &= u'\frac{\partial L}{\partial u'} - L \\
&= u'\frac{uu'}{\sqrt{1+u'^2}} - u\sqrt{1+u'^2} \\
&= c_1,
\end{aligned}$$

which implies

$$\frac{u^2}{1+u'^2} = c_1^2.$$

For $c_1 \neq 0$ this is written as

$$u' = \sqrt{\frac{u^2}{c_1^2} - 1}.$$

Using separation of variables this integrates to

$$x = \int \frac{du}{\sqrt{\frac{u^2}{c_1^2} - 1}} = c_1 \ln\left(\frac{u + \sqrt{u^2 - c_1^2}}{c_1}\right) + c_2.$$

Then we rearrange terms to find

$$\begin{aligned}
c_1 \exp\left(\frac{x-c_2}{c_1}\right) &= u + \sqrt{u^2 - c_1^2}, \\
c_1 \exp\left(-\frac{x-c_2}{c_1}\right) &= \frac{c_1^2}{u + \sqrt{u^2 - c_1^2}}.
\end{aligned}$$

It follows that

$$c_1\left(\exp\left(\frac{x-c_2}{c_1}\right)+\exp\left(-\frac{x-c_2}{c_1}\right)\right)=u+\sqrt{u^2-c_1^2}+\frac{c_1^2}{\sqrt{u^2-c_1^2}}=2u.$$

Hence we obtain an explicit optimum, the *catenary:*

$$u(x)=c_1\cosh\left(\frac{x-c_2}{c_1}\right),$$

where the constants c_1 and c_2 are given from the boundary conditions

$$u(x_0)=y_0, \qquad u(x_1)=y_1.$$

■

■ **Example 7.5 — Rolling ball.** For the rolling ball problem we found an action integral as

$$J(u)=\int_{x_0}^{x_1}\sqrt{\frac{1+u'^2}{u}}dx,$$

and again the Lagrangian does not explicitly depend on x. Therefore we compute the Hamiltonian

$$\begin{aligned}
H(u,u') &= u'\frac{\partial L}{\partial u'}-L\\
&= u'\frac{2u'/u}{2\sqrt{\frac{1+u'^2}{u}}}-\sqrt{\frac{1+u'^2}{u}}\\
&= \frac{u'^2}{\sqrt{u(1+u'^2)}}-\sqrt{\frac{1+u'^2}{u}}\\
&= -\frac{1}{\sqrt{u(1+u'^2)}}\\
&= c_1,
\end{aligned}$$

leading to the equation

$$u(1+u'^2)=\tilde{c}_1.$$

This is solved using the transformation

$$u'=\tan\psi, \qquad \text{with} \qquad 1+u'^2=\sec^2\psi$$

and

$$u=\frac{\tilde{c}_1}{\sec^2\psi}=\tilde{c}_1\cos^2\psi. \tag{7.5}$$

The last relation implies

$$du=-2\tilde{c}_1\cos\psi\,\sin\psi\,d\psi.$$

Further from $y' = \tan\psi$ we find the differential element for dx as

$$\begin{aligned} dx &= \frac{1}{\tan\psi} du \\ &= \cot\psi du \\ &= -2\tilde{c}_1 \cos^2\psi d\psi \\ &= -\tilde{c}_1(1+\cos(2\psi))d\psi. \end{aligned}$$

We integrate the last equation with respect to ψ to find

$$x = c_2 - \tilde{c}_1 \left(\psi + \frac{1}{2}\sin(2\psi)\right), \tag{7.6}$$

where c_2 is a constant of integration. The combined equations (7.5) and (7.6) give then an explicit parametric representation of the brachystochrone, which is also called the *cycloid curve*, parameterized in ψ. The two integration constants $\tilde{c}_1$ and c_2 are found from the boundary conditions. ■

7.4 Systems

The entire variational calculus can be lifted to systems of variables. We will not delve into this in detail, but just briefly mention the analogy. Consider

$$\vec{u} = \underbrace{(u_1,\ldots,u_m)}_{R^m}, \qquad \underbrace{\nabla u_1,\ldots,\nabla u_m}_{\mathbb{R}^{m\times n}}, \qquad \underbrace{x_1,\ldots,x_n}_{\mathbb{R}^n}$$

and a Lagrangian $L(\nabla\vec{u},\vec{u},x)$ with

$$L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}.$$

The energy functional is

$$J(\vec{u}) = \int_\Omega L(\nabla\vec{u},\vec{u},x)dx$$

and the Euler-Lagrange equation becomes an *Euler-Lagrange system*

$$-\nabla_x\Big(\nabla_{p_k} L(\nabla\vec{u},\vec{u},x)\Big) + L_{z_k}(\nabla\vec{u},\vec{u},x) = 0, \qquad \text{for all} \qquad k = 1,\ldots,m.$$

7.5 Hamilton's Principle

Variational Calculus originated in mechanics and the mechanical formulation is widespread in science. It is worthwhile to study variational calculus from the point of moving objects in space and to make the connection to what we have done in the previous sections. The principle of energy minimization is then called Hamilton's Principle. Let us start with an example of a moving body in three dimensions.

■ **Example 7.6** We describe a moving particle in 3-dimensional space through their x,y,z coordinates

$$r(t) = (x(t), y(t), z(t)).$$

The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \qquad \text{and} \qquad V(x,y,z,t), \qquad \text{respectively.}$$

We define the Lagrangian as balance between kinetic and potential energy

$$L = T - V.$$

When a particle moves from $r(t_0)$ to $r(t_1)$ we define the *action* or *action integral* as

$$J(r) = \int_{t_0}^{t_1} L(t, r, \dot{r}) dt.$$

Hamilton's principle now ascertains that a physical orbit is a minimizer of this energy. ■

To consider a general mechanical system, we introduce *generalized coordinates* $q(t) = (q_1(t), \ldots, q_n(t))$. These coordinates often include location, momentum and energy terms. The derivative $\dot{q}(t)$ describes the time change of these quantities. We define a corresponding kinetic energy as

$$T(q,\dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} c_{j,k}\dot{q}_j\dot{q}_k = \dot{q}^T C\dot{q}, \qquad C = (c_{jk})_{j,k},$$

and general potential energy $V(t,q)$. Then the general Lagrangian is

$$L(t,q,\dot{q}) = T(q,\dot{q}) - V(t,q).$$

The following Theorem is a basic axiom, which relates the mathematical formulation of generalized coordinates and kinetic and potential energies to physical reality:

Theorem 7.5.1 — Hamilton's Principle. The motion of a mechanical system $q(t)$ is a critical point of the action

$$J(q) = \int_{t_0}^{t_1} L(t,q,\dot{q}) dt,$$

i.e., a local minimum, or maximum, or a saddle point.

Assume that L is twice continuously differentiable and define a set of all smooth paths from $q(t_0) = q_0$ to $q(t_1) = q_1$:

$$S = \{q \in C^2(t_0,t_1) : q(t_0) = q_0,\ q(t_1) = q_1\},$$

then $q \in S$ is a critical point of J, if it satisfies the *Euler-Lagrange system*

$$-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial q_k} = 0, \qquad k = 1,\ldots,n, \tag{7.7}$$

where the derivation is the same as the derivation of the cousin equation (7.2). If we compare this mechanical Euler-Lagrange equation (7.7) to the previously derived Euler-Lagrange equation (7.2), we see that they have the same structure, but they convey different meanings. While in (7.2) we are looking for distributions in space, here in (7.7) we are concerned about the optimal path of a mechanical system.

■ **Example 7.7 — Harmonic Oscillator.** A standard example of a variational mechanical system is the harmonic oscillator. It can be realized through a mass attached to a spring, which moves frictionless up and down. Let $x(t)$ denote the height relative to the resting position of the mass m. Then kinetic and potential energy are given as

$$T = \frac{1}{2}m\dot{x}^2, \qquad V = \frac{1}{2}kx^2,$$

where k denotes the spring constant. The Lagrangian becomes

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

and the corresponding Euler-Lagrange equation is

$$-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0,$$

leading to

$$m\ddot{x} = -kx,$$

which is the differential equation for a harmonic oscillator. ■

7.6 The Second Variation

In the general case, we were looking at a Lagrangian $L(p,z,x)$ with $L \in C^2$ to minimize an energy

$$J(u) = \int_\Omega L(\nabla u, u, x)dx.$$

We found a necessary condition in the first variation

$$\delta J(u) = 0$$

with

$$\begin{aligned} \delta J(u) &= \frac{d}{d\varepsilon}J(u+\varepsilon v)\Big|_{\varepsilon=0} \\ &= \int_\Omega \left[-\nabla_x(\nabla_p L(\nabla u,u,x)) + L_z(\nabla u,u,x)\right] v dx. \end{aligned}$$

We compute the second variation in the same way. In this calculation it is beneficial to use Einstein's summation convention for repeated indices. Also for this calculation we use ∂_z for $\frac{\partial}{\partial z}$. From (7.1) we find

$$\frac{d}{d\varepsilon}J(u+\varepsilon v) = \int_\Omega \partial_{p_i}L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\partial_{x_i}v + \partial_z L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\, v\, dx.$$

We differentiate again to obtain

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}J(u+\varepsilon v) &= \int_\Omega \partial_{p_j}\partial_{p_i}L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\partial_{x_j}v\,\partial_{x_i}v\,dx\\ &\quad+\int_\Omega \partial_z\partial_{p_i}L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\,v\,\partial_{x_i}v\,dx\\ &\quad+\int_\Omega \partial_{p_j}\partial_z L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\partial_{x_j}v\,v\,dx\\ &\quad+\int_\Omega \partial_z\partial_z L(\nabla u+\varepsilon\nabla v, u+\varepsilon v, x)\,v\,v\,dx.\end{aligned}$$

Setting $\varepsilon=0$, we obtain the second variation

$$\begin{aligned}\delta^2 J(u) &= \frac{d^2}{d\varepsilon^2}J(u+\varepsilon v)\Big|_{\varepsilon=0}\\ &= \int_\Omega (\partial_{p_j}\partial_{p_i}L)\partial_{x_j}v\,\partial_{x_i}v+2(\partial_{p_i}\partial_z L)\partial_{x_i}v\,v+(\partial_z^2 L)v^2\,dx.\end{aligned}$$

Now we revert back to vector notation and find

$$\delta^2 J=\int_\Omega(\nabla v)^T(\mathrm{Hess}_p L)\nabla v+2\left(\nabla_p\frac{\partial}{\partial z}L\right)\cdot\nabla v\,v+\left(\frac{\partial^2}{\partial z^2}L\right)v^2\,dx,$$

where $\mathrm{Hess}_p L$ denotes the Hessian matrix of $L(p,z,x)$ with respect to derivative in p. Hence for a local minimum we require

$$\delta^2 J(u)\geq 0, \tag{7.8}$$

for each test function $v\in C_c^\infty(\Omega)$.

We consider now a particular test function, which is not in $C_c^\infty(\Omega)$, but it gives us a good understanding of the convexity condition that we need later. We define

$$v(x)=\varepsilon\gamma\left(\frac{x\cdot\xi}{\varepsilon}\right)\zeta(x),\qquad \zeta\in C_c^\infty(\Omega),\qquad \xi\in\mathbb{R}^n,$$

with

$$\gamma(x)=\begin{cases} x & 0\leq x\leq 0.5\\ 1-x & 0.5\leq x\leq 1\end{cases},\qquad \gamma(x+1)=\gamma(x)\quad\text{periodic}.$$

Then $|\gamma'|=1$ a.e. and

$$v_{x_i}(x)=\gamma'\left(\frac{x\cdot\xi}{\varepsilon}\right)\xi_i\zeta+\underbrace{\varepsilon\gamma\left(\frac{x\cdot\xi}{\varepsilon}\right)\zeta_{x_i}(x)}_{O(\varepsilon)}.$$

Substituting this test function in the condition for the second variation (7.8) gives the condition that

$$0\leq\int_\Omega \xi^T(\mathrm{Hess}_p L)\xi\,\gamma'^2\zeta^2+O(\varepsilon)dx.$$

Now sending $\varepsilon \to 0$ and appreciating that ζ is an arbitrary test function, we obtain that

$$\xi^T(\text{Hess}_p L)\xi \geq 0 \qquad \text{for all} \quad \xi \in \mathbb{R}^n.$$

A more common way to write down this *convexity condition* is

$$\sum_{i,j=1}^{n} L_{p_i p_j}(\nabla u, u, x)\xi_i \xi_j \geq 0. \qquad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{7.9}$$

We will encounter this condition again in the proof for the existence of a minimizer. If applied to the PDE context, this condition is also known as *ellipticity condition*.

7.7 Existence of a Minimizer

Given $L \in C^2$, when does

$$\begin{aligned} J(u) &= \int_\Omega L(\nabla u, u, x)dx, \\ u &= g \qquad \text{on} \qquad \partial\Omega \end{aligned}$$

have a minimizer? To answer this question we need a few new (and some old) concepts such as coercivity, compactness, lower semicontinuity, and convexity.

7.7.1 Coercivity

We first need a condition that says something like " $J(u)$ is large for large values of u". To make this precise, we define a coercivity condition.

Definition 7.7.1 The functional $J(u)$ satisfies a *coercivity condition*, if there exist $\alpha, \beta > 0, 1 < q < \infty$ such that

$$L(p,z,x) \geq \alpha |p|^q - \beta, \qquad \text{for all } p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \Omega. \tag{7.10}$$

To “coerce” also means to intimidate or to bully somebody. However, coercivity is not meant to intimidate the reader, it rather "bullies" the Lagrangian L to become large for large values of p.[1]

If we integrate the coercivity condition (7.10) we find a condition for J:

$$J(u) \geq \alpha \|\nabla u\|_q^q - \beta |\Omega|$$

and $J(u) \to \infty$ as $\|\nabla u\|_q \to \infty$. Since the $\|\nabla u\|_q$-norm arises, it is natural to work from now on in $W^{1,q}(\Omega)$.

Definition 7.7.2 We introduce an *admissible set*

$$\mathscr{A} = \{u \in W^{1,q}(\Omega) : u = g \quad \text{on} \quad \partial\Omega\}.$$

[1] The word coercivity is spoken as “co-er-civity”.

7.7.2 Compactness

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies a coercivity condition, then it does attain a minimum. This is not necessarily the case for $J(u)$. For example set

$$m := \inf_{u \in \mathscr{A}} J(u)$$

and chose a minimizing sequence $u_k \in \mathscr{A}$ with

$$J(u_k) \to m \qquad \text{as} \qquad k \to \infty.$$

We want to show that $\{u_k\}$ or a subsequence $\{u_{kj}\}$ converges to a minimizer u. So in $W^{1,q}$ we need some form of compactness. Using coercivity, we can show that

- $\{u_n\}$ lies in a bounded set in $W^{1,q}(\Omega)$, i.e. $\{u_n\}$ is bounded in $L^q(\Omega)$,
- and $\{\nabla u_n\}$ is bounded in $L^q(\Omega)$.

By the reflexive weak compactness result Corollary 4.5.3 or by the Alaoglu weak* compactness Theorem 4.5.2 we find weakly convergent subsequences

$$\begin{aligned} u_{kj} &\rightharpoonup u \quad \text{in} \quad L^q(\Omega), \\ \nabla u_{kj} &\rightharpoonup \chi \quad \text{in} \quad L^q(\Omega). \end{aligned}$$

We will show later that indeed $\chi = \nabla u$, that $u|_{\partial\Omega} = g$ in the sense of traces (Theorem 5.4.1) and that $u_{kj} \rightharpoonup u$ in $\mathscr{A}$. Before we can prove this, we need one more property, which is *lower semi-continuity*.

7.7.3 Lower semi-continuity

In general, the functional $J(u)$ will not be continuous with respect to weak convergence, i.e. it is not clear if $\lim_{j\to\infty} J(u_{kj}) = J(u)$ and we need a weaker form of continuity. As we are looking for minimizers of $J(u)$, it is suffiucient to require

$$J(u) \le \liminf_{j\to\infty} J(u_{kj}).$$

If that is true, then $J(u) \le m$. But since, by definition, $m \le J(u)$ we conclude $m = J(u)$.

Definition 7.7.3 The functional $J(u)$ is *(weakly) lower semi-continuous* on $W^{1,q}(\Omega)$ when

$$J(u) \le \liminf_{k\to\infty} J(u_k),$$

for all $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$.

This leaves the question: when is $J(u)$ weakly lower semi-continuous? The answer is given through convexity.

7.7.4 Convexity

Theorem 7.7.1 — Weak lower semi-continuity. Assume L is bounded below and convex in p, i.e., the Hessian $\mathrm{Hess}_p L$ is positive semi definite

$$\sum_{i,j=1}^{n} L_{p_i p_j} \xi_i \xi_j \geq 0 \qquad \text{for all} \qquad \xi \in \mathbb{R}^n.$$

Then $J(u)$ is weakly lower semi-continuous in $W^{1,q}(\Omega)$.

Proof. 1. Choose a sequence $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$ and set

$$l := \liminf_{k\to\infty} J(u_k).$$

This exists since $J(u)$ is bounded below. We must show that $J(u) \leq l$.

2. The above sequence can be seen as a family of linear forms on the dual of $W^{1,q}$. Since each member is bounded when applied to a test-function (weak convergence) the uniform boundedness principle (Theorem 3.3.3) applies and it follows that

$$\sup_k \|u_k\|_{W^{1,q}(\Omega)} < \infty.$$

Hence $\{J(u_k)\}$ is a bounded sequence of real numbers and we can find a subsequence such that

$$\lim_{k\to\infty} J(u_k) = l.$$

Since $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ (similar to the Rellich-Kondrachov result), we have $u_k \to u$ in $L^q(\Omega)$, i.e. $u_k \to u$ a.e. in Ω.

3. We cite a result from measure theory:

Theorem 7.7.2 — Egoroff's Theorem. Let $\{u_k\}, u$ be measurable and assume $u_k \to u$ a.e. in Ω, with $|\Omega| < \infty$. Then for each $\varepsilon > 0$ there exists a measurable subset $E \subset \Omega$ such that $|\Omega \backslash E| \leq \varepsilon$ and $u_k \to u$ uniformly in E.

We apply Egoroff's Theorem to the subsequence we just constructed in item 2. For each $\varepsilon > 0$ we can find a set $E_\varepsilon \subset \Omega$ with $|\Omega \backslash E_\varepsilon| \leq \varepsilon$ such that

$$u_k \to u \qquad \text{uniformly on} \qquad E_\varepsilon.$$

Define

$$F_\varepsilon = \left\{ x \in \Omega : |u(x)| + |\nabla u(x)| \leq \frac{1}{\varepsilon} \right\}.$$

Then

$$|\Omega \backslash F_\varepsilon| \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

We set $G_\varepsilon = E_\varepsilon \cap F_\varepsilon$ and we also have

$$|\Omega \backslash G_\varepsilon| \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

4. If L is convex in the variable p, then we us a second order Taylor expansion

$$L(p,z,x) = L(q,z,x) + \nabla_p L(q,z,x) \cdot (p-q) + \frac{1}{2} \underbrace{\xi^T (\mathrm{Hess}_p L) \xi}_{\geq 0}$$

for some increment vector ξ. Then

$$L(p,z,x) \geq L(q,z,x) + \nabla_p L(q,z,x)(p-q)$$

and

$$\begin{aligned} J(u_k) &= \int_\Omega L(\nabla u_k, u_k, x) dx \\ &\geq \int_{G_\varepsilon} L(\nabla u_k, u_k, x) dx \\ &\geq \int_{G_\varepsilon} L(\nabla u, u_k, x) dx + \int_{G_\varepsilon} \nabla_p L(\nabla u, u_k, x)(\nabla u_k - \nabla u) dx. \end{aligned}$$

On G_ε we know that

$$\lim_{k\to\infty} \int_{G_\varepsilon} L(\nabla u, u_k, x) dx = \int_{G_\varepsilon} L(\nabla u, u, x) dx.$$

Moreover, $\nabla_p L(\nabla u, u_k, x)$ is uniformly bounded on G_ε, and we know from step 1. that $\nabla u_k \rightharpoonup \nabla u$. This implies that

$$\lim_{k\to\infty} \int_{G_\varepsilon} \nabla_p L(\nabla u, u_k, x)(\nabla u_k - \nabla u) dx = 0.$$

Then

$$l = \lim_{k\to\infty} J(u_k) \geq \int_{G_\varepsilon} L(\nabla u, u, x) dx \qquad \text{for all} \qquad \varepsilon > 0.$$

Letting $\varepsilon \to 0$ and using the monotone convergence result (Theorem 2.2.1) we find

$$l \geq \int_\Omega L(\nabla u, u, x) dx = J(u).$$

■

7.7.5 Existence

Theorem 7.7.3 — Existence of a Minimizer. Assume L is coercive, convex in p and the admissible set $\mathscr{A}$ is not empty. Then there exists at least one minimizer $u \in \mathscr{A}$ with

$$J(u) = \min_{w\in\mathscr{A}} J(w).$$

Proof. 1. Set $m := \inf_{w\in\mathscr{A}} J(w)$. If $m = +\infty$, then $J(w) = +\infty$ for all $w \in \mathscr{A}$ and we are done. So we assume now that $m < \infty$ and select a minimizing sequence $\{u_k\}$, $J(u_k) \to m$ for $k \to \infty$.

2. J is coercive and we can always choose $\beta = 0$, by possibly adding a constant to L, i.e. $L(p,z,x) \geq \alpha |p|^q$. Then

$$J(u_k) \geq \alpha \int_\Omega |\nabla u_k|^q dx$$

and since $\{J(u_k)\}$ converges, we have

$$\sup_k \|\nabla u_k\|_q < \infty.$$

3. Take another $w \in \mathscr{A}$, then $w|_{\partial\Omega} = g$ and $u_k - w \in W_0^{1,q}(\Omega)$. Hence a Poincaré-inequality applies

$$\begin{aligned} \|u_k\|_q &\leq \|u_k - w\|_q + \|w\|_q \\ &\leq c_1 \|\nabla u_k - \nabla w\|_q + c_2 \\ &\leq c_3, \end{aligned}$$

for appropriate constants $c_1, c_2, c_3 > 0$, where we used step 2 in the last estimate. This implies that $\{u_k\}$ is bounded in $W^{1,q}$.
4. Using reflexive weak compactness for $u_k \rightharpoonup u$ and $\nabla u_k \rightharpoonup \chi$ in $L^q(\Omega)$ and uniqueness of weak limits, we find a subsequence such that

$$u_{kj} \rightharpoonup u \qquad \text{in} \quad W^{1,q}(\Omega).$$

On the boundary we use a trace argument (Theorem 5.4.1). It justifies restriction of a function in $W^{1,q}$ onto a smooth domain boundary. See also [7, 22]. We get

$$(u_{kj} - w)\Big|_{\partial\Omega} = 0 \qquad \Longrightarrow \qquad u \in \mathscr{A}.$$

5. Finally, since $J(u)$ is weakly lower semi-continuous we have

$$m \leq J(u) \leq \liminf_{j\to\infty} J(u_k) = m.$$

■

7.7.6 Uniqueness

We prove uniqueness of the minimizer only for the case that L does not depend on z and we assume uniform convexity in p:

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p,x)\xi_i\xi_j \geq \theta |\xi|^2, \qquad \text{for} \quad \theta > 0 \quad \text{and for all} \quad \xi \in \mathbb{R}^n. \tag{7.11}$$

In other words

$$\xi^T (\mathrm{Hess}_p L(p,x))\xi \geq \theta |\xi|^2, \text{ for all } \quad \xi \in \mathbb{R}^n.$$

Theorem 7.7.4 — Uniqueness of Minimizer. Suppose that (7.11) holds. Then the minimizer $u \in \mathscr{A}$ of $J(u)$ is unique.

Proof. Assume the minimizer is not unique and we have two minimizers $u, \tilde{u} \in \mathscr{A}$, with $u \neq \tilde{u}$. We will show that the mean value of those two has even less energy, which would be a contradiction. We claim

$$v := \frac{u+\tilde{u}}{2} \in \mathscr{A} \qquad \text{and} \qquad J(v) < \frac{J(u)+J(\tilde{u})}{2}. \tag{7.12}$$

Using the uniform convexity (7.11) we estimate for arbitrary $p, q \in \mathbb{R}^n$ that

$$L(p,x) \geq L(q,x) + \nabla_p L(q,x)(p-q) + \frac{\theta}{2}|p-q|^2.$$

We set

$$p = \nabla u \qquad \text{and} \qquad q = \frac{1}{2}(\nabla u + \nabla \tilde{u}),$$

and we integrate the above inequality over Ω.

$$\begin{aligned} J(u) \;&\geq\; J(v) + \int_\Omega \nabla_p L\left(\frac{1}{2}(\nabla u + \nabla \tilde{u}), x\right)\left(\nabla u - \frac{1}{2}(\nabla u + \nabla \tilde{u})\right) dx \\ &\quad + \int_\Omega \frac{\theta}{2}\left|\nabla u - \frac{1}{2}(\nabla u + \nabla \tilde{u})\right|^2 dx \\ &\geq\; J(v) + \int_\Omega \nabla_p L\left(\frac{1}{2}(\nabla u + \nabla \tilde{u}), x\right)\frac{1}{2}(\nabla u - \nabla \tilde{u}) dx + \frac{\theta}{8}\int_\Omega |\nabla u - \nabla \tilde{u}|^2 dx. \end{aligned}$$

Similarly, we set

$$p = \nabla \tilde{u} \qquad \text{and} \qquad q = \frac{1}{2}(\nabla u + \nabla \tilde{u}),$$

and obtain a similar estimate

$$J(\tilde{u}) \geq J(v) + \int_\Omega \nabla_p L\left(\frac{1}{2}(\nabla u + \nabla \tilde{u})\right)\frac{1}{2}(\nabla \tilde{u} - \nabla u) dx + \frac{\theta}{8}\int_\Omega |\nabla \tilde{u} - \nabla u|^2 dx.$$

Adding these two equations and dividing by 2 yields

$$\frac{J(u)+J(\tilde{u})}{2} \geq J(v) + \frac{\theta}{8}\int_\Omega |\nabla u - \nabla \tilde{u}|^2 dx.$$

Now, if $\nabla u \neq \nabla \tilde{u}$ on a set of non-zero measure, then we obtain the contradiction (7.12). Hence $\nabla u = \nabla \tilde{u}$ a.e., which implies, $u = \tilde{u}$ in $\mathscr{A}$.

■

7.8 Minimizer is a Weak Solution of the E-L Equations

So far we simply assumed that the mimizer of $J(u)$ and the solution of the Euler-Lagrane equation coincide. However, as we are arguing in infinite dimensional function spaces, the relation is actually not so clear. We need additional arguments to really prove that the energy minimizer is indeed a weak solution of the Euler-Lagrange equation. We bascically need a growth bound for L and its derivatives. For some $q > 1$ we assume

$$\begin{aligned} |L(p,z,x)| &\leq c\,(|p|^q + |z|^q + 1)\,, \\ |\nabla_p L(p,z,x)| &\leq c\left(|p|^{q-1} + |z|^{q-1} + 1\right), \\ |L_z(p,z,x)| &\leq c\left(|p|^{q-1} + |z|^{q-1} + 1\right). \end{aligned} \tag{7.13}$$

We let q' denote the conjugate index of q. These bounds basically define the underlying function space $W^{1,q}(\Omega)$, and as we will see, it also defines a natural relation between the minimizer u and a small perturbation v. In this framework, it is convenient to slightly modify the definition of a weak solution by reducing the set of test functions. In the general weak formulation, we would employ the dual space $(W^{1,q}(\Omega))^*$, which is some non-trivial measure space. Instead, we restrict test functions to be in $W_0^{1,q}(\Omega)$:

Definition 7.8.1 We say that $u \in \mathscr{A}$ is a *weak solution* of the Euler-Lagrange equation

$$\begin{aligned} -\nabla(\nabla_p L) + L_z &= 0, \\ u\Big|_{\partial\Omega} &= g, \end{aligned}$$

if

$$\int_\Omega \nabla_p L(\nabla u, u, x) \cdot \nabla v + L_z(\nabla u, u, x)\, v\, dx = 0,$$

for all $v \in W_0^{1,q}(\Omega)$.

Theorem 7.8.1 — Weak solution. Assume (7.13) and assume that $u \in \mathscr{A}$ is a minimizer of $J(u)$, i.e.

$$J(u) = \min_{w \in \mathscr{A}} J(w).$$

Then u is a weak solution of the Euler-Lagrange equation,

$$-\nabla(\nabla_p L(\nabla u, u, x)) + L_z(\nabla u, u, x) = 0.$$

Proof. Notice that $\mathscr{A} \subset W^{1,q}(\Omega)$, where q is now the exponent from assumption (7.13). We consider a perturbation $v \in W_0^{1,q}(\Omega)$ and study $V(\tau) = J(u + \tau v)$. Because of the bounds in (7.13), $V(\tau)$ is bounded for all $\tau \geq 0$.

For $\tau \neq 0$ we define

$$L^{(\tau)} := \frac{1}{\tau}\Big(L(\nabla u + \tau \nabla v, u + \tau v, x) - L(\nabla u, u, x)\Big)$$

and write the differential quotient for V as

$$\frac{V(\tau) - V(0)}{\tau} = \int_\Omega L^{(\tau)}(x)\,dx.$$

As $\tau \to 0$ we have

$$L^{(\tau)} \to \nabla_p L(\nabla u, u, x) \cdot \nabla v + L_z(\nabla u, u, x) v,$$

hence

$$\begin{aligned} \frac{V(\tau) - V(0)}{\tau} &\to \int_\Omega \nabla_p L \cdot \nabla v + L_z v \, dx \\ &= \int_\Omega \Big[-\nabla(\nabla_p L) + L_z \Big] v dx. \end{aligned} \tag{7.14}$$

We want to show that the limit of the differential quotient exists, hence we estimate the right hand side, using assumption (7.13) and a Hölder inequality with (q, q'):

$$\begin{aligned} \left| \int_\Omega \nabla_p L \cdot \nabla v + L_z v dx \right| &\leq \int_\Omega c \left(|\nabla u|^{q-1} + |u|^{q-1} + 1 \right) \left(|\nabla v| + |v| \right) dx \\ &\leq c \left(\| \, |\nabla u|^{q-1} \|_{q'} + \| u^{q-1} \|_{q'} + 1 \right) \|v\|_{W^{1,q}} \\ &= c \left(\|u\|_{W^{1,q}}^{q-1} + 1 \right) \|v\|_{W^{1,q}} \\ &< \infty, \end{aligned}$$

using $q' = \frac{q}{q-1}$. Hence $V'(\tau)$ exists, and since it has a minimum at $\tau = 0$, we have $V'(0) = 0$. This implies from (7.14) that

$$\int_\Omega \Big[-\nabla(\nabla_p L) + L_z \Big] v dx = 0,$$

for all test functions $v \in W_0^{1,q}(\Omega)$. Hence u is a weak solution. ■

7.9 Solutions of the E-L-equations are (sometimes) Minimizers

It is also not clear if solutions of the Euler-Lagrange equations are minimizers of $J(u)$. For this we need to come back to convexity, which we used already for the uniqueness above. But we need a bit more:

Theorem 7.9.1 Assume that $L(p, z, x)$ is convex in p as defined in (7.11) and in addition the map $z \mapsto L(p, z, x)$ is convex for all p, x. Then each weak solution of the Euler-Lagrange equation is a minimizer of $J(u)$.

Proof. Assume $u \in W^{1,q}(\Omega)$ is a weak solution of

$$\begin{aligned} -\nabla(\nabla_p L) + L_z &= 0, \\ u\Big|_{\partial\Omega} &= g, \end{aligned}$$

then $u \in \mathscr{A}$. Consider another $w \in \mathscr{A}$. Since L is convex in both variables p and z we have

$$L(r, y, x) \geq L(p, z, x) + \nabla_p L(p, z, x) \cdot (r - p) + L_z(p, z, x)(y - z).$$

We choose

$$p = \nabla u, \quad r = \nabla w, \quad z = u, \quad y = w,$$

and integrate over Ω

$$J(u) + \underbrace{\int_\Omega \nabla_p L(\nabla u, u, x)(\nabla w - \nabla u) + L_z(\nabla u, u, x)(w-u)dx}_{=0} \le J(w).$$

Notice that the bracket equals zero, since u is a weak solution and we test with the test function $w - u \in W_0^{1,q}(\Omega)$. Hence $J(u) \le J(w)$, and u is a minimizer. ■

7.10 Constraints

From optimization theory we are quite familiar with constraint optimization problems and the Method of Lagrange multipliers is a powerful tool. The variational calculus developed here also deals with optimization, and it is a natural extension to include constraints. We will see that also in the abstract case, we can introduce Lagrange multipliers, and a modified Lagrangian, such that modified Euler-Lagrange equations result. Before doing the detailed analysis, let us review the classical Lagrange Multiplier approach.

7.10.1 Review: Lagrange Multiplier Method

We are interested to find the extreme values of a real function $f(x,y), (x,y) \in \mathbb{R}^2$ under the constraint $g(x,y) = k$. We assume f and g are smooth functions and we define the k-level set of g as

$$G := \{(x,y) : g(x,y) = k\}.$$

The formal construction is as follows:

- We first draw the level-curve G and a whole lot of level curves $\{f(x,y) = c\}$ for various c.
- We find the largest value of c such that the level curves of f intersect the level curve of g.
- We find that at the maximum c the level sets touch each other. If they do not touch, then we can make c even bigger.
- Hence, at the intersection point (a,b), the level curves have the same tangent, i.e. they have the same gradient:

  $$\nabla f(a,b) = \lambda \nabla g(a,b),$$

  where λ is an unknown constant of proportionality, called the *Lagrange multiplier*.

Theorem 7.10.1 — Method of Lagrange Multiplier. To find the minima of a smooth real function $f(\vec{x}), \vec{x} \in \mathbb{R}^n$ subject to constraints $g(\vec{x}) = k$, we

1. find all values $(\vec{x}, \lambda)$ such that

   $$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x}), \qquad g(\vec{x}) = k$$

   and
2. evaluate $f(\vec{x})$ at all those points to find the minimum.

Note that the equations have $(n+1)$ unknowns and $(n+1)$ equations. Hence a solution is expected. Also note that λ can be multidimensional, such that each constraint gets its own Lagrange multiplier. Let us try to develop a similar principle in the abstract framework.

7.10.2 Isoperimetric Constraints

Given a Lagrangian $L(\nabla u, u, x)$ we are now interested in a constraint minimization problem to minimize

$$J(u) = \int_{\Omega} L(\nabla u, u, x) dx \tag{7.15}$$

under the *integral constraint* or *isoperimetric constraint*

$$V(u) = \int_{x_0}^{x_1} g(\nabla u, u, x) dx = V_0. \tag{7.16}$$

Let us perform a similar perturbation approach as we used for the First Variation in Section 7.2.2. We consider a perturbation of the minimizer $\bar{u}$ of the form $u = \bar{u} + \varepsilon\eta$. In doing this for an arbitrary perturbation η, we might violate the constraint (7.16). To compensate for this, we include a second small perturbation, which is used to push us back onto the constraint, i.e., we consider an ansatz

$$u = \bar{u} + \varepsilon_1\eta_1 + \varepsilon_2\eta_2,$$

where $\bar{u}$ is the minimizer with $V(\bar{u}) = V_0$ and $\eta_1, \eta_2 \in C_c^2$. We consider a function of two real variables:

$$G(\varepsilon_1, \varepsilon_2) = V(\bar{u} + \varepsilon_1\eta_1 + \varepsilon_2\eta_2).$$

By the implicit function theorem we can find functions $\varepsilon_1(\varepsilon_2)$ and $\varepsilon_2(\varepsilon_1)$ such that

$$G(\varepsilon_1(\varepsilon_2), \varepsilon_2) = V_0 = G(\varepsilon_1, \varepsilon_2(\varepsilon_1)),$$

provided

$$\nabla_\varepsilon G\Big|_{(0,0)} \neq 0.$$

We use this as a definition

Definition 7.10.1 A point $\bar{u}$ is called a *rigid extremum*, if $\bar{u}$ is an extremum of the constraint optimization problem (7.15, 7.16) and

$$\nabla_\varepsilon G\Big|_{(0,0)} \neq 0.$$

Now, for given minimizer $\bar{u}$, we write the above optimization problem (7.15, 7.16) as an optimization problem in $(\varepsilon_1, \varepsilon_2)$.

$$\begin{aligned} F(\varepsilon_1, \varepsilon_2) &= \min J(\bar{u} + \varepsilon_1\eta_1 + \varepsilon_2\eta_2) \\ G(\varepsilon_1, \varepsilon_2) &= V(\bar{u} + \varepsilon_1\eta_1 + \varepsilon_2\eta_2) = V_0. \end{aligned}$$

We can apply the standard Lagrange multiplier method from Theorem 7.10.1. There is a constant λ such that

$$\nabla_\varepsilon \Big(F - \lambda(G - V_0)\Big) = 0.$$

Let us compute the gradients.

$$\begin{aligned}\frac{\partial}{\partial\varepsilon_1}F\Big|_{(0,0)} &= \frac{\partial}{\partial\varepsilon_1}\int_\Omega L(\nabla\bar{u}+\varepsilon_1\nabla\eta_1+\varepsilon_2\nabla\eta_2,\bar{u}+\varepsilon_1\eta_1+\varepsilon_2\eta_2,x)dx\Big|_{(0,0)}\\ &= \int_\Omega\Big(\nabla_pL\cdot\nabla\eta_1+L_z\eta_1\Big)dx\\ &= \int_\Omega\eta_1\Big(-\nabla(\nabla_pL)+L_z\Big)dx.\end{aligned}$$

Similarly for G we get

$$\frac{\partial}{\partial\varepsilon_1}G\Big|_{(0,0)} = \int_\Omega\eta_1\Big(-\nabla(\nabla_pg)+g_z\Big)dx.$$

Hence we define a *constraint Lagrangian* as

$$\mathscr{L}(\nabla u,u,x) = L(\nabla u,u,x)-\lambda g(\nabla u,u,x)$$

and find the Eurler-Lagrange equation of the constraint Lagrangian:

$$-\nabla(\nabla_p\mathscr{L}(\nabla u,u,x))+\mathscr{L}_z(\nabla u,u,x)=0.$$

We call constraints to be

$$\begin{aligned}&\textit{holonomic constraints}\text{, if} && g(u,x)=0,\\ &\textit{nonholonomic constraints}\text{, if} && g(\nabla u,u,x)=0.\end{aligned}$$

More examples on holonomic and nonholonomic constraints can be found in many textbooks about mechanics and variational methods.

7.11 Summary: Variational Calculus

A variational problem is something like

$$\begin{aligned}\min\quad J(u) &= \int_\Omega L(\nabla u,u,x)dx, \qquad L\in C^2,\\ u\Big|_{\partial\Omega} &= g.\end{aligned}$$

The admissible set is

$$\mathscr{A}=\{u\in W^{1,q}(\Omega): u|_{\partial\Omega}=g\}.$$

1. Convexity: $\sum L_{p_ip_j}\xi_i\xi_j\geq 0$ for all $\xi\in\mathbb{R}^n \Longrightarrow J(u)$ is weakly lower semicontinuous
2. plus coercivity: $L(p,z,x)\geq\alpha|p|^q-\beta\Longrightarrow$ Each minimizing sequence $\{u_k\}$ has a convergent subsequence in $W^{1,q}$ (compactness)
3. plus $\mathscr{A}\neq\emptyset\Longrightarrow$ Existence of a minimizer
4. plus uniform convexity $\sum L_{p_ip_j}\xi_i\xi_j\geq\theta\|\xi\|^2\Longrightarrow$ Uniqueness of the minimizer
5. plus
$$\begin{aligned}|L(p,z,x)| &\leq c\left(|p|^q+|z|^q+1\right),\\ |\nabla_pL(p,z,x)| &\leq c\left(|p|^{q-1}+|z|^{q-1}+1\right),\\ |L_z(p,z,x)| &\leq c\left(|p|^{q-1}+|z|^{q-1}+1\right).\end{aligned}$$
$\Longrightarrow$ Minimizer is a weak solution of the Euler-Lagrange equation
6. plus $L(z,p,x)$ is convex in $(p,z)\Longrightarrow$ A weak solution of the Euler-Lagrange equations is a minimizer.

7.11.1 A Note on Weak Formulations

In this section, we found another definition of a weak formulation in Definition (7.8.1). In fact, so far we used four different definitions for a weak formulation:

1. In Chapter 4 on Dual Spaces, we defined weak convergence as convergence with respect to test functions from the dual space.
2. Also in Chapter 4, we defined weak* convergence, where the set of test functions is now the pre-dual space.
3. Later, in Chapter 5 on Sobolev Spaces, we defined a weak derivative as a derivative with respect to test functions in C_c^∞, and tempered distributions through the dual of the Schwartz space.
4. In this Chapter on Variational Calculus, we define weak solutions in $W^{1,q}$ through test functions in essentially the same space $W_0^{1,q}$.

These weak formulations are not equivalent and the choice of weak formulation is often guided by the methods that are available for the analysis. Indeed, the choice of appropriate function spaces is often an important and challenging part of analysis in applied mathematics. The basic idea of a weak formulation is always the same, we test an equation with test functions. However, a weak formulation that works for one model might have to be modified for a different situation.

7.12 Exercises

Exercise 7.1 (Minimizing) (level 1)
Minimize the functional

$$J(u) = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right) dx,$$

with the side conditions

$$y(0) = 2, \qquad y(1) = 3.$$

Give an argument that ensures that the result is indeed a minimum. ■

Exercise 7.2 (Constraints) (level 1)
For a function $y(x)$ with $y(0) = 0$ and $y(1) = 2$, find the extremals of

$$J(u) = \int_0^1 y'^2 dx$$

under the constraint

$$\int_0^1 y dx = L.$$

Are these extrema maxima or minima?

■

Exercise 7.3 (Geodesics) (level 2)
Geodesics in the plane are the shortest connections between two points. We use a variational method to show that these are straight lines: Let the curve $y(x)$ connect $(0,0)$ with $(1,1)$. Minimize the arc length

$$J(u) = \int_0^1 \sqrt{1+y'^2}\,dx$$

and find the shortest connection $\bar{y}(x)$. ■

Exercise 7.4 (Lagrangian) (level 2)
Find a Lagrangian $L(\nabla u, u, x)$ such that the functional $J(u) = \int L dx$ has the Euler-Lagrange equation

$$-\Delta u + \nabla\phi \cdot \nabla u = f$$

for given smooth functions ϕ and f.
■

Exercise 7.5 (Lagrangian in time and space) (level 2)
Find a Lagrangian $L(\nabla u, u_t, u, x, t)$ such that the functional $J(u) = \int_0^t \int L dx dt$ has the Euler-Lagrange equation

$$u_t - \Delta u + \varepsilon u_{tt} = 0$$

for a given parameter $\varepsilon > 0$.
■

Exercise 7.6 (Non-existence) (level 1)
Consider the energy functional

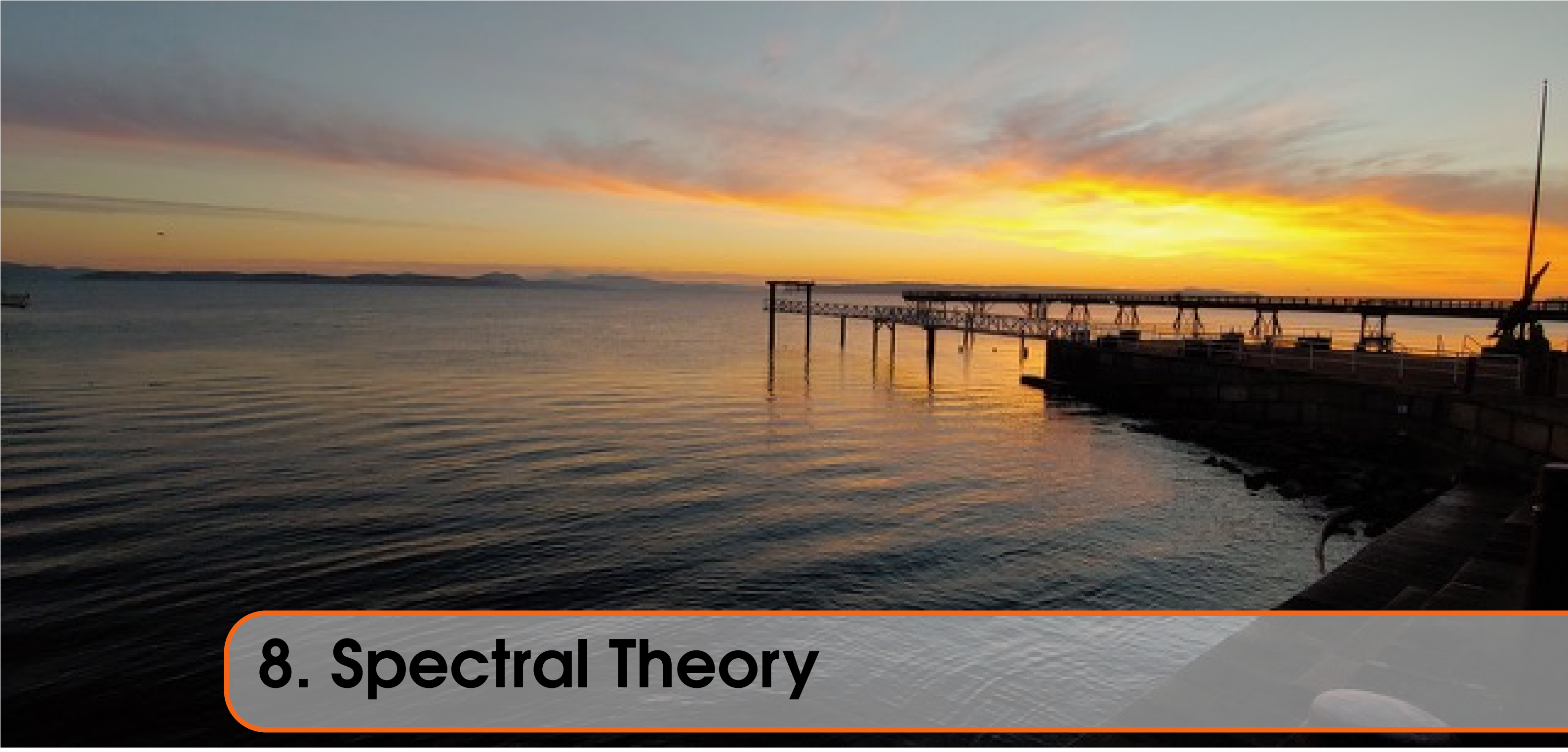
$$J(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx.$$

It turns out that the general theory for the existence of a minimizer (Theorem 7.20) does not work here. Why not?
■

Exercise 7.7 (Existence of PDEs) (level 3)
Let Ω be a smooth, bounded domain. Use variational methods and Sobolev embedding results to show the existence of a non-trivial solution $u \in H_0^1(\Omega)$ of the PDE

$$\begin{aligned} -\Delta u &= |u|^{q-1}u \\ u\big|_{\partial\Omega} &= 0 \end{aligned}$$

for $1 < q < \frac{n+2}{n-2}$ and $n > 2$. ■



8. Spectral Theory

Contents

8.1 Point, Continuous, and Residual Spectrum

When working with unbounded operators in infinite dimensional Hilbert spaces, the spectrum contains more than just eigenvalues. This relates to the fact that there is more than one way the operator $A_\lambda = A - \lambda I$ is not invertible.

Definition 8.1.1 Let X be a complex Banach space and $A : X \to X$ a linear operator with domain $D(A)$. We define

$$A_\lambda = A - \lambda I \qquad \text{and the } \textit{resolvent} \qquad R_\lambda(A) = \left(A - \lambda I\right)^{-1}.$$

Note that in several text books the resolvent is defined as $(\lambda I - A)^{-1}$, which adds a minus sign to all the calculations. I do not have any preference, but a choice needs to be made, and I am using $R_\lambda(A) = (A - \lambda I)^{-1}$.

Definition 8.1.2 Consider three conditions:
(a) $R_\lambda(A)$ exists
(b) $R_\lambda(A)$ is bounded
(c) the domain of $R_\lambda(A)$ is dense in X.
The *resolvent set* is then

$$\rho(A) = \{\lambda \in \mathbb{C} : (a),(b), \text{ and } (c) \text{ hold}\}.$$

Each $\lambda \in \rho(A)$ is called a *regular value*. The *spectrum*

$$\sigma(A) = \mathbb{C} \backslash \rho(A)$$

consists of three parts

$$\sigma(A) = \sigma_p(A) + \sigma_c(A) + \sigma_r(A) :$$

$$\begin{aligned} \text{the } \textit{point spectrum} \quad & \sigma_p(A) = \{\lambda \in \mathbb{C} : (a) \text{ does not hold}\} \\ \text{the } \textit{continuous spectrum} \quad & \sigma_c(A) = \{\lambda \in \mathbb{C} : (a),(c) \text{ hold but } (b) \text{ does not hold}\} \\ \text{the } \textit{residual spectrum} \quad & \sigma_r(A) = \{\lambda \in \mathbb{C} : (a) \text{ holds but } (c) \text{ does not hold}\}. \end{aligned}$$

If X is a Hilbert space we call $\dim(\{\text{Range}(A_\lambda)\}^\perp)$ the *deficiency* of $\lambda \in \mathbb{C}$.

■ **Example 8.1** On a finite dimensional space $A : \mathbb{R}^n \to \mathbb{R}^n$, we simply have

$$\sigma_p(A) = \{\text{eigenvalues}\}, \qquad \sigma_c(A) = \emptyset, \qquad \sigma_r(A) = \emptyset.$$

This fact will follow from Corollary 8.4.2, once we have further developed the spectral theory. ■

In the next sections we analyse the spectrum for more and more involved operators, starting with bounded operators, proceeding to unbounded operators, to self-adjoint operators, to compact operators and to Fredholm operators.

8.2 Bounded Operators

For linear bounded operators that map from X to X we shorten the notation as write $\mathscr{L}(X) = \mathscr{L}(X,X)$. An important tool in operator theory is the Neumann series:

Theorem 8.2.1 — Neumann series. Suppose $A \in \mathscr{L}(X), \|A\| < 1$. Then $(I-A)^{-1}$ exists and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k,$$

where the series converges in the operator norm.

Proof. We compute

$$\left\| \sum_{k=0}^{\infty} A^k \right\| \le \sum_{k=0}^{\infty} \|A^k\| \le \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1-\|A\|}.$$

To see the inverse property, we multiply $(I-A)$ with the Neumann series from left and from the right.

$$(I-A)\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - \sum_{k=0}^{\infty} A^{k+1} = A^0 = I,$$

and similarly

$$\sum_{k=0}^{\infty} A^k (I-A) = I.$$

Hence

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

■

Corollary 8.2.2 Let $A \in \mathscr{L}(X)$ and $\lambda \in \sigma(A)$, then

$$|\lambda| \leq \|A\|.$$

Proof. Consider $|\lambda| > \|A\|$. Then

$$R_\lambda(A) = (A-\lambda I)^{-1} = -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} = -\frac{1}{\lambda}\sum\left(\frac{A}{\lambda}\right)^k. \tag{8.1}$$

Since $\|\frac{A}{\lambda}\| < 1$ this resolvent exists. Hence $\lambda \in \rho(A)$. ■

Definition 8.2.1 The *spectral radius* of A is defined as

$$r_\sigma(A) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

Corollary 8.2.2 implies that

$$r_\sigma(A) \leq \|A\|,$$

and we also have the useful inclusion (see Figure 8.1)

$$\sigma(A) \subset B_{r_\sigma(A)}(0).$$

Also note that for normal operators ($A^*A = AA^*$) we have that

$$r_\sigma(A) = \|A\|.$$

The next result shows that the residual set $\rho(A)$ is an open set, and it gives an explicit representation of the resolvent as function of another resolvent.

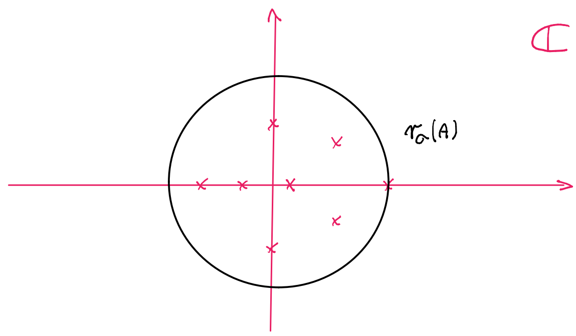


Figure 8.1: Sketch of point spectrum and spectral radius of a bounded operator.

Theorem 8.2.3 Consider $A \in \mathscr{L}(X)$ and $\lambda_0 \in \rho(A)$. Then

$$\lambda \in \overset{\circ}{B}_{\frac{1}{\|R_{\lambda_0}\|}}(\lambda_0)$$

implies that $\lambda \in \rho(A)$ and

$$R_\lambda(A) = \sum_{k=0}^{\infty} (\lambda - \lambda_0) R_{\lambda_0}(A)^{k+1}.$$

Proof. We write

$$A - \lambda I = A - \lambda_0 I - (\lambda - \lambda_0) I = (A - \lambda_0 I) \underbrace{\left(I - (\lambda - \lambda_0) R_{\lambda_0}(A)\right)}_{=B}. \tag{8.2}$$

By the choice of λ we have that

$$\|(\lambda - \lambda_0) R_{\lambda_0}(A)\| < 1,$$

hence B has a bounded inverse using the Neumann series

$$B^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k \left(R_{\lambda_0}(A)\right)^k.$$

Then from (8.2) we find

$$R_\lambda(A) = (A - \lambda I)^{-1} = B^{-1}(A - \lambda_0 I)^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k \left(R_{\lambda_0}(A)\right)^{k+1}.$$

■

An illustration of Theorem 8.2.3 is given in Figures 8.2 and 8.3. In Figure 8.2 we chose two values $\lambda_1, \lambda_2 \in \rho(A)$ and indicate that a ball of radius $(\|R_\lambda\|)^{-1}$ is still contained in the resolvent set. From this image we also conclude that the resolvent diverges to ∞ when we approximate the spectrum, as indicated in Figure 8.3.

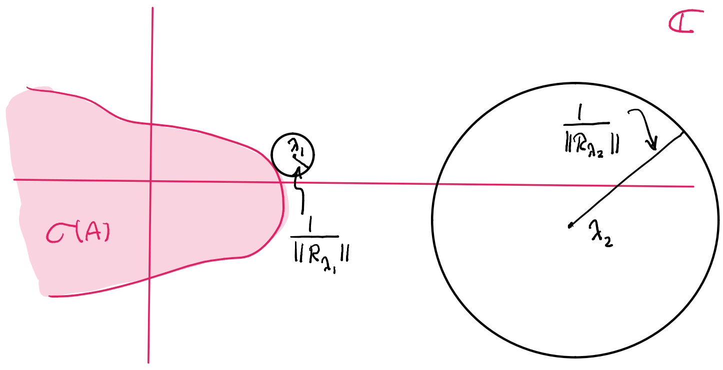


Figure 8.2: Illustration of Theroem 8.2.3. The inverse of the operator norm of the resolvent defines the radius of a ball that is still contained in the resolvent set.

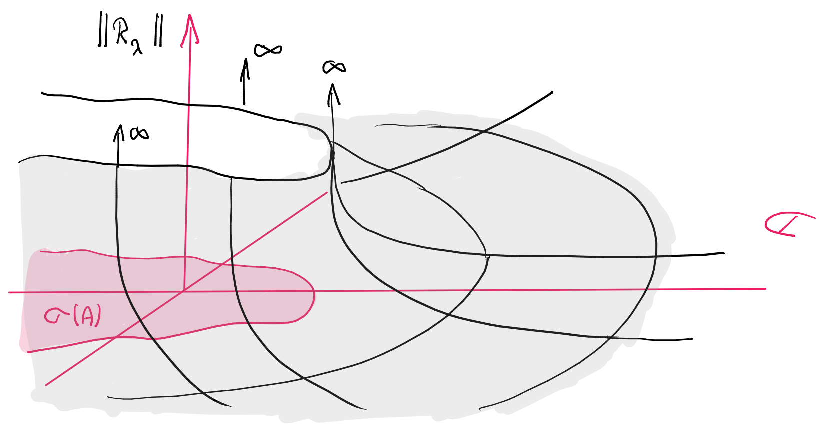


Figure 8.3: Illustration of the norm of the resolvent $\|R_\lambda\|$ as λ gets close to the spectrum $\sigma(A)$.

Corollary 8.2.4 For $A \in \mathscr{L}(X)$, the resolvent set $\rho(A)$ is open and the spectrum $\sigma(A)$ is closed.

Definition 8.2.2 Let $B(\lambda)$ denote a family of operators which depend on a parameter $\lambda \in \mathbb{C}$. We say the map $\lambda \mapsto B(\lambda)$ is *analytic in* $\lambda_0 \in \mathbb{C}$ if

$$\lim_{\lambda \to \lambda_0, \lambda \in \mathbb{C}} \frac{B(\lambda) - B(\lambda_0)}{\lambda - \lambda_0} \qquad \text{exists.}$$

Theorem 8.2.5 If $A \in \mathscr{L}(X)$ then $R_\lambda(A)$ is analytic in the resolvent set $\rho(A)$.

Proof. We compute the differential quotient at $\lambda_0 \in \rho(A)$ and use Theorem 8.2.3 plus the fact that $\rho(A)$ is open:

$$\begin{aligned}
\frac{R_\lambda(A) - R_{\lambda_0}(A)}{\lambda - \lambda_0} &= \frac{1}{\lambda - \lambda_0}\left(\sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1} - R_{\lambda_0}(A)\right) \\
&= \frac{1}{\lambda - \lambda_0} \sum_{k=1}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1} \\
&= \sum_{k=1}^{\infty} (\lambda - \lambda_0)^{k-1} R_{\lambda_0}(A)^{k+1} \\
&= R_{\lambda_0}(A) \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1} \\
&= R_{\lambda_0}(A) R_\lambda(A),
\end{aligned}$$

which exists and is bounded. ∎

Corollary 8.2.6 If $A \in \mathscr{L}(X)$ and $A \neq 0$ then $\sigma(A) \neq \emptyset$.

Proof. If $\sigma(A) = \emptyset$ then $R_\lambda(A)$ would be analytic in $\mathbb{C}$. Earlier in 8.1 we found

$$R_\lambda(A) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda} A\right)^k, \qquad \text{for} \quad \lambda \in \rho(A).$$

We chose $|\lambda| \geq 2\|A\|$ and find

$$\begin{aligned}
\|R_\lambda(A)\| &\leq \frac{1}{2\|A\|} \sum_{k=0}^{\infty} \left(\frac{A}{2\|A\|}\right)^k \\
&= \frac{1}{2\|A\|} \frac{1}{1 - \frac{\|A\|}{2\|A\|}} = \frac{1}{\|A\|}.
\end{aligned}$$

Hence $\|R_\lambda(A)\|$ is uniformly bounded on $\mathbb{C}$. By the Liouville Theorem every bounded analytic function in $\mathbb{C}$ is constant, i.e. for some c_1:

$$R_\lambda(A) = c_1 I.$$

Then

$$\begin{aligned}(A-\lambda I)^{-1} &= c_1 I\\ I &= c_1(A-\lambda I)\\ \Longrightarrow \qquad A &= \frac{1-c_1\lambda}{c_1}I,\end{aligned}$$

which depends on λ, unless $c_1 = 0$. But if $c_1 = 0$, then A is unbounded, which is a contradiction. ■

Theorem 8.2.7 Let $A : X \to X$ be a linear operator. Then the spectral radius is

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Proof. We know from Theorem 8.2.5 that the map $\lambda \mapsto R_\lambda(A)$ is analytic and by the Neumann series (8.1) we have

$$R_\lambda(A) = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n}.$$

This converges for

$$\frac{\|A^n\|}{\lambda^n} < 1, \text{ i.e., for } \lambda > \|A^n\|^{\frac{1}{n}}.$$

Taking the lim sup, this implies that

$$r(A) \geq \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

On the other hand we have for $\lambda \in \mathbb{C}$ that

$$\begin{aligned}(\lambda^n I - A^n) &= (\lambda I - A)(\lambda^{n-1}I + \lambda^{n-2}A + \cdots + A^{n-1})\\ &= (\lambda^{n-1}I + \lambda^{n-2}A + \cdots + A^{n-1})(\lambda I - A).\end{aligned}$$

Hence if $\lambda I - A$ is invertible, then also $\lambda^n I - A^n$ and vice versa. This implies that for $\lambda \in \sigma(A)$, the n-th iterate $(\lambda^n I - A^n)$ is not invertible and $\lambda^n \in \sigma(A^n)$. This implies that

$$|\lambda^n| \leq \|A^n\|, \text{ i.e. } |\lambda| \leq \|A^n\|^{\frac{1}{n}}.$$

Taking the infimum, we find

$$r(A) \leq \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Taking both arguments together we have

$$r(A) \leq \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r(A).$$

■

8.3 Adjoint Operators

In this section we prove the spectral theorem for symmetric operators. For this we need the concept of an adjoint operator.

Definition 8.3.1 Let $A : X \to Y$ be linear with dense domain $D(A)$. We define the *adjoint operator* on the dual spaces $A^x : Y^* \to X^*$ by

$$A^x(y) = w \qquad \text{if and only if} \qquad y(Au) = w(u), \qquad y \in Y^*, w \in X^*, u \in X,$$

i.e., $A^x y$ has the action of $y \circ A$ on $u \in X$. We call $(A^x, D(A^x))$ the *adjoint operator*, where $D(A^x)$ denotes the natural domain of definition of A^x.

Adjoint operators for bounded operators A are easily controlled.

Theorem 8.3.1 Let $A \in \mathscr{L}(X,Y)$. Then $D(A^x) = Y^*$ and A^x is bounded with

$$||A^x|| = ||A||.$$

Proof. This is a direct consequence of the Corollay 4.1.4 of the Hahn-Banach Theorem. ∎

In Hilbert spaces the adjoint can be expressed through the inner product. In that case we use the more common notation of A^* for the adjoint.

Definition 8.3.2 Let $A : H_1 \to H_2$ be a linear map between Hilbert spaces. Then $A^* : H_2 \to H_1$ such that

$$(v, Au)_{H_2} = (A^* v, u)_{H_1}.$$

$(A^*, D(A^*))$ is called the *Hilbert adjoint*, or simply the *adjoint* of A.

The formal difference between A^x and A^* is that A^x lives on the dual spaces and A^* on the Hilbert spaces:

$$A^x : H_2^* \to H_1^*, \qquad A^* : H_2 \to H_1.$$

By the Riesz representation theorem 4.2.1, these maps can be identified in a natural way.

Let us consider range and null space of the adjoint operator.

Theorem 8.3.2 Assume $(A, D(A))$ is given on a Hilbert space H with a dense domain $D(A)$. Then

$$(\text{Range}(A))^{\perp} = N(A^*).$$

If in addition $\text{Range}(A)$ is closed, then the reverse is also true:

$$\text{Range}(A) = N(A^*)^{\perp}.$$

Proof. If $z \in (\text{Range}(A))^{\perp}$ then $(z, Au) = 0$ for all $u \in H$. This implies $(A^* z, u) = 0$, i.e. $A^* z = 0$ and $z \in N(A^*)$. On the other hand, if $z \in N(A^*)$ then $(z, Au) = (A^* z, u) = 0$ and $z \in (\text{Range}(A))^{\perp}$. Finally, if $\text{Range}(A)$ is closed, then

$$\text{Range}(A) = \text{Range(A)}^{\perp\perp} = N(A^*)^{\perp}.$$

■

Definition 8.3.3 A is *symmetric* if $(Au,v) = (u,Av)$ for all $u,v \in D(A)$. A is *self-adjoint* if it is symmetric and $D(A) = D(A^*)$.

■ **Example 8.2** Consider $A = -\Delta$ on $L^2(\Omega)$, where Ω is a smooth domain. In the case of Dirichlet boundary conditions, we define A on

$$D(A) = \{u \in H_0^1(\Omega) \cap H^2(\Omega) : u\big|_{\partial\Omega} = 0\}.$$

Then we have

$$(u,Av) = -\int_\Omega u\Delta v\, dx = \int_\Omega \nabla u \nabla v dx = -\int_\Omega \Delta u\, v\, dx = (Au,v).$$

Hence $A^* = A$ and we can chose $D(A^*) = D(A)$ to obtain a self-adjoint operator. ■

■ **Example 8.3** Let $A = \frac{\partial}{\partial x}$ in $L^2(\mathbb{R})$ and $D(A) = H_0^1(\mathbb{R})$. Then

$$(u,Av) = \int uv' dx = -\int u' v dx = -(Au,v).$$

Hence $A^* = -A$ and $D(A^*) = D(A)$ and we call A to be *skew-adjoint*. ■

Before we prove the important spectral theorem for symmetric operators, we need a small technical result on bounds for the resolvent:

Proposition 8.3.3 Assume $A : X \to Y$ is a linear and surjective operator, which is bounded below $\|Ax\| \geq \delta \|x\| > 0$ for all $x \in X$. Then A^{-1} exists and is bounded.

Proof. Since A is surjective, for each $y \in Y$ there exists an $x \in X$ such that $Ax = y$. We call this $x = A^{-1}y$. Then

$$\|A^{-1}\| = \sup_{y\in Y} \frac{\|A^{-1}y\|}{\|y\|} = \sup_{x\in X} \frac{\|A^{-1}(Ax)\|}{\|Ax\|} = \sup_{x\in X} \frac{\|x\|}{\|Ax\|} \leq \frac{1}{\inf_x \frac{\|Ax\|}{\|x\|}} \leq \frac{1}{\delta}.$$

The inverse is well defined, since when there are two x_1, x_2 with $Ax_1 = Ax_2 = y$, then

$$\|x_1 - x_2\| = \|A^{-1}y - A^{-1}y\| = 0,$$

since A^{-1} is bounded. ■

Corollary 8.3.4 Assume $A - \lambda I$ is invertible and bounded below away from 0, then

$$\|R_\lambda(A)\| \leq \frac{1}{\|A - \lambda I\|}.$$

Theorem 8.3.5 — Spectral Theorem for Symmetric Operators. Let $(A, D(A))$ be a densely defined symmetric operator on a Hilbert space H. Then

1. (Ax, x) is real for all $x \in D(A)$.
2. All eigenvalues of A are real.
3. Eigenvectors to distinct eigenvalues are orthogonal.
4. The continuous spectrum is real: $\sigma_c(A) \subset \mathbb{R}$.

Proof.

1. On a complex Hilbert space we have $(a,b) = \overline{(b,a)}$. Then
$$\overline{(Ax,x)} = (x, Ax) = (Ax, x)$$
and $(Ax, x) \in \mathbb{R}$.
2. Let (λ, φ) be an eigenpair of A (i.e. λ is an eigenvalue and φ is a corresponding eigenvector). Then
$$(A\varphi, \varphi) = \lambda \|\varphi\|^2$$
and λ must be real.
3. Let $(\lambda_1, \varphi_1), (\lambda_2, \varphi_2)$ be two eigenpairs. Then
$$\lambda_1(\varphi_1, \varphi_2) = (A\varphi_1, \varphi_2) = (\varphi_1, A\varphi_2) = \lambda_2(\varphi_1, \varphi_2)$$
and either $\lambda_1 = \lambda_2$ or $\varphi_1 \perp \varphi_2$.
4. Let $\lambda \in \sigma_c(A)$ and $\lambda = \gamma + i\mu$, $\gamma, \mu \in \mathbb{R}$. We want to show that $\mu = 0$. We compute
$$\begin{aligned}\|(A-\lambda I)x\|^2 &= (Ax - \gamma x - i\mu x, Ax - \gamma x - i\mu x)\\ &= (Ax - \gamma x, Ax - \gamma x) + (i\mu x, \gamma x) + (i\mu x, i\mu x)\\ &\quad -(i\mu x, Ax) - (Ax, i\mu x) + (\gamma x, i\mu x)\\ &= \|Ax - \gamma x\|^2 - i(\mu x, \gamma x) + i(\gamma x, \mu x)\\ &\quad + i\mu(x, Ax) - i\mu(Ax, x) + \mu^2\|x\|^2\\ &= \|Ax - \gamma x\|^2 + \mu^2\|x\|^2.\end{aligned}$$
If $\mu > 0$, then $\|(A-\lambda I)x\|$ is bounded below, away from 0. Then, by Corollary 8.3.4, the inverse $R_\lambda(A)$ exists and is bounded. But in this case $\lambda \notin \sigma_c(A)$, which is a contradiction. Hence we must have $\mu = 0$ and λ is real.

■

8.4 Fredholm Alternatives

Definition 8.4.1 A linear operator $A : X \to Y$ has *finite rank*, if its range is finite dimensional.

Proposition 8.4.1 If A has finite rank on a Hilbert space H, and if for $\lambda \neq 0$
$$\inf\{\|(A-\lambda I)x\| : \|x\| = 1\} = 0,$$
then $\lambda \in \sigma_p(A)$.

Proof. Consider a sequence $\{x_n\}_n$ in H with $\|x_n\| = 1$ such that
$$\|(A-\lambda I)x_n\| \to 0, \qquad \text{for} \quad n \to \infty.$$

Since the range of A is finite, there exists a convergent subsequence in the range of A:

$$||Ax_{n_j} - y|| \to 0, \qquad \text{for} \quad n \to \infty.$$

Then

$$x_{n_j} = \frac{1}{\lambda}\Big(\underbrace{(\lambda I - A)x_{n_j}}_{\to 0} + \underbrace{Ax_{n_j}}_{\to y}\Big) \to \frac{1}{\lambda}y.$$

For the norm of x_{n_j} we have

$$1 = ||x_{n_j}|| \to 1 = \frac{1}{\lambda}||y||, \qquad \text{and} \qquad ||y|| \neq 0.$$

Furthermore,

$$Ax_{n_j} \to y \qquad \text{and} \qquad Ax_{n_j} \to \frac{1}{\lambda}Ay,$$

which implies that $Ay = \lambda y$ and $\lambda \in \sigma_p(A)$. ■

Corollary 8.4.2 Let H be a Hilbert space and $A : H \to H$ a linear map with finite rank. Then for each $\lambda \in \mathbb{C}$ one of the following cases holds

1. $\lambda \in \rho(A)$
2. $\lambda \in \sigma_p(A)$ is an eigenvalue of finite multiplicity.

Proof. If $\lambda \notin \sigma_p(A)$ then, by the previous Proposition 8.4.1, there exists a $c > 0$ such that

$$||(A - \lambda I)x|| \geq c||x|| \qquad \text{for all} \quad x \in H.$$

Consider $y \in \overline{\text{Range}(A - \lambda I)}$. Then there exists a sequence $\{x_n\}_n$ with

$$(A - \lambda I)x_n \to y, \qquad \text{as} \quad n \to \infty.$$

Then

$$||x_n - x_m|| \leq c^{-1}||(A - \lambda I)x_n - (A - \lambda I)x_m||$$

and $\{x_n\}_n$ is a Cauchy sequence with a limit in H: $x_n \to x$. Consequently, $(A - \lambda I)x = y$ and $R_\lambda(A)y$ exists. Hence $\text{Range}(A - \lambda I) = \text{domain}(R_\lambda(A))$ is closed. Since $\lambda \notin \sigma_p(A)$ it is also not in $\sigma_p(A^*)$ and

$$\text{Range}(A - \lambda I) = (\ker(A - \lambda I)^*)^\perp = H.$$

Now let

$$Ty := \{\text{ unique element } x \text{ such that } \quad (A - \lambda I)x = y\}.$$

Then $(A - \lambda I)Ty = y$ and

$$c||Ty|| \leq ||(A - \lambda I)Ty|| = ||y||,$$

which implies that T is bounded. In fact $T = (A - \lambda I)^{-1}$ so $\lambda \in \rho(A)$. If $\lambda \in \sigma_p(A)$ then the eigenspace must be finite dimensional, since A has finite rank. ■

Theorem 8.4.3 — Finite-Rank Fredholm Alternative. Let $G \subset \mathbb{C}$ and consider an analytic map $\lambda \mapsto F(\lambda) \in \mathscr{L}(H)$, $\lambda \in G$. Assume $F(\lambda)$ is of finite rank and that

$$\mathrm{Range}(F(\lambda)) \subset M, \qquad \text{with} \quad \dim(M) < \infty.$$

Then one of the two alternatives holds. Either

1. $(I - F(\lambda))^{-1}$ exists for no $\lambda \in G$, or
2. $(I - F(\lambda))^{-1}$ exists for every $\lambda \in G \backslash S$, where S is a discrete subset of G. In this case the map $\lambda \mapsto (I - F(\lambda))^{-1}$ is analytic in $G \backslash S$ and if $\lambda \in S$ then $F(\lambda)\Phi = \Phi$ has a finite dimensional family of solutions.

Proof. We consider $\{\psi_i\}_{i=1\dots n}$ a finite basis of the finite dimensional space $M \subset H$. Then F has a spectral representation

$$F(\lambda)\phi = \sum_{i=1}^{n} (\gamma_i(\lambda), \phi)\psi_i,$$

and the maps $\lambda \mapsto \gamma_i(\lambda)$ are analytic in G. We define

$$\Lambda_{ij}(\lambda) := (\gamma_i(\lambda), \psi_j).$$

The inverse $(I - F(\lambda))^{-1}$ does not exist when $F(\lambda)\phi = \phi$ has a nontrivial solution. We write this equation in components for $\phi = \sum_{i=1}^{n} a_i \psi_i$ as

$$\phi = \sum_{i=1}^{n} a_i \psi_i = F(\lambda)\phi = \sum_{i=1}^{n} \left(\gamma_i(\lambda), \sum_{j=1}^{n} a_j \psi_j \right) \psi_i.$$

Using orthogonality of the eigenfunctions, we find

$$a_i = \sum_{j=1}^{n} (\gamma_i(\lambda), \psi_i) a_j = \sum_{j=1}^{n} \Lambda_{ij} a_j,$$

which we write in matrix notation as

$$(I - \Lambda(\lambda))a = 0, \qquad a = (a_1, \dots, a_n), \quad \Lambda = (\Lambda_{ij}).$$

For a non-trivial solution we need

$$d(\lambda) = \det(I - \Lambda(\lambda)) = 0.$$

By the assumption $d(\lambda)$ is analytic in $\lambda \in G$, hence $d(\lambda)$ is either identically zero in G (case 1.), or the zeroes form a discrete set.

Finally, since $F(\lambda)$ has finite rank, we can only have finitely many solutions of $F(\lambda)\phi = \phi$. ■

Theorem 8.4.4 — Compact Fredholm Alternative. Let $G \subset \mathbb{C}$ and consider an analytic map $\lambda \mapsto B(\lambda) \in \mathscr{L}(H)$, $\lambda \in G$. Assume $B(\lambda)$ is compact for each $\lambda \in G$. Then one of the two alternatives holds. Either

1. $(I - B(\lambda))^{-1}$ exists for no $\lambda \in G$, or
2. $(I - B(\lambda))^{-1}$ exists for every $\lambda \in G \backslash S$, where S is a discrete subset of G. In this case the map $\lambda \mapsto (I - B(\lambda))^{-1}$ is analytic in $G \backslash S$ and if $\lambda \in S$ then $B(\lambda)\Phi = \Phi$ has a finite dimensional family of solutions.

Proof-Sketch. Use a spectral representation of $B(\lambda)$ and approximate $B(\lambda)$ by considering finite sums. Use the previous result and let n go to infinity. ■

■ **Example 8.4 — Spectrum of an integral operator.** Compute the spectrum of

$$u \mapsto Ku = \int_0^1 e^{-\gamma(x-y)} u(y) dy$$

in

$$D(K) = (L^2(0,1))_+ = \{u \in L^2(0,1) : u \geq 0\}.$$

Since the kernel of the integral operator

$$k(x,y) = e^{-\gamma(x-y)} \in L^2((0,1) \times (0,1)),$$

the operator K is a compact Hilbert-Schmidt operator. Hence it has a discrete spectrum of eigenvalues with a possible limit point at 0.

To solve $Ku = \lambda u$, that is

$$\int_0^1 e^{-\gamma(x-y)} u(y) dy = \lambda u(x),$$

we multiply by the kernel $e^{-\gamma(z-x)}$ and integrate:

$$\begin{aligned} \int_0^1 e^{-\gamma(z-x)} \int_0^1 e^{-\gamma(x-y)} u(y)\, dy\, dx &= \lambda \int_0^1 e^{-\gamma(z-x)} u(x) dx \\ &= \lambda^2 u(z). \end{aligned}$$

On the other hand we compute directly

$$\begin{aligned} \int_0^1 e^{-\gamma(z-x)} \int_0^1 e^{-\gamma(x-y)} u(y)\, dy\, dx &= \int_0^1 \int_0^1 e^{-\gamma z + \gamma y} u(y) dy\, dx \\ &= \int_0^1 e^{-\gamma(z-y)} u(y) dy \\ &= \lambda u(z). \end{aligned}$$

Hence for $u \neq 0$ we have $\lambda^2 = \lambda$, which implies $\lambda = 0$ or $\lambda = 1$.

If $\lambda = 0$, then we have

$$\int_0^1 e^{-\gamma(x-y)} u(y) dy = 0$$

and since $u \geq 0$ this implies that $u = 0$ a.e. in $[0,1]$. Since we work in L^2 this means $u = 0$, and u is not an eigenfunction. Hence $\lambda = 0$ is not an eigenvalue.

If $\lambda = 1$, then we need to solve

$$\int_0^1 e^{-\gamma(x-y)} u(y) dy = u(x).$$

We try $u(x) = e^{-\gamma x}$:

$$\int_0^1 e^{-\gamma(x-y)} u(y) dy = e^{-\gamma x} \int_0^1 e^{\gamma y} e^{-\gamma y} dy = e^{-\gamma x} = u(x).$$

Hence $u(x) = e^{-\gamma x}$ is an eigenfunction of K with eigenvalue 1. The spectrum is

$$\sigma(K) = \{1\}.$$

Moreover, since $k(x,y) > 0$ we can apply the Krein-Rutman Theorem [13] (not covered here), and find that $\lambda = 1$ has multiplicity one with a unique positive eigenfucntion $u(x) = e^{-\gamma x}$.

■

Theorem 8.4.5 — Spectral Fredholm Alternative. Assume $A : H \to H$ is a compact linear map. Then $\sigma(A)$ is compact and the only possible limit point is $\lambda = 0$. Furthermore, given $\lambda \in \mathbb{C} \backslash \{0\}$ there are two alternatives. Either

1. $\lambda \in \rho(A)$, or
2. $\lambda \in \sigma_p(A)$ is an eigenvalue of finite multiplicity.

Proof. Let $G := \mathbb{C} \backslash \{0\}$ and $B(\lambda) = \frac{1}{\lambda} A$. We apply the previous Compact Fredholm Alternative. Note that $\frac{1}{\lambda} \to \infty$ is possible, i.e., $\lambda \to 0$ is a possible limit point. ■

Theorem 8.4.6 — Hilbert-Schmidt. Let $A : H \to H$ be linear, compact and self-adjoint. Then all eigenvalues λ_i are real and there exists an orthonormal set of eigenfucntions $\{\psi_i\}_i$ such that A has a spectral representation

$$A\phi = \sum_{i=1}^{\infty} \lambda_i (\phi, \psi_i) \psi_i.$$

Proof. All eigenvalues are real, since A is self-adjoint. Based on the Spectral Fredholm Alternative the eigenvalues form a disctrete set, each with finite multiplicity. Thus we enumerate them according to their multiplicity $\{\lambda_i\}_i$ with eigenvectors $\{\psi_i\}_i$, which are orthogonal. We can always normalize them to norm 1.

Now we set

$$M := \overline{\operatorname{span}\{\psi_i, i = 1, \ldots, \infty\}}$$

and we show that $\operatorname{Range}(A) \subset M$.

First, since A is compact, we have $AM = M$. For $\phi \in M^\perp$ we have that $(A\phi, \psi) = (\phi, A\psi) = 0$ for all $\psi \in M$. Hence $A\phi \in M^\perp$ as well. This implies that both M and $M^\perp$ are invariant under A. We define

$$\hat{A} := A\Big|_{M^\perp}$$

and we will show that $\hat{A} = 0$. Indeed, $\hat{A}$ is self adjoint and compact (same as A), and each spectral value is an eigenvalue. But each eigenvector belongs already to M. Hence $\sigma_p(\hat{A}) = \emptyset$. Hence $\hat{A} = 0$. It follows that $\{\psi_i\}_i$ is a basis of Range(A) and we write

$$A(u) = \sum_{i=1}^{\infty} (\psi_i, Au)\psi_i = \sum_{i=1}^{\infty} (A\psi_i, u)\psi_i = \sum_{i=1}^{\infty} \lambda_i(\psi_i, u)\psi_i.$$

■

8.5 Summary of Spectral Theory

$$\begin{aligned} A &: X \to X, \qquad \text{linear} \\ \sigma(A) &= \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A) \\ \sigma_p(A) &= \{\text{eigenvalues}\} \\ \sigma_c(A) &= \{R_\lambda(A) \text{ exists with dense domain, but is unbounded}\} \\ \sigma_r(A) &= \{R_\lambda(A) \text{ exists, but the domain is not dense}\} \end{aligned}$$

- Neumann series $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$
- spectral radius $r_\sigma(A) = \sup_{\lambda \in \sigma(A)} |\lambda| \le \|A\|$
- resolvent set $\rho(A) = \mathbb{C} \backslash \sigma(A)$. $\rho(A)$ is open.

Spectral Theorems:

1. A symmetric on H-space: $\sigma_p(A) \subset \mathbb{R}, \sigma_c(A) \subset \mathbb{R}$; eigenvectors are orthogonal.
2. A compact on H-space. $\sigma(A) = \sigma_p(A)$; $\lambda \in \sigma_p(A)$ has finite multiplicity. $\sigma_p(A)$ is a discrete set with possible limit point at 0.
3. A compact and self-adjoint: In this case we have the properties of Item 1. and Item 2. plus the spectral representation

$$Au = \sum_{i=1}^{\infty} \lambda_i(\psi_i, u)\psi_i, \qquad \{\psi_i\}_i \text{ orthonormal basis of eigenvectors.}$$

8.6 Exercises

Exercise 8.1 (Classical Sturm-Liouville problems) (level 1)
Find the point spectrum and the corresponding eigenfunctions in $L^2([0,\pi])$ for the operators A and B with

$$A = -\frac{d^2}{dx^2}, \qquad D(A) = \{f \in L^2(0,\pi); Af \in L^2(0,\pi), f(0) = 0, f(\pi) = 0\},$$
$$B = -\frac{d^2}{dx^2}, \qquad D(B) = \{f \in L^2(0,\pi); Bf \in L^2(0,\pi), \frac{d}{dx}f(0) = 0, \frac{d}{dx}f(\pi) = 0\}.$$

■

Exercise 8.2 (Adjoint spectrum) (level 2)
Consider a linear operator A on a Banach space with adjoint A^*. Show that

$$\lambda \in \sigma_p(A) \qquad \Longrightarrow \qquad \bar{\lambda} \in \sigma_r(A^*) \cup \sigma_p(A^*).$$

■

Exercise 8.3 (Left and right shift) (level 3)
Consider the Hilbert space l^2, which consist of sequences $x = (x_1, x_2, \dots)$ with bounded norm

$$\|x\|_2^2 = \sum_{i=1}^{\infty} |x_i|^2.$$

We define two operators, a left-shift L and a right-shift R as

$$L(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots), \qquad R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

1. Find the adjoints of L and R.
2. Find $\sigma(L), \sigma_p(L), \sigma_c(L), \sigma_r(L)$. Recommended steps:
   - Find the norm of L and use it to find the spectral radius of L. Define a ball that contains the spectrum $\sigma(L)$.
   - Find the point-spectrum of L.
   - Show that $\sigma_r(L) = \emptyset$.
   - Use the fact that the spectrum is a closed set to identify all three components.
3. Find $\sigma(R), \sigma_p(R), \sigma_c(R), \sigma_r(R)$. Recommended steps:
   - Find the norm of R , the spectral radius of R, and define a ball that contains the spectrum $\sigma(R)$.
   - Find the point-spectrum of R.
   - Use the previous results and the result from Exercise 22 to find the residual spectrum of R.
   - Find the continuous spectrum of R.
   - Use the fact that the spectrum is a closed set to identify all three components.

■

Exercise 8.4 (skew adjoint) (level 2)

1. Show that the operator on $L^2(0,1)$ given by

$$A = \frac{\partial}{\partial x}, \qquad D(A) = \{u \in H^1([0,1]); u(0) = u(1)\}$$

is skew adjoint ($A^* = -A, D(A^*) = D(A)$).
2. Show that the point spectrum of A is purely imaginary.

9. Semigroup Theory

Contents

9.1 Introduction

The key idea of semigroup theory is to write a PDE as an ODE in a Banach space. Now we learned a lot about Banach spaces and their operators, and we are ready to do exactly that. For example, consider a reaction diffusion equation

$$u_t = D\Delta u + f(u)$$

which we write as

$$u_t = Au + f(u), \qquad u(t,\cdot) \in H^2(\Omega).$$

Some ideas to think about:

- Consider the linear case $\dot{u} = Au$. If A were a matrix and $u(t) \in \mathbb{R}^n$ then we could simply use the matrix exponential to solve the differential equation

  $$u(t) = e^{At}u_0.$$

  The exponential matrix (or matrix exponential)

  $$e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}t^n \tag{9.1}$$

  is well defined by its power series. In fact, by the Caley-Hamilton theorem e^{At}, for given t, is only a finite series, and there is no worry about convergence.
- If A is a bounded operator we can still use the same series (9.1) to define an operator exponential. The series might be infinite, but it still converges in the operator norm and is bounded by $e^{\|A\|}$.
- But what if A is an unbounded operator, for example $A = \Delta$? If we write a formula like (9.1), then u would need to be C^∞ such that all derivatives make sense.
- The question arises if there is a better way to define e^A if A is unbounded? Here are a few ideas:
- **Idea 1:** Use diagonalization. For a matrix $A \in \mathbb{R}^{n\times n}$ we can find its Jordan normal form J and an orthogonal transformation Q such that $A = Q^{-1}JQ$. Then

  $$e^A = e^{Q^{-1}JQ} = Q^{-1}e^J Q$$

  and we have methods to compute e^J. How about unbounded Banach spaces? Something like a Jordan form does not exist in an infinite dimensional context.
- **Idea 2:** We use the limit definition of an exponential

  $$e^A = \lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n.$$

  Again we are facing the problem of using higher orders of A^n, restricting the available function spaces.
- **Idea 3:** We could use the inverse limit formula for the exponential

  $$e^A = \lim_{n\to\infty}\left(I - \frac{A}{n}\right)^{-n},$$

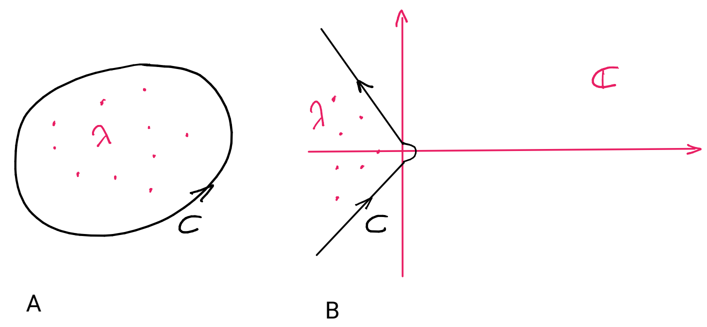


Figure 9.1: Illustration of idea 4. Using the Cauchy integral formula over a path C the encircles the spectrum of A might become a useful definition of an operator exponential.

which we need to read in the correct operator-way as

$$e^A = \lim_{n\to\infty} \left[\left(I - \frac{A}{n} \right)^{-1} \right]^n = \lim_{n\to\infty} [-nR_n(A)]^n$$

using the resolvent. This looks complicated, but it actually works! For an unbounded operator the resolvent, when it exists, is very well behaved, and taking it to some large power is no problem.

- **Idea 4:** Using the Chauchy-integral formula for the exponential

$$e^A = \frac{1}{2\pi i} \int_C e^\lambda (\lambda I - A)^{-1} d\lambda,$$

where $C \subset \mathbb{C}$ is a closed, simple, rectifiable, positive oriented curve that encloses the spectrum of A. This method seems very far fetched, but as we will see, it works too! In particular for analytic semigroups. In Figure 9.1 we illustrate this idea for a bounded operator on the left (A) and for an analytic semigroup on the right (B).

- A proper definition will arise from the properties of the matrix exponential, which we call the semigroup properties, and the concept of an infinitessimal generator A.

Definition 9.1.1 Let X be a Banach space. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators in X is called a *strongly continuous semigroup* or C^0-*semigroup*, if it satisfies

1. $T(t+s) = T(t)T(s), \qquad t,s \geq 0,$ (semigroup property)
2. $T(0) = I$,
3. For all $u \in X$ the map $t \mapsto T(t)u$ is continuous in X. Note that it is sufficient to check continuity at $t = 0$.

■ **Example 9.1** Let $X = C^0(\mathbb{R})$ and consider the left shift $T(t)u(x) = u(x+t)$. $T(t)$ is a strongly continuous semigroup. Let us check the conditions. Firstly $T(t+s)u(x) = u(x+s+t)$ while $T(t)T(s)u(x) = T(t)u(x+s) = u(x+s+t)$, which confirms item 1. At $t = 0$ we have $T(0)u(x) = u(x)$, i.e. $T(0) = I$ and item 2. is satisfied. Since $u \in C^0(\mathbb{R})$ we have continuity

$$\lim_{t\to 0} \|T(t)u\|_\infty = \lim_{t\to 0} \|u(t+x)\|_\infty = |u(x)|,$$

Let us now look at the simple first order, linear, partial differential equation

$$u_t - u_x = 0. \tag{9.2}$$

This equation on $\mathbb{R}$ is solved by a traveling wave $u(x,t) = u(x+t) = T(t)u(x)$. Hence $T(t)$ is the *solution semigroup* of (9.2). In this sense we could say

$$T(t) = e^{t\frac{\partial}{\partial x}}.$$

Note that equation (9.2) needs $u \in C^1$, while $T(t)$ only needs $u \in C^0$. ▪

We prove a technical Lemma

Proposition 9.1.1 Let $\omega : [0,\infty) \to \mathbb{R}$ be bounded on any finite interval and subadditive $\omega(t_1+t_2) \leq \omega(t_1)+\omega(t_2)$, then

$$\inf_{t>0} \frac{\omega(t)}{t} = \lim_{t\to\infty} \frac{\omega(t)}{t},$$

where the limit might be equal to $-\infty$.

Proof. Let $\omega_0 := \inf_{t>0} \frac{\omega(t)}{t}$ and consider $\gamma > \omega_0$. Then there exists a $t_0 > 0$ such that $\frac{\omega(t_0)}{t_0} < \gamma$. We chose a $t > t_0$ which we write as $t = nt_0 + r$. Then by subadditivity we have

$$\frac{\omega(t)}{t} \leq \frac{n\omega(t_0)+\omega(r)}{t}.$$

As we take the limit $n \to \infty$ we get $t \to \infty$ and $\frac{n}{t} \to \frac{1}{t_0}$. The second term on the right hand side satisfies

$$\frac{\omega(r)}{t} \leq \sup_{s\in[0,t_0)} \frac{\omega(s)}{t} \to 0 \text{ for } t \to \infty.$$

Then

$$\limsup_{t\to\infty} \frac{\omega(t)}{t} \leq \frac{\omega(t_0)}{t_0} < \gamma,$$

for all $\gamma > \omega_0$. Hence $\lim_{t\to\infty} \frac{\omega(t)}{t}$ exists and equals ω_0. ■

The next result gives us an important exponential growth bound .

Theorem 9.1.2 Let $\{T(t)\}\ t \geq 0$ be a strongly continuous semigroup of bounded linear operators on a Banach space X. Then

$$\omega_0 = \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t}$$

exists (or is $-\infty$). For every $\gamma > \omega_0$ there is a constant $M_\gamma > 0$ such that

$$\|T(t)\| \leq M_\gamma e^{\gamma t}.$$

The growth rate ω_0 identifies the type of the semigroup, which is exponentially decaying for $\omega_0 < 0$ and exponentially growing for $\omega_0 > 0$.

Proof. The function $\ln \|T(t)\|$ is subadditive,

$$\ln \|T(t+s)\| \leq \ln \|T(t)\| + \ln \|T(s)\|$$

and $\|T(t)\|$ is bounded on bounded intervals. Hence by the previous Proposition 9.1.1 the limit ω_0 exists. For each $\gamma > \omega_0$ and for $t > t_0$ for some $t_0 > 0$ we then have

$$\frac{\ln \|T(t)\|}{t} \leq \gamma \qquad \text{for all} \quad t > t_0,$$

which implies

$$\|T(t)\| \leq e^{\gamma t} \qquad \text{for all} \quad t > t_0.$$

We set

$$M_\gamma = \max \left\{ 1, \sup_{t \in [0,t_0]} \|T(t)\| e^{-\gamma t} \right\}.$$

■

■ **Example 9.2** We consider again the left-shift $T(t)u(x) = u(x+t)$ in $C^0(\mathbb{R})$. Then

$$\|T(t)\| = \sup_{\|u\|_\infty = 1} \|T(t)u\|_\infty = \sup_{\|u\|_\infty = 1} \|u(x+t)\|_\infty = 1.$$

Then $\ln \|T(t)\|/t = 0/t = 0 = \omega_0$. The semigroup shows no growth or decay, it is simply a shift to the left. ■

9.2 The Infinitessimal Generator

The infinitessimal generator arises if we have a semigroup and we would like to know what the corresponding operator is, i.e. we know e^{At} and want to determine A. If the operator exponential exists, we can formally compute

$$\frac{d}{dt} e^{At} \bigg|_{t=0} = A.$$

We do the same in the general case:

Definition 9.2.1 Let $\{T(t)\}$ be a strongly continuous semigroup. The *infinitessimal generator* is defined as

$$Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h}.$$

The domain $D(A)$ is the set of all $x \in X$ where the above limit exists.

■ **Example 9.3** Again, we consider the shift-semigroup on $C^1(\mathbb{R})$: $T(t)u(x) = u(x+t)$. The above differential quotient becomes

$$Au = \lim_{h \to 0^+} \frac{T(h)u(x) - u(x)}{h} = \lim_{h \to 0^+} \frac{u(x+h) - u(x)}{h} = \frac{\partial}{\partial x} u.$$

Hence the generator is $A = \frac{\partial}{\partial x}$ with domain $D(A) = C^1(\mathbb{R})$. ■

Let us collect some important properties of the generator.

Theorem 9.2.1 — Fundamental Theorem for Semigroups. Let $\{T(t)\}$ be a strongly continuous semigroup on a Banach space X with infinitessimal generator A. Then

1. For all $x \in X$ we have
$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x.$$
2. For all $x \in X$ we have
$$\int_0^t T(s)xds \in D(A), \qquad \text{and} \qquad A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$
3. For $x \in D(A)$ we have $T(t)x \in D(A)$. The function $t \mapsto T(t)x$ is differentiable and
$$\frac{d}{dt}T(t)x = A(T(t)x) = T(t)(Ax).$$
4. For $x \in D(A)$ we have
$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Proof. 1. First note that since $T(t)$ is strongly continuous. We find
$$\lim_{h\to 0}\left|\frac{1}{h}\int_0^h T(s)x - T(0)xds\right| \leq \lim_{h\to 0}\frac{1}{h}\int_0^h ds \sup_{s\in[0,h]}\|(T(s)x - T(0)x\| = 0,$$
hence
$$\lim_{h\to 0}\frac{1}{h}\int_0^h (T(s)x - T(0)x)ds = 0.$$
Then, using the semigroup property, we find
$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds = \lim_{h\to 0} T(t)\frac{1}{h}\int_0^h T(s)xds = T(t)\underbrace{T(0)}_{I}x = T(t)x.$$

2. We compute
$$\begin{aligned}
&\frac{T(h)-I}{h}\int_0^t T(s)xds \\
=\ & \frac{1}{h}\int_0^t (T(s+h) - T(s))xds \\
=\ & \frac{1}{h}\left(\int_0^{t-h} T(s+h)xds - \int_h^t T(s)xds + \int_{t-h}^t T(s+h)xds - \int_0^h T(s)xds\right) \\
=\ & \frac{1}{h}\left(\int_h^t T(s)xds - \int_h^t T(s)xds + \int_t^{t+h} T(s)xds - \int_0^h T(s)xds\right) \\
=\ & \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds \\
\to\ & T(t)x - x, \qquad \text{for} \quad h \to 0.
\end{aligned}$$

3. Writing down the time derivatives gives

$$\underbrace{\frac{T(t+h)x - T(t)x}{h}}_{\to \frac{\partial T(t)x}{\partial t}} = \underbrace{\frac{T(h)-I}{h}}_{\to A} T(t)x = T(t)\underbrace{\frac{T(h)-I}{h}}_{\to A} x.$$

4. Finally, to prove the last relationship, we integrate item 3. and use the Fundamental Theorem of Calculus.

■

R Note that the most important part of the Fundamental Theorem is item 3., where the semigroup $T(t)$ and the generator A are combined into an abstract differential equation. This is the reason why we develop semigroup theory. Items 1., 2. and 4. then tell us that in this abstract case we can still do differential calculus like we are used to.

Theorem 9.2.2 Let A be an infinitessimal generator. Then $D(A)$ is dense in X and A is closed.

Proof. From Theorem (9.2.1) we have

$$x = \lim_{h\to 0} \frac{1}{h}\int_0^h T(s)xds, \qquad \text{and} \qquad \int_0^h T(s)xds \in D(A),$$

hence $D(A)$ is dense in X. To show that A is closed we consider a convergent sequence $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$. We need to show $y = Ax$. Indeed, from Theorem (9.2.1) we know that

$$T(h)x_n - x_n = \int_0^h T(s)Ax_n ds.$$

The first term converges for $n \to \infty$ to $T(h)x - x$, while the right hand side converges to $\int_0^h T(s)yds$. Which implies

$$Ax = \lim_{h\to 0^+} \frac{T(h)x - x}{h} = \lim_{h\to 0} \frac{1}{h}\int_0^h T(s)yds = y.$$

■

R We are starting to build the *Semigroup Triangle*, which summarizes the relationships between semigroup $T(t)$, generator A and resolvent $R_\lambda(A)$. It is a tool that helps to visualize the various aspects that are discussed here. As seen in Figure 9.2 we identify the relation from T to A, and of course the relations between A and $R_\lambda(A)$. The question marks are relations that we fill in in the next sections.

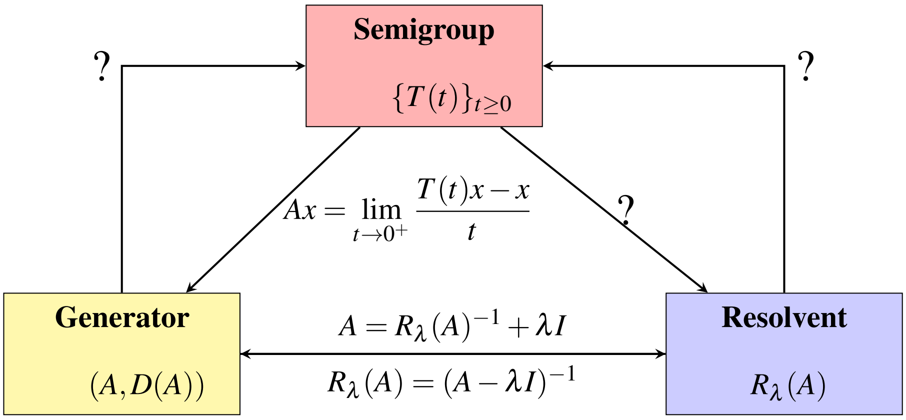


Figure 9.2: Part I of the *Semigroup Triangle*

9.3 Solutions to Abstract ODEs

We consider the abstract ODE on a Banach space X with unbounded operator A as

$$u_t = Au + f(t), \qquad u(0) = u_0 \in X, \tag{9.3}$$

where we assume $u_0 \in D(A)$, $f \in C([0,T],X)$. For $T > 0$ we are interested in a *classical solution*

$$u \in C^1([0,T],X) \cap C([0,T],D(A)).$$

Theorem 9.3.1 — Variation of Constants Formula. Assume A is infinitessimal generator of a C^0-semigroup $\{T(t)\}$. If u is a classical solution of (9.3), then

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds. \tag{9.4}$$

Proof. This is, in fact, the classical variation of constant formula. Using the properties of the generator in Theorem (9.2.1), we directly compute for $g(s) = T(t-s)u(s)$ that

$$\begin{aligned}
\frac{dg}{ds} &= -AT(t-s)u(s) + T(t-s)\dot{u}(s) \\
&= -AT(t-s)u(s) + T(t-s)(Au(s) + f(s)) \\
&= T(t-s)f(s).
\end{aligned}$$

Hence

$$g(t) - g(0) = u(t) - T(t)u_0 = \int_0^t T(t-s)f(s)ds.$$

■

Definition 9.3.1 Given

$$\dot{u} = Au + f(t), \qquad \text{in} \quad X,$$

and $f \in L^1((0,T),X)$. Then u is called a *mild solution* if it satisfies the variation of constant formula (9.4).

R Note that a mild solution does not need any differentiation.

R Note that a classical solution is automatically a mild solution. However, a mild solution is not necessarily a weak or classical solution. But mild solutions can be used to find those.

The next result answers the question when a mild solution is classical.

Theorem 9.3.2 Assume $f \in C([0,T],X)$ and $u_0 \in D(A)$. Assume that f satisfies at least one of the following conditions: either $f \in W^{1,1}([0,T],X)$ or $f \in L^1(0,T,D(A))$. Then a mild solution u is also a classical solution.

Proof. By Theorem (9.2.1) the term $T(t)u_0$ is differentiable and $T(t)u_0 \in D(A)$, hence the first term on the right hand side of (9.4) is classical. Now we study the integral term

$$v(t) := \int_0^t T(t-s)f(s)ds. \tag{9.5}$$

We need to show that

$$v \in \underbrace{C^1([0,T],X)}_{(1)} \cap \underbrace{C([0,T],D(A))}_{(2)}.$$

We compute

$$\begin{aligned}
\underbrace{\frac{T(h)-I}{h}}_{(3)} v &= \frac{1}{h}\left(\int_0^t T(h+t-s)f(s)ds - \int_0^t T(t-s)f(s)ds\right) \\
&= \frac{1}{h}\left(\underbrace{\int_0^{t+h} T(h+t-s)f(s)ds}_{v(t+h)} - \int_t^{t+h} T(h+t-s)f(s)ds - \underbrace{\int_0^t T(t-s)f(s)ds}_{v(t)}\right) \\
&= \underbrace{\frac{1}{h}(v(t+h)-v(t))}_{(4)} - \frac{1}{h}\int_t^{t+h} T(h+t-s)f(s)ds.
\end{aligned}$$

From Theorem (9.2.1) we evaluate the limit of the last term

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(h+t-s)f(s)ds = f(t).$$

The limits of (3) and (4) need some more attention. In fact, if (1) is true, then (4) converges

$$\lim_{h\to 0}\frac{1}{h}(v(t+h)-v(t)) = \dot{v}(t).$$

In this case (3) converges to Av and (2) is true, i.e (1) implies (2).

If (2) is true then (3) converges

$$\lim_{h\to 0}\frac{T(h)-I}{h}v(t) = Av(t).$$

Moreover, if (3) converges, then also (4), which implies (1). Hence (1) implies (2). Also, if (4) converges, then also (3). We find that (1) implies (2) and vice versa. We break this circle conclusion by the additional conditions on f.

If $f\in L^1(0,T,D(A))$, then (3) converges by the definition of $v(t)$ in (9.5). If $f\in W^{1,1}([0,T],X)$ we use the convolution identity

$$v(t) = \int_0^t T(s)f(t-s)ds$$

and differentiate

$$\dot{v}(t) = T(t)f(0)+\int_0^t T(s)\dot{f}(t-s)ds,$$

which is well defined. Hence (1) holds. Altogether we have a classical solution of $\dot{u} = Au+f(t)$, $u(0) = u_0$. ■

9.4 The Hille-Yosida Theorem

The Hille-Yosida Theorem is the first result that will answer the question: When is an operator a generator of a C^0-semigroup?

Theorem 9.4.1 — Hille-Yosida. The operator $A: X\to X$ is generator of a C^0-semigroup $\{T(t)\}$ with exponential bound

$$\|T(t)\| \le Me^{\omega t}, \tag{9.6}$$

if and only if

1. $D(A)$ is dense in X and A is closed.
2. Each $\lambda > \omega$ is in the resolvent set $\rho(A)$ and the resolvent is estimated as

$$\|R_\lambda(A)^n\| \le \frac{M}{(\lambda-\omega)^n}.$$

Proof. **Step 1: Item 1. is necessary** (semigroup $\Longrightarrow$ Item 1.)
We already know from Theorem (9.2.2) that $D(A)$ needs to be dense and A needs to be closed. Hence item 1. is necessary.
Step 2: Item 2. is necessary (semigroup $\Longrightarrow$ Item 2.)
Assume $\{T(t)\}$ is a semigroup with the exponential bound (9.6). We define a strange looking integral operator, called $I_n(\lambda)$, and we will show that it corresponds to $(-R_\lambda(A))^n$.

The integral representation will then allow us to prove the desired resolvent estimate. Here we go

$$I_n(\lambda)x := \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt, \qquad \lambda \in \mathbb{C}, \mathrm{Re}\lambda > \omega, n \geq 1.$$

We will show that

$$I_n(\lambda) = (-R_\lambda(A))^n \qquad \text{and} \qquad \|I_n(\lambda)\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^n}.$$

The inequality is straight-forward by using the exponential bound and integration by parts $(n-1)$-times:

$$\begin{aligned}
\|I_n(\lambda)\| &\leq \frac{M}{(n-1)!}\int_0^\infty \left|t^{n-1}e^{-\lambda t}e^{\omega t}\right| dt \\
&= \frac{M}{(n-1)!}\int_0^\infty t^{n-1}\left|e^{(\omega-\lambda)t}\right| dt \\
&= \frac{M}{(n-1)!}\int_0^\infty t^{n-1}e^{(\omega-\mathrm{Re}\lambda)t}\,dt \\
&= \frac{M(-1)^{n-1}}{(\omega-\mathrm{Re}\lambda)^{(n-1)}}\int_0^\infty e^{(\omega-\mathrm{Re}\lambda)t}\,dt \\
&= \frac{M(-1)^{n-1}}{(\omega-\mathrm{Re}\lambda)^n}(-1) && (9.7) \\
&= \frac{M}{(\mathrm{Re}\lambda-\omega)^n}. && (9.8)
\end{aligned}$$

For $x \in D(A)$ and $n > 1$ we find

$$\begin{aligned}
I_n(\lambda)Ax &= \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)Ax\,dt \\
&= \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}\frac{d}{dt}(T(t)x)\,dt \\
&= -\frac{1}{(n-2)!}\int_0^\infty t^{n-2}e^{-\lambda t}T(t)x\,dt + \frac{\lambda}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)x\,dt \\
&= -I_{n-1}(\lambda)x + \lambda I_n(\lambda)x. && (9.9)
\end{aligned}$$

Similarly, for $n = 1$ we get

$$\begin{aligned}
I_1(\lambda)Ax &= \int_0^\infty e^{-\lambda t}T(t)Ax\,dt \\
&= \int_0^\infty e^{-\lambda t}\frac{d}{dt}(T(t)x)\,dt \\
&= -e^0T(0)x + \lambda\int_0^\infty e^{-\lambda t}T(t)x\,dt \\
&= -x + \lambda I_1(\lambda)x. && (9.10)
\end{aligned}$$

Hence we get recursive relations between the I_n. From (9.9) we find

$$I_n(\lambda)(Ax - \lambda x) = -I_{n-1}(\lambda)x$$

and since T and A commute, we also have

$$(Ax - \lambda x)I_n(\lambda) = -I_{n-1}(\lambda)x.$$

Moreover

$$I_1(\lambda)(A - \lambda I) = -I = (A - \lambda I)I_1(\lambda).$$

This implies that

$$R_\lambda(A)x = (A - \lambda I)^{-1}x = -I_1(\lambda)x = -\int_0^\infty e^{-\lambda t}T(t)xdt. \tag{9.11}$$

This equation is an important relation in its own right, as it describes the resolvent as a Laplace transform of the semigroup!

Starting with I_1 we then argue recursively to find

$$I_n(\lambda) = (-R_\lambda(A))^n.$$

Step 3: Item 1. and 2. are sufficient (Item 1.+2. $\Longrightarrow$ semigroup). Now we come back to one of the ideas that we discussed in the introduction to this chapter; the idea to define an operator exponential as an inverse exponential limit

$$T_n(t) := \left(I - \frac{t}{n}A\right)^{-n}.$$

We show $T_n \to T$. We rewrite $T_n(t)$ as

$$T_n(t) = \left(\frac{t}{n}\right)^{-n}\left(-R_{\frac{n}{t}}(A)\right)^n,$$

which is bounded by Item 2. for $\frac{n}{t} > \omega$. This is particularly bounded for large n and small t. To show that $T_n(0) = I$, it is sufficient to show that

$$\lim_{t\to 0}\left(I - \frac{t}{n}A\right)^{-1} = I,$$

since the n-time iteration would then also be the identity. We use the relation for $u \in D(A)$

$$u = \left(I - \frac{t}{n}A\right)^{-1}\left(I - \frac{t}{n}A\right)u = \left(I - \frac{t}{n}A\right)^{-1}u - \frac{t}{n}\left(I - \frac{t}{n}A\right)^{-1}Au.$$

Then

$$\left\|\left(I - \frac{t}{n}A\right)^{-1}u - u\right\| = \frac{t}{n}\left\|\left(I - \frac{t}{n}A\right)^{-1}Au\right\| \le Ct\|Au\| \to 0,$$

for $t \to 0$, since $\|Au\|$ is bounded for $u \in D(A)$. Hence $T_n(0)u = u$ for a dense subset $D(A)$, and since $T_n(0)$ is continuous, we have $T_n(0) = I$ on X.

Next we want to show that indeed

$$T(t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n}.$$

We compute

$$\frac{d}{dt}T_n(t) = -n\left(I - \frac{t}{n}A\right)^{-n-1}\left(-\frac{A}{n}\right) = A\left(I - \frac{t}{n}A\right)^{-n-1}. \tag{9.12}$$

Again we argue for a dense subset, but this time for $D(A^2) \subset D(A) \subset X$, where it can be shown that $D(A^2)$ is also dense by repeating the arguments in Theorem (9.2.1). Let $u \in D(A^2)$. We show that $\{T_n\}$ form a Cauchy sequence.

$$\begin{aligned}
& T_n(t)u - T_m(t)u \\
= & \int_0^t \frac{d}{ds}\left(T_m(t-s)T_n(s)u\right) ds \\
= & \int_0^t \left(-\dot{T}_m(t-s)T_n(s)u + T_m(t-s)\dot{T}_n(s)u\right) ds \\
= & \int_0^t \left(-A\left(I - \frac{t-s}{m}A\right)^{-m-1}\left(I - \frac{s}{n}A\right)^{-n} u + \left(I - \frac{t-s}{m}A\right)^{-m} A\left(I - \frac{s}{n}A\right)^{-n-1} u\right) ds \\
= & \int_0^t \left(I - \frac{t-s}{m}A\right)^{-m-1}\left(I - \frac{s}{n}A\right)^{-n-1}\left[-A\left(I - \frac{s}{n}A\right) + \left(I - \frac{t-s}{m}A\right)A\right] u ds \\
= & \int_0^t \underbrace{\left(I - \frac{t-s}{m}A\right)^{-m-1}}_{\text{bounded}}\underbrace{\left(I - \frac{s}{n}A\right)^{-n-1}}_{\text{bounded}}\left[\frac{s}{n} - \frac{t-s}{m}\right] A^2 u \, ds.
\end{aligned} \tag{9.13}$$

Then

$$\begin{aligned}
\|T_n(t)u - T_m(t)u\| &\leq C\|A^2u\| \int_0^t \frac{s}{n} + \frac{t-s}{m} ds \\
&= \frac{Ct^2}{2}\left[\frac{1}{n} + \frac{1}{m}\right]\|A^2u\|.
\end{aligned}$$

The last term is bounded, since $u \in D(A^2)$ and it goes to zero for $n, m \to \infty$. Hence

$$T(t) = \lim_{n\to\infty} T_n(t)$$

exists.

Step 4: It remains to show that this $T(t)$ is indeed our semigroup and that A is its infinitessimal generator.

We know already that $\lim_{t\to 0} T_n(t) = I$, hence $T(t)$ is strongly continuous. For the semigroup property we use equation (9.13) for $n = m$, without the integral, to obtain

$$\frac{d}{ds}\left(T_n(t-s)T_n(s)u\right) = \frac{2s-t}{n}\left(I - \frac{t-s}{n}A\right)^{-n-1}\left(I - \frac{s}{n}A\right)^{-n-1} A^2 u.$$

In the limit as $n \to \infty$ this becomes

$$\frac{d}{ds}T(t-s)T(t) = 0.$$

Hence $T(t-s)T(s)$ is independent of s. Then it equals the value at $s = 0$, which is $T(t)$.

Finally, taking the limit as $n \to \infty$ in (9.12) leads to

$$\frac{d}{dt}T(t) = AT(t).$$

■

The proof of the Hille-Yosida Theorem gave us two important formulas

Corollary 9.4.2 Let A be the generator of a strongly continuous semigroup $T(t)$, then

$$T(t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} \tag{9.14}$$

and

$$R_\lambda(A) = -\int_0^\infty e^{-\lambda t}T(t)dt. \tag{9.15}$$

R The formula for the semigroup (9.14) is reminiscent to the discrete implicit Euler scheme for a numerical solver of $\dot{u} = Au$. Consider a small time increment $\Delta t = \frac{t}{n}$. Then the implicit Euler scheme is

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} = Au(t+\Delta t),$$

which gives

$$\left(\frac{1}{\Delta t}I - A\right)u(t+\Delta t) = \frac{u(t)}{\Delta t}.$$

This is written as

$$u(t+\Delta t) = (I - \Delta t A)^{-1}u(t).$$

An iteration of this scheme and taking $\Delta t \to 0$ gives us something like (9.14).

After all this work, we reward ourselves with a beautiful definition.

Definition 9.4.1 If $T(t)$ is a strongly continuous semigroup with generator A, then we write

$$T(t) = e^{At}.$$

R In the semigroup triangle Figure 9.3 we are able to add the connection from A to T and also the connection from T to R_λ. The only missing link is the arrow from R_λ to T. This can be filled in for analytic semigroups, which we develop in Section 9.8.

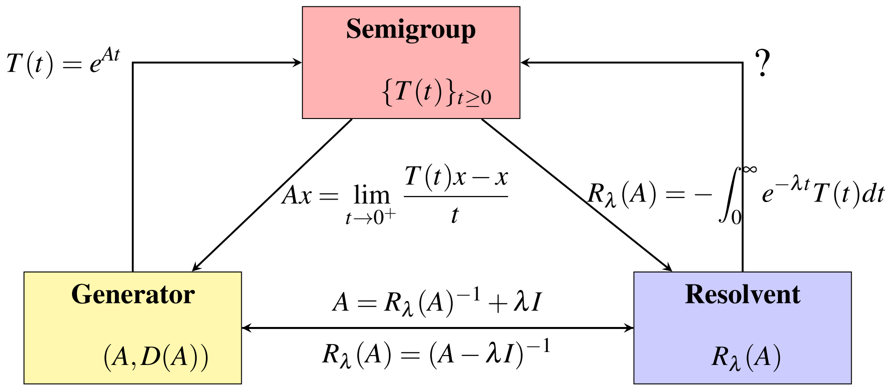


Figure 9.3: Almost complete *semigroup triangle*.

9.5 The Lumer-Phillips Theorem

A special case arises when the constant M_γ =1. In this case we only need the resolvent estimate for $n = 1$:

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda - \omega}$$

and the rest follows through iteration

$$\|R_\lambda(A)^n\| \leq \frac{1}{(\lambda - \omega)^n}.$$

Corollary 9.5.1 The operator A is a generator of a strongly continuous semigroup $\{T(t)\}$ with

$$\|T(t)\| \leq e^{\omega t},$$

if

1. $D(A)$ is dense and A is closed.
2. For all $\lambda > \omega$ we have

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda - \omega}.$$

Here condition 2. is much easier to check than condition 2. in the Hille-Yosida Theorem. We give this case a special name.

Definition 9.5.1 The semigroup $\{T(t)\}$ is called a *quasi contraction* if $\|T(t)\| \leq e^{\omega t}$, and it is called a *contraction* if $\|T(t)\| \leq 1$ (i.e. $\omega = 0$).

In addition to the Hille-Yosida Theorem for semigroups, there is an alternative for contractive semigroups on Hilbert spaces, the Lumer-Phillips theorem.

Theorem 9.5.2 — Lumer-Phillips. Let A be a linear operator on a Hilbert space. Assume

1. $D(A)$ is dense.
2. For all $x \in D(A)$, $\text{Re}(x, Ax) \leq \omega(x,x)$ for some $\omega \geq 0$.
3. There exists a $\lambda_0 > \omega$ such that $A - \lambda_0 I$ is onto.

Then A generates a quasi contraction semigroup with

$$||e^{At}|| \leq e^{\omega t}.$$

Proof. For $\lambda > \omega$ we estimate

$$||(A - \lambda I)x||\,||x|| \geq \text{Re}(x, (\lambda I - A)x) \geq (\lambda - \omega)(x,x),$$

which implies

$$||(A - \lambda I)x|| \geq (\lambda - \omega)||x||.$$

Hence $A - \lambda I$ is bounded below, away from 0, and by Corollary 8.3.4 the resolvent $R_\lambda(A)$ exists and

$$||R_\lambda(A)|| \leq \frac{1}{\lambda - \omega}.$$

Hence for any $\lambda_0 > \omega$ the map $A - \lambda_0 I$ is onto. Then $R_{\lambda_0}(A)$ has range X and is continuous. This implies that $A - \lambda_0 I$ is closed, which means that A is closed. By the Corollary 9.5.1 to the Hille-Yosida Theorem, A generates a strongly continuous semigroup $T(t)$ with

$$||T(t)|| \leq e^{\omega t}.$$

■

There is an immediate corollary, which is often called the Lumer-Phillips theorem [17]

Corollary 9.5.3 — Lumer-Phillips. Let A be a linear operator on a Hilbert space. Assume

1. $D(A)$ is dense.
2. For all $x \in D(A)$, $\text{Re}(x, Ax) \leq \omega(x,x)$ for some $\omega \geq 0$.
3. The resolvent set $\rho(A) \cap (\omega, \infty) \neq \emptyset$.

Then A generates a quasi contraction semigroup with

$$||e^{At}|| \leq e^{\omega t}.$$

■ **Example 9.4** Assume A is self-adjoint on a Hilbert space H with dense domain $D(A)$. Let $\{\lambda_i\}_i$ denote the (real) eigenvalues and assume

$$\omega > \max\{\text{Re}(\lambda), \lambda \in \sigma(A)\}. \tag{9.16}$$

Let $\{\phi_i\}_i$ denote an orthonormal basis of eigenvectors. Then

$$(x, Ax) = \sum_{i=1}^{\infty} \lambda_i (x, \phi_i)(x, \phi_i) \leq \omega \sum_{i=1}^{\infty} (x, \phi_i)(x, \phi_i) = \omega ||x||^2.$$

Then for $\lambda_0 > \omega$ the resolvent $R_{\lambda_0}(A)$ exists and $A - \lambda_0 I$ is onto. By the Lumer-Phillips theorem A generates a quasi contraction semigroup. Note here that the spectral bound (9.16) is essential. ■

■ **Example 9.5** Let $A = \frac{d}{dx}$ on $L^2(0,1)$ with $u(1) = 0$. As usual, we put the boundary condition into the domain

$$D(A) = \{u \in H^1(0,1), u(1) = 0\}.$$

$D(A)$ is dense in $L^2(0,1)$, since H^1 is dense in L^2 and functions in L^2 are only defined up to a set of measure zero. Hence the condition on one of the boundary points is not relevant.

We compute the spectrum of A. For $\varphi \in L^2(0,1)$ we like to solve the resolvent equation $(A - \lambda I)u = \varphi$. This is written as an ODE

$$u' = \lambda u + \varphi, \qquad u(1) = 0,$$

which is a linear ODE that can be solved for any $\lambda \in \mathbb{C}$. Hence $(A - \lambda I)^{-1}$ exists for all $\lambda \in \mathbb{C}$, and we can chose a spectral bound of $\omega = 0$. To apply Lumer-Phillips we need one more estimate

$$(u, Au) = \int_0^1 u(x)u'(x)dx = -\frac{1}{2}(u(0))^2 \leq 0.$$

Hence A generates a contraction semigroup $T(t)$ with

$$\|T(t)\| \leq 1.$$

■

9.6 Application to PDEs

In this section we study a number of partial differential equations that are relevant in many other areas of applied mathematics, such as reaction-diffusion equations, wave equations, the Schrödinger equation, and integral equations. We show how the semigroup theory can be used to find solutions for these models.

9.6.1 The Reaction Diffusion Equation

On a smooth domain $\Omega \subset \mathbb{R}^n$ we consider a reaction-diffusion equation

$$\begin{aligned} u_t &= \Delta u + f(t) && \text{on} \quad \Omega, \\ u(x,0) &= u_0 && \in L^2(\Omega), \\ u(x,t) &= 0 && x \in \partial\Omega, \end{aligned} \tag{9.17}$$

with $f \in C^0([0,\infty])$. We chose the Hilbert space $X = L^2(\Omega)$ and the domain for $A = \Delta$ as

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

This set is dense in $L^2(\Omega)$. On this Hilbert space $L^2(\Omega)$ the operator Δ is self-adjoint and for $u \in D(A)$ we have

$$(u, \Delta u) = \int_\Omega u\Delta u dx = -\int_\Omega \nabla u \nabla u dx = \int \Delta u u ds = (\Delta u, u).$$

This also shows that

$$\mathrm{Re}(u, \Delta u) \leq 0.$$

Hence we chose the spectral bound in the Lumer-Phillips Theorem as $\omega = 0$. Now we choose a $\lambda_0 > 0$ and show that $(A - \lambda_0 I)$ is onto. Given $\varphi \in L^2(\Omega)$ we aim to find $u \in D(A)$ such that

$$\Delta u - \lambda_0 u = \varphi, \qquad u|_{\partial\Omega} = 0.$$

This leads to the theory of elliptic partial differential equations on a smooth domain with Dirichlet boundary conditions, which are solved elsewhere (see Gilbarg and Trudinger [8]). Here we simply assume that such an λ_0 exists. Then we can apply Lumer-Phillips to $(A, D(A))$, and A generates a strongly continuous contraction semigroup $e^{\Delta t}$. A solution of the reaction-diffusion equation (9.17) can then be found using the variation of constant formula:

$$u(t) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} f(s) ds$$

and, using the same arguments as we did in the discussion of mild solutions, we have

$$u \in C^1([0,\infty), L^2(\Omega)) \cap C^0([0,\infty), D(A)).$$

9.6.2 The Wave Equation

On a smooth bounded domain $\Omega \subset \mathbb{R}^n$ we consider the wave equation

$$\begin{array}{rcll} u_{tt} & = & \Delta u & \text{on} \quad \Omega, \\ u(x,t) & = & 0 & \text{on } \partial\Omega, \\ u(x,0) & = & u_0(x), & \\ u_t(x,0) & = & u_1(x). & \end{array} \tag{9.18}$$

We introduce a variable for the velocity $v(x,t) = u_t(x,t)$ and write the wave equation as a system for (u,v) as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

This is a differential equation on $X = H_0^1(\Omega) \times L^2(\Omega)$. The matrix will become our generator with the following domain:

$$A := \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \qquad D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H^1(\Omega),$$

and the domain is dense.

Proposition 9.6.1 $(A, D(A))$ generates a strongly continuous semigroup of contractions.

Proof. To apply the Lumer-Philips result, we consider the inner product on X

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle = \int_\Omega \nabla u \nabla f + vg \, dx.$$

Then

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, A \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} v \\ \Delta u \end{pmatrix} \rangle = \int \nabla u \nabla v + v \Delta u \, dx = 0.$$

Hence we chose the spectral bound as $\omega = 0$. Next we claim that each $\lambda > 0$ satisfies $\lambda \in \rho(A)$. Given $(f,g) \in H_0^1(\Omega) \times L^2(\Omega)$ we solve for (u,v) the equation

$$(A - \lambda I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

We get two equations

$$v - \lambda u = f, \qquad \Delta u - \lambda v = g.$$

Solving the first equation for v and substituting this into the second equation gives

$$\Delta u - \lambda^2 u = g + \lambda f,$$

where $g + \lambda f \in L^2(\Omega)$. Again we employ elliptic solution theory [8] and get a unique solution for $\lambda^2 > 0$. This allows us to apply the Lumer-Phillips Theorem and obtain a solution semigroup of the wave equation

$$T(t) = \exp\left(\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} t\right).$$

■

9.6.3 The Schrödinger Equation

The Schrödinger equation for a given real potential $V(x)$ in $\mathbb{R}^n$ is

$$u_t = i(\Delta u - V(x)u),$$

where $u(x,t)$ is complex valued wave function.

Proposition 9.6.2 Assume $n \leq 3$ and let $V \in L^2(\mathbb{R}^n)$. Then A given by

$$Au := i(\Delta u - V(x)u), \qquad D(A) = H^2(\mathbb{R}^n)$$

is skew adjoint .

Proof. We compute

$$\begin{aligned} (u, Au) &= \int u\, \overline{i(\Delta u - V(x)u)} dx \\ &= \int u(-i(\Delta \bar{u} - V(x)\bar{u}))dx \\ &= \int -iu\Delta \bar{u} + iV(x)u\bar{u} dx \\ &= \int -i(\Delta u - V(x)u)\bar{u} dx \\ &= -(Au, u). \end{aligned}$$

■

If $V(x)$ is bounded, then A is closed and Stones Theorem for skew-adjoint operators applies [19] and we get a contractions semigroup. If $V(x)$ is unbounded, then all hell breaks loose and we quickly enter areas of active research.

9.6.4 Integral Equations

On a bounded domain $\Omega \subset \mathbb{R}^n$ we consider the integral equation

$$u_t = \int_\Omega k(x,y)u(y,t)dy = Au,$$

with a kernel $k(x,y) \in L^2(\Omega\times\Omega)$ and $\int_\Omega k(x,y)dx = 1$, and with domain of definition

$$D(A) = L^2(\Omega) = X.$$

Then A is a closed, compact Hilbert-Schmidt operator with full domain. From spectral theory we know that A has a discrete point spectrum and no other parts of the spectrum. This means that the resolvent set is dense in $\mathbb{C}$, and we happily find one λ_0 such that $A-\lambda_0 I$ is onto. We estimate the norm

$$\begin{aligned} |(u,Au)| &= \left|\int_\Omega\int_\Omega k(x,y)u(x)u(y)dxdy\right| \\ &\le \left|\int_\Omega \|k(\cdot,y)\|_2\|u\|_2 u(y)dy\right| \\ &\le \|k\|_2\|u\|_2\|u\|_2 \\ &= \|k\|_2\|u\|_2^2. \end{aligned}$$

Hence we choose $\omega = \|k\|_2$. By Lumer-Phillips, A generates a quasi contraction semigroup

$$\|T(t)\| \le e^{\omega t}.$$

9.7 Bounded Perturbations

Semigroup thoery is particulalry powerful if we consider nonhomogeneous equations, where the leading order differential operator is perturbed by an operator of lower order. There is a huge variety of perturbation results for semigroups [6, 18, 19] and here, we will focus on the example of bounded perturbations. We consider

$$u_t = Au + Bu,$$

where $(A,D(A))$ is a generator and B is bounded. We want to show that $(A+B,D(A))$ is also a generator, and possibly get a relation of $e^{(A+B)t}$ to e^{At}. The corresponding proof will use a switch of equivalent norms, which we prepare first:

Proposition 9.7.1 Let A be a linear operator on a Banach space X with $(0,\infty)\in\rho(A)$. If for each $\lambda\in(0,\infty)$

$$\|\lambda^n R_\lambda(A)^n\| \le M, \qquad \text{for all } n\ge 0,$$

then there exists an equivalent norm $|\cdot|_\diamond$ such

$$\|x\| \le |x|_\diamond \le M\|x\|, \qquad \text{for all } x\in X,$$

and

$$|\lambda^n R_\lambda(A)^n|_\diamond \le 1.$$

Proof. For $\mu > 0$ we define

$$\|x\|_\mu := \sup_{n\geq 0} \|\mu^n R_\mu(A)^n x\|.$$

Then for $n = 0$ we get $\|x\|$ such that

$$\|x\| \leq \|x\|_\mu \leq M\|x\| \tag{9.19}$$

and

$$\|\mu R_\mu(A)x\|_\mu = \sup_{n\geq 0} \|\mu^n R_\mu(A)^n \mu R_\mu(A)x\| \leq \|x\|_\mu.$$

Hence the operator norm

$$\|\mu R_\mu(A)\|_\mu \leq 1.$$

For $0 < \lambda \leq \mu$ we set $y = R_\lambda(A)x$ and we get

$$\begin{aligned} R_\mu(A)(x-(\mu-\lambda)y) &= R_\mu(A)\Big(x+(A-\mu I)y - \underbrace{(A-\lambda I)y}_{=x}\Big) \\ &= R_\mu(A)(A-\mu I)y \\ &= y. \end{aligned}$$

Evaluating y in the μ-norm we get

$$\begin{aligned} \|y\|_\mu &= \left\|\mu R_\mu(A)\frac{x-(\mu-\lambda)y}{\mu}\right\|_\mu \\ &\leq \left\|\frac{x}{\mu} - \frac{\mu-\lambda}{\mu}y\right\|_\mu \\ &\leq \frac{1}{\mu}\|x\|_\mu + \left(1-\frac{\lambda}{\mu}\right)\|y\|_\mu. \end{aligned}$$

This implies that

$$\frac{\lambda}{\mu}\|y\|_\mu \leq \frac{1}{\mu}\|x\|_\mu,$$

leading to

$$\lambda\|y\|_\mu \leq \|x\|_\mu.$$

Hence also the operator norm

$$\|\lambda R_\lambda(A)\|_\mu \leq 1, \qquad \text{for all } 0 < \lambda \leq \mu.$$

Now we take the limit $\mu \to \infty$ and define the corresponding limit norm

$$|x|_\diamond := \lim_{\mu\to\infty} \|x\|_\mu = \lim_{\mu\to\infty} \sup_{n\geq 0} \|\mu^n R_\mu(A)^n x\|$$

and

$$|\lambda R_\lambda(A)|_\diamond \leq 1.$$

Taking the limit $\mu \to \infty$ in (9.19) we get the equivalence of the $|\cdot|_\diamond$ norm. ■

Theorem 9.7.2 — Bounded perturbations. Assume $(A, D(A))$ is a generator on X of a strongly continuous semigroup $\{T(t)\}$ with exponential growth bound

$$\|T(t)\| \leq Me^{\omega t}.$$

If B is a bounded linear operator on X, then $(A+B, D(A))$ is generator of a strongly continuous semigroup $\{S(t)\}$, denoted by $S(t) = e^{(A+B)t}$, with exponential bound

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t}.$$

Proof. We use the previously defined equivalent norm $|\cdot|_\diamond$, such that

$$|T(t)|_\diamond \leq e^{\omega t}, \qquad |R_\lambda(A)|_\diamond \leq \frac{1}{\lambda-\omega}, \quad \lambda > \omega.$$

For $\lambda > \omega + |B|_\diamond$ we find

$$|BR_\lambda(A)|_\diamond \leq \frac{|B|_\diamond}{\lambda-\omega} < 1,$$

which implies that $I + BR_\lambda(A)$ is invertible. We set

$$\begin{aligned} V &:= R_\lambda(A)(I+BR_\lambda(A))^{-1} \\ &= R_\lambda(A)(I-(-BR_\lambda(A)))^{-1} \\ &= \sum_{k=0}^{\infty} R_\lambda(A)(-BR_\lambda(A))^k, \end{aligned}$$

where we used the Neumann series in the last step. Now we claim that $V = R_\lambda(A+B)$. Indeed,

$$\begin{aligned} (A+B-\lambda I)V &= [(A-\lambda I)+B]R_\lambda(A)(I+BR_\lambda(A))^{-1} \\ &= (I+BR_\lambda(A))(I+BR_\lambda(A))^{-1} \\ &= I. \end{aligned}$$

For the reverse case, we employ the above Neumann series

$$\begin{aligned} V(A+B-\lambda I)x &= \sum_{k=0}^{\infty} R_\lambda(A)(-BR_\lambda(A))^k((A-\lambda I)+B)x \\ &= x + R_\lambda(A)Bx + \sum_{k=1}^{\infty} R_\lambda(A)(-BR_\lambda(A))^k((A-\lambda I)+B)x \\ &= x + R_\lambda(A)Bx + \sum_{k=1}^{\infty} \underbrace{R_\lambda(A)(-B)R_\lambda(A)(-B)\cdots}_{k \text{ times}} R_\lambda(A)(A-\lambda I)x \\ &\quad + \sum_{k=1}^{\infty} \underbrace{R_\lambda(A)(-B)R_\lambda(A)(-B)\cdots}_{k \text{ times}} R_\lambda(A)Bx \\ &= x + R_\lambda(A)Bx + \sum_{k=1}^{\infty}(-R_\lambda(A)B)^k x - \sum_{k=2}^{\infty}(-R_\lambda(A)B)^k x \\ &= x. \end{aligned}$$

Then

$$
\begin{aligned}
|R_\lambda(A+B)|_\diamond &= |V|_\diamond \\
&= |R_\lambda(A)|_\diamond |(BR_\lambda(A)+I)^{-1}|_\diamond \\
&\leq \frac{1}{\lambda-\omega}\left|\sum_{k=0}^{\infty}(-BR_\lambda(A))^{-k}\right|_\diamond \\
&\leq \frac{1}{\lambda-\omega}\sum_{k=0^\infty}|B|_\diamond^k|R_\lambda(A)|_\diamond^k \\
&\leq \frac{1}{\lambda-\omega}\sum_{k=0}^{\infty}\frac{|B|_\diamond^k}{(\lambda-\omega)^k} \\
&= \frac{1}{\lambda-\omega}\frac{1}{1-\frac{|B|_\diamond}{\lambda-\omega}} \\
&= \frac{1}{\lambda-\omega-|B|_\diamond}.
\end{aligned}
$$

Now we apply the Corollary 9.5.1 of the Hille-Yosida Theorem and conclude that $A+B$ is a generator of a strongly continuous semigorup $\{S(t)\}$ with

$$|S(t)|_\diamond \leq e^{(\omega+|B|_\diamond)t},$$

which, upon reverting to the original norm, gives

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t}.$$

■

R The semigroup $\{S(t)\}$ solves the ODE $\dot{u} = Au + Bu$ and it can be related to the semigroup $\{T(t)\}$ with the variation of constants formula

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xdx.$$

R The shift in norm is a very useful trick, since we get rid of the constant M in the Hille-Yosida theorem. This means, that in the $|\cdot|_\diamond$-norm we need the resolvent estimate only for $n = 1$ and not for all n. In fact, whenever such estimates arise in the next sections, we implicitly assume to have chosen the $|\cdot|_\diamond$ norm to begin with.

■ **Example 9.6** Solve the integro-PDE in $L^2(\Omega)$, where Ω is bounded and smooth:

$$
\begin{aligned}
u_t &= D\Delta u + \int_\Omega k(x,y)u(y,t)dy, \\
u|_{\partial\Omega} &= 0. \\
k &\geq 0, \quad k \in L^2, \quad \int_\Omega k(x,y)dx = 1.
\end{aligned}
$$

The operator $D\Delta$ is generator of the heat-equation semigroup $\{e^{D\Delta t}\}$ with Dirichlet boundary conditions. The integral operator $Ku = \int_\Omega k(x,y)u(y,t)dy$ is a compact Hilbert-Schmidt operator, hence bounded. We directly apply the above perturbation result Theorem 9.7.2 and there exists a strongly continuous semigroup

$$S(t) = e^{(D\Delta+K)t}$$

on $L^2(\Omega)$ with

$$\|S(t)\| \leq Me^{(\omega+M\|K\|)t}.$$

For each $u_0 \in D(A)$ and for each $T > 0$ we have

$$S(t)u_0 \in C^1([0,T], L^2(\Omega)) \cap C([0,T], D(A)).$$

■

9.8 Analytic Semigroups

Analytic semigroup are a special breed of semigroups, as they allow for more regularity of the solutions. The Laplacian will, in many situations, lead to an analytic semigroup, hence it is of relevance in many applications. We have made use of complex analysis occasionally within this book. Now it is time to dig in even deeper.

Proposition 9.8.1 — Cauchy Integral Formula. Let $f(z)$ be an analytic function in a domain $D(f) \subset \mathbb{C}$ and $\gamma \subset D(f)$ a closed positively oriented curve (i.e., oriented counter-clockwise). Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a} dz = f(a).$$

It turns out that analytic semigroup can be characterized by the spectrum of the generator.

Definition 9.8.1 Let $(A, D(A))$ be a closed linear operator on a Banach space X. A is called *sectorial* if there exists an angle $0 < \delta < \frac{\pi}{2}$ such that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \setminus \{0\},$$

satisfies

$$\Sigma_{\frac{\pi}{2}+\delta} \subset \rho(A),$$

and if for each $\varepsilon \in (0, \delta)$ there exists an $M_\varepsilon > 0$ such that

$$\|R_\lambda(A)\| \leq \frac{M_\varepsilon}{|\lambda|}, \qquad \text{for all } \lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}} \setminus \{0\}. \tag{9.20}$$

In Figure 9.4 we illustrate the spectrum of A and the sector $\Sigma_{\frac{\pi}{2}+\delta}$ which is spanned by the angles $\frac{\pi}{2} + \delta$ and $-\frac{\pi}{2} - \delta$, excluding the origin.

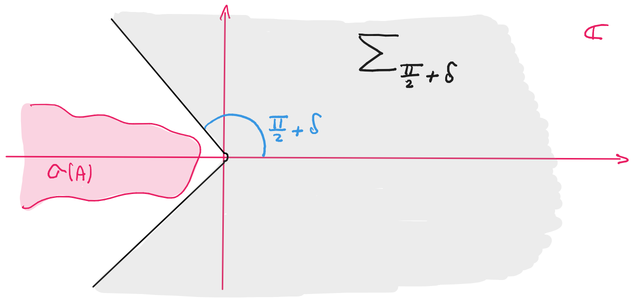


Figure 9.4: Schematic showing the sector $\Sigma_{\frac{\pi}{2}+\delta}$

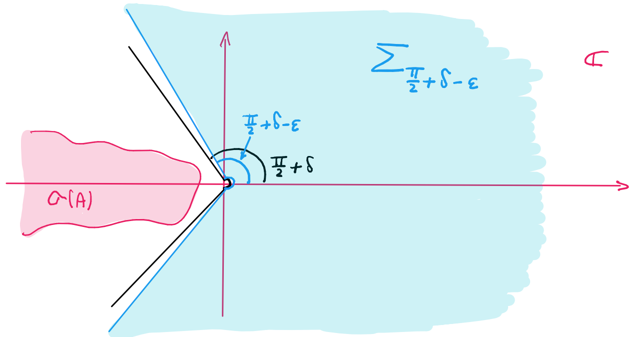


Figure 9.5: Uniform resolvent estimates in a slightly smaller sector $\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}$

The estimate above is a uniform resolvent estimate on a sector that is slightly smaller than $\Sigma_{\frac{\pi}{2}+\delta}$. The constant M_ε will grow as we get closer and closer to the boundary of $\Sigma_{\frac{\pi}{2}+\delta}$, i.e. as $\varepsilon \to 0$, and it can not be expected that M_ε stays bounded as we reach the boundary of $\Sigma_{\frac{\pi}{2}+\delta}$. See Figure 9.5.

We use the Cauchy Integral Formula to represent the semigroup.

Definition 9.8.2 — Cauchy's representation of a semigroup. Let $(A, D(A))$ be densely defined and sectorial with angle δ. We define an operator family $\{T(z)\}$ with complex argument z as $T(0) = I$ and for $z \in \Sigma_\delta$

$$T(z) = \frac{1}{2\pi i} \oint_\gamma e^{\mu z}(\mu I - A)^{-1} d\mu \tag{9.21}$$

$$= \frac{i}{2\pi} \oint_\gamma e^{\mu z} R_\mu(A) d\mu, \tag{9.22}$$

where γ is a piecewise smooth curve in $\Sigma_{\frac{\pi}{2}+\delta}$ connecting

$$\infty e^{-i(\frac{\pi}{2}+\delta')} \quad \text{to} \quad \infty e^{i(\frac{\pi}{2}+\delta')},$$

for some $\delta' \in (|\arg z|, \delta)$.

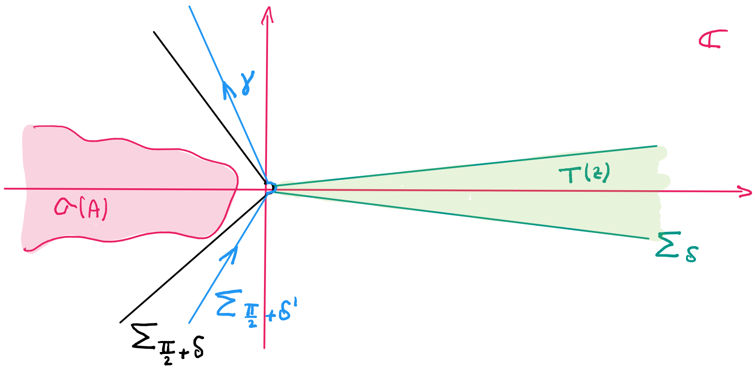


Figure 9.6: The semigroup $T(z)$ is defined in a small sector Σ_δ around the positive x-axis.

Notice the level of strangeness in this definition. Remember that for a semigroup, we were looking at an operator family that depends on a scalar variable t, which is often understood as time. In this context, the semigroup property is a very natural assumption, which says that a system evolving to $t+s$ must first evolve to t and then s units more. So what does it mean for T to depend on a complex variable z? Is this a complex time? Well, of course not, but the positive real axis is a part of the complex plane, and for a real world application we can simply restrict $T(z)$ to $T(t)$ and keep our physical meaning. In addition we find that close to the real axis, we can extend the semigroup to a semigroup with complex arguments.

Notice that $T(z)$ in (9.22) is defined on a much much smaller sector Σ_δ than the sector we considered earlier $\Sigma_{\frac{\pi}{2}+\delta}$. As we show in Figure 9.6 for small δ the sector Σ_δ, is just a small stripe above and below the positive real axis.

Theorem 9.8.2 Let $(A, D(A))$ be a densely defined sectorial operator with angle δ. Then for all $z \in \Sigma_\delta$ the maps $T(z)$ are bounded linear operators on X with the following properties

1. $\|T(z)\|$ is uniformly bounded in each smaller sector $\Sigma_{\delta'}$, $0 < \delta' < \delta$.
2. The map $z \mapsto T(z)$ is analytic in Σ_δ, i.e. we call $\{T(z)\}$ and *analytic semigroup*.
3. For all $z_1, z_2 \in \Sigma_\delta$ with $z_1 + z_2 \in \Sigma_\delta$ we have

$$T(z_1 + z_2) = T(z_1)T(z_2).$$

4. The map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'}$, $0 < \delta' < \delta$.

Proof. 1. We use the representation of the resolvent as a Laplace transform of T:

$$(A - \lambda I)^{-1} = -\int_0^\infty e^{-\lambda t} T(t) dt.$$

The right hand side is an analytic expression in λ, whenever the integral exists. Hence

$$\lambda \mapsto e^{\lambda z} R_\lambda(A)$$

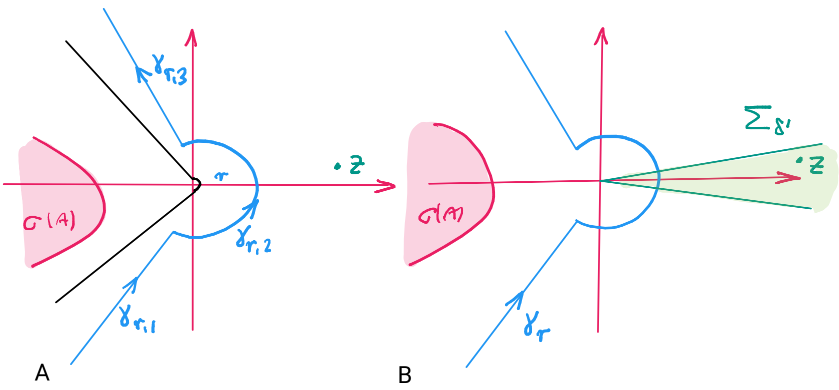


Figure 9.7: The construction of the path γ_r and the sector Σ_δ as used in the proof.

is analytic whenever the resolvent exists, that is in any sector $\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}$ for some small $\varepsilon > 0$. Then, by complex analysis results, the path integral (9.22) is independent of the choice of the path γ, as long as it stays inside the sector and is closed, simple and positively oriented. Hence we make a particular choice. For a given radius $r > 0$ we define

$$\begin{aligned}
\gamma_{r,1} &= \{-\rho e^{-i(\frac{\pi}{2}+\delta-\varepsilon)} : -\infty \le \rho \le -r\},\\
\gamma_{r,2} &= \{re^{-i\alpha} : -\left(\frac{\pi}{2}+\delta-\varepsilon\right) \le \alpha \le \frac{\pi}{2}+\delta-\varepsilon\},\\
\gamma_{r,3} &= \{\rho e^{i(\frac{\pi}{2}+\delta-\varepsilon)} : r \le \rho \le \infty\},
\end{aligned}$$

where $\varepsilon = \frac{1}{2}(\delta - \delta')$. A sketch of the different parts of this construction can be seen in Figure 9.7.

We fix a point $z \in \Sigma_{\delta'}$ and set $r = \frac{1}{|z|}$. Then for $\mu \in \gamma_{r,3}$ with $z \in \Sigma_{\delta'}$ we have

$$\mu z = |\mu z| e^{i(\arg\mu + \arg z)}.$$

By the special choice of ε we have

$$\arg\mu = \frac{\pi}{2} + \delta - \varepsilon = \frac{\pi}{2} + \delta - \frac{\delta - \delta'}{2} = \frac{\pi}{2} + \frac{\delta + \delta'}{2}$$

and $|\arg z| \le \delta'$. This means that

$$\arg\mu + \arg\, z \ge \frac{\pi}{2} + \frac{\delta + \delta'}{2} - \delta' = \frac{\pi}{2} + \frac{\delta - \delta'}{2} = \frac{\pi}{2} + \varepsilon.$$

On the other hand

$$\arg\mu + \arg z \le \frac{\pi}{2} + \delta - \varepsilon + \delta' \le \frac{\pi}{2} + \frac{\pi}{2} - \varepsilon + \frac{\pi}{2} = \frac{3\pi}{2} - \varepsilon.$$

So

$$\frac{\pi}{2} + \varepsilon \le \arg\mu + \arg z \le \frac{3\pi}{2} - \varepsilon.$$

Therefore,

$$\frac{1}{|\mu z|}\mathrm{Re}(\mu z) = \cos(\arg\mu + \arg z) \leq \cos\left(\frac{\pi}{2}+\varepsilon\right) = -\sin\varepsilon,$$

and we find

$$|e^{\mu z}| \leq e^{-|\mu z|\sin\varepsilon}.$$

We get a very similar estimate on the branch $\gamma_{r,1}$. Then on $\gamma_{r,1}\cup\gamma_{r,3}$ we find

$$\|e^{\mu z}R_\mu(A)\| \leq e^{-|\mu z|\sin\varepsilon}\frac{M_\varepsilon}{|\mu|}.$$

On $\gamma_{r,2}$ we have $|\mu| = r = \frac{1}{|z|}$, hence $|\mu z| = 1$ and

$$\|e^{\mu z}R_\mu(A)\| \leq \frac{eM_\varepsilon}{|\mu|} = eM_\varepsilon|z|.$$

Together on $\gamma_r = \gamma_{r,1}\cup\gamma_{r,2}\cup\gamma_{r,3}$ we then have

$$\begin{aligned}
\left\|\int_{\gamma_r} e^{\mu z}R_\mu(A)d\mu\right\| &\leq \sum_{k=1}^{3}\left\|\int_{\gamma_{r,k}} e^{\mu z}R_\mu(A)d\mu\right\| \\
&\leq 2M_\varepsilon\int_{\frac{1}{|z|}}^{\infty}\frac{1}{\rho}e^{-\rho|z|\sin\varepsilon}d\rho + eM_\varepsilon|z|\int_{-(\frac{\pi}{2}+\delta-\varepsilon)}^{\frac{\pi}{2}+\delta-\varepsilon} r\,d\alpha \\
&\leq 2M_\varepsilon\int_{\frac{1}{|z|}}^{\infty}\frac{1}{\rho}e^{-\rho|z|\sin\varepsilon}d\rho + 2\pi eM_\varepsilon,
\end{aligned}$$

where we used $r = \frac{1}{|z|}$ in the last integral. We now substitute $\rho|z| \to \rho$ in the remaining integral to obtain

$$\left\|\int_{\gamma_r} e^{\mu z}R_\mu(A)d\mu\right\| \leq 2M_\varepsilon\int_1^\infty\frac{1}{\rho}e^{-\rho\sin\varepsilon}d\rho + 2\pi eM_\varepsilon,$$

which is uniformly bounded and independent of z. It does depend on δ' through $\varepsilon = \frac{1}{2}(\delta-\delta')$. This proves item 1.

2. Since for $z\in\Sigma_\delta$ the map $z\mapsto e^{\mu z}R_\mu(A)$ is analytic, so is the map

$$z\mapsto \frac{i}{2\pi}\oint_\gamma e^{\mu z}R_\mu(A)d\mu$$

and we have item 2.

3. To check the semigroup property, we choose two paths. The first path is $\gamma^1 = \gamma_r$ from before and the second path is a shift of γ_r as $\gamma^2 = \gamma_r + c$ for some real $c > 0$. A sketch of these paths is given in Figure 9.8. We first note a useful identity

$$R_\mu(A)R_\lambda(A)(\mu-\lambda) = R_\mu(A)R_\lambda(A)((A-\lambda I)-(A-\mu I)) = R_\mu(A)-R_\lambda(A).$$

To see that resolvents commute with $(A-\mu I)$ we can use the Neumann series representation. We use the previous identity for $z_1, z_2\in\Sigma_{\delta'}$ in the definition of $T(z_1)$

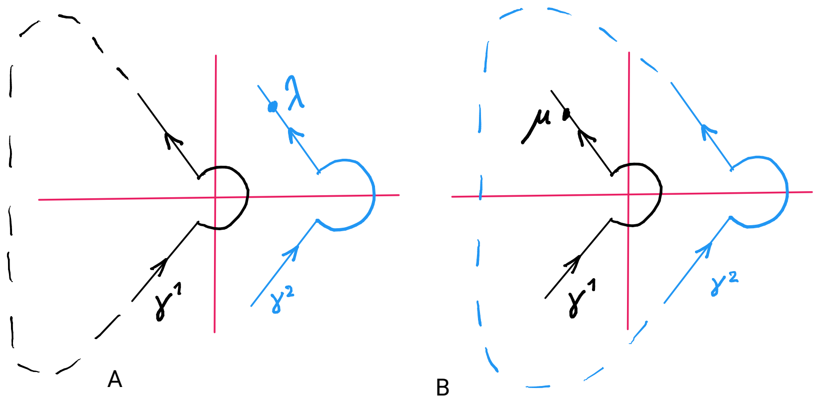


Figure 9.8: The semigroup $T(z)$ is defined in a small sector Σ_δ around the positive x-axis.

and $T(z_2)$ from (9.22)

$$\begin{aligned} T(z_1)T(z_2) &= \frac{i^2}{(2\pi)^2}\oint_{\gamma^1}\oint_{\gamma^2} e^{\mu z_1}e^{\lambda z_2}R_\mu(A)R_\lambda(A)\,d\lambda\,d\mu \\ &= \frac{1}{(2\pi i)^2}\oint_{\gamma^1}\oint_{\gamma^2} e^{\mu z_1}e^{\lambda z_2}\frac{R_\mu(A)-R_\lambda(A)}{\mu-\lambda}\,d\lambda\,d\mu \\ &= \frac{1}{2\pi i}\oint_{\gamma^1} e^{\mu z_1}R_\mu(A)\left(\frac{1}{2\pi i}\oint_{\gamma^2}\frac{e^{\lambda z_2}}{\mu-\lambda}d\lambda\right)d\mu \\ &\quad -\frac{1}{2\pi i}\oint_{\gamma^2} e^{\lambda z_2}R_\lambda(A)\left(\frac{1}{2\pi i}\oint_{\gamma^1}\frac{e^{\mu z_1}}{\mu-\lambda}d\mu\right)d\lambda. \end{aligned}$$

We now extend the paths γ^1 and γ^2 such that they form a closed loop. If we take a $\lambda \in \gamma^2$, then it lies outside of the closed loop γ^1 (see Figure 9.8 A), hence we have

$$\frac{1}{2\pi i}\oint_{\gamma^1}\frac{e^{\mu z_1}}{\mu-\lambda}d\mu = 0, \qquad \text{for all } \lambda \in \gamma^2,$$

since the pole λ of the integrand is not inside the enclosed domain of γ^1.
In the other case, for $\mu \in \gamma^1$ the closed loop γ^2 would go around the pole at μ (see Figure 9.8 B), hence we apply the Cauchy integral formula

$$\frac{1}{2\pi i}\oint_{\gamma^2}\frac{e^{\lambda z_2}}{\mu-\lambda}d\lambda = -e^{\mu z_2}.$$

Using these identities in the formula above we find

$$\begin{aligned}
T(z_1)T(z_2) &= -\frac{1}{2\pi i}\oint_{\gamma^1} e^{\mu z_1}e^{\mu z_2}R_\mu(A)d\mu \\
&= \frac{i}{2\pi}\oint_{\gamma^1} e^{\mu(z_1+z_2)}R_\mu(A)d\mu \\
&= T(z_1+z_2),
\end{aligned}$$

which proves item 3.

4. To show strong continuity, we consider the map $z \mapsto T(z)$ and show continuity at $z=0$. The semigroup property item 3. does the rest. We come back to the path γ_r defined earlier, but now we chose specific radius $r=1$, i.e. the path γ_1. Note that 0 is inside this path, hence we have

$$\frac{1}{2\pi i}\oint_{\gamma_1}\frac{e^{\mu z}}{\mu}d\mu = e^0 = 1.$$

We also note the following identity

$$(\mu I - A)^{-1}Ax = \mu I(\mu I - A)^{-1}x - x$$

and obtain

$$\begin{aligned}
T(z)x - x &= \frac{1}{2\pi i}\oint_{\gamma_1} e^{\mu z}\left((\mu I - A)^{-1} - \frac{1}{\mu}\right)xd\mu \\
&= \frac{1}{2\pi i}\oint_{\gamma_1}\frac{e^{\mu z}}{\mu}(\mu I - A)^{-1}Axd\mu.
\end{aligned}$$

Now

$$\begin{aligned}
\left\|\frac{e^{\mu z}}{\mu}(\mu I - A)^{-1}Ax\right\| &\leq \frac{M_\varepsilon}{|\mu|^2}\max\{e, e^{-|\mu z|\sin\varepsilon}\}\|Ax\| \\
&\to \frac{eM_\varepsilon}{|\mu|^2}\|Ax\|, \text{ for } z \to 0.
\end{aligned}$$

Then, by the Lebesgue dominated convergence theorem (Theorem 2.2.3) we find

$$\lim_{z\to 0, z\in\Sigma_{\delta'}} T(z)x - x = \frac{1}{2\pi i}\oint_{\gamma_1}\underbrace{\lim_{z\to 0, z\in\Sigma_{\delta'}}\frac{e^{\mu z}}{\mu}(\mu I - A)^{-1}Ax}_{\text{uniformly bounded and analytic}}\, d\mu = 0,$$

proving item 4.

■

In the Hille-Yosida Theorem, we defined the semigroup $T(t) = e^{At}$ as the limit of the negative exponential sequence

$$T(t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n},$$

Now, in (9.22), we define a semigroup through a complex path integral. Are these definitions the same, at least for real arguments $T(t)$? The answer is the next theorem.

Theorem 9.8.3 Let $\{T(z)\}$ be the analytical semigroup defined by (9.22). Then the sectorial operator $(A, D(A))$ is its generator.

Proof. Let $\{T(z)\}$ be defined through (9.22) and denote by $(B, D(B))$ the generator of $\{T(t)\}$. We show that $R_\lambda(B) = R_\lambda(A)$ for a very particular $\lambda \in \rho(A)$. Because if $R_\lambda(A) = R_\lambda(B)$, then $(A - \lambda I) = (B - \lambda I)$ and $A = B$.

Set $\lambda = |\omega| + 2$, where ω is the growth rate of $T(t)$, and use the Laplace transform formula for the resolvent

$$R_\lambda(B)x = -\int_0^\infty e^{-\lambda t} T(t) x dt.$$

We again employ the specific path γ_1 with radius $r = 1$ near zero. Then

$$\begin{aligned} -\int_0^{t_0} e^{-\lambda t} T(t) x dt &= \frac{-i}{2\pi} \int_0^{t_0} \oint_{\gamma_1} e^{-\lambda t} e^{\mu t} R_\mu(A) x \, d\mu \, dt \\ &= \frac{1}{2\pi i} \oint_{\gamma_1} \frac{e^{t_0(\mu - \lambda)} - 1}{\mu - \lambda} R_\mu(A) x d\mu. \end{aligned}$$

We close γ_1 as shown in Figure 9.9. Now, γ_1 is clockwise, hence we get a minus sign. Also, γ_1 encloses the point $\lambda = |\omega| + 2$. Hence we get

$$-\int_0^{t_0} e^{-\lambda t} T(t) x dt = R_\lambda(A)x + \frac{1}{2\pi i} \oint_{\gamma_1} \frac{e^{t_0(\mu - \lambda)}}{\mu - \lambda} R_\mu(A) d\mu.$$

The last term is estimated as

$$\left\| \oint_{\gamma_1} \frac{e^{t_0(\mu-\lambda)}}{\mu - \lambda} R_\mu(A) x d\mu \right\| \le e^{-t_0} \|x\| \underbrace{\left| \oint \frac{M_\varepsilon}{|\mu - \lambda||\mu|} d\mu \right|}.$$

Since $\lambda = |\omega| + 2$, we have $\lambda > 2$, and on γ_1 we have $\max\{\mathrm{Re}(\mu)\} = 1$, which implies $\mathrm{Re}(\lambda - \mu) \ge 1$. Hence $e^{t_0(\mu - \lambda)} \le e^{-t_0}$. And since $|\lambda - \mu| > |\mu|$, the under-braced integral is bounded by $\int_1^\infty \frac{M_\varepsilon}{\mu^2} d\mu$, which is bounded. Hence for $t_0 \to \infty$, we get indeed

$$R_\lambda(B)x = R_\lambda(A)x,$$

for all $x \in X$. ■

■ **Example 9.7** Again we consider the heat equation semigroup with homogeneous Dirichlet boundary conditions.

$$A = \frac{d^2}{dx^2}, \qquad D(A) = \{f \in H^2([0, L]) : f(0) = f(L) = 0\}.$$

Using Fourier analysis we know that the spectrum is

$$\sigma(A) = \sigma_p(A) = \left\{ -\frac{n^2\pi^2}{L^2}, n = 1, 2, 3, \cdots \right\}.$$

$(A, D(A))$ is densely defined and sectorial for any angle $0 < \delta < \frac{\pi}{2}$. Hence $T(t) = \exp(t \frac{d^2}{dx^2})$ is a analytic semigroup in a sector Σ_δ around the positive real axis. ■

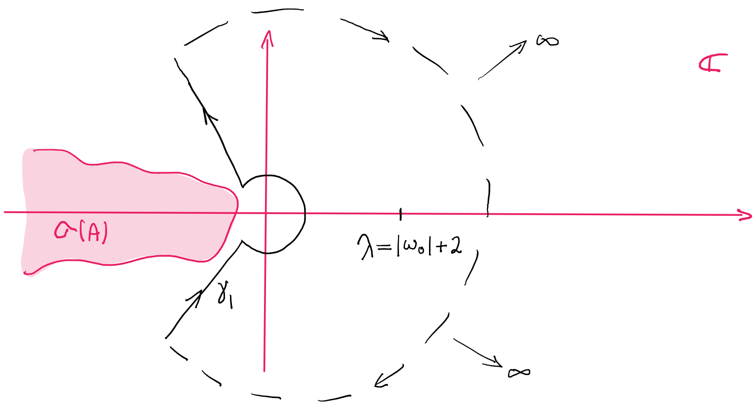


Figure 9.9: The construction of the path γ_1 for the proof of Theorem 9.8.3. The curve γ_1 is closed going in a large circle to the right and letting this circle radius go to ∞.

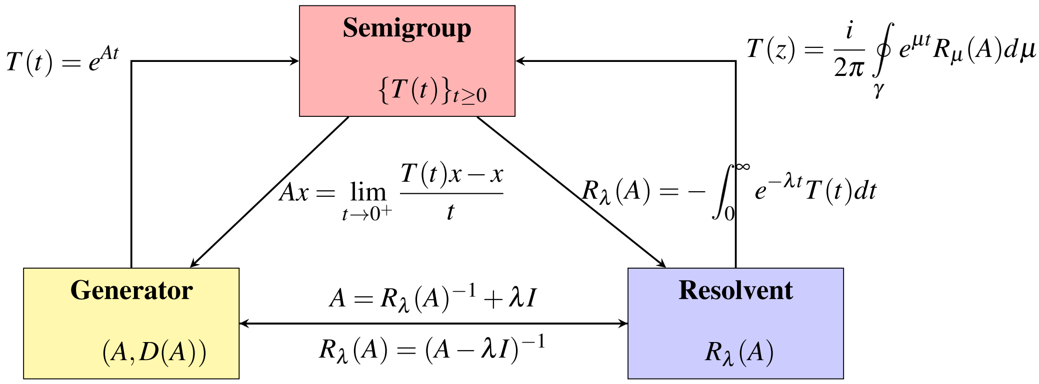


Figure 9.10: Full *Semigroup Triangle*. Note that the relationship from R_λ to T is only valid for analytic semigroups.

And the reverse of Theorem 9.8.3 is true as well, which we cite without proof. A proof, which is quite a bit more involved, can be found in the book of Engel and Nagel [6] Theorem 4.6, p 95.

Theorem 9.8.4 If A generates an analytic semigroup $\{T(z)\}$ then A is sectorial.

R We now complete our *Semigroup Triangle*. In Figure 9.8. We fill in all connections that we identified. Note that the connection between the resolvent R_λ and the semigroup T via the Cauchy integral formula is only valid for analytic semigroups.

9.9 Supplemental Material

Theorem 9.9.1 If A generates an analytic semigroup $\{e^{At}\}$ then there are constants $c > 0$ and ω such that

$$\|Ae^{At}\| \leq c\frac{e^{\omega t}}{t}, \qquad t > 0.$$

Proof. Consider the manipulation

$$\begin{aligned}
(A - \lambda I)R_\lambda(A) &= I \\
AR_\lambda(A) - \lambda R_\lambda(A) &= I \\
AR_\lambda(A) &= I + \lambda R_\lambda(A).
\end{aligned}$$

Then for an analytic semigroup, we have

$$\begin{aligned}
Ae^{At} &= \frac{i}{2\pi}\oint\limits_{\gamma} e^{\lambda t} AR_\lambda(A)d\lambda \\
&= \frac{i}{2\pi}\oint\limits_{\gamma} e^{\lambda t}(I + \lambda R_\lambda(A))d\lambda \\
&= \frac{i}{2\pi}\oint\limits_{\gamma} e^{\lambda t}\lambda(A - \lambda I)^{-1}d\lambda \\
&= \frac{i}{2\pi}\oint\limits_{\gamma'} e^{\lambda'}\frac{\lambda'}{t}\left(A - \frac{\lambda'}{t}I\right)^{-1}\frac{d\lambda'}{t},
\end{aligned}$$

where for $t > 0$ we used the substitution $\gamma' := \gamma t$ and $\lambda' = \lambda t$. This leads to the estimate

$$\begin{aligned}
|Ae^{At}| &\leq \frac{1}{2\pi t}\oint\limits_{\gamma'} |e^{\lambda'}|\left|\frac{\lambda'}{t}\right|\frac{1}{|\omega - \frac{\lambda'}{t}|}d\lambda' \\
&= \frac{1}{2\pi t}\oint\limits_{\gamma'} |e^{\lambda'}|\left|\frac{\lambda'}{t}\right|\frac{\omega - \frac{\lambda'}{t}}{|\omega - \frac{\lambda'}{t}|}\frac{1}{\omega - \frac{\lambda'}{t}}d\lambda' \\
&= \frac{1}{2\pi t}e^{\omega t}\left|\frac{\omega t}{t}\right| \\
&= \frac{\omega}{2\pi}\frac{e^{\omega t}}{t},
\end{aligned}$$

where we used Cauchy's integral formula in the second to last step.

■

9.9.1 Perturbations of Analytic Semigroups

Theorem 9.9.2 — Perturbations with operators dominated by A. Let $(A, D(A))$ be a sectorial operator with growth bound M_ε for $\varepsilon \in (0, \delta)$ as defined in (9.20). A generates an analytic semigroup. Let B be an operator satisfying

1. B is closed and $D(A) \subset D(B)$.
2. There exist constants $a, b > 0$ with $a < (1 + M_\varepsilon)^{-1}$, such that

$$\|(B - bI)x\| \leq a\|Ax\|, \qquad x \in D(A).$$

Then $A + B$ also generates an analytic semigroup.

Proof. Consider

$$\dot{x} = (A + B)x \tag{9.23}$$

and define

$$y(t) = x(t)e^{-bt}.$$

Then

$$\dot{y} = \dot{x}e^{-bt} - bxe^{-bt} = (A + B)xe^{-bt} - bxe^{-bt} = (A + \tilde{B})y,$$

with

$$\tilde{B} = B - bI.$$

This means that $x(t)$ solves (9.23) if and only if y solves $\dot{y} = (A + \tilde{B})y$, and $e^{(A+B)t}$ is an analytic semigroup if and only if $e^{(A+\tilde{B})t}$ is an analytic semigroup. Now, $\|\tilde{B}x\| \leq a\|Ax\|$, hence, without loss of generality, we can assume $b = 0$, and use B instead of $\tilde{B}$.

Since A generates an analytic semigroup, we have

$$\|R_\lambda(A)\| \leq \frac{M_\varepsilon}{|\lambda|}, \qquad \text{for all } \lambda \in \Sigma_{\frac{\pi}{2}+\delta-\varepsilon},$$

where $\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}$ is a sector as defined above, for some small $\varepsilon > 0$.

Then

$$\|BR_\lambda(A)x\| \leq a\|AR_\lambda(A)x\|$$

and since $AR_\lambda(A) = I + \lambda R_\lambda(A)$, we find

$$\|BR_\lambda(A)x\| \leq a\left|1 + \lambda\frac{M_\varepsilon}{|\lambda|}\right|\|x\| \leq a(1 + M_\varepsilon)\|x\|.$$

Hence $BR_\lambda(A)$ is bounded and for $a < (1 + M_\varepsilon)^{-1}$ we have

$$\|BR_\lambda(A)\| < 1.$$

Then $I+BR_\lambda(A)$ is invertible and we write

$$\begin{aligned} A+B-\lambda I &= (I+B((A-\lambda I)^{-1})(A-\lambda I) \\ (I+BR_\lambda(A))^{-1}(A+B-\lambda I) &= A-\lambda I \\ R_\lambda(A)(I+BR_\lambda(A))^{-1} &= R_\lambda(A+B). \end{aligned}$$

This manipulation implies that

$$\rho(A)\subset\rho(A+B),$$

which means that $A+B$ is sectorial as well and we use the same angle δ as for A. Moreover,

$$\|R_\lambda(A+B)\|\leq\|R_\lambda(A)(I+BR_\lambda(A))^{-1}\|\leq\frac{\tilde{M}}{|\lambda|}.$$

Hence $A+B$ is also sectorial and it generates an analytic semigroup defined by the Cauchy integral representation.

■

9.9.2 Regularity of Mild Solutions

We consider the inhomogeneous initial value problem

$$\begin{aligned} \dot{u}(t) &= Au(t)+f(t), \\ u(0) &= u_0, \end{aligned} \tag{9.24}$$

where A is a sectorial generator. We know already that $e^{At}u_0$ is analytic in t for $t>0$. Moreover $e^{At}u_0\in D(A^n)$ for each $n>0$ and

$$\|Ae^{At}u_0\|\leq C\frac{\|u_0\|}{t}.$$

Hence we prove a technical lemma:

Proposition 9.9.3 Let $(A,D(A))$ generate an analytic semigroup in X and assume $f\in C^{0,\theta}([0,T],X)$ for some $0<\theta\leq 1$. Define

$$w(t)=\int_0^t e^{A(t-s)}(f(s)-f(t))ds$$

then $w(t)\in D(A)$ for every $t\in[0,T]$ and $Aw\in C^{0,\theta}([0,T],X)$.

Proof. By Theorem 9.9.1 we chose constants $M,C>0$ such that

$$\|e^{At}\|\leq M,\qquad \|Ae^{At}\|\leq\frac{C}{t}\qquad \text{for all } t\in(0,T].$$

Then

$$\left\|\int_0^t Ae^{A(t-s)}(f(s)-f(t))ds\right\|\leq\int_0^t\frac{C}{t-s}L(t-s)^\theta ds=\frac{CLt^\theta}{\theta},$$

where L is the Hölder constant of f. This implies $w \in D(A)$. To show the Hölder continuity in t we compute

$$
\begin{aligned}
\|Ae^{At} - Ae^{As}\| &= \left\| \int_s^t A^2 e^{A\tau} d\tau \right\| \\
&\leq \int_s^t \|A^2 e^{A\tau}\| d\tau \\
&= \int_s^t \|Ae^{\frac{A\tau}{2}} Ae^{\frac{A\tau}{2}}\| d\tau \\
&\leq \int_s^t \frac{2C}{\tau}\frac{2C}{\tau} d\tau \\
&= 4C^2 \frac{t-s}{st}.
\end{aligned}
$$

Then

$$
\begin{aligned}
Aw(t+h) - Aw(t) &= A\int_0^{t+h} e^{A(t+h-s)}(f(s)-f(t+h))ds - \int_0^t e^{A(t-s)}(f(s)-f(t))ds \\
&= \underbrace{A\int_0^t \left(e^{A(t+h-s)} - e^{A(t-s)}\right)(f(s)-f(t))ds}_{I_1} \\
&\quad \underbrace{+A\int_0^t e^{A(t+h-s)}(f(t)-f(t+h))ds}_{I_2} \\
&\quad \underbrace{+A\int_t^{t+h} e^{A(t+h-s)}(f(s)-f(t+h))ds}_{I_3}.
\end{aligned}
$$

We begin with I_1, where we use the substitution $s = t - h\tau$ in the third step:

$$
\begin{aligned}
\|I_1\| &\leq \int_0^t 4C^2 \frac{(t+h-s)-(t-s)}{(t+h-s)(t-s)} L(t-s)^\theta ds \\
&= 4C^2 Lh \int_0^t (t+h-s)^{-1}(t-s)^{\theta-1} ds \\
&= 4C^2 Lh \int_0^{\frac{t}{h}} \frac{h^{\theta-1}\tau^{\theta-1}}{1+\tau} h d\tau \quad = \quad 4C^2 h^\theta \int_0^{\frac{t}{h}} \frac{\tau^{\theta-1}}{1+\tau} d\tau \\
&\leq 4C^2 h^\theta \frac{\tau^\theta}{\theta}\bigg|_0^{\frac{t}{h}} \quad \leq \quad Ch^\theta.
\end{aligned}
$$

For I_2 we find

$$
\begin{aligned}
\|I_2\| &\leq \left\| A\int_0^t e^{A(t+h-s)}(f(t)-f(t+h))ds \right\| \\
&\leq \left\| (e^{A(t+h)} - e^{At})(f(t+h)-f(t)) \right\| \\
&\leq Mh^\theta.
\end{aligned}
$$

Finally I_3 is estimated as

$$\begin{aligned}
\|I_3\| &\leq \int_t^{t+h} \frac{C}{t+h-s} L(t+h-s)^\theta ds \\
&\leq C\int_t^{t+h} (t+h-s)^{\theta-1} ds \quad = \quad \left| \frac{C}{\theta}(t+h-s)^\theta \Big|_t^{t+h} \right| \\
&= \frac{C}{\theta} h^\theta .
\end{aligned}$$

■

Theorem 9.9.4 — Regularity. Let A be a generator of an analytic semigroup and $u(t)$ a mild solution of (9.24) with $f \in C^{0,\theta}([0,T],X)$. Then for each $0 < \delta < T$ we have

1. The mild solution satisfies
$$Au, \dot{u} \in C^{0,\theta}([\delta,T],X).$$
2. If $u_0 \in D(A)$, then
$$Au, \dot{u} \in C^0([0,T],X).$$
3. If in addition $u_0 = 0$ and $f(0) = 0$, then
$$Au, \dot{u} \in C^{0,\theta}([0,T],X).$$

Proof. We write the mild solution as

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}(f(s) - f(t))ds + \int_0^t e^{A(t-s)} f(t)ds.$$

Upon applying A to the mild solution we find

$$Au = \underbrace{Ae^{At}u_0}_{C^{0,\theta}} + \underbrace{Aw}_{C^{0,\theta}} + A\int_0^t e^{A(t-s)} f(t)ds.$$

For the last term we have from the Fundamental Theorem for Semigroups 9.2.1 that

$$A\int_0^t e^{A(t-s)} f(t)ds = (e^{At} - I)f(t),$$

and we know that $If(t) = f(t) \in C^{0,\theta}$. We still need to check the term $e^{At}f(t)$. For $t \geq \delta, h > 0$ we estimate

$$\begin{aligned}
\|e^{A(t+h)}f(t+h) - e^{At}f(t)\| &\leq \|e^{A(t+h)}\|\|f(t+h) - f(t)\| + \|e^{A(t+h)} - e^{At}\|\|f(t)\| \\
&\leq C_1 h^\theta + C_2 \frac{h}{\delta} \quad \leq \quad C_3 h^\theta .
\end{aligned}$$

This proves that $Au \in C^{0,\theta}([\delta,T],X)$, and since u is a mild solution it also implies $\dot{u} \in C^{0,\theta}([\delta,T],X)$, and we have item 1. In the last inequality, we clearly see that $\delta > 0$ needs to be imposed.

To show continuity at time 0, we consider for $t \to 0$

$$\|e^{At} f(t) - f(0)\| \leq \|e^{At} f(0) - f(0)\| + \|e^{At}\| \|f(t) - f(0)\|.$$

The first term of the right hand side goes to zero since e^{At} is strongly continuous and the second term goes to zero since $f(t)$ is Hölder continuous. Hence we proved item 2.

Finally, for item 3. we estimate

$$\begin{aligned}
\|(e^{A(t+h)} - e^{At}) f(t)\| &= \left\| \int_t^{t+h} A e^{A\tau} f(t) d\tau \right\| \\
&\leq \int_t^{t+h} \|A e^{A\tau} (f(t) - \underbrace{f(0))}_{=0}\| d\tau \\
&\leq c \int_t^{t+h} \frac{1}{\tau} t^\theta d\tau \quad \leq C \int_t^{t+h} \tau^{\theta-1} d\tau \\
&= c((t+h)^\theta - t^\theta) \quad \leq \quad ch^\theta.
\end{aligned}$$

■

The previous result can essentially be visualized as

$$\underbrace{\dot{u}}_{C^{0,\theta}} = \underbrace{Au}_{C^{0,\theta}} + \underbrace{f(t)}_{C^{0,\theta}}.$$

For an analytic semigroup $\dot{u}$ and Au are as good as f. It also means that

$$u \in C^1((0,T], C^{0,\theta}), \qquad u \in C((0,T), C^{2,\theta}).$$

Hence u is twice continuously differentiable in x with Hölder continuous second derivative. This is called *regularity*.

Higher regularity results are possible with analytic semigroups and this is an entire research area in itself. See the excellent book by A. Lunardi [18].

9.10 Semigroup Summary

- A strongly continuous semigroup (or C^0-semigroup) satisfies
$$\begin{aligned} T(t+s) &= T(t)T(s), \qquad s,t \geq 0, \\ T(0) &= I, \\ t &\mapsto T(t) \quad \text{is continuous at 0.} \end{aligned}$$
- If $\{T(t)\}$ is a strongly continuous semigroup, then
$$\|T(t)\| \leq Me^{\omega t}.$$
- The infinitessimal generator A is defined as
$$Ax := \lim_{h \to 0^+} \frac{T(h)x - x}{h}.$$
It satisfies
$$\frac{d}{dt} T(t) = A\,T(t)x, \qquad T(0) = x_0.$$

- The inhomogeneous problem

$$\dot{u} = Au + f(t), \quad u(0) = u_0$$

has *mild solutions* that are given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds.$$

- **Hille-Yosida**: A is generator if and only if
  1. A is closed and $D(A)$ is dense.
  2. For all $\lambda > \omega$ we have a resolvent estimate

$$\|R_\lambda(A)^n\| \le \frac{M}{(\lambda-\omega)^n}.$$

From the proof we found

$$T(t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} = e^{At}$$

and also

$$R_\lambda(A)x = -\int_0^\infty e^{-\lambda t}T(t)xdt.$$

- If $M = 1$ in Hille-Yosida, then we only need

$$\|R_\lambda(A)\| \le \frac{1}{\lambda-\omega} \quad \text{for all } \lambda > \omega.$$

- **Lumer-Phillips:** A generator of a quasi-contraction semigroup

$$\|T(t)\| \le e^{\omega t},$$

if the following conditions are satisfied
  1. A is closed and densely defined
  2. $\mathrm{Re}(x,Ax) \le \omega(x,x)$
  3. There exists a $\lambda_0 > \omega$ in the resolvent set.
- **Perturbations:** A generator, B bounded, then $A+B$ is a generator with

$$\|S(t)\| \le Me^{(\omega+M\|B\|)t}.$$

- **Analytic semigroups:** For sectorial operators we can find a sector $\Sigma_{\frac{\pi}{2}+\delta}$ which splits the spectrum on the left from a large part of the resolvent set on the right.
  - Analytic semigroups have sectorial generators and sectorial generators generate analytic semigroups.
  - The semigroup can be written with a Cauchy integral formula

$$T(z) = \frac{i}{2\pi}\oint_\gamma e^{\mu z}R_\mu(A)d\mu,$$

  thereby completing the *Semigroup Triangle* Figure 9.8.
  - Analytic semigroups satisfy

$$\|Ae^{At}\| \le c\frac{e^{\omega t}}{t}.$$

  - Analytic semigroups regularize.
  - Perturbations with operators that are dominated by A.

9.11 Exercises

Exercise 9.1 (Continuity) (level 1)
Assume $T(t)$ is a C^0-semigroup. Show that $t \mapsto T(t)$ is continuous for all $t > 0$. ■

Exercise 9.2 (Shift) (level 2)
Consider the shift semigroup $T(t)u = u(x+t)$ on

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}), \lim_{x \to \infty} |f(x)| = 0\}.$$

Show that $T(t)$ is a C^0-semigroup. ■

Exercise 9.3 (Generator uniquely defines a semigroup) (level 1)
Suppose $T(t)$ and $S(t)$ have the same generator A. Show that the semigroups are identical. ■

Exercise 9.4 (Dense domain) (level 2)
Let $T(t)$ be a C^0-semigroup with generator A on the Banach space X. Consider $\phi \in C_c^\infty(0,\infty)$.

1. Show that for each $n > 0$: $\int_0^\infty \phi(s)T(s)x\,ds \in D(A^n)$ for all $x \in X$.
2. Use this property to show that $\bigcap_n D(A^n)$ is dense in X. ■

Exercise 9.5 (Shift semigroup) (level 3)
Consider the two operators A and B given as

$$A = \frac{\partial}{\partial x}, \qquad D(A) = \{u \in H^1(0,1); u(1) = 0\}.$$

$$B = \frac{\partial}{\partial x}, \qquad D(B) = \{u \in H^1(0,1); u(0) = 0\}.$$

1. Show that A generates a C^0-semigroup.
2. Show that B does not generate a C^0-semigroup.
3. Can you give an intuitive explanation why this is the case? ■

Exercise 9.6 (Fractional Powers) (level 2)
For an analytic semigroup $T(t)$ with generator A, we can define the negative fractional powers of A for $\alpha > 0$ as

$$(-A)^{-\alpha} := -\frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda I + A)^{-1} d\lambda,$$

where Γ is a curve connecting $e^{-i\theta}\infty$ with $e^{i\theta}\infty$, with 0 on the one side and $\sigma(-A)$ on the other. Use a Cauchy-integral argument to show that within this definition we have

$$(-A)^{-1} = -A^{-1}.$$

■

Exercise 9.7 (Projection semigroup) (level 3)
Given a Hilbert space H with orthonormal basis $\{\psi_i\}_{i=1,2,\ldots}$. For a given $n \in \mathbb{N}$ the projection operator is defined as

$$P_n u = \sum_{i=1}^{n} (u, \psi_i) \psi_i.$$

1. Show that $P_n : H \to H$ is generator of a strongly continuous semigroup of quasi contractions.
2. Find an explicit representation of the semigroup $e^{P_n t}$.

■

Exercise 9.8 (Reaction diffusion equation) (level 3)
Use a fixed-point argument to solve the reaction diffusion equation on a smooth domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} u_t &= \Delta u + f(u) \qquad \text{on} \quad \Omega, \\ u(0,x) &= u_0(x), \\ u(t,x) &= 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{9.25}$$

We assume
(A1) The solution space is

$$X = C^0([0,T], H_0^1(\Omega)),$$

with some $T > 0$ small enough. The norm on H_0^1 will be denoted by the double line notation

$$\|u\| = \|u\|_{H_0^1(\Omega)}.$$

(A2) We know that Δ generates a C^0- semigroup $T(t)$ with norm

$$\|T(t)\| \leq 1, \qquad \text{for all} \qquad t \geq 0.$$

(A3) We assume that f is linearly bounded and Lipschitz continuous, i.e., there exist constants $c_1, c_2 > 0$ such that for each $u, v \in H_0^1$:

$$\begin{aligned} \|f(u)\| &\leq c_1(1 + \|u\|), \\ \|f(u) - f(v)\| &\leq c_2 \|u - v\|. \end{aligned}$$

1. For a given $\phi \in X$ write down the definition of the norm of ϕ in X.
2. Define a mild solution of the above problem (9.25).

3. For each $0 < t < T$ define an operator Q on H_0^1 as

$$v \mapsto Qv := T(t)u_0 + \int_0^t T(t-s)f(v(s,x))ds.$$

4. Denote $m := 2\|u_0\|$ and show that, for t small enough, there is a radius $R > 0$ such that

$$Q : B_R(0) \to B_R(0),$$

where $B_R(0)$ denotes the closed ball of radius R in $H_0^1(\Omega)$.
5. Show that, for t small enough, the map Q is a k-contraction in $B_R(0)$. Find this k.
6. Use a Fixed Point Theorem to show the existence of a unique mild solution of (9.25).

■

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"*Mathematics has contributed to all levels of scientific discovery and technological progress. While immensely useful as a tool in applications, mathematics in itself shows deep structures and uncanny beauty.*"

The textbook on Elements of Applied Functional Analysis lies at the interface of pure and applied mathematics. Abstract methods are motivated through applications in physics, engineering, and biology. The first chapters on basic functional analysis provide a concise introduction to the subject matter. They cover Banach and Hilbert spaces, operators, dual spaces, Hahn-Banach theorems, and spectral theory.

One of the work-horses of applied mathematics are *partial differential equations* (PDEs). The solution theory of PDEs uses Sobolev spaces, which we cover here. In this context we find a rather useful graphical representation of function spaces in the *Rainbow of Function Spaces*. Fixed-point theorems come next. They enable us to find solutions to abstract equations as fixed points of operators. The Calculus of Variations makes extensive use of methods from functional amalysis, and it is a framework to solve general optimization problems. Finally, semigroup theory allows us to understand a PDE as an abstract differential equation in a Banach space. The relation between the semigroup, its generator, and the resolvent is illustrated in the *Semigroup Triangle*.

"*Applied mathematics will always need pure mathematics just as anteaters will always need ants.*" – Paul Halmos

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